18. Connectedness

1 Motivation

Connectedness is the sort of topological property that students love. Its definition is intuitive and easy to understand, and it is a powerful tool in proofs of well-known results.

Roughly speaking, a connected topological space is one that is "in one piece". The way we will define this is by giving a very concrete notion of what it means for a space to be "in two or more pieces", and then say a space is connected when this is not the case.

We will also explore a stronger property called path-connectedness. A path-connected space is one in which you can essentially walk continuously from any point to any other point.

Along the way we will see some novel proof techniques and mention one or two well-known results as easy corollaries. We will also have occasion to define one of the more novel "standard" spaces that topologists use, called the Topologist's Sine Curve.

2 Basic definitions and examples

Recall that the notation $A \sqcup B$ (ie. $\sqcup$ instead of $\cup$) is used to denote a union in which A and B are disjoint.

Definition 2.1. *A topological space $(X, \mathcal{T})$ is said to be disconnected if there exist disjoint nonempty subsets $A, B \subseteq X$ such that $X = A \sqcup B$, and $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. If $(X, \mathcal{T})$ is not disconnected, it is said to be connected.*

Just like with compactness we will often refer to subsets of topological spaces being connected, and in doing so we mean that the subset with its subspace topology is connected.

Before going on, we state some of the many equivalent forms of this definition. The proof that these are all equivalent is basically immediate.

Proposition 2.2. *The following are equivalent for a topological space $(X, \mathcal{T})$.*

1. *$(X, \mathcal{T})$ is disconnected.*
2. *There exist nonempty, disjoint, open sets $A, B \subseteq X$ such that $X = A \sqcup B$.*
3. *There exist nonempty, disjoint, closed sets $A, B \subseteq X$ such that $X = A \sqcup B$.*
4. *There is a nontrivial clopen subset of X. That is, there is a subset $A \subseteq X$ that is both open and closed, and A is not X or $\emptyset$.*

Proof. **Exercise.** $\square$

A pair of sets $A, B \subseteq X$ witnessing that X is disconnected is often called a disconnection of X.

Without further ado, here are see some examples. These results should all feel true given your natural intuition about spaces being in one or more "pieces", though some of their proofs are not obvious (and will follow from our subsequent discussions).

Example 2.3. 1. $\mathbb{R}_{\text{usual}}$ is connected, as is $\mathbb{R}^n_{\text{usual}}$ for all n, and even $\mathbb{R}^{\mathbb{N}}_{\text{prod}}$.

2. Any interval in $\mathbb{R}$ is connected (ie. any set of the form (a, b), $(a, b]$, $[a, b)$, or $[a, b]$ for $a < b \in \mathbb{R}$).

3. S^1 (the unit circle in $\mathbb{R}^2$) is connected.

4. $\mathbb{R}^2 \setminus \{(0, 0)\}$ with its usual subspace topology is connected.

5. More generally, if $A \subseteq \mathbb{R}^2$ is countable, then $\mathbb{R}^2 \setminus A$ is connected. In particular, $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is connected. (Careful, this is not the set of all points with both coordinates irrational; it is the set of points such that at least one coordinate is irrational.)

6. Any hyperconnected space is trivially connected. (Recall that a space is hyperconnected if any pair of nonempty open sets intersect.) In particular, for any set X, $(X, \mathcal{T}_{\text{indiscrete}})$ is connected, as are $(\mathbb{R}, \mathcal{T}_{\text{ray}})$, $(\mathbb{R}, \mathcal{T}_7)$ and any other particular point topology on any set, the co-countable and co-finite topologies on uncountable and infinite sets, respectively, etc.

7. $\mathbb{R}_{\text{Sorgenfrey}}$ is disconnected. In fact any zero dimensional space (that is not indiscrete) is disconnected, as is easy to see. (Recall that a topological space is zero dimensional if it has a basis consisting of clopen sets.)

8. If X is a set with more than one point, $(X, \mathcal{T}_{\text{discrete}})$ is disconnected.

9. $\omega + 1$, ω_1 and $\omega_1 + 1$ are all disconnected, since in each space the minimal element of the order is clopen as a singleton. More generally, any well-order with its order topology is disconnected (provided that it contains more than one point).

