Microlocal Analysis, Sharp Spectral Asymptotics and Applications

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Short Contents

Preface lxvi

Introduction lxvii

Volume I. Semiclassical Microlocal Analysis and Local and Microlocal Semiclassical Asymptotics 1

I Semiclassical Microlocal Analysis 1

1 Introduction to Microlocal Analysis 2

2 Propagation of Singularities in the Interior of the Domain 126

3 Propagation of Singularities near the Boundary 196

II Local and Microlocal Semiclassical Spectral Asymptotics in the Interior of the Domain 284

4 General Theory in the Interior of the Domain 285

5 Scalar Operators in the Interior of the Domain. Rescaling Technique 406

6 Operators in the Interior of Domain. Esoteric Theory 519

III Local and Microlocal Semiclassical Spectral Asymptotics near the Boundary 620

7 Standard Local Semiclassical Spectral Asymptotics near the Boundary 621

8 Standard Local Semiclassical Spectral Asymptotics near the Boundary. Miscellaneous 740

Volume II. Functional Methods and Eigenvalue Asymptotics 1

IV Estimates of the Spectrum 799

9 Estimates of the Negative Spectrum 800

10 Estimates of the Spectrum in the Interval 843

V Asymptotics of Spectra 892

11 Weyl Asymptotics of Spectra 893

12 Miscellaneous Asymptotics of Spectra 1060

Volume III. Magnetic Schrödinger and Dirac Operators. 1 1238

VI Smooth Theory in Dimensions 2 and 3 1238

13 Schrödinger Operator. Standard Theory Dimensions 2 and 3 1239

14 2D-Schrödinger Operator with the Strong Degenerating Magnetic Field 1419

15 2D-Schrödinger Operator with the Strong Magnetic Field near the Boundary 1553

VII Smooth Theory in Dimensions 2 and 3 (Continued) 1650

16 agnetic Schrödinger Operator: Short Loops, Pointwise Spectral Asymptotics and Asymptotics of Dirac Energy 1651

17 Dirac Operator with the Strong Magnetic Field 1799

Volume IV. Magnetic Schrödinger and Dirac Operators. 2 1

VIII Non-Smooth Theory and Higher Dimensions 1881

18 2D- And 3D-Magnetic SchrÖdinger Operator with Irregular Coefficients 1882

19 Multidimensional Magnetic Schrödinger Operator. Full-Rank Case 1984

20 Multidimensional Magnetic Schrödinger Operator. Non-Full-Rank Case 2102

IX Magnetic Schrödinger Operator in Dimension 4 2204

21 4D-Schrödinger Operator with the Strong Degenerating Magnetic Field 2205

22 Generic 4D-Schrödinger Operator with the Strong Magnetic Field 2313

X Eigenvalue Asymptotics for Schrödinger and Dirac Operators with the Strong Magnetic Field 2377

23 Eigenvalue asymptotics. 2D Case 2378

24 Eigenvalue asymptotics. 3D Case 2448

Volume V. Applications to Quantum Theory and Miscellaneous Problems 1

XI Application to Multiparticle Quantum Theory 2511

25 No Magnetic Field Case 2512

26 The Case of External Magnetic Field 2600

27 The Case of Self-Generated Magnetic Field 2742

28 The Case of Combined Magnetic Field 2819

Bibliography 2930

XII Articles 3003

Semiclassical Dirichlet to Neumann Operator 3004

Spectral Asymptotics for Fractional Laplacians 3031

Spectral Asymptotics for Dirichlet to Neumann operator in the Domains with Edges 3049

Asymptotics of the Ground State Energy in the Relativistic Settings 3076

Asymptotics of the Ground State Energy in the Relativistic Settings and with Self-Generated Magnetic Field 3095

Complete Semiclassical Spectral Asymptotics for Periodic and Almost Periodic Perturbations of Constant Operator 3118

Complete Differentiable Semiclassical Spectral Asymptotics 3142

Bethe-Sommerfeld Conjecture in Semiclassical Settings 3154

Scott and Thomas-Fermi approximations to electronic density 3175

Pointwise spectral asymptotics out of diagonal near boundary 3241

Pointwise spectral asymptotics out of diagonal near degeneration 3288

Pointwise spectral asymptotics near singularity 3315

29 Introduction 3316

30 Toy-model operator 3320

31 General case 3335

100 years of Weyl's Law 3346

Index 3437

Contents

Preface **lxvi**

Introduction **lxvii**

1. Problems to Study lxvii
2. Weyl's Law lxix
3. Methods lxx
4. Methods. II lxxiv
5. What is in the Book lxxvi

Volume I. Semiclassical Microlocal Analysis and Local and Microlocal Semiclassical Asymptotics lxxvi
Part I. Semiclassical Microlocal Analysis lxxvi
Part II. Local and Microlocal Semiclassical Spectral Asymptotics in the Interior of the Domain lxxx
Part III. Local and Microlocal Semiclassical Spectral Asymptotics near the Boundary . . lxxxiv
Volume II. Functional Methods and Eigenvalue Asymptotics lxxxvi
Part IV. Estimates of the Spectrum. lxxxvi
Part V. Asymptotics of Spectra lxxxix
Volume III. Magnetic Schrödinger and Dirac Operators. 1 xciv
Part VI. Smooth Theory in Dimensions 2 and 3 . xcv
Part VII. Smooth Theory in Dimensions 2 and 3 (Continued) xcvii
Volume IV. Magnetic Schrödinger and Dirac Operators. 2 xcviii
Part VIII. Non-smooth Theory and Higher Dimensions xcviii
Part IX. Magnetic Schrödinger Operator in Dimension 4 c

Part X. Eigenvalue Asymptotics for Schrödinger and Dirac Operators with the Strong Magnetic Field ci
Volume V. Applications to Quantum Theory and Miscellaneous Problems cii
Part XI. Asymptotics of the Ground State Energy of Heavy Atoms and Molecules cii
Part XII. Articles cvi
6. How LSSA are Applied to Classical Asymptotics cvii
7. Paths cix
8. About the Author cx
Elements of Biography cx
My Way to Spectral Asymptotics and Around cx
Weyl Conjecture (66–67 AW) cx
Aftermath (68–79 AW) cxiii
Multiparticle Quantum Theory (79 AW–88 AW) cxv
New Dawn (88–93 AW) cxv
Etc (93 AW–113 AW) cxv
Aftermath (113 AW–present time) cxvi

Volume I. Semiclassical Microlocal Analysis and Local and Microlocal Semiclassical Asymptotics 1

I Semiclassical Microlocal Analysis 1

1 Introduction to Microlocal Analysis 2
1.1 Pseudodifferential Operators 3
1.1.1 Calculus of Pseudodifferential Operators 3
h-Fourier Transform 5
h-DO Representation via h-Fourier Transform; Quantization of Different Flavors 6
Symbols and h-Pseudodifferential Operators; qp-, pq-, Weyl (Symmetric) Quantization. Double Symbols and Their Quantization 7

	Negligible Functions, Symbols, Operators. Going from one Type of Quantization to Another. Symbol and Operator classes .	9
	Algebra: Conjugate and Product of Operators. Symmetric Pseudodifferential Operators	12
	Quantization: Abstract Operator Point of View .	14
	Matrix Pseudodifferential Operators	15
	Ellipticity, Parametrix, Resolvent. Functional Calculus	15
	Traces and Hilbert-Schmidt Norms	16
	Pseudodifferential Operators Applied to an Exponent	19
1.1.2	Analysis of Pseudodifferential Operators	22
	Introduction	22
	Estimates	24
	Remainder Estimates	27
	Classes of Symbols and Pseudodifferential Operators	28
	Compactness	28
	Rescaling	29
	Gårding Inequalities	30
	Parametrix and Inverse	39
	Resolvent	41
	Analytic Functional Calculus of Pseudodifferential Operators	42
	Smooth Functional Calculus of Pseudodifferential Operators	44
	Families of Commuting Operators	47
	Norms and bBounds	47
1.1.3	Pseudodifferential Operators with Analytic Symbols and Logarithmic Uncertainty Principle . . .	48
	Pseudodifferential operators with Infinitely Smooth Symbols	50
	Pseudo-differential Operators with Analytic Symbols	51
1.1.4	Other Classes and Calculi of Pseudodifferential Operators	53
	Classical Pseudodifferential Operators	53
	Classes $\mathsf{S}^m_{\sigma,\varsigma}$ and $\mathbf{\Psi}^m_{\sigma,\varsigma}$	56
	Beals-Fefferman and Hörmander Operators	60
	Global Operators	60

h-Pseudodifferential Operators 61
1.2 Lagrangian Distributions and Fourier Integral Operators 62
1.2.1 Motivation 62
Oscillatory Solutions to Evolution Equations . . . 62
Blow-up 65
Maslov Canonical Operator 65
Propagator 67
1.2.2 Lagrangian Distributions and Manifolds 68
Definition 68
Main Properties 70
Maslov Index 75
Other Properties 76
Complex Phase 77
Coherent States 78
1.2.3 Fourier Integral Operators 79
Final Remarks 87
1.2.4 Metaplectic Operators 88
1.2.5 Germs 90
1.3 Wave Front Sets and Related Topics 93
1.3.1 Principal Notions 93
Philosophy 93
Smooth (Standard) Theory 94
Analytic Theory 96
1.3.2 Properties 96
1.3.3 Propagator 100
1.3.4 Alternative Theories 101
1.4 Elliptic Boundary Value Problems and Associated Operators 101
1.4.1 Pseudodifferential Operators with Transmission Property 101
Causality 101
Transmission Property 102
Pseudodifferential Operators in a Domain 104
1.4.2 Related Operators 105
Boundary (Poisson) Operators 105
Trace and Singular Green Operators 107
Boutet de Monvel Algebra 110
Ellipticity and Parametrices 110
Resolvent and Functional Calculus. I 112

1.4.3 Applications to Elliptic Boundary Value Problems 113
Parametrix Construction 113
Dirichlet-to-Neumann and Similar Operators . . . 116
Adjoint Operator; Self-Adjoint Operator 117
Projectors 119
Semiclassical Theory 120
1.A Appendices 121
1.A.1 Operator $e^{itQ(D)}$ 121
1.A.2 Almost Analytic Compactly Supported Symbols . 122
1.A.3 Stationary Phase Method 123
1.A.4 Function in the Box 124

2 Propagation of Singularities in the Interior of the Domain 126
2.1 Energy Estimates Approach 127
2.1.1 Main Theorem 127
2.1.2 Cauchy Problem. I 133
2.1.3 Cauchy Problem. II 136
2.1.4 Properties of Fundamental Solutions 138
2.1.5 Overdetermined Systems 143
Overdetermined Systems 143
Multi-Time Systems 143
2.2 Geometric Interpretation 145
2.2.1 Discussion of the Microhyperbolicity Condition . 145
Scalar Case 145
Cones of Directions of Microhyperbolicity 145
Examples 148
Microhyperbolicity: Algebraic Theory 149
2.2.2 Main Theorem 150
Main Theorem 150
Discussion 153
2.2.3 Examples of Propagation 154
Constant Coefficients Case 154
Scalar Principal Type (Again) 155
Operators with Completely Factorizing Characteristic Symbol: Non-Involutive Case . . 155
Operators with Completely Factorizing Characteristic Symbol: Involutive Case 159

Operators with not Factorizing Characteristic Symbol. I 160
Operators with not Factorizing Characteristic Symbol. II 161
Operators with not Factorizing Characteristic Symbol. III 163
Operators with not Factorizing Characteristic Symbol. IV 167
2.3 Operators with Rough Symbols 167
2.3.1 Main Theorem 168
Statement of the Main Theorem 168
Proof of the Main Theorem 169
2.3.2 Corollaries 175
2.4 Heisenberg Approach to Propagation of Singularities along Long Bicharacteristics 176
2.4.1 Evolution of Operators 177
Heisenberg Approach 177
Solving Heisenberg Equations. Standard Case . . 180
Discussion 181
Properties of Hamiltonian Flow 182
Solving Heisenberg Equations for Long-Time Evolution 183
2.4.2 Propagation of Singularities 187
Scalar Principal Symbol Case 187
Diagonalization 188
Heisenberg Evolution for Matrix Principal Symbols 191
Main Theorem 193

3 Propagation of Singularities near the Boundary 196
3.1 Energy Estimates Approach 197
3.1.1 Statement of the Main Theorem 197
Preliminary Notes 197
Finally, the Main Theorem 203
3.1.2 Proof of the Main Theorem 204
Reduction to a Special Case 204
Elliptic Theory 210
Main Theorem Refined 215
Proof of Theorem 3.1.13. Step 1 216

Proof of Theorem 3.1.13. Step 2 218
Proof of Theorem 3.1.13. Step 3 222
Proof of Proposition 3.1.14 223
Proof of Proposition 3.1.5 226
3.2 Geometric Interpretation 226
3.2.1 Main Theorem 226
Notations and Definitions 226
Finally, Main Theorem 229
3.2.2 Discussion and Examples 231
General Remarks 231
Constant Coefficients Case 231
Transversal Points 231
Elliptic Points 233
Bicharacteristics tangent to the boundary 234
3.3 Corollaries of Theorem 3.1.7 239
3.3.1 Finite Speed of Propagation and Related Topics . 239
Finite Speed of Propagation 239
Decomposition of the Fundamental Solution . . . 243
3.3.2 Normal Singularity Property 245
Preliminary Analysis 245
Main Results 248
3.4 Energy Estimates Approach Revisited 252
3.4.1 Theorem 3.1.7(ii) Revisited 253
Energy Estimates 253
Condition (3.4.13) 255
Examples 256
Theorems 3.1.13(ii), 3.1.7(ii) and 3.2.4(ii) Generalized 258
3.4.2 Main Theorem 258
Discussion 258
Symbol q: General Construction 261
Energy Estimates. I 262
Energy Estimates. II 263
Energy Estimates. III 264
Results: Single Glancing Point 266
Further Discussion 266
3.5 Propagation of Singularities along Long Transversally Reflecting Bicharacteristics 269

3.5.1 Evolution of Operators 269
Standard Theory 270
Reflections 271
3.5.2 Long Time Evolution 274
Main Theorem 274
Proof of the Main Theorem 276
3.5.3 Branching Billiards and Propagation 279
3.A Appendices 280
3.A.1 On Solutions to Linear Matrix Equations 280
3.A.2 Structure of Gliding Points 282

II Local and Microlocal Semiclassical Spectral Asymptotics in the Interior of the Domain 284

4 General Theory in the Interior of the Domain 285
4.1 Introduction 285
4.1.1 Main Objects 285
4.1.2 Kind of Assumptions to be Used 288
4.1.3 Plan of the Chapter 289
4.2 Preliminary Analysis and Tauberian Method 290
4.2.1 General Estimates 290
4.2.2 Semiclassical Spectral Gaps. I 293
First Standard Set of Assumptions 293
Second Standard Set of Assumptions 295
Estimates to Propagator 295
Estimates to Projector 298
Estimates without Cut-offs 302
Semiclassical Spectral Gaps under Reduced Smoothness Assumptions 304
4.2.3 Main Tauberian Estimate 309
Main Tauberian Formula 309
Asymptotics with Mollification 310
Asymptotics with Averaging 311
Operator Valued Functions 313
4.2.4 Applications 313
Scheme of Things 313

So, what We Get? 316
4.2.5 Estimates in the Hilbert Scale 317
Hilbert Scale 317
Estimates in the Hilbert Scale 319
4.3 The Method of Successive Approximations for Propagator 320
4.3.1 The Method of Successive Approximations in the Standard Ccase 320
Pilot-Construction 320
Transformation of the Successive Approximation Formula 322
Minor Modifications 324
4.3.2 Successive Approximations in the Hilbert Scale . 326
4.3.3 Calculations 331
4.3.4 Exploiting Microhyperbolicity and Increasing T . 337
Exploiting Microhyperbolicity 337
4.4 General Basic Theorems 340
4.4.1 General Theory: $T \asymp h^{\frac{1}{2}-\delta}$ 341
4.4.2 General Microhyperbolic Theory: $T \asymp 1$ 345
Asymptotics with Mollification 345
Asymptotics without Mollification 346
Improved Asymptotics without Mollification . . . 348
Useful Formulae 351
4.5 Long Term Propagation and Sharper Remainder Estimates 353
4.5.1 Preliminary Discussion 353
Increasing T 353
Synthesis 354
Sharp and Sharper 355
4.5.2 Standard Sharp Theory 356
Analysis of Wave Front Sets 356
Periodic Trajectories and Loops 358
Results 360
4.5.3 Sharper Remainder Estimates 364
Analysis as $|t| \le T_0$ 364
Successive Approximations 365
Increasing T 366
Tauberian Estimates 366
Finally, Results 367
4.5.4 Discussion 368

Local Results 368
Simple Improvements 369
Sharper Remainder Estimates 371
Matrix Case and Branching Bicharacteristics . . . 373
4.6 Operators with Rough Coefficients 373
4.6.1 Discussion: Rationale 374
4.6.2 Estimates 377
4.6.3 Successive Approximations and Calculations . . . 378
4.6.4 Main Theorems 383
4.6.5 Applications: Bracketing Method 385
4.6.6 Final Remarks 388
4.7 Operators with Irregular Coefficients 390
4.7.1 Preliminary Analysis 390
4.7.2 Tauberian Estimates 391
4.7.3 Weyl Asymptotics 394
4.7.4 Sharper Asymptotics 399
4.A Appendices 399
4.A.1 On the Traces of Almost Analytic Functions . . . 399
4.A.2 Smoothing of $\mathscr{C}^m_\vartheta$ Functions 402

5 Scalar Operators in the Interior of the Domain. Rescaling Technique 406
5.1 Introduction 406
5.1.1 Pilot: Schrödinger Operator 406
5.1.2 Plan of the Chapter 408
5.2 Spectral Asymptotics for Scalar Operators 410
5.2.1 From Condition "No Critical Point" to "$\mathsf{rank}\,\mathsf{Hess}\,a \geq 2$ at Critical Points." I 411
Standard Asymptotics 411
Sharp and Sharper Asymptotics 417
Near-Critical Zone 418
Sharp and Sharper Asymptotics (End) 422
5.2.2 From Condition "No Critical Point" to "$\mathsf{rank}\,\mathsf{Hess}\,a \geq 2$ at cCritical Points. " II 422
5.2.3 From Condition "$\mathsf{rank}\,\mathsf{Hess}\,a \geq 2$ at Critical Points" to "$\mathsf{rank}\,\mathsf{Hess}\,a \geq 1$ at Critical Points" 433
5.2.4 From Condition "$\mathsf{rank}\,\mathsf{Hess}\,a \geq 1$ at Critical Points" to a Weak Non-Degeneracy cCondition 439

5.2.5 From Microlocal to Local Spectral Asymptotics . 445
5.3 Local and Pointwise Spectral Asymptotics for the Schrödinger and Dirac Operators 445
5.3.1 Schrödinger Operator 446
Main Assumptions 446
Asymptotics with Spatial Mollification 447
Asymptotics without Spatial Mollification 451
Asymptotics without Spatial Mollification and Short Loops 452
5.3.2 Dirac Operator 461
Assumptions 461
Calculations 463
Asymptotics for $m \geq c_0$ 468
Asymptotics for $m \leq c_0$ 473
Further Improvements 479
Sharp Remainder Estimates 483
5.4 Operators with Irregular Coefficients 487
5.4.1 Preliminary Remarks 487
5.4.2 Schrödinger Operator 488
Reminder 488
Getting Rid of Condition (5.4.2) 489
From Condition (5.4.2) to Condition (5.4.3) . . . 492
Case $d \leq s+1$ Reloaded 493
Dropping Condition (5.4.3) 496
Sharp Remainder Estimates 497
Applying Results of Section 4.7 499
Applying Results of Section 4.7. II 499
5.4.3 Dirac Operator 501
Getting Rid of Condition (5.4.2) 501
From Condition (5.4.2) to Condition (5.4.3) . . . 503
Dropping Condition (5.4.3) 503
Sharp Remainder Estimate 504
5.4.4 General oOperators 504
Dropping Condition $|\nabla_\xi a| \neq 0$ 504
Case $\nu(t) = o(t)$ 507
Case $\nu(t) = o(t)$. II 509
Sharp Remainder Estimates 510
5.A Appendices 511

5.A.1 Spectral Kernel Calculations for Some Model Operators . . . 511
Case $d = 1$ . . . 511
Case $d = 2$ . . . 513
5.A.2 On Pauli Matrices . . . 514

6 Operators in the Interior of Domain. Esoteric Theory 519
6.1 Spectral Asymptotics for Families of Commuting Operators 520
6.1.1 Introduction . . . 520
6.1.2 General Theory . . . 522
Elliptic Arguments . . . 522
Successive Approximations . . . 523
Basic Results (General Case) . . . 524
The Microhyperbolicity Condition . . . 525
Exploiting Microhyperbolicity . . . 526
Tauberian Arguments . . . 528
Principal Results . . . 529
Possible Generalizations, Remarks . . . 530
6.1.3 Operators with Scalar Symbols . . . 532
Preliminary Remarks . . . 532
Asymptotics with Mollifications . . . 532
Principal Results . . . 534
6.2 Scalar Operators with Periodic Hamiltonian Flow . . . 536
6.2.1 Introduction . . . 536
6.2.2 Operator B . . . 543
Scalar Case . . . 543
Matrix Case . . . 549
6.2.3 Long Term Evolution . . . 550
Main Theorem . . . 550
Corollaries . . . 551
Spectral Gaps (the First Attempt) . . . 556
6.2.4 Exploiting Microhyperbolicity . . . 558
Extending Time Interval . . . 558
Tauberian Arguments . . . 558
Removing Cut-off ψ . . . 562
6.3 Scalar Operators with Periodic Hamiltonian Flow (End) 562
6.3.1 Scalar a and Subperiodic Trajectories . . . 562
Structure of Subperiodic Trajectories . . . 562

Propagation of Singularities near Π 564
Successive Approximations 566
Exploiting Microhyperbolicity 570
Tauberian Estimates 571
6.3.2 The case of Degenerate Scalar Operator B 572
Asymptotics with mMollification 572
Estimates without Mollification 573
6.3.3 Some Remarks 576
6.3.4 Spectral Gaps Reloaded 578
6.3.5 Families of Commuting Operators Approach . . . 579
Operators with Completely Periodic Evolution . . 579
Operators with the Global dDecomposition 582
General oOperators 584
6.3.6 Strong Perturbations 585
6.3.7 Operators with Massive set of Periodic tTrajectories 587
6.3.8 Examples 590
Laplacian 590
Harmonic Oscillator 591
Coulomb Potential 592
6.4 Asymptotics for Dirac Energy 593
6.4.1 Introduction 593
6.4.2 Estimates 595
Special Case 595
Smooth Case 597
Singular homogeneous case 599
Singular Homogeneous Case: Using ξ-Microhyperbolicity 600
6.4.3 Calculations 601
Constant Coefficients Case 601
General ξ-Microhyperbolic Case 602
6.4.4 Scalar Case 604
Smooth case. I 604
Smooth case. II 606
Singular Homogeneous Case 606
6.4.5 Rescaling Technique 609
Schrödinger Operator 609
6.5 Mollification with Respect to Non-Spectral Parameters . 612
6.5.1 Linear Dependence 612
6.5.2 Microhyperbolic Case 614

6.5.3 Averaging 616
6.A Appendices 616
6.A.1 Hamiltonian Flows of Commuting Operators . . . 616
6.A.2 One Property of Real Residues 617
6.A.3 Eigenvalues of Some Model Operators 618

III Local and Microlocal Semiclassical Spectral Asymptotics near the Boundary 620

7 Standard Local Semiclassical Spectral Asymptotics near the Boundary 621
7.1 Preliminary Analysis 622
7.1.1 General Case 622
7.1.2 Elliptic Case 627
7.1.3 Operators with Minimal Smoothness Conditions . 634
7.1.4 Estimates for the Schrödinger Operator 639
7.2 Method of Successive Approximations 643
7.2.1 Approximating the Free Space Term 643
Standard Calculations Revised 643
Calculating Restriction to the Boundary 644
7.2.2 Approximations for the Reflected Wave 648
7.2.3 Calculations 655
Starting Point 655
One Dimensional Parametrices 655
One Dimensional Parametrices (End) 657
Calculations: what We Got 664
Calculations: Taking Traces 665
Calculations: Mollification with Respect to τ . . . 666
7.2.4 Exploiting Microhyperbolicity 667
7.2.5 Conclusion 671
7.3 General Basic Theorems 671
7.3.1 Asymptotics with Mollification 672
7.3.2 Asymptotics under Microhyperbolicity Condition 673
7.3.3 Asymptotics in the Elliptic Zone 675
7.3.4 Local Asymptotics 679
7.4 Sharper Asymptotics 681
7.4.1 Discussion 681

7.4.2 General Theory 683
Sharp Asymptotics 683
Sharper Asymptotics 684
7.4.3 Billiard Flow for Schrödinger Operator 685
Definition of the Flow 685
Growth of the Billiard Flow. I 686
Growth of the Billiard Flow. II 690
Non-Periodicity Condition and Results. I 695
Non-Periodicity Condition and Results. II 697
Non-Periodicity Condition for Branching Billiards 700
7.5 Operators with Irregular Coefficients. Seeley's Method . 705
7.5.1 Discussion 706
Rescaling Technique 706
Propagation 707
What Then? 708
Mollification 709
7.5.2 $O(h^{1-d})$ Remainder Estimate for Scalar Operators 710
Inner Zone 710
Moving Closer to the Boundary 715
Boundary Strip 716
Synthesis 716
7.5.3 $O(h^{1-d})$ Remainder Estimate for Matrix Operators 719
Covering 720
Estimates 723
7.5.4 Sharp Remainder Estimate 724
Near Boundary Zone 724
Inner Zone 726
7.5.5 Less Sharp Remainder Estimates 729
7.A Appendices 731
7.A.1 Sets Associated with the Microhyperbolicity Condition 731

8 Standard Local Semiclassical Spectral Asymptotics near the Boundary. Miscellaneous 740
8.1 Pointwise Spectral Asymptotics 741
8.1.1 Preliminary Analysis 741
Discussion 741
Toy-Model 743

8.1.2 Schrödinger Operator 745
General Theory 745
Calculations and Main Theorem 748
8.1.3 Generalizations 749
8.2 Scalar operators and Rescaling 751
8.2.1 Schrödinger Operator. I 751
General Theory 751
Sharp Remainder Estimates 755
Special Case $d = 1$ 755
Generalizations 757
8.2.2 Schrödinger Operator. II 758
Preliminaries 758
Microhyperbolicity Conditions 759
Analysis in the "Elliptic" Zone 760
Asymptotics as $V \approx 0$ 762
8.2.3 Schrödinger Operator. III 764
8.3 Operators with Periodic Hamiltonian Flows 765
8.3.1 Discussion and Plan 765
8.3.2 Simple Hamiltonian Flow 767
Inner Asymptotics 767
Asymptotics near Boundary 770
Discussion 772
8.3.3 Branching Hamiltonian Flow with "Scattering" . 774
Analysis in X_2 774
Analysis in X_1 776
8.3.4 Two Periodic Flows 781
Examples and Discussion 781
Reduction to the Boundary 782
Analysis of the Evolution 784
8.4 Spectral Asymptotics on Subspaces 787
8.4.1 Examples and Discussion 787
8.4.2 Analysis away from the Boundary 791
8.4.3 Analysis near the Boundary. I 792
8.4.4 Analysis near the Boundary. II. Seeley's Approach 796

Volume II. Functional Methods and Eigenvalue Asymptotics 1

IV Estimates of the Spectrum 799

9 Estimates of the Negative Spectrum 800
9.1 Main Theorem . . . 801
9.1.1 Discussion . . . 801
9.1.2 Main Theorem Setup . . . 803
Scaling Functions . . . 803
Operators . . . 805
Regular Zone . . . 806
Singular Zone . . . 807
9.1.3 Proof of the Main Theorem . . . 809
9.1.4 Riesz Means Estimates . . . 816
9.2 Examples to Main Theorems . . . 818
9.2.1 Regular Zone: Standard Remainder Estimate . . . 818
9.2.2 Regular Zone: Sharper Asymptotics . . . 820
9.2.3 Singular Zone . . . 821
9.2.4 Riesz Means . . . 825
9.3 Estimates for Quadratic Forms . . . 826
9.A Appendices . . . 833
9.A.1 Variational Estimates . . . 833
Rozenblioum Estimates . . . 833
Bracketing . . . 834
Neumann Type Problems . . . 835
Some Useful Estimate . . . 839
9.A.2 Riesz Means Estimates . . . 840

10 Estimates of the Spectrum in the Interval 843
10.1 Statement of the Main Theorems . . . 844
10.1.1 Preliminary Remarks . . . 844
10.1.2 Birman-Schwinger Principle . . . 847
Main Formula . . . 847
Partition and Scaling . . . 848
Calculations in the Regular Zone . . . 853
Analysis in the Singular Zone . . . 855

10.1.3 Main Theorems 856
General Operators 856
Dirac Operator 858
10.2 Functional-Analytic Arguments and Proofs of the Main Theorems 861
10.2.1 Self-Adjointness of the Dirac Operator 861
10.2.2 Abstract Theory 866
10.2.3 Proof of Theorem 10.1.7 874
10.2.4 Proof of Theorem 10.1.8 875
Functional-Analytic Arguments 875
Estimates for a Number of Crossing Eigenvalues . 878
Estimates for a Number of Crossing Eigenvalues (End) 884
Estimates for a Number of Remaining Eigenvalues 885
10.2.5 From Rough to Irregular Operators 890

V Asymptotics of Spectra 892

11 Weyl Asymptotics of Spectra 893
11.1 Semiclassical Eigenvalue Asymptotics 894
11.1.1 General Analysis 895
Standard Remainder Estimates 895
Non-Standard Remainder Estimates 903
11.1.2 Examples 905
11.1.3 Non-Semi-Bounded oOperators 917
11.2 Large Eigenvalues for Operators with Weakly Singular Potentials 919
11.2.1 General Scheme 919
11.2.2 Weak Singularity 928
11.2.3 Main Theorems 929
11.2.4 Examples 933
11.3 Large Eigenvalues for Operators Wwith Weakly Singular Potentials (End) 947
11.3.1 Operators with Non-Periodic Flows 947
Existence of the Billiard Flow 948
Properties of Hamiltonian Billiard Flows and Asymptotics 949

Examples 951
11.3.2 Operators with Periodic Flows 957
Regular Operators on Closed Manifolds 957
11.3.3 Singular operators on Closed Manifolds 966
11.3.4 Manifolds with the Boundary 968
11.3.5 The Indefinite Case 968
11.3.6 Conditions to Lower Order Terms 971
11.4 Large Eigenvalues for Operators in Essentially Unbounded Domains 975
11.4.1 Coercivity Condition and Standard Asymptotics . 975
11.4.2 Sharp Remainder Estimates 979
11.4.3 Examples 981
11.4.4 Indefinite Perturbations 985
11.4.5 Operators with Indefinite Principal Symbols . . . 987
11.5 Large Eigenvalues for Operators in Essentially Unbounded Domains (End 988
11.5.1 Standard Results 988
11.5.2 Strongly Non-Periodic Hamiltonian Flow 993
11.5.3 Periodic Hamiltonian Flow 997
11.5.4 Singularities and Degenerations 1000
11.5.5 Quasihomogeneous Potentials 1007
11.5.6 Quickly Growing Potentials 1009
11.5.7 Slowly Growing Potentials 1013
11.5.8 Non-Dirichlet Boundary Conditions 1014
11.6 Asymptotics of Eigenvalues near -0 1016
11.6.1 Standard Results 1017
11.6.2 Strongly Non-Periodic Hamiltonian fFlow 1020
11.6.3 Periodic Hamiltonian Flow 1021
11.6.4 Singularities 1023
11.6.5 Quasihomogeneous Potentials 1026
11.6.6 Quickly Decreasing Potentials 1026
11.6.7 Slowly Decreasing Potentials 1029
11.6.8 General Boundary cConditions 1030
11.6.9 Non-sSemi-Bounded Operators 1031
11.7 Multiparameter Asymptotics 1037
11.7.1 Problem Set-up 1037
11.7.2 Semiclassical Asymptotics with a Vector Semiclassical Parameter 1038

11.7.3 Asymptotics of Eigenvalues near $+\infty$ 1040
11.7.4 Asymptotics of Eigenvalues near -0 and Related asymptotics 1044
11.7.5 Asymptotics of Eigenvalues near $+0$ 1048
11.7.6 Asymptotics of Eigenvalues near $-\infty$ 1051
11.8 Distributions of Spectra for Negative Order Operators . . 1055

12 Miscellaneous Asymptotics of Spectra 1060
12.1 Introduction 1060
12.2 Operators in Domains with Thick Cusps 1065
12.2.1 Problem Set-up 1065
12.2.2 Sharp Asymptotics 1068
Standard Remainder Estimate 1068
Improved Remainder Estimates 1075
12.2.3 Effective Asymptotics 1078
Scheme of Effective Asymptotics 1078
Simplifying the Answer 1091
Main Results 1092
12.3 Operators with Degenerate Potentials Growing at Infinity 1096
12.3.1 Problem Set-up 1096
12.3.2 Sharp Asymptotics 1102
12.3.3 Effective Asymptotics 1109
Simplifying the Answer 1109
Main Theorems 1112
Discussion 1113
12.4 Operators with Singular Potentials Decaying at Infinity . 1114
12.4.1 Problem Set-up 1114
12.4.2 Sharp Asymptotics 1117
12.4.3 Effective Asymptotics 1121
Simplifying the Answer 1121
Main tTheorems 1123
12.5 Spectral Asymptotics for Maximally Hypoelliptic Operators 1124
12.5.1 Maximally Hypoelliptic Operators 1124
12.5.2 Getting sStarted 1127
12.5.3 Weyl Asymptotics 1129
12.5.4 Sharp Asymptotics 1133
12.5.5 Calculations 1141
12.5.6 Operators with Negative Principal Symbols . . . 1147

12.5.7 Coefficients in Spectral Asymptotics 1150
12.6 Riesz Means Asymptotics for Operators Singular at a Point 1152
12.6.1 Problem Set-up 1152
12.6.2 The Scott Correction Term 1155
12.6.3 Improved Asymptotics: the First Attempt 1157
12.6.4 Adjustment for $\vartheta < 1$ 1163
12.6.5 The General Case 1165
12.6.6 Propagation of Singularities 1172
12.6.7 Improved Remainder Estimates 1180
12.6.8 Final Remarks 1184
12.7 Sharp Spectral Asymptotics for Neumann Laplacian in Thin Cusps 1187
12.7.1 Problem Set-up 1187
12.7.2 Weyl Part of Asymptotics 1191
12.7.3 Cusp pPart of Asymptotics 1192
12.7.4 Main Results. I 1194
12.7.5 Improved Estimate as $d'' = 1$ 1195
12.7.6 Maxwell System 1199
Reminder 1199
Thin Cusps. Heuristics 1202
Thin cusps. Results 1204
Thick Cusps. Sketch 1205
12.7.7 Generalizations 1206
Higher Order Operators–Discussion 1206
More General cCusps 1208
Domains with Spikes 1208
12.7.8 Operators with Degenerate Potentials 1209
12.8 Periodic Operators 1210
12.8.1 Periodic Operators 1210
12.8.2 Statement of the Problem 1212
12.8.3 Reformulation of the Problem 1213
12.8.4 Reduction to Operators with Operator-Valued Symbols 1214
12.8.5 Main Theorem: E inside Spectral Gap 1216
12.8.6 E on the Bottom of Spectral Gap 1223
12.8.7 E on the Top of Spectral Gap 1224
12.8.8 E on the Top of Spectral Gap. II. Improved Asymptotics 1229

12.8.9 Case of Positive $(A - E)$ 1232
12.8.10 Generalizations and Modifications 1232
12.A Appendices 1234
12.A.1 Variational Estimates as $m = 1$, $d = 1, 2$ 1234

Volume III. Magnetic Schrödinger and Dirac Operators. 1 **1238**

VI Smooth Theory in Dimensions 2 and 3 **1238**

13 Schrödinger Operator. Standard Theory Dimensions 2 and 3 **1239**
13.1 Introduction 1239
13.2 Preliminary Analysis 1241
13.2.1 Settings 1241
13.2.2 Toy-Model Case 1242
Calculations 1242
Magnetic Weyl Expression vs (Standard) Weyl Expression 1246
Dynamics 1249
13.2.3 Rough Estimates 1251
Functional Analytic Arguments 1251
Propagation of sSingularities 1256
13.3 Microlocal Canonical Form 1260
13.3.1 Symplectic Canonical Form 1260
Symplectic Canonical Form: $\varepsilon = 0$ 1260
13.3.2 Operator Reduction 1277
13.3.3 Reduced Problems 1281
13.4 Weak Magnetic Field 1282
13.4.1 Analysis of Singularities 1283
13.4.2 Refined Estimates 1289
Elliptic Estimates 1290
Propagation of Singularities 1291
Estimates. I 1294
Estimates. II 1297
13.4.3 Calculations 1299

13.4.4 Synthesis 1303
13.4.5 Miscellaneous Modifications of $\mathsf{R}^{\mathsf{W}\prime}$ 1306
13.5 Moderate Magnetic Field 1310
13.5.1 Decomposition of the Fundamental Solution and Propagation of Singularities 1310
13.5.2 Calculations 1314
13.5.3 Estimates 1318
Estimates in the Zone $\{\gamma \geq \bar{\gamma}\}$ 1318
Estimates in the Zone $\{\gamma \leq \bar{\gamma}\}$ 1325
Synthesis 1327
Synthesis: Estimates with Mollifications 1329
13.5.4 Effective Asymptotics 1330
General Scheme 1330
Implementation: Strongest Non-Degeneracy Condition 1333
Implementation: $d = 3$ 1334
Implementation: $d = 2$ 1339
Implementation: $d = 2$, II 1340
Implementation: $d = 2$, III 1344
13.6 Strong Magnetic Field 1349
13.6.1 Existence of Spectral Gaps 1350
13.6.2 Calculations in Spectral Gaps as $d = 2$ 1355
13.6.3 Final Remarks as $d = 2$ 1357
13.6.4 Case $d = 3$ 1360
13.7 Propagation of Singularities and Sharper aAsymptotics . 1362
13.7.1 Case $d = 2$ 1362
Classical Dynamics and Heuristics 1362
Propagation of Singularities 1365
Spectral Asymptotics 1367
13.7.2 Case $d = 3$ 1370
Preliminaries and Improved Reduction 1371
Weak Magnetic Field 1373
Moderate Magnetic Field 1382
Strong Magnetic Field 1387
13.7.3 Supersharp Estimates as $d = 3$ 1388
Framework 1388
Case $\mu h \leq 1$ 1390
Schrödinger-Pauli Operator as $\mu h \geq 1$ 1394

Schrödinger-Pauli Operator as $\mu h \geq 1$ reloaded . 1395
13.8 Certain gGeneralizations 1397
13.8.1 Getting Rid of the Condition "$v \leq -\epsilon$" 1397
Formal Rescaling 1397
Justification: $d = 3$ 1398
Justification: $d = 2$, $\nu = 1$ 1400
Justification: $d = 2$, $\nu < 1$ 1404
13.8.2 Getting Rid of the Condition "$F \geq \epsilon$" 1405
Case $d = 3$ 1405
Case $d = 2$ 1409
13.8.3 Getting Rid of the cCondition "$B(0,1) \subset X$" . . . 1409
Case $d = 3$ 1409
Case $d = 3$, Improved Remainder Estimates . . . 1412
Case $d = 2$ 1414
13.A Appendices 1414
13.A.1 Two Integral Formulae 1414
13.A.2 Trajectories of the Classical Particles in a Magnetic Field 1416
Case $V = 0, \boldsymbol{F} = \mathsf{const}$ 1416
Case $\nabla V = \mathsf{const}, \boldsymbol{F} = \mathsf{const}$ 1416
Case $V = \frac{1}{4}k^2|x|^2, \boldsymbol{F} = \mathsf{const}$ 1417
Case $V = \frac{1}{4}k^2x_3^2, \boldsymbol{F} = \mathsf{const}$ 1418

14 2D-Schrödinger Operator with the Strong Degenerating Magnetic Field 1419
14.1 Preliminary Analysis 1420
14.1.1 Set-up 1420
14.1.2 Zones and Rescaling: the First Blood 1422
14.2 Classical Dynamics 1423
14.2.1 Toy-mModel 1423
14.2.2 Another Toy-Model 1429
14.2.3 Toy-Models Perturbed 1435
14.2.4 General Case 1436
Remarks 1436
Regular Zone 1437
Degeneration Zone 1439
Hamiltonian mMaps 1441
14.3 Quantum Dynamics 1443

14.3.1 Preliminary Notes. Forbidden Zone 1443
14.3.2 Regular Zone 1446
14.3.3 Degeneration Zone 1452
General Analysis 1452
Near Periodic Sub-Zone 1456
Near Periodic Sub-Zone. End 1458
14.3.4 Periodic Zone 1460
Toy-Model 1460
Another Example 1464
General Case 1465
General Case. II 1467
14.4 Calculations 1470
14.4.1 Regular Zone 1471
14.4.2 Degeneration Zone 1479
Singularity at $t = 0$ 1479
Extending Time Interval 1482
14.4.3 Periodic and Near-Periodic Orbits 1484
Toy-Model 1484
Generic Settings 1491
General Settings. II 1496
14.5 Very Strong Magnetic Field 1500
14.5.1 Regular Zone 1500
14.5.2 Degeneration Zone 1502
Non-Periodic Zone 1502
Periodic Zone 1506
14.5.3 Calculations 1507
14.6 Main Theorem and Generalizations 1507
14.6.1 Main Theorem 1507
14.6.2 Generalization: Vanishing V 1509
14.7 Strong Magnetic Field Reloaded 1511
14.7.1 Statement of the Problem 1511
14.7.2 Simple Rescaling 1512
14.7.3 Estimates 1514
Regular Zone 1514
Degeneration Zone 1517
Extended Formerly Forbidden Zone 1518
Synthesis 1519
14.7.4 Calculations 1520

Calculations. I 1520
Calculations. II 1521
14.7.5 Analysis under Condition (14.7.5) 1523
Estimates 1523
Calculations 1526
14.8 Superstrong Magnetic Field 1531
14.8.1 Preliminary Analysis 1531
14.8.2 Transitional Superstrong Magnetic Field 1532
14.8.3 Genuine Superstrong Magnetic Field 1535
Degeneration Zone 1535
Regular Zone: Case $\mathfrak{z} \neq 1$ 1536
Regular Zone: Case $\mathfrak{z} = 1$ 1538
Regular Zone: Case of Condition (14.7.5) 1539
14.8.4 Calculations and Main Results 1539
Degeneration Zone and Case $\mathfrak{z} \notin (2\mathbb{Z}^+ + 1)$ 1540
Calculations under Conditions $(14.7.4)_\pm$ 1540
Calculations under Condition (14.7.5) 1541
14.A Eigenvalues of 1D-Operators 1541
General Observations 1541
14.A.1 Asymptotics as $\eta \to +\infty$ 1542
14.A.2 Case of $\mathfrak{z} = 1$ 1545
14.A.3 Generalizations 1550
14.A.4 Open Problems 1551
14.A.5 Maple for Trajectories 1552

15 2D-Schrödinger Operator with the Strong Magnetic Field near the Boundary **1553**
15.1 Preliminary Discussion 1555
15.1.1 Inner and Boundary Zones 1555
15.1.2 Classical Dynamics near the Boundary 1556
15.1.3 Spectrum of the Toy-Model Operator 1561
15.2 Weak Magnetic Field 1565
15.2.1 Precanonical Form 1565
15.2.2 Propagation of Singularities 1566
No-Critical Point Case 1566
Analysis in the Boundary Zone 1570
Analysis in the Transitional Zone 1576
Analysis in the Inner Zone 1577

15.2.3 From Tauberian to Magnetic Weyl Formula . . . 1582
15.3 Moderate Magnetic Field . . . 1585
15.3.1 Most Non-Degenerate Case . . . 1585
15.3.2 Generic Case. Analysis in Inner Zone . . . 1589
15.3.3 Generic Case. Analysis in Transitional Zone . . . 1592
15.3.4 Propagation of Singularities in Transitional Zone 1596
15.3.5 Calculations . . . 1601
General Discussion . . . 1601
Calculations . . . 1603
15.3.6 Final Results . . . 1605
15.4 Strong Magnetic Field . . . 1607
15.4.1 Preliminary Analysis . . . 1607
15.4.2 Almost Spectral Gaps . . . 1608
15.4.3 Landau Level . . . 1612
Preliminary Analysis . . . 1612
Non-Degenerate Case . . . 1614
Generic Case . . . 1615
15.4.4 From Tauberian to Magnetic Weyl Formula . . . 1620
Inner Zone . . . 1620
Boundary Zone . . . 1621
Transitional Zone . . . 1622
15.4.5 Justification: Reduction to Toy-Model Operator . 1622
Reduction: Step 1 . . . 1622
Reduction: Step 2 . . . 1627
Reduction: Step 3 . . . 1627
15.4.6 Final Results . . . 1628
15.5 Generalizations . . . 1630
15.5.1 Robin Boundary Value Problem . . . 1630
15.5.2 Boundary Meets Degeneration of V . . . 1630
15.5.3 Domains with Corners . . . 1631
Weak Magnetic Field Case . . . 1631
Moderate Magnetic Field Case . . . 1633
Intermediate Magnetic Field Case . . . 1635
Strong Magnetic Field Case . . . 1635
15.5.4 Boundary Meets Degeneration of F . . . 1636
Set-up . . . 1636
Weak Magnetic Field Case . . . 1637
Moderate Magnetic Field Case . . . 1638

Strong and Superstrong Magnetic Field cCases . . 1639
15.5.5 Case $h \geq 1$ 1639
15.A Appendices: Eigenvalues of $L(\eta)$ 1639
15.A.1 Basic Properties 1639
15.A.2 More Properties 1641
15.A.3 Estimates of $\lambda_{*,n}(\eta)$ as $\eta \to +\infty$ 1642
15.A.4 Asymptotics of $\lambda_{*,n}(\eta)$ as $\eta \to +\infty$ 1645
15.A.5 Remarks 1647

VII Smooth Theory in Dimensions 2 and 3 (Continued) 1650

16 agnetic Schrödinger Operator: Short Loops, Pointwise Spectral Asymptotics and Asymptotics of Dirac Energy 1651
16.1 Introduction 1651
16.2 Pointwise Asymptotics: 2D Toy-Model 1655
16.2.1 Toy-Model in 2D: Propagator 1655
16.2.2 Tauberian Estimate 1658
Preparatory Estimate 1658
Tauberian Estimate 1664
Micro-Averaging 1665
16.2.3 Calculations 1666
Correction Term 1666
Estimating Correction Term 1667
Calculating Correction Term 1668
Micro-Averaging 1669
16.2.4 Strong and Superstrong Magnetic Field 1670
Tauberian Estimate 1670
Micro-Averaging 1671
16.2.5 Weyl and Magnetic Weyl Approximations 1671
16.2.6 Geometric Interpretation 1674
16.3 Pointwise Asymptotics: General 2D-Operators 1676
16.3.1 Classical Dynamics 1677
16.3.2 Semiclassical Approximation to $U(x, y, t)$ 1681
16.3.3 Semiclassical Approximation to $e(x, x, \tau)$ 1684
16.3.4 Stationary Phase Calculations 1686
16.3.5 Approximation by a Toy-Model Operator 1687

Weak Magnetic Field Case 1687
Successive Approximations 1688
Justification 1690
Reduction to the Toy-Model Operator 1692
16.3.6 Strong Magnetic Field 1693
16.3.7 Main Theorem 1693
16.3.8 Problems 1694
16.4 Dirac Energy: 2D-Estimates 1695
16.4.1 Tauberian Formula 1695
16.4.2 Strong Magnetic Field 1700
16.5 Dirac Energy: 2D-Calculations 1703
16.5.1 Toy Model 1703
Transformations 1703
Case $\alpha^2 \gg \mu h$ 1705
Improvement 1708
Case $\alpha^2 \not\gg \mu h$ 1711
16.5.2 General Operators 1713
16.5.3 Perturbations 1717
General Scheme 1717
Improvement 1721
Results. I 1721
Results. II 1724
16.5.4 Strong Magnetic Field 1725
16.5.5 Problems and Remarks 1727
16.6 Pointwise Asymptotics: 3D Toy-Model 1728
16.6.1 Toy-Model in 3D: Propagator 1728
16.6.2 Tauberian Estimates 1730
16.6.3 Weyl Estimates 1731
16.6.4 Micro-Averaging 1732
16.6.5 Transitional and Strong Magnetic Field 1734
Tauberian Estimate 1734
Micro-Averaging 1735
16.6.6 Magnetic Weyl Approximation 1736
Pointwise Asymptotics 1736
Micro-Averaging 1737
16.6.7 Geometric Interpretation 1738
16.7 Pointwise Asymptotics: General 3D-Operators 1739
16.7.1 Set-up 1739

16.7.2 Classical Dynamics 1740
16.7.3 Semiclassical Approximation 1744
16.7.4 Tauberian Estimates 1745
16.7.5 Weyl Estimates 1747
16.7.6 Successive Approximations 1748
Generalized Toy-Model Approximation, $\mu h \le 1$. 1749
Toy-Model Approximation, $\mu h \le 1$ 1752
Magnetic Weyl Approximation 1757
16.7.7 Transitional and Strong Magnetic Field 1757
16.7.8 Micro-Averaging 1758
Tauberian Estimates 1758
Weyl and Magnetic Weyl Estimates 1759
16.8 Dirac Energy: 3D-Estimates 1759
16.8.1 Tauberian Formula 1759
16.8.2 Strong Magnetic Field 1760
16.9 Dirac Energy: 3D-Calculations 1764
16.9.1 Toy Model 1764
Transformations 1764
Case $\lambda^2 \gg \mu h$ 1765
Case $\lambda^2 \not\gg \mu h$ 1769
16.9.2 General Operators 1772
16.9.3 Perturbations 1777
Perturbations. I 1777
Perturbations. II 1781
Perturbations. III 1783
16.9.4 Strong Magnetic Field 1784
16.9.5 Problems and Remarks 1786
16.10Estimates of Expression (16.1.4) 1787
16.10.1Basics 1787
16.10.2Case $d = 2$ 1789
Tauberian Asymptotics 1789
Weyl Approximation 1792
Magnetic Weyl Approximation 1793
16.10.3Case $d = 3$ 1794
Tauberian Estimates 1795
Weyl Asymptotics 1797
Magnetic Weyl Asymptotics 1797
16.10.4Problems 1798

17 Dirac Operator with the Strong Magnetic Field 1799
17.1 Preliminary Analysis 1799
17.1.1 Toy-Model Operator 1799
17.1.2 Rough estimates 1806
17.2 Microlocal Canonical Form 1807
17.2.1 Principal Symbol Reduction 1807
17.2.2 Final Reduction 1812
17.2.3 Reduced Propagator 1816
17.3 Weak Magnetic Field 1817
17.3.1 Analysis of Singularities. I 1817
17.3.2 Analysis of Singularities. II 1820
17.3.3 Calculations 1825
17.3.4 Synthesis 1829
17.4 Temperate Magnetic Field 1831
17.4.1 Separation of Variables and Decomposition 1831
17.4.2 Further Reduction 1836
17.4.3 Tauberian Remainder Estimates 1842
17.4.4 Calculations 1843
Preliminary arguments 1843
From Tauberian to More Explicit Expression . . . 1844
Effective asymptotics. I 1846
Effective asymptotics. II 1848
Effective asymptotics. III 1849
17.5 Strong Magnetic Field 1855
17.5.1 Additional Reduction 1855
17.5.2 Estimates 1859
Estimates as $d = 3$ 1859
Estimates as $d = 2$ 1860
Estimates as $d = 2$ and h is Large 1862
17.6 Semiclassical Spectral Gaps 1863
17.6.1 Existence of Spectral Gaps 1863
17.6.2 Calculations in the Spectral Gaps 1869
17.6.3 Main Theorem 1872
17.7 Generalizations 1874
17.7.1 Propagation of Singularities and Improved Asymptotics 1874
Case $d = 3$ 1874
Case $d = 2$ 1874

17.7.2 Getting Rid of the Condition "$v^{\#} \le -\epsilon_0$" 1875
Case $d = 3$ 1875
Case $d = 2$ 1877
17.7.3 Getting Rid of the Condition "$F \gtrsim \epsilon_0$" 1878
Case $d = 3$ 1878
Case $d = 2$ 1878
17.7.4 Other Generalizations 1879

Volume IV. Magnetic Schrödinger and Dirac Operators. 2 **1**

VIII Non-Smooth Theory and Higher Dimensions **1881**

18 2D- And 3D-Magnetic SchrÖdinger Operator with Irregular Coefficients **1882**
18.1 Introduction 1882
18.1.1 Preliminary 1882
18.1.2 Smooth Case: Survey 1886
18.1.3 Non-Smooth Case: Heuristics as $d = 2$ 1887
18.1.4 Non-Smooth Case: Heuristics as $d = 3$ 1888
18.1.5 Plan of the Chapter 1889
18.1.6 Standard Results Rescaled 1890
18.2 $d = 2$: Weak Magnetic Field 1894
18.2.1 Heuristics 1894
Toy-Model 1894
Propagation of Singularities 1896
18.2.2 Propagation of Singularities: Rigorous Results . . 1898
18.2.3 Main Theorem 1901
18.3 $d = 2$: Canonical Form 1904
18.3.1 Reduction 1904
Main pPart of the Operator. I 1904
Main Part of the Operator. II 1908
Potential 1909
Final Reduction 1913
18.3.2 Calculations 1914

Calculations. I 1914
Calculations. II 1916
18.3.3 Strong Magnetic Field $\mu \geq \epsilon(h|\log h|)^{-1}$ 1917
18.4 $d = 2$: Tauberian Theory 1919
18.4.1 Decomposition 1919
18.4.2 Estimate under Assumption (18.2.15) 1920
18.4.3 Estimate under Assumption (18.2.16) 1924
18.4.4 Case $\mu \geq \epsilon(h|\log h|)^{-1}$ 1925
18.5 $d = 2$: Calculations and Main Theorems 1928
18.5.1 Calculations 1928
Step I under Assumption (18.2.15) 1928
Step I under Assumption (18.2.16) 1932
Step II under Assumption (18.2.15) 1933
Step II under Assumption (18.2.16) 1936
Calculations: $\mu \gtrsim h^{-1}|\log h|^{-1}$ 1937
Comparisons 1937
18.5.2 Main Theorems 1939
18.5.3 Generalizations 1941
Vanishing V 1941
Other Generalizations 1943
18.6 $d = 3$: Weak Magnetic Field 1944
18.6.1 Preliminary Remarks 1944
18.6.2 Heuristics 1946
18.6.3 Variable Parameter ε 1947
18.6.4 Rigorous Analysis 1948
18.6.5 Outer Zone 1953
18.6.6 Analysis under Assumption (18.2.15) 1956
18.6.7 Analysis under Assumption (18.2.16) 1958
18.6.8 Weakly-Degenerate Case 1959
18.6.9 Main Theorems 1960
18.7 $d = 3$: Canonical Form 1961
18.7.1 Canonical Form. Preliminary Remarks 1961
18.7.2 Canonical Form. I 1963
18.7.3 Canonical Form. II 1965
18.7.4 Canonical Form. III. $\mu \geq \epsilon h^{-1}|\log h|^{-1}$ 1968
18.8 $d = 3$: Tauberian tTheory 1968
18.8.1 Decomposition 1968
18.8.2 Reduction in the Zone $\mathcal{Z}_1$ 1969

18.8.3 Analysis in the Zone $\mathcal{Z}_{\mathsf{inn}}$ 1971
18.9 $d = 3$: Calculations and Main Theorems 1975
18.9.1 Assembling what We Got 1975
18.9.2 Correction Term 1976
18.9.3 Approximation Error 1977
18.9.4 Main Theorems 1978
18.9.5 Generalizations 1980
Vanishing V 1980
Vanishing F 1982
Other Generalizations 1983

19 Multidimensional Magnetic Schrödinger Operator. Full-Rank Case **1984**
19.1 Introduction 1984
19.1.1 Preface 1984
19.1.2 Canonical Form 1985
19.1.3 Resonances 1988
19.1.4 Dynamics and Microhyperbolicity 1990
19.1.5 Regularity Assumptions and Mollification 1990
19.1.6 Plan of the Chapter 1991
19.2 Weak Magnetic Field 1992
19.2.1 Heuristics. Classical dynamics 1992
Smooth theory. Classical Dynamics 1992
Microhyperbolicity 1999
Non-Smooth Theory 2001
19.2.2 Semiclassical Propagation 2001
General Theory 2002
Microhyperbolic Theory 2004
19.2.3 Tauberian Theory 2007
19.2.4 Main Theorem 2008
19.2.5 Special Case of Constant g^{jk}, F_{jk} 2010
19.3 Canonical Form 2012
19.3.1 Toy-Model 2012
19.3.2 General Case: Framework 2014
19.3.3 Reducing the Main Part 2014
19.3.4 Reducing Next Terms 2020
19.3.5 Intermediate and Strong Magnetic Field Cases . . 2023
Intermediate Magnetic Field Case 2023

Strong Magnetic Field Case 2024
19.4 Temperate Magnetic Field 2024
19.4.1 Mid-Range Propagation 2025
19.4.2 Short-Range Theory: Framework 2028
19.4.3 Toy-Model: Some Classes of Pseudodifferential Operators 2029
19.4.4 Short-Term Estimates 2037
19.4.5 Calculations 2039
Preliminary Remarks 2039
Successive Approximation Method 2041
Mollified by τ Asymptotics 2041
Unmollified Asymptotics. I 2042
Unmollified Asymptotics. II 2047
Unmollified Asymptotics. III 2049
19.4.6 Main Theorem 2055
19.4.7 Special Case of Constant g^{jk}, F_{jk} 2056
Framework; Mid- and Long-Range Propagation . 2056
Short-Range Theory 2057
Calculations 2058
Main Theorem 2059
19.5 Intermediate Magnetic Field 2060
19.5.1 Framework and Special Cases 2060
19.5.2 Partition of the Energy Space 2062
19.5.3 Propagation 2065
19.5.4 Propagation and Microhyperbolicity 2068
19.5.5 Short-Range Estimates 2069
19.5.6 Calculations 2069
19.5.7 Main Theorems 2070
19.6 Intermediate and Strong Magnetic Field 2071
19.6.1 Framework 2071
19.6.2 Hermitian Decomposition 2071
19.6.3 Propagation 2072
Special Case of Constant g^{jk}, F_{jk} 2072
Intermediate Magnetic Field: General Operators . 2074
Strong Magnetic Field Case. Reduction 2074
Strong Magnetic Field Case. Propagation 2080
19.6.4 Short-Range Estimates 2083
Intermediate Magnetic Field 2083

Strong Magnetic Field 2083
19.6.5 Calculations 2084
19.6.6 Main Theorems 2085
19.7 Simple Degeneration 2086
19.7.1 Framework 2087
19.7.2 Weak Magnetic Case 2088
19.7.3 Temperate Magnetic Field 2091
Tauberian Remainder: the Toy-Model 2091
Tauberian Estimates: the General Case 2093
Calculations: the Toy-Model 2095
Calculations: the General Case 2096
19.7.4 Intermediate Magnetic Field 2099
19.7.5 Conclusion to Section 2099
19.7.6 Final Remarks 2099
Vanishing V 2099
Pointwise Asymptotics 2100

20 Multidimensional Magnetic Schrödinger Operator. Non-Full-Rank Case 2102
20.1 Introduction 2102
20.1.1 Preface 2102
20.1.2 Assumptions 2104
20.1.3 Canonical Form 2105
20.1.4 Microhyperbolicity 2107
20.1.5 Plan of the Chapter 2108
20.2 Weak Magnetic Field Case 2108
20.2.1 Preliminary Remarks 2109
20.2.2 Tauberian Estimate 2110
Outer Zone: General Case 2110
Outer Zone: Special Case $q = 1$ 2112
Conclusion 2115
20.2.3 Main Theorems 2115
General Case 2115
Microhyperbolic Case 2117
Special Case of Constant g^{jk}, F_{jk} 2119
20.3 Canonical Form 2120
20.3.1 Pilot-Model 2120
20.3.2 General Case: Framework 2121

20.3.3 Reduction. Main Part 2123
20.3.4 Reduction. Junior Terms 2125
20.3.5 Temperateg and Stronger Magnetic Field 2129
20.4 Temperate Magnetic Field: Estimates 2130
20.4.1 Intermediate Zone 2130
General Arguments 2130
Case $q \geq 2$ 2134
Case $q \geq 2$ (Some Calculations) 2137
Case $q = 1$ 2138
Case $q = 1$ (Calculations) 2140
20.4.2 Inner Zone 2141
Case $q \geq 2$ 2141
Case $q = 1$ 2143
20.4.3 Cases $q = 1, 2$ Revised 2145
Microhyperbolicity Assumption 2145
Special Case of Constant g^{jk}, F_{jk} 2147
Case of f_j Having Constant Multiplicities 2147
Case of Constant f_j 2150
20.4.4 Improved Remainder Estimates 2151
20.5 Stronger Magnetic Field: Estimates 2155
20.5.1 Intermediate Magnetic Field 2155
General Settings 2155
Microhyperbolic Case 2155
Special Case of Constant g^{jk}, F_{jk} 2156
Case of f_j Having Constant Multiplicities 2156
Case of Constant f_j 2157
Improvement without Microhyperbolicity 2157
20.5.2 Intermediate-Strong Magnetic Field 2157
20.5.3 Strong Magnetic Field 2159
20.6 Intermediate Magnetic Field: Calculations 2160
20.6.1 Weak Magnetic Field Redone 2160
20.6.2 Decomposition 2162
Decomposition. Part I 2162
Decomposition. Part II 2167
20.6.3 Inner Zone: General Settings 2170
20.6.4 Improved Error Estimates 2174
Microhyperbolicity Assumption 2174
Case of f_j Having Constant Multiplicities 2177

Number-Theoretical Arguments 2179
20.6.5 $q = 1$: Asymptotics with Correction 2180
20.6.6 Main tTheorems 2185
Microhyperbolicity Assumption 2185
General Case 2186
Case of f_j Having Constant Multiplicities 2188
20.7 Stronger Magnetic Field: Calculations and Main Results 2190
20.7.1 Intermediate Magnetic Field 2191
Calculations 2191
Main Results 2191
20.7.2 Intermediate-Strong Magnetic Field 2192
Calculations 2192
Main Theorems 2193
20.7.3 Strong Magnetic Field 2193
20.8 Degenerating Magnetic Field 2195
20.8.1 Weak Magnetic Field 2195
20.8.2 Temperate Magnetic Field 2197
General Case 2197
Results under Microhyperbolicity or Non-Degeneracy Assumptions 2198
Other Improved Results 2200
20.8.3 Intermediate and Intermediate-Strong Magnetic Field 2201
20.8.4 Strong Magnetic Field 2201
Schrödinger Operator 2202
Schrödinger-Pauli Operator 2203

IX Magnetic Schrödinger Operator in Dimension 4 2204

21 4D-Schrödinger Operator with the Strong Degenerating Magnetic Field 2205
21.1 Introduction 2205
21.1.1 Framework 2205
21.1.2 Plan of the Chapter 2206
21.2 Weak Magnetic Field 2207
21.2.1 Geometry of Degenerating Magnetic Field 2207

21.2.2 Classical dDynamics 2209
Examples 2209
Precise Calculations 2211
General Case. I 2211
General Case. II 2215
General Case. III 2216
21.2.3 Quantum Dynamics. I. Outer Zone 2219
Short sScale Dynamics 2219
Dynamics and Remainder Estimate. I 2222
Dynamics and Remainder Estimate. II 2224
21.2.4 Quantum Dynamics. II. Inner Zone 2226
Dynamics and Remainder Estimate. III 2226
Dynamics and Remainder Estimate. IV 2228
Dynamics and Remainder Estimate. V 2231
21.3 Temperate and Intermediate Magnetic Field. Canonical Forms 2232
21.3.1 Precanonical Form 2232
Precanonical Form. I 2232
Precanonical Form. II 2234
Precanonical Form. III 2236
21.3.2 Canonical Form away from Λ 2238
Canonical Form away from Λ. I 2238
Canonical Form away from Λ. II 2240
Canonical Form away from Λ. III 2243
21.3.3 Canonical Form in the Strictly Outer Zone 2244
Canonical Form in the Strictly Outer Zone. I . . 2244
Canonical Form in the Strictly Outer Zone. II . . 2248
Canonical Form in the Strictly Outer Zone. III . 2249
21.4 Remainder Estimates 2249
21.4.1 Estimates in the Strictly Outer Zone 2250
Estimates in the Strictly Outer Zone. I 2250
Estimates in the Strictly Outer Zone. II 2252
Estimates in the Strictly Outer Zone. III 2254
Estimates in the Strictly Outer Zone. IV 2256
Estimates in the Strictly Outer Zone. V 2259
Estimates in the Strictly Outer Zone. VI 2260
21.4.2 Estimates in the Near Outer Zone 2264
Estimates in the Near Outer Zone. I 2264

Estimates in the Near Outer Zone. II 2268
Estimates in the Near Outer Zone. III 2272
21.4.3 Estimates in the Inner Zone 2274
Estimates in the Inner Zone. I 2274
Estimates in the Inner Zone. II 2275
Estimates in the Inner Zone. III 2276
Estimates in the Inner Zone. IV 2279
Estimates in the Inner Zone. V 2284
Conclusion 2285
21.5 Calculations and Main Results 2285
21.5.1 Temperate Magnetic Field 2286
Temperate Magnetic Field. I 2286
Temperate Magnetic Field. II 2289
Temperate Magnetic Field. III 2292
21.5.2 Final Calculations 2293
Final cCalculations. I 2293
Final Calculations. II 2295
Final Calculations. III 2296
Final Calculations. IV 2298
Final Calculations. V 2299
Final Calculations. VI 2301
21.5.3 Stronger Magnetic Field 2303
Stronger Magnetic Field. I 2303
Stronger Magnetic Field. II 2305
Stronger Magnetic Field. III 2306
21.5.4 Remarks 2308
Special Case 2308
About Term $C\mu^2 h^{-2}$ in the remainder estimate . 2309
21.5.5 Strong and Superstrong Magnetic Field Cases . . 2309
Easy Analysis 2310
Difficult Analysis 2311
Etc 2312

22 Generic 4D-Schrödinger Operator with the Strong Magnetic Field 2313
22.1 Introduction 2313
Plan of the Chapter 2315
22.2 Geometry and Preliminary Analysis 2316

22.2.1 Geometry 2316
22.2.2 Microhyperbolicity Condition 2317
Preliminary Analysis 2317
Microhyperbolicity Condition. I 2318
Microhyperbolicity Condition. II 2319
22.2.3 Analysis near Σ: Geometry 2320
Preliminary Remarks 2320
Geometry. I 2321
22.3 Weak Magnetic Field Case 2321
22.3.1 Weak Magnetic Field Case. I 2322
22.3.2 Weak Magnetic Field Case. II 2327
22.3.3 Weak Magnetic Field Case. III 2328
22.4 Temperate Magnetic Field Case: Estimates 2330
22.4.1 Canonical Form 2330
22.4.2 General Estimates at Regular Points 2330
General Estimates at Regular Points. I 2330
General Estimates at Regular Points. II 2333
22.4.3 Sharp Asymptotics at Regular Points 2334
Sharp Asymptotics at Regular Points. I 2334
Sharp Asymptotics at Regular Points. II 2335
Sharp Asymptotics at Regular Points. III 2336
22.4.4 General Asymptotics at Resonances 2336
General Asymptotics at Resonances. I 2336
22.4.5 Sharp Asymptotics at Resonances 2345
22.4.6 General Estimates near Σ 2347
22.4.7 Sharp Asymptotics near Σ 2349
Sharp Asymptotics near Σ. I 2349
Sharp Asymptotics near Σ. II 2350
Sharp Asymptotics near Σ. III 2352
22.4.8 Summary 2354
22.5 Calculations 2355
22.5.1 Temperate Magnetic Field 2355
Temperate Magnetic Field. I 2356
Temperate Magnetic Field. II 2357
Temperate Magnetic Field. III 2358
Temperate Magnetic Field. IV 2360
Temperate Magnetic Field. V 2361
Temperate Magnetic Field. VI 2362

Conclusion 2363
22.5.2 Intermediate Magnetic Field 2364
Strong Magnetic Field. I 2364
Intermediate Magnetic Field. II 2367
Intermediate Magnetic Field. III 2368
Intermediate Magnetic Field. IV 2369
Intermediate Magnetic Field. V 2370
Intermediate Magnetic Field. VI 2372
Intermediate Magnetic Field. VII 2373
Intermediate Magnetic Field. VIII 2374
22.6 Vanishing V Case 2374
22.6.1 Generic Case 2374
22.6.2 General Case 2374

X Eigenvalue Asymptotics for Schrödinger and Dirac Operators with the Strong Magnetic Field 2377

23 Eigenvalue asymptotics. 2D Case 2378
23.1 Introduction 2378
23.2 Asymptotics with Fixed Spectral Parameter 2380
23.2.1 Schrödinger Operator 2380
Estimates of the Spectrum 2380
Basic Results 2383
Power Singularities 2384
Improved Remainder Estimates 2389
Degenerations 2391
Power sSingularities. II 2394
Exponential Singularities 2399
23.2.2 Schrödinger-Pauli Operators 2403
23.2.3 Dirac Operator 2405
Preliminaries 2405
Asymptotics. I 2407
Asymptotics. II 2409
23.3 Asymptotics of Large Eigenvalues 2410
23.3.1 Singularities at the Point 2410
Schrödinger Operator 2410

Schrödinger-Pauli Operator 2412
Dirac Operator 2413
Miscellaneous Singularities 2414
23.3.2 Singularities at Infinity 2417
Power Singularities: Schrödinger Operator 2417
Power Singularities: Schrödinger-Pauli Operator . 2420
Power Singularities: Dirac Operator 2421
Exponential Singularities 2422
23.4 Asymptotics of Small Eigenvalues 2425
23.4.1 Operators Stabilizing at Infinity 2425
23.4.2 Operators Stabilizing at Infinity. II 2431
23.4.3 Case $F \to \infty$ as $|x| \to \infty$ 2432
23.4.4 Case $F \to 0$ as $|x| \to \infty$ 2435
23.5 Multiparameter Asymptotics 2438
23.5.1 Asymptotics of Large Eigenvalues 2438
23.5.2 Asymptotics of the Small Eigenvalues 2441
23.5.3 Case of $\tau \to +0$ 2442
23.5.4 Case of $\tau \to -\infty$ 2444
23.A Appendices 2445
23.A.1 Self-Adjointness of the Dirac Operator 2445

24 Eigenvalue asymptotics. 3D Case **2448**
24.1 Introduction 2448
24.2 Asymptotics with Fixed Spectral Parameter 2449
24.2.1 Schrödinger Operator 2450
Estimates of the Spectrum 2450
Power Singularities 2452
Improved Remainder Estimates 2455
Power Singularities. II 2456
Exponential Singularities 2458
24.2.2 Schrödinger-Pauli Operators 2459
24.2.3 Dirac Operator 2461
Preliminaries 2461
Asymptotics 2462
24.3 Asymptotics of Large Eigenvalues 2463
24.3.1 Singularities at the Point 2464
Schrödinger Operator 2464
Schrödinger-Pauli Operator 2465

Miscellaneous Singularities 2466
24.3.2 Singularities at iInfinity 2468
Power sSingularities: Schrödinger Operator 2469
Power Singularities: Schrödinger-Pauli Operator . 2470
Exponential Singularities 2471
24.4 Asymptotics of Small Eigenvalues 2472
24.4.1 Miscellaneous Asymptotics 2472
Case $F \gtrsim 1$ as $|x| \to \infty$ 2472
Case $F \to 0$ as $|x| \to \infty$ 2474
24.4.2 Case $\mathsf{rank}\, F_\infty = d-1$. Fast Decaying Potential . 2475
Preliminary aAnalysis 2476
Main Theorem 2478
Generalizations 2484
Possible Generalizations 2486
24.4.3 Case $\mathsf{rank}\, F_\infty = d-1$. Slow Decaying Potential . 2487
Main Theorem (Statement) 2487
Proof of Theorem 24.4.22: Propagation of Singularities 2488
Proof of Theorem 24.4.22: Estimates 2489
Discussion 2489
24.5 Multiparameter Asymptotics 2493
24.5.1 Asymptotics of Large Eigenvalues 2493
24.5.2 Asymptotics of the Small Eigenvalues 2495
24.5.3 Case of $\tau \to +0$ 2496
24.5.4 Case of $\tau \to -\infty$ 2497
24.A Appendices 2498
24.A.1 1D Schrödinger Operator 2498
24.A.2 1D Schrödinger Operator. II 2503
24.A.3 Examples of Vector Potential 2506

Volume V. Applications to Quantum Theory and Miscellaneous Problems 1

XI Application to Multiparticle Quantum Theory 2511

25 No Magnetic Field Case 2512
25.1 Introduction . . . 2512
25.1.1 Framework . . . 2512
25.1.2 Problems to Consider . . . 2514
25.1.3 Thomas-Fermi Theory . . . 2515
25.1.4 Main Results Sketched and Plan of the Chapter . 2517
25.2 Reduction to Semiclassical Theory . . . 2518
25.2.1 Lower Estimate . . . 2519
25.2.2 Upper Estimate . . . 2522
25.2.3 Remarks and Dirac Correction . . . 2524
25.3 Thomas-Fermi Theory . . . 2525
25.3.1 Existence . . . 2526
25.3.2 Properties . . . 2528
25.4 Application of Semiclassical Methods . . . 2534
25.4.1 Asymptotics of the Trace . . . 2534
25.4.2 Upper Estimate for E . . . 2538
Estimating $|\lambda_N - \nu|$ . . . 2538
Estimating D-Term . . . 2541
Finally, a Theorem . . . 2543
25.4.3 Improved Asymptotics . . . 2544
25.4.4 Corollaries and Discussion . . . 2547
Estimates for $\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}})$ . . . 2547
Estimates for Distance between Nuclei in the Free Nuclei Model . . . 2548
25.5 Negatively Charged Systems . . . 2550
25.5.1 Estimates of the Correlation Function . . . 2550
25.5.2 Excessive Negative Charge . . . 2557
25.5.3 Estimate for Ionization Energy . . . 2561
25.6 Positively Charged Systems . . . 2562
25.6.1 Estimate from above for Ionization Energy . . . 2562
25.6.2 Estimate from below for Ionization Energy . . . 2566

25.6.3 Estimate for Excessive Positive Charge 2576
25.A Appendices 2580
25.A.1 Electrostatic Inequalities 2580
25.A.2 Hamiltonian Trajectories 2581
25.A.3 Some Spectral Function Estimates 2582
25.A.4 Calculation of Scott, Dirac and Schwinger correction terms 2583
Scott correction term 2583
Dirac correction term 2584
Schwinger correction term 2585
General case 2586
25.A.5 Baxter's Lemma 2588
25.A.6 Dependence of Thomas-Fermi energy of the nuclear coordinates 2589
Regularized Thomas-Fermi theory 2589
Proof of Theorem 25.3.8 2593
Electronic contribution to the Born-Oppenheimer energy 2596
Auxiliary statements 2597

26 The Case of External Magnetic Field **2600**
26.1 Introduction 2600
26.1.1 Framework 2600
26.1.2 Problems to Consider 2601
26.1.3 Magnetic Thomas-Fermi Theory 2603
26.1.4 Main Results Sketched and Plan of the Chapter . 2604
26.2 Magnetic Thomas-Fermi Theory 2606
26.2.1 Framework and Existence 2606
26.2.2 Properties 2608
26.2.3 Positive Ions 2614
26.3 Applying Semiclassical Methods: $M = 1$ 2619
26.3.1 Heuristics 2619
Total Charge 2620
Semiclassical D-Term 2622
$|\lambda_N - \nu|$ and another D-Term 2624
Trace 2626
Discussion 2626
26.3.2 Smooth Approximation 2627

Trivial Arguments 2627
Formal Expansion 2631
Expansion: Justification 2633
26.3.3 Rough Approximation 2635
Properties of Mollification 2635
Charge Term 2637
Trace Term 2640
Semiclassical D-Term: Local Theory 2643
Semiclassical D-Term: Global Theory 2646
26.4 Applying Semiclassical Mmethods: $M \geq 2$ 2648
26.4.1 Scaling Functions in Zone $\mathcal{X}_2$ 2648
26.4.2 Zone $\mathcal{X}_2$: Semiclassical N-Term 2653
26.4.3 Zone $\mathcal{X}_2$: Semiclassical D-Term 2657
26.4.4 Semiclassical T-Term 2660
Semiclassical T-Term: Zone $\mathcal{X}_1$ Extended 2660
Semiclassical T-Term: Zone $\mathcal{X}_2$ 2662
26.4.5 Zone $\mathcal{X}_3$ 2666
26.5 Semiclassical Analysis in the Boundary Strip for $M \geq 2$. 2666
26.5.1 Properties of W_B^{TF} if $N = Z$ 2667
26.5.2 Analysis in the Boundary Strip $\mathcal{Y}$ for $N \geq Z$ . . . 2672
26.5.3 Analysis in the Boundary Strip $\mathcal{Y}$ for $N < Z$ . . . 2675
Case $B \geq (Z-N)_+^{\frac{4}{3}}$ 2675
Case $B \leq (Z-N)_+^{\frac{4}{3}}$ 2680
26.5.4 Summary 2681
26.6 Ground State Energy 2682
26.6.1 Lower Estimates 2682
26.6.2 Upper Estimate: General Scheme 2684
26.6.3 Upper Estimate as $M = 1$ 2684
Estimate for $|\lambda_N - \nu|$ 2684
Estimate for D-Terms 2686
Summary 2687
26.6.4 Upper Estimate as $M \geq 2$ 2688
Estimate for $|\lambda_N - \nu|$ 2688
Estimate for D-Terms for Almost Neutral sSystems 2691
Estimate for D-Terms for Positively Charged sSystems 2693
Summary 2698

26.7 Negatively Charged Systems 2699
26.7.1 Estimates of the Correlation Function 2699
26.7.2 Excessive Negative Charge 2701
26.7.3 Estimate for Ionization Energy 2704
26.8 Positively Charged Systems 2710
26.8.1 Upper Estimate for Ionization Energy: $M = 1$ . . 2710
26.8.2 Lower Estimate for Ionization Energy: $M = 1$ . . 2714
26.8.3 Estimates for Ionization Energy: $M \geq 2$ 2724
26.8.4 Free Nuclei Model 2724
Preliminary Arguments 2725
Minimal Distance 2725
Estimate of Excessive Positive Charge 2730
26.A Appendices 2731
26.A.1 Electrostatic Inequalities 2731
26.A.2 Very Strong Magnetic Field Case 2736
26.A.3 Riemann Sums and Integrals 2737
26.A.4 Some Spectral Ffunction Estimates 2738
26.A.5 Zhislin's Theorem for Constant Magnetic Field . . 2739

27 The Case of Self-Generated Magnetic Field 2742
27.1 Introduction 2742
27.2 Local Semiclassical Trace Asymptotics 2744
27.2.1 Toy-Model 2744
Statement of the Pproblem 2744
Preliminary Analysis 2745
Estimates 2748
27.2.2 Microlocal Analysis Unleashed 2751
Sharp Estimates 2751
Application 2756
Classical Dynamics and Sharper Estimates 2759
27.2.3 Local Theory 2762
Localization and Estimate from above 2762
Estimate from below 2765
27.2.4 Rescaling 2767
Case $\kappa \leq 1$ 2768
Case $1 \leq \kappa \leq h^{-1}$ 2770
27.3 Global Trace Asymptotics in the Case of Coulomb-Like Singularities 2772

27.3.1 Problem 2772
27.3.2 Estimates of a Minimizer 2773
Preliminary Analysis 2773
Estimates to a Minimizer. I 2776
Estimates to a Minimizer. II 2779
Estimates to a Minimizer. III 2781
27.3.3 Basic Trace Estimates 2782
27.3.4 Improved Trace Estimates 2785
Improved Tauberian Estimates 2785
Improved Weyl Estimates 2788
27.3.5 Single Singularity 2790
Coulomb Potential 2790
Main Theorem 2795
27.3.6 Several Singularities 2796
Decoupling of Singularities 2796
Main Results 2799
Problems and rRemarks 2800
27.4 Asymptotics of the Ground State Energy 2800
27.4.1 Problem 2800
27.4.2 Lower Estimate 2802
27.4.3 Upper Estimate 2805
27.4.4 Main Theorems 2807
27.4.5 Free Nuclei Model 2809
27.5 Miscellaneous Problems 2811
27.5.1 Excessive Negative Charge 2811
27.5.2 Estimates for Ionization Energy 2812
27.5.3 Free Nuclei Model: Excessive Positive Charge . . 2813
27.A Appendices 2814
27.A.1 Minimizers and Ground States 2814
27.A.2 Zhislin's Theorem 2815
27.A.3 L. Erdös–J. P. Solovej's Lemma 2815

28 The Case of Combined Magnetic Field 2819
28.1 Introduction 2819
28.1.1 Plan of the Chapter 2820
28.1.2 Unfinished Business 2820
28.2 Local Semiclassical Trace Asymptotics: Preparation . . . 2821
28.2.1 Toy-Model 2821

28.2.2 Formal Semiclassical Theory 2821
Semiclassical Theory: $\beta h \lesssim 1$ 2821
Semiclassical Theory: $\beta h \gtrsim 1$ 2825
28.2.3 Estimate from below 2825
Basic Estimates 2825
Estimates to a Minimizer: $\beta h \lesssim 1$ 2827
Estimates to a Minimizer: $\beta h \gtrsim 1$ 2829
28.3 Microlocal Analysis Unleashed: $\beta h \lesssim 1$ 2831
28.3.1 Rough Estimate to a Minimizer 2831
28.3.2 Microlocal Analysis 2835
28.3.3 Advanced Estimate to a Minimizer 2840
Tauberian Estimate 2840
Calculating Tauberian Expression 2843
Estimating $|\partial^2 A'|$ 2846
28.3.4 Trace Term Asymptotics 2847
General Microlocal Arguments 2847
Strong Non-Degenerate cCase 2850
Non-Degenerate Case 2851
Degenerate case 2852
28.3.5 Endgame 2853
Upper Estimate 2853
Lower Estimate 2854
Weak Magnetic Field Approach. 2856
Main Theorem 2856
28.3.6 N-Term Asymptotics 2857
Introduction 2857
Strong Non-Degenerate Case 2858
Non-Degenerate Case 2859
Degenerate Case 2859
28.3.7 D-Term Estimate 2860
28.4 Microlocal Analysis: $\beta h \gtrsim 1$ 2866
28.4.1 Estimate to a Minimizer 2866
28.4.2 Trace Term Asymptotics 2867
28.4.3 Endgame 2869
28.4.4 N-Term Asymptotics and D-Term Estimates . . . 2869
28.5 Global Trace aAsymptotics in the Case of Thomas-Fermi Potential: $B \le Z^{\frac{4}{3}}$ 2870
28.5.1 Introduction 2870

28.5.2 Estimates to a Minimizer 2871
Preliminary Analysis 2871
Estimates to a Minimizer: Interior Zone 2874
Estimates to a Minimizer: Exterior Zone 2875
28.5.3 Trace aAsymptotics 2876
28.5.4 Endgame 2882
Main Theorem: $M = 1$ 2882
28.5.5 N-Term Asymptotics and D-Term Estimate 2885
Case $M = 1$ 2885
Case $M \geq 2$ 2887
28.5.6 More Estimates to a Minimizer 2889
Case $M = 1$ 2890
Case $M \geq 2$ 2892
28.5.7 Endgame: $M \geq 2$ 2894
28.6 Global Trace Asymptotics in the Case of Thomas-Fermi Potential: $Z^{\frac{4}{3}} \leq B \leq Z^3$ 2895
28.6.1 Trace Estimates 2895
28.6.2 Estimates to a Minimizer 2899
28.6.3 N-Term Asymptotics and D-Term Estimates . . . 2902
28.7 Applications to Ground State Energy 2903
28.7.1 Preliminary Remarks 2903
28.7.2 Estimate from above: $B \leq Z^{\frac{4}{3}}$ 2905
28.7.3 Main Theorems: $B \leq Z^{\frac{4}{3}}$ 2909
Ground State Energy Asymptotics 2909
Ground State Density Asymptotics 2912
28.7.4 Main Theorems: $Z^{\frac{4}{3}} \leq B \leq Z^3$ 2914
Ground State Energy Asymptotics 2914
Ground State Density Asymptotics 2915
28.A Appendices 2917
28.A.1 Generalization of Lieb-Loss-Solovej Estimate . . . 2917
28.A.2 Electrostatic Inequality 2921
28.A.3 Estimates for $(hD_{x_j} - \mu x_j)e(x, y, \tau)|_{x=y}$ for Toy-Model Operator 2922
Calculations 2922
Case $\alpha \geq \mu^2 h$ 2924
Case $\alpha \leq \mu^2 h$ 2925
Case $\mu h \geq \epsilon_0$ 2927
Tauberian Estimates 2929

Bibliography **2930**

XII Articles 3003

Semiclassical Dirichlet to Neumann Operator **3004**
1 Introduction . . . 3005
2 Reduction to Semiclassical Spectral Asymptotics . . . 3011
3 Semiclassical Spectral Asymptotics . . . 3014
4 Relation to Dirichlet Boundary Condition . . . 3020
A Appendix . . . 3022
A.1 Standard Semiclassical Asymptotics . . . 3022
A.2 Tauberian Theorem . . . 3022
A.3 Propagator . . . 3023
A.4 Propagation of Singularities . . . 3024
A.5 Method of Successive Approximations . . . 3026
A.6 Two Term Expansion . . . 3027
Bibliography . . . 3029

Spectral Asymptotics for Fractional Laplacians **3031**
1 Problem Set-up . . . 3031
2 Preliminary aAnalysis . . . 3032
3 Propagation of Singularities near the Boundary . . . 3034
4 Reflection of Singularities from the Boundary . . . 3036
4.1 Toy-Model . . . 3036
4.2 General Case . . . 3039
5 Main rResults . . . 3040
5.1 From Tauberian to Weyl Asymptotics . . . 3040
5.2 Discussion . . . 3042
6 Global Theory . . . 3043
A Variational Estimates for Fractional Laplacian . . . 3045
A.1 Variational Estimates for Fractional Laplacian . . . 3045
A.2 μ-Transmission Property . . . 3046
Bibliography . . . 3047

Spectral Asymptotics for Dirichlet to Neumann operator in the Domains with Edges **3049**
1 Introduction . . . 3049

2 Dirichlet-to-Neumann Operator 3052
2.1 Toy-Model: Dihedral Angle 3052
2.2 General Case 3054
3 Microlocal Analysis 3055
3.1 Propagation of Singularities near Edge 3055
3.2 Reflection of Singularities from the Edge 3059
4 Main Results 3062
4.1 From Tauberian to Weyl Asymptotics 3062
4.2 Main Theorems 3063
4.3 Discussion 3063
A Appendix 3065
A.1 Planar Toy-Model 3065
Preparatory Results 3065
A.2 Spectrum 3072
Bibliography 3074

Asymptotics of the Ground State Energy in the Relativistic Settings 3076
1 Introduction 3076
2 Functional Analytic Arguments 3078
2.1 Estimate from below 3078
2.2 Estimate from above 3082
3 Semiclassical Methods 3082
3.1 Trace Term 3083
3.2 Trace tTerm. II 3087
3.3 Trace Term. III 3088
3.4 N- and D-Terms 3088
3.5 Dirac Term 3090
4 Main Theorems 3090
A Some Inequalities 3092
Bibliography 3093

Asymptotics of the Ground State Energy in the Relativistic Settings and with Self-Generated Magnetic Field 3095
1 Introduction 3095
2 Local Semiclassical Trace Asymptotics 3098
2.1 Set-up 3098

2.2 Functional Analytic Arguments 3098
Estimates 3098
Properties of a Minimizer 3100
2.3 Microlocal Analysis and Local Theory 3105
Microlocal Analysis Unleashed 3105
Local Theory and Rescaling 3105
3 Global Trace Asymptotics in the Case of Coulomb-Like Singularities 3106
3.1 Estimates to a Minimizer 3106
3.2 Trace Estimates 3109
4 Main Results 3112
4.1 Asymptotics of the Ground State Energy 3112
4.2 Related Problems 3114
A Some iInequalities 3115
Bibliography 3116

Complete Semiclassical Spectral Asymptotics for Periodic and Almost Periodic Perturbations of Constant Operator 3118
1 Introduction 3118
1.1 Preliminary Remarks 3118
1.2 Main Theorem 3121
1.3 Plan of the Paper 3124
2 Proof of the Main Theorem 3125
2.1 Preliminary Analysis 3125
2.2 Gauge Transformation 3128
2.3 Non-Resonant Zone 3129
2.4 Gauge Transformation 3129
2.5 Resonant Zone 3131
2.6 End of the Proof 3134
3 Generalizations and Discussion 3135
3.1 Matrix Operators 3135
3.2 Perturbations 3136
3.3 Differentiability 3138
Bibliography 3140

Complete Differentiable Semiclassical Spectral Asymptotics 3142
1 Introduction 3142
2 Proofs 3145

2.1 Preliminary Remarks 3145
2.2 Propagation and Local Energy Decay 3145
2.3 Traces and the End of the Proof 3150
2.4 Discussion 3151
Bibliography 3152

Bethe-Sommerfeld Conjecture in Semiclassical Settings 3154
1 Introduction 3154
1.1 Preliminary Remarks 3154
1.2 Main Theorem (Statement) 3156
1.3 Idea of the Proof and the Plan of the Paper . . . 3157
2 Reduction of Operator 3159
2.1 Reduction 3159
2.2 Classification of Resonant Points 3162
2.3 Structure of Operator $\mathcal{A}$ 3163
3 Proof of Theorem 1.3 3164
3.1 Choosing γ^* 3164
3.2 Non-Resonant Points 3165
3.3 Resonant Points 3168
4 Discussion 3169
4.1 Improving υ 3169
4.2 Condition (1.14) 3170
4.3 Differentiability 3170
4.4 Bethe-Sommerfeld Conjecture for Almost Periodic Perturbations 3170
Bibliography 3171

Scott and Thomas-Fermi approximations to electronic density 3175
1 Introduction 3175
2 Estimates of eigenvalues 3181
3 Estimates of projectors 3187
4 Electronic Density: Scott approximation 3196
5 Proof of main Theorem 1.7 3200
A Properties of eigenfunctions of Coulomb-Schrödinger operator 3204
A.1 General 3204
A.2 Zeroes 3205

A.3 Estimates 3207
B Properties of eigenvalues and eigenfunctions of Coulomb-Schrödinger operator in relativistic settings 3213
Bibliography 3220
Upper estimates for electronic density in heavy atoms and molecules 3222
1 Introduction 3222
2 Main intermediate inequality 3224
3 Estimates of the averaged electronic density 3228
4 Estimates of the correlation function 3230
5 Proof of Theorem 1.1 3236
Bibliography 3239

Pointwise spectral asymptotics out of diagonal near boundary **3241**
1 Introduction 3241
2 Asymptotics inside domain 3245
2.1 Tauberian asymptotics 3245
2.2 Weyl asymptotics 3246
2.3 Improvements and generalizations 3250
3 Microlocal methods 3252
3.1 Propagation 3252
3.2 Spectral estimates 3255
4 Successive approximations 3259
4.1 Successive approximations inside domain 3259
Standard successive approximations 3259
Improved successive approximations 3261
4.2 Successive approximations near boundary 3261
Standard successive approximations 3261
Improved successive approximations 3262
5 Geometric optics method 3270
5.1 Constructing solution 3270
5.2 Spectral asymptotics 3281
6 Synthesis and final remarks 3285
Bibliography 3286

Pointwise spectral asymptotics out of diagonal near degeneration **3288**

1 Introduction 3288
2 Toy-model 3292
2.1 Hamiltonian trajectories 3292
2.2 Calculations 3294
2.3 Generalized toy-model 3300
3 General case 3303
3.1 Preliminary remarks 3304
3.2 Propagator as an oscillatory integral 3305
3.3 Perturbation methods 3308
3.4 Proof of Theorem 1.2 3311
4 Generalizations and final remarks 3311
Bibliography 3313

Pointwise spectral asymptotics near singularity **3315**

29 Introduction **3316**
1 Toy-model operator 3316
2 General case 3318
3 Plan of the paper. 3319

30 Toy-model operator **3320**
1 Hamiltonian trajectories 3320
2 Estimates 3323
3 Scaling 3332

31 General case **3335**
1 Standard results for perturbed operator 3335
2 Perturbations of $e_h(x, y, 0)$ 3339
3 Near singularity 3342
3.1 Non-semiclassical subzone 3342
3.2 Intermediate subzone 3344
Bibliography 3344

100 years of Weyl's Law **3346**
1 Introduction 3346
A Bit of History 3346
Method of the Hyperbolic Operator 3349
2 Local Semiclassical Spectral Asymptotics 3351

Asymptotics Inside the Domain 3351
Propagation of Singularities 3352
Successive Approximation Method 3354
Recovering Spectral Asymptotics 3357
Second Term and Dynamics 3360
Rescaling Technique 3362
Operators with Periodic Trajectories 3365
Boundary Value Problems 3367
Preliminary Analysis 3367
Propagation of Singularities 3368
Successive Approximations Method 3371
Recovering Spectral Asymptotics 3372
Second Term and Dynamics 3373
Rescaling Technique 3374
Operators with Periodic Billiards 3374
3 Global Asymptotics 3375
Weyl Asymptotics 3376
Regular Theory 3376
Singularities 3381
Non-Weyl Asymptotics 3385
Partially Weyl Theory 3385
Domains with Thick Cusps 3386
Neumann Laplacian in Domains with Ultra-Thin Cusps 3389
Operators in $\mathbb{R}^d$ 3390
Maximally Hypoelliptic Operators 3391
Trace Asymptotics for Operators with Singularities 3391
Periodic Operators 3393
4 Non-Smooth Theory 3394
Non-Smooth Symbols and Rough Microlocal Analysis 3394
Non-Smooth Boundaries 3396
Aftermath 3397
2 Magnetic Schrödinger Operator 3397
Introduction 3397
Standard Theory 3398
Preliminaries 3398
Canonical Form 3399
Asymptotics: Moderate Magnetic Field 3399

Asymptotics: Strong Magnetic Field 3400
2D case, Degenerating Magnetic Field 3401
Preliminaries 3401
Moderate and Strong Magnetic Field 3402
Strong and Superstrong Magnetic Field 3403
2D Case, near the Boundary 3404
Moderate Magnetic Field 3404
Strong Magnetic Field 3406
Pointwise Asymptotics and Short Loops 3407
Case $d = 2$ 3408
Case $d = 3$ 3409
Related Asymptotics 3410
Magnetic Dirac oOperators 3410
3 Magnetic Schrödinger Operator. II 3411
Higher Dimensions 3411
General Theory 3411
Case $d = 4$: More Results 3413
Non-Smooth Theory 3413
Global Asymptotics 3414
Case $d = 2r$ 3415
Case $d > 2r$. I 3416
Case $d > 2r$. II 3417
4 Applications to Multiparticle Quantum Theory 3418
Problem Set-up 3418
Reduction to One-Particle Problem 3419
Estimate from below 3419
Estimate from above 3421
Semiclassical Approximation 3422
Estimate from below 3422
Estimate from above 3423
More Precise Estimates 3424
Ramifications 3425
Adding Magnetic Field 3426
Adding External Magnetic Field 3426
Adding Self-Generated Magnetic Field 3428
Combining External and Self-Generated Magnetic Fields 3429
Bibliography 3430

Presentations 3435

Index **3437**

Preface

The Problem of the *Spectral Asymptotics*, in particular the problem of the *Asymptotic Distribution of the Eigenvalues*, is one of the central problems in the *Spectral Theory of Partial Differential Operators*; moreover, it is very important for the *General Theory of Partial Differential Operators*.

I started working in this domain in 1979 after R. Seeley [1] justified a remainder estimate of the same order as the then hypothetical second term for the Laplacian in domains with boundary, and M. Shubin and B. M. Levitan suggested me to try to prove Weyl's conjecture. During the past almost 40 years I have not left the topic, although I had such intentions in 1985, when the methods I invented seemed to fail to provide the further progress and only a couple of not very exciting problems remained to be solved. However, at that time I made the step toward local semiclassical spectral asymptotics and rescaling, and new much wider horizons opened.

So I can say that this book is the result of 40 years of work in the Theory of Spectral Asymptotics and related domains of Microlocal Analysis and Mathematical Physics (I started analysis of *Propagation of singularities* (which plays the crucial role in my approach to the spectral asymptotics) in 1975).

Victor Ivrii,
Toronto, July 9, 2023.

Introduction

1. Problems to Study

Finding the *Spectral Asymptotics*, in particular the *Asymptotic Distribution of Eigenvalues*, is one of the central problems of the *Spectral Theory of Partial Differential Operators*, and is very important for the *General Theory of Partial Differential Operators*.

Apart from applications to Quantum Mechanics, Radio Physics, Continuum Media Mechanics (Elasticity, Acoustics, Hydrodynamics, Theory of Shells), etc., there are also applications to mathematics itself and deep though non-obvious links to Differential Geometry, Dynamical Systems Theory, Ergodic Theory and Number Theory; even the term *Spectral Geometry* was coined. All these circumstances make this topic very attractive for a mathematician.

The original problem was formulated independently by A. Sommerfeld [1] and H. A. Lorentz [1] in 1910 who stated the *Weyl's Law* as a conjecture based on the book of Lord Rayleigh "The Theory of Sound" (1887) (see in details in W. Arendt and others [1]).

This conjecture was proven in 1911 by then young mathematician, a disciple of David Hilbert, specialist in partial differential and integral equations Hermann Weyl who published several papers [1–5] devoted to eigenvalue asymptotics for the Laplace operator (and also the elasticity operator) in a bounded domain with a regular boundary. After this article a huge number of various papers devoted to spectral asymptotics were published among the authors of which were numerous prominent mathematicians.

Also, in [4] Weyl published his famous *Weyl conjecture*.

Much later, in 1950, H. Weyl in his paper [7] returned to this topic.

The theory was developed in two directions: on one hand, the theory was extended to consider more general operators, boundary conditions and

geometrical domains; on the other hand, the asymptotics were improved and more accurate remainder estimates were derived. In the latter extensions the links with differential geometry, dynamical systems theory and ergodic theory appeared. Even the theory of eigenvalue asymptotics for the Laplace (or Laplace-Beltrami) operator has a long, dramatic and still unfinished history.

Apart from asymptotics with respect to the *spectral parameter*, asymptotics with respect to other parameters appeared; the most important among them are (in my opinion) *semiclassical asymptotics*, i. e. asymptotics with respect to a small parameter h (the Planck constant in physics), tending to $+0$. For a long time these asymptotics were in the shadows: most attention was paid to the eigenvalue asymptotics for operators on compact manifolds (with or without boundary).

The results which had been obtained in that case were then reproved for operators in $\mathbb{R}^d$ such as the Schrödinger operator $h^2\Delta + V(x)$ with fixed $h > 0$ and with $V(x) \to +\infty$ as $|x| \to \infty$ where Δ is a (positive) Laplacian; less attention was paid to semiclassical asymptotics (i.e., asymptotics of the number of eigenvalues below some fixed level λ as $h \to +0$).

The asymptotics of small negative eigenvalues were considered in the case of fixed h and $V(x)$ decreasing at infinity as $|x|^{2m}$ with $m \in (-1, 0)$; under reasonable conditions in this case the discrete spectrum of the operator has accumulation point -0 and the essential spectrum coincides with $[0, +\infty)$.

The result of the development of the theory was that at a certain time there existed four parallel (though not equally developed and not equally respected)[1)] theories, the statements in each of which had to be proven separately.

This pluralism was over from ≈1985 because all other results are easily derived from the *Local Semiclassical Spectral Asymptotics* (*LSSA* in what follows)[2)].

[1)] While the achievements in the first theory were broadly known by mathematicians, the other results were familiar to the narrow circle of PDE-folk and mathematical physicists, many of whom (including myself, it is a real shame for me) even thought that only the first topic is of the prime interest.

[2)] I started to study semiclassical spectral asymptotics at the time when I foolishly assumed that the first theory was completely finished and I had nothing to do; so I decided to extend the results to semiclassical asymptotics before leaving the area. When I told my first results to M. Z. Solomyak, he asked if those results could be deduced from the results of the classical theory by the Birman-Schwinger principle. A quick check

One of the main tools here is a *rescaling technique* (see Section *6. How LSSA are Applied to Classical Asymptotics*) I invented approximately in the same time.

The semiclassical spectral asymptotics are the main object of this book, and all the other results are obtained as applications.

We also consider applications to the multiparticle quantum theory the topic I began working on in 1990 with M.I.Sigal.

Alternative introduction the reader can find in V. Ivrii [28]: "100 years of Weyl's law"; see also my talk [9].

2. Weyl's Law

Let us formulate here the Weyl's law for a non-Magnetic Schrödinger operator

$$H = h^2\Delta + V(x), \tag{0.1}$$

where $\Delta = -\partial_{x_1}^2 - ... - \partial_{x_d}^2$, $h \ll 1$ and $V(x)$ is the real valued function. Assuming that H is a self-adjoint operator, let $E(\lambda)$ is its spectral projector and $e(x, y, \lambda)$ its Schwartz kernel. Then *Weyl's law* states

$$\mathsf{N}(\lambda) \approx (2\pi h)^{-d} \iint_{\{|\xi|^2+V(x)\le\lambda\}} dxd\xi = (2\pi h)^{-d}\varpi_d \int \big(\lambda - V\big)_+^{d/2}\, dx, \tag{0.2}$$

where $N(\lambda) \coloneqq \mathsf{Tr}(E(\lambda))$ is an *eigenvalue counting function* i.e. a number of eigenvalues of H lesser than λ provided $(-\infty, \lambda) \cap \mathsf{Spec}_{\mathsf{ess}}(H) = \emptyset$, otherwise $N(\lambda) = +\infty$, ϖ_d is a volume of the unit ball in $\mathbb{R}^d$. There are also *local Weyl's law*

$$\mathsf{Tr}(E(\lambda)\psi) \approx (2\pi h)^{-d}\varpi_d \int \big(\lambda - V\big)_+^{d/2}\psi(x)\, dx \tag{0.3}$$

for $\psi \in \mathscr{C}_0^\infty$ and *pointwise Weyl's law*

$$e(x, x, \lambda) \approx (2\pi h)^{-d}\varpi_d\big(\lambda - V\big)_+^{d/2}. \tag{0.4}$$

gave a very surprising answer: using "classical" asymptotics one can prove semiclassical asymptotics only under very unnatural restrictions; but using semiclassical asymptotics one can extend "classical" asymptotics to more general operators and problems.

3. Methods

In his papers, starting from the second one, H. Weyl applied the *variational method* (*Dirichlet-Neumann bracketing*) he invented; later this method was improved in various directions by many outstanding mathematicians and mathematical physicists (R. Courant, M. Sh. Birman, and others). This method is simple and works in the general situations (even for non-semibounded operators) and seems to be the best method if one does not care about precise remainder estimates.

Later *Tauberian methods* appeared (first due to T. Carleman). The idea is to consider some function $F(H,t)$ of an operator in question H and some auxiliary parameter t. Then

$$\mathsf{Tr}(F(H,t)) = \int F(\tau,t)\, d_\tau\, \mathsf{Tr}(E(\tau)), \tag{0.5}$$

where $E(\tau)$ is the spectral projector. One then tries to construct the left-hand expression by means of PDE theory and then recover $\mathsf{Tr}(E(\tau))$ with some error by means of an appropriate Tauberian theorem. The choice of the function $F(H,t)$ is very important here. Taking $F(H,t) = \exp(-tA)$ (for a positive operator H), $F(H,t) = (t-H)^{-1}$ and $F(H,t) = H^t$ one obtains respectively the *heat equation, resolvent and ζ-function methods*, which are usually simple in their PDE part but difficult in the Tauberian part and almost always fail to provide a decent remainder estimate.

On the other hand, the *hyperbolic operator method* due to B. M. Levitan and V. G. Avakumovič is difficult in its PDE part but is almost trivial in the Tauberian part and provides accurate remainder estimates. By this method all the asymptotics with the most accurate remainder estimates were obtained. This method is based on the function $U(t) = F(H,t) = \exp(itH)$ which satisfies $(D_t - H)U = 0$, $U(0) = I$, and therefore the Schwartz kernel $u(x,y,t)$ of the operator is the fundamental solution to the Cauchy problem (or the initial-boundary value problem) for the operator $D_t - H$ (where $D_t = -i\partial_t$, etc.). The Schwartz kernel is connected to the eigenvalue counting function of the operator H by the formula

$$\int u(x,x,t)\, dx = \int \exp(it\lambda)\, d_\lambda \mathsf{N}(\lambda). \tag{0.6}$$

In the case of a matrix operator H, $u(x,y,t)$ is a matrix-valued function and should be replaced by its trace in the left-hand expression. Then by

the means of the inverse Fourier transform we can recover $\mathsf{N}(\lambda)$ provided we have constructed $u(x,y,t)$ by means of the methods of partial differential operator theory.

In fact, we are never able (excluding some very special and rare cases) to construct $u(x,y,t)$ precisely and for all the values $t \in \mathbb{R}$. Usually (now we assume that H is an elliptic first-order pseudodifferential operator) the fundamental solution is constructed approximately (modulo smooth functions) for t belonging to some interval $[-T,T]$ with $T>0$. As a consequence we obtain, modulo $O(\lambda^{-s})$ with any arbitrarily chosen exponent s, an expression for

$$F_{t\to\tau}\Big(\chi_T(t)\int u(x,x,t)\,dx\Big) = \int \hat{\chi}_T(\tau-\lambda)\,d_\lambda \mathsf{N}(\lambda), \tag{0.7}$$

where χ is a fixed smooth function supported in $[-1,1]$, $\chi_T(t)=\chi(t/T)$ and the hat as well as $F_{t\to\tau}$ mean the Fourier transform. Then, if we know the left-hand expression, we are able, by the Tauberian theorem due to B. M. Levitan, to recover $\mathsf{N}(\lambda)$ approximately by the formula

$$\mathsf{N}(\lambda) = \int_{-\infty}^{\lambda} \big(F_{t\to\tau}\chi_T(t)\sigma(t)\big)(\tau)\,d\tau + O\big(\lambda^{d-1}\big), \tag{0.8}$$

where d is the dimension of the domain and

$$\sigma(t) = \int u(x,x,t)\,dx. \tag{0.9}$$

The explicit construction of $u(x,x,t)$ in this situation leads us to the formula

$$\mathsf{N}(\lambda) = \kappa_0\lambda^d + O\big(\lambda^{d-1}\big) \tag{0.10}$$

with the leading coefficient

$$\kappa_0 = (2\pi)^{-d}\int_{H(x,\xi)<1} dx d\xi, \tag{0.11}$$

where $H(x,\xi)$ is the principal symbol of the operator H.

We see that the crucial step in this approach is the construction of the fundamental solution. This construction by means of Fourier integral operators is now standard and well known as long as we consider scalar operators or operators with the constant multiplicities of the eigenvalues of

the principal symbol. We construct $u(x,y,t)$ on a compact set K contained in the interior of our domain X (and T depends on the distance between K and ∂X).

If one of those assumptions is violated, then the construction is much more sophisticated and is possible only under some very restrictive conditions. In the presence of the boundary (but only in the case of the constant multiplicities of the eigenvalues of the principal symbol) this construction was implemented in papers due to V. M. Babich, B. M. Levitan, R. Seeley, R. Melrose, D. Vassiliev and others. But it is possible to avoid all these troubles by means of another approach suggested by V. Ivrii [8] (see also L. Hörmander [1] (vol. 4)) based on the investigation of the propagation of singularities for $u(x,y,t)$ and the construction of an "approximation" (in a rather exotic sense) for the distribution which leads to an approximation in a reasonable sense for $\sigma(t)$ for $|t| \le T$ with appropriate $T > 0$.

For the h-pseudodifferential operators, considered in this book, this approach is essentially simpler and more transparent because there is a selected parameter h; this case will be discussed below. This method allows us to prove the asymptotics (0.8) for an arbitrary self-adjoint m-th order elliptic operator with $m > 0$ and the spectral parameter λ^m on a compact manifold with or without boundary (in the former case the boundary conditions are also supposed to be elliptic), scalar or matrix, semibounded from below or not semibounded at all (in this case $\mathsf{N}(\lambda)$ is replaced by $\mathsf{N}^{\pm}(\lambda)$ which is the number of eigenvalues lying between 0 and $\pm\lambda^m$); the formula for κ_0 should be changed if necessary.

The two-term asymptotics

$$\mathsf{N}(\lambda) = \kappa_0\lambda^d + \kappa_1\lambda^{d-1} + o\big(\lambda^{d-1}\big), \tag{0.12}$$

suggested by H. Weyl [4] (who also gave a formula for κ_1) fails to be true without some additional condition. Surely this asymptotic expansion is wrong for $d = 1$ and for the Laplace-Beltrami operator on the sphere $\mathbb{S}^d$; this fact is associated with the high multiplicities of its eigenvalues. Moreover, it remains wrong in the case when this Laplace-Beltrami operator is perturbed by a potential or even by a symmetric first-order operator with small coefficients; in these cases all the eigenvalues of high multiplicities will be split into narrow eigenvalue clusters separated by spectral gaps.

On the other hand, under conditions of a global nature the asymptotics (0.12) is valid. For a scalar operator on a compact manifold without boundary the relevant condition is:

The measure of the {*set of all the points of the cotangent bundle periodic with respect to the Hamiltonian flow generated by the principal symbol*} *equals* 0

(see J. J. Duistermaat and V. Guillemin [1]). The condition is more complicated for matrix operators.

For a scalar second-order operator on a compact manifold with the boundary one needs only to consider trajectories which are transversal to the boundary and which reflect according the laws of geometrical optics. There are some points of the cotangent bundle through which infinitely long trajectories do not pass but the measure of these dead-end points vanishes and we need not take them into account.

However, for higher-order operators as well as for matrix operators the trajectories reflected from the boundary can branch and in this case it is necessary to follow every branch. This circumstance guides us to a situation which is essentially more complicated: the measure of the set of all these dead-end points does not necessarily vanish, so the following additional condition appears:

the measure of {*the set of all dead-end points*} *equals* 0*;*

this condition is not automatically fulfilled now.

Let us clarify, for scalar first-order operators on a manifold without boundary, the link between the asymptotics (0.12) and periodic Hamiltonian trajectories. It is well known that singularities of the solutions of hyperbolic equations propagate along Hamiltonian trajectories. This fact guides us to the conclusion that the singular support of $\sigma(t)$ is contained in the set of all the periods of the Hamiltonian trajectories including $t = 0$ (this is the so-called *Poisson relation*); in particular, $t = 0$ is an isolated point of the singular support (this fact remains true also in the much more general situation).

Hence, if there is no periodic trajectory with the period not exceeding T, then we know that in the interval $[-T, T]$ the distribution $\sigma(t)$ is singular only at 0 and hence we know $\sigma(t)$ in this interval modulo a smooth function. The Tauberian theorem implies that the remainder in asymptotics (0.12) does not exceed $C\lambda^{d-1}T^{-1} + O\bigl(\lambda^{d-2}\bigr)$ with a constant C, which does not depend on T; however "O" here is not necessarily uniform with respect to T. Therefore, if $t = 0$ were the only period (surely I know that it is impossible!), then we could choose T arbitrarily large and obtain the remainder estimate $o\bigl(\lambda^{d-1}\bigr)$.

In the general (realistic) case one should consider a partition of unity by means of two pseudodifferential operators Q_j ($j = 1, 2$) for every chosen T, such that the support of the first operator contains no periodic point with a period not exceeding T, and the measure of the support of the symbol of the second operator is less than ε with arbitrarily chosen $\varepsilon > 0$. Applying the Tauberian theorem to each term

$$\mathsf{N}_j(\lambda) = \int (Q_j e)(x, x, \lambda)\, dx, \tag{0.13}$$

we obtain the remainder estimate $C\lambda^{d-1}T^{-1} + O\bigl(\lambda^{d-2}\bigr)$ for $j = 1$ and $C\varepsilon\lambda^{d-1} + O\bigl(\lambda^{d-2}\bigr)$ for $j = 2$. Therefore the total remainder estimate for $\mathsf{N}(\lambda) = \mathsf{N}_1(\lambda) + \mathsf{N}_2(\lambda)$ is $C\lambda^{d-1}(\varepsilon + T^{-1}) + O\bigl(\lambda^{d-2}\bigr)$ with arbitrarily large T and arbitrarily small $\varepsilon > 0$; this estimate again implies (0.12).

Moreover, under certain more restrictive conditions to the Hamiltonian flow, one can improve the remainder estimate in (0.12) up to $O\bigl(\lambda^{d-1}(\log \lambda)^{-1}\bigr)$ or even $O\bigl(\lambda^{d-1-\delta}\bigr)$ with a small exponent $\delta > 0$.

It appeared later that the presence of periodic trajectories and, moreover, the presence of eigenvalue clusters allow two-term asymptotics of the form

$$\mathsf{N}(\lambda) = \kappa_0\lambda^d + f(\lambda)\lambda^{d-1} + o\bigl(\lambda^{d-1}\bigr) \tag{0.14}$$

with an explicitly calculable function $f(\lambda)$ which is bounded and oscillating as $\lambda \to +\infty$ with the characteristic "period" of oscillations $\asymp 1$. In particular, this fact permits us to obtain the asymptotic distribution of eigenvalues inside clusters. In addition, under some assumptions including the assumption that all the trajectories are periodic, one can obtain the asymptotics (0.14) with the remainder estimate $O\bigl(\lambda^{d-2}\bigr)$!

4. Methods. II

The theory presented above has major exceptions. First, it could happen that in some part of our domain we should not apply the semiclassical spectral asymptotics (rescaled or not) but instead we simply need to estimate the main part of the asymptotics using methods different from those explained (usually those are variational methods). So, our domain is split into two parts: a *semiclassical zone* (aka a *regular zone*) and a *singular zone*; this partition depends on the parameters. F. e., if we consider the Schrödinger operator $h^2\Delta - |x|^{-1}$, then the semiclassical zone is $\{x\colon |x| \gtrsim h^{-2}\}$ and the

singular zone is $\{x\colon |x| \lesssim h^{-2}\}$. Definitely joining results in these two zones requires some work.

Second, it may happen that (in some part of the domain) we need to split variables x and ξ to two parts: $x = (x'; x'')$, $\xi = (\xi'; \xi'')$, and only consider (x', ξ') as *Weyl variables*, which means that one should consider the operator in question as a (partial) differential operator with respect to x' with operator-valued coefficients (acting in some auxiliary Hilbert space) and apply the Weyl procedure to this operator. F.e., if we consider Laplacian in the domain has a cusp: $\{x = (x', x'')\colon x'' \in f(x')\Omega\}$, where Ω is a bounded domain in $\mathbb{R}^{d''}$ with a smooth boundary, and $f(x')$ decays as $|x'| \to \infty$, then for $|x'| \ge g(\lambda)$ we can consider x' as Weyl variables and an auxiliary space is $\mathbb{K} = \mathscr{L}^2(\Omega)$.

In the more general case one should divide the phase space into a few parts, one of which should be removed from consideration altogether while in the other parts the "Weylization" (preceded by a certain transform) should be made only with respect to certain variables.

The previous remarks are not a survey, even an incomplete one. Their purpose is only to motivate this book in general and the following part of this introduction in particular. Nor are the comments in each chapter complete.

As the best surveys I recommend the books of M. S. Birman and M. Z. Solomyak [5] and G. Rozenblioum, M. Z. Solomyak and M. Shubin [1], despite being outdated.

It also should be noted that though this book demands from the reader only a good and detailed knowledge of the theory of pseudodifferential operators and a familiarity with the theory of Fourier integral operators (see, e.g., the books of L. Hörmander [1], F. Treves [1], M. Taylor [7] and M. Shubin [3]); an elementary familiarity with the theory of spectral asymptotics (see the same books) is also very desirable. As an elementary introduction to Semiclassical analysis I recommend M. Zworski [1], A. Martinez [1], B. Helffer [2].

It should be noted that this book has few intersections with the books of S. Levendorskii [6] and Yu. Safarov and D. Vassiliev [3] which are also devoted to spectral asymptotics. I recommend both of those books, especially the second one.

Finally, I would like to mention that there are some problems in this book. There are problems the solutions of which are not known to me. There are problems the solutions of which are known to me but which cannot be published here or elsewhere by me because of lack of time due to my

efforts on *this* book. I welcome readers to solve these problems either by the method suggested or by some other method and to publish their solutions. Finally, there are problems I consider rather simple exercises.

The structure of this book and the numbering of statements and formulas is clear from the table of contents and from the Section below.

5. What is in the Book

Let us now discuss the contents and principal ideas of this book. It consists of five volumes:

Volume I covers the general microlocal and local theory.

Volume II covers the general functional-analytic theory which in combination with the microlocal theory of Volume 1 leads us to spectral asymptotics in domains with singularities. Then we apply these results to more specific problems. We also consider here some problems not covered by the general theory.

Volume III is devoted to Schrödinger and Dirac operators with the strong magnetic field.

Volume IV is also devoted to Schrödinger and Dirac operators with the strong magnetic field.

Volume V is devoted to the asymptotics of the ground state energy of heavy atoms and molecules and related topics. It also contains some latest articles, which are not included into the main body of the book.

Volume I. Semiclassical Microlocal Analysis and Local and Microlocal Semiclassical Asymptotics

Part I. Semiclassical Microlocal Analysis

There are no spectral asymptotics yet in Part I, "Semiclassical Microlocal Analysis"; we develop the apparatus (both basic notions and technique) of the semiclassical microlocal analysis here.

Chapter 1. Introduction to Microlocal Analysis. This chapter contains the necessary survey of the theory of the h-pseudodifferential operators (in *Section 1.1*), the functional calculus of these operators (in *Section 1.4*), the theory of the h-Fourier integral operators (in *Section 1.2*) and the theory of singularities (wave front sets and related topics, in *Section 1.3*). There are no new non-trivial results in this chapter but there is a rather obvious but very important new

Definition 0.1. A family of functions $u = u_h$, depending on a small parameter $h \in (0, h_0]$ (and maybe depending on some other parameters of an arbitrary nature), tempered (i.e., with $\|u\| \le Ch^{-n}$ with some constant C and exponent n where here and below $\|.\|$ means the $\mathscr{L}^2$-norm) is *negligible in the box*

$$\Pi = \{(x,\xi) : |x_j - \bar{x}_j| \le \gamma_j,\ |\xi_j - \bar{\xi}_j| \le \rho_j\ \forall j = 1,\dots,d\},$$

if there exists a function $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^{2d})$, equal to 1 in the unit cube $\{(x,\xi) : |x_j| \le 1, |\xi_j| \le 1 \quad \forall j = 1,\dots,d\}$ and such that $\|\phi_\Pi^w(x, hD)u\| \le Ch^s$, where $\phi_\Pi(x,\xi) = \phi(\gamma^{-1}(x-\bar{x}), \rho^{-1}(\xi - \bar{\xi}))$; note that the exponent s enters into the definition of negligibility and here and below we use a notation $\gamma^{-1}x = (\gamma_1^{-1}x_1, \dots, \gamma_d^{-1}x_d)$ etc.

In the standard definition, leading to the notion of an *wave front set*, one takes a *fixed* box Π (i.e., with constant $(\bar{x},\bar{\xi})$ and $(\gamma_1,\dots,\gamma_d;\rho_1,\dots,\rho_d)$); this definition is too crude for our needs and we admit variable (i.e., depending on parameters) $(\bar{x},\bar{\xi})$ and $(\gamma_1,\dots,\gamma_d;\rho_1,\dots,\rho_d)$, satisfying the *microlocal uncertainty principle*

$$\gamma_j\rho_j \ge h^{1-\delta} \qquad \forall j = 1,\dots,d \tag{0.15}$$

with an arbitrarily small exponent $\delta > 0$. An important special case is $\gamma_1 = \dots = \rho_d \ge Ch^{\frac{1}{2}(1-\delta)}$ [3)]

[3)] We can weaken (0.15) to the *logarithmic uncertainty principle*

$$\gamma_j\rho_j \ge C_s h|\log h| \qquad \forall j = 1,\dots,d \tag{0.16}$$

with important special case $\gamma_1 = \dots = \rho_d \ge C(h|\log h|)^{\frac{1}{2}}$ albeit not to the *standard uncertainty principle*

$$\gamma_j\rho_j \ge h \qquad \forall j = 1,\dots,d. \tag{0.17}$$

Chapter 2. Propagation of Singularities in the Interior of the Domain. This chapter is devoted to the propagation of singularities (or, more precisely, of the negligibility) of solutions in the interior of the domain. The key definition of the *microhyperbolicity of an operator P at* the given *point z* of the cotangent bundle *in the direction* $\mathcal{T} \in T_z T^*X$ appears in *Section 2.1*, and the main theorem concerning propagation of singularities is proven there; the theorem is formulated in terms of real-valued functions ϕ_k, such that the operator P under consideration is microhyperbolic at every point $z \in \Omega$ in the directions of the Hamiltonian fields $\nabla^{\#}\phi_k$, generated by ϕ_k; here and in what follows $\Omega \subset T^*X$ is a phase domain under consideration. This theorem is proven by means of the energy estimates.

We then apply this theorem to the operator $P = hD_t - A^w(x, hD_x, h)$ and to the analysis of the fundamental solution $u(x, y, t)$ of the Cauchy problem for this operator, establishing the finite speed of propagation of singularities (in the domain $\Omega \subset T^*X$). In particular, under the microhyperbolicity condition for the operator A on the energy level λ we prove that the singularities of $u(x, y, t)$ leave the diagonal $\{x = y, \xi = \eta\}$; this statement yields the negligibility of

$$\sigma_Q(t) = \int (Q_x u)(x, x, t) dx \tag{0.18}$$

in boxes of the form $\{(t, \tau) : Ch^{1-\delta} \le \pm t \le T_0, |\tau - \lambda| \le \epsilon_0\}$ with small enough positive constants T_0 and ϵ_0 and with arbitrarily small exponent $\delta > 0$; here and in what follows Q is a cutting h-pseudodifferential operator supported in Ω.

The results of Section 2.1 are reformulated in *Section 2.2* in the terms of the *generalized bicharacteristics* of P, which, in the case of constant multiplicities of the eigenvalues $\alpha_j(x, \xi)$ of the symbol $a(x, \xi) = A(x, \xi, 0)$, are the standard bicharacteristics of the symbols $\tau - \alpha_j(x, \xi)$.

Also results of Section 2.1 are extended in *Section 2.3* to operators with rough symbols.

In *Section 2.4* the classical theorem on the propagation of singularities along bicharacteristics for operators with constant multiplicities of the characteristics is proven in a more general and refined form. Namely, for the operator $P = hD_t - A^w(x, hD_x, h)$ with self-adjoint operator $A^w(x, hD_x, h)$ and with constant multiplicities of the eigenvalues $\alpha_j(x, \xi)$ of the symbol $a(x, \xi)$ we replace the usually used negligibility in a small fixed neighborhood by negligibility in the γ-neighborhood of a Hamiltonian trajectory with

$\gamma \geq h^{\frac{1}{2}(1-\delta)}$; it is assumed that this trajectory is not too long (i.e., $T \leq ch^{-\delta'}$ with a small enough exponent $\delta' > 0$); the symbol $A(x, hD_x, h)$ and its derivatives up to large enough order K and the elements of the Jacobi matrix of the corresponding Hamiltonian flow are also assumed to be not too large (i.e., their absolute values or norms are also less than $ch^{-\delta'}$). The proof is based on the Heisenberg equation. In what follows we sometimes return to the investigation of the propagation of singularities.

