MAT137 - Term 2, week 12, lecture 1

- This is our last lecture! We did it!
- **Reminder:** Your final exam is on Tuesday! (We know that's awful, but we had no control over it.)
- **Reminder:** Problem Set E is available on the course website. Use it to practise the concepts you haven't had a chance to practise on graded problem sets.
- **Reminder:** There are lots of extra office hours before the exam. See the course website!
- Also make sure you have done all the problems from the last two tutorials. They're good practise for Taylor series problems.
- Today we're still talking about applications of Taylor series.
- Please take a few minutes to fill out your course evaluation! They really do matter.

Problem. We want to compute the value of

$$A = \int_0^1 x^{17} \sin(x)\, dx.$$

There are two ways you can do this:

1. Integrate by parts 17 times to find an antiderivative.
2. Use power series to find an antiderivative.

Use whichever one you think is faster.

Follow-up problem. Estimate the value of A with an error smaller than 0.001. (At least convince yourself of how to do this.)

Limits with power series

Problem 1. Compute these limits by writing out the first few terms of the Maclaurin series of numerator and denominator:

1. $\lim_{x\to 0} \dfrac{\sin x - x}{x^3}$
2. $\lim_{x\to 0} \dfrac{6\sin x - 6x + x^3}{x^5}$
3. $\lim_{x\to 0} \dfrac{x^2 \sin x^2 - x^4}{e^{x^8} - 1}$

Problem 2. Find a value of $a \in \mathbb{R}$ such that the following limit exists and is not 0. Then compute the limit.

$$\lim_{x\to 0} \frac{e^{\sin x} - e^x + ax^3}{x^4}$$

Hint: Recall that earlier we showed that

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

A story – Using Taylor series to estimate π

Taylor series are great for estimating and approximating things, and the number that has famously been estimated the most is π.

To use Taylor series to estimate π, we would like a function...

- ...that is analytic and whose Taylor series we know (or can easily compute), and
- ...that outputs π at some value, or maybe a constant multiple of π.

Can you think of such a function?

$f(x) = \arctan(x)$ is the function for the job!
We know its Taylor series, and we know that $\arctan(1) = \dfrac{\pi}{4}$.

Using Taylor series to estimate π

Exercise:

1. Write down a series whose sum equals $\frac{\pi}{4}$.
2. Note that the series you wrote down is alternating.

   Now use the alternating series estimation technique you learned to figure out how many terms of the series you need to approximate π to an error of less than 0.001.

This series converges to $\frac{\pi}{4}$ *very* slowly.

To get π accurate to 10 decimal places, you need over five *billion* terms.

Using Taylor series to estimate π

Mathematicians have done much better over the years.

In 1706, John Machin proved this boring-looking formula:

$$\frac{\pi}{4} = 4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right).$$

Luckily, he later met Brook Taylor, who told him about his fancy new series.
If you write out the right side of this equation as a series, it converges *much* faster than the one you found earlier.

For example, after 11 terms of the series, the partial sum agrees with π up to 15 decimal places.

Approximating π with series

Machin's formula and several others like it were the primary tools for approximating π for a long time.

In 1910, the now-legendary Indian mathematician Srinivasa Ramanujan proved this (somehow):

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{99^2} \sum_{n=0}^{\infty} \frac{(4n)!\,(1103 - 26390n)}{(n!)^4\, 369^{4n}}.$$

This series yields eight correct decimal digits of π *per term*.

This series is the basis for the current fastest algorithms for appoximating π.

Approximating π with series

As far as I know, this is the best one of these series currently known:

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n \, (6n)! \, (13591409 + 545140134n)}{(3n)! \, (n!)^3 \, (640320)^{3n+3/2}}.$$

This series yields a little more than 14 correct digits of π per term, on average.

We currently know a bit more than the first 22 trillion digits of π.