5. Sequences, weak T-axioms, and first countability

1 Motivation

Up to now we have been mentioning the notion of sequence convergence without actually defining it. So in this section we will define sequence convergence in a general topological space, along with a few fundamental properties of topological spaces that naturally come along with sequence convergence. In particular, we will challenge a fact about sequences that seems so obvious that you have taken for it granted in the contexts in which you are used to working with them.

Recall that we used sequence convergence in $\mathbb{R}^n$ as intuition for how to define closures. We even related them directly to closures in that context in a Big List problem. You were asked to prove the following:

Exercise 1.1. Let A be a subset of $\mathbb{R}^n$ with its usual topology. Show that $x \in \overline{A}$ if and only if there exists a sequence of elements of A that converges to x.

I presented this fact in lecture and asked if you thought this was true in a general topological space. If not, what property or properties of $\mathbb{R}^n_{\text{usual}}$ guarantee this equivalence?

2 Convergence

We begin with the definition we have been hinting at previously. Recall that a sequence in a set X is just a function $a : \mathbb{N} \to X$, though we often conflate the function with its image as a set. We usually denote a sequence by $\{a_n\}_{n=1}^{\infty}$ or simply $\{a_n\}$ if the range of the indexing is clear, rather than the more proper $\{a(n)\}$. We assume the reader is familiar with sequences from at least one previous course.

Definition 2.1. *Let $(X, \mathcal{T})$ be a topological space. A sequence $\{x_n\}_{n=1}^{\infty}$ is said to converge to a point $x \in X$ if for every open set U containing x, there is an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$.*

In this case we write $\lim\limits_{n\to\infty} x_n = x$, or more commonly we simply write $x_n \to x$.

Recalling the language of "tails", the way I would actually say this is "$\{x_n\}$ converges to x if there is a tail of the sequence in any open set containing x".

Remark 2.2. Just as with the definition of closures, given a basis $\mathcal{B}$ of $\mathcal{T}$, it suffices to only consider basic open sets U in the above definition.

We have seen or hinted at some examples before, so we remind ourselves and in some cases we will be rigorous where we could not have been be previously.

Example 2.3.

1. In $\mathbb{R}_{\text{usual}}$ and the Sorgenfrey line, $\frac{1}{n} \to 0$.

2. In $\mathbb{R}_{\text{usual}}$, $-\frac{1}{n} \to 0$. But this is not true in the Sorgenfrey line, since $[0, 1)$ is an open set containing 0 and containing no elements of the sequence.

3. Trivially, a constant sequence $x, x, x, x, \ldots$ converges to x in any topological space. The same is true of a sequence that is constant on some tail.

4. In $(X, \mathcal{T}_{\text{discrete}})$, the only sequences that converge are ones with constant tails. To see this, note that if $x_n \to x$ for some $x \in X$, then a tail of the sequence must be inside the open set $\{x\}$.

5. In $(X, \mathcal{T}_p)$, the particular point topology at p in a set X, things become weirder. Again, sequences with constant tails converge trivially. If $\{x_n\}$ is a sequence that does not have a constant tail and x is a point in X, then if the sequence wants to converge to x it must respect the fact that $\{x, p\}$ is an open set. That means if the sequence is not eventually constantly equal to x, the only other value it can have after some point is p.

   For example, this means the sequence $x, p, x, p, x, p, \ldots$ converges to x. (It doesn't converge to p, since $\{p\}$ is an open set containing p and not containing any tail of the sequence.)

   This turns out to be a complete characterization of convergent sequences in this topology: a sequence in $(X, \mathcal{T}_p)$ converges if and only if it is eventually constant, or eventually "constant or p".

6. Weirder still, consider $(X, \mathcal{T}_{\text{indiscrete}})$. This is the next part in our ongoing story of the indiscrete topology being awful. Here, every sequence (yes, *every sequence*) converges to every point in the space. In $(\mathbb{R}, \mathcal{T}_{\text{indiscrete}})$, the sequence $7, 7, 7, 7, 7, \ldots$ converges to π. It also converges to 7, e, $1,000,000$, and every other real number. This is awful.

   Remember in first year calculus when your instructor went to the trouble of mentioning that limits of functions and limits of sequences are unique? This is one of the scenarios in which that can fail.

