Introduction to Real Analysis

Assignment 9, due November 22

Problem 1 of 5. Let

$$f_n(x) := x \sin\left(\frac{x}{x^2 + 1/n}\right).$$

(1) Show that $(f_n(x))$ converges point-wise on $\mathbb{R}$.

(2) Show that $(f_n'(x))$ converges point-wise on $\mathbb{R}$.

(3) Show that $(f_n'(x))$ does not converge uniformly on any interval $[-a, a]$, $a > 0$.

Problem 2 of 5. Assume that the series $\sum_{n=0}^{\infty} na_n$ converges absolutely.

(1) Prove that the series $\sum_{n=0}^{\infty} a_n \sin(nx)$ converges for every $x \in \mathbb{R}$.

(2) Denote $f(x) := \sum_{n=0}^{\infty} a_n \sin(nx)$. Prove that f is differentiable on $\mathbb{R}$ and find a formula for the derivative.

Problem 3 of 5.

(1) Assume that for a sequence (a_n) the limit

$$R := \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

exists. Prove that R is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$.
Hint: Show that the power series diverges for $|x| > R$ and converges for $|x| < R$.

(2) Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} 3^n (n^3 - 1/n) x^n.$$

Problem 4 of 5. For $|x| < 1$, compute the value

$$f(x) := \sum_{n=2}^{\infty} \frac{x^n}{n(n-1)}.$$

Justify all the steps of your computation.
Hint: What is $f''(x)$?

Problem 5 of 5 ((Uniqueness Theorem for power series)). Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is defined in some interval $(-R, R)$, $R > 0$. Assume that for some sequence $(x_k) \subset (-R, R)$, with $\lim_{k\to\infty} x_k = 0$ and $x_k \neq 0$, we have $f(x_k) = 0$.

(1) Snow that for any $n \geq 0$, $f^{(n)}(0) = 0$. **Hint:** Use Mean Value Theorem and induction on n. This problem is very similar to Problem 5.3.4.

(2) Conclude that $f(x) = 0$ for all $x \in (-R, R)$.