What we need to know

1) Sets.

$\emptyset, x \in A$, when two sets are equal, subsets.

Standard sets: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ *Quantors*: $\forall, \exists, \exists!$

Basic set operations: $A \cup B = \{x : x \in A \text{ or } x \in B\}$

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

$A \backslash B = \{x : x \in A \text{ and } x \notin B\}$

Examples: $\{x\colon \forall \varepsilon > 0\ |x - a| < \varepsilon\} = \{a\}$

$\{x\colon \exists b < a\colon x < b\} = \{x\colon x < a\}$

$\{x\colon \forall b > a\colon x < b\} = \{x\colon x \leq a\}$

2) Cartesian products.

$A \times B = \{(a, b)\colon a \in A, b \in B\}$ - set of pairs.

Note: $(a_1, b_1) = (a_2, b_2)$ iff $a_1 = a_2$ and $b_1 = b_2$.

Notation: $A^n \coloneqq A \times A \times \cdots \times A, n$ times

Examples: $\mathbb{R}^n$, $\mathbb{Z}^n$

3) Relations

$R \subset A \times B$ is called a *relation* between elements of A and B

Examples (easy): empty and full relations

4) Functions

Functions are rigorously defined by their graphs.

A relation $R \subset A \times B$ is called a *function* from A to B if

$\forall a \in A\ \exists!\, b \in B : (a, b) \in R$.

Notation: $R\colon A \to B$

Example + notation: a function $R\colon \mathbb{N} \to A$ is called a *sequence*, $a_n \coloneqq R(n)$

Can compose functions: $f\colon A \to B, g\colon B \to C$, then $g \circ f\colon A \to C$

is defined by $(a, c) \in g \circ f$ iff $\exists b \in B\colon (a, b) \in f$ and $(b, c) \in g$.

Special types of functions:

Surjection: each point has a pre image $\forall b \in B\ \exists a \in A : (a, b) \in R$.

Injection: images of different points are different

$(a_1, b) \in R$ and $(a_2, b) \in R \Rightarrow a_1 = a_2$

Bijection or **one-to-one map:** both injection and surjection. Has inverse.

5) Equivalence relations

A relation $R \subset A \times A$ is called an *equivalence relation* if it is *reflexive*($(a, a) \in R$), *symmetric*($(a, b) \in R \Rightarrow (b, a) \in R$), and *transitive*($(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$).

The same as partition of the set A into disjoint subsets. Indeed, each partition defines an equivalence relation: two elements are equivalent iff they belong to the same partition. And vice versa, different equivalency classes are disjoint.

Notation: $a \sim b$. The set of equivalency classes: $A/\sim$.

6) Example of equivalence: Z_p.

Let p$\in \mathbb{N}, p > 1$. $m, n \in \mathbb{Z}$. $m \equiv n (mod\, p)$ iff $m - n$ is divisible by p.

It is an equivalence relation -- check.

Moreover, addition is well-defined on Z_p an equivalency class of the sum depends only on the equivalency class of summands. So Z_p becomes a *commutative group* with respect to addition.

Multiplication is well-defined also, but it is not always a group -- not every element has an inverse. It is a commutative group with respect to multiplication iff *p* is a prime number. Moreover, in this case it is a *field.*

7) Rational numbers.

More interesting example: rational numbers. Start with $\mathbb{Z} \times \mathbb{N}$.

Define an equivalence relation $(p_1, g_1) \sim (p_2, g_2)$ iff $p_1 g_2 = p_2 g_1$.

$\mathbb{Q}$ becomes a field. Need to check, that addition and multiplication are well $-$ defined, and satisfy the axioms of the field.

Review: some extra material

8) **What do we need to know from Calculus**
   Most of the results will be reproved. But I will assume that you know how to work with exponential, logarithmic, trigonometric, and inverse trigonometric functions, take their derivatives.
9) **What do we need to know from Linear Algebra.**
   There will be some review when you need it, but you need to know the definitions of
   - Linear, or vector, space.
   - Linear independence, basis, dimension of vector space.
   - Linear transformations, their matrix representation.
   - Matrix operations: addition, multiplication.
   - Determinant.
   - Solving linear systems by elimination.