12. Metric spaces and metrizability

1 Motivation

By this point in the course, this section should not need much in the way of motivation. From the very beginning, we have talked about $\mathbb{R}^n_{\text{usual}}$ and how relatively easy it is to prove things about it due to the fact that the topology is defined by a distance function. We have the Euclidean distance function (a distance function is more properly called a metric) $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d(x, y) = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2},$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. This metric allowed us to define ε-balls:

$$B_\varepsilon(x) = \{\, y \in \mathbb{R}^n \,:\, d(x, y) < \varepsilon \,\}.$$

We discovered that the collection of all such ε-balls forms a basis, and the topology this basis generates is the usual topology. Everything was defined in terms of d.

We later saw that $\mathbb{R}^n_{\text{usual}}$ is Hausdorff, and the proof was easy. Given two points x and y, we can put a ball around each of them with radius $\frac{1}{2}d(x, y)$, and everything works as expected. We similarly saw that $\mathbb{R}^n_{\text{usual}}$ is regular, normal, and first countable, and all of those proofs were also relatively easy because of the fact that the topology is generated by ε-balls.

It turns out that any space whose topology is defined by ε-balls in this way is equally nice, for the most part. These are metric spaces. In this section we will formally define them and explore their properties. As you will see, they are about as well-behaved as we could hope. They are the ideal toward which other topological spaces aspire. In fact, they are almost so nice as to not be very interesting from the point of view of the properties we have studied so far.

We will also explore how we can tell if a given topological space is a metric space. Of course, if we are given a basis for a topology made of ε-balls for some metric we will know it is a metric space, but what about when we do not have such a convenient description? Is the Sorgenfrey Line a metric space, for example? We know that the usual distance function on $\mathbb{R}$ does not generate this topology, but what if some other, weirder distance function does do so? Is ω_1 a metric space? Is a discrete or indiscrete space a metric space? We will answer all of these questions.

Finally, we will discuss two interesting properties that metric spaces can have.

2 Metric spaces

The core definition here is that of a metric, or a distance function. The properties in the following definition are what a sensible notion of distance should satisfy.

Definition 2.1. *Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a metric on X if it satisfies the following properties:*

1. *For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.*
2. *$d(x, y) \geq 0$ for all $x, y \in X$.*
3. *(Symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$.*
4. *(Subadditivity) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.*

In this case, the pair (X, d) is called a metric space. If (X, d) is a metric space, $x \in X$, and $\varepsilon > 0$, the set

$$B_\varepsilon(x) = \{ y \in X \,:\, d(x, y) < \varepsilon \}$$

is called the ε-ball centred at x.

The first two properties in the above definition are collectively called being positive definite. You may also recognize the last property as the triangle inequality, though the name "subadditivity" generalizes better in other contexts.

Some examples:

Example 2.2.

1. The usual metric on $\mathbb{R}$ is defined by $d(x, y) = |x - y|$.
2. The usual metric or Euclidean metric on $\mathbb{R}^n$ is defined as in the Motivation section above. This of course generalizes the definition of the usual metric on $\mathbb{R}$.
3. The square metric on $\mathbb{R}^2$ is defined by

   $$d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\},$$

   where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Make sure to draw some pictures and get a feeling for how this metric acts in comparison with the usual metric. What does an ε-ball according to this metric look like? (There is a hint in its name!)

   This metric also generalizes in the obvious way to $\mathbb{R}^n$. Make sure to take a moment to imagine what ε-balls look like in $\mathbb{R}^3$, for example.

   It is worth noting that this metric is sometimes called the supremum metric, for reasons we may encounter later.
4. The taxicab metric on $\mathbb{R}^2$ is defined by

   $$d(x, y) = |x_1 - y_1| + |x_2 - y_2|.$$

   Again, draw a picture to get a feeling for this metric. What do the ε-balls according to this metric look like?

5. Let X be a set. Define the function $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

   This is called the discrete metric on X.

6. Let $X = C[0, 1]$, the set of all continuous functions $f : [0, 1] \to \mathbb{R}$. We define three metrics $d_1, d_2, d_\infty : X \times X \to \mathbb{R}$ by:

$$d_1(f, g) = \int_0^1 |f(x) - g(x)|\, dx$$

$$d_2(f, g) = \sqrt{\int_0^1 \big(f(x) - g(x)\big)^2\, dx}$$

$$d_\infty(f, g) = \max\{\, |f(x) - g(x)| \,:\, x \in [0, 1]\,\}$$

   These metrics, and many others like them (you can define d_p for any $p \in (0, \infty) \cup \{\infty\}$), are the subject of much real and functional analysis. We will not be dealing with them a great deal in this course, but they are worth bringing up as important definitions in other fields.

