MAT327 - Lecture 2

Friday, May 10th, 2019

Definition : Refinements

Let X be a set, let $\mathcal{T}_1$, $\mathcal{T}_2$ be two topologies on X. We say that $\mathcal{T}_1$ is **finer** than $\mathcal{T}_2$ if $\mathcal{T}_1 \supseteq \mathcal{T}_2$. We also say that $\mathcal{T}_2$ is **coarser** than $\mathcal{T}_1$. Most commonly, we'll say that $\mathcal{T}_1$ refines $\mathcal{T}_2$.

The way I remember this is just by remembering that the discrete topology is the finest on a space, and the indiscrete topology is the coarsest. From there, I can figure out on my own that the finer topology is the 'bigger' one, and the coarser topology is the 'smaller' one.

Example :

Recall the topologies on $\mathbb{R}$ we defined last lecture. The following holds:

$$\begin{array}{ccccccc} & & & \supsetneq & \mathbb{R}_{\text{co-finite}} & \supsetneq & \\ \mathbb{R}_{\text{discrete}} & \supsetneq & \mathbb{R}_{\text{usual}} & & & & \mathbb{R}_{\text{indiscrete}} \\ & & & \supsetneq & \mathbb{R}_{\text{ray}} & \supsetneq & \end{array}$$

(You don't wanna know how hard this was to draw.)

Now, if $\mathcal{T}_1 \not\subseteq \mathcal{T}_2$ and $\mathcal{T}_1 \not\supseteq \mathcal{T}_2$, we say that $\mathcal{T}_1$ and $\mathcal{T}_2$ are **incomparable**.

Not just in $\mathbb{R}$, but in general, we always have that $(X, \mathcal{T}_{\text{discrete}})$ is the finest topology on a set X. That is, for any topology $\mathcal{T}$ on X, we have that $\mathcal{T}_{\text{discrete}} \supseteq \mathcal{T}$.

Similarly, if $\mathcal{T}$ is any topology on X, then $\mathcal{T} \supseteq \mathcal{T}_{\text{indiscrete}}$.

Topological Bases

We have seen that topologies can be defined explicitly by specifying their open sets. But this won't always work. Even in our usual topology on $\mathbb{R}^n$, we have only a rough idea of what open sets look like. They look like open blobs that don't contain any of their boundary.

The advantage of a topological basis is that it's usually easier to define one than to define a topology by specifying its open sets. The tradeoff is that bases are harder to work with once they are defined.

As we discussed in the last lecture, there is a sense in which the open intervals (a, b) are the fundamental building blocks of $\mathbb{R}_{\text{usual}}$. We now explore this idea.

Definition : Basis

Let X be a set, and let $\mathcal{B} \subseteq \mathcal{P}(X)$. We say that $\mathcal{B}$ is a **basis** on X if the following holds:

1. $\mathcal{B}$ covers X, (that is, $\bigcup \mathcal{B} = X$)
2. For every $B_1, B_2 \in \mathcal{B}$, for every $x \in B_1 \cap B_2$, there exists a set $B \in \mathcal{B}$ such that $x \in B \subseteq (B_1 \cap B_2)$.

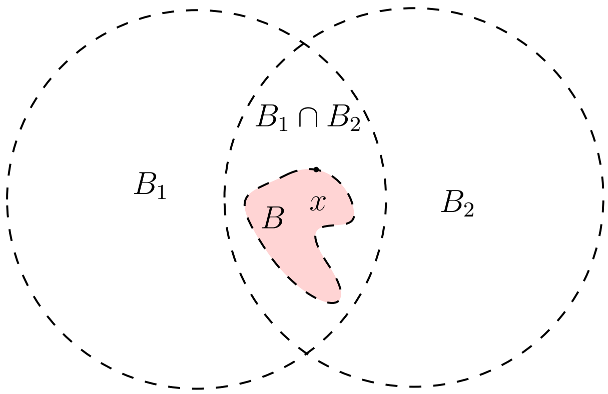

Figure: Property (2) is illustrated here. The idea is that we can always find a set in our basis that lies in the intersection $B_1 \cap B_2$ and contains our point x. This idea is related to $\mathcal{B}$ being closed under finite intersections, but it's a strictly weaker condition. Note that if B is closed under finite intersections, this condition follows immediately by letting $B = B_1 \cap B_2$.

Example :

Let S be any set. The set $\mathcal{B} = \{\{x\} : x \in S\}$ is a basis on S.

Proof. That $\bigcup \mathcal{B} = S$ is obvious. Now let $B_1 = \{b_1\}$ and $B_2 = \{b_2\}$ be any two singletons in $\mathcal{B}$. If $B_1 \neq B_2$ we are done, as their intersection is empty and condition (2) is vacuously satisfied.

Otherwise, if $B_1 = B_2$, their intersection is just $\{b_1\}$, for which we can let $B = B_1$ and $b_1 \in B = B_1 \cap B_2$. ∎

Example :

The set $\mathcal{B} = \{(a, \infty) : a \in \mathbb{R}\}$ is a basis on $\mathbb{R}$.

Proof. To show that $\bigcup \mathcal{B} = \mathbb{R}$, let $x \in \mathbb{R}$. Take $a = x - 1$, for which $x \in (a, \infty)$, and $(a, \infty) \in \mathcal{B}$ by definition. Therefore $x \in \bigcup \mathcal{B}$, and so $\bigcup \mathcal{B} = \mathbb{R}$. (I'm going to skip the $\supseteq$ direction altogether as there's nothing to do for it.)