Chapter 3. Propagation of Singularities near the Boundary. Here we realize approximately the same program for operators near the boundary on which boundary conditions are given. In *Section 3.1* we give the key definition of the *microhyperbolicity of the problem* (P, B) *at the point* $z \in T^*\partial X$ *in the multidirection* $\mathcal{T} = (\mathcal{T}', \zeta_1, \dots, \zeta_n) \in T_z T^*\partial X \times \mathbb{R}^n$, where n is the number of the different characteristic points z_k of the operator P, lying over z. Then we prove the main theorem concerning the propagation of singularities; this theorem is formulated in terms of real-valued functions $\phi_k(x, \xi')$, coinciding at $\{x_1 = 0\}$ and such that the problem (P, B) is microhyperbolic in the domain $\Omega \subset T^*\partial X$ under consideration in the multidirection $(\nabla^{\#}\phi, \partial_{x_1}\phi_1, \dots, \partial_{x_1}\phi_n)$, where we assume that $X = \{x_1 = 0\}$ locally and $x' = (x_2, \dots, x_d)$, $\phi = \phi_k(0, x, \xi')$.

Then in *Section 3.2* we discuss some geometrical aspects of the propagation of singularities; however the general results are not very precise.

Next, in *Section 3.3* we deduce the same corollaries as in Section 2.1 (concerning the finite speed of the propagation of singularities for the operator $P = hD_t - A^w(x, hD_x, h)$ and the negligibility of $\sigma_Q(t)$ in the same boxes as before provided (A, B) is microhyperbolic at the level λ) from this theorem.

Further, in *Section 3.4* we develop a more refined energy method and prove that there is no propagation along creeping rays.

Finally, *Section 3.5* is devoted to the propagation of singularities along "long trajectories"; here we consider only h-differential second-order operators and assume that at the point where a trajectory meets the boundary and reflects the trajectory is transversal enough to the boundary (with angle greater than $h^{\delta'}$); moreover, we assume that a certain non-degeneracy condition on the boundary operator is also fulfilled at such reflection points. It should be noted that the number of reflection points is not too large (less

than $h^{-\delta''}$ with exponent $\delta'' = \delta''(\delta') > 0$, $\delta''(\delta') \to +0$ as $\delta' \to +0$).

Part II. Local and Microlocal Semiclassical Spectral Asymptotics in the Interior of the Domain

The main goal of Part II is to obtain asymptotics as $h \to +0$ for

$$\mathcal{E}_Q(\lambda_1, \lambda_2) = \int (Q_x e)(x, x, \lambda_1, \lambda_2)\, dx \tag{0.19}$$

with

$$e(x, y, \lambda_1, \lambda_2) = e(x, y, \lambda)\big|_{\lambda=\lambda_1}^{\lambda=\lambda_2}, \tag{0.20}$$

where $e(x, y, \lambda)$ is a Schwartz kernel of $E(\lambda)$ (which is a spectral projector of A) and either $Q = Q^w(x, hD_x)$ is an h-pseudodifferential operator with compactly supported symbol or $Q = \psi(x)$ is a compactly supported function; we speak about *microlocal spectral asymptotics* and *local spectral asymptotics* respectively. Here and below the subscript "x" means that the corresponding operator acts with respect to x, etc.

We also consider the same expressions mollified with respect to the spectral parameter.

Chapter 4. General Theory in the Interior of the Domain. Here we consider the case when $\mathsf{supp}(Q)$ is disjoint from the boundary of the domain X.

Section 4.2 contains a preliminary analysis and the results obtained there are rather trivial. First of all, microlocal estimates $\|\mathcal{E}_Q(\lambda_1, \lambda_2)\| \le Ch^{-d}$ are derived in the general case and estimates $\|\mathcal{E}_Q(\lambda_1, \lambda_2)\| \le Ch^s$ are derived in the case when the interval (λ_1, λ_2) is *classically forbidden* in a neighborhood of $\mathsf{supp}(Q)$ (i.e., the interval is disjoint from the spectrum of the symbol $A(x, \xi, h)$ for the indicated (x, ξ)); one can weaken this condition provided the multiplicities of the eigenvalues of the symbol $a(x, \xi)$ are constant. We obtain similar local estimates for operators elliptic in the standard sense.

Then in *Section 4.3* we apply the *method of successive approximations* to $u(x, y, t)$; we take an unperturbed operator $\bar{A} = A(y, hD_x, 0)$ with the symbol depending only on ξ. This method turns out to be applicable in the rather short time interval $\{|t| \le h^{\frac{1}{2}+\delta}\}$ with arbitrary small exponent $\delta > 0$.

This immediately yields in *Section 4.4* complete semiclassical asymptotics for the spectral means

$$\int \phi\big(\frac{1}{L}(\lambda - \bar{\lambda})\big)\, d_\lambda \mathcal{E}_Q(\lambda_1, \lambda) \tag{0.21}$$

with any $L \ge h^{\frac12-\delta}$ and with $\phi \in \mathscr{C}_0^K(\mathbb{R})$; further, one can take $L \ge h^{1-\delta}$ provided the eigenvalues of $A(x,\xi)$ have constant multiplicities. This yields *complete asymptotics* for $\mathcal{E}_Q(\lambda_1,\lambda_2)$ in the case when both levels λ_1 and λ_2 are classically forbidden but the whole interval (λ_1,λ_2) is not; in all these cases the leading term is $\kappa_0 h^{-d}$. Then we consider the main case when $\bar\lambda$ is not a classically forbidden level and A is microhyperbolic at this level.

Combining the results of Section 4.3 with the statement from Section 2.1 concerning the negligibility of $\sigma_Q(t)$, we obtain complete asymptotics of $\sigma_Q(t)$ in the zone $\{(t,\tau) : |t| \le T_0, |\tau-\bar\lambda| \le \epsilon_0\}$ with small positive constants T_0 and ϵ_0. For $|\tau-\bar\lambda| \le \epsilon_0$ we get asymptotics in the interval $[-h^{\frac12+\delta}, h^{\frac12+\delta}]$. It was proven in Section 2.1 that $\sigma(t)$ is negligible in $[-T_0,-h^{1-\delta}]\cup[h^{1-\delta},T_0]$, and these intervals overlap! We immediately get asymptotics of (0.21) for $L \ge h^{1-\delta}$. Moreover, the Tauberian method yields the estimate

$$\|\mathcal{E}_Q(\lambda_1,\lambda_2) - \kappa_{0Q}(\lambda_1,\lambda_2)h^{-d}\| \le Ch^{1-d} \tag{0.22}$$

for $\lambda_1,\lambda_2 \in (\bar\lambda-\frac12\epsilon_0, \bar\lambda+\frac12\epsilon_0)$ and miscellaneous related asymptotics in which the possible smallness of $\mathsf{supp}(Q)$ is taken into account. The same method also yields asymptotics of the Riesz means of order $\vartheta > 0$ with remainder estimate $Ch^{1-d+\vartheta}$. However, all these asymptotics are due to the simple Tauberian estimate

$$\|\mathcal{E}_Q(\lambda_1,\lambda_2)\| \le Ch^{1-d} \qquad \forall \lambda_1,\lambda_2 \in [\bar\lambda-\epsilon_0, \bar\lambda+\epsilon_0] : |\lambda_1-\lambda_2| \le h \tag{0.23}$$

and asymptotics of (0.21) with $L \ge h$ and remainder estimate $Ch^{1-d}\big(h/L\big)^s$ with an arbitrarily large exponent s. Actually estimates of this type seem to be the most important for our internal needs.

In *Section 4.5* we derive sharper remainder estimates under assumptions of the global nature. Applying the results of Section 2.3 under a condition forbidding periodic trajectories of length not exceeding T, we obtain complete asymptotics of $\sigma_Q(t)$ in the zone indicated above with T_0 replaced by T; then the same arguments yield the estimate

$$\|\mathcal{E}_Q(\lambda_1,\lambda_2) - \kappa_{0Q}(\lambda_1,\lambda_2)h^{-d} - \kappa_{1Q}(\lambda_1,\lambda_2)h^{1-d}\| \le C\varepsilon h^{1-d} + Ch^{1-d+\delta'} \tag{0.24}$$

with $\epsilon = T^{-1}$ and small enough exponent $\delta' > 0$; under some more general and realistic conditions we obtain an estimate with a more complicated

expression for ε. Those adjustments are also achieved for Riesz means (the first term in the remainder estimate is $Ch^{1-d+\vartheta}T^{-1-\vartheta}$) and for expression (0.21) with $L \geq hT^{-1}$ (the first term of the remainder estimate is $Ch^{1-d}T^{-1}\big(h/LT\big)^{s}$).

Furthermore, in *Section 4.6* we generalize these results to operators with rough coefficients. Then for operators with irregular coefficients we construct two operators with rough coefficients, bracketing them, and close enough to provide the same principal parts of the asymptotics.

Considering $a(x,\xi)$ as the symbol of an m-th order differential operator and assuming that it is elliptic in the standard sense, we switch from microlocal to local asymptotics. It allows us in certain cases to consider operators with non-smooth ($\mathscr{C}^l$) coefficients by the bracketing method.

Finally, in Section 4.7 we consider operators with non-smooth coefficients directly by the perturbation method.

We discuss Chapters 4–6 so comprehensively because its results and methods are highly important for the parts of the book which follow. The scheme of the arguments in the main stream of these chapters is the following:

Construction (by means of the method of successive approximations) of $u(x,y,t)$ *and hence* $\sigma_Q(t)$ *in the short time-interval* $\{|t| \leq T_1\}$*—step-by-step extension of this interval by different methods—application of the Tauberian method;*

in what follows this scheme is used repeatedly. An important general idea is that

In the theory of spectral asymptotics the step-by-step approach with improvement is much better than more bold attempts to do everything in one shot.

It should be mentioned that we do not use the results of Chapter 3 in Chapters 4–6 and that the results of Sections 4.2–4.4 are obtained for standard operators as well as for operators with operator-valued symbols; however, the results for operators with operator-valued symbols are applied only in Chapter 12 (except Section 12.6).

Chapter 5. Scalar Operators in the Interior of the Domain. Rescaling Technique. Here we consider scalar operators (and sometimes operators with the similar properties).

In *Section 5.2* we analyze operators with constant multiplicities of the eigenvalues of the principal symbol. By means of *an admissible partition of unity and rescaling with respect to x with a change of h on every element of this partition* we replace the microhyperbolicity condition by another much less restrictive condition (which is void for Riesz means). The method applied here is very important and will be used very often in this book. We call it the *rescaling method* or *partition-rescaling method* in what follows. To get the best possible results one should combine this method with a refined analysis of the propagation of singularities along trajectories which are long *after* rescaling but are usually of bounded length or even short *before*: time and energy are rescaled automatically.

Section 5.3 may be summarized simply: for the Schrödinger, Schrödinger-Pauli and Dirac operators in dimension $d \geq 2$ one can simply drop the microhyperbolicity condition. Here we apply the same rescaling method as before with some simplifications and modifications.

For these operators we also consider *pointwise asymptotics*, i.e. asymptotics of $e(x,x;\lambda_1,\lambda_2)$ or $(Q_{x,y}e)(x,x;\lambda_1,\lambda_2)$ without mollification with respect to the spatial variables. In this case not only periodic trajectories but also loops play important role: namely, trajectory returns to the same point but from the different direction.

In *Section 5.4* we generalize previous results to operators with rough and irregular symbols.

Chapter 6. Operators in the Interior of Domain. Esoteric Theory. In this chapter we consider more special questions.

First, in *Section 6.1* we extend the above results to the families of commuting operators.

Further, *Sections 6.2 and 6.3* have a more special character. Here we consider scalar operators with periodic Hamiltonian flow of the principal symbol, or some weak perturbation of an operator of this type; there automatically appears an additional parameter $\eta \in [h^n, h^\delta]$ with arbitrarily large n and arbitrarily small $\delta > 0$ (this parameter equals h in the "normal" situation); for $\eta \leq \epsilon h$ with a small constant $\epsilon > 0$ there arise spectral zones of length $\leq C\eta$ separated by lacunary zones of length $\asymp h$ in the semiclassical

approximation to the spectrum; it is proven that $\mathcal{E}_Q(\lambda_1,\lambda_2)=O(h^s)$ with arbitrarily large s provided the interval (λ_1,λ_2) lies in a lacunary zone; in the case when λ_1 and λ_2 belong to different lacunary zones we obtain complete non-Weyl asymptotics for $\mathcal{E}_Q(\lambda_1,\lambda_2)$.

Turning to the long-time propagation of singularities (with $|t|\le \epsilon\eta^{-1}$) we prove that singularities propagate along Hamiltonian trajectories of the symbol $a(x,\xi)$, drifting in the direction of the Hamiltonian field generated by an auxiliary symbol $b(x,\xi)$ with drift speed $\asymp\eta$. Now turning to $\mathcal{E}_Q(\lambda_1,\lambda_2)$ we finally derive, under certain conditions, the estimate

$$\bigl|\mathcal{E}_Q(\lambda_1,\lambda_2)-\kappa_{0Q}(\lambda_1,\lambda_2)h^{-d}-\kappa_{1Q}(\lambda_1,\lambda_2)-f_Q(\lambda_1,\lambda_2)h^{1-d}\bigr|\le C(\eta+h)h^{1-d}, \tag{0.25}$$

where the quickly oscillating function f is explicitly calculated. It arises from singularities of $\sigma_Q(t)$ different from $t=0$. Nevertheless the last term in the asymptotics is of the order of the second one and its variation on the interval of length $\asymp h$ is of the same order $\asymp h^{1-d}$ as the variation of the first term[4)]. For $\eta\ge h^{1-\delta}$ we obtain asymptotics (0.24) with $\varepsilon=\eta$ under certain assumptions.

In *Section 6.4* we consider asymptotics of expression

$$I:=\iint \omega(x,y)|e(x,y,\tau)|^2\,dxdy \tag{0.26}$$

with $\omega(x,y):=\Omega(x,y;x-y)$, where function $\Omega(x,y,z)$ is smooth with respect to x and y and smooth positively homogeneous of degree κ with respect to $z\ne 0$. This expression generalize the *Dirac energy* used in Chapters 25–28 (in which case $\omega(x,y)=|x-y|^{-1}$).

Finally, in *Section 6.5* we consider improved asymptotics with mollifications with respect to some parameter, which is different from a spectral parameter.

Part III. Local and Microlocal Semiclassical Spectral Asymptotics near the Boundary

Chapter 7. Standard Local Semiclassical Spectral Asymptotics near the Boundary. This chapter is essentially parallel to Chapter 4:

[4)] As it should be because the decrease of the last term along the spectral gap should compensate for the increase of the first term; along the cluster both terms grow and if $\eta\ll h$ then the last term grows more quickly.

the content of each of *Sections 7.1–7.5* is very similar to the content of its counterpart from 4.2–4.6. We underline only the following "shortcomings":

(a) We now consider only standard operators, not operators with operator-valued symbols.

(b) We do not weaken the microhyperbolicity condition for operators with constant multiplicities of the eigenvalues of the principal symbol; we do it only for the Schrödinger operator in Section 7.4.

(c) In Section 7.4 boundary is also irregular and we cannot smooth it up by a change of coordinates because then coefficients become too irregular. Instead we use (generalized) Seeley's method.

It also should be noted that if the interval $[\lambda_1, \lambda_2]$ is classically forbidden for the symbol $a(x,\xi)$ but is not forbidden for the 1D-problem (on the semiaxis $\mathbb{R}^+ \ni t$) $\big(a(x', D_t, \xi'); b(x', D_t, \xi')\big)$ (as above we assume that $X = \{x_1 > 0\}$ locally), then in the above estimates and asymptotics one should replace d by $(d-1)$; sometimes we are even able to get two-term asymptotics.

Chapter 8. Standard Local Semiclassical Spectral Asymptotics near the Boundary. Miscellaneous. This chapter contains results similar to those of Chapters 5 and 6.

In *Section 8.1* we consider pointwise asymptotics (i.e. asymptotics of $e(x,x;\lambda_1,\lambda_2)$) near the boundary. Because there are short loops (when a trajectory, reflected from the boundary, goes in the exactly opposite direction and returns to the same point x) such asymptotics with a decent remainder estimate has an additional term of the *boundary layer type* of the width h, i.e. the term of the magnitude h^{-d} if $\operatorname{dist}(x,\partial X) \lesssim d$ and decaying away from ∂X.

In *Section 8.2* we consider scalar operators and develop a rescaling technique for them.

In *Section 8.3* we consider operators with periodic Hamiltonian flows but with a twist. First, as we know from Section 6.2 a quantum periodicity may be violated by the phase shift; now however there may be an additional phase shift at the reflections from the boundary. Second, in the case of branching billiards the periodicity could be partial: only some branches return to the original point (x,ξ) while other branches do not, so there is an energy decay along periods.

Finally, in *Section 8.4* we consider spectral asymptotics on subspaces (when the main Hilbert space is not $\mathscr{L}^2(X, \mathbb{C}^D)$ but it subspace which is defined by underdetermined elliptical boundary value problem (f.e. a space of solenoid vector fields)). Such generalization in the case without boundary is rather trivial.

Volume II. Functional Methods and Eigenvalue Asymptotics

Part IV. Estimates of the Spectrum.

The character of Part IV is essentially different from the character of the previous parts. There are only a few micro-analytical arguments in this part but there are many functional-analytical arguments.

The substance of these arguments is the following: in Parts II and III we considered $\mathcal{E}_\psi(0)$ for a function ψ supported in $B(0, \frac{1}{2})$ (where $B(y, r)$ is the ball with radius r and center y in $\mathbb{R}^d$) and all conditions (apart from the self-adjointness condition and conditions associated with Hamiltonian trajectories) were assumed to be fulfilled only in $B(0, 1)$. We had asymptotics with remainder estimates with constants depending only on the constants, used in these conditions.

Hence those estimates yield a similar estimate in the case in which ψ is supported in $B(y, \frac{1}{2}\gamma(y))$ and it is assumed that after the rescaling (an affine transformation in $\mathbb{R}^d$) transforming $B(y, \gamma(y))$ into $B(0, 1)$ and multiplication by $\varrho^{-1}(y)$ for the operator in question the conditions of Part II or III will be fulfilled with the parameter h replaced by a new parameter $h_{\text{eff}} = h\rho(y)^{-1}\gamma(y)^{-1}$ [5)].

One should assume here that $h_{\text{eff}} \le 1$ (i.e., that $\rho(y)\gamma(y) \ge h$) and one can rewrite the conditions mentioned above for the operator before rescaling. Let us assume that $\gamma(y) > 0$ and $|\nabla\gamma(y)| \le 1$; then one can take a γ-admissible partition of unity and under the condition $\rho(y)\gamma(y) \ge h$ one can sum the derived estimates and obtain an estimate of the type (0.28) or (0.29) below.

Chapter 9. Estimates of the Negative Spectrum. We assume here that the operator is semibounded from below. Then under assumption

[5)] Moreover, in Chapters 13–22 the results which also could be used in the similar arguments a parameter μ_{eff} also appears.

$\rho(y)\gamma(y) \ge h$ everywhere the partition-rescaling technique results in

$$\bigl|\mathsf{N}^-(0) - \mathcal{N}^-(0)\bigr| \le CR_1 \tag{0.27}$$

with the same (Weyl or non-Weyl) principal part $\mathcal{N}^-(0)$ as before and with some remainder estimate R_1. However, there are many important cases in which this condition $\rho(y)\gamma(y) \ge h$ is fulfilled only in a subdomain X' of the domain X (as we mentioned we call X' semiclassical or regular zone); then summing with respect to a partition of unity in X' we obtain estimates only for $\mathcal{E}_\psi(0)$ (but not for $\mathsf{N}^-(0)$) with an admissible function ψ such that $\mathsf{supp}(\psi) \subset X'$ and $\psi = 1$ outside of $X'_{3/2} = \{y\colon \rho(y)\gamma(y) \le \frac{3}{2}h\}$. Thus, in order to obtain an estimate for $\mathsf{N}^-(0) = \mathcal{E}_1(0)$, we need to derive an estimate for $\mathcal{E}_{1-\psi}(0)$.

On the other hand, if we had $X = X'' \supset \{y\colon \rho(y)\gamma(y) \le 2h\}$, then we very often would have some estimates for $\mathsf{N}^-(0)$ of a completely different (variational) origin which have the form $\mathsf{N}^-(0) \le CR_2$.

In this book we almost exclusively apply the Rozenblioum variational estimate, but one can use other estimates of this type. It appears that in the general case one can prove these estimates together and derive the estimate

$$\bigl|\mathsf{N}^-(0) - \mathcal{N}^-(0)\bigr| \le C\bigl(R_1 + R_2 + R_{12}\bigr), \tag{0.28}$$

where $\mathcal{N}^-(0)$ and R_1 (associated with $\mathcal{E}_\psi(0)$) are exactly the same as before in the zone X', the estimate R_2 is derived for a self-adjoint operator A'' defined in the domain X'' where A'' coincides with A in the zone $\{y\colon \rho(y)\gamma(y) \le \frac{7}{4}h\}$ (and the boundary conditions also coincide at the common part of the boundaries of X and $\partial X''$), and R_{12} is an additional term, depending on the auxiliary construction.

The proper construction using the *Birman-Schwinger principle* with an appropriate auxiliary function makes this term less than $C(R_1 + R_2)$. This construction is done in Chapter 9.

First of all, in *Section 9.1* the principal theorems are formulated and proven.

Next, in *Section 9.2* we illustrate these theorems in the simpler cases and in *Section 9.3* we reformulate them for quadratic forms. Here we have two important ideas:

(a) *rescaling with multiplication and a change of parameters* (we mentioned this idea in the discussion of Section 5.3) and

(b) *combining the asymptotics derived in Parts II and III with estimates of the variational origin which makes it possible to consider a domain containing not only a semiclassical zone X' but also a singular zone X''.*

This task is done in this chapter for semi-bounded operators.

Chapter 10. Estimates of the Spectrum in the Interval. Here we consider the non-semibounded case and our analysis here is essentially parallel to the analysis of Chapter 9; a very important role is played by the following form of the *Birman-Schwinger principle:*

$$\mathsf{N}\big(A;[\lambda_1,\lambda_2)\big) = \mathsf{N}\big(A-\bar t J;[\lambda_1,\lambda_2)\big) - \mathsf{N}\big(-J^{-1}(A-\lambda I);[0,\bar t)\big)\Big|_{\lambda=\lambda_1}^{\lambda=\lambda_2}, s \tag{0.29}$$

which applies (under some hypotheses) in the case of a positively defined matrix-valued function J and for $\bar t>0$ [6], where $\mathsf{N}(A;[\lambda_1,\lambda_2))$ is the number of eigenvalues of the operator A in the interval $[\lambda_1,\lambda_2)$ [7] and similarly $\mathsf{N}(-J^{-1}(A-\lambda I);[0,\bar t))$ is the number of eigenvalues of the spectral problem

$$\big((A-\lambda I)+tJ\big)u=0 \tag{0.30}$$

in the interval $[0,\bar t)$.

One can iterate this formula, at each step replacing the operator A by $A-t_1J_1-\dots-t_{k-1}J_{k-1}$, the number $\bar t$ by t_k and the function J by J_k with $k=1,\dots,n$; here $t_1,\dots,t_n$ *are not necessarily of the same sign*. As a result we obtain the following more complicated formula

$$\begin{aligned}\mathsf{N}\big(A;[\lambda_1,\lambda_2)\big) =& \mathsf{N}\big(A-t_1J_1-\dots-t_{k-1}J_{k-1};[\lambda_1,\lambda_2)\big) - \\ & \mathsf{N}\big(-J_1^{-1}(A-\lambda I);[0,t_1)\big)\Big|_{\lambda=\lambda_1}^{\lambda=\lambda_2} - \dots \\ & \mathsf{N}\big(-J_n^{-1}(A-\lambda I-t_1J_1-\dots-t_{n-1}J_{n-1});[0,t_n)\big)\Big|_{\lambda=\lambda_1}^{\lambda=\lambda_2},\end{aligned}$$

to the right-hand expression of which we apply arguments, similar to those discussed above (in the discussion of Chapter 9)[8].

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[6] A similar form applies to the case $\bar t<0$.

[7] Provided $\mathsf{Spec}_{\mathsf{ess}}(A)\cap[\lambda_1,\lambda_2)=\emptyset$; otherwise $\mathsf{N}(A;[\lambda_1,\lambda_2))=\infty$.

[8] Actually, we multiply the last equality by $\varphi_1(t_1)\cdots\varphi_n(t_n)$ with appropriate functions φ_k and integrate on $(t_1,\dots,t_n)\in\mathbb{R}^n$ (this rather technical trick is useful).

It should be noted that the reader who is interested only in semibounded operators has no need to read Chapter 10. It should also be noted that Chapters 9 and 10 would be absolutely trivial if we assumed that the singular zone is empty; however in this case we would obtain a much more restricted series of results in Chapters 11, 23 and 24 and we would be ready for Chapter 12 neither technically nor psychologically.

Part V. Asymptotics of Spectra

In Part V we return to spectral asymptotics, now based on Parts II, III and IV.

Chapter 11. Weyl Asymptotics of Spectra. The essence of Chapter 11 is very simple: *we substitute an operator A depending on certain parameters into the assertions of Chapters 9 and 10 (and even Chapters 4–8 after partition of unity, rescaling, multiplication and change of parameters), check that the conditions of these assertions are uniformly fulfilled, and as a result we obtain asymptotics with respect to these parameters*.

The arguments of this Chapter are in terms of "arithmetics", dynamical systems theory (when we check conditions associated with the Hamiltonian trajectories) and rather simple functional analysis (when we check conditions associated with the singular zone).

So *having started with (micro)local semiclassical asymptotics we come to asymptotics with respect to quite different parameters (e.g., the spectral parameter)*. Actually, semiclassical asymptotics are also treated but our results are not the same as before and one should distinguish the actual semiclassical parameter h and the effective semiclassical parameter $h_{\mathsf{eff}} = h/(\rho\gamma)$. The latter depends on the point and arises always, no matter what parameters were in the original problem.

In *Section 11.1* we treat semiclassical asymptotics again, but now we are interested in $\mathsf{N}_h(\lambda_1, \lambda_2)$ and in situations when there are singularities in the domain and coefficients, exits to infinity of different shapes (or even the domain $\mathbb{R}^d$) and other non-regular phenomena.

In Sections 11.2–11.5 we investigate *asymptotics with respect to a large spectral parameter*. Namely, in *Sections 11.2* and *11.3* we consider cases in which a domain may have singularities on the boundary (edges, vertices, conical points, spikes, etc.), cusps (i.e., exits to infinity) which are not too massive and also cases in which coefficients may have singularities which are

not too strong (a good example is the Schrödinger operator with potential similar to $\pm|x|^{2m}$ near the origin with $m \in (-1, 0)$).

Further, in *Section 11.4* stronger singularities of coefficients are allowed but a *coercivity* condition is assumed (a good example is the Schrödinger operator with potential similar to $+|x|^{2m}$ near the origin with $m \le -1$).

Furthermore, in *Section 11.5* we consider the situation in which a domain is "essentially unbounded" or even coincides with $\mathbb{R}^d$; e.g., we consider the Schrödinger operator in $\mathbb{R}^d$ with potential similar to $+|x|^{2m}$ near infinity with $m > 0$. We are able to consider certain situations when the potential does not tend to $+\infty$ along some directions (along these directions, $V(x) = 0$ or even $V(x)$ tends to $-\infty$, for example) but the arising "canyons" are assumed to be narrow and shallow enough to provide the standard asymptotics.

Next, in *Section 11.6* we analyze eigenvalues tending to -0 which is the bottom of the essential spectrum; e.g., we consider the Schrödinger operator in $\mathbb{R}^d$ with potential similar to $-|x|^{2m}$ near infinity with $m \in (-1, 0)$; for this operator the essential spectrum is $[0, +\infty)$ and $(-\infty, 0)$ contains the discrete spectrum tending to -0, the asymptotics of which is the object of our interest.

Next, in *Section 11.7* we consider asymptotics with several parameters: when we have different semiclassical parameters with respect to different variables or when we have semiclassical and spectral parameters at the same time.

It should be noted, that in all Sections 11.1–11.7 there are three cases: when singularities are rather week and affect neither principal part nor remainder estimate, when singularities are too strong and we cannot derive any meaningful results this way, and the intermediate case, when we derive asymptotics with the same principal part albeit with deteriorated remainder estimate. Two latter cases are partially covered in Chapter 12.

In rather different *Section 11.8* the distribution of the spectra of negative-order pseudodifferential operators on compact closed manifolds is derived.

Chapter 12. Miscellaneous Asymptotics of Spectra. The character of this chapter is quite different: we improve and generalize some results of Chapter 11, but the results of Chapters 4–8 are insufficient for our purposes in this case and we are forced to repeat some arguments of Chapter 4 and even those of Chapter 2 in certain specific situations. Basically we apply the

ideas discussed above. The asymptotics obtained are more or less non-Weyl (or, more precisely, "partially Weyl").

In *Section 12.2* elliptic operators in domains with *thick cusps* are discussed: a cusp is thick if the area of its boundary is infinite, and the standard Weyl asymptotics with the remainder estimate $O\bigl(\tau^{\frac{1}{m}(d-1)}\bigr)$ fails to hold. Moreover, if the volume of the cusp is infinite, then $\mathsf{N}(\tau) = O\bigl(\tau^{\frac{d}{m}}\bigr)$ is no longer true[9)].

For simplicity let us consider the Dirichlet Laplacian in a domain which for $|x'| > c$ coincides with $\{(x', x'') : x'' \in f(x')\Omega\}$ with $\Omega \Subset \mathbb{R}^{d''}$. Here Ω and $f(x')$ describe the shape of the cusp's cross section and its thickness. Making a change of variables $(x', x'') \mapsto (x', x''/f(x'))$, which transforms the cusp into the cylinder $\mathbb{R}^{d'} \times \Omega$, one can consider the transformed operator as a Schrödinger operator in $\mathbb{R}^{d'}$ with operator-valued coefficients.

The auxiliary space is $\mathbb{H} = \mathscr{L}^2(\Omega)$. Applying the results of Chapter 4 concerning operators with operator-valued symbols we obtain non-Weyl asymptotics. However, this asymptotics is not very effective in the sense that the principal part contains a term

$$(2\pi)^{-d'} \iint \boldsymbol{n}(x', \xi'; \lambda)(1 - \psi_0(x'))\, dx' d\xi' \tag{0.31}$$

(with $\psi_0 \in \mathscr{C}_0^\infty$), thus depending on the eigenvalue counting function $\boldsymbol{n}(x', \xi')$ for the family of the auxiliary operators in $\mathbb{H}$ while we would like to have just one such operator.

Observe that one can study the propagation of singularities for the transformed operator, treated as an operator with an operator-valued symbol. We then obtain some results for the original operator as well. Namely, we obtain that for a not very remote part of the cusp in Weyl asymptotics one can take take much larger T (depending on τ, $|x'|$) and drastically improve the remainder estimate. These asymptotics are not very effective either because in addition to the Weyl principal part they contain plenty of extra terms.

On the other hand, under a stabilization condition in the remote part $\{x\colon |x'| \ge r\}$ of the cusp ($r = r(\tau) \to \infty$ as $\tau \to +\infty$) the Schrödinger operator is rather close to $\Delta' + |x'|^{2m}\Delta''$ provided $f(x') = |x'|^{-m}$; for the latter operator we can express the right-hand expression of (0.31) through

[9)] This is correct as long as we consider $\mathsf{N}(\lambda)$; for its Riesz means $\mathsf{N}_\vartheta(\lambda)$ the definition of the thickness depends on ϑ.

eigenvalue counting function $\boldsymbol{n}(.)$ for the Dirichlet Laplacian Δ'' in Ω. Then for the Schrödinger operator which we really got we have an asymptotics which is not very effective either; its main part depends only on $\boldsymbol{n}(.)$ but one needs to add plenty of extra terms.

So, we get two different asymptotics, both of which are not effective. However, combining them we obtain much more effective asymptotics because in the result most of these extra terms turn out to be superficial since they strongly depend on parameter $r(\tau)$, which we can select almost arbitrarily; comparing these terms for different choices of this parameter we can conclude that they must vanish. In the best possible case one has only one auxiliary operator instead of a family. Thus, the ideas are basically the same as in many forthcoming sections and chapters:

We take two different approaches (on one hand, we apply the theory of operators with operator-valued symbols to derive spectral asymptotics, and on the other, we apply the operator-valued theory only in order to get better propagation results) and then we combine the asymptotics obtained.

In *Section 12.3* we consider the class of elliptic operators in $\mathbb{R}^d$. A good example of such operators is the Schrödinger operator with potential V, failing to tend to $+\infty$ along directions $x \in \Lambda$, where Λ is a conical set of the "bad" directions. So, the domain $\{x\colon V(x) \le \tau\}$ contains cusps and if the "canyons" are not narrow and shallow enough the Weyl asymptotics with the standard remainder estimate fails.

Let us assume that Λ is a smooth d'-dimensional manifold and let us transform it to $\mathbb{R}^{d'}$. Then the ideas of Section 12.2 yield results of the same type (heuristically one can say that we apply the results of the Section 12.2 but the cusp depends on τ and widens when τ increases).

In *Section 12.4* we generalize the results of Section 11.6 to singular potentials, decaying at infinity in the case when singularity is strong enough; again non-Weyl correction terms appear; moreover, if singularity is strong enough the principal term in the asymptotics is non-Weyl.

In *Section 12.5 maximally hypoelliptic operators* on compact manifolds are treated. We assume that the operator uniformly degenerates on the symplectic conic manifold Λ. Microlocally $\Lambda = \{(x,\xi)\colon x'' = \xi'' = 0\}$. The same basic ideas as above are applied. We treat the operator in question as $A(x', D')$ in $\mathscr{L}^2(\mathbb{R}^{d'}, \mathbb{H})$ with $\mathbb{H} = \mathscr{L}^2(\mathbb{R}^{d''})$. Instead of the remote part of

the cusp we have now a tight (depending on the spectral parameter) vicinity of Λ: $\{(x,\xi)\colon |x'|+|\xi'|\le \varepsilon(\tau)\}$ with $\varepsilon(\tau)\to +0$ as $\tau\to+\infty$.

In *Section 12.6* we consider semiclassical asymptotics of the spectral Riesz means for operators with singularities at disjoint points. In this case the asymptotics of $\mathsf{N}^-(h)$ are Weyl as are the asymptotics of $\mathsf{N}^-_\vartheta(h)$ with $\vartheta<\bar\vartheta$ where ϑ is the order of the Riesz mean. So, we are interested in the case $\vartheta\ge\bar\vartheta$. One very early idea was to use the equality

$$\mathsf{Tr}\big(E^1_\vartheta(\tau)-E^0_\vartheta(\tau)\big)=\mathsf{const}\int_0^1 \mathsf{Tr}(A^1-A^0)E^t_{\vartheta-1}(\tau)\,d\tau, \tag{0.32}$$

where $A^t=tA^1+(1-t)A^0$ and the superscript $E^t(\tau)$ is related to A^t and $E_\vartheta(\tau)\coloneqq \Gamma(\vartheta+1)^{-1}\,\mathsf{Tr}\big((\tau-A)^\vartheta_+\big)$. Then taking appropriate partitions and applying the standard Weyl approximation to some elements and this formula (or its more sophisticated and general form) to the other elements we obtain the final answer as a sum of the Weyl expression and the term(s) associated with the model operator A^0. In comparison with purely Weyl asymptotics correction terms appear. They are similar to *Scott correction term* for $-|x|^{-1}$-like potential in dimension 3 for $\vartheta=1$.

We pick A^0 as an operator with positively homogeneous coefficients and get a more or less simple expression for the non-Weyl term(s). Actually this idea is adjusted in different ways, one of which is an investigation of the very long range (in the local scale) propagation of singularities which allows to improve significantly the remainder estimate.

In *Section 12.7* we apply the same idea as in Section 12.2 to the Neumann Laplacian in domain with cusps. However situation is very different now because the Neumann Laplacian in Ω of the cusp has the lowest eigenvalue 0 and the Schrödinger operator described there breaks into direct sum of two: one with the potential growing as in Section 12.2 in $\mathscr{L}^2(\mathbb{R}^{d'},\mathbb{H}\ominus\mathbb{K})$ and a special one in $\mathscr{L}^2(\mathbb{R}^{d'},\mathbb{K})$ coinciding here with $\mathsf{const}|\nabla\log f(x')|^2$. Here $\mathbb{K}=\mathbb{C}^1$ (or $\mathbb{C}^p$ in more general examples). This term is due to the the change of the density when we change variables.

As a result, unless the cusp is really very thin, the Neumann Laplacian has an essential spectrum $[0,\infty)$ or $[c_*,\infty)$. However if $|\log f|\asymp |x'|^m$, $|\nabla\log f|\asymp |x'|^{m-1}$ with $m>1$, then the essential spectrum is empty and we derive sharp spectral asymptotics for $\mathsf{N}(\tau)$ which may contain non-Weyl correction or even principal term.

Finally, in *Section 12.8* we consider periodic operator A^0; its spectrum consists of *spectral bands* separated by *spectral gaps*. Let λ belong to a

spectral gap (or its border). Consider perturbed operator $A^t := A^0 + tW$ with W non-negative and decaying fast enough at infinity. This operator can have discrete eigenvalues inside spectral gaps. We study asymptotics for the number $N(\tau)$ of such eigenvalues crossing λ as t changes from 0 to τ and $\tau \to \pm\infty$.

Volume III. Magnetic Schrödinger and Dirac Operators. 1

This Volume is entirely devoted to the study of the *magnetic Schrödinger operator*[10),11)]

$$H = (-ih\nabla - \mu \boldsymbol{A}(x))^2 + V(x), \tag{0.33}$$

and also Schrödinger-Pauli operator[10)]

$$H = ((-ih\nabla - \mu \boldsymbol{A}(x)) \cdot \boldsymbol{\sigma})^2 + V(x), \tag{0.34}$$

and Dirac operators[10)]

$$H = ((-ih\nabla - \mu \boldsymbol{A}(x)) \cdot \boldsymbol{\sigma}) + \sigma_0 M + V(x) \tag{0.35}$$

and

$$H = ((-ih\nabla - \mu \boldsymbol{A}(x)) \cdot \boldsymbol{\sigma}) + V(x) \tag{0.36}$$

with a small semiclassical parameter h and large *magnetic intensity parameter* (a coupling constant), responsible for the interaction of the particle with the magnetic field, μ. Here $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_d)$ where $\sigma_1, \dots, \sigma_d$ are Pauli matrices, $\sigma_0, \sigma_1, \dots, \sigma_d$ are Dirac matrices, $\boldsymbol{A}$ is *magnetic vector potential*. Here for odd d operator (0.36) is not a special case of (0.35) with $M = 0$ since the former exists for D multiple of $2^{(d-1)/2}$ and the latter for D multiple of $2^{(d-1)/2}$; D is a dimension of matrices.

In this Volume we consider the most interesting cases $d = 2, 3$ and also assume that $\boldsymbol{A}(x)$, $V(x)$ (and metrics g) are smooth.

[10)] Actually we consider more general form.

[11)] From now we denote operator by H.

Part VI. Smooth Theory in Dimensions 2 and 3

Chapter 13. Schrödinger Operator. Standard Theory. Here we consider the Schrödinger operator, assuming that the the unit ball under consideration is entirely contained in the domain X and that the intensity of the magnetic field nowhere vanishes in this ball.

In *Section 13.2* a preliminary analysis is done and, in particular, $e(x,x,\lambda)$ is calculated explicitly in the case when the metrics and the scalar potential are constant and the vector potential is linear (so the intensity of the magnetic field is also constant) and $X = \mathbb{R}^d$; these formulae are good approximations for the general case, which we consider. We prove the estimates $|\mathcal{E}_\psi(0)| \le ch^{-d}$ in the general case and $|\mathcal{E}_\psi(0)| \le \mu^{-s}$ for $\mu h \ge C_0$; hence for the Schrödinger operator (but neither for the Schrödinger-Pauli nor for the Dirac operators!) we assume in what follows that $\mu h \le c$.

In *Section 13.3* we reduce the operator in question microlocally to canonical forms which are really different in the cases $d = 2$ and $d = 3$. For $d = 2$ we obtain, modulo $O(\mu^{-s})$, $\mu^{-1}h$-pseudodifferential operators with respect to $x' = x_1$. For $d = 3$ we obtain, also modulo $O(\mu^{-s})$, $(\mu^{-1}h, h)$-pseudodifferential operators with respect to $x' = (x_1, x_2)$. In both cases x_d enters in these forms only through the harmonic oscillator.

Based on this reduction, in *Section 13.4* we analyze the propagation of singularities with respect to x' in the case of the *weak-to-moderate magnetic field* $\mu \le h^{\delta-1}$ and, under some restrictions, we prove that on the intervals $[-T,-T_0]$ and $[T_0,T]$ the distribution $\sigma_\psi(t)$ is negligible, where either $T_0 \asymp \mu^{-1}$ or $T_0 \asymp h^{1-\delta}$, while on the interval $[-2T_0, 2T_0]$ for $\sigma_\psi(t)$ the standard formulae of Section 4.3 hold where $T \asymp 1$ for $d = 3$ and $T \asymp \mu$ or larger under *non-degeneracy* assumptions for $d = 2$; in the latter case singularities propagate along *cyclotrons* which are circles of radii $\asymp \mu^{-1}$ which however drift with a speed also $\asymp \mu^{-1}$. For $d = 3$ singularities propagate along helices of radii $\asymp \mu^{-1}$ winding around *magnetic lines*.

We always assume that there is an appropriate cutting near the energy level 0 with respect to the dual variable τ. Applying the Tauberian, method we obtain asymptotics for $\mathcal{E}_\psi(0)$ with the principal part $\asymp h^{-d}$ and the remainder term $O(h^{-2})$ for $d = 3$ and $O(\mu^{-1}h^{-1})$ or even better for $d = 2$. We also analyze less restrictive conditions in this and the following sections and obtain asymptotics in these cases with the same or worse remainder estimates. Thus, in this section *the canonical form is applied only for the study of the long-time propagation of singularities in order to extend the time*

interval and improve the remainder estimate in the Weyl-type asymptotics.

Next, in *Section 13.5* the case of the *moderate-to-strong magnetic field* $h^{-\delta} \le \mu \le C_0 h^{-1}$ is considered. Then the remainder term $O(\mu^{-s})$ in the microlocal canonical form is negligible and, after decomposition with respect to Hermitian functions of $\sqrt{\mu h^{-1}} x_d$, we obtain families of $\mu^{-1}h$-pseudodifferential operators with respect to $x' = x_1$ for $d = 2$ and $(\mu^{-1}h, h)$-pseudodifferential operators with respect to $x' = (x_1, x_2)$ for $d = 3$. These operators depend on the number n of the Hermitian function. Applying the results of Chapter 4 to these families of operators, we obtain asymptotics for $\mathcal{E}_\psi(0)$ under certain conditions with the same remainder estimates as in Section 13.4. Thus, now *the canonical form is used fully*. These asymptotics are not completely effective (i.e., one can calculate some terms only "in principle" but not "in fact").

The frameworks of this section and those of Section 13.4 overlap and under all of them both asymptotics are valid. These asymptotics are of different types, but comparing them, we can eliminate "superficial terms", obtaining effective asymptotics for $\mu \le C_0 h^{-1}$. These two ideas—*Application of the canonical form for the analysis of propagation of singularities and the extension of the interval for* $\sigma_\psi(t)$ and *comparison of the asymptotics obtained by different methods*—are very useful and we use them heavily in many chapters of the book.

Next, in *Section 13.6* we consider the case of the *strong magnetic field* $\mu \ge \epsilon_0 h^{-1}$. Then instead of V we should impose the same conditions to $V_n^* = V + (2n+1)\mu F$ for the Schrödinger operator and to $V_n^* = V + 2n\mu F$ for the Schrödinger-Pauli operator (with $n = \in \mathbb{Z}^+$ for $d = 2$, $n = 0$ for $d = 3$) and derive asymptotics with the principal part $\asymp \mu h^{1-d}$ and remainder estimates $O(1)$ for $d = 2$, $O(\mu h^{-1})$ for $d = 3$.

Further, in the case $d = 2$ it is possible that $V_n^* \le -\epsilon\mu h$ and $V_{n+1}^* \ge \epsilon\mu h$ everywhere in the ball in question; then $\tau = 0$ belongs to the *spectral gap* and in this case remainder estimate is $O(\mu^{-s})$.

Furthermore, in *Section 13.7* we improve results of Sections 13.4–13.7 under assumptions about the long-time dynamics (along *drift lines* for $d = 2$ and *magnetic lines* for $d = 3$).

Some possible generalizations (weakening or even dropping some assumptions) are discussed in *Section 13.8.*

Chapter 14. 2D-Schrödinger Operator with the Strong Degenerating Magnetic Field. In this chapter we consider the same operators in dimension 2 only albeit intensity of magnetic field F vanishes along lines with $\nabla F \neq 0$.

Section 14.1 is devoted to a preliminary analysis; in *Section 14.2* we study the classical dynamics which is rather non-trivial with short periodic trajectories appearing in $\mu^{-\frac{1}{2}}$-vicinity of these degeneration lines.

In *Section 14.3* we study related quantum dynamics and prove the remainder estimate $O(\mu^{-\frac{1}{2}}h)$. Then in *Section 14.4* we calculate asymptotics with the same magnitude of the principal part and with a correction term due to aforementioned short periodic trajectories. Further, in *Section 14.5* we formulate main theorems (all as $\mu \le h^{-1}$).

Finally, in *Sections 14.5* and *14.6* we analyze cases of the *strong magnetic field* $h^{-1} \le \mu \le h^{-2}$ and *superstrong magnetic field* $\mu \ge h^{-2}$ respectively.

Chapter 15. 2D-Schrödinger Operator with the Strong Magnetic Field near Boundary. In this chapter we consider the same 2-dimensional operators as in Chapter 13 albeit near smooth boundary which causes a significant modification of the dynamics: while inner "particles" (aka *bulk electrons*) travel along the same cyclotrons as before, near-boundary "particles" (aka *edge electrons*) "jump" along the boundary with an average speed $\asymp 1$. Drifting, bulk particles may collide with the boundary and become edge particles and edge particles may leave the boundary and become bulk particles so the global dynamics could be really complicated.

We analyze different cases (weak, moderate, strong magnetic field) in *Sections 15.2–15.4* respectively. There is a significant difference between operators with Dirichlet and Neumann boundary conditions , especially in the case of the strong magnetic field.

Also the former spectral gaps could be (at least partially) filled with *edge energy levels* in which case we need to study their distribution as well. They are less dense than eigenvalues in the *spectral bands*.

As usual, the first *Section 15.1* is devoted to the preliminary analysis and the last *Section 15.5* to some generalizations.

Part VII. Smooth Theory in Dimensions 2 and 3 (Continued)

Chapter 16. Magnetic Schrödinger Operator: Short Loops, Pointwise Spectral Asymptotics and Asymptotics of Dirac Energy. In

this chapter we study the same operators as in Chapter 13 but under different angle: we are interested in the *pointwise spectral asymptotics*, i.e. asymptotics of $e(x,x,\lambda)$ without mollification with respect to x, and also asymptotics of the expression (0.26); the results needed in Chapters 25–28.

As we know, for pointwise spectral asymptotics the important role play short loops (recall that in contrast to periodic trajectory the loop returns to the same spatial point albeit from the different direction) and for operators in question there are plenty of them for $d=2$.

The first half of the chapter is devoted to the case $d=2$: in *Section 16.2* we analyze a toy-model when explicit construction is possible and derive formulae which we would like to prove in the general case. The short loops cause either correction term if magnetic field is not too strong or even completely different asymptotics if magnetic field is sufficiently strong.

Then in *Section 6.2* we derive pointwise asymptotics, and in *Sections 16.4* and *16.5* we derive asymptotics of the expression (0.26).

In *Sections 16.6–16.9* we do the same as $d=3$; then effects of short loops is mitigated by a "free" movement along magnetic lines.

Finally, in *Section 16.9* we derive estimate of

$$\iint \bigl(e(x,x,\tau)-h^{-d}\mathcal{N}_x(\tau)\bigr)\bigl(e(y,y,\tau)-h^{-d}\mathcal{N}_x(\tau)\bigr)\omega(x,y)\,dxdy, \tag{0.37}$$

where $h^{-d}\mathcal{N}_x(\tau)$ is a Weyl expression for $e(x,x,\tau)$ for $d=2,3$.

Chapter 17. Dirac Operator with the Strong Magnetic Field. Here we consider operators (0.35) and (0.36) and our analysis is very similar to one of Chapter 13 but some significant differences and complications appear. *Sections 17.1–17.7* run in parallel with Sections 13.2–13.8.

We should mention that here as in Section 5.3 our results are proven uniformly with respect to $M\ge 0$.

Volume IV. Magnetic Schrödinger and Dirac Operators. 2

Part VIII. Non-smooth Theory and Higher Dimensions

In this Part we consider magnetic Schrödinger operator in dimensions 2, 3 and higher under rather low smoothness conditions.

Chapter 18. 2D- and 3D-magnetic Schrödinger Operator with Irregular Coefficients. Here we generalize Chapter 13 to the case when the coefficients are not very smooth. It leads us to the necessity to a microlocal canonical form, which, due to the lack of the smootness, is not only incomplete but also the lowest computed term in which is modified.

Section 18.1 contains a preliminary analysis, *Sections 18.2–18.5* cover $d = 2$ (Case of weak magnetic field, Canonical form, Tauberian theory and Calculations and main theorems–respectively).

Sections 18.6–18.9 cover $d = 3$ (exactly in the same order as for $d = 2$).

Chapter 19. Multidimensional Magnetic Schrödinger Operator. I. Full-rank case. Here we consider $d \ge 4$, assuming that $\mathsf{rank}(F_{jk})(x) = d$ at each point x, where (F_{jk}) is a skew-symmetric *magnetic intensity matrix*, $F_{jk} = (\partial_k A_j - \partial_j A_k)$ [12]. This is the main case if $d = 2r$ is even. Then this matrix has eigenvalues $\pm i f_j(x)$ with $j = 1, \dots, r$ and $f_j(x) > 0$.

This chapter does not completely generalize Chapter 18 since smoothness conditions here are higher. We also need two *non-resonance* conditions:

$$(0.38)_{2,3} \quad f_j(x) \ne f_k(x) \quad \forall j \ne k \qquad \text{and} \qquad f_j(x) \ne f_k(x) + f_l(x) \quad \forall j, k, l$$

at each point x (where the second condition covers both cases $k \ne l$ and $k = l$).

Section 19.1 contains a preliminary analysis, *Section 19.2* covers the case of the weak magnetic field.

In *Section 19.3* we reduce our operator to a microlocal canonical form, which again is incomplete but there are two obstacles: insufficient smoothness but also a lack of higher order non-resonance conditions since we do not impose them.

Sections 19.4–19.6 cover cases of the weak-to-moderate, moderate, moderate-to-strong, and strong magnetic field respectively.

In *Section 19.7* we consider the case of a simple violation of *non-degeneracy* assumption.

Chapter 20. Multidimensional Magnetic Schrödinger Operator. II. Non-full-rank Case. Here we consider $d \ge 4$, now assuming that $\mathsf{rank}(F_{jk})(x) = 2r < d$ at each point x. This is the main case if $d = 2r + 1$ is odd. Then this matrix has eigenvalues $\pm i f_j(x)$ with $j = 1, \dots, r$ and $f_j(x) > 0$ and eigenvalue 0 of the constant multiplicity $d - 2r$.

[12] For non-Euclidean metrics we need to lift one of two indices.

This chapter does not completely generalize Chapter 18 since smoothness conditions here are higher. We also need two *non-resonance* conditions $(0.38)_{2,3}$.

Section 20.1 contains a preliminary analysis, *Section 20.2* covers the case of the weak magnetic field.

In *Section 20.3* we reduce operator to a microlocal canonical form with the same caveats as in Section 19.3.

In *Sections 20.4* and *20.5* we derive remainder estimates in the cases of the moderate and strong magnetic field respectively; in *Sections 20.6* and *20.7* we calculate asymptotics and prove main theorems.

In *Section 20.8* we consider the cases of a simple violation of main assumptions.

Part IX. Magnetic Schrödinger Operator in Dimension 4

Chapter 21. 4D-Schrödinger Operator with a Strong Degenerating Magnetic Field. We start analysis of the generic operators in dimension 4. Then there are two magnetic intensities $f_1(x)$ and $f_2(x)$ and in the generic case they do not vanish simultaneously. We consider a vicinity of $\Sigma = \{x\colon f_1(x) = 0\}$ and in the generic case $\nabla f_1|_\Sigma \neq 0$. However it is not a simple lift of Chapter 14 because now $\nabla f_1|_\Sigma$ belongs to $\operatorname{Ker} F$ not in every point of Σ but only on submanifold $\Lambda \subset \Sigma$, $\dim \Lambda = 1$.

Section 21.1 contains a preliminary analysis, *Section 21.2* covers the case of the weak magnetic field.

Sections 21.3–21.5 cover the case of the stronger magnetic field: in *Section 21.3* we reduce operator to a microlocal canonical form, in *Section 21.4* we prove a remainder estimate and in *Section 21.5* we calculate asymptotics and prove main theorems.

Chapter 22. Generic 4*D*-Schrödinger Operator with the Strong Magnetic Field. In this chapter we conclude our analysis of the generic operators in dimension 4. Now we assume that neither magnetic intensity vanishes but non-resonance conditions $(0.38)_{2,3}$ are violated. More precisely, in the generic case $(0.38)_2$ can be violated on manifold of codimension 2 and $(0.38)_3$ can be violated on manifold of codimension 1 and these manifolds do not intersect.

Sections 22.1 and *22.2* contain a preliminary analysis, *Section 22.3* covers the case of the weak magnetic field.

Sections 22.4–22.5 cover the case of stronger magnetic field: in *Section 22.4* we prove a remainder estimate and in *Section 22.5* we calculate asymptotics and prove main theorems.

In *Section 22.6* we consider a case of V, vanishing somewhere.

Part X. Eigenvalue Asymptotics for Schrödinger and Dirac Operators with the Strong Magnetic Field

Chapter 23. Eigenvalue Asymptotics. 2D case and Chapter 24. Eigenvalue Asymptotics. 3D case. These chapters are devoted to the distribution of spectra for the Schrödinger and Dirac operators with the strong magnetic field. The idea is exactly the same as in Chapter 11, but now we consider the case when we apply the results of Chapters 13, 17 and 19 after the partition-rescaling-multiplication procedure. So we now have two local effective parameters: h_{eff} and μ_{eff}, and on elements with $\mu_{\mathsf{eff}} \gg 1$ we refer to the results of Chapters 13 and 17 while on elements with $\mu_{\mathsf{eff}} \le \mathsf{const}$ we refer to the results of Chapters 4 and 7.

In *Sections 23.2* and *24.2* we consider the case of a fixed spectral parameter.

In *Sections 23.3* and *24.3* we are interested in asymptotics with the spectral parameter tending to $+\infty$.

In *Sections 23.4* and *24.4* we are interested in asymptotics with the spectral parameter tending to ± 0. In particular, we consider the Schrödinger and Dirac operators in $\mathbb{R}^d$ with constant metric coefficients, vanishing scalar potential and constant non-vanishing magnetic field.

(a) Schrödinger and Dirac operators in $\mathbb{R}^2$ with constant metric coefficients, vanishing scalar potential and constant non-vanishing magnetic field have pure point spectra of infinite multiplicity. If we perturb these operators by a scalar potential, tending to 0 at $-\infty$, then each of these eigenvalues of infinite multiplicity decomposes to a sequence of eigenvalues of finite multiplicities, tending to a point of the essential spectrum (which is an eigenvalue of the unperturbed operator) and we are interested in the asymptotic distribution of eigenvalues in each sequence.

We also consider generalizations to even-dimensional full-rank case, and for magnetic field either growing, or decaying at infinity, or stabilizing to positively homogeneous of order 0.

(b) For $d = 3$ we obtain asymptotics of eigenvalues near the boundary of the essential spectrum. If the potential decreases slowly enough, the results of Chapters 13 and 17 could be applied.

(c) For $d = 3$ we obtain asymptotics of eigenvalues near the boundary of the essential spectrum. If the potential decreases quickly enough, the results of Chapters 13 and 17 fail to apply so we use the theory of spectral asymptotics for operators with operator-valued symbols: if the magnetic field is directed along x_3 we introduce an auxiliary space $\mathbb{H} = \mathscr{L}^2(\mathbb{R}_{x_3})$ and consider x' as the main variable.

(d) We reload case (b) to obtain sharper asymptotics, using the theory of spectral asymptotics for operators with operator-valued symbols as in the case (c).

In *Sections 23.5* and *24.5* we consider asymptotics with respect to all three parametrs τ, μ, h.

Volume V. Applications to Quantum Theory and Miscellaneous Problems

Part XI. Asymptotics of the Ground State Energy of Heavy Atoms and Molecules

In this Part we consider an application to Thomas-Fermi Theory. Consider a large (heavy) atom or molecule; it is described by *Multiparticle Quantum Hamiltonian*

$$\mathsf{H}_N = \sum_{1\le n\le N} H_V(x_n) + \sum_{1\le n<k\le N} \frac{1}{|x_n - x_k|}, \tag{0.39}$$

where H is *one-particle quantum Hamiltonian*, Planck constant $\hbar = 1$, electron mass $= \frac{1}{2}$, electron charge $= -1$, y_m is a location of m-th nuclei and Z_m its charge, M is fixed, but $Z_m \asymp N \to \infty$.

This operator acts on the space $\wedge_{1\le j\le N}\mathscr{L}^2(\mathbb{R}^3, \mathbb{C}^2)$ of totally antisymmetric functions $\Psi(x_1, \varsigma_1; \dots; x_N, \varsigma_N)$ because electrons are Fermions, $x_n = (x_n^1, x_n^2, x_n^3)$ is a coordinate and $\varsigma_n \in \{-\frac{1}{2}, \frac{1}{2}\}$ is a *spin* of n-th particle. We identify $\mathbb{C}^2$-valued function $\psi(x)$ on $\mathbb{R}^3$ with scalar-valued $\psi(x, \varsigma)$

If electrons were not interacting between themselves, but the field potential was $-W(x)$, then they would occupy lowest eigenvalues and ground state

wave functions would be (anti-symmetrized) $\phi_1(x_1,\varsigma_1)\phi_2(x_2,\varsigma_2)\dots\phi_N(x_N,\varsigma_N)$ where ϕ_n and λ_n are eigenfunctions and eigenvalues of H_W.

Then the local electron density would be $\rho_\Psi = \sum_{1\le n\le N} |\phi_n(x)|^2$ and according to pointwise Weyl law (if there is no magnetic field)

$$\rho_\Psi(x) \approx \frac{1}{3\pi^2}(W+\nu)_+^{\frac{3}{2}}, \tag{0.40}$$

where $\nu = \lambda_N$; here we assume that $H_V = \Delta - V(x)$ as it is in Chapter 25.

This density would generate potential $-|x|^{-1} * \rho_\Psi$ and we would have $W \approx V - |x|^{-1} * \rho_\Psi$.

Replacing all approximate equalities by strict ones we arrive to Thomas-Fermi equations:

$$V - W^{\mathsf{TF}} = |x|^{-1} * \rho^{\mathsf{TF}}, \tag{0.41}$$

$$\rho^{\mathsf{TF}} = \frac{1}{3\pi^2}(W^{\mathsf{TF}}+\nu)_+^{\frac{3}{2}}, \tag{0.42}$$

$$\int \rho^{\mathsf{TF}}\,dx = N, \tag{0.43}$$

where $\nu \le 0$ is called *chemical potential* and in fact approximates λ_N.

Thomas-Fermi theory has been rigorously justified (with pretty good error estimates).

Chapter 25. No Magnetic Field Case. In this chapter we assume that there is no magnetic field: $H_V = \Delta - V(x)$ with

$$V(x) = \sum_{1\le m\le M} \frac{Z_m}{|x-\mathsf{y}_m|}, \tag{0.44}$$

where y_m is a *position* and Z_m is a *charge* of m-th nuclei, M is fixed and $Z_1 \asymp Z_2 \asymp \dots \asymp Z_M \asymp N$.

Section 25.1 is an Introduction and in *Section 25.2* we justify a reduction of the original multiparticle problem to one-particle one. This is done mainly by methods of the classical mathematical physics (functional analysis).

In *Section 25.3* we expose *Thomas-Fermi theory* described by (0.41)–(0.43).

In *Section 25.4* we, based on previous sections and chapters, prove our main results. First of all, the *ground state energy* is given by

$$\mathsf{E}_N = 2\frac{(6\pi^2)^{\frac{5}{3}}}{15\pi^2}\int \Big(\rho^{\mathsf{TF}\,\frac{5}{3}} - V\rho^{\mathsf{TF}}\Big)\,dx - \frac{1}{2}\iint \rho^{\mathsf{TF}}(x)\rho^{\mathsf{TF}}(y)|x-y|^{-1}\,dxdy + O(Z^2). \tag{0.45}$$

Using results of Section 12.6 *Riesz Means Asymptotics for Operators Singular at a Point*, we improve the remainder estimate to $O(Z^{\frac{5}{3}})$ but we need to include the *Scott correction term*

$$\sum_{1\le m\le M} S_0 Z_m^2, \tag{0.46}$$

and even to $O(Z^{\frac{5}{3}-\delta})$ but we need to include *Dirac and Schwinger correction terms* $\sim Z^{\frac{5}{3}}$.

In *Section 25.5* we consider *negatively charged systems* (i.e. those with $Z := Z_1 + ... + Z_m < N$) and estimate *excessive negative charge* $(N - Z)$, when system is able to bind N electrons, i.e. $\mathsf{E}_N < \mathsf{E}_{N-1}$.

In *Section 25.6* we consider *positively charged systems* (i.e. those with $Z > N$) and estimate or find asymptotics $\mathsf{I}_N \approx \nu_N$ for an *ionization energy* $\mathsf{I}_N := \mathsf{E}_{N-1} - \mathsf{E}_N$.

We also consider systems with free $\mathsf{y}_1, \dots, \mathsf{y}_M$ and include in the total energy the *internuclear energy*

$$\sum_{1\le m<m'\le M} \frac{Z_m Z_{m'}}{|\mathsf{y}_m - \mathsf{y}_{m'}|}, \tag{0.47}$$

and minimize the ground state energy by $\mathsf{y}_1, \dots, \mathsf{y}_M$ and recover all aforementioned results.

We also estimate *excessive positive charge* $(Z - N)$ for which system does not disintegrate into separate atoms.

Chapter 26. The Case of External Magnetic Field. Here we assume that there is a constant magnetic field with a magnetic potential $\boldsymbol{A}(x) = B(-\frac{1}{2}x_2, \frac{1}{2}x_1, 0)$ and then one-particle Hamiltonian is Schrödinger-Pauli operator (0.34). We assume that the magnetic field is not hyperstrong:

$B \ll N^3$. Still it may be sufficiently strong to affect pointwise Weyl formula, which needs to be modified according to Chapter 13 (the results of Chapter 14 are not needed here).

Basically there are two principally cases $B \ll Z^{\frac{4}{3}}$ of a moderate magnetic field and $Z^{\frac{4}{3}} \ll B \ll Z^3$ of a strong magnetic field and a transitional case.

Section 26.1 is an Introduction (which is parallel to Sections 25.1 and 25.2, arguments and results of which require almost no modification).

Section 26.2 is parallel to Section 25.3 but instead of Section 25.4 we have four *Sections 26.3–26.6*, covering single nucleus case $M = 1$, multiple nuclei case $M \geq 2$ with analysis "inside molecule" and "near the molecule edge"and the synthesis, respectively.

Sections 26.7 and *26.8* are similar to Sections 25.5 and 25.6.

Chapter 27. The Case of Self-Generated Magnetic Field. Here we consider the same Schrödinger-Pauli operator but magnetic field is underdetermined a priory and its energy

$$\alpha^{-1} \int |\nabla \times \boldsymbol{A}|^2 \, dx \tag{0.48}$$

is included in the total energy. Here we assume that $\alpha Z_m \leq \kappa^*$ with some constant $\kappa^* > 0$. We recover the same results as in Chapter 25 but with the *Scott correction term*

$$\sum_{1 \leq m \leq M} S(\alpha Z_m) Z_m^2 \tag{0.49}$$

instead of (0.46).

Section 27.1 is an Introduction. *Sections 27.2* and *27.3* provide a replacement for Sections 12.6: while Section 27.2 treats a single singularity, Section 27.3 deals with a molecular case and the necessity to "decouple" singularities.

In *Section 27.4* as in Section 25.4 the asymptotics of the ground state energy is recovered and *Section 27.5* is similar to Sections 25.5 and *Section 25.6*.

Chapter 28. The Case of Combined Magnetic Field. This chapter combines two previous ones: there is an external constant magnetic field

$\boldsymbol{A}_0(x) = B(-\frac{1}{2}x_2, \frac{1}{2}x_1, 0)$ and an unknown self-generated magnetic field $\big(\boldsymbol{A} - \boldsymbol{A}_0\big)(x)$, and the energy of the latter

$$\alpha^{-1} \int |\nabla \times (\boldsymbol{A} - \boldsymbol{A}_0(x))|^2 \, dx \tag{0.50}$$

should be added to the total energy.

Section 28.1 is an Introduction. Since we need a lot of microlocal arguments, they spread over six sections: *Sections 28.2–28.4* cover the local theory and *Sections 28.5* and *28.6* the global theory under different assumptions.

However applications to the ground state energy, excessive negative and positive charges and ionization energy need only small modifications of our previous arguments and squeeze into a single *Section 28.7*.

Part XII. Articles

I decided to put results of the last few years in this volume as separate articles. So far those are[13)]

- Joint paper with A. Hassel [1] *Spectral asymptotics for the semiclassical Dirichlet to Neumann operator*.
- My paper [29] *Spectral asymptotics for fractional Laplacians*.
- My paper [32] *Spectral asymptotics for Dirichlet to Neumann operator in the domains with edges*.
- My paper [30] *Asymptotics of the ground state energy in the relativistic settings*.
- My paper [31] *Asymptotics of the ground state energy in the relativistic settings and with self-generated magnetic field*.
- My paper [33] *Complete semiclassical spectral asymptotics for periodic and almost periodic perturbations of constant operator*.
- My paper [34] *Complete differentiable semiclassical spectral asymptotics*.

[13)] While I do not plan to change the main body of this book, I intend to add new articles, as soon as I write them.

- My paper [34] *Bethe-Sommerfeld conjecture in semiclassical settings.*
- My paper [34] *Pointwise spectral asymptotics out of the diagonal near boundary.*
- My paper [34] *Pointwise spectral asymptotics out of the diagonal near degeneration.*
- My paper [28] *100 years of Weyl's law.*

See also the List of my presentations with links to them.

6. How LSSA are Applied to Classical Asymptotics

Finally, let us discuss three examples which illustrate Chapter 11 and show how LSSA yield asymptotics with respect to the spectral parameter. This basically explains

Example 0.2. Let X be a compact Riemannian manifold without boundary and let Δ be a (positive) Laplace-Beltrami operator. Let us consider the Schrödinger operator $A_h = h^2\Delta + V(x)$; it is well known that

$$\mathsf{N}^-(A_h) = \kappa_0 h^{-d} + O\big(h^{1-d}\big) \tag{0.51}$$

with Weyl constant κ_0 provided 0 is not a critical value of $V(x)$ [14]. The Birman-Schwinger principle implies that $\mathsf{N}^-(A_h) = \mathsf{N}(h^{-2})$ where $\mathsf{N}(\lambda)$ is the number of eigenvalues $\mu \in [0,\lambda)$ of the spectral problem $\big(\Delta + \mu V(x)\big)u = 0$ (counting their eigenvalues).

These two equalities immediately yield the asymptotics

$$\mathsf{N}(\lambda) = \kappa_0 \lambda^{\frac{d}{2}} + O\big(\lambda^{\frac{1}{2}(d-1)}\big) \tag{0.52}$$

provided 0 is not a critical value of $V(x)$ (the condition which was claimed not to be necessary).

Try to obtain this result directly in the case when $V(x)$ vanishes at some point! It is striking that nobody had observed this non-trivial new result which trivially follows from two well known facts.

[14] It is shown in this book that one can skip this condition for $d \geq 2$ and for manifolds with boundary.

Example 0.3. Let us consider the Schrödinger operator $A = \Delta + V(x)$ in $\mathbb{R}^d$ where Δ is the (positive) Laplacian and the real-valued potential $V(x)$ satisfies the following conditions:

$$\big|D^\alpha V\big| \le \langle x\rangle^{2m-\alpha} \qquad \forall \alpha : |\alpha| \le K, \tag{0.53}$$

and

$$V \ge \epsilon_0 |x|^{2m} \qquad \forall x : |x| \ge c, \tag{0.54}$$

where $\langle x\rangle = \big(|x|^2+1\big)^{\frac{1}{2}}$ here and in what follows. Let us take a γ-admissible partition of unity with $\gamma(x) = \frac{1}{4}\langle x\rangle$. If we make the rescaling transforming $B\big(\bar{x}, \gamma(\bar{x})\big)$ to $B(0,1)$ and multiply $A-\lambda$ by λ^{-1}, we obtain a Schrödinger operator for $|x| \le C\lambda^{\frac{1}{2m}}$ in $B(0,1)$ with the standard restrictions to the potential and with $h = \gamma(\bar{x})^{-1}\lambda^{-\frac{1}{2}}$.

Let us apply corresponding results of Chapter 4 [15]. Then we obtain Weyl asymptotics for spatial means of $e(x,x,\lambda)$ in $B\big(\bar{x}, \gamma(\bar{x})\big)$ with the remainder estimate $O\big(h^{1-d}\big) = O\big(\lambda^{\frac{1}{2}(d-1)}\gamma(\bar{x})^{d-1}\big)$; there is no additional factor because $e(x,x,\lambda)$ is a density rather than a function. Summing with respect to the partition of unity in $\{x\colon |x| \le C\lambda^{1/2m}\}$, we obtain the remainder estimate $O\big(\lambda^{(d-1)l}\big)$ with $l = (1+m)/2m$.

On the other hand, let us consider the ball $B\big(\bar{x}, \gamma(\bar{x})\big)$ with $|x| \ge C\lambda^{\frac{1}{2m}}$. After rescaling and multiplication of $A-\lambda$ by $\rho^{-2}(\bar{x})$ with $\rho(\bar{x}) = \langle x\rangle^m$ we obtain a Schrödinger operator with the standard restrictions to the potential $v(x) \ge \epsilon$ and with $h = \gamma(\bar{x})^{-1}\rho(\bar{x})^{-1}$; then we derive the estimate $|e(x,x,\lambda)| \le Ch^s\gamma(\bar{x})^{-d}$ with an arbitrary exponent s and hence the contribution of the domain $\{x\colon |x| \ge C\lambda^{\frac{1}{2m}}\}$ to the remainder is $O(\lambda^{-s})$ with another, also arbitrarily large, exponent s. Combining with the contribution of the previous zone, we obtain the asymptotics

$$\mathsf{N}(\lambda) = \mathsf{N}(\lambda) + O\big(\lambda^{(d-1)l}\big) \tag{0.55}$$

with

$$\mathsf{N}(\lambda) = \kappa_0 \int \big(\lambda - V(x)\big)_+^{\frac{d}{2}}\, dx \tag{0.56}$$

as $\lambda \to +\infty$ and $\mathsf{N}(\lambda) \asymp \lambda^{dl}$.

[15] In which case we are able to skip assumption that V is non-degenerate, i.e., $|\nabla V| \ge \epsilon_0 |x|^{2m-1}$ for $|x| \ge c$.

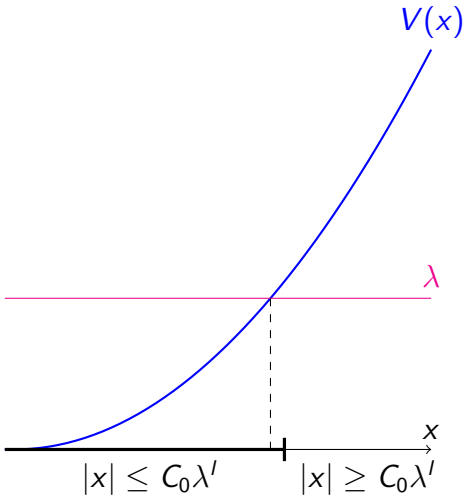
Example 0.4. Let us consider the Schrödinger operator in $\mathbb{R}^d$ with potential $V(x)$ satisfying (0.53) with $m \in (-1, 0)$ and now let $\lambda < 0$ be a small parameter. Then for the same ρ and γ as before the rescaling of $B\big(\bar{x}, \gamma(\bar{x})\big)$ into $B(0,1)$ and multiplication of $A-\lambda$ by $\rho(\bar{x})^{-2}$ produce a Schrödinger operator with standard restrictions to the potential and with $h = \gamma(\bar{x})^{-1}\rho(\bar{x})^{-1}$ provided $|x| \le C|\lambda|^{\frac{1}{2m}}$. Hence for the spatial mean of $e(x,x,\lambda)$ in $B\big(\bar{x}, \gamma(\bar{x})\big)$ the Weyl formula holds with remainder estimate $O\big(h^{1-d}\big) = O\big(\rho(\bar{x})^{d-1}\gamma(\bar{x})^{d-1}\big)$.

On the other hand, for $|x| \ge C|\lambda|^{\frac{1}{2m}}$ rescaling and multiplication by $|\lambda|^{-1}$ produce a Schrödinger operator with standard restrictions to the potential $v(x) \ge \epsilon$ and with $h = \gamma(\bar{x})^{-1}|\lambda|^{-\frac{1}{2}}$. Hence the estimate $|e(x,x,\lambda)| \le Ch^s\gamma(\bar{x})^{-d}$ again holds.

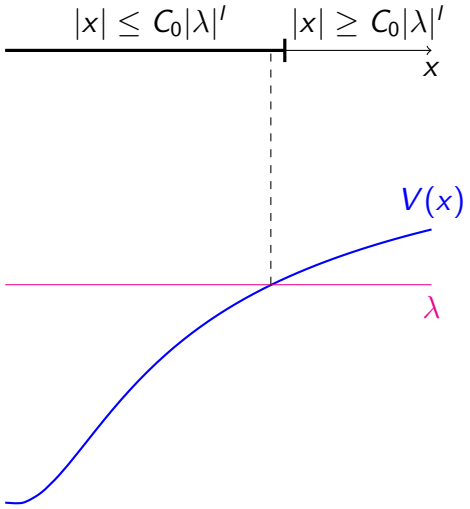
Summing with respect to a partition of unity we obtain asymptotics (0.55)–(0.56) as $\lambda \to -0$ with the same $l = (1+m)/2m$ as in Example 0.3; moreover $\mathsf{N}(\lambda) \asymp |\lambda|^{dl}$ provided

$$V \le -\epsilon|x|^{2m} \qquad \text{for} \quad x : |x| \ge c \tag{0.57}$$

in some non-empty open cone in $\mathbb{R}^d$.



(a) Illustration to Example 0.3



(b) Illustration to Example 0.4

7. Paths

For those interested in the basics of the theory I recommend the following sequence of chapters: 1 ⟶ 2 ⟶ 4 ⟶ 11 and then 1 ⟶ 2 ⟶ 4 ⟶

$9 \longrightarrow 11$; in addition, I suggest considering only the Schrödinger operator in Chapter 11 and only the standard (not the operator-valued) case in Chapters 2 and 4.

Those interested in domains with boundaries (still on a rather elementary level) should read also Chapter 3 and then Chapter 7.

Those interested in magnetic field theory should start reading Chapters 13 (and maybe 17) and 23 (but Chapters 3 and 7 are not necessary for this).

Those interested in the results of Chapter 12 should consider references to every its section.

Those interested in the results of other chapters should look at the map and read the references to each chapter they are interested in.

8. About the Author

Elements of Biography

Elements of biography:

http://weyl.math.toronto.edu/victor/victor_ivrii-bio.html ,

http://weyl.math.toronto.edu/victor/victor_ivrii-vita.pdf

and

http://en.wikipedia.org/wiki/Victor_Ivrii

(the latter is authored by someone and, while it looks accurate, it may contain some errors).

My Way to Spectral Asymptotics and Around

Weyl Conjecture (66–67 AW)

Background. It was Winter '78-'79 when Michael Shubin and B. M. Levitan suggested me to prove Weyl conjecture. R. Seeley just proved remainder estimate $O(\lambda^{(d-1)/2})$. He considered (as anyone would at this time) $\sigma(t) = \mathsf{Tr}(\cos(t\Delta^{\frac{1}{2}}))$ and respectively $u(x,y,t)$ was the Schwartz kernel of $\cos(t\Delta^{\frac{1}{2}})$, satisfying initial-boundary value problem

$$(D_t^2 - \Delta_x)u = 0, \tag{0.58}$$

$$u|_{t=0} = \delta(x-y), \qquad D_t u|_{t=0} = 0, \tag{0.59}$$

$$u|_{x\in\partial X} = 0, \tag{0.60}$$

where for simplicity we discuss the Dirichlet Laplacian; the Neumann Laplacian is considered in the same way. So it was the wave equation rather than the Schrödinger equation (which came later, with semiclassics).

I did not messed up with spectral asymptotics before but I was an established expert in the propagation of singularities, and here I used the energy estimates which is much more robust and general method than the explicit construction of the parametrix.

My idea was to invent a new approach because (as I thought) if Seeley's method worked for Weyl conjecture then Seeley would prove it!

I thought wrong! Couple of years later D. Vassiliev who was *a secret physicist* at that time gave the proof of Weyl conjecture using Seeley's method. And much later I combined Seeley's approach with my own.

But it was the best mistake I ever made! Because the method I invented worked in many situations Seeley' method did not, f.e. for general systems.

Normal Singularity. First, I conjectured that the singularity of $\sigma(t)$ at 0 is normal i.e. $(tD_t)^n\sigma(t)$ at 0 have the same order of singularity for any n exactly as for manifolds without boundary.

I proved this conjecture by implicit method of propagation of singularities in Spring '79. At this moment I had a powerful method of the energy estimates with the theorem stated in the form:

If Ω is a domain in the phase space and ϕ satisfies microhyperbolicity condition and Pu is smooth in $\Omega\cap\{\phi<0\}$, and u is smooth in $\partial\Omega\cap\{\phi<0\}$, then u is smooth in $\Omega\cap\{\phi<0\}$.

Sure, it looks similar to Holmgren uniqueness theorem! Using this technique plus rescaling arguments I proved that singularity of $\sigma(t)$ at $t=0$ is normal.

Successive Approximations. My next idea was rather crazy: to calculate $u(x,y,t)$, using successive approximations, first going to the coordinate system where boundary was planar and getting operator with variable coefficients even if it originally was not, then to the problem (0.58)–(0.59) and apply successive approximation method, freezing coefficients of Δ at point y.

This looks insane because such perturbation decreases smoothness by 2 while parametrix to the problem

$$\begin{aligned}&(D_t^2-\Delta)u=f,\\ &u|_{t=0}=u_t|_{t=0}=0,\\ &u|_{\partial X}=0\end{aligned}$$

increases it only by 1, so each next term in this approximation approach is more singular than the previous one! However, the perturbation contains factors (x_j-y_j) which are of magnitude t due to the finite propagation speed and each parametrix contains factor t due to the Duhamel integral, so in fact, each next term in successive approximations acquires an extra factor t^2D_t. *Big deal, so what?*

But then the same is true for $\sigma(t)$ as well, but for $\sigma(t)$ I knew already that near 0 each multiplication by t compensates one differentiation and this allowed me to justify the successive approximation method for $\sigma(t)$ near 0 without justification it for u. So, a complete asymptotics (with respect to smoothness) of $\sigma(t)$ near 0 was done. This would imply Seeley's result.

Other Singularities. It was early August '79 and I wrote to Michael Shubin about my progress. His answer came two weeks later (no email at that time!): "So what? Other singularities (at $t\neq 0$) are much more difficult!"

But at this moment I already had a solution! Actually other singularities were easier. I have proven that *if the set of periodic geodesic billiards has measure* 0 *then two-term (Weyl) asymptotics holds*.

To tackle "other" singularities of $\sigma(t)$ I analyzed Duistermaat-Guillemin method, purging all irrelevant FIO stuff.

As a result arguments became very simple: Fix arbitrary $T>0$. A set Π_T of geodesic billiards, periodic with periods $\le T$, is a closed set. Then due to our assumption it is a closed nowhere dense set of measure 0. And the set Λ_T of all dead-end billiards of the length $\le T$ is also of this type. Recall that dead-end billiards are those which become tangent to the boundary or behaving badly.

Let $I=Q_1+Q_2$, where Q_1 is a pdo with symbol in ε-vicinity of $\Pi_T\cup\Lambda_T$ and the small vicinity of the boundary and Q_2 has a symbol vanishing in the vicinity of this set. Then $\sigma(t)=\sigma_{Q_1}(t)+\sigma_{Q_2}(t)$ with $\sigma_Q(t)=\mathsf{Tr}(\cos(t\Delta^{\frac12})Q)$.

Here $\sigma_{Q_2}(t)$ has no "other" singularities on $[-T, T]$ and Tauberian methods let me to recover asymptotics of $\mathsf{Tr}(E(\lambda)Q_2)$ with the remainder estimate $CT^{-1}\lambda^{\frac{d-1}{2}} + C'_T$ with C which does not depend on T or ε.

On the other hand, I recovered asymptotics of $\mathsf{Tr}(E(\lambda)Q_1)$ with the remainder estimate $C\varepsilon\lambda^{\frac{d-1}{2}} + C'_{T,\varepsilon}$ where $\varepsilon = \mathsf{mes}(\mathsf{supp}(Q_1))$ is arbitrarily small. So, I recovered asymptotics of $N(\lambda) = \mathsf{Tr}(E(\lambda))$ with the remainder estimate

$$C\bigl(\varepsilon + \frac{1}{T}\bigr)\lambda^{\frac{d-1}{2}} + C'_{T,\varepsilon}$$

and the rest was a second-year calculus exercise!

Aftermath (68–79 AW)

Using this method I instantly proved asymptotics with the remainder estimate $O(\lambda^{\frac{d-1}{m}})$ for m-th order elliptic systems on manifolds without boundaries, and later on manifolds with the boundaries.

When I was peacefully exploiting my method and harvesting results and even published a book two events happened:

Corners, Edges Etc. I got from RZh Matematika (a Russian version of Math. Reviews, but better) a some weird paper (K. Otsuka [1]), deriving two-term Weyl asymptotics for the Euclidean Laplacian in polygons. The author was spending a lot of efforts to consider wave equation near vertices and I realized that this was a completely unnecessary job.

So I decided to do a proper job and I with my student Svetlana Fedorova proved ('84) Weyl formula for Laplace-Beltrami operators in domains with edges, vertices, conical points, cuts etc. But what is more important: rescaling technique was invented!

Going Semiclassic. I decided to go cheap and prove some semiclassic asymptotics. I said "cheap" because at this time I believed as many did that really great mathematicians like H. Weyl, R. Courant, L. Hörmander, J. J. Duistermaat, V. Guillemin, R. Seeley and myself (smile!) study the classical $\mathsf{N}(\lambda)$ on compact manifolds while less great mathematicians study $N(\lambda)$ for the Schrödinger operator with growing potential, semiclassics, etc. So I proved some semiclassical results and told M. Z. Solomyak about them.

He asked: "Why you just do not deduct it from classical asymptotics by a *cheap trick*[16)]?"

I tried to follow this advice with a rather surprising result. Semiclassical results derived this way were less general than I had already; however, working the opposite way I derived more general classical results than I had. It was an eye-opener: *Semiclassical Asymptotics are the most important!* So, I began to study semiclassical asymptotics as a prime target. And in this framework my method became not only more general, but also more natural and transparent, and it worked better with the rescaling technique.

Going Ballistic. First, I discovered that the rescaling applied to the semiclassic produces a bunch of new results. In particular, it allowed to eliminate $|V| + |\nabla V| \neq 0$ (if $d \geq 2$) as a precondition for the sharp semiclassical asymptotics for the Schrödinger operator, which using cheap trick got instantly new results for the classical asymptotics.

Further, I studied degenerations and singularities of different kinds, horns and cusps, other degenerations and singularities. I derived classical asymptotics, asymptotics of eigenvalues for operators, generalizing the Schrödinger operator with the potential growing at infinity, and for operators, generalizing the Schrödinger operator with potential slowly decaying at infinity, and these operators had their degenerations and singularities too. *It was a complete Results Explosion! And it was so damn easy!*

I was feeling like a prospector who found a place with native ores of gold lying just on the surface, ready to be picked. These results were subject of my ICM-1986 talk[17)].

But often remainder estimates were not as good as in the non-degenerate or non-singular cases and I felt that the reason was not my lack of skill but a more profound one.

Going Deeper. I modified my method to treat operators with symbols which were operators in auxiliary spaces and treated only some variables as semiclassical variables and others as non-semiclassical variables, and I reexamined some singularities and degenerations and cusps and derived more sharp asymptotics than before. But these asymptotics either contained

[16)] It was a colloquial name for the Birman-Schwinger principle.

[17)] My ICM-1978 was devoted to propagation of singularities. By Soviet bureaucracy I was not allowed to go to ICMs, further, the permission to send a talk for ICM-1978 came too late and, while this talk was published, it was not read. 8 years later I was smarter–and sent the talk without any permission. L. Hörmander read it. I visited Berkeley many times since and in 2014 I visited Helsinki the first time–36 years too late!

non-Weyl correction terms of the magnitude of the remainder estimates derived on the previous stage or even were non-Weyl in their main parts.

I also considered the Schrödinger and Dirac operators with the strong magnetic field. This was longer, slower and much more difficult process than before and its results appeared in my research monograph of '98.

Multiparticle Quantum Theory (79 AW–88 AW)

In '90 Michael Sigal suggested me to work together to justify the Scott correction term for or asymptotics of the ground state energy for molecules consisting of very large atoms (for atoms even more precise results were already justified by C. Fefferman and L. Seco, using harmonic analysis). My part was to derive sharp asymptotics for the trace of the Schrödinger operator with a Coulomb type singularity. My first attempt was a (partial) success: the remainder estimate was sufficiently sharp to justify the Scott correction, but not sharp enough to justify Dirac and Schwinger corrections or even to approach to them.

A bit later I was able to find more subtle arguments and succeeded to derive estimates, sufficiently sharp to justify Dirac and Schwinger corrections. Further, I considered this problem in the presence a strong external magnetic field.

New Dawn (88–93 AW)

Next few years I have been working on sharp spectral asymptotics for operators with not very regular coefficients. The major tool is the same kind of the analysis as before, applied to the mollified operator. Mollification scale depends on h and equals to $Ch|\log h|$ in the simplest case. This is related to the *Logarithmic Uncertainty Principle*.

These arguments were applied also to the Schrödinger operator with the strong magnetics field.

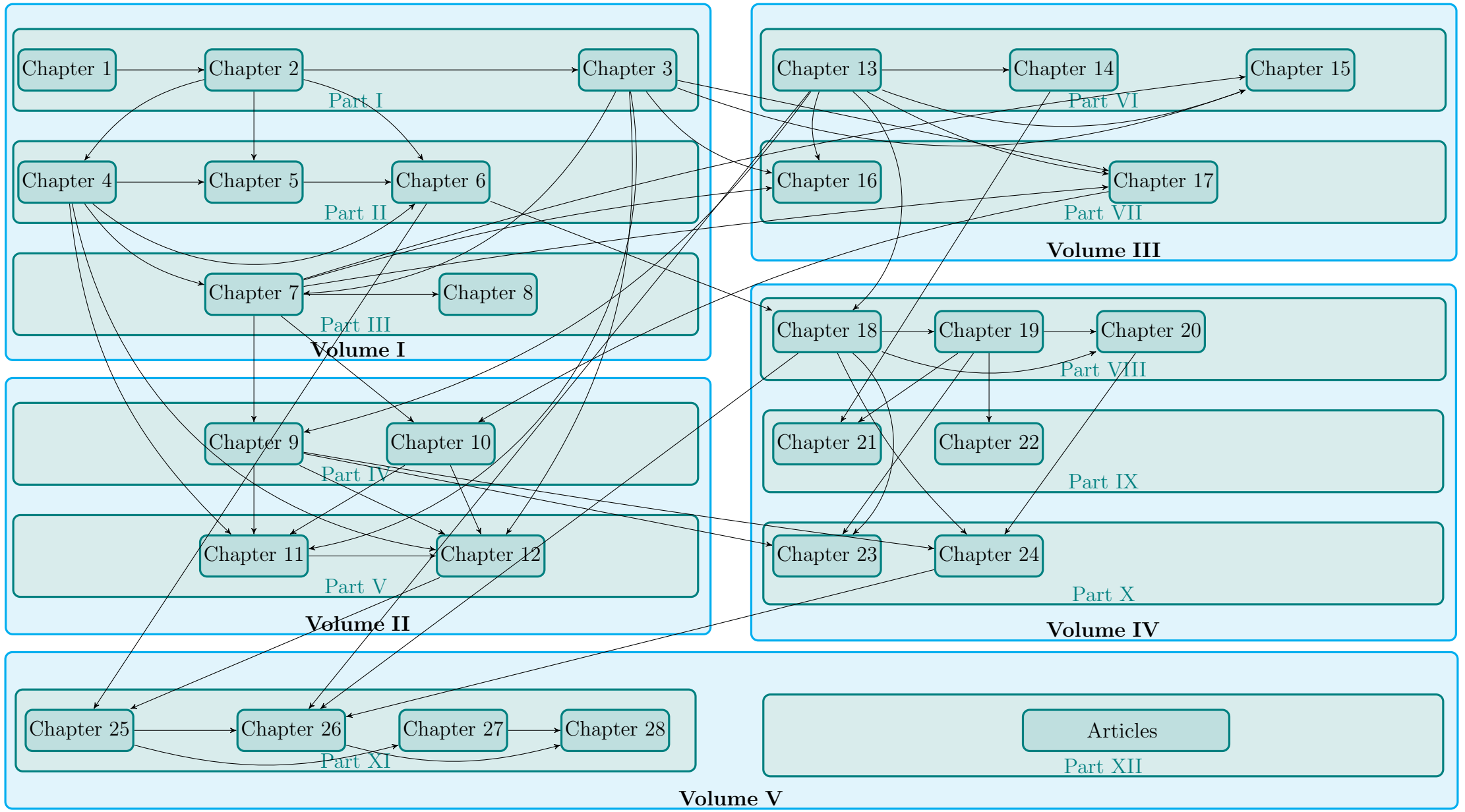
Etc (93 AW–113 AW)

I did many other things later (apart of improving my old results and writing this monograph, which I started in 96 AW). The whole Volumes III and IV (except Chapter 13) and also Chapters 27 and 28 are brand new.

Further, many sections of the "old" chapters are brand new, so chapters became much larger and the material was rearranged.

Aftermath (113 AW–present time)

As I mentioned, I decided not to change the main body of the monograph, but to add new results as separate articles (actually, I even moved two results from the main body to *Part XII. Articles*).


Chapter 1
Chapter 2
Chapter 3
Part I
Chapter 4
Chapter 5
Chapter 6
Part II
Chapter 7
Chapter 8
Part III
Volume I
Chapter 9
Chapter 10
Part IV
Chapter 11
Chapter 12
Part V
Volume II
Chapter 13
Chapter 14
Chapter 15
Part VI
Chapter 16
Chapter 17
Part VII
Volume III
Chapter 18
Chapter 19
Chapter 20
Part VIII
Chapter 21
Chapter 22
Part IX
Chapter 23
Chapter 24
Part X
Volume IV
Chapter 25
Chapter 26
Chapter 27
Chapter 28
Part XI
Articles
Part XII
Volume V

Microlocal Analysis, Sharp Spectral Asymptotics and Applications. I

Semiclassical Microlocal Analysis and Local and Microlocal Semiclassical Asymptotics

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Part I

Semiclassical Microlocal Analysis

Chapter 1

Introduction to Microlocal Analysis

In this chapter we give a survey of the theory of h-pseudodifferential operators (in Section 1.1), h-Fourier integral operators (in Section 1.2), the notion of wave front sets and related topics (in Section 1.3). Proofs are mostly sketchy since all the results of this chapter are trivial consequences of the results proven in L. Hörmander's monograph [1] or they can be proven by trivial modifications of the arguments used in that book.

Further, (in Section 1.4) we give a concise introduction to L. Boutet de Monvel calculus and its application to elliptic boundary value problems.

What is Microlocal Analysis?

Microlocal Analysis is a theory of *Pseudo-Differential Operators* and *Fourier Integral Operators* (PDOs and FIOs). It draws on many rich traditions in mathematics including the links between analysis and geometry and has a powerful application to partial differential equations. For example, all the significant progress in linear PDEs for the last 50 years has been based on microlocal analysis and many important advances in non-linear PDEs over this period were also based on microlocal analysis.

Pseudo-differential operators are generalization of (partial) differential operators (DOs) and also appear as parametrices (almost inverse) of elliptic differential operators (and parabolic differential operators unless we stick with "classical" pseudodifferential operators). 0-order PDOs are singular in-

tegral operators. There is a very nice calculus of pseudodifferential operators which make them extremely useful tool.

Fourier integral operators appear as propagators of hyperbolic and similar equations. While from the Quantum Mechanics point of view pseudodifferential operators are observables, Fourier integral operators are canonical transformations.

Microlocal analysis comes in few flavors. One of the possible classifications is: semi-classical with explicit asymptotics with respect to a small parameter h (Plank constant or wavelength) and classical (standard or more general) where there is no explicit parameter and asymptotics is with respect to the smoothness (or something like this). However, there is still implicit small parameter and microlocal analysis is asymptotical by its nature.

We will consider semi-classical pseudodifferential operators and Fourier integral operators: one can always get from there to standard pseudodifferential operators and Fourier integral operators but semi-classical theory has more applications.

Less formal description of (semi-classical) microlocal analysis is "a semi-classical limit of quantum mechanics" or "high-frequency limit of electrodynamics" (etc). This is really difficult and exciting topic to understand how quantum mechanics yields classical mechanics as Plank constant h goes to 0 (in contrast to trivial "Special relativity yields Newton mechanics as light speed $c \to \infty$"; the fundamental difference is that quantum and corresponding classical objects are mathematically completely different while special relativity and Newton mechanics objects are exactly the same). So, *Microlocal* means local both in coordinates and momenta. There are direct and deep connections with Global Analysis.

1.1 Pseudodifferential Operators

1.1.1 Calculus of Pseudodifferential Operators

Officially Microlocal Analysis was born in 1965 after two articles of L. Hörmander [2] and J. J. Kohn and L. Nirenberg [1] were published where pseudodifferential operators were introduced by formula

$$(Au)(x) = \int a(x,\xi) e^{i\langle x,\xi\rangle} \hat{u}(\xi)\, d\xi \tag{1.1.1}$$

or by alternative formula

$$\widehat{(Au)}(\xi) = \int a(x,\xi) e^{-i\langle x,\xi\rangle} u(x)\, dx \tag{1.1.2}$$

where

$$\hat{u}(\xi) = F_{x\mapsto\xi} u = (2\pi)^{-d} \int e^{-i\langle x,\xi\rangle} u(x)\, dx \tag{1.1.3}$$

is a Fourier transform.

Formulae (1.1.1) and (1.1.2) are not exactly equivalent. Really, if $a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ is a polynomial then (1.1.1) defines a differential operator $A(x,D) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$ while (1.1.2) defines differential operator $A(x,D) = \sum_{|\alpha|\le m} D^\alpha a_\alpha(x)$, $D = -i\partial_x$. In the first case we differentiate first and multiply by coefficients after while in the second case we reverse this order. Following terminology originated from physics we call (1.1.1) by qp-quantization and (1.1.2) by pq-quantization.

In these articles *symbol* $a(x,\xi)$ was supposed to be either positive homogeneous of degree m or the asymptotic sum of such

$$a \sim \sum_{j=0}^{\infty} a_{m-j}(x,\xi) \tag{1.1.4}$$

with a_{m-j} positively homogeneous of degree $m-j$. Later more general classes were introduced.

However one can notice that essentially the same operators were introduced by A. Calderon-A. Zygmund and S. G. Mihlin as singular integral operators. Also quantization procedure was introduced first by H. Weyl, who used yet another formula

$$(Au)(x) = (2\pi)^{-d} \iint a(\frac{x+y}{2},\xi) e^{i\langle x-y,\xi\rangle} u(y)\, dyd\xi \tag{1.1.5}$$

which is called *symmetric* or *Weyl* quantization. The procedure $\mathsf{symbol} \mapsto \mathsf{operator}$ is called *quantization*.

Even if there are different methods of quantization we get the same classes of pseudodifferential operators. So, each pseudodifferential operator has pq-, qp- and Weyl-symbols and the navigation between different symbols will be given.

Moving from differential operators to pseudodifferential we lose property $\mathsf{supp}(Au) \subset \mathsf{supp}(u)$ but retain $\mathsf{sing\,supp}(Au) \subset \mathsf{sing\,supp}(u)$.

Microlocal Analysis of this type is not exact in the sense that we consider infinitely smooth functions and infinitely smoothing operators as negligible. Microlocal Analysis is asymptotic in the sense that we will consider (1.1.4)-type decomposition of symbols and decomposition on functions which are more and more smooth.

The most important properties of such calculus are:

Remark 1.1.1. The most important properties of such calculus are:

(i) Product of operators of orders m_1 and m_2 has degree $m_1 + m_2$ and its principal symbol is equal to the product of principal symbols.

(ii) Commutator of operators of orders m_1 and m_2 has degree $m_1 + m_2 - 1$ and its principal symbol is equal to the Poisson bracket of principal symbols.

(iii) Elliptic operator (i.e. operator with non-vanishing principal symbol) has parametrix (or almost inverse): $AB \equiv BA \equiv I$ modulo infinitely smoothing operators.

(iv) There is a functional calculus of 0-order pseudodifferential operators.

This section is devoted to the calculus of pseudodifferential operators and to major inequalities. However we will deal mainly with h-pseudodifferential operators $a(x, hD)$ with small parameter $h \to 0$ (in this case homogeneity is not assumed). This makes asymptotic nature of Microlocal Analysis more explicit. The price to pay: we have not functions or operators but families of them depending on h. Temperate function or operator (or rather the family!) is with the norm $O(h^{-l})$ for some l. Negligible function or operator (or rather the family!) is with the norm $O(h^s)$ for any s.

h-Fourier Transform

Important role in our constructions plays h-Fourier transform given by

$$F_h : u(x) \mapsto (F_h u)(\xi) = (2\pi h)^{-d} \int e^{-ih^{-1}\langle x,\xi\rangle} u(x)\, dx \tag{1.1.6}$$

and its inverse is

$$F_h^{-1} : v(\xi) \mapsto (F_h^{-1} v)(x) = \int e^{ih^{-1}\langle x,\xi\rangle} v(\xi)\, d\xi. \tag{1.1.7}$$

We could put factor $(2\pi h)^{-\frac{d}{2}}$ in both expressions (instead of $(2\pi h)^{-d}$ and 1, to get unitary operators $\mathcal{F}_h$ and $\mathcal{F}_h^*$ in $\mathscr{L}^2(\mathbb{R}^d)$.

Physicists call $u(x)$ a coordinate representation and $\mathcal{F}_h u$ a momentum representation because $\mathcal{F}(hD_{x_j})u = \xi_j \mathcal{F}_h$, $D_{x_j} = -i\partial_{x_j}$ and hD_{x_j} is momentum operator. In physics q and p is reserved for coordinates and momenta.

The only operators commuting with multiplication by $e^{ih^{-1}\langle x,\xi\rangle}$ for every ξ are operators of multiplication $u(x)$ by a function $\nu(x)$. Operator norm of such operator $\nu(x)$ in $\mathscr{L}^2(\mathbb{R}^d)$ is $\|\nu\|_\infty = \operatorname{ess\,sup} |\nu(x)|$.

Respectively, the only operators commuting with all shifts are *multiplicators* :

$$\mu(hD) = F_h^{-1}\mu(\xi)F_h. \tag{1.1.8}$$

Operator norm of such operator $\mu(hD)$ in $\mathscr{L}^2(\mathbb{R}^d)$ is $\|\mu\|_\infty = \operatorname{ess\,sup} |\mu(\xi)|$.

h-DO Representation via h-Fourier Transform; Quantization of Different Flavors

Now we will construct much broader algebra including both operators of multiplication and multiplicators (with admissible *symbols* $a(x)$, $b(\xi)$. The trouble is that these operators do not commute! So, how to quantize $a(x,\xi) = \nu(x)\mu(\xi)$?

The first way is $\nu(x)\mu(hD)$ which would be given by formula

$$\big(a(\overset{2}{x}, \overset{1}{hD})u\big)(x) = (2\pi h)^{-d} \iint e^{ih^{-1}\langle x-y,\xi\rangle} a(x,\xi)u(y)\, dyd\xi; \tag{1.1.9}$$

we can do it in the reverse order $\nu(x)\mu(hD)$:

$$\big(a(\overset{1}{x}, \overset{2}{hD})u\big)(x) = (2\pi h)^{-d} \iint e^{ih^{-1}\langle x-y,\xi\rangle} a(y,\xi)u(y)\, dyd\xi. \tag{1.1.10}$$

One can see easily that

(1.1.11) As $a(x,\xi) = \sum_\alpha a_\alpha(x)\xi^\alpha$ formulae (1.1.9) and (1.1.10) define operators $\sum_\alpha a_\alpha(x)(hD)^\alpha$ and $\sum_\alpha (hD)^\alpha a_\alpha(x)$ respectively;

(1.1.12) If $a(x,hD)$ is defined by (1.1.9) or (1.1.10) then $F_h a(x,hD)F_h^{-1} = a(-hD_\xi,\xi)$ where $a(-hD_\xi,\xi)$ is defined by method (1.1.10) or (1.1.9) respectively.

Formulae (1.1.9), (1.1.10) are called qp- and pq-quantization respectively to reflect order in which x (aka q) and hD_x (aka p) are applied.

Should we look for something more symmetric? The first guess $\frac{1}{2}\big(\nu(x)\mu(hD)+\mu(hD)\nu(x)\big)$ is not that bad but

$$\big(a^{\mathsf{w}}(x,hD)u\big)(x)=(2\pi h)^{-d}\iint e^{ih^{-1}\langle x-y,\xi\rangle}a\big(\frac{x+y}{2},\xi\big)u(y)\,dyd\xi \tag{1.1.13}$$

is way better. One can prove easily that

(1.1.14) For $a(x,\xi)=\sum a_\alpha(x)\xi^\alpha$ formula (1.1.13) gives h-DO;

$$a^{\mathsf{w}}(x,hD)=\sum_{\alpha,\beta}2^{-|\alpha|-|\beta|}\frac{(\alpha+\beta)!}{\alpha!\beta!}(hD)^\alpha a_{\alpha+\beta}(x)(hD)^\beta; \tag{1.1.15}$$

(1.1.16) If $a(x,hD)$ is defined by (1.1.13) then $F_h a(x,hD)F_h^{-1}=a(-hD_\xi,\xi)$ where $a(-hD_\xi,\xi)$ is defined by the same method (1.1.13).

Formula (1.1.13) is called *Weyl* or *symmetric quantization*. Apart of p and q equality it has yet another advantage: if $a(x,\xi)$ is real valued then $a(x,hD)$ is symmetric and often self-adjoint operator.

Symbols and h-Pseudodifferential Operators; qp-, pq-, Weyl (Symmetric) Quantization. Double Symbols and Their Quantization

Now we can define using formulae (1.1.9), (1.1.10) or (1.1.13) pseudodifferential operators for more general classes of symbols than it was before: we simply plug an (admissible) general symbol $a(x,\xi)$ in such formula. For a sake of generality and simplicity in some arguments we, however, introduce quantization *double symbols* $a(x,\xi,y)$ by formula

$$\big(a(\overset{3}{x},\overset{2}{hD},\overset{1}{x})u\big)(x)=(2\pi h)^{-d}\iint e^{ih^{-1}\langle x-y,\xi\rangle}a(x,\xi,y)u(y)\,dyd\xi \tag{1.1.17}$$

which for $a(x,\xi,y)=a(x,\xi)$, $a(x,\xi,y)=a(y,\xi)$, $a(x,\xi,y)=a(\frac{x+y}{2},\xi)$ gives $a(\overset{2}{x},\overset{1}{hD})$, $a(\overset{1}{x},\overset{2}{hD})$ and $a^{\mathsf{w}}(x,hD)$ respectively.

Surely there are two shortcomings of this way: completely lost symmetry between x and hD and very non-unique symbol $\leftrightarrow$ operator correspondence. However such symbols are technically useful.

Operator $A = a(\overset{3}{x}, \overset{2}{hD}, \overset{1}{x})$ defined by (1.1.17) has Schwartz kernel

$$\mathcal{K}_A(x,y) = (2\pi h)^{-d} \iint e^{ih^{-1}\langle x-y,\xi\rangle} a(x,\xi,y)\, d\xi \tag{1.1.18}$$

Recall that each linear operator $A : \mathscr{C}_0^\infty(Y) \to \mathscr{D}'(X)$ where $\mathscr{D}'(Y)$ is the space of distributions over $\mathscr{D}(X) = \mathscr{C}_0^\infty(X)$ has a (unique) Schwartz kernel $\mathcal{K}_A \in \mathscr{D}'(X \times Y)$ such that $\langle A\phi, \psi\rangle = \langle \mathcal{K}_A, \phi \otimes \psi\rangle$ where $(\phi \otimes \psi)(x,y) = \phi(x)\psi(y)$.

For an infinitely smooth and fast decaying with respect to ξ symbol this Schwartz kernel belongs to $\mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and operator is infinitely smoothing and can be extended to operator $\mathscr{E}'(\mathbb{R}^d) \to \mathscr{C}^\infty(\mathbb{R}^d)$ where $\mathscr{E}'$ the space of distributions over $\mathscr{E} = \mathscr{C}^\infty$. However it is not uniform with respect to h and in contrast to classical theory (to follow) smoothness of the function (distribution) is not the subject of our prime interest.

Let us consider operator defined by (1.1.18) in $\mathscr{L}^2(\mathbb{R}^d)$ (our usual playground). The following very weak statements are nevertheless useful:

Proposition 1.1.2. *If*

$$|\partial_\xi^\alpha a| \le M(1+|\xi|)^{-s} \qquad \forall \alpha : |\alpha| \le K \tag{1.1.19}$$

then

$$|\mathcal{K}_A(x,y)| \le C_{s,K} M h^{-d}\big(1 + h^{-1}|x-y|\big)^{-K} \tag{1.1.20}$$

Proof. Obviously statement holds as $K = 0$.

Let us multiply $\mathcal{K}_A$ by $(x_j - y_j)$, observe that $e^{ih^{-1}\langle x-y,\xi\rangle}(x_j - y_j) = i\partial_{\xi_j} e^{ih^{-1}\langle x-y,\xi\rangle}(x_j - y_j)$ and integrate by parts with respect to ξ_j. We will get that

(1.1.21) $\mathcal{K}_A(x,y)(x_j - y_j)$ has the Schwartz kernel defined by (1.1.18) with symbol $a(x,\xi,y)$ replaced by $-ih\partial_{\xi_j} a(x,\xi,y)$.

Then (1.1.20) holds with $K = 1$. Repeating the same argument K times we will get (1.1.20) for any K. □

Obviously, $\mathcal{K}_A(x,y)(x_j-y_j)$ is the Schwartz kernel of commutator $[x_j, A]$.

Theorem 1.1.3. *If* (1.1.19) *(or* (1.1.20)*) holds for* $K > d$ *then operator norm of* A *in* $\mathscr{L}^2(\mathbb{R}^d)$ *does not exceed* CM.

Proof. There is a theorem in Real Analysis that the operator norm in $\mathscr{L}^p(\mathbb{R}^d)$ does not exceed

$$\Big(\sup_x \int |\mathcal{K}(x,y)|dy\Big)^{\frac{1}{p}} \Big(\sup_y \int |\mathcal{K}(x,y)|dx\Big)^{\frac{p-1}{p}}. \tag{1.1.22}$$

The proof of it based on the trivial proofs for $p = 1, \infty$ and then interpolation.

It follows from (1.1.20) with $K > d$ that both integrals in (1.1.22) do not exceed CM. □

Later we will replace assumption that a decays in ξ by a more natural one.

Negligible Functions, Symbols, Operators. Going from one Type of Quantization to Another. Symbol and Operator classes

Now when we noticed that factor (x_j-y_j) in the double symbol leads to extra h in the operator norm estimate, we can assuming some smoothness reduce $a(\overset{3}{x}, \overset{2}{hD}, \overset{1}{x})$ to $b'(\overset{2}{x}, \overset{1}{hD})$ or to $b''(\overset{1}{x}, \overset{2}{hD})$ or to $b^{\mathrm{w}}(x, hD)$ (modulo operators with small norms).

Namely, to go to $b'(\overset{2}{x}, \overset{1}{hD})$ we need to get rid of y. The simplest way to do this is to use the Taylor expansion

$$a(x,\xi,y) = \sum_{\alpha:|\alpha|<N} \frac{1}{\alpha!}(y-x)^\alpha(\partial_y^\alpha a)(x,\xi,y=x) + \sum_{\alpha:|\alpha|=N} (y-x)^\alpha \tilde{a}_\alpha(x,\xi,y) \tag{1.1.23}$$

and then using arguments of the proof of Proposition 1.1.2 one can replace $(y-x)^\alpha$ by $(-ih\partial_\xi)^\alpha$. Then the regular terms (with $|\alpha| < N$) in (1.1.23) will produce

$$b' = b''_N(x,\xi) = \sum_{\alpha:|\alpha|<N} \frac{(-ih)^{|\alpha|}}{\alpha!}(\partial_y^\alpha \partial_\xi^\alpha a)(x,\xi,x) \tag{1.1.24}$$

while the remainder terms (with $|\alpha| = N$) will produce operator with operator norm not exceeding CMh^N provided

$$|\partial_y^\alpha \partial_\xi^\nu a(x,y,\xi)| \le M(1+|\xi|)^{-s} \qquad \forall \alpha : |\alpha| \le N, \forall \nu : |\nu| \le N+d+1 \tag{1.1.25}$$

(with some $s > d$).

Using similar arguments (but with permuted x and y) one can reduce $a(\overset{3}{x}, \overset{2}{hD}, \overset{1}{x})$ to $b''(\overset{1}{x}, \overset{1}{hD})$ with

$$b'' = b''_N(x,\xi) = \sum_{\alpha:|\alpha|<N} \frac{(ih)^{|\alpha|}}{\alpha!} (\partial_x^\alpha \partial_\xi^\alpha a)(x,\xi,x) \tag{1.1.26}$$

provided

$$|\partial_x^\alpha \partial_\xi^\nu a(x,y,\xi)| \le M(1+|\xi|)^{-s} \qquad \forall \alpha : |\alpha| \le N, \forall \nu : |\nu| \le N+d+1. \tag{1.1.27}$$

Furthermore, decomposing in both x, y at the midpoint $\frac{1}{2}(x+y)$ one can reduce $a(\overset{3}{x}, \overset{2}{hD}, \overset{1}{x})$ to $b^{\mathrm{w}}(x, hD)$ with

$$b = b_N(x,\xi) = \sum_{\alpha,\beta:|\alpha|+|\beta|<N} \frac{i^{|\alpha|-|\beta|}(h/2)^{|\alpha|+|\beta|}}{\alpha!\beta!} (\partial_x^\alpha \partial_y^\beta \partial_\xi^{\alpha+\beta} a)(x,\xi,x) \tag{1.1.28}$$

provided

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\nu a(x,y,\xi)| \le M(1+|\xi|)^{-s} \qquad \forall \alpha,\beta : |\alpha|+|\beta| \le N, \forall \nu : |\nu| \le N+d+1. \tag{1.1.29}$$

So we proved

Theorem 1.1.4. *(i) Let condition* (1.1.25) *be fulfilled with* $s > d$. *Then* $a(\overset{3}{x}, \overset{2}{hD}, \overset{1}{x}) \equiv b'(\overset{2}{x}, \overset{1}{hD})$ *modulo operator with operator norm*[1)] *not exceeding* $C_{N,s} M h^N$ *where* $b' = b'_N$ *is defined by* (1.1.24).

[1)] All the norms are by default $\mathscr{L}^2$-norms and all the operator norms are by default operator norms in $\mathscr{L}^2$.

(ii) Let condition (1.1.27) *be fulfilled with* $s > d$. *Then* $a(\overset{3}{x}, \overset{2}{hD}, \overset{1}{x}) \equiv b''(\overset{1}{x}, \overset{2}{hD})$ *modulo operator with operator norm not exceeding* $C_{N,s}Mh^N$ *where* $b'' = b''_N$ *is defined by* (1.1.26).

(iii) Let condition (1.1.29) *be fulfilled with* $s > d$. *Then* $a(\overset{3}{x}, \overset{2}{hD}, \overset{1}{x}) \equiv b^{\mathsf{w}}(x, hD)$ *modulo operator with operator norm not exceeding* $C_{N,s}Mh^N$ *where* $b = b_N$ *is defined by* (1.1.28).

We refer to b', b'', b as to pq-, qp- and *Weyl symbols*. .

Definition 1.1.5. (i) We call operator *negligible* if its operator norm does not exceed Ch^N with some constant C.

(ii) We call double symbol negligible if it satisfies (1.1.25) or (1.1.27) or (1.1.29) and leads to $b_N = 0$ or $c_N = 0$ or $e_N = 0$ respectively.

(iii) We call a family of functions u_h depending on parameter h *negligible* if $\|u_h\| \le Ch^N$.

(iv) We call a family of functions u_h depending on parameter h *admissible* if $\|u_h\| \le C$. Sometimes we admit the family with $\|u_h\| \le Ch^{-l}$.

Surely these definitions depend on N; we will just assume that N is large enough. Note that if a satisfies one of the conditions (1.1.25),(1.1.27),(1.1.29), it is not generally true for b', b'', b. More robust formulae will be provided later.

Definition 1.1.6. Fix s and $N \ge 1$.

(i) Denote by $\mathsf{S}^s_N(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ the class of double symbols

$$a(x,\xi,y) = \sum_{j=0}^{N-1} a_j(x,y,\xi)h^j \tag{1.1.30}$$

with a_j satisfying condition (1.1.29) with N replaced by $(N-j)$ and with some constant M. Denote by $\mathsf{S}^s_N(\mathbb{R}^d \times \mathbb{R}^d)$ the class of corresponding ordinary symbols.

(ii) Denote by $\boldsymbol{\Psi}^s_N(\mathbb{R}^d)$ the class of operators obtained (modulo negligible) from double symbols $a \in \mathsf{S}^s_N(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ by quantization procedure (1.1.17).

(iii) According to Theorem 1.1.3 we get the same class if we start from ordinary symbols $b \in \mathsf{S}^s_N(\mathbb{R}^d \times \mathbb{R}^d)$ and apply quantization procedures (1.1.9),(1.1.10),(1.1.13); in this case we refer to b as qp-symbol, pq-symbol and Weyl symbol respectively.

(iv) We call $a_0(x,\xi)|_{h=0}$ or $a_0(x,\xi,x)|_{h=0}$ the principal symbol. It does not depend on method of quantization.

Remark 1.1.7. Several questions arise immediately:

Can two different symbols generate the same operator (modulo negligible)? We have seen this non-uniqueness for double symbols but how about ordinary symbols? We will see later that they are uniquely determined. Note that if we allow dependence a_j on h decomposition of the symbol is not unique for sure but we are talking about symbol itself. In particular, the principal symbol is defined uniquely.

Algebra: Conjugate and Product of Operators. Symmetric Pseudodifferential Operators

There are two operations which are not completely trivial: multiplication and finding adjoint (finding resolvent and functional calculus will follow later). Since we study now bounded operators, we do not need to study the domain of the adjoint operator:

$$(Au, v) = (u, A^* v) \qquad \forall u, v \in \mathscr{L}^2(\mathbb{R}^d) \tag{1.1.31}$$

where $(.,.)$ is an inner product in $\mathscr{L}^2(\mathbb{R}^d)$. Adjoint is rather easy:

$$\bigl(a(\overset{3}{x}, \overset{2}{hD}, \overset{1}{x})\bigr)^* = a^\dagger(\overset{1}{x}, \overset{2}{hD}, \overset{3}{x}) \tag{1.1.32}$$

where $a^\dagger$ means a complex or Hermitian conjugate to a. In particular

$$\bigl(a(\overset{1}{x}, \overset{2}{hD})\bigr)^* = a^\dagger(\overset{2}{x}, \overset{1}{hD}), \tag{1.1.33}_1$$

$$\bigl(a(\overset{2}{x}, \overset{1}{hD})\bigr)^* = a^\dagger(\overset{1}{x}, \overset{2}{hD}), \tag{1.1.33}_2$$

$$\bigl(a^{\mathsf{w}}(x, hD)\bigr)^* = a^{\dagger\mathsf{w}}(x, hD), \tag{1.1.33}_3$$

(1.1.34) $a^\dagger = a$ implies that $a^{\mathsf{w}}(x, hD)$ is symmetric.

Product is more tricky but we can restrict ourselves by $a(\overset{2}{x}, \overset{1}{hD})b(\overset{1}{x}, \overset{2}{hD})$. One can see easily that

$$a(\overset{2}{x}, \overset{1}{hD})b(\overset{1}{x}, \overset{2}{hD}) = c(\overset{3}{x}, \overset{2}{hD}, \overset{1}{x}), \qquad c(x,\xi,y) = a(x,\xi)b(y,\xi). \tag{1.1.35}$$

Further, using (1.1.35) and (1.1.24), (1.1.26) and (1.1.28) one can see easily that

$$a(x, hD)b(x, hD) \equiv (a \bullet b)(x, hD) \tag{1.1.36}$$

with the same quantization method where (with indicated quantization method)

$$a \bullet_{qp,L} b = \sum_{\alpha:|\alpha|<N} \frac{(-ih)^\alpha}{\alpha!}(\partial_\xi^\alpha a)(\partial_x^\alpha b), \tag{1.1.37}_1$$

$$a \bullet_{pq,L} b = \sum_{\alpha:|\alpha|<N} \frac{(ih)^\alpha}{\alpha!}(\partial_x^\alpha a)(\partial_\xi^\alpha b), \tag{1.1.37}_2$$

$$a \bullet_{\mathsf{w},L} b = \sum_{\alpha,\beta:|\alpha|+|\beta|<N} \frac{i^{|\beta|-|\alpha|}(h/2)^{|\beta|+|\alpha|}}{\alpha!\beta!}(\partial_\xi^\alpha \partial_x^\beta a)(\partial_x^\alpha \partial_\xi^\beta b). \tag{1.1.37}_3$$

In particular, the principal symbol of the product is equal to the product of the principal symbols:

$$a(x, hD)b(x, hD) \equiv (ab)(x, hD) \qquad \mathsf{mod}\ h\boldsymbol{\Psi}_{N-1}. \tag{1.1.38}$$

Using above formulae one can check easily that the *principal symbol of the commutator is equal to* $-ih \times$ *Poisson bracket of principal symbols*:

$$[a(x, hD), b(x, hD)] \equiv -ih\{a, b\}(x, hD) \qquad \mathsf{mod}\ h^2\boldsymbol{\Psi}_{N-2} \tag{1.1.39}$$

where

$$\{a, b\} = \langle \partial_\xi a, \partial_x b\rangle - \langle \partial_x a, \partial_\xi b\rangle. \tag{1.1.40}$$

Furthermore, using above formulae one can check easily that

$$a^{\mathsf{w}}(x, hD)b^{\mathsf{w}}(x, hD) + b^{\mathsf{w}}(x, hD)a^{\mathsf{w}}(x, hD) \equiv 2(ab)^{\mathsf{w}}(x, hD) \qquad \mathsf{mod}\ h^2\boldsymbol{\Psi}_{N-2}, \tag{1.1.41}$$

$$[a^{\mathsf{w}}(x, hD), b^{\mathsf{w}}(x, hD)] \equiv -ih\{a, b\}(x, hD) \qquad \mathsf{mod}\ h^3\boldsymbol{\Psi}_{N-3}. \tag{1.1.42}$$

Remark 1.1.8. Formulae $(1.1.33)_3$, (1.1.39) and (1.1.42) hold for Weyl quantization only.

Quantization: Abstract Operator Point of View

This approach is due to Hermann Weyl (1927) who was looking for a quantization method in the flat space. In quantum mechanics observables should be self-adjoint operators. Unfortunately, self-adjointness of unbounded operators is a very subtle thing and one can assure only symmetry of the quantization of randomly chosen real-valued symbol (classical observable).

Let $\mathscr{H}$ be a Hilbert space and Q_j, P_j $j = 1, \dots, d$ be a self-adjoint unbounded operators satisfying commutation relations

$$[P_j, P_k] = 0, \qquad [Q_j, Q_k] = 0, \qquad [P_j, Q_k] = -ih\delta_{jk} \tag{1.1.43}$$

This is how physicists usually write it but it is bizarre: for unbounded operators we need to indicate domains and even if we assume that (1.1.43) hold on everywhere dense linear subset of $\mathscr{H}$ it would not be enough. Compare:

(1.1.44) Two unbounded self-adjoint operators commute if their resolvents commute ($\Leftrightarrow$ their spectral projectors commute $\Leftrightarrow$ their generated groups commute).

To write commutation relations properly we consider unitary groups $U_j(t) = e^{itQ_j}$, $V_j(t) = e^{itP_j}$ and assume that

$$\begin{aligned} &U_j(t)U_k(s) = U_k(s)U_j(t), \qquad V_j(t)V_k(s) = V_k(s)V_j(t), \\ &\qquad\qquad V_j(t)U_k(s) = e^{i\delta_{jk}hts}U_k(s)V_j(t) \qquad \forall j,k \quad \forall t,s \in \mathbb{R}. \end{aligned} \tag{1.1.45}$$

One can prove a following

Theorem 1.1.9. *If P_j, Q_k $(j, k = 1, \dots, d)$ are self-adjoint operators satisfying* (1.1.45) *then there exists a unitary operator $F : \mathscr{H} \to \mathscr{L}^2(\mathbb{R}^d, \mathbb{H})$ with some auxiliary Hilbert space $\mathbb{H}$ such that $FP_jF^* = hD_j$, $FQ_jF^* = x_j$ (operators of multiplication by).*

Surely one can transform it to momenta representation $FP_jF^* = \xi_j$, $FQ_jF^* = -hD_{\xi_j}$. We call $Q_j = x_j$, $P_j = hD_j$ *canonical*.

Let us introduce $\boldsymbol{Q} = (Q_1, \dots, Q_d)$, $U(\boldsymbol{s}) = e^{ih\langle \boldsymbol{s}, \boldsymbol{Q}\rangle} = U_1(s_1)\cdots U_d(s_d)$, $\boldsymbol{P} = (P_1, \dots, P_d)$ $V(\boldsymbol{t}) = e^{ih\langle \boldsymbol{t}, \boldsymbol{P}\rangle} = V_1(t_1)\cdots V_d(t_d)$ as $\boldsymbol{s} = (s_1, \dots, s_d)$ and $\boldsymbol{t} = (t_1, \dots, t_d)$.