   Why is this happening? In this extreme setting, none of the points in the space are *topologically distinguishable* from one another. That is, given any two points, there is no open set that contains one but not the other (because the only nonempty open set is X itself). So sequences that are trying to converge in sensible ways do not know that $7, \pi, e, 1,000,000$, etc. are different points.

7. Another weird example. Consider $(\mathbb{R}, \mathcal{T}_{\text{ray}})$. In this case, the two sequences we discussed earlier converge as they do in $\mathbb{R}_{\text{usual}}$. That is, $\frac{1}{n} \to 0$ and $-\frac{1}{n} \to 0$.

The similarities do not go much further, however. Recall that in this space, the only nontrivial open sets are of the form (a, ∞). So every open set contains everything above any element of itself. What does this mean for sequences? It means for example that every sequence that is bounded below converges to every point below all the values in a tail of the sequence. For example, the sequence $\{\frac{1}{n}\}$, which we already know converges to 0, also converges to -1, -7, $-1,000,000$, etc. More generally, for example, any sequence of strictly positive numbers converges to any non-positive number. And so on.

Weird stuff. This topology is quite pathological with respect to sequence convergence.

8. What about the "co-" topologies? Let X be an infinite set, and consider $(X, \mathcal{T}_{\text{co-finite}})$. In this space, it seems like any sequence should converge to any point because any co-finite subset of X necessarily contains a tail of any sequence. But wait! That only applies to sequences that take on infinitely many values.

   For example, the sequence $0, 1, 0, 1, 0, 1, \ldots$ does not converge in $(\mathbb{R}, \mathcal{T}_{\text{co-finite}})$. On the other hand, the sequence $1, 2, 3, 4, \ldots$ converges to any real number. Try to precisely classify the convergent sequences in $(\mathbb{R}, \mathcal{T}_{\text{co-finite}})$ (this will be on the Big List).

   On the other hand, again assuming that X is infinite (countable or uncountable), consider $(X, \mathcal{T}_{\text{co-countable}})$. In this setting, the only sequences that converge are eventually constant (and the eventually constant sequences converge only to their constant values). Indeed, assuming $\{x_n\}$ is not eventually constant and $x \in X$, the set

$$(X \setminus \{\, x_n \,:\, n \in \mathbb{N} \,\}) \cup \{x\}$$

   is an open set containing x but no tail of the sequence.

3 When is sequence convergence not weird?

By now I hope the reader is convinced that sequence converge can be weird. In this section we will formalize some properties of topological spaces called "separation axioms" that will ensure things are less weird. One of these, the Hausdorff property, is particularly important. We will approach it through some weaker properties first.

Earlier we remarked that points in an indiscrete space are not *topologically distinguishable*, meaning that given any two points, there are no open sets distinguishing them. Our first property will ensure this never happens.

Definition 3.1. *A topological space $(X, \mathcal{T})$ is said to be $\underline{T_0}$ (or much less commonly said to be a Kolmogorov space), if for any pair of distinct points $x, y \in X$ there is an open set U that contains one of them and not the other.*

One would usually express this by saying that $(X, \mathcal{T})$ is T_0 provided that given a pair of distinct points, one of them has an open neighbourhood not containing the other.

This is a very weak property that we will not discuss much, as it is rarely useful to ask for a space to be just T_0; se will usually ask for more. We introduce this property because it is the weakest of the T-axioms, of which we will see several throughout the course. The only space of any consequence we have seen that fails to be T_0 is the indiscrete space.

Next, a property that we foreshadowed while discussing closed sets, though the definition may not seem familiar at first.

Definition 3.2. *A topological space $(X, \mathcal{T})$ is said to be $\underline{T_1}$ (or much less commonly said to be a <u>Fréchet space</u>) if for any pair of distinct points $x, y \in X$, there exist open sets U and V such that U contains x but not y, and V contains y but not x.*

This is a very slight strengthening of T_0.

Exercise 3.3. Show that the following spaces we have discussed so far in the course are T_0 but not T_1.

1. $(\mathbb{R}, \mathcal{T}_{\text{ray}})$
2. $(X, \mathcal{T}_p)$
3. $X = \{0, 1\}$ with the topology $\mathcal{T} = \{\emptyset, X, \{0\}\}$.

