Complex Analysis

Assignment 7, due November 23

Problem 1 of 5.

(1) Let f and g be two analytic maps of the unit disk. Assume that g is conformal (analytic and injective), that $f(\mathbb{D}) \subset g(\mathbb{D})$, and that $f(0) = g(0)$. Show that for any $r \leq 1$, $f(r\mathbb{D}) \subset g(r\mathbb{D})$ and that $|f'(0)| \leq |g'(0)|$.
Hint: Consider the function $h := g^{-1} \circ f$.

(2) Let f be an analytic and bounded by 1 function in the unit disk. Show that for all z, $0 < |z| < 1$, we have

$$|f(z) - f(0)| \leq |z| \frac{1 - |f(0)|^2}{1 - |f(0)||z|}.$$

For which f can the equality be attained?
Hint: Use the previous part with the function $g(z) = (z + f(0))/(1 + \overline{f(0)}z)$.

Problem 2 of 5 (A version of Weierstrass convergence theorem.). Let (f_n) be a sequence of functions analytic in a region Ω. Assume that $\sum_{n=1}^{\infty} |f_n(z)|$ converges locally uniformly in Ω. Show that for any k, the series of k-th derivatives, $\sum_{n=1}^{\infty} |f_n^{(k)}(z)|$ also converges locally uniformly in Ω.

Problem 3 of 5. Let f be a function analytic in the closed unit disk. Show that there exists $M > 0$ such that for any $A \in \mathbb{C}$ with $|A| > M$, the equation $f(z) = Az^n$ has exactly n solutions in the unit disk counted with multiplicity.

Problem 4 of 5. Let $p(z) = z^d + a_{d-1}z^{d-1} + \ldots a_0$ be a polynomial, $d \geq 1$. Show that there exists c such that $|c| = 1$ and $|p(c)| \geq 1$.

Problem 5 of 5.

(1) A region Ω is called *star-shaped* if there exists a point $z_0 \in \Omega$ such that for any other point $z \in \Omega$, the segment $[z_0, z]$ is a subset of Ω. Prove that Ω is simply connected.

(2) A region Ω is called *almost convex* if if for any point $z \notin \Omega$ one can find a ray $l \subset \Omega^c$ such that $z \in l$. Prove that Ω is simply connected.