Now let us quantize symbol $a(q,p)$ which we represent as

$$a(q,p)=\iint \hat{a}(s,t)e^{i\langle s,q\rangle}e^{i\langle t,p\rangle}\,ds\,dt \tag{1.1.46}$$

where $\hat{a}$ is a Fourier transform of a with respect to all variables. We should plug into (1.1.46) P, Q instead of p, q but we need to select an order: one can replace $e^{i\langle s,q\rangle}e^{i\langle t,p\rangle}$ either by $U(s)V(t)$, or take them in the opposite order which is equivalent to taking $e^{ih\langle t,s\rangle}U(s)V(t)$, or to take a "symmetric product" $e^{\frac{i}{2}h\langle t,s\rangle}U(s)V(t)$. So, we get

$$\begin{aligned} a(Q,P)&=\iint \hat{a}(s,t)e^{ih\varsigma\langle t,s\rangle}U(s)V(t)\,ds\,dt = \\ &(2\pi)^{-2d}\iint\iint a(q,p)e^{i(h\varsigma\langle t,s\rangle-\langle p,t\rangle-\langle q,s\rangle)}U(s)V(t)\,ds\,dt\,dp\,dq. \end{aligned} \tag{1.1.47}$$

One can check easily that

(1.1.48) For canonical Q_j, P_j this formula with $\varsigma=0,1,\frac{1}{2}$ gives qp-, pq-and Weyl quantizations respectively as we introduced them before.

Matrix Pseudodifferential Operators

We can construct the same theory for matrix pseudodifferential operators (i.e. pseudodifferential operators with matrix symbols). More precisely, we can consider symbols belonging to $\mathcal{L}(\mathbb{K},\mathbb{H})$ which is the space of bounded linear operators from $\mathbb{H}$ to $\mathbb{K}$ (where $\mathbb{H}$ and $\mathbb{K}$ two auxiliary Hilbert spaces). Then all the above formulae hold albeit with Hermitian conjugate rather than complex conjugate in (1.1.32)–(1.1.33)$_3$.

All the product formulae including (1.1.38) remain true but (1.1.39)–(1.1.42) fail unless one of the factors is a scalar operator $\lambda(x,hD)\otimes I$.

One can consider also symbols which are unbounded operators with fixed domain and "strength" (and it has important applications) but it brings tons of difficulties with justification.

Ellipticity, Parametrix, Resolvent. Functional Calculus

Since symbols of the classes we considered so far decay as $|\xi|\to\infty$, they cannot be elliptic, having inverse of the same class. It does not preclude ellipticity of symbols $\big(\lambda I-a(x,\xi)\big)$ with $\lambda\in\mathbb{C}$ and therefore construction of the

resolvent and then functional calculus. However we postpone construction until the class of operators we study is extended.

Traces and Hilbert-Schmidt Norms

Let us start from the formal calculations. It is known from Real Analysis (see f.e. P. D. Lax [1]) that if operator $A \in \mathcal{L}\big(\mathscr{L}^2(\mathbb{R}^d,\mathbb{H}),\mathscr{L}^2(\mathbb{R}^d,\mathbb{H})\big)$ belongs to the trace class then its trace is equal to

$$\mathsf{Tr}(A) = \int \mathsf{tr}(\mathcal{K}_A(x,x))\,dx \tag{1.1.49}$$

where "tr" means a "matrix trace" i.e. the trace in $\mathcal{L}(\mathbb{H},\mathbb{H})$.

One can check easily that

$$\mathsf{Tr}(a(\overset{3}{x},\overset{2}{hD},\overset{1}{x})) = (2\pi h)^{-d}\iint \mathsf{tr}(a(x,\xi,x))\,dxd\xi. \tag{1.1.50}$$

Meanwhile the Hilbert-Schmidt norm of an operator $A \in \mathcal{L}\big(\mathscr{L}^2(\mathbb{R}^{d'},\mathbb{H}'),\mathscr{L}^2(\mathbb{R}^d,\mathbb{H})\big)$ is $\|A\|_{\mathsf{HS}} = (\mathsf{Tr}(A^*A))^{1/2} = (\mathsf{Tr}(AA^*))^{1/2}$. It is known from Real Analysis that A is a Hilbert-Schmidt operator if and only if $\mathcal{K} \in \mathscr{L}^2\big(\mathbb{R}^d\times\mathbb{R}^{d'},\mathsf{HS}(\mathbb{H}',\mathbb{H})\big)$ where $\mathsf{HS}(\mathbb{H}',\mathbb{H})$ is the space of Hilbert-Schmidt operators from $\mathbb{H}$ to $\mathbb{H}'$ equipped with the corresponding norm; further

$$\|A\|^2_{\mathsf{HS}} = \iint \mathsf{tr}(\mathcal{K}_A(x,y)^\dagger \mathcal{K}_A(x,y))\,dxdy. \tag{1.1.51}$$

Let us prove

Proposition 1.1.10. *For any quantization of symbol* $a(x,\xi)$

$$\|a(x,hD)\|^2_{\mathsf{HS}} = (2\pi h)^{-d}\iint \mathsf{tr}((a^\dagger a))(x,\xi)\,dxd\xi \tag{1.1.52}$$

Proof. This is obvious for pq- and qp-quantizations but holds also for Weyl quantization; even more generally for any $\varsigma\in\mathbb{R}$ and $\varsigma' = 1-\varsigma$

$$\begin{aligned}&\iint \mathsf{tr}(\mathcal{K}_A(x,y)^\dagger\mathcal{K}_A(x,y))\,dxdy = \\ &(2\pi h)^{-2d}\iint\iint \mathsf{tr}(a^\dagger(\varsigma x+\varsigma' y,\xi)a(\varsigma x+\varsigma' y,\eta))e^{ih^{-1}\langle x-y,\eta-\xi\rangle}\,dxdy\,d\xi d\eta = \\ &\qquad (2\pi h)^{-2d}\iint\iint \mathsf{tr}(a^\dagger(z,\xi)a(z,\eta))e^{ih^{-1}\langle z',\eta-\xi\rangle}\,dzdz'\,d\xi d\eta\end{aligned}$$

(after obvious change of variables) which obviously equals to the right-hand expression of (1.1.52). □

To justify all these formulae we need

Proposition 1.1.11. *Let us consider operator A with a double symbol $a = a(x,y,\xi)$ such that*

$$\|D_x^\alpha D_y^\beta D_\xi^\nu a\| \le M\Phi(\xi)^{-m}\phi(x)^{-l} \qquad \forall \alpha,\beta,\nu : |\alpha|+|\beta| \le N+K, |\nu| \le N+K \qquad \forall (x,y,\xi) \in \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d \tag{1.1.53}$$

where $\phi(x) = (1+\sum_j x_j^2/\bar\gamma_j^2)^{\frac12}$, $\Phi(\xi) = (1+\sum_j \xi_j^2/\bar\rho_j^2)^{\frac12}$ with $\bar\rho_j \ge 1$, $\bar\gamma_j \ge 1$ and $N \ge 0$. Alternatively, let us consider operator A with a symbol $a(x,\xi)$ which satisfies (1.1.53) *and apply any method of quantization.*

Then its Schwartz kernel satisfies

$$\int \|\mathcal{K}_A(x,y)\|^2 dy \le CM^2\bar\rho_1\cdots\bar\rho_d h^{-d} \qquad \forall x\in\mathbb{R}^d \tag{1.1.54}$$

provided $m > d/2$, $l = 0$ and

$$\iint \|\mathcal{K}_A(x,y)\|^2\, dxdy \le CM^2\bar\rho_1\cdots\bar\rho_j\bar\gamma_1\cdots\bar\gamma_d h^{-d} \tag{1.1.55}$$

provided $m > d/2$, $l > d/2$.

Proof. The conditions on $\bar\gamma_j, \rho_j$ and $\bar\rho_j$ imply that the weight functions $\Phi(\xi)^m$ and $\phi(x)^l$ are tempered with respect to the metrics $\sum_j dx_j^2 + h^2 d\xi_j^2$. Then Theorem 18.5.5 of [1] implies that the operator $\Phi(hD)^m A^* A \Phi(hD)^m$ is a scalar h-pseudodifferential operator with a symbol satisfying (1.1.27) with M replaced by CM^2; hence it is a bounded operator from $\mathscr{L}^2(\mathbb{R}^d)$ to $\mathscr{L}^2(\mathbb{R}^d)$ and its norm does not exceed CM^2.

Then Sobolev embedding theorem implies that the norm of the operator $A^*A\Phi(hD)^m$ from $\mathscr{L}^2(\mathbb{R}^d)$ to $\mathscr{L}^\infty(\mathbb{R}^d,\mathbb{H})$ does not exceed $CM^2\bar\rho_1^{\frac12}\cdots\bar\rho_d^{\frac12}h^{-\frac{d}{2}}$ and therefore

$$\int |\mathcal{K}_{A^*A\Phi(hD)^m}(x,y)|^2\, dy \le CM^4\bar\rho_1\cdots\bar\rho_d h^{-d} \qquad \forall x\in\mathbb{R}^d.$$

But the left-hand expression is equal to

$$\int |\Phi(hD_y)^m \mathcal{K}_{A^*A}(x,y)|^2\, dy$$

and the same arguments imply that

$$|\mathcal{K}_{A^*A}(x,y)| \le CM^2 \bar\rho_1 \cdots \bar\rho_d h^{-d} \qquad \forall x,y \in \mathbb{R}^d. \tag{1.1.56}$$

On the other hand,

$$\mathcal{K}_{A^*A}(x,x) = \int \mathcal{K}_A^\dagger(x,y)\mathcal{K}_A(x,y)\, dy = \int \|\mathcal{K}_A(x,y)\|^2\, dy \tag{1.1.57}$$

since the range of $\mathcal{K}_A(x,y)$ is a matrix-column. This equality and (1.1.56) imply (1.1.54).

Moreover, (1.1.56) holds with A replaced by $A\phi(x)^l$ and hence the inequality for A remains true if we integrate with respect to $x \in \mathbb{R}^d$ provided $l < d/2$. This inequality and (1.1.57) imply (1.1.55). □

Theorem 1.1.12. *(i) Let A operator be an operator with a symbol a (simple or double) satisfying* (1.1.29) *with Hilbert-Schmidt rather than operator norm. Let $m > d/2$, $l > d/2$. Then A is Hilbert-Schmidt operator and its Hilbert-Schmidt norm does not exceed $CM(\bar\rho_1 \cdots \bar\rho_d \bar\gamma_1 \cdots \bar\gamma_d)^{1/2} h^{-d/2}$;*

(ii) Let A operator be an operator with a symbol a (simple or double) satisfying (1.1.53) *with trace norm rather than operator norm. Let $m > d$, $l > d$. Then A is trace class operator and its trace norm does not exceed $CM(\bar\rho_1 \cdots \bar\rho_d \bar\gamma_1 \cdots \bar\gamma_d) h^{-d}$.*

Proof. Assume first that $\min(\dim \mathbb{H}, \dim \mathbb{H}') = 1$. Then there is no difference between Hilbert-Schmidt, trace or operator norms for $\mathcal{L}(\mathbb{H}', \mathbb{H})$ operators.

Without any loss of the generality one can assume that $m = l$. Obviously (i) and (ii) hold as m is large enough. Further, both (i) and (ii) hold for $\Lambda_m = (1 + |x|^2 + h^2 D^2)^{-m/2}$. One can see easily that $A = B\Lambda_m + C$ where B is an operator with the symbol $a(1 + |x|^2 + |\xi|^2)^{m/2}$ and C is an operator with the symbol satisfying (1.1.53) with m, K replaced by $m+1, K-1$. Then "induction" with respect to m proves the Theorem in this restricted case.

The general case can be forced to the restricted one considering operators A_w with symbols $a(x,\xi)w$ with $w \in \mathbb{H}$, $\|w\| = 1$. □

Remark 1.1.13. All the conclusions of Proposition 1.1.11 and this Theorem remain true with constants not depending on $\bar{x}$, $\bar{\xi}$ if we replace $\Phi(\xi)$ and $\phi(x)$ by $\Phi(\xi-\bar{\xi})$ and $\phi(x-\bar{x})$ respectively, with arbitrary $(\bar{x},\bar{\xi})\in\mathbb{R}^{2d}$.

Note that the shift $x\to x-\bar{x}$, $\xi\to\xi-\bar{\xi}$ is equivalent to the transformation $A\in T^*AT$ with the operator T: $(Tu)(x)=e^{-i\langle x,\bar{\xi}\rangle}u(x+\bar{x})$ which preserves all the $\mathscr{L}^p$-norms.

Pseudodifferential Operators Applied to an Exponent

We need to study pseudodifferential operators applied to an exponent $u=f(x)e^{ih^{-1}\phi(x)}$ where currently

(1.1.58) *phase* $\phi(x)$ is a real-valued function,

$\phi(x)$ and *amplitude* $f(x)$ satisfy

$$|\partial^\alpha\phi|\le c\quad\forall\alpha:|\alpha|\le N+1,\quad |\partial^\alpha f|\le M_1\qquad\forall\alpha:|\alpha|\le N \tag{1.1.59}$$

where for f it is a norm in the auxiliary space $\mathbb{H}$.

Let us consider

$$\begin{aligned}a(\overset{2}{x},h\overset{1}{D})\bigl(f(x)e^{ih^{-1}\phi(x)}\bigr)=&\\ (2\pi h)^{-d}\iint a(x,\xi)f(y)e^{ih^{-1}[\langle x-y,\xi\rangle+\phi(y)]}\,d\xi dy=&\\ (2\pi h)^{-d}e^{ih^{-1}\phi(x)}\iint a(x,\xi)f(y)e^{ih^{-1}[\langle x-y,\xi\rangle+\phi(y)-\phi(x)]}\,d\xi dy&\end{aligned} \tag{1.1.60}$$

which after substitution $\xi\mapsto\nabla\phi(x)+\xi$ becomes

$$\begin{aligned}(2\pi h)^{-d}e^{ih^{-1}\phi(x)}\iint a(x,\nabla\phi(x)+\xi)f(y)e^{ih^{-1}[\langle x-y,\xi\rangle+\varphi(x,y)]}\,d\xi dy=&\\ e^{ih^{-1}\phi(x)}a(\overset{2}{x},\nabla\phi(x)+h\overset{1}{D}_y)e^{ih^{-1}\varphi(x,y)}\Big|_{y=x}&\end{aligned}$$

with

$$\varphi(y,x)=\phi(y)-\phi(x)-\langle y-x,\nabla\phi(x)\rangle. \tag{1.1.61}$$

If an original operator was h-differential operator of order m the last expression would be $e^{ih^{-1}\phi(x)}\sum_{n\le m} b_n(x)h^n$ where differentiating $e^{ih^{-1}\varphi(x,y)}$ ones we would need to differentiate factor $\big(\nabla\phi(y)-\nabla\phi(x)\big)$ at least ones to prevent getting 0 as $y=x$; so, at least one factor h would not be compensated by h^{-1} arising from the differentiation of exponent.

We can apply *stationary phase method*. We will learn it later in full but now we need its poor man's approach.

Step 1. Let us replace $\varphi(y,x)$ by

$$\varphi_N(x,y)=\sum_{\alpha:2\le|\alpha|\le N}\frac{1}{\alpha!}(y-x)^\alpha\partial_x^\alpha\phi(x). \tag{1.1.62}$$

Note that

$$e^{ih^{-1}\varphi(x,y)}-e^{ih^{-1}\varphi_N(x,y)}= ih^{-1}[\varphi(y)-\varphi_N(x,y)]\int_0^1 e^{ih^{-1}[t\varphi(y)+(1-t)\varphi_N(x,y)]} \tag{1.1.63}$$

and one can decompose $\phi(y)-\phi_N(x,y)$ into the sum $(y-x)^\alpha b_\alpha(x,y)$ with $|\alpha|=N+1$. Then replacing $(y-x)^\alpha$ by $(ih\partial_\xi)^\alpha$ applied to $e^{ih^{-1}\langle x-y,\xi\rangle}$ and integrating $N+1$ times by parts we will obviously get that this error is $O(h^{N-d})$.

Similarly, one can replace f by

$$f_{N-1}(x,y)=\sum_{\alpha:|\alpha|\le N-1}\frac{1}{\alpha!}(y-x)^\alpha\partial_x^\alpha f(x). \tag{1.1.64}$$

One can skip Step 1 assuming extra smoothness of ϕ and f.

Step 2. Now let us assume that ϕ and f are smooth enough (or we plug ϕ_N and f_{N-1} instead) and let us replace $f(y)e^{ih^{-1}\varphi(a,y)}$ by its Taylor decomposition

$$\sum_{\alpha:|\alpha|<L}\frac{1}{\alpha!}(y-x)^\alpha\partial_y^\alpha\big(f(y)e^{ih^{-1}\varphi(x,y)}\big)\Big|_{y=x}$$

with an error of the similar form but $|\alpha|=L$. Then the error will be

$$\sum_{\substack{k,\alpha:k\le L,\\ |\alpha|\ge N+(2k-L)_+}} c_{k\alpha}(x,y)(x-y)^\alpha h^{-k}e^{ih^{-1}\varphi(x,y)}$$

where L is large enough and $k_+ = \max(k,0)$. Replacing $(y-x)^\alpha$ by $(ih\partial_\xi)^\alpha$ applied to $e^{ih^{-1}\langle x-y,\xi\rangle}$ and integrating $|\alpha|$ times by parts we will obviously get that this error is $O(h^{|\alpha|-d-k})$ which is $O(h^{N-k})$ as $|\alpha| = L = 2N$.

Consider now "regular terms": $\partial_y^\alpha \big(f(y)e^{ih^{-1}\varphi(x,y)}\big)\big|_{y=x}$ which contain no exponent and lead to the answer

$$\sum_{\alpha:|\alpha|<2N} \frac{1}{\alpha!}(2\pi h)^{-d} \iint a(x,\nabla\phi(x)+\xi)(y-x)^\alpha e^{ih^{-1}\langle x-y,\xi\rangle}\times \big(\partial_y^\alpha f(y)e^{ih^{-1}\varphi(x,y)}\big)\big|_{y=x} d\xi\, dy = \sum_{\alpha:|\alpha|<2N} \frac{(-ih)^{|\alpha|}}{\alpha!} a^{(\alpha)}\big(x,\nabla\phi(x)\big)\big(\partial_y^\alpha f(y)e^{ih^{-1}\varphi(x,y)}\big)\big|_{y=x} e^{ih^{-1}\phi(x)} \tag{1.1.65}$$

where here and below

$$a^{(\alpha)} = \partial_\xi^\alpha a, \quad a_{(\beta)} = \partial_x^\beta a, \quad a_{(\beta)}^{(\alpha)} = \partial_\xi^\alpha \partial_x^\beta a.$$

So we arrive to

Theorem 1.1.14. *Let a satisfy* (1.1.25) *and ϕ, f satisfy* (1.1.58) *and* (1.1.59). *Then*

$$\|a(x, \overset{2}{h}D, \overset{1}{h}D)\big(f(x)e^{ih^{-1}\phi(x)}\big) - g(x)e^{ih^{-1}\phi(x)}\| \le CM_1 h^{N-d} \tag{1.1.66}$$

with

$$g(x) = \frac{(-ih)^{|\alpha|}}{\alpha!} a^{(\alpha)}(x,\nabla\phi(x))\,\partial_y^\alpha\big(f_N(y)e^{ih^{-1}\varphi_N(x,y)}\big)\big|_{y=x} \tag{1.1.67}$$

where for smooth enough f, ϕ one does not need to replace f, φ by f_N, φ_N.

Remark 1.1.15. Assuming instead of (1.1.58) that

$$\operatorname{Im}\phi(x) \ge 0 \tag{1.1.68}$$

one would be able to prove the same statement provided $a(x,\xi)$ is analytic with respect to ξ; otherwise one needs to make sense of $a(x,\nabla\phi(x))$. One can do it replacing $a(x,\xi)$ by its *quasianalytic* continuation from $\mathbb{R}^d$:

$$\tilde a(x,\xi) = \sum_{\beta:|\beta|\le M} \frac{1}{\beta!} a^{(\beta)}(x,\operatorname{Re}\xi)(\operatorname{Im}\xi)^\beta \tag{1.1.69}$$

and using fact that $|\nabla \operatorname{Im}\phi| \le c_0 |\operatorname{Im}\phi|^{1/2}$. Then (1.1.67) is preserved with $a^{(\alpha)}(x,\nabla\phi(x))$ replaced by $\tilde a^{(\alpha)}(x,\nabla\phi(x))$.

Remark 1.1.16. Note that

$$\partial_y^\alpha \bigl(f_N(y)e^{ih^{-1}\varphi_N(x,y)}\bigr)\big|_{y=x} = \sum_{\substack{m,\alpha_1:|\alpha_1|\ge 2,\dots,\alpha_m:|\alpha_m|\ge 2,\\ \beta:\alpha_1+\dots+\alpha_m+\beta=\alpha}} \kappa_{\alpha_1,\dots,\alpha_m,\beta}(\nabla_1^\alpha\phi)\cdots(\nabla_m^\alpha\phi)\nabla^\beta f\cdot h^{-m} \tag{1.1.70}$$

with absolute constants $\kappa_{\alpha_1,\dots,\alpha_m,\beta}$ and therefore in (1.1.66) one can take

$$g=\sum_{n<N} h^n \mathcal{L}_n f \tag{1.1.71}$$

where

$$\mathcal{L}_n=\sum \kappa_{\alpha_1,\dots,\alpha_m,\beta}\frac{i^{-|\alpha|}}{\alpha!}a_l^{(\alpha)}(x,\nabla\phi)(\nabla_1^\alpha\phi)\cdots(\nabla_m^\alpha\phi)\nabla^\beta \tag{1.1.72}$$

with the sum over set

$$\bigl\{l,\alpha,m,\alpha_1,\dots,\alpha_m,\beta:|\alpha_1|\ge 2,\dots,|\alpha_m|\ge 2,\ \alpha_1+\dots+\alpha_m+\beta=\alpha,|\alpha|=m+n+l\bigr\} \tag{1.1.73}$$

(which implies that $\frac{1}{2}|\alpha|+|\beta|\le n-l$). In particular

$$\mathcal{L}_0 = a_0(x,\nabla\phi), \tag{1.1.74}$$

$$\begin{aligned}\mathcal{L}_1 &= -i\sum_j a_0^{(j)}(x,\nabla\phi)\partial_j - \frac{i}{2}\sum_{j,k} a_0^{(jk)}(x,\nabla\phi)(\partial_j\partial_k\phi)+a_1(x,\nabla\phi)\\ &= -i\sum_j a_0^{(j)}(x,\nabla\phi)\partial_j - \frac{i}{2}\sum_j\bigl(\partial_j a_0^{(j)}(x,\nabla\phi)\bigr)+a^s(x,\nabla\phi)\end{aligned} \tag{1.1.75}$$

(recall that a^s is subprincipal symbol of operator).

For ϕ satisfying (1.1.68) rather than (1.1.58) one should replace a by $\tilde{a}$.

1.1.2 Analysis of Pseudodifferential Operators

Introduction

We need to generalize our results to symbols which do not satisfy any decay as $|\xi|\to\infty$ conditions. Namely we assume that (1.1.29) in the matrix sense

is satisfied with $s = 0$:

$$\|\partial_x^\alpha \partial_y^\beta \partial_\xi^\nu a(x,\xi,y)\| \le M \qquad \forall \alpha,\beta,\nu : |\alpha|+|\beta| \le N+K, |\nu| \le N+K \tag{1.1.76}$$

where $\|.\|$ is the norm in $\mathcal{L}(\mathbb{H}', \mathbb{H})$ and define operators $a(\overset{3}{x}, \overset{2}{hD}, \overset{1}{x})$ by (1.1.17) while for ordinary symbols satisfying (1.1.76)

$$\|\partial_x^\alpha \partial_\xi^\nu a(x,y,\xi)\| \le M \qquad \forall \alpha,\nu : |\alpha| \le N+K, |\nu| \le N+K \tag{1.1.77}$$

we define $a(\overset{2}{x}, \overset{1}{hD})$, $a(\overset{1}{x}, \overset{2}{hD})$ and $a^{\mathsf{w}}(x, hD)$ by (1.1.9), (1.1.10) and (1.1.13) respectively. Here $K = K(d)$ is large enough.

In principle, according to Section 18.4 of L. Hörmander [1], the different types of quantization are unnecessary; namely an operator $a = a(\overset{3}{x}, \overset{2}{hD}, \overset{1}{x})$ can be rewritten in each form $a_{(1)}(\overset{2}{x}, \overset{1}{hD})$, $a_{(2)}(\overset{1}{x}, \overset{2}{hD})$, $a^{\mathsf{w}}_{(3)}(x, hD)$ with the symbols $a_{(j)}$ are given by formulas

$$a_{(1)}(x,\xi) = \Gamma_x e^{ih\langle D_y, D_\xi\rangle} a(x,\xi,y), \tag{1.1.78$_1$}$$

$$a_{(2)}(x,\xi) = \Gamma_x e^{-ih\langle D_x, D_\xi\rangle} a(x,\xi,y), \tag{1.1.78$_2$}$$

$$a_{(3)}(x,\xi) = \Gamma_x e^{\frac{ih}{2}\langle D_y - D_x, D_\xi\rangle} a(x,\xi,y), \tag{1.1.78$_3$}$$

where $e^{ih\langle D_y, D_\xi\rangle}$ etc are defined explicitly through corresponding Fourier transforms and Γ_x is the operator of restriction to the diagonal $\{x = y\}$.

Moreover, due to the similar arguments it is not necessary to introduce operators with multiple symbols:

$$A = \int \cdots \int \hat{a}(\alpha_{(1)}, \dots, \alpha_{(n+1)}, \beta_{(1)}, \dots, \beta_{(n)}) \times e^{i\langle \alpha_{(1)}, x\rangle} e^{i\langle \beta_{(1)}, hD\rangle} \cdots e^{i\langle \beta_{(n)}, hD\rangle} e^{i\langle \alpha_{(n+1)}, x\rangle} d\alpha_{(1)} \dots d\alpha_{(n+1)} d\beta_{(1)} \dots d\beta_{(n)} \tag{1.1.79}$$

where $a(x_{(1)}, \dots, x_{(n+1)}, \xi_{(1)}, \dots, \xi_{(n)})$ is a multiple symbol, $\hat{a}$ its complete Fourier transform (with respect to all variables) and $x_{(j)}, \xi_{(j)}, \alpha_{(j)}, \beta_{(j)} \in \mathbb{R}^d$.

The following simple exact formulae are useful:

Lemma 1.1.17. *(i) Let $b(x,\xi)$ be a linear scalar symbol. Then for an arbitrary symbol a and any type of quantization*

$$[\mathsf{Op}(a), \mathsf{Op}(b)] = -ih\,\mathsf{Op}(\{a,b\}) \tag{1.1.80}$$

where $[.,.]$ is the commutator and $\{.,.\}$ is Poisson bracket.

(ii) Moreover, for Weyl quantization

$$\mathsf{Op}^{\mathrm{w}}(a)\,\mathsf{Op}^{\mathrm{w}}(b)+\mathsf{Op}^{\mathrm{w}}(b)\,\mathsf{Op}^{\mathrm{w}}(a)=\mathsf{Op}^{\mathrm{w}}(ab+ba) \tag{1.1.81}$$

for a linear symbol b; further (1.1.80) *holds even for a quadratic scalar symbol b.*

Proof. By direct calculation. □

Expressions $(1.1.78)_1$–$(1.1.78)_3$ are precise but non-local with respect to the corresponding variables. To remedy this we need

Proposition 1.1.18. *Suppose that $a(x,y,\xi)$ satisfies* (1.1.76)*. Then*

(i) For $h\in(0,1]$ symbols $a_{(j)}$ defined by $(1.1.78)_1$-$(1.1.78)_3$ satisfy (1.1.77) *with M replaced by CM with $C=C(d)$ and K replaced by $K/2$.*

(ii) Further, as in Theorem 1.1.4 for every $L\le N-1$ symbols $\bigl(a_{(j)}-a_{(j,L)}\bigr)$ satisfy (1.1.77) *with N, M, and K replaced by $(N-L)$, CMh^L with $C=C(L,d)$ and $K/2$ respectively where $a_{(j,L)}$ are b'_L, b''_L and b_L defined by* (1.1.24), (1.1.26) *and* (1.1.28) *respectively.*

Proof. Proof follows immediately from Theorem 18.4.10 and formula (18.4.8) of L. Hörmander [1] or alternatively from (1.A.5). Note that expressions for $a_{(j),L}$ are just Taylor decompositions (without remainder terms) for corresponding exponents. □

Estimates

Now we need to prove the estimates for operator norms:

Theorem 1.1.19. *Let* (1.1.76) *be fulfilled with $N=0$, $K>2d$. Then for $h\in(0,1]$ the $\mathcal{L}\bigl(\mathscr{L}^2(\mathbb{R}^d,\mathbb{H}'),\mathscr{L}^2(\mathbb{R}^d,\mathbb{H})\bigr)$-norm of operator $A=a(\overset{3}{x},\overset{2}{hD},\overset{1}{x})$ does not exceed CM.*

Proof. Proof follows immediately Theorem 18.6.2 of L. Hörmander [1]. However let us sketch it. While it is sufficient to cover only case $h=1$ (otherwise one can apply rescaling $x\mapsto xh$ [2].) Let us conduct proof for any $h\in(0,1]$.

[2] I.e. $x_{\mathsf{new}}=x_{\mathsf{old}}h^{-1}$.

Let us consider partition of unity in $\mathbb{R}^d_\xi$:

$$\sum_\mu \psi_\mu^2 = 1 \tag{1.1.82}$$

where

(1.1.83) ψ_μ are real-valued, supported in $B_\mu = \{|\xi - \xi_\mu| \le 1\}$ and such that $|\partial^\alpha \psi_\mu| \le c_\alpha \ \forall \alpha, \mu$

and moreover

(1.1.84) Multiplicity of covering $\mathbb{R}^d$ by $\bar{B}_\mu = \{|\xi - \xi_\mu| \le 2\}$ does not exceed $C = C(d)$ [3)].

Then

$$\|Au\|^2 = \sum_\mu \|\psi_\mu(hD)Au\|^2 = \sum_\mu \|\psi_\mu(hD)\sum_{\nu,\lambda} A_\nu \phi_\lambda(hD)u\|^2 \tag{1.1.85}$$

with $\phi_\lambda = \psi_\lambda^2$ and $A_\nu = (a\phi_\nu)(\overset{3}{x}, \overset{2}{hD}, \overset{1}{x})$; so in fact we quantize symbol $a(x,\xi,y)\phi_\nu(\xi)$.

Note first that the operator norm of A_ν does not exceed CM because its symbol satisfies (1.1.76) and is supported in $\{(x,\xi) : \xi \in B_\nu\}$; if $\xi_\nu = 0$ it would be in the framework of Theorem 1.1.3 but one can either notice that all arguments of the proof of this theorem remain valid or just notice that

$$e^{-ih^{-1}\langle x,\eta\rangle} b(\overset{3}{x}, \overset{2}{hD}, \overset{1}{x}) e^{ih^{-1}\langle x,\eta\rangle} = b(\overset{3}{x}, \overset{2}{hD} + \eta, \overset{1}{x}) \tag{1.1.86}$$

(so one can shift support to $\{|\xi| \le 1\}$).

Let us prove that

(1.1.87) Operator norm of $\psi_\mu A_\nu$ does not exceed $CM(1 + |\xi_\mu - \xi_\nu|)^{-K}$.

[3)] That means that no point belongs to more than L of these sets. One can prove existence of such partition easily.

Really, let for simplicity $\bar{\xi} = \xi_\mu$, $\bar{\eta} = \xi_\nu$ and subscript denote coordinate. Note that

$$(\bar{\xi}_j - \bar{\eta}_j)\psi_\mu(hD)A_\nu = \psi'_\mu(hD)A_\nu + \psi_\mu(hD)A'_\nu \tag{1.1.88}$$

where $\psi'_\mu(\xi) = (\bar{\xi}_j - \xi_j)\psi_\mu(\xi)$ and A'_ν is a quantization of

$$\Big(a(x,\xi,y)(\xi_j - \bar{\eta}_j) - ih\partial_{x_j}a(x,\xi,y)\Big)\phi_\lambda(\xi)$$

and therefore operator norm of each term in the right-hand expression of (1.1.88) does not exceed CM. Thus (1.1.87) is proven for $K = 1$. Repeating these arguments we can prove it for any K.

Similarly,

(1.1.89) Operator norm of $\psi_\mu A_\nu \psi_\lambda$ does not exceed

$$CM(1 + |\xi_\mu - \xi_\nu| + |\xi_\nu - \xi_\lambda|)^{-K}.$$

Then as $K > d$

$$\begin{aligned}\sum_\nu \|\psi_\mu(hD)A_\nu\phi_\lambda(hD)u\| \le{}& \\ CM\sum_\nu \big(1 + |\xi_\mu - \xi_\nu| + |\xi_\nu - \xi_\lambda|\big)^{-K}&\|\psi_\lambda(hD)u\| \le \\ CM\big(1 + |\xi_\mu - \xi_\lambda|\big)^{d-K}&\|\psi_\lambda(hD)u\|\end{aligned} \tag{1.1.90}$$

and plugging into (1.1.85) we get

$$\begin{aligned}\|Au\|^2 \le C^2M^2\sum_\mu\Big(\sum_\lambda (1 + |\xi_\mu - \xi_\lambda|)^{d-K}\|\psi_\lambda(hD)u\|\Big)^2 \le& \\ \sum_{\lambda,\mu,\nu}\big(1 + |\xi_\mu - \xi_\lambda|\big)^{d-K}\big(1 + |\xi_\mu - \xi_\nu|\big)^{d-K}\|\psi_\lambda(hD)u\|\cdot\|\psi_\nu(hD)u\| \le& \\ C^2M^2\sum_{\lambda,\nu}\big(1 + |\xi_\nu - \xi_\lambda|\big)^{d-K}\|\psi_\lambda(hD)u\|^2 \le& \\ C^2M^2\sum_\lambda \|\psi_\lambda(hD)u\|^2 = C^2M^2\|u\|^2&\end{aligned} \tag{1.1.91}$$

provided $K > 2d$; between the second and the third lines we summed with respect to μ and estimated $\|\psi_\lambda(hD)u\|\cdot\|\psi_\nu(hD)u\|$ by the half-sum of squares; then in one part we permuted λ and ν.

So, we proved that $\|Au\| \le CM\|u\|$ for any u. □

Remark 1.1.20. Actually we need a bit weaker assumption than (1.1.76) with $N = 0$, $K > 2d$; it is enough to assume that the corresponding estimate holds as $|\alpha| \le K$, $|\beta| \le K$, $|\nu| \le K$ with $K > d$. Then in (1.1.89) we get

$$CM(1 + |\xi_\mu - \xi_\nu|)^{-K}(1 + |\xi_\nu - \xi_\lambda|)^{-K}.$$

This remark holds for all statements of this section.

Remainder Estimates

Theorem 1.1.19 and Proposition 1.1.18(ii) immediately imply

Theorem 1.1.21. *Let* (1.1.76) *be fulfilled with* $K > 2d$. *Then for* $h \in (0, h_0]$ *and* $L \le N - 1$ *the* $\mathcal{L}\big(\mathscr{L}^2(\mathbb{R}^d, \mathbb{H}'), \mathscr{L}^2(\mathbb{R}^d, \mathbb{H})\big)$*-norm of operator* $\mathsf{Op}\big(a_{(j)} - a_{(j,L)}\big)$ *with* $a_{(j,L)}$ *defined by* (1.1.24), (1.1.26) *and* (1.1.28) *does not exceed* CMh^L.

Theorem 1.1.22. *Let* a *and* b *be* $\mathcal{L}(\mathbb{H}'', \mathbb{H}')$*- and* $\mathcal{L}(\mathbb{H}', \mathbb{H})$*-valued ordinary symbols satisfying* (1.1.77) *with* $M = M_a$ *and* $M = M_b$ *respectively. Let* $a(x, hD)$, $b(x, hD)$ *and* $(a \bullet_L b)(x, hD)$ *be quantizations of symbols* a, b *and* $a \bullet_L b$ *where* $L \le N - 1$ *and here and in what follows* $a \bullet_{pq,L} b$, $a \bullet_{qp,L} b$, $a \bullet_{\mathsf{w},L} b$ *are defined by* $(1.1.37)_1$–$(1.1.37)_3$ *for* qp*-,* pq*- and Weyl quantization respectively.*

Then the $\mathcal{L}\big(\mathscr{L}^2(\mathbb{R}^d, \mathbb{H}''), \mathscr{L}^2(\mathbb{R}^d, \mathbb{H})\big)$*-norm of the operator*

$$a(x, hD)b(x, hD) - (a \bullet_L b)(x, hD)$$

does not exceed $Ch^L M_a M_b$.

Proof. Proof follows immediately from Proposition 1.1.18(ii) and Theorem 1.1.16. □

Remark 1.1.23. Again there are sharp but non-local formulae

$$(1.1.92)_1 \qquad a \bullet_{qp} b = \Gamma_{x,\xi} e^{ih\langle D_y, D_\xi\rangle}\big(a(x, \xi)b(y, \eta)\big),$$

$$(1.1.92)_2 \qquad a \bullet_{pq} b = \Gamma_{x,\xi} e^{-ih\langle D_x, D_\eta\rangle}\big(a(x, \xi)b(y, \eta)\big),$$

$$(1.1.92)_3 \qquad a \bullet_{\mathsf{w}} b = \Gamma_{x,\xi} e^{\frac{ih}{2}\langle D_y, D_\xi\rangle - \frac{ih}{2}\langle D_x, D_\eta\rangle}\big(a(x, \xi)b(y, \eta)\big),$$

with $\Gamma_{x,\xi}$ is the operator of restriction to the diagonal $\{x = y, \xi = \eta\}$.

Classes of Symbols and Pseudodifferential Operators

From now on we in use

Definition 1.1.24. (i) Denote by $\mathsf{S}_{h,N}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{H}', \mathbb{H})$ and $\mathsf{S}_{h,N}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{H}', \mathbb{H})$ classes of double and ordinary symbols in the form (1.1.30)

$$a(x,\xi,y) = \sum_{j=0}^{N-1} a_j(x,y,\xi)h^j$$

with a_j satisfying condition (1.1.76) with N replaced by $(N-j)$ and with some constant M.

(ii) Denote by $\mathbf{\Psi}_{h,N}(\mathbb{R}^d, \mathbb{H}', \mathbb{H})$ the class of operators obtained from double symbols $a \in \mathsf{S}_{h,N}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ by quantization; operators of this class are defined modulo negligible i.e. with the operator norm not exceeding Mh^N.

(iii) The smallest M such that (1.1.76) with N replaced by $(N-j)$ holds for all $j = 0, \dots, N$ and (ii) also holds, the norm of A: $\|\!|A|\!\| = \|A\|_{\mathbf{\Psi}}$.

We preserve the notions of the principal and subprincipal symbol.

Remark 1.1.25. Discussing trace or Hilbert-Schmidt operators we need to use trace or Hilbert-Schmidt norms both in (1.1.76) and for negligible operator estimate as well. Discussing compact operators (see below) we need to assume that both symbols and negligible operator are compact.

Compactness

Theorem 1.1.26. *Let A operator be an operator with a symbol a (simple or double) satisfying* (1.1.76) *and*

$$\|D_x^\alpha D_y^\beta D_\xi^\nu a(x,\xi,y)\| \le \epsilon(R) \qquad \forall (x,\xi,y) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d : |x| + |y| + |\xi| \ge R \qquad \forall \alpha, \beta, \nu : |\alpha| \le K, |\beta| \le K, |\nu| \le K \tag{1.1.93}$$

$\epsilon(R) \to 0$ *as* $R \to \infty$, $N \ge 0$.

Further, let us assume that $a(x,\xi,y)$ is a compact operator for each (x,y,ξ).

Then A is compact operator.

Proof. We need to prove that one can approximate A by compact operators (in operator norm).

Let A_R be an operator with the symbol $a\varphi_R$ with

$$\varphi_R = \chi\big(\frac{1}{R^2}(|x|^2+|y|^2+|\xi|^2)\big),$$

$\chi \in \mathscr{C}_0^\infty(\mathbb{R})$ equal 1 on $(-1,1)$. Note that $(A-A_R)$ is a quantization of the symbol $a(1-\varphi_R)$ which satisfies (1.1.76) with M replaced by $C\epsilon(R)$. Therefore its operator norm does not exceed $C\epsilon(R)$.

Let P_n be a sequence of finite-dimensional projectors in $\mathbb{H}$ converging to I strongly (i.e. $P_n w \to w$ for each $w \in \mathbb{H}$. Then $a(x,y,\xi)P_n \to a(x,y,\xi)$ in operator norm for each (x,y,ξ) and due to (1.1.76)

$$\|D_x^\alpha D_y^\beta D_\xi^\nu a(x,\xi,y)(I-P_n)\| \le \epsilon_R(n) \qquad \forall (x,\xi,y)\in \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d : |x|+|y|+|\xi|\le R \qquad \forall \alpha,\beta,\nu : |\alpha|\le K/2, |\beta|\le K/2, |\nu|\le K/2$$

with $\epsilon_R(n)\to 0$ as $n\to\infty$. Similar inequalities then hold for $a\varphi_R$. Therefore operator norm of $A_R(I-P_n)$ does not exceed $C\epsilon_R(n)$.

On $A_R P_n$ is Hilbert-Schmidt (and even the trace class) operator due to Theorem 1.1.12.

Therefore operator A can be approximated in operator norm by compact operators $A_R P_n$ and therefore is compact as well. □

Rescaling

We will need to consider operators with symbols which could be reduced to those considered before by rescaling $x \mapsto x\boldsymbol{\gamma}^{-1}$, $\xi \mapsto \xi\boldsymbol{\rho}^{-1}$ with $\boldsymbol{\rho} = (\rho_1,\dots,\rho_d)$ and $\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_d)$ which means $x_j \mapsto x_j\gamma_j^{-1}$, $\xi_j \mapsto \xi_j\rho_j^{-1}$ and $h \mapsto \boldsymbol{h} = (h_1,\dots,h_d)$ with $h_j = h/(\rho_j\gamma_j)$ [4]. To reach after this rescaling (1.1.76) or (1.1.77) with $h\le 1$ we need to assume that

$$\|D_x^\alpha D_y^\beta D_\xi^\nu a\| \le M\boldsymbol{\rho}^{-\nu}\boldsymbol{\gamma}^{-\alpha-\beta} \quad \forall (x,y,\xi)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d \qquad \forall \alpha,\beta,\nu : |\alpha|+|\beta|\le N+K, |\nu|\le N+K \tag{1.1.94}$$

or

$$\|\partial_x^\alpha\partial_\xi^\nu a(x,y,\xi)\| \le M\boldsymbol{\rho}^{-\nu}\boldsymbol{\gamma}^{-\alpha} \qquad \forall \alpha,\nu : |\alpha|\le N+K, |\nu|\le N+K \tag{1.1.95}$$

[4] I.e. $x_{j,\mathsf{new}} = x_{j,\mathsf{old}}\gamma_j^{-1}$ etc.

respectively with

$$\hbar = \max_j h/(\rho_j\gamma_j) \le 1 \qquad (\iff \rho_j\gamma_j \ge h). \tag{1.1.96}$$

Remark 1.1.27. (i) Under these assumptions Theorems 1.1.21, 1.1.22 remain true with the remainder estimate $CM\hbar^L$. Condition (1.1.96) is known as *an uncertainty principle*.

(ii) To get negligible remainder estimate $O(h^s)$ one should assume that $\hbar = O(h^\delta)$ with an arbitrarily small exponent $\delta > 0$ i.e.

$$\rho_j\gamma_j \ge h^{1-\delta}. \tag{1.1.97}$$

Condition (1.1.97) is known as *a (standard) microlocal uncertainty principle*. Later we will introduce *a logarithmic uncertainty principle*.

Remark 1.1.28. Using the same methods one can generalize Theorem 1.1.22 to the case when a, b satisfy (1.1.95) with M_a, ρ_a, γ_a and M_b, ρ_b, γ_b respectively such that $\rho_{a,j}\gamma_{a,j} \ge h$, $\rho_{b,j}\gamma_{b,j} \ge h$ $\forall j$ and with exactly the same methods of quantizations of all symbols; in this case an operator norm of the remainder does not exceed $C\hbar^L M_a M_b$ with

$$\hbar = h \max_{1\le j\le d} \frac{1}{\rho_{a,j}\gamma_{b,j}}, \tag{1.1.98}_1$$

$$\hbar = h \max_{1\le j\le d} \frac{1}{\rho_{b,j}\gamma_{a,j}}, \tag{1.1.98}_2$$

$$\hbar = h \max_{1\le j\le d} \Big(\frac{1}{\rho_{a,j}\gamma_{b,j}} + \frac{1}{\rho_{b,j}\gamma_{a,j}}\Big) \tag{1.1.98}_3$$

for qp-, pq- and Weyl quantizations respectively.

Gårding Inequalities

By *Gårding inequalities* we call statements of the type "if symbol is positive then operator is (almost) positive". The following theorem will be our workhorse:

Theorem 1.1.29. *Let a be an $\mathcal{L}(\mathbb{H},\mathbb{H})$-valued Hermitian symbol satisfying* (1.1.77) *with $N = 0$ and such that*

$$\langle a(x,\xi)w, w\rangle \ge 0 \qquad \forall (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \quad \forall w \in \mathbb{H} \tag{1.1.99}$$

where here $\langle .,. \rangle$ *means the inner product in* $\mathbb{H}$. *Then for* $h \in (0, h_0]$

$$(a^{\mathsf{w}}(x, hD)u, u) \geq -CMh\|u\|^2 \qquad \forall u \in \mathscr{L}^2(\mathbb{R}^d, \mathbb{H}) \tag{1.1.100}$$

where here and below $(.,.)$ *and* $\|.\|$ *mean the inner product and the norm in* $\mathscr{L}^2(\mathbb{R}^d, \mathbb{H})$ *and* $h_0 = \mathsf{min}_{1\leq j\leq d}\, \rho_j\gamma_j$.

Proof. This is an adapted proof of Theorem 18.6.4 L. Hörmander [1]. Without any loss of the generality one can assume that $M = 1$.

First of all we need a really fine partitions of unity. Consider

$$1 = \sum_{\nu} \phi_\nu^2(x), \quad 1 = \sum_{\mu} \psi_\mu^2(\xi) \tag{1.1.101}$$

where all functions are real, supported in $B(x_\nu, 1)$ or $B(\xi_\mu, 1)$ which are balls of radius 1 with the centers at the lattice points x_ν, ξ_μ and such that

$$|\partial_x^\alpha \phi_\nu| \leq c_\alpha, \quad |\partial^\alpha \psi_\mu| \leq c_\alpha \qquad \forall \alpha, \nu, \mu \tag{1.1.102}$$

uniformly with respect to x, ξ, ν, μ.

Let us introduce $\hat{\phi}_\nu = \phi_\nu(h^{-\frac{1}{2}}x)$, $\hat{\psi}_\mu = \psi_\mu(h^{\frac{1}{2}}D)$, and consider equality

$$\begin{aligned}(Au, u) = \sum_{\nu} \big(\hat{\phi}_\nu^2 Au, u\big) = \\ \sum_{\nu} \big(\hat{\phi}_\nu Au, \hat{\phi}_\nu u\big) = \sum_{\nu} \big(A\hat{\phi}_\nu u, \hat{\phi}_\nu u\big) - \sum_{\nu} \big(\hat{\phi}_\nu [A, \hat{\phi}_\nu]u, u\big).\end{aligned} \tag{1.1.103}$$

Note that operator $[A, \hat{\phi}_\nu]$ is skew-symmetric, and therefore

$$\big(\hat{\phi}_\nu [A, \hat{\phi}_\nu]\big)^* + \hat{\phi}_\nu [A, \hat{\phi}_\nu] = -\frac{1}{2}\big[[A, \hat{\phi}_\nu], \hat{\phi}_\nu\big] \tag{1.1.104}$$

and (1.1.103) becomes

$$(Au, u) = \sum_{\nu} \big(A\hat{\phi}_\nu u, \hat{\phi}_\nu u\big) + \frac{1}{2}\sum_{\nu} \big(\big[[A, \hat{\phi}_\nu], \hat{\phi}_\nu\big]u, u\big). \tag{1.1.105}$$

We claim that

(1.1.106) *Operator norm of* $\sum_\nu \big[[A, \hat{\phi}_\nu], \hat{\phi}_\nu\big]$ *does not exceed* Ch.

On the heuristic level, calculating commutators formally, one can notice that every time a factor $h^{\frac12}$ appears, so the second commutator has an extra factor h.

To justify (1.1.106) if $a(x,\xi)$ is fast-decaying with respect to ξ one can notice that the Schwartz kernel of the second commutator is

$$\mathcal{K}_{[[A,\hat\phi_\nu],\hat\phi_\nu]}(x,y) = \mathcal{K}_A(x,y)\bigl(\phi_\nu(h^{-\frac12}x) - \phi_\nu(h^{-\frac12}y)\bigr)^2 \tag{1.1.107}$$

and therefore for $K \ge 2$

$$\sum_\nu |\mathcal{K}_{[[A,\phi_\nu],\phi_\nu]}(x,y)| \le \\ C(1+h^{-1}|x-y|)^{-d-3}\times h^{-1}|x-y|^2 \le Ch(1+h^{-1}|x-y|)^{-d-1}.$$

The proof in the general situation repeats one of Theorem 1.1.9.

After (1.1.106) is proven, (1.1.105) implies

$$(Au,u) \ge \sum_\nu (A\hat\phi_\nu u, \hat\phi_\nu u) - Ch\|u\|^2 \tag{1.1.108}$$

and therefore it is sufficient to prove (1.1.100) for u replaced by $\hat\phi_\nu u$. The same arguments work for the second partition. Therefore it is sufficient to prove (1.1.100) for u replaced by $\hat\psi_\mu\hat\phi_\nu u$.

Remark 1.1.30. We used the finest x- and ξ- partitions for which (1.1.106) is still valid.

Furthermore, now one can replace $a(x,hD)$ by its approximation on support of $\hat\psi_\mu\hat\phi_\nu$:

$$\bar a(x,hD) = \\ a(\bar x,\bar\xi) + \sum_j \bigl(\partial_{x_j}a\bigr)(\bar x,\bar\xi)(x_j-\bar x_j) + \bigl(\partial_{\xi_j}a\bigr)(\bar x,\bar\xi)(hD_j-\bar\xi_j) + \\ C_0|x-\bar x|^2 + C_0|hD-\bar\xi|^2 \tag{1.1.109}$$

where overbar replaces index $_{\mu\nu}$. Really, the quadratic term of the Taylor decomposition is $O(h)$ on the support of $\hat\phi_\nu$, $\hat\psi_\mu$. To justify it we can freeze $x = x_\nu$ first (before introduction ψ_μ), and $hD = \xi_\mu$ after it. Proofs repeat one of Theorem 1.1.19.

Consider first the case of the scalar operator a. Then one can rewrite quadratic operator $\bar{a}(x, hD)$ as

$$\bar{a}(x, hD) = C_0|hD - \eta - \bar{\xi}|^2 + C_0|x - y - \bar{x}|^2 + b$$

with

$$\eta = -\frac{1}{2C_0}\nabla_\xi a, \quad y = -\frac{1}{2C_0}\nabla_x a, \quad b = a - \frac{1}{4C_0}(|\nabla_x a|^2 + |\nabla_x a|^2)$$

calculated at $x = \bar{x}$, $\xi = \bar{\xi}$.

Now without any loss of the generality one can assume that $\bar{x} + y = \bar{\xi} + \eta = 0$ (otherwise we can make a shift). So, we have a harmonic oscillator $C_0(|x|^2 + |hD|^2)$ plus b. Here $b \ge 0$ (provided we picked up a large enough constant C_0) due to the known fact that if the second derivatives of f are bounded and f is nonnegative then $f \ge C_0|\nabla f|^2$.

On the other hand, harmonic oscillator is non-negative operator; in fact its lower bound is Cdh; we will analyze it later. So, our reduced operator $\bar{a}(x, hD)$ satisfies (1.1.100) and thus our operator a satisfies it as well. Later we reexamine scalar case to improve sharp Gårding inequality.

Consider general (matrix) case now. Then

$$\bigl\langle \bigl(a(\bar{x}, \bar{\xi}) + L(\bar{x}, \bar{\xi}; y, \eta)\bigr)v, v\bigr\rangle + c(|y|^2 + |\eta|^2)\|v\|^2 \ge 0 \qquad \forall v \in \mathbb{H} \quad \forall y, \eta \in \mathbb{R}^d \tag{1.1.110}$$

where

$$L(\bar{x}, \bar{\xi}; y, \eta) = \langle (\nabla_x a)(\bar{x}, \bar{\xi}), y\rangle + \langle (\nabla_\xi a)(\bar{x}, \bar{\xi}), \eta\rangle.$$

We can plug $2y, 0$ instead of y, ξ. Then we L becomes an operator of multiplication and then

$$\bigl(\bigl(a(\bar{x}, \bar{\xi}) + 2L(\bar{x}, \bar{\xi}; x - \bar{x}, 0)\bigr)u, u\bigr) + 4c\|xu\|^2 \ge 0;$$

here u is an arbitrary $\mathbb{H}$-valued function. Plugging in (1.1.110) $0, 2\eta$ instead of y, η and making h-Fourier transform (recall that such transform does not break Weyl quantization) we will get

$$\bigl(\bigl(a(\bar{x}, \bar{\xi}) + 2L(\bar{x}, \bar{\xi}; 0, hD - \bar{\xi})\bigr)u, u\bigr) + 4c\|hDu\|^2 \ge 0.$$

Adding two last inequalities we will arrive to

$$(\bar{a}(x, hD)u, u) \ge 0$$

as long as $C \ge 4c$. It concludes the proof. □

In the scalar case (i.e., for $\mathbb{H} = \mathbb{C}$) one can prove a better estimate.

Theorem 1.1.31. *Let a be a scalar real valued symbol satisfying* (1.1.77) *with $N = 0$ and such that*

$$a(x,\xi) \ge 0. \tag{1.1.111}$$

Then

$$(a^{\mathrm{w}}(x,hD)u,u) \ge -CMh^2\|u\|^2 \qquad \forall u \in \mathscr{L}^2(\mathbb{R}^d). \tag{1.1.112}$$

Proof. This is very precise result and it looks like no tight partition of unity can save us because if we take scale h^δ the double commutator has the norm $\asymp h^{2-2\delta}$ which will be too large for any $\delta > 0$. However, we will run rescaling method. The proof is really difficult and idea rich.

We will follow the proof of Theorem 18.6.8 of L. Hörmander [1]. We will run induction with respect to d.

Step 1. Non-degenerate case. First, let us consider ball $B(\bar z,\epsilon)$ of radius ϵ with the center 0 in $\mathbb{R}^{2d}$ where $\epsilon > 0$ is a small enough constant. If $a(\bar z) \ge \kappa$ then $a(z) \ge \kappa/2$ in $B(\bar z,\epsilon)$ where $\kappa > 0$ is an arbitrarily small constant and $\epsilon = \epsilon(\kappa) > 0$ and then

$$(a^{\mathrm{w}}(x,hD)\phi(x,hD)u, \phi(x,hD)u) \ge -C_0h^2\|u\|^2 \tag{1.1.113}$$

provided

$$\operatorname{supp}(\phi) \subset B(\bar z,r), \qquad |\nabla^\alpha\phi| \le c_\alpha r^{-|\alpha|} \qquad \forall \alpha \tag{1.1.114}$$

with $r = \epsilon$.

Assume now that $|\nabla^2 a|(\bar z) = 1$ and $a(\bar z) < \kappa$ with a small enough constant $\kappa > 0$. Then without any loss of the generality one can assume that

$$\partial^2_{\xi_1} a(\bar z) \ge \kappa. \tag{1.1.115}$$

Really, if one of the second derivatives with respect to ξ_j is greater than κ by absolute value we can reach our inequality by rotation (and multiplication of a by some constant); if it is true for one of the second derivatives with respect to x_j can reach the previous case by rotation and h-Fourier transform; if one of the mixed x,ξ-derivatives we can reach the previous case by gauge transformation:

$$a^{\mathrm{w}}(x,hD) \mapsto e^{-\frac{i}{2}h^{-1}\langle Qx,x\rangle} a^{\mathrm{w}}(x,hD) e^{\frac{i}{2}h^{-1}\langle Qx,x\rangle}$$

with symmetric real matrix Q, transforming the symbol $a(x,\xi)$ to $a(x,\xi+Qx)$. One can prove easily that

$$e^{-\frac{i}{2}h^{-1}\langle Qx,x\rangle}a^{\mathrm{w}}(x,hD)e^{\frac{i}{2}h^{-1}\langle Qx,x\rangle} = b^{\mathrm{w}}(x,hD), \qquad b(x,\xi)=a(x,\xi+Qx). \tag{1.1.116}$$

Then under assumption (1.1.115) in $B(\bar{z},\epsilon)$

$$a(x,\xi) = b(x,\xi)^2 + a_1(x,\xi'), \qquad x'=(x_2,\dots,x_d),\ \xi'=(\xi_2,\dots,\xi_d) \tag{1.1.117}$$

where $a_1 = a(x,t(x,\xi'),\xi') \ge 0$ and $\xi_1 = t(x,\xi')$ is a solution to equation $\partial_{\xi_1} a = 0$ and a, b are real-valued. Further, one can prove easily that derivatives of a_1, b up to order k are estimated by derivatives of a up to order $k+2$ (multiplied by a some constant).

Then

$$a^{\mathrm{w}}(x,hD) \equiv b^{\mathrm{w}}(x,hD)^* b^{\mathrm{w}}(x,hD) + a_1^{\mathrm{w}}(x,hD') + h^2\boldsymbol{\Psi} \tag{1.1.118}$$

where $\boldsymbol{\Psi}$ denotes the class of bounded h-pseudodifferential operators. Therefore inequality (1.1.112) for $a^{\mathrm{w}}(x,hD)$ is reduced to the same inequality for $a_1^{\mathrm{w}}(x,hD')$; this is an operator-valued function and one needs to prove (1.1.112) for each x_1 with an inner product in $\mathscr{L}^2(\mathbb{R}^{d-1})$. Reduction is done.

Step 2. Rescaling. Assume now that

$$|\nabla^\alpha a| \le c_\alpha r^{4-|\alpha|} \qquad \forall \alpha : |\alpha| \le K \tag{1.1.119}$$

and that

$$\text{Either } a(\bar{z}) \asymp r^4 \text{ or } |\nabla^2 a(\bar{z})| \asymp r^2 \text{ with } r \ge Ch^{\frac{1}{2}}. \tag{1.1.120}$$

In this case we can rescale: $x \mapsto x_{\mathrm{new}} = r^{-1}x$, $\xi \mapsto \xi_{\mathrm{new}} = r^{-1}\xi$ and thus $B(\bar{z},\epsilon r) \mapsto B(r^{-1}\bar{z},\epsilon))$, $h \mapsto h_{\mathrm{new}} = r^{-2}h$ and $a(x,\xi) \mapsto a_{\mathrm{new}}(x,\xi) = r^{-4}a(rx,r\xi)$. Then we get $-Cr^4h_{\mathrm{new}}^2 = -Ch^2$ in the right-hand side of our target inequality.

Step 3. Choice of scale. Let us introduce *scaling function*

$$r(z) = \epsilon_0\big(a(z) + |\nabla^2 a(z)|^2\big)^{\frac{1}{4}} + Ch^{\frac{1}{2}} \tag{1.1.121}$$

in $\mathbb{R}^{2d}$. Furthermore

(1.1.122) If $f(z) \ge 0$ and $|\nabla^4 f| \le c$ in $\mathbb{R}^n$ then in each point

$$|\nabla f(z)| \le C f(z)^{\frac12} |\nabla^2 f|^{\frac12} + C f^{\frac34}, \tag{1.1.123}$$

$$|\nabla^3 f(z)| \le C |\nabla^2 f|^{\frac12} + C f^{\frac14} \tag{1.1.124}$$

with an appropriate constant C.

One can prove (1.1.123), (1.1.124) basing on Taylor decomposition up to fourth order terms. Further, one can prove then that for an appropriate constant $\epsilon_0 > 0$

$$|\nabla_z r| \le \frac{1}{2}, \tag{1.1.125}$$

(1.1.119) holds and either (1.1.120) holds or $r \le 2Ch^{\frac12}$.

Then we conclude that

(1.1.126) Inequality (1.1.112) holds for symbol a replaced by $a_\phi = a\phi$ where ϕ satisfies (1.1.120) for some $\bar z$ and $r = r(\bar z)$.

Indeed, if $r(\bar z) \ge 2Ch^{\frac12}$ it follows from Step 2. If $r(\bar z) \le 2Ch^{\frac12}$ then $h_{\mathsf{new}} \asymp 1$ and our inequality is trivial (after rescaling).

Step 4. Partition. Now we need an *r-admissible partition*. Namely, as *scaling function* $r(z)$ satisfies (1.1.125), there is a partition of unity $1 = \sum_\nu \psi_\nu^2$, such that

(1.1.127) Each ψ_ν satisfies (1.1.114), is supported in the ball $B(z_\nu, \frac12 r(z_\nu))$ and balls $B(z_\nu, r(z_\nu))$ cover $\mathbb{R}^n$ with multiplicity not exceeding $C(d)$.

To construct such partition one needs to construct first covering of $\mathbb{R}^n$ by balls $B(z_\nu, \frac13 r(z_\nu))$ with the distance $|z_\nu - z_\mu| \ge \frac14 \max\big(r(z_\nu), r(z_\mu)\big)$; then we have multiplicity condition fulfilled. Let $\psi'_\nu(z) = \chi((z - z_\nu)/r(z_\nu))$ with smooth function χ, supported in $B(0, \frac12)$ and equal 1 in $B(0, \frac13)$. Then $\phi = \sum_\nu \psi_\nu(z)^2$ is a smooth function, satisfying (1.1.114) in $B(z_\nu, \frac23 r(z_\nu))$ for each ν and disjoint from 0. Finally $\psi_\nu = \phi^{-\frac12}\psi'_\nu$ satisfy (1.1.127).

More general scaling should reflect the different scales with respect to x, ξ and even more general scaling can be introduced by *temperate metric* (see Definition 18.5.1 in L. Hörmander [1], Vol. III).

Now one can quantize ψ'_ν and construct instead operators $\hat{\psi}'_\nu = {\psi'_\nu}^{\mathsf{w}}$, $\hat{\phi} = \sum_\nu \hat{\psi}'_\nu \hat{\psi}'_\nu$ and $\hat{\psi}_\nu = \hat{\psi}'_\nu \hat{\phi}^{-\frac{1}{2}}$. One can see easily that $\hat{\psi}'_\nu$ and $\hat{\phi}$ are Hermitian operators, $h_\nu^{-1}(\hat{\psi}^*_\nu - \hat{\psi}_\nu)$ satisfies (1.1.114) with $h_\nu = h/r(z_\nu)^2$ and

$$I = \sum \hat{\psi}^*_\nu \hat{\psi}_\nu. \tag{1.1.128}$$

Then we arrive to decomposition

$$(Au, u) = \sum_\nu (\hat{\psi}^*_\nu \hat{\psi}_\nu Au, Au) = \sum_\nu (A_\nu \hat{\psi}_\nu u, \hat{\psi}_\nu u) - \frac{1}{2}(Tu, u) + (Ru, u)$$

with

$$T = \hat{\psi}^*_\nu [A_\nu, \hat{\psi}_n] \; - \; [A_\nu, \hat{\psi}^*_\nu]\hat{\psi}_\nu, \qquad R = \sum_\nu \psi^*_\nu \psi_\nu (A - A_\nu)$$

where $A_\nu = (A\phi_\nu)^{\mathsf{w}}(x, hD)$, ϕ_ν satisfies (1.1.114), is supported in $B(z_\nu, \frac{3}{4}r(z_\nu))$ and equal 1 in $B(z_\nu, \frac{2}{3}r(z_\nu))$.

One can see easily that operator norms of each term in the sum defining T, R does not exceed $r(z_\nu)^4 h_\nu^2 = h^2$. Then using arguments of the proofs of Theorems 1.1.9, 1.1.22 one can prove easily that both (Tu, u) and (Ru, u) do not exceed $Ch^2\|u\|^2$ (surely we have a more tricky partition but it is not a grave problem. We are skipping this part).

On the other hand, due to Step 2 $(A_\nu \hat{\psi}_\nu u, \hat{\psi}_\nu u) \ge -Ch^2 \|\hat{\psi}_\nu u\|^2$ and $\sum_\nu \|\hat{\psi}_\nu u\|^2 = \|u\|^2$ due to (1.1.128).

Combining these inequalities we arrive to (1.1.112). □

It looks like Theorem 1.1.31 is very sharp. Surely it is. But in many cases it gives a wrong lower bound! Namely, let us look in $d = 1$ at Harmonic oscillator $H = x^2 + (hD)^2$. Its spectrum is $\{h, 3h, 5h, ...\}$ and $(Hu, u) \ge h\|u\|^2$ while Theorem 1.1.31 claims that $(Hu, u) \ge -Ch^2\|u\|^2$. Actually formally this theorem is not applicable since H has unbounded symbol and is unbounded operator but one can overcome this easily. Let us discuss another inequality for scalar operators which covers this gap even in general it is less sharp than Theorem 1.1.31. It also involves some *symplectic* notions.

Theorem 1.1.32. *Let $a(x,\xi)$ be scalar, real-valued and satisfy* (1.1.77). *Let*

$$a(x,\xi) + h\,\mathsf{Tr}^+(\mathsf{Hess}^{\#}\, a)(x,\xi) \ge 0 \tag{1.1.129}$$

where

$$\mathsf{Hess}\, a = \begin{pmatrix} a_{xx} & a_{x\xi} \\ a_{\xi x} & a_{\xi\xi}\end{pmatrix}, \tag{1.1.130}$$

$$\mathsf{Hess}^{\#}\, a = \begin{pmatrix} a_{\xi x} & a_{\xi\xi} \\ -a_{xx} & -a_{x\xi}\end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix} \mathsf{Hess}\, a$$

are $2d\times 2d$-matrices called <u>Hessian</u> and <u>skew-Hessian</u> of a respectively, $\mathsf{Tr}^+(\mathsf{Hess}^{\#})a$ is a half-sum of absolute values of eigenvalues of $\mathsf{Hess}^{\#}\, a$. Then for any $\epsilon>0$

$$(a^{\mathsf{w}}(x,hD)u,u) \ge -(\epsilon h - C_\epsilon h^2)\|u\|^2 \tag{1.1.131}$$

for all u, $h\le 1$.

Proof. Let sketch the proof. First of all we can always make ε-partition of unity in $\mathbb{R}^{2d}$ with arbitrarily small constant $\epsilon>0$ (actually since our remainder estimate is $o(h)$ we can take even $L_\epsilon h^{\frac12}$-partition with large enough L_ϵ).

Let us look at Step 1 of the proof of Theorem 1.1.31. Repeating it as long as $|\nabla^2 a|$ or $|\nabla^2 a_1|$ etc are disjoint from 0 we will get that we can reduce symbol to $b_1^2+\ldots+b_I^2+a_I$ where $|\nabla^2 a_I|$ is small enough and then it would be sufficient to prove (1.1.131) for symbol $a'=g-h\,\mathsf{Tr}^+\,\mathsf{Hess}^{\sigma}\, g$, $g=b_1^2+\cdots+b_I^2$. Furthermore, in this construction $\nabla b_1,\ldots\nabla b_I$ are linearly independent.

Now it would be sufficient to check (1.1.129) on manifold $\Sigma=\{(x,\xi): b_1=\ldots=b_I=0\}$ and one can check easily that on Σ non-zero eigenvalues of $\mathsf{Hess}^{\#}\, g$ are those of skew-symmetric matrix $(\{b_j,b_k\})_{j,k=1,\ldots,I}$ (and thus these eigenvalues are purely imaginary).

Further, one can replace system $b_1,\ldots,b_I$ by $b_1',\ldots,b_I'$, $b_j'=\sum_k \alpha_{jk}b_k$ with orthogonal real matrix (α_{jk}) in the way that at some point $\bar z\in\Sigma$ matrix $(\{b_j',b_k'\})$ is in the canonical form: $\{b_j',b_k'\}=\rho_m$ as $j=2m-1, k=2m$, $m=1,\ldots,r$, $\{b_j',b_k'\}=-\rho_m$ as $j=2m, k=2m-1$, $m=1,\ldots,r$ and $\{b_j',b_k'\}=0$ otherwise where $\pm i\rho_m$ $(m=1,\ldots,r)$ are non-zero eigenvalues of

$(\{b_j, b_k\})$ and $\rho_m > 0$. Therefore it is sufficient to prove (1.1.131) as $I = 2$. Now it is reduced to

$$\|B_1 u\|^2 + \|B_2 u\|^2 + i([B_1, B_2]u) \geq -Ch^2\|u\|^2 \tag{1.1.132}$$

because replacing B_j by $B_j + \frac{h}{2}\mu_j$ we can always get rid of terms $(h\mu_j B_j u, u)$ with pseudodifferential operators μ_1, μ_2. However the left-hand expression of (1.1.132) is exactly $\|(B_1 + iB_2)u\|^2$ as long as B_j are symmetric. □

Inequality (1.1.131) (Melin's inequality) gives a correct lower bound of the spectrum of $a(x, hD)$. Really, for quadratic symbol

$$a(x,\xi) = \sum_{\alpha,\beta:|\alpha|+|\beta|=2} c_{\alpha\beta} x^\alpha \xi^\beta$$

which is non-negative, the bottom of the spectrum of $a^{\mathrm{w}}(x, hD)$ is exactly $h\,\mathsf{Tr}^+(a)$.

There is more limited but more precise statement which we will not prove:

Theorem 1.1.33. *Let $b_1, \dots, b_k$ be real valued symbols with $\nabla b_1, \dots, \nabla b_I$ linearly independent at $\Sigma = \{b_1 = \dots = b_I = 0\}$. Furthermore, let matrix $\beta = (\{b_j, b_k\})$ have a constant rank on Σ and let $a = b_1^2 + \dots + b_I^2 + hb_0$ with $b_0 = -h\,\mathsf{Tr}^+(\beta)$ on Σ. Then* (1.1.131) *holds.*

Remark 1.1.34. (i) Theorems 1.1.31–1.1.33 remain true even if only the principal symbol a_0 is scalar.

(ii) A generalization of Melin's inequality to 2×2-matrix case is known. See f.e. R. Brummelhuis [2].

Parametrix and Inverse

Now let us consider parametrix construction.

Theorem 1.1.35. *Let $a \in \mathsf{S}_{h,N}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{H}', \mathbb{H})$. Assume that $\|\!|a|\!\| \leq c$ and its principal symbol a_0 is left invertible or right invertible at every point $(x,\xi) \in \Omega \subset \mathbb{R}^d \times \mathbb{R}^d$:*

$$b_0(x,\xi)a_0(x,\xi) = I \qquad \text{or} \qquad a_0(x,\xi)b_0(x,\xi) = I \tag{1.1.133}$$

with

$$\|\!|b_0|\!\| \le c. \tag{1.1.134}$$

Then there exists a symbol $b \in \mathsf{S}_{h,N}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{H}, \mathbb{H}')$ *with* $\|\!|b|\!\| \le C = C(d, N, c)$ *such that respectively*

$$b \bullet_N a = I \qquad or \qquad a \bullet_N b = I \quad in \ \ \Omega. \tag{1.1.135}$$

Proof. One can define b_n recursively by

$$b_n = -\Big(\sum_{m<n} b_m h^m \bullet a\Big)_n b_0, \qquad b_n = -b_0\Big(a \bullet \sum_{m<n} b_m h^m\Big)_n \tag{1.1.136}$$

in the case of left (right) invertible symbol a_0. Note that in this construction b_n depends only on $D_x^\alpha D_\xi^\nu a_k$ with $|\alpha| \le n-k, |\nu| \le n-k$ at this point. In particular, if $\mathbb{H}' = \mathbb{H}$, $a_0 = I, a_1 = ... = a_n = 0$ in Ω then in this construction $b_0 = I, b_1 = ... = b_n = 0$ in Ω as well. □

Remark 1.1.36. (i) If $a_0(x,\xi)$ is invertible (i.e. both left and right invertible) then $a_0 b_0 = b_0 a_0 = I$ with $b_0 = a_0^{-1}$. Condition (1.1.134) is then replaced by

$$\|a_0^{-1}(x,\xi)\| \le c. \tag{1.1.137}$$

(ii) Further, both recursive formulae in (1.1.136) are equivalent. Really, let b and b' be left and right inverse symbols in the sense of (1.1.135). Then $b \equiv b \bullet Na \bullet_N b' \equiv b'$ implies that $b_n = b'_n$ for all n which is not obvious from (1.1.136). Furthermore one can see easily that in this case b is defined uniquely in Ω.

We would call operator $\mathsf{Op}(a)$ *left elliptic*, *right elliptic* or *elliptic* in the corresponding cases and we will call operator $\mathsf{Op}(b)$ a *left parametrix*, *right parametrix* or *parametrix* in Ω of operator $\mathsf{Op}(a)$.

Corollary 1.1.37. *If* a_0 *is either left invertible, or right invertible, or invertible in* $\mathbb{R}^d \times \mathbb{R}^d$ *and* (1.1.134) *holds then for* $h \le h_1$ *operator* $\mathsf{Op}(a)$ *is either left invertible, or right invertible, or invertible and the corresponding inverse belongs to the same class.*

We know that if the principal symbol of inverse operator is inverse to the principal symbol. We leave to the reader an easy proof of the following useful formula:

$$\begin{aligned}(1.1.138)\quad \bigl(A^{-1}\bigr)^s = -a_0^{-1}a^s a_0^{-1} - \\ \frac{i}{2}\sum_j a_0^{-1}\Bigl((\partial_{\xi_j}a_0)a_0^{-1}(\partial_{x_j}a_0) - (\partial_{x_j}a_0)a_0^{-1}(\partial_{\xi_j}a_0)\Bigr)a_0^{-1};\end{aligned}$$

further, if a_0 is a scalar symbol the last line vanishes and

$$(1.1.139)\qquad \bigl((\lambda - A)^{-1}\bigr)^s = -a^s a_0^{-2}.$$

Resolvent

As $\mathbb{H}' = \mathbb{H}$ we can apply these arguments to the resolvent construction. Let A be a quantization of $a \in \mathsf{S}_{h,N}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{H}, \mathbb{H})$. The we define

$$(1.1.140)\qquad \Sigma(a) = \overline{\bigcup_{(x,\xi)} \mathsf{Spec}\bigl(a_0(x,\xi)\bigr)}.$$

In other words, $\Sigma(a)$ is a minimal closed subset in $\mathbb{C}$ such that $\bigl(\lambda - a_0(x,\xi)\bigr)$ is invertible as $\lambda \notin \Sigma(a)$ and $\bigl\|\bigl(\lambda - a_0(x,\xi)\bigr)^{-1}\bigr\| \le C = C(\lambda)$ for all $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$. Obviously

(1.1.141) Let $a \in \mathsf{S}_{h,N}$. Then $\Sigma(a) \subset \{\lambda \in \mathbb{C}, |\lambda| \le C_0\}$ and also $\bigl\|\bigl(\lambda - a_0(x,\xi)\bigr)^{-1}\bigr\| \le C_0|\lambda|^{-1}$ as $|\lambda| \ge C_0$;

(1.1.142) Let $a(x_0,\xi)$ be Hermitian for all (x,ξ). Then $\Sigma(a) \subset \mathbb{R}$ and also $\bigl\|\bigl(\lambda - a_0(x,\xi)\bigr)^{-1}\bigr\| \le C_0|\operatorname{Im}\lambda|^{-1}$.

Now it follows immediately from Theorem 1.1.35, Remark 1.1.36 and Corollary 1.1.37 that

Theorem 1.1.38. *Let A be a pseudodifferential operator with symbol $a \in \mathsf{S}_{h,N}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{H}, \mathbb{H})$. Then*

(i) For arbitrarily small $\epsilon > 0$ $\mathsf{Spec}(A)$ is contained in ϵ-vicinity of $\Sigma(a)$ as $h \le h_0(\epsilon)$;

(ii) Let $r(\lambda) = r_{(N)}(\lambda, h)$ *be a result of standard parametrix construction (truncated modulo* $O(h^N)$*). Then as* λ *does not belong* ϵ*-vicinity of* $\Sigma(a)$

$$\|(A-\lambda)^{-1} - \mathsf{Op}\big(r(\lambda)\big)\| \le C|\lambda|^{-N-1}h^N. \tag{1.1.143}$$

We will consider a more precise statement in (1.1.152).

Remark 1.1.39. (i) The kind of inverse statement follows from (1.1.164).

(ii) Furthermore, as operator A is Hermitian (i.e. $A = a^{\mathsf{w}}(x, hD)$ with Hermitian $a(x,\xi)$) all such statements are really about the distance to the spectrum).

Analytic Functional Calculus of Pseudodifferential Operators

Let f be an analytic in $\mathcal{G} \supset \mathsf{Spec}(A)$ function and ℓ a closed contour in $\mathcal{G}$ going around spectrum of A exactly once in counter-clock-wise direction (ℓ is not necessarily connected). Then according to Real Analysis

$$f(A) = \frac{1}{2\pi i}\oint_{\ell} f(\lambda)(\lambda - A)^{-1}\,d\lambda. \tag{1.1.144}$$

Plugging into Theorem 1.1.38 we arrive to

Theorem 1.1.40. *Let* $A = \mathsf{Op}(a)$, $a \in \mathsf{S}_{h,N}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{H}, \mathbb{H}, \mathbb{H})$ *and* $\Sigma(a)$ *be defined by* (1.1.140)*. Further, let* $\mathcal{G}$ *be a vicinity of* $\Sigma(a)$ *and* $f(\lambda)$ *analytic in* $\mathcal{G}$*. Then for* $h \le h_1$ *(with small enough constant* $h_1 > 0$*)* $f(A) \in \mathbf{\Psi}_{h,N}$ *and coincides modulo* $O(h^N)$ *with*

$$f^{\bullet}(a) = \frac{1}{2\pi i}\oint_{\ell} f(\lambda) r_{(N)}(\lambda, h)\,d\lambda \tag{1.1.145}$$

with $r_{(N)}(\lambda, h)$ *defined in Theorem 1.1.38(ii).*

Furthermore, the principal symbol of $f(a)$ *is* $f(a_0)$*.*

Formula (1.1.139) implies that if a_0 is a scalar symbol then

$$\big(f(A)\big)^s = -a^s f'(a_0). \tag{1.1.146}$$

Remark 1.1.41. (i) Assume instead that $\Sigma(a) \subset \mathcal{G} \cup \mathcal{G}'$ where $\mathcal{G} \cap \mathcal{G} = \emptyset$. Then (1.1.144) defines $f(A)\Pi_{\mathcal{G}}$ where $\Pi_{\mathcal{G}} = \chi_{\mathcal{G}}(A)$ is also defined by (1.1.144) with $f = 1$ is a projector, $\chi_{\mathcal{G}}$ is a characteristic function of $\mathcal{G}$. In particular, as a_0 is Hermitian elliptic symbol we conclude that $|A|^t$ and $\theta(\pm A)|A|^t$ are pseudodifferential operators where θ is a Heaviside function and $t \in \mathbb{C}$.

(ii) One can define $\Sigma(a,\Omega)$ and run the whole construction on the (full) symbol rather than operator level.

We need however to consider a more delicate case and to be more precise. Namely let us now assume that

$$\|(\tau - a_0(x,\xi))^{-1}\| \le c\,\mathsf{dist}(\tau,\Sigma)^{-p} \qquad \forall (x,\xi)\in \mathbb{R}^d\times\mathbb{R}^d\ \forall \lambda\in\mathbb{C} : |\lambda|\le C_0 \tag{1.1.147}$$

with an exponent $p\ge 1$ where $\Sigma(a)\subset\Sigma\subset\{|\lambda|<C_0\}$.

Remark 1.1.42. Condition (1.1.147) is satisfied with $\Sigma=\Sigma(a)$ provided either

$$\mathbb{H}=\mathbb{C}^r\otimes\mathbb{H}_0,\quad a_0(x,\xi)=b(x,\xi)\otimes I,\quad p=r \tag{1.1.148}$$

or

$$a_0(x,\xi)^\dagger = a_0(x,\xi)\quad \forall (x,\xi),\quad p=1. \tag{1.1.149}$$

Note that in the resolvent construction each new term norm gains factor $O(h\|(\tau-a_0(x,\xi))^{-1}\|^2)$. Therefore as $\mathsf{dist}(x,\Sigma)\ge\eta^{1/p}$ with

$$\eta \ge h^{\frac12-\delta} \tag{1.1.150}$$

with an arbitrarily small exponent $\delta>0$ the resolvent construction is justified and the error is $O(h^N\eta^{-(2N+1)})$. Further, after rescaling $x\mapsto x/\eta$, $\xi\mapsto\xi/\eta$, $h\mapsto\hbar=h/\eta^{2p}$ $\quad j$-th term satisfies (1.1.77) with $M_j=h^j\eta^{-2j-1}$.

In particular

$$\|(\tau-A)^{-1}\|\le C\eta^{-1},\quad \|(\tau-A)^{-1}-\mathsf{Op}\big((\tau-a_0)^{-1}\big)\|\le Ch\eta^{-3}. \tag{1.1.151}$$

Also we conclude that

(1.1.152) $\mathsf{Spec}(A)$ is contained in $Ch^{1/2p}$ vicinity of Σ.

Combining with (1.1.144) we arrive to

Theorem 1.1.43. *Let* (1.1.47) *be fulfilled. Let f be analytic in $\mathcal{G}$ and ℓ be a closed contour in $\mathcal{G}$ going around spectrum of A exactly once in counter-clock-wise direction on the distance no less than $\epsilon_0\eta^{1/p}$ from Σ. Then*

$$\|f(A)-\mathsf{Op}\big(f^\bullet(a)\big)\|\le Ch^N\eta^{-2N-1}\|f\|_{\mathscr{L}^1(\ell)} \tag{1.1.153}$$

where symbol $f^\bullet$ is defined by (1.1.145) *and each term in its decomposition does not exceed $Ch^j\eta^{-2j-1}\|f\|_{\mathscr{L}^1(\ell)}$.*

Further,

$$\|f(A)\| \le C\eta^{-1}\|f\|_{\mathscr{L}^1(\ell)}, \quad \|f(A) - \mathsf{Op}\big(f(a_0)\big)\| \le CMh\eta^{-3}. \tag{1.1.154}$$

Smooth Functional Calculus of Pseudodifferential Operators

As operator A is Hermitian (or unbounded self-adjoint) one can construct $f(A)$ with smooth function f. Let us look for a smooth function of a Hermitian pseudodifferential operator. Since one can define $f(A)$ through propagator e^{itA} let us consider such operators first but we need to consider them for large $t \in \mathbb{R}$ as well.

Let us apply analytic theory to $\varphi_t(\tau) = e^{it\tau}$ under condition (1.1.49) (we do not need a stronger condition at this moment). Then $p = 1$. To avoid exponential growth with respect to t we must bound $|t\,\mathsf{Im}\,\tau|$; so let ℓ be a boundary of the rectangle $\{|\,\mathsf{Re}\,\tau| \le c, |\,\mathsf{Im}\,\tau| \le |t|^{-1}\}$, where we assume that $\Sigma(a) \subset [-\frac{1}{2}c, \frac{1}{2}c]$. Then $\|\varphi_t\|_{\mathscr{L}^1(\ell)} \asymp 1$ and therefore

$$\|e^{itA} - \mathsf{Op}(\varphi_t^\bullet(a))\| \le Ch^s \tag{1.1.155}$$

and

$$\|\,\mathsf{Op}(\varphi_t^\bullet(a))\| \le C|t|, \qquad \|\,\mathsf{Op}(\varphi_t^\bullet(a)) - \mathsf{Op}(e^{ita_0})\| \le Ch|t|^3 \tag{1.1.156}$$

provided $t \in \mathbb{R}, 1 \le |t| \le h^{(\delta-1)/2}$.

Moreover, note that the terms $r_n(\tau)$ in the decomposition of the symbol $r(\tau)$ are sums of terms of the form

$$(\tau - a_0)^{-1} b_1 (\tau - a_0)^{-1} \cdots (\tau - a_0)^{-1} b_{j-1} (\tau - a_0)^{-1} h^n$$

with $j \le 2n+1$, $b_k = b_{njk} \in \mathsf{S}_{h,N-(2n+1)+j}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{H}, \mathbb{H})$ do not depend on τ and the number of these terms does not exceed $K_0(n)$.

Then we conclude that the terms $\varphi_{tn}^\bullet$ in the decomposition of the symbol $\varphi_t^\bullet$ are sums of terms of the form $g(t) * b_1 g(t) * \cdots * b_{j-1} g(t) h^n$ where "$*$" means convolution and

$$g(t) = \frac{1}{2\pi i} \oint_\ell (\tau - a_0)^{-1} e^{it\tau}\, d\tau = e^{ita_0};$$

hence $\varphi_{tn}^\bullet$ is a sum of terms of the form

$$t^{j-1} h^n \int \cdots \int_{\Delta_{j-1}} e^{itz_1 a_0} b_1 e^{itz_2 a_0} \cdots b_{j-1} e^{itz_j a_0}\, dz' \tag{1.1.157}$$

with $z_j = 1 - z_1 - ... - z_{j-1}$, $dz' = dz_1 \cdots dz_{j-1}$ and
$\Delta_{j-1} = \{z' \in \mathbb{R}^{j-1} : z_k > 0 \quad \forall k, z_1 + ... z_{j-1} < 1\}$. Recall that $j \le 2n+1$.

Note that for any first-order differentiation ∂ the following formula holds:

$$\partial e^{ita_0} = \int_0^t e^{it'a_0}(i\partial a_0)e^{i(t-t')a_0}\, dt'. \tag{1.1.158}$$

Applying this formula to (1.1.157) we conclude easily that

$$\| \mathsf{Op}(\varphi_t^\bullet(a))\| \le C, \qquad \| \mathsf{Op}(\varphi_t^\bullet(a)) - \mathsf{Op}(e^{ita_0})\| \le Ch\gamma^{-2} \tag{1.1.159}$$

provided $|t| \le h^{(\delta-1)/2}$. So we have improved estimates (1.1.156).

Therefore we arrive to

Theorem 1.1.44. *Under condition* (1.1.149) e^{itA} *is a pseudodifferential operator with the symbol satisfying* (1.1.95) *with* $\boldsymbol{\rho} = \boldsymbol{\gamma} = |t|^{-1}\mathbf{1}$ *provided* $|t| \le ch^{(\delta-1)/2}$, $\mathbf{1} = (1, 1, ..., 1)$.

Remark 1.1.45. (i) One can see easily that symbol e^{ita_0} satisfies (1.1.95) with $\boldsymbol{\rho} = \boldsymbol{\gamma} = |t|^{-1}\mathbf{1}$ provided $|t| \le ch^{(\delta-1)/2}$ and one cannot improve this statement;

(ii) Note that for a scalar symbol a_0 the symbol $\varphi_{tn}^\bullet$ is a symbol of the form

$$e^{ita_0} \sum_{0\le n\le N,\, j\le 2n} b_{nj} h^n t^j. \tag{1.1.160}$$

Now we can consider smooth functions of Hermitian operator. Assume that $A^* = A$ and $f \in \mathscr{C}^N(\mathbb{R})$ satisfies

$$\langle\tau\rangle^2 |D_\tau^j f(\tau)| \le M\eta^{-j} \quad \forall j \le N \qquad \text{for } \eta \in [h^{\frac{1}{2}(1-\delta)}, 1],\ \delta > 0 \tag{1.1.161}$$

with $\langle\tau\rangle = (1 + |\tau|^2)^{1/2}$. Then $f(A)$ is given by the formula

$$f(A) = \int_{\mathbb{R}} \hat f(t) e^{itA}\, dt \tag{1.1.162}$$

where (1.1.161) implies that

$$|\hat f(t)| \le CM(1 + \eta|t|)^{-N}. \tag{1.1.163}$$

Consider integral (1.1.162) separately on $[-h^{(\delta'-1)/2}, h^{(\delta'-1)/2}]$ and on the remaining part of $\mathbb{R}$ where $\delta' > 0$ is a small enough exponent; then (1.1.163) implies that the operator norm of the second integral does not exceed

$$C\big(h^{(\delta'-1)/2}\eta\big)^{1-N}\eta^{-1} = O(h^s)$$

while in the first integral we can replace e^{itA} by $\mathsf{Op}(\varphi_t^\bullet(a))$. One can consider $\hat f(t)\varphi_t^\bullet(a))$ for $|t| \ge h^{(\delta'-1)/2}$ as a small and thus quantizable (generating a negligible operator); so one can take the integral on $\mathbb{R}$ again; the operator norm of the new error also is $O(h^s)$. We obtain then the sum of terms of the form

$$h^n \int_{\mathbb{R}} \mathcal{J}_{nj}(\tau) f(\tau)\, d\tau \tag{1.1.164}$$

where

$$\mathcal{J}_{nj}(\tau) = \mathsf{Res}_{\mathbb{R}}\, J_{nj} = \frac{1}{2\pi i}\big(J_{nj}(\tau + i0) - J_{nj}(\tau - i0)\big)$$

and $J_{nj}(\tau)$ are operator-valued functions of the form (1.1.157). Therefore $\mathcal{J}_{nj}(\tau)$ are distributions supported in $\mathcal{C} = \big\{(\tau, x, \xi) : \tau \in \mathsf{Spec}\big(a_0(x,\xi)\big)\big\}$; so $\mathsf{supp}(\mathcal{J}_{nj}) \subset \{|\tau| \le C_0\}$. Moreover, $|J_{nj}(\tau)| \le C|\,\mathsf{Im}\,\tau|^{-j} \quad \forall \tau \in \mathbb{C}$ and hence

$$\Big|\int J_{nj}(\tau \pm i0) f(\tau)\, d\tau\Big| \le C\|f\|_{\mathscr{C}^j} \tag{1.1.165}$$

for every $f \in \mathscr{C}_0^j([-2c, 2c])$. Therefore (1.1.158) implies that the term (1.1.164) does not exceed $Ch^n\eta^{-j}$. Moreover, applying $D^\alpha_{x,\xi}$ to (1.1.157) we obtain a function of the same type with $j + |\alpha|$ instead of j. So we obtain the first assertion of the following statement:

Theorem 1.1.46. *(i) Let conditions* (1.1.149) *and* (1.1.161) *be satisfied. Then the described procedure leads to a construction of quantizable symbol* $f^\bullet(a)$ *such that*

$$\|\,\mathsf{Op}(f^\bullet(a)) - \mathsf{Op}(f(a_0))\| \le CMh\eta^{-3}.$$

In particular as A *is Hermitian (f.e.* $a^\dagger = a$ *and we use Weyl quantization)* (1.1.153) *and* (1.1.154) *hold.*

(ii) Moreover, for scalar symbol a_0

$$\|f(A) - \mathsf{Op}\big(f(a_0)\big)\| \le CMh\eta^{-2}. \tag{1.1.166}$$

The proof of the second assertion is based on the fact that in this case

$$\int_{-2c}^{2c} |J_{nj}(\tau \pm i\eta)|\, d\tau \le C\eta^{1-j}$$

provided $j > 1$ and hence in the right-hand expression in (1.1.165) one can replace j by $(j-1)$.

Finally, for a scalar symbol a_0 formula (1.1.146) remains true.

Remark 1.1.47. Instead of (1.1.161) we can assume that $f_1 = \langle\tau\rangle^{-2l} f \in \mathscr{C}_b^K$ and derive Theorem 1.1.46 with $M = \|f_1\|_{\mathscr{C}_b^K}$ where

$$\mathscr{C}_b^K = \{g : \|g\|_K = \sum_{|\alpha|\le K} \sup |D^\alpha g| < \infty\};$$

surely in this case K and C depend on l also. In fact, $f(A) = (A^2 + I)^l f_1(A)$ where f_1 satisfies (1.1.161) with $\eta = 1$.

Families of Commuting Operators

One can easily derive results similar to those obtained in two previous sections for a family $A^1, \dots, A^n$ of commuting operators; these operators have commuting principal symbols; instead of spectrum one should consider a joint spectrum and introduce $\Sigma(a^1, \dots, a^n) \subset \mathbb{C}^n$; respectively one should consider $f(A^1, \dots, A^n)$ where either f is analytic in the vicinity of $\Sigma(a^1, \dots, a^n)$ or all operators $A^1, \dots, A^n$ are Hermitian.

We leave details to the reader.

Norms and bBounds

Gårding inequalities imply that for Hermitian operator its lower bound is not less than $\inf \Sigma(a_0) - CMh$ and its upper bound is not greater than $\sup \Sigma(a_0) + CMh$; this is a better result than $Ch^{1/2}$ error due to (1.1.152).

On the other hand, one can easily generalize Theorem 1.1.14 to operators of the type considered here: using arguments of the proof of Theorem 1.1.19 one can prove easily that $a(x, hD)\big(e^{ih^{-1}\phi} f\big)$ is negligible as $\operatorname{supp}(a)$ is disjoint from the set $\{x, \nabla\phi(x)\}$.

Let us consider $u = (2\pi h)^{-d/4} e^{ih^{-1}\phi} w$ with $\phi = \langle x, \bar\xi\rangle + i|x - \bar x|^2/2$ and $w \in \mathbb{H}$; one can prove easily that

$$(a^{\mathrm{w}}(x, hD)u, u) = \langle a(\bar x, \bar\xi) w, w\rangle + O(h)$$

and therefore we arrive to

Theorem 1.1.48. *For Hermitian operator operator*

$$|\inf \mathsf{Spec}(A) - \inf \Sigma(a)| \le CMh \tag{1.1.167}$$

and

$$|\sup \mathsf{Spec}(A) - \sup \Sigma(a)| \le CMh. \tag{1.1.168}$$

Remark 1.1.49. (i) In the general case one can apply Theorem 1.1.48 to operator $B = \frac{1}{2}(A + A^*)$;

(ii) If $\dim \mathbb{H} \ge 2$ it may happen that $\Sigma(a)$ is not connected. Let $\mathcal{G}$ be an open set in $\mathbb{C}$ such that $\Sigma(a) \cup \mathcal{G} \ne \emptyset$ and $\mathsf{dist}(\mathcal{G}, \Sigma(a) \setminus \mathcal{G}) \ge \epsilon_0$. Then one can apply (i) to $A\Pi_{\mathcal{G}}$ with projector $\Pi_{\mathcal{G}}$ introduced in Remark 1.1.41(i). This generalization makes sense for Hermitian operators as well as general ones.

(iii) Further, we can apply Theorem 1.1.46 to A^*A resulting in

$$|\, \|A\| - \sup_{(x,\xi)} \|a_0(x,\xi)\| \,| \le CMh. \tag{1.1.169}$$

1.1.3 Pseudodifferential Operators with Analytic Symbols and Logarithmic Uncertainty Principle

So far we considered pseudodifferential operators with finitely smooth symbols $N < \infty$. How one needs to modify results as $N = \infty$? Surely it concerns only results with the remainder estimates depending on N (unlike Gårding inequalities). The first answer is rather easy: all sums became asymptotic and all such remainder estimates are "$L_N h^N$ for all $N < \infty$".

The next question is "how L_N depend on N?" The answer we got already f.e. for the product of $a^w(x, hD)$ and $b^w(x, hD)$ is

$$L_N \le C_N \sum_{|\alpha|+|\beta|\le N} \|D_x^\alpha D_\xi^\beta a\|_{\mathscr{L}^\infty} \times \sum_{|\alpha|+|\beta|\le N} \|D_x^\alpha D_\xi^\beta b\|_{\mathscr{L}^\infty}$$

and if we look at the regular terms containing $\frac{1}{\alpha!\beta!}$ etc we can think that $C_N = C_0^N (N!)^{-1}$. Inequality (1.A.5) and formulae $(1.1.92)_1$–$(1.1.92)_3$ immediately prove this remainder estimate for all types of quantization.

To exploit this observation we need to consider operators with ultra-smooth symbols: those whose derivatives derivatives grow with the controlled rate: f.e. with

$$\|D_x^\alpha D_\xi^\beta a\|_{\mathscr{L}^\infty} \le M\varrho^{|\alpha|+|\beta|}(|\alpha|!)^\sigma(|\beta|!)^\varsigma \tag{1.1.170}$$

with $\sigma \ge 1$, $\varsigma \ge 1$ and ϱ independent on N. To be consistent we should assume that junior terms satisfy

$$\|D_x^\alpha D_\xi^\beta a_n\|_{\mathscr{L}^\infty} \le M\varrho^{|\alpha|+|\beta|+2n}(|\alpha|!)^\sigma(|\beta|!)^\varsigma(n!)^{\sigma+\varsigma-1} \tag{1.1.171}$$

Then the remainder estimate is $c^N h^N (N!)^{\sigma+\varsigma-1}$. One can see easily that there is a minimal value $N = N_h = \epsilon_0 h^{-1/(\sigma+\varsigma-1)}$ when estimate hits the minimum

$$\exp\bigl(-\epsilon h^{-1/(\sigma+\varsigma-1)}\bigr) \tag{1.1.172}$$

and this is "supersmall" remainder estimate.

We will reconstruct the theory of two previous subsections for such symbols and remainder estimates and with this notion of negligible. Obviously the best possible case is $\sigma = \varsigma = 1$ which corresponds to analytic symbols. One can object that there are no compactly supported symbols and therefore there is no microlocalization. However M. Bronstein observed that we need to satisfy estimates only as $|\alpha| + n \le N_h$, $|\beta| + n \le N_h$ and there are very regular families of symbols satisfying a bit weaker but still allowing quantization condition (see Proposition 1.A.2).

Further, this theory brings fruits even for the old notion of negligible $O(h^s)$. Namely, if estimate (1.1.171) holds only after rescaling $x_j \mapsto x_j/\gamma_j$, $\xi_j \mapsto \xi_j/\rho_j$, then the remainder estimate would be $O(\exp(-\epsilon\hbar^{1/(\sigma+\varsigma-1)})) = O(h^s)$ as $\hbar \le \epsilon s^{-1}|\log h|^{-(\sigma+\varsigma-1)}$.

Recall that $\hbar = \sup_j h/(\rho_j\gamma_j)$; so the previous condition becomes

$$\rho_j\gamma_j \ge C_s h|\log h|^{\sigma+\varsigma-1}. \tag{1.1.173}$$

As $\sigma = \varsigma = 1$ we arrive to the most important case

$$\rho_j\gamma_j \ge C_s h|\log h| \tag{1.1.174}$$

which we call *logarithmic uncertainty principle*.

Pseudodifferential operators with Infinitely Smooth Symbols

Definition 1.1.50. Let $\mathsf{S}_{h,\infty}$ be a class of the symbols satisfying (1.1.77) for all N (with constant depending on N). Denote by $\mathbf{\Psi}_{h,\infty}$ the corresponding class of pseudodifferential operators.

We immediately arrive to

Theorem 1.1.51. *Let $a_j \in \mathsf{S}_{h,\infty}$. Then there exist $a \in \mathsf{S}_{h,\infty}$ such that*

$$a \sim \sum_j h^j a_j \tag{1.1.175}$$

which means that for any N

$$a - \sum_{0\le j\le N-1} h^j a_j \in h^N \mathsf{S}_{h,\infty}. \tag{1.1.176}$$

Proof. Proof is the standard asymptotic exercise (very similar to the proof that for any sequence of numbers c_k there exists a function f such that $f^{(k)}(0) = c_k$). Namely let ϕ be a smooth function, supported in $(-2,2)$ and equal 1 in $(-1,1)$. Let us consider

$$a = \sum_j h^j a_j \phi(hl_j)$$

with $l_j \to \infty$ as $j \to \infty$. Then a is defined properly for each $h > 0$ (the sum is finite). We leave to the reader to prove that a is a proper symbol in the following steps: prove that the first of inequalities holds

$$|\partial^\alpha \sum_{j\ge N} h^j a_j \phi(hl_j)| \le C_N h^N \qquad \forall \alpha : |\alpha| \le N \tag{1.1.177}$$

$$|\partial^\alpha \sum_{j< N} h^j a_j (\phi(hl_j) - 1)| \le C_N h^N \qquad \forall \alpha : |\alpha| \le N \tag{1.1.178}$$

with proper constants C_N provided l_j grow fast enough; then one can prove easily that the second inequality holds with proper constants C_N. □

Then all the constructions of this chapter are valid with $N = \infty$ as well.

Pseudo-differential Operators with Analytic Symbols

We are interested *only* in the case of analytic symbols with $\sigma = \varsigma = 1$ leaving more general case of Gevrey symbols to the reader; the standard theory was developed by L. Boutet de Monvel and P. Krée [1], pp. 302–306 and semiclassical theory does not differ much. We follow this article here for analysis of symbols.

We introduce class $\mathsf{S}_{h,\mathsf{an}}$ of symbols

$$a(x,\xi,h) = \sum_{n:\le N} a_n(x,\xi,h)h^n, \quad N = N(C,h) = \epsilon_0 C^{-1}h^{-1}, \tag{1.1.179}$$

$$\|D^\alpha_{x,\xi} a_n\| \le MC^{|\alpha|+2n}|\alpha|!n! \qquad \forall \alpha, n : 2|\alpha| + n \le N + 2K, n \le N \tag{1.1.180}$$

where C, M could be arbitrarily large and ϵ_0, K depend only on d.

Equivalent but more technically convenient is

Definition 1.1.52. $\mathsf{S}_{h,\mathsf{an}} = \bigcup_{\varrho>0} \mathsf{S}_{h,\mathsf{an}}(\varrho)$ where $a \in \mathsf{S}_{h,\mathsf{an}}(\varrho)$ if *the formal norm* defined by *a formal power series* (1.1.80) converges for $|\zeta| < \varrho$:

$$P(a,\zeta) = \sum_{\alpha,n} \frac{\epsilon_1^{-1}\epsilon_0^{-n}}{(2n+|\alpha|)!}\zeta^{2n+|\alpha|} \sup_z \|\nabla^\alpha_z a(z)\| \tag{1.1.181}$$

where $z = (x,\xi)$ and we select $\epsilon_0, \epsilon_1 > 0$ later.

Proposition 1.1.53. *For small enough ϵ_0, ϵ_1 norm $P(a,\zeta)$ has following properties:*

$$P(a+b,\zeta) \lll P(a,\zeta) + P(b,\zeta), \tag{1.1.182}$$

$$P(a \bullet b,\zeta) \lll P(a,\zeta)\cdot P(b,\zeta), \tag{1.1.183}$$

where $\bullet$ is calculated for $N = \infty$ and we consider here formal symbols i.e. sum $\sum a_n h^n$ etc and $\sum_n A_n\zeta^n \lll \sum_n B_n\zeta^n$ means that $A_n \le B_n$ for all n.

Proof. We prove the second estimate. From obvious

$$\|\nabla^\alpha (a\bullet b)_n\| \le \sum_{\substack{l+m+|\gamma|=n,\,\gamma,\,\alpha'\ge\gamma,\,\alpha''\ge\gamma,\\ \alpha'+\alpha''=\alpha+2\gamma}} \frac{\alpha!}{(\alpha'-\gamma)!(\alpha''-\gamma)!\gamma!}\|\nabla^{\alpha'} a_l\|\cdot\|\nabla^{\alpha''} b_m\|$$

we conclude that

$$P(a\bullet b,\zeta)\lll \sum_{l,m,\gamma,\alpha',\alpha''} \frac{(\alpha'+\alpha''-2\gamma)!}{(\alpha'-\gamma)!(\alpha''-\gamma)!(2m+2l+|\alpha'|+|\alpha''|)!\gamma!}\times \\ \|\nabla^{\alpha'}a_l\|\cdot\|\nabla^{\alpha''}b_m\|\times \epsilon_1^{-1}\epsilon_0^{l+m+|\gamma|}\zeta^{2l+2m+|\alpha'|+|\alpha''|}$$

which equals

$$\sum_{l,m,\gamma,\alpha',\alpha''} \frac{(\alpha'+\alpha''-2\gamma)!}{(\alpha'-\gamma)!(\alpha''-\gamma)!}\cdot\frac{(2l+|\alpha'|)!(2m+|\alpha''|)!}{(2m+2l+|\alpha'|+|\alpha''|)!}\cdot\frac{\epsilon_1\epsilon_0^{|\gamma|}}{\gamma!}\times \\ \frac{\epsilon_1^{-1}\epsilon_0^{l}}{(2l+|\alpha'|)!}\|\nabla^{\alpha'}a_l\|\zeta^{2l+|\alpha'|}\cdot\frac{\epsilon_1^{-1}\epsilon_0^{m}}{(2m+|\alpha''|)!}\|\nabla^{\alpha''}b_m\|\zeta^{2m+|\alpha''|}$$

where in the first line the first and the second factor do not decrease if we replace γ by 0 and l,m by 0 as well; meanwhile $\sum_\gamma \frac{1}{\gamma!}\epsilon_0^{|\gamma|}\le e^{2d\epsilon_0}$ and Proposition 1.1.53 follows from estimate[5)]

$$\frac{(\alpha'+\alpha'')!}{\alpha'!\alpha''!}\le\frac{(|\alpha'|+|\alpha''|)!}{|\alpha'|!(|\alpha''|)!}\impliedby \frac{|\alpha'|!}{\alpha'!}\cdot\frac{|\alpha''|!}{\alpha''!}\le\frac{|\alpha'+\alpha''|!}{(\alpha'+\alpha'')!}$$

with the trivial from the combinatorial point of view last inequality. □

Then we arrive immediately to

Theorem 1.1.54. *Let $a,b\in \mathsf{S}_{h,\mathsf{an}}(\varrho)$. Then $a\bullet b\in \mathsf{S}_{h,\mathsf{an}}(\varrho)$ and with $N=N(\varrho^{-1},h)$)*

$$\|\operatorname{Op}(a)\operatorname{Op}(b)-\operatorname{Op}(a\bullet_N b)\|\le C_0\bar{P}(a,\varrho)\bar{P}(b,\varrho)\exp(-\epsilon_2\varrho h^{-1}) \tag{1.1.184}$$

where C_0 depends only on d and $\bar{P}(a,\varrho)$ is an actual sum of $P(a,\varrho)$.

Finally, consider two-sided parametrix.

Theorem 1.1.55. *Let $a\in \mathsf{S}_{h,\mathsf{an}}$ satisfy* (1.1.137)*. Then there exist $b\in \mathsf{S}_{h,\mathsf{an}}$ such that $a\bullet b=b\bullet a=I$.*

Proof. Note first that the following statement holds:

[5)] In [1] more general inequality is considered to be "easy". Still We prove this one (1.A.5).

(1.1.185) Let $a \in \mathsf{S}_{h,\mathsf{an}}(\varrho)$ with $\varrho > 0$. Assume that $\mathsf{sup}_{(x,\xi)} \|a_0(x,\xi)\| \le c$ and $\mathsf{sup}_{(x,\xi)} \|a_0(x,\xi)^{-1}\| \le c$. Then $a_0^{-1} \in \mathsf{S}_{h,\mathsf{an}}(\varrho')$ with some $\varrho' > 0$.

Then $a \bullet a_0^{-1} = I - hr$ with $r \in \mathsf{S}_{h,\mathsf{an}}(\varrho'')$ and $b = r' \bullet a$ with

$$r' \sim I + \sum_{n\ge 1} h^n \underbrace{r \bullet r \bullet \cdots \bullet r}_{n \text{ factors}}$$

and one can see easily that

$$P(r',\zeta) \lll 1 + \sum_{n\ge 1} \bigl(\zeta P(r,\zeta)\bigr)^n \tag{1.1.186}$$

which converges as $|\zeta| < \varrho'''$ with $\varrho''' \bar{P}(r,\varrho''') < 1$. □

1.1.4 Other Classes and Calculi of Pseudodifferential Operators

Let us describe now more traditional classes and calculi of pseudodifferential operators without visible parameter h. Instead of it a hidden parameter $\hbar = h_{\mathsf{eff}}$ appears. Now the notion of *order* appears; in semiclassical theory it was masked: instead of m-th order operator we used 0-th order one multiplied by h^{-m}. Another new property which comes with certain classes is invariance with respect to change of coordinates.

Classical Pseudodifferential Operators

Classical pseudodifferential operators were the first to be introduced historically:

Definition 1.1.56. (i) Symbol $a \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ belongs to $\mathsf{S}^m_{\mathsf{cl}}$ with $m \in \mathbb{R}$ if

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} \langle \xi \rangle^{m-|\beta|} \qquad \forall \alpha, \beta \tag{1.1.187}$$

and

$$a \sim \sum_{j\ge 0} a_j \tag{1.1.188}$$

with symbols $a_j \in \mathscr{C}^\infty\big(\mathbb{R}^d \times (\mathbb{R}^d \setminus 0)\big)$ positively homogeneous of degrees $(m-j)$ in the sense that

$$|D_x^\alpha D_\xi^\beta \big(a(x,\xi) - \sum_{j\le n} a_j(x,\xi)\big)| \le C_{n\alpha\beta}\langle\xi\rangle^{m-n=|\beta|} \qquad \forall n,\alpha,\beta. \tag{1.1.189}$$

(ii) Meanwhile $\mathsf{S}^{-\infty}$ is a class of symbols satisfying (1.1.86) for all m and such symbols are *negligible*[6)].

(iii) Class $\boldsymbol{\Psi}^m_{\mathsf{cl}}$ consists of pseudodifferential operators of the form $\mathsf{Op}(a)$ with $h=1$ and $a \in \mathsf{S}^m_{\mathsf{cl}}$. Such operators are defined modulo negligible operators where negligible means *infinitely smoothing* operator i.e. continuous from $\mathscr{H}^t$ to $\mathscr{H}^r$ for all $t, r \in \mathbb{R}$.

Here $a^0 = a_0$ is *principal symbol* and with Weyl quantization $a^s = a_1$ is *subprincipal symbol*. Here and below $\mathscr{H}^r$ are equipped with the standard norms $\|u\|_{\mathscr{H}^r} = \|(I+|D|^2)^{r/2}u\|_{\mathscr{L}^2}$.

One can consider $m \in \mathbb{C}$; then one needs to replace m by $\mathsf{Re}\, m$ in the right-hand expressions of (1.1.187) and (1.1.189). We do not list any properties of them except that

Theorem 1.1.57. *(i) Let $A \in \boldsymbol{\Psi}^{m_1}_{\mathsf{cl}}$, $B \in \boldsymbol{\Psi}^{m_2}_{\mathsf{cl}}$. Then $AB \in \boldsymbol{\Psi}^{m_1+m_2}_{\mathsf{cl}}$; if one of operators is scalar then $[A,B] \in \boldsymbol{\Psi}^{m_1+m_2-1}_{\mathsf{cl}}$.*

(ii) Let $A \in \boldsymbol{\Psi}^m_{\mathsf{cl}}$ with principal symbol a_0 and $|a_0^{-1}(x,\xi)| \le c$ for all (x,ξ). Then there exists a parametrix $B \in \boldsymbol{\Psi}^{-m}_{\mathsf{cl}}$: $AB \equiv BA \equiv I$ (modulo negligible); similar statements hold for left and right parametrices.

(iii) Let $m=0$ and either f be analytic in the vicinity of $\mathsf{Spec}(A)$ or f be smooth and A be Hermitian. Then $f(A) \in \boldsymbol{\Psi}^0_{\mathsf{cl}}$. Moreover let $m<0$ be an integer; then $f(A) - f(0) \in \boldsymbol{\Psi}^m_{\mathsf{cl}}$ (as m not being integer powers $ml-k$ with $l,k \in \mathbb{Z}^+$ will appear in decomposition (1.1.188).

*(iv) $\boldsymbol{\Psi}^m_{\mathsf{cl}}$ is invariant with respect to change of coordinates (*see *Teorem 1.1.62 below for more details).*

Remark 1.1.58. Since "negligible" now means "infinitely smoothing" then parametrix does not mean exact invertibility and $\mathsf{Spec}(A)$ does not shrink to $\Sigma(a)$ defined by (1.1.40) as $m=0$ or to 0 as $m<0$.

[6)] So actually symbol classes S^m_{cl} and $S^m_{\sigma,\varsigma}$ defined below are considered modulo $\mathsf{S}^{-\infty}$.

Theorem 1.1.59. *(i) Let* $A \in \boldsymbol{\Psi}^m_{\mathsf{cl}}(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ *be self-adjoint and moreover*

$$A - \epsilon(I + |D|^2)^{m/2} \geq 0 \tag{1.1.190}$$

in operator sense. Then $A^l \in\in \boldsymbol{\Psi}^{ml}_{\mathsf{cl}}(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ *for any* $l \in \mathbb{R}$ *(and even* $l \in \mathbb{C}$*); further, its principal symbol is* a_0^l *and if* a_0 *is scalar, its subprincipal symbol is* $l a^s a_0^{l-1}$.

(ii) Let $A \in \boldsymbol{\Psi}^m_{\mathsf{cl}}(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ *be self-adjoint and moreover*

$$\|Au\| \geq \epsilon \|(I + |D|^2)^{m/2} u\| \qquad \forall u \in \mathscr{L}(\mathbb{R}^d, \mathbb{H}); \tag{1.1.191}$$

Consider orthogonal (in $\mathscr{L}^2(\mathbb{R}^d, \mathbb{H})$*) projectors* $\Pi^{\pm}$ *on positive and negative subspaces of* A*. Then* $\Pi^{\pm} \in\in \boldsymbol{\Psi}^0_{\mathsf{cl}}$*; their principal symbols are* $\pi^{\pm}(x, \xi)$ *(orthogonal projectors to positive and negative subspaces of* $a_0(x, \xi)$*.*

Proof. Note first that assumptions (1.1.190) and (1.1.191) imply respectively that $a_0(x, \xi) - \epsilon|\xi|^m \geq 0$ and $\|a_0^{-1}(x, \xi)\| \leq c|\xi|^{-m}$.

(i) Without any loss of the generality one can assume that $\mathsf{Re}\, m \geq 0$ (otherwise we substitute A by A^{-1} and l by $-l$) and $-1 < \mathsf{Re}\, l < 0$ (otherwise $l = l'n' + n$ with l' satisfying this assumption and $n, n' \in \mathbb{Z}$). Then

$$A^l = -\frac{1}{2\pi i} \int_{-\infty i}^{+\infty i} \zeta^l (\zeta - A)^{-1}\, d\zeta \tag{1.1.192}$$

along $i\mathbb{R}$. One can see easily that operator $(|\zeta| + 1)(\zeta - A)^{-1}$ belongs to $\boldsymbol{\Psi}^0_{1,0}$ uniformly with respect to $\zeta \in i\mathbb{R}$ (see discussion of $\mathsf{S}^m_{\sigma,\varsigma}$ in the next two subsubsections) and therefore $A^l \in \boldsymbol{\Psi}^0_{1,0}$.

Decomposing symbol of $(\zeta - A)^{-1}$ into asymptotic series

$$r(x, \xi, \zeta) \sim \sum_{n \geq 0} r_n(x, \xi, \zeta)$$

with $r_n(x, \lambda\xi, \lambda^m\zeta) = \lambda^{-m-1} r_n(x, \xi, \zeta)$, $r_0 = \big(\zeta - a_0(x, \xi)\big)^{-1}$ and r_n a finite sum terms in the form

$$\big(\zeta - a_0(x, \xi)\big)^{-1} b_1(x, \xi) \big(\zeta - a_0(x, \xi)\big)^{-1} b_2(x, \xi) \cdots b_j(x, \xi) \big(\zeta - a_0(x, \xi)\big)^{-1}$$

with $(j+1)$ factors $\bigl(\zeta - a_0(x,\xi)\bigr)^{-1}$ we calculate easily that symbol of A^l belongs to $\mathsf{S}^{ml}_{\mathsf{cl}}$; furthermore its principal and subprincipal symbols equal

$$a^0_{(l)} = -\frac{1}{2\pi i}\int_{-\infty i}^{+\infty i} \zeta^l(\zeta - a_0)^{-1}\,d\zeta, \tag{1.1.193}$$

$$a^s_{(l)} = -\frac{1}{2\pi i}\int_{-\infty i}^{+\infty i} \zeta^l(\zeta - a_0)^{-1}a^s\zeta^l(\zeta - a_0)^{-1}\,d\zeta, \tag{1.1.194}$$

which implies the rest of Statement (i).

(ii) Observe that orthogonal projectors $\Pi^{\pm}$ for A and $A(A^2)^{-1/2}$ coincide and $A(A^2)^{-1/2} \in \boldsymbol{\Psi}^0_{\mathsf{cl}}$ due to (i). Then Statement (ii) follows from the functional calculus (Theorem 1.1.57(iii)). □

Classes $\mathsf{S}^m_{\sigma,\varsigma}$ and $\boldsymbol{\Psi}^m_{\sigma,\varsigma}$

If we consider symbols satisfying only (1.1.187) we get classes of symbols and operators denoted by $\mathsf{S}^m_{1,0}$ and $\boldsymbol{\Psi}^m_{1,0}$; all other properties remain. Next generalization is given by

Definition 1.1.60. (i) Symbol $a \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ belongs to $\mathsf{S}^m_{\sigma,\varsigma}$ with $m \in \mathbb{R}$, $0 \le \varsigma < \sigma \le 1$ if

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^{m-\sigma|\beta|+\varsigma|\alpha|} \qquad \forall \alpha,\beta. \tag{1.1.195}$$

(ii) Class $\boldsymbol{\Psi}^m_{\sigma,\varsigma}$ consists of pseudodifferential operators of the form $\mathsf{Op}(a)$ with $h = 1$ and $a \in \mathsf{S}^m_{\mathsf{cl}}$. Such operators are defined modulo negligible operators where negligible means infinitely smoothing operator.

If we consider $|\xi| \asymp \rho \gg 1$ and rescale $x \mapsto x\rho^\varsigma$, $\xi \mapsto \xi\rho^\sigma$ we get $\hbar$-pseudodifferential operator with $\hbar = \rho^{\sigma-\varsigma}$ and to have it small for $\rho \gg 1$ one needs to set $\varsigma < \sigma$.

We state only few most important results here:

Theorem 1.1.61. *(i) Let $a \in \mathsf{S}^{m_1}_{\sigma,\varsigma}$, $b \in \mathsf{S}^{m_2}_{\sigma,\varsigma}$. Then $a \bullet b \in \mathsf{S}^{m_1+m_2}_{\sigma,\varsigma}$ and*

$$a \bullet b \equiv ab \qquad \mathsf{mod}\ \mathsf{S}^{m_1+m_2-\sigma+\varsigma}_{\sigma,\varsigma}. \tag{1.1.196}$$

(ii) If one of symbols a, b *is scalar then* $(a \bullet b - b \bullet a) \in \mathsf{S}^{m_1+m_2-\sigma+\varsigma}_{\sigma,\varsigma}$ *and*

$$a \bullet b - b \bullet a \equiv \{a, b\} \qquad \mathsf{mod}\ \mathsf{S}^{m_1+m_2-\sigma+\varsigma}_{\sigma,\varsigma}. \tag{1.1.197}$$

(iii) Further, if one of symbols a, b *is scalar then*

$$a \bullet_{\mathsf{w}} b + b \bullet_{w} a \equiv 2ab \qquad \mathsf{mod}\ \mathsf{S}^{m_1+m_2-2\sigma+2\varsigma}_{\sigma,\varsigma}, \tag{1.1.198}$$

$$a \bullet_{\mathsf{w}} b - b \bullet_{w} a \equiv -i\{a, b\} \qquad \mathsf{mod}\ \mathsf{S}^{m_1+m_2-3\sigma+3\varsigma}_{\sigma,\varsigma}. \tag{1.1.199}$$

(iv) Let $A \in \boldsymbol{\Psi}^m_{\sigma,\varsigma}$ *with principal symbol* a_0 *and* $|a^{-1}(x,\xi)| \le c$ *for all* $(x,\xi) : |\xi| \ge c$. *Then there exists a parametrix* $B \in \boldsymbol{\Psi}^{-m}_{\sigma,\varsigma}$: $AB \equiv BA \equiv I$ *(modulo negligible); similar statements hold for left and right parametrices.*

(v) Let $m = 0$ *and either* f *be analytic in the vicinity of* $\mathsf{Spec}(A)$ *or* f *be smooth and* A *be Hermitian. Then* $f(A) \in \boldsymbol{\Psi}^0_{\sigma,\varsigma}$. *Moreover, let* $m < 0$. *Then* $f(A) - f(0) \in \boldsymbol{\Psi}^m_{\sigma,\varsigma}$.

Until now we did not discuss change of variables with a good reason: semiclassical pseudodifferential operators of subsection 1.1.2 did not possess this property (but those of Subsection 1.1.1 did).

Theorem 1.1.62. *(i) Let* $a \in \mathsf{S}^m_{\sigma,\varsigma}$ *with*

$$0 \le 1 - \sigma \le \varsigma < \sigma \le 1. \tag{1.1.200}$$

Let G *and* G' *be open domains in* $\mathbb{R}^d$ *and let* $\varkappa : G \to G'$ *be a diffeomorphism (onto* G'*). Then there exists a symbol* $a_{(\varkappa)} \in \mathsf{S}^m_{\sigma,\varsigma}$ *such that for every pair of functions* $f, g \in \mathscr{C}^\infty_0(G')$

$$\varkappa^* f a(x, hD) g \varkappa^{*-1} \equiv f' a_{(\varkappa)}(x, D) g' \tag{1.1.201}$$

modulo negligible where $\varkappa^*$ *is the induced map:* $\varkappa^* u = u \circ \varkappa$, $f' = f \circ \varkappa^{-1}, g' = g \circ \varkappa$.

(ii) Moreover, in T^*G' *the symbol* $a_{(\varkappa)}$ *is defined uniquely modulo negligible: in particular for* qp*-quantization*

$$a_{(\varkappa)}(x,\xi) \sim \Gamma_x \sum_\alpha \frac{1}{\alpha!} a^{(\alpha)}\bigl(\varkappa(x),\, {}^{\mathsf{t}}\varkappa'^{-1}(x)\xi\bigr) \times \\ (hD_y)^\alpha \exp\bigl(i\langle \varkappa(y) - \varkappa(x) - \varkappa'(x)(y-x), \xi\rangle\bigr) \tag{1.1.202}$$

where $\varkappa'(x)$ *is the Jacobi matrix of* $\varkappa$ *at* x *and* ${}^{\mathsf{t}}$ *means matrix transposition here,* $a^{(\alpha)}(x,\xi) = \partial^\alpha_\xi a(x,\xi)$.

Proof. Note that we need $1-\sigma \le \varsigma$ in order to have $a\bigl(\varphi(x),\, \check{\varkappa}'(x)\xi\bigr)$ belonging to $\mathsf{S}^m_{\sigma,\varsigma}$. Really, differentiating this symbol with respect to x through ξ we gain factor $\langle\xi\rangle^{-\sigma}$ because of definition of the class but also $\check{\varkappa}'(x)\xi$ i.e. we get factor $\langle\xi\rangle^{1-\sigma}$ which should not exceed $\langle\xi\rangle^{\varsigma}$.