10. $\mathbb{R} \setminus \{0\}$ (with its usual subspace topology) is disconnected. If you have been doing the exercises on the Big List, you will recognize that 0 (or indeed any real number) is a cut point of $\mathbb{R}_{\text{usual}}$.

11. $\mathbb{R}^2 \setminus \{\text{the } x\text{-axis}\}$ is disconnected.

12. $\mathbb{R}^{\mathbb{N}}_{\text{box}}$ is disconnected, though it may not seem as though it should be. The collections of bounded and unbounded sequences form a disconnection. Checking this will be on the Big List.

We prove the most important of these results first, and from its proof derive a much more general result without much difficulty. This proof should remind you, at least superficially, of the creeping along proof that $[0, 1]$ is compact.

Theorem 2.4. $\mathbb{R}_{usual}$ *is connected.*

Proof. This is a proof by contradiction, so we begin by assuming that $\mathbb{R}$ is disconnected. Then there is an open subset X such that $\mathbb{R} \setminus X$ is also open, and both are nonempty. Let $a \in X$ and $b \in \mathbb{R} \setminus X$, and suppose without loss of generality that $a < b$.

Define a subset A of X by:

$$A := \{ x \in X : x < b \}$$

or in other words, $A = X \cap (-\infty, b]$. Note that A is nonempty since $a \in A$, and clearly A is bounded above (by b, for example), so A has a least upper bound; call it $p = \sup A$.

Suppose $p \in X$, which in particular implies that $p < b$. Since X is open there is some $\epsilon > 0$, which we may assume is less than $b - p$, such that $(p - \epsilon, p + \epsilon) \subseteq X$. But then any $t \in (p, p + \epsilon)$ would be in A, contradicting the fact that p is an upper bound of A.

This means $p \in \mathbb{R} \setminus X$ which by assumption is also open, and so again there is $\epsilon > 0$ such that $(p - \epsilon, p + \epsilon) \subseteq \mathbb{R} \setminus X$. But then any number $t \in (p - \epsilon, p)$ would be a strictly smaller upper bound for A, contradicting the fact that p is the least upper bound of A.

So in summation we have shown that p cannot be in X or in $\mathbb{R} \setminus X$, which is certainly impossible. □

Not so hard, right? Examining this argument brings up an interesting question though: what properties of $\mathbb{R}_{\text{usual}}$ did we actually use in the proof? It looks like we used the fact that the topology is generated by a metric, but if you think about it carefully you will discover that this was not actually necessary. For example when we assumed $p \in X$, we obtained an open subset U (which happened to be a metric ball) such that $p \in U \subseteq X$, and found an element larger than p inside U. We were really using the fact that $\mathbb{R}$ is a "dense" linear order. We also made use of the ability to find a least upper bound of A. These observations lead us to the following definitions for a linear order.

Definition 2.5. *Let* $(L, \leq)$ *be a linear order.*

- $(L, \leq)$ *is called* Dedekind complete *if every nonempty subset of* L *that is bounded above has a least upper bound (all in the sense of* $\leq$*).*

- $(L, \leq)$ *is said to have a* gap *if there exist elements* $a < b \in L$ *such that* $(a, b) = \emptyset$*. Conversely,* $(L, \leq)$ *has no gaps if between every pair of elements of* L *there is another element of* L*.*

And these in turn lead us to the following theorem. Well, half of the following theorem.

Theorem 2.6. *Let* $(L, \leq)$ *be a linear order, thought of as a topological space with its order topology. Then* L *is connected if* and only if *it is Dedekind complete and has no gaps.*

Proof. The $(\Leftarrow)$ direction of this proof is exactly the one we just gave for $\mathbb{R}$.

$(\Rightarrow)$. **Exercise.** □

I present Theorem 2.6 not because it is of critical importance for us, but because it is a good illustration of how some topological properties get defined in the first place. We did a proof which looked simple, then stared at it until we were able to extract the precise conditions required to repeat the same argument in a more general context. We even got a nice surprise at the end, which is that we actually *characterized* connectedness in linear orders, instead of just finding condition that implies connectedness in linear orders.

One direct benefit of Theorem 2.6 for us is that it lets us state the following result essentially for free.

Theorem 2.7. *A subspace of* $\mathbb{R}$ *is connected if and only if it is an interval.*

Proof. **Exercise.** This should be very easy given the previous result.

