Line integrals. Green Theorem. Path independence and the existence of a primitive.

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Integral of a complex-valued function.

$f : [a,b] \to \mathbb{C} = x(t) + iy(t)$

$\int_a^b f(t)\,dt := \int_a^b x(t)\,dt + i\int_a^b y(t)\,dt.$

Linear: $\int_a^b (\alpha f(t) + \beta g(t))\,dt = \alpha\int_a^b f(t)\,dt + \beta\int_a^b g(t)\,dt \quad (\alpha, \beta \in \mathbb{C}).$

Additive: $\int_a^b f(t)\,dt + \int_b^c f(t)\,dt = \int_a^c f(t)\,dt$

Change of variables: $t \mapsto g(t)$ piecewise differentiable, increasing on $[c,d]$, $g(c) = a$, $g(d) = b$: $\int_a^b f(t)\,dt = \int_c^d f(g(s))\,g'(s)\,ds$

Change of orientation: $\int_a^b f(t)\,dt = -\int_b^a f(a+b-t)\,dt$

Lemma. $\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt$

Remark: $f : [a,b] \to \mathbb{R}$: use $-|f| \le f \le |f|$.

Proof. Change of phase trick.

$M := \int_a^b f(t)\,dt$

$|M|^2 = \overline{M}\int_a^b f(t)\,dt = \operatorname{Re}\overline{M}\int_a^b f(t)\,dt = \int_a^b \operatorname{Re}\overline{M} f(t)\,dt \le$

$\int_a^b |\overline{M}|\,|f(t)|\,dt \le |M|\int_a^b |f(t)|\,dt \Rightarrow$

$|M| \le \int_a^b |f(t)|\,dt$ if $M \ne 0$

($M = 0$ - obvious) ∎

Line (contour) integral.

Let γ be a piece-wise smooth curve. f - a function continuous on $\gamma[a,b]$.

$z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, t \in [a,b]$ - parameterization.

Def. (Line integrals).

$$\oint_\gamma f(z)\,dx := \int_a^b f(z(t))\,x'(t)\,dt$$

$$\oint_\gamma f(z)\,dy := \int_a^b f(z(t))\,y'(t)\,dt$$

$$\oint_\gamma f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt = \oint_\gamma f(z)\,dx + i\int_\gamma f(z)\,dy.$$

Properties. 1) Independent of parameterization.

$\gamma : [a,b] \to \mathbb{C}$, $s : [c,d] \to [a,b]$ - increasing, piecewise-differentiable. $s(c) = a$, $s(d) = b$

Then $\oint_\gamma f(z)\,dz = \int_a f(z(s))\,z'(s)\,ds = \int_c f(z(s(t)))\,z'(s)\,s'(t)\,dt = \int_c^d f(z(s(t)))\,(z(s(t)))'\,dt.$

2) Change of orientation: $s(c)=b,\ s(d)=a$

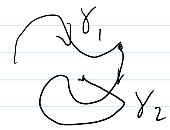
$\gamma:[a,b]\to\mathbb{C}$, $s:[c,d]\to[a,b]$ - decreasing, piecewise-differentiable.

Then $\oint_\gamma f(z)\,dz = \int_a^b f(z(s))\,z'(s)\,ds = \int_d^c f(z(s(t)))\,z'(s)\,s'(t)\,dt = -\int_c^d f(z(s(t)))\,(z(s(t)))'\,dt.$

Notation: γ^- $(-\gamma)$ – γ traversed in the opposite direction.

3) Additivity: $\gamma_1:[a,b]\to\mathbb{C}$, $\gamma_2:[b,c]\to\mathbb{C}$, $\gamma_1(b)=\gamma_2(b)$.

Define: $(\gamma_1+\gamma_2)(t)=\begin{cases}\gamma_1(t), & a\le t\le b\\ \gamma_2(t), & b\le t\le c\end{cases}$ on $[a,c]$.



$\gamma+(-\gamma)\neq 0$

$\oint_{\gamma_1+\gamma_2} f(z)\,dz = \oint_{\gamma_1} f(z)\,dz + \oint_{\gamma_2} f(z)\,dz$ – by direct computation.