Earlier we saw that in an indiscrete space, a constant sequence like $x, x, x, x, \ldots$ converges to points other than x (to all of the other points, in fact). This can also happen in T_0 spaces. For example in (3) above, the sequence $0, 0, 0, 0 \ldots$ converges to both 0 and 1. This cannot happen in a T_1 space, however.

Exercise 3.4. Let $(X, \mathcal{T})$ be a T_1 space, and let $x \in X$. Show that the constant sequence $x, x, x, x, \ldots$ converges to x and to no other point.

The sense in which we have foreshadowed the T_1 property is with regard to singletons being closed. It turns out this property is equivalent to the property that every singleton is closed. We can characterize this in more ways, even.

Exercise 3.5. Let $(X, \mathcal{T})$ be a topological space. Prove that the following are equivalent.

1. $(X, \mathcal{T})$ is T_1.
2. For every $x \in X$, $\{x\}$ is closed.
3. Every finite subset of X is closed.
4. For every subset $A \subseteq X$, $A = \bigcap \{ U \subseteq X : U \text{ is open and } A \subseteq U \}$.

Okay, so T_1 spaces seem pretty nice. Singletons being closed is a property which feels nice, somehow. T_1 spaces can still exhibit weird sequence behaviour though.

Example 3.6. Let X be an infinite set. Then the space $(X, \mathcal{T}_{\text{co-finite}})$ is T_1. Indeed, given distinct points $x, y \in X$, the open sets $U = X \setminus \{y\}$ and $V = X \setminus \{x\}$ witness this. However, as we saw in Example 2.3.8, sequences in this space can converge to many different points. For example, in $(\mathbb{R}, \mathcal{T}_{\text{co-finite}})$ the sequence $1, 2, 3, 4, \ldots$ converges to all points, since any co-finite subset of X contains a tail of the sequence.

This should make you a little bit uncomfortable. Your intuition from the usual topology on $\mathbb{R}^n$ says that a convergent sequence should converge to only one point. The next definition is what guarantees this. It is by far the most important of the weaker separation axioms.

Definition 3.7. *A topological space $(X, \mathcal{T})$ is said to be $\underline{T_2}$, or more commonly said to be a Hausdorff space, if for every pair of distinct points $x, y \in X$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.*

Theorem 3.8. *Let $(X, \mathcal{T})$ be a Hausdorff space. Then every sequence in X converges to at most one point.*

Proof. Suppose X is Hausdorff and let $\{x_n\}$ be a sequence in X. Suppose $x_n \to x$ and $y \neq x$. Then there are disjoint open sets U and V such that $x \in U$ and $y \in V$. By definition of convergence, some tail of the sequence is in the set U. But then that tail (and therefore all tails of the sequence) is disjoint from V, meaning $x_n \not\to y$. □

It really seems as though this implication should reverse, but unfortunately it does not.

Example 3.9. Let X be an uncountable set and consider $(X, \mathcal{T}_{\text{co-countable}})$. Then every sequence converges to at most one point, but the space is not Hausdorff. To see this, on the one hand we noted in Example 2.3.8 that the only convergent sequences in this topology are eventually constant, and that they converge only to their constant values. On the other hand, since X is uncountable, any two co-countable sets intersect (check this!), and therefore X cannot be Hausdorff since there simply are no pairs of distinct, nonempty, disjoint open sets.

There are two ways to "fix" the fact that this implication does not reverse. One is to define a property which fills the gap (ie. a property P such that "limits of convergent sequences are unique" + $P \Leftrightarrow$ Hausdorff). There is no precise property for this job, though one of the properties we will define shortly gives an implication rather than an equivalence.

The other solution is to use an entirely different—and better—notion of a sequence, called a *net*. We will not focus on nets in this course, though you will have a supplementary section of notes on the topic if you are interested. Nets are somehow the "correct" way to talk about a lot of things, including sequence-like structures, "closeness", and so on.

Anyway, on to some examples related to the Hausdorff property.

Example 3.10.

