MAT137 - Week 8, Lecture 1

- **Reminder:** Problem Set 3 is available on the course website, and is due **Thursday, October 31 by 11:59pm**.
  - Don't leave the submission process until the last minute.

- Today's lecture will assume you have watched up to and including video 3.12.

  For Thursday's lecture, watch videos 3.13 through 3.18.

  (This looks like a lot of videos, but they're short and are probably high school review for many of you.)

Derivative of cos

In video 3.11, you saw a derivation of a formula for the derivative of the sin function.

Problem. Let $g(x) = \cos x$.

Derive a formula for its derivative directly from the definition of the derivative as a limit.

Hint: Use the "$h \to 0$" version of the definition of the derivative, and imitate the derivation in Video 3.11.

Hint: This identity may come in handy:

$$\forall a, b \in \mathbb{R}, \quad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

Trigonometric derivatives

You now know that:

$$\frac{d}{dx}\sin x = \cos x \quad \text{and} \quad \frac{d}{dx}\cos x = -\sin x$$

Problem. Evaluate the following limits:

1. $\displaystyle\lim_{h\to 0}\frac{\cos(7(x+h)) - \cos(7x)}{h}$.
2. $\displaystyle\lim_{h\to 0}\frac{\cos(7x+h) - \cos(7x)}{h}$.

Hint: They are not equal.

Trigonometric derivatives

Again, you now know that:

$$\frac{d}{dx}\sin x = \cos x \quad \text{and} \quad \frac{d}{dx}\cos x = -\sin x$$

From these two results, quickly derive the formulas for the derivatives of...

1. $\tan(x)$
2. $\sec(x)$
3. $\csc(x)$
4. $\cot(x)$

(This is what I do in my head every time for the latter three functions. I don't like remembering things.)

Explicit functions

An equation like $y = x^2 + \sin x$ expresses a relationship between values of x and y.

More specifically, it says that the values of y that satisfy the equation are related to the values of x that satisfy the equation by a function.

(The function is $f(x) = x^2 + \sin x$.)

If we want to figure out how y varies when x varies, we can simply differentiate f, in this case getting

$$\frac{dy}{dx} = 2x + \cos x.$$

Implicit functions

The equation $x^2 + y^2 = 1$ also expresses a relationship between values of x and y.

In this case though, the values of y cannot be expressed as an explicit function of x.

That is, there is no function f such that the equation $y = f(x)$ encapsulates the relationship expressed by the equation above.

But we still might want to ask how the values of y that satisfy this relationship vary as x varies.

Implicit functions (part 2)

In the particular case of $x^2 + y^2 = 1$, we know that by splitting into two cases—when y is non-negative or non-positive—the relationship in each case can be expressed by an explicit function:

When $y \geq 0$, we know $y = \sqrt{1 - x^2}$.
When $y \leq 0$, we know $y = -\sqrt{1 - x^2}$.

Implicit functions (part 3)

We can also differentiate both of these functions to find out how y varies when x varies:

When $y \geq 0$, we find $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{1-x^2}} \left(= \dfrac{-x}{y} \right)$.
When $y \leq 0$, we find $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{1-x^2}} \left(= \dfrac{-x}{-\sqrt{1-x^2}} = \dfrac{-x}{y} \right)$.

So, no matter what x is, it turns out that $\dfrac{dy}{dx} = \dfrac{-x}{y}$.

This equation *accounts for both cases*.

Implicit differentiation

Instead of splitting up the cases, we could have done all of this at once by *implicitly differentiating* the original equation $x^2 + y^2 = 1$, as you saw in video 3.12.

To do this you differentiate both sides of the equation, and treat y as though it's a function of x.

So for example if you see a y^2, you apply the Chain Rule:

$$\frac{d}{dx}\left(y^2\right) = 2y\,y'.$$

In our case we get:

$$2x + 2y\,y' = 0 \implies y' = -\frac{x}{y}.$$

Notice that the RHS of this formula doesn't make sense when $y = 0$. That makes sense, since *y cannot be thought of as a function of x* around those points.

Implicit differentiation (part 2)

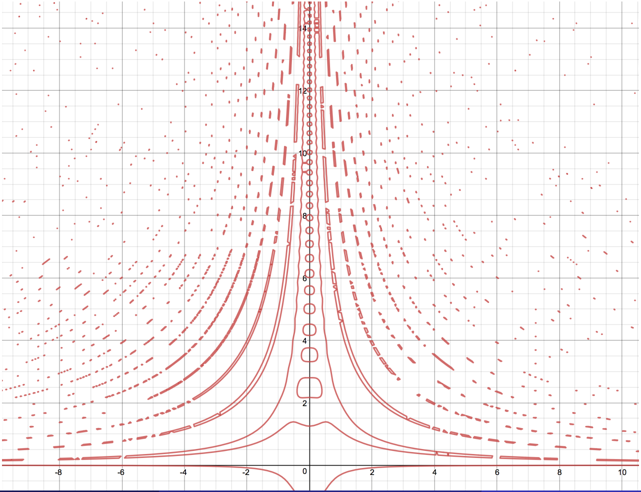
In general, given some complex relationship between x and y, like this...

$$\tan^2(xy) = 3x^2y + \sec(y^2)$$

...it's just impossible to split into cases in which you can express y as an explicit function of x, so implicit differentiation is our only tool that always works.

By the way, here's what that curve looks like. (Desmos doesn't render it very well, sadly.)

Explore this graph here: graph



Implicit differentiation exercises

Problem 1. The equation

$$\sin(x+y) + xy^2 = 0$$

defines a function $y = h(x)$ near $(0,0)$.

Using implicit differentiation, compute

1. $h(0)$
2. $h'(0)$
3. $h''(0)$
4. $h'''(0)$

Problem 2. For the horrible curve from earlier:

$$\tan^2(xy) = 3x^2y + \cos(y^2)$$

...try to compute y'.

Implicit differentiation exercises (part 2)

Here's what the first curve on the last slide looks like.

Explore this graph here: graph