4) Linearity $\alpha,\beta\in\mathbb{C}$ $\oint_\gamma(\alpha f(z)+\beta g(z))\,dz = \alpha\oint_\gamma f(z)\,dz + \beta\oint_\gamma g(z)\,dz.$

A very important example.

$\gamma:[0,2\pi]\to\mathbb{C}$ $\quad z(t)=a+re^{it}$.

$n\in\mathbb{Z}$: $\oint_\gamma (z-a)^n\,dz = \int_0^{2\pi} r^n e^{int}\, rie^{it}\,dt = ir^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt =$

$ir^{n+1}\int_0^{2\pi}(\cos(n+1)t + i\sin(n+1)t)\,dt = \begin{cases} ir^{n+1}\frac{1}{n+1}(\sin(n+1)t - i\cos(n+1)t)\Big|_0^{2\pi} = 0, & n\neq -1\\ 2\pi i, & n=-1.\end{cases}$

$$\boxed{\oint_\gamma (z-a)^n\,dz = \begin{cases}2\pi i, & n=-1\\ 0, & n\neq -1.\end{cases}}$$

Heuristically. $(z-a)^n = \left(\frac{(z-a)^{n+1}}{n+1}\right)'$ $\quad n\neq -1$

$(z-a)^{-1} = (\log(z-a))'$

$\int (z-a)^{-1} = 2\pi i - 0$

Length and integral with respect to length

Real notation:	Complex notation.
$\ell(\gamma) := \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$	$\ell(\gamma) := \int_a^b \lvert z'(t)\rvert\,dt$
$\int_\gamma f\,ds := \int_a^b f(x(t),y(t))\sqrt{x'(t)^2+y'(t)^2}\,dt$	$\int_\gamma f(z)\lvert dz\rvert := \int_a^b f(z(t))\lvert z'(t)\rvert\,dt$

Independent of parameterization and orientation.

Lemma $\left|\oint_\gamma f(z)dz\right| \le \int_\gamma |f(z)||dz| \le \sup_{z\in\gamma}|f|\, \ell(\gamma)$.

Proof $\left|\oint f(z)dz\right| = \left|\int_a^b f(z(t))\, z'(t)\, dt\right| \le \int_a^b |f(z(t))||z'(t)|dt \overset{(=)}{}$

$\overset{(=)}{} \int_\gamma |f(z)||dz|$ and $\overset{(\le)}{} \sup_{z\in\gamma}|f| \int_a^b |z'(t)|dt = \sup_{z\in\gamma}|f|\,\ell(\gamma)$ ∎

Let γ be an arc.

Lemma. Let $f_n, f : \gamma \to \mathbb{C}$ - continuous, $f_n \rightrightarrows f$ uniformly on γ.

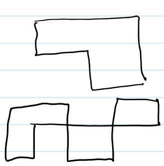
Then $\oint_\gamma f_n(z)dz \to \oint_\gamma f(z)dz$

Proof. $\left|\oint_\gamma f_n(z)dz - \oint_\gamma f(z)dz\right| = \left|\oint_\gamma (f_n(z)-f(z))dz\right| \le \sup_\gamma |f_n(z)-f(z)|\,\ell(\gamma) \to 0$ ∎

Corollary. Let $f_n, f : \gamma \to \mathbb{C}$, continuous, $\sum_{n=1}^\infty f_n = f$ uniformly on γ. Then $\oint_\gamma f(z)dz = \sum_{n=1}^\infty \oint_\gamma f_n(z)dz$

Notations: γ is

1) simple, if $\gamma : [a,b] \to \mathbb{C}$ - injective
2) closed, if $\gamma(a) = \gamma(b)$
3) closed simple, if $\gamma(a) = \gamma(b)$, $\gamma : [a,b) \to \mathbb{C}$ injective
4) polygonal arc, if $\gamma = \gamma_1 + \gamma_2 + \dots + \gamma_n$, where each γ_k is parallel to one of the axes.