For property (2), let (a, ∞) and (b, ∞) be in $\mathcal{B}$. As we've proven in the previous lecture, these sets are closed under finite intersections, so property (2) is implied.

To see this, assume without a loss of generality that $a < b$, then their intersection is just (b, ∞). For every $x \in (b, \infty)$, we can simply let $B = (b, \infty)$, so that $x \in (b, \infty)$. ■

Note that so far, bases have had nothing to do with topologies. We've defined them in such a way that they only relate to the set they are constructed over (the set we've been calling X all this time). It still remains to be shown that every basis defines a topology, which was our original goal.

Theorem : The Basis Topology

Let $\mathcal{B}$ be a basis on a set X. Define:

$$\mathcal{T}_{\mathcal{B}} = \left\{ \bigcup \mathcal{C} : \mathcal{C} \subseteq \mathcal{B} \right\}$$

Then $\mathcal{T}_{\mathcal{B}}$ is a topology.

Proof. Clearly $\emptyset$ and X are in $\mathcal{T}_{\mathcal{B}}$, which follows from letting $\mathcal{C} = \emptyset$ and $\mathcal{C} = \mathcal{B}$ respectively. In the first case, $\bigcup \emptyset = \emptyset$.[1]

In the second case, we know that $\bigcup \mathcal{B} = X$, since $\mathcal{B}$ is a basis.

We'll next prove that $\mathcal{T}_{\mathcal{B}}$ is closed under arbitrary unions, as we'll need this to prove that $\mathcal{T}_{\mathcal{B}}$ is closed under finite intersections.

Let $\{U_i : i \in I\}$ be a subset of $\mathcal{T}_{\mathcal{B}}$ with indexing set I. By definition of $\mathcal{T}_{\mathcal{B}}$, $U_i = \bigcup \mathcal{V}_i$ for some $\mathcal{V}_i \subseteq \mathcal{B}$.

We then have that:

$$\bigcup_{i \in I} U_i = \bigcup_{i \in I} \left(\bigcup \mathcal{V} i \right) = \bigcup \left(\bigcup_{i \in I} \mathcal{V}_i \right) \in \mathcal{T}_{\mathcal{B}}$$

The notation in that last step gave me a lot of trouble. When we don't index our union, we're just taking all the elements of all the sets inside the collection. However, in this case, we're taking a different kind of union. Now, we're taking each collection $\mathcal{V}_i$, bunching together all of their sets to get a much bigger collection of sets, which is $\bigcup_{i \in I} \mathcal{V}_i$, and only then taking the union over this big collection.

It shouldn't be too hard to convince yourself that you get the same result.

Finally, we prove that $\mathcal{T}_{\mathcal{B}}$ is closed under finite intersections. It suffices to prove this for two elements U and $V \in \mathcal{T}_{\mathcal{B}}$, as we can apply an induction argument to larger finite intersections.

[1] Make sure you're comfortable with what this means. On the left side of this equality, we are thinking of $\emptyset$ as an empty collection of **subsets** of X. In the second case, we are thinking of it as an empty collection of **elements** of X.

Suppose $U = \bigcup \mathcal{C}$ and $V = \bigcup \mathcal{O}$ are elements of $\mathcal{T}_{\mathcal{B}}$. Then:

$$U \cap V = \left(\bigcup \mathcal{C}\right) \cap \left(\bigcup \mathcal{O}\right)$$

$$= \bigcup \{C \cap O : C \in \mathcal{C} : O \in \mathcal{O}\}$$

This last equality might also require you to stare at it for a bit to convince yourself it's true, but I promise it's nothing too bad. If you try to write out the proof of this it will fall apart very fast.

As this is just an arbitrary union of these $(C \cap O)$'s, it suffices to show that each $C \cap O$ is an element of $\mathcal{T}_{\mathcal{B}}$. This follows from the fact that $\mathcal{T}_{\mathcal{B}}$ is closed under arbitrary unions which we've already shown.

Indeed, fix some $C \in \mathcal{C}$ and some $O \in \mathcal{O}$. If their intersection is empty we are done. Otherwise, pick some point $x \in C \cap O$. By property (2) of the basis, there must exist some basis element $B_x \in \mathcal{B}$ such that

$$x \in B_x \subseteq C \cap O$$

Repeating this for every $x \in C \cap O$:

$$C \cap O \subseteq \bigcup_{x \in C \cap O} B_x \subseteq C \cap O$$

Which implies that

$$C \cap O = \bigcup_{x \in C \cap O} B_x$$

Where the collection $\{B_x : x \in C \cap O\}$ which we are taking the union over, is just a subset of $\mathcal{B}$. Therefore $C \cap O \in \mathcal{T}_{\mathcal{B}}$.

And that concludes the first 'hard' proof in this course. ∎

During lecture, the student in front of me asked a very good question. Remember how in linear algebra, there was a notion of a basis being minimal? That is, if your basis was any smaller, it wouldn't generate the space, and if it was any bigger, you'd have a redundancy.

The student's question was if a similar notion existed for topological bases, if there was a property analogous to linear independence.

Ivan's answer was that, given a topology $\mathcal{T}$ on a set X, finding a basis for $\mathcal{T}$ isn't always possible. He also said that we won't be concerning ourselves too much with bases being minimal because it isn't a very useful thing to know.

There is, however, a nice property that bases have that agrees with the intuition we've built up so far; the basis generated by a topology is the 'smallest' topology containing that basis. This is somewhat analogous to a theorem in linear algebra that says the span of a set of vectors is the smallest vector space containing that set. See big list exercise 2.6.

Other than that, see big list exercises 2.1 and 2.2 for an example of reducing the size of a basis while it remains a basis.