2. Bases of topologies

1 Motivation

In the previous section we saw some examples of topologies. We described each of them by explicitly specifying all of the open sets in each one. This is not be a feasible strategy for all—or even most—topologies we may wish to describe in the future. Even when we can explicitly specify all of the open sets in a topology, there is usually another way to describe the topology that will be easier to understand.

For example, recall that we described the usual topology on $\mathbb{R}$ explicitly as follows:

$$\mathcal{T}_{\text{usual}} = \{ U \subseteq \mathbb{R} : \forall x \in U, \exists \delta > 0 \text{ such that } (x - \delta, x + \delta) \subseteq U \},$$

We then remarked that the open sets in this topology are precisely the familiar open intervals, along with their unions. We also know that a topology is by definition closed under arbitrary unions, so that if two intervals $(1, 2)$ and $(7, 8)$ are in some topology $\mathcal{T}$ on $\mathbb{R}$, we would necessarily have that their union $(1, 2) \cup (7, 8)$ is in $\mathcal{T}$ as well. So it seems like the entire collection of sets in $\mathcal{T}_{\text{usual}}$ can be specified by declaring that *just* the usual open intervals are open. Once these "special sets" are known to be open, we get all the other sets for free by taking unions.

For another example, let X be a nonempty set. Consider the discrete topology $\mathcal{T}_{\text{discrete}} = \mathcal{P}(X)$ on X—the topology consisting of *all* subsets of X. Bearing in mind again that $\mathcal{T}_{\text{discrete}}$ must be closed under unions, it seems as though declaring that all of the singletons $\{x\}$, for $x \in X$, are open is enough to specify the entire topology. Every element $U \in \mathcal{P}(X)$ is a union of singletons, after all:

$$U = \bigcup \{ \{x\} : x \in U \}.$$

Again in this case, specifying a much smaller collection of sets in the topology effectively specifies all the open sets via taking unions of the special ones.

These special collections of sets are called <u>bases</u> of topologies.

Exercise 1.1. Let X be a nonempty set, and let $\mathcal{B} = \{ \{x\} : x \in X \}$. Show that if $\mathcal{T}$ is a topology on X and $\mathcal{B} \subseteq \mathcal{T}$, then $\mathcal{T}$ is the discrete topology on X. Let $\mathcal{T}$ be a topology on $\mathbb{R}$ containing all of the usual open intervals. Is $\mathcal{T}$ the usual topology?

Remark 1.2. Recall the following notation, which we will use frequently throughout this section. If $\mathcal{A}$ is a collection of sets, then

$$\bigcup \mathcal{A} = \bigcup_{X \in \mathcal{A}} X.$$

In words, $\bigcup \mathcal{A}$ is the set containing all the elements of all the sets in $\mathcal{A}$.

2 Definition

Definition 2.1. *Let X be a set. A collection of sets $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a basis on X if the following two properties hold:*

1. *$\mathcal{B}$ covers X. That means: $\forall x \in X, \exists B \in \mathcal{B}$ such that $x \in B$. Or, more concisely, $X = \bigcup \mathcal{B}$.*
2. *$\forall B_1, B_2 \in \mathcal{B}, \forall x \in B_1 \cap B_2, \exists B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.*

In words, the second property says: given a point x in the intersection of two elements of the basis, there is some element of the basis containing x and contained in this intersection.

As we hinted earlier, a basis is a "special" collection of sets, in the sense that it specifies a topology. How does it specify a topology? By taking unions, as we hinted at earlier. More specifically, if you start with a basis on X and add to it all possible unions of sets from the basis, the resulting collection is a topology on X.

Definition 2.2. *Let X be a set and $\mathcal{B}$ a basis on X. We define*

$$\mathcal{T}_{\mathcal{B}} = \left\{ \bigcup \mathcal{C} \,:\, \mathcal{C} \subseteq \mathcal{B} \right\} \cup \{\emptyset\}.$$

Then $\mathcal{T}_{\mathcal{B}}$ is called the topology generated by $\mathcal{B}$.

(Note that I specifically include the empty set in the definition above for the sake of clarity. The $\emptyset$ is a subset of every set, and $\bigcup \emptyset = \emptyset$, so $\emptyset$ is already a member of the set on the left above.)

We note here that since a topology must be closed under unions, every element of the set $\mathcal{T}_{\mathcal{B}}$ we just described *must* be in any topology containing $\mathcal{B}$. That is, once we declare that all the sets from $\mathcal{B}$ are open, all unions of elements of $\mathcal{B}$ must therefore also be open. It turns out that upon adding all of those, the result is a topology. This also justifies the definite article: **the** topology generated by $\mathcal{B}$.

It remains to be proved that $\mathcal{T}_{\mathcal{B}}$ is actually a topology. We will delay that until after we see some examples of bases and the topologies they generate.