(continuous)

Def. Vector field $\begin{pmatrix} p(z) \\ q(z) \end{pmatrix}$ (differential form $p\,dx + q\,dy$) is exact or path-independent in a region D if for any two arcs $\gamma, \gamma' \subset D$, with the same start point and the same endpoint we have $\oint_\gamma p(z)dx + q(z)dy = \oint_{\gamma'} p(z)dx + q(z)dy$

Theorem. TFAE:

1) $\begin{pmatrix} p(z) \\ q(z) \end{pmatrix}$ is exact.
2) $\forall \gamma \subset D$, closed : $\oint_\gamma p\,dx + q\,dy = 0$
3) $\forall \gamma \subset D$, polygonal, closed : $\oint_\gamma p\,dx + q\,dy = 0$
4) $\exists V : D \to \mathbb{C}$, real-differentiable,

$$\frac{\partial U}{\partial x} = p, \quad \frac{\partial U}{\partial y} = q. \quad \left(\nabla U = (p, q); \ dU = p\,dx + q\,dy \right)$$

U is called <u>potential</u> of $\binom{p}{q}$. $\binom{p}{q}$ is called <u>gradient field</u> of U.

<u>Proof</u>. 1) $\Rightarrow$ 2) - consider $\gamma + \gamma$, then

$$\oint_\gamma \underset{\text{indep.}}{\overset{\text{path-}}{=}} \oint_{\gamma+\gamma} \underset{\text{addidivity}}{=} 2\oint_\gamma \Rightarrow \oint_\gamma = 0$$

2) $\Rightarrow$ 1) consider the closed arc $\gamma - \gamma'$.

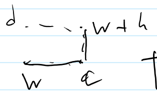
$$\int_{\gamma - \gamma'} = 0 \Rightarrow \int_\gamma - \int_{\gamma'} = 0.$$

2) $\Rightarrow$ 3) - obvious.

3) $\Rightarrow$ 4) Fix $w_0 \in D$. For $w \in D$, define

$U(w) := \oint_\gamma P(z)\,dx + Q(z)\,dy$ for any polygonal path γ from w_0 to w. Does not depend on γ, by 3).

Let $B(w, r) \in D$, take $h = c + id$, $|h| < r$.



Then let $\gamma_h = [w, w+c] \cup [w+c, w+h]$ - polygonal.

If $U(w) = \oint_\gamma p(z)\,dx + q(z)\,dy$ then

$U(w+h) = \oint_{\gamma + \gamma_h} \dots$

$$U(w+h) - U(w) = \int_{\gamma_h} p(z)\,dx + q(z)\,dy = \int_0^c p(w+x)\,dx + \int_0^d q(w+c+iy)\,dy$$

So
$$\left| U(w+h) - U(w) - p(w)c - q(w)d \right| = \left| \int_0^c p(w+x) - p(w)\,dx + \int_0^d q(w+c+iy) - q(w)\,dy \right| \le c \sup_{w' \in B(w,r)} |p(w') - p(w)| + d \sup_{w' \in B(w,r)} |q(w') - q(w)|$$

As $r \to 0$, the bound is $o(|h|)$.

So V is real-differentiable, $\frac{\partial V}{\partial x} = p$, $\frac{\partial V}{\partial y} = q$.

4) $\Rightarrow$ 1)
$$\oint_\gamma p\,dx + q\,dy = \oint_\gamma \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy = \int_a^b \left(\frac{\partial U}{\partial x} x'(t) + \frac{\partial U}{\partial y} y'(t) \right) dt =$$
$U(\gamma(b)) - U(\gamma(a))$ ∎

For complex differential:

$f(z)\,dz = f(z)\,dx + i\,f(z)\,dy$.

It is exact if and only if $\exists F$:

$\frac{\partial F}{\partial x} = f(z)$, $\frac{\partial F}{\partial y} = i f(z)$. I.e. $\frac{\partial F}{\partial z} = \frac{1}{2}\left(f(z) - i \cdot i f(z)\right) = f(z)$

$\frac{\partial F}{\partial \bar z} = \frac{1}{2}\left(f(z) + i \cdot i f(z)\right) = 0$.