1. $\mathbb{R}_{\text{usual}}$, the Sorgenfrey line, and $(X, \mathcal{T}_{\text{discrete}})$ for any set X are all Hausdorff.
2. The Furstenberg topology on $\mathbb{Z}$ discussed in the Big List is Hausdorff (you proved this, in fact, though I spelled out the property in the question rather than giving its name).
3. $(X, \mathcal{T}_{\text{indiscrete}})$ is almost always not Hausdorff. Under what circumstances is it Hausdorff?
4. The co-finite topology on an infinite set and the co-countable topology on an uncountable set are both not Hausdorff.
5. $(\mathbb{R}, \mathcal{T}_{\text{ray}})$ is not Hausdorff, since any two nonempty open sets intersect.

The overwhelming majority of topological spaces you will encounter, outside of some pretty specialized contexts, are Hausdorff. This property is sort of the minimum "niceness" standard that mathematicians expect from a space they are working in. In fact, the Hausdorff property was part of the original definition of a topological space. (The original definition was created by Felix Hausdorff himself!)

The one exception to this is the following topology, which is very important to the field of algebraic geometry. It is usually defined in more abstract contexts, but here we will just define it on $\mathbb{R}^n$ to get the idea across.

Example 3.11. The Zariski topology on $\mathbb{R}^n$ is the topology in which the basic open sets are the complements of zero sets of polynomials. That is, given a polynomial $p : \mathbb{R}^n \to \mathbb{R}$, define

$$U_p = \{\, x \in \mathbb{R}^n \,:\, p(x) \neq 0 \,\}\,.$$

Then $\mathcal{B} := \{\, U_p \,:\, p \text{ is a polynomial} \,\}$ is a basis. The topology it generates is the Zariski topology. In this topology any two nonempty open sets intersect (check this!), so it is very far from being Hausdorff.

4 Sequences and closures

Between comments in lecture notes and Big List problems we have essentially covered the material in this section already, but I collect it here for your convenience. As we have hinted at before, sequences have something to do with closures because sequences "get close to" their limit points.

Proposition 4.1. *Let $(X, \mathcal{T})$ be a topological space, let $A \subseteq X$, and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of elements of A. If $a_n \to a$, then $a \in \overline{A}$.*

Proof. **Exercise** (which you should have done by now, if you are doing the exercises). □

Just as with the last theorem, it really, really seems like this implication should reverse. If $a \in \overline{A}$ (ie. if a is close to A) it feels like we should be able to find a sequence of points from A that converges to A. Unfortunately, this is not the case.

Example 4.2. Consider $(\mathbb{R}, \mathcal{T}_{\text{co-countable}})$. Let $A = \mathbb{R}\setminus\{7\}$. Then $7 \in \overline{A}$, but there is no sequence from A that converges to 7.

On the one hand, any open set that contains 7 is co-countable and therefore contains some other element of $\mathbb{R}$, or in other words contains an element of A. This shows that $7 \in \overline{A}$. On the other hand, as we remarked in Example 2.3.8 the only sequences that converge in this topology are eventually constant, and therefore no sequence of elements of A can converge to 7.

Again, as with the Theorem 3.8, the problem here is that sequences are not expressive enough to capture the notion of "being close to" a set. Our main solution will be to define a property that fills the gap, which is what the next section is about. (But again, the ideal solution is really to define the notion of net convergence, which is "better" than sequence convergence.)

5 First countability

This section describes the property of topological spaces that allows sequences capture all the information we feel they should capture. Before we define it we need a new sort of object, which is useful in many settings.

Definition 5.1. *Let $(X, \mathcal{T})$ be a topological space and let $x \in X$. A local basis at x is a collection of open sets $\mathcal{B}_x \subseteq \mathcal{T}$ satisfying the following two properties:*

1. *$x \in B$ for all $B \in \mathcal{B}_x$.*
2. *For any open set U containing x, there is a $B \in \mathcal{B}_x$ such that $B \subseteq U$.*

Example 5.2.

1. In $\mathbb{R}_{\text{usual}}$, fix a point x. Then $\{\, (a,b) \subseteq \mathbb{R} : a < x < b \,\}$ is a local basis at x. In fact we can "do better" by finding a countable local basis, in the familiar way:

$$\{\, (a,b) \subseteq \mathbb{R} : a < x < b, \text{ and } a, b \in \mathbb{Q} \,\}.$$

   Even better still is:

$$\mathcal{B}_x = \left\{\, (x - \tfrac{1}{n}, x + \tfrac{1}{n}) : n \in \mathbb{N} \,\right\}.$$

   These are the local bases we will usually think about in $\mathbb{R}_{\text{usual}}$.