   Before going on, take a moment to think about how these three metrics are similar to the taxicab metric, the usual metric, and the square metric on $\mathbb{R}^2$, respectively.

Exercise 2.3. Prove that all of the function in the preceding examples are actually metrics.

3 Metric topologies

The point of defining metric spaces is of course that they come pre-packaged with a nice topology. We formally define that topology here.

Proposition 3.1. *Let (X, d) be a metric space. Then the collection*

$$\mathcal{B}_d := \{\, B_\varepsilon(x) \,:\, x \in X,\, \varepsilon > 0\,\}$$

is a basis on X.

Proof. This is a proof you have already done for the usual metric on $\mathbb{R}^n$, but we will repeat it here for the sake of completeness.

To prove that $\mathcal{B}_d$ is a basis, we must show that it covers X, and that for every $B_1, B_2 \in \mathcal{B}_d$ and every $x \in B_1 \cap B_2$, there is a $B \in \mathcal{B}_d$ such that $x \in B \subseteq B_1 \cap B_2$. It is immediate that $\mathcal{B}_d$ covers X, since $x \in \mathcal{B}_1(x)$ for every $x \in X$.

Now, let $x \in B_{\varepsilon_1}(x_1) \cap B_{\varepsilon_2}(x_2)$. Define

$$\varepsilon := \min\{\varepsilon_1 - d(x, x_1), \varepsilon_2 - d(x, x_2)\}.$$

The reader is strongly encouraged at this stage to draw a picture of the situation. The definition of ε here might look confusing, but the situation will seem very simple from a picture.

Note that $\varepsilon > 0$ since it is the smaller of two positive numbers. We will show that $B_\varepsilon(x) \subseteq B_{\varepsilon_1}(x_1) \cap B_{\varepsilon_2}(x_2)$, which will finish the proof.

Let $y \in B_\varepsilon(x)$. Then

$$\begin{aligned} d(x,y) < \varepsilon \leq \varepsilon_1 - d(x,x_1) \\ \Rightarrow \quad d(x,y) + d(x,x_1) < \varepsilon_1 \\ \Rightarrow \quad d(y,x_1) \leq d(x,y) + d(x,x_1) < \varepsilon_1 \\ \Rightarrow \quad y \in B_{\varepsilon_1}(x_1). \end{aligned}$$

Where on the second to last line we made use of the triangle inequality. A very similar argument shoes that $y \in B_{\varepsilon_2}(x_2)$, as required. □

Having established that $\mathcal{B}_d$ is a basis on X, we state our main definition.

Definition 3.2. *Let (X, d) be a metric space. The topology generated by the basis $\mathcal{B}_d$ on X is called the* metric topology *(or more properly the* metric topology generated by d*) on X.*

Remark 3.3. This remark is for the very pedantic reader. Strictly speaking, a "topological space" is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology on X. A "metric space" is a pair (X, d) where X is a set and d is a metric on X. These two objects are not the same, even if the topology $\mathcal{T}$ is the metric topology generated by d. We now know that given a metric space (X, d), there is a canonical topological space associated to it. On the other hand, given a topological space $(X, \mathcal{T})$, there may be many metrics on X (ie. many metric spaces whose underlying set is X) that have this space associated to them. We will explore this a bit later.

All of this is to say that a "metric space" does not have a topology strictly speaking, though we will often refer to metric spaces as though they are topological spaces.

Example 3.4.

1. All three of the metrics on $\mathbb{R}^2$ we defined in Example 2.2 generate the usual topology on $\mathbb{R}^2$. Make sure to convince yourself of this using a picture (a formal proof should not be necessary once you understand your picture).

2. The discrete metric on a set X generates the discrete topology on X.

4 Metrizability

Now that we have defined metric topologies, we want to describe some of the nice properties they have. Before that, however, we want to give a name to those topological spaces whose topologies are metric topologies. As we have seen, every metric space has a canonical topology associated with it, but a given topology on a set can be generated by many different metrics.

