MAT137 - Term 2, week 2, lecture 3

- **Reminder:** Problem Set 6 is due on **Monday**.
- Problem Set 7 is due on Thursday, 30 January.
- Today's lecture will assume you have watched up to and including video 8.2.

  For next Tuesday's lecture, please watch videos 8.3 and 8.4.

Using Riemann sums

Theorem

Let f be an integrable function defined on an interval $[a, b]$.

Let $P_1, P_2, P_3, \ldots$ be a sequence of partitions of $[a, b]$ such that

$$\lim_{n\to\infty} \|P_n\| = 0.$$

Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} S^*_{P_n}(f).$$

Note that we don't need to specify which x_i^*'s we are using in each sub-interval of each partition. The result is true no matter which points we choose.

Example

Let's compute an integral using Riemann sums.

Consider the function $f(x) = x^2$ defined on $[0, 1]$. Is this function integrable?

Recall: It is a theorem that if f is continuous on $[a, b]$, then it is integrable on $[a, b]$.

For each $n \geq 1$, let P_n be the partition that splits $[0, 1]$ up into n equal sub-intervals. Write down this partition.

Fix an n and consider the partition P_n. For each i, what is Δx_i? What is $\|P_n\|$? What is $\lim_{n\to\infty} \|P_n\|$?

Example (part 2)

The fact that $\lim_{n\to\infty} \|P_n\| = 0$ means we can use this sequence of partitions to compute the integral.

Next we have to compute Riemann sums for these partitions. To do that, we need a point x_i^* in each sub-interval of each partition.

Let's use the right endpoints.

For a given P_n, give an expression for x_i^*, and then an expression for $f(x_i^*)$.

Write down the Riemann sum $S_{P_n}^*(f)$ and factor out any unecessary terms from the sum.

Example (part 3)

We now need to compute:

$$\lim_{n\to\infty} S^*_{P_n}(f) = \lim_{n\to\infty} \left[\frac{1}{n^3} \sum_{i=1}^{n} i^2 \right].$$

Do to this, recall the following formula:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Use this formula to evaluate the the limit and obtain the value of the integral.

Homework exercise: Repeat the same steps, but instead choose the *left* endpoints for the x_i^* rather than the right endpoints. Verify that you get the same answer.

Warm-up questions for FTC

Let f be a continuous function defined on some interval $[a, b]$.

Question 1. The notation $\int f(x)\,dx$ represents...?

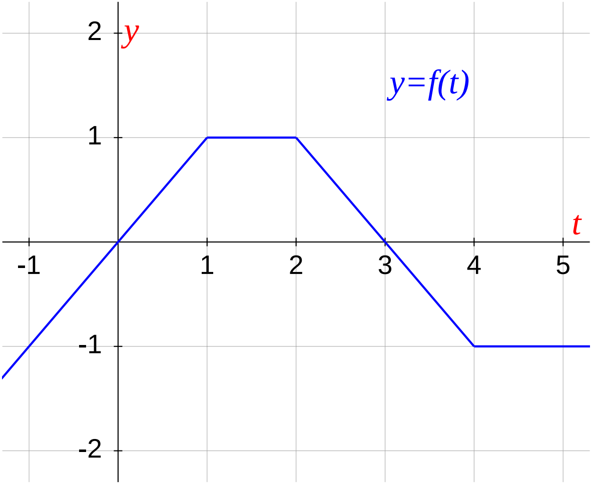
1. A number.
2. A function (and if so, a function of what?).
3. A collection of functions.
4. None of the above.

Question 2. What do these notations represent (same options)...?

$$\int_a^b f(x)\,dx \qquad \text{and} \qquad \int_a^x f(t)\,dt$$

True or False? $\int f(x)\,dx$ and $\int f(\heartsuit)\,d\heartsuit$ mean the same thing.

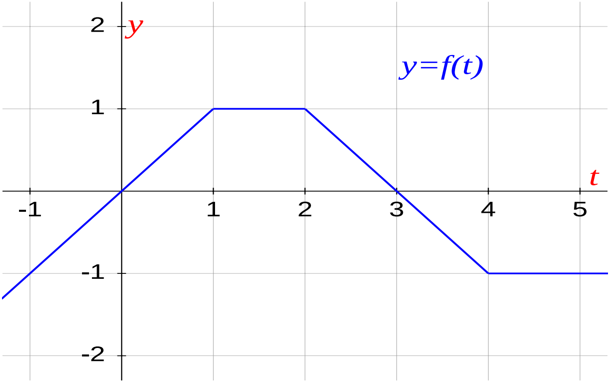
Towards FTC



Compute:

1. $\displaystyle\int_0^1 f(t)\,dt.$
2. $\displaystyle\int_0^2 f(t)\,dt.$
3. $\displaystyle\int_0^3 f(t)\,dt.$
4. $\displaystyle\int_0^4 f(t)\,dt.$
5. $\displaystyle\int_0^5 f(t)\,dt.$

Towards FTC (continued)



Call $F(x) = \int_0^x f(t)\, dt$. This is a new function.

- Sketch the graph of $y = F(x)$.
- Using the graph you just sketched, sketch the graph of $y = F'(x)$.

Defining functions with integrals

Let f be an integrable function defined on an interval $[a, b]$.
Let c and d be elements of $[a, b]$.

Problem 1. Consider the following expressions.

1. $\displaystyle\int_c^x f(x)\,dx$
2. $\displaystyle\int_c^x f(u)\,du$
3. $\displaystyle\int_c^x f(c)\,dt$
4. $\displaystyle\int_c^x f(t)\,dt$

Which of these expressions make sense? Do any of them equal one another?

Problem 2. Define two functions F_c and F_d on $[a, b]$ by:

$$F_c(x) = \int_c^x f(t)\,dt \quad \text{and} \quad F_d(x) = \int_d^x f(t)\,dt.$$

What is the relationship between the values of F_c and F_d?