Here is one thing to be cautious of though. This theorem implies that $(0,1)$ is connected, for example. When you think about $(0,1)$ you may think it is not Dedekind complete, since $(0,1)$ is bounded in $\mathbb{R}$ and yet has no upper bound in $(0,1)$. Remember, however, that $(0,1)$, when thought of as a linear order in isolation, has no upper bound. So, for example, the set $\left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\} \subseteq (0,1)$ is unbounded in this order. The fact that it has no upper bound in $(0,1)$ does not "break" Dedekind completeness. □

Before we go on to talk about properties of connectedness and more complex connected spaces, we state the most useful alternative characterization of connectedness. This may not seem like the most useful way of thinking of connectedness yet, but it will after using it for a bit shortly.

Proposition 2.8. *A topological space* $(X, \mathcal{T})$ *is connected if and only if every continuous function* $f : X \to \{0,1\}$ *is constant (where* $\{0,1\}$ *has the discrete topology).*

Proof. This is quite simple. Note that a function that maps into $\{0,1\}$ is either constant or surjective.

($\Rightarrow$). We prove this by contrapositive. If $f : X \to \{0,1\}$ is surjective, then $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are disjoint, nonempty, open sets, and therefore form a disconnection of X.

($\Leftarrow$). We also prove this by contrapositive. Suppose $A, B \subseteq X$ are disjoint, nonempty, open sets that form a disconnection of X. Define $f : X \to \{0,1\}$ by

$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases}$$

Then it is easy to check that f is continuous and surjective. □

3 Properties of connectedness

We have an established series of tests to which we subject new topological properties, and connectedness should be no exception. Our first question should be whether connectedness is a topological invariant. It is, of course, and in fact we can even show more.

Proposition 3.1. *Let $(X, \mathcal{T})$ be connected, $(Y, \mathcal{U})$ be a topological space, and suppose $f : X \to Y$ is a continuous surjection. Then Y is connected.*

Proof. Suppose for the sake of contradiction that Y is disconnected. Then by Proposition 2.8 there exists a continuous surjection $g : Y \to \{0, 1\}$. But then $g \circ f : X \to \{0, 1\}$ is also a continuous surjection (check this!) contradicting the assumption that X is connected. □

Compactness also had this property. As we remarked then, a topologist would phrase the result of this proposition as "continuous images of connected spaces are connected". See how little we had to do as a result of Proposition 2.8?

Proposition 3.2. *Connectedness is not hereditary.*

Proof. $\mathbb{R}_{\text{usual}}$ is connected, but $\{0, 1\} \subseteq \mathbb{R}$ is discrete with its subspace topology, and therefore not connected. □

Proposition 3.3. *Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$ be connected subsets. Then neither $A \cap B$ nor $A \cup B$ need be connected.*

Proof. Consider the graphs of the functions $f(x) = x^2 - 1$ and $g(x) = -x^2 + 1$, as subsets of $\mathbb{R}^2_{\text{usual}}$. Both of these subsets are connected (being continuous images of $\mathbb{R}$), but their intersection is the two-point discrete set $\{(-1, 0), (1, 0)\}$.

The case for unions is even simpler. Simply take two different points in $\mathbb{R}_{\text{usual}}$. They are connected as singletons, but their union is a two-point discrete set. □

The previous result is not particulary interesting, but it leads us to somewhere nice. We just saw that a union of two connected sets need not be connected, but a union of connected sets that share a point in common *is* connected. This should be pretty intuitive, and it is another example of a proof made very easy by Proposition 2.8.

Proposition 3.4. *Let $(X, \mathcal{T})$ be a topological space, and suppose $\mathcal{A} = \{ A_\alpha : \alpha \in I \}$ is a nonempty collection of nonempty connected subsets of X, with the additional property that $\bigcap \mathcal{A} = \bigcap_{\alpha \in I} A_\alpha \neq \emptyset$. Then $\bigcup \mathcal{A}$ is connected.*

Proof. Fix a point $a \in \bigcap \mathcal{A}$. Now let $f : \bigcup \mathcal{A} \to \{0, 1\}$ be a continuous function. Our goal is to show that f is constant.