Example 2.3.

1. Let X be a set, and let $\mathcal{B} = \{ \{x\} : x \in X \}$. Then $\mathcal{B}$ is a basis on X, and $\mathcal{T}_{\mathcal{B}}$ is the discrete topology.
2. The collection $\mathcal{A} = \{ (a, \infty) \subseteq \mathbb{R} : a \in \mathbb{R} \}$ of open rays is a basis on $\mathbb{R}$, for somewhat trivial reasons. $\mathcal{A}$ covers $\mathbb{R}$ since for example $x \in (x-1, \infty)$ for any x. Moreover, given any two elements of $\mathcal{A}$, their intersection is again an element of $\mathcal{A}$. (ie. $\mathcal{A}$ is closed under pairwise intersections), and therefore it follows inductively that the intersection of finitely many elements of $\mathcal{A}$ is again an element of $\mathcal{A}$. This makes the second property in the definition of a basis trivial to satisfy. Can you tell what topology $\mathcal{A}$ generates?

3. The collection $\mathcal{B} = \{\,(a,b) \subseteq \mathbb{R} \,:\, a < b\,\}$ of open intervals is a basis on $\mathbb{R}$.

4. The collection $\mathcal{B}_2 = \left\{\, B_\epsilon(x) \subseteq \mathbb{R}^2 \,:\, x \in \mathbb{R}^2,\, \epsilon > 0 \,\right\}$ is a basis on $\mathbb{R}^2$.

5. The collection $\mathcal{B} = \{\,[a,b) \subseteq \mathbb{R} \,:\, a < b\,\}$ of "half-open" intervals is a basis on $\mathbb{R}$. So is $\mathcal{B}' = \{\,(a,b] \subseteq \mathbb{R} \,:\, a < b\,\}$.

6. Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces, and define:

$$\mathcal{B} = \mathcal{T}_1 \times \mathcal{T}_2 = \{\, U \times V \subseteq X_1 \times X_2 \,:\, U \in \mathcal{T}_1,\, V \in T_2 \,\}\,.$$

   Then $\mathcal{B}$ is a basis on $X_1 \times X_2$.

   Exercise 2.4. Prove that the collection $\mathcal{B}$ just above is actually a basis.

7. The collection

$$\mathcal{B} = \left\{\, (a,b) \times (c,d) \subseteq \mathbb{R}^2 \,:\, a < b,\, c < d \,\right\}$$

   is a basis on $\mathbb{R}^2$.

   Exercise 2.5.

   (a) Prove that the collection $\mathcal{B}$ just above is actually a basis.

   (b) Let $\mathcal{C}$ be the basis on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ obtained from two copies of $(\mathbb{R}, \mathcal{T}_{\text{usual}})$ as in example 6 above. Does $\mathcal{C} = \mathcal{B}$?

8. A topology $\mathcal{T}$ on a set X is itself a basis on X: First, $X \in \mathcal{T}$ and so $\mathcal{T}$ covers X. Second, the intersection of two sets in $\mathcal{T}$ is again in $\mathcal{T}$ (because $\mathcal{T}$ is closed under finite intersections), and so the second property in the definition of a basis is trivially satisfied.

Now that we have some intuition for bases, we can slog through the proof we delayed earlier.

Proof that $\mathcal{T}_\mathcal{B}$ is a topology. We check the three properties that define a topology.

1. $\emptyset \in \mathcal{T}_\mathcal{B}$ by definition, and $X \in \mathcal{T}_\mathcal{B}$ since $\mathcal{B}$ covers X (ie. $X = \bigcup \mathcal{B} \in \mathcal{T}_\mathcal{B}$).

2. $\mathcal{T}_\mathcal{B}$ is closed under arbitrary unions. What follows is relatively simple argument hidden under a mess of notation, so go through it slowly. If you like, you can first consider a special case in which we only look at two elements of $\mathcal{T}_\mathcal{B}$ (rather than an arbitrary collection of elements of $\mathcal{T}_\mathcal{B}$) to get an idea of what is happening.

   Let V_α, $\alpha \in I$, be an arbitrary collection of elements of $\mathcal{T}_\mathcal{B}$, where I is some indexing set. We want to show that

$$\bigcup_{\alpha \in I} V_\alpha \in \mathcal{T}_\mathcal{B}.$$

By definition of $\mathcal{T}_{\mathcal{B}}$, for each $\alpha \in I$ we have that $V_\alpha = \bigcup \mathcal{C}_\alpha$ for some $\mathcal{C}_\alpha \subseteq \mathcal{B}$. Immediately we can see:

$$\bigcup_{\alpha\in I} V_\alpha = \bigcup_{\alpha\in I}\left(\bigcup \mathcal{C}_\alpha\right) = \bigcup\left(\bigcup_{\alpha\in I}\mathcal{C}_\alpha\right) \in \mathcal{T}_{\mathcal{B}},$$

where the last set is an element of $\mathcal{T}_{\mathcal{B}}$ since $\bigcup_{\alpha\in I}\mathcal{C}_\alpha \subseteq \mathcal{B}$.

