7. Subspaces

1 Motivation

In this section we will build perhaps our most useful tool for defining new topological spaces, and for analyzing properties of the ones we define: how to define a topology on a subset of a topological space in a way that "agrees" with the topology of the larger space.

Many of the topologies we have used as examples and testing grounds for our various results are useful almost exclusively for this purpose. For example you are *very* unlikely to discover a practical mathematical application for the co-countable topology on $\mathbb{R}$ outside of this or another similar course; its best usage is as an easily-defined example of a non-first countable topological space. Topological spaces found in common mathematical applications actually arise as subspaces of relatively simple spaces like $\mathbb{R}^n_{\text{usual}}$, or as spaces that are homeomorphic or locally homeomorphic to subspaces of $\mathbb{R}^n_{\text{usual}}$, subspaces of various function spaces, etc.

More importantly for the study of topology itself, subspaces give us another important way to analyze how properties "move around". We have already dealt with continuous functions and homeomorphisms in this regard. We discussed how the various topological properties we have defined thus far are preserved by homeomorphisms and how some of them are preserved by continuous functions. Now we will be able to analyze which properties pass to subspaces.

Our main definition in this chapter is a natural one, in the sense that it agrees with your intuition for very simple cases. For example you should not find it surprising that the "usual topology" on the open interval $(0, 1) \subseteq \mathbb{R}$ is the one generated by the basis $\mathcal{B} = \{ (a, b) : 0 < a < b < 1 \}$.

2 Subspace topologies

As promised, this definition gives us a way of defining a topology on a subset of a topological space that "agrees" with the topology on the larger space in a very strong way.

Definition 2.1. *Let $(X, \mathcal{T})$ be a topological space, and let $Y \subseteq X$ be any subset. We define the* <u>*subspace topology*</u> *$\mathcal{T}_Y$ on Y (we will sometimes write $\mathcal{T}_{subspace}$ for the sake of clarity), by:*

$$\mathcal{T}_Y = \{ U \cap Y : U \in \mathcal{T} \} .$$

We might call $\mathcal{T}_Y$ the topology "induced by" or "inherited from" $\mathcal{T}$. We call the topological space $(Y, \mathcal{T}_Y)$ a <u>*subspace*</u> *of $(X, \mathcal{T})$.*

Exercise 2.2. Verify that the subspace topology is in fact a topology. (This should take almost no work.)

You may find it odd at this point in the course that we specify the subspace topology explicitly like this rather than specify a basis for it. Since the definition that works is so simple, there is no need for this; we can describe a subspace topology without having a specified basis for the larger topology.

If we do happen to have a basis for the larger topology though, it induces a basis on the subspace topology in the obvious way:

Proposition 2.3. *Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B}$ be a basis on X that generates $\mathcal{T}$. Let $Y \subseteq X$. Then the collection*

$$\mathcal{B}_Y = \{ B \cap Y \,:\, B \in \mathcal{B} \}$$

is a basis on Y that generates the subspace topology $\mathcal{T}_Y$ on Y.

Proof. As with the earlier exercise, the proof here is almost immediate.

Clearly $\mathcal{B}_Y \subseteq \mathcal{T}_Y$. Given an open set $U \cap Y \in \mathcal{T}_Y$ and a point $y \in U \cap Y$, there is an element $B \in \mathcal{B}$ such that $y \in B \subseteq U$, since $\mathcal{B}$ generates $\mathcal{T}$. Then $B \cap Y \in \mathcal{B}_Y$, and we have $y \in B \cap Y \subseteq U \cap Y$. □

Before we go much further, here are some examples.

Example 2.4.

1. Any subspace of a discrete space is discrete.

2. Any subspace of an indiscrete space is indiscrete.

3. Let $(\mathbb{R}, \mathcal{T}_7)$ be the reals with the particular point topology at 7. If $A \subseteq \mathbb{R}$ contains 7, then the subspace topology on A is also the particular point topology on A. If A does not contain 7, then the subspace topology on A is discrete.

4. The subspace topology on $(0, 1) \subseteq \mathbb{R}$ induced by the usual topology on $\mathbb{R}$ is the topology generated by the basis $\mathcal{B}_{(0,1)} = \{ (a, b) \,:\, 0 \le a < b \le 1 \} = \{ B \cap (0, 1) \,:\, B \in \mathcal{B} \}$, where $\mathcal{B}$ is the usual basis of open intervals for $\mathbb{R}_{\text{usual}}$.