Consider in qp-quantization

$$\varkappa^*(A\varkappa^{*-1}u)(x) = (2\pi)^{-d}\int a\bigl(\varkappa(x),\xi\bigr)e^{i\langle\varkappa(x)-y,\xi\rangle}u\bigl(\varkappa^{-1}(y)\bigr)\,dyd\xi$$

where we accommodated f and g' in $a(x,\xi)$ and u respectively. To be closer to the semiclassical arguments let us consider integral over zone $\{|\xi| \asymp h^{-1}\}$ only and change $\xi \mapsto \xi h$ arriving to

$$(2\pi h)^{-d}\int a\bigl(\varkappa(x),\xi\bigr)e^{ih^{-1}\langle\varkappa(x)-\varkappa(y),\xi\rangle}|\det\varkappa'(y)|u(y)\,dyd\xi$$

where we also changed variables of integration $y_{\mathsf{old}} = \varkappa(y)$. Plugging $\varkappa(x)-\varkappa(y) = \varkappa'(x)(x-y)-\varphi(x,y)$ we get $\exp\bigl(ih^{-1}\langle x-y-\varphi_1(x,y),\, \check{\varkappa}(x)\xi\rangle\bigr)$ with $\varphi_1(x,y) = \varkappa'^{-1}\varphi(x,y)$ and changing variables $\xi_{\mathsf{old}} = \varkappa'(x)\xi$ we get

$$(2\pi h)^{-d}\int a\bigl(\varkappa(x),\, \check{\varkappa}'^{-1}(x)\xi\bigr)e^{ih^{-1}\langle x-y+\varphi_1(x,y),\xi\rangle}\times \\ |\det\varkappa'(y)|\cdot|\det\varkappa'(x)|^{-1}u(y)\,dyd\xi.$$

Note that

$$e^{ih^{-1}\langle\varphi_1(x,y),\xi\rangle} \sim \sum_n \frac{1}{n!}h^{-n}\langle\varphi_1(x,y),\xi\rangle^n$$

with each term $O(|x-y|^{2n}h^{-n})$ and the similar error estimate and then integrating $2n$ times by parts with respect to ξ we get also factor h^{2n}; however these $2n$ derivatives could be applied to a symbol which leads to the factor $h^{2n(\sigma-1)}$ in the estimates[7]; so the total factor in estimates is $h^{n(2\sigma-1)}$. Noting that $\sigma > 1/2$ we prove that what we got is an asymptotic expansion.

Further, these arguments imply that it is sufficient to prove (1.1.202) for operators D^β only; however for them this formula is obvious. □

Corollary 1.1.63. *Let $a \in \mathsf{S}^m_{\mathsf{cl}}$. Then*

[7] Differentiating power of $\langle\varphi_1(x,y),\xi\rangle$ causes no extra powers of h.

(i) For pq-quantization

$$a_{(\varkappa)n} = \sum_{0\le k\le n} (\mathcal{L}_{n-k} a_k)\circ \Psi \tag{1.1.203}$$

where $\Psi : T^*G \to T^*G'$ is a symplectomorphism, induced by $\varkappa$:

$$\Psi(x,\xi) = (\varkappa(x), {}^t\varkappa'^{-1}(x)\xi), \tag{1.1.204}$$

$\mathcal{L}_k$ are linear $2k$-th degree differential operators with respect to ξ only, depending on $\varkappa$; in particular

$$\mathcal{L}_0 = I, \qquad \mathcal{L}_k 1 = 0 \quad \forall k \ge 1. \tag{1.1.205}$$

(ii) For any method of quantization $\mathcal{L}_0 = I$ and

$$\mathcal{L}_k = \sum_{|\beta|<|\alpha|,\, 1\le|\alpha|+|\beta|\le 2k} l_{k\alpha\beta}\partial_\xi^\alpha \partial_x^\beta \quad \forall k \ge 1. \tag{1.1.206}$$

(iii) Moreover, for Weyl quantization operators are of order not exceeding $(2k-1)$.

Proof. Statement (i) is trivial, (ii) follows from (i) and formulae describing transitions between quantizations.

To prove (iii) one needs to consider only differential operators; moreover, one needs to consider only operators with symbols $a(x,\xi) = b(x)\langle \ell,\xi\rangle^m$ with $\ell \in \mathbb{R}^d$. Then $a^{\mathrm{w}}(x,D)$ is a symmetric product of $\langle \ell, D_x\rangle$ (m copies) and $b(x)$. Therefore $\varkappa^* a^{\mathrm{w}}(x,D)\varkappa^{*-1}$ is a symmetric product of

$$\varkappa^*\langle \ell, D_x\rangle \varkappa^{*-1} = \langle \varkappa'^{-1}(x)\ell, D_x\rangle = j(x)^{-1/2} L^{\mathrm{w}}(x,D) j(x)^{1/2}$$

(m copies) and $\varkappa^* b \varkappa^{*-1} = b(\varkappa(x))$ with $L(x,\xi) = \langle \ell, {}^t\varkappa'^{-1}(x)\xi\rangle$ and $j(x) = |\det \varkappa'|$. Then (1.1.198) implies (iii). □

Remark 1.1.64. (i) In particular, above arguments imply that

$$a^0_{(\varkappa)} \circ \Psi, \tag{1.1.207}$$

$$a^s_{(\varkappa)} = \bigl(a^s - \frac{i}{2}\sum_j \partial_{\xi_j} a^0 \cdot \partial_{x_j} \log|\det \varkappa' \circ \varkappa^{-1}|\bigr)\circ \Psi \tag{1.1.208}$$

(ii) With the change of variables comes the theory of pseudodifferential operators on manifolds, and acting on sections of the bundles (with finite or infinite-dimensional fibers) over these manifolds rather than functions.

Beals-Fefferman and Hörmander Operators

Even more general classes of symbols (and operators associated with them) are due to R. Beals and R. Beals–C. Fefferman and are described by

$$\|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C_{\alpha\beta}\mu(x,\xi)\Phi(x,\xi)^{-|\beta|}\phi(x,\xi)^{-|\alpha|} \tag{1.1.209}$$

two scaling functions $\Phi(x,\xi)$, $\phi(x,\xi)$ and a weight function $\mu(x,\xi)$ such that

$$c^{-1} \le \frac{\Phi(x,\xi)}{\Phi(y,\eta)} \le c,\ c^{-1} \le \frac{\phi(x,\xi)}{\phi(y,\eta)} \le c,\ c^{-1} \le \frac{\mu(x,\xi)}{\mu(y,\eta)} \le c \tag{1.1.210}$$
$$\Longleftarrow\ |x-y| \le c^{-1}\phi(x,\xi),\ |\xi-\eta| \le \Phi(x,\xi).$$

In this case effective semiclassical parameter is $\hbar = 1/(\Phi\phi)(x,\xi)$ and we need condition $\Phi\phi \ge 1$.

Assume that weight function μ is identically 1. It is proven that $\mathsf{Op}^{\mathsf{w}}(a)$ is a bounded operator in $\mathscr{L}^2(\mathbb{R}^d)$ (or from in $\mathscr{L}^2(\mathbb{R}^d,\mathbb{H})$ to $\mathscr{L}^2(\mathbb{R}^d,\mathbb{H}')$ in case of operator valued symbols). Further, if we want negligible operators to be infinitely smoothing we need a condition

$$(\Phi\phi)(x,\xi) \ge (1+|\xi|^2)^{\delta/2}, \qquad \delta > 0; \tag{1.1.211}$$

further, if we want negligible operators to act from $\mathscr{S}'(\mathbb{R}^d)$ to $\mathscr{S}(\mathbb{R}^d)$ we need a condition

$$(\Phi\phi)(x,\xi) \ge (1+|\xi|^2+|x|^2)^{\delta/2}, \qquad \delta > 0. \tag{1.1.212}$$

Taking $\mu(x,\xi) = (\Phi\phi)(x,\xi)^m$ we get operators of order m; then principles "$\mathsf{ord}(ab) = \mathsf{ord}(a) + \mathsf{ord}(b)$" and "$\mathsf{ord}(\{a,b\}) = \mathsf{ord}(a) + \mathsf{ord}(b) - 1$" hold.

Without any loss of the generality we can assume that each of functions $a = \Phi$, $a = \phi$ and $a = \mu$ satisfies (1.1.209) with a weight function in the right-hand expression Φ, ϕ and μ respectively; otherwise one can replace them with equivalent functions satisfying this condition. Then one can introduce corresponding spaces $\mathscr{H}^s = \mathscr{H}^s_{\Phi\phi,\mu}$ with norms $\|u\|_{\mathscr{H}^s} = \|\,\mathsf{Op}(\Phi\phi)^s\,\mathsf{Op}(\mu)u\|_{\mathscr{L}^2}$.

The most general classes due to L. Hörmander use temperate metrics instead of temperate scaling functions L. Hörmander [1].

Global Operators

In particular one can select $\Phi(x,\xi) = \phi(x,\xi) = (1+|x|^2+|\xi|^2)^{1/2}$ and consider *global symbols* of order m:

$$\|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C_{\alpha\beta}(1+|x|^2+|\xi|^2)^{m-(|\alpha|+|\beta|)/2} \qquad \forall \alpha,\beta. \tag{1.1.213}$$

Operators of this class are invariant with respect to Fourier transform.

One can consider classical version of such symbols: they should admit decomposition (1.1.188) with a_j positively homogeneous of degrees $2(m-j)$ with respect to (x,ξ); operators of this class are invariant with respect to Fourier transform.

The simplest example is harmonic oscillator $|D|^2+|x|^2$; in the framework of this calculus it has an order 1.

Slightly more general class of operators is described by scaling functions $\Phi(x,\xi)=(1+|\xi|^2+|x|^{2\nu})^{1/2}$, $\phi(x,\xi)=(1+|\xi|^2+|x|^{2\nu})^{1/2\nu}$:

$$\|D_x^\alpha D_\xi^\beta a(x,\xi)|\le C_{\alpha\beta}(1+|x|^{2\nu}+|\xi|^2)^{m(1+1/\nu)-|\alpha|/(2\nu)-|\beta|/2}\quad \forall\alpha,\beta. \tag{1.1.214}$$

One can consider classical version of such symbols: the should admit decomposition (1.1.188) with a_j positively $(1,\nu)$-quasihomogeneous of degrees $(m-j)(\nu+1)$ with respect to (x,ξ): $a_j(\lambda x,\lambda^\nu\xi)=\lambda^{(m-j)(\nu+1)}a_j(x,\xi)$ as $\lambda>0$. The simplest example is an anharmonic oscillator $|D|^2+|x|^{2\nu}$; in the framework of this calculus it has an order $2\nu/(\nu+1)$. In this case one can prove statements similar to Theorems 1.1.57 and 1.1.59.

Both propagation of singularities and spectral asymptotics demonstrate that discussed operators behave like operators of prescribed orders on compact manifolds.

Furthermore, one can consider symbols and operators "global" with respect to some variables and ordinary with respect to others or with vector $\nu\in\mathbb{R}^{+d}$.

h-Pseudodifferential Operators

One can consider classes of operators described in this subsection and introduce semiclassical parameter h: i.e. replace operator $a(x,D)$ by $a(x,hD)$. Surely it affects the spaces as well: f.e. standard $\mathscr{H}^s$ should be replaced by spaces with the norm $\|u\|_s=\|(1+h^2|D|^2)^{s/2}u\|_{\mathscr{L}^2}$.

Then effective semiclassical parameter would be $\hbar=h/(\Phi\phi)(x,\xi)$ (for Beals-Fefferman operators). h-pseudodifferential operators with $\mathsf{S}^m_{\sigma,\varsigma}$ symbols admit a smooth change of coordinates provided $0\le 1-\sigma\le\varsigma<\sigma\le 1$; h-pseudodifferential operators with classical symbols also admit a smooth change of coordinates.

1.2 Lagrangian Distributions and Fourier Integral Operators

1.2.1 Motivation

Oscillatory Solutions to Evolution Equations

Let us consider evolution equation

$$hD_t u + a(x,t,hD_x)u = 0 \tag{1.2.1}$$

with the scalar symbol a and a real-valued a_0 and let us find oscillatory solutions

$$u(x,t) = e^{ih^{-1}\phi(x,t)} f(x,t). \tag{1.2.2}$$

Plugging (1.2.2) into (1.2.1) and applying Theorem 1.1.14 and Remark 1.1.16 we find that

$$\big(hD_t + a(x,t,hD_x)\big)u \sim \Big(\big(a_0(x,\nabla\phi) + \phi_t\big) + \sum_{n\ge 1} h^n \mathcal{L}_n\Big) f \times e^{ih^{-1}\phi(x,t)} \tag{1.2.3}$$

where $\mathcal{L}_n$ are differential operators of order n; in particular,

$$\mathcal{L}_1 = -i\partial_t - i\sum_j (\partial_{\xi_j} a_0)(x,\nabla\phi)\partial_{x_j} + K, \tag{1.2.4}$$

where for qp-quantization

$$K = \frac{i}{2}\sum_{j,k} (\partial^2_{\xi_j\xi_k} a_0)(x,\nabla\phi)\partial^2_{x_j x_k}\phi + a_1(x,\nabla\phi) = -\frac{i}{2}\sum_j \partial_{x_j} a_0^{(j)}(x,\nabla\phi)\big) + a^s(x,\nabla\phi) \tag{1.2.5}$$

where only in the last sum we first plug $\xi = \nabla\phi(x,t)$ and differentiate after; recall that $a_0^{(j)} = \partial_{\xi_j} a_0$.

To eliminate the first term in the right hand expression of (1.2.3) we need to impose condition to ϕ:

$$\phi_t + a_0(x,\nabla\phi) = 0 \tag{1.2.6}$$

which is called *eikonal equation* as ϕ is called *eikonal*. For a simplicity of the terminology we assume that

(1.2.7) a does not depend on t.

Note that this is *Hamilton-Jacobi equation* for *classical Hamiltonian dynamics*

$$\frac{d}{dt}(x,\xi) = \nabla^{\#} a_0(x,\xi) \iff \begin{cases} \dfrac{dx}{dt} = \nabla_\xi a_0(x,\xi), \\ \dfrac{d\xi}{dt} = -\nabla_x a_0(x,\xi), \end{cases} \tag{1.2.8}$$

where $\nabla^{\#} b = (\nabla_\xi b, -\nabla_x b)$ is a *Hamiltonian field* generated by b. We refer to V. I. Arnold [2] for details. For such equations

$$\frac{d}{dt}\phi = \langle \xi, a'_{0\xi}(x,\xi)\rangle, \qquad \nabla\phi = \xi \tag{1.2.9}$$

along trajectories of (1.2.8).

Equation (1.2.8) defines a *Hamiltonian flow* Ψ_t on the *phase space* $T^*X \ni (x,\xi)$ ($X = \mathbb{R}^d$); however after ϕ is selected we have a flow Φ_t on the *configuration space* $X \ni x$:

$$\frac{dx}{dt} = \nabla_\xi a_0(x, \nabla\phi(x,t)). \tag{1.2.10}$$

Remark 1.2.1. It is known that Ψ_t is a *symplectomorphism* and so preserves the volume form on the phase space; however the similar statement is not true for Φ_t and $\sum_j \partial_{x_j} a_0^{(j)}(x, \nabla\phi)$ is a divergence of this flow:

$$\sum_j \partial_{x_j} a_0^{(j)}(x,\nabla\phi) = \frac{d}{dt} \log |\det d\Phi_t|. \tag{1.2.11}$$

To eliminate the second term in the right hand expression of (1.2.3) we need to impose condition to f:

$$\bigl(\partial_t + \sum_j (\partial_{\xi_j} a_0)(x,\nabla\phi) + iK\bigr) f = 0 \tag{1.2.12}$$

which is equivalent to equation along trajectories of Φ_t

$$\Big(\frac{d}{dt} + iK\Big) f = 0 \tag{1.2.13}$$

and its called *transport equation*. Together with (1.2.11) it implies that

$$\Big(\frac{d}{dt} - 2\operatorname{Im} ia^s\Big)\big(|f|^2 \cdot |\det d\Phi_t|\big) = 0; \tag{1.2.14}$$

in particular for real a^s we get a kind of *energy conservation*

$$\frac{d}{dt}\big(|f|^2 \cdot |\det d\Phi_t|\big) = 0. \tag{1.2.15}$$

To eliminate all other term in (1.2.3) we pick up

$$f \sim \sum_{n\ge 0} f_n h^n \tag{1.2.16}$$

where f_0 we already defined and all other terms are defined recursively from transport equations

$$\Big(\frac{d}{dt} + iK\Big) f_n = -i \sum_{r\ge 2} \mathcal{L}_r f_{n+1-r}. \tag{1.2.17}$$

Remark 1.2.2. This construction is obviously extended to the matrix operators provided a_0 is scalar. Furthermore, this construction is extended even to more general matrix operators provided

$$\dim\ker\big(\phi_t + a_0(x, \nabla\phi)\big) = \text{const}. \tag{1.2.18}$$

So, we found asymptotic solutions to a equation (1.2.1) with initial condition

$$u|_{t=0} = e^{ih^{-1}\phi_0} f_0(x). \tag{1.2.19}$$

However *this solution is only local* because of the blow-up.

Blow-up

The trouble is that solution to eikonal equation (1.2.6) is only local. The geometric reason is very transparent: while Ψ_t is a symplectomorphism,

$$\Phi_t : x \to \pi_x \Psi_t(x, \nabla\phi_0(x)) \tag{1.2.20}$$

is a smooth map but may cease to be a diffeomorphism for $|t| \ge t_0$; here $\pi_x : (x,\xi) \to x$ is a projection map.

At the moment when Φ_t ceases to be a diffeomorphism the energy density $|f|^2$ surges due to (1.2.14) and this is the reason why propagators fail to be bounded operators from $\mathscr{L}^p(\mathbb{R}^d)$ to $\mathscr{L}^p(\mathbb{R}^d)$ for $p \ne 2$.

One can describe it in a different way:

$$\Lambda_0 = \{(x, \nabla\phi_0(x)),\ x \in X\} \subset T^*X \tag{1.2.21}$$

is a *Lagrangian submanifold*, i.e. submanifold of the maximal dimension (thus d) on which *symplectic form*

$$\sigma = \sum_j (d\xi_j \wedge dx_j) \tag{1.2.22}$$

identically vanishes. Then since Ψ_t is a symplectomorphism $\Lambda_t = \Psi_t\Lambda_0$ is a Lagrangian manifold as well. If $\pi_x : \Lambda_t \to \mathbb{R}^d$ is a local diffeomorphism one can (locally) describe Λ_t by (1.2.21) with ϕ_0 replaced by ϕ_t. Blow-up happens when $\mathsf{rank}(d\pi_x|_{\Lambda_t}) < d$.

Very geometrical discussion of blow-up (caustics, focussing) the reader can find in V. I. Arnold [3]. Extensive analytic theory could be found in V. M. Babich [1].

Maslov Canonical Operator

To tackle blow-ups V. Maslov [1] (see also M. Fedoryuk, V. Maslov [1]) suggested *Maslov canonical operator* method. Idea of this method is to pass to the different representation near the blow-up. While we do not use this method in our book, we sketch it here.

So, let us go to p-representations of (1.2.2):

$$\mathcal{F}(u)(\xi) = (2\pi h)^{-d/2} \int e^{ih^{-1}(-\langle x,\xi\rangle + \phi(x))} f(x)\, dx \tag{1.2.23}$$

where we ignore t-dependence. To calculate oscillatory integral (1.2.23) we need to apply *stationary phase formula* (see Appendix 1.A.3):

$$\nabla_x\big(-\langle x,\xi\rangle+\phi(x)\big)=0 \iff \xi=\nabla_x\phi(x) \tag{1.2.24}$$

and under the nondegeneracy assumption

$$\det \mathsf{Hess}\,\phi\neq 0 \tag{1.2.25}$$

we get

$$\mathcal{F}(u)(\xi)\sim e^{ih^{-1}\tilde\phi(\xi)}\big(\tilde f(\xi)+h\tilde f_1(\xi)+\dots\big) \tag{1.2.26}$$

with $x(\xi)$ defined by (1.2.24) (locally) uniquely under assumption (1.2.25); if there are several solutions to (1.2.24) we arrive to the sum of several expressions of (1.2.26) type; further

$$\tilde\phi(\xi)=-\langle\xi,x(\xi)\rangle+\phi\big(x(\xi)\big), \tag{1.2.27}$$

and

$$\tilde f_0(\xi)=e^{\frac{\pi i}{4}\,\mathsf{sgn}\,\mathsf{Hess}\,\phi}|\det \mathsf{Hess}\,\phi|^{-1/2}f_0\big|_{x=x(\xi)}. \tag{1.2.28}$$

Transition from $(x,\phi(x))$ to $(\xi,\tilde\phi(\xi))$ described above is called *Legendre transformation*.

Remark 1.2.3. The meaning of the second factor in (1.2.28) is also obvious in virtue of $|\det \mathsf{Hess}\,\phi|=|\det(\xi'(x))|$ where ξ' is a Jacobi matrix of $x\to\xi=\nabla_x\phi$. Due to this factor when we pass from x to ξ coordinates amplitude behaves as a *half-density* rather than a function: $|\tilde f|^2\,d\xi=|f|^2\,dx$ in the corresponding points where $d\xi$ and dx are Euclidean measures on $\mathbb{R}^d_x$ and $\mathbb{R}^d_\xi$ respectively.

Finally, in the first factor of (1.2.28) $\mathsf{sgn}\,b$ is a *signature* of a real symmetric matrix b, which is the number of its positive eigenvalues minus number of its negative ones. This factor manifests itself prominently when we return back to x after blow-up.

Similar formulae hold for partial h-Fourier transform and for (partial) inverse h-Fourier transform. Furthermore, locally on every Lagrange manifold Λ one can introduce coordinates $(x_{\mathcal{J}},\xi_{\mathcal{J}'})$ where $x_{\mathcal{J}}=(x_{j_1},\dots,x_{j_r})$, $\{j_1,\dots,j_r\}=\mathcal{J}\subset\{1,\dots,d\}$, $\mathcal{J}'=\{1,\dots,d\}\setminus\mathcal{J}$ and therefore we can always find an appropriate representation. To every such representation we

attribute locally a phase function and also an amplitude; moreover, modulo factor i^p (with $p \in \mathbb{Z}/4\mathbb{Z}$) this amplitude can be considered as half-density on Λ.

To understand the role of the first factor in (1.2.28) let us consider case $d = 1$. Then this factor is $\pm i$ if $\pm\frac{d\xi}{dx} > 0$. F.e. if Λ is as on picture below.

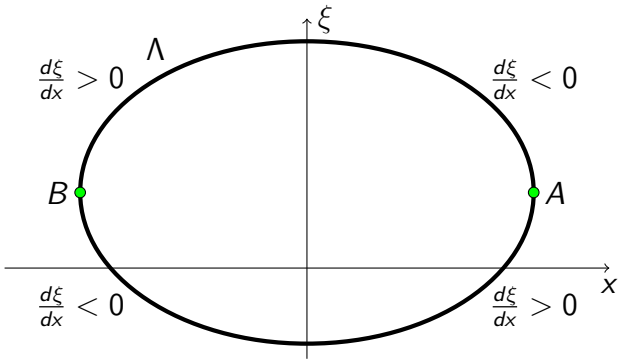

Figure 1.1: Maslov index

Here A and B are blow-up points, so signature when passing from $\{\frac{d\xi}{dx} > 0\}$ to $\{\frac{d\xi}{dx} < 0\}$ increases by -2 and thus factor $e^{-\frac{\pi i}{2}} = -i$ is acquired. Therefore after going counter-clockwise around Λ factor -1 is acquired.

More generally: consider multidimensional case. Let Λ be a Lagrangian submanifold in T^*X and $z, z' \in \Lambda$ such that $\operatorname{rank} d\pi_x(z) = \operatorname{rank} d\pi_x(z') = d$. Consider path $\ell(z', z)$ from z to z'. One can prove that this path could be deformed to the path along which $\operatorname{rank} d\pi_x = d$ in all but a discrete set of points $z_1, \dots, z_r$ and $\operatorname{rank} d\pi_x(z_j) = d - 1$.

Near point z_j one can select k such that $\operatorname{rank} d\pi_{x_{\hat{k}},\xi_k}(z_j) = d$; here $x_{\hat{k}}$ means "all coordinates but k-th". At this points *Maslov index* of the path grows (decreases) by 2 if we pass from $\{\frac{d\xi_k}{dx_k} < 0\}$ to $\{\frac{d\xi_k}{dx_k} > 0\}$ (in the opposite direction, respectively). This procedure introduces Maslov index of the path $\ell(z', z)$ and allows to introduce Maslov index of the closed path ℓ which does not depend on the choice of initial point.

The most important conclusion we carry from this construction, that solution for any time is expected to be (a sum of terms)

$$(2\pi h)^{-m} \int e^{ih^{-1}\varphi(x,\theta)} f(x,\theta,h)\, d\theta \tag{1.2.29}$$

with $\theta = \xi_{\mathcal{J}'}$, $\varphi(x,\theta) = \langle x_{\mathcal{J}'}, \theta\rangle + \phi(x_{\mathcal{J}}, \theta)$, $m = \#\mathcal{J}'$.

Propagator

Another approach is to consider initial phase $\langle x - y, \xi\rangle$ and initial function $(2\pi h)^{-d}$ and construct local solution in the form

$$\mathcal{K}(x,y,t) = (2\pi h)^{-d} e^{ih^{-1}(\phi(x,\xi,t) - \langle y,\xi\rangle)} b(x,\xi,t,h) \tag{1.2.30}$$

and this would be a Schwartz kernel of the propagator $e^{-ih^{-1}tA}$.

Then for any $t = t_1 + t_2 + ... + t_m$ we can construct a Schwartz kernel of propagator $e^{-ih^{-1}tA} = e^{-ih^{-1}t_1A} \cdots e^{-ih^{-1}t_mA}$

$$\mathcal{K}(x,y,t) = (2\pi h)^{-dm} \int e^{ih^{-1}\varphi(x,\theta,y,t)} b(x,\theta,y,t,h)\, d\theta \tag{1.2.31}$$

where we dumped all intermediate variables into $\theta = (\xi, x_{(1)}, \xi_{(1)}, \dots, x_{(m-1)}, \xi_{(m-1)})$ and $\varphi = \phi(x,\xi,t_1) - \langle x_{(1)}, \xi\rangle + \phi(x_{(1)}, \xi_{(1)}) - \langle x_{(2)}, \xi_{(1)}\rangle + ... - \langle y, \xi_{(m-1)}\rangle$.

Further, applying it to initial function $e^{ih^{-1}\psi} f_0$ we get a solution in the form

$$u(x,t) = (2\pi h)^{-dm} \int e^{ih^{-1}\varphi(x,\theta,t)} f(x,\theta,t,h)\, d\theta \tag{1.2.32}$$

where here we dumped y into θ as well and included $\psi(y)$ into φ. Expressions (1.2.32) and (1.2.31) will be used to define Lagrangian distributions and Schwartz kernels of Fourier integral operators.

Remark 1.2.4. There is a disparity between Lagrangian distributions and Lagrangian distributions which are Schwartz kernels of operators in $\mathscr{L}^2(X)$: the former with m-dimensional parameter $theta$ have factor $h^{-m/2}$ while the latter have factor $h^{-(m+d)/2}$ where $d = \dim(X)$.

Later we will consider Lagrangian distributions which are Schwartz kernels of operators $\mathscr{L}^2(X)$ to $\mathscr{L}^2(Y)$ and those will have factor $h^{-(m+d_X)/2}$ where $d_X = \dim(X)$ and $d_Y = \dim(Y)$.

1.2.2 Lagrangian Distributions and Manifolds

Definition

We start with the theory of Lagrangian distributions which are given by (1.2.32) without t and with ϕ satisfying non-degeneracy condition below.

Definition 1.2.5. A smooth real-valued function $\phi(x,\theta)$ defined for $(x,\theta) \in G \Subset \mathbb{R}^d \times \mathbb{R}^r$ is called *non-degenerate* if

$$(x,\theta) \in \bar{G}, \phi'_\theta(x,\theta) = 0 \implies \operatorname{rank}(\phi''_{\theta x}, \phi''_{\theta\theta}) = r \tag{1.2.33}$$

where $\phi''_{\theta x}$, $\phi''_{\theta\theta}$ are $d \times r$ and $r \times r$ matrices of the second derivatives, so $(\phi''_{\theta x}, \phi''_{\theta\theta})$ is $(d+r) \times r$ matrix.

The quantified version of (1.2.33) is

(1.2.34) If $|\phi'_\theta(x,\theta)| \le \epsilon$, then the matrix $(\phi''_{\theta x}, \phi''_{\theta\theta})$ has a minor (determinant) of dimension r, the absolute value of which is not less than ϵ with a small enough constant $\epsilon > 0$.

Let $\mathbb{H}$ be an auxiliary Hilbert space as before.

Definition 1.2.6. An $\mathbb{H}$-*valued Lagrangian distribution* is a distribution of the form

$$u(x) = (2\pi h)^{-\frac{r}{2}} \int a(x,\theta) e^{ih^{-1}\phi(x,\theta)}\, d\theta \tag{1.2.35}$$

with the smooth, real-valued and nondegenerate phase function $\phi \in \mathscr{C}^{N+K}(\bar{G})$ and smooth *amplitude* $a \in \mathscr{C}_0^{N+K}(G,\mathbb{H})$.

The quantitative versions of these conditions are:

$$\operatorname{diam} G \le c, \quad \operatorname{dist}(\operatorname{supp}(a), \partial G) \ge \epsilon, \tag{1.2.36}$$

$$|D_x^\alpha D_\theta^\nu \phi| \le L \quad \forall \alpha,\nu : |\alpha| \le N+K, |\nu| \le N+K \quad \forall (x,\theta) \in G, \tag{1.2.37}$$

$$\|D_x^\alpha D_\theta^\nu a\| \le M \quad \forall \alpha,\theta : |\alpha| \le N+K, |\nu| \le N+K \quad \forall (x,\theta). \tag{1.2.38}$$

Definition 1.2.7. More generally, let us consider amplitudes of the form (1.1.30) where a_j satisfy the previous conditions with N replaced by $(N-j)$. The class of these amplitudes we denote by $\dot{S}_{h,N}(G,\mathbb{H})$ and the least M for which these estimates hold we denote by $|||a|||$.

It is rather obvious that the same Lagrangian distribution could be defined (modulo $O(h^\infty)$) by different phase functions and amplitudes and the first question to answer is "How one can characterize phase functions defining the same Lagrangian distributions"?

Definition 1.2.8. *Lagrangian manifold generated by a phase function* ϕ is

$$\Lambda_\phi = \{(x, \phi'_x(x,\theta)) : (x,\theta) \in \Sigma_\phi\} \subset T^*\mathbb{R}^d \tag{1.2.39}$$

where

$$\Sigma_\phi = \{(x,\theta) \in G, \phi'_\theta(x,\theta) = 0\}. \tag{1.2.40}$$

Then $\Lambda = \Lambda_\phi$ and Σ_ϕ are respectively $\mathscr{C}^{N+K-2}$ and $\mathscr{C}^{N+K-1}$ manifolds of dimension d. Furthermore

(1.2.41) Λ_ϕ is a *Lagrangian manifold* which means that $\dim(\Lambda) = d$ and the restriction of the symplectic form $\sigma = \sum_j d\xi_j \wedge dx_j$ to Λ equals 0,

(1.2.42) The map $\iota : \Sigma_\phi \ni (x,\theta) \to (x, \phi'_x(x,\theta)) \in \Lambda_\phi$ is a local diffeomorphism.

One can prove easily that

$$r - \operatorname{rank} \phi''_{\theta\theta}(x,\theta) = d - \operatorname{rank} \pi'_{\Lambda,x}(\iota(x,\theta)), \tag{1.2.43}$$

where $\pi_{\Lambda,x} : \Lambda \ni (x,\xi) \to x \in \mathbb{R}^d$ and $\pi'_{\Lambda,x}$ is its differential.

Main Properties

For a justification of all our future calculations we need

Proposition 1.2.9. *Let conditions* (1.2.34) *and* (1.2.36)–(1.2.38) *be fulfilled with* $K = K_0(d,r)$, $N = 0$. *Then the* $\mathscr{L}^2(\mathbb{R}^d, \mathbb{H})$*-norm of* u *does not exceed* $C\|\!|a|\!\|$ *with* $C = C(d,r,c,\epsilon)$.

Proof. The proof of this proposition follows from the arguments of Chapter 21.1 of L. Hörmander [1]. Namely, consider

$$\|u\|^2 = (2\pi h)^{-r} \iiint \langle a(x,\theta),\, a(x,\theta')\rangle e^{ih^{-1}(\phi(x,\theta)-\phi(x,\theta'))}\, dx d\theta d\theta' \tag{1.2.44}$$

which is an oscillatory integral with the phase function $\varphi(z) = \phi(x,\theta) - \phi(x,\theta')$, $z = (\theta,\theta',x)$ (and with the "deficient" power of $(2\pi h)$). One can see easily that the (quantified) nondegeneracy condition for ϕ implies (quantified) condition $\varphi'_z = 0 \implies \operatorname{rank} \varphi''_{zz} \ge 2r$. Then sthe tationary phase method reduces this integral to the finite sum of (oscillatory) integrals with d variables and 0-th power of $(2\pi h)$. □

Remark 1.2.10. Since $(hD)^\alpha u$ is a distribution of the same form again we obtain that its $\mathscr{L}^2(\mathbb{R}^d, \mathbb{H})$-norm also does not exceed $C\|\!|a|\!\|$ with $C = C(d,r,N,c,\epsilon)$ provided $|\alpha| \le N$ in the conditions.

Then the stationary phase method immediately implies

Proposition 1.2.11. *Let ι be a one-to-one map, and let $\pi_{\Lambda,x}$ be a diffeomorphism*[8)]*. Then for $K = K(d,r)$ a distribution u of the form* (1.2.35) *can be rewritten in the following form:*

$$u(x) = \tilde{a}(x)e^{ih^{-1}\tilde{\phi}(x)} + U(x) \tag{1.2.45}$$

where

$$\tilde{\phi}(x) = \phi(\iota^{-1}\pi'^{-1}_{\Lambda,x}(x)), \tag{1.2.46}$$

$\tilde{a} \in \dot{S}_{h,N}(\pi'_{\Lambda,x}\iota G)$ *with* $K = K(d,0)$, $\|\tilde{a}\| \le C\|a\|$ *where the norms are calculated in the corresponding classes,*

$$\tilde{a}_0(x) = e^{\frac{\pi i}{4}\operatorname{sgn}(\phi''_{\theta\theta})}|\det(\phi''_{\theta\theta})|^{-\frac{1}{2}}a_0\big|_{(x,\theta)=\iota^{-1}\pi'^{-1}_{\Lambda,x}(x)}, \tag{1.2.47}$$

and the $\mathscr{L}^2(\mathbb{R}^d,\mathbb{H})$-norm of U does not exceed $Ch^N\|a\|$ with $C = C(d,r,N,c,\epsilon)$; moreover the same is true for $(hD)^\alpha U$ provided $|\alpha| \le k$ and we add k to K and take C also depending on k[9)]*.*

Remark 1.2.12. One can see easily that $\Lambda = \{(x,\tilde{\phi}'_x(x)) : x \in \pi'_{\Lambda,x}\iota G\}$ in this case.

Definition 1.2.13. A point $z \in \Lambda$ is called *regular* provided

$$\operatorname{rank}\pi'_{\Lambda,x}(z) = d.$$

Therefore Proposition 1.2.11 states that if all points of a Lagrangian manifold are regular then one can take $r = 0$ in (1.2.35) (we assume that ι is a one-to-one map and that $K = K(d,r)$).

Let us consider the case when $\operatorname{rank}\pi'_{\Lambda,x}(z) = k$ with $0 \le k \le d-1$. Then there exists a partition of variables $x = (x',x'') \in \mathbb{R}^k \times \mathbb{R}^{d-k}$ (preceded by a change of enumeration of the coordinates if necessary) such that if $\xi = (\xi',\xi'')$ is the corresponding partition of the dual variables in $T^*_x\mathbb{R}^d$ then

$$\operatorname{rank}\pi'_{\Lambda,(x',\xi'')}(z) = d \tag{1.2.48}$$

[8)] I.e., let $\pi_{\Lambda,x}$ be a one-to-one map with $\operatorname{rank}\pi'_{\Lambda,x} = d$ at every point of Λ; then $|\det\pi'_{\Lambda,x}| \ge \epsilon$ at every point of Λ.

[9)] Such functions we call (k,N)-negligible or simply negligible.

where $\pi'_{\Lambda,(x',\xi'')} : \Lambda \ni (x,\xi) \to (x',\xi'')$ and the prime means differential (i.e. we assume that the absolute value of the determinant of this matrix is not less than ϵ).

Then one can prove easily that the $r^* \times r^*$ matrix

$$\Phi = \begin{pmatrix} \phi''_{\theta\theta} & \phi''_{\theta x''} \\ \phi''_{x''\theta} & \phi''_{x''x''} \end{pmatrix} \tag{1.2.49}$$

with $r^* = r + d - k$ is non-degenerate. Let us consider function

$$v(x',\xi'') = (2\pi h)^{\frac{k-d}{2}} \int e^{ih^{-1}\langle x'',\xi''\rangle} u(x)\, d\xi''$$

which is a partial h^{-1}-Fourier transform according to Maslov's terminology. Then v is a distribution of the form (1.2.35) with $r, x, \theta, a(x,\theta)$ and $\phi(x,\theta)$ replaced by $r^*, y = (x',\xi'')$, $\theta^* = (\theta, x'')$, $a^*(y,\theta^*) = a(x,\theta)$ and $\phi^*(y,\theta^*) = \phi(x,\theta) - \langle x'',\xi''\rangle$ respectively.

Surely the new amplitude is not compactly supported with respect to ξ'' but it can easily be improved by multiplying by the cutting function $\zeta(\xi'')$ with a fixed $\zeta \in \mathscr{C}_0^\infty(\mathbb{R}^{d-k})$ supported in the ball of appropriate radius R and equal to 1 in the ball of radius $\frac{1}{2}R$; one can prove easily by the stationary phase method that the error function V as well as $\langle\xi\rangle^k V$ are negligible.

Applying Proposition 1.2.11 one can rewrite v in the form (1.2.45) with the phase function

$$S(x',\xi'') = \phi\big(\iota^{-1}\pi^{-1}_{\Lambda,(x',\xi'')}(x',\xi'')\big) - \langle x'',\xi''\rangle \tag{1.2.50}$$

with x'' calculated at the point $x = \pi_{\Lambda,x}\pi^{-1}_{\Lambda,(x',\xi'')}(x',\xi'')$ instead of the phase function $\tilde\phi$. Hence we obtain

Proposition 1.2.14. *Let conditions* (1.2.36)–(1.2.38) *be satisfied with* $K = K(d,r)$*; let* ι *be a one-to-one map and for all* $z \in \Lambda_\phi$*. Finally. let condition* (1.2.48) *be fulfilled.*

Then one can rewrite (modulo negligible functions) a distribution u *of the form* (1.2.35) *in the same form with* θ *replaced by* ξ''*, with the phase function*

$$\psi(x,\xi'') = S(x,\xi'') + \langle x'',\xi''\rangle \tag{1.2.51}$$

and with a new amplitude b *satisfying the same inequality for amplitude norms, with the same constants here and in the negligibility condition as before.*

One can see easily that

$$\Lambda_\phi = \Lambda_\psi = \{(x,\xi) : x'' = -S'_{\xi''}(x',\xi''), \xi' = S'_{x'}(x'',\xi'')\}; \tag{1.2.52}$$

so despite ever changing phase function, corresponding set $\Sigma_.$ and dimension of θ, Lagrangian manifold Λ_ϕ stays the same.

Also any distribution of the form (1.2.35) can be rewritten modulo negligible functions as the sum of distributions of the same form with $r \le d$; recall that amplitudes can be chosen compactly supported. Here assumption that ι is one-to-one is no more needed.

The arguments used in the proofs of Propositions 1.2.11 and 1.2.14 imply that the terms in the decomposition of the amplitude $b(x',\xi'')$ are defined by the formula

$$b_n = \sum_{0\le j\le n} (\mathcal{L}_{n-j} a_j) \circ \iota^{-1} \pi^{-1}_{\Lambda,(x',\xi'')} \tag{1.2.53}$$

where $\mathcal{L}_j$ are $(2j)$-th order linear differential operators depending on ϕ. In particular,

$$b_0 = e^{\frac{\pi i}{4}\operatorname{sgn}\Phi} |\det \Phi|^{-\frac12} a_0 \circ \iota^{-1} \pi^{-1}_{\Lambda,(x',\xi'')}. \tag{1.2.54}$$

This implies that

Corollary 1.2.15. *Classes of distributions defined by different phase functions coincide modulo negligible functions provided these functions generate the same Lagrangian manifold and we take appropriate K.*

On the other hand, it is well known[10)] that if Λ is a Lagrangian manifold and if condition (1.2.48) is fulfilled at $z \in \Lambda$ then Λ is generated in a neighborhood of z by the phase function (1.2.51) with S defined uniquely modulo an additive constant.

Therefore we conclude that

Corollary 1.2.16. *For a compact subset Λ_0 of the Lagrangian $\mathscr{C}^M$ manifold Λ, the spaces $I_{h,N,k}(\mathbb{R}^d, \Lambda_0, \mathbb{H})$ of Lagrangian distributions are well defined.*

[10)] L. Hörmander [1], Chapter XXV and Fedoryuk-Maslov [1].

Here we assume that in the ϵ-neighborhood of each point $z \in \Lambda_0$ the Lagrangian manifold coincides with $\{l_j(x,\xi) = 0 \quad \forall j = 1, \dots, d\}$ with all the derivatives up to order $K+N$ of the functions l_j not exceeding c and with the absolute value of some d-dimensional minor of the $d \times 2d$ matrix $(l_{jx_k}, l_{j\xi_k})_{j,k=1,\dots,d}$ greater than or equal to ϵ; then locally after an appropriate partition of coordinates in $\mathbb{R}^d \times \mathbb{R}^d$ Lagrangian manifold Λ is defined by (1.2.52) with appropriate function S.

Let us consider the density δ_Σ on the manifold $\Sigma = \Sigma_\phi$ which is the push-forward of the Dirac measure δ in $\mathbb{R}^r$ under the map $(x,\theta) \to \phi'_\theta(x,\theta)$; then

$$\delta_\Sigma = \frac{|dz|}{\left|\frac{D(\lambda,\phi'_{\theta'})}{D(x,\theta)}\right|}$$

where $z = (z_1, \dots, z_d)$ is an arbitrary local coordinate system on Λ continued as $\mathscr{C}^1$ functions in a neighborhood of Λ and $|dz|$ is the Lebesgue measure on Λ. Then (1.2.54) implies that

$$a_0 \delta_\Sigma^{\frac{1}{2}} \circ \iota^{-1} = \tilde{a}_0 \delta_{\tilde{\Sigma}}^{\frac{1}{2}} \circ \tilde{\iota}^{-1}$$

provided the manifold Λ in a neighborhood of z is generated by two phase functions ϕ and $\tilde{\phi}$ with coinciding r, $\operatorname{sgn}\Phi$ and with the coinciding values on Λ [11)] and a_0 and $\tilde{a}_0$ are the leading terms of the symbols a and $\tilde{a}$ and (a,ϕ) and $(\tilde{a},\tilde{\phi})$ define the same Lagrangian distribution (modulo negligible functions).

More generally, (1.2.54) implies that the half-density

$$a^0 = e^{\frac{\pi i}{4}\operatorname{sgn}\Phi} a_0 \delta_\Sigma^{\frac{1}{2}} \circ \iota^{-1} \tag{1.2.55}$$

does not depend (modulo a factor i^p with $p \in \mathbb{Z}/4\mathbb{Z}$) on the choice of ϕ generating Λ where here

$$\Phi = \begin{pmatrix} \phi''_{\theta\theta} & \phi''_{\theta x} \\ \phi''_{x\theta} & \phi''_{xx} \end{pmatrix}$$

is a $(d+r) \times (d+r)$ matrix (see Proposition 25.1.5 of L. Hörmander [1]).

Definition 1.2.17. $\mathbb{H}$-valued half-density a^0 defined on Λ modulo factor i^p with $p \in \mathbb{Z}/4\mathbb{Z}$ is called *principal symbol of the Lagrangian distribution* u.

[11)] The values of the functions at Λ are well defined since Λ and Σ are identified by means of ι; one can prove easily that two phase functions with coinciding r, $\operatorname{sgn}\Phi$ and Λ coincide at Λ provided they coincide at z.

Maslov Index

Locally on Λ a branch of i^p can be chosen uniquely; however it is not always possible globally (see below). To avoid this "modulo factor" one can use *Maslov bundle* $\mathcal{M}$ defined on Λ; so principal symbol is actually $\mathbb{H}\otimes\mathcal{M}$-valued (see L. Hörmander [1], Definition 21.6.5). However notion of Maslov index seems to be more transparent.

Namely, assume that $\iota : \Sigma \to \Lambda$ one-to-one map. Let us consider some path ℓ without self-intersections; consider two regular points z_1 and z_2. Then

$$\operatorname{sgn} \phi''_{\theta\theta}\big|_{\iota^{-1}z_1}^{\iota^{-1}z_2} = 2\operatorname{ind}_{\mathsf{M}} \ell(z_1, z_2) \tag{1.2.56}$$

where $\operatorname{ind}_{\mathsf{M}}$ is the Maslov index and $\ell(z_1, z_2)$ is any curve from z_1 to z_2.

In the general case one can define Maslov index of any path and then one can select branch i^p globally if and only if

$$\operatorname{ind}_{\mathsf{M}} \ell \equiv 0 \qquad \operatorname{mod} 4\mathbb{Z} \tag{1.2.57}$$

for every closed curve ℓ on Λ (more precisely, for every class $[\ell]$ of homotopic closed curves since Maslov indices of homotopic closed curves coincide). This condition means exactly that Maslov bundle is trivial.

Since $d\phi = \langle \xi, dx\rangle$ on Λ [12] it follows that the global unique continuous definition of a^0 is impossible unless

$$\int_\ell \langle \xi, dx\rangle = 0 \tag{1.2.58}$$

for every closed curve ℓ on Λ (more precisely, for every class $[\ell]$ of homotopic closed curves since these integrals for homotopic closed curves coincide): for a Lagrangian manifold the exterior form $\langle \xi, dx\rangle|_\Lambda$ is closed[13].

Definition 1.2.18. On the other hand, under quantization conditions (1.2.57) and (1.2.58) the symbol a^0 can be chosen globally uniquely for u and conversely starting from a^0 one can define $u \in I_{h,N,k}(\mathbb{R}^d, \Lambda, \mathbb{H})$ with principal symbol a^0. The isomorphic map $a^0 \mapsto u$ is called the *Maslov canonical operator*.

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[12] In fact, $d\phi = \langle \phi'_x, dx\rangle + \langle \phi'_\theta, d\theta\rangle = \langle \xi, dx\rangle$.

[13] In fact, $d\langle \xi, dx\rangle = -\sigma$.

Remark 1.2.19. The same arguments imply that if we want to consider not all the values of $h \in (0, h_0]$ but only a certain sequence tending to $+0$ then we can replace conditions (1.2.57) and (1.2.58) by one quantization condition

$$\frac{1}{4}\,\mathsf{ind}_M[\ell] + (2\pi h)^{-1}\int_{[\ell]} \langle \xi, dx\rangle \equiv 0 \qquad \mathsf{mod}\ \mathbb{Z} \tag{1.2.59}$$

for every class $[\ell]$ of homotopic closed curves and for every h from this sequence.

Condition (1.2.59) is called *Bohr-Sommerfeld quantization condition*. This condition plays a crucial role in calculation of eigenvalues of one-dimensional operators.

Other Properties

In conclusion we give some other properties of Lagrangian distributions:

Theorem 1.2.20. *Let u be a distribution of the form* (1.2.35) *and ϕ, a satisfy conditions* (1.2.36)–(1.2.38) *with $M = K + N$, $a \in \dot{S}_{h,N}(G, \mathbb{H})$ with $K = K(d, r, k)$. Moreover, let us assume that for every $n \le s$ the amplitude a_n has a $(2s - 2n + 1)$-th order zero at Σ_ϕ. Then*

(i) There exists an amplitude $a' \in \dot{S}_{h,N}(G, \mathbb{H})$ with $a'_n = 0$ for $n \le s$ such that $\|a'\| \le C\|a\|$ and a' defines u modulo negligible functions[14]*.*

(ii) The terms in the decomposition of a' are defined by the formula

$$a'_n = \sum_{0\le j\le n-s-1} \mathcal{L}_j a_{n-s-j} \tag{1.2.60}$$

where $\mathcal{L}_j$ are $2j$-th order linear differential operators.

Proof. Obviously $a_n \sim \sum_{|\alpha|\ge(2s-2n+1)} b_{n\alpha}(\nabla_\theta\phi)^\alpha$ with $b_{n\alpha}$ being $|\alpha|$-th order differential operators applied to a_n. Integration $|\alpha|$ times by parts concludes the proof. □

Theorem 1.2.21. *Let u bes a distribution of the form* (1.2.35) *and ϕ satisfy condition* (1.2.37) *with $M = K + N$. Let $a \in \dot{S}_{h,N}(G, \mathbb{H})$ with $K = K(d, r, k)$.*

Let $P \in S_{h,N}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{H}', \mathbb{H})$. Then

[14] More precisely, the $\mathscr{L}^2(\mathbb{R}^d, \mathbb{H}')$-norm of $(hD)^\alpha(u - v)$ does not exceed $C_\alpha\|a\| \cdot h^{N+1}$ for $|\alpha| \le k$.

(i) There exists a symbol $b \in \dot{S}_{h,N}(G, \mathbb{H}')$ *such that* $|||b||| \le C|||a||| \cdot |||P|||$ *(we recall that all norms are calculated in the corresponding classes) and* b *defines a Lagrangian distribution* v *with the same phase function coinciding modulo negligible functions with* $P(x, hD, h)u$ [15]*. Moreover,*

$$b_0(x,\theta) = P_0(x, \phi'_x(x,\theta))a_0(x,\theta) \tag{1.2.61}$$

and therefore

$$b^0(x,\xi) = P^0(x,\xi)a^0(x,\xi) \quad \text{on } \Lambda. \tag{1.2.61}'$$

(ii) Finally, if $P^0(x,\xi) = 0$ *on* Λ *then* $v = hw$ *where* w *is also a Lagrangian distribution with the principal symbol*

$$c^0 = -i\{P^0, a^0\} + Ka^0 \quad \text{on } \Lambda \tag{1.2.62}$$

where the Poisson bracket does not depend on a smooth continuation of a^0 *from* Λ *and* K *is a symbol depending on* P^0 *and* Λ*.*

Proof. (i) Statement (i) follows immediately from Theorem 1.1.14.

(ii) Since all objects engaged in (1.2.62) are invariant with respect to action of unitary Fourier integral operator $u \mapsto \mathcal{F}^* u, P \mapsto \mathcal{F}^* P \mathcal{F}$ (see next subsection) then one can assume without any loss of the generality that $u = b(x)$ and $\Lambda = \{\xi = 0\}$. In this case formula (1.2.62) becomes trivial. □

Remark 1.2.22. Obviously all these statementss remain true if we replace $\mathbb{H}$-valued distributions and symbols (amplitudes) by $\mathcal{L}(\mathbb{H}, \mathbb{H}_0)$-valued ones where $\mathbb{H}_0$ is also an auxiliary Hilbert space.

Complex Phase

The similar construction works provided ϕ is complex-valued function satisfying condition $\operatorname{Im}\phi \ge 0$. Then one should redefine

$$\Sigma_\phi = \{(x,\theta) : \phi'_\theta(x,\theta) = 0,\ \operatorname{Im}\phi(x,\theta) = 0\} \tag{1.2.63}$$

[15] More precisely, the $\mathscr{L}^2(\mathbb{R}^d, \mathbb{H}')$-norm of $(hD)^\alpha(P(x, hD, h)u - v)$ does not exceed $C|||a||| \cdot |||P|||h^{N+1}$ for $|\alpha| \le k$.

(instead of (1.2.40)), while preserving (1.2.39) for Λ. However Λ is not a Lagrangian manifold anymore, but a "real part" of a *positive Lagrangian ideal* (Section 25.2 of L. Hörmander [1]). In this case all non-degeneracy conditions are supposed to be fulfilled at the real points (i.e. of Σ and Λ).

We only mention that such distributions allow to generalize construction of the oscillatory solutions for operators with scalar principal symbols with $\operatorname{Im} a^0(x,\xi) \le 0$ and with initial data of the same type as before. Then Hamiltonian trajectories while going into complex domain where $\operatorname{Im}\phi > 0$ would never become real again and they do not contribute in Λ_t.

Coherent States

Out of all such Lagrangian distributions a special role is played by those with $\phi(x,\theta) = \phi(x)$ with

$$\operatorname{Im}\phi(x) \ge \epsilon|x-y|^2 \tag{1.2.64}$$

(i.e. $\phi = 0 \iff x = y$ and moreover $\operatorname{Im}\phi''(y) > 0$). In this case

$$\Lambda = \{(y, \nabla\phi(y))\} \tag{1.2.65}$$

consists of a single point.

The simplest example of such distribution is the one with quadratic phase functions

$$\phi(x) = \bar{\phi} + \langle x-y, \eta\rangle + \frac{1}{2}\langle Q(x-y), (x-y)\rangle \tag{1.2.66}$$

where Q is symmetric matrix and $\operatorname{Im} Q$ is positive.

Lagrangian distribution with the general phase function $\phi(x)$ satisfying (1.2.64) could be reduced to one with the quadratic phase function and with amplitudes

$$f_h(x) \sim \sum_{\alpha, n \le |\alpha|/3} c_{\alpha n}(x-y)^\alpha h^{-n} \tag{1.2.67}$$

where here $n \in \mathbb{Z}$ may be negative.

Such functions are called *Gaussian beams* (due to the obvious reason) or *coherent states* (not sure why). However, we add an extra factor $h^{-d/4}$ since

$$\|e^{ih^{-1}\phi} f\| \le C h^{d/4} \tag{1.2.68}$$

provided ϕ satisfies (1.2.64).

The remarkable property of such distributions is that such form is preserved in the propagation with the scalar real-valued a^0 for arbitrarily long time (despite all the caustics). In this case $y(t), \eta(t)$ and $\bar{\phi}(t)$ evolve according to the Hamiltonian equation (1.2.8) (with (y,η) instead of (x,ξ)) and $d\bar{\phi}(t) = \langle \eta(t), dy(t)\rangle$ and $Q(t)$ evolves according to

$$\frac{d}{dt}Q(t) = -Q(t)A(t)Q(t) - B(t)Q(t) - Q(t)B^T(t) - C(t), \tag{1.2.69}$$

with

$$A(t) = a^{0\prime\prime}_{\eta\eta}(y(t),\eta(t)), \quad B(t) = a^{0\prime\prime}_{y\eta}(y(t),\eta(t)), \quad C(t) = a^{0\prime\prime}_{yy}(y(t)\eta(t)), \tag{1.2.70}$$

where $B^T(t) = a^{0\prime\prime}_{y\eta}(y(t),\eta(t))$ means transposed matrix.

The proof of (1.2.69)–(1.2.70) is based on the linearization of the Hamiltonian system at $(y(t),\eta(t))$

$$\frac{d}{dt}\delta x = A(t)\delta x + B(t)\delta\xi, \qquad \frac{d}{dt}\delta\xi = -B^T(t)\delta x - C(t)\delta\xi, \tag{1.2.71}$$

which preserves both forms

$$\sum_j \delta\xi_j \wedge \delta x_j \qquad \text{and} \qquad \mathsf{Im}\sum_j \delta\xi_j \cdot \delta x_j^\dagger; \tag{1.2.72}$$

and therefore on the tangent plane to Λ_t the first one is 0 and the second one is positive definite and therefore this plane is given by $\delta\xi = Q(t)\delta x$ with symmetric $Q(t)$, $\mathsf{Im}\, Q(t) > 0$.

Coherent states surely look more like particles than waves.

Another (closely related) property is that $\mathcal{T}u$ is a coherent state for any Fourier integral operator $\mathcal{T}$ (see next subsection).

1.2.3 Fourier Integral Operators

Let us now proceed to the theory of h-Fourier integral operators. Generally speaking (see Section 25.2 of L. Hörmander [1]) Fourier integral operators are operators whose Schwartz kernels are Lagrangian distributions in $\mathbb{R}^d \times \mathbb{R}^m$ corresponding to Lagrangian manifold Λ^T with respect to symplectic form

$\sigma = \sigma_X + \sigma_Y$. However we consider a slightly more narrow but far the most important (and the only one we need in this book) case of Λ being a *canonical graph*. In this case automatically $m = d$.

Definition 1.2.23. A Lagrangian manifold Λ in $X \times Y = \mathbb{R}^d \times \mathbb{R}^d$ equipped with a symplectic form

$$\sigma_X - \sigma_Y = \sum_j d\xi_j \wedge dx_j - \sum_j d\eta_j \wedge dy_j \tag{1.2.73}$$

is a *canonical graph*, if projectors $\pi_X : \Lambda \to T^*X$ and $\pi_Y : \Lambda \to T^*Y$ are local diffeomorphisms and therefore $\Psi\pi_X\pi_Y^{-1} : T^*Y \to T^*X$ is a local diffeomorphism (and then $\Lambda = \mathsf{graph}(\Psi)$ is its graph).

On the other hand, let $\mathcal{X}$ and $\mathcal{Y}$ be domains in phase spaces T^*X and T^*Y equipped with symplectic forms $\sigma_X = \sum_j d\xi_j \wedge dx_j$ and $\sigma_Y = \sum_j d\eta_j \wedge dy_j$ respectively; here $X = \mathbb{R}^d \ni x$ and $Y = \mathbb{R}^d \ni y$.

Let $\Psi : \mathcal{Y} \to \mathcal{X}$ be a diffeomorphism such that

$$|D^\alpha_{y,\eta}\Psi(y,\eta)| \le c \qquad \forall \alpha : |\alpha| \le M \quad \forall (y,\eta) \in \mathcal{Y} \tag{1.2.74}$$

and the same inequalities remain true for the inverse diffeomorphism Ψ^{-1} and $(x,\xi) \in \mathcal{X}$.

Remark 1.2.24. Then the following statements are equivalent:

(i) Ψ is a symplectomorphism, i.e., Ψ preserves the symplectic structure ($\Psi_*\sigma_Y = \sigma_X$) or (what is the same) Ψ preserves Poisson brackets ($\{f \circ \Psi, g \circ \Psi\} = \{f,g\} \circ \Psi$ for all functions f, g on Ω_X; one need check only $f, g \in \{x_1, \dots, x_d, \xi_1, \dots, \xi_d\}$).

(ii) $\Lambda = \mathsf{graph}(\Psi)$ is a Lagrangian manifold with respect to the symplectic form $\sigma = \sigma_X - \sigma_Y$ or (what is the same) $\Lambda^T = \{(x,\xi,y,\eta) : (x,\xi,y,-\eta) \in \Lambda\}$ is a Lagrangian manifold with respect to the symplectic form $\sigma^T = \sigma_X + \sigma_Y$.

Definition 1.2.25. Let $\Psi : \mathcal{Y} \to \mathcal{X}$ be a symplectomorphism, Λ its graph, Λ_0 a compact subset of Λ with $\mathsf{dist}(\Lambda_0, \bar{\Lambda} \setminus \Lambda) \ge \epsilon > 0$. A *Fourier integral operator* is an operator A with Schwartz kernel $\mathcal{K}_A(x,y)$ such that $(2\pi h)^{d/2}\mathcal{K}_A(x,y) \in I_{h,N,k}\big(\mathbb{R}^d \times \mathbb{R}^d, \Lambda_0^T, \mathcal{L}(\mathbb{H}', \mathbb{H})\big)$.

On the other hand, consider $\phi(x,y,\theta)$ is a phase function satisfying (1.2.36)–(1.2.37) with (x,y) instead of x and generating Lagrangian manifold Λ^T then

$$\mathcal{K}_A(x,y) = (2\pi h)^{-\frac{r+d}{2}} \int a(x,y,\theta) e^{ih^{-1}\phi(x,y,\theta)}\, d\theta \tag{1.2.75}$$

with $a \in \dot{S}_{h,N}(G_0, \mathcal{L}(\mathbb{H}', \mathbb{H}))$ where G_0 is an admissible domain in $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^r$.

Then locally

$$\Lambda = \big\{(x, \phi'_x(x,y,\theta), y, -\phi'_y(x,y,\theta)) : \phi'_\theta(x,y,\theta) = 0\big\}. \tag{1.2.76}$$

Example 1.2.26. (i) If the differential of the map $\Lambda \ni (x,\xi,y,\eta) \to (x,\eta)$ is non-degenerate then we can take $\phi(x,y,\theta) = S(x,\eta) - \langle y,\eta\rangle$ with $\theta = \eta$ and then

$$\mathcal{K}_A(x,y) = (2\pi h)^{-\frac{d}{2}} \int a(x,y,\eta) e^{ih^{-1}(S(x,\eta) - \langle y,\eta\rangle)}\, d\eta \tag{1.2.77}$$

and in this case $\Lambda = \{(x, S'_x(x,\eta), S'_\eta(x,\eta), \eta)\}$.

(ii) In particular, for $S(x,\eta) = \langle y,\eta\rangle$ we obtain an h-pseudodifferential operator with compactly supported symbol.

(iii) The map $\Psi : (y,\eta) = (y', y'', \eta', \eta'') \to (y', \eta', \eta', -y'')$ with $y' \in \mathbb{R}^k$, $y'' \in \mathbb{R}^{d-k}$ corresponds to the phase function

$$\phi(x,y,\eta') = \langle x' - y', \eta'\rangle - \langle x'', y''\rangle$$

and for a symbol equal to 1 we obtain the operator which is the partial h-Fourier transform:

$$(\mathcal{F}'' u)(x) = (2\pi h)^{(k-d)/2} \int e^{-ih^{-1}\langle x'', y'\rangle} u(x', y'')\, dy'' \tag{1.2.78}$$

(surely 1 is not a compactly supported function but we can save this by multiplying F'' on the left or right by an h-pseudodifferential operator with compactly supported symbol (see Theorem 1.2.29(ii) below)). With this choice of the scalar factor $\mathcal{F}''$ is an unitary operator.

(iv) Suppose $\mathcal{U}$ and $\mathcal{V}$ are domains in X and Y respectively and $\psi : \mathcal{V} \to \mathcal{U}$ is a diffeomorphism; let us introduce the operator $A : u \to P(fug \circ \psi^{-1})$ with $f, g \in C_0^M(\mathcal{V})$ and an h-pseudodifferential operator P with compactly supported symbol. Then A corresponds to the symplectomorphism $\Psi : (y,\eta) \to (\psi(y), {}^t\psi'^{-1}(y)\eta)$ where the notation of Section 1.1 is used.

(v) The special case of a symplectomorphism described in (i) is a map $(y,\eta)\to(y,\eta-\psi')$ with a real-valued scalar function ψ; the corresponding Fourier integral operator is of the form $Pe^{ih^{-1}\psi}$ (or $e^{ih^{-1}\psi}P$) where P is an h-pseudodifferential operator with compactly supported symbol.

Remark 1.2.27. Egorov's theorem (see L. Hörmander [1] states that every symplectomorphism locally is a composition of those described in Example 1.2.26(i), (iii) and (iv); moreover, microlocally (see the next Subsection) every Fourier integral operator is a product of those described in (i), (iii) and (iv).

Theorem 1.2.28. *Let $\phi(x,y,\theta)$ be a phase function which satisfies conditions* (1.2.36)–(1.2.37) *and suppose Λ is the Lagrangian manifold given by* (1.2.76). *Assume that Λ is a canonical graph, i.e.*

(1.2.79) *At every point $(x,\xi,y,\eta)\in\Lambda$ the differentials of the maps $\pi_X:\Lambda\ni(x,\xi,y,\eta)\to(x,\xi)$ and $\pi_Y:\Lambda\ni(x,\xi,y,\eta)\to(y,\eta)$ have rank $2d$* [16].

Further, let $a\in\dot{\mathsf{S}}_{h,N}(G,\mathcal{L}(\mathbb{H}',\mathbb{H}))$ with $N=0$, $K=K(d,r)$ and A is an operator with Schwartz kernel defined by (1.2.75).

Then $\|A\|\le C\|\!|a|\!\|$ where $\|.\|$ means the $\mathcal{L}\bigl(\mathscr{L}^2(\mathbb{R}^d,\mathbb{H}'),\mathscr{L}^2(\mathbb{R}^d,\mathbb{H})\bigr)$-norm and $C=C(d,r,c,\epsilon)$ [17].

Proof. By the arguments of Section 25.2 of L. Hörmander [1]. First, due to Theorem 1.2.29(i),(ii) below A^*A is an h-Fourier integral operator with the identical canonical map, i.e. pseudodifferential operator. Applying Theorem 1.1.19 we arrive to the statement. □

Applying this Theorem one can prove easily that all the statements of Proposition 1.2.14, Corollaries 1.2.15 and 1.2.16 and Theorems 1.2.20 and 1.2.21 (with P staying either to the left or to the right of A) remain true for Fourier integral operators; one should only replace (1.2.35) by (1.2.74) and the $\mathscr{L}^2(\mathbb{R}^d,\mathbb{H})$-norm by the $\mathcal{L}\bigl(\mathscr{L}^2(\mathbb{R}^d,\mathbb{H}'),\mathscr{L}^2(\mathbb{R}^d,\mathbb{H})\bigr)$-norm and in (1.2.61)

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[16] Equivalently, at every point $(x,y,\theta)\in\Sigma_\phi$ the matrix $\begin{pmatrix}\phi''_{\theta\theta}&\phi''_{\theta x}\\ \phi''_{x\theta}&\phi''_{xx}\end{pmatrix}$ is non-degenerate. In quantitative terms: $|\det\begin{pmatrix}\phi''_{\theta\theta}&\phi''_{\theta x}\\ \phi''_{x\theta}&\phi''_{xx}\end{pmatrix}|\ge\epsilon>0\quad\forall(x,y,\theta)\in\Sigma_\phi$.

[17] Moreover, the same is true for the operator $(hD)^\alpha A(hD)^\beta$ provided $|\alpha|+|\beta|\le k$ and K is replaced by $K+k$; in the conditions and C depends on k now.

one should replace $P_0(x,\phi'_x)$ by $P_0(y,-\phi'_y)$ in the case when P is to the right of A; formulas (1.2.61)$'$ and (1.2.62) should be changed in the same manner.

Moreover, we obtain the following statements which one can prove in the same manner

Theorem 1.2.29. *(i) Let A be a Fourier integral operator with the phase function $\phi(x,y,\theta)$ and symbol $a(x,y,\theta)$. Then the adjoint operator A^* is also a Fourier integral operator with the phase function $-\phi(y,x,\theta)$ and symbol $a^\dagger(y,x,\theta)$ and*

$$\Psi_{A^*} = \Psi_A^{-1} \tag{1.2.80}$$

where here and below Ψ_A is the symplectomorphism corresponding to A.

(ii) Let A and B be Fourier integral operators with the phase functions $\phi_A(x,y,\theta')$ and $\phi_B(y,z,\theta'')$ and symbols $a(x,y,\theta')$ and $b(y,z,\theta'')$ respectively. Then AB is also a Fourier integral operator with phase function $\phi_A(x,z,\theta')+\phi_B(z,y,\theta'')$ and symbol $a(x,z,\theta')b(z,y,\theta'')$, where now $\theta=(\theta',z,\theta'')\in\mathbb{R}^r$, $r=r'+d+r''$.

Moreover, if ϕ_A and ϕ_B satisfy (1.2.36), (1.2.37) *and* (1.2.79) *then the new phase function also satisfies those conditions (maybe with other constants c and ϵ) and*

$$\Psi_{AB} = \Psi_A \circ \Psi_B. \tag{1.2.81}$$

Further, if $a\in \dot{S}_{h,N}(G'',\mathbb{H}'',\mathbb{H}')$ and $b\in \dot{S}_{h,N}(G',\mathbb{H}',\mathbb{H})$ respectively and we remove from the product ab all terms with degree of h greater than N then the $\mathcal{L}\big(\mathscr{L}^2(\mathbb{R}^d,\mathbb{H}''),\mathscr{L}^2(\mathbb{R}^d,\mathbb{H})\big)$-norm of the error does not exceed $C\|\!|a|\!\|\cdot\|\!|b|\!\|h^{N+1}$; moreover, footnote [17)] remains true.

Furthermore,

$$c^0(x,\xi,z,\zeta) = a^0(x,\xi,y,\eta)b^0(y,\eta,z,\zeta) \tag{1.2.82}$$

for $(x,\xi)=\Psi_A(y,\eta)$, $(y,\eta)=\Psi_B(z,\zeta)$.

(iii) Let A be a Fourier integral operator with phase function $\phi(x,y,\theta')$ which satisfies (1.2.36), (1.2.37) *and* (1.2.79) *and with symbol $a(x,y,\theta')$ and let u be a Lagrangian distribution with phase function $\psi(y,\theta'')$ which satisfies* (1.2.36), (1.2.37) *and* (1.2.33) *and with amplitude $b(y,\theta'')$.*

Then Au *is a Lagrangian distribution with phase function* $\psi'(x,\theta) = \phi(x,y,\theta') + \psi(y,\theta'')$ *and amplitude* $b(x,\theta) = a(x,y,\theta')b(y,\theta'')$*, where* $\theta = (\theta', y, \theta'') \in \mathbb{R}^r$*,* $r = r' + d + r''$*. Moreover,* $\psi'(x,\theta)$ *satisfies* (1.2.36)*,* (1.2.37) *and* (1.2.33) *and*

$$\Lambda_{Au} = \Psi_A(\Lambda_u) \tag{1.2.83}$$

where here Λ_u *is the Lagrangian manifold corresponding to the Lagrangian distribution* u*.*

Further, if $a \in \dot{S}_{h,N}(G'', \mathbb{H}', \mathbb{H})$ *and* $b \in \dot{S}_{h,N}(G', \mathbb{H})$ *and we remove from the product* ab *all terms with degree of* h *greater than* N *then the* $\mathscr{L}^2(\mathbb{R}^d, \mathbb{H}')$*-norm of the error does not exceed* $C\|a\| \cdot \|b\| h^{N+1}$*; moreover, the natural modification of footnote* [17] *remains true.*

Furthermore

$$d^0(x,\xi) = a^0(x,\xi,y,\eta)b^0(y,\eta) \tag{1.2.82}'$$

for $(x,\xi) = \Psi_A(y,\eta)$ *where* c^0 *and* d^0 *are the principal symbols of* AB *and* Au *respectively.*

Theorem 1.2.30. *Let* A *be a Fourier integral operator with phase function* ϕ *satisfying* (1.2.36)*,* (1.2.37) *and* (1.2.79) *with* $K = K(d,r)$ *and with symbol* $a \in \dot{S}_{h,N}(G, \mathbb{H}', \mathbb{H})$ *with* $K = K(d,r)$*.*

Moreover, let $p \in S_{h,N}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{H}'', \mathbb{H}')$ *and* $q \in S_{h,N}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{H}, \mathbb{H}'')$*.*

Then there exist symbols $b \in \dot{S}_{h,N}(G, \mathbb{H}'', \mathbb{H})$ *and* $b' \in \dot{S}_{h,N}(G, \mathbb{H}', \mathbb{H}')$ *with* $\|b\| \le C\|P\| \cdot \|a\|$ *and* $\|b'\| \le C\|P\| \cdot \|a\|$ *such that the* $\mathcal{L}\big(\mathscr{L}^2(\mathbb{R}^d, \mathbb{H}''), \mathscr{L}^2(\mathbb{R}^d, \mathbb{H})\big)$*-norm of the operator* $p(x, hD, h)A - B$ *and the* $\mathcal{L}\big(\mathscr{L}^2(\mathbb{R}^d, \mathbb{H}'), \mathscr{L}^2(\mathbb{R}^d, \mathbb{H}'')\big)$*-norm of the operator* $Aq(x, hD, h) - B'$ *do not exceed* $C\|P\| \cdot \|a\| h^{N+1}$ *and, moreover, footnote* [17] *remains true. Here* B *and* B' *are Fourier integral operators with phase function* ϕ *and symbols* b *and* b' *respectively.*

Moreover,

$$b_0(x,y,\theta) = p_0(x, \phi'_x(x,\theta)) \cdot a_0(x,y,\theta), \tag{1.2.84}$$

$$b'_0(x,y,\theta) = a_0(x,y,\theta) \cdot q_0(x, -\phi'_y(x,\theta)) \tag{1.2.84}'$$

and hence

$$b^0(x,\xi,y,\eta) = p_0(x,\xi) \cdot a^0(x,\xi,y,\eta), \tag{1.2.85}$$

$$b'^0(x,\xi,y,\eta) = a^0(x,\xi,y,\eta) \cdot q_0(y,\eta) \tag{1.2.85}'$$

where the principal symbol of a Fourier integral operator equals the principal symbol of its Schwartz kernel at the ^{T}-transposed point.

In what follows we assume that the domains and ranges of all symbols and operators agree.

Corollary 1.2.31. *(i) Let A and B be Fourier integral operators with symplectomorphisms Ψ and Ψ^{-1} respectively and let $p(x, hD, h)$ be an h-pseudodifferential operator.*

Then modulo negligible operators $q(x, hD, h) \equiv Bp(x, hD, h)A$ is also an h-pseudodifferential operator and

$$q_n = \sum_{0\le j\le n} \mathcal{L}_{n-j}(p_j)\circ\Psi \tag{1.2.86}$$

where $\mathcal{L}_j$ are $(2j)$-th degree linear differential operators of the form

$$\mathcal{L}_j(q) = \sum_{k,|\alpha|\le 2j} c_{jk\alpha}(D^\alpha_{x,\xi}q)c'_{jk\alpha}$$

with operator-valued coefficients $c_{jk\alpha}$ and $c'_{jk\alpha}$ (for scalar A and B we obtain scalar differential operators); in particular,

$$q_0(x,\xi) = b^0((x,\xi),\Psi(x,\xi))\cdot p_0(\Psi(x,\xi))\cdot a^0(\Psi(x,\xi),(x,\xi)). \tag{1.2.87}$$

(ii) Assume that the operator A is elliptic for (x,ξ,y,η) with $(x,\xi) = \Psi_A(y,\eta)$ and $(y,\eta)\in\mathcal{Y}'\Subset\mathcal{Y}$, i.e., $a^0(x,\xi,y,\eta)$ is invertible and the norm of the inverse symbol does not exceed c (as well as $|||a|||$).

Then there exists a Fourier integral operator B corresponding to the symplectomorphism Ψ_A^{-1} such that $BA\equiv I$ in $\mathcal{Y}'$ (i.e., the equality is true modulo the sum of an h-pseudodifferential operator with symbol vanishing in this neighborhood and a negligible operator) and $AB\equiv I$ in $\mathcal{X}' = \Psi_A(\mathcal{Y}')$.

In particular, if A is a scalar operator and $P = p(x, hD, h)$ is an h-pseudodifferential operator then in $\mathcal{Y}$ for the principal symbol of $Q\equiv BPA$ the following equality holds:

$$q_0 = p_0\circ\Psi. \tag{1.2.88}$$

Let us note that for a given symplectomorphism Ψ and domain $\mathcal{Y}'$ an elliptic symbol does not necessarily exist because of obstacles of a topological nature. However, such symbols certainly exist (even in the scalar case) provided quantization conditions (1.2.57) and (1.2.58) are fulfilled; in particular, such symbols exist provided $\mathcal{Y}'$ is simply connected. In this case A^*A is an h-pseudodifferential operator with a Hermitian symbol and with the positive definite principal symbol in the vicinity of $\mathcal{Y}'$. Then we arrive to

Theorem 1.2.32. *Let $\Psi : \mathcal{Y} \to \mathcal{X}$ be a local symplectomorphism satisfying condition* (1.2.74) *with graph satisfying* (1.2.57) *and* (1.2.58). *Suppose $\mathcal{Y}' \Subset \mathcal{Y}$ and let $\mathcal{X}' = \Psi(\mathcal{Y}')$. Then there exists a scalar Fourier integral operator A corresponding to Ψ such that $A^*A \equiv I$ and $AA^* \equiv I$ in neighborhoods of $\mathcal{Y}'$ and $\mathcal{X}'$ respectively.*

One can easily rewrite all these assertions in the uniform form which we used before. Using a partition of unity in $\mathbb{R}^d \times \mathbb{R}^d$ and Corollary 1.2.31 one can prove

Theorem 1.2.33. *Let A and B be scalar Fourier integral operators with symplectomorphisms Ψ and Ψ^{-1} respectively and let p be a symbol supported in $G \Subset \mathbb{R}^d \times \mathbb{R}^d$ belonging to $\mathsf{S}_{h,N}(\mathbb{R}^d, \mathbb{H}', \mathbb{H})$ after rescaling $x \mapsto x/\rho$, $\xi \mapsto \xi/\rho$, $\rho \in (0,1]$. Let $\hbar = h\rho^{-2} \in (0,1]$.*

Then there exists symbol supported in $G' = \Psi(G)$ and belonging to $\mathsf{S}_{h,N}(G', \mathbb{H}', \mathbb{H})$ after the same rescaling such that $\|\!|q|\!\|_\rho \le C\|\!|p|\!\|_\rho$ (where $\|\!|.|\!\|_\rho$ denotes the norm of the rescaled symbol)

$$\|q(x, hD, h) - Bp(x, hD, h)A\| \le C\rho^{-4d}\hbar^{N+1}\|\!|p|\!\| \tag{1.2.89}$$

with $C = C(d, c, N, G, A, B)$ and $K = K(d)$ in the conditions to Ψ, A, B. Moreover, footnote[17] *remains true.*

Remark 1.2.34. (i) One can easily replace in (1.2.89) factor ρ^{-4d} by the factor $(\rho^{-2d}\,\mathsf{mes}\,\mathsf{supp}(P)+1)$. I do not know whether it is possible to replace the factor ρ^{-4d} by 1 here. However we have no need of these improvements.

(ii) Since when transforming pseudodifferential operators by Fourier integral operators we mess x and ξ, in this and similar theorems one needs to replace in the definition of $\mathsf{S}_{h,\dots,N}$ range $|\alpha| \le N + K$, $|\beta| \le N + K$ by $|\alpha| + |\beta| \le 2N + K$.

However we are not concerned about N and we will skip N in notations at all (or assume $N = \infty$).

The following statement is well known (see, e.g., L. Hörmander [1]):

Theorem 1.2.35. *Let $A \in \boldsymbol{\Psi}_{h,N}(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ and suppose its principal symbol a_0 is real-valued and scalar. Let $P \in \boldsymbol{\Psi}_{h,N}(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ be an operator with compactly supported symbol. Then $e^{ith^{-1}A}P$ and $Pe^{ith^{-1}A}$ are Fourier integral operators (for any $t \in \mathbb{R}$) with symplectomorphism $\Psi = \Phi_t$ where $\Phi_t = \exp(tH_{a_0})$ is the Hamiltonian flow generated by the Hamiltonian field $-\nabla^{\#}a_0$; recall that $\nabla^{\#}a_0 = (\nabla_\xi a_0, -\nabla_x a_0)$.*

In conclusion let us note that the notations for negligible functions and negligible operators do not agree completely: if $h^{d/2}\mathcal{K}_A$ is an (N,k)-negligible function then A is an $(N - \frac{d}{2}, k)$-negligible operator; conversely, if A is an (N,k)-negligible operator then $h^{d/2}\mathcal{K}_A$ is an $(N, k - \lfloor \frac{d}{2} \rfloor - 1)$-negligible function. However this fact should not lead to confusion.

Final Remarks

Remark 1.2.36. (i) So far we constructed propagator $U(t) = e^{-ih^{-1}tA}$ of operator A which does not depend on t. Construction of propagator $U(t,t')$ instead of $U(t-t')$ of t-dependent A is done by a similar way.

(ii) As in Subsection 1.2.2 we considered Lagrangian distributions with the complex phase, we can construct h-Fourier integral operators with the complex phase. They appear as parametrix for operators with scalar principal symbol a with $\operatorname{Im} a \le 0$.

(iii) Since any function could be decomposed into coherent states due to

$$\delta(x) = (2\pi h)^{-3d/2} \iint e^{ih^{-1}\langle x,\xi\rangle - \frac{1}{2}h^{-1}|x-z|^2}\, d\xi dz \tag{1.2.90}$$

one can construct propagator constructing solutions with the initial data $e^{ih^{-1}\langle x-y,\xi\rangle - \frac{1}{2}h^{-1}|x-y-z|^2}$ (instead of the usual procedure of plane waves decomposition.

(iv) One can construct the theory for "analytic" phase functions and symbols using Fourier-Bros-Iagolnitzer transform (see f.e. A. Martinez [1]).

(v) Non semiclassical theory is well known.

1.2.4 Metaplectic Operators

Let us consider operators with quadratic symbols:

$$(1.2.91)_1 \quad A = \sigma(FZ, Z) = \frac{1}{2}\sum_{i,j}\bigl(\alpha_{ij}x_ix_j + \alpha'_{ij}(x_ihD_j + hD_ja_i) + \alpha''_{ij}h^2D_iD_j\bigr)$$

$$(1.2.92)_1 \qquad a = \sigma(Fz, z) = \frac{1}{2}\sum_{i,j}\bigl(\alpha a_{ij}x_ix_j + 2\alpha'_{ij}x_i\xi_j + \alpha''_{ij}\xi_i\xi_j\bigr)$$

where $\sigma(z,z') = \langle\xi, x'\rangle - \langle x, \xi'\rangle$ for $z = (x,\xi)$ and $z' = (x',\xi')$ is the skew-symmetric bilinear *symplectic form*, $Z = (x, hD)$ is a vector-valued operator from $\mathscr{H}^1(\mathbb{R}^d) := \{u\colon u, x_ju, D_ju \in \mathscr{L}^2(\mathbb{R}^d, \mathbb{C}^{2d})\ \forall j = 1, \dots, d\}$ to $\mathscr{L}^2(\mathbb{R}^d, \mathbb{C}^{2d})$, $F = \begin{pmatrix} -{}^t\alpha' & -\alpha'' \\ \alpha & \alpha' \end{pmatrix}$ is the *skew-Hessian* of a, $\alpha = (\alpha_{ij})$, $\alpha' = (\alpha'_{ij})$, $\alpha'' = (\alpha''_{ij})$ are $d \times d$ matrices and (α), (α'') are assumed to be symmetric and all the coefficients here and below are assumed to be real.

Moreover, let us consider operators with linear symbols:

$$(1.2.91)_{\frac12} \qquad A = \sigma(\ell, Z) = \sum_j(\alpha_jx_j + \alpha'_jhD_j),,$$

$$(1.2.92)_{\frac12} \qquad a = \sigma(\ell, z) = \sum_j(\alpha_jx_j + \alpha'_j\xi_j),$$

where $\ell = (-\alpha, \alpha')$, $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha' = (\alpha'_1, \dots, \alpha'_d)$ and the subscript means the genuine order of the operator in the framework of the "correct" calculus.

One can see easily that

$$(1.2.93) \qquad \sigma(Fz, z') = -\sigma(z, Fz'), \qquad \forall z, z'$$

$$(1.2.94) \qquad \sigma(\nabla^\# f, \nabla^\# g) = \{f, g\}$$

and

(1.2.95) $\nabla^\# a = -Fz$, $\nabla^\# a = -\ell$ for quadratic and linear symbols respectively.

Let $\mathfrak{A}_1$ and $\mathfrak{A}_{\frac12}$ be the spaces of operators with real-valued quadratic and real-valued linear symbols respectively and let $\mathfrak{A}_0 = \mathbb{R}$. Then

(1.2.96) $\mathfrak{A} = \mathfrak{A}_1$ and $\tilde{\mathfrak{A}} = \mathfrak{A}_1 \oplus \mathfrak{A}_{\frac12} \oplus \mathfrak{A}_0$ are Lie algebras with Lie operation $A, B \mapsto ih^{-1}[A, B]$ (moreover, $\tilde{\mathfrak{A}}$ is a graded Lie algebra)

and

$$\dim \mathfrak{A} = d(2d+1), \qquad \dim \tilde{\mathfrak{A}} = (d+1)(2d+1). \tag{1.2.97}$$

On the other hand, there are the natural isomorphisms $\mathfrak{A}_j \simeq \mathfrak{B}_j$, etc., where $\mathfrak{B}_1$ is the space of all real $2d \times 2d$ matrices satisfying (1.2.93), $\mathfrak{B}_{\frac12} = \mathbb{R}^d$, $\mathfrak{B}_0 = \mathbb{R}$ equipped by induced Lie operations $F, F' \mapsto -[F, F']$ and $(F; \ell; c), (F'; \ell'; c') \mapsto (-[F, F']; -Fl' + F'l; \sigma(l, l'))$ on $\mathfrak{B}$ and $\tilde{\mathfrak{B}}$ respectively.

The following statements are either well known or easy (see, e.g., Leray [1]):

Proposition 1.2.37. *(i) Let $A \in \tilde{\mathfrak{A}}$ and $\mathfrak{D}(A) = \{u \in \mathscr{L}^2(\mathbb{R}^d)\colon Au \in \mathscr{L}^2(\mathbb{R}^d)\}$. Then A is a self-adjoint operator.*

(ii) The sets $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$ of all operators of the form $e^{ih^{-1}A}$ with $A \in \mathfrak{A}$, $A \in \tilde{\mathfrak{A}}$ are Lie groups and $\dim \mathfrak{G} = d(2d+1)$, $\dim \tilde{\mathfrak{G}} = (d+1)(2d+1)$.

(iii) The set $\mathscr{G}$ of all matrices of the form $\exp(-F)$ with $F \in \mathfrak{B}$ coincides with the Lie group of all symplectic matrices (i.e., matrices S such that $\sigma(Sz, Sz') = \sigma(z, \zeta')$); further $\dim \mathscr{G} = d(2d+1)$.

Moreover, the set $\tilde{\mathscr{G}}$ of all transformations of the form $\exp(-\tilde{F})|_{\mathbb{R}^{2d}}$ with $\tilde{F} \in \tilde{\mathfrak{B}}$ coincides with the Lie group of all symplectic affine transformations of $\mathbb{R}^{2d}$ (i.e., transformations of the form $z \to Sz + L$ with $S \in \mathscr{G}$, $L \in \mathbb{R}^{2d}$). For $\tilde{F} = (F; \ell; c)$ we have

$$S = exp(-F), \quad L = \int_0^1 \exp(-tF)\ell \, dt;$$

further, $\dim \tilde{\mathscr{G}} = d(2d+3)$.

(iv) The isomorphisms $\mathfrak{A} \simeq \mathfrak{B}$ and $\tilde{\mathfrak{A}} \simeq \tilde{\mathfrak{B}}$ generate natural maps $\theta : \mathfrak{G} \to \mathscr{G}$ and $\tilde{\theta} : \tilde{\mathfrak{G}} \to \tilde{\mathscr{G}}$ where θ is two-sheeted covering and for any $g \in \tilde{\mathscr{G}}$, $\tilde{\theta}^{-1} g \simeq \mathbb{S}^1 \subset \mathbb{C}$.

(v) Suppose $G \in \mathfrak{G}$ or $G \in \tilde{\mathfrak{G}}$ and let $P = p^{\mathrm{w}}(x, D)$ be a pseudodifferential operator. Then

$$G^* P G = q^{\mathrm{w}}(x, D), \qquad q(x, \xi) = q(S(x, \xi)) \tag{1.2.98}$$

where $S = \theta(G) \in \mathscr{G}$ or $S = \tilde{\theta}(G) \in \tilde{\mathscr{G}}$ respectively.

One can find the complete description of spectra of such operators in M. Zaretskaya [1].

Proposition 1.2.38. *(i) Let $G \in \mathfrak{G}$ (or $G \in \tilde{\mathfrak{G}}$) and let P be an h-pseudodifferential operator with compactly supported symbol. Then PG and GP are Fourier integral operators with symplectomorphism $S = \theta(G)$ (or $S = \tilde{\theta}(G)$ respectively). They can be defined by a quadratic phase function (with linear part in the second case). Moreover,*

$$GZ = (SZ)G. \tag{1.2.99}$$

(ii) On the other hand, (1.2.99) *with linear (or affine) transformation S yields that $G \in \mathfrak{G}$, $S = \theta(G)$ (or $G \in \tilde{\mathfrak{G}}$, $S = \tilde{\theta}(G)$ respectively).*

Definition 1.2.39. (i) Elements of $\mathfrak{G}$ are *metaplectic operators*.

(ii) Elements of $\tilde{\mathfrak{G}}$ are *generalized metaplectic operators*.

Example 1.2.40. (i) Partial h-Fourier transforms (with an appropriate numerical factor) are metaplectic operators.

Further, operators of the form $(Gu)(x) = |\det \beta|^{1/2} u(\beta x)$ with non-degenerate matrix β are metaplectic operators.

Finally, operators of the form $(Gu)(x) = e^{\frac{1}{2} ih^{-1}\langle Wx, x\rangle} u(x)$ with symmetric matrix W are metaplectic operators.

Conversely, all metaplectic operators are products of operators of these three types.

(ii) Operators of the form $(Gu)(x) = e^{ih^{-1}(\langle bx\rangle + \phi)} u(x-a)$ are (all) generalized metaplectic operators with linear phase functions.

(iii) All generalized metaplectic operators are products of operators of form (i) and (ii).

1.2.5 Germs

Lemma 1.1.17 yields the following

Proposition 1.2.41. *Let $X = \mathbb{R}^d$ and $p_j(x,\xi)$ with $j = 1, \dots, s-1$ be scalar linear functions on T^*X. Then*

$$(p_1 \cdots p_s)^{\mathrm{w}}(x, hD) = \frac{1}{s!} \sum_{|J|=s} p_{j_1}^{\mathrm{w}}(x, hD) \cdots p_{j_s}^{\mathrm{w}}(x, hD) \tag{1.2.100}$$

where the summation is taken over all the permutations J.

Definition 1.2.42. The right-hand expression of (1.2.100) is a *symmetric product* of (operators) $p_1^{\mathrm{w}}(x, hD), \dots, p_s^{\mathrm{w}}(x, hD)$.

Moreover $(1.1.37)_3$ implies

Proposition 1.2.43. *Let $p_j(x,\xi,h)$, $j = 1, \dots, s > 1$ be scalar symbols and suppose $p_{j0}(x,\xi)$ vanish at $(\bar{x},\bar{\xi}) \in T^*\mathbb{R}^d$. Let q be the Weyl symbol of the symmetric product of $p_1^{\mathrm{w}}(x, hD, h), \dots, p_s^{\mathrm{w}}(x, hD, h)$. Then*

$$D^{\alpha}_{x,\xi} q_n(\bar{x},\bar{\xi}) = 0 \qquad \forall n \ge 1,\ \alpha : |\alpha| + 2n \le s. \tag{1.2.101}$$

Remark 1.2.44. These equalities remain true for other types of quantization only provided $|\alpha| + 2n < s$.

Let $p_j(x,\xi,h)$ be the symbols described in Proposition 1.2.43 and let $\tilde{p}_j(x,\xi,h)$ be Weyl symbols of $Bp_j^{\mathrm{w}}(x, hD, h)A$ where A and B are Fourier integral operators corresponding to symplectomorphisms Ψ and Ψ^{-1} respectively such that $AB \equiv I$ and $BA \equiv I$ in $\mathcal{X} \ni (\bar{x},\bar{\xi})$ and $\mathcal{Y} \ni (\bar{y},\bar{\eta}) = \Psi^{-1}(\bar{x},\bar{\xi})$ respectively.

Then symbols $\tilde{p}_{j0} = p_{j0} \circ \Psi$ vanish at $(\bar{y},\bar{\eta})$. On the other hand,

$$\prod_{0\le j\le s} \bigl(Bp_j^{\mathrm{w}}(x, hD, h)A\bigr) \equiv B\Bigl(\prod_{0\le j\le s} p_j^{\mathrm{w}}(x, hD, h)\Bigr)A$$

and the same is also true for a symmetric product.

Applying Corollary 1.2.31(i) and Pproposition 1.2.43 to both sides of the latter equality we obtain that the degrees of the operators $\mathcal{L}_j$ in (1.2.86) do not exceed $(2j-1)$ for $j \ge 1$. Moreover, the general case of Corollary 1.2.31(i) can be easily reduced to the case considered (when A and B are scalar and mutually inverse in the indicated sense) by replacing A and B by $A' = AR_1$ and $B' = R_2B$ respectively with h-pseudodifferential operators R_1 and R_2. So we arrive to

Proposition 1.2.45. *For Weyl quantization* $\mathcal{L}_j$ *in* (1.2.86) *(and hence in* (1.2.74) *also) are* $(2j-1)$*-th degree operators for* $j \ge 1$*.*

Remark 1.2.46. One can observe easily that this assertion fails for other types of the quantization (it fails even for $j = 1$).

Definition 1.2.47. (i) Let $P = p(x, hD, h)$ be an operator with symbol p such that

$$\partial_x^\alpha \partial_\xi^\beta p_n(\bar{x}, \bar{\xi}) = 0 \qquad \forall \alpha, \beta, n : l|\alpha| + r|\beta| + n(l+r) < m \tag{1.2.102}$$

in $\mathbb{R}^d$ where $l > 0, r > 0, (\bar{x}, \bar{\xi}) \in T^*\mathbb{R}^d$ [18]. Then the operator

$$q(z, hD_z, h) \coloneqq \sum_{\alpha,\beta,n: l|\alpha|+r|\beta|+n(l+r)=m} \frac{1}{\alpha!\beta!} \big(\partial_x^\alpha \partial_\xi^\beta p_n\big)(\bar{x}, \bar{\xi}) \ell_{\alpha\beta}(z, hD_z) h^n \tag{1.2.103}$$

is called an (l, r)*-germ of operator* P *at the point* $(\bar{x}, \bar{\xi})$*;* here $\ell_{\alpha\beta}(z, \zeta) = z^\alpha \zeta^\beta$ and we use the same type of quantization for p and $z^\alpha \zeta^\beta$.

(ii) This procedure is called *microlocalization*.

One can show easily that

(1.2.104) The microlocalization does not depend on the type of quantization.

Remark 1.2.48. (i) One can also introduce *anisotropic microlocalizations* (with vector l and r); moreover it is possible to consider the case in which some components of l and r vanish, but we have no need of it.

(ii) One can see easily that the product of operators corresponds to the product of their microlocalizations; (l, r) are supposed to be the same and then degrees (m) sum.

Corollary 1.2.31, Theorem 1.1.35 and Proposition 1.2.43 imply

Proposition 1.2.49. *Let* (1.2.102) *be fulfilled. Then*

[18] Without any loss of generality one can assume that $l + r = 1$.

(i) Let A and B be scalar Fourier integral operators corresponding to symplectomorphisms Ψ and Ψ^{-1} respectively such that $BA \equiv I$ on $\mathcal{Y}$ and $AB \equiv I$ on $\mathcal{X} \ni (\bar{x}, \bar{\xi})$.

Then for $l = r$ the operator $\tilde{P} = BPA$ satisfies (1.2.102) *at $(\bar{y}, \bar{\eta}) = \Psi(\bar{x}, \bar{\xi})$ and the germ of operator $\tilde{P}$ at point $(\bar{y}, \bar{\eta})$ is linked to the the germ of operator P at point $(\bar{x}, \bar{\xi})$ by the equality*

$$\tilde{q}(z, hD_z, h) = (q \circ \Psi'_{(\bar{y},\bar{\eta})})(z, hD_z, h) = G^* q(z, hD_z, h) G \tag{1.2.105}$$

where $\Psi'_{(\bar{y},\bar{\eta})}$ is the differential of Ψ at $(\bar{y}, \bar{\eta})$ (so $\Psi'_{(\bar{y},\bar{\eta})}$ is a linear symplectomorphism in $\mathbb{R}^d \times \mathbb{R}^d$) and G is the corresponding metaplectic operator.

(ii) The above statement remains true provided $l < r$ and $Au = u \circ \varkappa$, $Bu = u \circ \varkappa^{-1}$ where $\varkappa$ is a diffeomorphism[19]*.*

1.3 Wave Front Sets and Related Topics

1.3.1 Principal Notions

Philosophy

In this book we consider various objects depending on a parameter (or parameters) $\mathfrak{m} \in \mathfrak{M}$ of an arbitrary nature (i.e., we consider *families* of the individual objects); one of these parameters is the *semiclassical parameter* $h \in (0, 1]$[20] and the exact nature of other parameters is as a rule unessential[21].

Objects not depending on the parameters (i.e., individual objects) are called *fixed* while numbers depending on parameters are also called *parameters*[22].

If $\mathscr{B}$ is a Banach space of individual objects then $\underline{\mathscr{B}}$ is the Banach space of these objects uniformly bounded (in which case $\|u\|_{\underline{\mathscr{B}}} = \sup_{\mathfrak{m}\in\mathfrak{M}} \|u_{\mathfrak{m}}\|_{\mathscr{B}}$

[19] In this case $(Gu)(x) = |\det \beta|^{\frac{1}{2}} u(\beta x)$ with $\beta = \varkappa'(\bar{y})$.

[20] Surely, h can run over a certain subset of $(0, 1]$ with an accumulation point 0.

[21] For example, for h-differential operators one can treat coefficients as parameters belonging to certain functional spaces. On the other hand, we can consider coefficients also depending on h via other parameters.

[22] There is no room for any confusion: fixed numbers are called *constants* and some of these constants are called *exponents.*

for $u = (u_{\mathtt{m}})_{\mathtt{m}}$); if $\lambda = (\lambda_{\mathtt{m}})$ is a (numerical) parameter then $\lambda\underline{\mathscr{B}}$ is the space of families of the form λu with $u \in \underline{\mathscr{B}}$ and $\|v\|_{\lambda\underline{\mathscr{B}}} = \|\lambda^{-1}u\|_{\underline{\mathscr{B}}}$.

In what follows we speak of *objects but mean families of these objects* [23].

In this book as *admissible* we consider *tempered* functions, i.e., functions $u \in h^{-L}\underline{\mathscr{L}}^2(\mathbb{R}^d, \mathbb{H})$ where $\mathbb{H}$, $\mathbb{H}'$, etc., are auxiliary Hilbert spaces. Sometimes we replace $\mathbb{H}$ in this definition by $\mathcal{L}(\mathbb{H}, \mathbb{H}')$ [24].

Similarly, as *admissible* we consider *tempered* operators, i.e., operators $A \in h^{-L}\underline{\mathcal{L}}\big(\mathscr{L}^2(\mathbb{R}^d, \mathbb{H}), \mathscr{L}^2(\mathbb{R}^{d'}, \mathbb{H}')\big)$.

Functions $u \in h^s\underline{\mathscr{L}}^2(\mathbb{R}^d, \mathbb{H})$ or $u \in h^s\underline{\mathscr{L}}^2\big(\mathbb{R}^d, \mathcal{L}(\mathbb{H}, \mathbb{H}')\big)$ and operators $A \in h^s\underline{\mathcal{L}}\big(\mathscr{L}^2(\mathbb{R}^d, \mathbb{H}), \mathscr{L}^2(\mathbb{R}^{d'}, \mathbb{H}')\big)$ are called *negligible;* in these cases we write $u \equiv 0$ and $A \equiv 0$ respectively. Exponents l and s enter into these definitions (so in fact we speak about l-admissible or s-negligible functions, etc.).

Smooth (Standard) Theory

As an *admissible box* we consider a parallelepiped

$$B = \big\{(x,\xi) : \ |x_j - \bar{x}_j| \le \gamma_j, \ |\xi_j - \bar{\xi}_j| \le \rho_j \ \forall j = 1, \dots, d\big\} \tag{1.3.1}$$

where $(\bar{x}, \bar{\xi})$, $\boldsymbol{\rho} = (\rho_1, \dots, \rho_d)$, $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_d)$ are parameters such that

$$\infty \ge \gamma_j > 0, \quad \infty \ge \rho_j > 0, \quad \rho_j\gamma_j \ge h^{1-\delta} \qquad \forall j = 1, \dots, d \tag{1.3.2}$$

where here and in what follows $\delta > 0$ is always an arbitrarily small positive exponent; so an admissible box is not necessarily fixed.

Definition 1.3.1. (i) Let u be an admissible function and B an admissible box. Then u *is negligible in* B ($u \equiv 0$ in B) if there exists a symbol $\varphi \in \mathsf{S}_{h,\rho,\gamma,N}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C})$ with $K = K(d, l, s, \delta)$ in the definition of the class such that $\varphi = 1$ in B and $\varphi(x, hD, h)u \equiv 0$ (this agrees with the definition given in the introduction)[25].

(ii) Similarly, let A be an admissible operator and $B = B \times B''$ an admissible box. Then A is *negligible in* B ($A \equiv 0$ in B) if there exist symbols $\varphi' \in$

[23] This casualness does not lead to any confusion because we use the word *fixed* for individual objects.

[24] In what follows we use plain letters instead of bold.

[25] Here and below extra indices $\boldsymbol{\rho}, \boldsymbol{\gamma}$ mean that φ would belong to the standard class $\mathsf{S}_{h,N}$ after rescaling $x_j \mapsto x_j/\gamma_j$, $\xi_j \mapsto \xi_j/\rho_j$.

$\mathsf{S}_{h,\rho',\gamma',N}(\mathbb{R}^{d'}\times\mathbb{R}^{d'},\mathbb{C})$ and $\varphi''\in \mathsf{S}_{h,\rho'',\gamma'',N}(\mathbb{R}^{d''}\times\mathbb{R}^{d''},\mathbb{C})$ with $K=K(d,I,s,\delta)$ in the definition of the class such that $\varphi'=1$ in B', $\varphi''=I$ in B'' and $\mathsf{Op}(\varphi'')A\,\mathsf{Op}(\varphi')\equiv 0$.

(iii) On the other hand, we call an admissible function u *essentially supported in* B if $\varphi(x,hD,h)u\equiv u$ for some ϕ with a symbol, supported in B.

Also we call an operator A *essentially supported in* B if $\mathsf{Op}(\phi')A\,\mathsf{Op}(\phi'')\equiv A$ for some ϕ', ϕ'' with symbols, supported in B' and B'' respectively.

Remark 1.3.2. (i) Surely these definitions depend on exponents I and s; the results of Section 1.1 imply that these definitions do not depend on the choice of K large enough;

(ii) Though all admissible (negligible) functions are Schwartz kernels of admissible (negligible) operators the converse assertion fails to be true.

Definition 1.3.3. (i) Let u be an admissible function. Let us introduce the set $\mathsf{WF}^s(u)\subset T^*\mathbb{R}^d$: $(\bar{x},\bar{\xi})\notin \mathsf{WF}^s(u)$ if and only if $u\equiv 0$ in some fixed box centered at $(\bar{x},\bar{\xi})$ (so, the point $(\bar{x},\bar{\xi})$ is also fixed);

(ii) Similarly, let A be an admissible operator. Let us introduce $\mathsf{WF}^s(A)\subset T^*(\mathbb{R}^{d'}\times\mathbb{R}^{d''}) : (\bar{x},\bar{\xi},\bar{y},\bar{\eta})\notin \mathsf{WF}^s(A)$ iff $A\equiv 0$ in some fixed box centered in $(\bar{x},\bar{\xi},\bar{y},\bar{\eta})$.

(iii) In both cases $\mathsf{WF}^s(.)$ are called *wave front sets*.

Remark 1.3.4. (i) In the previous book I called these sets *oscillation front sets* and denote them as OF^s to avoid confusion with the standard wave front sets. Here on the contrary I would like to emphasize similarity and I use the standard term wave front set and notation WF^s.

(ii) Still, there is a very essential difference: negligible functions are not smooth anymore. This leads to certain problems, usually related to the restrictions of functions. Therefore we can consider as a main space

$$\mathscr{H}^m=\bigl\{u : \|u\|_m=\|(1+h|D|)^{m/2}u\|<\infty\bigr\} \tag{1.3.3}$$

where $m\in\mathbb{R}$. This leads to the modification of all our definitions and to sets $\mathsf{WF}^{s,m}(u)$.

(iii) Respectively for operators we consider $\mathcal{L}(\mathscr{H}^m, \mathscr{H}^k)$ of (uniformly with respect to h) bounded operators from $\mathscr{H}^k$ to $\mathscr{H}^m$ and to wave front sets $\mathsf{WF}^{s,m,k}(A)$.

(iv) The notion of wave (oscillation) front set is rather rough and it would be desirable to give a more refined and sophisticated definition also based on Definition 1.3.1 and leading to no difficulty in the advanced development of the theory. Maybe non-standard analysis will be useful? However we simply use Definition 1.3.1 instead of Definition 1.3.3 if the latter is too rough for our needs.

Analytic Theory

In the framework of the theory of operators with analytic symbols we replace the standard microlocal uncertainty principle $\rho_j\gamma_j \ge h^{1-\delta}$ by the logarithmic uncertainty principle $\rho_j\gamma_j \ge C_s h|\log h|$ and thus we replace (1.3.2) by

$$\infty \ge \gamma_j > 0, \quad \infty \ge \rho_j > 0, \quad \rho_j\gamma_j \ge Ch|\log h| \qquad \forall j = 1, \dots, d \tag{1.3.4}$$

Remark 1.3.5. This leads to the question: can we go further? The answer is *No. Microlocal Analysis starts from logarithmic uncertainty principle*; see Appendix 1.A.4.

1.3.2 Properties

The results of Sections 1.1 and 1.2 imply immediately

Theorem 1.3.6. *(i) Let $p \in \mathsf{S}_{h,\rho,\gamma,N}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{H}', \mathbb{H})$ with $K = K(d, s, \delta)$. Then $P = p(x, hD, h)$ is negligible in every box $B = B' \times B''$ with*

$$\begin{aligned} B' &= \{(x,\xi) : |x_j - \bar{x}_j| \le \gamma'_j,\ |\xi_j - \bar{\xi}_j| \le \rho'_j \qquad \forall j = 1, \dots, d\}, \\ B'' &= \{(y,\eta) : |y_j - \bar{y}_j| \le \gamma''_j,\ |\eta_j - \bar{\eta}_j| \le \rho''_j \qquad \forall j = 1, \dots, d\} \end{aligned} \tag{1.3.5}$$

provided there exist γ_j, ρ_j such that

$$\gamma'_j \asymp \gamma''_j, \quad \rho'_j \asymp \rho''_j, \tag{1.3.6}$$

$$\min(\rho'_j, \rho_j) \cdot \min(\gamma'_j, \gamma_j) \ge h^{1-\delta} \quad \forall j \tag{1.3.7}$$

and

$$\sum_{1\le j\le d}\big(\frac{1}{\gamma_j}(|x_j-y_j|+|x_j-z_j|)+\frac{1}{\rho_j}(|\xi_j-\eta_j|+|\xi_j-\zeta_j|)\big)\ge\epsilon \tag{1.3.8}$$
$$\forall(x,\xi)\in B',\quad \forall(y,\eta)\in B'',\quad \forall(z,\zeta)\in \mathsf{supp}(p);$$

recall that ϵ is an arbitrary positive constant.

(ii) Further, if $p\in \mathsf{S}_{h,\rho,\gamma,\mathsf{an}}(\mathbb{R}^d\times\mathbb{R}^d,\mathbb{H}',\mathbb{H})$ [26] *one can replace* (1.3.7) *by*

$$\min(\rho_j',\rho_j)\cdot\min(\gamma_j',\gamma_j)\ge Ch|\log h|\quad\forall j. \tag{1.3.9}$$

Theorem 1.3.7. *(i) Let A be a Fourier integral operator corresponding to the symplectomorphism Ψ and let all the conditions on phase functions and symbols be fulfilled with $M=K=K(d,r,s,\delta)$ uniformly with respect to all the parameters. Then A is negligible in every box $B'\times B''$ of the form* (1.3.5) *such that*

$$\gamma_j'\asymp\gamma_j''\asymp\gamma,\quad \rho_j'\asymp\rho_j''\asymp\rho,\quad \gamma=\rho\ge\epsilon h^{(1-\delta)/2}, \tag{1.3.10}$$

$$\mathsf{dist}\big((x,\xi),\Psi(y,\eta)\big)+\mathsf{dist}\big((x,\xi),(z,\phi_z'(z,z',\theta)\big)\ge\epsilon\rho \tag{1.3.11}$$
$$\forall(x,\xi)\in B',\quad\forall(y,\eta)\in B'',$$
$$\forall(z,z',\theta)\in\mathsf{supp}(a)\cap\{|\phi_\theta'(z,z',\theta)|\le\epsilon\rho\}.$$

(ii) Let $Au=(fu)\circ\varkappa^{-1}$ where $\varkappa$ is a $\mathscr{C}^K$ diffeomorphism from $Y\subset\mathbb{R}^d$ to $X\subset\mathbb{R}^d$ and $f\in\mathscr{C}_0^K(Y)$ uniformly with respect to all the parameters, $K=K(d,s,\delta)$. Then A is negligible in every box $B'\times B''$ of the form (1.3.5) *such that*

$$\gamma_j'\asymp\gamma_j''\asymp\gamma,\ \rho_j'\asymp\rho_j''\asymp\rho,\ \gamma\le\rho,\ \min(\gamma,1)\cdot\min(\rho,1)\ge h^{1-\delta}, \tag{1.3.12}$$

$$\frac{1}{\gamma}\big(|x-\varkappa(y)|+|y-z|\big)+\frac{1}{\rho}|{}^t\varkappa'\xi-\eta|\ge\epsilon \tag{1.3.13}$$
$$\forall(x,\xi)\in B',\quad\forall(y,\eta)\in B'',\quad\forall z\in\mathsf{supp}(f).$$

[26] We define these classes in the manner similar to [25].

(iii) Let u be a Lagrangian distribution with Lagrangian manifold Λ and let all the conditions on Λ, phase functions and amplitudes be fulfilled with $K = K(d, r, s, \delta)$ uniformly with respect to all the parameters. Then u is negligible in every box B of the form (1.3.5) *provided*

$$\gamma_j' \asymp \gamma, \quad \rho_j' \asymp \rho \quad \forall j, \quad \gamma = \rho \ge \epsilon h^{(1-\delta)/2}, \tag{1.3.14}$$

$$\begin{aligned} \operatorname{dist}\big((x,\xi),(z,\phi_z'(z,\theta)\big) \ge \epsilon\rho \\ \forall (x,\xi) \in \Pi \quad \forall (z,\theta) \in \operatorname{supp}(a) \cap \{|\phi_\theta'(z,\theta)| \le \epsilon\rho\}. \end{aligned} \tag{1.3.15}$$

(iv) Moreover, let $\Lambda = \{x' = \xi'' = 0\}$ be a model Lagrangian manifold where $x = (x'; x'') = (x_1, \dots, x_{d'}; x_{d'+1}, \dots, x_d)$, etc., with $0 \le d' \le d$. Then u is negligible in every box of the form (1.3.5) *such that*

$$\min(\rho_j, 1) \cdot \min(\gamma_j, 1) \ge \epsilon h^{1-\delta} \quad \forall j, \tag{1.3.16}$$

$$\sum_{1 \le j \le d'} \frac{1}{\gamma_j}|x_j| + \sum_{d'+1 \le j \le d} \frac{1}{\rho_j}|\xi_j| \ge \epsilon. \tag{1.3.17}$$

Using statement (iv) one can easily find boxes in which operators A: $(Au)(x') = (f(hD)u)(x', 0)$ and B: $(Bu)(x') = \int f(x)u(x)|_{x=(x',x'')}\, dx''$ with compactly supported smooth function f are negligible (see also Theorem 1.3.13. It is also easy to prove that wave front sets introduced here have the following properties similar to properties of the standard wave front sets:

Theorem 1.3.8. *$\mathsf{WF}^s(.)$ is a closed set; conversely, if $M \subset T^*\mathbb{R}^d$ is a closed set and $s > 0$ then there exists $u \in \mathscr{L}^2(\mathbb{R}^d)$ such that $\mathsf{WF}^s(u) = M$.*

Theorem 1.3.9. *Let u, v etc be admissible with $l = 0$ functions and A, B etc be admissible with $l = 0$ operators. Further, let $Q \in \boldsymbol{\Psi}_{h,N}$ be an operator with symbol supported in $\{|x| + |\xi| \le C_0\}$. Then for $s > 0$*

$$\begin{aligned} \mathsf{WF}^s(AQu) \subset \mathsf{WF}^s(A) \circ \mathsf{WF}^s(u) = \\ \big\{(x,\xi : \exists (y,\eta) \in \mathsf{WF}^s(u) : (x,\xi,y,\eta) \in \mathsf{WF}^s(A)\big\}, \end{aligned} \tag{1.3.18}$$

$$\begin{aligned} \mathsf{WF}^s(AQB) \subset \mathsf{WF}^s(A) \circ \mathsf{WF}^s(B) = \\ \big\{(x,\xi,z,\zeta) : \exists (y,\eta) : (x,\xi,y,\eta) \in \mathsf{WF}^s(A), (y,\eta,z,\zeta) \in \mathsf{WF}^s(B)\big\}; \end{aligned} \tag{1.3.19}$$

$$\mathsf{WF}^s(u \otimes v) = \mathsf{WF}^s(u) \times \mathsf{WF}^s(v). \tag{1.3.20}$$

Theorem 1.3.10. *Let* $P = p(x, hD, h)$ *with* $p \in \mathsf{S}_{h,\rho,\gamma,N}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{H}', \mathbb{H})$ *with* $K = K(d, s, \delta)$ *such that*

$$\min(1, \rho_j) \cdot \min(1, \gamma_j) \geq \epsilon h^{1-\delta} \qquad \forall j. \tag{1.3.21}$$

Then

$$\mathsf{WF}^s(P) = \{(x, \xi, x, \xi) : (x, \xi) \in \mathsf{supp}(p)\} \tag{1.3.22}$$

and for every admissible function u

$$\mathsf{WF}^s(Pu) \subset \mathsf{WF}^s(u) \subset \mathsf{WF}^s(Pu) \cup \mathsf{Char}(P) \tag{1.3.23}$$

provided K *also depends on* I *and* $\mathsf{Char}(P)$ *is the set of all the points* (x, ξ) *at which the symbol* $p(x, \xi, 0)$ *is not invertible with the norm of the inverse symbol not exceeding* C*.*

Theorem 1.3.11. *Let* A *be an admissible operator with Schwartz kernel* $\mathcal{K}_A$*. Then the following inclusions hold:*

$$\mathsf{WF}^s(A) \subset \mathsf{WF}^s(\mathcal{K}_A)^T \tag{1.3.24}$$

and for admissible $\mathcal{K}_A$

$$\mathsf{WF}^s(\mathcal{K}_A)^T \subset \mathsf{WF}^{s+\frac{d}{4}}(A) \tag{1.3.25}$$

with $d = d' + d''$ *provided* $\mathcal{K}_A \in h^{-I}\mathscr{L}^2$*.*

Theorem 1.3.12. *(i) Let* u *be a Lagrangian distribution with Lagrangian manifold* $\Lambda \subset T^*\mathbb{R}^d$*. Then*

$$\mathsf{WF}^s(u) \subset \Lambda \tag{1.3.26}$$

provided $K = K(d, r, s)$*;*

(ii) Let A *be a Fourier integral operator with Lagrangian manifold* $\Lambda \subset T^*\mathbb{R}^{2d}$*. Then*

$$\mathsf{WF}^s(u) \subset \Lambda^T. \tag{1.3.27}$$

Theorem 1.3.13. *(i) Let* $a \in \mathsf{S}_{h,N}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{H}', \mathbb{H})$ *with* $K = K(d, I, s)$ *and* $\mathsf{supp}(a) \subset \{|\xi''| \leq c\}$*. Let*

$$v(x') = \big(a(x, hD, h)u\big)(x', 0). \tag{1.3.28}$$

Then

$$\mathsf{WF}^s(v) \subset \big\{(x', \xi') : \exists \xi'' : (x', 0, \xi', \xi'') \in \mathsf{WF}^{s+\frac{d''}{2}}(u)\big\}; \tag{1.3.29}$$

(ii) Further, let $a \in \mathsf{S}_{h,N}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{H}', \mathbb{H})$ *with* $K = K(d, l, s)$ *and* $\mathsf{supp}(a) \subset \{|x''| \le c\}$. *Let*

$$v(x') = \int_{\mathbb{R}^{d''}} (a(x, hD, h)u)(x', x'')dx''. \tag{1.3.30}$$

Then

$$\mathsf{WF}^s(v) \subset \big\{(x', \xi') : \exists x'' : (x', x'', \xi', 0) \in \mathsf{WF}^s(u)\big\}. \tag{1.3.31}$$

Remark 1.3.14. (i) Let $u \in \mathscr{L}^2(X, \mathbb{H})$ with $X \subset \mathbb{R}^d$. Then Definition 1.3.1(i) remains reasonable provided

$$B_{\epsilon x} = \{|x_j - \bar{x}_j| \le (1+\epsilon)\gamma_j \quad \forall j = 1, \dots, d\} \subset X$$

with some small positive constant ϵ; in this case one should replace the cutting operator $\varphi(x, hD, h)$ by $\varphi(x, hD, h)\psi(x)$ where $\psi = 1$ in Π, ψ is supported in $\Pi_{\epsilon x}$ and belongs to the same symbol class.

Therefore Definition 1.3.3(i) remains reasonable provided $\mathsf{dist}(\bar{x}, \complement X) \ge \epsilon$.

(ii) Let $A \in \mathcal{L}\big(\mathscr{L}^2(Y, \mathbb{H}'), \mathscr{L}^2(X, \mathbb{H})\big)$ with $X \subset \mathbb{R}^d$ and $Y \subset \mathbb{R}^{d'}$ then Definition 1.3.1(ii) remains reasonable provided $B'_{\epsilon x} \subset X$ and $B''_{\epsilon y} \subset Y$; one should only take cutting operators of the form $\mathsf{Op}(\varphi')\psi'$ and $\psi''\,\mathsf{Op}(\varphi'')$.

So Definition 1.3.3(ii) remains reasonable provided $\mathsf{dist}(\bar{x}, \complement X) \ge \epsilon$ and $\mathsf{dist}(\bar{y}, \complement Y) \ge \epsilon$.

(iii) Let $v \in \mathscr{L}^2(X, \mathbb{H})$ be a fixed function. Let $u = u_h = \phi(h|D|)v$ be a family of functions where $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^+)$, $\phi \ne 0$. Then $(\mathsf{WF}(u))_{\mathsf{cone}} = \mathsf{WF}(v)$ where M_{cone} is the cone hull (with respect to ξ) of the set M and $\mathsf{WF}_{\mathsf{cl}}(v)$ is a standard wave front set.

1.3.3 Propagator

Theorem 1.3.12(ii) and the construction of propagator as a Fourier integral operator imply immediately

Theorem 1.3.15. *Let* $A \in \boldsymbol{\Psi}_{h,N}$ *with a scalar principal symbol* a^0, *which is real. Then*

$$\mathsf{WF}^s(e^{-ith^{-1}A}) = \big\{(x, \xi; y, \eta) : \ (x, \xi) = \Psi_t(y, \eta)\big\} \tag{1.3.32}$$

where Ψ_t *is a forward Hamiltonian flow described by* (1.2.8).

In particular,

$$\mathsf{WF}^s(e^{-ith^{-1}A}f) = \Psi_t \circ \mathsf{WF}^s(f). \tag{1.3.33}$$

1.3.4 Alternative Theories

We are sending the reader to any Microlocal Analysis textbook for the theory of wave front sets associated with the classical pseudodifferential operators. In this case $h = 1$ and $A \in \boldsymbol{\Psi}^1_{\mathsf{cl}}$.

Further, one can con construct the theory of global operators (whose symbols satisfy (1.1.213) or (1.1.214) with $m = 1$: order of operator *must* be 1. In this case we assume that modulo 0-th order symbol a is positively quasihomogeneous as $|x| + |\xi| \ge C$. Then Hamiltonian flow generated by quasihomogeneous principal symbol acts on $\mathbb{R}^{2d}/R$ where R means a group $(x, \xi) \mapsto (\lambda x, \lambda^\nu \xi)$ with any $\lambda > 0$; see V. Ivrii, [12].

1.4 Elliptic Boundary Value Problems and Associated Operators

1.4.1 Pseudodifferential Operators with Transmission Property

The purpose of this Subsection is to introduce pseudodifferential operators in the half-space and then in the domains with the smooth boundaries.

Causality

Consider classical double symbol $a(x, y, \xi)$ such that

$$\|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x, y, \xi)\| \le M(1 + |\xi|)^m \qquad \forall \alpha, \beta, \gamma : |\alpha| + |\beta| \le N + K, |\gamma| \le N + K \tag{1.4.1}$$

and a vector $\nu \in \mathbb{R}^d \setminus 0$. Assume that

(1.4.2) $a(x, y, \xi + \tau\nu)$ has an analytic continuation to $\tau \in \mathbb{C}_+$ [27] such that

$$\|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x, y, \xi + \tau\nu)\| \le M(1 + |\xi| + |\tau|)^m \qquad \forall \alpha, \beta, \gamma : |\alpha| + |\beta| \le N + K, |\gamma| \le N + K. \tag{1.4.3}$$

[27] Where we here and in what follows use notations $\mathbb{C}_\pm := \{\tau \in \mathbb{C} : \pm\tau > 0\}$.

We arrive immediately to

Proposition 1.4.1. *Let symbol a satisfy* (1.4.1)–(1.4.2) *and let $\mathcal{K}_A(x,y)$ be a Schwartz kernel of operator $A = a(\overset{3}{x}, \overset{2}{hD}, \overset{1}{x})$. Then*

$$\operatorname{supp}(\mathcal{K}_A(x,y)) \subset \bigl\{(x,y) : \langle x-y, \nu\rangle \le 0\bigr\}. \tag{1.4.4}$$

Proof. Without any loss of the generality one can assume that $\nu = (1,0,\dots,0)$. Then the proof follows from deforming the contour of integration with respect to ξ_1 in expression (1.1.18) for $K_A(x,y)$. □

We refer to property (1.4.4) as a*causality*. One can prove easily that

Proposition 1.4.2. *(i) If a double symbol a satisfies* (1.4.1)–(1.4.2) *then the same is true for all three ordinary symbols (pq-, qp- and Weyl) of operator A.*

(ii) If ordinary symbols a, b satisfy (1.4.1)–(1.4.2) *then the same is true for $a \bullet b$ for any method of quantization;*

(iii) If an ordinary symbol a satisfies (1.4.1)–(1.4.2) *then the same is true for $a^\dagger$ and vector $-\nu$ instead of ν.*

Transmission Property

While causality is a useful and deserving study property, it is too restrictive for our needs. Instead we introduce *transmission property* essentially following L. Boutet de Monvel [1]:

Definition 1.4.3. (i) Symbol $a \in \mathsf{S}^m_{\sigma,\varsigma}$ with $1 \ge \sigma > \varsigma \ge 0$ has a *transmission property* with respect to half-space $\dot{\mathbb{R}}^+ \times \mathbb{R}^{d-1} = \{x_+ \ge 0\}$ if one can decompose a and all its derivatives in $\{|\xi_1| \ge C|\xi'| + C\}$:

$$a \sim \sum_{k\ge 1} a_k(x',\xi')(\xi_1 + i0)^{m-k} \tag{1.4.5}$$

with $a_k \in \mathsf{S}^{-k}_{\sigma,\varsigma}(\mathbb{R}^d \times \mathbb{R}^{d-1}, \mathbb{H}', \mathbb{H})$ where asymptotic decomposition (1.4.5) means that

$$\|\partial_x^\alpha \partial_\xi^\gamma \bigl(a - \sum_{1\le k\le r-1} a_k(x',\xi')(\xi_1+i0)^{m-k}\bigr)\big|_{x_1=0}\| \le M_{r\alpha\gamma}(1+|\xi'|)^{r+\varsigma|\alpha|}|\xi_1|^{m-r-\sigma|\gamma|} \tag{1.4.6}$$

for all r, α, γ.