Definition 4.1. *A topological space $(X, \mathcal{T})$ is said to be metrizable if there is a metric d on X that generates $\mathcal{T}$.*

Due to the fact that very different looking metrics can generate the same topology, we usually talk about metrizable spaces rather than about metric spaces. The particular details of a metric are often not important to us. We care about the topologies they generate.

As a topological property, metrizability is very well-behaved, as the following few propositions demonstrate.

Proposition 4.2. *Metrizability is a topological invariant.*

Proof. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be a topological spaces, and let $f : X \to Y$ be a homeomorphism. Suppose $d : X \times X \to \mathbb{R}$ is a metric that generates $\mathcal{T}$. Define $\rho : Y \times Y \to \mathbb{R}$ by

$$\rho(y_1, y_2) := d(f^{-1}(y_1), f^{-1}(y_2)).$$

That is, we define the distance between points in Y to equal the distance of the corresponding points in X, where the correspondence is the homeomorphism.

We first check that ρ is a metric on Y. ρ is positive definite since f is injective and d is positive definite. ρ is symmetric since d is symmetric. It remains to check that ρ is subadditive, which is left to the reader as an easy exercise.

We now check that ρ generates $\mathcal{U}$, which is to say that the basis $\mathcal{B}_\rho$ generates $\mathcal{U}$. To check that $\mathcal{B}_\rho \subseteq \mathcal{U}$, let $U = B_\varepsilon(y) \in \mathcal{B}_\rho$ be a ε-ball according to ρ. Then $U = f\left(B_\varepsilon(f^{-1}(y))\right)$ (where this ε-ball is according to d), which is open since f is open.

Next, let $U \in \mathcal{U}$ and $y \in U$. We need to find an $\varepsilon > 0$ such that $B_\varepsilon(y) \subseteq U$. Again we use the fact that in this metric, f does not change distances. $f^{-1}(U)$ is an open subset of X containing $f^{-1}(y)$, so since X is metrizable there is an $\varepsilon > 0$ such that $B_\varepsilon(f^{-1}(y)) \subseteq f^{-1}(U)$. But then $B_\varepsilon(y) \subseteq U$, as required. □

Proposition 4.3. *Metrizability is finitely productive.*

Proof. This proof is just like what happens in $\mathbb{R}^n_{\text{usual}}$. Let $(X_1, d_1), \dots, (X_n, d_n)$ be metric spaces. Then any of the following three metrics on $X_1 \times \cdots \times X_n$ generates the product topology. In each case, let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$.

$$d(x, y) = \sqrt{d_1(x_1, y_1)^2 + \cdots + d_n(x_n, y_n)^2}$$

$$d(x,y) = \max\{d_1(x_1,y_1),\ldots,d_n(x_n,y_n)\}$$

$$d(x,y) = d_1(x_1,y_1) + \cdots + d_n(x_n,y_n)$$

Checking that all of these metrics generates the product topology on $X_1 \times \cdots \times X_n$ is left as an easy, if tedious, exercise for the reader. Pictures and analogies with $\mathbb{R}^2$ are strongly encouraged. □

Proposition 4.4. *Metrizability is hereditary.*

Proof. Let (X,d) be a metric space, and let $A \subseteq X$. Use the same metric on A! It is easy to check that it generates the subspace topology. □

Before we go on, notice that the last two propositions open up a lot more examples of metric spaces for us. Before this we basically only knew $\mathbb{R}^n_{\text{usual}}$ and discrete spaces, but now we have any products and subspaces of those to work with. For example the circle $S^1 \subseteq \mathbb{R}^2$, the torus $S_1 \times S_1 \subseteq \mathbb{R}^3$, the set $GL(3,\mathbb{R})$ of invertible 3×3 matrices with real coefficients (seen as a subspace of $\mathbb{R}^9$), $\omega + 1$ (which we have seen is homeomorphic to a subspace of $\mathbb{R}_{\text{usual}}$), etc. are all metrizable spaces.

5 Properties of metrizable spaces

We started this note by promising that metric spaces are very nice. We will justify that claim in this section.

Proposition 5.1. *Every metrizable space is T_2, T_3, and T_4.*

Proof. **Exercise.** (Note that in Section 9 of the lecture notes we proved that $\mathbb{R}^n_{\text{usual}}$ is normal. Exactly the same proof shows that every metrizable space is normal. The proof that a metrizable space is Hausdorff is more or less immediate from a picture.)

