MAT137 - Term 2, week 10, lecture 1

- These slides are representative of what I would have brought to class with me, were in-person not lectures not cancelled. Please feel free to work on these questions, as I think you will find them useful practice.

  It is likely that Asif will cover some of these questions during his online lectures. For those or any other questions from these slides, please feel free to ask about them on Piazza or during online office hours.

- This lecture assumes you have watched up to and including video 13.19.

  For the next lecture, watch videos 14.1 and 14.2.

The Ratio Test

Theorem (Ratio Test)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence with <u>non-zero</u> terms.

Suppose also that $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right|$ exists, and equals a real number L.

Then:

1. *IF $0 \leq L < 1$, THEN the series $\sum a_n$ converges absolutely.*
2. *IF $L > 1$, THEN the series $\sum a_n$ diverges.*
3. *IF $L = 1$, THEN we can't conclude anything about $\sum a_n$.*

Note: Note that $L = 0$ is fine here, in contrast to the LCT. Don't mix them up!

Follow-up question: Is it possible to conclude from the Ratio Test that a series conditionally converges?

Ratio test – Convergent or divergent?

For the following series, try to use the ratio test to determine whether they converge.

If the ratio test is inconclusive, try another test.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!^2\, 3^{n+1}}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$
4. $\displaystyle\sum_{n=2}^{\infty} \frac{n!}{n^n}$
5. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$
6. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

A new test

To prove the Ratio Test, the idea was to "compare" (informally speaking) a series $\sum a_n$ with a geometric series.

For any geometric series $\sum b_n$, where $b_n = r^n$ for some r, we have that

$$\left| \frac{b_{n+1}}{b_n} \right| = \frac{|r|^{n+1}}{|r|^n} = |r|,$$

and that $\sum b_n$ converges if and only if $|r| < 1$. The proof of the ratio test exploits this idea.

Another way to characterize a geometric series is as follows:

$$\text{For every } n \in \mathbb{N}, \quad \sqrt[n]{|b_n|} = \sqrt[n]{|r|^n} = |r|.$$

The Root Test

So we can use the same idea to create a new test.

Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence such that $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L$.

What can you say about $\sum a_n$ when L is greater than, less than, or equal to 1?

(Don't write a formal proof. Just go with what your intuition tells you.)

This is called the "Root Test", and it's actually strictly stronger than the Ratio Test!