MAT137 - Term 2, week 7, lecture 3

- Today's lecture will assume you have watched up to and including video 12.10.

  For next Tuesday's lecture, watch videos 13.1 through 13.4.

The most important improper integrals

Use the definition of improper integral to determine for which values of $p \in \mathbb{R}$ each of the following improper integrals converges.

1. $\displaystyle\int_1^\infty \frac{1}{x^p}\, dx$
2. $\displaystyle\int_0^1 \frac{1}{x^p}\, dx$.
3. $\displaystyle\int_0^\infty \frac{1}{x^p}\, dx$

The Limit Comparison Test

This is the second (and much more important) comparison test you learned:

Theorem (Limit Comparison Test (LCT))

Let $a \in \mathbb{R}$, and let f, g be <u>*positive*</u> *functions that are integrable on $[a, b]$ for every $b > a$.*

Suppose also that $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)}$ exists and equals a <u>*positive*</u> *constant.*

Then:

$$\int_a^\infty f(x)\,dx \text{ converges} \quad \Longleftrightarrow \quad \int_a^\infty g(x)\,dx \text{ converges}.$$

The analogous result is true for integrals of functions that are unbounded at a point.

Extending the Limit Comparison Test

Problem. Let f, g be positive, continuous functions defined on $[1, \infty)$.

Suppose that $\lim_{x \to \infty} \dfrac{f(x)}{g(x)} = 0$.

- First, suppose $\int_1^\infty f(x)\,dx$ converges.
  Can you conclude anything about $\int_1^\infty g(x)\,dx$?
- Next, suppose $\int_1^\infty g(x)\,dx$ converges.
  Can you conclude anything about $\int_1^\infty f(x)\,dx$?

Hint: Try to take what you know about $\lim_{x \to \infty} \dfrac{f(x)}{g(x)}$ and use it to find a relationship between $f(x)$ and $g(x)$.

Homework: Now suppose $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$.
Answer the same questions as above.

Using the Limit Comparison Test

The idea of the LCT is always to try to isolate the "dominant" behaviour of the integrand, and compare it to that.

Determine whether the following improper integrals converge or diverge.

1. $\displaystyle\int_1^\infty \frac{1 - e^{-x}}{x}\,dx$
2. $\displaystyle\int_1^\infty \frac{x^3 + 7x^2 + 3x + 1}{\sqrt{x^9 + 6x^4 + 10}}\,dx$
3. $\displaystyle\int_0^1 \frac{\sin(x)}{x^{3/2}}\,dx$
4. $\displaystyle\int_0^1 \frac{3\cos x}{x + \sqrt{x}}\,dx$