□

So metric spaces, like the order topologies we learned about recently, are all very separative. That is, they are good at separating things from one another with open sets. The ability to "draw balls" around things is very powerful.

The behaviour of metric spaces with respect to countability properties is a little more subtle.

Proposition 5.2. *Every metric space is first countable.*

Proof. The proof that $\mathbb{R}^n_{\text{usual}}$ is first countable amounted to noticing that the collection

$$\mathcal{B}_x = \left\{ B_{\frac{1}{n}}(x) \,:\, n \in \mathbb{N} \right\}$$

is a countable local basis at x. The same proof works in any metrizable space. Simply choose a metric that generates the topology, and form the same collection of $\frac{1}{n}$-balls around any point. □

After that, the natural questions to ask are whether metric spaces have the other countability properties: second countability, separability, and the countable chain condition. Earlier we learned that every discrete space is metrizable, and so since there are uncountable discrete spaces (like $\mathbb{R}_{\text{discrete}}$, for example) we can immediately conclude that not *every* metrizable space necessarily has any of these properties.

What *is* nice however is that for a metrizable space, these three properties are the same.

Proposition 5.3. *Let $(X, \mathcal{T})$ be metrizable. Then the following are equivalent.*

1. *$(X, \mathcal{T})$ is separable.*
2. *$(X, \mathcal{T})$ is second countable.*
3. *$(X, \mathcal{T})$ has the countable chain condition.*

Proof. We have done some of this already. On the Big List, you showed that every second countable topological space (metrizable or otherwise) is separable, and that every separable topological space is ccc. In the numbering of this Proposition, that means we already know (2) ⇒ (1) ⇒ (3) for any topological space.

All that remains to be shown is that every metrizable space with the countable chain condition is second countable. This proof (with hints) will appear on the Big List. □

6 Putting things together

We now know a great deal about metrizable spaces. Hopefully enough that you believe metrizable spaces are very nice. We know some "basic" metric spaces: $\mathbb{R}^n_{\text{usual}}$ and any discrete space, along with any subspaces and any finite products of those.

The question we should be asking ourselves now is: What other spaces are metrizable? If these are the nicest sorts of spaces, we should hope to be able to identify them easily. The question of whether a given topological space is metrizable turns out to be a very rich and interesting one that has motivated a great deal of research over the years. There is a whole group of theorems called "metrization theorems" that concern which combinations of other topological properties imply metrizability. We will learn about one of these, the Urysohn Metrization Theorem, later in the course. Its proof is very interesting.

For now though, the results from the previous section give us some tools for easily checking that a space is *not* metrizable.

Example 6.1.

1. If X has more than two points, then $(X, \mathcal{T}_{\text{indiscrete}})$ is not metrizable, since it is not Hausdorff.

2. Any non-first countable space is not metrizable. In particular, $\omega_1 + 1$, $\mathbb{R}_{\text{co-countable}}$ and $\mathbb{R}_{\text{co-finite}}$ are not metrizable.

3. The Sorgenfrey line is not metrizable, since it is separable but not second countable.
4. The Sorgenfrey Square $X = \mathbb{R}_{\text{Sorgenfrey}} \times \mathbb{R}_{\text{Sorgenfrey}}$ is not metrizable for the same reason as the previous example, but also because $\mathbb{R}_{\text{Sorgenfrey}}$ is non-metrizable and homeomorphic to $\mathbb{R} \times \{0\}$ as a subspace of X.

It turns out that ω_1 is also not metrizable, though we do not yet have the tools to prove it easily. Recall that since it has an order topology, it is T_2, T_3, and T_4. We also saw that it is first countable, not separable (and therefore not second countable), and we used Zorn's Lemma to prove that it does not satisfy the countable chain condition. So far it looks like it could be metrizable. We will return to this question later in the course.

7 Two properties metric spaces can have

In this last section we mention two properties that metric spaces can have, which do not make sense for other topological spaces.

The first one is really a property of metric spaces (X, d) rather than of metrizable spaces. Recall that different metrics on the same set can generate the same topology. This property *does* care about the specific metric being used. It is still useful for us in topology, as we will soon see.

