Theorem. *Let* $f : A \mapsto \mathbb{R}$ *and* $c \in A$*. Then* f *is differentiable at* c *if there exists* $\phi : A \mapsto \mathbb{R}$*,* continuous *at* c*, such that for* $x \in A$

$$f(x) = f(c) + (x - c)\phi(x).$$

In this case, $\phi(c) = f'(c)$*.*

Proof. Let f be differentiable at c. Define

$$\phi(x) = \begin{cases} \frac{f(x)-f(c)}{x-c}, & x \neq c \\ f'(c), & x = c \end{cases}$$

Then

$$\lim_{x \to c} \phi(x) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) = \phi(c),$$

so ϕ is continuous at a and

$$f(c) + (x - c)\phi(x) = f(c) + (x - c)\frac{f(x) - f(c)}{x - c} = f(x).$$

On the other hand, if for some ϕ continuous at c,

$$f(x) = f(c) + (x - c)\phi(x),$$

then, for $x \neq c$

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \phi(x) = \phi(c),$$

so f is differentiable at c and $f'(c) = \phi(c)$. □

Corollary. *If* $f : A \mapsto \mathbb{R}$ *is differentiable at* $c \in A$*, it is continuous at* c*.*

Proof. If f is differentiable at c then for some $\phi : A \mapsto \mathbb{R}$, continuous at c, $f(x) = f(c) + (x - c)\phi(x)$. Just f is an algebraic combination of a constant $f(c)$ and two functions continuous at c: $(x - c)$ and $\phi(x)$. Thus it is also continuous. □