5. There is nothing special about $(0, 1)$ in the example above, among open intervals. All the subspaces $\big((a, b), \mathcal{T}_{(a,b)}\big)$ (for $a < b \in \mathbb{R}$) are homeomorphic to one another. To check this, verify that $f : (0, 1) \to (a, b)$ given by $f(x) = (b - a)x + a$ is a homeomorphism, and conclude the claim above. This result extends to open balls in $\mathbb{R}^n_{\text{usual}}$.

6. Going even further, every subspace in the previous example is homeomorphic to $\mathbb{R}_{\text{usual}}$ itself. To verify this, note that $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a homeomorphism, and conclude the result.

7. Consider $\mathbb{Z}$ as a subset of $\mathbb{R}$ with its usual topology. Then for any $n \in \mathbb{Z}$,

$$(n - \tfrac{1}{2}, n + \tfrac{1}{2}) \cap \mathbb{Z} = \{n\}.$$

   That is, $\{n\}$ is open in the subspace topology on $\mathbb{Z}$ induced by $\mathbb{R}_{\text{usual}}$. Therefore $(\mathbb{Z}, \mathcal{T}_{\text{subspace}}) = (\mathbb{Z}, \mathcal{T}_{\text{discrete}})$.

   In general, a subspace of a topological space whose subspace topology is discrete is called a discrete subspace. We have just shown that $\mathbb{Z}$ is a discrete subspace of $\mathbb{R}$. Similarly $\mathbb{N}$ and $\left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ are discrete subspaces of $\mathbb{R}_{\text{usual}}$.

8. $\mathbb{Q}$ and $Y := \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$ are not discrete subspaces of $\mathbb{R}_{\text{usual}}$. The subspace topology on $\mathbb{Q}$ is generated by the basis

$$\mathcal{B}_{\mathbb{Q}} = \{ (a, b) \cap \mathbb{Q} : a < b \in \mathbb{R} \}.$$

   Notice that this is not the same collection as $\{ (a, b) \cap \mathbb{Q} : a < b \in \mathbb{Q} \}$, though this collection is a basis which generates the same topology as the subspace topology, by virtue of the fact that every irrational number is the limit point of a sequence of rationals.

   The subspace topology on Y is not discrete because $\{0\}$ is not open. Indeed, given any open subset U of $\mathbb{R}_{\text{usual}}$ containing 0, we know that U contains infinitely many members of Y. Of course, $\{y\}$ is open in the subspace topology on Y for all $0 \neq y \in Y$. Y is a very important space in topology, and we will give it a better name and talk about it much more later on.

9. Let $Y = [a, b] \subseteq \mathbb{R}$ with the Lower Limit topology. Then $\{b\}$ is open in the subspace topology on Y, since $[a, b] \cap [b, b + 1) = \{b\}$. No other singleton is open though.

10. The subspace topology on the unit circle $S^1 := \left\{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \right\}$ is generated by the basis of "open arcs" on the circle. That is, given $a < b \in \mathbb{R}$, let

$$U(a, b) := \left\{ (1, \theta) \in \mathcal{S}^1 : a < \theta < b \right\}$$

   where the points are written in polar coordinates. Each of these is simply the intersection of an open ball in $\mathbb{R}^2$ with the circle. Then the collection

$$\mathcal{B}_{S^1} := \{ U(a, b) : a < b \in \mathbb{R} \}$$

   generates the subspace topology on S^1. You may get the feeling that this subspace feels a lot like $\mathbb{R}_{\text{usual}}$ or perhaps like $[0, 1]$ with its subspace topology, just "bent" around the circle. You would not be far off, as the following example demonstrates.

11. Let $Y = S^1 \setminus \{(0, 1)\} \subseteq \mathbb{R}^2$ be the unit circle without its top point (here, $(0, 1)$ is a point written in Cartesian coordinates). Then Y with its subspace topology inherited from the

usual topology on $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}_{\text{usual}}$. The function that usually witnesses this is called the "stereographic projection" map $f : Y \to \mathbb{R}$. Given a point $x \in Y$, let ℓ_x be the line joining x and $(0, 1)$. Then we define $f(x)$ to equal the point where ℓ_x intersects the x-axis. The map f essentially takes Y, flattens it out, and then stretches it to cover the whole real line.

Try to give an explicit formula for f. This is a nice exercise in high school geometry.