So F is complex-differentiable and $F'(z) = f(z)$.

Theorem. $f(z)\,dz$ is exact iff $\exists$ analytic F such that $F' = f$. F is called antiderivative.

$\forall \gamma \subset D$ from z_1 to z_2, $\oint_\gamma f(z)\,dz = F(z_2) - F(z_1)$.

Remark (important) Not all analytic functions are exact!

$f(z) = \frac{1}{z}$ in $D = \mathbb{C} \setminus \{0\}$. $\oint_\gamma \frac{dz}{z} = 2\pi i \neq 0$.

But locally all analytic functions have antiderivative.

Theorem (Local behavior). Continuous vector field is exact in $D = B(z_0, r)$ iff for any rectangle $R \subset B(z_0, r)$,

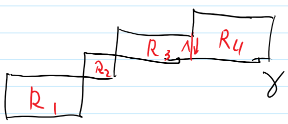
$$\oint_{\partial R} p\,dx + q\,dy = 0$$

Proof. Need to proof only "if" ($\Leftarrow$) part.

Consider polygonal closed curve γ in D. Let $D' \subset D$ be the interior of γ, $\partial D' = \gamma$.

Cut it interior into rectangles $R_1 \cup R_2 \ldots \cup R_n = D'$. ($D'$ is open, but not always a region)

Note that $\oint_\gamma p\,dx + q\,dy = \sum_{j=1}^{n} \oint_{\partial R_j} p\,dx + q\,dy = 0$



The same holds for $D = B(z_0, r) \setminus \{z_1, \ldots, z_n\}$ - finite collection of points.

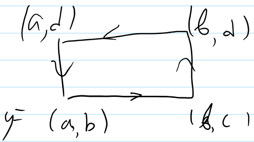
Need to consider any rectangle R: $\partial R \subset D$.

Assume that $f \in \mathcal{A}(B(z_0,r))$ and f' is continuous. Then we can use Green theorem to show that f has antiderivative.

Thm (Green). Let D be a domain, ∂D - piecewise differentiable curve. Orient ∂D counterclockwise. Let $\binom{p}{q}$ - continuous, and both p and q continuously differentiable in $D \cup \partial D$. Then

$$\oint_{\partial D} p\,dx + q\,dy = \iint_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy.$$

For $D = R = [a,b]\times[c,d]$, it is very easy:



$$\int_a^b p(x,c)\,dx - \int_a^b p(x,d)\,dx + \int_c^d q(b,y)\,dy - \int_c^d q(a,y)\,dy =$$

$$\int_c^d \big(q(b,y) - q(a,y)\big)\,dy - \int_a^b \big(p(x,d) - p(x,c)\big)\,dx \underset{FTC}{=} \int_a^b\int_c^d \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy.$$

Complex form: If f is continuously real-differentiable in $D \cup \partial D$, then

$$\oint_{\partial D} f(z)\,dz = \oint_{\partial D} f\,dx + if\,dy = \iint_D \left(\frac{\partial(if)}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy =$$

$$i\iint\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dx\,dy = \boxed{2i\iint \frac{\partial f}{\partial \bar z}\,dx\,dy}$$

Corollary. If $f \in \mathcal{A}(B(z_0,r))$ and f' is continuous, then $\exists F \in \mathcal{A}(B(z_0,r)) : F' = f$.

Proof. We need to prove that for any rectangle $R \subset B(z_0,r)$, $\oint_{\partial R} f(z)\,dz = 0$. But by Green Theorem, $\oint_{\partial R} f(z)\,dz = 2i\iint_R \frac{\partial f}{\partial \bar z}\,dx\,dy = 0$ ∎

Classical Cauchy Theorem:

If $f \in \mathcal{A}(B(z_0,r))$ and f' is continuous, then $\forall \gamma \subset B(z_0,r)$, closed $\oint f(z)\,dz = 0$

If $f \in \mathcal{A}(B(z_0, r))$ and f is continuous,
then $\forall \gamma \subset B(z_0, r)$, closed $\oint_\gamma f(z)\,dz = 0$.