Class of such symbols is denoted by $\mathsf{S}^m_{\sigma,\varsigma,\mathsf{trans}}(X\times\mathbb{R}^d,\mathbb{H}',\mathbb{H})$ where here $X=\dot{\mathbb{R}}^+\times\mathbb{R}^{d-1}:=\{x\colon x_1>0\}$.

(ii) Operator $A\in\boldsymbol{\Psi}^m_{\sigma,\varsigma}$ with $1\ge\sigma>\varsigma\ge 0$ has a *transmission property* with respect to a half-space $X=\dot{\mathbb{R}}^+\times\mathbb{R}^{d-1}$ if $A\equiv\mathsf{Op}(a)$ modulo infinitely smoothing operator where $a\in\mathsf{S}^m_{\sigma,\varsigma}$ satisfies (i).

Class of such operators is denoted by $\boldsymbol{\Psi}^m_{\sigma,\varsigma,\mathsf{trans}}(X,\mathbb{H}',\mathbb{H})$.

(iii) Finally, $\mathsf{S}^m_{\mathsf{cl},\mathsf{trans}}$ and $\boldsymbol{\Psi}^m_{\mathsf{cl},\mathsf{trans}}$ denote classes of classical symbols and operators with transmission property.

Remark 1.4.4. (i) Surely one can introduce similar classes with r, $|\alpha|$, ν bounded by $N+K$.

(ii) Obviously transmission property is preserved under affine changes of variables preserving half-space $\mathbb{R}^+\times\mathbb{R}^{d-1}$. Therefore one can introduce transmission property with respect to any half-space $X=\{x:\langle x,\nu\rangle>c\}$.

One can prove easily that

Proposition 1.4.5. *For $X=\{x:\langle x,\nu\rangle>c\}$ the following properties hold:*

(i) If $A_1\in\boldsymbol{\Psi}^{m_1}_{\sigma,\varsigma,\mathsf{trans}}(X,\mathbb{H}'',\mathbb{H}')$ and $A_2\in\boldsymbol{\Psi}^{m_2}_{\sigma,\varsigma,\mathsf{trans}}(X,\mathbb{H}',\mathbb{H})$ then $A_1A_2\in\boldsymbol{\Psi}^{m_1+m_2}_{\sigma,\varsigma,\mathsf{trans}}(X,\mathbb{H}',\mathbb{H})$.

(ii) If $A\in\boldsymbol{\Psi}^m_{\sigma,\varsigma,\mathsf{trans}}(X,\mathbb{H}',\mathbb{H})$ is elliptic and $m\in\mathbb{Z}$ then it has a parametrix $A'\in\boldsymbol{\Psi}^{-m}_{\sigma,\varsigma,\mathsf{trans}}(X,\mathbb{H}',\mathbb{H})$.

(iii) Therefore if $A\in\boldsymbol{\Psi}^m_{\sigma,\varsigma,\mathsf{trans}}(X,\mathbb{H},\mathbb{H})$ and $m\le 0$ then $f(A)-f(0)\in\boldsymbol{\Psi}^m_{\sigma,\varsigma,\mathsf{trans}}(X,\mathbb{H},\mathbb{H})$ as well for all "admissible" functions f (in the sense of the functional calculus of Section 1.1.

(iv) If $A\in\boldsymbol{\Psi}^m_{\sigma,\varsigma,\mathsf{trans}}(X,\mathbb{H}'',\mathbb{H}')$ and $\varkappa$ is a diffeomorphism preserving X then $A_\varkappa=\varkappa_*^{-1}A\varkappa_*$ also belongs to this class;

(v) If $A\in\boldsymbol{\Psi}^m_{\sigma,\varsigma,\mathsf{trans}}(X,\mathbb{H}',\mathbb{H})$ with then $A^*\in\boldsymbol{\Psi}^m_{\sigma,\varsigma,\mathsf{trans}}(X',\mathbb{H},\mathbb{H}')$ with an opposite half-space $X'=\{\langle x,\nu\rangle\le c\}$.

(vi) Classes S^m and $\boldsymbol{\Psi}^m$ for half-space X' coincide with those for X if and only if $m\in\mathbb{Z}$.

Remark 1.4.6. Statement (ii) and a pseudo-locality allow us to extend the notion of transmission property to any bounded open domain $X \subset \mathbb{R}^d$ with a smooth boundary in such way that all statements of Proposition 1.4.5 still hold. We need to assume that either X is a half-space or ∂X is bounded or to assume that ∂X is "uniformly $\mathscr{C}^\infty$" (the notion we do not want to exploit).

Pseudodifferential Operators in a Domain

However we need to explain how actually one applies such operators to functions defined in such domain (or in an open half-space). The main idea is to introduce action of operator A by

(1.4.7) $A_X = \eth_X A F_X$ where F_X is an operator of extension from X to $\mathbb{R}^d$ by 0 and $\eth_X$ is an operator of restriction from $\mathbb{R}^d$ to X.

This formula would definitely brake composition and related operations. We can take different extension operators F_X but since we develop mainly $\mathscr{L}^2$-theory we will take $F_X = \eth_X^*$ which is continuation by 0.

(1.4.8) For $s \ge 0$ let us introduce $\mathscr{H}^s(X) = \{u : \exists v \in \mathscr{H}^s(\mathbb{R}^d) : \ \eth_X v = u\}$ with $\|u\|_{\mathscr{H}^s(X)} = \inf_{v:\eth_X v = u} \|v\|_{\mathscr{H}^s(\mathbb{R}^d)}$.

As $u = \eth_X v$ with $v \in \mathscr{H}^s(\mathbb{R}^d)$ we get $F_X u = v^+ = v - v^-$ with $v^+ = v\theta_X(x)$ and $v^- = v\theta_{\complement X}(x)$ where $\theta_Y(x)$ denotes characteristic function of Y and therefore $A_X u = \eth_X v^+ = \eth_X A v - \eth_X A v^-$.

Let us consider first the case of half-space.

Proposition 1.4.7. *Let $X = \{x\colon x_1 > 0\}$ and $A \in \boldsymbol{\Psi}^m_{\sigma,\varsigma,\mathsf{trans}}(X, \mathbb{H}', \mathbb{H})$.*

Then for any $j \in \mathbb{Z}^+$ operator $D_1^j \eth_X A$ is a bounded operator from $\mathscr{L}^2\big(\mathbb{R}^-, \mathscr{H}^s(\mathbb{R}^{d-1}, \mathbb{H})\big)$ to $\mathscr{L}^2\big(\mathbb{R}^+, \mathscr{H}^{s-m+j}(\mathbb{R}^{d-1}, \mathbb{H}')\big)$.

Proof. Obviously $D_1^j A$ would be a bounded operator from $\mathscr{L}^2\big(\mathbb{R}, \mathscr{H}^s(\mathbb{H})\big)$ to $\mathscr{L}^2\big(\mathbb{R}^-, \mathscr{H}^{s-m+j}(\mathbb{R}^{d-1}, \mathbb{H}')\big)$ if we cut of it in zone $\{|\xi_1| \ge C|\xi'|\}$ replacing $a(x,\xi)$ by $a(x,\xi)\varphi(\xi_1/|\xi'|)$ with $\varphi \in \mathscr{C}_0^\infty(\mathbb{R})$. Therefore we need to consider only $a(x,\xi) = a_k(x,\xi')(\xi_1 + i0)^{m-k}$.

On the other hand, operator A with such symbol has a causality property and therefore $\eth_X A$ restricted to $\mathscr{L}^2\big(\dot{\mathbb{R}}^-, \mathscr{H}^s(\mathbb{H})\big)$ is simply 0. □

Observing that

$$\bigcap_{j\ge 0} \mathscr{H}^{j}(\mathbb{R}^{+}, \mathscr{H}^{s-j}(\mathbb{R}^{d-1}, \mathbb{H}')) \subset \mathscr{H}^{s}(X, \mathbb{H}') \qquad \text{as} \quad s \ge 0 \tag{1.4.9}$$

we conclude that the following statement holds for half-space $X = \{x\colon x_1 > 0\}$ and therefore for any bounded domain with a smooth boundary.

Proposition 1.4.8. *Let $A \in \mathbf{\Psi}^{m}_{\sigma,\varsigma,\mathsf{trans}}(X, \mathbb{H}', \mathbb{H})$. Then for any $s \ge \max(m, 0)$ A_X is a bounded operator from $\mathscr{H}^{s}(X, \mathbb{H}))$ to $\mathscr{H}^{s-m}(X, \mathbb{H}))$.*

Now we want to consider product a $A_X B_X$. According to (1.4.7)

$$A_X B_X = \eth_X A F_X \eth_X B F_X = (AB)_X - \eth_X A (I - F_X \eth_X) B F_X. \tag{1.4.10}$$

One can apply proposition 1.4.7 to such operator. We investigate it later.

1.4.2 Related Operators

Boundary (Poisson) Operators

Let us start from equation

$$A(x, D)u = f \qquad \text{in } X \tag{1.4.11}$$

where A is a differential operator of order m. To apply a pseudodifferential parametrix to it we need to rewrite it first as an equation in $\mathbb{R}^d$. Consider continuation $F_X u$ to $\mathbb{R}^d$ by 0. Then

$$A F_X u = F_X f + \sum_{j\le m-1} \delta^{(j)}_{\partial X}(x) g_j(x) \tag{1.4.12}$$

with $g_j = A_j(x, D)u$ where $A_j(x, D)$ are differential operators of order $(m - j - 1)$, $j = 0, \dots, m-1$.

More precisely, if X coincides locally with a half-space $\{x\colon x_1 > 0\}$ then $\delta^{(j)}_{\partial X}(x) = \delta^{(j)}(x_1)$ and $A_j(x, \xi) = \frac{1}{(j+1)!} D^{j+1}_{\xi_1} P(x, \xi)$ provided we use qp-quantization. Then

$$\sum_{j\le m-1} \delta^{(j)}(x_1) g_j(x) = \sum_{k\le j\le m-1} \delta^{(j-k)}(x_1) \eth_0 \partial^{k}_{x_1} g_j (-1)^k = \sum_{k\le j\le m-1} \delta^{(j-k)}(x_1) \eth_0 \partial^{k}_{x_1} \bigl(A_j(x, D) u_j\bigr) (-1)^k$$

where $\mathfrak{d}_0$ is an operator of restriction to ∂X.

Assume that A is elliptic; let A' be its parametrix. Then $A' \in \boldsymbol{\Psi}^{-m}_{\mathsf{cl},\mathsf{trans}}$. Applying $\mathfrak{d}_X A'$ to (1.4.12) we arrive to

$$u \equiv \mathfrak{d}_X A' F_X f + \sum_{j\le m-1} \mathfrak{d}_X A'\bigl(\delta^{(j)}_{\partial X}(x) g_j(x)\bigr) \tag{1.4.13}$$

where the first term is exactly $A'_X f$. Consider other terms. To do this we need to analyze $\mathfrak{d}_X A' \delta^{(j)}(x_1) g'_j(x')$ where $g'_j = \mathfrak{d}_0 L_j(x,D)u$ and L_j are differential operators of order $(m-1-j)$.

Departing from the previous considerations and notations we need to study operators of the type $P : g \to A(\delta^{(j)} \otimes g)$ where $g = g(x')$ and A is operator of order $(m-j)$. This operator has a Schwartz kernel

$$\mathcal{K}_P(x,y') = (2\pi)^{-d} \int b_j(x,y',\xi) e^{i\langle x'-y',\xi'\rangle + i x_1\xi_1}\, d\xi \tag{1.4.14}$$

where $b_j = (-\partial_{y_1} + i\xi_1)^j a(x,y,\xi)$; one can rewrite it as

$$\mathcal{K}_P(x,y') = (2\pi)^{1-d} \int k(\langle\xi'\rangle x_1, x', y', \xi') e^{i\langle x'-y',\xi'\rangle}\, d\xi' \tag{1.4.15}$$

with

$$k(t,x',y',\xi') = (2\pi)^{-1} \int b_j(x,y',\xi) e^{it\xi_1/\langle\xi'\rangle}\, d\xi_1. \tag{1.4.16}$$

Remark 1.4.9. While this construction works for any symbol a, for symbols with transmission property resulting function $k(t,x',y',\xi')$ is "nice"; one can prove easily that for $a \in \boldsymbol{\Psi}^{m-1}_{\sigma,\varsigma,\mathsf{trans}}$ with $m \in \mathbb{Z}$

$$\begin{aligned}\|D^\alpha_{x'} D^\beta_{y'} D^\gamma_{\xi'} D^j_t k(t,x',y',\xi')\| \le \\ C_{\alpha\beta\gamma jr}(1+t)^{-s}(1+|\xi'|)^{m+\varsigma(|\alpha|+|\beta|)-\sigma|\gamma|} \qquad \forall \alpha,\beta,\gamma,j,r.\end{aligned} \tag{1.4.17}$$

Definition 1.4.10. (i) Operator with the Schwartz kernel $\mathcal{K}_P(x,y')$ given modulo $\mathscr{C}^\infty(\bar X \times \partial X)$ by (1.4.15) with $k(t,x',y',\xi')$ satisfying (1.4.17) is called *Poisson operator of order* m.

Let $\boldsymbol{P}^m_{\sigma,\varsigma}$ denote the class of such operators. Further, by $\boldsymbol{P}^m_{\mathsf{cl}}$ we denote class of operators with functions $k \sim \sum k_s$ where k_s are positively homogeneous of degrees $(m-s)$ with respect to ξ'.

(ii) One can define $\boldsymbol{P}^m_{\sigma,\varsigma}$ and $\boldsymbol{P}^m_{\mathsf{cl}}$ for any half-space and for any domain X with the smooth compact boundary in the natural way.

Trace and Singular Green Operators

Let us introduce other classes of operators:

Definition 1.4.11. (i) Operator with the Schwartz kernel $K_T(x', y)$ defined modulo $\mathscr{C}^\infty(\partial X \times \bar{X})$ by (1.4.15) with x_1 replaced by y_1 and with $k(t, x', y', \xi')$ satisfying (1.4.17) with m replaced by $(m+1)$ is called *trace operator of order* m.

Let $\boldsymbol{T}^m_{\sigma,\varsigma}$ denote the class of such operators. Further, by $\boldsymbol{T}^m_{\mathsf{cl}}$ we denote class of operators with functions $k \sim \sum k_s$ where k_s are positively homogeneous of degrees $(m+1-s)$ with respect to ξ'.

(ii) One can define $\boldsymbol{T}^m_{\sigma,\varsigma}$ and $\boldsymbol{T}^m_{\mathsf{cl}}$ for any half-space and for any domain X with the smooth compact boundary.

Sometimes Poisson operators are called *coboundary operators* and trace operators are called *boundary operators*.

Definition 1.4.12. (i) Operator with the Schwartz kernel $\mathcal{K}_G(x, y)$ given modulo $\mathscr{C}^\infty(\bar{X} \times \bar{X})$ by

$$\mathcal{K}_G(x, y) = (2\pi)^{1-d} \int k(\langle\xi'\rangle x_1, x', \langle\xi'\rangle y_1, y', \xi') e^{i\langle x'-y', \xi'\rangle} d\xi' \tag{1.4.18}$$

with $k(t, x', t', y', \xi')$ satisfying

$$\begin{aligned} &\|D^\alpha_{x'} D^\beta_{y'} D^\gamma_{\xi'} D^j_{t,t'} k(t, x', t', y', \xi')\| \le \\ &\qquad C_{\alpha\beta\gamma jr}(1+t+t')^{-s}(1+|\xi'|)^{m+1+\varsigma(|\alpha|+|\beta|)-\sigma|\gamma|} \qquad \forall \alpha, \beta, \gamma, j, r. \end{aligned} \tag{1.4.19}$$

is called *singular Green operator of order* m.

Let $\boldsymbol{G}^m_{\sigma,\varsigma}$ denote the class of such operators. Further, by $\boldsymbol{G}^m_{\mathsf{cl}}$ we denote class of operators with functions $k \sim \sum k_s$ where k_s are positively homogeneous of degrees $(m+1-s)$ with respect to ξ'.

(ii) One can define $\boldsymbol{G}^m_{\sigma,\varsigma}$ and $\boldsymbol{G}^m_{\mathsf{cl}}$ for any half-space and for any bounded domain X with the smooth boundary.

One can prove (see L. Boutet de Monvel [1]) the following statements:

Proposition 1.4.13. *Let $P \in \boldsymbol{P}_{\mathsf{cl}}(X)$, $T \in \boldsymbol{T}_{\mathsf{cl}}(X)$, $G, G' \in \boldsymbol{G}_{\mathsf{cl}}(X)$. Then $\mathcal{K}_P$, $\mathcal{K}_T$ and $\mathcal{K}_G$ are smooth in $\partial X \times \bar{X}$, $\bar{X} \times \partial X$ and $\bar{X} \times \bar{X}$ respectively with the exception of the diagonal of $\partial X \times \partial X$ (i.e. the set $\{(x,x),\ x \in \partial X\}$).*

Proposition 1.4.14. *(i) Let $P \in \boldsymbol{P}^m_{\mathsf{cl}}(X, \mathbb{H}', \mathbb{H})$. Then P is a bounded operator from $\mathscr{H}^{s-1/2}(\partial X, \mathbb{H})$ to $\mathscr{H}^{s-m}(X, \mathbb{H}')$.*

Further, as $X = \{x\colon x_1 > 0\}$ P is a bounded operator from $\mathscr{H}^{s-1/2}(\partial X, \mathbb{H})$ to

$$\bigcap_{j\ge 0} \mathscr{H}^{j}\bigl(\mathbb{R}^+, \mathscr{H}^{s-m-j}(\mathbb{R}^{d-1}, \mathbb{H}')\bigr). \tag{1.4.20}$$

(ii) Let $T \in \boldsymbol{T}^m_{\mathsf{cl}}(X, \mathbb{H}', \mathbb{H})$. Then T is a bounded operator from $\mathscr{H}^{s}(X, \mathbb{H})$ to $\mathscr{H}^{s-m-1/2}(\partial X, \mathbb{H}')$.

(iii) Let $G \in \boldsymbol{G}^m_{\mathsf{cl}}(X, \mathbb{H}', \mathbb{H})$. Then G is a bounded operator from $\mathscr{H}^{s}(X, \mathbb{H})$ to $\mathscr{H}^{s-m}(X, \mathbb{H}')$.

Further, as $X = \{x\colon x_1 > 0\}$ P is a bounded operator from $\mathscr{H}^{s}(X, \mathbb{H})$ to (1.4.20).

Proposition 1.4.15. *(i) $P \in \boldsymbol{P}^m_{\mathsf{cl}}(X, \mathbb{H}', \mathbb{H})$ if and only if $P^* \in \boldsymbol{T}^{m+1}_{\mathsf{cl}}(X, \mathbb{H}, \mathbb{H}')$.*

(ii) $G \in \boldsymbol{G}^m_{\mathsf{cl}}(X, \mathbb{H}'\mathbb{H})$ if and only if $G^* \in \boldsymbol{G}^m_{\mathsf{cl}}(X, \mathbb{H}, \mathbb{H}')$.

Proposition 1.4.16. *Let $P \in \boldsymbol{P}_{\mathsf{cl}}(X)$, $T \in \boldsymbol{T}_{\mathsf{cl}}(X)$, $G, G' \in \boldsymbol{G}_{\mathsf{cl}}(X)$ and let $A, A' \in \boldsymbol{\Psi}_{\mathsf{cl},\mathsf{trans}}(X)$ with $\mathsf{ord}(A), \mathsf{ord}(A') \in \mathbb{Z}$, $B \in \boldsymbol{\Psi}_{\mathsf{cl}}(\partial X)$* [28].

(i) Then as $\mathsf{ord}A' \le 0$

$$A_X P, GP, PB \in \boldsymbol{P}_{\mathsf{cl}}(X), \tag{1.4.21}$$

$$TA'_X, TG, BT \in \boldsymbol{T}_{\mathsf{cl}}(X), \tag{1.4.22}$$

$$A_X G, GA'_X, GG', PT \in \boldsymbol{G}_{\mathsf{cl}}(X), \tag{1.4.23}$$

$$TP \in \boldsymbol{\Psi}_{\mathsf{cl}}(\partial X) \tag{1.4.24}$$

and

$$A_X A'_X - (AA')_X \in \boldsymbol{G}_{\mathsf{cl}}(X) \tag{1.4.25}$$

and in each statement the rule "the order of the product equals to the sum of the orders" holds;

[28] Here and below we deliberately skip $\mathbb{H}$, $\mathbb{H}'$ and $\mathbb{H}''$ in our notations as otherwise it would be too tedious.

(ii) On the other hand, as $m' = \mathsf{ord}A' \ge 1$

$$TA'_X = \sum_{0\le j\le m'-1} B_j \eth_{\partial X} D_1^j + T', \tag{1.4.26}$$

$$GA'_X - (GA')_X = \sum_{0\le j\le m'-1} T''_j \eth_{\partial X} D_1^j + G', \tag{1.4.27}$$

$$A_X A'_X - (AA')_X = \sum_{0\le j\le m'-1} T'''_j \eth_{\partial X} D_1^j + G' \tag{1.4.28}$$

with operators $B_j \in \boldsymbol{\Psi}_{\mathsf{cl}}^{m+m'-1-j}(\partial X)$ *and* T''_j, $T'''_j \in \boldsymbol{T}_{\mathsf{cl}}^{m+m'-1-j}(X)$ *and* T', G' *belonging to the corresponding classes.*

Proposition 1.4.17. *Let conditions of proposition 1.4.16 be fulfilled. Assume that locally* $X = \{x_1 > 0\}$.

(i) Let $F_j : v \to (\delta^{(j)} \otimes v)$. *Then* $\eth_X A F_j, G F_j \in \boldsymbol{P}_{\mathsf{cl}}(X)$ *and* $T F_j \in \boldsymbol{\Psi}_{\mathsf{cl}}(\partial X)$ *and the rule "the order of the product equals to the sum of the orders" holds if we assign an order* $(j+1)$ *to operator* F_j.

(ii) Let $\eth_{\partial X}$ *be an operator of restriction to* ∂X. *Then* $\eth_{\partial X} G \in \boldsymbol{T}_{\mathsf{cl}}(X)$ *and* $\eth_{\partial X} A_X \in \boldsymbol{T}_{\mathsf{cl}}(X)$ *provided* $\mathsf{ord}(A) < 0$.

On the other hand, for $m = \mathsf{ord}(A) \ge 0$

$$\eth_{\partial X} A_X = \sum_{0\le j\le m} Q_j \eth_{\partial X} D_1^j + T, \qquad Q_j \in \boldsymbol{\Psi}_{\mathsf{cl}}^{m-j}(\partial X),\ T \in \boldsymbol{T}_{\mathsf{cl}}^m(X). \tag{1.4.29}$$

The rule "the order of the product equals to the sum of the orders" holds in both cases if we assign an order 0 *to* $\eth_{\partial X}$.

(iii) $x_1 G, G x_1 \in \boldsymbol{G}_{\mathsf{cl}}(X)$, $x_1 P \in \boldsymbol{P}_{\mathsf{cl}}(X)$ *and* $T x_1 \in \boldsymbol{T}_{\mathsf{cl}}(X)$ *and the rule "the order of the product equals to the sum of the orders" holds if we assign an order* -1 *to an operator of multiplication by* x_1.

Theorem 1.4.18. *Let* $\kappa : X \to X'$ *be a diffeomorphism of* $U \ni 0$ *to* $V \ni 0$ *such that* $\kappa(X) = X'$ *with* $X = U \cap \{x_1 > 0\}$, $X = V \cap \{x_1 > 0\}$. *Then induced map* $A \to \kappa_*^{-1} A \kappa_*$ *transforms* $\boldsymbol{P}_{\mathsf{cl}}^m(X)$, $\boldsymbol{T}_{\mathsf{cl}}^m(X)$ *and* $\boldsymbol{G}_{\mathsf{cl}}^m(X)$ *to* $\boldsymbol{P}_{\mathsf{cl}}^m(X')$, $\boldsymbol{T}_{\mathsf{cl}}^m(X')$ *and* $\boldsymbol{G}_{\mathsf{cl}}^m(X')$ *respectively.*

Remark 1.4.19. Similar statements (may be with some modifications) hold for $\boldsymbol{\Psi}_{\sigma,\varsigma}^m$, $\boldsymbol{P}_{\sigma,\varsigma}^m$, $\boldsymbol{T}_{\sigma,\varsigma}^m$ and $\boldsymbol{G}_{\sigma,\varsigma}^m$.

Boutet de Monvel Algebra

In L. Boutet de Monvel [1] the notion of trace and singular Green operator are extended to include operators as in the right hand expressions of (1.4.26) and (1.4.27) respectively. Further, operators of the type $A+G$ are called *Green operators* in contrast to singular Green operators with $A=0$.

Furthermore, the matrix operators of the form $\mathcal{A}=\begin{pmatrix} A_X+G & P\\ T & B\end{pmatrix}$ from $\mathbb{H}\oplus\mathbb{H}_b$ to $\mathbb{H}'\oplus\mathbb{H}'_b$ are introduced where $\mathbb{H}_b$ and $\mathbb{H}'_b$ are also axillary Hilbert spaces.Recall that $B\in \boldsymbol{\Psi}_{\mathsf{cl}}(\partial X, \mathbb{H}', \mathbb{H}_b)$; T is a trace operator of extended type.

This leads to *Boutet de Monvel algebra* $\mathfrak{B}(X)$ of such operators.

Moreover, the notion of ellipticity is introduced and parametrices for elliptic Boutet de Monvel operators are constructed; these parametrices are also Boutet de Monvel operators; the usual rule $\mathsf{ord}(\mathcal{A}^{-1})=-\mathsf{ord}(\mathcal{A})$ holds.

There are easy and natural generalizations to manifolds and to sections of the Hilbert bundles over such manifolds.

We refer to L. Boutet de Monvel [1] for the further reading; the only things we am interested are Green operators of order 0 (see next Subsubsection) and applications to elliptic boundary problems (see Subsection 1.4.3).

One can also prove similar results for operators of (σ,ς)-type.

Ellipticity and Parametrices

Consider $\mathcal{A}=A_X+G$ operators with $A\in \boldsymbol{\Psi}^0_{\mathsf{cl},\mathsf{trans}}(X,\mathbb{H}',\mathbb{H})$ and $G\in \boldsymbol{G}^0_{\mathsf{cl}}(X,\mathbb{H}',\mathbb{H})$. We want to construct parametrix of the same type. Obviously ellipticity of A is necessary (but not sufficient):

(1.4.30) For each $(x,\xi)\in S^*X$ symbol $a_0(x,\xi)$ is invertible and satisfies $\|a_0(x,\xi)^{-1}\|\le c$.

On the other hand if condition (1.4.30) is fulfilled what we need is to construct parametrix locally in the vicinity of every point $\bar{x}\in\partial X$ and therefore we can assume that $X=\{x\colon x_1>0\}$.

Consider first the "constant" and homogeneous case when $a=a_0(\xi)$ and in definition (1.4.18) of $\mathcal{K}_G$ one can take $k(x_1|\xi'|,y_1|\xi'|,\xi')$ and this function is positively homogeneous of order 1 with respect to the third argument: $|\xi'|\kappa(x_1|\xi'|,y_1|\xi'|,\xi'/|\xi'|)$.

In this case we can make a Fourier transform with respect to $x' \to \xi'$ and the analysis of $A_X + G$ is reduced to the analysis of 1-dimensional operator $a_0(D_1; \xi')_{\mathbb{R}^+} + g_0(\xi')$ where $g_0(\xi')$ is one dimensional integral operator on $\mathbb{R}^+$ with Schwartz kernel $\kappa(x_1, y_1; \xi')$; we need to consider only $|\xi'| = 1$ because the general case is reduced by an obvious change of variables x_1 and y_1.

Obviously we need to assume that $a_0(D_1; \xi')_{\mathbb{R}^+} + g_0(\xi') : \mathscr{L}^2(\mathbb{R}^+, \mathbb{H}) \to \mathscr{L}^2(\mathbb{R}^+, \mathbb{H}')$ is invertible for all ξ'.

Definition 1.4.20. Green operator $A_X + G$ with $A \in \boldsymbol{\Psi}^0_{\mathsf{cl},\mathsf{trans}}(X, \mathbb{H}', \mathbb{H})$ and $G \in \boldsymbol{G}^0_{\mathsf{cl}}(X, \mathbb{H}', \mathbb{H})$ is *elliptic* if

(i) a is elliptic in the sense of (1.4.30) and

(ii) For each $(x', \xi') \in S^*\partial X$ operator

$$a_0(D_1; x', \xi')_{\mathbb{R}^+} + g_0(x', \xi') : \mathscr{L}^2(\mathbb{R}^+, \mathbb{H}) \to \mathscr{L}^2(\mathbb{R}^+, \mathbb{H}')$$

is invertible and $\|\big(a_0(D_1; x', \xi') + g_0(x', \xi')\big)^{-1}\| \le c$ where to construct $g_0(x', \xi')$ we locally make $X = \{x\colon x_1 > 0\}$, take principal symbol of k (i.e. remove all the terms with an order of homogeneity less than 1) and freeze $y' = x'$.

Definition 1.4.21. One-dimensional integral operator g_0 is *principal symbol* of G.

One can see easily that under these assumptions an inverse operator is of the same type as the original one

$$\big(a_0(D_1; x', \xi')_{\mathbb{R}^+} + g_0(x', \xi')\big)^{-1} = a_0^{-1}(D_1; x', \xi')_{\mathbb{R}^+} + g_0'(x', \xi')$$

and smoothly depends on (x', ξ'). This leads to an obvious construction of the singular Green operator $G' \in \boldsymbol{G}^0_{\mathsf{cl}}(X)$ with the principal symbol g_0'. One can see easily that operator $(A'_X + G')$ where A' is the parametrix of A satisfies

$$(A_X + G)(A'_X + G') \equiv (A'_X + G')(A_X + G) \equiv I \tag{1.4.31}$$

modulo $\boldsymbol{G}^{-1}_{\mathsf{cl}}(X)$. Therefore one can adjust G' (by $\boldsymbol{G}^{-1}_{\mathsf{cl}}(X)$ operator) so that (1.4.31) holds modulo operators with smooth in $\bar{X} \times \bar{X}$ symbol.

Therefore we have proved

Theorem 1.4.22. *Operator* $(A_X + G)$ *with* $A \in \boldsymbol{\Psi}^0_{\mathsf{cl},\mathsf{trans}}(X, \mathbb{H}', \mathbb{H})$ *and* $G \in \boldsymbol{G}^0_{\mathsf{cl}}(X, \mathbb{H}', \mathbb{H})$ *has a parametrix* $(A'_X + G')$ *with* $A' \in \boldsymbol{\Psi}^0_{\mathsf{cl},\mathsf{trans}}(X, \mathbb{H}, \mathbb{H}')$ *and* $G' \in \boldsymbol{G}^0_{\mathsf{cl}}(X, \mathbb{H}, \mathbb{H}')$ *if and only if* $(A_X + G)$ *is elliptic in the sense of Definition 1.4.20.*

Resolvent and Functional Calculus. I

Now one can consider resolvent $R(\zeta)$ of operator of this type. Similarly to arguments for classical pseudodifferential and h-pseudodifferential operators we have:

Theorem 1.4.23. *Consider operator* $(A_X + G)$ *with* $A \in \boldsymbol{\Psi}^0_{\mathsf{cl}}(X, \mathbb{H}, \mathbb{H})$ *and* $G \in \boldsymbol{G}^0_{\mathsf{cl}}(X)$. *Let*

$$\Sigma(a) = \overline{\bigcup_{(x,\xi)\in S^*X} \mathsf{Spec}\big(a_0(x,\xi)\big)} \tag{1.4.32}$$

and

$$\Sigma_b(a_0, g_0) = \overline{\bigcup_{(x,\xi)\in S^*\partial X} \mathsf{Spec}\big(a_0(D_1; x', \xi')_{\mathbb{R}^+} + g_0(x', \xi')\big)}. \tag{1.4.33}$$

Then for any vicinity Ω *of* $\Sigma(a_0, g_0) = \Sigma(a) \cup \Sigma_b(a_0, g_0)$ *one can adjust* $(A_X + G)$ *by an operator with infinitely smooth Schwartz kernel so that the spectrum of the new* $(A_X + G)$ *would be contained in* Ω.

Theorem 1.4.24. *Consider operator* $(A_X + G)$ *with* $A \in \boldsymbol{\Psi}^0_{\mathsf{cl}}(X, \mathbb{H}, \mathbb{H})$ *and* $G \in \boldsymbol{G}^0_{\mathsf{cl}}(X)$. *Let* $f(z)$ *be analytic function in the vicinity* Ω *of* $\mathsf{Spec}(A_X + G)$. *Then*

(i) $f(A_X + G)$ *also is operator of this type. Further, pseudodifferential part of it could be constructed as a (formal)* $f(A)$ *and thus principal pseudodifferential symbol is* $f(a_0)$.

(ii) Furthermore, its principal singular Green symbol (in the sense of definition 1.4.21) is a singular Green part of $f\big(a_0(D_1; x', \xi')_{\mathbb{R}^+} + g_0(x', \xi')\big)$;

(iii) In particular, as $A = 0$ *pseudodifferential part of* $f(A_X + G)$ *is* $f(0)$ *and* $f(G) - f(0) \in \boldsymbol{G}^0_{\mathsf{cl}}(X)$.

Further, the resolvent based construction leads to

Theorem 1.4.25. *Consider operator* $(A_X + G)$ *with* $A \in \boldsymbol{\Psi}^0_{\mathsf{cl}}(X, \mathbb{H}, \mathbb{H})$ *and* $G \in \boldsymbol{G}^0_{\mathsf{cl}}(X, \mathbb{H}.\mathbb{H})$.

(i) Assume that it is Hermitian, positive and disjoint from 0 *in operator sense operator.*

Then $(A_X + G)^l$ *is operator of the same type for any* $l \in \mathbb{R}$ *(and even* $l \in \mathbb{C}$*); further, its principal pseudodifferential symbol is* a_0^l *and its principal singular Green symbol is a singular Green part of* $\bigl(a_0(D_1; x', \xi')_{\mathbb{R}^+} + g_0(x', \xi')\bigr)^l$.

(ii) Assume that it is Hermitian and disjoint from 0 in the operator sense operator.

Consider orthogonal (in $\mathscr{L}^2(X, \mathbb{H})$*) projectors* $\Pi^{\pm}$ *on positive and negative subspaces of* A. *Then* $\Pi^{\pm}$ *are operators of the same type; their principal pseudodifferential symbols are* $\pi^{\pm}(x, \xi)$ *(orthogonal projectors to positive and negative subspaces of* $a_0(x, \xi)$*) and their singular Green principal symbols are singular Green parts of the corresponding projectors for* $\bigl(a_0(D_1; x', \xi')_{\mathbb{R}^+} + g_0(x', \xi')\bigr)$.

(iii) In particular, $\Pi^{\pm} \in \boldsymbol{G}^0_{\mathsf{cl}}(X, \mathbb{H}, \mathbb{H})$ *(i.e. pseudodifferential part is* 0*) if and only if* $\pm a_0(x, \xi) < 0$ *for all* (x, ξ).

Remark 1.4.26. Arguments leading to the proof of Theorem 1.1.46 allows us to construct $f(A_X + G)$ as $(A_X + G)$ is Hermitian and f is smooth in the vicinity of $\mathsf{Spec}(A_X + G)$.

1.4.3 Applications to Elliptic Boundary Value Problems

Parametrix Construction

Let us consider $m = \mathsf{ord}A > 0$. In this case $(A_X + G)$ is not necessarily elliptic (i.e. $(A_X + G) : \mathscr{H}^s(X, \mathbb{H}) \to \mathscr{H}^{s-m}(X, \mathbb{H}')$ fails to be invertible) and one needs to consider in addition to $(A_X + G)u = f$ boundary value problem $Tu = g$ where $T : \mathscr{H}^s(X, \mathbb{H}) \to \mathscr{H}^{s-\mu-1/2}(X, \mathbb{H}_b)$ is a trace operator of extended type.

Remark 1.4.27. (i) We are interested only in the case when A is a differential operator, $G = 0$ and $T = B$ with

$$B = \sum_{0\le j\le m-1} B_j \eth_{\partial X} D_1^j \qquad \text{with} \quad B_j \in \boldsymbol{\Psi}^{\mu-j}_{\mathsf{cl}}(\partial X). \tag{1.4.34}$$

(ii) There could be more than one boundary conditions of different orders but one can always make them of the same order (say $\mu = (m-1)$) by multiplying by elliptic pseudodifferential operators on the boundary and then join then by taking $\mathbb{H}_b = \mathbb{H}_{b1} \oplus ... \oplus \mathbb{H}_{bk}$.

Assume that A is elliptic in the sense that $\|a_0(x,\xi)^{-1}\| \le c|\xi|^{-m}$. Let us consider in $(x',\xi') \in T^*\partial X$; without any loss of the generality one can assume that locally $X = \{x_1 > 0\}$.

Consider space $\mathsf{Ker}\, a_0(D_1; x', \xi')_{\mathbb{R}^+}$ of exponentially decaying solutions of $a_0(D_1; x', \xi')$. Alternatively one can consider $\mathcal{Y}(x',\xi') = \eth_{(m-1)} \mathsf{Ker}\, a_0(D_1; x', \xi')_{\mathbb{R}^+}$ where $\eth_{(m-1)} v = (v, D_1 v, ..., D_1^{m-1} v)$.

Definition 1.4.28. We call (A, B) *elliptic boundary value problem* if

(i) A is elliptic in the sense that $\|a_0(x,\xi)^{-1}\| \le c|\xi|^{-m}$;

(ii) Map $(b_j(x',\xi), ... b_{m-1}(x',\xi') : \mathcal{Y}(x',\xi') \to \mathbb{H}_b$ is invertible with uniformly bounded inverse for all (x',ξ') where b_j are principal symbols of B_j are positively homogeneous with respect to ξ' of degrees $\mu - j$.

Assume that (A, B) is elliptic. Let

$$\beta(x',\xi) : \mathbb{H}_b \to \mathcal{Y}(x',\xi') \subset \mathbb{H}^m = \underbrace{\mathbb{H} \oplus ... \mathbb{H}}_{m \text{ copies}} \tag{1.4.35}$$

be an inverse map and $\beta_j(x',\xi')$ $(j = 0, ..., m-1)$ be its components; then $\beta_j(x',\xi')$ are positively homogeneous of degrees $(j-\mu)$ with respect to ξ'.

Consider also maps

$$\kappa_j(x',\xi') : \mathbb{H} \to \mathsf{Ker}\, a_0(D_1; x', \xi') : D_t^k \kappa_j(t, x', \xi')|_{t=0} = \delta_{jk} \qquad j, k = 0, ..., m-1 \tag{1.4.36}$$

which has a Schwartz kernel $k_j(x_1|\xi'|, x', \xi')$ with $k_j(t, x', \xi')$ positively homogeneous of degree $-j$ with respect to ξ' and satisfying (1.4.17) as $|\xi'| = 1$.

Then operator

$$\kappa(x',\xi') = \sum_j \kappa_j(x',\xi')\beta_j(x',\xi') : \mathbb{H}_b \to \mathsf{Ker}\, a_0(D_1; x',\xi') \tag{1.4.37}$$

has a Schwartz kernel $k(|\xi'|x_1, x', \xi') = \sum_j \kappa_j(|\xi'|x_1, x', \xi')\beta_j(x',\xi')$ with $k(t, x', \xi')$ positively homogeneous of degree $-j$ with respect to ξ' and satisfying (1.4.17) as $|\xi'| = 1$.

Consider $P \in \boldsymbol{P}^{-\mu}_{\mathsf{cl}}(X, \mathbb{H}_b, \mathbb{H})$ with the principal symbol $k(|\xi'|x_1, x', \xi')$. Then

$$A_X P \equiv 0 \qquad \mathsf{mod}\ \boldsymbol{P}^{m-\mu-1}_{\mathsf{cl}}(X, \mathbb{H}', \mathbb{H}_b), \tag{1.4.38}$$

$$BP \equiv I \qquad \mathsf{mod}\ \boldsymbol{\Psi}^{-1}_{\mathsf{cl}}(\partial X, \mathbb{H}_b, \mathbb{H}_b). \tag{1.4.39}$$

Meanwhile let $A' \in \boldsymbol{\Psi}^{-m}_{\mathsf{cl}}(X, \mathbb{H}, \mathbb{H}')$ be a parametrix of A. Then $A_X A'_X \equiv I$ modulo infinitely smoothing operator because A is a differential operator. On the other hand, $BA'_X \in \boldsymbol{T}^{\mu-m}_{\mathsf{cl}}(X, \mathbb{H}_b, \mathbb{H}')$ and therefore $G = -PBA'_X \in \boldsymbol{G}^{-m}_{\mathsf{cl}}(X, \mathbb{H}', \mathbb{H}')$ and $R = A'_X + G$ satisfies

$$A_X R \equiv I \qquad \mathsf{mod}\ \boldsymbol{\Psi}^{-\infty}_{\mathsf{cl}}(X, \mathbb{H}', \mathbb{H}'), \tag{1.4.40}$$

$$BR \equiv 0 \qquad \mathsf{mod}\ \boldsymbol{T}^{\mu-m-1}_{\mathsf{cl}}(X, \mathbb{H}_b, \mathbb{H}'). \tag{1.4.41}$$

Therefore

(1.4.42) One can adjust R by element of $\boldsymbol{G}^{-m-1}_{\mathsf{cl}}(X, \mathbb{H}, \mathbb{H}')$ and P by element of $\boldsymbol{P}^{-\mu-1}_{\mathsf{cl}}(X, \mathbb{H}, \mathbb{H}_b)$ so that (1.4.38)–(1.4.41) hold modulo operators with infinitely smooth Schwartz kernels.

In other words,

$$\begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} R & P \end{pmatrix} \equiv \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{1.4.43}$$

and $\mathfrak{R} = (R, B)$ is a *left parametrix of boundary value problem* (A, B).

On the other hand, if $Au \equiv f$ then due to (1.4.12) and consequent analysis

$$A\digamma_X u \equiv \digamma_X f + \sum_{j\le m-1} \delta^{(j)}_{\partial X}(x) g_j$$

modulo $F_X\mathscr{C}^\infty(\bar{X},\mathbb{H}',\mathbb{H})$ where $g_j = \eth_{\partial X,j}u$. Applying $\eth_X A'$ we arrive to

$$u \equiv A'_X f + \eth_X A' \sum_{j\le m-1} \delta^{(j)}_{\partial X}(x) g_j \equiv A'_X f + \sum_j P'_j g_j \tag{1.4.44}$$

modulo $\mathscr{C}^\infty(\bar{X},\mathbb{H})$ where $P'_j \in \boldsymbol{P}^{-j}_{\mathsf{cl}}(X,\mathbb{H},\mathbb{H})$. Further,

$$g_k = \eth_{\partial X,k}u \equiv T_k f + \sum_j Q_{kj} g_j \tag{1.4.45}$$

where $T_k \in \boldsymbol{T}^{-m+k}_{\mathsf{cl}}(X,\mathbb{H},\mathbb{H}')$ and $Q_{kj} \in \boldsymbol{\Psi}^{k-j}_{\mathsf{cl}}(\partial X,\mathbb{H},\mathbb{H})$. Furthermore

$$Bu \equiv \sum_k B_k g_k. \tag{1.4.46}$$

One can prove easily that ellipticity as in Definition 1.4.28 implies that if $f\equiv 0$, $Bu\equiv 0$ then (1.4.45)–(1.4.46) imply that $g_k \equiv 0$. Therefore $\mathfrak{R} = (R,B)$ is a *right parametrix of boundary value problem* as well: $RA+PB \equiv I$ and therefore

(1.4.47) $\mathfrak{R}$ is a *parametrix of boundary value problem*.

Meanwhile

(1.4.48) $R = A'_X + G$ is a *parametrix of* A *under boundary condition* $Bu\equiv 0$.

Dirichlet-to-Neumann and Similar Operators

On the other hand, as $Au\equiv 0$ we get

$$\eth_{\partial X,j}u \equiv \eth_{\partial X,j}PBu \equiv L_j Bu \qquad \text{with } \ L_j \in \boldsymbol{\Psi}^{-\mu+j}(\partial X,\mathbb{H},\mathbb{H}_b) \tag{1.4.49}$$

Example 1.4.29. Consider Laplace-Beltrami operator $\Delta_X = \sum_{j,k} D_j g^{jk} D_k$ with symmetric real positive definite matrix (g^{jk}) (considered as a Riemannian metrics). Then $\Delta_X u \equiv 0$ implies that $\sum g^{jk} n_j D_k u|_{\partial X} = i\Delta^{1/2}_{\partial X} u|_{\partial X}$ where n_j are elements on the unit normal to ∂X (with respect to this metrics) directed into X, $\Delta_{\partial X}$ is defined in the same way according to induced metrics on ∂X.

The properly defined Laplace-Beltrami operator (which is self-adjoint in $\mathscr{L}^2(X)$ with Riemannian density) differs from this one by lower order terms.

Adjoint Operator; Self-Adjoint Operator

Consider operator A_B which is operator A with the domain

$$\mathscr{D}(A_B) = \{u \in \mathscr{H}^m(X, \mathbb{H}) : Bu = 0\}. \tag{1.4.50}$$

We can introduce adjoint operator A_B^* in the standard functional analysis way and we would say that A_B is self-adjoint if $A_B^* = A_B$ (as $\mathbb{H}' = \mathbb{H}$). However we will give a slightly more restrictive definition. As far as we know, nobody explored the gap.

So, for elliptic differential operator A let us consider

$$(Au, v)_X - (u, A^*v)_X = (\mathcal{B}\eth_{(m-1),\partial X}u, \eth_{(m-1),\partial X}v)_{\partial X} = \sum_{j+k\le m-1} (\mathcal{B}_{jk}\eth_{j,\partial X}u, \eth_{k,\partial X}v)_{\partial X} \tag{1.4.51}$$

where A^* is a *formally adjoint* differential operator, we pass from the left-hand expression to the middle one by integration by parts and $\mathcal{B}_{jk}$ are differential operators of order $(m-1-j-k)$ on ∂X. Recall that $\eth_{(m-1),\partial X}u = (\eth_{0,\partial X}u, \dots, \eth_{m-1,\partial X})$ is a function with values in $\mathbb{H}^m$.

Consider the middle expression of (1.4.51). It is a bilinear form of $\eth_{(m-1),\partial X}u$ and $\eth_{(m-1),\partial X}v$. Let us consider boundary value problem $Bu = 0$ for A; Recall that actually

$$Bu = B\eth_{(m-1),\partial X}u = \sum_{0\le j\le m-1} B_j\eth_{j,\partial X}u \qquad \text{with } B_j \in \boldsymbol{\Psi}^{\mu-j}_{\mathsf{cl}}(\partial X, \mathbb{H}_b, \mathbb{H}); \tag{1.4.52}$$

there should be no confusion that we excluded $\eth_{(m-1),\partial X}$ from B.

Further, let us consider for A^* the boundary conditions $B'v = 0$ of the same type (1.4.52).

Definition 1.4.30. Let us consider boundary conditions $Bu = 0$ for operator A and $B'u = 0$ for A^*; both conditions are of (1.4.52) type. Then $B' = 0$ is an *adjoined boundary value problem* for A^* if

$$BU = 0, B'V = 0 \implies (\mathcal{B}U, V)_{\partial X} = 0, \tag{1.4.53}$$

$$(\mathcal{B}U, V)_{\partial X} = 0 \qquad \forall U : BU = 0 \implies B'V = 0 \tag{1.4.54}$$

$$(\mathcal{B}U, V)_{\partial X} = 0 \qquad \forall V : B'V = 0 \implies BU = 0 \tag{1.4.55}$$

and moreover

$$\mathcal{B} = \beta B + (\beta' B')^* \qquad (1.4.56)$$
$$\text{with } \beta \in \boldsymbol{\Psi}^{m-1-\mu}_{\mathsf{cl}}(\partial X, \mathbb{H}'^{m}, \mathbb{H}_b),\ \beta' \in \boldsymbol{\Psi}^{m-1-\mu}_{\mathsf{cl}}(\partial X, \mathbb{H}^{m}, \mathbb{H}'_b).$$

One can prove easily that principal symbols of both β and β' should satisfy

$$\operatorname{Ker} \beta_0(x', \xi') = 0, \qquad \operatorname{Ker} \beta'_0(x', \xi') = 0. \qquad (1.4.57)$$

Definition 1.4.31. Let $A = A^*$, $\mathbb{H}' = \mathbb{H}$. Then $\mathcal{B}$ is skew-self-adjoint. We call $Bu = 0$ *self-adjoint boundary value problem* for A if $B' = B$ satisfies definition 1.4.30.

Example 1.4.32. Let $A = \sum_{j,k}(D_j - V_j)g^{jk}(D_k = V_k) + V$ with symmetric real positive definite matrix $(g^{jk}(x))$ (considered as a Riemannian metrics) and real-valued $V_j(x)$, $V(x)$. Then $A^* = A$. On the other hand,

$$(Au, v)_X - (u, Av)_X = i(Lu, v)_{\partial X} + i(u, Lv)_{\partial X} \qquad (1.4.58)$$
$$\text{with } Lu = \sum_{j,k} \nu_j g^{jk}(D_k - A_k)$$

where $\nu = (\nu_1, \dots, \nu_d)$ is an inner unit normal to ∂X. Therefore

(i) Dirichlet boundary condition $u|_{\partial X} = 0$ is self-adjoint;

(ii) Neumann boundary condition $Lu|_{\partial X} = 0$ is self-adjoint;

(iii) Adjoint to boundary condition $(L - iM)u|_{\partial X} = 0$ with $M \in \boldsymbol{\Psi}^1_{\mathsf{cl}}(\partial X)$ is $(L - iM^*)v|_{\partial X} = 0$; in particular it is self-adjoint if and only if $M = M^*$.

One can see easily that this problem is elliptic if and only if

$$|\big(\sum_{j,k} \tilde{g}^{jk}\xi_j\xi_k\big)^{1/2} + M_1(x', \xi')| \ge \epsilon \qquad (x', \xi') \in S^*\partial X \qquad (1.4.59)$$

where $\tilde{g}^{jk}$ is an induced metrics on ∂X.

Further, one can prove easily that (under condition (1.4.59)) A_B is semi-bounded from below if and only if

$$\big(\sum_{j,k} \tilde{g}^{jk}\xi_j\xi_k\big)^{1/2} + M_1(x', \xi') \ge \epsilon \qquad (x', \xi') \in S^*\partial X. \qquad (1.4.60)$$

As $d = 2$ one can satisfy (1.4.59) and fail (1.4.60) even with differential operator M.

Projectors

Standard resolvent construction (like in the proof of Theorem 1.1.59) yields two following theorems (as $\mathbb{H} = \mathbb{H}'$):

Theorem 1.4.33. *Let A be an elliptic differential operator and B an elliptic boundary value problem for it. Let X be compact. Then for arbitrarily small $\epsilon > 0$ $\operatorname{Spec}(A_B)$ with the exception of finite subset is contained in ϵ-vicinity of $\Sigma(a) \cup \Sigma(a_b)$ where as before*

$$\Sigma(a) = \overline{\bigcup_{(x,\xi)\in T^*X} \operatorname{Spec} a_0(x,\xi)}$$

and

$$\Sigma(a_b) = \overline{\bigcup_{(x',\xi')\in T^*\partial X} \operatorname{Spec} a_0(D_1,x',\xi')_{b_0(D_1,x',x')}} \tag{1.4.61}$$

where a_0 and b_0 are principal symbols of A and B and $a_0(D_1,x',\xi')_{b_0(D_1,x',\xi')}$ is boundary-value problem on $\mathbb{R}^+$;

Theorem 1.4.34. *Let A be an elliptic formally self-adjoint differential operator and B an elliptic boundary value problem for it. Let A_B be self-adjoint.*

Let $\Pi^\pm$ be orthogonal projectors onto positive and negative invariant subspaces of A_B. Then

(i) $\Pi^\pm \equiv \pi^\pm(x,D)_X \mod \boldsymbol{G}^0_{\mathsf{cl}}(X,\mathbb{H},\mathbb{H})$ where $\pi^\pm(x,D) \in \boldsymbol{\Psi}^0_{\mathsf{cl},\mathsf{trans}}(X,\mathbb{H},\mathbb{H})$ and their principal symbols $\pi_0^\pm(x,\xi)$ are projectors onto positive and negative subspaces of $a_0(x,\xi)$.

(ii) $\pi^\pm = 0$ if and only if $\pm a_0(x,\xi)$ is negative definite at every $(x,\xi) \in T^*X \setminus 0$. This condition is equivalent to $\Sigma(a) \subset \mathbb{R}^\mp$. In this case $\Pi^\pm \in \boldsymbol{G}^0_{\mathsf{cl}}(X,\mathbb{H},\mathbb{H})$ and its Schwartz kernel is

$$\mathcal{K}_{\Pi^\pm}(x,y) = (2\pi)^{1-d} \int e^{i\langle x'-y',\xi'\rangle} \kappa^\pm(x_1|\xi'|, \frac{1}{2}(x'+y'),\xi')\, d\xi' \tag{1.4.62}$$

where modulo lower order symbol $\kappa^\pm(x_1|\xi'|,x',\xi')$ is the Schwartz kernel of the spectral projector $\pi^\pm(x',\xi)$ of $a_0(D_1,x',\xi')_{b_0(D_1,x',\xi')}$.

(iii) $\Pi^{\pm} \equiv 0$ *modulo infinitely smoothing operator is and only if* $\pm a_0(x,\xi)$ *is negative definite at every* $(x,\xi) \in T^*X \setminus 0$ *and* $a_0(D_1, x', xi')_{b_0(D_1,x',\xi')}$ *is negative definite at every* $(x',\xi') \in T^*\partial X \setminus 0$. *This condition is equivalent to* $\Sigma(a) \cup \Sigma(a_b) \subset \mathbb{R}^{\mp}$.

Note however that $|A|$ does not have a transmission property unless m is even and $|A|^l$ does not have a transmission property unless ml is even.

Example 1.4.35. Consider operator A_B from Example 1.4.32(iii) with violated condition (1.4.62). Without any loss of the generality one can assume that $g^{1k} = \delta_{1k}$ as $x \in \partial X$ [29]. Then modulo Schwartz kernel of operator G_{cl}^{-1} one can define $\mathcal{K}_{\Pi^{\pm}}(x,y)$ by (1.4.62) with

$$\kappa^{-}(x_1|\xi'|, x', \xi') \equiv 2\lambda(x',\xi) e^{-\lambda(x',\xi')x_1} \theta\big(-M_1(x',\xi') - \lambda(x',\xi)\big) \tag{1.4.63}$$

with $\lambda(x',\xi) = \big(\sum_{j,k} \tilde{g}^{jk}\xi_j\xi_k\big)^{1/2}$.

Semiclassical Theory

Remark 1.4.36. Obviously all things stay the same for differential and pseudodifferential operators with D replaced by hD (and multiplied by h^{-s} if it was operator belonging to Ψ^s_{cl}). Then in the right-hand expressions of (1.4.15) of $\mathcal{K}_P(x,y')$ and $\mathcal{K}_T(x',y)$ and (1.4.15) of $\mathcal{K}_G(x,y)$ one should

(i) Replace x_1 and y_1 by $h^{-1}x_1$ and $h^{-1}y_1$ respectively,

(ii) Replace $e^{i\langle x'-y',\xi'\rangle}$ by $e^{ih^{-1}\langle x'-y',\xi'\rangle}$ and $d\xi'$ by $h^{1-d}d\xi'$ and

(iii) Multiply them by h^{-s} as $\mathcal{K}_P(x,y')$ is defined and by h^{-s-1} as $\mathcal{K}_T(x',y)$ and $\mathcal{G}_T(x,y)$ are defined.

Example 1.4.37. Consider operator A_B from example 1.4.32(iii). Then in the semiclassical theory $a_0 = \sum_{j,k} g^{jk}(\xi_j - V_j)(\xi_k - V_k) + V$ and positivity of a_0 condition means that also $V \ge \epsilon$.

Ellipticity condition (1.4.59) and positivity condition (1.4.62) are replaced by

$$|\lambda(x',\xi') + M_1(x',\xi')| \ge \epsilon \qquad \forall (x',\xi') \in T^*\partial X \tag{1.4.64}$$

[29] One can always achieve it by taking $x_1 = \mathsf{dist}(x, \partial X)$ in the vicinity of ∂X where dist means the Riemannian distance.

and

$$\lambda(x',\xi') + M_1(x',\xi') \ge \epsilon \qquad \forall (x',\xi') \in T^*\partial X \tag{1.4.65}$$

respectively and (1.4.63) is replaced by

$$\kappa^-(h^{-1}x_1|\xi'|, x', \xi') \equiv 2h^{-1}\lambda(x',\xi)e^{-h^{-1}\lambda(x',\xi')x_1}\theta\big(-M_1(x',\xi') - \lambda(x',\xi)\big) \tag{1.4.66}$$

with

$$\lambda(x',\xi') = \big(\sum_{j,k} \tilde{g}^{jk}(\xi_j - V_j)(\xi_k - V_k) + V\big)^{1/2}. \tag{1.4.67}$$

1.A Appendices

1.A.1 Operator $e^{itQ(D)}$

Let $q(D) = \langle QD, D\rangle$ be a nondegenerate real quadratic form in $\mathbb{R}^d$ with real symmetric matrix Q; then

$$k(x,y) = \mathcal{K}_{e^{iq(D)}}(x,y) = (2\pi)^{d/2}|\det Q'|^{1/2} e^{\frac{\pi i d}{4} - \frac{i}{4}q'(x-y)} \tag{1.A.1}$$

is the Schwartz kernel of $e^{iq(D)}$ where $q'(z) = \langle Q^{-1}z, z\rangle$.

We claim first that

$$\|e^{iq(D)}f\|_{\mathscr{L}^\infty} \le C \sum_{|\alpha|\le (d+2)/2} \|D^\alpha f\|_{\mathscr{L}^\infty}. \tag{1.A.2}$$

Indeed, an estimate for $\|\int_{B(x,1)} k(x,y)f(y)\,dy\|$ is obvious; to estimate a similar integral over $\mathbb{R}^d\backslash B(x,1)$ we note that $\nabla_y k(x,y) = \frac{i}{2}Q^{-1}(y-x)k(x,y)$; then $k(x,y) = \omega_{-1}(x-y)\nabla_y k(x,y)$ with positive homogeneous of degree s functions ω_s; then after integration by parts

$$\int_{|x-y|\ge 1} k(x,y)f(y)\,dy = \int \omega_{-2}(x-y)k(x,y)f(y)\,dy + \int \omega_{-1}(x-y)k(x,y)\nabla f(y)\,dy$$

and we continue repeating this procedure until we get ω_{-d-1} in every term.

Estimate (1.A.2) after rescaling $x \mapsto |t|^{-1/2}x$, $D \mapsto |t|^{1/2}D$ implies immediately

$$\|e^{tiq(D)}f\|_{\mathscr{L}^\infty} \le C \sum_{|\alpha|\le(d+2)/2} |t|^{|\alpha|/2}\|D^\alpha f\|_{\mathscr{L}^\infty}. \tag{1.A.3}$$

Using Taylor decomposition (with respect to t at $t=0$) and applying (1.A.3) to the remainder estimate we arrive to

$$\|\Big(e^{tiq(D)} - \sum_{n\le N-1} \frac{1}{n!}\big(itQ(D)\big)^n\Big)f\| \le C|t|^N \frac{1}{N!} \sum_{|\alpha|\le(d+2)/2} |t|^{|\alpha|/2}\|D^\alpha Q(D)^N f\|_{\mathscr{L}^\infty}. \tag{1.A.4}$$

Finally, we get immediately

$$\sum_{|\beta|=K} \|D^\beta\Big(e^{tiq(D)} - \sum_{n\le N-1} \frac{1}{n!}\big(itq(D)\big)^n\Big)f\| \le C|t|^N \frac{1}{N!} \sum_{|\beta|=K,|\alpha|\le(d+2)/2} |t|^{|\alpha|/2}\|D^{\alpha+\beta} q(D)^N f\|_{\mathscr{L}^\infty}. \tag{1.A.5}$$

1.A.2 Almost Analytic Compactly Supported Symbols

Proposition 1.A.1. *For any n there exists a function ϕ_n, such that*

$$\operatorname{supp}(\phi_n) \subset [0,1], \tag{1.A.6}$$

$$\phi_n \ge 0\ , \quad \int_{-\infty}^{\infty} \phi_n\, dx = 1, \tag{1.A.7}$$

$$|\partial^k \phi_n| \le c(cn)^k \qquad \forall k \le n \tag{1.A.8}$$

with c which does not depend on n.

Proof. Let us fix $\phi = \phi_1$ supported in $[0, \frac{1}{2}]$ and satisfying (1.A.7). Then

$$\phi_n(t) = \underbrace{n\phi(nt) * n\phi(nt) * \cdots * n\phi(nt)}_{n \text{ times}} * \phi(t) \tag{1.A.9}$$

is a proper function. □

1.A.3 Stationary Phase Method

Let us consider

$$\mathcal{I}(x) = e^{ih^{-1}\phi(\theta)} f(\theta) d\theta \tag{1.A.10}$$

with smooth and real-valued ϕ, smooth and compactly supported f and $\theta \in \mathbb{R}^N$. Then under assumption

$$|\nabla \phi| \ge \epsilon_0 \qquad \text{on} \quad \mathsf{supp}(f) \tag{1.A.11}$$

$\mathcal{I} = O(h^s)$ for any s. Proof is trivial: decomposing $f = f_1 + ... + f_N$ with $|\partial_{\theta_k}\phi| \ge \epsilon_1$ on $\mathsf{supp}(f_k)$ and integrating by parts.

Therefore in the general case we need to consider f supported in the vicinity of the set

$$\mathcal{C} = \{\theta : \nabla \phi(\theta) = 0\} \tag{1.A.12}$$

of stationary points of ϕ.

Assume now that

$$|\nabla \phi| \le \epsilon_0 \implies |\det \mathsf{Hess}\, \phi| \ge \epsilon_0 \qquad \text{on} \quad \mathsf{supp}(f) \tag{1.A.13}$$

which is a quantitative version of $\mathsf{rank}\, \mathsf{Hess}\, \phi = N$ on $\mathcal{C} \cap \mathsf{supp}(f)$. Then $\mathcal{C} \cap \mathsf{supp}(f) = \{\theta^1, ..., \theta^r\}$ is a finite set. So we can consider f supported in the vicinity a single non-degenerate stationary point $\bar{\theta}$. Without any loss of the generality we can assume that $\bar{\theta} = 0$. Changing variables we can achieve $\phi(\theta) = \frac{1}{2}\varsigma_1\theta_1^2 + ... + \frac{1}{2}\varsigma_N\theta_N^2$ where ς_j are eigenvalues of $\mathsf{Hess}\, \phi(0)$; then $f(0)$ is preserved. Note that

(1.A.14) If $f = O(|\nabla\phi|^l)$ then one can rewrite $\mathcal{I} = h^l \mathcal{I}'$ where $\mathcal{I}'$ is given by the same formula (1.A.10) with $f' = \mathcal{L}_l f$ with depending on ϕ linear differential operator $\mathcal{L}_l$ of order l.

One can prove it using integration by parts.

Decomposing f into Taylor series at $\theta = 0$ we arrive to need to consider only case $f = 1$; then applying formula

$$\int e^{\frac{i}{2}h\langle Q\theta,\theta\rangle}\, d\theta = (2\pi h)^{N/2} |\det Q|^{-1/2} e^{\frac{\pi i\, \mathsf{sgn}\, Q}{4}} \tag{1.A.15}$$

we arrive to the *stationary phase principle*

Proposition 1.A.2. *Consider* (1.A.10) *with smooth and real-valued* ϕ*, smooth and compactly supported* f *and* $\theta \in \mathbb{R}^N$*. Then under assumption* (1.A.13)

$$\mathcal{I} \asymp \sum_{z_k \in \mathcal{C}_k} e^{ih^{-1}\phi(\theta_k)} \times |\det \mathsf{Hess}\, \phi(\theta_k)|^{-1/2} e^{\frac{\pi i}{4}\operatorname{sgn} \mathsf{Hess}\, \phi(\theta_k)} \Big(\sum_{n\ge 0} f_{k,n} h^n\Big) \tag{1.A.16}$$

with $f_{k,0} = f(\theta_k)$ *and* $f_{k,n} = (\mathcal{L}_n f)(\theta_k)$.

Proof. Proof of (1.A.10) is trivial: first, as Q is diagonal it is a product of one-dimensional integrals; further, changing $z = |\lambda_j|^{1/2} h^{-1/2}\theta$ we reduce it to $\int_{-\infty}^{\infty} e^{\pm iz^2/2}\, dz$. The contour of integration of the latter could be deformed to $e^{\pm i\pi/4}\mathbb{R}$; so we get $e^{\pm i\pi/4}\int_{-\infty}^{\infty} e^{-z^2/2}\, dz$ and the latter integral is well-known.□

1.A.4 Function in the Box

Consider function u such that

(1.A.17) $\|u\| \asymp 1$ such that is negligible in $\{|x| \ge \gamma\}$ and $F_{x\to h^{-1}}u$ is negligible in $\{|\xi| \ge \rho\}$.

Does such function exists?

Obviously the same answer remains if instead of "$F_{x\to h^{-1}\xi}u$ is negligible in $\{|\xi| \ge \rho\}$" we request "$\mathsf{supp}\big(F_{x\to h^{-1}}u\big) \subset \{|x| \le \rho\}$".

However then (see f.e. (3.3) of B. Paneah [2] and the references there):

$$\|u\|_{\{|x|\ge\gamma\}} \ge \epsilon e^{-\epsilon h^{-1}\rho\gamma}\|u\| \tag{1.A.18}$$

with $\epsilon = \epsilon(d)$. Therefore

(1.A.19) $\|u\| \asymp 1$ and $\|u\|_{\{|x|\ge\gamma\}} \le Ch^s$ imply $e^{-\epsilon h^{-1}\rho\gamma} \le Ch^s$ which is equivalent to $h^{-1}\rho\gamma \ge \epsilon s|\log h|$; so we must assume $\rho\gamma \ge \epsilon sh|\log h|$.

On the other hand,

(1.A.20) Consider a coherent state $u = (2\pi\alpha)^{-d/2}e^{-|x|^2/2\alpha^2}$. Then $F_{x\to h^{-1}\xi} = (2\pi\beta)^{-d/2}e^{-|\xi|^2/2\beta^2}$ with $\beta = h/\alpha$. Picking $\alpha = \gamma(s|\log h|)^{-1/2}$ we get that u satisfies (1.A.17) as long as $\beta \le \rho(s|\log h|)^{-1/2}$ which holds as $\rho\gamma \ge Cs|\log h|$.

Comments

We have sent the reader to the book of L. Hörmander [1] for details, proofs and advanced results. One can find relatively easy exposition in the monographs of F. Treves [1] and M. Taylor [7].

One can find interesting approaches in the monograph of M. Shubin [3]. Unfortunately there seem to be no comprehensive and comprehensible book, covering relation between pseudodifferential operators and *singular integral operators* due to Calderon-Zygmund-Mihlin.

Chapter 2

Propagation of Singularities in the Interior of the Domain

In this chapter we consider microlocal propagation of singularities (i.e., wave front sets) in the interior of a domain. In general in this book we consider two types of microlocal operator, namely, microelliptic and microhyperbolic. For microelliptic operators[1)] there is no propagation of singularities at all and we prove the different variants of this fact which we need by means of either a parametrix construction or the Gårding inequality; sometimes we combine both methods.

In this chapter we treat microhyperbolic operators. There are different approaches. One based on Fourier integral operators we already considered. While delivering the most information about solutions it also imposes the most conditions to operator. We will apply different approaches instead.

In Sections 2.1–2.3 we use the most general approach based on the energy estimates. First of all, in Section 2.1 we prove the main general Theorem 2.1.2 and certain of its corollaries which will be necessary for our study of spectral asymptotics; this Theorem is formulated in the terms of auxiliary real-valued functions $\phi_j(x,\xi)$ with $j = 1, \dots, J$.

Next, in Section 2.2 we reformulate this Theorem in more geometric terms, namely, in the terms of bicharacteristics and generalized bicharacteristics and consider a host of different examples.

Finally, in Section 2.4 we generalize this Theorem to the case of the

[1)] I.e. operators elliptic in the vicinity of the point in question; so all operators are microelliptic everywhere except their characteristic varieties.

"rough" symbols.

In Section 2.4 we consider Heisenberg approach for the propagation of singularities for scalar and related operators. It will allow us to consider a large time interval.

We will use not only results but also the methods of this chapter to prove different type of propagation later; in particular in perturbed case of the periodic Hamiltonian flow the very long-time propagation results will be obtained by the same and other methods.

2.1 Energy Estimates Approach

2.1.1 Main Theorem

Let us start with the key notion and key statement of this chapter:

Definition 2.1.1. Let $p \in \mathsf{S}_{h,1}(T^*\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ be a Hermitian symbol. Then p is *microhyperbolic at the point* $z = (x, \xi) \in T^*\mathbb{R}^d$ *in the direction* $\ell = (\ell_x, \ell_\xi) \in T_z T^*\mathbb{R}^d$ iff

$$-\langle (\ell p)(z)w, w\rangle \ge \epsilon_0 \|w\|^2 - c_1 \|p(z)w\|^2 \qquad \forall w \in \mathbb{H} \tag{2.1.1}$$

where $\ell p = (\langle \ell_x \partial_x\rangle + \langle \ell_\xi, \partial_\xi\rangle)p$ and $\epsilon_0 > 0$ and c_1 are appropriate constants.

Using this definition we are going to prove the following main Theorem:

Theorem 2.1.2. *Suppose* $P \in \mathbf{\Psi}_h(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ *and assume that*

$$p(z) = P_0(z) \quad \text{is a Hermitian symbol.} \tag{2.1.2}$$

Let $\Omega \Subset T^*\mathbb{R}^d$ *and let* $\phi_1, \dots, \phi_J$ *be real-valued* $\mathscr{C}^N_b$ *functions in* $T^*\mathbb{R}^d$ *such that at every point* $z \in \bar{\Omega}$ *the symbol* p *is microhyperbolic in the directions* $\nabla^\# \phi_1, \dots, \nabla^\# \phi_J$ *where* $\nabla^\# \phi = \langle (\nabla_\xi \phi), \nabla_x\rangle - \langle (\nabla_x \phi), \nabla_\xi\rangle$ *is the Hamiltonian field generated by* ϕ.

Let u *be tempered, i.e.,*

$$\|u\| \le h^{-I}, \tag{2.1.3}$$

and

$$\mathsf{WF}^{s+1}(Pu) \cap \Omega \cap \{\phi_1 \le 0\} \cap \dots \cap \{\phi_J \le 0\} = \emptyset, \tag{2.1.4}$$

$$\mathsf{WF}^{s}(u) \cap \partial\Omega \cap \{\phi_1 \le 0\} \cap \dots \cap \{\phi_J \le 0\} = \emptyset. \tag{2.1.5}$$

Then

$$\mathsf{WF}^s(u) \cap \Omega \cap \{\phi_1 \le 0\} \cap \dots \cap \{\phi_J \le 0\} = \emptyset. \tag{2.1.6}$$

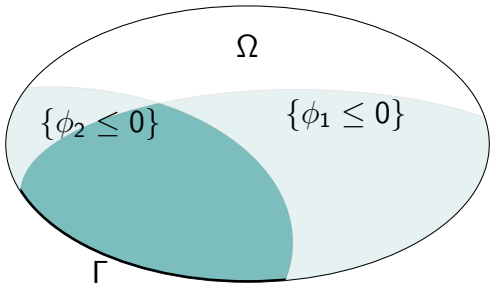


Figure 2.1: Domains to Theorem 2.1.2

This picture shows domains Ω and $\{\phi_j \le 0\}$: so the dark shaded area is where we assume that Pu is negligible and we deduct that u is negligible while we also assume that u is negligible in the small vicinity of the thick line $\Gamma = \partial\Omega \cap \{\phi_1 \le 0\} \cap \{\phi_2 \le 0\}$.

It looks like Holmgren uniqueness Theorem but now Ω resides in the phase space rather than configuration space and "$u \equiv 0$" replaces "$u = 0$" which does not have sense with the relation to phase space.

Proof. Let $q_1, q_2, q_3 \in \mathsf{S}_h(\mathbb{R}^d, \mathbb{C})$ be arbitrary symbols such that $q_1 = 1$ in $\Omega_\epsilon \cap \{\phi_1 \le \epsilon\} \cap \dots \cap \{\phi_J \le \epsilon\}$, $q_2 = 1$ in $(\partial\Omega)_\epsilon \cap \{\phi_1 \le \epsilon\} \cap \dots \cap \{\phi_J \le \epsilon\}$ and $\mathsf{supp}(q_3) \subset \Omega_{\frac{1}{2}\epsilon} \cap \{\phi_1 \le \frac{1}{2}\epsilon\} \cap \dots \cap \{\phi_J \le \frac{1}{2}\epsilon\}$.

Let $Q_k = \mathsf{Op}^{\mathsf{w}}(q_k)$ and $N = N(s, l)$ in the definition of the symbol classes, etc.

One can easily see that in order to prove the Theorem it is sufficient to prove that (2.1.3) and

$$\|Q_1 P u\| \le h^{s+1}, \qquad \|Q_2 u\| \le h^s \tag{2.1.7}_{1,2}$$

yield

$$\|Q_3 u\| \le C h^s \tag{2.1.8}$$

where $\|.\|$ as usual means the $\mathscr{L}^2(\mathbb{R}^d, \mathbb{H})$-norm and C depends on $d, s, l, \epsilon_0, c_1, \epsilon$,

$$c_2 = \|\!|P|\!\| + \sum_k \|\!|q_k|\!\| + \sum_j \|\phi_j\|_{\mathscr{C}^\infty};$$

however,

(2.1.9) C depends neither on Ω nor even its diameter.

We prove (2.1.8) under the additional assumption

$$\|Q_1 u\| \le h^{s-\delta}; \tag{2.1.10}$$

with any exponent $\delta > 0$ which does not depend s; we can take $\delta = \frac{1}{2}$ here; obviously, the general case is reduced to this one by "induction" on s in (2.1.8).

Let $\chi \in \mathscr{C}^\infty(\mathbb{R})$ be a fixed real-valued function such that

$$\begin{aligned}\mathsf{supp}(\chi) \subset (-\infty, 0],\ \chi > 0 \text{ on } \quad (-\infty, 0),\ \chi' \ge c_3\chi \text{ and}\\ \chi_1 = \chi^{\frac{1}{2}},\ \chi_2 = (-\chi')^{\frac{1}{2}} \in \mathscr{C}^\infty(\mathbb{R})\end{aligned} \tag{2.1.11}$$

where the prime means derivative and the constant c_3 can be chosen later (such a function χ exists for every c_3).

We can replace ϕ_j by $(\phi_j - \frac{1}{4}\epsilon)$ and ϵ by $\frac{3}{4}\epsilon$; hence it is sufficient to prove (2.1.8) only for $Q_3 = \mathsf{Op}^{\mathsf{w}}(q_3)$ with

$$q_3(z) = \zeta(z) \prod_{1\le k\le J} \chi(\phi_k(z)) \tag{2.1.12}$$

where $\zeta \in \mathscr{C}_0^N(\Omega_\epsilon)$, $0 \le \zeta \le 1$, $\zeta = 1$ in $\Omega_{\frac{2}{3}\epsilon}$; there exists such a function with $\mathscr{C}^\infty$-norm not exceeding $C_1 = C_1(d, N, \epsilon)$.

Let us consider the main identity of the energy estimate method

$$\begin{aligned}-\mathsf{Re}\ i([P, Q]u, u) =\\ \mathsf{Re}\, i((Q + Q^*)Pu, u) + \mathsf{Re}\, i((P^* - P)Qu, u).\end{aligned} \tag{2.1.13}$$

Let us plug $Q = Q_3^2$; then $Q = Q^*$ and the absolute value of the first term in the right hand expression in (2.1.13) does not exceed $\|Q_3 Pu\| \cdot \|Q_3 u\|$.

Then (2.1.3) and (2.1.6) imply that $\|Q_3 Pu\| \le 2h^{s+1}$: indeed, $q_1 = 1$ in the ϵ_1-neighborhood of $\mathsf{supp}(q_3)$ with $\epsilon_1 = \epsilon_1(d, \epsilon_0, c_1, J, \epsilon) > 0$). Therefore the absolute value of the first term in the right hand expression in (2.1.13) does not exceed $2h^{s+1}\|Q_3 u\|^2$.

Meanwhile the second term in the right hand expression in (2.1.13) equals

$$\mathsf{Re}\, i((P^* - P)Q_3 u, Q_3 u) + \mathsf{Re}\, i([P^* - P, Q_3]Q_3 u, u). \tag{2.1.14}$$

Then (2.1.2) yields that the absolute value of the first term here does not exceed $C_2 h\|Q_3 u\|$ where C_2 does not depend on c_3. Further, (2.1.2), (2.1.3)

and (2.1.10) imply that the absolute value of the second term does not exceed $Ch^{s+\frac{3}{2}}\|Q_3u\|^2$ where C depends on c_3. Thus (2.1.13) implies that

$$|-\operatorname{Re} ih^{-1}([P,Q]u,u)| \le C_2\|Q_3u\|^2 + Ch^{2s}. \tag{2.1.15}$$

Now our goal is to estimate $-\operatorname{Re} ih^{-1}([P,Q]u,u)$ from below. Let us note that the principal symbol of the operator $-ih^{-1}[P,Q]$ is equal to

$$\sum_{1\le j\le J} 2\{p,\phi_j\}r_j^2 - \{p,\zeta^2\}\prod_{1\le k\le J}\chi^2(\phi_k) \tag{2.1.16}$$

where

$$r_j = \zeta\chi_1(\phi_j)\chi_2(\phi_j)\prod_{k\ne j}\chi(\phi_k)$$

and χ_1, χ_2 are defined by (2.1.11).

Since $q_2 = 1$ in the ϵ_1-neighborhood of the support of the second term in (2.1.16) we conclude from (2.1.3), $(2.1.7)_{1,2}$, (2.1.10) and (2.1.15) that

$$|\sum_{1\le j\le J}(S_jR_ju,R_ju)| \le C_2\|Q_3u\|^2 + Ch^{2s} \tag{2.1.17}$$

where $R_j = \operatorname{Op}^{\mathrm{w}}(r_j)$, $S_j = \operatorname{Op}^{\mathrm{w}}(\{p,\phi_j\})$.

The microhyperbolicity of p in the directions $\nabla^{\#}\phi_j$ yields that there exists a symbol q_4 vanishing in the ϵ_1-neighborhood of the supports of r_j for all j such that $\|\!|q_4|\!\| \le C$ and the following inequalities are fulfilled:

$$\langle(\{p,\phi_j\} + cp^{\dagger}p + (q_4-\epsilon_0)I)w,w\rangle \ge 0 \qquad \forall w\in\mathbb{H}\quad \forall z\in T^*\mathbb{R}^d\quad \forall j = 1,\dots,J. \tag{2.1.18}$$

Then passing from symbols to operators we obtain by means of the Gårding inequality (Theorem 1.1.29) that

$$\operatorname{Re}(S_jv,v) + c_1\|Pv\|^2 + \operatorname{Re}(Q_4v,v) \ge (\epsilon_0 - Ch)\|v\|^2. \tag{2.1.19}$$

Let us plug $v = R_ju$ here. Observe that (2.1.3), (2.1.6) and (2.1.10) yield that

$$\|Pv\| \le \|R_jPu\| + \|[R_j,P]u\| \le Ch^s. \tag{2.1.20}$$

Moreover, observe that $|(Q_4v, v)| \le Ch^{2s}$ because the supports of Q_4 and R_j are disjoint and that $\|v\| \le Ch^{s-\frac{1}{2}}$ in virtue of (2.1.10) and (2.1.3). Therefore

$$\mathsf{Re}(S_jR_ju, R_ju) \ge \epsilon_0\|R_ju\|^2 - Ch^{2s}. \tag{2.1.21}$$

This inequality and (2.1.17) yield that

$$\sum_{1\le j\le J} \|R_ju\| \le C_2\|Q_3u\|^2 + Ch^{2s}. \tag{2.1.22}$$

Moreover, (2.1.11) yields that $q_3^2 \le \epsilon_3^{-1}r_j^2$ with $\epsilon_3 = c_3^{-1}$. Therefore the Gårding inequality yields that

$$\|Q_3v\|^2 \le \epsilon_3\|R_jv\|^2 + Ch\|v\|^2.$$

Let us plug $v = Q_5u$ here where $q_5 = 1$ in the ϵ_1-neighborhood of $\mathsf{supp}(q_5)$; then this estimate and (2.1.3) and (2.1.6) yield that

$$\|Q_3u\|^2 \le c_3^{-1}\|R_ju\|^2 + Ch^{2s}.$$

This estimate, (2.1.6) again, the possibility of choosing ϵ_3 arbitrarily small and the fact that C_2 does not depend on c_3 (and ϵ_3) together imply (2.1.8).□

Remark 2.1.3. (i) In fact, we could prove this Theorem for $J = 1$ from which the general case follows. Indeed, one can reduce $J \ge 2$ to $J = 1$ considering $\phi = f(\phi_1, \dots, \phi_J)$ where $f = f_\varepsilon \in \mathscr{C}^\infty$ such that $\partial_{t_j}f(t) > 0$ $\forall j$ $\forall t = (t_1, \dots, t_J)$ and

$$\{t : f(t) \le -\varepsilon\} \subset \{t_1 \le 0, \dots, t_J \le 0\} \subset \{t : f(t) \le \varepsilon\}$$

with arbitrarily small constant $\varepsilon > 0$.

(ii) We can replace "$\phi_j \le 0$" by "$\phi_j < 0$" in (2.1.4)–(2.1.6). We can use one type of inequalities for some indices j and another type fo other indices.

(iii) The Theorem and Statement (ii) here remain true if we assume that p is microhyperbolic in $\bar\Omega$ in the direction $\nabla^\#\psi$ with a real-valued function ψ with $\|\psi\|_{\mathscr{C}^\infty} \le c_2$ and that p is *non-strictly microhyperbolic* in $\bar\Omega$ in the directions $\nabla^\#\phi_j$ for $j = 1, \dots, J$; the non-strict microhyperbolicity means that for $\ell = \nabla^\#\phi_j$

$$-\langle((\ell p)(z)w, w\rangle \ge -c_1\|p(z)w\|^2 \qquad \forall w \in \mathbb{H}; \tag{2.1.23}$$

we reduce this case to one considered above replacing ϕ_j by $(\phi_j + \varepsilon\psi)$ with small enough $\varepsilon > 0$.

We need however to assume that sets $\{\phi_j \le t\}$ depend continuously on t at $t = 0$ i.e.

$$\overline{\{z, \phi_j(z) < 0\}} = \{z, \phi_j(z) \le 0\}$$

and

$$\overline{\{z, \phi_j(z) > 0\}} = \{z, \phi_j(z) \ge 0\}.$$

In what follows we need

Proposition 2.1.4. *Statement "*$(2.1.7)_{1,2}$ & $(2.1.3) \implies (2.1.8)$*" remains true if in the definition of* c_2 *and hence in the definition of* C *we replace* $\|\!|P|\!\|$ *by* $\|\!|P - \bar{P}|\!\|$ *with an arbitrary Hermitian matrix* $\bar{P} \in \mathcal{L}(\mathbb{H}, \mathbb{H})$.

Proof. Let $P = \bar{P} + P'$, $\|\!|P'|\!\| \le c_2 - 1$. Without any loss of the generality one can assume that $\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_2$, $\bar{P}\mathbb{H}_j \subset \mathbb{H}_j$ and $\|\bar{P}w_1\| \le c_2\|w_1\|$, $\|\bar{P}w_2\| \ge c_2\|w_2\| \quad \forall w_j \in \mathbb{H}_j$. Then

$$P = \begin{pmatrix} P_1 & P_{12} \\ P_{21} & P_2 \end{pmatrix}$$

and the principal symbol p_2 of P_2 is invertible and the norm of the inverse symbol does not exceed 1; then its parametrix $R_2 \in \mathbf{\Psi}_h(\mathbb{R}^d, \mathbb{H}_2, \mathbb{H}_2)$ exists with $\|\!|R_2|\!\| \le C_0 = C_0(d, N, c, c_1, c_2)$. Thus the treatment of the propagation of singularities for P is reduced to that for $P'' = P_1 - P_{12}R_2P_{21} \in \mathbf{\Psi}_h(\mathbb{R}^d, \mathbb{H}_1, \mathbb{H}_1)$ with $\|\!|P''|\!\| \le C_0$.

The principal symbol $p'' = p_1 - p_{12}p_2^{-1}p_{21}$ of P'' is Hermitian because $p_{12} = p_{21}^{\dagger}$. Moreover, the equality

$$\langle \{p'', \phi\} w_1, w_1 \rangle = \langle \{p, \phi\} w, w \rangle$$

holds for $w = (w_1, w_2) = (w_1, -p_2^{-1}p_{21}w_1)$ and hence condition (2.1.1) for p implies the same condition for p'' with the constants c_1 and ϵ_0 replaced by C_0c_1 and $C_0^{-1}\epsilon_0$. □

2.1.2 Cauchy Problem. I

Let us now consider the Cauchy problem for the operator

$$P = hD_t + A \tag{2.1.24}$$

with $A \in \mathscr{C}^1\big([0,\,T], \mathbf{\Psi}_h(\mathbb{R}^d, \mathbb{H}, \mathbb{H})\big)$ where

(2.1.25) $a(t,x,\xi) = A_0(t,x,\xi)$ is a Hermitian symbol for all (t,x,ξ).

We assume that $\Omega \Subset \mathbb{R} \times T^*\mathbb{R}^d$ and $\phi_j \in \mathscr{C}^1\big([0,\,T], \mathscr{C}^N_b(T^*\mathbb{R}^d)\big)$ are real-valued functions such that

$$\langle \{p, \phi_j\}(t,z)w, w\rangle \geq \epsilon_0 \|w\|^2 \quad \forall w \in \mathbb{H}. \tag{2.1.26}$$

Then we replace (2.1.13) by the identity

$$\begin{aligned} h\,\mathsf{Re}(Qu,u)_T - \mathsf{Re}\, i([P,Q]u,u)_{[0,T]} = \\ h\,\mathsf{Re}(Qu,u)_0 + \mathsf{Re}\, i((Q+Q^*)u,u)_{[0,T]} + \mathsf{Re}\, i((P-P^*)Qu,u)_{[0,T]} \end{aligned} \tag{2.1.27}$$

where $(.,.)_{[0,T]}$ and $(.,.)_t$ are inner products in $\mathscr{L}^2((0,\,T)\times\mathbb{R}^d, \mathbb{H})$ and in $\mathscr{L}^2(\mathbb{R}^d, \mathbb{H})$ for the indicated t respectively; for the norms similar notation will be used. Repeating the proof of Theorem 2.1.2 with the obvious modifications we obtain

Theorem 2.1.5. *Let conditions* (2.1.24) *and* (2.1.25) *be fulfilled and suppose* $\Omega \Subset \mathbb{R}\times T^*\mathbb{R}^d$. *Let* $\phi_j \in \mathscr{C}^1([0,\,T], \mathscr{C}^\infty_b(T^*\mathbb{R}^d))$ *be real-valued functions satisfying* (2.1.26) *in* Ω_ϵ *where* M_ϵ *denotes* ϵ*-vicinity of* M.

Let $q_1, q_2, q_3 \in \mathscr{C}([0,\,T], \mathsf{S}_h(T^*\mathbb{R}^d, \mathbb{C}))$ *be symbols such that*

$$\begin{aligned} &q_1 = 1 \quad \textit{in} \qquad \Omega_\epsilon \cap \{\phi_1 \leq \epsilon\} \cap \dots \cap \{\phi_J \leq \epsilon\},\\ &q_2 = 1 \quad \textit{in} \quad (\partial\Omega)_\epsilon \cap \{\phi_1 \leq \epsilon\} \cap \dots \cap \{\phi_J \leq \epsilon\} \end{aligned}$$

and

$$\mathsf{supp}(q_3) \subset \Omega_{\frac{\epsilon}{2}} \cap \{\phi_1 \leq \frac{\epsilon}{2}\} \cap \dots \cap \{\phi_J \leq \frac{\epsilon}{2}\}.$$

Let $Q_k = \mathsf{Op}^{\mathsf{w}}(q_k)$ *and* $N = N(s,l)$ *in the definition of symbol classes, etc. Then*

$$\sup_{t\in[0,T]} \|u\|_t \leq 1, \qquad \|Q_1 u\|_0 \leq h^s, \tag{2.1.28}$$

$$\int_0^T (\|Q_1 Pu\|_t + h\|Q_2 u\|_t)\,dt \leq h^{s+1} \tag{2.1.29}$$

imply that

$$\sup_{t\in[0,T]} \|Q_3 u\|_t \le Ch^s \tag{2.1.30}$$

with the same type constant C as in Theorem 2.1.2[2)].

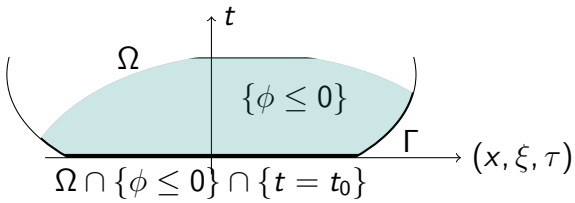


Figure 2.2: Domains to Theorem 2.1.5 $\Gamma = \partial\Omega \cap \{\phi \le 0\} \cap \{t_0 \le t \le T\}$ $\Omega \cap \{\phi \le 0\} \cap \{t_0 \le t \le T\}$.

Here instead of Ω we consider strip $\Omega \cap \{t_0 \le t \le T\}$ and initial condition are on the bottom lid $\{t = t_0\}$ while the top lid $\{t = T\}$ is free. So, we assume that $Pu \equiv 0$ at $\Omega \cap \{\phi \le 0\} \cap \{t_0 \le t \le T\}$ and conclude that $u \equiv 0$ there, assuming also that $u \equiv 0$ at and also that $u|_{t=0} \equiv 0$ at

Remark 2.1.6. (i) Theorem 2.1.5 remains true if we in the conditions and in the conclusion replace $\sup_{t\in[0,T]} \|.\|_t$ by $\frac{1}{\sqrt{T}}\|.\|_{[0,T]}$.

(ii) Moreover, Proposition 2.1.4 remains true[3)].

Let us note that if

$$\|\nabla_z a\| \le c \qquad \forall (t,z) \in [0,T]\times T^*\mathbb{R}^d \tag{2.1.31}$$

then (2.1.26) holds for a function $\phi = t + \psi(z)$ provided $\|\nabla_z \psi\| \le c^{-1}(1-\epsilon)$.

Corollary 2.1.7. *Let conditions* (2.1.24), (2.1.25) *and* (2.1.31) *be fulfilled. Then the statements of Theorem 2.1.5 and Remark 2.1.6 hold for $\Omega = [0,T]\times T^*\mathbb{R}^d$, $q_k = q_k(z)$ $(k = 1,3)$, $q_2 = 0$ provided*

$$\operatorname{dist}(\operatorname{supp}(1-q_1), \operatorname{supp}(q_3)) \ge CT + \epsilon$$

with an arbitrarily small constant $\epsilon > 0$.

[2)] Surely with $\|P\|$ replaced by $\|A\|$ in the definition of c_2.

[3)] Surely with $\|P - \bar{P}\|$ replaced by $\|A - \bar{A}\|$ in the definition of c_2.

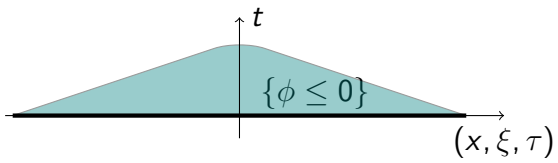


Figure 2.3: To the proof of Corollary 2.1.7

One can prove this Corollary using $\psi = -\epsilon\big(|x-\bar{x}|^2+|\xi-\bar{\xi}|^2+\varepsilon^2\big)^{1/2}$ with an arbitrarily small constant ϵ and the domain $\{t+\psi\le 0\}$ is drawn here.

We will need the following

Definition 2.1.8. Let $x=(x',x'')\in\mathbb{R}^{d'}\times\mathbb{R}^{d''}$ be a partition of variables, $d=d'+d''$. Let u be an admissible (in the previous sense) function. Let us introduce $\mathsf{WF}^{s'}(u)\subset T^*\mathbb{R}^{d'}\times\mathbb{R}^{d''}$. Namely, $(\bar{x},\bar{\xi}')\notin\mathsf{WF}^{s'}(u)$ iff there exists a symbol $q(x,\xi')\in\mathscr{C}^1(\mathbb{R}^{d''},\mathsf{S}_h(T^*\mathbb{R}^{d'},\mathbb{C}))$ such that $q(\bar{x},\bar{\xi}')\ne 0$ and $\|q(x,hD')u\|\le Ch^s$ where $\|.\|$ means the $\mathscr{L}^2(\mathbb{R}^d,\mathbb{H})$-norm.

We will call $\mathsf{WF}^{s'}(u)$ *partial wave front set*.

Partial wave front sets are useful when studying boundary value problems, including those in the domains with edges or conical points.

Using the natural partition (t,x) and in the definition replacing the $\mathscr{L}^2(\mathbb{R}^d,\mathbb{H})$-norm by the $\mathscr{L}^2([0,T]\times\mathbb{R}^d,\mathbb{H})$-norm we obtain

Corollary 2.1.9. *Under the assumptions of Corollary 2.1.7*

$$\begin{aligned}\mathsf{WF}^{s'}(u)\subset\big\{(t,x,\xi):\exists(t',y,\eta)\in\mathsf{WF}^{(s+1)\prime}(Pu)\quad\text{with}\quad \pm t'\in[0,\pm t]\\ \text{and}\quad \mathsf{dist}((x,\xi),(y,\eta))\le c|t-t'|\}\cup\\ \{(t,x,\xi):\exists(y,\eta)\in\mathsf{WF}^{s}((u|_{t=0})\quad\text{with}\quad \mathsf{dist}((x,\xi),(y,\eta))\le c|t|\big\};\end{aligned}$$

this assertion holds for intervals $[0,T]$, $[-T,0]$, $[-T,T]$*; in particular in the last case we obtain that*

$$\pi'\,\mathsf{WF}^{s}(u)\big|_{t=0}\subset\mathsf{WF}^{s'}(u)\big|_{t=0}\subset\mathsf{WF}^{s}(u\big|_{t=0})\cup\mathsf{WF}^{(s+1)\prime}(Pu)\big|_{t=0}\tag{2.1.32}$$

where $\pi':(t,x,\tau,\xi)\to(t,x,\xi)$ *is a natural map.*

Corollary 2.1.9 states the finite speed of propagation of singularities. The second of its assertions also is useful: it justifies the application of Theorem 2.1.2 for the propagation of singularities of solutions to the Cauchy problem without loss in the order of "smallness".

2.1.3 Cauchy Problem. II

Let us now consider the Cauchy problem for the operator

$$P = B^{-1}hD_t + A \tag{2.1.33}$$

where $A \in \boldsymbol{\Psi}_h(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ satisfying (2.1.25) and

(2.1.34) B is a self-adjoint operator in $\mathbb{H}$ such that

$$\langle Bw, w\rangle \geq \|w\|^2 \qquad \forall w \in \mathfrak{D}(B).$$

Let $u^{\pm} = \theta(\pm t)u$ where here and in what follows θ is the Heaviside function. Then

$$Pu^{\pm} = (Pu)^{\pm} \mp ih\delta(t)B^{-1}u_0, \qquad u_0 = \eth_0 u \tag{2.1.35}$$

where $\eth_0$ is the operator of restriction to $\{t = 0\}$. Then we obtain easily

Proposition 2.1.10. *Let conditions* (2.1.33), (2.1.25) *and* (2.1.34) *be fulfilled and* $N = N(s, l)$. *Then*

$$\mathsf{WF}^{s+1}(Pu^{\pm}) \subset \mathsf{WF}^{s+1}(Pu) \cap \{\pm t \geq 0\} \cup \pi_0^{-1}(\mathsf{WF}^{(s+1)\prime}(Pu)\big|_{t=0} \cup \mathsf{WF}^{s+\frac{1}{2}}(\eth_0 u)) \tag{2.1.36}$$

where $\pi_0 : (0, x, \tau, \xi) \to (x, \xi)$ *is a natural map.*

Remark 2.1.11. (i) Obviously function $\phi = t$ satisfies the microhyperbolicity condition in the domain $\Omega' \times \{|\tau| \leq 2c'\}$ if and only if

$$\|B^{-\frac{1}{2}}w\| + \|a(x, \xi)w\| \geq \epsilon_0\|w\| \qquad \forall w \in \mathbb{H} \quad \forall (x, \xi) \in \Omega'. \tag{2.1.37}$$

More precisely, the microhyperbolicity condition (for $\phi = t$) at some energy level $\tau \in [-2c', 2c']$ [4)] yields (2.1.37) while (2.1.37) yields the microhyperbolicity condition at every level $\tau \in [-2c', 2c']$; without any loss of generality we assume that $c' = c$ here.

(ii) One can check easily that replacing in (2.1.37) $B^{-\frac{1}{2}}$ by $B^{-\delta}$ with an exponent $\delta > 0$ we obtain exactly the same condition (2.1.37).

[4)] I.e., the standard microhyperbolicity condition for the operator $A - B^{-1}\tau I$.

Observe that if conditions (2.1.37) and (2.1.31) are fulfilled then microhyperbolicity condition (2.1.1) is fulfilled for $\phi = t + \psi(x,\xi)$ with any real-valued function ψ such that $|\nabla_{x,\xi}\psi| \le \epsilon_2$ with $\epsilon_2 = \epsilon_2(\epsilon_0, c) > 0$. This observation, the uniform version of Proposition 2.1.10, Theorem 2.1.2 and the fact that operators P and $\varphi(hD_t)$ commute imply

Proposition 2.1.12. *Let conditions* (2.1.31), (2.1.33), (2.1.34), (2.1.25) *and* (2.1.37) *be fulfilled. Let* $T \in [\epsilon, c_1]$, $T' = (1+\epsilon)T$ *and* $Q_j = \mathsf{Op}(q_j)$ *with* $q_j \in \mathsf{S}_h(T^*\mathbb{R}, \mathbb{C})$ *such that*

$$\mathsf{dist}(\complement\Omega', \mathsf{supp}(1-q_1), \mathsf{supp}(q_2)) \ge C_0 T \tag{2.1.38}$$

with $C_0 = C_0(c, \epsilon_0)$. *Let* $\chi \in \mathscr{C}_0^N([-1,1])$ *and* $\varphi \in C_0^N([-c,c])$ *be fixed functions,* $\chi_T(t) = \chi(\frac{t}{T})$. *Then*

(i) The inequalities

$$\|u\|_{[-T',T']} \le h^{-l}, \qquad \|Q_1 P u\|_{[-T',T']} \le h^{s+1}, \tag{2.1.39}$$

$$\|B^{-\frac{1}{2}} Q_1 u\|_0 \le h^{s+\frac{1}{2}} \tag{2.1.40}$$

imply that

$$\|\varphi(hD_t)\chi_T(t) Q_2 u\| \le C h^s \tag{2.1.41}$$

where C *depends on* d, s, l, c, ϵ_0, c_1, c_2, *and* ϵ, $c_2 = \|\!|A - \bar{A}|\!\| + \sum_k \|\!|q_k|\!\|$, $\bar{A}$ *is an arbitrarily chosen (constant) Hermitian operator in* $\mathbb{H}$ *and* $N = N(s,l)$ *in the conditions.*

(ii) In particular, for $\|u\| \le 1$ *we obtain that*

$$\begin{aligned} &\mathsf{WF}^s(u) \cap \big\{|t| \le T, |\tau| \le c, (x,\xi) \in \Omega'\big\} \\ &\quad \subset \big\{(t,x,\xi) : \exists (t',y,\eta) \in \mathsf{WF}^{s+1}(Pu) \quad \textit{with} \quad \pm t' \in [0, \pm t] \\ &\qquad\qquad \textit{and} \ \ \mathsf{dist}((x,\xi),(y,\eta)) \le C_0|t-t'|\big\} \cup \\ &\quad \big\{(t,x,\xi) : \exists (y,\eta) \in \mathsf{WF}^{s+\frac{1}{2}}(B^{-\frac{1}{2}} u\big|_{t=0}) \cup \partial\Omega' \\ &\qquad\qquad \textit{with} \ \ \mathsf{dist}((x,\xi),(y,\eta)) \le C_0|t|\big\}. \end{aligned} \tag{2.1.42}$$

So we state here the finite speed of propagation of singularities for energy levels $\tau \in [-c, c]$.

2.1.4 Properties of Fundamental Solutions

Now let us treat the (approximate) fundamental solution to the Cauchy problem for operator (2.1.33).

Let us consider $\mathcal{L}(\mathbb{H},\mathbb{H})$-valued distribution $U(t,x,y)$ such that an operator $\boldsymbol{U}:\mathscr{L}^2(\mathbb{R}^d,\mathbb{H})\to\mathscr{L}^2(\mathbb{R}^{d+1},\mathbb{H})$ with Schwartz kernel $U(t,x,y)$ satisfies the following estimates in the $\mathcal{L}\bigl(\mathscr{L}^2(\mathbb{R}^{d+1},\mathbb{H}),\mathscr{L}^2(\mathbb{R}^d,\mathbb{H})\bigr)$

$$\|\boldsymbol{U}\|\le M, \tag{2.1.43}$$

$$\|\bar\chi_{T_0}(t)\,\mathsf{Op}(\zeta)P\boldsymbol{U}\|\le Mh^{s+1} \tag{2.1.44}$$

and the estimate in the $\mathcal{L}\bigl(\mathscr{L}^2(\mathbb{R}^d,\mathbb{H}),\mathscr{L}^2(\mathbb{R}^d,\mathbb{H})\bigr)$-norm

$$\|\,\mathsf{Op}(\zeta)(\bar\partial_0\boldsymbol{U}-I)\|\le Mh^{s+\frac12} \tag{2.1.45}$$

where $\zeta=1$ in $\Omega'_\epsilon\Subset T^*\mathbb{R}^d$, here and below $\bar\chi$ (and χ as well) is a fixed function

$$\bar\chi\in\mathscr{C}_0^\infty(\mathbb{R}),\qquad \bar\chi=1\quad\text{on}\quad(-\frac12,\frac12),\qquad \bar\chi_T(t)=\bar\chi(t/T) \tag{2.1.46}$$

$\epsilon\le T_0\le c_1$, $|\!|\!|\zeta|\!|\!|\le c_1$ and the constant M will be discussed later.

In particular, the Schwartz kernel $u(x,y,t)$ of $e^{ih^{-1}BA}$ cut outside of $\{|t|\le T,|\xi|\le C_0\}$ satisfies these assumptions. Then Proposition 2.1.12 yields

Proposition 2.1.13. *Let conditions* (2.1.31)–(2.1.34), (2.1.25), (2.1.37) *and* (2.1.43)–(2.1.45) *be fulfilled with* $M\ge1$, $N=N(s,l)$.

Let $\varphi\in\mathscr{C}_0^N([-c,c])$ *be a fixed function and* $Q_k=\mathsf{Op}(q_k)$ *where* $q_k\in \mathsf{S}_h(T^*\mathbb{R}^d,\mathbb{H},\mathbb{H})$ *satisfy condition*

$$\mathsf{dist}\bigl(\mathsf{supp}(q_1),\mathsf{supp}(q_2)\cup\complement\Omega'\bigr)\ge C_0T \tag{2.1.47}$$

with $C_0=C_0(c)$ *and* $T=T_0/2$. *Then*

$$\|\varphi(hD_t)\bar\chi_{T/2}(t)Q_1\boldsymbol{U}Q_2\|\le CMh^s. \tag{2.1.48}$$

This Proposition and Proposition 1.1.11 yield

Corollary 2.1.14. *Let conditions* (2.1.31)–(2.1.34), (2.1.25), (2.1.37) *and* (2.1.43)–(2.1.45) *be fulfilled with* $M\ge1$, $N=N(s,l)$. *Let* $q_k\in\mathsf{S}_h(T^*\mathbb{R}^d,\mathbb{H},\mathbb{H})$ *and* $\mathsf{diam}(\mathsf{supp}(q_k))\le c$, $k=1,2$. *Then*

(i) If (2.1.43) *holds then for* $T \in [\epsilon, T_0]$ *estimate*

$$\|F_{t\to h^{-1}\tau}\chi_T(t)Q_{1x}U\,{}^tQ_{2y}\| \le CMh^{-d} \qquad \forall \tau : |\tau| \le c \quad \forall x, y \in \mathbb{R}^d \tag{2.1.49}$$

holds where here and in what follows ${}^tQ$ *is a dual operator to* Q *defined by* ${}^tQu = (Q^*u^\dagger)^\dagger$, *"†" is a natural antilinear map from* $\mathbb{H}$ *to the dual space* $\mathbb{H}^\dagger$ *and inversely, and operators with respect to* y *are written to the right of the function*[5]*.*

(ii) Further, under assumption (2.1.47) *the following estimate holds:*

$$\|F_{t\to h^{-1}\tau}\chi_T(t)Q_{1x}U\,{}^tQ_{2y}\| \le CMh^{s} \qquad \forall \tau : |\tau| \le c \quad \forall x, y \in \mathbb{R}^d \tag{2.1.50}$$

Remark 2.1.15. (i) If $Q = q(\overset{3}{x}, \overset{2}{hD}, \overset{1}{x})$ then ${}^tQ = q(\overset{1}{x}, \overset{2}{-hD}, \overset{3}{x})$;

(ii) C is the same type of constant as before.

Let us assume that a is microhyperbolic in Ω' with respect to $\nabla^\# \phi$. Let us apply Theorem 2.1.2 with $\phi_1 = \pm t$ and $\phi_2 = \pm(\phi - \epsilon_1 t)$ with a small enough constant ϵ_1 to $u^\pm$ and equation (2.1.35) replacing P by $\pm P$. We then obtain immediately

Proposition 2.1.16. *Let conditions* (2.1.31)–(2.1.34), (2.1.25), (2.1.37) *and* (2.1.43)–(2.1.45) *be fulfilled with* $M \ge 1$, $N = N(s, l)$. *Let* $\phi \in \mathscr{C}_b^N(T^*\mathbb{R}^d)$ *be a real-valued function such that* $\|\phi\|_{\mathscr{C}^\infty} \le c_1$ *and* a *is microhyperbolic in* Ω' *with respect to* $\nabla^\# \phi$.[6]

Let $q_k \in S_h(T^*\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ *satisfy the conditions*

$$\pm\phi(x,\xi) \le \pm\phi(y,\eta) + \epsilon_1 T - \epsilon \qquad \forall (x,\xi) \in \operatorname{supp}(q_1)\ \forall (y,\eta) \in \operatorname{supp}(q_2) \tag{2.1.51}$$

and

$$\operatorname{dist}\big(\operatorname{supp}(q_1), \complement\Omega'\big) \ge C_0 T + \epsilon \tag{2.1.52}$$

[5] This is a convenient notation because it agrees with the matrix theory notation even if it contradicts to the standard operator theory notation.

[6] Then p is also microhyperbolic in $\Omega = \{(t, x, \tau, \xi) : (x, \xi) \in \Omega', |\tau| \le 2\epsilon_1\}$ with respect to $\nabla^\# \phi$ where $\epsilon_1 = \epsilon_1(\epsilon_0, c, c_1)$.

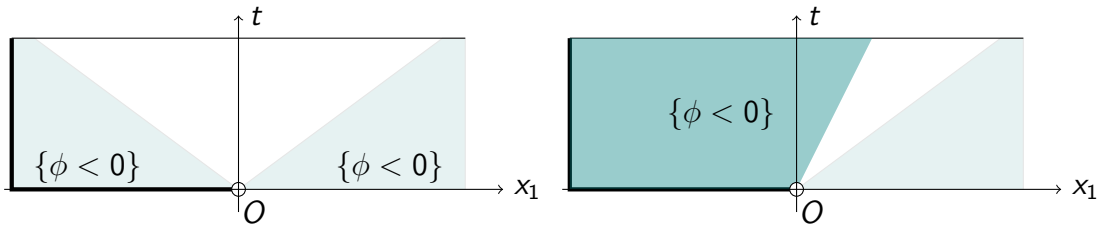
where $T \in [\epsilon, T_0]$ *is arbitrary and* $T_0 = T_0(c) > 0$ *is a small enough constant,* $\epsilon > 0$ *is an arbitrarily small constant.*

Moreover, let $\chi^{\pm}$ *and* $\varphi \in \mathscr{C}_0^N((-2\epsilon_1, 2\epsilon_1))$ *be fixed functions where here and below*

$$\chi^+ \in \mathscr{C}_0^\infty([\frac{1}{2}, 1]), \qquad \chi^-(t) = \chi(-t), \qquad \chi = \chi^+ + \chi^-. \tag{2.1.53}$$

Then

$$\|\varphi(hD_t)\chi_T^{\pm}(t)Q_1 U Q_2\| \le CMh^s. \tag{2.1.54}$$



(a) teal shaded zone is free of WFs (b) now dark-shaded zone is also free of WFs

Figure 2.4: Illustration to the proof of Proposition 2.1.16 and Corollary 2.1.17; (x', ξ, τ) are not shown. All zones are open.

Representing q_1 and q_2 as sums of symbols with small enough supports and applying Proposition 1.1.11 we obtain immediately

Corollary 2.1.17. *Let conditions* (2.1.31)–(2.1.34), (2.1.25), (2.1.37) *and* (2.1.43)–(2.1.45) *be fulfilled with* $M \ge 1$, $N = N(s, l)$.

Let $\phi \in \mathscr{C}_b^N(T^*\mathbb{R}^d)$ *be a real-valued function such that* $\|\phi\|_{\mathscr{C}^\infty} \le c_1$ *and* a *is microhyperbolic in* Ω' *with respect to* $\nabla^{\#}\phi$. *Let* $\operatorname{diam}(\operatorname{supp}(q_k)) \le c$, $k = 1, 2$. *Then*

(i) The estimate

$$\|F_{t\to h^{-1}\tau}\chi_T^{\pm}(t)\Gamma(Q_{1x}U\,{}^tQ_{2y})\| \le CMh^s \qquad \forall \tau : |\tau| \le \varepsilon_0 \tag{2.1.55}$$

holds where here and in what follows

$$(\Gamma_x u)(x) = u(x, x) \qquad \textit{and} \qquad (\Gamma u) = \int (\Gamma_x u)(x)dx. \tag{2.1.56}$$

(ii) Moreover, if $\phi = \phi(x)$ *and* $\mathsf{supp}(q_k) \subset \{|\xi| \le c_1\}$ $\forall k = 1, 2$ *then the following estimate holds:*

$$\|F_{t\to h^{-1}\tau}\chi_T^{\pm}(t)\Gamma_x(Q_{1x}U\,{}^tQ_{2y})\| \le CMh^s \quad \forall \tau : |\tau| \le \varepsilon_0 \quad \forall x \in \mathbb{R}^d. \tag{2.1.57}$$

Remark 2.1.18. Since condition to sign $\pm$ disappeared when we passed from Proposition 2.1.16 to Corollary 2.1.17, estimates (2.1.55) and (2.1.57) hold for both signs $\pm$ and thus for function χ instead of $\chi^{\pm}$.

Actually this Corollary easily implies a much stronger version of itself:

Theorem 2.1.19. *Let conditions* (2.1.31)–(2.1.34), (2.1.25), (2.1.37) *and* (2.1.43)–(2.1.45) *be fulfilled with* $M \ge 1$, $N = N(s, l)$.

Let $\phi \in \mathscr{C}_b^N(T^*\mathbb{R}^d)$ *be a real-valued function such that* $\|\phi\|_{\mathscr{C}^\infty} \le c_1$ *and assume* a *is microhyperbolic in* Ω' *with respect to* $\nabla^{\#}\phi$.

Let $q_k \in \mathsf{S}_h(T^*\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ *satisfy* (2.1.52), *and let* $\mathsf{diam}(\mathsf{supp}(q_k)) \le c$, $k = 1, 2$. *Let* $T_0 = T_0(c, \epsilon_0) > 0$ *be a small enough constant. Then*

(i) For $T \in [h, T_0]$ *the following estimate holds:*

$$\|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma(Q_{1x}U\,{}^tQ_{2y})\| \le CMh^{1-d}\Big(\frac{h}{T}\Big)^s \qquad \forall \tau : |\tau| \le \varepsilon_0. \tag{2.1.58}$$

(ii) Moreover, if $\phi = \phi(x)$ *and* $\mathsf{supp}(q_k) \subset \{|\xi| \le c_1\}$ $\forall k = 1, 2$ *then the following estimate following holds:*

$$\|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma_x(Q_{1x}U\,{}^tQ_{2y})\| \le CMh^{1-d}\Big(\frac{h}{T}\Big)^s \qquad \forall \tau : |\tau| \le \varepsilon_0 \quad \forall x \in \mathbb{R}^d \tag{2.1.59}$$

Proof. Let us observe that we need to prove estimates (2.1.58) and (2.1.59) only provided $\mathsf{diam}(\mathsf{supp}(q_k)) \le \epsilon_2$, $k = 1, 2$ with a small enough constant $\epsilon_2 = \epsilon_2(d, \epsilon_0, c, c_1) > 0$ (otherwise one can make a partition of unity).

Then without any loss of generality one can assume that $\phi = x_1$: in fact, we can reduce the case $\phi = \phi(x)$ to this one by a change of the coordinate system (and (2.1.59) remains true); meanwhile we can reduce the most general case to this one by replacing U by $U' = \mathcal{T}_x U\,{}^t\mathcal{T}_y$ where $\mathcal{T}$ is a scalar Fourier integral operator such that $\mathcal{T}^*\mathcal{T} \equiv I$ in a neighborhood of $\mathsf{supp}(q_1)$,

$\mathcal{T}\mathcal{T}^* \equiv I$ in a neighborhood of $\Psi_{\mathcal{T}}(\mathsf{supp}(q_1))$ and $\phi = x_1 \circ \Psi_{\mathcal{T}}$ where $\Psi_{\mathcal{T}}$ is the corresponding symplectomorphism.

For a parameter $r \in [h,1]$ let us consider a real-valued r-admissible partition of unity $\sum_k \psi_k^2$ in the vicinity Ω'' of $\mathsf{supp}(q_1)$ [7]. Obviously, for $c_3 = c_3(d,c)$ such functions exist.

Let us replace Q_1 and Q_2 by $\psi_k Q_1$ and $\psi_k Q_2$ respectively and make the *rescaling* (dilatation) $t' = t/r$, $x' = x/r$, $h' = h/r$ (in which case $h'D_{x'} = hD_x$ and $h'D_{t'} = hD_t$).

Now we are in precisely the framework of Corollary 2.1.17 and therefore we obtain the estimate (2.1.57) in the new variables. Then in the old variables we obtain the estimate

$$\|F_{t\to h^{-1}\tau}\chi_{\pm r\mathcal{T}_0}(t)\Gamma_x(\psi_k^2(x)Q_{1x}U\,{}^tQ_{2y})\| \le CMr^{1-d}\bigl(\frac{h}{r}\bigr)^s \qquad \forall \tau : |\tau| \le \varepsilon_0 \quad \forall x \in \mathbb{R}^d \quad \forall k = 1,\dots,K. \tag{2.1.60}$$

Remark 2.1.20. (i) The factor r^{-d} in the right hand expression appears because

(2.1.61) The Schwartz kernel is a function with respect to x and a density with respect to y and therefore the restriction of the Schwartz kernel to the diagonal is also a density.

(ii) The factor r in the right hand expression appears because

$$F_{t\to h^{-1}\tau} = rF_{t'\to h'^{-1}\tau}. \tag{2.1.62}$$

These two simple observations are very important because the rescaling technique will be used very often.

Finally, let us sum with respect to k and pick $r = T/\mathcal{T}_0$. We get estimate (2.1.59). □

Moreover, we need

Remark 2.1.21. In addition to conditions of Theorem 2.1.19 let us assume that

$$\|B_1^{2j}\varphi(hD_t)\chi_{\mathcal{T}}(t)\boldsymbol{U}(t)B_1^{2j}\| \le M \tag{2.1.63}$$

[7] As usual this means that $\psi_k(x)$ $(k = 1,\dots,K)$ are real-valued functions such that $\sum_k \psi_k^2(x) = 1$ in Ω'', $\mathsf{diam}(\mathsf{supp}(\psi_k)) \le r$, $|D_x^\alpha \psi_k| \le c_3 r^{-|\alpha|}$ $\forall \alpha : |\alpha| \le K$; $K = c_3(1+r^{-d})$ and the multiplicity of the covering by $\mathsf{supp}(\psi_k)$ is less than c_3.

where $\varphi = 1$ at $(-c, c)$ and B_1 is an operator satisfying (2.1.34). Then the standard interpolation arguments imply that every estimate obtained remains true if we multiply the left-hand expression by B_1^j on both sides and replace the s-degree of h or $\frac{h}{T}$ in the right hand expression by the corresponding $s/2$-degree.

2.1.5 Overdetermined Systems

Overdetermined Systems

In certain cases we have an extra conditions to solutions of the evolution equation: f.e. for the standard Maxwell system $\partial_t E = \nabla \times H, \partial_t H = -\nabla \times E$ there are extra conditions $\nabla \cdot E = \nabla \cdot H = 0$. To cover this case we consider original system with such extra conditions.

Remark 2.1.22. Let us note that if in Theorem 2.1.2 we know that

$$\|Q_1 P' u\| \le h^s \tag{2.1.64}$$

then one can weaken microhyperbolicity condition (2.1.1):

$$-\langle((\ell p)(z)w, w\rangle \ge \epsilon_0\|w\|^2 - c_1\|p(z)w\|^2 - \|p'(z)w\|^2 \qquad \forall w \in \mathbb{H} \tag{2.1.1}'$$

where p' is the principal symbol of $P' \in \mathbf{\Psi}_h(T^*X, \mathbb{H}', \mathbb{H})$.

Indeed, replacement of (2.1.1) by (2.1.1)$'$ leads to the additional term $C\|q_1 p' w\|^2$ in the left-hand expression of (2.1.18) and then to the additional term $C\|Q_1 P' v\|^2$ in the left-hand expression of (2.1.21). Other details of the proof do not change.

Multi-Time Systems

We will need to consider such systems to analyze systems of commuting operators. Namely let us consider the solution of the *multi-time Cauchy problem*

$$hD_{t_j} U \equiv -A_j \mathbf{U} \qquad \text{in } \Omega \quad (j = 1, \dots, m), \tag{2.1.65}$$

$$\mathbf{U}\big|_{t_1 = \dots = t_m = 0} \equiv I \tag{2.1.66}$$

where $A_j = A_j(x, hD, h)$ commute modulo negligible operators:

$$[A_j, A_k] \equiv 0 \qquad \forall j, k = 1, \dots, m. \tag{2.1.67}$$

Then Proposition 2.1.13 and Corollary 2.1.14 remain true with $\bar{\chi}(t)$ and $\chi(t)$ understood as the products of the corresponding one-dimensional functions with $t = (t_1, \dots, t_m)$, $\tau = (\tau_1, \dots, \tau_m)$.

Moreover, Proposition 2.1.16 remains true under microhyperbolicity condition $(2.1.1)'$ with $P = A_\beta = \beta_1 A_1 + \dots + \beta_m A_m$, $P'_j = A_j$ for some $\beta \in \mathbb{R}^m$ with $|\beta| = 1$ and χ supported in the vicinity of β.

Repeating the proof of Theorem 2.1.19 with rescaling rule

$$F_{t \to h^{-1}\tau} = r^m F_{t' \to h'^{-1}\tau} \tag{2.1.62}'$$

instead of (2.1.62) we arrive to

Theorem 2.1.23. *Let conditions* (2.1.31), (2.1.25), (2.1.43) *and* (2.1.65)–(2.1.65) *be fulfilled for* $j = 1, \dots, m$ *with* $M \ge 1$, $N = N(s, l)$.

Suppose $\phi \in \mathscr{C}^N_b(T^*\mathbb{R}^d)$ *is a real-valued function such that* $\|\phi\|_{\mathscr{C}^\infty} \le c_1$ *and the microhyperbolicity condition* $(2.1.1)'$ *is fulfilled in* Ω' *with* $\mathcal{T} = \nabla^\# \phi$, $p = a_\beta = \beta_1 a_1 + \dots + \beta_m a_m$ *and* $p'_j = a_j$.

Let $q_k \in \mathsf{S}_h(T^*\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ *satisfy* (2.1.52) *and let* $\operatorname{diam}(\operatorname{supp}(q_k)) \le c$, $k = 1, 2$. *Let* $\chi \in \mathscr{C}^\infty$ *be a fixed function supported in the neighborhood of* β *and* $T_0 = T_0(c, \epsilon_0) > 0$ *be a small enough constant. Then*

(i) For $T \in [h, T_0]$ *the estimate*

$$\|F_{t \to h^{-1}\tau} \chi_{\pm T}(t) \Gamma(Q_{1x} U\, {}^t Q_{2y})\| \le C M h^{m-d} \Big(\frac{h}{T}\Big)^s \qquad \forall \tau : |\tau| \le \varepsilon_0 \tag{2.1.58}'$$

holds for a sufficiently small constant $\epsilon_0 = \epsilon_0(c, 0)$.

(ii) Moreover, if $\phi = \phi(x)$ *and* $\operatorname{supp}(q_k) \subset \{|\xi| \le c_1\}$ $\forall k = 1, 2$ *then the estimate*

$$\|F_{t \to h^{-1}\tau} \chi_{\pm T}(t) \Gamma_x(Q_{1x} U\, {}^t Q_{2y})\| \le C M h^{m-d} \Big(\frac{h}{T}\Big)^s \qquad \forall \tau : |\tau| \le \varepsilon_0 \quad \forall x \in \mathbb{R}^d \tag{2.1.59}'$$

holds.

(iii) If an appropriate function $\phi = \phi_\beta$ *exists for every* $\beta \in \mathbb{S}^{m-1}$ *then the corresponding estimate* $(2.1.58)'$ *or* $(2.1.59)'$ *holds for every function* $\chi \in \mathscr{C}^\infty_0\big(B(0,1) \setminus B(0, \frac{1}{2})\big)$.

2.2 Geometric Interpretation

2.2.1 Discussion of the Microhyperbolicity Condition

Scalar Case

In this Section we consider a more geometrical version of Theorem 2.1.2, the main Theorem of the previous section. Then we will discuss various special cases of this main Theorem. We assume throughout this section that $\Omega \Subset T^*\mathbb{R}^d$ and

(2.2.1) Symbol $p \in \mathscr{C}^1_b\big(\bar{\Omega}, \mathcal{L}(\mathbb{H},\mathbb{H})\big)$ is Hermitian and microhyperbolic at each point $z = (x,\xi) \in \Omega$ in the direction $\nabla^{\#}x_1$.

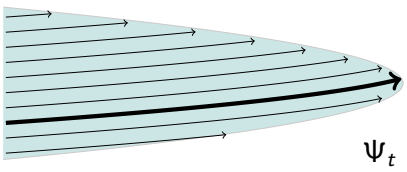


Figure 2.5: Scalar principal type; colored zone is $\{\phi < 0\}$

So x_1 will play a role of time. In the simplest case of scalar principal symbol $p \in \mathscr{C}^2$ microhyperbolicity condition $\{p,\phi\} > 0$ means exactly that $\frac{d}{dt}\phi > 0$ along Hamiltonian trajectories (as $p = 0$) and then Theorem 2.1.2 immediately implies that wave fronts propagate along such trajectories as shown on the picture to the left: one can find ϕ strongly increasing along Hamiltonian trajectories such that $\{\phi < 0\}$ is an arbitrarily narrow "finger" there.

