- **Reminder:** Test 1 will take place **tomorrow**. See the course website for details.
- **Reminder:** Problem Set A is on the website now. It contains material that is not covered by Problem Sets 1 and 2, but that is covered by Test 1. It is not to be submitted, but it is very good practise. Do these problems before Test 1.
- Today's lecture will assume you have watched up to and including video 3.9.

  For tomorrow's lecture, watch video 3.10.

Quick warm-up

Let $a \in \mathbb{R}$. Let f be a function with domain $\mathbb{R}$.
Assume f is differentiable everywhere.

What can we conclude?

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. f is continuous at a.
4. $f'(a)$ exists.
5. $\lim_{x \to a} f'(x)$ exists.
6. f' is continuous at a.

Proving the quotient rule.

Recall the quotient rule from the videos, stated formally here:

Theorem

Let $c \in \mathbb{R}$. Let f and g be functions defined at c and near c, and assume that $g(x) \neq 0$ for all x near c.

Define a function h by $h(x) = \dfrac{f(x)}{g(x)}$.

If f and g are differentiable at c, then h is differentiable at c, and

$$h'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}.$$

First, use the definition of $h'(c)$ to write down the limit you need to prove.

Then write a formal proof for this theorem.

Recall a trick from the product rule.

In order to prove the product rule, we had to compute a similar limit, and to do that we did a simple "trick" of adding zero in a creative way:

$$\begin{aligned}
&\frac{f(x)g(x) - f(c)g(c)}{x - c} \\
=&\frac{f(x)g(x) - {\color{red}f(c)g(x) + f(c)g(x)} - f(c)g(c)}{x - c} \\
=&\frac{f(x) - f(c)}{x - c}\, g(x) + f(c)\frac{g(x) - g(c)}{x - c}
\end{aligned}$$

A similar (but not identical) trick will help you with this proof.

Be careful to explicitly justify any limits you evaluate in your proof.

Check your proof of the quotient rule

1. Did you use the *definition* of the derivative?
2. Are there only equations and no words? If so, you haven't written a proof.
3. Does every step follow logically from the previous steps (with explanation)?
4. Did you assume anything you couldn't assume?
5. Did you assume at any point that a function is differentiatiable? If so, did you justify it?
6. Did you assume at any point that a function is continuous? If so, did you justify it?

If you answered "no" to Q6 above, your proof cannot be fully correct.

Critique this proof

$$
\begin{aligned}
h'(c) &= \lim_{x\to c} \frac{h(x)-h(c)}{x-c} = \lim_{x\to c} \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x-c} \\
&= \lim_{x\to c} \frac{f(x)g(c)-f(c)g(x)}{g(x)g(c)\,(x-c)} \\
&= \lim_{x\to c} \frac{f(x)g(c)-f(c)g(c)+f(c)g(c)-f(c)g(x)}{g(x)g(c)\,(x-c)} \\
&= \lim_{x\to c} \left(\left[\frac{f(x)-f(c)}{x-c} g(c) - f(c) \frac{g(x)-g(c)}{x-c} \right] \frac{1}{g(x)g(c)} \right) \\
&= \left[f'(c)g(c) - f(c)g'(c) \right] \frac{1}{g(c)g(c)}
\end{aligned}
$$

Critique this other proof

$$h'(x) = \lim_{y \to x} \frac{h(y) - h(x)}{y - x} = \lim_{t \to x} \frac{\frac{f(y)}{g(y)} - \frac{f(x)}{g(x)}}{y - x}$$

$= \dots$ (assume the rest of the algebra is here)

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

When $x = c$: $\quad h'(c) = \dfrac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}$