Note that for each $\alpha \in I$, the restriction of f to A_α is also continuous and therefore constant since A_α is connected. Pick two points $x, y \in \bigcup \mathcal{A}$. Then $x \in A_\alpha$ and $y \in A_\beta$ for some $\alpha, \beta \in I$. Then by the remark we just made, we have:

$$f(x) = f(a) = f(y),$$

since $a \in A_\alpha$ and $a \in A_\beta$. Since we can do this for any two points, it easily follows that f must be constant. □

This result is extremely useful for the issue of productivity. It basically lets you glue together connected sets and guarantees you will get a connected set at the end as long as all the pieces share a point.

Proposition 3.5. *Connectedness is finitely productive.*

Proof. As usual, we prove that a product of two connected spaces is connected, from which the general result follows inductively. During this argument, you are strongly encouraged to follow along with a picture of the case where $X = Y = \mathbb{R}_{\text{usual}}$.

Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be connected topological spaces, and we show that $X \times Y$ with its product topology is connected. We will make use of the elementary fact that for all $x \in X$, $Y \simeq \{x\} \times Y$, and similarly for all $y \in Y$, $X \simeq X \times \{y\}$ (where these subsets of $X \times Y$ have their subspace topologies inherited from the product topology).

Let $f : X \times Y \to \{0, 1\}$ be a continuous function. We will show that f is constant by showing that $f((x_1, y_1)) = f((x_2, y_2))$ for all $(x_1, y_1), (x_2, y_2) \in X \times Y$. Fix two such points arbitrarily.

Then note that $A_1 = \{x_1\} \times Y$ and $A_2 = X \times \{y_2\}$ are connected subsets of $X \times Y$ (being homeomorphic to Y and X, respectively) that contain (x_1, y_1) and (x_2, y_2), respectively. Also note that $(x_1, y_2) \in A_1 \cap A_2$. Therefore by Proposition 3.4, $A_1 \cup A_2$ is a connected subset of $X \times Y$. Then f must be constant when restricted to $A_1 \cup A_2$, and in particular this means that $f((x_1, y_1)) = f((x_2, y_2))$, as required. □

Corollary 3.6. *$\mathbb{R}^n_{usual}$ is connected for all $n > 1$, as are all metric balls in these spaces.*

What about higher products? Well, there's good news.

Proposition 3.7. *Connectedness is productive.*

Proof. **Exercise.** This is not as scary as it looks. The proof is the same in spirit as the proof given above for finite products. There is just a bit more bookkeeping to do. □

4 Some applications

The first and most notable application of the theory we have built around connectedness is the following result.

Proposition 4.1. *Let $(X, \mathcal{T})$ be a connected topological space and suppose $f : X \to \mathbb{R}_{usual}$ is a continuous function. If $a < b$ are two distinct numbers in $f(X)$, then there is an $x \in X$ such that $a < f(x) < b$.*

Proof. By proposition 3.1, $f(X)$ is connected. By Theorem 2.7 this means $f(X)$ is an interval, from which the result follows immediately. □

(Take a moment to think about how you can generalize the proposition. In particular, what sort of object can you put in place of $\mathbb{R}_{\text{usual}}$ that keeps the result true?) This is a slightly more general version of a result with which you should already be very familiar...

Corollary 4.2 (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function, such that $f(a) < f(b)$. Then for every $v \in (f(a), f(b))$, there is a $c \in (a, b)$ such that $f(a) = v$.*

From Proposition 2.2, we can see that if $(X, \mathcal{T})$ is a connected topological space and $A \subseteq X$ is a nonempty clopen subset, then it must be that $A = X$. A notable use of this technique can be found in complex analysis. We will briefly outline this below. We think of the complex plane as a topological space by identifying $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way.

Definition 4.3. *Let $D \subseteq \mathbb{C}$ be an open, connected set. A function $f : D \to \mathbb{C}$ is called* <u>*analytic*</u> *if for every $z \in D$ there is an $\epsilon > 0$ such that f has a complex power series representation on $B_\epsilon(z)$.*

Theorem 4.4. *Let $D \subseteq \mathbb{C}$ be an open, connected set, and let $f : D \to \mathbb{C}$ be an analytic function. Let $A = \{\, z \in D : f(z) = 0 \,\}$ be its zero set. If A is non-discrete (as a topological subspace), then $f(z) = 0$ for all $z \in D$.*

Proof. We just sketch out the proof here. Define the set:

$$B = \left\{ z \in D \,:\, f^{(k)}(z) = 0 \text{ for all } k \geq 0 \right\},$$

(where by $f^{(k)}$ we mean the k^{th} derivative of f). The proof actually shows that $B = D$ (which obviously implies $A = D$ since $B \subseteq A$) in three steps. First show that $B \neq \emptyset$, then that B is both open and closed.