Now we want to provide a similar geometric description in the case of the matrix[8)] symbol p.

Cones of Directions of Microhyperbolicity

Let us introduce the closed invariant subspace $\mathbb{H}_\rho(p,z) \subset \mathbb{H}$ of the matrix $p(z)$ corresponding to the interval $[-\rho,\rho]$ of its spectrum where $\rho > 0$ and $z \in \omega$; recall that $p(z)$ is Hermitian.

Let us define for $\rho > 0$, $\varepsilon > 0$ the sets

$$K_{\rho\varepsilon}(p,z) = \{\ell \in T_z\Omega \setminus 0 : -\langle((\ell^0 p)(z)w, w\rangle \ge \varepsilon\|w\|^2 \quad \forall w \in \mathbb{H}_\rho(p,z)\} \tag{2.2.2}$$

[8)] It is more convenient to use term "matrix" for an operator acting in the auxiliary spaces.

where $\ell^0 = \ell/|\ell|$. Moreover, let us introduce the set

$$K(p,z) = \bigcup_{\rho>0,\varepsilon>0} K_{\rho\varepsilon}(p,z) = \bigcup_{\rho>0} K_{\rho 0}(p,z) \tag{2.2.3}$$

where the second equality is due to (2.2.1) (and we temporarily extended (2.2.2) to $\varepsilon = 0$). Then the following assertion is obvious:

Lemma 2.2.1. *(i) If $|\ell| \le c$ satisfies* (2.1.1) *then $\ell \in K_{\rho\varepsilon}(p,z)$ with $\varepsilon = \rho = \rho(\epsilon_0, c_1) > 0$. On the other hand, if $|\ell| \ge \rho$ and $\ell \in K_{\rho\varepsilon}(p,z)$ then* (2.1.1) *holds with $\epsilon_0 = \epsilon_0(\rho,\varepsilon) > 0$ and $c_1 = c_1(\rho,\varepsilon)$.*

(ii) $K_{\rho\varepsilon}(p,z)$ is a conical strictly convex respectively closed subset of $T_z\Omega \setminus 0$ where here and below 0 means the 0-section of this bundle.

(iii) For $\rho' < \rho$ and $\varepsilon' < \varepsilon$ the set $K_{\rho',\varepsilon'}(p,z)$ contains the ε''-neighborhood of $K_{\rho\varepsilon}(p,z) \cap \{|\ell| = 1\}$ with $\varepsilon'' = \varepsilon''(\epsilon_0, c_1, \rho, \rho', \varepsilon, \varepsilon')$.

(iv) Hence $K(p,z)$ is a conical convex open subset of $T_z\Omega \setminus 0$;

(v) $\ell \in \partial K(p,z)$ if and only if

$$\lim_{\rho\to+0} \inf_{w\in\mathbb{H}_\rho(p,z),\, \|w\|=1} \langle (\ell p)(z)w, w\rangle = 0.$$

Definition 2.2.2. Let $M(z)$ be *set-valued function*; then

(i) As $z_n \to z$ we define for such functions limits

$$\lim_{z_n\to z} M(z_n) := \bigcap_{\varepsilon>0} \bigcap_n \big(M(z_n)\big)_\varepsilon$$

where M_ε is ε-vicinity of M; for conical set we set $M_\varepsilon = \big((M\cap\{|\zeta| = 1\})_\varepsilon\big)_{\mathsf{cone}}$ instead where $M_{\mathsf{cone}} = \{t\zeta, \zeta \in M, t > 0\}$ is a conical hull of M;

(ii) We say that $M(z)$ is *upper semicontinuous*, *lower semicontinuous* and *continuous* at z if for any sequence $z_n \to z$ $\lim_{z_n\to z} M(z_n) \subset M(z)$, $\lim_{z_n\to z} M(z_n) \supset M(z)$ and $\lim_{z_n\to z} M(z_n) = M(z)$ respectively.

Obviously, $K(p,z)$ introduced by (2.2.2)–(2.2.2) is lower semicontinuous; moreover

Lemma 2.2.3. *(i)*[9] $\{(z,\ell) \in T\Omega : \ell \in K_{\rho\varepsilon}(p,z)\}$ *is a respectively closed subset of* $T\Omega \setminus 0$ *where* 0 *means the* 0*-section of this bundle.*

Furthermore, let the index set $\mathfrak{M}$ *be a Hausdorff topological space and assume that for all* $\mathfrak{m} \in \mathfrak{M}$ *the symbol* $p_{\mathfrak{m}}$ *satisfies* (2.2.1) *and that the map* $\mathfrak{M} \ni \mathfrak{m} \to p_{\mathfrak{m}} \in \mathscr{C}^1_b(T^*\mathbb{R}^d)$ *is continuous; then the set* $\{(\mathfrak{m},z,\ell) \in \mathfrak{M} \times T\Omega : \ell \in K_{\rho\varepsilon}(p,z)\}$ *is a respectively closed subset of* $\mathfrak{M} \times (T\Omega \setminus 0)$.

(ii) Statement (iii) of Lemma 2.2.1 remains true if we replace $K_{\rho'\varepsilon'}(p,z)$ *by* $\{(z,\ell) \in T\Omega : \ell \in K_{\rho\varepsilon}(p,z)\}$ *and even by* $\{(\mathfrak{m},z,\ell) \in \mathfrak{M} \times T\Omega : \ell \in K_{\rho\varepsilon}(p,z)\}$ *provided the index set is a metric space, and for all* $\mathfrak{m} \in \mathfrak{M}$ *the symbol* $p_{\mathfrak{m}}$ *satisfies* (2.2.1) *and the map* $\mathfrak{M} \ni \mathfrak{m} \to p_{\mathfrak{m}} \in \mathscr{C}^1_b(T^*\mathbb{R}^d)$ *is uniformly continuous (surely in this case* ϵ *depends on this map also).*

(iii)[10] $\{(\mathfrak{m},z,\ell) \in \mathfrak{M} \times T\Omega : \ell \in K(p,z)\}$ *is an open subset of* $\mathfrak{M} \times (T\Omega \setminus 0)$ *provided the additional conditions at the end of Statement (ii) are fulfilled.*

(iv)[11] $(z,\ell) \in \partial\{(z,\ell) \in T\Omega : \ell \in K(p,z)\}$ *iff*

$$\lim_{\rho\to+0,z'\to z} \inf_{w\in\mathbb{H}_\rho(p,z'),\,\|w\|=1} \langle (\ell p)(z')w, w\rangle = 0.$$

Definition 2.2.4. (i) We call $K(p,z)$ *microhyperbolicity cones*.

(ii) Furthermore, let us introduce the *propagation cones* $K^{\#}_{\rho\varepsilon}(p,z)$ and

$$K^{\#}(p,z) = \bigcap_{\rho>0,\varepsilon>0} K^{\#}_{\rho\varepsilon}(p,z) = \bigcap_{\rho>0} K^{\#}_{\rho 0}(p,z) \tag{2.2.4}$$

where for any set $G \subset T^*\mathbb{R}^d$ the dual cone $G^{\#}$ is defined by

$$G^{\#} = \{\zeta \in T^*\mathbb{R}^d \setminus 0, \sigma(\zeta,\zeta') \le 0 \quad \forall \zeta' \in G\}$$

and $\sigma(.,.)$ is a bilinear symplectic form[12].

Since $K(p,z)$ is lower semicontinuous (see definition 2.2.2) we conclude that

(2.2.5) $K^{\#}(p,z)$ is upper semicontinuous.

[9] Cf. Lemma 2.2.1(ii).
[10] Cf. Lemma 2.2.1(iv).
[11] Cf. Lemma 2.2.1(v).
[12] Surely, $G^{\#}$ is always conic convex closed set in $T^*\mathbb{R}^d$.

Examples

Example 2.2.5. (i) If p is elliptic at z, i.e., if

$$\|p(z)w\| \ge \epsilon\|w\| \qquad \forall w \in \mathbb{H} \tag{2.2.6}$$

then $K(p,z) = T^*\mathbb{R}^d \setminus 0$ and $K^{\#}(p,z) = 0$.

(ii) Suppose that for $z \in \Omega$ the symbol $p(z)$ is unitarily equivalent to

$$\begin{pmatrix} q(z)(\xi_1 - \lambda(z')) & 0 \\ 0 & p''(z) \end{pmatrix}, \qquad \mathbb{H} = \mathbb{H}' \oplus \mathbb{H}'' \tag{2.2.7}$$

where the symbols q and p'' are bounded and Hermitian, λ is a real-valued function with $z' = (x,\xi')$, $\xi' = (\xi_2, \dots, \xi_d)$ in this example and p'' is elliptic i.e. satisfies (2.2.6) on $\mathbb{H}''$,

$$\langle q(z)w, w\rangle \ge \epsilon\|w\|^2 \qquad \forall w \in \mathbb{H}'. \tag{2.2.8}$$

Then in a neighborhood of $\Sigma = \{\xi_1 = \lambda(z')\}$ all those symbols and functions (including the symbol of the operator effecting the unitary equivalence) can be chosen as smooth as p; moreover, for $z \in \Sigma$

$$K(p,z) = \{(\ell', \eta_1) \in T^*\mathbb{R}^d : \eta_1 > (\ell'\lambda)(z')\} \tag{2.2.9}$$

and

$$K^{\#}(p,z) = (\nabla^{\#}(\xi_1 - \lambda))_{\mathsf{cone}}. \tag{2.2.10}$$

(iii) Suppose that for $z \in \Omega$, $p(z)$ is unitarily equivalent to the symbol

$$\begin{pmatrix} p_1(z)\otimes I_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & p_n(z)\otimes I_n & 0 \\ 0 & \cdots & 0 & p''(z) \end{pmatrix} \tag{2.2.11}$$

where $\mathbb{H} = \mathbb{H}'_1 \oplus \dots \oplus \mathbb{H}'_n \oplus \mathbb{H}''$, $\mathbb{H}'_j = \mathbb{H}_j \otimes \mathbb{H}^0_j$, $\mathbb{H}_j$ are finite-dimensional spaces, $p_j(z)$ are matrices in $\mathbb{H}_j$ and I_j are the identity operators in $\mathbb{H}^0_j$. Let us assume that all the symbols are Hermitian and smooth, the symbol effecting the unitary equivalence is smooth too and p'' satisfies (2.2.6) at $\mathbb{H}''$. Then

$$K(p,z) = \bigcap_{1\le j\le n} K(p_j, z) \tag{2.2.12}$$

and

$$K^{\#}(p,z) = \text{convex hull}\Big(\bigcup_{1\le j\le n} K^{\#}(p_j,z)\Big) = \sum_{1\le j\le n} K^{\#}(p_j,z) \tag{2.2.13}$$

where $\sum G_j \coloneqq \{\zeta = \zeta_1 ... + \zeta_n \colon z_1 \in G_1, ..., z_n \in G_n\}$ is an *arithmetical sum of sets*.

Microhyperbolicity: Algebraic Theory

In the framework of (2.2.1) and (2.2.11) let us consider the *characteristic symbol*

$$g(z) = \prod_{1\le j\le n} \det p_j(z). \tag{2.2.14}$$

Proposition 2.2.6. *Let g have a zero of order exactly r at $\bar z \in \Omega$.*

(i) Then[13)]

$$(H^r_{x_1} g)(\bar z) \ne 0 \tag{2.2.15}$$

and $\ell \in K(p,\bar z)$ if and only if

$$(\ell^j H^{r-j}_{x_1} g)(\bar z)\cdot (H^r_{x_1} g)^{-1}(\bar z) > 0 \qquad \forall j = 1, ..., r. \tag{2.2.16}$$

Moreover, $\ell \in \overline{K(p,\bar z)}$ if and only if the corresponding non-strict inequalities are fulfilled.

(ii) Further, $K(p,\bar z)$ coincides with the connected component of the set $\{\zeta \colon g_{\bar z}(\zeta) \ne 0\}$ where $g_{\bar z}(\zeta)$ is a germ of g at $\bar z$:

$$g_{\bar z}(\zeta) = \sum_{|\alpha|=r} \frac{1}{\alpha!}(\partial^\alpha_z g)(\bar z)\zeta^\alpha \tag{2.2.17}$$

(iii) Furthermore, $g_{\bar z}(\zeta)$ is a hyperbolic polynomial of ζ with respect to $\nabla^{\#} x_1$.

[13)] Here and below sometimes we use notation $H_f g \coloneqq (\nabla^{\#} f)(g) = \{f, g\}$.

Proof. Let us first consider the special case $r = r_1 + ... + r_n$ where $r_j = \dim \mathbb{H}_j$. Then (2.2.1) easily implies that $p_1(\bar z) = ... = p_n(\bar z) = 0$ and (2.2.15) holds. In this case $\ell \in K(p, z)$ iff $-(\ell p)(\bar z)$ is a positive definite matrix, i.e., iff $\big((-\ell - tH_{x_1})p\big)(\bar z)$ is a positive definite matrix for every $t \ge 0$ and that is obviously equivalent to (2.2.16). Moreover, Statements (ii) and (iii) are also obvious in this case.

On the other hand, if $r < r_1 + ... + r_n$ then in a neighborhood of $\bar z$ the finite-dimensional symbol

$$p' = \begin{pmatrix} p_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_n \end{pmatrix}$$

is unitarily equivalent to the matrix $\begin{pmatrix} p''' & 0 \\ 0 & p^{IV} \end{pmatrix}$ where p^{IV} satisfies (2.2.6) on the corresponding space and the dimension of the first component of the decomposition is r. It is obvious that condition (2.2.15) or (2.2.16) for $g = \det p'$ is fulfilled if and only if the same condition is fulfilled for $g''' = \det p'''$. Moreover, the germs of g and g''' in $\bar z$ coincide modulo of a constant factor. Finally, it is obvious that the microhyperbolicity conditions for p and p''' are equivalent. □

2.2.2 Main Theorem

Main Theorem

To prove the main result of this section Theorem 2.2.10 we need first to reformulate Theorem 2.1.2 in the following way:

Proposition 2.2.7. *Let conditions* (2.1.2) *and* (2.2.1) *be fulfilled. Consider* $\bar z = (\bar x, \bar \xi) \in \Omega$, *a proper convex closed cone* $G \subset \{(x, \xi) \in T^*\mathbb{R}^d, x_1 > 0\} \cup 0$ *and* $\bar t < x_1$. *Assume that for some* $\rho > 0$, $\varepsilon > 0$

$$K^{\#}_{\rho\varepsilon}(p, z) \subset G \qquad \forall z \in (\bar z - G) \cap \{\bar t \le x_1 \le \bar x_1\} \tag{2.2.18}$$

where $\bar z - G := \{\bar z - z, z \in G\}$.

Let u *be temperate and let*

$$\mathsf{WF}^{s+1}(Pu) \cap (\bar z - G) \cap \{t \le x_1 \le \bar x_1\} \cap \Omega = \emptyset, \tag{2.2.19}$$

$$\mathsf{WF}^{s}(u) \cap (\bar z - G) \cap \{t \le x_1 \le \bar x_1\} \cap \partial\Omega = \emptyset, \tag{2.2.20}$$

$$\mathsf{WF}^{s}(u) \cap (\bar z - G) \cap \{x_1 = t\} \cap \Omega = \emptyset. \tag{2.2.21}$$

Then

$$\mathsf{WF}^s(u) \cap (\bar z - G) \cap \{t \le x_1 \le \bar x_1\} \cap \Omega = \emptyset. \tag{2.2.22}$$

Proof. Due to our conditions $G = \{(x,\xi) : x_1 \ge \psi(x',\xi)\}$ with continuous positive homogeneous of degree 1 convex function ψ. One can smoothen ψ replacing it by $\psi_\eta = \psi * \eta^{-2d}\omega(z/\eta)$ with the smooth nonnegative $\omega \in \mathscr{C}_0^\infty$ such that $\int \omega\, dz' = 1$.

Then $\{x_1 \ge \psi_\eta(x',\xi) + \beta\} \subset G \subset \{x_1 \ge \psi_\eta(x',\xi) - \beta\}$ for arbitrarily small $\beta > 0$ and small enough $\eta = \eta(\beta) > 0$. Furthermore, due to (2.2.18) and properties of cones $K_{\rho\varepsilon}(p,z)$ we conclude that $\nabla^\# \phi \in K_{\rho'\varepsilon'}(p,z)$ for all $z \in \{\phi \le \beta\} \cap \{x_1 \ge \bar t\}$ where $\phi(x,\xi) = x_1 - \bar x_1 + \psi(x',\xi)$, $\rho' = \rho/2$, $\varepsilon' = \varepsilon/2$ and $\beta > 0$ is small enough.

Then Theorem 2.1.2 yields the Proposition. □

We are going to take $G = K^\#_{\rho\varepsilon}(p,\bar z)$ and apply Proposition 2.2.7 for $\bar t = \bar x_1 - \eta$ with $\eta = \eta(\rho,\varepsilon)$ so small that (2.2.18) holds with ρ,ε replaced by $\rho/2, \varepsilon/2$. Furthermore, we are going to repeat this procedure for any $z \in (\bar z - G) \cap \{x_1 = \bar t\}$ and so on, getting the domain $\mathcal{G}_{\rho\varepsilon}$ which is the union of small cut-off cones. Finally, we are going to consider its limit as $\rho \to +0, \varepsilon \to +0$. To codify this construction let us give the key

Definition 2.2.8. (i) *A generalized bicharacteristics (generalized (ρ,ε)-bicharacteristics)* of p is a Lipschitz curve $\{z = z(t)\}$ such that for a.a. t, $z(t)$ is differentiable and

$$\frac{dz}{dt} \in K^\#(p,z), \tag{2.2.23}$$

$$\frac{dz}{dt} \in K^\#_{\rho\varepsilon}(p,z) \tag{2.2.24}$$

respectively.

Since $K^\#_{\rho\varepsilon}(p,z) \subset \{|z'| \le Cx_1\}$ then one can take x_1 as a parameter t (let us recall that $z = (x,\xi)$).

(ii) Let $\mathcal{K}^\pm(p,z,\Omega)$ ($\mathcal{K}^\pm_{\rho\varepsilon}(p,z,\Omega)$) be the union of all the generalized bicharacteristics (generalized (ρ,ε)-bicharacteristics) of p lying in $\bar\Omega$ and going out from the point $z \in \bar\Omega$ in the direction of increasing $\pm t$.

Lemma 2.2.9. *The introduced sets have the following properties:*

(i) $\mathcal{K}^{\pm}(p,z,\Omega) \subset \mathcal{K}^{\pm}_{\rho\varepsilon}(p,z,\Omega) \subset \mathcal{K}^{\pm}_{\rho'\varepsilon'}(p,z,\Omega)$ *for* $0<\rho<\rho', 0<\varepsilon<\varepsilon'$*; moreover,* $\mathcal{K}^{\pm}_{\rho'\varepsilon'}(p,\bar z,\Omega)$ *contains the* $\eta|x_1-\bar x_1|$*-vicinity of every point* $z=(x,\xi)\in\mathcal{K}^{\pm}_{\rho\varepsilon}(p,\bar z,\Omega)$ *intersected with* Ω *where* $\eta=\eta(\epsilon_0,c_1,c_2,\rho,\rho',\varepsilon,\varepsilon')>0$.

(ii) $z'\in\mathcal{K}^{\pm}(p,z,\Omega)$ *if and only if* $z\in\mathcal{K}^{\mp}(p,z',\Omega)$.

(iii) If $z''\in\mathcal{K}^{\pm}(p,z',\Omega)$ *and* $z'\in\mathcal{K}^{\pm}(p,z,\Omega)$ *then* $z''\in\mathcal{K}^{\pm}(p,z,\Omega)$*; Statements (ii) and (iii) remain true for* $\mathcal{K}^{\pm}_{\rho\varepsilon}$ *as well.*

(iv) $\mathcal{K}^{\pm}(p,z,\Omega)$ *and* $\mathcal{K}^{\pm}_{\rho\varepsilon}(p,z,\Omega)$ *are closed sets.*

(v) $\mathcal{K}^{\pm}(p,z,\Omega)=\bigcap\limits_{\rho>0,\varepsilon>0}\mathcal{K}^{\pm}_{\rho\varepsilon}(p,z,\Omega)$.

(vi) If $z_n\to z$, $z'_n\to z'$ *and* $p_n\to p$ *in* $\mathscr{C}^1_b$ *as* $n\to+\infty$ *and if* $z'_n\in\mathcal{K}^{\pm}(p_n,z_n,\Omega)$ *then* $z'\in\mathcal{K}^{\pm}(p,z,\Omega)$ [14].

Proof. Statements (i)–(iii) are obvious; we prove (v); and (iv) and (vi) are proven in the same way. Let us consider $\mathcal{K}^{+}(p,z,\Omega)$. Then (i) implies that $\mathcal{K}^{+}(p,z,\Omega)$ is contained in $\bigcap_{\rho>0,\varepsilon>0}\mathcal{K}_{\rho\varepsilon}(p,z,\Omega)$.

Let us prove the reverse inclusion. If $z'\in\bigcap_{\rho>0,\varepsilon>0}\mathcal{K}^{+}_{\rho\varepsilon}(p,z,\Omega)$ then there exist sequences $\rho_n\to+0,\varepsilon_n\to+0$ and generalized (ρ_n,ε_n)-bicharacteristics $z=z_n(t)$ such that $z_n(0)=z$, $z_n(t_n)=z'$; since one can take $t=x_1+\mathsf{const}$ then one can assume that $t_n=t_0$.

Since $|\frac{dz_n(t)}{dt}|$ is uniformly bounded for all n and $t\in[0,t_0]$ then by the Ascoli-Arzelà Theorem one can assume that $z_n(t)\to z(t)$ as $n\to+\infty$ uniformly with respect to $t\in[0,t_0]$ where $z=z(t)$ is a Lipschitz curve; since $z(t)$ is a function with bounded variation, by the Lebesgue Theorem (f.e. A. N. Kolmogorov, S. V. Fomin [1]) it is differentiable a.e. and we need only to prove that $\frac{dz}{dt}\in K^{\#}(p,z(t))$ for a.a. t.

It is obvious that for every $\rho>0,\varepsilon>0$ there exist $\epsilon>0$ and n_0 such that $z_n-z_n(t')\in\pm K^{\#}_{\rho\varepsilon}(p,z(t'))$ for all $n\ge n_0$ and $t\in[t'-\epsilon,t'+\epsilon]\cap[0,t_0]$ with $\pm(t-t')>0$. Then after the limit transition we obtain that $z-z(t')\in\pm K^{\#}_{\rho\varepsilon}(p,z(t'))$ for every $\rho>0,\varepsilon>0$ and indicated t,t'. This yields that if $z(t)$ is differentiable at $t=t'$ then $\frac{dz}{dt}\in K^{\#}_{\rho\varepsilon}(p,z(t))$ for $t=t'$ and for all $\rho>0,\varepsilon>0$ and hence $\frac{dz}{dt}\in K^{\#}(p,z(t))$ for $t=t'$. □

[14] However, even for $p_n=p$ it is possible that the set $\mathcal{K}^{\pm}(p,z,\Omega)$ is larger than the set of such limits.

Now we can prove the central Theorem of this section:

Theorem 2.2.10. *Suppose* $\Omega \Subset T^*\mathbb{R}^d$ *and conditions* (2.1.2) *and* (2.2.1) *are fulfilled. Moreover, let* $\bar z \in \Omega$. *Then if* u *is temperate and*

$$\mathsf{WF}^{s+1}(Pu) \cap \mathcal{K}^{\mp}(p, \bar z, \Omega) = \emptyset \tag{2.2.25}$$

and

$$\mathsf{WF}^{s}(u) \cap \mathcal{K}^{\mp}(p, \bar z, \Omega) \cap \partial\Omega = \emptyset \tag{2.2.26}$$

then

$$\mathsf{WF}^{s}(u) \cap \mathcal{K}^{\mp}(p, \bar z, \Omega) = \emptyset. \tag{2.2.27}$$

Proof. Let us prove this statement with $\mathcal{K}^{\mp}_{\rho\varepsilon}$ instead of $\mathcal{K}^{\mp}$. Then assertions (v) and (i) of Lemma 2.2.9 yield the statement in its original form. Let us consider $\mathcal{K}^{-}_{\rho\varepsilon}$; replacing t, P by $-t, -P$ one can reduce the $\mathcal{K}^{+}_{\rho\varepsilon}$ case to this one.

It is easy to see that if $\rho' > \rho > \rho'', \varepsilon' > \varepsilon > \varepsilon''$ then for $\bar z = (\bar x, \bar\xi) \in \Omega$ and $\bar t = \bar x_1 - \eta$ with small enough $\eta = \eta(\epsilon_0, \rho, \rho', \rho'', \varepsilon, \varepsilon', \varepsilon'', c, c_1, c_2) > 0$ the following inclusions hold:

$$\mathcal{K}^{-}_{\rho'\varepsilon'}(p, \bar z, \Omega) \supset (\bar z - K^{\#}_{\rho\varepsilon}(p, \bar z)) \cap \{x_1 \ge \bar t\} \supset \mathcal{K}^{-}_{\rho''\varepsilon''}(p, \bar z, \Omega) \cap \{x_1 \ge \bar t\}.$$

Then the assertions of the Theorem for the domain $\Omega_n = \{z \in \Omega, -c_4 + \eta n > x_1\}$ instead of Ω are proven easily for an appropriate constant c_4 by means of induction with respect to n: for $n = 0$ the assertion is trivial since $\Omega_0 = \emptyset$ for appropriate c_4, and the induction step is provided by Proposition 2.2.7. Therefore this assertion holds for Ω as well. □

Discussion

Remark 2.2.11. (i) It is easy to see that in the assumptions of Theorem 2.1.2 the function ϕ_j increase along the generalized (ρ, ε)-bicharacteristics provided $\rho > 0, \varepsilon > 0$ are small enough; hence if $\phi(\bar z) < 0$ and $z \in \mathcal{K}^{-}_{\rho\varepsilon}(p, \bar z, \Omega)$ then $\phi(z) < 0$. Therefore Theorem 2.1.2 and Theorem 2.2.10 are equivalent.

(ii) One can generalize Theorem 2.2.10 to overdetermined systems. Assume that $\mathbb{H}' = \mathbb{H}^m$ and $P' = (P'_1, \dots, P'_m)$ where P'_j are self-adjoint and commute between themselves and with P. Then one should define $\mathbb{H}_\rho(z)$ as an invariant subspace of $\mathbb{H}$ corresponding to joint spectrum of $p(z), p'_1(z), \dots, p'_m$ residing in $[-\rho, \rho]^{m+1}$.

Remark 2.2.12. (i) "Equations" of (2.2.23) type are called *ordinary differential inclusions*. They are very well known in the control theory. However our case is rather special: one can see easily that either in the generic point $\nabla^{\#} g \neq 0$ and then in such points $K^{\#}$ are 1-dimensional or at least $g = g_1^{r_1} g_2^{r_2} \cdots g^k r_k$ and again g redefined as $g_1 g_2 \cdots g_k$ has the above property. So in our case in in generic points we have (up to parametrization[15)]) ordinary differential equations.

(ii) Let $\Sigma^{(r)}(p)$ be a set of points in which $g(z)$ has a zero exactly of order not exceeding r. One can see easily that $K(p,z)$ and $K^{\#}(p,z)$ are continuous functions of $z \in \Sigma^{(r)}(p) \setminus \Sigma^{(r-1)}(p)$.

(iii) Let us define $L(p,z)$ for $z \in \Sigma^{(r)}(p)$ recurrently in the following way:

$$z \in \Sigma^{(1)}(p) \implies L(p) = K^{\#}(p,z), \tag{2.2.28}$$

$$z \in \Sigma^{(r)}(p) \setminus \Sigma^{(r-1)}(p) \implies L(p,z) = \bigcup_{\{z_n\} \subset \Sigma^{(r-1)},\ z_n \to z} \lim_{z_n \to z} L(p, z_n) \cup \left(K^{\sigma}(p,z) \cap T_z \Sigma^{(r)}\right) \tag{2.2.29}$$

where for any set $M \ni z$ we define a conical subset in $T_z\Omega$:

$$\zeta \notin T_z M \iff \exists \varepsilon > 0 \text{ such that } \operatorname{dist}(z + t\zeta, M) \geq \varepsilon t \ \forall t \in (0, \varepsilon). \tag{2.2.30}$$

Due to (2.2.5) $L(p,z) \subset K^{\#}(p,z)$. Obviously, $L(p,z)$ is conic but not necessarily convex set. One can prove that actually generalized bicharacteristics satisfy a.e.

$$\frac{dz}{dt} \in L(p,z). \tag{2.2.31}$$

2.2.3 Examples of Propagation

We consider $\mathcal{K}^{-}(p,z)$ in different cases.

Constant Coefficients Case

Example 2.2.13. Let $p = p(\xi)$; then $K(x,\xi) = K(\xi) = \mathbb{R}^d_x \oplus \Gamma(\xi)$ with a conic set $\Gamma(\xi) \subset \mathbb{R}^d_\xi$ and $K^{\#}(\xi) = \Gamma'(\xi) \oplus \{0\}$ where $\mathbb{R}^d_x \supset \Gamma'(\xi)$ is a dual cone to $\Gamma(\xi)$.

[15)] And we can take x_1 as a parameter due to (2.2.1).

Finally, $\mathcal{K}^{\pm}(x,\xi) = \big(x \pm \Gamma'(\xi)\big) \times \{\xi\}$. One can look at the phenomena of *conical refraction* (especially in Crystal Optics and Magnetic Hydrodynamics in R. Courant & D. Hilbert [2]. General theory is exposed in M F. Atiyah, R. Bott & L. Gårding [1]

Scalar Principal Type (Again)

Example 2.2.14. (i) If p satisfies (2.2.6) then $\mathcal{K}^{\pm}(p,z,\Omega) = z$;

(ii) If in Ω

$$p(z) = \begin{pmatrix} g(z)q(z) & 0 \\ 0 & p''(z) \end{pmatrix},$$

$p''(z)$ and $q(z)$ satisfy (2.2.6) and (2.2.8) respectively on the corresponding subspaces, $g(z)$ is a real-valued scalar function with $dg \neq 0$ and if $g(\bar{z}) = 0$ then $\mathcal{K}^{\pm}(p,\bar{z},\Omega) = \gamma^{\pm}(p,\bar{z})$ is an interval of the *semibicharacteristics* of g lying in $\bar{\Omega}$, i.e., of the curve $z = z(t)$ with

$$\frac{dz}{dt} = \nabla^{\#} g(z), \qquad z(0) = \bar{z} \qquad (\pm t \geq 0). \tag{2.2.32}$$

In particular, if $g = \tau + a(x,\xi)$ where a is a scalar real-valued symbol and (x,ξ) is replaced by (t,x,τ,ξ) we obtain the system

$$\frac{dz}{dt} = \nabla^{\#} a(z), \qquad z = (x,\xi), \qquad \tau = -a(z) = \mathsf{const}. \tag{2.2.33}$$

This case will be analyzed very carefully in Section 2.4.

Until the end of subsection we are dealing in the framework of Example 2.2.5(iii).

Operators with Completely Factorizing Characteristic Symbol: Non-Involutive Case

We assume that characteristic symbol $g(z)$ defined by (2.2.14) completely factorizes:

$$g = \prod_{1\leq j\leq n} g_j(z), \qquad g_j = 0 \implies H_{g_j} \neq 0. \tag{2.2.34}$$

In this case

$$\Sigma := \mathsf{Char}\, p = \bigcup_{1\le j\le n} \Sigma_j, \quad \Sigma_j = \{z : g_j(z) = 0\}.$$

Then at $\Sigma_j \setminus \bigcup_{k\neq j} \Sigma_k$ wave front sets propagate along bicharacteristics of g_j and the question is what happens at $\Sigma_j \cap \Sigma_k$ with $j \neq k$. Obviously

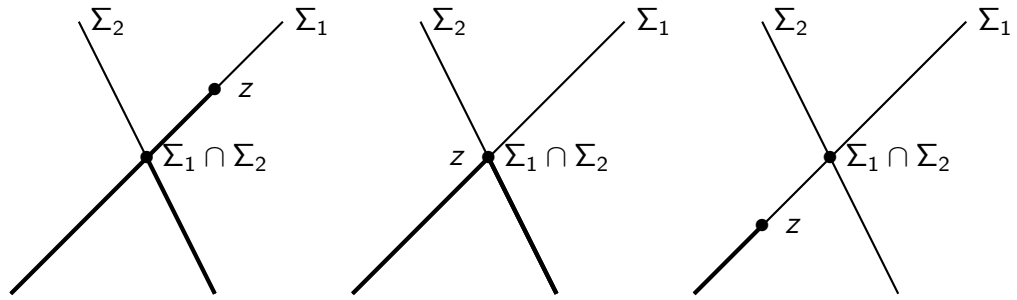
$$\begin{aligned} &(2.2.35)\quad g_1 = ... = g_n = 0 \implies \\ &\qquad K^{\#}(z) = \{\alpha_1 \nabla^{\#} g_1 + \alpha_2 \nabla^{\#} g_2(z) + ... + \alpha_n \nabla^{\#} g_n(z), \quad \alpha \in \mathbb{R}^{+\,n}\}. \end{aligned}$$

Let us consider the series of *microlocal examples*. So all statements below are true for guaranteed small time.

Example 2.2.15. Let $n = 2$ and

$$(2.2.36)\qquad g_j = g_k = 0 \implies \{g_j, g_k\} \neq 0 \qquad (j \neq k).$$

Then as $z \in \Sigma_1 \cap \Sigma_2$ $L(z) = \{\alpha_1 \nabla^{\#} g_1 + \alpha_2 \nabla^{\#} g_2, \quad \alpha \in \mathbb{R}^+ \times 0 \cup 0 \times \mathbb{R}^+\}$ and $\mathcal{K}^-(g, z) = \gamma_1^-(z) \cup \gamma_2^-(z)$ where $\gamma_j^-(z)$ are semibicharacteristics of g_j. The case of $z \in \Sigma_1 \setminus \Sigma_2$ is shown at pictures:



(a) $\gamma_1^-(z)$ hits $\Sigma_1 \cap \Sigma_2$ (b) $z \in \Sigma_1 \cap \Sigma_2$ (c) $\gamma_1^-(z) \cap \Sigma_1 \cap \Sigma_2 = \emptyset$

Figure 2.6: Thick lines show $\mathcal{K}^-(z)$

However Theorem 2.2.10 fails to notice that while branching at $\Sigma_1 \cap \Sigma_2$ the singularity loses its power by $1/2$: let $z^* \in \Sigma_1 \cap \Sigma^2$, $z_j^{\pm} \in \gamma_j^+(z^*)$ and different from z^*. Let $\gamma_j(z, z')$ be a segment of the bicharacteristics with the ends z and z'.

Proposition 2.2.16. *In the framework of Example 2.2.15*

$$\begin{aligned}(2.2.37)\quad &\gamma_1(z_1^-, z_1^+)\cap \mathsf{WF}^{s+1}(Pu)=\emptyset,\quad z_1^-\notin \mathsf{WF}^{s}(u),\\ &\gamma_2(z_2^-, z^*)\cap \mathsf{WF}^{s+1/2}(Pu)=\emptyset, z_2^-\notin \mathsf{WF}^{s-1/2}(u)\implies\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad z_1^+\notin \mathsf{WF}^{s}(u).\end{aligned}$$

Proof. To prove this let us notice first that due to (2.2.36)

(2.2.38) $g_j=g_k=0=0 \implies H_{g_j}$ and $\nabla^{\#}g_k$ are linearly independent for all $j\neq k$

and then that multiplying p by elliptic symbols from the left and right one can achieve p in the form (2.2.11) with $n=2$ and $p_j=g_j$. Then $P=\begin{pmatrix} P_1 & hB_1\\ hB_2 & P_2\end{pmatrix}$ where $P_j\in \mathbf{\Psi}_h$ are operators with the principal parts g_j and $B_j\in\mathbf{\Psi}_h$. Let R_j be parametrices of P_j with the properly directed propagation; then one can run method of successive approximations and thus needs to consider terms Q^nR where $Q=\begin{pmatrix} 0 & hR_1B_1\\ hR_2B_2 & 0\end{pmatrix}$ and $R=\begin{pmatrix} R_1 & 0\\ 0 & R_2\end{pmatrix}$.

Notice that Q^2 is diagonal with diagonal elements $h^2R_1B_1R_2B_2$ and $h^2R_2B_2R_1B_1$ and QR has diagonal elements 0 and non-diagonal ones $hR_1B_1R_2$ and $hR_2B_2R_1$. So one should consider two these later operators.

Further, due to (2.2.36) one can assume without any loss of the generality that $P_1=\xi_1$ and $P_2=\xi_1-x_1$ and then one can take

$$(2.2.39)\qquad (R_ju)(x)=h^{-1}\int^{x_1} e^{ih^{-1}(\phi_j(x_1)-\phi_j(y_1))}u(y_1,x')\,dy_1$$

with $\phi_1=0$ and $\phi_2=\frac{1}{2}x_1^2$. Then stationary phase method imply that $\|R_1B_1R_2\|\asymp h^{-3/2}$ which easily implies (2.2.37). □

Remark 2.2.17. (i) Arguments of the above proof imply that if $n\geq 3$, condition (2.2.36) is fulfilled and p is diagonal, then singularities propagate along branching bicharacteristics and while switching the branch magnitude acquires factor $h^{1/2}$; so as l (in the condition $\|u\|=O(h^{-l})$) and s (in the WF) are fixed, only $\lfloor 2(l+s)\rfloor$ switches should be allowed.

See V. Kucherenko [1] for the original proof (where $n\geq 2$ was considered).

(ii) $\mathcal{K}^-$ is a different matter. Note that

(2.2.40) If $J \subset \{1, \dots, n\}$ and $g_j(z) = 0 \iff j \in J$ then

$$L(z) = \Big\{ \sum_{j \in J} \alpha_j \nabla^{\#} g_j, \quad \alpha = (\alpha_k)_{k \in J} \in \operatorname{Ker} \mathcal{C}_J \cap \mathbb{R}^{+\#J} \Big\}$$

where $\mathcal{C}_J = (\{g_j, g_k\})_{j,k \in J}$.

If we know that

(2.2.41) For each $J \subset \{1, \dots, n\}$ $L(z) = 0$ where $L(z)$ is now defined by (2.2.40)

then every generalized bicharacteristics is also a branching bicharacteristics. However, if assumption (2.2.41) [16] fails and one can chose non-vanishing $\alpha \in \mathscr{C}^1$ such that $\alpha(z) \in L(z)$ then bicharacteristics of $g' = \sum_{j \in J} \alpha_j g_j$ on $\Sigma_J = \{g_j = 0 \ j \in J\}$ are generalized bicharacteristics of p but not branching bicharacteristics of $\{g_j\}_{j \in J}$.

Example 2.2.18. Let $n = 2$, (2.2.38) holds but $\{g_1, g_2\}$ vanishes at some points. Note that $L(z) = K(z)$ as $g_1 = g_2 = \omega = 0$. Further, assume that

(2.2.42) $g_1 = g_2 = \omega = 0 \implies \{g_1, \omega\} \cdot \{g_2, \omega\} > 0$ with $\omega = \{g_1, g_2\}$.

Then while Σ_1 and Σ_2 intersect transversally due to (2.2.34), bicharacteristics of g_j are tangent to Σ_{3-j} as $g_1 = g_2 = \omega = 0$. Note due to (2.2.42) one can consider ω as a "time" parameter along bicharacteristics as ω is close to 0; one can see easily that generalized bicharacteristics are ordinary bicharacteristics with branching at $\Sigma_1 \cup \Sigma_2$ and that there are no more than two branching on each of bicharacteristics. More precisely, every maximally extended bicharacteristics passing near but not through $\Lambda = \Sigma_1 \cap \Sigma_2 \cap \{\omega = 0\}$ has either two (as on Figure 2.7(a)) or none (as on Figure 2.7(b)) branching points and each time trajectory switches branches the magnitude of solution acquires factor $h^{1/2}$. On the other hand, such trajectory passing through Λ has exactly one branching point (as on Figure 2.7(c)).

Model: $g_1 = \xi_1$, $g_2 = \xi_1 - \xi_2 - x_1^2/2$.

Conjecture 2.2.19. *As trajectory switches branches on Λ the magnitude of solution acquires factor $h^{1/3}$.*

[16] As $n = 3$ (2.2.41) fails iff $\{g_1, g_2\} \cdot \{g_2, g_3\} > 0$ and $\{g_2, g_3\} \cdot \{g_3, g_1\} > 0$ (then the third inequality of this sort holds as well).

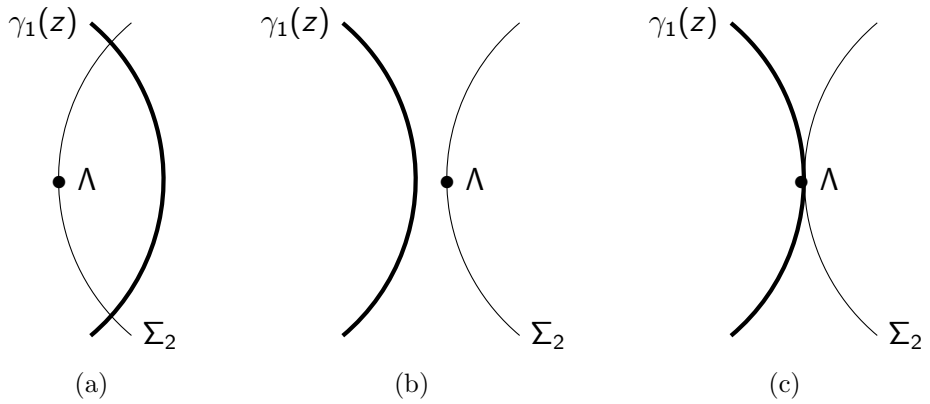


Figure 2.7: Thick lines show $\mathcal{K}^-(z)$

Proof of this conjecture would require a study of oscillatory integrals with degenerate phase functions.

Example 2.2.20. Let $n = 2$, (2.2.38) holds and

$$g_1 = g_2 = \omega = 0 \implies \{g_1, \omega\} \cdot \{g_2, \omega\} < 0 \quad \text{with } \omega = \{g_1, g_2\}. \tag{2.2.43}$$

In this case bicharacteristics of g_j passing through Λ do not hit Σ_{3-j} again. On the other hand if branching on $\Sigma_1 \cap \Sigma_2 \setminus \Lambda$ occurs then due to Proposition 2.2.16 factor $O(h^{1/2})$ is acquired. Therefore again singularities propagate along branching bicharacteristics.

However again as in Remark 2.2.17(ii) bicharacteristics of $g' = \sum_j \alpha_j g_j$ (with $\alpha_j > 0$ defined from $\sum_j \alpha_j \{g_j, \omega\} = 0$) are generalized bicharacteristics of p but not a branching bicharacteristics of $\{g_1, g_2\}$.

Operators with Completely Factorizing Characteristic Symbol: Involutive Case

Example 2.2.21. Assume that

(2.2.44) For each $J \subset \{1, \dots, n\}$ $\nabla^{\#} g_j$, $j \in J$ are linearly independent on $\Sigma_J = \{g_j = 0 \ \forall j \in J\}$.

and

$$g_j = g_k = 0 \implies \{g_j, g_k\} = 0 \qquad \forall j \neq k. \tag{2.2.45}$$

Then Σ_J are involutive manifolds of codimensions $r = \#J$:

$$\nabla^{\#} g_j \in T_z \Sigma_J \qquad \forall z \in \Sigma_J \ \forall j \in J. \tag{2.2.46}$$

Therefore due to Frobenius Theorem Σ_J is locally stratified into submanifolds of dimension r:

$$\Lambda_J(z) = \bigl\{ e^{\sum_{j\in J} t_j \nabla^{\#} g_j},\ t \in B(0,\epsilon) \subset \mathbb{R}^r \bigr\}. \tag{2.2.47}$$

Let

$$\Lambda^-_J(z) = \bigl\{ e^{\sum_{j\in J} t_j \nabla^{\#} g_j},\ t \in B(0,\epsilon) \cap \mathbb{R}^{-r} \bigr\} \tag{2.2.48}$$

which is diffeomorphic locally to $\mathbb{R}^r$.

Then $\mathcal{K}^-(p,z) = \Lambda^-_{J(z)}(z)$ with $J(z) = \{j : g_j(z) = 0\}$. In particular, if $J' = \{j\}$ and $r \geq 2$ we get *limiting bicharacteristics* $\gamma_j(z) \subset \Sigma_J$.

In particular, if $\Sigma_{J'} \ni z' \to z \in \Sigma_J$ and $J' = J(z') \subsetneq J$ then $\mathcal{K}^-(z') \to \Lambda^-_{J'}(z)$.

Operators with not Factorizing Characteristic Symbol. I

Consider case when g has a double characteristic point but is not completely factorizing in its vicinity. Then due to microhyperbolicity condition (2.2.1) microlocally

$$g = -g_1^2 + \sum_{2\leq j\leq n} g_j^2 \tag{2.2.49}$$

with smooth functions g_j and $n = 3, 4$ [17]. Let $\Sigma = \{g = 0\}$ and

$$\Sigma^0 = \{g = \nabla^{\#} g = 0\} = \{g_j = 0 \quad \forall j = 1, \dots, n\}. \tag{2.2.50}$$

Then the eigenvalues of $\mathsf{Hess}^{\#} g$ calculated at Σ^0 play the crucial role in the behavior of bicharacteristics and propagation of singularities[18]. As $z \in \Sigma^0$ consider the second order germ $g(z;\zeta)$ of g; then due to (2.2.1)

(2.2.51) At each point $z \in \Sigma^0$ germ $g(z;\zeta)$ is a hyperbolic polynomial with respect to $\nabla^{\#} x_1$

---

[17] $n = 3, 4$ in the generic case of real or complex matrix symbol p, respectively.

[18] Some eigenvalues could be 0; the rest is coming in pairs belonging to $(\mathbb{R} \cup i\mathbb{R}) \setminus 0$, and quartets belonging to $\mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ and eigenvalues λ, $-\lambda$, $\lambda^\dagger$ and $-\lambda^\dagger$ have the same Jordan structure.

and therefore

(2.2.52) At each point $z \in \Sigma^0$ germ $\mathsf{Hess}^{\#}(z)$ either has only purely imaginary eigenvalues or it has two simple real eigenvalues $\pm\lambda(z)$ and all other eigenvalues are purely imaginary; further, all non-zero purely imaginary eigenvalues $\pm i\lambda_j$ come in pairs (so the multiplicity of $i\lambda_j$ and $-i\lambda_j$ coincide and associated Jordan cells have height 1 while Jordan cells associated with eigenvalue 0 "normally" have heights at most 2; as there are no real eigenvalues there always is at least one Jordan cell of height 2 associated with eigenvalue 0 and even one cell of the height 4 may appear.

One can deduct the proof from Williamson Theorem (see V. Arnold [2]) or find it in V. Ivrii & V. Petkov [1] or L. Hörmander [6] where the crucial role of this distinction in the Cauchy problem is uncovered.

We just note that non-zero eigenvalues of $\mathsf{Hess}^{\#} g$ coincide with those of the product $\mathsf{diag}(1,-1,\dots,-1)(\{g_j,g_k\})_{j,k=1,\dots,n}$.

Let us assume that

(2.2.53) $\nabla^{\#} g_1,\dots,\nabla^{\#} g_n$ are linearly independent on Σ^0,

which is equivalent to

(2.2.54) Σ^0 is $\mathscr{C}^\infty$-submanifold and $T_z(\Sigma^0) = \mathsf{Ker}\,\mathsf{Hess}^{\#}(z)$ at each point $z \in \Sigma^0$.

Operators with not Factorizing Characteristic Symbol. II

In this subsubsection we assume that

(2.2.55) At no point $z \in \Sigma^0$ matrix $\mathsf{Hess}^{\#}(z)$ has real non-zero eigenvalues.

Then without any loss of the generality one can assume that

(2.2.56) $g(x,\xi) = -\xi_1^2 + B(x,\xi') \quad$ with $\quad B(x,\xi') \ge 0, \qquad \xi' = (\xi_2,\dots,\xi_d)$

and

(2.2.57) $$|\partial_{x_1} B| \le cB.$$

Indeed, we can achieve (2.2.56) by multiplying g by disjoint from 0 symbol and symplectic change of variables. Observe that if $\partial_{x_1}^2 B > 0$ at some point

$z \in \Sigma^0$ then $\mathsf{Hess}^{\#} g(z)$ would have non-zero real eigenvalues; therefore $\partial^2_{x_1} B = 0$ on Σ_0 and then in virtue (2.2.53) $\Sigma^0 = \{\xi_1 = h_2(x', \xi') = ... = h_n(x', \xi') = 0\}$ which implies (2.2.57).

Observe that at level 0 (i.e. as $g(x, \xi) = 0$), $|\xi_1| = B^{\frac{1}{2}}$ and along bicharacteristics $|d\xi_1/dx_1| \le C$ and therefore we arrive to Statement (i) below. Then generalized bicharacteristics cannot enter or leave Σ^0 and therefore $dz/dt \in T_z\Sigma^0$ along generalized bicharacteristics residing on Σ^0; further, $dz/dt \in \mathsf{Ran}\, \mathsf{Hess}^{\#}(z) = (T_z\Sigma^0)$ at point $z \in \Sigma^0$. Then we arrive to Statement (ii) below.

Proposition 2.2.22. *Let conditions* (2.1.2), (2.2.1), (2.2.54) *and*(2.2.55) *be fulfilled. Then*

(i) Let $z \in \Sigma = \{g = 0\}$ *and* $\mathsf{dist}(z, \Sigma^0) = \rho > 0$*; consider* $z(t)$ *bicharacteristics of* g *with* $z(0) = z$*. Then*

$$c^{-1}\rho \le \mathsf{dist}(z(t), \Sigma^0) \le c\rho \quad \text{and} \quad c^{-1}\rho|t| \le \mathsf{dist}(z, z(t)) \le c\rho|t| \tag{2.2.58}$$

as $|t| \le \epsilon\rho^{-1}$.

(ii) On Σ^0 *generalized bicharacteristics of* p *are defined by*

$$\frac{dz}{dt} \in K^{\#}(p, z) \cap T_z\Sigma^0 \cap (T_z\Sigma^0)^{\perp} \tag{2.2.59}$$

which is (2.2.31).

Assume now that

$\sigma|_{\Sigma^0}$ has a constant rank. (2.2.60)

Then there exists foliation of Σ^0 into isotropic submanifolds[19] $\Lambda(z)$:

$$T_z\Lambda(z) = T_z\Sigma^0 \cap (T_z\Sigma^0)^{\perp} \qquad \forall z \in \Sigma^0 \tag{2.2.61}$$

and generalized bicharacteristics reside inside the leaves of this foliation.

Consider one leaf $\Lambda(z)$. Without any loss of the generality one can assume that

$$\Lambda(z) = \{\xi'' = \xi''' = x''' = 0\} \quad \text{with} \quad \xi' = (\xi''; \xi''') = (\xi_1, ..., \xi_k; \xi_{k+1}, ..., \xi_d). \tag{2.2.62}$$

[19] I.e. manifolds such that $\sigma|_{\Lambda} = 0$.

Let us consider symbol $g_2(x'';\xi'',x''',\xi''')$ which is a 2-germ of $g(x,\xi)$ at Σ^0; obviously $g_2(x'';\xi'',x''',\xi''')$ is a second order homogeneous polynomial with respect to (ξ'',x''',ξ''') and let us set $x'''=\xi'''=0$; we get $g'(x'',\xi'')=g_2(x'';\xi'',0,0)$ a second order homogeneous polynomial with respect to ξ''. Obviously all these polynomials are strictly hyperbolic.

Then

(2.2.63) $\mathcal{K}^-(z)\subset\Lambda(z)$ and is the *dependence conoid* for $g'(x',\xi')$ [20)]

Problem 2.2.23. We leave to the reader

(i) To prove that the generic case is $k=1,2$; so $k=1$ only if $n=k+2m$ is odd (and thus equals 3).

(ii) Investigate limiting bicharacteristics (residing on Σ^0) in the cases of Σ^0 involutive submanifold (i.e. $\mathsf{rank}\,\sigma|_{\Sigma^0}=2d-2n$ which is equivalent $\{g_j,g_k\}|_{\Sigma^0}=0\ \forall j,k$) and Σ^0 not involutive submanifold (i.e. $\mathsf{rank}\,\sigma|_{\sigma^0}>2d-2n$); in particular, consider the case of Σ^0 symplectic submanifold (i.e. $\mathsf{rank}\,\sigma|_{\Sigma^0}=2d-n$).

(iii) For each limiting bicharacteristics $\gamma(z)$ prove existence of solution u with $\mathsf{WF}(Pu)=\emptyset$, $WF(u)=\gamma(z)$ (in the vicinity of a given point $z\in\Sigma^0$).

For detail see V. Ivrii [4].

Example 2.2.24. The toy-model symbol with such properties is

$$\frac{1}{2}\Big(-\xi_1^2+\sum_{2\le j\le k}\xi_j^2+\sum_{k+1\le j\le k+m}\omega_j^2(x_j^2+\xi_j^2)\Big) \tag{2.2.64}$$

with $k\ge 1$, $m\ge 0$ and $\omega_j>0$; then $n=k+2m$.

Operators with not Factorizing Characteristic Symbol. III

In this subsubsection in addition to (2.2.53) we assume that

(2.2.65) At each point $z\in\Sigma^0$ matrix $\mathsf{Hess}^{\#}(z)$ has two real non-zero eigenvalues $\pm\lambda$, $\lambda=1$ [21)];

[20)] See any course in partial differential equations; R. Courant and D. Hilbert [2] is my favorite.

[21)] Obviously the latter could be achieved by multiplying g by λ^{-1}.

Then we can assume without any loss of the generality that

$$g(x,\xi) = x_1\xi_1 + B(x',\xi') + \mathcal{O}_n(\Sigma_0) \tag{2.2.66}_n$$

with $x' = (x_2,\dots,x_d)$ etc, $\Sigma^0 = \{(x,xi)\colon x_1 = \xi_1 = 0, (x',\xi') \in \Sigma'\}$, Σ' a smooth manifold,

$$B(x'\xi') = \mathcal{O}_2(\Sigma'), \qquad B(z') \asymp \mathsf{dist}(z',\Sigma')^2 \tag{2.2.67}$$

$\mathcal{O}_n(\Lambda)$ denotes a space of $\mathscr{C}^\infty$-functions vanishing on Λ with all their derivatives of order less than n and so far $n = 3$.

One can prove it easily from microhyperbolicity (in this case in the direction $\nabla^\#(x_1+\xi_1)$) and conditions (2.2.53) and (2.2.65).

Proposition 2.2.25. *In appropriate symplectic coordinates* $(2.2.66)_n$ *and* (2.2.67) *hold with arbitrarily large* $n = \infty$.

Proof. Let us make induction with respect to n. Assume that $(2.2.66)_n$ has been proven and let us prove it with n replaced by $(n+1)$. Consider undesirable terms in the right-hand expression of $(2.2.66)_n$:

$$\sum_{j,k:j+k=l} b_{jk}(x',\xi')x_1^j\xi_1^k, \qquad b_{jk} = O_{n-l}(\Sigma') \tag{2.2.68}$$

with $l = 1,\dots,n$. We do not care about $l = 0$ as we include this term in $B(x'\xi')$.

Let us make almost symplectomorphism $x_j \mapsto x_j + \partial_{\xi_j}S$, $\xi_j \mapsto \xi_j - \partial_{x_j}S$ with

$$S = \sum_{j,k:j+k=l} s_{jk}(x',\xi')x_1^j\xi_1^k, \qquad s_{jk} = O_{n-l}(\Sigma'). \tag{2.2.69}$$

Then canonical relations will be fulfilled modulo $\mathcal{O}_{2n-4}$ and they could be fixed by a symplectomorfism $(x,\xi) \mapsto (x,\xi) + \mathcal{O}_{2n-3}(\Sigma^0))$ which does not spoil $(2.2.66)_{n+1}$ as $2n-3 \ge n$.

One can prove easily that by such transformation one can eliminate all terms in (2.2.68) except those with $j = k$: $b_{jj}(x',\xi')(x_1\xi_1)^j$ and those latter terms could be eliminated by multiplying $g(x,\xi)$ by $\bigl(1+\mathcal{O}_{n-2}(\Sigma^0)\bigr)$. Thus we can escalate l from $l = 1$ to $l = n+1$ but then we get $\mathcal{O}_{n+1}(\Sigma^0)$.

Therefore we can achieve $(2.2.66)_n$ for any n and since transition from n to $n+1$ is achieved by $(x,\xi) \mapsto (x,\xi)+\mathcal{O}_{n-1}(\Sigma^0))$ we can take an asymptotic limit and achieve $(2.2.66)_\infty$. □

Assume for a moment that instead of $(2.2.66)_\infty$ we have an exact equality

$$g(x,\xi) = x_1\xi_1 + B(x',\xi'). \tag{2.2.70}$$

Then $B'(x',\xi')$ is a movement integral and we have the following picture

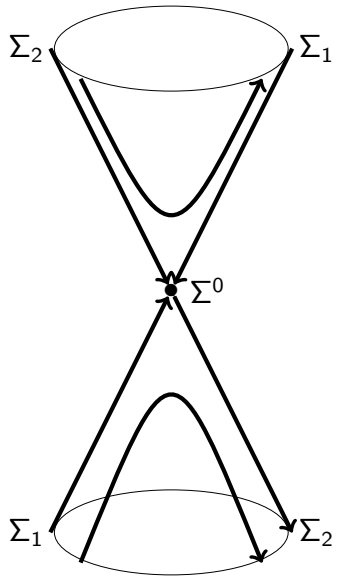


Figure 2.8: Bicharacteristics

Therefore bicharacteristics with $B(x',\xi') \neq 0$ miss Σ^0 while bicharacteristics with $B(x',\xi') = 0$ and thus with $x_1\xi_1 = 0$ have limiting points on Σ^0 as $t \to +\infty$ (for $x_1 = 0$) and as $t \to -\infty$ (for $\xi_1 = 0$) and on $\{B(x',\xi') = 0\}$ picture of $K^-(z)$ is as on Figure 2.6.

More precisely: consider a toy-model (2.2.71). One can look at section of the circular cone by planes parallel to its axis (as on Figure 2.8 as $k = 2$, $m = 0$); as $m \geq 1$ one needs to remember about extra circular motions in (x_j,ξ_j)-planes ($j = k+1,\dots,k+m$) which was present in the previous subsubsection as well.

Example 2.2.26. The toy-model symbol with such properties is

$$\frac{1}{2}\Big(\lambda x_1\xi_1 + \sum_{2\leq j\leq k} \xi_j^2 + \sum_{k+1\leq j\leq k+m} \omega_j^2(x_j^2+\xi_j^2)\Big) \tag{2.2.71}$$

with $k \geq 1$, $m \geq 0$ and $\omega_j > 0$; then $n = k + 2m$. In this case

$$\Sigma^0 = \{x_1 = \xi_1 = \xi' = \xi'' = x'' = 0\}, \tag{2.2.72}$$
$$x = (x_1; x'; x''; x''') = (x_1; x_2 \dots, x_k; x_{k+1}, \dots, x_{k+2m}; x_{k+2m+1}, \dots, x_n), \text{ etc}$$

Σ^0 has codimension $k+2m+1$ and $\operatorname{rank}\sigma|_{\Sigma^0} = 2m$. Then ξ_j ($j = 2,\dots,k$), $(x_j^2+\xi_j^2)$ ($j = k+1,\dots,k+m$) and x_j,ξ_j ($j \geq k+m+1$) are motion integrals while x_1 and ξ_1 are exponents $e^{\pm\lambda t}$.

Let us prove that this happens in the general case as well:

Proposition 2.2.27. *Let conditions* (2.1.2), (2.2.1), (2.2.53), *and* (2.2.65) *be fulfilled. Then*

(i) There exist $(\dim \Sigma^0 + 1)$*-dimensional smooth submanifolds* $\Sigma_j \subset \Sigma = \{g = 0\}$ *and real valued smooth symbols* $\eta_j (j = 1, 2)$ *such that* $\Sigma^0 = \Sigma_1 \cap \Sigma_2$,

$$d\eta_j \notin T_z\Sigma^0 \qquad \text{for} \quad z \in \Sigma^0, \tag{2.2.73}$$

$$\{\eta_1, \eta_2\} = 1 \qquad \text{on} \quad \Sigma^0 \tag{2.2.74}$$

and

$$\eta_j = 0 \quad \text{and} \quad \nabla^{\#} g = \big(\eta_{3-j} + O(|B|^{\infty})\big)\nabla^{\#}\eta_j \qquad \text{on} \quad \Sigma_j \tag{2.2.75}$$

and moreover,

- *(a) as* $z \in \Sigma_j$ *bicharacteristics of* g *have limiting points on* Σ^0 *as* $t \to (-1)^j\infty$*;*
- *(b) These bicharacteristics (modulo parametrization) coincide with bicharacteristics of* η_j*;*
- *(c) further, for* $z \in \Sigma_j$ $\mathcal{K}^-(z)$ *consists of bicharacteristics of* η_j *and* η_{3-j} *with branching on* Σ^0 *exactly as on Figure 2.6.*

(ii) Let $z \in \{g = 0\} \setminus (\Sigma_1 \cup \Sigma_2)$. *Then* $\mathcal{K}^-(z) = \gamma_g^-(z)$ *and it does not have limiting points on* Σ^0.

Proof. We sketch the proof; for details see Proposition 3.8 in V. Ivrii [1].

Consider bicharacteristics which start as $t = 0$ at $z \in \Sigma \cap \{x_1 + \xi_1 = 0\}$, $B(z) = \rho^2$. Using (2.2.53) and canonical form $(2.2.66)_\infty$ one can prove easily that for $0 < t < |\log \rho| - C_0$

$$|x_1| \asymp \rho e^t, \qquad C_0^{-1}\rho e^{-t} - C_0(\rho e^t)^n \le |\xi_1| \le C_0\rho e^{-t} + C_0(\rho e^t)^n,$$

and

$$C_0^{-1}\rho^2 - C_0(\rho e^t)^n \le B(x', \xi') \le C_0\rho^2 + C_0(\rho e^t)^n$$

and then that

$$|D_z^\alpha \Phi_t(z)| \le C_\alpha \qquad \forall z \in \Sigma \cap \{x_1 + \xi_1 = 0\} \setminus \Sigma^0, \quad 0 < t < \frac{1}{2}|\log B(z)| - C.$$

Then as $B(z) \to 0$ we get in the limit $\gamma_1^{\pm}(z)$ for $z \in \Sigma^0$ with signs $\pm$ corresponding two possible signs of x_1. Further, one can prove easily that

$\gamma_1(z) = \gamma_1^+(z) \cup \gamma_2^-(z)$ is a smooth curve and that $\Sigma_1 := \bigcup_{z\in\Sigma^0} \gamma_1(z)$ is a smooth manifold.

Similarly, but with t replaced by $-t$ we get $\gamma_2^\pm(z)$, $\gamma_2(z)$ and Σ_2. The proof of other statements is rather trivial. □

Remark 2.2.28. Observe that limiting bicharacteristics are of the form $\gamma_1^\pm(z) \cup \gamma_2^\mp(z)$ with $z \in \Sigma^0$.

Problem 2.2.29. It would be interesting to investigate if one always can achieve (2.2.70) rather than just $(2.2.66)_\infty$.

Operators with not Factorizing Characteristic Symbol. IV

Problem 2.2.30. (i) It would be interesting to consider symbols satisfying condition

(2.2.76) At each point $z \in \Sigma^0$ germ $\mathsf{Hess}^\#(z)$ has a Jordan cell of the height 4 associated with eigenvalue 0

with the toy-model

$$\frac{1}{2}\Big(-2\xi_1\xi_2 - \alpha\xi_2^2 + x_2^2 + \sum_{3\le j\le k} \xi_j^2 + \sum_{k+1\le j\le k+m} \omega_j^2(x_j^2+\xi_j^2)\Big) \tag{2.2.77}$$

with $\alpha = 0$, $k \ge 3$. Note that at $\xi_1 = \xi_2 = x_2 = 0$ non-zero eigenvalues of $\mathsf{Hess}^\#(g)$ are $\pm\alpha^{1/2}$ and $\pm i\omega_j$ so as $\alpha < 0$ we are in the framework of Proposition 2.2.22 and as $\alpha > 0$ we are in the framework of Proposition 2.2.27. One can analyze pretty easily bicharacteristics of (2.2.77) in all three cases of $\alpha = \mathsf{const}$.

(ii) Further, it would be really interesting to consider variable α f.e. $\alpha = kx_1$ with $k \ne 0$ when type is changed on Σ^0.

2.3 Operators with Rough Symbols

Our goal now is to generalize Theorem 2.1.2 and then some of its corollaries to the case when instead of belonging to $\boldsymbol{\Psi}_h$ operator under consideration

belongs to $\boldsymbol{\Psi}_{h,\boldsymbol{\rho},\boldsymbol{\gamma},N}$ or $\boldsymbol{\Psi}_{h,\boldsymbol{\rho},\boldsymbol{\gamma},\mathsf{an}}$ with $\boldsymbol{\rho} \le \mathbf{1}$, $\gamma \le \mathbf{1}$ satisfying an appropriate form of the uncertainty principle.

Operators with rough symbols serve as proxies (or approximations) for operators with irregular symbols and some rather limited results we derive for those later in Section 27.2.

2.3.1 Main Theorem

Statement of the Main Theorem

Let us analyze assumptions of Theorem 2.1.2 and look at its proof and applications. We do not need to have ϕ_j "rough", and smooth and even analytic functions work well for applications. On the other hand, we consider there $[P, Q_j]$ and we need to assume that these operators also belong to $\boldsymbol{\Psi}_{h,\boldsymbol{\rho},\boldsymbol{\gamma}}$ or $\boldsymbol{\Psi}_{h,\boldsymbol{\rho},\boldsymbol{\gamma},\mathsf{an}}$, i.e. in the smooth case we need to assume that

$$\|\partial_\xi^\alpha \partial_x^\beta \partial_{x,\xi}^\gamma p_m(x,\xi)\| \le c_{m\alpha\beta} \boldsymbol{\rho}^{-\alpha} \boldsymbol{\gamma}^{-\beta} \varepsilon^{-m} \qquad \forall m, \alpha, \beta : |\alpha| + |\beta| + 2m \le N \ \forall \gamma : |\gamma| \le 1 \tag{2.3.1}$$

with $\boldsymbol{\rho}, \boldsymbol{\gamma}$ satisfying uncertainty principle (1.1.97):

$$\varepsilon = \min_j \rho_j \gamma_j \ge h^{1-\delta}. \tag{2.3.2}$$

In the proof we need also assume that $h^{-1}(P - P^*)$ belongs to the same class, i.e. in addition to (2.1.2) we need to assume that

$$\|\partial_\xi^\alpha \partial_x^\beta (p_m - p_m^\dagger)(x,\xi)\| \le c_{m\alpha\beta} \boldsymbol{\rho}^{-\alpha} \boldsymbol{\gamma}^{-\beta} \varepsilon^{1-m} \qquad \forall m, \alpha, \beta : m \ge 1, |\alpha| + |\beta| + 2m \le N \tag{2.3.3}$$

Then the proof of Theorem 2.1.2 implies

Theorem 2.3.1. *Suppose* $P \in \boldsymbol{\Psi}_{h,\boldsymbol{\rho},\boldsymbol{\gamma}}(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ *satisfying* (2.1.2), (2.3.1) *and* (2.3.3) *with* $\boldsymbol{\rho}, \boldsymbol{\gamma}$ *satisfying uncertainty principle* (2.3.2).

Let $\Omega \Subset T^*\mathbb{R}^d$ *and let* $\phi_1, \dots, \phi_J$ *be real-valued* $\mathscr{C}_b^N$ *functions in* $T^*\mathbb{R}^d$ *such that at every point* $z \in \Omega_\epsilon$ *the symbol* p *is microhyperbolic in the directions* $H_{\phi_1}, \dots, H_{\phi_J}$ *Let* u *be tempered, i.e.,* $\|u\| \le h^{-I}$.

Then (2.1.4) *and* (2.1.5) *imply* (2.1.6).

Skipping easy details, instead let us go to a more difficult task: case of $\boldsymbol{\Psi}_{h,\boldsymbol{\rho},\boldsymbol{\gamma},\mathsf{an}}$ and logarithmic uncertainty principle (1.1.174). Then we need to replace (2.3.1) and (2.3.2) respectively by

$$\|\partial_\xi^\alpha \partial_x^\beta \partial_{x,\xi}^\gamma p_m(x,\xi)\| \le c(m!|\alpha|!|\beta|!)^\sigma \boldsymbol{\rho}^{-\alpha}\boldsymbol{\gamma}^{-\beta}\varepsilon^{-m} \qquad \forall m,\alpha,\beta : |\alpha|+|\beta|+2m \le N\ \forall \gamma : |\gamma| \le 1 \tag{2.3.4}$$

and

$$\|\partial_\xi^\alpha \partial_x^\beta (p_m - p_m^\dagger)(x,\xi)\| \le c(m!|\alpha|!|\beta|!)^\sigma \boldsymbol{\rho}^{-\alpha-m}\boldsymbol{\gamma}^{-\beta}\varepsilon^{-m+1} \qquad \forall m,\alpha,\beta : m \ge 1, |\alpha|+|\beta|+2m \le N \tag{2.3.5}$$

with $N = \epsilon\hbar^{-1}$, $\hbar = h/\varepsilon$ and

$$\min_j \rho_j\gamma_j \ge \varepsilon = Ch|\log h|^\sigma \qquad \sigma \ge 1. \tag{2.3.6}$$

Theorem 2.3.2. *Suppose* $P \in \boldsymbol{\Psi}_{h,\boldsymbol{\rho},\boldsymbol{\gamma}}(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ *satisfying* (2.1.2), (2.3.4) *and* (2.3.5) *with* $\boldsymbol{\rho}, \boldsymbol{\gamma}$ *satisfying logarithmic uncertainty principle* (2.3.6) *with* $\sigma \ge 1$ *and large enough* C [22].

Let $\Omega \Subset T^*\mathbb{R}^d$ *and let* $\phi_1, \dots, \phi_J$ *be real-valued* $\mathscr{C}_b^N$ *functions in* $T^*\mathbb{R}^d$ *such that at every point* $z \in \Omega_\epsilon$ *the symbol* p *is microhyperbolic in the directions* $H_{\phi_1}, \dots, H_{\phi_J}$ *Let* u *be tempered, i.e.,* $\|u\| \le h^{-l}$. *Then* (2.1.4), (2.1.5) *imply* (2.1.6).

Proof of the Main Theorem

Remark 2.3.3. (i) Recall that assumptions (2.1.4), (2.1.5) mean that for appropriate scalar symbols q_1 and q_2 equal to 1 in the vicinities of $\bar\Omega \cap \{\phi_1 \le \epsilon\} \cap \dots \cap \{\phi_J \le \epsilon\}$ and $\partial\Omega \cap \{\phi_1 \le \epsilon\} \cap \dots \cap \{\phi_J \le \epsilon\}$ respectively and for Q_1, Q_2 their Weyl quantizations

$$\|\bar Q_1 Pu\| \le Ch^{s+1}, \quad \|\bar Q_2 u\| \le Ch^s. \tag{2.3.7}$$

(ii) Without any loss of the generality one can assume that

$$\|\bar Q_1 u\| \le Ch^{s-\delta} \tag{2.3.8}$$

with any fixed exponent $\delta > 0$. Otherwise we will be able to prove Theorem in steps reducing ϵ and shrinking $\mathsf{supp}(q_1)$.

[22] C must depend on all other constants including ϵ in the definition of WF when we are looking at "ϵ-vicinity."

(iii) Without any loss of the generality one can assume $J = 1$: one can always find ϕ satisfying microhyperbolicity condition such that

$$\bar{\Omega} \cap \{\phi_1 \le 0\} \cap \dots \cap \{\phi_J \le 0\} \subset \bar{\Omega} \cap \{\phi < 0\}$$

and

$$\bar{\Omega} \cap \{\phi < \epsilon\} \subset \bar{\Omega} \cap \{\phi_1 \le \epsilon\} \cap \dots \cap \{\phi_J \le \epsilon\}.$$

(iv) Without any loss of the generality one can assume that ϕ is analytic and symbols $\bar{q}_j$ introduced in (i) satisfy

$$|\partial_x^\alpha \partial_\xi^\beta \phi| \le c^{|\alpha|+|\beta|+1}(|\alpha+\beta|)! \tag{2.3.9}$$

$$|\partial_x^\alpha \partial_\xi^\beta f| \le c(cN)^{(|\alpha|+|\beta|-K)_+} \qquad \forall \alpha, \beta : |\alpha| + |\beta| \le N \tag{2.3.10}$$

with arbitrarily large fixed K [23) 24)].

With $Q = Q^*$ of the same type let us repeat (2.1.13):

$$\begin{aligned} -\operatorname{Re} i([P,Q]u,u) = \\ \operatorname{Re} i((Q+Q^*)Pu,u) + \operatorname{Re} i((P^*-P)Qu,u). \end{aligned} \tag{2.3.11}$$

As in the proof of Theorem 2.1.2 we pick up $Q = q^{\mathsf{w}}(x, hD)$ with

$$q = \zeta(x,\xi)^2 \chi\big(\phi(x,\xi)\big) \tag{2.3.12}$$

where

$$\zeta = 1 \text{ in } \Omega \cap \Big\{\phi < \frac{\epsilon}{2}\Big\} \quad \text{and} \quad \operatorname{supp}(\zeta) \subset \Omega_\epsilon \cap \big\{\phi < \epsilon\big\}. \tag{2.3.13}$$

Let $\chi(t) = 0$ for $t \ge \epsilon/4$, both χ and ζ satisfy (2.3.10). One can prove easily that

(2.3.14) Let both χ and ζ satisfy (2.3.10). Then q also satisfies (2.3.10) (with different constants).

[23)] I.e. (2.3.10) holds for $f = \bar{q}_j$.

[24)] Symbols q_j satisfying (2.3.10) and supported in arbitrary small constant vicinities of the corresponding sets exist due to Proposition 1.A.2; however as $\sigma = 1$ such symbols depend on h but conditions are satisfied in the uniform manner.

Let us consider $[P,Q]$ first. With $O(h^K)$ error in (2.3.11) one can replace $[P,Q]$ by $\zeta^{\mathsf{w}} r^{\mathsf{w}} \zeta^{\mathsf{w}}$ with

$$r = \zeta_1 \big(p \bullet \chi(\phi) - \chi(\phi) \bullet p\big)\zeta_1 \tag{2.3.15}$$

with ζ_1 of the same type (2.3.13) as ζ but with ϵ replaced by 3ϵ.

Really, the error would not exceed $C_0^{N+1} N^N h^N \bar{\varepsilon}^{-N} h^{-I}$ with

$$\bar{\varepsilon} = \min_j \min(\rho_j, \gamma_j) \ge \varepsilon$$

and with fixed I and this expression does not exceed $C_0(C_0 N\hbar)^N h^{-I}$ with $\hbar = h/\varepsilon$ and with $N = \epsilon_1 \hbar^{-1/\sigma}$ it does not exceed $e^{-\epsilon_2 \hbar^{-1/\sigma}} h^{-I} = O(h^K)$ as $\hbar \le C|\log h|^{-1}$ with large enough constant C which follows from the logarithmic uncertainty principle (2.3.6).

Consider

$$p \bullet q - q \bullet p + ih\{p,q\} = \sum_{\substack{\alpha,\beta:|\alpha+\beta|\ge 2,\\ |\alpha|-|\beta|\in 2\mathbb{Z}+1}} \frac{2}{\alpha!\beta!}(-i)^{|\alpha|-|\beta|}\big(\frac{h}{2}\big)h^{|\alpha+\beta|}\big(D_\xi^\alpha D_x^\beta p\big)\cdot\big(D_x^\alpha D_\xi^\beta q\big) \tag{2.3.16}$$

and plug $q = \chi(\phi)$. Note that

$$\frac{1}{\nu!}D^\nu \chi(\phi) = \sum_{k\le|\nu|} \sum_{\substack{\nu_1,\dots,\nu_k:\\ |\nu_1|\ge 1,\dots,|\nu_k|\ge 1,\\ \nu_1+\dots+\nu_k=\nu}} \chi^{(k)}(\phi) \prod_{1\le j\le k} \frac{1}{\nu_j!} D^{\nu_j}\phi \tag{2.3.17}$$

where $\chi^{(k)}(t) = \partial_t^k \chi(t)$. Therefore

(2.3.18) Let χ satisfy (2.3.10). Then modulo negligible one can rewrite the right hand expression of (2.3.16) as

$$\sum_{1\le l+n\le N} \chi^{(l+1)}(\phi) r_{ln}(x,\xi)\hbar^{n+l} h \tag{2.3.19}$$

with $\phi = \phi(x,\xi)$ and

$$\|\partial_\xi^\alpha \partial_x^\beta r_{ln}\| \le c_1^{l+n+|\alpha|+|\beta|}(l!)^{\sigma-1}(n!\alpha!\beta!)^\sigma \boldsymbol{\rho}^{-\alpha}\boldsymbol{\gamma}^{-\beta} \qquad \forall l,n,\alpha,\beta : l+n+|\alpha|+|\beta| \le N. \tag{2.3.20}$$

We will be very particular choosing χ. Namely, we consider χ satisfying (2.3.10) and then replace in the formulae above $\chi(t)$ by

$$\chi_j(t) = \hbar^{2j} \int_t^\infty \bigl(\chi^{(j)}(t')\bigr)^2 \, dt'; \tag{2.3.21}$$

one can see easily that

(2.3.22) $\hbar^{-2j} N^{-2j} \chi_j(\phi)$ and $N^{-j}\chi^{(j)}(\phi)$ satisfy (2.3.10) uniformly with respect to $j \le N = \epsilon_1 \hbar^{-1/\sigma}$;

therefore every formula above holds uniformly as well.

Then (2.3.18) implies that

(2.3.23) Modulo negligible one can rewrite the right hand expression of (2.3.16) with $q = \chi_j(\phi)$ as

$$\sum_{1\le l_1+l_2+n\le N} \chi^{(l_1+j)}(\phi)\chi^{(l_2+j)}(\phi) r_{l_1 l_2 n}(x,\xi)\hbar^{2j+l_1+l_2+n} h \tag{2.3.24}$$

with

$$\|\partial_\xi^\alpha \partial_x^\beta r_{l_1 l_2 n}\| \le c_1^{l_1+l_2+n+|\alpha|+|\beta|} (l_1! l_2!)^{\sigma-1} (n!\alpha!\beta!)^\sigma \cdot \boldsymbol{\rho}^{-\alpha}\boldsymbol{\gamma}^{-\beta} \qquad \forall l, n, \alpha, \beta : l_1 + l_2 + n + |\alpha| + |\beta| \le N. \tag{2.3.25}$$

Now using "revert" formula

$$pq \equiv \sum_{\alpha,\beta:} \frac{2}{\alpha!\beta!} i^{|\alpha|-|\beta|} \bigl(\frac{h}{2}\bigr) h^{|\alpha|+|\beta|} \bigl(D_\xi^\alpha D_x^\beta p\bigr) \bullet \bigl(D_x^\alpha D_\xi^\beta q\bigr) \tag{2.3.26}$$

one can rewrite (2.3.24) as

$$\sum_{1\le l_1+l_2+n\le N} \hbar^{l_1+j}\chi^{(l_1+j)}(\phi) \bullet r'_{l_1 l_2 n}(x,\xi) h^n \bullet \hbar^{l_2+j}\chi^{(l_2+j)}(\phi) \tag{2.3.27}$$

where $r'_{l_1 l_2 n}$ also satisfy (2.3.25).

Then

$$\ell_j \coloneqq h^{-1}\Bigl(p \bullet \chi_j(\phi) - \chi_j(\phi) \bullet p + ih\{p, \chi_j(\phi)\}\Bigr)\zeta^2 \equiv \sum_{1\le l_1+l_2+n\le N} t_{l_1+1} \bullet r'_{l_1 l_2 n}(x,\xi) h^n \bullet t_{l_2+j} + \ell'_j \tag{2.3.28}$$

with $t_m = \chi^{(m)}(\phi)\zeta\hbar^m$ and

$$\ell'_j = \zeta^2\Big(\sum_{1\le l_1+l_2+n\le N} \hbar^{l_1+j}\chi^{(l_1+j)}(\phi)\bullet r'_{l_1l_2n}(x,\xi)h^n\bullet\hbar^{l_2+j}\chi^{(l_2+j)}(\phi)\Big)- \sum_{1\le l_1+l_2+n\le N} \zeta\hbar^{l_1+j}\chi^{(l_1+j)}(\phi)\bullet r'_{l_1l_2n}(x,\xi)h^n\bullet\zeta\hbar^{l_2+j}\chi^{(l_2+j)}(\phi) \tag{2.3.29}$$

and therefore

$$|(L_ju,u)| \le c_2\|T_ju\|\cdot\Big(\sum_{1\le l\le N-j} c_2^l(l!)^{\sigma-1}\cdot\|T_{j+l}u\|\Big)+ c_2\Big(\sum_{1\le l\le N-j} c_2^l(l!)^{\sigma-1}\cdot\|T_{j+l}u\|\Big)^2 + C_2\hbar\|T_ju\|^2 + |(L'_ju,u)| \tag{2.3.30}$$

with $T_m = \mathsf{Op}^{\mathsf{w}}(t_m)$, $L_j = \mathsf{Op}^{\mathsf{w}}(\ell_j)$ and $L'_j = \mathsf{Op}^{\mathsf{w}}(\ell'_j)$.

One can prove easily that due to (2.3.8) and (2.3.22)

$$|(L'_ju,u)| \le C_3h^s\hbar^jN^j\Big(\sum_{0\le l\le N-j} c_3^l(l!)^{\sigma-1}\cdot\|T_{j+l}u\|\Big) + C_3h^{2s}\hbar^{2j}N^{2j}. \tag{2.3.31}$$

Meanwhile

$$\{p,\chi_j(\phi)\} = -\chi^{(j)}(\phi)^2\hbar^{2j}\{p,\phi\} \tag{2.3.32}$$

and due to microhyperbolicity condition

$$\{p,\phi\} = b^2 + f^\dagger p + pf \tag{2.3.33}$$

where

(2.3.34) both b, f satisfy (2.3.4) type estimates and b is Hermitian and elliptic.

Therefore modulo (2.3.27)-type expression one can rewrite (2.3.32) as

$$\hbar^j\chi^{(j)}(\phi)\bullet b\bullet b\bullet\hbar^j\chi^{(j)} + \chi^{(j)}\bullet f^\dagger\bullet p + p\bullet f\bullet\hbar^j\chi^{(j)}. \tag{2.3.35}$$

Repeating arguments leading to (2.3.30)–(2.3.30) we conclude that

$$-ih^{-1}([P,Q']\,\mathsf{Op}^{\mathsf{w}}(\zeta^2)u,u) \ge \epsilon_1\|T_ju\|^2 - (2.3.30) - (2.3.31). \tag{2.3.36}$$

Here we used the (2.3.8) and the first of estimates (2.3.7) exactly as in the proof of Theorem 2.1.2.

Also, without any loss of the generality we can assume that

$$-ih^{-1}(p_1^\dagger - p_1) = \tilde{b}^2 + f^\dagger p + pf \tag{2.3.37}$$

where both b, f satisfy (2.3.4) type estimates and $\tilde{b}$ is Hermitian and elliptic. Really, due to (2.3.33) we can achieve this by multiplying operator P by an equivalent operator $e^{k\phi}Pe^{-k\phi}$ with large enough $k \in \mathbb{R}^+$.

Then the second term in the right hand expression of (2.3.11) multiplied by h^{-1} is less than $-\epsilon_1(Q_j u, u) + Ch^{2s+\delta}$ due to Gårding inequality with the semiclassical parameter $\hbar$.

Then (2.3.11) implies

$$\epsilon_1\|T_j u\|^2 + \epsilon_1(Q_j u, u) \le (2.3.30) + (2.3.31) + ch^s\|Q_j u\| + Ch^{2s}\hbar^{2j}N^{2j}. \tag{2.3.38}$$

Note, that $q_j^2 \le (c\hbar N)^j q_j$ and using to Gårding inequality with the semiclassical parameter h we conclude that

$$\|Q_j u\|^2 \le C(cN\hbar)^j(Q_j u, u) + Ch^{2s+\delta} \tag{2.3.39}$$

again due to (2.3.8).

Then (2.3.38) implies that

$$\mu_j := \epsilon_1\|T_j u\| + \epsilon_1(Q_j u, u)^{1/2} \tag{2.3.40}$$

satisfy inequalities

$$\mu_j \le \sum_{1\le l\le N-j} \mu_l c_3^l (l!)^{\sigma-1} + f_n \qquad \text{with} \quad f_n = C_2(cN\hbar)^j h^s \tag{2.3.41}$$

where the last term is greater than $Ch^{s+\delta}$. Then[25)]

$$\mu_j \le C_4 \sum_{0\le l\le N-j} c_4^l (l!)^{\sigma-1} f_{j+l}. \tag{2.3.42}$$

Plugging f_{j+l} and $N = \epsilon_1\hbar^{1/\sigma}$ one can see easily that $\mu_j \le Ch^s$; picking up $j = 0$ we conclude that (2.1.6) holds. Theorem 2.3.2 is proven.

[25)] Really, solving (2.3.41) for $f_j = 0$ and for $j < k$ only one can prove by induction with respect to $j = k, \dots, 0$ that $\mu_j \le C_4 c_4^{k-j}((k-j)!)^{\sigma-1}\mu_k$, which implies (2.3.42) for the original inhomogeneous system.

2.3.2 Corollaries

After propagation of singularities Theorem is proven, it implies few corollaries similar to those of Theorem 2.1.2.

Remark 2.3.4. In the regularity conditions of Theorem 2.3.2 assertions of Remark 2.1.3 and Proposition 2.1.4 remain true.

Theorem 2.3.5. *For operator $P = hD_t + A$ with the regularity of A as in Theorem 2.3.2 rather than of 2.1.2*

(i) Assertion of Theorem 2.1.5 remains true: (2.1.28) *and* (2.1.29) *imply* (2.1.30)*; Remark 2.1.6 also remains true.*

(ii) Assertions of Corollaries 2.1.7 and 2.1.9 (finite speed of propagation) remain true.

(iii) Assertions of Propositions 2.1.10 and 2.1.12 remain true.

(iv) Assertions of Proposition 2.1.13 and Corollary 2.1.14 remain true.

(v) Assertions of Proposition 2.1.16, Corollary 2.1.17 and Remark 2.1.18 remain true.

(vi) Furthermore, as $\rho = 1$ assertions of Theorem 2.1.19 and Remark 2.1.21 remain true. Moreover, if $\rho \le 1$ then estimate (2.1.59) *is replaced by*

$$\|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma_x(Q_{1x}U\,{}^tQ_{2y})\| \le CMh^{1-d}\Big(\frac{h}{\rho T}\Big)^s \qquad \forall \tau : |\tau| \le \varepsilon_0 \quad \forall x \in \mathbb{R}^d \tag{2.3.43}$$

where $\rho = \min_j \rho_j$.

(vii) Assertions of Remark 2.1.22 remains true.

(viii) Finally, as $\rho = 1$ assertion of Theorem 2.1.23(ii) with the same modification as in (vi) remains true. Moreover, if $\rho \le 1$ then estimate $(2.1.59)'$ *is replaced by*

$$\|F_{t\to h^{-1}\tau}\chi_{\pm T}(t)\Gamma_x(Q_{1x}U\,{}^tQ_{2y})\| \le CMh^{m-d}\Big(\frac{h}{\rho T}\Big)^s \qquad \forall \tau : |\tau| \le \varepsilon_0 \quad \forall x \in \mathbb{R}^d \tag{2.3.43$'$}$$

where $\rho = \min_j \rho_j$.

(ix) Further, if $\phi = \phi(x')$ *wuth* $\xi' = (\xi_1, \dots, \xi_k)$ *then in (vi), (viii) one can take* $\rho = \min_{j\le k} \rho_j$.

(x) Meanwhile modified in the same way Statements Theorems 2.1.19(i) and 2.1.23(i) remain true as $\rho \le \gamma$.

Proofs of all these statements just repeats those of those we refer to.

Moreover, let us assume that as $h \to +0$ symbol $p(x, \xi, h)$ tends to some limit which we denote also by p. Let us define $K(p, z)$, $K^{\#}(p, z)$ and $\mathcal{K}^{\pm}(p, z)$ for this limiting symbol. Then

Theorem 2.3.6. *For operator* A *with the regularity as in Theorem 2.3.2 rather than of 2.1.2*

(i) Assertions of Lemma 2.2.1, Proposition 2.2.7 remain true; further, Definition 2.2.8 make sense and assertions of Lemma2.2.9 remain true.

(ii) Moreover, assertion of Theorem 2.2.10 remains true: (2.2.25) *and* (2.2.26) *imply* (2.2.27).

Again, proofs of all these statements just repeats those of those we refer to.

2.4 Heisenberg Approach to Propagation of Singularities along Long Bicharacteristics

The results of Section 2.2 imply the well known fact that singularities for operators with scalar principal symbols propagate along bicharacteristics. In this section we improve and generalize this result in two directions: namely, we treat large time intervals depending on h and we replace the description of the singularities by means of wave front sets with a more refined description related to boxes depending on h.

2.4.1 Evolution of Operators

Heisenberg Approach

We start with the Cauchy problem for the operator

$$P = hD_t + A \tag{2.4.1}$$

where

(2.4.2) $A = A^w(x, hD_x, h)$ is an operator with a Hermitian Weyl symbol and hence A is a self-adjoint operator.

Solution to $Pu = 0$ is given by $u_t = e^{-ih^{-1}tA}u_0$ where $u_t = u(., t)$; then

$$e^{-ih^{-1}tA}Qu_0 = e^{-ih^{-1}tA}Qe^{ih^{-1}tA} \cdot e^{-ih^{-1}tA}u_0 = Q^t u_t \tag{2.4.3}$$

with

$$Q^t = e^{-ih^{-1}tA}Q^0 e^{ih^{-1}tA}. \tag{2.4.4}$$

Therefore one can apply observable $Q^0 = Q$ to the initial state or alternatively apply evolved observable Q^t to the evolved state u_t and the result would be the same. Equivalently one can apply observable $Q^0 = Q$ to the evolved state u_t or alternatively apply evolved observable Q^{-t} to the initial state u_0 and the result would be the same. In the latter case physicists would say that we use the *Heisenberg representation*.

Since WF are intimately related to pseudodifferential operators, it is obvious that the investigation of the propagation of singularities of solutions to this Cauchy problem can be reduced to the treatment of the operator Q^t where $Q^0 = \mathsf{Op}^w(q^0)$ is a pseudodifferential operator. The crucial question is if Q^t is a pseudodifferential operator.

We already know that for a standard pseudodifferential operator A such that and

(2.4.5) $a = A_0$ is a scalar real-valued symbol.

and Q and finite time t the *propagators* $e^{\pm ih^{-1}tA}$ are Fourier integral operators corresponding to the symplectomorphisms $\varphi_{\pm t}$ where $\varphi_t : z(0) \to z(t)$ and $z(t)$ is a solution of the Hamiltonian system (2.2.33).

This implies that Q^t is also a pseudodifferential operator. However, we use these results only as a heuristic preliminary.

Definition (2.4.4) implies that Q^t is a solution of the Cauchy problem

$$\partial_t Q^t = -ih^{-1}[A, Q^t], \tag{2.4.6}$$

$$Q^t\big|_{t=0} = Q^0. \tag{2.4.7}$$

Let us *assume* that Q^t is a pseudodifferential operator with Weyl symbol q^t where for now we treat symbols as formal power series; we will *justify* this assumption later.

Then the Cauchy problem (2.4.6)–(2.4.7) implies the similar Cauchy problem for symbols; namely, for a term q_n^t in the decomposition of q^t we obtain the equation

$$\partial_t q_n^t + \{a, q_n^t\} + i[a^s, q_n^t] = \sum_{\substack{0\le k\le n-1\\ j+|\alpha|+|\beta|=n-k}} \frac{1}{2^{|\alpha|+|\beta|}\alpha!\beta!}\big((-i)^{|\alpha|-|\beta|}A^{(\alpha)}_{j(\beta)}q^{t(\beta)}_{k(\alpha)} - i^{|\alpha|-|\beta|}q^{t(\beta)}_{k(\alpha)}A^{(\alpha)}_{j(\beta)}\big) \tag{2.4.8$_n$}$$

with initial condition

$$q_n^t\big|_{t=0} = q_n^0. \tag{2.4.9$_n$}$$

Recall that $\{.,.\}$ are Poisson brackets, $[.,.]$ is the commutator in $\mathcal{L}(\mathbb{H},\mathbb{H})$, a^s is the subprincipal symbol of A (i.e., $a^s = A_1$), A_n are terms in the decomposition of the Weyl symbol of A and $b^{(\alpha)}_{(\beta)} = \partial_x^\beta \partial_\xi^\alpha b$.

Now we throw of our assumption that Q^t is an h-pseudodifferential operator. We needed this assumption *only* to derive problems (2.4.8)$_n$–(2.4.9)$_n$. So, let us consider this problem, where for now "symbol" means "function" rather than "symbol of an operator."

Let us start with the principal symbol q_0^t which satisfies the following *transport equation*

$$\partial_t q_0^t + \{a, q_0^t\} + i[a^s, q_0^t] = 0 \tag{2.4.8$_0$}$$

where $[.,.]$ means the matrix (i.e., operator in the auxiliary space $\mathbb{H}$) commutator.

Moreover, if

(2.4.10) a^s is also a scalar symbol

we obtain the following equation:

$$\partial_t q_0^t + \{a, q_0^t\} = 0 \tag{2.4.11}$$

which can be rewritten in the form

$$\partial_t (q_0^t \circ \Psi_t) = 0 \tag{2.4.12}$$

where $\Psi_t : T^*\mathbb{R}^d \to T^*\mathbb{R}^d$ is a symplectomorphism; namely, $\Psi_t : z(0) \to z(t)$ where $z(t)$ is a solution of the dynamical system

$$\frac{dz(t)}{dt} = H_a(z(t)) \tag{2.4.13}$$

and hence

$$q_0^t = q_0^0 \circ \Psi_{-t}. \tag{2.4.14}$$

Definition 2.4.1. The family Ψ_t is called the *Hamiltonian flow generated by the Hamiltonian* a.

In the general case (i.e. without assumption (2.4.10)) one can rewrite equation $(2.4.8)_0$ in the form

$$\partial_t \big(R_t^{-1}(q_0^t \circ \Psi_t)R_t\big) = 0 \tag{2.4.15}$$

where R_t is a differentiable matrix family in $\mathbb{H}$ satisfying the Cauchy problem

$$\partial_t R_t = -i(a^s \circ \Psi_t)R_t, \qquad R_0 = I. \tag{2.4.16}$$

These matrices are unitary since a^s is Hermitian. Therefore $R_t^\dagger(q_0^t \circ \Psi_t)R_t = q_0^0$ and

$$q_0^t = (R_t q_0^0 R_t^\dagger) \circ \Psi_{-t}. \tag{2.4.17}$$

Note that

$$R_{t,t'} = R_t R_{t'}^\dagger \tag{2.4.18}$$

satisfies the Cauchy problem

$$\partial_{t'} R_{t,t'} = -i(a^s \circ \Psi_t)R_{t,t'}, \qquad R_{t,t'} = I. \tag{2.4.19}$$

Solving Heisenberg Equations. Standard Case

As long as T is bounded, and $a^{(\alpha)}_{(\beta)} \circ \Psi_t$ are bounded at (x,ξ) for $\pm t \in [0,\, T]$ so are derivatives of $\Psi_t(x,\xi)$ with respect to (x,ξ). Further, since $d\Psi_t$ is a linear symplectomorphism and therefore $\det d\Psi_t = 1$, derivatives of $\Psi_{-t'}$ remain bounded at $\Psi_t(x,\xi)$ as $0 \le \pm t' \le \pm t \le T$.

Assume that these properties hold for all $z \in \Omega = B(\bar z, \varepsilon)$ and that $a^{(\alpha)}_{n(\beta)} \circ \Psi_t$ are bounded at (x,ξ) for $\pm t \in [0,\, T]$. Further, assume that

$$q^0 \in \mathsf{S}_{h,\varepsilon,\varepsilon}, \qquad \mathsf{supp}(q^0) \subset \Omega. \tag{2.4.20}$$

Then it immediately follows from $(2.4.8)_n$ that

$$q^t \in \mathsf{S}_{h,\varepsilon,\varepsilon}, \qquad \mathsf{supp}(q^0) \subset \Omega_t = \Psi_t\Omega. \tag{2.4.21}$$

Moreover, $B(\bar z_t,\, C_0^{-1}\varepsilon) \subset \Omega_t \subset B(\bar z_t,\, C_0\varepsilon)$ where $\bar z_t = \Psi_t(\bar z)$.

Then as ε satisfies microlocal uncertainty principle

$$\varepsilon \ge h^{(1-\delta)/2} \tag{2.4.22}$$

one can quantize q^t.

Thus $Q^t = q^{t\,\mathsf{w}}(x, hD)$ satisfies (2.4.6)–(2.4.7) modulo $O(h^N)$ [26] operators. Therefore

$$\partial_t\Big(e^{ih^{-1}tA} Q^t e^{-ih^{-1}tA}\Big) \equiv 0 \tag{2.4.23}$$

also modulo $O(h^N)$ [26] operators. Then

$$e^{ih^{-1}tA} Q^t e^{-ih^{-1}tA} \equiv Q^0 \tag{2.4.24}$$

and finally

$$Q^t \equiv e^{-ih^{-1}tA} Q^0 e^{ih^{-1}tA}. \tag{2.4.25}$$

This proves that $Q^t = q^{t\,\mathsf{w}}$ differs from Q^t defined by (2.4.4) by $O(h^N)$ and therefore

Theorem 2.4.2. *Let $A \in \boldsymbol{\Psi}_h$ be an operator with the Hermitian and scalar principal symbol. Let Ψ_t be a corresponding Hamiltonian flow.*

Furthermore, let $Q^0 \in \boldsymbol{\Psi}_{h,\varepsilon,\varepsilon}$ where $\varepsilon \in (0,1]$ satisfies (2.4.22)*. Then*

[26] In the operator norm.

(i) For $\pm t \in [0, T]$ *with bounded* T *operator* Q^t *defined by* (2.4.4) *belongs to* $\boldsymbol{\Psi}_{h,\varepsilon,\varepsilon}$ *and is defined by the procedure described in the previous section.*

(ii) In particular, $\mathsf{supp}(q^t) = \Psi_t(\mathsf{supp}(q^0))$.

In the same way one can prove

Theorem 2.4.3. *Let* $A \in \boldsymbol{\Psi}_{h,\mathsf{an}}$ *be an operator with the Hermitian and scalar principal symbol. Let* Ψ_t *be a corresponding Hamiltonian flow.*

Furthermore, let $Q^0 \in \boldsymbol{\Psi}_{h,\varepsilon,\varepsilon,\mathsf{an}}$ *where* $\varepsilon \in (0, 1]$ *satisfies logarithmic uncertainty principle*

$$\varepsilon \geq C(h|\log h|)^{1/2}. \tag{2.4.26}$$

Then

(i) For $\pm t \in [0, T]$ *with bounded* T *operator* Q^t *defined by* (2.4.4) *belongs to* $\boldsymbol{\Psi}_{h,\varepsilon,\varepsilon,\mathsf{an}}$ *and is defined by the procedure described in the previous section.*

(ii) In particular $\mathsf{supp}(q^t) = \Psi_t(\mathsf{supp}(q^0))$.

Remark 2.4.4. Actually in the symbol construction we need only that conditions to q^0 be fulfilled only in Ω and conditions to symbol of A in $\bigcup_{\pm t\in[0,T]} \boldsymbol{\Psi}_t(\Omega)$.

Discussion

Now we want to extend the results of the previous subsubsection to the case of unbounded (but not too large) T. Then derivatives of Ψ_t may cease to be bounded. Usually in applications they have either exponential (general case) or polynomial (completely integrable case) growth with respect to t. Also in the applications it is more natural to assume that A belongs to $\boldsymbol{\Psi}_{h,\rho,\rho}$ with unbounded but not to large ρ. In this theory since different components of (x, ξ) are messed up during Hamiltonian motion, one needs to assume that all components of $\boldsymbol{\rho}$ and $\boldsymbol{\gamma}$ are equal.

For the sake of simplicity we consider $t \in [0, T]$ but exactly the same results hold as $-t \in [0, T]$.

Properties of Hamiltonian Flow

First we need to establish the growth properties of the Hamiltonian flow. It appears that everything is defined by the growth of $d\Psi_t$ and by the growth of F along trajectories.

Lemma 2.4.5. *Let $\Psi_t : \mathbb{R}^d \to \mathbb{R}^d$ be a flow defined by the dynamical system*

$$\frac{dz}{dt} = F(z). \tag{2.4.27}$$

Let $\bar{z} \in \mathbb{R}^d$, $T > 0$ and $z(t) = \Psi_t(\bar{z})$. Let us assume that

$$|(\partial_z^\alpha F)(z(t))| \le c\mu\rho^{-|\alpha|} \qquad \forall t \in [0, T] \quad \forall \alpha : 1 \le |\alpha| \tag{2.4.28}$$

and

$$|d\Psi_{t-t'}(z(t'))| \le c\omega \qquad \forall t, t' \in [0, T] \tag{2.4.29}$$

where $d\Psi_t$ is a differential of Ψ_t (with respect to z) and μ, ρ, ω are (positive) parameters such that

$$\mu\rho T\omega \ge 1, \qquad \omega \ge 1. \tag{2.4.30}$$

Then

$$|(\partial_z^\alpha \Psi_t)| \le C_\alpha \omega \eta^{1-|\alpha|} \qquad \forall t \in [0, T] \; \forall \alpha : 1 \le |\alpha| \tag{2.4.31}$$

with

$$\eta = \rho^2\omega^{-2}\mu^{-1}T^{-1}. \tag{2.4.32}$$

Proof. Let us use induction with respect to $n \ge 1$; for $n = 1$ the estimate (2.4.31) coincides with (2.4.29). Let us assume that (2.4.31) holds for all α with $|\alpha| < n$ and let us consider $z(t, u) = \Psi_t(u)$.

Differentiating the initial system n times with respect to u we obtain that

$$\frac{dz_k^{(\alpha)}}{dt} = \sum_{1 \le j \le d} F'_{kj}(z) z_j^{(\alpha)} + f_{k,\alpha} \tag{2.4.33}$$

where F_k and z_k are components of F and z respectively and $F'_{kj} = \partial_{z_j} F_k$, $z_k^{(\alpha)} = \partial_u^\alpha z$,

$$f_{k,\alpha} = \sum_{2 \le |I| \le r} \sum_{\substack{\alpha_1, \dots, \alpha_s : s = |I|, \\ |\alpha_i| \ge 1, \alpha_1 + \dots + \alpha_s = \alpha}} \bar{c}_{d, I, \alpha_1, \dots, \alpha_s} F_I(z(t)) z_{i_1}^{(\alpha_1)} \cdots z_{i_s}^{(\alpha_s)},$$

where $I = (i_1, \dots, i_s) \in \{1, \dots, d\}^s$, $F_I = \partial_{z_{i_1}} \dots \partial_{z_{i_s}} F$, and $\bar{c}_\dots$ are absolute constants.

Then (2.4.28), (2.4.30), and the induction assumption imply that

$$|f_{k,\alpha}| \le C\mu(\mu T)^{n-2}\omega^{2n-2}\rho^{2-2n} \tag{2.4.34}$$

On the other hand, (2.4.33) and the equality $z_k^{(\alpha)} = 0$ for $t = 0$ and $|\alpha| \ge 2$ imply that

$$z^{(\alpha)}(t) = \int_0^t (d\Psi)_{t-t'}(z(t'))f_\alpha(t')dt'$$

where $z^{(\alpha)}$ and f_α are vectors with components $z_k^{(\alpha)}$ and $f_{k,\alpha}$ respectively; this formula, (2.4.29) and (2.4.34) imply (2.4.31) for $|\alpha| = n$ where we change the constant C if necessary. □

Remark 2.4.6. (i) For Hamiltonian system (2.4.13) condition (2.4.28) means precisely that

$$|a_{(\beta)}^{(\alpha)}\big(z(t)\big)| \le c\mu\rho^{1-|\alpha|-|\beta|} \tag{2.4.35}$$
$$\forall t : [0, T] \quad \forall \alpha, \beta : 2 \le |\alpha| + |\beta| \le n + 1.$$

(ii) If for system (2.4.13) condition (2.4.29) is fulfilled for $t' = 0$ then this condition is fulfilled for all $t' \in [0, t]$ with ω replaced by ω^{2d}; in fact,

$$d\Psi_{t-t'}(z(t')) = d\Psi_t(z(0))\big(d\Psi_{t'}(z(0))\big)^{-1}$$

and since $\det d\Psi_t = 1$ we conclude that $(d\Psi_t)^{-1}$ is a cofactor matrix.

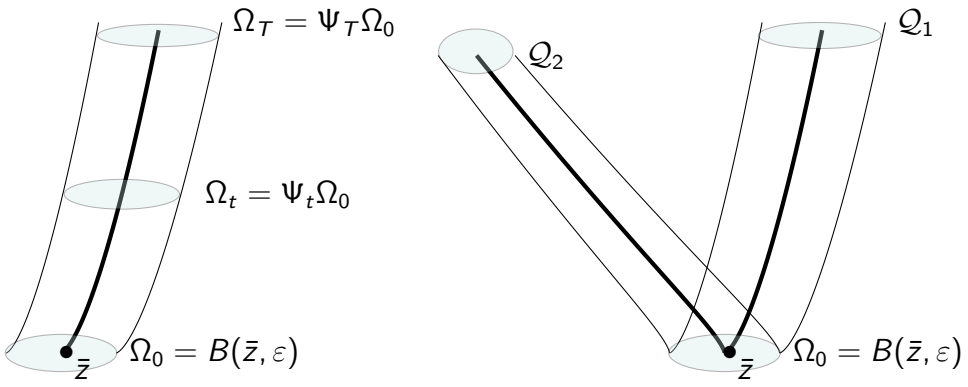
Solving Heisenberg Equations for Long-Time Evolution

Let us consider Hamiltonian trajectory $z(t)$ with $z(0) = \bar{z}$ and the evolution of $\Omega_0 = B(0, \varepsilon)$. Our analysis will deal only with the Hamiltonian tube

$$\mathcal{Q} := \bigcup_{0 \le t \le T} \Psi_t \Omega_0$$

where $\Omega_0 = B(\bar{z}, \varepsilon)$. We want to estimate q_n^t and their derivatives as $z \in \Omega_t$ but we are going to deal with $r_n^t := q_n^t \circ \Psi_t$ in Ω_0.

We assume that Hamiltonian flow Ψ_t satisfies (2.4.27)–(2.4.30). As a^s is scalar we have $r_0^t = q_0^0$. In the general case however formula is (2.4.18) and



(a) (b)

Figure 2.9: Hamiltonian tubes: in (a) for $D_t - \lambda(x, D_x)$, and in (b) for $D_t - A(x, D_x)$ with eigenvalues of $a(x, \xi)$ of constant mutiplicities.

we need to estimate derivatives of $R_{t,t}$; Recall that since a^s is Hermitian, $R_{t,t'}$ is unitary. Assuming that

$$\|D^\alpha_{x,\xi} a^s\| \le C\mu\omega\rho^{-1-|\alpha|} \qquad \forall \alpha : 1 \le |\alpha| \tag{2.4.36}$$

and

$$\mu T \ge 1, \quad \omega \ge 1, \quad \rho \le 1 \tag{2.4.37}$$

we conclude that

$$\|D^\alpha_{x,\xi}(a^s \circ \Psi_t)\| \le C\mu\rho^{-2}\omega^2\eta^{1-|\alpha|} \qquad \forall \alpha : 1 \le |\alpha| \tag{2.4.38}$$

and then repeating in the simplified way proof of Lemma 2.4.5 we arrive to

Lemma 2.4.7. *In the framework of Lemma 2.4.5 assume that* (2.4.35) *and* (2.4.37) *are fulfilled. Further, let a^s be a Hermitian symbol satisfying* (2.4.36) *(in the Hamiltonian tube $\mathcal{Q}$). Then*

$$\|D^\alpha_{x,\xi} R_{t,t'}\| \le C\eta^{-|\alpha|} \qquad \forall (x,\xi) \in \Omega_{t'}\ \forall t, t' \in [0, T]\ \forall \alpha. \tag{2.4.39}$$

Therefore we arrive to

Corollary 2.4.8. *In the framework of Lemma 2.4.7 $q_0^t \circ \Psi_t$ satisfies*

$$\|D^\gamma_{x,\xi}(q_0^t \circ \Psi_t)\| \le C\varepsilon^{-1}\varsigma^{1-|\gamma|} \tag{2.4.40}$$
$$\forall (x,\xi) \in \Omega_0\ \forall t \in [0, T]\ \forall \gamma : 1 \le |\gamma|$$

with

$$\varsigma = \min(\varepsilon, \eta). \tag{2.4.41}$$

Consider now equation $(2.4.8)_n$ defining q_n with $n \ge 1$; one can rewrite it as

$$\partial_t r_n^t + i[a^s \circ \Psi_t, r_n^t] = f_n^t := \sum_{\substack{0\le k\le n-1\\ j+|\alpha|+|\beta|=n-k}} \frac{1}{2^{|\alpha|+|\beta|}\alpha!\beta!}\bigl((-i)^{|\alpha|-|\beta|}A_{j(\beta)}^{(\alpha)}q_{k(\alpha)}^{t(\beta)} - i^{|\alpha|-|\beta|}q_{k(\alpha)}^{t(\beta)}A_{j(\beta)}^{(\alpha)}\bigr)\circ\Psi_t \tag{2.4.42}$$

with the initial condition $r_n^0 = q_n^0$; observe that we can rewrite (2.4.42) as

$$\partial_t\bigl(R_t^{-1}(q_n^t\circ\Psi_t)R_t\bigr) = R_t^{-1}f_n^tR_t. \tag{2.4.43}$$

Further, we can rewrite $q_{k(\beta)}^{t(\alpha)}\circ\Psi_t$ as the sum of $b_{\alpha,\beta;\alpha',\beta'}D_{x,\xi}^{\alpha',\beta'}(q_k^t\circ\Psi_t)$ with $1\le|\alpha'|+|\beta'|\le|\alpha|+|\beta|$ and with the the coefficients satisfying

$$\|D_{x,\xi}^{\gamma}b_{\alpha,\beta;\alpha',\beta'}\| \le C\omega^{|\alpha|+|\beta|+|\gamma|}\eta^{-|\gamma|-|\alpha|-|\beta|+|\alpha'|+|\beta'|}. \tag{2.4.44}$$

Meanwhile

$$\|D_{x,\xi}^{\gamma}\bigl(A_{k(\beta)}^{(\alpha)}\circ\Psi_t\bigr)\| \le C\mu\omega^k\eta^{-|\gamma|}\rho^{-k-|\alpha|-|\beta|} \tag{2.4.45}$$

provided we assumed that

$$\|A_{k(\beta)}^{(\alpha)}\| \le C\mu\omega^k\rho^{-k-|\alpha|-|\beta|} \tag{2.4.46}$$

in the respective range of α,β,γ,k. Then the induction with respect to n yields

Proposition 2.4.9. *Let $\Psi_t:\mathbb{R}^d\to\mathbb{R}^d$ be a Hamiltonian flow defined by a. Let us assume that in the Hamiltonian tube $\mathcal{Q}$ conditions* (2.4.35) *and* (2.4.29) *are fulfilled with ρ and $\omega\ge 1$ satisfying* (2.4.30) *and* (2.4.37).

Further, let us assume that in the Hamiltonian tube $\mathcal{Q}$ all terms of Weyl symbol of A satisfy (2.4.46) *and that $q^0\in \mathsf{S}_{h,\varepsilon,\varepsilon}$. Then for all n*

$$\|D_{x,\xi}^{\gamma}(q_n^t\circ\Psi_t)\| \le C(\mu T\rho^{-2})^n\varsigma^{-2n-|\gamma|} \qquad \forall(x,\xi)\in\Omega_0\ \forall t\in[0,T]\ \forall k,\gamma. \tag{2.4.47}$$

Proof. Assuming that (2.4.47) is fulfilled for $k < n$ we conclude that

$$\|D^{\gamma}_{x,\xi} f^t_n\| \le C\mu\rho^{-2}\varsigma^{-2n-|\gamma|} \tag{2.4.48}$$

which due to Lemma 2.4.7 implies (2.4.47). □

It follows from (2.4.47) and Lemma 2.4.5 that

$$\|D^{\gamma}_{x,\xi} q^t_n\| \le C(\mu T\rho^{-2})^n \varsigma^{-2n-|\gamma|}\omega^{|\gamma|} \qquad \forall (x,\xi)\in\Omega_t\ \forall t\in[0,T]\ \forall k,\gamma. \tag{2.4.49}$$

To have $\sum_n q^t_n h^n$ decaying we need to assume

$$\varsigma \ge (\mu T)^{1/2}\rho^{-1}h^{(1-\delta)/2} \tag{2.4.50}$$

and to perform quantization we need

$$\varsigma \ge \omega h^{(1-\delta)/2}; \tag{2.4.51}$$

plugging $\varsigma = \varepsilon$ and $\varsigma = \eta$ in both we get

$$\varepsilon \ge \max\bigl((\mu T)^{1/2}\rho^{-1},\omega\bigr)h^{(1-\delta)/2} \tag{2.4.52}$$

and

$$\rho \ge (\mu T)^{1/2}\max\bigl(\omega^{2/3}h^{(1-\delta)/6}, \omega^{3/2}h^{(1-\delta)/4}\bigr); \tag{2.4.53}$$

Thus we (almost) proved

Theorem 2.4.10. *Let $A \in \mu\rho\boldsymbol{\Psi}_{h,\cdot,\cdot}(\mathbb{R}^d,\mathbb{H},\mathbb{H})$ be a self-adjoint operator with the scalar principal symbol satisfying in the Hamiltonian tube $\mathcal{Q}$ conditions* (2.4.29), (2.4.35), (2.4.46) *with ρ and $\omega \ge 1$ satisfying* (2.4.30), (2.4.37) *and* (2.4.53).

Let $q^0 \in \mathsf{S}_{h,\varepsilon,\varepsilon}$ be supported in Ω_0 with ε satisfying (2.4.52). *Let q^t be a symbol described in Proposition 2.4.9.*

Then for $t\in[0,T]$ operator $Q^t = \mathsf{Op}^{\mathsf{w}}(q^t)$ satisfies

$$\|Q^t - e^{-ih^{-1}tA}Q^0 e^{ih^{-1}tA}\| \le C\mu T h^s. \tag{2.4.54}$$

Remark 2.4.11. As we consider $\mathsf{S}_{h,\cdot,\cdot,N}$ and $\boldsymbol{\Psi}_{h,\cdot,N}$ with finite N one should take $s = \delta'(\delta)N$ where $\delta' > 0$ as $\delta > 0$.

Proof of Theorem 2.4.10. Obviously

$$\|\partial_t Q^t + ih^{-1}[A, Q^t]\| \le C\mu h^s$$

which is equivalent to

$$\|\partial_t(e^{ih^{-1}tA} Q^t e^{-ih^{-1}tA})\| \le C\mu h^s$$

because the operator A is self-adjoint.

On the other hand, $\|Q^0 - Q\| \le Ch^s$. Then

$$\|e^{ih^{-1}tA} Q^t e^{-ih^{-1}tA} - Q^0\| \le C\mu T h^s$$

(note that $\mu T \ge 1$) which is equivalent to (2.4.54). □

2.4.2 Propagation of Singularities

Scalar Principal Symbol Case

Theorem 2.4.10 immediately implies

Theorem 2.4.12. *Let $A \in \mu\rho \boldsymbol{\Psi}_{h,.,.}(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ be a self-adjoint operator with the scalar principal symbol satisfying in the Hamiltonian tube $\mathcal{Q}$ conditions* (2.4.29), (2.4.35), (2.4.46) *with ρ and $\omega \ge 1$ satisfying* (2.4.30), (2.4.37) *and* (2.4.53). *Let ε satisfy* (2.4.52).

Let us assume that <u>*either*</u>

(a) $q \in \mathsf{S}_{h,\varepsilon,\varepsilon}$ is supported in Ω_0, $q' \in \mathsf{S}_{h,\varsigma/\omega,\varsigma/\omega}$ and for some $t = t^* \in [0, T]$

$$\mathsf{dist}(z', \Psi_t(z)) \ge \epsilon\varsigma/\omega \qquad \forall z \in \mathsf{supp}(q)\ \forall z' \in \mathsf{supp}(q') \tag{2.4.55}$$

<u>*or*</u>

(b) The assumptions of (a) are fulfilled with permuted q and q' and Ψ_t replaced by Ψ_{-t}.

Then

(i) For for $t = t^* \in [0, T]$, $Q = \mathsf{Op}(q)$ and $Q' = \mathsf{Op}(q')$ the following inequality holds:

$$\|Qe^{ih^{-1}tA} Q'\| \le C\mu T h^s \tag{2.4.56}$$

(ii) Moreover, let $U(x, y, t)$ be the Schwartz kernel of $e^{ih^{-1}tA}$ and $\mathsf{supp}(q) \cup \mathsf{supp}(q') \subset \{|\xi| \le c\}$. Then the following estimate holds:

$$\|Q_x U(., ., t^*)^t Q'_y\| \le C' h^{s-d} T \qquad \forall x, y \in \mathbb{R}^d. \tag{2.4.57}$$

Diagonalization

We need to generalize this Theorem 2.4.12. Namely, we consider operator whose principal symbol is not scalar but whose spectrum in some interval consists of few rather disjoint eigenvalues

$$\mathsf{Spec}\, a(z) \cap [-\tau_1 - \varrho, -\tau_2 + \varrho] = \{\lambda_1(z), \dots, \lambda_m(z)\} \cap [-\tau_1, -\tau_2] \qquad \forall z \in \Omega \tag{2.4.58}$$

and for every j

$$\mathsf{Spec}\, a(z) \cap [\lambda_j(z) - \varrho, \lambda_j(z) + \varrho] = \lambda_j(z) \qquad \forall z \in \mathcal{Q}_j \tag{2.4.59}$$

where λ_j are assumed to be uniformly continuous and $\mathcal{Q}_j = \bigcup_{t\in[0,T]} \Psi_{\lambda_j,t}\Omega_0$ is a Hamiltonian tube generated by λ_j and the lower bound to ϱ will be defined later.

We want to reduce this case to one of Theorem 2.4.12. To do this we need a series of axillary statements.

Proposition 2.4.13. *Let $a \in \mathsf{S}_{h,\gamma,\gamma}$ be Hermitian symbol and the conditions* (2.4.58) *and* (2.4.59) *be fulfilled. Let Ω be simple connected. Then there exist unitary symbols $r(z), r_1(t,z), \dots, r_m(t,z)$ defined in $\Omega, \Omega_1(t), \dots, \Omega_m(t)$ respectively where $\Omega_j(t) = \Psi_{\lambda_j,t}\Omega$ such that*

(i) In Ω

$$r_j(0,z) = r(z) \qquad \forall j = 1, \dots, m, \tag{2.4.60}$$

$$r^\dagger(z)a(z)r(z) = a^0 = \begin{pmatrix} \lambda_1(z)I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \lambda_m(z)I & 0 \\ 0 & \cdots & 0 & a'(z) \end{pmatrix}, \tag{2.4.61}$$

and

$$\mathsf{Spec}\, a'(z) \cap [\lambda_j(z) - \frac{\varrho}{2}, \lambda_j(z) + \frac{\varrho}{2}] = \emptyset \qquad \forall j = 1, \dots, m \tag{2.4.62}$$

(ii) and, moreover, for every j in $\Omega_j(t)$

$$r_j^\dagger(t,z)a(z)r_j(t,z) = a^0_{j,t} = \begin{pmatrix} b_j'(t,z) & 0 & b_j''(t,z) \\ 0 & \lambda_j(z) & 0 \\ b_j'''(t,z) & 0 & b_j^{IV}(t,z) \end{pmatrix}, \tag{2.4.63}$$

$$\mathsf{Spec}\, a_j'(t,z) \cap [\lambda_j(z) - \frac{\varrho}{2}, \lambda_j(z) + \frac{\varrho}{2}] = \emptyset \tag{2.4.64}$$

with

$$a'_j(t,z) = \begin{pmatrix} b'_j(t,z) & b''_j(t,z) \\ b'''_j(t,z) & b^{IV}_j(t,z) \end{pmatrix}$$

and

$$\|\partial_t^k r^{(\alpha)}_{j(\beta)}\| \le C\varrho^{-|\alpha|-|\beta|-k}\gamma^{-|\alpha|-|\beta|-2k} \qquad \forall \alpha,\beta,k. \tag{2.4.65}$$

Proof. Let us define

$$\pi_j(z) = \frac{1}{2\pi i}\int_{\ell_j} (\zeta - a(z))^{-1}\, d\zeta \tag{2.4.66}$$

for $z \in \Omega_j(t)$ where ℓ_j is a closed smooth curve contained in the ring $\{\frac{1}{8}\gamma \le |\zeta - \lambda_j(z)| \le \frac{1}{4}\gamma\} \subset \mathbb{C}$ and going one time around $\lambda_j(z)$ in the counter-clockwise direction; then $\pi_j(z)$ is the orthogonal projector to the eigenspace of $a(z)$ corresponding to the eigenvalue $\lambda_j(z)$. Here we used condition (2.4.59).

Then in $\Omega_j(t)$ symbol $\pi(z)$ satisfies (2.4.65); here we used conditions (2.4.59) again and an assumption that $a \in \mathsf{S}_{h,\gamma,\gamma}$. Then one can find $r, r_j(t)$ satisfying (2.4.65) with $k = 0$.

It should be noted that the periodicity or approximate periodicity of some trajectories of the Hamiltonian flows would make $\mathcal{Q}_j$ not simple connected possibly prevent the existence of r_j not depending on t. However it would not happen during time $T \asymp \gamma^2\varrho$. Indeed, we can replace Ω by its $\gamma\varrho$-vicinity and note that the speed does not exceed $c\gamma^{-1}$.

