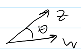
Def. $T: \mathbb{C} \to \mathbb{C}$ $\mathbb{R}$-linear invertible map

angle-preserving if $\forall w, z \in \mathbb{C}$ $|w||z|\langle Tw, Tz\rangle = |Tw||Tz|\langle w,z\rangle$

$\langle w, z\rangle$ - scalar product $= \operatorname{Re} w \operatorname{Re} z + \operatorname{Im} w \operatorname{Im} z = \operatorname{Re} z\bar{w} = \operatorname{Re} \bar{z} w$

$\left(\dfrac{\langle w, z\rangle}{|w||z|} = \cos\theta \right)$



Examples $Tz = az$, $Tz = a\bar{z}$.

complex linear, complex anti-linear

Lemma (angle-preserving for linear maps)

The following are equivalent for $\mathbb{R}$-linear map $T: \mathbb{C} \to \mathbb{C}$:

1) T is angle preserving
2) $\exists a \in \mathbb{C}\setminus\{0\}$: $Tz = az$ $\forall z \in \mathbb{C}$ $(T = M_a)$ or $Tz = a\bar{z}$ $\forall z \in \mathbb{C}$ $\left(T = M_a \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\right)$
3) $\exists s > 0: \langle Tw, Tz\rangle = s\langle w, z\rangle$ $\forall z, w \in \mathbb{C}$.

$\Leftarrow$? $|Tw| = s|w|$

$\langle T(w+z), T(w+z)\rangle = s^2\langle w+z, w+z\rangle$

$\langle T(w-z), T(w-z)\rangle = s^2\langle w-z, w-z\rangle$

$4\langle Tw, Tz\rangle = 4s^2\langle w,z\rangle$.

$\langle Tw, Tw\rangle = s\langle w, w\rangle$ $\forall w \Rightarrow$

$\langle Tw, Tz\rangle = s\langle w, z\rangle$ $\forall w, z$

$\lim_{h\to 0} \dfrac{|f(z_0+h) - f(z_0)|}{|h|} = \dfrac{|T(h) + o(|h|)|}{|h|} = \lim_{h\to 0} \dfrac{|Th|}{|h|}$

$\dfrac{|Th|}{|h|} = s$ $\quad \alpha h$ $\quad \alpha \in \mathbb{R}\setminus\{0\}$

Proof $1) \Rightarrow 2)$ Let $a = T1$.

Consider $Sz := a^{-1} \cdot Tz$ - angle-preserving.

$S1 = 1$

$\langle Si, S1\rangle = \langle Si, 1\rangle = 0 \Rightarrow$ $Si = ri$ for some $r \in \mathbb{R}$, $r \neq 0$ (invertible $\Rightarrow Si \neq 0$).

$S(1+i) = S1 + Si = 1 + ri$ $\quad S(1-i) = 1 - ri$

$\langle 1+i, 1-i\rangle = 0 \Rightarrow \langle 1+ri, 1-ri\rangle = \operatorname{Re}\left((1+ri)\overline{(1-ri)}\right) = 1 - r^2 = 0 \Rightarrow r = \pm 1$.

$r = 1 \Rightarrow$ $Sz = \operatorname{Re} z\, S1 + \operatorname{Im} z\, Si = z \Rightarrow Tz = aSz = az$

$r = -1 \Rightarrow Sz = \bar{z} \Rightarrow Tz = a\bar{z}$.

$2) \Rightarrow 3)$ $\langle Tz, Tw\rangle = \operatorname{Re} Tz\overline{Tw} = |a|^2 \operatorname{Re} z\bar{w} = |a|^2\langle z, w\rangle$

$3) \Rightarrow 1)$ $|Tz| = \sqrt{s}|z|$, $|Tw| = \sqrt{s}|w|$. Plug in ∎

Conformal maps.

Piecewise smooth arcs.

Real notation $\qquad$ Complex notation.

$\begin{pmatrix} x(t) \\ y(t)\end{pmatrix}$ $t \in [a, b]$ $\qquad z(t)$

Tangent:

$\begin{pmatrix} x'(t) \\ y'(t)\end{pmatrix} \neq \vec{0}$ $\qquad z'(t) \neq 0$.

Piecewise smooth: $S \subset [a, b]$ - finite

$\forall t \in [a, b]\setminus S$, $z'(t) \neq 0$.

f - real differentiable map at z_0, $z(t) = z_0$, $z'(t) \neq 0$.

T - differential of f at z_0 $\left(\dfrac{|f(z_0+h) - f(z_0) - T(z)h|}{|h|} \to 0\right)$

Then $(f(z(t)))' = T(z'(t))$ - chain rule.

Tangent map or differential.

Def. f is called angle-preserving at z_0 if its tangent map at z_0 is angle-preserving.

Lemma. The following are equivalent:

1) f is angle-preserving at z_0.

2) Either $\frac{\partial f}{\partial \bar z} = 0$ $\frac{\partial f}{\partial z} \neq 0$ or $\frac{\partial f}{\partial \bar z} \neq 0$, $\frac{\partial f}{\partial z} = 0$

3) T satisfies $\langle Tw, Tz \rangle = s \langle w, z \rangle$ for some $s > 0$.

Proof. This is just a restatement of our previous Lemma ∎

Theorem. Let f be continuously real differentiable function in a region D. Then f is angle-preserving $\forall z \in D$ if and only if $f \in A(D)$, $f'(z) \neq 0$ $\forall z \in D$ or $\bar f \in A(D)$, $(\bar f)'(z) \neq 0$ $\forall z \in D$.

Proof. By previous Lemma, know "if" part.

Also know: $\forall z \in D$ $\frac{\partial f}{\partial z} \neq 0, \frac{\partial f}{\partial \bar z} = 0$; or $\frac{\partial f}{\partial z} = 0, \frac{\partial f}{\partial \bar z} \neq 0$.

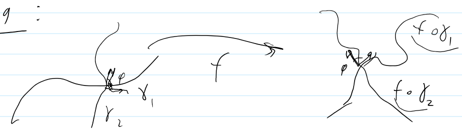
Consider: $\dfrac{\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar z}}{\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar z}}$ - well defined, can be +1 or -1.

By connectivity of D - it is a constant!

If $= 1 \Rightarrow f \in A(D)$

If $= -1 \Rightarrow \bar f \in A(D)$ ∎

Geometric meaning:



If γ_1 and γ_2 intersect at angle φ, then $f \circ \gamma_1$ and $f \circ \gamma_2$ intersect at angle φ.