12. The idea in the previous examples extends to all dimensions in the following nice way. Let $S^n = \left\{ (x_1, x_2, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1 \right\}$ be the unit sphere in $\mathbb{R}^{n+1}$, and let $Y_n = S^n \setminus \{(0, 0, \ldots, 0, 1)\}$ be S^n without its "top" point. Then Y with its subspace topology is homeomorphic to $\mathbb{R}^n$ with its usual topology.

    This is relatively easy to see for Y_2, but more or less impossible to picture for any higher n. This idea will be very important for us at the end of the course, when we show that for example S^1 is the *one point compactification* of $\mathbb{R}$.

3 Easy results

Having now defined and explored some subspaces, many intuitive facts present themselves. The proofs of all of these facts are very straightforward, and are left as exercises in the Big List. None of them should require any creativity or deep thinking beyond understanding the definitions involved. You are encouraged not to try to memorize this list of facts, since they are all so natural, and so straightforward to prove if necessary.

Also, from this point forward we are going to be a little more loose with our notation. Instead of saying that $(Y, \mathcal{T}_{\text{subspace}})$ is a subspace of $(X, \mathcal{T})$, we will usually just say "Y is a subspace of X" when the topologies are easily understood from context.

Proposition 3.1. *Let $(X, \mathcal{T})$ be a topological space and let Y be a subspace of X. If U is an open subset of Y and Y is an open subset of X, then U is an open subset of X.*

Proposition 3.2. *Let $(X, \mathcal{T})$ be a topological space, let Y be a subspace of X, and let A be a subset of Y. Then the subspace topology A inherits from Y is equal to the subspace topology it inherits from X.*

Proposition 3.3. *Let $(X, \mathcal{T})$ be a topological space, and let A be a subspace of X. For any $B \subseteq A$, $\text{cl}_A(B) = A \cap \text{cl}_X(B)$, where $\text{cl}_X(B)$ denotes the closure of B computed in X, and similarly $\text{cl}_A(B)$ denotes the closure of B computed in the subspace topology on A.*

Proposition 3.4. *Let $(X, \mathcal{T})$ be a topological space, and let A be a subspace of X. Then the inclusion map $i : A \to X$ given by $i(x) = x$ is continuous.*

Proposition 3.5. *If $f : X \to Y$ is a continuous function and $A \subseteq X$ is a subspace, then $f \restriction A : A \to Y$, the restriction of f to A, is continuous.*

(Note that this generalizes the previous proposition, since an inclusion map is simply the restriction of the identity function.)

Proposition 3.6. *If $f : X \to Y$ is a homeomorphism and A is a subspace of X. Then $B := f(A)$ is a subspace of Y, and $f \restriction A : A \to B$ is a homeomorphism.*

Proposition 3.7. *Let B be a subspace of Y, and let $f : X \to B$ be a continuous function. Then $f : X \to Y$ is continuous (we are not altering the function here at all, just expanding the space we think of as its codomain).*

The following lemma says that you can "glue" or "paste" together two continuous functions defined on closed or open subspaces, as long as they agree on the intersection of those subspaces.

Lemma 3.8 (Pasting lemma)**.** *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let $A, B \subseteq X$ be both closed (or both open) subsets of X such that $X = A \cup B$, thought of as subspaces. Suppose $f : A \to Y$ and $g : B \to Y$ are continuous functions that agree on $A \cap B$ (ie. $f(x) = g(x)$ for all $x \in A \cap B$). Define $h : X \to Y$ by*

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

Then h is continuous.

Remark 3.9. The above proposition need not be true if A and B are not both closed or both open. For a very simple example of this, let $X = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$, thought of with its subspace topology from $\mathbb{R}_{\text{usual}}$. Let $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ (which is open) and $B = \{0\}$ (which is not open). Then A and B are disjoint, discrete subspaces of X, and so *any* functions $f : A \to Y$ and $g : B \to Y$ are continuous and satisfy the assumptions of the lemma. Choosing $Y = \mathbb{R}$ and letting $f(x) = x$, $g(0) = -7$ contradicts the conclusion of the lemma, since for example the sequence $\{h(\frac{1}{n})\} = \{f(\frac{1}{n})\} = \{\frac{1}{n}\}$ doesn't converge to $h(0) = g(0) = -7$ in $\mathbb{R}_{\text{usual}}$.

We prove one of the propositions above, just as an example.

