- These slides are representative of what I would have brought to class with me, were in-person not lectures not cancelled. Please feel free to work on these questions, as I think you will find them useful practice.

  It is likely that Asif will cover some of these questions during his online lectures. For those or any other questions from these slides, please feel free to ask about them on Piazza or during online office hours.

- This lecture assumes you have watched up to and including video 14.8.

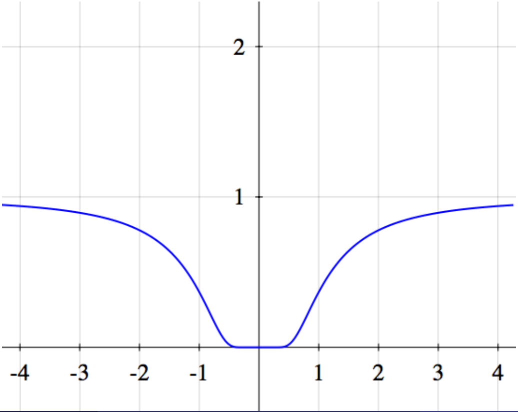
  For the next lecture, watch videos 14.9 and 14.10.

A very pathalogical example

Consider the following function:

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Here's the relevant part of its graph:



A very pathological example (continued)

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

It is a a tedious (but purely computational) exercise to check that:

$$f^{(k)}(0) = 0 \quad \text{for all } k = 0, 1, 2, \ldots$$

You may assume this without proof.

Exercise:

- What is the Taylor series of this function at 0?
- Where does this series converge absolutely?
- For which points x does the function equal its Taylor series?

Lagrange's Remainder Theorem

Reminder of the theorem from the video:

Theorem

Let f be C^{n+1} on an interval I containing a point a.

Then for any $x \in I$, we have:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some number c in between a and x.

This is a consequence of the MVT.
Note that the value of c depends on both n and x.

The easiest way to use this theorem is to put an upper bound M on $\left|f^{(n+1)}(c)\right|$, so you don't have to care about the particular c.

The sine function is analytic

In this exercise you'll prove that $f(x) = \sin(x)$ is analytic on $\mathbb{R}$.

1. (A warm up, just to set a goal.) Fix a real number a, and write down the Taylor series for $\sin(x)$ centred at a.

2. Fix an $x \in \mathbb{R}$ and a non-negative integer n, and use Lagrange's theorem to write down an expression for the remainder $R_n(x)$.
(Remember to quantify your variables.)

3. Find a positive number M such that $|f^{(n+1)}(c)| < M$, no matter what c is.
(Hint: This is easy and you definitely know how to do it already.)

4. Prove that $\lim_{n\to\infty} |R_n(x)| = 0$.

5. Prove that $\lim_{n\to\infty} R_n(x) = 0$.

We conclude that $\sin(x)$ is analytic on all of $\mathbb{R}$!

Let $\alpha \in \mathbb{R}$. Let $f(x) = (1 + x)^{\alpha}$.

1. Find a formula for its derivatives $f^{(n)}(x)$.
2. Write its Taylor series at 0. Call it $S(x)$.
3. What is special about this series when $\alpha \in \mathbb{N}$?
4. Now assume $\alpha \notin \mathbb{N}$. Calculate the radius of convergence of the series $S(x)$.

arcsin

1. Use the Binomial series to write

$$g(x) = \frac{1}{\sqrt{1-x^2}}$$

as a power series centered at 0.

2. Write

$$h(x) = \arcsin x$$

as a power series centere at 0.

Hint: Compute $h'(x)$. Then use what you already know so you do not have to compute any more derivatives.

3. Find a formula for $h^{(n)}(0)$ without computing any more derivatives.