MAT137 - Term 2, week 2, lecture 2

- **Reminder:** Problem Set 6 is due on **Monday**, 20 January.
- Problem Set 7 is due on Thursday, 30 January.
- Today's lecture will assume you have watched up to and including video 7.12.

  For tomorrow's lecture, please watch videos 8.1 and 8.2

The "ε–characterization" of integrability

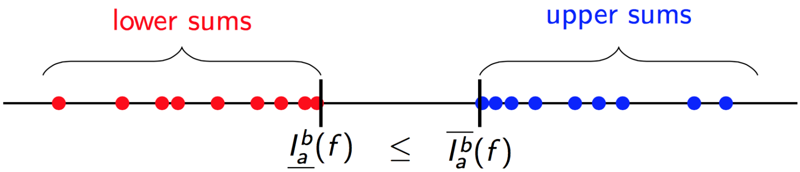
Theorem

Let f be a bounded function on $[a, b]$. Then:

f is integrable on $[a, b]$

IF AND ONLY IF

$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$ such that $U_P(f) - L_P(f) < \varepsilon$



"ε–characterization" $\implies$ integrable

Prove this claim.

Claim

Let f be a bounded function on $[a, b]$.

IF $\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$ such that $U_P(f) - L_P(f) < \varepsilon$.
THEN f is integrable on $[a, b]$

1. Recall the definition of "f is integrable on $[a, b]$".
2. For any partition P, order the quantities $U_P(f)$, $L_P(f)$, $\overline{I_a^b}(f)$, $\underline{I_a^b}(f)$.
3. For any partition P, order the quantities $U_P(f) - L_P(f)$, $\overline{I_a^b}(f) - \underline{I_a^b}(f)$, and 0.

integrable $\implies$ "ε–characterization"

Prove this claim.

Claim

Let f be a bounded function on $[a, b]$.

IF f is integrable on $[a, b]$
THEN $\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$ such that $U_P(f) - L_P(f) < \varepsilon$.

1. Show that $\forall \varepsilon > 0$, there is a partition P s.t. $U_P(f) < \overline{I_a^b}(f) + \frac{\varepsilon}{2}$.
2. Show that $\forall \varepsilon > 0$, there is a partition P s.t. $L_P(f) > \underline{I_a^b}(f) - \frac{\varepsilon}{2}$.
3. Recall the definition of "f is integrable on $[a, b]$".
4. Assume f is integrable on $[a, b]$. Fix $\varepsilon > 0$.
   Show there are partitions P_1 and P_2 s.t. $U_{P_1}(f) - L_{P_2}(f) < \varepsilon$.
5. Using P_1 and P_2 from the previous step, construct a partition P such that $U_P(f) - L_P(f) < \varepsilon$.

Norms of partitions

Consider the interval $[0, 7]$. What are the norms of the following partitions?

1. $P = \{0, 1, 2, 3, 4, 5, 6, 7\}$.
2. $P = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, 3, 4, 5, 6, 7\}$.
3. $P = \{0, 1, 2, 3, 7\}$
4. $P = \{0, 2, 4, 6, 7\}$
5. $P = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, 5, \frac{11}{2}, 6, \frac{13}{2}, 7\}$

Describe a simple sequence of partitions $P_1, P_2, P_3, \ldots$ such that $\|P_n\| \to 0$ as $n \to \infty$.