Proof of Proposition 3.2. Let $(X, \mathcal{T})$ be a topological space, Y a subspace of X, and A a subset of Y. Let $\mathcal{T}_{Y,A}$ be the subspace topology A inherits from Y, and let $\mathcal{T}_{X,A}$ be the subspace topology A inherits from X. We want to show that $\mathcal{T}_{Y,A} = \mathcal{T}_{X,A}$.

So let $U \in \mathcal{T}_{Y,A}$. By definition $U = V \cap A$ for some V open in the subspace topology on Y. But then $V = W \cap Y$ for some $W \in \mathcal{T}$. So, $U = (Y \cap W) \cap A = W \cap A \in \mathcal{T}_{X,A}$, since $A \subseteq Y$.

On the other hand, let $U \in \mathcal{T}_{X,A}$. By definition $U = W \cap A$ for some $W \in \mathcal{T}$. Again since $A \subseteq Y$, this is the same as $U = W \cap Y \cap A = (W \cap Y) \cap A$. Since $W \cap Y$ is an element of the subspace topology on Y, we have that $U \in \mathcal{T}_{Y,A}$, as required. □

4 Hereditary properties

Now that we have some basics down, we can explore how topological properties "move" between spaces and their subspaces. That is, we will define a property of topological properties that means it plays nicely with subspaces.

Definition 4.1. *A topological property ϕ is said to be hereditary if every subspace of a topological space with ϕ has ϕ.*

That is, ϕ is hereditary if whenever X is a space with property ϕ and Y is a subspace of X, then Y has ϕ.

This definition is a little abstract, so we study some easy examples.

Proposition 4.2. *The Hausdorff property is hereditary.*

Proof. Not much to do here. Suppose X is Hausdorff and Y is a subspace of X. Fix two distinct point $x, y \in Y$. Then $x, y \in X$, and so there exist open subsets U and V of X, containing x and y respectively, such that $U \cap V = \emptyset$. But then $U \cap Y$ and $V \cap Y$ are open subsets of Y with its subspace topology, containing x and y respectively, and $(U \cap Y) \cap (V \cap Y) = \emptyset$, as required. □

The following properties are also all hereditary. The proofs are all very straightforward.

1. T_0 and T_1.
2. Countable.
3. First countable.
4. Second countable.

It may seem as though all topological invariants are hereditary, but this is definitely not the case.

Example 4.3. Separability and ccc are not hereditary properties.

For a very nice example of this, consider $(\mathbb{R}, \mathcal{T}_7)$. Then $\overline{\{7\}} = \mathbb{R}$, since every nonempty open set contains 7. In particular, this means the space is separable. However $X := \mathbb{R} \setminus \{7\}$ as a subspace is discrete and uncountable, and therefore not separable. This same example also shows that ccc is not hereditary.

However, every subspace of $\mathbb{R}_{\text{usual}}$ *is* separable and ccc. This is easy to see because on the one hand every second countable space is separable and ccc (you proved every second countable space is separable in a Big List problem, and the latter proof should be clear), and on the other hand second countability is hereditary.

So to be clear, since $\mathbb{R}_{\text{usual}}$ is second countable, every one of its subspaces is also second countable and therefore separable and ccc.

So while separable and ccc are not hereditary, they do seem to be hereditary when we restrict our attention to $\mathbb{R}_{\text{usual}}$, or more generally to any second countable space. This presents us with another property of topological spaces that is worth defining.

Definition 4.4. *Let ϕ be a topological property. A topological space $(X, \mathcal{T})$ is said to be hereditarily ϕ if X and every subspace of X has ϕ.*

In light of this definition, the argument given at the end of the previous example proves that $\mathbb{R}_{\text{usual}}$ is hereditarily separable. Really we proved the following more general fact.

Proposition 4.5. *Every second countable topological space is hereditarily separable and hereditarily ccc.*

The property of being "hereditarily ϕ" is sort of a meta-topological property. These are topological invariants though, as is straightforward—if somewhat abstract—to prove.

Proposition 4.6. *Let ϕ be a topological invariant. Then the property of being hereditarily ϕ is also a topological invariant.*

Proof. Suppose ϕ is a topological invariant, and let $(X, \mathcal{T})$ be a topological space that is hereditarily ϕ. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a homeomorphism. We wish to show that $(Y, \mathcal{U})$ is hereditarily ϕ.

So let $B \subseteq Y$ be a subspace. Then $f^{-1}(B)$ is a subspace of X and therefore has ϕ. But B is homeomorphic to $f^{-1}(B)$ (since f^{-1} is also a homeomorphism), and therefore B has ϕ since ϕ is a topological invariant. □