This implies statement of Proposition. □

So, we proved that symbols $r, r_j \in \mathsf{S}_{h,\rho,\rho}$ with $\rho = \gamma\varrho$; therefore we need to assume that

$$\rho = \gamma\varrho \ge h^{(1-\delta)/2}. \tag{2.4.67}$$

Proposition 2.4.14. *In the framework of Proposition 2.4.13 assume that condition* (2.4.67) *is fulfilled. Further, let* $a' \in \nu\mathsf{S}_{h,\gamma,\gamma}$ *with*

$$h(\gamma\varrho)^{-2} \le \nu \le \varrho h^{\delta}. \tag{2.4.68}$$

Then there exist symbols $r, r_{1t}, \dots, r_{mt} \in \mathsf{S}_{h,\rho,\rho}(T^*\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ *satisfying* (2.4.60) *and such that* $\gamma^{2k}\varrho^k\partial_t^k r_{j,t} \in \mathsf{S}_{h,\rho,\rho}(T^*\mathbb{R}^d, \mathbb{H}, \mathbb{H})$, $\forall k$, *with* $\rho = \gamma\varrho$ *and the corresponding norms of these operators do not exceed* C, *and*

(i) Modulo operators with symbols vanishing in Ω

$$R^*R \equiv RR^* \equiv I, \tag{2.4.69}$$

$$R^*AR \equiv \begin{pmatrix} A_1 I & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & A_m I & 0 \\ 0 & \dots & 0 & A' \end{pmatrix} \tag{2.4.70}$$

(ii) and modulo operators with symbols vanishing in $\Omega_j(t)$

$$R^*_{j,t}R_{j,t} \equiv R_{j,t}R^*_{j,t} \equiv I, \tag{2.4.71}$$

$$-ihR^*_{j,t}(\partial_t R_{j,t}) + R^*_{j,t}AR_{j,t} \equiv \begin{pmatrix} B'_{j,t} & 0 & B''_{j,t} \\ 0 & A_{j,t} & 0 \\ B'''_{j,t} & 0 & B^{IV}_{j,t} \end{pmatrix} \tag{2.4.72}$$

where $R_{j,t} = \mathsf{Op}(r_{j,t})$, $R = \mathsf{Op}(r)$ *and*

(2.4.73) *Symbols of operators* $A_{j,t}, A', B'_{j,t}, \dots, B^{IV}_{j,t}$ *are equal to* $\lambda_j I, a', b'_{j,t}, \dots, b^{IV}_{j,t}$ *modulo* $\nu S_{h,\rho,\rho}$.

Proof. Let us use induction with respect to n:

$$V \in \nu^n \varrho^{1-n} \mathsf{S}_{h,\rho,\rho}(\Omega, \mathbb{H}, \mathbb{H}) \tag{2.4.74}$$

and

$$\partial_t^k V_{j,t} \in \gamma^{-2k}\gamma^{-2k}\nu^n \varrho^{1-n-k} \mathsf{S}_{h,\rho,\rho}(\Omega_j(t), \mathbb{H}, \mathbb{H}) \tag{2.4.75}$$

where V and $V_{j,t}$ are the differences between the symbols of the left-hand and right hand expressions in (2.4.70) and (2.4.72) respectively while (2.4.69) and (2.4.71) are assumed to be fulfilled with much higher precision.

Proposition 2.4.13 implies this assertion for $n = 1$ as long as $\nu \ge h/(\gamma\varrho)^2$.

In order to make the induction step it is sufficient to construct Hermitian symbols I and $I_{j,t}$ satisfying (2.4.60), (2.4.74), (2.4.75) [27], and

$$[I, a^0] = -iV \qquad \text{in } \Omega, \tag{2.4.76}$$

[27] So we plug $r = I$ etc in these equations.

and

$$[I_{j,t}, a^0_{j,t}] = -iV_{j,t} \qquad \text{in } \Omega_j(t) \tag{2.4.77}$$

where a^0 and $a^0_{j,t}$ are defined by (2.4.61) and (2.4.63) respectively.

Indeed, let us replace $R_{j,t}$ by $R_{j,t}(I + iL_{j,t})(I + L'_{j,t})$ with $L_{j,t} = \mathsf{Op}(l_{j,t})$, $L'_{j,t} = \mathsf{Op}(l'_{j,t})$ with Hermitian symbols $l'_{j,t}$ satisfying (2.4.63) with n replaced by $(n+1)$ and providing (2.4.71) after the induction step; then one needs to pick up $L'_{j,t} = -\frac{1}{2}L^*_{j,t}L_{j,t}$.

Moreover, let us replace R by $R(I + iL)(I + L')$ with $L' = -\frac{1}{2}L^*L$; then the induction step will be made obviously.

On the other hand, the construction of I and $I_{j,t}$ satisfying (2.4.76) and (2.4.77) is obvious and, moreover, Hermitian symbols I with diagonal blocks equal to 0 and $I_{j,t}$ of the form

$$I_{j,t} = \begin{pmatrix} 0 & * & 0 \\ * & 0 & * \\ 0 & * & 0 \end{pmatrix}$$

are uniquely defined from (2.4.76) and (2.4.77). □

Heisenberg Evolution for Matrix Principal Symbols

Let us introduce the operators $\tilde{R}_{j,t} = R_{j,t}\pi_j$, $\tilde{R}_j = R\pi_j$ where π_j are projectors corresponding to the block decomposition in (2.4.61) and $j = 1, \dots, m+1$. Then (2.4.69) and (2.4.70) imply that

$$\tilde{R}^*_j \tilde{R}_k \equiv \delta_{jk} I \qquad \text{in } \Omega \quad (j, k = 1, \dots, m+1), \tag{2.4.78}$$

$$\tilde{R}^*_{j,t} \tilde{R}_{j,t} \equiv I \qquad \text{in } \Omega_j(t) \quad (j = 1, \dots, m+1), \tag{2.4.79}$$

$$A\tilde{R}_j \equiv A_j \tilde{R}_j \qquad \text{in } \Omega \quad (j = 1, \dots, m+1), \tag{2.4.80}$$

and

$$A\tilde{R}_{j,t} - ih^{-1}[A_{j,t}, Q_{j,t}] \equiv \tilde{R}_{j,t} A_{j,t} \qquad \text{in } \Omega_j(t) \quad (j = 1, \dots, m) \tag{2.4.81}$$

with $A_{m+1} = A'$ and $Q_{j,t} = e^{-ih^{-1}tA_j} Q^0 e^{ih^{-1}tA_j}$.

Let us consider Cauchy problem (2.4.6)–(2.4.7) for A_j. Then in order to fit into theory of the previous Subsection 2.4.1, conditions (2.4.30), (2.4.32),

(2.4.37), (2.4.41), (2.4.52) and (2.4.53) must be fulfilled with $\mu = \rho^{-1}$ and $\rho = \gamma\varrho$:

$$T \ge \max(\omega^{-1}, \varrho\gamma), \quad \omega \ge 1, \quad \varrho\gamma \le 1, \tag{2.4.82}$$

$$\eta = \varrho^3\gamma^3\omega^{-2}T^{-1}, \quad \varsigma = \min(\varepsilon, \eta), \tag{2.4.83}$$

$$\varepsilon \ge \max\bigl((\varrho^{-3}\gamma^{-3}T)^{1/2}, \omega\bigr)h^{(1-\delta)/2}, \tag{2.4.84}$$

$$\varrho\gamma \ge T^{1/3}\max\bigl(\omega^{4/9}h^{(1-\delta)/9}, \omega h^{(1-\delta)/6}\bigr) \tag{2.4.85}$$

while ν satisfies (2.4.68) and also the following version of (2.4.38)

$$\nu \le \min\bigl(\varrho h^{\delta}, \omega\varrho^{-1}\gamma^{-1}h\bigr), \tag{2.4.86}$$

$\nu = h(\gamma\varrho)^{-2}$ shouls also fit into this last inequality

$$\omega\gamma\varrho \ge 1, \quad \gamma^2\varrho^3 \ge h^{1-\delta}. \tag{2.4.87}$$

Then Theorem 2.4.10 yields the following

Proposition 2.4.15. *Let $a \in \mathsf{S}_{h,\gamma,\gamma}$* [28] *be Hermitian symbol and the conditions* (2.4.58), (2.4.59) *be fulfilled. Let A be a self-adjoint operator and $A - \mathsf{Op}^w(a) \in \nu\mathsf{S}_{h,\gamma,\gamma}$. Let domain Ω be simple connected. Further, let conditions* (2.4.29) *and* (2.4.82)–(2.4.87) *be fulfilled. Finally, let*

$$Q_j = \tilde{R}_j Q \tilde{R}_j^* \qquad (j = 1, \dots, m) \tag{2.4.88}$$

and let $Q_{j,t}$ be a solution of (2.4.81) *with initial data $Q_{j,0} = Q_j$. Then*

$$\|Q_j^t - e^{-ih^{-1}tA}Qe^{ih^{-1}tA}\| \le Ch^s T \tag{2.4.89}$$

for $Q_j^t = \tilde{R}_{j,t}Q_{j,t}\tilde{R}_{j,t}^*$.

As $m = 1$ and we are interested in the left-hand expression of (2.4.90) only we get

Corollary 2.4.16. *Let conditions of Proposition 2.4.15 be fulfilled with $m = 1$. Then*

$$\|F_{t\to h^{-1}\tau}\chi_T(t)\bigl(Q^t e^{-ih^{-1}tA} - e^{-ih^{-1}tA}Q\bigr)\| \le Ch^s \qquad \forall \tau \in [\tau_1, \tau_2] \tag{2.4.90}$$

where χ is our standard function supported in <u>either</u> $[-1, 1]$ <u>or</u> $[-1, 0]$ <u>or</u> $[0, -1]$ as conditions of Proposition 2.4.15 are fulfilled <u>either</u> for all $t \in [-T, T]$ <u>or</u> for all $t \in [-T, 0]$ <u>or</u> for all $t \in [0, T]$ respectively and $Q^t = Q_1^t = \tilde{R}_{1,t}Q_{1,t}\tilde{R}_{1,t}^*$.

[28] All conditions need to be fulfilled only in $\mathcal{Q}_j$.

Main Theorem

Theorem 2.4.17. *Let* $a \in \mathsf{S}_{h,\gamma,\gamma}$ [28] *be Hermitian symbol and the conditions* (2.4.58), (2.4.59) *be fulfilled. Let* A *be a self-adjoint operator and* $A - \mathsf{Op}^w(a) \in \nu \mathsf{S}_{h,\gamma,\gamma}$ [28]. *Let domain* Ω *be simple connected. Further, let conditions* (2.4.29) *and* (2.4.82)–(2.4.87) *be fulfilled.*

Furthermore let us assume that <u>*either*</u>

(a) $q \in \mathsf{S}_{h,\varepsilon,\varepsilon}$ *is supported in* Ω_0, $q' \in \mathsf{S}_{h,\varsigma/\omega,\varsigma/\omega}$ *and for* $t \in [t^* - \kappa, t^* + \kappa] \subset [0, T]$ *condition*

$$\mathsf{dist}(z', \Psi_t(z)) \geq \varsigma/\omega \qquad \forall z \in \mathsf{supp}(q)\ \forall z' \in \mathsf{supp}(q') \tag{2.4.91}$$

is fulfilled for Hamiltonian flows generated by λ_j *for all* $j = 1, \dots, m$ <u>*or*</u>

(b) The assumptions of (a) are fulfilled with permuted q *and* q' *and* $\Psi_{\lambda_j,t}$ *replaced by* $\Psi_{\lambda_j,-t}$.

Finally, let $U(x, y, t)$ *be the Schwartz kernel of* $e^{ih^{-1}tA}$; *assume that* $\mathsf{supp}(q) \cup \mathsf{supp}(q') \subset \{|\xi| \leq c\}$, $\chi \in \mathscr{C}^\infty([-1, 1])$ *and*

$$\kappa\varrho \geq h^{1-\delta}. \tag{2.4.92}$$

Then the following estimate holds:

$$\|F_{t\to h^{-1}\tau}\chi_{\kappa/2}(t - t^*)Q_x U(t, ., .)\, {}^tQ'_y\| \leq C' h^s \qquad \forall x, y \in \mathbb{R}^d \quad \forall \tau \in [\tau_1, \tau_2] \tag{2.4.93}$$

Proof. Assume that we are in the framework of assumption (a) (the other case is considered similarly).

Theorem 2.4.12 microlocalizes $e^{ih^{-1}tA_j}Q$ in $\Psi_{\lambda_j,t}(\mathsf{supp}(q))$ and equality (2.4.80) is fulfilled in its vicinity. Therefore modulo negligible

$$e^{ih^{-1}tA}\tilde{R}_j Q'' \equiv e^{ih^{-1}tA_j}\tilde{R}_j Q'' \qquad \forall j = 1, \dots, m \tag{2.4.94}$$

as long as $0 \leq t \leq T$ and Q'' has a symbol supported in vicinity of $\mathsf{supp}(q)$. Recall that if we are looking for energy levels $\tau \in [\tau_1, \tau_2]$

$$\Psi_t(z) = \bigcup_{j=1,\dots,m} \Psi_{\lambda_j,t}(z) \tag{2.4.95}$$

Therefore in virtue of Theorem 2.4.12 estimate (2.4.93) holds for $U_j(x,y,t)$, the Schwartz kernels of $e^{ih^{-1}tA}\tilde{R}_j$ with $1 \le j \le m$.

Next, using ellipticity of A_{m+1} on energy levels $\tau \in [\tau_1, \tau_2]$ in the vicinity of $\mathsf{supp}(q)$ we conclude that (2.4.93) holds $j = m+1$ as well.

Finally these results combined with

$$\sum_{j=1,\dots,m+1} \tilde{R}_j \tilde{R}_j^* Q \equiv Q \tag{2.4.96}$$

imply (2.4.93). □

Comments

Theorem 2.1.2 was first proven for wave front sets in V. Ivrii [1]. Theorem 2.3.2 was first proven under steally more restrictive conditions in M. Bronstein, V. Ivrii [1].

The description of propagation of singularities in terms of generalized bicharacteristics is due to S. Wakabayashi [2].

Similar and also different results for equations with multiple characteristics could be found in V. Ivrii [4], [5]. Pseudoconvexity method based on the second commutators to study propagation of singularities was used in V. Ivrii [9].

For wave front sets there is a paper due to A. Volovoi [1] where a parametrix construction by means of Fourier integral operators for long bicharacteristics was examined very carefully; his approach is essentially more complicated. Moreover in standard microlocal analysis (when there is no exterior parameter such as h or a frequency) these results are senseless.

D. Robert [5], D. Robert and M. Combescure [1] used coherent states to describe a long term propagation.

The more detailed description of singularities can be made using polarization wave front sets:

Definition 2.4.18. Let $\|u\| = O(h^{s-\delta})$. Then

$$\mathsf{PWF}^s(u) = \Big\{(x,\xi;v) \in T^*\mathbb{R}^d \times \mathbb{H} : v \in \mathsf{Ker}\, q_0(x,\xi) \\ \forall q \in \mathbf{\Psi}_h \ \text{ such that } \ \|qu\| = O(h^s)\Big\}$$

is called *polarization wave front set*.

Description of polarization wave front sets for systems with constant multiplicities of characteristics follows from our results: their (x, ξ) components travel along bicharacteristics and their v components belongs to $\mathsf{Ker}\, P_0(x, \xi)$ and "rotates". The definition of polarization wave front set is due to N. Dencker, who also investigated propagation of them not only in the case of characteristics of constant multiplicities, but also in the framework of conical refraction [1] and [3], [4].

Chapter 3

Propagation of Singularities near the Boundary

In this chapter we perform the same task as in the previous Chapter 2 but near the smooth boundary. In Section 3.1 we prove Theorem 3.1.7 which is the main theorem of the energy estimates method and is formulated in the terms of some auxiliary functions exactly as Theorem 2.1.2 was. However the phase space T^*X is "glued" at the boundary ∂X and we really need several such auxiliary functions and the notion of the microhyperbolicity of the operator in the direction is replaced by the notion of the microhyperbolicity of the boundary value problem in the multidirection.

In Section 3.2 we prove Theorem 3.2.4 which is the geometric reincarnation of Theorem 3.1.7 exactly like Theorem 2.2.10 was the the geometric reincarnation of Theorem 2.1.2. However geometric picture becomes much more complicated even in the case of the scalar operators.

In Section 3.3 we prove some statements similar to those of the second part of sSection 2.1.

Theorem 3.2.4 is not very precise: for example it cannot exclude creeping rays even in the case of wave equation. So, in sSection 3.4 we reinvent the energy estimates method and prove (under much more restrictive assumptions) Theorems 3.4.4 and 3.4.5 which improve Theorem 3.1.7 and in combination with it provide in some cases more precise description of the propagation.

Finally, in Section 3.5 we study propagation of singularities along long bicharacteristic billiards which are "almost transversal to the boundary" and may contain large number of reflections (even with branching); this section is similar to Section 2.4.

3.1 Energy Estimates Approach

3.1.1 Statement of the Main Theorem

Preliminary Notes

In this section our geometric domain is $X = \mathbb{R}^+ \times X' = \mathbb{R}^+ \times \mathbb{R}^{d-1} \ni (x_1; x') = (x_1; x_2, \dots, x_d)$ with the boundary $\partial X = 0 \times X'$ and interior $\dot{X} = X \setminus \partial X$. We consider the operator

$$P = \sum_{0 \le k \le m} A_k(x, hD')(hD_1)^{m-k} \tag{3.1.1}$$

where

$$A_k \in \mathbf{\Psi}'_{h,N}(X, \mathbb{H}, \mathbb{H}) = \bigcap_{0 \le j \le N} \mathscr{C}^j_b\big(\mathbb{R}^+, \mathbf{\Psi}_{h,N-j}(X', \mathbb{H}, \mathbb{H})\big). \tag{3.1.2}$$

Let $a_k(x, \xi')$ and

$$p(x, \xi) = \sum_{0 \le k \le m} a_k(x, \xi')\xi_1^{m-k} \tag{3.1.3}$$

be the principal symbols of A_k and P respectively. We assume that

(3.1.4) $p(z)$ is a Hermitian matrix for every $z \in T^*X$,

i.e., that the system in question is *symmetric in the principal*.

Let $\Omega \Subset T^*X$, $\Omega' \Subset T^*\partial X$, $\Omega|_{\partial X} = \iota^{-1}\Omega'$ [1] and let us assume further that

$$\|a_0(z)w\| \ge \epsilon_0 \|w\| \qquad \forall w \in \mathbb{H} \quad \forall z \in \Omega, \tag{3.1.5}$$

i.e., that *the boundary is not characteristic* in Ω.

Then (3.1.4) yields the equality

$$\langle p(z, D_1)u, v\rangle_+ - \langle u, p(z, D_1)v\rangle_+ = i\langle T(z)\eth u, \eth v\rangle_m \tag{3.1.6}$$
$$\forall z \in T^*\partial X \qquad \forall u, v \in \mathscr{S}(\mathbb{R}^+, \mathbb{H})$$

where $\langle ., . \rangle_+$ and $\langle ., . \rangle_m$ are inner products in $\mathscr{L}^2(\mathbb{R}^+, \mathbb{H})$ and $\mathbb{H}^m$ respectively, $\eth u = \eth_{(m)} u = (u, D_1 u, \dots, D_1^{m-1}u)|_{x_1=0}$ is the m-trace of u at ∂X and $T(z) \in$

[1] Recall that $\iota : T^*X|_{\partial x} \to T^*\partial X$ is a natural map.

$\mathcal{L}(\mathbb{H}^m, \mathbb{H}^m)$ is a matrix with blocks

$$T_{jk}(z) = \begin{cases} a_{m-j-k-1} & \text{for } j+k \le m+1 \\ 0 & \text{for } j+k \ge m+2 \end{cases} \qquad j,k = 1,\dots,m \tag{3.1.7}$$

i.e.,

$$T(z) = \begin{pmatrix} a_{m-1} & a_{m-2} & \dots & a_1 & a_0 \\ a_{m-2} & a_{m-3} & \dots & a_0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & a_0 & \dots & 0 & 0 \\ a_0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Then equality (3.1.6) implies the estimate

$$\|T(z)\underline{w}\| \ge \epsilon \|\underline{w}\| \qquad \forall \underline{w} \in \mathbb{H}^m \quad \forall z \in \Omega' \tag{3.1.8}$$

with a constant $\epsilon = \epsilon(m, c, \epsilon_0, c_2) > 0$, $c_2 = \max_{1\le k\le m} \sup_{z\in\Omega} \|a_k(z)\|$. Without any loss of the generality we may replace ϵ by ϵ_0.

Let $B = (B_1, \dots, B_m)$ be a boundary operator, $B_k \in \mathbf{\Psi}_{h,N-k}(\partial X, \mathbb{H}_b, \mathbb{H})$ where $\mathbb{H}_b$ is another auxiliary Hilbert space with an inner product $\langle .,. \rangle_b$ and a norm $\|.\|_b$.

Let us assume that there exist symbols $\beta_k \in \mathscr{C}_b^N(\partial X, \mathcal{L}(\mathbb{H}_b, \mathbb{H}))$ and $\beta = (\beta_1, \dots, \beta_m)$ such that

$$\langle T(z)\underline{w}, \underline{w}\rangle_m + 2\,\mathrm{Re}\,\langle b(z)\underline{w}, \beta(z)\underline{w}\rangle_b \le 0 \qquad \forall \underline{w} \in \mathbb{H}^m \quad \forall z \in \Omega' \tag{3.1.9}$$

where b_k and b are the principal symbols of B_k and B respectively; in this case the boundary value problem (P, B) is called *dissipative* in Ω'. Moreover, if

$$\langle T(z)\underline{w}, \underline{w}\rangle_m + 2\,\mathrm{Re}\,\langle b(z)\underline{w}, \beta(z)\underline{w}\rangle_b \le -\epsilon_0\|\underline{w}\|^2 \qquad \forall \underline{w} \in \mathbb{H}^m \quad \forall z \in \Omega' \tag{3.1.9}^*$$

then the boundary value problem (P, B) is called *strictly dissipative* in Ω'.

In what follows the most important case is $\mathbb{H} = \mathbb{C}^{\mathsf{D}}$; one can easily prove then that if $\mathsf{D}_\pm$ and $\bar{\mathsf{D}}_\pm$ are the numbers of positive (negative) eigenvalues of the matrices a_0 and T respectively then equality (3.1.6) yields that

$$\bar{\mathsf{D}}_\pm = \begin{cases} \dfrac{1}{2} m\mathsf{D} & \text{for even } m, \\ \dfrac{1}{2}(m-1)\mathsf{D} + \mathsf{D}_\pm & \text{for odd } m. \end{cases} \tag{3.1.10}$$

Moreover, the most important case is $\mathbb{H}_b = \mathbb{C}^{\bar{\mathsf{D}}_+}$ when condition (3.1.9) is equivalent to the pair of conditions

$$\operatorname{rank} b(z) = \bar{\mathsf{D}}_+ \qquad \forall z \in \bar{\Omega}' \tag{3.1.11}$$

and

$$\langle T(z)\underline{w}, \underline{w}\rangle_m \le 0 \qquad \forall \underline{w} \in \operatorname{Ker} b(z) \quad \forall z \in \bar{\Omega}'. \tag{3.1.12}$$

Furthermore, condition (3.1.9)* is equivalent to (3.1.11) and

$$\langle T(z)\underline{w}, \underline{w}\rangle_m \le -\epsilon_0\|\underline{w}\|^2 \qquad \forall \underline{w} \in \operatorname{Ker} b(z) \quad \forall z \in \bar{\Omega}'. \tag{3.1.12*}$$

Let us notice that (3.1.6) implies that

$$|\eta|^m\|w\| \le C(\|p(z,\eta)w\| + \|w\|) \qquad \forall w \in \mathbb{H} \quad \forall z \in \Omega' \quad \forall \eta \in \mathbb{R}. \tag{3.1.13}$$

Let $\Sigma_\rho = \Sigma_\rho(P)$ be the set of all the points $z \in T^*X$ at which the ellipticity condition

$$\|p(z)w\| \ge \rho\|w\| \qquad \forall w \in \mathbb{H} \tag{3.1.14}$$

is violated; here and below $\rho > 0$ is an arbitrarily small constant.

Further, let $\Sigma'_\rho = \iota\Sigma_\rho|_{\partial X}$ be the set of all the points $z \in \Omega'$ at which the ellipticity condition

$$\|p(z,\eta)w\| \ge \rho\|w\| \qquad \forall w \in \mathbb{H} \quad \forall \eta \in \mathbb{R} \tag{3.1.14$'$}$$

is violated.

For $z = (x', \xi')$ let us introduce the linear space

$$\mathbb{H}'(z) = \{v \in \mathscr{L}^2(\mathbb{R}^+, \mathbb{H}) : p(z, D_1)v = 0\}. \tag{3.1.15}$$

Then for $z \notin \Sigma'_\rho$ the following estimates hold:

$$\|e^{\varkappa x_1} D_1^j v\|_+ \le C_j\|v\|_+ \qquad \forall j \ge 0 \quad \forall v \in \mathbb{H}'(z) \tag{3.1.16}$$

with an appropriate exponent $\varkappa = \varkappa(m, c, c_1, \epsilon_0, \rho) > 0$ and constants $C_j = C_j(m, c, c_1, \epsilon_0, \rho)$.

Furthermore, let us introduce the sets $\Sigma_{b,\rho} = \Sigma_{b,\rho}(P, B)$: $z \in \Sigma_{b,\rho}$ if *either* $z \in \Sigma'_\rho$ *or* the following condition

$$\|b(z)\eth_m v\|_{\mathbb{H}'} \ge \rho\|v\|_+^2 \qquad \forall v \in \mathbb{H}'(z) \tag{3.1.17}$$

is violated.

Finally, let

$$\Sigma = \Sigma(P) = \bigcap_{\rho>0} \Sigma_\rho, \quad \text{and} \quad \Sigma_b = \Sigma_b(P,B) = \bigcap_{\rho>0} \Sigma_{b,\rho}\,; \tag{3.1.18}$$

it is obvious that

(3.1.19) Σ and Σ_b are closed sets and that the space $\mathbb{H}'(z)$ depends continuously (in the natural sense) on $z \in \Omega' \setminus \Sigma'$ where $\Sigma' = \iota\Sigma|_{\partial X}$.

Definition 3.1.1. A point $z \in \Sigma \subset T^*X$ ($z \in T^*X \setminus \Sigma$) is called a *characteristic* (*elliptic* respectively) point of the operator P.

A point $z \in \Sigma_b \subset \Omega'$ ($z \in \Omega' \setminus \Sigma_b$) is called a *characteristic* (*elliptic* respectively) point of the problem (P,B).

By means of standard elliptic arguments of Section 1.4 one can easily prove the following semiclassical result

Theorem 3.1.2. *Let condition* (3.1.6) *be fulfilled,* $N = N(d,s,M,l)$ *and let* u *be temperate in the sense*

$$\|D_1^j u\|_X \le h^{-M} \qquad \forall j = 1,\dots,m+l. \tag{3.1.20}$$

Then

$$\mathsf{WF}_b^{s,m+l}(u) \cap \Omega' \subset \mathsf{WF}_b^{s,l}(Pu) \cup \mathsf{WF}^{s-\frac12}(B\eth_h u) \cup \Sigma_b \tag{3.1.21}$$

where here and in what follows

$$\mathsf{WF}_b^{s,l}(u) \coloneqq \bigcup_{0\le j\le l} \mathsf{WF}^{s-j\,\prime}(D_1^j u)|_{\partial X} \tag{3.1.22}$$

is called the boundary wave front set of order (s,l) of u *and* $\eth_h u = \eth_{h(m)} u = (u, hD_1 u, \dots, (hD_1)^{m-1}u)|_{\partial X}$.

Let us recall that $\mathsf{WF}^{t\,\prime}$ denotes a partial wave front set (see Definition 2.1.8). Later we will prove this theorem. Let us now assume that

$$\begin{aligned} \mathbb{H} =& \mathbb{C}^{\mathsf{D}_1}\otimes\mathbb{H}^0_1 \oplus \dots \oplus \mathbb{C}^{\mathsf{D}_M}\otimes\mathbb{H}^0_M \oplus \mathbb{H}_0 \\ p =& \quad p_1\otimes I_{\mathbb{H}^0_1} \oplus \dots \oplus p_M \otimes \quad I_{\mathbb{H}^0_M} \oplus p_0 \end{aligned} \tag{3.1.23}$$

and

(3.1.24) p_0 on $\mathbb{H}_0$ satisfies condition (3.1.14)$'$ with $\epsilon = \epsilon_0$

where $\mathbb{H}^0_1, \dots, \mathbb{H}^0_M, \mathbb{H}_0$ are auxiliary Hilbert spaces and $p_1, \dots, p_M, p_0$ are symbols on $\mathbb{C}^{D_1}, \dots, \mathbb{C}^{D_M}, \mathbb{H}_0$ respectively.

Then at every point $z \in \Omega'$ the estimate (3.1.16) holds with an appropriate exponent $\varkappa > 0$ and constants C_j depending on $p(z, .)$. Moreover, the space $\mathbb{H}'(z)$ is a lower semicontinuous function of z in the sense of Definition 2.2.2.

Let us introduce the *characteristic polynomial*

$$g(z, \eta) = \det p_1(z, \eta) \cdots \det p_M(z, \eta). \tag{3.1.25}$$

Definition 3.1.3. For $z \in \Omega' \subset T^*\partial X$ let us denote by $\eta_1(z), \dots, \eta_n(z)$ all the real distinct roots of $g(z, \eta)$ where also $n = n(z)$. Then $\mathcal{T} = (\ell', \nu_1, \dots, \nu_n) \in T_z T^*\partial X \times \mathbb{R}^n$ is called *multidirection*.

Definition 3.1.4. Let conditions (3.1.1)–(3.1.6), (3.1.23) and (3.1.9) be fulfilled.

Then the problem (P, B) is *microhyperbolic at the point* $z \in \Omega' \subset T^*\partial X$ *in the multidirection* $\mathcal{T} = (\ell', \nu_1, \dots, \nu_n)$ if the following two conditions are fulfilled:

(i) For every $k = 1, \dots, n$ the operator P is microhyperbolic at the point $z_k = (z, \eta_k)$ in the direction $\ell_k = (\ell, \nu_k, 0) \in T_{z_k}(T^*X|_{\partial X}) \subset (T_{z_k} T^*X)|_{\partial X}$;

(ii) The following estimate holds

$$\begin{aligned} -\operatorname{Re} \langle (\mathcal{T}' p)(z, D_1) v, v \rangle_+ + \operatorname{Re} i \langle (\mathcal{T}' b)(z) \eth v, \beta(z) \eth v \rangle_b \geq \\ \epsilon \|v\|^2_+ - c_1 (c_1 \|b(z) \eth v\|'^2 - \langle T(z) \eth v, \eth v \rangle_m) \qquad \forall v \in \mathbb{H}'(z) \end{aligned} \tag{3.1.26}$$

(compare with estimate (2.1.1) in Definition 2.1.1).

In what follows we will prove the following

Proposition 3.1.5. *Let the problem* (P, B) *be microhyperbolic at the point* z *in the multidirection* $\mathcal{T}$. *Further, let the problem* $(\tilde{P}, \tilde{B})$ *also satisfy conditions* (3.1.1)–(3.1.6), (3.1.23) *and* (3.1.9) *and suppose that*

$$\|\tilde{p} - p\|_{\mathscr{C}^1_b} \leq \varepsilon, \quad \|\tilde{b} - b\|_{\mathscr{C}^1_b} \leq \varepsilon, \quad |\tilde{z} - z| \leq \varepsilon, \ |\tilde{\mathcal{T}} - \mathcal{T}| \leq \varepsilon$$

with a small enough constant $\varepsilon > 0$.

Then the problem $(\tilde{P}, \tilde{B})$ *is also microhyperbolic at the point* $\tilde{z}$ *in the multidirection* $\tilde{\mathcal{T}}$.

Remark 3.1.6. Obviously the structure of $\iota^{-1}\tilde{z}\cap\Sigma(\tilde{P})$ does not necessarily coincide with that of $\iota^{-1}z\cap\Sigma(P)$ and it can happen that $\tilde{n}(\tilde{z})>n(z)$. However one can avoid this difficulty by either of two methods:

(i) If z_{ki} with $i=1,\dots,t_k$ $(t_k\ge 0)$ are points of $\iota^{-1}\tilde{z}\cap\Sigma(\tilde{P})$ close to z_k then the condition $|\tilde{\mathcal{T}}-\mathcal{T}|\le\varepsilon$ is understood in the sense that
$|\tilde{\ell}_{ki}-\ell_k|\le\varepsilon\quad\forall i=1,\dots,l_k\quad\forall k=1,\dots,n$ [2)];

(ii) The condition that η_k $(k=1,\dots,n)$ are all the real roots of $g(z,\eta)$ is replaced by the condition that the set $\{\eta:(z,\eta)\in\Sigma_\epsilon\}$ is contained in the union of the intervals $(\eta_k-\mu,\eta_k+\mu)$ $(k=1,\dots,n)$ where $|\eta_k-\eta_j|\ge 3\mu$ $\forall j\ne k$ and $0<\mu\le\bar{\mu}=\bar{\mu}(c,\epsilon_0,c_1,c_3)$ where ϵ_0 and c_1 are constants from the microhyperbolicity conditions (2.1.1) with z,ℓ replaced by z_k,ℓ_k (with $k=1,\dots,n$), (3.1.6) and (3.1.26), and $c_3=\max_j\|a_j\|_{\mathscr{C}^2_b}+\max_j\|b_j\|_{\mathscr{C}^2_b}$.

In this case $\varepsilon=\varepsilon(d,m,M,\mathrm{D}^*,c,\epsilon_0,c_1,c_2,\epsilon)$, $\mathrm{D}^*=m\max_j\mathrm{D}_j$.

[2)] I.e. $|\tilde{\ell}-\ell|\le\varepsilon$ and $|\tilde{\nu}_{ki}-\nu_k|\le\varepsilon\quad\forall i=1,\dots,l_k\quad\forall k=1,\dots,n$.

Finally, the Main Theorem

Now we can formulate the main result of this section:

Theorem 3.1.7. *Let conditions* (3.1.1)–(3.1.6), (3.1.23), (3.1.9) *be fulfilled and let* $N = N(d, m, s, M, l, \mathsf{D}^*)$.

Further, let $z^* \in \Omega' \subset T^*\partial X$ *and* $\phi_j \in \mathscr{C}^N(T^*\partial X)$, $\phi_{kj} \in \mathscr{C}^N(T^*X' \times \mathbb{R}^+)$ *with* $j = 1, \dots, J$, $k = 1, \dots, n$ *be real-valued functions such that*

$$\phi_{1j} = \dots = \phi_{nj} = \phi_j \qquad \text{at} \quad T^*\partial X \quad \forall j \tag{3.1.27}$$

and for every $j = 1, \dots, J$ *the problem* (P, B) *is microhyperbolic at the point* z^* *in the multidirection* $\mathcal{T}_j = (\nabla^{\#}\phi_j, -\partial_{x_1}\phi_{1j}, \dots, -\partial_{x_1}\phi_{nj})(z^*)$.

Furthermore, let u *satisfy* (3.1.20) *and*

$$\bigl(\mathsf{WF}_b^{s+1,l}(Pu) \cup \mathsf{WF}^{s+\frac{1}{2}}(B\mathfrak{d}_h u)\bigr) \cap \Omega' \cap \{\phi_1 \le t_1\} \cap \dots \cap \{\phi_J \le t_J\} = \emptyset, \tag{3.1.28}$$

$$\mathsf{WF}^{s+1}(Pu) \cap \Omega'_k \cap \{\phi_{k1} \le t_1\} \cap \dots \cap \{\phi_{kJ} \le t_J\} = \emptyset \quad \forall k = 1, \dots, n, \tag{3.1.29}$$

$$\mathsf{WF}_b^{s,0}(u) \cap \partial\Omega' \cap \{\phi_1 \le t_1\} \cap \dots \cap \{\phi_J \le t_J\} = \emptyset \tag{3.1.30}$$

and

$$\mathsf{WF}^{s}(u) \cap \partial\Omega_k \cap \{\phi_{k1} \le t_1\} \cap \dots \cap \{\phi_{kJ} \le t_J\} = \emptyset \quad \forall k = 1, \dots, n \tag{3.1.31}$$

where Ω' *and* Ω_k *are small enough neighborhoods of* z *and* z_k *in* $T^*\partial X$ *and* T^*X *respectively not depending on* $u, t_1, \dots, t_J$ *such that*

$$\iota\Omega_k|_{\partial X} = \Omega' \qquad \forall k = 1, \dots, n. \tag{3.1.32}$$

(i) Then

$$\mathsf{WF}_b^{s,m+l}(u) \cap \Omega' \cap \{\phi_1 \le t_1\} \cap \dots \cap \{\phi_J \le t_J\} = \emptyset\,. \tag{3.1.33}$$

(ii) Moreover, if condition $(3.1.9)^*$ *is fulfilled then one can weaken condition* (3.1.28) *replacing in it* $\mathsf{WF}^{s+\frac{1}{2}}(B\mathfrak{d}_h u)$ *by* $\mathsf{WF}^{s}(B\mathfrak{d}_h u)$*:*

$$\bigl(\mathsf{WF}_b^{s+1,l}(Pu) \cup \mathsf{WF}^{s}(B\mathfrak{d}_h u)\bigr) \cap \Omega' \cap \{\phi_1 \le t_1\} \cap \dots \cap \{\phi_J \le t_J\} = \emptyset \tag{3.1.28}'$$

and at same time obtain in addition that

$$\mathsf{WF}^{s-i}(D_1^i u|_{\partial X}) \cap \Omega' \cap \{\phi_1 \le t_1\} \cap \dots \cap \{\phi_J \le t_J\} = \emptyset \qquad \forall i = 0, \dots, m+l-1. \tag{3.1.34}$$

Remark 3.1.8. (i) Assumption (3.1.29) is referring to points of $T^*\dot{X}$.

(ii) Assumptions (3.1.6), (3.1.20), (3.1.28) and (3.1.30) imply (3.1.30) again with $\mathsf{WF}^{s,0}(u)$ replaced by $\mathsf{WF}^{s,m+l}(u)$.

(iii) The uniform version of Theorem 3.1.7 (such as was proven in the proof of Theorem 2.1.2) is obvious but very long and tedious to formulate.

3.1.2 Proof of the Main Theorem

In order to prove Theorems 3.1.2 and 3.1.7 and Proposition 3.1.5 we first of all reduce the general case to a special one: first to $m = 1$ and then to the block-diagonal form with each block corresponding to a single point $\iota^{-1}z^*$ and an elliptic block of the special type.

Reduction to a Special Case

Step 1. Reduction to $m = 1$. First of all, *one can assume that* $m = 1$. Indeed, let us set $\underline{u} = (u_j)_{j=1,\dots,m}$ with $u_j = (hD_1)^{j-1}u$. Then in the intersection with the vicinities of $\iota^{-1}z^*$ or z^*

$$\mathsf{WF}^s(u) = \mathsf{WF}^s(\underline{u}), \qquad \mathsf{WF}^{s,m+l\,\prime}(u) = \mathsf{WF}^{s,l+1\,\prime}(\underline{u}),$$
$$\mathsf{WF}^t(Pu) = \mathsf{WF}^t(\underline{P}\,\underline{u}), \qquad \mathsf{WF}^{s,l\,\prime}(Pu) = \mathsf{WF}^{s,l\,\prime}(\underline{P}\,\underline{u})$$

provided $|s-t| \le 1$ where $\underline{P} = \underline{A}_0 hD_1 + \underline{A}_1$,

$$\underline{A}_0 = (A_{m-j-k+1})_{j,k=1,\dots,m} \tag{3.1.35}$$

is an operator with principal symbol $T(z)$, $\underline{A}_1 = -\underline{A}_0 L$,

$$L = (L_{jk})_{j,k=1,\dots,m}, \qquad L_{jk} = \delta_{j+1,k} - A_0' A_{m-k+1}\delta_{j,m} \tag{3.1.36}$$

where A_0' is a parametrix of A_0 here (condition (3.1.6) guarantees its existence) and $\delta_{j,k}$ is the Kronecker symbol as usual. Then modulo operators with symbols vanishing in the vicinity of z^*

$$\underline{A}_1 = (-A_{m-j-k+2} + A_{m-j+1}\delta_{k,1} + A_{m-k+1}\delta_{j,1})_{j,k=1,\dots,m}. \tag{3.1.37}$$

Observe that condition (3.1.6) for P implies the same condition for $\underline{P}$. Moreover, condition (3.1.4) for P is equivalent to the same condition for $\underline{P}$. Furthermore, condition (3.1.23) for P is equivalent to to the same condition for $\underline{P}$ with D_j replaced by $m\mathsf{D}_j$.

One can prove easily that

$$\Sigma_{C_1^{-1}\rho}(P) \subset \Sigma_\rho(\underline{P}) \subset \Sigma_{C_1\rho}(P)$$

for appropriate $C_1 = C_1(c, \epsilon_0, c_1, m)$ and arbitrary $\rho > 0$.

Moreover, $B\eth_{h(m)}u = B\eth_{h(1)}\underline{u}$ and hence

(3.1.38) Condition (3.1.9) or $(3.1.9)^*$ for (P, B) is equivalent to to the same condition for $(\underline{P}, B)$;

from this one can prove easily that

$$\Sigma_{C_1^{-1}\rho}(P, B) \subset \Sigma_\rho(\underline{P}, B) \subset \Sigma_{C_1\rho}(P, B)$$

for appropriate $C_1 = C_1(c, \epsilon_0, c_1, c_2, m)$ and arbitrary $\rho > 0$.

Finally, one can prove easily the following inequalities

$$|\langle(\ell p)(z)w, w\rangle - \langle(\ell \underline{p})(z)\underline{w}, \underline{w}\rangle_m| \le C|\ell| \cdot \|\underline{p}(z)\underline{w}\|\|\underline{w}\| \qquad \forall \underline{w} \in \mathbb{H}^m : w = w_1$$

and

$$|\langle(\ell' p)(z, D_1)v, v\rangle_+ - \langle(\ell' \underline{p})(z, D_1)\underline{v}, \underline{v}\rangle_+| \le C|\ell'| \cdot \|\underline{p}(z, D_1)\underline{v}\|_+ \cdot \|\underline{v}\|_+ \qquad \forall \underline{v} \in \mathscr{S}(\mathbb{R}^+, \mathbb{H}^m) : v = v_1$$

where as usual $\|.\|_+$ and $\langle ., .\rangle$ is a norm and an inner product in $\mathscr{L}^2(\mathbb{R}^+, .)$.

Hence

(3.1.39) $\underline{P}$ is microhyperbolic at z in the direction ℓ if and only if P has the same property; moreover, $(\underline{P}, B)$ is microhyperbolic at z in the multidirection $\mathcal{T}$ if and only if (P, B) has the same property.

The reduction step is complete:

Proposition 3.1.9. *Under reduction* (3.1.35)–(3.1.37) *conditions* (3.1.6), (3.1.4), (3.1.23), (3.1.9) *or* $(3.1.9)^*$, *and microhyperbolicity conditions are preserved.*

Furthermore all wave front sets and Σ_* *are are essentially preserved as described above.*

Step 2. Isolation of characteristic roots. Let us now show that

Proposition 3.1.10. *(i) In the proofs of Theorem 3.1.7 and Proposition 3.1.5 one can assume without any loss of the generality that*

$$|\det p_j(z^*,\eta)| \ge \epsilon \qquad \forall \eta \in \mathbb{C} : |\eta-\eta_j| \ge \mu \tag{3.1.40}$$

where $\mu > 0$ is arbitrarily small, $\epsilon > 0$ depends on μ, η_j are not necessarily distinct anymore and

$$p_0 = \begin{pmatrix} 0 & p_0' \\ p_0'' & 0 \end{pmatrix}, \qquad p_0' = \rho(\eta-\alpha), \quad p_0'' = (\eta-\alpha^\dagger)\rho^\dagger \tag{3.1.41}$$

with the matrix symbols α and ρ such that

$$\|(\eta-\alpha)w\| \ge \epsilon\|w\|, \quad \|\rho w\| \ge \epsilon\|w\| \qquad \forall w \in \mathbb{H} \quad \forall \eta \in \mathbb{C} : \operatorname{Im}\eta \le \mu. \tag{3.1.42}$$

(ii) In the proof of Theorem 3.1.2 one can assume without any loss of the generality that

$$p_0 = \begin{pmatrix} 0 & p_0' \\ p_0'' & 0 \end{pmatrix}, \quad p_0' = \eta-\alpha', \quad p_0'' = \eta-\alpha'' \tag{3.1.41$'$}$$

and (3.1.42) *holds for $\alpha = \alpha'$ and $\alpha = -\alpha''$.*

(iii) Also in this reduction D^* does not increase.

In order to make the reduction step let us first consider the separate symbols $p_j(z,\eta)$ with $j = 1,\dots,J$. Let η_{jk} be all the real distinct roots of $\det p_j(z^*,\eta)$, $k = 1,\dots,n_j$. Then η_{jk} are all the real distinct eigenvalues of $L_j(z^*)$ where $L_j(z) = -a_{0j}(z)^{-1}a_{1j}(z)$ [3)].

Hence there exist symbols $r_j(z)$ defined, smooth and invertible such that in Ω''

$$r_j^{-1}L_j r_j = \tilde{L}_j = \begin{pmatrix} \tilde{L}_{j1} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \tilde{L}_{jn_j} & 0 \\ 0 & \dots & 0 & \tilde{L}_{j0} \end{pmatrix},$$

where

[3)] Let us recall that $p_j(z,\eta) = a_{0j}(z)\eta + a_{1j}(z)$) and $z \in \Omega''$ where Ω'' is a small enough neighborhood of z^* in $T^*X' \times \mathbb{R}^+$ which we shrink if necessary.

(3.1.43) For $k \ge 1$ all the eigenvalues of $\tilde{L}_{jk}$ are equal to η_{jk} and all the eigenvalues of $\tilde{L}_{j0}$ are non-real.

Then

$$r_j^\dagger a_{1j} r_j = r_j^\dagger a_{0j} r_j \tilde{L}_j$$

and taking the Hermitian conjugate we obtain that

$$(r_j^\dagger a_{0j} r_j)\tilde{L}_j = \tilde{L}_j^\dagger (r_j^\dagger a_{0j} r_j).$$

Since $\tilde{L}_j$ is of the block-diagonal type and since $\eta \ne \zeta^\dagger$ if η and ζ belong to the spectra of different blocks (where $\zeta^\dagger$ is a complex conjugate number) we conclude that $r_j^\dagger a_{0j} r_j$ is also of the block-diagonal type and hence $r_j^\dagger a_{1j} r_j$ is also of this type.

Therefore multiplying $p(z,\eta)$ on the left and on the right by $r^\dagger(z)$ and $r(z)$ respectively we obtain

$$p = \oplus \sum_{1\le j\le M, 1\le k\le n_j} p_{jk} \otimes I_{\mathbb{H}_j^0} \ \oplus p_0 \tag{3.1.44}$$

provided

$$r = \begin{pmatrix} r_1 \otimes I_{\mathbb{H}_j^0} & \dots & 0 & 0 \\ \vdots & \ddots & r_M \otimes I_{\mathbb{H}_M^0} & \\ 0 & \dots & 0 & I \end{pmatrix};$$

here $\det p_{jk}(z^*,\eta)$ has no complex root except $\eta_{jk} \in \mathbb{R}$ and p_0 satisfies (3.1.14)$'$; after this reduction step the space $\mathbb{H}_0$ can be larger than before.

Similarly, let us consider $L_0(z) = -a_{00}(z)^{-1} a_{10}(z)$ [4]; then the spectrum of $L_0(z)$ resides in $\{\eta \in \mathbb{C}, |\operatorname{Im}\eta| \ge \epsilon\}$ and hence there exists a smooth and invertible symbol $r_0(z)$ such that in Ω''

$$r_0^{-1} L_0 r_0 = \tilde{L}_0 = \begin{pmatrix} L_0' & 0 \\ 0 & L_0'' \end{pmatrix}$$

where the spectrum of L_0' resides in $\{\eta \in \mathbb{C}, \operatorname{Im}\eta \ge \epsilon\}$ and the spectrum of L_0'' resides in $\{\eta \in \mathbb{C}, \operatorname{Im}\eta \le -\epsilon\}$.

Then using the same arguments as before (but taking into account that $\eta \ne \zeta^\dagger$ provided η and ζ belong to the spectrum of the same block) we obtain that both symbols $r_0^\dagger a_{00} r_0$ and $r_0^\dagger a_{10} r_0$ are of the form $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$.

[4] Let us recall that $p_0(z,\eta) = a_{00}(z)\eta + a_{10}(z)$.

Then multiplying (3.1.44) on the left and on the right by $r^\dagger$ and r respectively with

$$r = \begin{pmatrix} I & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & I & 0 \\ 0 & \dots & 0 & r_0 \end{pmatrix},$$

we obtain p_0 of the form (3.1.41).

Thus, the second step of the reduction is provided by multiplying P on the left and on the right by $\mathsf{Op}(r)^*$ and $\mathsf{Op}(r)$ respectively where r is now the product of the two matrices constructed in the two sub-steps of this step.

Step 3. Reduction of Lower Order Terms. In the final reduction step we obtain

$$P = \begin{pmatrix} P_1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & P_n & 0 \\ 0 & \dots & 0 & P_0 \end{pmatrix}, \qquad P_0 = \begin{pmatrix} 0 & P_0' \\ P_0'' & 0 \end{pmatrix} \tag{3.1.45}$$

with $P_0'' = P_0'^*$ where for $k = 1, \dots, n$ the principal symbols of P_k are of the form

$$p_k = \oplus \sum_{1 \le k \le K_j} p_{kj} \otimes I_{\mathbb{H}^0_{jk}}, \tag{3.1.46}$$

p_{jk} are the same as in (3.1.44), and p_0' and p_0'' (the principal symbols of P_0' and P_0'' respectively) are of the same type as in (3.1.41).

In order to make this reduction step we assume that (3.1.45) is fulfilled modulo operators with symbol $h^n(F_0\eta + F_1)$; we then apply induction on n. For $n = 1$ these approximate equalities are fulfilled.

In order to make the induction step we multiply P on the right and on the left by $(I + h^n Q^*)$ and $(I + h^n Q)$ respectively; then the step in the first equality (3.1.45) will be made provided for $j \neq k$

$$q_{kj}^\dagger a_{0k} + a_{0j} q_{jk} = F_{0jk}, \tag{3.1.47}_1$$

$$q_{kj}^\dagger a_{1k} + a_{1j} q_{jk} = F_{1jk} \tag{3.1.47}_2$$

where a_{0k}, a_{1k} are diagonal blocks and q_{jk}, F_{0jk}, F_{1jk} are blocks of the corresponding symbols, $Q = \mathsf{Op}(q)$.

Since both F_0 and F_1 are Hermitian matrices we need to fulfill these equalities only for $j < k$; for $j > k$ they will then be fulfilled automatically. Finding $q_{kj}^\dagger$ from the first equality and substituting into the second one we obtain after multiplication by a_{0j}^{-1} that we need only to solve the equation

$$-q_{jk}L_k + L_j q_{jk} = F'_{jk} \tag{3.1.48}$$

for $j < k$ where $L_k = -a_{0k}^{-1}a_{1k}$ and F'_{jk} is obtained from F_{0jk} and F_{1jk}. This equation is solvable since the spectra of L_j and L_k are disjoint (see Appendix 3.A.1). The choice of q_{jk} for $j = k$ is arbitrary.

In order to make the induction step in the second equality in (3.1.45) we need to solve the equations

$$q_{(21)}^\dagger \rho^\dagger + \rho q_{(21)} = F_{0(11)}, \tag{3.1.49}_1$$

$$q_{(21)}^\dagger \alpha^\dagger \rho^\dagger + \rho\alpha q_{(21)} = F_{1(11)} \tag{3.1.49}_2$$

and similar equations for q_{12} where $\rho = p'_0$, $q_{(jk)}$ are blocks of

$$q_{00} = \begin{pmatrix} q_{(11)} & q_{(12)} \\ q_{(21)} & q_{(22)} \end{pmatrix},$$

$F_{0(jk)}$ and $F_{1(jk)}$ are similar blocks of the matrices F_{000} and F_{100}, and these blocks are Hermitian for $j = k$. The solution of the first equation is of the form

$$q_{(21)} = \rho^{-1}(\frac{1}{2}F_{0(11)} + i\Lambda)$$

where Λ is an arbitrary Hermitian symbol. Substituting this equality to the second equation we obtain that

$$\Lambda\alpha'^\dagger - \alpha'\Lambda = iF'_{(11)} \tag{3.1.50}$$

where $\alpha' = \rho\alpha'\rho^{-1}$ and $F'_{(11)}$ is some given Hermitian symbol. We can solve this equation because the spectra of α' and $\alpha'^\dagger$ are disjoint (see Appendix 3.A.1 again). Moreover, one can choose $q_{(11)}$ and $q_{(22)}$ arbitrarily. The induction step is done.

The reduction is complete.

Elliptic Theory

The following series of assertions is crucial in the proof of Theorem 3.1.2 and is important in the proof of Theorem 3.1.7:

Proposition 3.1.11. *Let $\alpha \in \mathscr{C}^{l}_{b}\big(\mathbb{R}, \mathbf{\Psi}_h(\mathbb{R}^d, \mathbb{H}, \mathbb{H})\big)$ be an operator with the principal symbol α_0 such that*

$$\|(\eta - \alpha_0(z))w\| \geq \epsilon\|w\| \qquad \forall w \in \mathbb{H} \quad \forall \eta \in \mathbb{C} : \operatorname{Im}\eta \leq \epsilon \quad \forall z \in \mathbb{R} \times T^*\mathbb{R}^d \tag{3.1.51}$$

and let $l \geq 1$. Then for $h \in (0, h_0]$ with $h_0 = h_0(d, c, \epsilon, c_2) > 0$, $c_2 = \|\alpha\|$ there exists an operator family

$$G \in \mathscr{C}^{l+1}\big(\Delta, \mathcal{L}(\mathscr{L}^2(\mathbb{R}^d, \mathbb{H}), \mathscr{L}^2(\mathbb{R}^d, \mathbb{H}))\big)$$

with $\Delta = \{(x_1, y_1) \in \mathbb{R}^2, x_1 \geq y_1\}$ such that

(i) The following relations hold:

$$G(x_1, x_1) = I, \tag{3.1.52}$$

$$\|D^j_{x_1} D^k_{y_1} G(x_1, y_1)\| \leq Ch^{-j-k} e^{-\epsilon h^{-1}(x_1 - y_1)} \qquad \forall j, k : j + k \leq q + 1 \tag{3.1.53}$$

where here and below $C = C(d, s, M, l, \epsilon, c_2)$.

(ii) If the operators $G_\pm : \mathscr{L}^2(\mathbb{R}^{d+1}, \mathbb{H}) \to \mathscr{L}^2(\mathbb{R}^{d+1}, \mathbb{H})$ are given by the formulae

$$(G_+ f)(x_1) = +ih^{-1} \int_{-\infty}^{x_1} G(x_1, y_1) f(y_1)\, dy_1, \tag{3.1.54}$$

$$(G_- f)(x_1) = -ih^{-1} \int_{x_1}^{\infty} G(y_1, x_1) f(y_1)\, dy_1 \tag{3.1.54}$$

then $G_+^* = G_-$ and

$$\|G_\pm\| \leq C, \tag{3.1.55}$$

$$PG_+ = G_+ P = I, \tag{3.1.56}$$

$$P^* G_- = G_- P^* = I \tag{3.1.56}$$

where $P = hD_1 - \alpha$ and then $P^* = hD_1 - \alpha^*$.

Moreover, if f is tempered (i.e., $\|D_1^j f\| \leq h^{-M} \ \ \forall j = 0, \dots, l$) then

$$\mathsf{WF}^{s,l+1\prime}(G_\pm f) = \mathsf{WF}^{s,l\prime}(f). \tag{3.1.57}$$

(iii) If the operator family $G_t : \mathscr{L}^2(\mathbb{R}^d, \mathbb{H}) \to \mathscr{L}^2(\mathbb{R}^+_t \times \mathbb{R}^d, \mathbb{H})$ *with* $\mathbb{R}^+_t = \{x_1, x_1 \ge t\}$ *is given by the formula*

$$(G_t f)(x_1) = G(x_1, t) f, \tag{3.1.58}$$

then

$$\|G_t\| \le C h^{\frac{1}{2}}, \tag{3.1.59}$$

$$PG_t = 0, \qquad \eth_t G_t = I \tag{3.1.60}$$

where $\eth_t u = u|_{x_1 = t}$ *and, moreover, if* f' *is tempered then*

$$\mathsf{WF}^{s, l+1\prime}(G_t f') = \{t\} \times \mathsf{WF}^{s-\frac{1}{2}}(f'). \tag{3.1.61}$$

Proof. Let us note that for $h \in (0, h_0]$ with an appropriate constant $h_0 > 0$ the following estimate holds:

$$\|(\eta - \alpha)v\| \ge \frac{1}{2}\epsilon \|v\| \qquad \forall v \in \mathscr{L}^2(\mathbb{R}^d, \mathbb{H}) \quad \forall \eta \in \mathbb{C} : \mathsf{Im}\, \eta \le \frac{1}{2}\epsilon \quad \forall x_1 \in \mathbb{R}.$$

Then for operators

$$G_{(1)}(x_1, y_1) = e^{ih^{-1}(x_1 - y_1)\alpha_{y_1}} \qquad \text{with} \quad \alpha_{y_1} = \alpha|_{x_1 = y_1}$$

and

$$G_{(2)}(x_1, y_1) = e^{ih^{-1}(x_1 - y_1)\alpha_{x_1}}$$

(3.1.52) and (3.1.53) with $j = k = 0$ are fulfilled. Hence if we define $G'_\pm$ by formulae $(3.1.54)_\pm$ with $G_{(1)}$ and $G_{(2)}$ respectively instead of G then (3.1.55) would be fulfilled while $(3.1.56)_\pm$ would be fulfilled only partially and approximately:

$$PG'_+ = I + R_+, \quad P^* G'_- = I + R_-, \quad \|R_\pm\| \le Ch.$$

Then for $h \in (0, h_0]$ with an appropriate constant $h_0 > 0$ the operators $G_\pm = G'_\pm (I + R_\pm)^{-1}$ satisfy $(3.1.56)_\pm$ partially

$$PG_+ = P^* G_- = I \tag{3.1.56}'$$

and are of the form $(3.1.54)_\pm$ with appropriate functions $G = G_\pm(x_1, y_1)$ satisfying (3.1.52)–(3.1.53); however these functions may be different for different indices $\pm$.

But then $(3.1.56)'$ yields that $G_+^* P^* = G_-^* P = I$ and these equalities and $(3.1.56)'$ again imply that $G_+ = G_-^*$ and then $(3.1.56)_\pm$ hold; therefore the functions $G_\pm$ in formulae $(3.1.54)_\pm$ coincide. This implies (3.1.60) and (3.1.53).

It remains to prove (3.1.57) and (3.1.61). In fact, it is sufficient to prove that $\mathsf{WF}^{s\prime}(G_\pm f)$ and $\mathsf{WF}^{s\prime}(G_t f')$ are contained in the right-hand sets of (3.1.57) and (3.1.61) respectively; these inclusions and (3.1.56) and (3.1.60) imply (3.1.57) and (3.1.61).

Furthermore, one can see easily that it is sufficient to prove these inclusions for $G'_\pm f$ and $G'_t f'$ where G'_t is given by (3.1.58) with G replaced by $G_{(1)}$.

Note that

$$e^{ih^{-1}\tau\alpha} Q = \sum_{0\le k\le n} \frac{1}{k!}\partial_\lambda^k \bigl(e^{ih^{-1}\tau\lambda\alpha} Q e^{ih^{-1}\tau(1-\lambda)\alpha}\bigr)\big|_{\lambda=0} + \frac{1}{n!}\int_0^1 (1-\lambda)^n \partial_\lambda^{n+1}\bigl(e^{ih^{-1}\tau\lambda\alpha} Q e^{ih^{-1}\tau(1-\lambda)\alpha}\bigr)\, d\lambda = \sum_{0\le k\le n} \frac{1}{k!}(\mathsf{Ad}^k_{ih^{-1}\alpha} Q)\tau^k e^{ih^{-1}\tau\alpha} + \frac{1}{n!}\int_0^1 (1-\lambda)^n (\mathsf{Ad}^{n+1}_{ih^{-1}\alpha} Q)\tau^{n+1} e^{ih^{-1}\tau(1-\lambda)\alpha}\, d\lambda$$

where Ad_M is an operator of commutation with M and Ad^k_M are defined by $(\mathsf{Ad}^0_M Q) = Q$, $(\mathsf{Ad}^1_M Q) = [M, Q]$ and $(\mathsf{Ad}^k_M Q) = [M, (\mathsf{Ad}^{k-1}_M Q)]$ for $k \ge 1$.

It is easy to deduce from (3.1.53) that if Q is a scalar pseudodifferential operator then the norm of the last term does not exceed $C_n h^n e^{-\frac{1}{2}\epsilon\tau h^{-1}}$ and hence modulo negligible operators

$$e^{ih^{-1}\tau\alpha} Q \equiv \sum_{0\le k\le n} \frac{1}{k!}(\mathsf{Ad}^k_{ih^{-1}\alpha} Q)\tau^k e^{ih^{-1}\tau\alpha}$$

with large enough $n = n(s)$. This equality obviously implies the inclusions in question. □

Theorem 3.1.12. *Let Ω be a domain in $\mathbb{R}\times T^*\mathbb{R}^{d-1}$ and $\tilde\Omega$ its ϵ-vicinity. Further, let $\Omega' = \Omega\cap\{x_1 = 0\}$, $\Omega_+ = \Omega\cap\{x_1 \ge 0\}$, $\tilde\Omega' = \tilde\Omega\cap\{x_1 = 0\}$ and $\tilde\Omega_+ = \tilde\Omega\cap\{x_1 \ge 0\}$.*

Furthermore, let $\alpha \in \mathscr{C}^l(\mathbb{R}, \mathbf{\Psi}_h(\mathbb{R}^d, \mathbb{H}, \mathbb{H}))$ *satisfy* (3.1.51) *for all* $z \in \tilde{\Omega}$ *and let* $l \ge 1$.

Then there exist operators

$$G_0 \in \mathcal{L}\big(\mathscr{L}^2(\partial X, \mathbb{H}), \mathscr{L}^2(X, \mathbb{H})\big) \quad \text{with} \quad \|G_0\| \le Ch^{1/2}$$

and

$$G \in \mathcal{L}\big(\mathscr{L}^2(X, \mathbb{H}), \mathscr{L}^2(X, \mathbb{H})\big) \quad \text{with} \quad \|G_0\| \le C$$

with the same type of constant C *as before such that for tempered* u_0, f *(i.e., with* $\|u_0\|_{\partial X} \le h^{-M}, \|D_1^j f\|_X \le h^{-M}\ \forall j \le l$*) the following relations hold:*

$$\mathsf{WF}^{s,l+1\,\prime}(G_0 u_0) \subset \mathsf{WF}^{s-\frac{1}{2}}(u_0) \cap \tilde{\Omega}, \tag{3.1.62}$$

$$\mathsf{WF}^{s,l+1\,\prime}(Gf) \subset \mathsf{WF}^{s,l\,\prime}(f) \cap \tilde{\Omega}_+, \tag{3.1.63}$$

$$\mathsf{WF}^{s,l\,\prime}(PG_0 u_0) \cap \Omega_+ = \emptyset, \tag{3.1.64}$$

$$\mathsf{WF}^{s-\frac{1}{2}}(\eth G_0 u_0 - u_0) \cap \Omega = \emptyset, \tag{3.1.65}$$

$$\mathsf{WF}^{s,l\,\prime}(PGf - f) \cap \Omega_+ = \emptyset, \tag{3.1.66}$$

$$\mathsf{WF}^{s-\frac{1}{2}}(\eth Gf) = \emptyset, \tag{3.1.67}$$

and

$$\mathsf{WF}^{s,l+1\,\prime}(G^* f) \subset \mathsf{WF}^{s,l\,\prime}(f) \cap \tilde{\Omega}_+, \tag{3.1.62}'$$

$$\mathsf{WF}^{s,l\,\prime}(P^* G^* f - f) \cap \Omega_+ = \emptyset. \tag{3.1.64}'$$

Moreover, if $\|D_1^j u\|_X \le h^{-M}$ *for all* $j \le l+1$ *then the following relations hold:*

$$\mathsf{WF}^{s,l+1\,\prime}(u - GPu - G_0 \eth u) \cap \Omega_+ = \emptyset, \tag{3.1.68}$$

$$\mathsf{WF}^{s,l+1\,\prime}(u - G^* P^* u) \cap \Omega_+ = \emptyset. \tag{3.1.68}'$$

Proof. Without any loss of the generality one can assume that (3.1.51) is satisfied in $\mathbb{R} \times T^*\mathbb{R}^d$. Indeed, (3.1.51) is satisfied in $\tilde{\Omega}'$ for a small enough constant $\epsilon > 0$ and hence $\bar{\alpha}(z) = \alpha(z)\varphi(z) + i(1 - \varphi(z))I$ satisfies (3.1.51) in $\mathbb{R} \times T^*\mathbb{R}^d$ provided φ is a smooth scalar function equal to 1 in the $\frac{1}{3}\epsilon$-neighborhood of Ω' and vanishing outside of the $\frac{2}{3}\epsilon$-neighborhood of Ω' and such that $0 \le \varphi \le 1$.

Let us define $\bar{G}_{\pm}$ and $\bar{G}_0$ according to the Proposition 3.1.11[5)] and let us set $\bar{G} = R\bar{G}_+T$ where here R and T are the operators of restriction from $\mathbb{R}^{d+1}$ to X and extension from X to $\mathbb{R}^{d+1}$ by 0 in $\mathbb{R}^{d+1}\setminus X$ respectively.

Then $\bar{G}^* = R\bar{G}_-T$ and operators $\bar{G}$ and $\bar{G}_0$ obviously satisfy (3.1.62)–(3.1.64)$'$ with $\Omega = \tilde{\Omega} = T^*\mathbb{R}^d$ and with $\Omega_+ = \tilde{\Omega}_+ = \mathbb{R}^+ \times T^*\mathbb{R}^d$. Moreover, the obvious equalities

$$\begin{aligned} PTu &= TPu - ih(\eth u)\delta(x_1),\\ P^*Tu &= TP^*u - ih(\eth u)\delta(x_1) \end{aligned}$$

combined with $(3.1.56)_{\pm}$ and $(3.1.54)_{\pm}$ imply the equalities

$$u = \bar{G}Pu + \bar{G}_0\eth u, \qquad u = \bar{G}^*P^*u;$$

these equalities combined with (3.1.57) and (3.1.60) imply (3.1.68) and (3.1.68)$'$ with $\Omega_+ = \mathbb{R}^+ \times T^*\mathbb{R}^d$.

Finally, $G = \mathsf{Op}(\varphi)\bar{G}\,\mathsf{Op}(\varphi)$ and $G_0 = \mathsf{Op}(\varphi)\bar{G}_0\,\mathsf{Op}(\varphi_0)$ are the operators in question provided $\varphi_0 = \varphi|_{x_1=0}$. □

Proof of Theorem 3.1.2. The simplifying reduction of Subsubsection *3.1.2.1 Reduction to a Special Case* implies that without any loss of the generality one can assume that $\Sigma(P) = \emptyset$ and

$$P = \begin{pmatrix} P_1 & 0\\ 0 & P_2\end{pmatrix}, \qquad \mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2$$

where $P_j = hD_1 - \alpha_j$ and α_1 and $\alpha_2^\dagger$ satisfy (3.1.51).

Therefore if $\Omega' \supset \omega$ and ω does not intersect $\mathsf{WF}_b^{s,l}(Pu)$ then Theorem 3.1.12 yields that

$$\omega \cap \mathsf{WF}_b^{s,l+1}(u_2) = \emptyset, \qquad \omega \cap \mathsf{WF}_b^{s,l+1}(u_1 - G_0v) = \emptyset,$$
$$\omega \cap \mathsf{WF}_b^{s,l+1}(u_1) \subset \mathsf{WF}^{s-\frac{1}{2}}(v)$$

where G_0 is constructed for P_1 and $v = \eth u_1$. Then

$$\omega \cap \mathsf{WF}^{s-\frac{1}{2}}(B\eth u) = \omega \cap \mathsf{WF}^{s-\frac{1}{2}}(B'v) \supset \omega \cap \mathsf{WF}^{s-\frac{1}{2}}(B'^*B'v)$$

where B' is the restriction of B to $\mathbb{H}_1$.

One can see easily that $\mathbb{H}'(z) = \mathbb{H}_1$ and hence the set of characteristic points of operator B'^*B' is exactly $\Sigma_b(P,B)$. Hence if $\omega \cap \mathsf{WF}^{s-1/2}(B\eth u) = \emptyset$ then $\omega \cap \mathsf{WF}^{s-1/2}(v) \subset \Sigma_b(P,B)$. Therefore $\omega \cap \mathsf{WF}_b^{s,l+1}(u) \subset \Sigma_b(P,B)$. □

[5)] We now use a bar for the operators constructed in that proposition, with $t = 0$ now in (3.1.60) and (3.1.61).

Main Theorem Refined

The rest of this section is almost entirely devoted to the proof of the following assertion which is slightly more precise than Theorem 3.1.7:

Theorem 3.1.13. *Let P be a reduced operator satisfying the conditions of Theorem 3.1.7, i.e., we suppose that conditions* (3.1.1) *with* $m=1$, (3.1.4), (3.1.6), (3.1.41), (3.1.42), (3.1.45) *and* (3.1.46) *are satisfied.*

Further, let $z^*\in \Omega' \Subset T^*\partial X$ and let us assume that each polynomial $\det p_k(z^*,\eta)$ with $k=1,\dots,n$ has only one complex root and this root is real (namely $\eta_k\in\mathbb{R}$).

Further, let boundary operator B satisfy condition (3.1.9).

Finally, let $\phi_j\in \mathscr{C}^N(T^*\partial X)$, $\phi_{kj}\in \mathscr{C}^N(\mathbb{R}^+\times T^*X')$ be real-valued functions satisfying (3.1.27) *($j=1,\dots,J$, $k=1,\dots,n$).*

Let us assume that for every $j=1,\dots,J$ the problem (P,B) is microhyperbolic at the point z^* in the multidirection $\mathcal{T}_j=(\nabla^{\#}\phi_j, -\partial_{x_1}\phi_{1j},\dots,-\partial_{x_1}\phi_{nj})(z^*)$. Let $N=N(d,s,M,l)$ and $l\ge 2\mathrm{D}^*-2$, $\mathrm{D}^*=\max_k \mathrm{D}_k$ where D_k are the dimensions of $\mathbb{H}_k$. Then

(i) Let u satisfy (3.1.20) *and*

$$\bigl(\mathsf{WF}_b^{s+1,l}(P_0u_0)\cup \mathsf{WF}^{s+\frac12}(B\eth u)\bigr)\cap \Omega'\cap\{\phi_1\le t_1\}\cap\dots\cap\{\phi_J\le t_J\}=\emptyset, \tag{3.1.69}$$

$$\mathsf{WF}^{s+1,l\,\prime}(P_ku_k)\cap\Omega_k''\cap\{\phi_{k1}\le t_1\}\cap\dots\cap\{\phi_{kJ}\le t_J\}=\emptyset \quad \forall k=1,\dots,n, \tag{3.1.70}$$

$$\mathsf{WF}^{s-\frac12}(\eth u_0)\cap\partial\Omega'\cap\{\phi_1\le t_1\}\cap\dots\cap\{\phi_J\le t_J\}=\emptyset \tag{3.1.71}$$

and

$$\mathsf{WF}^{s\,\prime}(u_k)\cap\partial\Omega_k''\cap\{\phi_{k1}\le t_1\}\cap\dots\cap\{\phi_{kJ}\le t_J\}=\emptyset \quad \forall k=1,\dots,n \tag{3.1.72}$$

where $u=(u_1,\dots,u_n,u_0)$, Ω' and Ω_k'' are small enough neighborhoods of z^* in $T^*\partial X$ and $\mathbb{R}^+\times T^*X'$ respectively such that $\Omega'=\Omega_k''|_{\partial X}$ $\forall k=1,\dots n$. Then

$$\mathsf{WF}^{s-\frac12}(\eth u_0)\cap\Omega'\cap\{\phi_1\le t_1\}\cap\dots\cap\{\phi_J\le t_J\}=\emptyset \tag{3.1.73}$$

and

$$\mathsf{WF}^{s,l+1\prime}(u_k)\cap\Omega_k''\cap\{\phi_{k1}\le t_1\}\cap\dots\cap\{\phi_{kJ}\le t_J\}=\emptyset \qquad \forall k=1,\dots,n. \tag{3.1.74}$$

(ii) Moreover, if condition $(3.1.9)^*$ *is satisfied then one can weaken condition* (3.1.69)*, replacing in it* $\mathsf{WF}_b^{s+1,l}(P_0u_0)$ *and* $\mathsf{WF}^{s+1/2}(B\eth u)$ *by* $\mathsf{WF}_b^{s+1/2,l}(P_0u_0)$ *and* $\mathsf{WF}^{s}(B\eth u)$ *respectively:*

$$\bigl(\mathsf{WF}_b^{s+\frac12,l}(P_0u_0)\cup\mathsf{WF}^{s}(B\eth u)\bigr)\cap \Omega'\cap\{\phi_1\le t_1\}\cap\dots\cap\{\phi_J\le t_J\}=\emptyset, \tag{3.1.69}'$$

and skip condition (3.1.71) *completely, while getting statement* (3.1.34) *in addition to* (3.1.73) *and* (3.1.74)*.*

Proof of Theorem 3.1.7. Let conditions (3.1.20) and (3.1.28)–(3.1.31) be fulfilled. Then there obviously exists $\varepsilon>0$ depending on $u, t_1,\dots,t_J$ such that if $\psi=\psi(x_1)$ is a smooth function equal to 1 for $x_1\le\frac12\varepsilon$ and vanishing for $x_1\ge\varepsilon$ and if Q is a scalar h-pseudodifferential operator with symbol equal to 1 in the $\frac12\varepsilon$-neighborhood of $\Sigma(P)$ and vanishing outside of the ε-neighborhood of $\Sigma(P)$ then $U=\psi u+Q(1-\psi)u$ satisfies (3.1.69)–(3.1.72) with $\Omega_k''=\tilde\imath\Omega_k$ where $\tilde\imath: T^*X\to\mathbb{R}^+\times T^*X'$ is a natural map.

Then Theorem 3.1.13 applied to U yields (3.1.33) and even (3.1.34) (under condition $(3.1.9)^*$) for u. □

Proof of Theorem 3.1.13. Step 1

In this proof we can assume without any loss of the generality that (3.1.73) and (3.1.74) hold with s replaced by $s-\frac12$, i.e., that

$$\mathsf{WF}^{s-1}(\eth u_0)\cap\Omega'\cap\{\phi_1\le t_1\}\cap\dots\cap\{\phi_J\le t_J\}=\emptyset, \tag{3.1.75}$$

$$\mathsf{WF}^{s-\frac12,l+1\prime}(u_k)\cap\Omega_k''\cap\{\phi_{k1}\le t_1\}\cap\dots\cap\{\phi_{kJ}\le t_J\}=\emptyset \qquad \forall k=1,\dots,n. \tag{3.1.76}$$

Let us construct operators $Q_k\in\boldsymbol{\Psi}_{h,N}(X,\mathbb{C})$ in the same manner as Q_3 in the proof of Theorem 2.1.2, i.e., let us set

$$Q_k=\mathsf{Op}^{\mathsf{w}}\bigl(\zeta_k(x,\xi')\prod_{1\le j\le J}\chi(\phi_{kj}(x,\xi'))\bigr) \tag{3.1.77}$$

where the real-valued function χ satisfies (2.1.8); for the sake of simplicity we assume that $t_1 = ... = t_J = 0$, ϕ_{kj} are replaced everywhere by $\phi_{kj} - \frac{1}{4}\varepsilon$, $\varepsilon > 0$ is small enough, $\zeta_k \in \mathscr{C}_0^N(\mathbb{R}^+ \times T^*X')$, $0 \le \zeta_k \le 1$, ζ_k are supported in the $\frac{1}{2}\varepsilon$-neighborhood of Ω_k'' and $\zeta_k = 1$ in the $\frac{1}{3}\varepsilon$-neighborhood of Ω_k'', and $\zeta_1 = ... = \zeta_n = \zeta$ on $T^*\partial X$.

Let us consider the following identities similar to (2.1.10):

$$-\operatorname{Re} i([P_k, Q_k']u_k, u_k) = \operatorname{Re} i((Q_k' + Q_k'^*)P_k u_k, u_k) + \operatorname{Re} i((P_k - P_k^*)Q_k' u_k, u_k) + \operatorname{Re} h(Q'^* T_k u_k, u_k)_{\partial X} \tag{3.1.78}$$

where $T_k = A_{0k}|_{\partial X}$ (now T is an operator with the principal symbol T_0 given by (3.1.5), so we have changed notation slightly), $Q_k' = Q_k^2$ and the index k in the last term is skipped because all the operators Q_k coincide at ∂X due to condition (3.1.27).

Repeating the arguments of the proof of Theorem 2.1.2 with the obvious modifications we obtain the following inequalities similar to (2.1.11):

$$|-\operatorname{Re} i([P_k, Q_k']u_k, u_k) - \operatorname{Re}(Q'^* T_k u_k, u_k)_{\partial X}| \le C_2 \|Q_k u_k\|^2 + Ch^{2s}. \tag{3.1.79}$$

Let us note that the principal symbol of the operator $-ih^{-1}[P_k, Q_k']$ equals

$$\sum_{1\le j\le J} 2\{p_k, \phi_{kj}\} r_{kj}^2 - \{p_k, \zeta_k\} \prod_{1\le j\le J} \chi(\phi_{kj})^2$$

where $r_{kj} = \zeta\chi_1(\phi_{kj})\chi_2(\phi_{kj}) \prod_{i\ne j} \chi(\phi_{ki})^2$ and the functions χ_1, χ_2 are defined in (2.1.11).

Then similarly to (2.1.17) we obtain the inequalities

$$\Big|\sum_{1\le j\le J} 2(S_{kj}R_{kj}u_k, R_{kj}u_k) - \operatorname{Re}(Q'^* T_k u_k, u_k)_{\partial X}\Big| \le C_2\|Q_k u_k\|^2 + C_2 h \|Q_k u_k\|_{\partial X}^2 + Ch^{2s} \tag{3.1.80}$$

where $R_{kj} = \operatorname{Op}(r_{kj})$, $S_{kj} = \operatorname{Op}^{\mathrm{w}}(\{p_k, \phi_{kj}\})$ and a boundary term (the second term) in the right-hand expression appears because the symbols $\{p_k, \phi_{kj}\}$ depend on ξ_1.

The microhyperbolicity of the symbols p_k in the directions $\nabla^{\#}\phi_{kj}$ yields the existence of symbols q_k'' vanishing in the ε-neighborhoods of the supports

of r_{kj} for all k, j such that

$$\begin{aligned}(3.1.81)\quad &\langle(\{p_k,\phi_{kj}\}+cp_k^\dagger p_k+(q_k''-\epsilon_0)I)w,w\rangle\ge 0\\ &\forall w\in\mathbb{H}_k\quad \forall(x,\xi')\in\mathbb{R}^+\times T^*X'\qquad \forall j=1,\dots,J\quad \forall k=1,\dots,n.\end{aligned}$$

Then passing from symbols to operators and applying the Gårding inequality we obtain an inequality similar to (2.1.19):

$$(3.1.82)\quad \mathsf{Re}(S_{kj}v,v)+c\|P_kv\|^2+\mathsf{Re}(Q_k''v,v)\ge\epsilon_0\|v\|^2-C_2h\|v\|^2_{\partial X}-Ch\|v\|^2.$$

More precisely, we obtain this inequality first only for v with $\eth v=0$; in order to get rid of this restriction we substitute $v-e^{-h^{-1}x_1}\eth v$ for v in (3.1.82) to derive (3.1.82) in the general case.

Substituting $v=R_{kj}u_k$ into (3.1.82) and repeating the arguments of the proof of Theorem 2.1.2 we obtain the inequalities

$$(3.1.83)\qquad \mathsf{Re}(S_{kj}R_{kj}u_k,R_{kj}u_k)\ge\epsilon_0\|R_{kj}u_k\|^2-C_2h\|R_ju_k\|^2_{\partial X}-Ch^{2s}$$

where the index k is omitted in the second term in the right-hand expression since R_{kj} on ∂X do not depend on k.

Proof of Theorem 3.1.13. Step 2

So far we have been just repeating the arguments of the proof of Theorem 2.1.2. However now we need some new ideas:

Proposition 3.1.14. *Let conditions* (3.1.1)–(3.1.6) *with* $m=1$ *and* (3.1.46) *be fulfilled and let us assume that* $\det p_{kj}(z^*,\eta)$ *has only one complex root and this root is real (namely,* $\eta_k\in\mathbb{R}$*). Then*

(i) For every $\epsilon>0$ *there exists a neighborhood* Ω''' *of* z^* *such that if* $\mathcal{Q}$ *is an operator with principal symbol* $\mathcal{Q}^0$ *supported in* Ω''' *and satisfying the inequality* $|\mathcal{Q}^0|\le 1$ *then the following estimate holds:*

$$(3.1.84)\qquad h\|\mathcal{Q}v\|^2_{\partial X}\le\epsilon\|w\|^2+C_\epsilon\sum_{0\le i\le\bar l}\|D_1^iP_kv\|^2+Ch\sum_{0\le i\le\bar l+1}\|D_1^lv\|^2$$

where we omit the subscript "X*" as before and* $\bar l=2\mathsf{D}^*-1$ *where* D^* *is the maximal dimension of the Jordan cell in the Jordan normal form of* $L_{kj}(z^*)$*,* $L_{kj}=-a_{0kj}^{-1}a_{1kj}$*; the maximum is taken for all* $j=1,\dots,J_k$ *and all the Jordan cells.*

(ii) Moreover,

$$\|v(0)\|^2 \le 2\epsilon\|v\|_+^2 + C_\epsilon \sum_{0\le l\le \bar l} \|D_1^l p_k(z, D_1)v\|_+^2 \qquad \forall v \in \mathscr{S}(\mathbb{R}^+, \mathbb{H}_k) \qquad \forall z \in \Omega'''. \tag{3.1.85}$$

This proposition[6)] will be proven later (see Subsubsection *3.1.2.7 Proof of Proposition 3.1.14*). Let us now finish the proof of Theorem 3.1.13. Taking in (3.1.84) $\mathcal{Q}$ with symbol equal to 1 in a neighborhood of the support of the symbol of Q_k and using (3.1.70), (3.1.75) and (3.1.76) we upgrade (3.1.83) to the estimate

$$\operatorname{Re}(S_{kj}R_{kj}u_k, R_{kj}u_k) \ge \frac{1}{2}\epsilon_0\|R_{kj}u_k\|^2 + c_4 h\|R_j u_k\|_{\partial X}^2 - Ch^{2s} \tag{3.1.86}$$

with an arbitrarily constant c_4 which will be chosen later.

So the final estimate at this step in the proof is

$$-\operatorname{Re} ih^{-1}([P_k, Q_k']u_k, u_k) \ge \frac{1}{2}\epsilon_0 \sum_{1\le j\le J} \|R_{kj}u_k\|^2 + c_4 h \sum_{1\le j\le J} \|R_j u_k\|_{\partial X}^2 - Ch^{2s} \quad \forall k = 1, \dots, n; \tag{3.1.87}$$

thus we estimated from below the left hand expression of (3.1.78).

Let us now consider the fourth term in the right hand expression of (3.1.78) and sum with respect to $k = 0, \dots, n$; we obtain

$$-\sum_{0\le k\le n} \operatorname{Re}(Q'^* T_k u_k, u_k)_{\partial X}. \tag{3.1.88}$$

Definitions of Q and Q' and (3.1.76) imply immediately that modulo terms with absolute values not exceeding $C_2 h\|Qu\|_{\partial X}^2 + Ch^{2s}$ (where C_2 does not depend on the choice of c_3 and c_4) this expression equals

$$-\sum_{0\le k\le n} \operatorname{Re}(T_k Qu_k, Qu_k)_{\partial X} = -\operatorname{Re}(TQu, Qu)_{\partial X} + \operatorname{Re}(T_0 Qu_0, Qu_0)_{\partial X}. \tag{3.1.89}$$

[6)] Which implies that the boundary operator enters into microhyperbolicity condition only through nonreal roots.

Let us transform (3.1.89) modulo terms with absolute values not exceeding $C_2 h\|Qu\|^2_{\partial X} + Ch^{2s}$ [7]; in these transformations we will use assumption (3.1.76) repeatedly. Let us recall that in a neighborhood of z^*

$$S_0 = -T_0 - \beta^\dagger b - b^\dagger \beta$$

is a non-negatively defined symbol due to assumption (3.1.9). Hence

$$-\operatorname{Re}(TQu, Qu)_{\partial X} \equiv \operatorname{Re}(SQu, Qu)_{\partial X} + 2\operatorname{Re}(BQu, \mathcal{B}Qu)_{\partial X} \equiv \\ -\operatorname{Re}(SQu, Qu)_{\partial X} + 2\operatorname{Re}(Q[B, Q]u, \mathcal{B}u)_{\partial X}$$

because (3.1.69) implies $\operatorname{Re}(QBu, \mathcal{B}Qu) \equiv 0$ where $S = \operatorname{Op}^{\mathrm{w}}(S_0)$ and $\mathcal{B} = \operatorname{Op}^{\mathrm{w}}(\beta)$.

On the other hand, the structure of P_0 (see (3.1.45)) yields that

$$|(T_0 Qu_0, Qu_0)_{\partial X}| \le C_2 \|Qu_0'\|_{\partial X} \cdot \|Qu_0''\|_{\partial X};$$

moreover, (3.1.69) and Proposition 3.1.11 imply that $\|Qu_0''\|_{\partial X} \le Ch^{s+\frac{1}{2}}$ and therefore this expression is equivalent to 0. Thus

$$-\sum_{0\le k\le n} \operatorname{Re}(T_k Qu_k, Qu_k)_{\partial X} \equiv \\ \operatorname{Re}(SQu, Qu)_{\partial X} + 2\operatorname{Re}(Q[B, Q]u, \mathcal{B}u)_{\partial X}. \tag{3.1.90}$$

Let S' be an h-pseudodifferential operator on ∂X with a non-negatively defined Weyl symbol coinciding with that of S in the neighborhood of z^*; then the Gårding inequality yields that

$$\operatorname{Re}(S'v, v)_{\partial X} \ge -C_2\|v\|^2_{\partial X} \tag{3.1.91}$$

and hence

$$\operatorname{Re}(SQu, Qu)_{\partial X} \ge -C_2\|Qu\|^2_{\partial X} - Ch^{2s} \tag{3.1.92}$$

where we again have used the fact that the symbol of Q is supported in a small neighborhood of z^*.

Let us now consider the second term in the right-hand expression in (3.1.90). Calculating the principal symbol of the operator $Q[B, Q]$ we see that this term is equivalent to

$$\sum_{1\le j\le J} h(L_j R_j u, R_j u)_{\partial X}$$

[7] Where C_2 does not depend on the choice of c_3 and c_4.

where the operators R_j are defined above and L_j are operators with principal symbols

$$L_{j0} = i\beta^\dagger\{b, \phi_j\} - i\{b^\dagger, \phi_j\}\beta.$$

Then (3.1.26) implies that in a neighborhood of z^* the symbols

$$L_{j0} + c(S_0 + b^\dagger b + \Pi) - 2\epsilon_0 I$$

are non-negative definite where

(3.1.93) Π is the projector to the orthogonal complement of $\mathbb{H}'(z^*)$.

Therefore Gårding inequality again implies that

$$\operatorname{Re}(L_j v, v)_{\partial X} = c\operatorname{Re}(Sv, v)_{\partial X} + c\|Bv\|^2_{\partial X} + c\|\Pi v\|^2_{\partial X} \geq (2\epsilon_0 - Ch)\|v\|^2_{\partial X} - \|\mathcal{Q}' v\|^2_{\partial X}$$

where the symbol of $\mathcal{Q}'$ vanishes in some neighborhood of z^*.

Setting $v = R_j u$ and multiplying by h we obtain that

$$h\operatorname{Re}(L_j R_j u, R_j u)_{\partial X} + ch\operatorname{Re}(SR_j u, R_j u)_{\partial X} + ch\|BR_j u\|^2_{\partial X} + ch\|\Pi R_j u\|^2_{\partial X} \geq \epsilon_0 h\|R_j u\|^2_{\partial X} - Ch^{2s}. \tag{3.1.94}$$

Moreover, condition (3.1.69) implies that the third term in the left-hand expression is equivalent to 0 and the second term is equivalent to $ch\operatorname{Re}(S''Qu, R_j' u)_{\partial X}$ where $S'' = S' + C_2 h$ and R_j' is an operator with principal symbol $\zeta\chi_4(\phi_j)\prod_{k\neq j}\chi(\phi_k)$; here $\chi_4 = (\chi')^2/\chi = (2\chi_1')^2$ is a smooth function because of (2.1.8).

Then (3.1.91) and the Cauchy inequality yield that

$$h|(S''Qu, R_j' u)_{\partial X}| \leq h(S''Qu, Qu)^{1/2}_{\partial X} \cdot (S'' R_j' u, R_j' u)^{1/2}_{\partial X} \leq \frac{1}{2}\epsilon_0 \operatorname{Re} h(SQu, Qu)_{\partial X} + C_2 h\|Qu\|^2_{\partial X} + Ch^{2s}. \tag{3.1.95}$$

On the other hand, (3.1.69) and Proposition 3.1.11 yield that $\|R_j u_0''\|^2 \leq Ch^{2s}$ and hence

$$\|\Pi R_j u\|^2_{\partial X} \leq C_2 \sum_{1\leq k\leq n} \|R_j u_k\|^2_{\partial X} + \varepsilon\|R_j u_0'\|^2_{\partial X} + C_\varepsilon h^{2s}$$

with arbitrary small $\varepsilon > 0$ provided $\Omega' = \Omega'_\varepsilon$ is narrow enough.

Then (3.1.90), (3.1.92) and (3.1.94) imply that

$$(3.1.96)\quad -\sum_{1\le k\le n} \mathsf{Re}(Q'^{*}T_k u_k, u_k) \ge \epsilon_0 h \sum_{\le j\le J} \|R_j u_0'\|^2_{\partial X} - C_2 h \sum_{1\le j\le J,\, 1\le k\le n} \|R_j u_k\|^2_{\partial X} - C_2 h\|Qu\|^2_{\partial X} - Ch^{2s}.$$

This inequality combined with (3.1.79), (3.1.87) and (3.1.95) yields after an appropriate choice of constant c_4 that

$$(3.1.97)\quad \sum_{1\le j\le J}\Big(h\|R_j u\|^2_{\partial X} + \sum_{1\le k\le n}\|R_{jk}u_k\|^2\Big) \le C_2\Big(h\|Qu\|^2_{\partial X} + \sum_{1\le k\le n}\|Q_k u_k\|^2\Big) + Ch^{2s}.$$

This estimate is similar to estimate (2.1.18) in the proof of Theorem 2.1.2. Repeating the remaining arguments of the proof of Theorem 2.1.2 together with the choice of constant c_3 in (2.1.8) we finally obtain that (3.1.73) and (3.1.74) with $l = -1$ hold. Then (3.1.70) implies (3.1.74) in the general form.

Theorem 3.1.13(i) has beens proven.

Proof of Theorem 3.1.13. Step 3

Let us assume now that condition $(3.1.9)^*$ is fulfilled. Then one can replace inequality (3.1.92) by

$$(3.1.92)^*\qquad \mathsf{Re}(SQu, Qu) \ge (\epsilon_0 - Ch)\|Qu\|^2_{\partial X} - Ch^{2s}$$

(we assume that (3.1.34) holds with $(s - \frac{1}{2})$ instead of s). Moreover, we know that $\|Qu_0''\|_{\partial X} \le Ch^s$ because of Proposition 3.1.11 and we can weaken condition (3.1.69) to $(3.1.69)'$.

But in this case with no analysis of the principal symbols of the commutators we obtain instead of (3.1.95) the following stronger inequality:

$$(3.1.96)^*\qquad -\sum_{1\le k\le n} \mathsf{Re}(Q'^{*}T_k u_k, u_k) \ge \frac{1}{2}\epsilon_0\|Qu\|^2_{\partial X} - Ch^{2s}.$$

This inequality combined with (3.1.79) and (3.1.87) implies that

$$\|Qu\|^2_{\partial X} + \sum_{1\le j\le J,\ 1\le k\le n} \|R_{jk}u_k\|^2 \le C \sum_{1\le k\le n} \|Q_k u_k\|^2 + Ch^{2s}$$

and repeating the remaining arguments of the proof of Theorem 2.1.2 we obtain (3.1.73), (3.1.74) and (3.1.34).

Theorem 3.1.13(ii) has beens proven.

Proof of Proposition 3.1.14

Proof of Proposition 3.1.14. Obviously, it is sufficient to prove this proposition for $P = hD_1 - L$ with $L \in \boldsymbol{\Psi}'_h(X, \mathbb{H}, \mathbb{H})$ where $\mathbb{H} = \mathbb{C}^r$ and $L_0(z^*)$ has exactly one complex eigenvalue (namely, $\eta \in \mathbb{R}$); now L_0 (the principal symbol of L) is not necessarily Hermitian. Obviously, estimates (3.1.84) and (3.1.85) do not depend on the lower order terms; moreover, these estimates are stable with respect to small perturbations of L_0 for every fixed $\epsilon > 0$. Therefore it is sufficient to prove them only in the case of constant $L = L_0(z^*)$.

Then (3.1.84) is reduced to (3.1.85) by a change of variables $x_{1\mathsf{new}} = h^{-1}x_1$. So we need to prove only (3.1.85) for $z = z^*$. Moreover, one can assume that $\eta = 0$; the general case is reduced to this one by the substitution $v = v_{\mathsf{new}} e^{i\eta x_1}$. We will apply the following

Lemma 3.1.15. *Let L be an $r \times r$ matrix such that $L^r = 0$. Then for every $\mu > 0$ there exist $r \times r$ matrices ρ_j with $j = 0, \dots, 2r-2$ such that*

$$\rho_j^\dagger = \rho_j \qquad \forall j, \tag{3.1.98}$$

$$-\sum_{0\le j\le 2r-2} \operatorname{Re}\langle \rho_j L^{j+1} w, w\rangle \ge \|Lw\|^2 - \mu\|w\|^2 \tag{3.1.99}$$

and

$$-\sum_{0\le k\le j\le 2r-2} \operatorname{Re}\langle i\rho_j L^{j-k} w, L^k w\rangle \le \mu\|w\|^2 \qquad \forall w \in \mathbb{H}. \tag{3.1.100}$$

This lemma will be proven later; now let us prove Proposition 3.1.14 in the reduced form. Let

$$P = D_1 - L, \qquad Q = \sum_{0\le k\le j\le 2r-2} \rho_j D_1^k L^{j-k};$$

then

$$PQ = \sum_{0\le j\le 2r-2} \rho_j (D_1^{j+1} - L^{j+1})$$

and hence

$$\operatorname{Re}\langle QPv, v\rangle_+ = -\sum_{0\le j\le 2r-2} \langle \rho_j L^{j+1} v, v\rangle_+ + \sum_{0\le j\le 2r-2} \langle \rho_j D_1^{j+1} v, v\rangle_+ =$$

$$-\sum_{0\le j\le 2r-2} \langle \rho_j L^{j+1} v, v\rangle_+ + \frac{1}{2} \sum_{0\le k\le j\le 2r-2} \operatorname{Re}\langle i\rho_j (D_1^{j-k} v)(0), (D_1^k v)(0)\rangle$$

where we have applied (3.1.98). Let us replace D_1^k and D_1^{j-k} in the last term by L^k and L^{j-k} respectively; then the absolute value of the error will not exceed

$$C\Big(\sum_{1\le j\le 2r-2} \|(D_1^{j-1} Pv)(0)\|^2 + \|v(0)\|^2\Big) \le \mu \|v\|_+^2 + C \sum_{0\le j\le 2r-2} \|D_1^j Pv\|_+^2$$

where the last inequality is due to the obvious estimates

$$\|v(0)\|^2 \le \|v\|_+ \cdot \|D_1 v\|_+ \quad \text{and} \qquad \|D_1 v\|_+ \le \|Pv\|_+ + C\|v\|_+.$$

Hence for every $\mu > 0$ the following estimate holds:

$$\sum_{0\le j\le 2r-2} \operatorname{Re}\langle -\rho_j L^{j+1} v, v\rangle_+ + \frac{1}{2} \sum_{0\le k\le j\le 2r-2} \operatorname{Re}\langle i\rho_j L^{j-k} v(0), L^k v(0)\rangle \le$$

$$\mu \|v\|_+^2 + C \sum_{0\le j\le r-2} \|D_1^j Pv\|_+^2.$$

These estimates combined with (3.1.100) imply (3.1.85). □

Proof of Lemma 3.1.15. By induction on $l = 1, \dots, r$ we will prove that for every l there exist Hermitian matrices ρ_j with $j = 0, \dots, 2l-2$ such that

$$(3.1.101)_l \quad -\sum_{0\le j\le 2l-2} \operatorname{Re}\langle \rho_j L^{j+1} w, w\rangle \ge \epsilon_l \|Lw\|^2 - \mu \|w\|^2 - C_{l\mu} \|L^l w\|^2$$

and

$$(3.1.102)_l \quad -\sum_{0\le k\le j\le 2l-2} \operatorname{Re}\langle i\rho_j L^{j-k} w, L^k w\rangle \le \mu \|w\|^2 + C_{l\mu} \|L^l w\|^2$$

$$\forall w \in \mathbb{H}$$

where $\epsilon_l > 0$ do not depend on arbitrarily small $\mu > 0$.

Then for $l = r$ we obtain the assertion of the lemma. On the other hand, $(3.1.101)_1$ with $\mu = 0$ and $(3.1.102)_1$ with arbitrarily small $\mu > 0$ hold for $\rho_0 = L - L^\dagger$.

Let $l = 1, \dots, r-1$; let us assume that $(3.1.101)_l$ and $(3.1.102)_l$ are fulfilled with some $\epsilon_l > 0$ and matrices ρ_j $(j = 0, \dots, 2l-2)$. Then $(3.1.101)_{l+1}$ and $(3.1.102)_{l+1}$ will be fulfilled with 2μ instead of μ and with $\epsilon_{l+1} = \frac{1}{2}\epsilon$ provided

$$(3.1.103)_{l+1} \quad \operatorname{Re}\langle -\rho_{2l-1}L^{2l} - \rho_{2l}L^{2l+1})w, w\rangle \geq \\ -\frac{1}{2}\epsilon_l\|Lw\|^2 - \mu\|w\|^2 + C_{l,\mu}\|L^{2l}w\|^2 - C'_{l+1,\mu}\|L^{2l+2}w\|^2$$

and

$$(3.1.104)_{l+1} \quad \operatorname{Re}\langle(-\rho_{2l}L^l w, L^l w\rangle \leq \mu\|w\|^2 - C_{l,\mu}\|L^l w\|^2 + C'_{l+1,\mu}\|L^{l+1}w\|^2 \\ \forall w \in \mathbb{H}.$$

Indeed, let us consider all the terms in the left hand expression in $(3.1.102)_{l+1}$ which are both new in comparison with $(3.1.102)_l$ and different from $(3.1.104)_{l+1}$; obviously the absolute values of these terms do not exceed $\varepsilon\|w\|^2 + C''_{;+1,\mu,\varepsilon}\|L^{l+1}w\|^2$ where $\varepsilon > 0$ is arbitrarily small, and the constant $C_{l,\mu}$ is fixed but the constant $C'_{l+1,\mu}$ is not fixed yet.

Therefore we need to prove only that for every $\varepsilon > 0$ and $l < r$ there exist matrices $\alpha = -\alpha^\dagger$ and $\beta = -\beta^\dagger$ such that

$$-\operatorname{Re}\langle\alpha L^{2l}w, w\rangle - \operatorname{Re}\langle\beta L^{2l+1}w, w\rangle \geq -\varepsilon\|w\|^2 + \|L^{2l}w\|^2 - C_\varepsilon\|L^{2l+2}w\|^2$$

and

$$\operatorname{Re}\langle i\beta L^l w, L^l w\rangle \geq -\varepsilon\|w\|^2 + \|L^l w\|^2 - C_\varepsilon\|L^{l+1}w\|^2 \qquad \forall w \in \mathbb{H}.$$

Finally, let us note that matrices $\alpha = L^{2l} - L^{\dagger 2l}$ and

$$\beta = 4\varepsilon^{-1}(L^{2l+1} - L^{\dagger 2l+1}) - i(I - L^\dagger L)$$

satisfy these inequalities. □

Proof of Proposition 3.1.5

Obviously it is sufficient to prove this proposition for the reduced operator. Moreover, it is sufficient to check only condition (ii) in Definition 3.1.4.

The arguments of the proof of Theorem 3.1.7 imply that if the operator P_k is microhyperbolic at (every) point $z_k \in \iota^{-1}z^*$ in the direction (ℓ', ν_k) then for every $t > 0$ the following estimate holds

$$-\operatorname{Re}\langle((\ell' p_k)(z^*, D_1)v, v\rangle_+ \geq \epsilon_0\|v\|_+^2 + t\|v(0)\|^2 - C_t \sum_{0\leq j\leq l} \|D_1^j p_k(z^*, D_1)v\|_+^2 \qquad \forall v \in \mathscr{S}(\mathbb{R}^+, \mathbb{H}).$$

Hence the same estimate holds for all $z : |z - z^*| \leq \varepsilon(t)$. Then the estimate

$$-\operatorname{Re}\langle((\ell' p_k)(z^*, D_1)v, v\rangle_+ \geq \epsilon_0\|v\|_+^2 + t\|v(0)\|^2 \qquad \forall v \in \mathbb{H}'_k(z)$$

holds where $\mathbb{H}'_k(z)$ is the space of all solutions of the equation $p_k(z, D_1)v = 0$ decreasing at $+\infty$.

Therefore for large enough $t > 0$ and small enough $\varepsilon > 0$ estimate (3.1.26) at the point $z = z^*$ implies the same estimate at points $z : |z - z^*| \leq \varepsilon$. Furthermore, $\mathbb{H}'(z^*) = \mathbb{H}'_0$ in our decomposition and $\mathbb{H}'(z) = \mathbb{H}'_0 \oplus \mathbb{H}'_1(z) \oplus \cdots \oplus \mathbb{H}'_n(z)$.

Hence condition (ii) is stable with respect to perturbations of z and hence it is stable with respect to perturbations of P, B also. Moreover, this condition is obviously stable with respect to ℓ' as well.

3.2 Geometric Interpretation

3.2.1 Main Theorem

Notations and Definitions

This section is similar to Section 2.2. Let us remind that the set

$$\mathfrak{N}(p, b) = \big\{(z', \mathcal{T}) : z' \in T^*\partial X, \text{ and the problem } (P, B) \text{ is microhyperbolic at the point } z' \text{ in the multidirection } \mathcal{T}\big\} \tag{3.2.1}$$

is open in the sense of Proposition 3.1.5 and Remark 3.1.6. Moreover, obviously

(3.2.2) The set $\mathfrak{K}(p,b,z') = \{\mathcal{T} : (z',\mathcal{T}) \in \mathfrak{N}(p,b)\}$ is convex for any $z' \in T^*\partial X$.

Let us introduce the set

(3.2.3) $K'(p,b,z') = \boldsymbol{\iota}\mathfrak{K}(p,b,z') =$
$\{\ell', \exists(\nu_1,...,\nu_n)$ such that $(\ell',\nu_1,...,\nu_M) \in \mathfrak{K}(p,b,z')\} \subset T_{z'}T^*\partial X$

where $\boldsymbol{\iota} : \mathcal{T} = (\ell,\nu_1,...,\nu_n) \to \ell$ is a natural map.

Since microhyperbolicity condition in Definition 3.1.4 consists of two conditions (i) and (ii) and the first condition does not include B while the second one does not contain $(\nu_1,...,\nu_n)$ we conclude that

$$K'(p,b,z') = K'(p,z') \cap K''(p,b,z') \tag{3.2.4}$$

where

(3.2.5) $K'(p,z') = \boldsymbol{\iota}\mathfrak{K}(p,z')$ and $\mathfrak{K}(p,z')$ is defined by (3.2.1) and (3.2.2) with the full microhyperbolicity condition replaced by condition (i) in Definition 3.1.4

while

(3.2.6) $K''(p,b,z')$ is the set of all directions $\ell' \in T_{z'}T^*\partial X$ satisfying condition (3.1.26).

Then

(3.2.7) $K''(p,b,z')$ is an open convex cone

and

$$K'(p,z') = \bigcap_{z_k \in \iota^{-1}z' \cap \Sigma} \pi K(p,z_k) \tag{3.2.8}$$

$$K'(p,b,z') = \bigcap_{z_k \in \iota^{-1}z' \cap \Sigma} \pi K(p,z_k) \cap K''(p,b,z') \tag{3.2.9}$$

where $\pi : T_{\iota^{-1}z'}T^*X \to T_{z'}T^*\partial X$ is a natural map.

Thus, $K''(p,b,z')$, $K'(p,z')$ and $K'(p,b,z')$ are subsets of $T_{z'}T^*\partial X$ while $K(p,z_k) \subset T_{z_k}T^*X$.

Now we can introduce the dual (with respect to bilinear form σ) cones $K''^{\#}(p,b,z')$, $K'^{\#}(p,z')$ and $K'^{\#}(p,b,z')$, all subsets of $T_{z'}T^*\partial X$ as well. Obviously

$$K'^{\#}(p,z') = \oplus \sum_{z_k \in \iota^{-1}z' \cap \Sigma} (K^{\#}(p,z_k) \cap T_{z'}T^*\partial X), \tag{3.2.10}$$

$$K'^{\#}(p,b,z') = \oplus \sum_{z_k \in \iota^{-1}\iota z' \cap \Sigma} (K^{\#}(p,z_k) \cap T_{z'}T^*\partial X) \oplus K'^{\#}(p,b,z'). \tag{3.2.11}$$

Finally, let

$$K^{\#}(p,b,z') = \oplus \sum_{z_k \in \iota^{-1}z' \cap \Sigma} K^{\#}(p,z_k) \oplus K'^{\#}(p,b,z') \tag{3.2.12}$$

(and one can introduce $K^{\#}(p,z')$ in the same way but without the last term).

Remark 3.2.1. (i) $K^{\#}(p,z_k)$, $K^{\#}(p,z')$ and $K^{\#}(p,b,z')$ are subsets of $T_{\iota^{-1}z'}T^*X$ while $K'^{\#}(p,z')$ and $K''^{\#}(p,z')$ are subsets of $T_{z'}T^*\partial X$.

(ii) Obviously these sets have the properties described in Lemma 2.2.3.

Without any loss of generality we can assume that $X = \{x_1 \geq 0\}$ locally. Further, let us assume that

(3.2.13) $\nabla^{\#}x_2 \in K^{\#}(p,z) \quad \forall z \in \bar{\Omega}$ such that $z \notin T^*X|_{\partial X}$ and
$\nabla^{\#}x_2 \in K^{\#}(p,b,z') \quad \forall z' \in \iota\bar{\Omega} \cap T^*\partial X$ where $\Omega \subset T^*X$ such that

$$z \in \Omega \cap T^*X|_{\partial X} \implies \iota^{-1}\iota z \in \Omega \tag{3.2.14}$$

This assumption is similar to (2.2.1).

Definition 3.2.2. Let $\Omega \subset T^*X$ be a domain satisfying (3.2.14). A *generalized bicharacteristic billiard* is a curve $\{z = z(t)\}$ in $\bar{\Omega} \cap T^*X$ (where $\bar{\Omega}$ is a closure of Ω) such that

(i) If $z(t) \notin T^*X|_{\partial X}$ for some t then $|z(t') - z(t'')| \leq C_0|t' - t''|$ for all $t', t'' : |t' - t| + |t'' - t| \leq \epsilon_0 \mathsf{dist}(z(t), T^*X|_{\partial X})$ where the constants $\epsilon_0 > 0$ and C_0 do not depend on $t, z(t)$;

(ii) In the general case $|z'(t') - z'(t'')| \le C_0|t' - t''|$ for all t', t'' where $z' = (x, \xi')$, $\xi' = (\xi_2, \dots, \xi_d)$;

(iii) For a.e. t if $z(t) \notin T^*X|_{\partial X}$ then the function $z(t)$ is differentiable at t and

$$\frac{dz}{dt} \in K^{\#}(p, z); \tag{3.2.15}$$

(iv) For a.e. t if $z(t) \in T^*X|_{\partial X}$ then the function $z'(t)$ is differentiable at t and

$$\frac{dz}{dt} \in K'^{\#}(p, b, z'). \tag{3.2.16}$$

We consider examples later. Here let us notice that generalized bicharacteristic billiards are the generalized bicharacteristics inside the domain (equation (3.2.15) coincides with (2.2.23)) but on the boundary they can jump along $\iota^{-1}\iota z$ and also experience other effects. Equivalent definition would be if we considered T^*X with points z_1, z_2 glued together iff $z_1, z_2 \in T^*X|_{\partial X}$ and $\iota z_1 = \iota z_2$.

Then one could rewrite (3.2.15), (3.2.16) as

$$\frac{dz}{dt} \in K^{\#}a(p, b, z). \tag{3.2.17}$$

where $K^{\#}(p, b, z) = K^{\#}(p, z)$ as $z \notin T^*X|_{\partial X}$ and $K^{\#}(p, b, z) = K^{\#}(p, b, \iota z)$ as $z \in T^*X|_{\partial X}$.

Definition 3.2.3. Let $\mathcal{K}^{\pm}(p, b, z, \Omega)$ be the union of all the generalized bicharacteristic billiards residing in $\bar{\Omega}$ and going out from the point $z \in \bar{\Omega}$ in the direction of increasing $\pm t$.

Finally, Main Theorem

Theorem 3.2.4. *Let $\Omega \Subset T^*X$, $\Omega' \Subset T^*\partial X$ and $\Omega|_{\partial X} = \iota^{-1}\Omega'$. Let (P, B) satisfy the conditions of Theorem 3.1.7 in Ω' and let p be symmetric in Ω.*

Further, let condition (3.2.13) *be fulfilled.*

(i) Let u *be admissible (i.e., satisfy* (3.1.20)*) and*

$$\mathsf{WF}^{s+1}(Pu) \cap \mathcal{K}^-(p,b,z,\Omega) = \emptyset, \tag{3.2.18}$$

$$\bigl(\mathsf{WF}_b^{s+1,l}(Pu) \cup \mathsf{WF}^{s+\frac12}(B\mathring{\partial}_h u)\bigr) \cap \iota\mathcal{K}^-(p,b,z,\Omega) = \emptyset, \tag{3.2.19}$$

and

$$\mathsf{WF}^{s}(u) \cap \partial\Omega \cap \mathcal{K}^-(p,b,z,\Omega) = \emptyset, \tag{3.2.20}$$

$$\mathsf{WF}_b^{s,0}(u) \cap \partial\Omega' \cap \iota\mathcal{K}^-(p,b,z,\Omega) = \emptyset. \tag{3.2.21}$$

Then

$$\mathsf{WF}^{s}(u) \cap \mathcal{K}^-(p,b,z,\Omega) = \emptyset, \tag{3.2.22}$$

$$\mathsf{WF}_b^{s,l+m}(u) \cap \iota\mathcal{K}^-(p,b,z,\Omega) = \emptyset. \tag{3.2.23}$$

(ii) Moreover, let condition $(3.1.9)^*$ *be fulfilled in* Ω'*. Then one can weaken the condition* (3.2.19) *replacing* $\mathsf{WF}^{s+\frac12}(B\mathring{\partial}_h u)$ *by* $\mathsf{WF}^{s}(B\mathring{\partial}_h u)$*:*

$$\bigl(\mathsf{WF}_b^{s+1,l}(Pu) \cup \mathsf{WF}^{s+\frac12}(B\mathring{\partial}_h u)\bigr) \cap \iota\mathcal{K}^-(p,b,z,\Omega) = \emptyset \tag{3.2.19}'$$

and skip condition (3.2.21) *completely and at the same time obtain in addition that*

$$\mathsf{WF}^{s-j}(D_1^j u|_{\partial X}) \cap \iota\mathcal{K}^-(p,b,z,\Omega) = \emptyset \qquad \forall j = 0,\dots,m+l-1. \tag{3.2.24}$$

Proof. The proof repeats without any essential modifications the proof of Theorem 2.2.10: first, we introduce $K^{\cdots}_{\rho\varepsilon}(\dots)$ versions of all sets $K^{\cdots}(\dots)$ introduced before and study their properties.

Further, using Theorem 3.1.7 we prove the counterpart of Proposition 2.2.7: namely, we consider point $z' \in \Omega$, and for arbitrarily small ρ,ε we establish Statements (i) and (ii) similar to those of Theorem 3.1.7 but with Ω' and Ω replaced by narrow vicinities of z' and $\iota^{-1}z'$ respectively and with $\{\phi_1 \le t_1\}\cap\dots\cap\{\phi_J \le t_J\}$ and $\{\phi_{k1} \le t_1\}\cap\dots\cap\{\phi_{kJ} \le t_J\}$ replaced by $\bigl(z' - K'^{\#}_{\rho\varepsilon}(p,b,z')\bigr)$ and $\bigl(\iota^{-1}z' - K^{\#}_{\rho\varepsilon}(p,b,z')\bigr)$ respectively.

Furthermore, we introduce $\mathcal{K}^{\pm}_{\rho\varepsilon}(p,b,z')$ and study their properties. We prove the counterpart of Lemma 2.2.9 and based on the above counterpart of Proposition 2.2.7 and on Proposition 2.2.7 itself we establish Theorem 3.2.4 but with $\mathcal{K}^-(p,b,z)$ replaced by $\mathcal{K}^-_{\rho\varepsilon}(p,b,z)$.

Finally, taking $\rho \to +0$, $\varepsilon \to +0$ we prove that $\mathcal{K}^-_{\rho\varepsilon}(p,b,z,\Omega)$ tends to $\mathcal{K}^-(p,b,z,\Omega)$ which concludes the proof. We leave all obvious but tedious details to the reader. □

3.2.2 Discussion and Examples

General Remarks

Propagation of singularities near boundary is very complicated. Even analysis of the scalar operators of the principal type becomes really complicated due to jumping along $\iota^{-1}(z')$ and especially due to bicharacteristics tangent to the boundary. Even case of elliptic P is as complicated as the general boundaryless case case (see Subsubsection *3.2.2.4 Elliptic Points* below).

Remark 3.2.5. (i) Under condition $(3.1.9)^*$ $K''(p,b,z) = \mathbb{R}^{2d-2}$ and hence there is no propagation linked with the boundary conditions. One can prove it easily following the proof of Theorem 3.1.13. This would make results marginally simpler.

However in this book we are interested in the symmetric, not strictly dissipative boundary conditions. In Section 3.4 we will show that one can replace $(3.1.9)^*$ by a weaker condition (3.4.18) which dallows symmetric boundary conditions and has the same effect.

(ii) To ensure that boundary value problem is not overdetermined, we assume that

(3.2.25) B is $\bar{\mathsf{D}}_+ \times$ D-matrix operator.

Constant Coefficients Case

Example 3.2.6. As $X = \{x \in \mathbb{R}^d, x_1 \ge 0\}$ and $p = p(\xi)$, $b = b(\xi)$ we have that

$$\mathcal{K}^-(p,b,z) = \big(z - K^{\#}(p,b,z)\big) \cap T^*X \qquad \text{as } z \in T^*X|_{\partial X} \tag{3.2.26}$$

$$\mathcal{K}^-(p,b,z) = \big(z - K^{\sigma}(p,z)\big) \cap T^*X \ \cup \bigcup_{z' \in (z - K^{\sigma}(p,z))} \mathcal{K}^-(p,b,z') \qquad \text{as } z \in T^*X|_{X\setminus\partial X} \tag{3.2.27}$$

where $K(p,b,z) \subset 0 \times \mathbb{R}^d_\xi$ and $K^{\#}(p,b,z) \subset \mathbb{R}^d_x \times \iota^{-1}\iota z$.

Transversal Points

Let $z' \in T^*\partial X$ and $z_k \in \iota^{-1}z'$. Consider two very special cases:

$$K^{\sigma}(p,z_k) \setminus 0 \subset \{\pm x_1 > 0\}. \tag{3.2.28}_{\pm}$$

Definition 3.2.7. (i) As $(3.2.28)_+$ is fulfilled z_k is *outgoing point* ;

(ii) As $(3.2.28)_-$ is fulfilled z_k is *incoming point* .

(iii) As $m = 1$ (i.e. $P = A_0(x, hD')hD_1 + A_1(x, hD')$) and P has a block-diagonal form (3.1.45), the corresponding component of u is called *outgoing and incoming components* respectively.

Obviously,

(3.2.29) As $m = 1$ and P has a block-diagonal form (3.1.45), condition $(3.2.28)_\pm$ is equivalent to

$(3.2.30)_\pm$ $\qquad \pm a_k(z')$ is positive definite.

Let us denote by u', u'' and u''' collections of all outgoing, incoming and other components respectively. Let $\mathsf{D}', \mathsf{D}'', \mathsf{D}'''$ be dimensions of u', u'', u'''. Let us rewrite boundary operator B in the form $B = \begin{pmatrix} B' & B'' & B''' \end{pmatrix}$. Then (3.2.10) and $(3.2.30)_+$ imply that

$$\operatorname{rank} b' = \mathsf{D}' \tag{3.2.31}$$

and therefore one can assume without any loss of the generality that

$$B = \begin{pmatrix} I & B_1'' & B_1''' \\ 0 & B_2'' & B_2''' \end{pmatrix} \tag{3.2.32}$$

(otherwise we multiply it by an elliptic operator from the left). Then microlocally one can exclude outgoing components, consider reduced operator P and boundary operator B:

$$P = \begin{pmatrix} P' & 0 & 0 \\ 0 & P'' & 0 \\ 0 & 0 & P''' \end{pmatrix}, \qquad B = \begin{pmatrix} I & B_1'' & B_1''' \\ 0 & B_2'' & B_2''' \end{pmatrix} \tag{3.2.33}$$

where removed components are dimmed.

One can see easily that all conditions are fulfilled including (3.1.11), (3.2.25) and $(3.1.9)^{8)}$. As we determine (u'', u''') from the reduced problem,

[8)] Which may even become $(3.1.9)^*$; we leave to the reader to find out when it happens.

we can find $\eth u'$ from the remaining boundary conditions and then find u' for $x_1 > 0$ solving Cauchy problem for P' with the initial data as $\{x_1 = 0\}$.

Consider now incoming component u''. Assume that we know u'' as $\{x_2 \le t - \varepsilon, x_1 \ge \epsilon\varepsilon\}$ while $x_2(z') = t$; recall that we assume that P is microhyperbolic in direction $\nabla^{\#} x_2$ where $\epsilon > 0$ and $\varepsilon > 0$ are small enough constants. Then u' is determined in the small vicinity of $\iota^{-1} z'$ from the "Cauchy" problem with data at $\{x_2 \le t - \varepsilon\}$. Thus, we can exclude incoming trajectories as well:

$$P = \begin{pmatrix} P'' & 0 \\ 0 & P''' \end{pmatrix}, \qquad B = \begin{pmatrix} B_2'' & B_2''' \end{pmatrix} \tag{3.2.34}$$

(but due to the different reason).

One can see easily that all conditions are fulfilled including (3.1.11), (3.2.25) and (3.1.9).

Elliptic Points

Consider now *elliptic component* u_0 of u; in accordance with (3.1.45) we denote it by (u', u'') where $p_0'' = (p_0')^{\dagger}$ and without any loss of the generality one can assume that

$$a_0(z') = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad a_1(z') = \begin{pmatrix} 0 & -\alpha \\ -\alpha^{\dagger} & 0 \end{pmatrix} \tag{3.2.35}$$

where α satisfies (3.1.51).

Then due to Theorem 3.1.12 the equation $P_0'' u'' \equiv f''$ alone determines u'' (modulo $O(h^s)$). In a certain sense u'' is similar to incoming component of the previous subsubsection. However, while we can remove it from B, we cannot remove it from P without removing u' because then assumption $p^{\dagger} = p$ would be violated. Let us rewrite B as $B = \begin{pmatrix} B' & B'' & B''' \end{pmatrix}$ again where again u''' denotes remaining components and $\mathsf{D}', \mathsf{D}'', \mathsf{D}'''$ are dimensions of components. Assuming that

$$\operatorname{rank} b' = \mathsf{D}' \tag{3.2.36}$$

we can assume without any loss of the generality that B has form (3.2.32) and then we can determine $\eth u'$ from $\eth u'''$ and then we can remove both u'' and u' from P and from B reducing them to P''' and B_2'''. One can see easily that all conditions are fulfilled including (3.1.11), (3.2.25) and (3.1.9).

Knowing u''' we can find $\eth u'$ from the boundary conditions and then u' according to Theorem 3.1.12 solving Cauchy problem.

However, this reduction requires condition (3.2.36). Assume now that $\mathsf{D}''' = 0$. Then what we have is $B'\eth u' = 0$. Due to Theorem 3.1.12 equation $P'u \equiv f'$ has no effect on $\eth u'$ and therefore in this case analysis of propagation for (P, B) is reduced to the analysis of propagation for B'.

Condition (3.3.17) below implies that $\beta' b'$ is symmetric while (3.1.9) implies that $\beta' b'$ is dissipative[9]; here β' is an elliptic symbol. Further, condition (3.1.9)* clearly implies that $\beta' B'$ is strictly dissipative and thus elliptic.

Finally microhyperbolicity condition for (P, B) is equivalent to microhyperbolicity condition for $\beta' b'$.

Remark 3.2.8. The boundary waves are often called surface waves and they are $O(h^s)$ on the distances $\geq C_s h^{-1}|\log h|$ from the boundary.

Example 3.2.9. The most famous example of the boundary waves in the elliptic zone are Rayleigh waves in the elasticity; see f.e. M. Taylor [4].

Bicharacteristics tangent to the boundary

Let us consider the simplest and the most interesting case. Assume that $\mathsf{D} = 2$, $\mathsf{D}_+ = 1$ (may be after removal transversal and removable elliptic components) . One can prove easily that in this case one can reduce p to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \xi_1 \pm \begin{pmatrix} I & 0 \\ 0 & r(x,\xi') \end{pmatrix} \tag{3.2.37}$$

with a scalar real-valued symbol $r(x,\xi')$ and then

$$g(x,\xi) = -\xi_1^2 + r(x,\xi'). \tag{3.2.38}$$

Further, let us assume that $g(x,\xi) = \det p(x,\xi)$ is of the real principal type (i.e. $g = 0 \implies |\nabla g| \geq \epsilon$) which now is equivalent to

$$|r(x,\xi') = 0| \leq \epsilon_0 \implies |\nabla_{x,\xi'} r(x,\xi')| \geq \epsilon_0 \tag{3.2.39}$$

[9] We call symbol ε *dissipative* if $i(\varepsilon - \varepsilon^\dagger)$ is nonnegative matrix and *strictly dissipative* if it is strictly positive matrix.

and also microhyperbolic with respect $\nabla^{\#}x_2$ which is equivalent to

$$|r(x,\xi')=0| \le \epsilon_0 \implies \pm\partial_{\xi_2} r(x,\xi') \ge \epsilon_0. \tag{3.2.40}_{\pm}$$

Then

$$\begin{aligned} T^*\partial X = \{(0,x',\xi')\} = \mathcal{H}\cup\mathcal{G}\cup\mathcal{G} \coloneqq \\ \{(x',\xi') : r(0,x',\xi')>0\}\cup\{(x',\xi') : r(0,x',\xi')=0\}\cup \\ \{(x',\xi') : r(0,x',\xi')<0\} \end{aligned} \tag{3.2.41}$$

where $\mathcal{H}$, $\mathcal{E}$ and $\mathcal{G}$ are called respectively *hyperbolic, elliptic, glancing zones*.

Analysis in the Hyperbolic Zone. In the hyperbolic zone $\mathcal{H}$ we have two transversal trajectories: an incoming trajectory with $\xi_2 = \pm r^{1/2}(x',\xi')$ and an outgoing trajectory with $\xi_2 = \mp r^{1/2}(x',\xi')$ (in the framework of $(3.2.40)_{\pm}$); reflection goes according to "incident angle = reflection angle" law.