Claim. $B \neq \emptyset$.

Proof. This is actually the tricky part. Since A is not discrete as a subspace, there is a point $z_0 \in A$ such that any open subset of $\mathbb{C}$ containing z_0 also contains other elements of A. Another way of saying this is that $z_0 \in \overline{A \setminus \{z_0\}}$. Since $\mathbb{C}$ is first countable, we can find a sequence $\{z_n\}_{n\in\mathbb{N}}$ in $A \setminus \{z_0\}$ that converges to z_0. Note that by definition of A, this means $f(z_0) = f(z_n) = 0$ for all $n \in \mathbb{N}$. We show that $z_0 \in B$.

Suppose not. Then there is a least $k \in \mathbb{N}$ such that $f^{(k)}(z_0) \neq 0$. Since f analytic, it is equal to a complex power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

on some ϵ-ball around z_0, where $a_n = \frac{f^{(n)}(z_0)}{n!}$ are the usual Taylor series coefficients. By definition of k, we in fact have:

$$f(z) = (z - z_0)^k g(z),$$

where g is an analytic function such that $g(z_0) \neq 0$. Now since $z_n \to z_0$, there is some tail of this sequence inside the ϵ-ball around z_0 we are considering, and so for all n in this tail:

$$0 = f(z_n) = (z_n - z_0)^k g(z_n).$$

Since $z_n \neq z_0$ for all n, this means $g(z_n) = 0$ for all n in this tail of the sequence. But then we must have $g(z_0) = 0$ since g is continuous, which is a contradiction. □

Claim. *B is open.*

Proof. Let $z \in B$. Then $f^{(k)}(z) = 0$ for all $k \geq 0$, and since f equals its Taylor series on an ϵ-ball around z, its Taylor series is identically zero on that ball. □

Claim. *B is closed.*

Proof. This is essentially the same argument as the previous claim, but instead you show that $D \setminus B$ is open. □

□

5 Path-connectedness

This property is in many ways even more intuitive than connectedness, since it formalizes the intuitive feeling of being able to walk from any point to any other point in a set. This gives us a useful proof technique.

Definition 5.1. *Let $(X, \mathcal{T})$ be a topological space. A path in X is a continuous function $p : [0,1] \to X$. More specifically, given two points $a, b \in X$, a path p in X such that $p(0) = a$ and $p(1) = b$ is called a path from a to b.*

The image of a path is what you intuitively think of as a path: essentially a curve in X. Note that the image of a path is always connected, by Proposition 3.1

Definition 5.2. *A topological space $(X, \mathcal{T})$ is called path-connected if for any distinct $a, b \in X$, there is a path from a to b.*

An important thing to note is that we can glue paths together to get new paths. In other words if in a space $(X, \mathcal{T})$ we have a path p_1 from a to b, and a path p_2 from b to c, then we can easily glue them together to make a path from a to c, in the following way. Define $p : [0,1] \to X$ by:

$$p(x) = \begin{cases} p_1(2x) & x \in [0, \frac{1}{2}] \\ p_2(2x-1) & x \in [\frac{1}{2}, 1] \end{cases}$$

p is simply the path in which we first follow p_1, then follow p_2, walking at "twice the speed" each time. Take a moment to convince yourself that this is well-defined, and that it is a path in X from a to c.

Example 5.3.

1. $\mathbb{R}_{\text{usual}}$ is path-connected. This is more or less immediate, since the closed interval between any two points is homeomorphic to $[0,1]$.

2. $\mathbb{R}^n_{\text{usual}}$ is path-connected. To see this, note that any two points can be connected via a straight path to the origin. Gluing these two paths together makes a path from one point to the other.

3. $\mathbb{R}_{\text{co-finite}}$ is path-connected. To see this most easily, let $a, b \in \mathbb{R}$ be distinct points and let $p : [0,1] \to \mathbb{R}_{\text{co-finite}}$ be any injection such that $p(0) = a$ and $p(1) = b$. Any closed subset of $\mathbb{R}_{\text{co-finite}}$ is finite and so its preimage under p is a finite subset of $[0,1]$ and therefore closed, showing that p is continuous.

