MAT137 - Week 8, lecture 1

- Reminder: Problem Set 3 is due this Thursday, November 1, at 11:59pm.
  - Don't leave the submission process until the last minute!

- In today's lecture we'll talk about implicit differentiation, exponentials and logarithms, and maybe some related rates.

- For tomorrow's lecture, watch videos 4.1 and 4.2.

Explicit functions

An equation like $y = x^2 + \sin(x)$ expresses a relationship between values of x and y.

More specifically, it says that the values of y that satisfy the equation are related to the values of x that satisfy the equation by a function.

(The function is $f(x) = x^2 + \sin(x)$.)

If we want to figure out how y varies when x varies, we can simply differentiate f, in this case getting

$$\frac{dy}{dx} = 2x + \cos(x).$$

Implicit functions

The equation $x^2 + y^2 = 1$ also expresses a relationship between values of x and y.

In this case though, the values of y cannot be expressed as an explicit function of x.

That is, there is no function f such that the equation $y = f(x)$ encapsulates all the information in the earlier equation.

But we still might want to ask how y varies when x varies.

Implicit functions

In the particular case of $x^2 + y^2 = 1$, we know that by splitting into two cases—when y is non-negative or non-positive—the relationship in each case can be expressed by an explicit function:

When $y \geq 0$, we know $y = \sqrt{1 - x^2}$.
When $y \leq 0$, we know $y = -\sqrt{1 - x^2}$.

We can also differentiate both of these functions to find out how y varies when x varies:

When $y \geq 0$, we find $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{1 - x^2}} \left(= \dfrac{-x}{y} \right)$.

When $y \leq 0$, we find $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{1 - x^2}} \left(= \dfrac{-x}{-\sqrt{1 - x^2}} = \dfrac{-x}{y} \right)$.

So, no matter what x is, it turns out that $\dfrac{dy}{dx} = \dfrac{-x}{y}$. This equation *accounts for both cases*.

Implicit differentiation

Instead of splitting up the cases, we could have done all of this at once by *implicitly differentiating* the original equation $x^2 + y^2 = 1$, as you saw in video 3.12.

To do this you differentiate both sides of the equation, and treat y as though it's a function of x.

So for example if you see a y^2, you apply the Chain Rule:

$$\frac{d}{dx}\left(y^2\right) = 2y\,y'.$$

In this case you'd get:

$$2x + 2y\,y' = 0 \implies y' = -\frac{x}{y}.$$

Notice that the RHS of this formula doesn't make sense when $y = 0$. That makes sense, since *y cannot be thought of as a function of x* around those points.

Implicit differentiation

In general, given some complex relationship between x and y, like this...

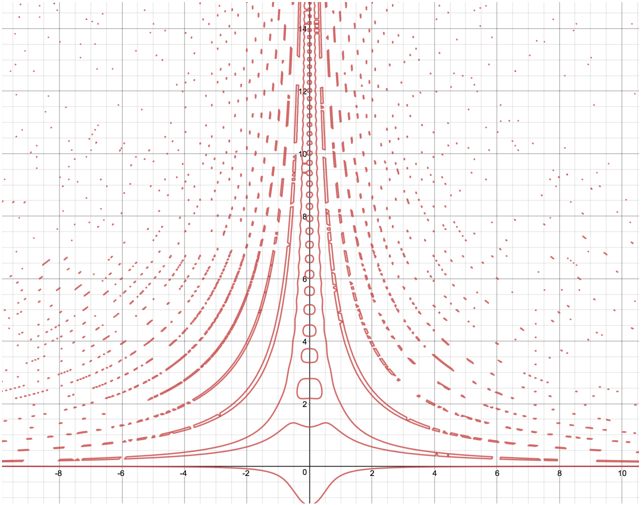
$$\tan^2(xy) = 3x^2y + \cos(y^2)$$

...it's just impossible to split into cases in which you can express y as an explicit function of x, so implicit differentiation is our only tool that always works.

Implicit differentiation

By the way, here's what that curve looks like.

Explore this graph here: graph



Implicit differentiation

Problem 1. The equation

$$\sin(x + y) + xy^2 = 0$$

defines a function $y = h(x)$ near $(0, 0)$.

Using implicit differentiation, compute

1. $h(0)$
2. $h'(0)$
3. $h''(0)$
4. $h''(0)$

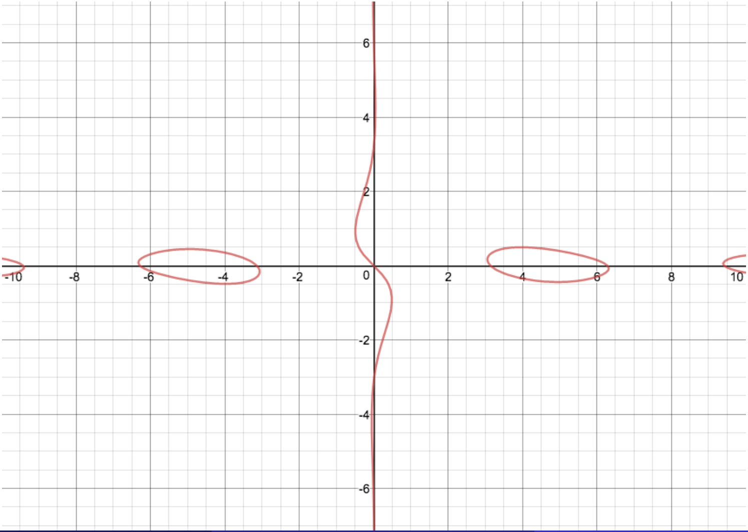
Problem 2. For the horrible curve from earlier:

$$\tan^2(xy) = 3x^2y + \cos(y^2)$$

...try to compute y'.

Implicit differentiation

Here's what the first curve looks like. Explore this graph here: graph



Some quick derivatives with exponentials and logarithms

Problem. Compute the derivatives of the following functions:

1. $f(x) = e^{\sin x + \cos x} \ln(x)$
2. $f(x) = \pi^{\tan x}$
3. $f(x) = \ln\left[e^x + \ln(\ln(\ln(x)))\right]$

Reminder: We know:

- $\dfrac{d}{dx} e^x = e^x$
- $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$
- $\dfrac{d}{dx} a^x = a^x \ln a$

True or false?

Problem. Let $f(x) = (x+1)^x$. Is the following formula true?

$$f'(x) = x \cdot (x+1)^{x-1}.$$

False! This formula is trying to use the power rule for a situation it can't be used for.

The power rule only applies to functions of the form $g(x) = x^{\text{constant}}$.

Logarithmic differentiation to the rescue! If we take log of both sides, we get:

$$\ln(f(x)) = \ln\left((x+1)^x\right) = x \ln(x+1),$$

which we can now differentiate implicitly and isolate for f'.

Logarithmic differentiation.

Problem. Compute the derivatives of the following functions:

1. $f(x) = (x+1)^x$.

2. $g(x) = x^{\tan(x)}$.

3. Now generalize these ideas into a new differentiation rule:

Let f and g be differentiable functions, and define h by

$$h(x) = [f(x)]^{g(x)}.$$

Derive a formula for $h'(x)$.

A very common error...

Problem. Calculate the derivative of

$$f(x) = (\sin x)^{\cos x} + (\cos x)^{\sin x}.$$

What is wrong with this answer?

$$\begin{aligned}
\ln f(x) =& (\cos x)\ln(\sin x) + (\sin x)(\ln \cos x) \\
\frac{d}{dx}[\ln f(x)] =& \frac{d}{dx}[(\cos x)\ln(\sin x)] + \frac{d}{dx}[(\sin x)(\ln \cos x)] \\
\frac{f'(x)}{f(x)} =& -(\sin x)\ln(\sin x) + (\cos x)\frac{\cos x}{\sin x} \\
& + (\cos x)\ln(\cos x) + (\sin x)\frac{-\sin x}{\cos x}
\end{aligned}$$

$$f'(x) = f(x)\left[-(\sin x)\ln(\sin x) + (\cos x)\ln(\cos x) + \frac{\cos^2 x}{\sin x} - \frac{\sin^2 x}{\cos x}\right]$$

A different type of logarithm

Problem. Compute the derivative of

$$f(x) = \log_{x+1}\left(x^2 + 1\right).$$

Hint: If you don't know where to start, remember the definition of the logarithm:

$$\log_a b = c \iff a^c = b.$$

You have now achieved full differentiation power!

With the tools you now know, you can more or less differentiate any function you can right down.

For example, you can compute the derivative of:

$$h(x) = \sqrt[3]{\frac{(\sin^6 x)\sqrt{x^7 + 6x + 2}}{3^x\left(x^{10} + 2x\right)^{10}}}$$

It will be long, but easy. Taking a log of both sides will turn the right side into a long sum, which is easy to differentiate.

Related rates

Idea of these problems: If you know a relationship between two quantities, you can derive a relationship between rates of change of those two quantities.

For example: If you know how the area A of a circle relates to its radius R ($A = \pi R^2$), and you know the area is changing at some rate $\frac{dA}{dt}$, then you can figure out the rate $\frac{dR}{dt}$ at which the radius must be changing.

You can do this by differentiating both sides of our relationship with respect to time:

$$A = \pi R^2 \implies \frac{d}{dt}[A] = \frac{d}{dt}\left[\pi R^2\right] \implies \frac{dA}{dt} = 2\pi R \frac{dR}{dt}$$

Related rates

Here are two classic related rates problem, to start us off.

Problem 1. A 10 foot ladder leans against a wall. The bottom of the ladder starts slipping away at a rate of 0.5 feet per second. How quickly is the top of the ladder dropping when the bottom is 4 feet from the wall?

Problem 2. A spherical balloon is being inflated with 1 cubic metre of air per hour. How quickly is its diameter increasing when it is 2 metres in diameter?