MAT137 - Week 8, Lecture 3

- **Reminder:** Problem Set 3 is available on the course website, and is due **Thursday, October 31 by 11:59pm**.
  - Don't leave the submission process until the last minute.
- Today's lecture will assume you have watched up to the end of playlist 3.

  For next Tuesday's lecture, watch videos 4.1 and 4.2.

Logarithmic differentiation.

Problem 1. Compute the derivatives of the following functions:

1. $f(x) = (x+1)^x$.
2. $g(x) = x^{\tan(x)}$.

Problem 2. Now generalize this technique into a new differentiation rule:

Let f and g be differentiable functions, and define h by

$$h(x) = [f(x)]^{g(x)}.$$

Derive a formula for $h'(x)$.

A very common error...

Problem. Calculate the derivative of

$$f(x) = (\sin x)^{\cos x} + (\cos x)^{\sin x}.$$

What is wrong with this answer?

$$\begin{aligned}
\ln f(x) =& (\cos x) \ln(\sin x) + (\sin x)(\ln \cos x) \\
\frac{d}{dx}\left[\ln f(x)\right] =& \frac{d}{dx}\left[(\cos x)\ln(\sin x)\right] + \frac{d}{dx}\left[(\sin x)(\ln \cos x)\right] \\
\frac{f'(x)}{f(x)} =& -(\sin x)\ln(\sin x) + (\cos x)\frac{\cos x}{\sin x} \\
& + (\cos x)\ln(\cos x) + (\sin x)\frac{-\sin x}{\cos x}
\end{aligned}$$

$$f'(x) = f(x)\left[-(\sin x)\ln(\sin x) + (\cos x)\ln(\cos x) + \frac{\cos^2 x}{\sin x} - \frac{\sin^2 x}{\cos x}\right]$$

A different type of logarithm

Problem. Compute the derivative of

$$f(x) = \log_{x+1}\left(x^2 + 1\right).$$

Hint: If you don't know where to start, remember the definition of the logarithm:

$$\log_a b = c \iff a^c = b.$$

You have now achieved full differentiation power!

With the tools you now know, you can more or less differentiate any function you can right down.

For example, you can compute the derivative of:

$$h(x) = \sqrt[3]{\frac{(\sin^6 x)\sqrt{x^7 + 6x + 2}}{3^x\left(x^{10} + 2x\right)^{10}}}$$

It will be long, but simple. Taking a log of both sides will turn the right side into a long sum, which is easy to differentiate.

Related rates

Idea of these problems: If you know a relationship between two quantities, you can derive a relationship between rates of change of those two quantities.

For example: If you know how the area A of a circle relates to its radius R ($A = \pi R^2$), and you know the area is changing at some rate $\frac{dA}{dt}$, then you can figure out the rate $\frac{dR}{dt}$ at which the radius must be changing.

You can do this by differentiating both sides of our relationship with respect to time:

$$A = \pi R^2 \implies \frac{d}{dt}[A] = \frac{d}{dt}\left[\pi R^2\right] \implies \frac{dA}{dt} = 2\pi R \frac{dR}{dt}$$

Related rates

These are two most classic related rates problem, to start us off. Everyone who learns calculus needs to do these problems once.

Problem 1. A 10 foot ladder leans against a wall. The bottom of the ladder starts slipping away at a rate of 0.5 feet per second. How quickly is the top of the ladder dropping when the bottom is 4 feet from the wall?

Problem 2. A spherical balloon is being inflated with 1 cubic metre of air per hour. How quickly is its diameter increasing when it is 2 metres in diameter?

Math party

The MAT137 TAs are having a party, and they wanted to rent a disco ball.

However, they are poor, and they could only afford a flashlight. At the party, one TA is designated the "human disco ball".

The human disco ball stands in the center of the room pointing the flashlight horizontally and spins around at 3 revolutions per second. (Yes, they really are that fast. Ask your TA to demonstrate this during your next tutorial!)

The room is a square with side length 8 meters. At what speed is the light from the flashlight moving across the wall when it is 2 meters away from a corner?

My favourite related rates problem

We didn't get to see this problem in class, but it's a great practise problem.

Two ants are taking a nap on the hands of a clock.

The first one is resting at the tip of the minute hand, which is 25 cm long (it's a big clock).

The second one is resting at the tip of the hour hand, which is half the length of the minute hand.

At what rate is the distance between the two ants changing at 3:30pm?