MAT137 - Term 2, week 6, lecture 2

- Today's lecture will be three hours in length.
- Next week is reading week! We did it!
- Today we're talking about:
  - Properties of sequences
  - Theorems about sequences
  - "The Big Theorem"
- Our next lecture is Tuesday, 25 February. For that lecture, please watch videos 12.1 through 12.8.

Let f be a function defined at least on $[1, \infty)$.
We define a sequence by $a_n = f(n)$.

1. IF f is increasing, THEN $\{a_n\}_{n=0}^{\infty}$ is increasing.
2. IF $\{a_n\}_{n=0}^{\infty}$ is increasing, THEN f is increasing.

(If you think one of them is true, try to prove it.
If you think one of them is false, give a counterexample.)

Monotonicty vs. boundedness vs. convergence

For each of the eight "???" boxes, construct an example sequence if possible.
If any of them is impossible, cite a theorem to justify why.

		convergent	divergent
monotonic	bounded	???	???
	unbounded	???	???
not monotonic	bounded	???	???
	unbounded	???	???

Quick review – True or False?

1. If a sequence is convergent, then it is bounded above.
2. If a sequence is convergent, then it is eventually monotonic.
3. If a sequence diverges and is increasing, then there exists $n \in \mathbb{N}$ such that $a_n > 100$.
4. If $\lim_{n\to\infty} a_n = L$, then $a_n < L + 1$ for all n.
5. If a sequence is non-decreasing and non-increasing, then it is convergent.
6. If a sequence is not decreasing and is not increasing, then it is convergent.
7. If $\lim_{n\to\infty} a_{2n} = L$, then $\lim_{n\to\infty} a_n = L$.

A recursively-defined sequence

Consider the sequence $\{a_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} & a_0 = 1 \\ \forall n \geq 1, & a_{n+1} = \dfrac{a_n + 2}{a_n + 3} \end{cases}$$

Compute a_1, a_2, and a_3.

Is this proof correct?

Let $\{a_n\}_{n=0}^{\infty}$ be the sequence in the previous slide.

Claim:

$\{a_n\}_{n=0}^{\infty}$ converges to $-1 + \sqrt{3}$.

Proof.

Let $L = \lim_{n\to\infty} a_n$.

Starting with the recurrence relation and taking limits of both sides, we get

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left[\frac{a_n + 2}{a_n + 3}\right] \implies L = \frac{L+2}{L+3} \implies L^2 + 2L - 2 = 0$$

Solving the quadratic yields $L = -1 \pm \sqrt{3}$.

Every term of the sequence is postivie, so L cannot be negative. So we conclude that $L = -1 + \sqrt{3}$. □

Another recursively-defined sequence

Consider the sequence $\{b_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} & b_0 = 1 \\ \forall n \geq 1, & b_{n+1} = 1 - b_n \end{cases}$$

1. Using the same technique as in the previous slide, compute the limit of the sequence.
2. **AFTER** you have computed the limit, compute the first five terms of the sequence by hand.
3. What happened?

The first recursive sequence, done correctly.

Consider the sequence $\{a_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} & a_0 = 1 \\ \forall n \geq 1, & a_{n+1} = \dfrac{a_n + 2}{a_n + 3} \end{cases}$$

1. Prove $\{a_n\}_{n=0}^{\infty}$ is bounded below by 0.
2. Prove $\{a_n\}_{n=0}^{\infty}$ is decreasing (use induction).
3. Prove $\{a_n\}_{n=0}^{\infty}$ is convergent (use a theorem).
4. Now the calculation in the earlier slide is correct and justified.

Some notation before The Big Theorem

Definition

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of *positive* numbers. We say that a_n is much smaller than b_n, or b_n grows much faster than a_n, if

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 0.$$

If b_n grows much faster than a_n, we denote it by writing $a_n \ll b_n$.

Computer scientists may be familiar with "little-o notation":

$$a_n \ll b_n \quad \iff \quad a_n \in o(b_n).$$

The Big Theorem

For any positive numbers a, b, and any real number $c > 1$,

$$\log_b(n) \ll n^a \ll c^n \ll n! \ll n^n.$$

One application of The Big Theorem is that it greatly simplifies limit calculations.

Compute the following limits.

1. $\lim_{n\to\infty} \dfrac{n! + 2e^n}{3n! + 4e^n}$
2. $\lim_{n\to\infty} \dfrac{2^n + (2n)^2}{2^{n+1} + n^2}$
3. $\lim_{n\to\infty} \dfrac{5n^5 + 5^n + 5n!}{n^n}$
4. $\lim_{n\to\infty} \dfrac{7n^{12}\,\log_{88}(n^2)\,n!}{5(n+1)^\pi\,(3n)^n}$

The Big Theorem – True or False?

Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be positive sequences.

1. IF $a_n \ll b_n$, THEN $\forall m \in \mathbb{N},\ a_m < b_m$.
2. IF $a_n \ll b_n$, THEN $\exists m \in \mathbb{N}$ s.t. $a_m < b_m$.
3. IF $a_n \ll b_n$, THEN $\exists n_0 \in \mathbb{N}$ s.t. $\forall m \in \mathbb{N},\ m \geq n_0 \implies a_m < b_m$.
4. IF $\forall m \in \mathbb{N},\ a_m < b_m$, THEN $a_n \ll b_n$.
5. IF $\exists m \in \mathbb{N}$ s.t. $a_m < b_m$, THEN $a_n \ll b_n$.
6. IF $\exists n_0 \in \mathbb{N}$ s.t. $\forall m \in \mathbb{N},\ m \geq n_0 \implies a_m < b_m$, THEN $a_n \ll b_n$.

Refining the Big Theorem

1. Construct a sequence $\{x_n\}_{n=0}^{\infty}$ such that

$$n^a \ll x_n \ll n^b \quad \text{for all } a < 2 \text{ and } b \geq 2$$

2. Construct a sequence $\{y_n\}_{n=0}^{\infty}$ such that

$$n^a \ll y_n \ll n^b \quad \text{for all } a \leq 2 \text{ and } b > 2$$

3. Construct a sequence $\{z_n\}_{n=0}^{\infty}$ such that

$$n^a \ll z_n \ll c^n \quad \text{for all } a > 0 \text{ and } c > 1$$

(i.e., construct a sequence that grows much faster than all polynomials, and grows much slower than all exponentials)