2. More generally, in $\mathbb{R}^n_{\text{usual}}$, the following collections are all local bases at x.

$$\{\, B_\epsilon(x) : \epsilon > 0 \,\}, \quad \{\, B_\epsilon(x) : \epsilon > 0 \text{ and } \epsilon \in \mathbb{Q} \,\}, \quad \mathcal{B}_x = \left\{\, B_{\frac{1}{n}}(x) : n \in \mathbb{N} \,\right\}$$

   Note that the latter two are countable while the first one is not.

3. In the Sorgenfrey line, fix a point x. Then both $\{\, [x, b) \subseteq \mathbb{R} \,:\, b > x \,\}$ and

$$\mathcal{B}_x = \left\{\, [x, x + \tfrac{1}{n}) \,:\, n \in \mathbb{N} \,\right\}$$

are local bases at x. (Show this.) Again, the latter is countable while the former is not.

4. In $(X, \mathcal{T}_{\text{discrete}})$, fix $x \in X$. In this space $\{x\}$ is open, and therefore must be an element of any local base at x. In fact, it turns out that $\mathcal{B}_x = \{\{x\}\}$ is a local base at x.

5. In general it should be easy to see that if $\{x\}$ is open in a topological space, then $\{\{x\}\}$ is a local base at x. As a result, for example, $\{\{p\}\}$ is a local base at p in $(X, \mathcal{T}_p)$. This is not true for other points in that space though. If $x \neq p$ in this space, then the smallest local base at x would be $\{\{x, p\}\}$.

Having gained some intuition for local bases, here is the new property promised earlier.

Definition 5.3. *A topological space $(X, \mathcal{T})$ is said to be first countable if every point in X has a countable local basis.*

This is the key property that allows sequences to capture all the information we feel they should capture. We make note of some examples. We have already shown most of what there is to show here.

Example 5.4.

1. $\mathbb{R}^n_{\text{usual}}$ and the Sorgenfrey line are first countable, as is every discrete space.

2. Spaces that are not first countable are relatively rare. The only ones we have encountered so far are $(X, \mathcal{T}_{\text{co-finite}})$ and $(X, \mathcal{T}_{\text{co-countable}})$, provided that X is uncountable.

Before we use first countability to prove the results we were hoping for, we establish an equivalent definition which looks like a slight strengthening, but actually is not.

Proposition 5.5. *Let $(X, \mathcal{T})$ be a first countable topological space. Then every $x \in X$ has a countable nested local basis. That is, for every $x \in X$, there is a local basis $\mathcal{B}_x = \{\, B_n \,:\, n \in \mathbb{N} \,\}$ such that $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$.*

Proof. **Exercise.** □

With this fact in hand, here are the payoffs.

Proposition 5.6. *Let $(X, \mathcal{T})$ be a first countable topological space, and let $A \subseteq X$. Then $x \in \overline{A}$ if and only if there is a sequence of elements of A converging to x.*

Proof. **Exercise.** □

Proposition 5.7. *Let $(X, \mathcal{T})$ be a first countable topological space with the property that every convergent sequence has a unique limit point. Then $(X, \mathcal{T})$ is Hausdorff.*

Proof. We prove this by contrapositive. Suppose $(X, \mathcal{T})$ is not Hausdorff. Then there must exist a pair of distinct point $x, y \in X$ such that for every pair of open sets U, V containing x and y respectively, $U \cap V \neq \emptyset$.

Since the space is first countable, by the previous Proposition we can fix countable nested local bases $\mathcal{B}_x = \{ B_n : n \in \mathbb{N} \}$ and $\mathcal{C}_y = \{ C_n : n \in \mathbb{N} \}$ at x and y. For each $n \in \mathbb{N}$, $B_n \cap C_n \neq \emptyset$, so fix a point x_n in this intersection. Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges to both x and y (check this!). □

First countability helps us in other ways that we will see later. For now, some exercises will be included in the Big List to help you familiarize yourself with this property.

