Introduction to Real Analysis

Assignment 8, due November 15

Problem 1 of 5. Assume f is continuous on $[-1, 1]$ and differentiable on $(-1, 0) \bigcup (0, 1)$. If $\lim_{x \to 0} f'(x) = L$, show that $f'(0)$ exists and equals L.

Problem 2 of 5.

(1) Let (f_n) be a sequence of continuous functions that converges uniformly to a function f on a compact set A. If $f(x) \neq 0$ on A, show that $(1/f_n)$ converges uniformly on A to $1/f$.

(2) Give an example of a sequence (f_n) of continuous functions that converges uniformly to a function f on a $(0, 1]$, $f(x) \neq 0$ on $(0, 1]$, but $(1/f_n)$ does not converge uniformly on $(0, 1]$ to $1/f$.

(3) Give an example of a sequence (f_n) of continuous functions that converges uniformly to a function f on a $[1, \infty)$, $f(x) \neq 0$ on $[1, \infty)$, but $(1/f_n)$ does not converge uniformly on $[1, \infty)$ to $1/f$.

Problem 3 of 5. Assume the sequence of functions $f_n(x)$ converges to a function $f(x)$ pointwise on a compact set A and assume that for each $x \in A$ the sequence $f_n(x)$ is increasing. Assume that f_n and f are continuous on A.

(1) Set $g_n := f - f_n$ and translate the preceding hypothesis into statements about the sequence (g_n).

(2) Let $\varepsilon > 0$ be arbitrary, and define
$$K_n := \{x \in A \,:\, g_n(x) \geq \varepsilon\}.$$
show that $K_{n+1} \subset K_n$.

(3) Show that each K_n is compact.

(4) Use the pointwise convergence of the sequence (g_n) to show that $\bigcap_{n=1}^{\infty} K_n = \emptyset$.

(5) Conclude that for some N, $K_N = \emptyset$.

(6) Derive that for $n \geq N$ and for all $x \in A$
$$|f_n(x) - f(x)| < \varepsilon.$$
This proves that (f_n) converges to f uniformly.

Problem 4 of 5. Let f be a continuous function on $\mathbb{R}$ and let (a_n) be a real sequence converging to zero. Let the sequence of functions $(f_n(x))$ be defined by $f_n(x) := f(x + a_n)$.

(1) Show that the sequence of functions $(f_n(x))$ converges to f uniformly on every bounded $A \subset \mathbb{R}$.

(2) Show that if the function f is uniformly continuous on $\mathbb{R}$, then $(f_n(x))$ converges to f uniformly on $\mathbb{R}$.

(3) Show that a function f is not uniformly continuous on $\mathbb{R}$ if and only if for some sequence (a_n), $f_n(x)$ does not converge to $f(x)$ uniformly.
Hint: You can use Theorem 4.4.5.

Problem 5 of 5. Let (f_n) be a sequence of bounded (not necessarily continuous functions) on a [0, 1].

(1) Construct such a sequence (f_n) converging pointwise to an unbounded function f on $[0, 1]$.

(2) Assume that (f_n) converges uniformly to a function f. Show that f is also bounded.