3. $\mathcal{T}_{\mathcal{B}}$ is closed under finite intersections. We will actually show that $\mathcal{T}_{\mathcal{B}}$ is closed under pairwise intersections, from which the general result follows inductively.

   Fix two elements $U = \bigcup \mathcal{A}$ and $V = \bigcup \mathcal{C}$ in $\mathcal{T}_{\mathcal{B}}$. We want to show $U \cap V \in \mathcal{T}_{\mathcal{B}}$. The first thing to note is:

$$U \cap V = \left(\bigcup \mathcal{A}\right) \cap \left(\bigcup \mathcal{C}\right) = \bigcup \{\, A \cap C \,:\, A \in \mathcal{A},\, C \in \mathcal{C} \,\}.$$

   (In words: in order for a point x to be in the intersection of $\bigcup \mathcal{A}$ and $\bigcup \mathcal{C}$, it must be in the intersection of some $A \in \mathcal{A}$ with some $C \in \mathcal{C}$.)

   To show that this set is in $\mathcal{T}_{\mathcal{B}}$, it suffices to prove the following containment, since we just showed in the previous part that $\mathcal{T}_{\mathcal{B}}$ is closed under arbitrary unions:

$$\{\, A \cap C \,:\, A \in \mathcal{A},\, C \in \mathcal{C} \,\} \subseteq \mathcal{T}_{\mathcal{B}}.$$

   So fix $A \cap C$ for some $A \in \mathcal{A}$ and $C \in \mathcal{C}$ (ie. fix an element of the set on the left side of the containment we have to show). We want to show that $A \cap C \in \mathcal{T}_{\mathcal{B}}$. If $A \cap C$ is nonempty, then for a given $x \in A \cap C$ there is a basis element $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq A \cap C$. This is by the second property in the definition of a basis. Repeating this for every $x \in A \cap C$ we get:

$$A \cap C \subseteq \left[\bigcup_{x\in A\cap C} B_x\right] \subseteq A \cap C,$$

   or in other words

$$A \cap C = \left[\bigcup_{x\in A\cap C} B_x\right] \in \mathcal{T}_{\mathcal{B}}.$$

   On the other hand if $A \cap C$ is empty, then $A \cap C \in \mathcal{T}_{\mathcal{B}}$ by part 1.

□

Pat yourself on the back for getting through that proof. It was not a complicated proof, but it was very heavy on notation and involves many “levels”: points, sets of points, collections of sets of points, and even a collection of collections of sets of points.

Here is another, equivalent way of defining $\mathcal{T}_{\mathcal{B}}$. This definition is less concrete, but occasionally easier to work with in proofs.

Definition 2.6. *Let X be a set and $\mathcal{B}$ a basis on X. Define*

$$\mathcal{T}'_{\mathcal{B}} = \{\, U \subseteq X \,:\, \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subseteq U \,\}\,.$$

We will continue using the original definition for now, but will come back to this new one shortly. In the meantime, you can prove that they actually define the same thing.

Exercise 2.7. Prove that the set $\mathcal{T}'_{\mathcal{B}}$ from Definition 2.6 is a topology by explicitly showing that all three properties of a topology are satisfied. Be sure to save yourself some work by simply showing that $\mathcal{T}'_{\mathcal{B}}$ is closed under pairwise intersections, from which closure under finite intersections follows inductively. This part (and only this part) may be a bit tricky. By the way, the definition of $\mathcal{T}'_{\mathcal{B}}$ should remind you of the explicit definition of the usual topology on $\mathbb{R}$ given at the beginning of this note.

Remark 2.8. Observe that $\mathcal{B} \subseteq \mathcal{T}_{\mathcal{B}}$ and $\mathcal{B} \subseteq \mathcal{T}'_{\mathcal{B}}$. To see this, fix some $U \in \mathcal{B}$. We first want to show that $U \in \mathcal{T}_{\mathcal{B}}$. Using the first definition, we can observe that $\{U\} \subseteq \mathcal{B}$ and therefore $U = \bigcup\{U\} \in \mathcal{T}_{\mathcal{B}}$. On the other hand, the fact that $U \in \mathcal{T}'_{\mathcal{B}}$ (using the second definition) is immediate from property 2 in the definition of a basis.

In other words, every element of a basis is open in the topology that it generates (in both senses of generating a topology we have seen). For this reason, we often call elements of the basis of a topology basic open sets.

Exercise 2.9 (Easier than it looks)**.** Prove that $\mathcal{T}'_{\mathcal{B}} = \mathcal{T}_{\mathcal{B}}$.

