MAT137 - Week 5, Lecture 3

- **Reminder:** Problem Set 2 is available on the course website, and is due **Thursday, 10 October by 11:59pm**.

- Today's lecture will assume you have watched up to and including video 2.15.

  (We're still mostly talking about limits. The interesting continuity material begins in the next videos.)

  For next Tuesday's lecture, watch videos 2.16 through 2.18.

Quick warm-up: Did you watch the videos?

Let $a \in \mathbb{R}$ and let f be a function. Assume $f(a)$ is **undefined**.

What can we conclude?

1. $\lim_{x \to a} f(x)$ exist
2. $\lim_{x \to a} f(x)$ doesn't exist.
3. No conclusion. $\lim_{x \to a} f(x)$ may or may not exist.

What else can we conclude?

4. f is continuous at a.
5. f is not continuous at a.
6. No conclusion. f may or may not be continuous at a.

A new theorem about limits

In one of the videos you learned about the Squeeze Theorem. Now you're going to prove a similar (and simpler) theorem.

Theorem

Let $a \in \mathbb{R}$. Let f and g be functions defined at least on an interval centred at a, except possibly at a.

If

- *for x close to a (but not a), $f(x) \geq g(x)$,*
- $\lim_{x \to a} g(x) = \infty$,

then $\lim_{x \to a} f(x) = \infty$

A new theorem about limits

When working on this, do the following things, in this order.

1. Replace the hypothesis in the first bullet point with a more precise mathematical statement (i.e., there should be a quantifier).
2. Write down the precise definition of the second hypothesis.
3. Write down the precise definition of what you have to prove.
4. Write down the structure that your proof must have.
5. Start thinking about the problem (i.e., do some rough work to figure out how to prove it).
6. Write the complete proof.

We didn't see any of the following slides in class, but I'm posting them here because it's a nice little proof to practice with. This isn't homework and we won't go over it next week.

Let's prove a new theorem.

Theorem

Let a be a real number, and let f and g be functions defined everywhere except possibly at a.

Assume:

- $\lim_{x \to a} f(x) = 0.$
- *g is bounded. That means:*

$$\exists M > 0 \text{ such that } \forall x \neq a,\ |g(x)| < M.$$

Then $\lim_{x \to a} [f(x)g(x)] = 0.$

- Write down the formal definition of what you have to prove.
- Before thinking about anything, write down the structure of the proof.
- Do some rough work to figure out how your assumptions relate to what you need to do.
- Then, write the proof.

Some things your proof should do.

- Is the structure of the proof correct?
  (ie. Do you start by fixing an arbitrary ε, then choose a δ that depends only on ε, etc.)
- Did you precisely say what δ is?
- Is your proof self-contained? (ie. Does it reference rough work that isn't written in the proof?)
- Are all of your variables defined? In the right order?
- Does each step follow logically from the previous steps? Have you explained why?
- Do you make sure not to start by assuming the conclusion?

What's wrong with this "proof"?

Proof.

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |f(x)g(x)| < \varepsilon.$$

$$\forall \varepsilon_1 > 0, \exists \delta_1 > 0 \text{ such that } 0 < |x - a| < \delta_1 \implies |f(x)| < \varepsilon_1.$$

$$\exists M > 0 \text{ such that } \forall x \neq a,\ |g(x)| < M.$$

$$|f(x)g(x)| = |f(x)||g(x)| < \varepsilon_1 M.$$

$$\varepsilon = \varepsilon_1 M \implies \varepsilon_1 = \frac{\varepsilon}{M}.$$

$$\text{Therefore } \delta = \delta_1.$$

□

Most of the right ideas are here, but this is not a proof. Make sure your proof doesn't have all of these problems.

What's wrong with this other proof?

- Since g is bounded, $\exists M > 0$ such that $\forall x \neq 0, |g(x)| \leq M$.
- Since $\lim_{x \to a} f(x) = 0$, there exists $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - 0| = |f(x)| < \varepsilon_1 = \frac{\varepsilon}{M}.$$

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$$|f(x)g(x)| = |f(x)||g(x)| \leq |f(x)|M < \varepsilon_1 M = \frac{\varepsilon}{M} M = \varepsilon$$

- In summary, by setting $\delta = \min\{\delta_1\}$, we find that
  if $0 < |x - a| < \delta$ then $|f(x)g(x)| < \varepsilon$.