Complex Analysis

Assignment 2, due October 1

Problem 1 of 5. Let g be a continuous function on a disk $B(z_0, \delta)$, and f be a complex-differentiable function at the point $g(z_0)$, which satisfies the relation $f(g(z)) = z$ for all $z \in B(z_0, \delta)$. Assume that $f'(g(z_0)) \neq 0$. Show that g is complex-differentiable at z_0 and

$$g'(z_0) = \frac{1}{f'(g(z_0))}.$$

Problem 2 of 5. Let f be defined in some disk $B(z_0, \delta)$ and real-differentiable at z_0. Assume that

$$\lim_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right|$$

exists. Show that either f or $\overline{f}$ is complex-differentiable at z_0.

Problem 3 of 5. Let A, B, C be real numbers, at least one of which is non-zero, and let f be analytic in a disk $B(z_0, \delta)$. Show that

$$A \operatorname{Re} f(z) + B \operatorname{Im} f(z) + C = 0 \qquad \text{for all } z \in B(z_0, \delta)$$

if and only if f is identically constant, say $f \equiv c$, where the constant c satisfies $A \operatorname{Re} c + B \operatorname{Im} c + C = 0$.

Problem 4 of 5. Let $\vec{s}$ and $\vec{t}$ be two fixed orthogonal unit vectors in $\mathbb{R}^2$, such that $\vec{t}$ makes a positive (i.e. counterclockwise) angle of $\pi/2$ with $\vec{s}$.

Recall that for a unit vector $\vec{u} = (u_1, u_2)$ and a real-differentiable function f, the *directional derivative* of f in the direction $\vec{u}$ at a point z is

$$\frac{\partial f}{\partial u}(z) := \lim_{h \to 0^+} \frac{f(z + hu) - f(z)}{h}.$$

Let f be a complex-valued, real-differentiable function on a disk $B(z_0, \delta)$. Show that

$$\frac{\partial f}{\partial t}(z) = i \, \frac{\partial f}{\partial s}(z) \qquad \text{for all } z \in B(z_0, \delta)$$

holds if and only if f is analytic in $B(z_0, \delta)$.

Problem 5 of 5. Find all harmonic functions of the form $\varphi\left(\dfrac{x^2 + y^2}{x}\right)$.