Complex Analysis

Assignment 1, due September 21

Problem 1 of 5. Let A be a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(1) Show that A is *Complex Linear*, i.e.

$$A(\lambda z) = \lambda A(z), \quad \forall \lambda \in \mathbb{C},\ z \in \mathbb{C}$$

if and only if A has a matrix M_w for some complex w. Show that in this case $A(z) = wz$.

(2) Show that A is *Complex Anti-Linear*, i.e.

$$A(\lambda z) = \overline{\lambda} A(z), \quad \forall \lambda \in \mathbb{C},\ z \in \mathbb{C}$$

if and only if A has a matrix $M_w \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for some complex w. Show that in this case $A(z) = w\overline{z}$.

Problem 2 of 5. Let $p > 1$ and $q = \frac{p}{p-1}$.

(1) Let $a \geq 0$ and $b \geq 0$. Show that the rectangle $\{(x, y) : 0 \leq x \leq a;\ 0 \leq y \leq b\}$ is contained in the union of subgraphs

$$\left\{(x, y) : 0 \leq x \leq a;\ 0 \leq y \leq x^{p-1}\right\} \cup \left\{(x, y) : 0 \leq y \leq b;\ 0 \leq x \leq y^{1/(p-1)}\right\}.$$

(2) Use the previous inclusion and integration to prove *Young's inequality*:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where $a \geq 0$ and $b \geq 0$.

(3) Use Young's inequality to show that if $a_j \geq 0$, $b_j \geq 0$ and

$$\sum_{j=1}^{n} a_j^p = \sum_{j=1}^{n} b_j^q = 1,$$

then

$$\sum_{j=1}^{n} a_j b_j \leq 1.$$

(4) Prove *Hölder inequality*:

$$\left| \sum_{j=1}^{n} z_j w_j \right| \leq \left(\sum_{j=1}^{n} |z_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |w_j|^q \right)^{\frac{1}{p}},$$

where z_j and w_j are arbitrary complex numbers. When $p = q = 2$, this is called *Cauchy inequality*.

Hint: Take $a_j = \frac{|z_j|}{\left(\sum_{j=1}^{n} |z_j|^p\right)^{\frac{1}{p}}}$ and $b_j = \frac{|w_j|}{\left(\sum_{j=1}^{n} |w_j|^q\right)^{\frac{1}{q}}}$ and apply the previous part. Be sure to deal with the case when one of the denominators is zero.

Problem 3 of 5. Problem 3 on Page 15.

Problem 4 of 5. Problem 4 on Page 16.

Problem 5 of 5. Problem 5 on Page 20.