   (The set-theoretically minded reader will see something interesting happening here. This proof relied on the fact that p was injective. It would have still worked if p was finite-to-one. But if X is a set with cardinality strictly less than the reals, such that there can be no injective or finite-to-one functions $p : [0,1] \to X$, this proof cannot work. It turns out that such an infinite set with its co-finite topology will necessarily be connected but not path-connected. For example, $(\mathbb{N}, \mathcal{T}_{\text{co-finite}})$ is connected—hyperconnected, in fact—but not path-connected.)

4. Let $\mathbb{M}$ be the set of all 7×7 matrices with real coefficients. Think of $\mathbb{M}$ as $\mathbb{R}^{49}_{\text{usual}}$ in the natural way. Let $\mathbb{L} \subseteq \mathbb{M}$ be the collection of matrices all of whose elements have the same absolute value. That is:

   $$\mathbb{L} = \{\, A = (a_{ij}) \in \mathbb{M} : |a_{ij}| = |a_{nm}| \text{ for all } 1 \leq i, j, m, n \leq 7 \,\}$$

   Then $\mathbb{L}$ is path-connected. Indeed, given any $A \in \mathbb{L}$, let c be the common absolute value of its entries. Let $p : [0,1] \to \mathbb{R}$ be a path from c to 0. Applying this path to each positive element of A, and a similar path from $-c$ to 0 to each negative entry of A, we can form a path joining A to the zero matrix. Therefore any two matrices can be joined by a path by gluing the paths that join each one to the zero matrix.

By now you may have guessed the following fact already.

Proposition 5.4. *Every path-connected topological space is connected.*

Proof. Suppose for the sake of contradiction that $(X, \mathcal{T})$ is path-connected but not connected. Then there is a nonempty open set $A \subseteq X$ such that $X \setminus A$ is also open and nonempty. Let $a \in A$ and $b \in X \setminus A$. Let $p : [0,1] \to X$ be a path from a to b. Then $p([0,1])$, the image of the path, is a connected subset of X, but the sets $p([0,1]) \cap A$ and $p([0,1]) \cap (X \setminus A)$ form a disconnection of it, which is a contradiction. □

We already have an example illustrating that the converse is not true: $(\mathbb{N}, \mathcal{T}_{\text{co-finite}})$ is connected but not path-connected. The proof that it is not path-connected is a little lengthy, and we leave it for the Big List.

Another much more important example of this is the following

Example 5.5 (Topologist's Sine Curve)**.** Be sure to draw a picture of this set. It is hard to understand without one.

The Topologist's Sine Curve is a subspace S of $\mathbb{R}^2_{\text{usual}}$ defined as follows. Let $f(x) = \sin(\frac{\pi}{x})$.

$$S := \{ (x, f(x)) \,:\, 0 < x \leq 1 \} \cup \{ (0, y) \,:\, -1 \leq y \leq 1 \}.$$

In other words, S is the graph of $\sin(\frac{\pi}{x})$ from 0 up to and including and 1, along with a section of the y-axis between -1 and 1, inclusive.

Note that S is a closed, bounded subset of $\mathbb{R}^2$, and is therefore compact. Be sure to convince yourself it is closed. It is best to do this by convincing yourself that every convergent sequence from S converges to something in S (which is equivalent to being closed since $\mathbb{R}^2$ is first countable). The section of the y-axis in S contains the limit points of the interesting convergent sequences from the graph part of S.

S is connected. Clearly the two parts of S (the two parts in the definition of S above) are both path-connected, so the only hope of disconnecting S is to separate the two parts. However, any open set containing the y-axis part necessarily intersects the graph part.

S is not path-connected, however. There are no paths from points in one part of S to the other part. Intuitively, a path that starts on the graph part is constrained to follow the graph, and in doing so can never get to the y-axis part. More technically, suppose p is a path from $(1,0)$ (in the graph part) to $(0,1)$ (in the y-axis part). In particular, this means $p(1) = (0,1)$. Then for each $\epsilon > 0$ there should be a $\delta > 0$ such that $d(p(t), (0,1)) < \epsilon$ whenever $1 - \delta < t \leq 1$. But this is clearly impossible for an ϵ smaller than 2, given the oscillatory behaviour of $\sin(\frac{\pi}{x})$.