- **Reminder:** Problem Set 6 is due on **Monday**, 20 January.
- Problem Set 7 is due on Thursday, 30 January.
- Today's lecture will assume you have watched up to and including video 7.9.

  For Thursday's class, watch videos 7.10 through 7.12.

# Lower and upper sums

Let $f$ be a **decreasing**, bounded function on $[a, b]$.
Let $P = \{x_0, x_1, \ldots, x_N\}$ be some partition of $[a, b]$.

Before you see the question, draw yourself a picture of such a function and partition.

Which of the following expressions equal $L_P(f)$? What about $U_f(P)$? (There may be more than one answer for each.)

1. $\displaystyle\sum_{i=0}^{N} \Delta x_i\, f(x_i)$
2. $\displaystyle\sum_{i=1}^{N} \Delta x_i\, f(x_i)$
3. $\displaystyle\sum_{i=0}^{N-1} \Delta x_i\, f(x_i)$
4. $\displaystyle\sum_{i=1}^{N} \Delta x_i\, f(x_{i+1})$
5. $\displaystyle\sum_{i=1}^{N} \Delta x_i\, f(x_{i-1})$
6. $\displaystyle\sum_{i=0}^{N-1} \Delta x_{i+1}\, f(x_i)$

Recall: $\Delta x_i = x_i - x_{i-1}$.

# Constant functions

I only briefly mentioned this slide in class. Seems like everyone understood the idea. But write it out just in case.

Let's see that these definitions do what we expect in the simplest possible case.

Let $f$ be the constant function 1, defined on $[0, 7]$.

Clearly, by looking at a picture, we see that the area under the graph of $f$ is 7. In other words:

$$\int_0^7 1\,dx = 7.$$

**Problem.** Fix an *arbitrary* partition $P = \{x_0, x_1, \ldots, x_N\}$ of $[0, 7]$, and *explicitly* compute $U_P(f)$ and $L_P(f)$.

# Challenging your intuition about upper and lower integrals

Let $f$ be a bounded function on $[a, b]$.

**Problem.** True or false?

1. There exists a partition $P$ of $[a, b]$ such that

$$\underline{I_a^b}(f) = L_P(f) \quad \text{and} \quad \overline{I_a^b}(f) = U_P(f).$$

2. There exist partitions $P$ and $Q$ of $[a, b]$ such that

$$\underline{I_a^b}(f) = L_P(f) \quad \text{and} \quad \overline{I_a^b}(f) = U_Q(f).$$

# The "$\varepsilon$–characterization" of integrability

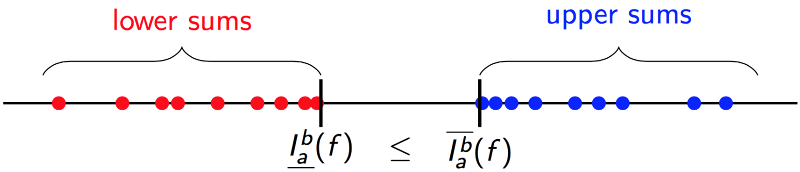
## Theorem

Let $f$ be a bounded function on $[a, b]$. Then:

$f$ is integrable on $[a, b]$

IF AND ONLY IF

$\forall \varepsilon > 0$, $\exists$ a partition $P$ of $[a, b]$ such that $U_P(f) - L_P(f) < \varepsilon$



# "$\varepsilon$–characterization" $\implies$ integrable

Prove this claim.

## Claim

Let $f$ be a bounded function on $[a, b]$.

IF $\forall\varepsilon > 0$, $\exists$ a partition $P$ of $[a, b]$ such that $U_P(f) - L_P(f) < \varepsilon$.
THEN $f$ is integrable on $[a, b]$

1. Recall the definition of "$f$ is integrable on $[a, b]$".
2. For any partition $P$, order the quantities $U_P(f)$, $L_P(f)$, $\overline{I_a^b}(f)$, $\underline{I_a^b}(f)$.
3. For any partition $P$, order the quantities $U_P(f) - L_P(f)$, $\overline{I_a^b}(f) - \underline{I_a^b}(f)$, and 0.

For next lecture we'll come back to this, and talk about the next slide.
Think about them in advance!

# integrable $\implies$ "$\varepsilon$–characterization"

Prove this claim.

**Claim**

Let $f$ be a bounded function on $[a, b]$.

IF $f$ is integrable on $[a, b]$
THEN $\forall \varepsilon > 0$, $\exists$ a partition $P$ of $[a, b]$ such that $U_P(f) - L_P(f) < \varepsilon$.

1. Show that $\forall \varepsilon > 0$, there is a partition $P$ s.t. $U_P(f) < \overline{I_a^b}(f) + \frac{\varepsilon}{2}$.
2. Show that $\forall \varepsilon > 0$, there is a partition $P$ s.t. $L_P(f) > \underline{I_a^b}(f) + \frac{\varepsilon}{2}$.
3. Recall the definition of "$f$ is integrable on $[a, b]$".
4. Assume $f$ is integrable on $[a, b]$. Fix $\varepsilon > 0$.
   Show there are partitions $P_1$ and $P_2$ s.t. $U_{P_1}(f) - L_{P_2}(f) < \varepsilon$.
5. Using $P_1$ and $P_2$ from the previous step, construct a partition $P$ such that $U_P(f) - L_P(f) < \varepsilon$.