Definition 7.1. *Let (X, d) be a metric space. We define the diameter of the space to be*

$$\operatorname{diam}(X) := \sup \{ d(x, y) \,:\, x, y \in X \} .$$

We allow $\operatorname{diam}(X) = \infty$ the set of distances above is unbounded. (X, d) is said to be bounded if $\operatorname{diam}(X) < \infty$.

This should seem like a reasonable definition. For example, we know that every discrete metric space is bounded, $(0, 1)$ with its usual metric is bounded, while $\mathbb{R}$ with its usual metric is not bounded. Boundedness of a metric space is very useful in certain proofs.

You should be asking yourself, so what? Why should we care about a property of the metric itself? We learned earlier that metrizable spaces are what we should study, so we should not care about properties of specific metrics.

The previous examples illustrate the nice fact here. Even though $\mathbb{R}$ and $(0, 1)$ with their usual metrics are unbounded and bounded, respectively, we already know they are homeomorphic as topological spaces. In fact, the following proposition says that we can assume every metrizable space is generated by a bounded metric.

Proposition 7.2. *Let $(X, \mathcal{T})$ be metrizable. Then $\mathcal{T}$ can be generated by a bounded metric.*

Proof. Suppose $d : X \times X \to \mathbb{R}$ is a metric that generates $\mathcal{T}$. Define another metric $\overline{d}$ on X by

$$\overline{d}(x, y) = \min\{1, d(x, y)\}.$$

Then $\overline{d}$ also generates $\mathcal{T}$. □

Exercise 7.3. Complete the proof of the previous Proposition.

Another bounded metric we could have used in the proof is

$$d_0(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Also verify that this one works in the proof.

The second property is one that should be familiar from first or second year calculus. The definition here is exactly the same as it was there, but this is the first time we have had the ability to express it in our more general context.

Definition 7.4. *Let (X, d) be a metric space (just a metric space, no topology needed for this definition). A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be a* <u>*Cauchy sequence*</u> *if*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } d(x_n, x_m) < \varepsilon \text{ for all } n, m > N.$$

Intuitively, a Cauchy sequence is one in which the terms get arbitrarily close to one another as you go farther out in the sequence.

Definition 7.5. *Let (X, d) be a metric space (this time thought of as a topological space with its metric topology). (X, d) is said to be* <u>*complete*</u> *if every Cauchy sequence in X converges.*

For example, you should already know that $\mathbb{R}_{\text{usual}}$ is complete. $(0, 1)$ and $\mathbb{Q}$, both seen as subspaces of $\mathbb{R}_{\text{usual}}$, are easily seen to not be complete.

Note in particular that this shows that completeness is not a topological invariant. As with boundedness, whether or not a sequence is Cauchy depends on the particular metric being used. If two metric spaces (thinking of them with specific, fixed metrics) are homeomorphic as topological spaces, the image of a Cauchy sequence need not be Cauchy. For example, we know that $\mathbb{R}_{\text{usual}}$ is homeomorphic to $(-\frac{\pi}{2}, \frac{\pi}{2})$ via arctan, but under that map the sequence $1, 2, 3, 4, \ldots$ in $\mathbb{R}$—which is obviously not Cauchy with respect to the usual metric on $\mathbb{R}$—will map to a Cauchy sequence in $(0, 1)$.

The problem here is that homeomorphisms, as nice as they are, are not good ways of comparing metric spaces. They compare topologies well, but they are not able to preserve properties that use the specifics of metrics. A function that preserves metric space structure is called an <u>isometry</u>. To be clear:

Definition 7.6. *Let (X_1, d_1) and (X_2, d_2) be metric spaces. A function $f : X_1 \to X_2$ is said to be an* <u>*isometry*</u> *if $d_1(x, y) = d_2(f(x), f(y))$ for all $x, y \in X_1$.*

Note that any isometry is injective. A bijective isometry is sometimes called a <u>*global isometry*</u> *or an* <u>*isometric isomorphism*</u>*. If such a function exists, we say X_1 and X_2 are* <u>*isometric*</u>*.*

A consequence of this is that isometric metric spaces are not only homeomorphic as topological spaces (since their topologies are generated by essentially the same metric), but the properties like boundedness and completeness that depend on the specifics of the metric are also preserved. Boundedness and completeness are metric space invariants, but not topological invariants.

Note in particular that while proving Proposition 4.2, the metric ρ we defined on Y made (Y, ρ) isometric to (X, d).