MAT327 - Lecture 3

Wednesday, May 15th, 2019

In his notes, Ivan mentions the following definition of the topology defined by a basis:

Definition :

$\mathcal{T}_\mathcal{B} = \{U \subseteq X : \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subseteq U\}$

Note that if $U = \emptyset$, the condition is vacuously satisfied.

This is very similar to the way open sets are defined on $\mathbb{R}^n$, and metric spaces in general.

We'll first prove that this is in fact a topology.

Proof. It's easy to see that both $\emptyset$ and X are in $\mathcal{T}_\mathcal{B}$.

For arbitrary unions, let $\mathcal{U} = \{U_i : i \in I\}$ be a collection of open sets with indexing set I. Pick some $x \in \bigcup \mathcal{U}$. Then $x \in U_i$ for some $i \in I$. So there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq U_i \subseteq \bigcup \mathcal{U}$.

For finite intersections, suppose that $U_1, U_2 \in \mathcal{T}_\mathcal{B}$. If their intersection is empty we are done as the condition on all $x \in U$ is vacuously satisfied.

Otherwise, fix some $x \in B_1 \cap B_2$. Then there exists $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. Therefore $x \in B_1 \cap B_2$.

By the finite intersection property of bases, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$, so we are done. ■

We'll now prove that both definitions of this topology are equivalent.

Define:

$$\mathcal{T}'_\mathcal{B} = \left\{ \bigcup \mathcal{C} : \mathcal{C} \subseteq \mathcal{B} \right\}$$

Which is just our definition from last lecture. We'll show that $\mathcal{T}_\mathcal{B} \subseteq \mathcal{T}'_\mathcal{B}$.

($\subseteq$) Let $U \in \mathcal{T}_\mathcal{B}$. For each $x \in U$ take B_x so that $x \in B_x \subseteq U$. Then $\bigcup_{x \in U} B_x = U$, so $U \in \mathcal{T}'_\mathcal{B}$.

($\supseteq$) Let $U \in \mathcal{T}'_\mathcal{B}$. Then U is just a union of basic open sets[1]. That is, $U = \bigcup \mathcal{C}$ for some $\mathcal{C} \subseteq \mathcal{B}$. Pick some $x \in U$. Then $x \in C$ for some $C \in \mathcal{C} \subseteq \mathcal{B}$. This C shows that U is open in $\mathcal{T}_\mathcal{B}$. ■

[1] This word will be used a lot in this course. Convince yourself that every element of a basis is open in the topology that it defines (under either definition). This is why we call elements of the basis **basic open sets.**

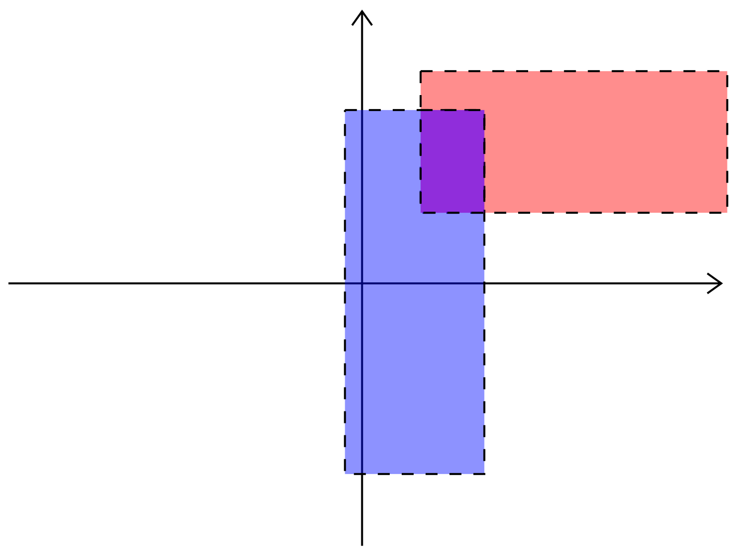
Example :

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces. Then the set:

$$\{A \times B : A \in \mathcal{T}_1, B \in \mathcal{T}_2\}$$

Is a basis (but not a topology) on $X_1 \times X_2$.

The reason they're not a topology is that "the union of two open rectangles is not necessarily a rectangle." The easiest way to see this is in $\mathbb{R}^2$:



Theorem : Important Lemma

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases on a set X. Then $\mathcal{T}_{\mathcal{B}_1} \subseteq \mathcal{T}_{\mathcal{B}_2}$ if and only if for all $x \in X$ and for all $B_1 \in \mathcal{B}_1$ containing x, there exists a set $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$.

