MAT137 - Term 2, week 7, lecture 2

- **Reminder:** Problem Set 8 is due today.
- Today's lecture will assume you have watched up to and including video 12.8.

  For tomorrow's lecture, watch videos 12.9 and 12.10.

What can you conclude?

This is the last slide we looked at on Tuesday.

Problem 1. Suppose f is continuous on $[1, \infty)$, and that $\lim_{x \to \infty} f(x) = 7$.

What can you conclude about $\int_1^\infty f(x)\,dx$? Try to prove your answer.

Problem 2. Suppose f is continuous on $[1, \infty)$, and suppose you know that $\int_1^\infty f(x)\,dx$ converges.

What can you conclude about $\lim_{x \to \infty} f(x)$?

What can you conclude? - Follow-up

Last class, we agreed that the assumption that $\lim_{x \to \infty} f(x) = 7$ implies that $\int_1^\infty f(x)\,dx$ diverges.

The same reasoning led us to say that in Problem 2, **if** $\int_1^\infty f(x)\,dx$ **converges and** $\lim_{x \to \infty} f(x)$ **exists**, then $\lim_{x \to \infty} f(x) = 0$

However, I said a stronger thing is true: the assumption that $\int_1^\infty f(x)\,dx$ converges alone implies that $\lim_{x \to \infty} f(x) = 0$.

This isn't true. There are examples of functions f such that $\int_1^\infty f(x)\,dx$ converges but $\lim_{x \to \infty} f(x)$ does not exist. Try to think of one! It will be tricky at this point, but easier later in the course.

Comparison tests

Determining whether an improper integral converges or diverges from the definition can be hard (because finding antiderivatives can be hard).

We want to be able to use things we know about simple functions to say things about more complicated functions, like we always do.

We have two powerful tools for doing this. These tools will be *very* useful for us when we talk about series as well.

The first of these is the Basic Comparison Test.

This is how I remember it:

- small hat $\implies$ small head
- big head $\implies$ big hat

However:

- small head $\not\Longrightarrow$ small hat
- big hat $\not\Longrightarrow$ big head

True or False?

Let $a \in \mathbb{R}$, and let f and g be continuous functions defined on $[a, \infty)$.

Assume that $\boxed{\forall x \geq a, \quad 0 \leq f(x) \leq g(x)}$.

What can we conclude?

1. IF $\int_a^\infty f(x)\,dx$ is convergent, THEN $\int_a^\infty g(x)\,dx$ is convergent.
2. IF $\int_a^\infty f(x)\,dx = \infty$, THEN $\int_a^\infty g(x)\,dx = \infty$.
3. IF $\int_a^\infty g(x)\,dx$ is convergent, THEN $\int_a^\infty f(x)\,dx$ is convergent.
4. IF $\int_a^\infty g(x)\,dx = \infty$, THEN $\int_a^\infty f(x)\,dx = \infty$.

True or False? – Part 2

Let $a \in \mathbb{R}$, and let f and g be continuous functions defined on $[a, \infty)$.

Assume that $\boxed{\forall x \geq a, \quad f(x) \leq g(x)}$.

What can we conclude?

1. IF $\int_a^\infty f(x)dx$ is convergent, THEN $\int_a^\infty g(x)dx$ is convergent.
2. IF $\int_a^\infty f(x)dx = \infty$, THEN $\int_a^\infty g(x)dx = \infty$.
3. IF $\int_a^\infty g(x)dx$ is convergent, THEN $\int_a^\infty f(x)dx$ is convergent.
4. IF $\int_a^\infty g(x)dx = \infty$, THEN $\int_a^\infty f(x)dx = \infty$.

True or False? – Part 3

Let $a \in \mathbb{R}$, and let f and g be continuous functions defined on $[a, \infty)$.

Assume that $\boxed{\exists M \geq a \text{ such that } \forall x \geq M, \quad 0 \leq f(x) \leq g(x)}$.

What can we conclude?

1. IF $\displaystyle\int_a^\infty f(x)dx$ is convergent, THEN $\displaystyle\int_a^\infty g(x)dx$ is convergent.
2. IF $\displaystyle\int_a^\infty f(x)dx = \infty$, THEN $\displaystyle\int_a^\infty g(x)dx = \infty$.
3. IF $\displaystyle\int_a^\infty g(x)dx$ is convergent, THEN $\displaystyle\int_a^\infty f(x)dx$ is convergent.
4. IF $\displaystyle\int_a^\infty g(x)dx = \infty$, THEN $\displaystyle\int_a^\infty f(x)dx = \infty$.

What can you conclude?

Let $a \in \mathbb{R}$, and let f be a continuous, positive function defined on $[a, \infty)$.

In each of the following cases, what can you conclude about $\int_a^\infty f(x)\,dx$? Is it convergent, divergent, or we can't conclude anything?

1. $\forall b \geq a$, $\exists M \in \mathbb{R}$ such that $\int_a^b f(x)\,dx \leq M$.
2. $\exists M \in \mathbb{R}$ such that $\forall b \geq a$, $\int_a^b f(x)\,dx \leq M$.
3. $\exists M > 0$ such that $\forall x \geq a$, $f(x) \leq M$.
4. $\exists M > 0$ such that $\forall x \geq a$, $f(x) \geq M$.