MAT137 - Week 11, Lecture 1

- **Reminder:** Problem Set 4 is due **this Thursday, by 11:59pm**.
- **Reminder:** Test 2 is scheduled for next Friday 29 November. Please read the vocabulary list posted on the course website before the test.
- Problem Set B is on the website now. It contains material that is not covered by Problem Sets 1 through 4, but that is covered by Test 2. It is not to be submitted, but it is very good practise. Do these problems before Test 2.
- Today's lecture will assume you have watched up to and including video 5.12.

  For Thursday's lecture, watch videos 6.1 and 6.2.

  For Friday's lecture, watch videos 6.3 through 6.7.

Proving difficult identities

Prove that there is some constant C such that for every $x \geq 0$,

$$\arcsin\left(\frac{1-x}{1+x}\right) + 2\arctan\left(\sqrt{x}\right) = C.$$

Hint: In other words, I'm asking you to prove that the function g defined by

$$g(x) = \arcsin\left(\frac{1-x}{1+x}\right) + 2\arctan\left(\sqrt{x}\right).$$

is constant on $[0, \infty)$.

Is this theorem true?

Theorem

Let $a < b$, and let f be a function defined on $[a, b]$.

IF

- *f is differentiable on (a, b)*
- *$\forall x \in (a, b)$, $f'(x) = 0$,*

THEN f is constant on $[a, b]$.

If you believe it's true, try to explain (or prove) why.

If you believe it's false, come up with a counterexample.

What's wrong with this "proof"?

Proof.

- Define a function g on $[0, \infty)$ by
$$g(x) = \arcsin\left(\frac{1-x}{1+x}\right) + 2\arctan\left(\sqrt{x}\right).$$
- g is differentiable on $(0, \infty)$. For any $x \in (0, \infty)$ we have
$$g'(x) = \text{ ... (derivative computations) ... } = 0.$$
- Therefore, by the MVT, g is constant on $(0, \infty)$.
- We will evaluate g at 0 to find the constant value.
- $g(0) = \text{ ... (computations) ... } = \frac{\pi}{2}$.
- Therefore $g(x) = \frac{\pi}{2}$ for all $x \geq 0$.

□

The goal for today

In video 5.9 you saw the proof that a function with zero derivative is constant (there were some details about endpoints which we explored in the previous question, but that was the idea).

Now you're going to see how the sign of the derivative of a function relates to whether it is increasing or decreasing.

Students commonly mistakenly believe that "the derivative is positive" is the *definition* of a function increasing.

Of course, none of you can make that mistake anymore because you know that the defintion of "increasing" has nothing to do with derivatives.

Warm-up for the proof

Let f be a function defined on an interval D.

1. Write the definition of "f is increasing on D".
2. Write the statement of the Mean Value Theorem. (Don't omit any hypotheses.)

Use the MVT to prove this theorem

Theorem

Let $a < b$, and let f be a differentiable function defined on (a, b).

- *IF* $\forall x \in (a, b), f'(x) > 0$,
- *THEN* f *is increasing on* (a, b).

1. From the statement, write the structure of the proof.
2. If you use a theorem, did you verify the hypotheses?
3. Are there words in your proof, or just equations?

What is wrong with this "proof"?

Theorem

Let $a < b$. Let f be a differentiable function defined on (a, b).

- *IF $\forall x \in (a, b), f'(x) > 0$,*
- *THEN f is increasing on (a, b).*

"Proof".

- From the MVT, we know $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.
- We know $b - a > 0$ and $f'(c) > 0$.
- Therefore $f(b) - f(a) > 0$, and so $f(b) > f(a)$.
- Therefore f is increasing.

This proof is:

- Total nonsense.
- The most common proof students write when we ask this on tests.