The proof is apparently an exercise in symbol-pushing. ■

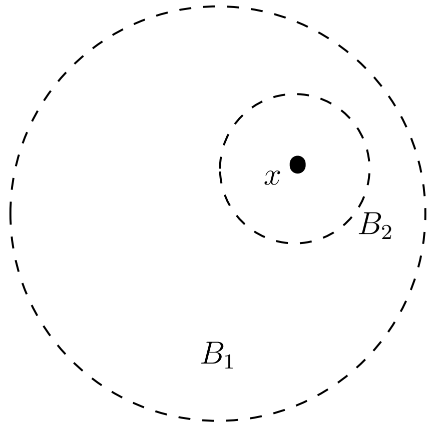


Figure: Given an open set $B_1 \in \mathcal{B}_1$ about the point x, we can find an open ball $B_2 \in \mathcal{B}_2$ contained in B_1 that also contains x.

A condition we might want for this theorem to be true is that $\mathcal{B}_1 \subseteq \mathcal{B}_2$, but the condition given in the theorem statement is actually a much stronger condition. Two bases on $\mathbb{R}^2$ are the set of all open rectangles and the set of all open balls. These two bases are disjoint, but they both generate $\mathcal{T}_{\text{usual}}$. This can be verified by this lemma.

This lemma also allows us to check whether a topology refines another one based on only their respective bases.

There's some useful corollaries to this.

Theorem : Corollary 1

Let $(X, \mathcal{T})$ be a topological space. A set $A \subseteq X$ is open if and only if for all $x \in A$, there exists some $U \in \mathcal{T}$ such that $x \in U \subseteq A$

Theorem : Corollary 2

Let $\mathcal{B}$ be a basis on a set X (independent of an topology), then $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$ if and only if $\mathcal{B} \subseteq \mathcal{T}$ and for all $U \in \mathcal{T}$, and for all $x \in U$, there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

This is a condition for checking when a particular topology is the same one as that generated by a fixed basis.

Both of these are once again, exercises in symbol pushing.

Closed Sets and Closures

Closed sets and closures are two ways of defining the same thing. We'll now answer our question from the beginning of the course, of what it meant for two points to be close to each other.

Our immediate intuition might be to think about closeness in terms of sequence convergence.

Definition : Sequence Convergence in $\mathbb{R}^n$

A sequence $(x_n)_{n\in\mathbb{N}}$ is said to converge to a point $x \in \mathbb{R}^n$ if for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, $||x_n - x|| < \epsilon$.

In terms of the usual topology we can define this as:

"For all open sets $U \in \mathbb{R}^n$ containing x, there exists an $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in U$."

Here we might think of U as being an open ball, in which case our usual notion of sequence convergence applies.

Definition : Closure of a Set

Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. We define the **closure of A in** $(X, \mathcal{T})$, denoted as $\overline{A}$, as:

$$\{x \in X : \text{For all open } U \text{ containing x, } U \cap A \neq \emptyset\}$$

Equivalently,

$$x \notin A \Leftrightarrow \text{ there exists an open set } U \text{ containing } x \text{ such that } U \cap A = \emptyset$$

So the closure's complement is the set of points that can be "separated" from A by putting them in an "open bubble". Another way to think about this, the closure of A is the set of all points (including those not in A) which are *close* to A.

Theorem : Properties of Closure

Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$. Then the following holds:

1. $A \subseteq \overline{A}$
2. $\overline{\overline{A}} = \overline{A}$
3. $\overline{A \cup B} = \overline{A} \cup \overline{B}$
4. $X \setminus A$ is open if and only if $A = \overline{A}$
5. $\overline{\emptyset} = \emptyset$, $\overline{X} = X$

Proof. (1) Follows immediately. ■

Proof. (2) By (1), $\overline{A} \subseteq \overline{\overline{A}}$. It remains to be shown that $\overline{\overline{A}} \subseteq \overline{A}$.

Fix some point $x \in \overline{\overline{A}}$. Let U be an open set containing x, by definition of $\overline{\overline{A}}$, there exists some $y \in \overline{A} \cap U$. So $U \cap A \neq \emptyset$ by definition of $\overline{A}$. ■

Proof. (4) ($\Rightarrow$) Assume that $X \setminus A$ is open. We know that $A \subseteq \overline{A}$. We have to show that $\overline{A} \subseteq A$. Suppose not, then $\overline{A} \setminus A \neq \emptyset$, so fix some $x \in \overline{A} \setminus A$. Then $x \in X \setminus A$, and $X \setminus A$ is open, but $(X \setminus A) \cap A \neq \emptyset$, contradicting the fact that $x \in \overline{A}$. □

($\Leftarrow$) Assume that $\overline{A} = A$, let $x \in X \setminus A = X \setminus \overline{A}$. Therefore there exists some open set $U \in X \setminus A$ containing x. As x was arbitrary, this shows that $X \setminus A$ is open. ■

Remark: This is an important trick to know for showing that a set is open, namely picking some arbitrary point in the set and showing that there exists an open ball which contains that point and remains inside the set. If this is true for every point, the set can be written as a union of these open balls (which is open!)