(Hint: Use the result of Exercise 2.7 and Remark 2.8).

3 Finally, topologies...

Example 3.1.

- The basis consisting of all the singletons in a set X (Example 2.3.1) generates the discrete topology on X.
- The basis consisting of all the open intervals in $\mathbb{R}$ (Example 2.3.3) generates the usual topology on $\mathbb{R}$. We can actually "do better" than this basis, in a certain sense. See the Big List for elaboration.
- The basis consisting of all "open balls" in $\mathbb{R}^2$ (Example 2.3.4) generates the usual topology on $\mathbb{R}^2$.
- Earlier (Example 2.3.8) we noted that a topology $\mathcal{T}$ on a set X is itself a basis on X. We can now conclude what we might expect, which is that $\mathcal{T}$ generates itself. That is, $\mathcal{T} = \mathcal{T}_{\mathcal{T}}$.

The basis given in Example 2.3.5 turns out to be a weird one, deserving of its own definition:

Definition 3.2. *Let $\mathcal{B} = \{ [a, b) \subseteq \mathbb{R} : a < b \}$. Then $\mathcal{B}$ is a basis on $\mathbb{R}$. The topology $\mathcal{S} := \mathcal{T}_{\mathcal{B}}$ it generates is called the lower limit topology on $\mathbb{R}$, and the corresponding topological space $(\mathbb{R}, \mathcal{S})$ is called the Sorgenfrey line. This is a space to which we will refer regularly throughout the course.*

We will come to realize that the Sorgenfrey line can be quite odd. Some introductory exercises surrounding it can be found on the Big List.

4 Comparing bases

We have mentioned that a topology can be generated by different bases. The point of dealing with bases rather than topologies themselves is that we can avoid having to explicitly specify *all* of the open sets in the space. The basis carries all the information we need to recover the rest of the topology. So the natural question to ask is: how can we tell from looking at two bases whether they generate the same topology?

Lemma 4.1. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bases on a set X, then $\mathcal{T}_{\mathcal{B}_1} \subseteq \mathcal{T}_{\mathcal{B}_2}$ if and only if for every $x \in X$ and $B_1 \in \mathcal{B}_1$ containing x, there is a $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$.*

Proof. **Exercise.** □

This is a slightly confusing statement, but it gets clearer the better you understand the definition of a basis.

James Munkres gives an excellent analogy for remembering how this goes in his *Topology*, page 81:

> It may be easier to remember if you recall the analogy between a topological space and a truckload of gravel. Think of the pebbles as the basis elements of the topology; after the pebbles are smashed to dust, the dust particles are the basis elements of the new topology. The new topology is finer than the old one, and each dust particle was contained inside a pebble, as [Lemma 4.1] states.

As a corollary of this lemma we can give a useful result that is essentially a restatement of the result of Exercise 2.9 in more useful-looking language. Given a topological space $(X, \mathcal{T})$ and a basis $\mathcal{B}$ on X, this corollary will allow us to (relatively) easily check whether $\mathcal{B}$ generates $\mathcal{T}$ (that is, whether $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$).

Corollary 4.2. *Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B}$ a basis on X. Then $\mathcal{B}$ generates $\mathcal{T}$ if and only if*

1. *$\mathcal{B} \subseteq \mathcal{T}$; and*
2. *for every $U \in \mathcal{T}$ and every $x \in U$, there is a $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

Proof. ($\Rightarrow$) Assume $\mathcal{B}$ generates $\mathcal{T}$, or in other words that $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$. It is clear that $\mathcal{B} \subseteq \mathcal{T}$ since $\mathcal{B} \subseteq \mathcal{T}_{\mathcal{B}}$. To show property (2), fix a set $U \in \mathcal{T}$ and a point $x \in U$. Since $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$, U is a union of elements from $\mathcal{B}$, say $U = \bigcup \mathcal{C}$ for some $\mathcal{C} \subseteq \mathcal{B}$. Then $x \in B_x$ for some $B_x \in \mathcal{C}$, and that set satisfies (2).

($\Leftarrow$) We will show that $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$ by showing containment both ways. On the one hand, $\mathcal{B} \subseteq \mathcal{T}$ by assumption and therefore $\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}$ since $\mathcal{T}$ is closed under arbitrary unions.

On the other hand, $\mathcal{T}$ is a basis for itself (ie. $\mathcal{T} = \mathcal{T}_{\mathcal{T}}$; see Examples 2.3.8 and 3.1) and so it follows from property (2) above that $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{B}}$ by Lemma 4.1. $\square$

Exercise 4.3. Come up with two different bases for the usual topology on $\mathbb{R}^2$ that we haven't mentioned yet.

Exercise 4.4. Does the usual topology on $\mathbb{R}$ refine the lower limit topology, vice-versa, or neither? (You will explore the lower limit topology more in some exercises on the Big List.)