MAT137 - Week 4, Lecture 3

- **Reminder:** Problem Set 2 is available on the course website, and is due **Thursday, 10 October by 11:59pm**.

- Today's lecture will assume you have watched up to and including video 2.11.

  For next Tuesday's lecture, watch videos 2.12 and 2.13.

Warm-up exercise (we'll use this stuff later)

For all of these questions, let ε and C be fixed positive real numbers.

1. Find **one** value of $\delta > 0$ such that $$|x| < \delta \implies Cx^2 < \varepsilon.$$
2. Find **all** values of $\delta > 0$ such that $$|x| < \delta \implies Cx^2 < \varepsilon.$$
3. Find **one** value of $\delta > 0$ such that $$|x| < \delta \implies |x+1| < 7.$$
4. Find **all** values of $\delta > 0$ such that $$|x| < \delta \implies |x+1| < 7.$$
5. Find **one** value of $\delta > 0$ such that $$|x| < \delta \implies \begin{cases} Cx^2 < \varepsilon \\ |x+1| < 7 \end{cases}.$$

Your first $\varepsilon - \delta$ proof.

Problem. Prove, directly from the formal definition of the limit, that

$$\lim_{x \to 3} (7x - 15) = 6.$$

Follow these steps, in order:

1. Write down the formal definition of what you're trying to prove. Without the definition, you can't prove anything.
2. Write down what the structure of the proof should be, without filling in any details. What variables must you define in what order, what must you assume and where, etc. This part should require little to no thinking.
3. Do some rough work to figure out what value of δ will work.
4. Finally, write the complete proof.

A more complicated $\varepsilon - \delta$ proof.

Problem. Prove, directly from the formal definition of the limit, that

$$\lim_{x \to 0} (x^3 + x^2) = 0.$$

We'll guide you towards writing this proof.

First:

1. Write down the formal definition of what you're trying to prove.
2. Write down what the structure of the proof should be, without filling in any details. What variables must you define in what order, what must you assume and where, etc. This part should require little to no thinking.

A sample proof.

Is the following proof correct? If not, what does it do well, and (more importantly) what does it do wrong?

Claim.

$\forall \varepsilon > 0, \quad \exists \delta > 0 \quad$ such that $\quad 0 < |x| < \delta \implies |x^3 + x^2| < \varepsilon.$

Proof.

Fix $\varepsilon > 0$. Let $\delta = \sqrt{\frac{\varepsilon}{|x+1|}}$.

Let $x \in \mathbb{R}$, and assume $0 < |x| < \delta$. Then we have

$$|x^3 + x^2| = x^2|x+1| < \delta^2|x+1| = \frac{\varepsilon}{|x+1|}|x+1| = \varepsilon.$$

Therefore $|x^3 + x^2| < \varepsilon$, as required. $\square$

Let's think about choosing δ for this proof.

In the warm-up exercise, you found that if ε and C are positive real numbers, then:

$$|x| < \min\left\{\sqrt{\frac{\varepsilon}{C}}, 6\right\} \implies \begin{cases} Cx^2 < \varepsilon \\ |x+1| < 7 \end{cases}$$

Problem. Find **one** value of $\delta > 0$ such that $|x| < \delta \implies |x^3 + x^2| < \varepsilon$.

Back to the proof.

I didn't actually show you this slide, but I told you to do this for next class.

Problem. Prove, directly from the formal definition of the limit, that

$$\lim_{x \to 0} (x^3 + x^2) = 0.$$

Now you have...

- ...written the formal definition for what you have to prove.
- ...written down the structure of the proof.
- ...figured out how to find a δ that works.

So, now, write the complete proof.