Example :

In $\mathbb{R}_{\text{usual}}$:

1. $\overline{(a, b)} = \overline{(a, b]} = \overline{[a, b)} = [a, b]$ for any interval consisting of points $a < b \in \mathbb{R}$.
2. $\overline{\{7\}} = \{7\}$
3. $\overline{\left\{\frac{1}{n} : n \in \mathbb{N}\right\}} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$

In general these aren't true. The following examples demonstrate this:

Example :

1. Consider the set $X = \{0,1\}, \mathcal{T} = \{\phi, X, \{1\}\}$. Here, $\overline{\{1\}} = X$.
2. More generally, consider $\mathbb{R}_7$. Here, even more surprisingly, $\overline{\{7\}} = \mathbb{R}$.
3. In $(X, \mathcal{T}_{\text{discrete}})$, $\overline{A} = A$ for any $A \subseteq X$. Ivan said to think of the discrete topology as a topology where all points are far apart.
4. In $(X, \mathcal{T}_{\text{indiscrete}})$, $\overline{A} = X$ for any $A \subseteq X$. Here, we think of the indiscrete topology as a horrid place where every two points are closely packed together.
5. In $\mathbb{R}_{\text{ray}}$, $\overline{\{7\}} = (-\infty, 7]$
6. In $\mathbb{R}_{\text{Sorgenfrey}}$, $\overline{\{7\}} = \{7\}$.
7. In $\mathbb{R}_{\text{Sorgenfrey}}$, let:

$$A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$$

$$B = \left\{-\frac{1}{n} : n \in \mathbb{N}\right\}$$

   Here $\overline{A} = A \cup \{0\}$, but $\overline{B} = B$.

Remark: In $\mathbb{R}_{\text{usual}}$, it doesn't matter which side a sequence approaches its limit point, we say it converges all the same. In $\mathbb{R}_{\text{Sorgenfrey}}$, the direction we approach the point matters. It won't converge if we approach from the left, but it will if we approach from the right. This is just another example of $\mathbb{R}_{\text{Sorgenfrey}}$ being worlds different from $\mathbb{R}_{\text{usual}}$, despite how similar they look.

Definition : Closed Sets

Let $(X, \mathcal{T})$ be a topological space, and let $C \subseteq X$. C is said to **closed in** $(X, \mathcal{T})$ if $X \setminus C$ is open in $\mathcal{T}$.

Note that being closed is not the same as being "not open". These are two related concepts with horrible names.

It follows immediately from de Morgan's laws that the finite unions and arbitrary intersections of closed sets are closed.

Just as we specified topologies by their open sets, it is actually also possible to define topologies based on their closed sets. This is what we were doing with the co-finite and co-countable topologies.

Example :

In $\mathbb{R}_{\text{usual}}$,

1. $(0, 1)$ is open.
2. $[0, 1]$ is closed.
3. $(0, 1]$ and $[0, 1)$ are neither closed nor open.
4. $\emptyset$ and $\mathbb{R}$ are both closed and open (called clopen).

Theorem :

A set A is closed if and only if $\overline{A} = A$.

The proof of this fact is essentially done for us. We've shown above (Properties of Closure #4) that $X \setminus A$ is open if and only if $A = \overline{A}$.

Theorem :

Let $(X, \mathcal{T}_X)$ be a topological space. Let $A \subseteq X$. Then:

$$\overline{A} = \bigcap \{C \subseteq X : C \text{ is closed and } A \subseteq C\}$$

That is, $\overline{A}$ is the smallest closed set containing A.

Proof. ($\subseteq$) let $x \in \overline{A}$. Let C be any closed set such that $C \supseteq A$. If $x \notin C$, then $X \setminus C$ is an open set containing x, but not intersecting A since $(X \setminus C) \subseteq (X \setminus A)$. This contradicts the assumption that $x \in \overline{A}$.

($\supseteq$) We've shown that $\overline{A}$ is closed and that $A \subseteq \overline{A}$, so we're done. ∎

One final definition:

Definition :

Let $(X, \mathcal{T}_X)$ be a topological space. We say that a subset $D \subseteq X$ is **dense** if $\overline{D} = X$.

What this says is, a dense set is "close" to the whole set.

Example :

The rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}_{\text{usual}}$.