MAT137 - Term 2, week 11, lecture 3

- These slides are representative of what I would have brought to class with me, were in-person not lectures not cancelled. Please feel free to work on these questions, as I think you will find them useful practice.

  It is likely that Asif will cover some of these questions during his online lectures. For those or any other questions from these slides, please feel free to ask about them on Piazza or during online office hours.

- This lecture assumes you have watched all the videos!

  This set of slides covers what would normally be two lectures. All the applications of Taylor series in one place!

Going the other way

This is a repetition of an earlier problem. We're exploring the same ideas here, so remind yourself of it.

Problem. What function is this the Taylor series of?

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n+2)}\, x^{n+2}$$

Hint: First, think about whether this series looks like it has been differentiated or integrated. If so, what series was differentiated or integrated to obtain it?

Follow-up problem. What is the value of the following series?

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n+2)\, 2^n}$$

Compute the exact values of these series

1. $\displaystyle\sum_{n=2}^{\infty} \frac{(-2)^n}{(2n+1)!}$ *Hint:* Think of sin. Notice where the sum starts.

2. $\displaystyle\sum_{n=0}^{\infty} (4n+1)\, x^{4n+2}$ *Hint:* $\dfrac{d}{dx}\left[x^{4n+1}\right] = ???$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{(2n)!}$ *Hint:* Write the first few terms. Combine e^x and e^{-x}.

4. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!(n+1)}$ *Hint:* Compute $\displaystyle\int x^{2n+1}\, dx$.

Problem. We want to compute the value of

$$A = \int_0^1 x^{17} \sin(x)\, dx.$$

There are two ways you can do this:

1. Integrate by parts 17 times to find an antiderivative.
2. Use power series to find an antiderivative.

Use whichever one you think is faster.

Follow-up problem. Estimate the value of A with an error smaller than 0.001.

Limits with power series

Problem 1. Compute these limits by writing out the first few terms of the Maclaurin series of numerator and denominator:

1. $\lim_{x \to 0} \frac{\sin x - x}{x^3}$
2. $\lim_{x \to 0} \frac{6 \sin x - 6x + x^3}{x^5}$
3. $\lim_{x \to 0} \frac{x^2 \sin x^2 - x^4}{e^{x^8} - 1}$

Problem 2. Find a value of $a \in \mathbb{R}$ such that the following limit exists and is not 0. Then compute the limit.

$$\lim_{x \to 0} \frac{e^{\sin x} - e^x + ax^3}{x^4}$$

Hint: Recall that earlier we showed that

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

More applications.

Exercise: Use Taylor series to help prove whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n}\right)$$

This can also be answered with the integral test if you don't want to think about Taylor series, but in mathematics it is your duty to find the laziest way of doing everything. Taylor series are easier, so use those.