Mathematical Introduction to Game Theory

Assignment 1, due September 20

Problem 1 of 5. Find the set of P-positions for the subtraction game with subtraction set $\{1,\ 3,\ 5\}$ and misere winning condition (i.e. the last player to move loses). Justify your answer.

Problem 2 of 5. Misere Empty and Divide. There are two boxes. Initially, one box contains m chips and the other contains n chips. Such a position is denoted by (m, n), where $m > 0$ and $n > 0$. The two players alternate moving. A move consists of emptying one of the boxes, and dividing the contents of the other between the two boxes with at least one chip in each box. There is a unique terminal position, namely $(1, 1)$. Last player to move loses. Show that the only P-positions are of the form

$$(3k - 1, 1),\ (1, 3k - 1),\ \text{or } (3k - 1, 3l - 1),$$

where $k > 0,\ l > 0$ are arbitrary natural numbers.

Problem 3 of 5. A subtraction game is played by the following rules:
If the number of chips in the pile is even, the player can take 1, 2, 9, 11, 24, 38, 41, or 132 chips from the pile.
If the number of chips in the pile is odd, the player can take 1, 10, 23, 37, or 131 chips from the pile.
Show that all starting positions with even number of chips are N-positions.
Hint: Use strategy stealing.

Problem 4 of 5. Find the Nim-sum of all numbers from 1 to $(2^n - 1)$, where $n > 1$ is a natural number. More precisely, compute

$$1 \oplus 2 \oplus \cdots \oplus (2^n - 1).$$

Hint: Show that there are exactly 2^{n-1} numbers with 1 as the k-th digit of binary expansion, for each $k = 1,\ 2, \cdots, n$.

Problem 5 of 5. The Raumschach 3D chess is played on a board that can be thought of as a cube sliced into five equal spaces across each of its three major coordinate planes. This sectioning yields a $5 \times 5 \times 5$ (125-cube) gamespace. The horizontal levels are denoted by capital letters A through E. Ranks and files of a level are denoted using standard chess notation. So, for example, the lowest leftmost cube in the lowest plane is denoted by Aa1, and the topmost rightmost cube at the upper plane is denoted by Ee5.

The *crippled 3D rook* can move along any coordinate direction, but only down, left, or to a level bellow. So, for example, such a rook can move from Bb2 only to Ab2, Ba2, or Bb1.

In the game, two players take turns moving a crippled 3D rook. The player who can not make a move, loses the game.

Which player will win if the starting position is Bc5? If the first player can win, what are the winning moves? Justify your answer.

Hint: This game is just a game of Nim in disguise.