Analysis in the Elliptic Zone. In the elliptic zone $\mathcal{E}$ propagation can happen only along boundary bicharacteristics. Namely, assuming that the boundary problem satisfies conditions (3.2.25) and (3.3.17) we have (after multiplication by an elliptic symbol) either

$$b(x'\xi') = \begin{pmatrix} i & \beta(x',\xi') \end{pmatrix} \tag{3.2.42}$$

or

$$b(x'\xi') = \begin{pmatrix} i\tilde{\beta}(x',\xi') & 1 \end{pmatrix} \tag{3.2.43}$$

where $\beta(x',\xi')$ and $\tilde{\beta}(x',\xi')$ are scalar real-valued symbols and the second case is not reduced to the first one near $\{\tilde{\beta}=0\}$.

Then in the elliptic zone the decaying (for $x_1>0$) solution of $p(0,x',D_1,\xi')u = 0$ is $u = e^{-(-r)^{1/2}x_1}(-i(-r)^{1/2},1)^T$ (up to a constant factor) with $r = r(0,x',\xi')$; so a prorogation of singularities for (P,B) is reduced there to a propagation for operator $Q(x',hD')$ on the boundary (applied to $v=\eth u_2$) and the principal symbol of Q is

$$q(x',\xi') = -(-r(0,x',\xi'))^{1/2} - \beta(x',\xi') \tag{3.2.44}$$

in the case (3.2.42). In the case (3.2.43)

$$q(x',\xi') = \tilde{\beta}(x',\xi')(-r(0,x',\xi'))^{1/2} + 1$$

and since we consider this case only near $\{\tilde{\beta} = 0\}$, operator Q is elliptic.

Therefore, in the case (3.2.42) the characteristic manifold of Q is

$$\Sigma_b \coloneqq \{(x',\xi') : \ \lambda(x',\xi') \coloneqq r(0,\xi,\xi) - \beta^2(x',\xi) = 0,\ \beta(x',\xi) > 0\} \tag{3.2.45}$$

and the propagation is going in the elliptic zone along the boundary along trajectories of $\lambda(x',\xi')$.

Analysis in the Glancing Zone. Consider now $(x',\xi') \in \mathcal{G}$. One can see easily that at such points

$$K^{\#}(p,b,(x',\xi')) \cap T^*X = \{\nabla^{\#} r(0,x',\xi')\}_{\mathsf{cone}}; \tag{3.2.46}$$

so we get generalized bicharacteristics going along the boundary: namely, bicharacteristics of $r(0,x',\xi')$ going along $\mathcal{G}$. We call them *boundary bicharacteristics*

The generalized bicharacteristic billiards of (P,B) thus should consist of those bicharacteristics and bicharacteristics of $r(x,\xi) - \xi_1^2$ and here we have two principally different cases:

Bicharacteristically Strongly Convex dDomain means that

$$x_1 = 0,\ g = 0,\ \{g,x_1\} = 0 \implies \{g,\{g,x_1\}\} < 0 \tag{3.2.47}$$

which is equivalent to

$$x_1 = 0,\ r(0,x',\xi') = 0 \implies \partial_{x_1} r(0,x',\xi') < 0. \tag{3.2.48}$$

In this case no bicharacteristic of g can pass through points of $\mathcal{G}$ and thus only boundary bicharacteristics are passing through $\mathcal{G}$. Further, along bicharacteristic billiards passing through points of $\mathcal{H}$ close to $\mathcal{G}$ symbol

$$\rho(x,\xi') \coloneqq r(x,\xi') - x_1 \cdot \partial_{x_1} r(0,x',\xi) = \xi_1^2 - x_1 \cdot \partial_{x_1} r(0,x',\xi) \tag{3.2.49}$$

preserves its magnitude (at any finite time interval); so these billiards make short jumps; the height (maximal value of the jump) is $\asymp \rho$, the length of

each jump and the incidence/reflection angles are $\asymp \rho^{1/2}$; thus the number of jumps during the finite time is $\asymp \rho^{-1/2}$.

There is a huge literature (see *Comments*). Let us mention only that it is referred as *whispering gallery* case and boundary bicharacteristics are called *gliding rays*.

Example 3.2.10. Consider

$$g(x,\xi) = \xi_2^2 - \xi_1^2 - x_1. \tag{3.2.50}$$

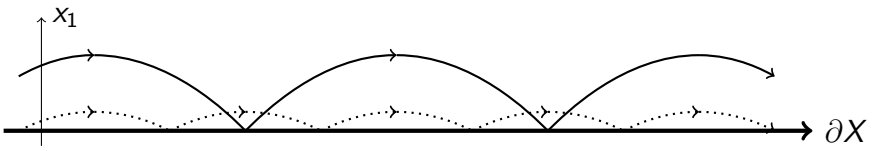


Figure 3.1: Bicharacteristically convex case: Example 3.2.10.

Bicharacteristically Strongly Concave Domain means that

$$x_1 = 0,\ g = 0,\ \{g, x_1\} = 0 \implies \{g, \{g, x_1\}\} > 0 \tag{3.2.51}$$

which is equivalent to

$$x_1 = 0,\ r(0, x', \xi') = 0 \implies \partial_{x_1} r(0, x', \xi') > 0. \tag{3.2.52}$$

In this case through each point $(x', \xi') \in \mathcal{G}$ passes both bicharacteristics of g and boundary bicharacteristics as well. The x-projection of bicharacteristic of g is tangent to ∂X; such bicharacteristics are called *grazing rays*.

Since boundary bicharacteristics resides in $\mathcal{G}$, we get that at every its point the generalized bicharacteristic billiard can leave the boundary along grazing ray. Such trajectories consisting of boundary bicharacteristics and may be grazing rays are called *creeping rays*.

Remark 3.2.11. (i) The truth is however that our results in this case are not sharp: singularities as we understand them (which are equivalent to smooth singularities) *do not* propagate along creeping rays. However analytic singularities *do* propagate along boundary bicharacteristics and creeping rays.

(ii) Further, Gevrey singularities with index $\kappa \le 3$ propagate along creeping rays while Gevrey singularities with index $\kappa > 3$ don't.

Namely, consider propagator $U(x,y,t)$ and assume that no other ray but a creeping ray only reaches x from y for time t; then $|U(x,y,t) = O(h^{\infty})$ and moreover $|\log u| \asymp sh^{-1/3}$ where $s \le t$ is a time spent by this ray creeping along boundary bicharacteristic. For more details see 3.5.3.

(iii) We will improve Theorems 3.1.7, 3.1.13 and thus 3.2.4 to exclude creeping rays.

Example 3.2.12. Consider

$$g(x,\xi) = \xi_2^2 - \xi_1^2 + x_1. \tag{3.2.53}$$

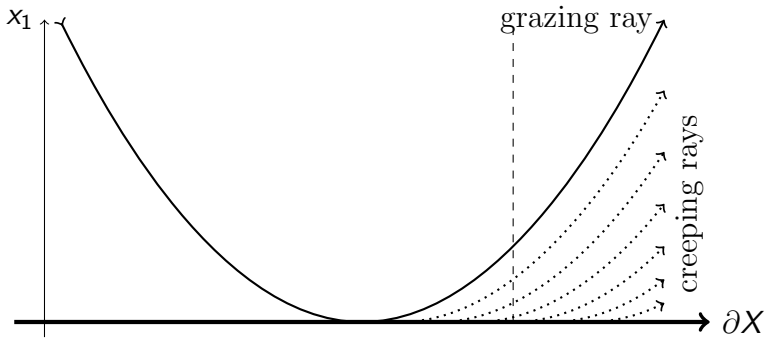
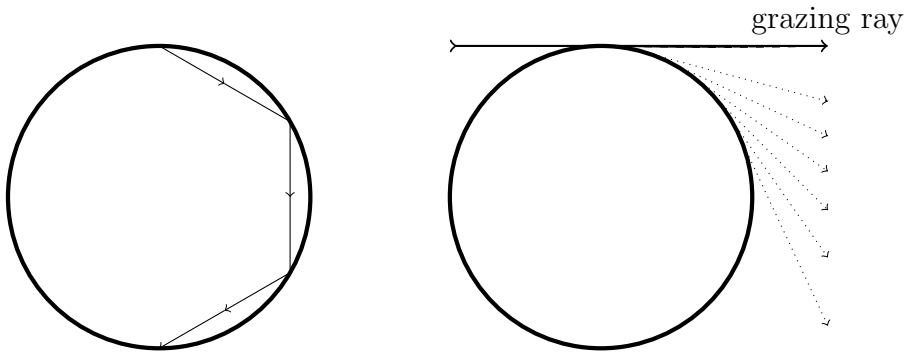


Figure 3.2: Bicharacteristically concave case: Example 3.2.10. Dotted lines show creeping rays and the dashed line shows the front of them at some moment

Remark 3.2.13. Consider $P = -\tau^2 + |\xi|^2$ in $X \times \mathbb{R}_t$ where $X \subset \mathbb{R}^d$ is a domain with the smooth boundary and (ξ,τ) are dual variables. Then notions of (strong) convexity and concavity coincide with ordinary geometric notions. On Figure 3.3 we consider interior (a) and exterior (b) of the circle; on (b) we also show front of the creeping rays "in the development".

We, however, distinguish *strong* and *strict* convexity and concavity; strong is a bit more restrictive as the boundaries of strictly convex and concave domains may be flat at some points (like $x_2 = x_1^4$).



(a) Convex case (b) Concave case

Figure 3.3: Straight rays and non-straight boundary

Domain which is neither Bicharacteristically Concave nor Bicharacteristically Convex. This is when the real troubles come. In this case bicharacteristic billiards can branch even if domain is just convex and the curvature $\{g,\{g,x_1\}\}$ vanishes at some points with all its derivatives.

3.3 Corollaries of Theorem 3.1.7

In this section we consider the following issues: the immediate corollaries of Theorem 3.1.7 like the finite speed of the propagation of singularities and the decomposition of a solution into two components (the free space solution and the reflected wave) and the structure in a neighborhood of $t=0$ of the restriction of the fundamental solution to the diagonal and the structure of some related functions (we are mainly interested in the *normality of the singularity*.

3.3.1 Finite Speed of Propagation and Related Topics

Finite Speed of Propagation

In this section we consider the operators

$$
\begin{aligned}
&P = hD_t + A, && B = (B_j)_{j=1,\dots,m},\\
&A \in \boldsymbol{\Psi}'^{m}_{h}(X,\mathbb{H},\mathbb{H}), && B_j \in \boldsymbol{\Psi}_h(\partial X,\mathbb{H}',\mathbb{H})
\end{aligned}
\tag{3.3.1}
$$

where $\boldsymbol{\Psi}'^{m}_{h,N}(X,\mathbb{H},\mathbb{H})$ is the class of operators of the form (3.1.1) with $A_k \in \boldsymbol{\Psi}'_{h,N}(X,\mathbb{H},\mathbb{H})$.

Surely Theorem 3.1.7 immediately yields an assertion about the finite speed of the propagation of singularities, but condition (3.1.23) of that theorem is not necessary for this special consequence, so we want to get rid of it and to prove the assertion directly.

Theorem 3.3.1. *Let P, B be operators of type* (3.3.1) *and let conditions* (3.1.4), (3.1.6) *and* (3.1.9) *be fulfilled in the domains $\Omega_+ \subset T^*(X'\times\mathbb{R}_t)\times\mathbb{R}^+_{x_1}$ and $\Omega \subset T^*(\partial X\times\mathbb{R}_t)$ respectively where $\Omega = \Omega_+ \cap \{x_1 = 0\}$; the matrix T is now defined by the operator A. Let us assume that*

$$(3.3.2)_1 \quad \|a_k(x,\xi')\| + \|\nabla_{x,\xi'}a_k(x,\xi')\| \le c \qquad \forall (t,x,\tau,\xi') \in \Omega_+ \quad \forall k = 0,\dots,m,$$

$$(3.3.2)_2 \quad \|\beta_k(x',\xi')\| + \|b_k(x,\xi')\| + \|\nabla_{x',\xi'}b_k(x,\xi')\| \le c \qquad \forall (t,x',\tau,\xi') \in \Omega \quad \forall k = 1,\dots,m$$

and

$$(3.3.2)_3 \quad |\tau| \le c \qquad \text{in } \Omega_+$$

where $a_k(x,\xi')$ are the principal symbols of A_k, $A = \Sigma_{0\le k\le m}A_k(hD_1)^k$. Let $N = N(d,m,s,M,l)$ and let u be temperate (i.e. satisfy condition (3.1.20)*). Then*

(i) If $\bar z = (\bar t,\bar x,\bar\tau,\bar\xi') \in \Omega_+$, $\bar t > 0$ and

$$(3.3.3) \qquad \mathrm{WF}^{s+1,l\prime}(Pu) \cap K_-(\bar z) \cap \Omega_+ = \emptyset,$$

$$(3.3.4) \qquad \mathrm{WF}^{s+\frac12}(B\eth_h u) \cap K_-(\bar z) \cap \Omega = \emptyset,$$

$$(3.3.5) \qquad \mathrm{WF}^{s,m+l\prime}(u) \cap K_-(\bar z) \cap \partial\Omega_+ = \emptyset$$

and if $K_-(\bar z)\cap\Omega_+$ is a bounded domain where

$$(3.3.6) \quad K_\pm(\bar z) = \{(t,x,\tau,\xi') \in T^*(X'\times\mathbb{R})\times\mathbb{R}^+ : \tau = \bar\tau, |x-\bar x| + |\xi'-\bar\xi'| \le \pm C_0(t-\bar t)\},$$

$C_0 = C_0(m,c)$, then

$$(3.3.7) \qquad \mathrm{WF}^{s,l\prime} \cap K_-(\bar z) \cap \Omega_+ = \emptyset.$$

(ii) Moreover, if condition $(3.1.9)^*$ *is fulfilled then one can replace superscript* $s+\frac{1}{2}$ *in* (3.3.4) *by* s *(see f.e.* $(3.1.28)'$*) and obtain at the same time that*

$$\mathsf{WF}^{s-j}(D_1^j u|_{\partial X}) \cap K_-(\bar z) \cap \Omega = \emptyset \qquad \forall j = 0, \dots, q+m-1. \tag{3.3.8}$$

Proof. It is sufficient to prove this theorem with $K_-(\bar z)$ replaced by $\{\phi \le 0\}$ for $\phi = t + \psi(x, \tau, \xi')$ for every smooth real-valued function ψ with

$$|\nabla_{x,\xi'}\psi| \le \varepsilon = \varepsilon(m, c) > 0; \tag{3.3.9}$$

the assertion modified in this way will be indicated by a prime.

Moreover, it is sufficient to consider the case $l = 0$ and one can assume that

$$\mathsf{WF}^{s-\frac{1}{2},m\prime} \cap \{\phi \le 0\} \cap \Omega_+ = \emptyset \tag{3.3.10}$$

and in the proof of (3.3.8) one can assume in addition that

$$\mathsf{WF}^{s-j-\frac{1}{2}}(D_1^j u|_{\partial X}) \cap \{\phi \le 0\} \cap \Omega = \emptyset \qquad \forall j = 0, \dots, m-1. \tag{3.3.11}$$

Let $q(t, x, \tau, \xi') = \zeta\chi(\phi - \delta)$ where $\zeta \in \mathscr{C}_0^\infty(\Omega_+)$ equals 1 in Ω_+ outside of the μ-neighborhood of $\partial\Omega_+$, χ is a function satisfying (2.1.8) and $\mu > 0$ is small enough. Let $Q = \mathsf{Op}^{\mathsf{w}}(q)$ and $Q' = Q^2$.

Let us apply the identity

$$\begin{aligned} -\operatorname{Re} i([P, Q']u, u) = \operatorname{Re} i((Q' + Q'^*)Pu, u) + \\ \operatorname{Re} i((P^* - P)Q'u, u) + \operatorname{Re} h(Q'^* T\eth_h u, \eth_h u)_{\partial X\times\mathbb{R}} \end{aligned} \tag{3.3.12}$$

(compare with (3.1.70)) where now the inner product $(.,.)$ is taken on $X \times \mathbb{R}$ and the matrix T is defined by (3.1.5); then after routine estimates we obtain the following inequality:

$$\begin{aligned} \|Q''u\|^2 \le \mathsf{Re}([A, \Psi]Q''u, Q''u) - \\ \mathsf{Re}\, ih(\mathcal{B}[B, \Psi]Q''\eth_h u, Q''\eth_h u)_{\partial X\times\mathbb{R}} + Ch^{2s} \end{aligned} \tag{3.3.13}$$

where Q'' and Ψ are pseudodifferentials operators with the principal symbols $q'' = \zeta\chi_1(\phi-\delta)\chi_2(\phi-\delta)$ and ψ respectively, χ_1 and χ_2 are defined in (2.1.8); moreover, under condition $(3.1.9)^*$ one can add the term $\epsilon_1\|Q''\gamma_h u\|^2_{\partial X\times\mathbb{R}}$ to

the left-hand expression in (3.3.13) where here and below $\epsilon_1 = \epsilon_1(m, c)$ and $C_1 = C_1(m, c)$.

On the other hand, (3.3.9) yields that the right-hand expression in (3.3.13) does not exceed

$$C\varepsilon \sum_{0\le j\le m} \|(hD_1)^j Q'' u\|^2 + Ch^{2s}$$

hence the theorem in question will be proven provided we prove the inequality

$$\sum_{0\le j\le m} \|(hD_1)^j Q'' u\| \le C_1 \|Q'' u\| + Ch^s.$$

This inequality follows from $(3.3.2)_3$ and the estimate

$$\sum_{0\le j\le m} \|(hD_1)^j v\| \le C_1\big(\|v\| + \|Av\| + \|\mathcal{Q}v\|\big) \tag{3.3.14}$$

where $\mathcal{Q} = \mathcal{Q}(x, D_x) \in \boldsymbol{\Psi}'^m_h(X, \mathbb{H}, \mathbb{H})$ is some operator with symbol vanishing in $\pi\Omega_+ \times \mathbb{R}_{\xi_1}$ where π is the map $(t, x, \tau, \xi) \to (x, \xi)$ and the norms are taken on X instead of $X \times \mathbb{R}$.

Estimate (3.3.14) is stable with respect to small variations of the principal symbol of A and arbitrary variations of the lower order terms and hence it is sufficient to prove the estimate

$$\sum_{0\le j\le m} \|(hD_1)^j v\|_+ \le C_1\big(\|v\|_+ + \|a(z, hD_1)v\|_+\big) \tag{3.3.15}$$
$$\forall z \in \pi\Omega_+ \quad \forall v \in \mathscr{S}(\mathbb{R}^+, \mathbb{H});$$

the previous estimate follows from this one by a partition of unity in $\pi\Omega_+$. In order to prove (3.3.15) we use the well known inequality

$$\|D_1^j v\|_+ \le C_{mj} \|D_1^m v\|_+^{\frac{j}{m}} \cdot \|v\|_+^{1-\frac{j}{m}} \qquad \forall j = 0, \dots, m; \tag{3.3.16}$$

this inequality yields that for every $\delta > 0$

$$\|\big(a(z, hD_1) - a_0(z)(hD_1)^m\big)v\|_+ \le \delta \|(hD_1)^m v\|_+ + C_\delta \|v\|_+$$

and hence condition (3.1.6) implies that

$$\|(hD_1)^m v\|_+ \le C_1\delta \|(hD_1)^m v\|_+ + C_\delta \|v\|_+ + C_\delta \|a(z, hD_1)v\|_+.$$

This inequality and (3.3.16) obviously imply (3.3.15). □

Let us replace condition (3.1.9) by the condition

$$\langle T(z)\underline{w},\underline{w}\rangle_m + 2\,\mathrm{Re}\,\langle \beta(z)b(z)\underline{w},\underline{w}\rangle = 0 \qquad \forall \underline{w}\in \mathbb{H}^m. \tag{3.3.17}$$

Such boundary problems are called *symmetric*.

Then one can replace t by $-t$, P by $-P$ and $K_-(\bar z)$ by $K_+(\bar z)$. Let us apply Theorem 3.3.1 to $u^{\pm} = \theta(\pm t)u$ instead of u and let us use the obvious analogue of Proposition 2.1.10; we then obtain

Theorem 3.3.2. *Let conditions* (3.3.1), (3.1.4), (3.1.6), (3.3.2) *and* (3.3.17) *be fulfilled and let u be temperate. Then if $\bar z = (\bar t, \bar x, \bar\tau, \bar\xi') \in \Omega_+$ and $\bar t > 0$,*

$$\mathsf{WF}^{s+1,l\prime}(Pu)\cap K_-(\bar z)\cap\{t\ge 0\}\cap\Omega_+ = \emptyset, \tag{3.3.18}$$

$$\mathsf{WF}^{s+\frac12,l\prime}(B\eth_h u)\cap K_-(\bar z)\cap\{t\ge 0\}\cap\Omega = \emptyset, \tag{3.3.19}$$

$$\mathsf{WF}^{s,m+l\prime}(u)\cap K_-(\bar z)\cap\{t\ge 0\}\cap\partial\Omega_+ = \emptyset \tag{3.3.20}$$

and

$$\mathsf{WF}^{s+1,l\prime}(Pu)\cap \pi(K_-(\bar z)\cap\{t=0\}\cap\Omega_+) = \emptyset, \tag{3.3.21}$$

$$\mathsf{WF}^{s+\frac12\prime}(B\eth_h u)\cap (\pi K_-(\bar z)\cap\{t=0\}\cap\Omega) = \emptyset, \tag{3.3.22}$$

$$\mathsf{WF}^{s,m+l\prime}(u|_{t=0})\cap \pi(K_-(\bar z)\cap\{t=0\}\cap\partial\Omega_+) = \emptyset \tag{3.3.23}$$

where in (3.3.21) *and* (3.3.22) *the wave front sets are calculated as wave front sets of functions on $X'\times(\mathbb{R}_t\times\mathbb{R}^+)$ and $\partial X\times\mathbb{R}_t$ respectively; then*

$$\mathsf{WF}^{s,m+l\prime}(u)\cap K_-(\bar z)\cap\{t\ge 0\}\cap\Omega_+ = \emptyset \tag{3.3.24}$$

This assertion remains true if we take $\bar t < 0$ and replace $K_-(\bar z)$ by $K_+(\bar z)$ and $\{t\ge 0\}$ by $\{t\le 0\}$.

Decomposition of the Fundamental Solution

At this point we assume that conditions (3.1.6) and $(3.3.2)_{1-3}$ are fulfilled for all $z\in T^*\mathbb{R}^d$. Let us assume that

(3.3.25) The operator A with domain

$$\mathfrak{D}(A) = \big\{u : D_1^j u\in \mathscr{L}^2(X,\mathbb{H}) \quad \forall j = 0,\dots,m \ \text{ and } \ B\eth_h u = 0\big\}$$

is a self-adjoint operator in $\mathscr{L}^2(X,\mathbb{H})$.

Let $U(t,x,y)$ be the Schwartz kernel of the operator $e^{-ith^{-1}A}$. Then

$$P_x U = 0, \tag{3.3.26}$$

$$B_x \eth_{hx} U = 0, \tag{3.3.27}$$

$$U\, {}^tP_y = 0, \tag{3.3.26)†}$$

$$U\, {}^t(B_y \eth_{hy}) = 0, \tag{3.3.27)†}$$

$$U|_{t=0} = \delta(x-y)I, \tag{3.3.28}$$

$$U(t,x,y) = U^\dagger(-t,y,x). \tag{3.3.29}$$

Then Theorem 3.3.2 immediately implies

Theorem 3.3.3. *Let conditions* (3.3.1), (3.1.4), (3.1.6), (3.3.17) *and* (3.3.25) *be fulfilled and* $N = N(d,m,M,s,l)$. *Then*

$$\mathsf{WF}^{s,l\prime}(U) \cap \{|\tau| \le c\} \subset \{|x-y| + |\xi' + \eta'| \le C_0|t|\} \tag{3.3.30}$$

where in the calculation of the wave front set U *is treated as a distribution on* $(\mathbb{R}_t \times X' \times X') \times \mathbb{R}^{+2}_{x_1,y_1}$.

Let us now assume that

(3.3.31) The operator A^0 with domain

$$\mathfrak{D}(A^0) = \bigl\{u : D_1^j u \in \mathscr{L}^2(X^0, \mathbb{H}) \quad \forall j = 0, \dots, m\bigr\}$$

is a self-adjoint operator[10)] in $\mathscr{L}^2(X^0,\mathbb{H})$, $X^0 = \mathbb{R}^d$ and coincides with A on X.

Let $U^0(t,x,y)$ be the Schwartz kernel of the operator $e^{-ith^{-1}A^0}$; then U^0 satisfies (3.3.26), (3.3.26)†, (3.3.28) and (3.3.29) with X replaced by X^0.

Then $U^1 = U - U^0$ satisfies equations (3.3.26), (3.3.26)†, (3.3.29) and

$$B_x \eth_{hx} U^1 = -B_x \eth_{hx} U^0, \tag{3.3.32}$$

$$U^1(\, {}^tB_y \eth_{hy}) = -U^0(\, {}^t(B_y \eth_{hy}), \tag{3.3.32)†}$$

$$U^1|_{t=0} = 0. \tag{3.3.33}$$

Hence $U^{1\pm} = \theta(\pm t)U^1$ satisfies (3.3.26), (3.3.26)† and (3.3.32), (3.3.32)† with U^0 replaced by $U^{0\pm}$. In particular, (3.3.26) and (3.3.26)† imply that

[10)] If condition (3.1.6) is fulfilled in the whole space X^0 then the operator is self-adjoint provided it is symmetric, i.e., provided its Weyl symbol is Hermitian.

(3.3.34) $U^{1\pm}$ is many times differentiable with respect to x_1, y_1.

Theorem 2.1.2 implies that for $y_1 > 0$

$$\mathsf{WF}^{s,l\prime}(\bar{\partial}_{hx} U^{0\pm}) \cap \{|\tau| \le c\} \subset \{|x'-y'| + |\xi'+\eta'| + y_1 \le \pm C_0 t\} \tag{3.3.35}$$

and hence Theorem 3.3.2 implies that

$$\mathsf{WF}^{s,l\prime}(U^{1\pm}) \cap \{|\tau| \le c\} \subset \{|x'-y'| + |\xi'+\eta'| + x_1 + y_1 \le \pm C_0 t\}; \tag{3.3.36}$$

moreover

(3.3.37) (3.3.36) holds as $y_1 \ge 0$.

Really, the only reason why we need this assumption is that otherwise $D_1^{m-1} U^{0\pm}$ has a jump as $x_1 = y_1$ and thus $\bar{\partial}_{hx} U^{0\pm}$ is not defined; however we can consider (3.3.28) with $\delta(x_1 - y_1)$ replaced by $\theta(x_1 - y_1)$ and get for modified $U^{1\pm}$ (3.3.36) with $l = 0$; then for modified $U^{1\pm}$ we get (3.3.36) with arbitrarily large l due to (3.3.26) and (3.3.26)†. Differentiating with respect to y_1 we return to original $U^{1\pm}$.

3.3.2 Normal Singularity Property

Preliminary Analysis

In this subsection we investigate the structure near $t = 0$ of the functions $\Gamma_x Q_{1x} U\, {}^t Q_{2y}$ and $\Gamma Q_{1x} U\, {}^t Q_{2y}$ [11] and some related functions where $Q_1, Q_2 \in \boldsymbol{\Psi}_h'^{m-1}(X \times \mathbb{R}, \mathbb{H}, \mathbb{H})$ are operators with symbols supported in a small neighborhood of $z^* \in T^*(\partial X \times 0)$. We now assume that condition (3.1.23) is fulfilled.

Let $\mathcal{R} \in \boldsymbol{\Psi}_h'^{m-1}(X \times \mathbb{R}, \mathbb{H}^m, \mathbb{H})$ be an operator implementing all the reductions of Subsubsection *3.1.2.1 Reduction to a Special Case*. Let us introduce $\underline{U} = \mathcal{R}_x U\, {}^t\mathcal{R}_y$ and let us consider its blocks $\underline{U}_{jk}$ with $j, k = 0, \dots, n$. Moreover, let us subdivide $\underline{U}_{00}$ into four subblocks

$$\underline{U}_{00} = \begin{pmatrix} \underline{U}_{00}^{(11)} & \underline{U}_{00}^{(12)} \\ \underline{U}_{00}^{(21)} & \underline{U}_{00}^{(22)} \end{pmatrix},$$

[11] Recall that $(\Gamma_x V)(x,t) = V(x,t)$ and $(\Gamma V)(t) = \int V(x,x,t)dx$.

and subdivide $\underline{U}_{0k}$ and $\underline{U}_{j0}$ with $j, k = 1, \dots, n$ into two subblocks

$$\underline{U}_{0k} = \begin{pmatrix} \underline{U}^{(1)}_{0k} \\ \underline{U}^{(2)}_{0k} \end{pmatrix}, \qquad \underline{U}_{j0} = \begin{pmatrix} \underline{U}^{(1)}_{j0} & \underline{U}^{(2)}_{j0} \end{pmatrix}.$$

Then

$$\Gamma_x Q_{1x} U\, {}^t Q_{2y} \equiv \sum_{0\le j,k\le n} \Gamma_x Q_{1,jx} \underline{U}_{jk}\, {}^t Q_{2,ky} \tag{3.3.38}$$

where $Q_{1,j}, Q_{2,j} \in \boldsymbol{\Psi}'_h(X \times \mathbb{R}, \mathbb{H}, \mathbb{H}_j)$ are operators with symbols supported in a small neighborhood of z^*.

Theorem 3.1.2, (3.3.26) and (3.3.26)† imply that

$$\mathsf{WF}^{s,l\prime\prime}(\underline{U}^{(2)}_{0k})^{\mathsf{T}} \cap z^* \times X'_y \times \mathbb{R}^{+2}_{x_1,y_1} = \emptyset, \tag{3.3.39}$$

$$\mathsf{WF}^{s,l\prime\prime}(\underline{U}^{(1)}_{j0})^{\mathsf{T}} \cap X'_x \times z^* \times \mathbb{R}^{+2}_{x_1,y_1} = \emptyset \tag{3.3.39}^*$$

where ${}^{\mathsf{T}}$ means that η is replaced by $-\eta$.

Therefore

(3.3.40) Assertion (3.3.36) remains true for $Q_{1,0x}\underline{U}_{0k}$, $\underline{U}_{j0}\, {}^t Q_{2,0y}$ and $Q_{1,0x}\underline{U}_{00}\, {}^t Q_{2,0y}$ replaced by $Q^{(1)}_{1,0x}\underline{U}_{0k}$, $\underline{U}^{(2)}_{j0}\, {}^t Q^{(2)}_{2,0y}$ and $Q^{(1)}_{1,0x}\underline{U}^{(12)}_{00}\, {}^t Q^{(2)}_{2,0y}$ respectively (where we use the obvious notation).

We need

Proposition 3.3.4. *The following formula holds:*

$$\Gamma'' Q_{1,x} U\, {}^t Q_{2,y} \equiv \sum_{1\le j\le n} \Gamma'' Q_{1,jx} \underline{U}_{jj}\, {}^t Q_{2,jy} + \sum_{0\le j,k\le n} h\Gamma_{x'} \mathcal{Q}_{jk}(V_{jk}), \tag{3.3.41}$$

where

$$V = \underline{U}|_{x_1=y_1=0}, \tag{3.3.42}$$

$$(\Gamma'_{x'} V)(x', t) = V(x', x', t), \qquad (\Gamma'' V)(x', t) = \int_{\mathbb{R}^+} V(x, x, t)\, dx_1. \tag{3.3.43}$$

and $\mathcal{Q}_{jk} \in \boldsymbol{\Psi}_h\big(\partial X \times \partial X \times \mathbb{R}^+, \mathcal{L}(\mathbb{H}, \mathbb{H}), \mathcal{L}(\mathbb{H}_j, \mathbb{H}_k)\big)$ *are operators with symbols supported in a small neighborhood of point* $z^{**} = (x^*, y^*, t^*, \xi^*, -\eta^*, \tau^*)$.

Proof. It is sufficient to prove that for $j \neq k$

$$\Gamma'' Q_{1,jx} \underline{U}_{jk}\, {}^t Q_{2,ky} \equiv h\mathcal{Q}_{jk}(V_{jk}); \tag{3.3.44}$$

the equality

$$\Gamma'' Q^{(1)}_{1,jx} \underline{U}^{(12)}_{00}\, {}^t Q_{2,ky} \equiv h\mathcal{Q}_{00}(V^{(12)}_{00}) \tag{3.3.45}$$

is proven in the same way.

Let us note that modulo functions negligible in the vicinity of z^{**} the following equalities are due to (3.3.26) and (3.3.26)†:

$$(hD_{x_1} - L_{jx})\underline{U}_{jk} \equiv 0, \tag{3.3.46}$$

$$\underline{U}_{jk}(-hD_{y_1} - {}^t L_{ky}) \equiv 0 \tag{3.3.47}$$

where $L_{jx} = L_j(x, hD'_x, hD_t) = -\underline{A}^{-1}_{0j}\underline{A}_{1j}$, $\underline{P}_j = \underline{A}_{0j} hD_1 + \underline{A}_{1j}$ are the diagonal blocks of the reduced operator and here M^{-1} means a parametrix of M in the neighborhood of z^* or z^{**}. Hence

$$h(D_{x_1} + D_{y_1})\underline{U}_{jk} \equiv L_{jx}\underline{U}_{jk} - \underline{U}_{jk}\, {}^t L_{ky} \tag{3.3.48}$$

and setting $W = \underline{U}_{jk}|_{x_1=y_1}$ we obtain that

$$hD_1 W \equiv \mathcal{N}_{jk}(W) \coloneqq L_{jx} W - W\, {}^t L_{ky}. \tag{3.3.49}$$

Let us consider operator $\mathcal{N}_{jk} = \mathcal{N}(x_1, x', y', hD_{x'}, hD_{y'})$ as an element of $\boldsymbol{\Psi}'_h\bigl(\mathbb{R}^+ \times X' \times X', \mathcal{L}(\mathbb{H}_j, \mathbb{H}_k), \mathcal{L}(\mathbb{H}_j, \mathbb{H}_k)\bigr)$.

Let us note that if λ_j and λ_k belong to the spectra of the principal symbols of L_j and L_k respectively with $j \neq k$ then $|\lambda_j - \lambda_k| \geq \epsilon > 0$ at z^{**}.

Therefore due to Appendix 3.A.1 the principal symbol of the operator $\mathcal{N}_{jk}$ is invertible at z^{**} and thus this operator is elliptic in a neighborhood of the point in question[12)]. Then (3.3.49) yields that

$$W \equiv {\mathcal{N}_{jk}}^{-1}(hD_1 W). \tag{3.3.50}$$

Hence

$$Q_{1,jx} W\, {}^t Q_{2,ky} \equiv -i\mathcal{Q}'_{jk}(hD_1 W).$$

[12)] The same remains true if we replace L_j and L_k by $L^{(1)}_0$ and $L^{(2)}_0$ respectively and hence all the forthcoming analysis also remains true.

Let us permute $\mathcal{Q}'_{jk} \in \boldsymbol{\Psi}'_h\bigl(X' \times X' \times \mathbb{R}^+, \mathcal{L}(\mathbb{H},\mathbb{H}), \mathcal{L}(\mathbb{H}_j,\mathbb{H}_k)\bigr)$ with hD_1 and substitute the right-hand expression of (3.3.40) instead of W into the term with the commutator. Moreover, let us repeat this procedure many times; we then obtain

$$Q_{1,jx} W\, {}^tQ_{2,ky} \equiv -ihD_1 \mathcal{Q}''_{jk}(W).$$

Integration on $x_1 \in \mathbb{R}^+$ and substitution $y' = x'$ gives us (3.3.44) with $\mathcal{Q}_{jk} = \mathcal{Q}''_{jk}|_{x_1=0}$. □

Main Results

Theorem 3.3.5. *Let P and B satisfy conditions* (3.3.1), (3.1.4), (3.1.6), (3.1.23), (3.3.17), (3.3.25) *and* (3.3.31). *Let us assume that the problem (P,B) is microhyperbolic at the point $z^* \in T^*(\partial X \times \mathbb{R})$, $t(z^*) = 0$ in the multidirection $\mathcal{T} = (\ell'_x, \ell'_\xi, \nu_1, \dots, \nu_n)$.*

Further, let $Q_1, Q_2 \in \boldsymbol{\Psi}'_h(X \times \mathbb{R}, \mathbb{H}, \mathbb{H})$ have symbols supported in a small enough neighborhood of point z^*. Finally, let $\chi \in \mathscr{C}_0^N([\frac12, 1])$ and $T > 0$ be a small enough constant, $N = N(d, m, l, s, \mathsf{D}^*)$ and let $|\alpha| \le l$, $|\beta| \le l$. Then

(i) If $\ell'_x = 0, \nu_1 = \dots = \nu_n = \nu$ then

$$\|\Gamma_x \chi_T(t) D_x^\alpha D_y^\beta Q_{1,x} U\, {}^tQ_{2,y}\| \le C' h^s \tag{3.3.51}$$

where here and below in this theorem C' depends on T.

(ii) If $\ell'_x = 0$ then

$$\|\Gamma'' \chi_T(t) D_x^\alpha D_y^\beta Q_{1,x} U\, {}^tQ_{2,y}\| \le C' h^s \tag{3.3.52}$$

and

$$\|\Gamma'_{x'} \eth_x \eth_y \chi_T(t) D_x^\alpha D_y^\beta Q_{1,x} U\, {}^tQ_{2,y}\| \le C' h^s. \tag{3.3.53}$$

(iii) If $\nu_1 = \dots = \nu_n = \nu$ then

$$\|\Gamma''' \chi_T(t) D_x^\alpha D_y^\beta Q_{1,x} U\, {}^tQ_{2,y}\| \le C' h^s \tag{3.3.54}$$

where $(\Gamma''' V)(x_1, t) := \int_{X'} (\Gamma_x V)(x_1, x', t)\, dx'$.

(iv) If $\ell'_x = \ell'_\xi = 0$ then

$$\|\eth_x \eth_y \chi_T(t) D_x^\alpha D_y^\beta Q_{1,x} U\, {}^tQ_{2,y}\| \le C' h^s. \tag{3.3.55}$$

(v) In the general case

$$\|\Gamma\chi_T(t)D_x^\alpha D_y^\beta Q_{1,x}U\,{}^tQ_{2,y}\| \le C'h^s \tag{3.3.56}$$

and

$$\|\Gamma'\eth_x\eth_y\chi_T(t)D_x^\alpha D_y^\beta Q_{1,x}U\,{}^tQ_{2,y}\| \le C'h^s. \tag{3.3.57}$$

Proof. Let us note that (iii) and (v) follow from (i) and (ii) respectively by means of the Fourier integral operator on x' with linear canonical transformation $\Psi : (x',\xi') \to \Psi(x',\xi')$ such that $(\Psi\ell')_\xi = 0$. Let us prove (i), (ii) and (iv).

Let $\underline{P}$ be a reduced operator and $\mathcal{R}$ an operator implementing this reduction. Let

$$\underline{U} = \mathcal{R}_x U\,{}^t\mathcal{R}_y, \quad \underline{U}^0 = \mathcal{R}_x U^0\,{}^t\mathcal{R}_y \quad \text{and} \quad \underline{U}^1 = \mathcal{R}_x U^1\,{}^t\mathcal{R}_y.$$

Then (3.3.26), (3.3.26)† and the ellipticity of P_0 yield that

$$\mathsf{WF}^{s,l'}(\underline{U}^0_{0k}) \cap \boldsymbol{\pi}(\Omega_+ \times \Omega_+^T) = \emptyset \qquad \forall k = 0,\dots,n, \tag{3.3.58}$$

$$\mathsf{WF}^{s,l'}(\underline{U}^0_{j0}) \cap \boldsymbol{\pi}(\Omega_+ \times \Omega_+^T) = \emptyset \qquad \forall j = 0,\dots,n, \tag{3.3.58†}$$

and

$$\mathsf{WF}^{s,l'}(\underline{U}^1_{0k}) \cap \boldsymbol{\pi}(\Omega_+ \times \Omega_+^T) \subset \{x_1 = 0\} \qquad \forall k = 0,\dots,n, \tag{3.3.59}$$

$$\mathsf{WF}^{s,l'}(\underline{U}^1_{j0}) \cap \boldsymbol{\pi}(\Omega_+ \times \Omega_+^T) \subset \{y_1 = 0\} \qquad \forall j = 0,\dots,n \tag{3.3.59†}$$

where Ω and Ω_+ are small enough neighborhoods of z^* in $T^*(\partial X \times \mathbb{R})$ and $T^*(X' \times \mathbb{R}) \times \mathbb{R}^+$ respectively and $\boldsymbol{\pi}$ is the map deleting the second copy of (t,τ) in $\Omega_+ \times \Omega_+^T$, etc.

Moreover, Theorem 3.1.2 yields that

$$\mathsf{WF}^{s,l'}(\underline{U}^{1(2)}_{0k}) \cap \boldsymbol{\pi}(\Omega_+ \times \Omega_+^T) = \emptyset \qquad \forall k = 0,\dots,n, \tag{3.3.60}$$

$$\mathsf{WF}^{s,l'}(\underline{U}^{1(1)}_{j0}) \cap \boldsymbol{\pi}(\Omega_+ \times \Omega_+^T) = \emptyset \qquad \forall j = 0,\dots,n. \tag{3.3.60†}$$

Let us note that Corollary 2.1.14 implies that the set

$$\mathsf{WF}^s(\underline{U}^0_{jk}) \cap \boldsymbol{\pi}(\Omega_+ \times \Omega_+^T) \times \mathbb{R}^2_{\xi_1,\eta_1} \cap \{|t| \le T\} \tag{3.3.61}$$

is contained in the C_0T-neighborhood of $\xi_1 = -\eta_1$. Furthermore, (3.3.26) and (3.3.26)† imply that this set is contained in a small neighborhood of

$\{\xi_1 = \zeta_j, \eta_1 = -\zeta_k\}$ (recall that ζ_j are characteristic roots) and hence this set(3.3.61) is empty for small $T > 0$:

$$\mathsf{WF}^s(\underline{U}^0_{jk}) \cap \boldsymbol{\pi}(\Omega_+ \times \Omega^T_+) \times \mathbb{R}^2_{\xi_1,\eta_1} \cap \{|t| \le T\} = \emptyset \qquad \forall j \ne k. \tag{3.3.62}$$

Moreover, Theorem 2.1.2 with $J = 2$ and $\phi_1 = t$ and

$$\phi_2 = \langle \ell'_\xi, x' - y' \rangle + \nu_k(x_1 - y_1) - \varepsilon t \tag{3.3.63}$$

with a small enough constant $\varepsilon > 0$ and Corollary 2.1.14 imply that

$$\begin{aligned} &\mathsf{WF}^s(\underline{U}^0_{kk}) \cap \boldsymbol{\pi}(\Omega_+ \times \Omega^T_+) \times \mathbb{R}^2_{\xi_1,\eta_1} \cap \\ &\qquad \bigl\{0 \le t \le T, \langle \ell'_\xi, x' - y' \rangle + \nu_k(x_1 - y_1) < \varepsilon t\bigr\} = \emptyset \quad \forall k = 1, \dots, n. \end{aligned} \tag{3.3.64}$$

Furthermore, (3.3.26), (3.3.26)† and the ellipticity of $\underline{P}_{jx}$ and ${}^t\underline{P}_{ky}$ for $|\xi_1| \ge c, |\eta_1| \ge c$ [13] combined with (3.3.62) and (3.3.64) imply that

$$\mathsf{WF}^{s,l\,\prime}(\underline{U}^0_{jk}) \cap \boldsymbol{\pi}(\Omega_+ \times \Omega^T_+) \cap \{|t| \le T\} = \emptyset \qquad \forall j \ne k \tag{3.3.62}'$$

and

$$\begin{aligned} &\mathsf{WF}^{s,l\,\prime}(\underline{U}^0_{kk}) \cap \boldsymbol{\pi}(\Omega_+ \times \Omega^T_+) \cap \\ &\qquad \bigl\{0 \le t \le T, \langle \ell'_\xi, x' - y' \rangle + \nu_k(x_1 - y_1) < \varepsilon t\bigr\} = \emptyset \quad \forall k = 1, \dots, n. \end{aligned} \tag{3.3.64}'$$

Therefore

$$\mathsf{WF}^{s,l\,\prime}(\eth_x \underline{U}^0_{jk}) \cap \boldsymbol{\pi}(\Omega \times \Omega^T_+) \cap \{|t| \le T\} = \emptyset \qquad \forall j \ne k, \tag{3.3.65}$$

and

$$\begin{aligned} &\mathsf{WF}^{s,l\,\prime}(\eth_x \underline{U}^0_{kk}) \cap \Omega \times \Omega^T_+ \cap \\ &\qquad \bigl\{0 \le t \le T, \langle \ell'_\xi, x' - y' \rangle - \nu_k y_1 < \varepsilon t\bigr\} = \emptyset \qquad \forall k = 1, \dots, n. \end{aligned} \tag{3.3.66}$$

Let us apply Theorem 3.1.13 with $J = 2$, $\phi_{j1} = \phi_1 = t$ and

$$\phi_{j2} = \langle \ell'_\xi, x' - y' \rangle + \nu_j x_1 - \varepsilon t, \qquad \phi_2 = \langle \ell'_\xi, x' - y' \rangle - \varepsilon t. \tag{3.3.67}$$

[13] Which is due to (3.1.6).

Then (3.3.65), (3.3.66), (3.3.26), (3.3.27) and (3.3.36) yield that

$$\mathsf{WF}^{s,l'}(\underline{U}^1_{jk}) \cap \boldsymbol{\pi}(\Omega_+ \times \Omega^T_+) \cap \bigl\{\{0 \le t \le T, \langle \ell'_\xi, x'-y'\rangle - \nu_k y_1 + \nu_j x_1 < \varepsilon t\bigr\} = \emptyset \quad \forall j,k = 1,\dots,n \tag{3.3.68}$$

and

$$\mathsf{WF}^{s,l'}(\underline{U}^1_{0k}) \cap \boldsymbol{\pi}(\Omega_+ \times \Omega^T_+) \cap \bigl\{0 \le t \le T, \langle \ell'_\xi, x'-y'\rangle - \nu_k y_1 < \varepsilon t\bigr\} = \emptyset \qquad \forall k = 1,\dots,n. \tag{3.3.69}$$

Similarly, (3.3.58)†, (3.3.26), (3.3.27) and (3.3.36) imply that

$$\mathsf{WF}^{s,l'}(\underline{U}^1_{j0}) \cap \boldsymbol{\pi}(\Omega_+ \times \Omega^T_+) \cap \bigl\{0 \le t \le T, \langle \ell'_\xi, x'-y'\rangle + \nu_j x_1 < \varepsilon t\bigr\} = \emptyset \qquad \forall j = 1,\dots,n \tag{3.3.70}$$

and

$$\mathsf{WF}^{s,l'}(\underline{U}^1_{00}) \cap \boldsymbol{\pi}(\Omega_+ \times \Omega^T_+) \cap \bigl\{0 \le t \le T, \langle \ell'_\xi, x'-y'\rangle < \varepsilon t\bigr\} = \emptyset. \tag{3.3.71}$$

These four relations combined with relations (3.3.62), (3.3.64) and (3.3.58) yield Statements (i), (ii) and (iv); moreover, in the proof of (3.3.52) we also use Proposition 3.3.4 which reduces contribution of the non-diagonal blocks to (3.3.53)-type expression. □

The arguments used in the proof of Theorem 2.1.19 yield immediately

Theorem 3.3.6. *Let conditions of assertions (i), (ii), (iii), (iv) or (v) of Theorem 3.3.5 be fulfilled. Then the left-hand expressions in* (3.3.51)*,* (3.3.52) *and* (3.3.53)*,* (3.3.54)*,* (3.3.55)*,* (3.3.56) *and* (3.3.57) *respectively do not exceed*

$$Ch^{1-d-|\alpha|-|\beta|}\bigl(\frac{h}{T}\bigr)^s \tag{3.3.72}$$

with constant C not depending on $T \in [h, T_0]$, where $T_0 > 0$ is a small constant.

For the forthcoming applications inequality (3.3.56) and its sidekick in Theorem 3.3.6 are the most important. Note that according to Proposition 3.3.4 the contributions of non-diagonal blocks to (3.3.52) and (3.3.56)

is given by (3.3.53) and (3.3.57)-type expressions respectively with an extra factor h. On the other hand, we do not need to integrate with respect to x_1 the contributions of the diagonal blocks but if we plug into these expressions U^1 instead of U and integrate we recover an extra factor T due to (3.3.36). Then we get immediately

Theorem 3.3.7. *Let conditions of assertions (ii) or (v) of Theorem 3.3.5 be fulfilled. Then the left-hand expressions in* (3.3.52), (3.3.56) *respectively with U replaced by U^1 do not exceed*

$$Ch^{2-d-|\alpha|-|\beta|}\bigl(\frac{h}{T}\bigr)^s \tag{3.3.73}$$

with constant C not depending on $T \in [h, T_0]$, where $T_0 > 0$ is a small constant.

In particular we get

Corollary 3.3.8. *Let conditions of assertions (ii) or (v) of Theorem 3.3.5 be fulfilled. Assume that P is elliptic at $\iota^{-1}z^*$. Then the left-hand expressions in* (3.3.52), (3.3.56) *respectively do not exceed* (3.3.73) *with constant C not depending on $T \in [h, T_0]$, where $T_0 > 0$ is a small constant.*

Really, due to the ellipticity of P U^0 is negligible in the vicinity in question. However it is not true for U^1 unless (P, B) is elliptic as well.

3.4 Energy Estimates Approach Revisited

The main purpose of this section is to prove theorem similar to Theorem 3.1.7 but showing that there is no propagation along creeping ways. In such theorem one needs to have function ϕ depending on ξ_1; in this case having separate function ϕ_j for each component does not make sense.

Then $\chi(\phi)$ would also depend on ξ_1; to avoid dealing with pseudodifferential operators containing hD_1 we however use Weierstrass theorem and rewrite this and other symbols as polynomials with respect to ξ_1 modulo $g(x,\xi)$; namely $\chi(\phi) = q(x,\xi)+g(x,\xi)\mu(x,\xi)$ where $g(x,\xi)$ is a determinant as usual and $q(x,\xi)$ is a polynomial with respect to ξ_1.

Since $g(x,\xi)I = \nu(x,\xi)p(x,\xi)$ then after quantization term an operator which applied to u we can properly bound. So we need to consider case when operator $Q(x,hD')$ we used before is replaced by

$$Q(x,hD) = \sum_{0\le j\le m} Q_k(x,hD')(hD_1)^j \tag{3.4.1}$$

which is a scalar symmetric operator as before.

Actually to a certain degree we did it before in Proposition 3.1.14.

Remark 3.4.1. Assume that in the vicinity of $(\bar{x}',\bar{\xi}') \in T^*\partial X$ operator P has a block form with different blocks corresponding to disjoint groups of characteristic roots $\xi_1 = \zeta_*(x,\xi')$ of $g(x,\xi_1,\xi')$; let $g_j(x,\xi)$ be corresponding characteristic polynomials; $j = 1,\dots,J$. Then

$$\phi(x,\xi) = \sum_{1\le j\le J} G_j(x,\xi)\phi_j(x,\xi), \qquad G_j(x,\xi) \coloneqq \prod_{k\ne j} g_k(x,\xi) \tag{3.4.2}$$

would deliver for each group its own function ϕ_j and therefore, at least in principle, we need just one function ϕ if we allow it to depend on ξ_1; conversely, if each group has just one real characteristic root $\xi_1 = \zeta_j(x,\xi')$ then our new approach does not deliver anything ne.

On the contrary, if $g_j(x,\xi)$ has several characteristic roots which cannot be separated (so, they coincide at $(\bar{x}',\bar{\xi}')$) our new approach separates them "mildly".

3.4.1 Theorem 3.1.7(ii) Revisited

To test this method we first prove statements (ii) of Theorems 3.1.7 and 3.1.13 under more general condition than (3.1.9)*.

Energy Estimates

To do this we replace our old $Q_k(x,hD')$ by $Q_{k,\mathsf{new}} = \alpha_k(x,hD)Q_k(x,hD')$ where an extra factor $\alpha(x,hD)$ is a polynomial with respect to hD_1

$$\alpha_k(x,hD) = \sum_{0\le j\le m} \bigl(\delta_{j0} + \alpha_{kj}(x,hD')\bigr)(hD_1)^j \tag{3.4.3}$$

and $\alpha_{kj}(x,\xi')$ are real-valued and $\mathsf{S}_{h,N}$-norms of α_{kj} do not exceed ε (which is a sufficiently small constant).

Then $Q_{kj,\mathsf{new}} \equiv \big(\delta_{j0} + \alpha_{kj}\big) Q_k \mod O(h)$ (with Q_k as in the proof of Theorem 3.1.13) and one can see easily that all the terms along the proof of Theorem 3.1.13 produce exactly the same result as in the proof of Statement (i) with the singular exception of the boundary term

$$h(A_{0k} Q_{k,\mathsf{new}} u_k, u_k)_{\partial X} \tag{3.4.4}$$

where index k indicate diagonal blocks of the corresponding operators and their symbols or components of u.

Really, the principal symbol of the commutator term is still $-\{p_k, q_k\}$ which is now replaced by

$$-\{p_k, \alpha_k q_k\} = -\alpha_k \{p_k, q_k\} - \{p_k, \alpha_k\} q_k. \tag{3.4.5}$$

Let us recall that in this proof $-\{p_k, q_k\}$ was basically replaced by $\chi_2^2(\phi_k)\{p_k, \phi_k\}$ with $\chi_2 = (-\chi')^{1/2}$ (all other terms were properly estimated) and then $\{p_k, \phi_k\}$ was replaced by a positive symbol plus a symbol of the form $\rho_k^\dagger p_k + p_k \rho_k$. This still is the case after multiplication by α_k because we can rewrite

$$\alpha_k(x, \xi) = \alpha'_k(x, \xi') + \tau_k(x, \xi) p_k(x, \xi), \tag{3.4.6}$$

$$\alpha'_k(x', \xi') = \sum_{0 \le j \le m} \alpha_{kj} (-a_{0k}^{-1} a_{1k})^j \tag{3.4.7}$$

and τ_k and $(\alpha'_k - I)$ are both small. On the other hand, rewriting $-\{p_k, \alpha_k\}$ in the form (3.4.6) we see that the only suspicious term is $-\alpha''_k q_k = -\alpha''_k \chi_1^2(\phi_k)$ where $\alpha''_k = \alpha''(x, \xi')$, $\chi_1 = \chi^{1/2}$ and we already know how to suppress such terms.

Further, boundary terms appearing from expression

$$(Q_{k,\mathsf{new}} u_k, P_k u) - (u_k, Q^*_{k,\mathsf{new}} P u_k) \tag{3.4.8}$$

are easy to estimate as well.

Therefore let us consider the mentioned above boundary term (3.4.4) and plug (3.4.6), (3.4.7) there; then obviously we need to consider only $h(T_k Q_k u_k, u_k)_{\partial X}$ where $T_k = T_k(x', hD')$ is an operator with the symbol

$$T_k(x', \xi') = \sum_{0 \le j \le m} \big(\delta_{j0} + \alpha_{kj}\big) a_{0k} (-a_{0k}^{-1} a_{1k})^j. \tag{3.4.9}$$

Further we can replace Q_k by $Q_k'^* Q_k'$ where Q_k' is an operator with the symbol $\chi_1(\phi)$. Then the boundary term becomes $h(T_k Q' u_k, Q' u_k)_{\partial X}$ (modulo terms we can easily estimate); we skip index k because we assume that functions ϕ_k (and thus operators Q_k, Q_k') as well) restricted to ∂X do not depend on k.

In virtue of arguments of Subsubsection *3.1.2.6 Proof of Theorem 3.1.13. Step 3* we conclude that

(3.4.10) Statements (ii) of Theorems 3.1.7 and 3.1.13 remain true provided

$$\langle Tw, w\rangle \le -\epsilon \|w\|^2 \qquad \forall w \in \operatorname{Ker} b \tag{3.4.11}$$

where T is a block-diagonal symbol with blocks defined by (3.4.9).

Obviously (3.1.9)* coincides with (3.4.11) with $\alpha_{kj} = 0$ $\forall j \ge 1$. Let us assume that the weaker condition (3.1.9) holds; then

$$a_0 = a_0^- + \beta^\dagger b + b^\dagger \beta \tag{3.4.12}$$

where a_0^- is non-positive definite matrix. So we want to construct small α_{kj} such that

$$\langle T'w, w\rangle \le -\epsilon \|w\|^2 \qquad \forall w \in \operatorname{Ker} b \cap \mathbb{K}_\epsilon \tag{3.4.13}$$

where T' is a block-diagonal symbol with blocks

$$T_k'(x', \xi') = \sum_{0\le j\le m} \alpha_{kj} a_{0k} (-a_{0k}^{-1} a_{1k})^j. \tag{3.4.14}$$

and $\mathbb{K}_\epsilon$ is an invariant subspace of a_0^- corresponding to eigenvalues in $[-\epsilon, 0]$.

Condition (3.4.13)

Obviously it is sufficient to establish (3.4.13) at one point; then it will be fulfilled in its vicinity. Further, one can skip assumption that α_{kj} are small since we can always can replace α_{kj} by $\varepsilon\alpha_{kj}$.

Note that

(3.4.15) One needs to satisfy (3.4.13) only for w with all incoming and outgoing components of w equal 0.

Indeed, for such components $\mp a_{0k} > 0$ and one can replace α_{0k} by $\alpha_{0k} \pm \epsilon$ without changing anything else.

Further, note that (3.4.13) implies (3.2.36) where $b = \begin{pmatrix} b' & b'' & b''' \end{pmatrix}$ with the reference to (3.2.35), (3.1.51). Really, if (3.2.36) is not satisfied one can pick up $w = \begin{pmatrix} w' & 0 & 0 \end{pmatrix}^T \in \mathsf{Ker}\, b$ and then $\langle Tw, w\rangle = 0$.

On the other hand, if condition (3.2.36) is satisfied we can exclude both components u', u'' as described in Subsubsection 3.2.2 and condition (3.4.13) will be fulfilled for the reduced problem (P, B) if it was fulfilled for the original one but only under extra assumption with $w' = w'' = 0$. Further, one can see easily that Statement (ii) of Theorem 3.1.13 holds for original problem (P, B) provided it holds for the reduced one.

Meanwhile, (3.4.13) for the original system fails unless in addition to (3.2.36) $\mathsf{rank}\, b'' = \mathsf{D}''$ (where we recall that $\mathsf{D}'' = \mathsf{D}'$) due to the same arguments as above. However, since our purpose is Statement (ii) of Theorem 3.1.13 rather than (3.4.13) itself, we do not need to add this twin of (3.2.36). Thus

(3.4.16) Only w with components corresponding to roots with $|\mathsf{Im}\, \zeta| \le \epsilon'$ equal 0 need to be considered.

Furthermore,

(3.4.17) One can replace in (3.4.9) $a_{0k}^{-1}a_{1k}$ by $(a_{0k}^{-1}a_{1k} - \eta_k)$ with any real (bounded) η_k.

Then adjusting α_{kj} one can fulfil (3.4.13). Thus, what we need is

(3.4.18) There exist real α_{kj} such that (3.4.11) is fulfilled for all $w \in \mathsf{Ker}\, b \cap \mathbb{K} \cap \mathsf{Ker}\, \pi$ where π is the a spectral projector for $a_0^{-1}a_1$ to subspace corresponding to non-real eigenvalues.

Examples

Example 3.4.2. Let us consider D-matrices of the form

$$a_0 = -\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix} \quad \text{and} \quad a_1 = \begin{pmatrix} 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 1 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix}; \tag{3.4.19}$$

i.e. $a_{0kl} = \delta_{\mathsf{D}+1-k-l}$, $a_{1kl} = \delta_{\mathsf{D}-k-l}$ [14].

One can see easily that elements of $a_0(-a_0^{-1}a_1)^j$ are $a_{jkl} = -\delta_{\mathsf{D}+1-j-k-l}$. Then

(i) Let $\mathsf{D} = 2n$. Then both positive and negative spectral subspaces of a_0 have dimension n and thus b should be $n \times \mathsf{D}$ matrix. Let us prove that

(3.4.20) (3.1.14) is fulfilled provided $\mathsf{rank}\, b_{II} = n$ with $b = \begin{pmatrix} b_I & b_{II}\end{pmatrix}$.

Indeed, selecting recurrently α_{2j+1} (from $j = 0$) and picking up $\alpha_{2j} = 0$ one can achieve

$$\langle T'w, w\rangle \le -\|w_I\|^2 - \varepsilon\|w_{II}\|^2 \tag{3.4.21}$$

with arbitrarily small $\varepsilon > 0$ where $w_I = (w_1, \dots, w_n)$, $w_{II} = (w_{n+1}, \dots, w_{\mathsf{D}})$ and thus we achieve (3.1.14) under condition $\mathsf{rank}\, b_{II} = n$.

On the other hand, since for any choice of α_j

$$\langle Tw, w\rangle = 0 \qquad \forall w : w_I = 0 \tag{3.4.22}$$

condition $\mathsf{rank}\, b_{II} = n$ is also necessary for (3.1.14).

(ii) Let $\mathsf{D} = 2n + 1$. Then positive and negative spectral subspaces of a_0 have dimensions n and $(n + 1)$ respectively and thus b should be $n \times \mathsf{D}$ matrix. Selecting recurrently α_{2j} (from $j = 0$) and picking up $\alpha_{2j+1} = 0$ one can achieve (3.4.21) where now $w_I = (w_1, \dots, w_{n+1})$, $w_{II} = (w_{n+2}, \dots, w_{\mathsf{D}})$. On the other hand, (3.4.22) holds. Therefore claim (3.4.20) holds in this case as well.

(iii) Let $\mathsf{D} = 2n + 1$ but a_0 is replaced by $-a_0$. Then positive and negative spectral subspaces of a_0 have dimensions $(n+1)$ and n respectively and thus b should be $(n + 1) \times \mathsf{D}$ matrix. Selecting recurrently α_{2j} (from $j = 0$) and picking up $\alpha_{2j+1} = 0$ one can achieve (3.4.21) in the same notations as in (ii). On the other hand, (3.4.22) holds. Therefore claim (3.4.20) holds in this case as well.

Example 3.4.3. In particular let $n = 1$, $\mathsf{D} = 2$. Then (3.1.9) and (3.4.18) together are equivalent to $b = (b_1\ 1)$ (or could be reduced to this form) with $\mathsf{Re}\, b_1 \le 0$.

[14] We use notation $\delta_l := \delta_{l,0}$.

One can easily consider the case when a_0 and a_1 are block-diagonal with blocks a_{0k} and $(a_{1k} + \eta_k a_{0k})$ of the types covered by example 3.4.2(i)–(iii).

Theorems 3.1.13(ii), 3.1.7(ii) and 3.2.4(ii) Generalized

So, we arrive to following generalization of Theorems 3.1.13(ii) and 3.1.7(ii):

Theorem 3.4.4. *Let conditions of Theorem 3.1.13 be fulfilled. Further, let boundary operator B satisfy* (3.1.9), (3.2.36) *and* (3.4.18).

Finally, let $\phi_j \in \mathscr{C}^N(T^*\partial X)$, $\phi_{kj} \in \mathscr{C}^N(\mathbb{R}^+ \times T^*X')$ be real-valued functions satisfying (3.1.27) *($j = 1, \dots, J$, $k = 1, \dots, n$). Let us assume that for every $j = 1, \dots, J$ the problem (P, B) is microhyperbolic at the point z^* in the multidirection $\mathcal{T}_j = (\nabla^{\#}\phi_j, -\partial_{x_1}\phi_{1j}, \dots, -\partial_{x_1}\phi_{nj})(z^*)$. Let $N = N(d, M, s, l)$ and $l \geq 2\mathsf{D}^* - 2$, $\mathsf{D}^* = \max_k \mathsf{D}_k$ where D_k are the dimensions of $\mathbb{H}_k$.*

Let u satisfy (3.1.20), (3.1.69)$'$, (3.1.70), (3.1.71), (3.1.72) *where $u = (u_1, \dots, u_n, u_0)$, Ω' and Ω''_k are small enough neighborhoods of z^* in $T^*\partial X$ and $\mathbb{R}^+ \times T^*X'$ respectively such that $\Omega' = \Omega''_k|_{\partial X}$ $\forall k = 1, \dots, n$.*

Then (3.1.34), (3.1.73) *and* (3.1.74) *hold.*

Theorem 3.4.5. *Let conditions of Theorem 3.1.13 be fulfilled. Further, let boundary operator B satisfy* (3.1.9), (3.2.36) *and* (3.4.18).

Furthermore, let u satisfy (3.1.20), (3.1.28)$'$ (3.1.29), (3.1.30) *and* (3.1.31) *where Ω' and Ω_k are small enough neighborhoods of z and z_k in $T^*\partial X$ and T^*X respectively not depending on $u, t_1, \dots, t_J$ such that $\Omega' = \Omega''_k|_{\partial X}$ $\forall k = 1, \dots n$.*

Then (3.1.33) *and* (3.1.34) *hold.*

Theorem 3.4.6. *Let conditions of Theorem 3.2.4 be fulfilled. Further, let boundary operator B satisfy* (3.1.9), (3.2.36) *and* (3.4.18). *Furthermore, let u satisfy* (3.2.18), (3.2.19)$'$ (3.2.20), *and* (3.2.21).

Then (3.2.22) *and* (3.2.24) *hold.*

3.4.2 Main Theorem

Discussion

We are interested in te case when $g = \det p$ is of the real principal type or more generally when p is block-diagonal with $p_k \otimes I$ blocks and $g_k = \det p_k$ of the real principal type. Then points where $\pm\{g_k, x_1\}(z_k) > 0$ are outgoing

and incoming points respectively and the corresponding components could be excluded from the analysis, we are interested in the tangent points z_k where $\{g_k, x_1\}$ vanishes. Then $(a_{0k}^{-1}a_{1k})(z)$ has Jordan cells of the height D_k as $z = \iota z_k$. We will prove (see Appendix 3.A.2) then that symbols a_{0k} and a_{1k} calculated at such points could be reduced to the types covered by example 3.4.2(i)–(iii).

This is still pretty complicated due to the interaction of the different components; so we start from the simplest case $n = 1$, $\mathsf{D} = 2$. Then modulo positive factor

$$g(x,\xi) = -\bigl(\xi_1 - \eta(x,\xi')\bigr)^2 + r(x,\xi'); \tag{3.4.23}$$

we have sign "$-$" since a_0 has two eigenvalues of different signs. Without any loss of the generality one can assume that $\eta = 0$ i.e.

$$g(x,\xi) = -\xi_1^2 + r(x,\xi'). \tag{3.4.24}$$

Anyway, points $(x'\xi') \in T^*\partial X$ with $r(0,x',\xi') > 0$, $r(0,x',\xi') = 0$ and $r(0,x',\xi') < 0$ are called *hyperbolic*, *elliptic* and *glancing* points respectively, and we are interested in the small vicinities of the glancing points. Microhyperbolicity with respect to $\nabla^{\#}x_2$ means exactly that

$$\varsigma\partial_{\xi_2} r(x,\xi') > 0 \tag{3.4.25}$$

with $\varsigma = \pm 1$ (depending on the structure of the system; see more precisely in Appendix 3.A.2.

Further, glancing points (i.e. those with $g = x_1 = \{g, x_1\} = 0$) could be of three types: *gliding* with $H_g^2 x_1 = -2\partial_{x_1} r < 0$ (recall that $H_g f = \{g, f\}$, see footnote [13)] from Chapter 2), *grazing* or *diffractive* with

$$H_g^2 x_1 = 2\partial_{x_1} r > 0, \tag{3.4.26}$$

and *degenerate* with $H_g^2 x_1 = -2\partial_{x_1} r = 0$.

We do not need any extra analysis in the gliding points but for diffractive points we need to prove that there is no propagation along boundary bicharacteristics. The results for degenerate points will follow.

So, we consider now only diffractive points.

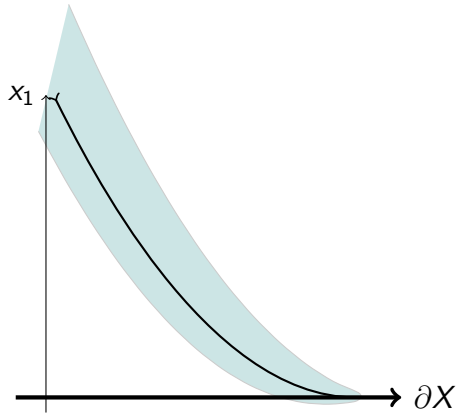


Figure 3.4: Diffractive case: $\{\phi \le 0\}$

To prove that singularities do not propagate along boundary bicharacteristics one needs to consider incoming bicharacteristics to a boundary point and chose function $\phi(x,\xi)$ such that $\{\phi < 0\}$ would be a narrow tongue around it and in the same time $\varsigma\{g,\phi\} > 0$. We already noticed that this is impossible if $\phi(x,\xi)$ does not depend on ξ_1 [15] So we need to take $\phi(x,\xi)$ depending on ξ_1. Due to Weierstrass theorem one needs to check ϕ as a linear function with respect to ξ_1 and from the heuristic point of view ϕ should not decrease at reflections from the boundary. However $\varsigma\xi_1 > 0$ at incoming and $\varsigma\xi_1 < 0$ at outgoing bicharacteristics; so we need to have

$$\varsigma\partial_{\xi_1}\phi < 0; \tag{3.4.27}$$

therefore adding $\phi_1(x,\xi')\xi_1$ to $\phi(x,\xi)$ with negative $\varsigma\phi_0$ helps to inequality $\varsigma\{g,\phi\} > 0$ by increasing $\varsigma\{g,\phi\}$ iff $\{g,\xi_1\} < 0$ which is exactly (3.4.26).

What actually we would like to prove in general is the following

Conjecture 3.4.7. *Let P be an operator with the Hermitean principal symbol p microhyperbolic in direction $\nabla^{\#}\xi_2$. Assume that*

$$p = \begin{pmatrix} p' & 0 \\ 0 & p'' \end{pmatrix} \tag{3.4.28}$$

Let $z' \in T^*\partial X$ and let $z_k \in \iota^{-1}z' \cap \mathsf{Char}(p')$, $k = 1,\dots,J$. Assume that

$$\mathcal{K}^-(p,z_k) \cap T^*X|_{\partial X} \cap \Omega_k = \{z_k\} \qquad \forall k = 1,\dots J. \tag{3.4.29}$$

Further, let boundary operator B satisfy (3.1.9), (3.2.36) *and* (3.4.18).

Furthermore, let u satisfy (3.1.20) *and*

$$z' \notin \mathsf{WF}^{s+1,l'}(Pu) \cup \mathsf{WF}^{s}(B\mathfrak{d}_h u), \tag{3.4.30}$$

$$\bigl(\mathcal{K}^-(p',z_k) \setminus z_k\bigr) \cap \mathsf{WF}^s(u) \cap \Omega_k = \emptyset \qquad \forall k = 1,\dots,J, \tag{3.4.31}$$

$$\mathcal{K}^-(p'',z') \cap T^*X|_{X\setminus\partial X} \cap \mathsf{WF}^s(u) \cap \Omega = \emptyset, \tag{3.4.32}$$

$$\bigl(\mathcal{K}^-(p'',z') \setminus \iota^{-1}z\bigr) \cap \bigl(\mathsf{WF}^{s,0\,\prime}(u) \cup \mathsf{WF}^s(u|_{\partial X})\bigr) \cap \Omega' = \emptyset \tag{3.4.33}$$

[15] Or under weaker condition "$\phi(0,x',\xi)$ does not depend on ξ_1".

where Ω' *and* Ω_k *are small enough neighborhoods of* z *and* z_k *in* $T^*\partial X$ *and* T^*X *respectively such that* $\Omega' = \Omega_k|_{\partial X}$ $\forall k = 1, \dots, J$.

Then

$$\bigl(\mathcal{K}^-(p'', z') \setminus \iota^{-1}z\bigr) \cap \bigl(\mathsf{WF}^{s,0\,\prime}(u) \cup \mathsf{WF}^s(u|_{\partial X})\bigr) \cap \Omega' = \emptyset. \tag{3.4.34}$$

We will be able to prove this conjecture under certain restrictions.

Symbol q: General Construction

In this subsubsection we consider the case $n = 1$ and Jordan cell of an arbitrary order m. Let us consider $\phi(x, \xi)$ which is the smooth and real-valued function as usual. We are interested only in ϕ on $\Sigma = \{g(x, \xi) = 0\}$; then due to Weierstrass theorem we can assume without any loss of the generality that

(3.4.35) $\phi(x, \xi)$ is a polynomial of degree $(m - 1)$ with respect to ξ_1.

Let consider $q(x, \xi) = \chi\bigl(\phi(x, \xi)\bigr)$. Again due to Weierstrass theorem we rewrite it as polynomial of degree $m - 1$ with respect to ξ_1 (modulo $g(x, \xi)$). Then slightly redefining $q(x, \xi)$ we get

$$q(x, \xi) = \sum_{0 \le j \le m-1} q_j(x, \xi')\xi_1^j = \chi\bigl(\phi(x, \xi)\bigr) + \mu(x, \xi)g(x, \xi). \tag{3.4.36}$$

Equality (3.4.36) reflects two "faces" of $q(x, \xi)$: the second one will be used to calculate commutator $\{p, q\}$.

$$\begin{aligned} -\{p, \chi(\phi)\}(x, \xi) = \chi_2^2\bigl(\phi(x, \xi)\bigr)\{p, \phi\}(x, \xi) + \mu'(x, \xi)g(x, \xi) = \\ \bigl(q''(x, \xi)\bigr)^2\{p, \phi\}(x, \xi) + \mu''(x, \xi)g(x, \xi) \end{aligned} \tag{3.4.37}$$

where $\chi_2 = (-\chi')^{1/2}$, $\chi_1 = \chi^{1/2}$ and here and below $q'(x, \xi)$, $q''(x, \xi)$ are polynomials of degree $(m - 1)$ with respect to ξ_1 constructed according to Weierstrass theorem from $\chi_1\bigl(\phi(x, \xi)\bigr)$ and $\chi_2\bigl(\phi(x, \xi)\bigr)$ respectively.

On the other, hand $g = Gp = pG$ where G is a matrix of cofactors and therefore

$$\{p, g\} = \{p, G\}p + \sum_j \bigl(p^{(j)}Gp_{(j)} - p_{(j)}Gp^{(j)}\bigr) \tag{3.4.38}$$

and

$$\{p,g\} = \frac{1}{2}\bigl(\{p,G\}p + p\{p,G\}\bigr) \tag{3.4.39}$$

with the matrix Poisson brackets are defined by usual formula $\{p,G\} = \sum_j \bigl(p^{(j)}G_{(j)} - p_{(j)}G^{(j)}\bigr)$ (in this multiplication order; as usual $^{(j)}$ and $_{(j)}$ denote derivatives with respect to ξ_j and x_j respectively) and thus

$$\begin{aligned} -\{p,q\} = \bigl(q'''(x,\xi')\bigr)^{\dagger}\{p,\phi\}(x,\xi')q'''(x,\xi')+ \\ \nu(x,\xi)p(x,\xi) + p(x,\xi)\nu^{\dagger}(x,\xi) \end{aligned} \tag{3.4.40}$$

with matrix symbols q''' and μ'''.

Energy Estimates. I

Actually we will need to make some minor adjustments multiplying q by cut-off functions $\zeta^2(x,\xi')$ equal 1 in vicinity $z' \in T^*\partial X$ and slightly shifting argument of χ; so finally we take

$$q = \chi\bigl(\phi(x,\xi) - \varepsilon\bigr)\zeta^2(x,\xi') \tag{3.4.41}$$

with arbitrarily small constant $\varepsilon > 0$. Then we arrive to the following inequality provided p is microhyperbolic in the direction $\nabla^{\#}\phi$:

$$\begin{aligned} -h^{-1}\operatorname{Re} i([P,Q]u,u)_X \ge \\ \epsilon_0\|Q'''u\|^2 - Ch\|Q^{IV}u\|^2 - C\|Q^{IV}Pu\|^2 - C\|Q^{V}u\|^2 - Ch\|Q'''u\|^2_{\partial X} \end{aligned} \tag{3.4.42}$$

where Q^{IV} and Q^{V} are operators with symbols $q^{IV}(x,\xi)$ and $q^{V}(x,\xi)$ which are produced by the Weierstrass theorem from $\chi(\phi - 2\varepsilon)$ and $\chi(\phi - 2\varepsilon)\zeta_1$ respectively with ζ_1 supported in ε-vicinity of $\operatorname{supp}(\nabla\zeta)$. We can also replace here Q^{IV} and Q^{V} by matrix operators not containing hD_1.

Let us assume that p is microhyperbolic in direction $\nabla^{\#}x_2$ and that $g(x,\xi)$ is of principal type in $\iota^{-1}z' \cap \{g = 0\} = z$ (which consists of just one point). Then (3.4.25) holds in vicinity of z with $\varsigma = \pm 1$.

Let us consider incoming bicharacteristics $\gamma^{-}(z)$; this bicharacteristics is a bicharacteristics of ςg issued from z into direction of $t < 0$. Assume that (3.4.29) holds, i.e.

$$\gamma^{-}(z) \cap T^*X|_{\partial X} \cap \Omega = z \tag{3.4.43}$$

where Ω is vicinity of z. This assumption follows from concavity condition (3.4.26) but is definitely weaker; however this assumption would be incompatible with the convexity condition.

Also assume that (3.4.30)–(3.4.33) and a weaker assumption of (3.4.34) hold

$$\bigl(\mathcal{K}^-(p'',z')\setminus \iota^{-1}z\bigr)\cap \bigl(\mathsf{WF}^{s-\delta,0\,\prime}(u)\cup \mathsf{WF}^{s-\delta}(u|_{\partial X})\bigr)\cap \Omega' = \emptyset \tag{3.4.44}$$

with small exponent $\delta > 0$.

Let ϕ be a real-valued function such that p is microhyperbolic in direction of $\nabla^{\#}\phi$ and

(3.4.45) $\{\phi < 0\}\cap\{g = 0\}\cap\Omega$ contains $\gamma^-(z)\cap\Omega$ and is contained in the small enough vicinity of $\gamma^{-1}(z)\cap\Omega$

Proposition 3.4.8. *Let p be microhyperbolic in direction $\nabla^{\#}\phi$ at point z_k and q be defined by* (3.4.41)*. Then under assumptions* (3.4.30)–(3.4.33) *and* (3.4.44) *all the terms but the first one in the right-hand expression of* (3.4.42) *do not exceed Ch^{2s}.*

Proof. Proof is obvious. Actually we do not need conditions to $B\bar{\partial}u$ or $\bar{\partial}u$ here. □

Energy Estimates. II

Now we need to consider boundary term (3.4.4) which is after division by h

$$\mathsf{Re}(A_0Qu,u)_{\partial X} = \mathsf{Re}\sum_{0\le j\le m-1} A_0(Q_j(hD_1)^j u,u)_{\partial X}\equiv \mathsf{Re}\sum_{0\le j\le m-1}(Q_jA_0(A_0^{-1}A_1)^j u,u)_{\partial X} \tag{3.4.46}$$

modulo terms not exceeding

$$\sum_{0\le j\le m-} \|Q^{VI}D_1^jPu\|_{\partial X}\cdot\|QVI\|_{\partial X}\le Ch^{2s}$$

where $Q^{VI} = q^{VI}(x',hD')$ with q^{VI} supported in the small vicinity of z' and the second inequality is due to $z'\notin \mathsf{WF}^{s+1/2-j}(\bar{\partial}D_1^jPu)$ (which is due to (3.4.30)) and (3.4.44).

Energy Estimates. III

Let us consider the most important case $m = 2$ first. Without any loss of the generality one can assume that g is given by (3.4.24). Note that under condition (3.4.25) microhyperbolicity condition is equivalent to

$$\varsigma\{g, \phi\} > 0. \tag{3.4.47}$$

Then there are only two terms in (3.4.46). Let us assume that in vicinity of z'

$$\varsigma^j q_j \ge 0 \tag{3.4.48}_j$$

where q_j are defined in (3.4.36) and also that

$$\varsigma^j \langle a_j w, w\rangle \le -\epsilon \|w\|^2 \qquad \forall w \in \mathsf{Ker}\, b. \tag{3.4.49}_j$$

Then Gårding inequality and $(3.4.48)_j$, $(3.4.49)_j$ imply that

$$\begin{aligned} -\mathsf{Re}(Q_j A_j u, u)_{\partial X} \ge \\ \epsilon\, \mathsf{Re}\, \varsigma^j (Q_j u, u)_{\partial X} - \mu \|Q_j u\|^2_{\partial X} - C\mu^{-1}\|Q^{VI} B u\|^2_{\partial X} - Ch^{2s} \end{aligned} \tag{3.4.50}$$

with arbitrarily small constant $\mu > 0$.

We can skip the third term in the right-hand expression due to $z' \notin \mathsf{WF}^s(B\eth u)$.

Further, since $\varsigma^j q_j \ge 2\varepsilon q_j^2$ Gårding inequality and $(3.4.48)_j$ imply that

$$\varsigma^j (Q_j u, u)_{\partial X} \ge 2\varepsilon \|Q_j u\|^2 - Ch^{2s}$$

and and we can skip the second term in the right-hand expression of (3.4.50) as well.

Then the arguments of the proof of Theorem 3.1.13(ii) imply that

$$\|Q''' u\| \le Ch^s \qquad \text{and} \qquad \varsigma^j (Q_j u, u)_{\partial X} \le Ch^{2s} \tag{3.4.51}$$

which implies $z' \notin \mathsf{WF}^s(\eth u)$ since at least one of operators Q_j is elliptic in z'. The latter statement is trivial because $q_0(z') = \chi(\phi(z) - \varepsilon)$ and $q_1 = \chi'(\phi(z))(\partial_{\xi_1}\phi(z))$ with $z = \iota^{-1} z' \cap \{g = 0\}$.

Then $Pu^0 = -ih\delta(x_1) A_0 u|_{\partial X} + (Pu)^0$ where u^0 is a continuation of u by 0 to $\{x_1 < 0\}$ and ellipticity of the "scalar version" of Q''' (which is different

from "matrix version" by multiple of P and has a symbol $\chi_2(\phi(z)-\varepsilon))$ implies that $z' \notin \mathsf{WF}^{s,0\,\prime}(u)$.

Let us consider $(3.4.49)_j$. Obviously $(3.4.49)_0$ is exactly $(3.1.9)^*$.

On the other hand, without any loss of the generality under (3.4.24) one can assume that at z'

$$a_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad a_1 = \begin{pmatrix} \varsigma & 0 \\ 0 & 0 \end{pmatrix} \tag{3.4.52}$$

and condition $(3.4.49)_1$ is nothing but (3.4.18) (assuming that (3.1.9) holds).

Furthermore, we actually do not need $(3.1.9)^*$: as (3.1.9) and (3.4.18) are fulfilled we can add to Q operator $\alpha_1 Q_0 h D_1$ which would lead to an extra boundary term as in the proof of Theorem 3.4.4.

Let us prove $(3.4.48)_j$. As $R > 0$

$$q_0 = \frac{1}{2}\bigl(\chi(\phi^+ - \varepsilon) + \chi(\phi^+ - \varepsilon)\bigr), \tag{3.4.53}$$

$$q_1 = \frac{1}{2R^{1/2}}\bigl(\chi(\phi^+ - \varepsilon) - \chi(\phi^+ - \varepsilon)\bigr) \tag{3.4.54}$$

where

$$\phi^{\pm} = \phi_0 \mp \varsigma\phi_1 R^{1/2} \tag{3.4.55}$$

and we assume that

$$-\varsigma\phi_1(z') > 0. \tag{3.4.56}$$

Moreover, as $R < 0$ one can prove easily that for all n

$$|q_0 - \chi(\phi_0 - \varepsilon)| \le C_n|R|^n, \qquad |q_1 - \chi'(\phi_0 - \varepsilon)\phi_1| \le C_n|R|^n$$

provided $|\chi^{(m)}| \le C_{m\delta}\chi^{1-\delta}$ for any m and $\delta > 0$ and $-\chi'$ has the same property.

Then redefining $q_0 := q_0 + C|r|^n$, $q_1 := q_1 + C\varsigma|r|^n$ for $r < 0$ we achieve $(3.4.48)_j$ without violating (3.4.36). Here functions q_0, q_1 and μ in (3.4.36) are very smooth and we do not infinite smoothness (but actually we can construct infinitely smooth correction).

Note that $\chi(t) = e^{1/t}$ as $t < 0$ and $\chi(t) = 0$ as $t > 0$ has all the properties we need.

Results: Single Glancing Point

So, we have proven

Theorem 3.4.9. *Under assumptions*[16] $\mathsf{D}=2$ *and* (3.4.26) *Conjecture 3.4.7 holds.*

Note that for systems with $\mathsf{D}=2$ satisfying (3.4.23) and (3.4.25) generalized bicharacteristic billiards are a.e. described by

$$\frac{dz}{dt} = \nabla^{\#} g + \kappa \nabla^{\#} x_1 \qquad \text{with} \quad \kappa = 0 \text{ as } \quad x_1 > 0. \tag{3.4.57}$$

Then along boundary bicharacteristics

$$\kappa = H_g^2 x_1 \cdot \left(H_{x_1}^2 g\right)^{-1} \tag{3.4.58}$$

(this formula is invariant if we replace g by eg with non-vanishing e).

Combing Theorems 3.4.9 and 3.4.6 we arrive to

Theorem 3.4.10. *Let* $\mathsf{D}=2$ *and* (3.4.23), (3.4.25) *be fulfilled. Then Theorem 3.4.4 remains true if in the definition* (3.4.57) *of the generalized bicharacteristic billiard we add an extra condition* $\kappa \le 0$ *thus replacing it by*

$$\frac{dz}{dt} = \nabla^{\#} g + \kappa \nabla^{\#} x_1 \qquad \text{with} \quad \kappa = 0 \text{ as } \quad x_1 > 0 \text{ and } \quad \kappa \le 0 \text{ as } \quad x_1 = 0. \tag{3.4.57}^*$$

Remark 3.4.11. Theorem 3.4.10 remain true with the obvious modification: p has a block diagonal form with the single block described there and all other blocks corresponding to transversal or elliptic components.

Further Discussion

Unfortunately Theorem 3.4.9 is almost everything we managed to prove by this energy estimate method. We were unable to cover multiple tangent points belonging to $\iota^{-1}z$ or higher multiplicity of ξ_1-root (with the exception of the special case). Neither it works for Neumann boundary conditions for wave operator and one needs to use direct parametrix construction (see *Comments*).

The real power of Theorem 3.4.9 comes when it works together with Theorem 3.2.4. Let us consider several examples:

[16] And all other assumptions listed in Conjecture 3.4.7.

Violation of Assumptions (3.2.36), (3.4.18). Assume that z' is a grazing point with $\mathsf{D}=2$ but (3.2.36) and (3.4.18) do not necessarily hold.

Consider in the vicinity of z' manifold $\mathcal{G}=\{z', r(z')=0\}$ where we assume that $g(z)$ is of the form (3.4.24) and (3.4.25), (3.4.26) are fulfilled. Further, assume that in the vicinity of z'

$$a_0=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \qquad a_1=\begin{pmatrix}\varsigma&0\\0&\varsigma r\end{pmatrix}. \tag{3.4.59}$$

Then boundary operator could be reduced to form $\begin{pmatrix}1 & b_2\end{pmatrix}$ with real-valued b_2 which vanishes in z' (otherwise we can apply Theorem 3.4.10 directly).

Then condition (3.2.15) is fulfilled at points of $\mathcal{G}$ with $b_2\neq 0$. Therefore Theorem 3.4.9 and thus Theorem 3.4.10 could be directly used there. On the other hand, Theorem 3.2.4 does allow propagation along boundary bicharacteristic of g (i.e. bicharacteristic of r) and also propagation in the elliptic zone along bicharacteristics of $\bigl((-r)^{1/2}+b_2\bigr)$ or those of

$$\varsigma(r+b_2^2), \qquad b_2<0. \tag{3.4.60}$$

However if we assume that $\gamma^-_{\varsigma r}(z')$ which is (boundary) bicharacteristics of ςr issued from z' in the direction $t<0$ leaves $\Lambda=\{b_2=0\}$ immediately, which is the case provided

$$x_1=r=b_2=0 \implies \{r,b_2\}(z')\neq 0 \tag{3.4.61}$$

then there is no propagation along it and thus Theorem 3.4.9 could be extended to this case:

Theorem 3.4.12. *Let in vicinity of z'* (3.4.59) *and* (3.4.24)–(3.4.26) *be fulfilled. Let boundary operator $b=\begin{pmatrix}1 & b_2\end{pmatrix}$ with real-valued b_2, $b_2(z')=0$. Let*

$$\bigl(\gamma^-_{\varsigma r}(z')\setminus z'\bigr)\cap\{b_2=0\}\cap\Omega'=\emptyset, \tag{3.4.62}$$

$$z'\notin \mathsf{WF}^{s+1,l'}(Pu)\cup \mathsf{WF}^{s+1/2}(B\eth u), \tag{3.4.63}$$

$$\bigl(\gamma^-_{\varsigma g}(z)\setminus z\bigr)\cap \mathsf{WF}^{s}(u)\cap\Omega=\emptyset, \qquad z=\iota^{-1}z'\cap\{g=0\} \tag{3.4.64}$$

$$\bigl(\gamma^-_{\beta}(z')\setminus z'\bigr)\cap \mathsf{WF}^{s-1/2}(\eth u)\cap\Omega'=\emptyset. \tag{3.4.65}$$

Then

$$z'\notin \mathsf{WF}^{s,l+1'}(u)\cup \mathsf{WF}^{s-1/2}(\eth u). \tag{3.4.66}$$

Further one can prove more global statement where (3.4.61) ensures (3.4.62) at every point of Λ.

Several Grazing Points. Let us assume that $\iota^{-1}z'$ consists of n grazing points, $n \ge 2$, each of them satisfying (3.4.23), (3.4.25), (3.4.26). Then we have $\varsigma_k = \pm$ and g_k, r_k, $\mathcal{G}_k$ with $k = 1, \dots, n$. Let

$$\Lambda^{(m)} = \bigcup_{\{J\subset\{1,\dots,n\},\#J=m\}} \Big(\bigcap_{k\in J} \mathcal{G}_k.\Big). \tag{3.4.67}$$

Let us assume that (3.4.63), (3.4.64) are replaced by

$$z' \notin \mathsf{WF}^{s+1,l'}(Pu) \cup \mathsf{WF}^s(B\eth u), \tag{3.4.63)*}$$

and

$$\big(\gamma^-_{\varsigma_k g_k}(z_k) \setminus z_k\big) \cap \mathsf{WF}^s(u) \cap \Omega_k = \emptyset, \qquad z_k = \iota^{-1}z' \cap \{g_k = 0\} \quad \forall k = 1, \dots, n. \tag{3.4.64)*}$$

respectively.

Assuming (3.2.36), (3.4.18) we conclude from Theorem 3.4.9 that

$$\big(\Lambda^{(m)} \setminus \Lambda^{(m+1)}\big) \cap \big(\mathsf{WF}^s \cup \mathsf{WF}^s(\eth u)\big) \cap \Omega' = \emptyset. \tag{3.4.68)$_m$}$$

holds with $m = 1$.

Let us apply Theorem 3.4.4. Let us consider a generalized bicharacteristics of p issued from z' in the direction $t < 0$. It consists of a boundary bicharacteristics defined by

$$\frac{dz''}{dt} \in K_b(z'') \tag{3.4.69}$$

where

$$K_b(z'') = \mathsf{convex\ hall}\big\{\nabla^{\#}\varsigma_j r_j(z'),\ j \in J\big\} \quad \text{as } z'' \in \Lambda^{(m)} \setminus \Lambda^{(m+1)} \text{ and } r_j(z'') = 0 \iff j \in J. \tag{3.4.70}$$

Then $z'' \notin \big(\mathsf{WF}^s \cup \mathsf{WF}^s(\eth u)\big)$ for every $z'' \in \Omega'$ such that z'' has a vicinity Ω'' such that $\mathcal{K}_b^-(z'') \cap \Omega'' \cap \big(\Lambda^{(m)} \setminus \Lambda^{(m+1)}\big) = \emptyset$ with $m = 1$.

But then $(3.4.68)_1$ implies $(3.4.68)_2$ provided $r_j = r_k = 0 \implies \{r_j, r_k\} \ne 0$ etc until $(3.4.68)_n$ under condition (2.2.41) for g_k replaced by $\varsigma_k r_k$.

Meet a Gliding Component. Let situation described in the previous paragraph hold for all points $z_k \in \iota^{-1}z'$ with the exception of z_1 such that $\mathsf{D}_1 = 2$ but it is gliding point i.e. $H_{g_1}x_1 < 0$ at z_1. Let g_1 be of the form (3.2.20) and let conditions (3.4.63)* and (3.4.64)* with $k = 2, \dots, n$ be fulfilled.

Let us assume that

$$\bigl(\gamma^-_{\varsigma_1 r_1}(z') \setminus z'\bigr) \cap \bigl(\mathsf{WF}^{s,l'}(u) \cup \mathsf{WF}^s(\eth u)\bigr) \cap \Omega' = \emptyset. \tag{3.4.71}$$

Then the previous arguments yield that

$$\{r_1 \neq 0\} \cap \bigl(\mathsf{WF}^{s,l'}(u) \cup \mathsf{WF}^s(\eth u)\bigr) \cap \Omega'' = \emptyset \tag{3.4.72}$$

where Ω'' is small enough vicinity of z'. Applying Theorem 3.2.4 again we conclude that $z' \notin \mathsf{WF}^{s,l'}(u) \cup \mathsf{WF}^s(\eth u)$ under condition (2.2.41) for g_k replaced by $\varsigma_k r_k$ $(k = 1, \dots, n)$.

On the other hand, two gliding components create a real mess in billiards due to jumps in different directions and after jump to some direction the length of the jump to other direction can change drastically.

3.5 Propagation of Singularities along Long Transversally Reflecting Bicharacteristics

In this section we obtain results similar to those of Section 2.4. However, the presence of the boundary makes things much more complicated and we need to assume that in the zone under analysis not only $a(x,\xi)$ has eigenvalues of the constant multiplicities but also that bicharacteristic billiards are transversal to the boundary. Further, we need to assume not only that symbol $A(x,\xi)$ is regular but it is also true for the function defining the boundary and coefficients of the boundary operator. More precisely, we assume that operator "incoming components $\to$ outgoing components" is a pseudodifferential operator with a regular symbol.

3.5.1 Evolution of Operators

So, as in Subsection 2.4.1 we are interested in operator (2.4.4)

$$Q^t = e^{-ih^{-1}tA} Q^0 e^{ih^{-1}tA} \tag{3.5.1}$$

where we assume that

(3.5.2) $A = A_B$ i.e. operator with domain

$$\mathfrak{D}(A_B) = \{u : u \in \mathscr{L}^2(X, \mathbb{H}),\ Au \in \mathscr{L}^2(X, \mathbb{H}),\ B\eth_{m-1}u = 0\} \tag{3.5.3}$$

is a self-adjoint in $\mathscr{L}^2(X, \mathbb{H})$ operator.

Standard Theory

Here we need to assume that $Q = Q(x, hD)$ is a pseudodifferential operator with the symbol $q(x, \xi)$ and to avoid boundary operators in domain with the boundary we assume that

$$\mathsf{dist}(\pi_x \mathsf{supp}(q), \partial X) \ge \varrho. \tag{3.5.4}$$

To describe a classical evolution of A we need to consider generalized bicharacteristic billiards; we need to assume that the free space evolution is according to Section 2.4; so we repeat assumptions (2.4.58) and (2.4.59):

$$\mathsf{Spec}\, a(z) = \{\lambda_1(z), \dots, \lambda_m(z)\} \qquad \forall z \in \Omega \tag{3.5.5}$$

and for every $j = 1, \dots, m$

$$\mathsf{Spec}\, a(z) \cap [\lambda_j(z) - \varrho, \lambda_j(z) + \varrho] = \lambda_j(z) \qquad \forall z \in \Omega \tag{3.5.6}$$

where λ_j are assumed to be uniformly continuous and $\Omega = \mathcal{K}^+_{[0,T]}(p, \Omega_0)$ with Ω_0 a ϱ-vicinity of $\mathsf{supp}(q)$. Here we parametrize bicharacteristics billiards by t and subscript means the range of t.

Moreover we need to assume that the reflection is transversal:

$$|\{\lambda_j, \phi\}(z)| \ge \rho \qquad \forall z = (x, \xi) :\ \phi(x) \le \varrho \tag{3.5.7}$$

where

$$X = \{x,\ \phi(x) > 0\}, \quad \partial X = \{x,\ \phi(x) = 0\}, \tag{3.5.8}$$
$$\phi(x) \le \varrho \implies |\nabla\phi| \ge \rho.$$

If there is no branching of billiards i.e. if

$$\#\big\{k : \exists z_k \in \iota^{-1}z',\ \lambda_k(z_k) = 0,\ \{\lambda_k, \phi\}(z_k) > 0\big\} = 1 \tag{3.5.9}$$
$$\forall z' \in \iota\mathcal{K}^+_{[0,T]}(p, \Omega_0) \cap T^*X|_{\partial X}$$

then

$$\mathcal{K}_t^+(p,z) = \Psi_t(z) \quad \text{for} \quad z \in \Omega_0 \tag{3.5.10}$$

where Ψ_t is a *Hamiltonian dynamics with the reflections at* ∂X. Otherwise this formula holds but now Ψ_t is a *Hamiltonian dynamics with the reflections and branching at* ∂X:

$$\Psi_t(z) = \{\Psi_{t,j}(z),\ j = 1, \dots, N(t)\} \tag{3.5.11}$$

where $\Psi_{t,j}$ are Hamiltonian maps.

We expect (correctly) that Q^t is pseudodifferential operators with the symbol supported in $\mathcal{K}_t^+$ but again we need to assume that

$$\mathsf{dist}(\pi_x \mathcal{K}_{\bar t}^+(p, \mathsf{supp}(q)), \partial X) \ge \varrho \tag{3.5.12}$$

for this particular $t = \bar t$; however we cannot assume that this is true on the whole interval $[0, \bar t]$.

Therefore the evolution would consist of alternating free space evolution along Hamiltonian trajectories where we apply results of Section 2.4 and reflections. These reflections happen when $\phi \circ \Psi_t(z) \le \varrho$.

If billiards branch then different branches have different reflection times and we consider different branches separately replacing operator Q^t by the sum of operators supported in different components of $\Psi_t(\Omega_0)$.

Reflections

Let us consider vicinity of the boundary: $\{x, \phi(x) \le \varrho\}$. Without any loss of the generality one can introduce coordinate system with $x_1 = \phi(x)$.

Let us consider point $(z', 0, \tau) \in T^*(\partial X \times \mathbb{R}_t)$ and $z_j = \iota^{-1} z' \cap \{\lambda_j = -\tau\}$. Let us consider a single incoming component with index j ignoring all other. So we consider trajectory of $\big(\tau + \lambda_j(z)\big)$ with $\{\lambda_j, \phi\} \le -\rho$. Let us consider all outgoing trajectories (those of $\big(\tau + \lambda_k(z)\big)$ with $\{\lambda_k, \phi\} \ge \rho$ where $k = 1, \dots, m$).

Let us consider point $(z^-, -T', \tau)$ belonging to the incoming trajectory such that $x_1 \asymp \varrho$ at z^- and let us consider operator $Q^- = Q^-(x, hD_x)$ with the symbol q^- supported in $\epsilon\varrho$-vicinity of z^-. Our goal is to consider evolution of this operator during time $T' \asymp \varrho/\rho$; this will be a short time evolution in the case of small parameters ρ and $\varrho \ll \rho$. There are three steps:

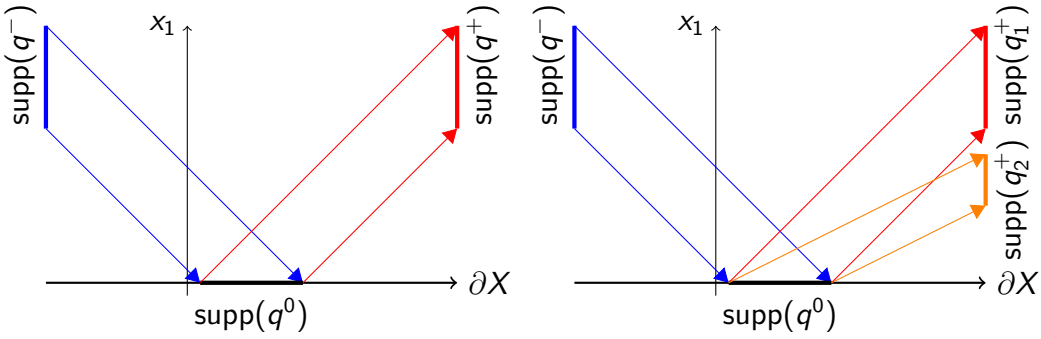
(i) We construct G_j^+ which transforms Cauchy data for u_j as $t = -T'$ to solution u_j of $P_j u_j \equiv 0$ as $t > 0$ and to prove that

(3.5.13) $\eth G_j^+ Q^- = Q^0 \eth G_j^+$ where $Q^0 = Q^0(x', t, hD'_x, hD_t)$ is an operator with symbol q^0 and

(3.5.14) $$\mathsf{supp}(q^0) = \big\{(z', t, \tau) : z' \in \iota\Psi_{t+T',j}(\mathsf{supp}(q^-) \cap \{\lambda_j = -\tau\})\big\};$$

(ii) Then we construct operators Q_k^0 such that $B\eth u \equiv 0$ implies $Q^0 u_j \equiv \sum_k Q_k^0 u_k$ where u_k are outgoing components of u as $k \geq 1$ and u_0 is an elliptic component and

(3.5.15) $$\bigcup_k \mathsf{supp}(q_k^0) = \mathsf{supp}(q^0).$$



(a) Simple reflection (b) Branching reflection

Figure 3.5: Reflections as $\eth G_j^+ Q^- = Q^0 \eth G_j^+$ and $G_k^+ Q_k^0 = \sum_k Q_k^+ G_k^+$

(iii) Finally, we construct parametrices G_k^+ corresponding to outgoing components and solving Cauchy problems $P_k u_k \equiv 0$ with Cauchy data on ∂X (with x_1 considered as "time") and prove that as $t \geq T'$

(3.5.16) $\gamma_{T'} G_k^+ Q_k^0 = Q_k^+ \gamma_{T'} G_k^+$ where γ_t is operation of fixing t, and $Q_k^+ = Q_k^+(x, hD_x)$ is an operator with the symbol q_k^+ and

(3.5.17) $$\mathsf{supp}(q_k^+) \cap \{t = T'\} = \{z : \exists t' \in [0, T'],\ \tau = -\lambda_k(z),$$
$$z' \in \mathsf{supp}(q_k^0) : z' \in \iota\Psi_{-T'+t',k}(z)\}.$$

To implement Steps (i) and (iii) we can consider Cauchy problems with respect to x_1 and note that one can always rewrite operator $Q_j^{\pm}(x, hD_x)$ as $Q_j^{\pm}(x, \pm T', hD_x)$ where $Q_j^{\pm}(x, t, hD_x)$ commutes with P_j and one can always rewrite $Q_j^{\pm}(x, t, hD_x)$ as

$$Q_j^0(x, t, hD_x', hD_t) + R_j(x, t, hD_x, hD_t)P_j(x, hD_x, hD_t).$$

Obviously

(3.5.18) As ρ, ϱ, T' are (small) constants, both operators G_j^+ and $\gamma_t G_k^+$ (with $t \in [T', T'']$) are Fourier integral operators with the symplectomorphisms $z \in T^*X \to (z', t, \tau) \in T^*(\partial X \times \mathbb{R}_t)$ and $(z', t, \tau) \in T^*(\partial X \times \mathbb{R}_t) \to z \in T^*X$ described above.

Note also that due to symmetry assumption

(3.5.19) Operator $\mathcal{B}$: "incoming components $\to$ outgoing components" is unitary modulo lower order terms.

This statement is true even if there are elliptic components. Now combining free propagation and reflection we arrive to

Theorem 3.5.1. *Let conditions* (3.5.2), (3.5.4)–(3.5.8) *be fulfilled with constants* ρ, ϱ, T. *Let* $t \in (0, T]$ *and* (3.5.12) *be fulfilled as well. Also assume that*

(3.5.20) $K^+_{[0,T]}(p, z)$ *consists of one non-branching billiard.*

Then for any $\tau_1 < \tau_2$ *with small enough* $|\tau_2 - \tau_1|$ *there exists pseudodifferential operator* Q^t *such that*

$$\|F_{t\to h^{-1}\tau}\bar{\chi}_{T'}(t-\bar{t})\big(e^{-ih^{-1}tA}Q - Q^t e^{-ih^{-1}tA}\big)\| \le Ch^s \qquad \forall \tau \in [\tau_1, \tau_2] \tag{3.5.21}$$

with small enough constant T' *and symbol of* Q^t *is supported in*

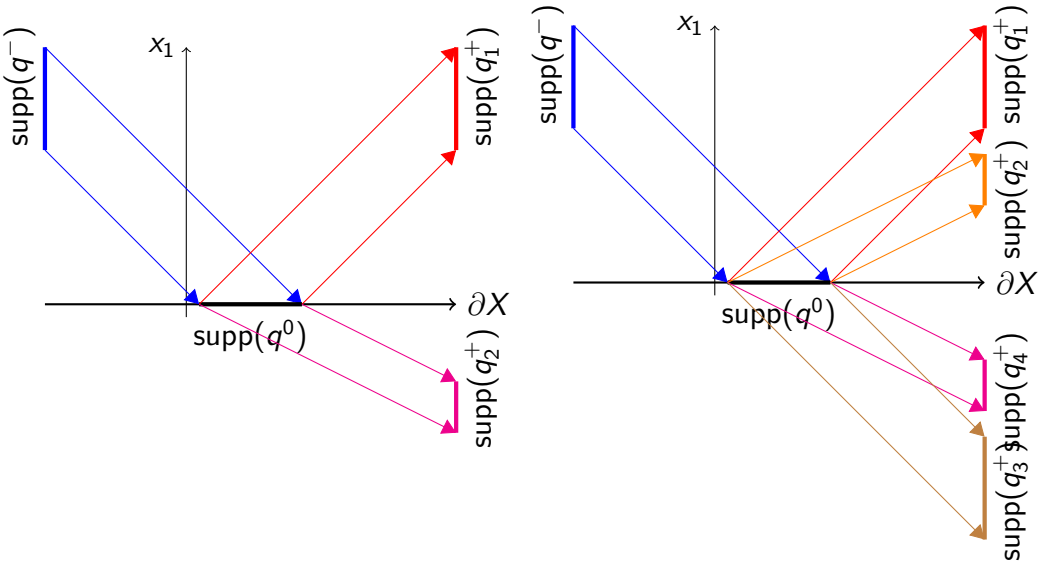
$$\bigcup_{\tau\in[\tau_1',\tau_2']} \mathcal{K}_t^+(a+\tau, \mathsf{supp}(q)) : \tag{3.5.22}$$

where $\tau_1' < \tau_1 < \tau_2 < \tau_2'$ *are arbitrary constants.*

Since we did not assume that A has a scalar principal symbol we cannot use the same operator Q^t for all τ but since λ_j are separated according to (3.5.5)–(3.5.6) we can exclude hD_t from Q^t in (3.5.21).

Remark 3.5.2. One can consider problems when operator is defined as $x_1 > 0$ and as $x_1 < 0$ with *transition conditions* at $x_1 = 0$. We refer to it as *reflection-refraction*.

We show it on Figure 3.6 below. The numbers of *reflected waves* and *refracted waves* do not necessarily coincide.



(a) Simple reflection-refraction (b) Branching reflection-refraction

Figure 3.6: Reflections and refractions

3.5.2 Long Time Evolution

Main Theorem

Now we want to take ρ, ϱ to be small parameters and T be a large parameter. Actually there are plenty of parameters here: large parameters time T, the bounds to derivatives of coefficients of operators A, B (which is natural to assume to be h-differential operators) and the bounds to number of reflections

along one billiard, the total number of billiards (due to branching) and the bounds to elements of $D\Psi_{t,j}$. Further, there are also upper bounds to derivatives of ϕ defining ∂X and the lower bounds to $|\nabla\phi|$, incidence and reflection angles. Furthermore, here are bounds to derivatives of the symbol of pseudodifferential operator $\mathcal{B}$: "$\mathsf{incoming\ components} \to \mathsf{outgoing\ components}$". Finally, there is a scaling parameter ε for operator Q.

To avoid an orgy of parameters we just assume (with no implications to applications) that $\varepsilon \ge h^{1/2-\delta}$ with an arbitrarily small exponent $\delta > 0$ while all other small parameters are not less than $h^{\delta'}$ and all large parameters are not larger than $h^{-\delta'}$ with $\delta' = \delta'(\delta) > 0$.

So, we assume that all the coefficients of A and B satisfy the inequalities in $\mathcal{K}^+_{[0,T]}(p, \mathsf{supp}(q))$

$$\|D^\alpha A_\beta\| \le c\rho^{-|\alpha|-1} \tag{3.5.23)$_1$}$$

$$\|D^\alpha B_\beta\| \le c_\alpha \rho^{-|\alpha|-1} \qquad \forall \alpha, \tag{3.5.23)$_2$}$$

and

$$|D^\alpha \phi| \le c_\alpha \rho^{-|\alpha|-1} \qquad \forall \alpha, \tag{3.5.24)$_1$}$$

$$|\phi(x)| \le \epsilon_0 \rho \implies |\nabla_x \phi(x)| \ge \epsilon_0, \tag{3.5.24)$_2$}$$

$$|\phi(x)| \le \epsilon_0 \rho \implies |\frac{d}{dt}\phi(x)| \ge \epsilon_0 \rho, \tag{3.5.24)$_3$}$$

along Hamiltonian trajectories comprising $\mathcal{K}^+_{[0,T]}(p, \mathsf{supp}(q))$, symbol of $\mathcal{B}$ satisfies

$$\|D^\alpha_{x',\xi',\tau} \mathcal{B}_n\| \le c_\alpha \rho^{-2n-|\alpha|-1} \qquad \forall \alpha. \tag{3.5.23)$_3$}$$

Finally we assume that

$$|D\Psi_{t,j}| \le c\rho^{-1} \tag{3.5.25}$$

again along trajectories comprising $\mathcal{K}^+_{[0,T]}(p, \mathsf{supp}(q))$.

Our goal is to prove

Theorem 3.5.3. *Let A, B be differential operators of orders m and not exceeding $(m-1)$ respectively. Let A_B be a self-adjoint operator.*

Let $Q \in \boldsymbol{\Psi}_{h,\varepsilon,\varepsilon}(X, \mathbb{H}, \mathbb{H})$ with $\varepsilon \ge h^{1/2-\delta}$. Let us assume that $T \le \rho^{-1}$ and in $\mathcal{K}^+_{[0,T]}(p, \mathsf{supp}(q))$ conditions (3.5.5), (3.5.6), $(3.5.23)_{1-3}$, $(3.5.24)_{1-3}$, (3.5.25) *be fulfilled with $\rho = h^{\delta'}$, $\delta' = \delta'(\delta) > 0$.*

Further, assume that

$$\mathsf{dist}(\pi_x \operatorname{supp}(q), \partial X) \ge \epsilon_0 \varepsilon, \tag{3.5.26)$_1$}$$

$$\mathsf{dist}(\pi_x \mathcal{K}_t^+(p, \operatorname{supp}(q)), \partial X) \ge \epsilon_0 \varepsilon. \tag{3.5.26)$_2$}$$

Finally, assume that (3.5.20) *is fulfilled.*

Then for any $\tau_1 < \tau_2$ *with* $|\tau_2 - \tau_1| \le \epsilon\rho$ *there exists operator* $Q^t \in \mathbf{\Psi}_{h,\varepsilon',\varepsilon'}(X, \mathbb{H}, \mathbb{H})$ *with* $\varepsilon' = \rho^r \varepsilon$ *such that* (3.5.21) *holds with* $T' = \epsilon_0 \rho^r$ *where* r *is large enough (but not depending on* δ*).*

Symbol of Q^t *is supported in* $\bigcup_{\tau\in[\tau_1,\tau_2]} \mathcal{K}_t^+(a+\tau, \operatorname{supp}(q))$.

Proof of the Main Theorem

First of all we note that the case of billiard not hitting the boundary is covered by Theorem 2.4.10. So we need to consider the case of the billiard hitting boundary at least once and possibly branching there. Each branch of such billiard consists of *hops* between boundary points but the first and the last hops are not complete: the former does not start at the boundary and the latter does not end here. Note that the length of every hop could be rather small: if everything but the angle with the boundary is bounded by a constant rather than a parameter, and angle is still bounded from below by ρ then the length of the hop could be as small as $\asymp \rho$ (for the bicharacteristically strongly convex domain as in the Figure 3.1. So the number of hops in such case would be $\asymp T\rho^{-1}$.

In the general case the length of the hop is bounded from below by ρ^3 and the number of hops is bounded from above by $T\rho^{-1} \asymp \rho^4$ (here exact fixed power is of no importance).

Let us consider first the Hamiltonian map Ψ_t along such billiard. Let $0 < t_1 < ... < t_N < t$ be moments when billiard hits the boundary and let us set $t_n' = t_n - \varsigma$, $t_n'' = t_n + \varsigma$ be moments "just before" and "just after" the hit. Then near z

$$\Psi_t = \Psi_{t,t_N''} \circ \Psi_{t_N'',t_N'} \circ \Psi_{t_N',t_{N-1}''} \circ \cdots \Psi_{t_1'',t_1'} \circ \Psi_{t_0',0} \tag{3.5.27}$$

where we use $\Psi_{t,t'}$ instead of $\Psi_{t-t'}$ to emphasize the moments on the billiards and

$$D\Psi_t = D\Psi_{t,t_N''} \cdot D\Psi_{t_N'',t_N'} \cdot D\Psi_{t_N',t_{N-1}''} \cdots D\Psi_{t_1'',t_1'} \cdot D\Psi_{t_0',0} \tag{3.5.28}$$

and we claim that

Proposition 3.5.4. $(m+1)$*-th order derivatives of* $\Psi_{t,0}$ *do not exceed* $C\rho^{-1-9m}$.

Proof. We apply induction with respect to m; as $m=0$ this just assumption (3.5.25). Let us consider $m \ge 1$.

Differentiating $(m+1)$-times $\Psi_{t,0} = \Psi_{t,s}\circ\Psi_{s,0}$ we get the following terms:

(i) Just one term when $\Psi_{t,s}$ is differentiated $(m+1)$ times.

(ii) No more than c_m terms when $\Psi_{t,s}$ is differentiated k times, $2 \le k \le m$. Such terms do not exceed

$$|D^k\Psi_{t,s}|\cdot|D^{r_1}\Psi_{s,0}|\cdot|D^{r_2}\Psi_{s,0}|\cdots|D^{r_k}\Psi_{s,0}|$$

with $r_1+...+r_k = m+1$, $r_i \ge 1$. However as time interval between s and t contains no hits the first factor due to Lemma 2.4.5 does not exceed $c\rho^{2-3k}$ and by induction assumption all $(i+1)$-th factor does not exceed $C\rho^{-8+9r_i}$ so the whole product does not exceed $C\rho^{2-3k-(9r_1-8)-\dots-(9r_k-8)} = C\rho^{2+5k-9(m+1)} = C\rho^{3-9m}$.

(iii) One term when $\Psi_{t,s}$ is differentiated just one time and $\Psi_{s,0}$ is differentiated $(m+1)$ times.

Repeating this procedure again and again we realize that there will be no more $2n+2 \le \rho^{-4}$ (ii)-type terms and their total contribution would not exceed $C\rho^{-1-9m}$. Therefore only terms when one of the factors gets all $(m+1)$ derivatives and all other factors get one derivative each need to be considered. So we get n terms of the type

$$|D\Psi_{t,s}|\cdot|D^{m+1}\Psi_{s,t}|\cdot|D\Psi_{s,0}| \tag{3.5.29}$$

where we used $D\Psi_{t,s} = D\Psi_{t,\tau}\cdot D\Psi_{\tau,s}$. Note that in (3.5.29) the first and the third factors do not exceed $c\rho^{-1}$ each and in the second factor either $t=t''_n$, $s=t'_n$ or $t=t'_{n+1}$, $s=t''_n$ and we set $t'_{N+1}=t$, $t''_0=0$.

So, we need to prove only that in this framework $|D^{m+1}\Psi_{s,t}|$ does not exceed $C\rho^{4-9m}$.

As $t=t'_{n+1}$, $s=t''_n$ the time interval between them contains no hits and due to Lemma 2.4.5 $|D^{m+1}\Psi_{s,t}|$ does not exceed $C\rho^{-1-3m}$.

Consider $t=t''_n$, $s=t'_n$. We can take $\varsigma>0$ arbitrarily small here; thus $t''_n = t_n+0$, $t'_n = t_n-0$. Then $\Psi_{t_n+0,t_n-0} = \Psi^+_k\Psi^-_j$ where Ψ^+_k :

$(x', \xi', t, \tau) \to (x, \xi)$ and $\Psi_j^- : (x, \xi) \to (x', \xi', t, \tau)$ correspond to outgoing and incoming bicharacteristic respectively. So we can study both of these symplectomorphisms separately.

Scaling ρ times with respect to x and ξ and changing coordinates so X locally is defined as $\{x_1 > 0\}$ we arrive to the standard situation with the sole exception that $-\{\lambda_j, x_1\} \asymp \zeta_j$ and $\{\lambda_k, x_1\} \asymp \zeta_k$ with $\zeta_j, \zeta_k \in [\rho, 1]$. Then rescaling with respect to x_1, ξ_1 only we arrive to the standard situation. One can prove easily that the inverse change of the coordinates provide the required estimate. □

Now consider point $\bar{z}$ and operator Q with the symbol supported in ε-vicinity of $\bar{z}$. By assumption $\tau_{1,2} = \bar{\tau} \mp \epsilon\rho$ for some $\bar{\tau}$. Without any loss of the generality one can assume that $|\lambda_j(\bar{z}) + \bar{\tau}| \le 2\epsilon\rho$ for some j [17].

Let us repeat arguments of Subsubsection *Heisenberg Evolution for Matrix Principal Symbols.3 Heisenberg Evolution for Matrix Principal Symbols.* In ρ^3-vicinity of $\bar{z}$ one can reduce A to the form

$$A \equiv \begin{pmatrix} A_j & 0 \\ 0 & A_{\hat{j}} \end{pmatrix}, \qquad Q \equiv \begin{pmatrix} Q_{jj} & Q_{j\hat{j}} \\ Q_{\hat{j}j} & Q_{\hat{j}\hat{j}} \end{pmatrix} \tag{3.5.30}$$

where the principal symbol of A_j is λ_j and eigenvalues of the principal symbol of $A_{\hat{j}}$ differ from $\bar{\tau}$ by $3\epsilon\rho$ at least.

Then one can redefine Q replacing two lower blocks by 0 [18] and then one can redefine A replacing A_j by whatever we want[19] (and we want it to be equal to λ_j^{w}):

$$A := A_{(j)} = \begin{pmatrix} A_j & 0 \\ 0 & \lambda_j^{\mathsf{w}} \end{pmatrix}, \qquad Q := Q_{(j)} = \begin{pmatrix} Q_{jj} & Q_{j\hat{j}}, \\ 0 & 0 \end{pmatrix}. \tag{3.5.31}$$

So we replaced A, Q by

$$A_{(j)} = \pi_j A + (I - \pi_j)\lambda_j^{\mathsf{w}}, \qquad Q_{(j)} = \pi_j Q \tag{3.5.32}$$

where π_j is the corresponding projector. While the global[20] reduction of A, Q to form (3.5.30) and definition (3.5.31) is a murky business, pseudodifferential operator π_j and thus $A_{(j)}$, $Q_{(j)}$ are well defined globally[20].

[17] Otherwise $F_{t\to h^{-1}\tau}\bar{\chi}_T(t)UQ \equiv 0$ for $\tau \in [\tau_1, \tau_2]$.

[18] Then $F_{t\to h^{-1}\tau}U(t)Q$ changes negligibly as $\tau \in [\tau_1, \tau_2]$.

[19] Then $e^{-ih^{-1}tA}Q$ changes negligibly.

[20] In the zone $\Sigma_j = \{|\lambda_j + \bar{\tau}| \le 3\epsilon\rho\}$.

However since $A_{(j)}$ has a scalar principal symbol, operator Q^t defined by (2.4.4) with A, Q replaced by $A_{(j)}$ and $Q_{(j)}$ respectively is a pseudodifferential operator as long as we do not hit the boundary. Therefore, Theorem 3.5.3 is proven as long there are no reflection points.

Now consider the act of reflection. Since we can push the free propagation almost until the boundary we can rescale as we did in the proof of Proposition 3.5.4. After this rescaling we have bicharacteristic transversal to the boundary and therefore we arrive to

$$\eth_{\partial X}\bigl(e^{-ih^{-1}(t-t')A_{(j)}}Q_j^{t'}\bigr) \equiv Q_j'\eth_{\partial X}e^{-ih^{-1}(t-t')A_{(j)}} \qquad \text{for } \ t\in[t_n-\epsilon\rho^2, t_n+\epsilon\rho^2] \tag{3.5.33}$$

where $Q_j' = Q_j'(x', hD', t, hD_t)$.

As $\mathsf{D}_j = 1$ its principal symbol is $\bigl(q_j^{t'}\circ \Psi_j^{-}{}^{-1}\bigr)$ and all other symbols are given by

$$Q_{jm}' = \sum_{|\alpha|+2l\le 2m} \ell_{m-m',\alpha} D_{x,\xi}^{\alpha}\bigl(Q_j^{t'}\circ\Psi_j^{-}{}^{-1}\bigr) \tag{3.5.34}$$

with

$$\|D_{x,\xi}^{\beta} b_{l,\alpha}\| \le c_{l,\beta}\rho^{-6l-3|\beta|}. \tag{3.5.35}$$

However as $\mathsf{D}_j \ge 2$ ℓ_j are linear maps from the space of $\mathsf{D}_j\times\mathsf{D}_j$-matrices to itself; in particular $q_j' = T_j^{\dagger}\bigl(q_j^{t'}\circ\Psi_j^{-}{}^{-1}\bigr)T_j$ with unitary matrix T_j.

In the same manner solving Cauchy problem with the initial data on ∂X we get the similar statement with the reverse time and j corresponding to the incoming bicharacteristics replaced by k corresponding to the outgoing one (which is an incoming bicharacteristics in the reverse time). Finally, as $\mathcal{B}$ mapping incoming component u_j to outgoing one u_k is a unitary operator (modulo negligible) we can rewrite $\mathcal{B}Q_j' \equiv Q_{kj}''\mathcal{B}$ and therefore as $t' = t_n-\epsilon\rho^2$, $\mathsf{supp}(\chi)\subset[t_n+\epsilon\rho^2, t_n+\epsilon\rho^2]$ (3.5.21) holds.

Combining with free space propagation we arrive to Theorem 3.5.1 as long as there is just one reflection point. However combining with Proposition 3.5.4 we arrive to Theorem 3.5.1 in the general case.

3.5.3 Branching Billiards and Propagation

Theorem 3.5.5. *Let A, B be differential operators of orders m and not exceeding $(m-1)$ respectively. Let $A_{\mathcal{B}}$ be a self-adjoint operator.*

Let $Q \in \mathbf{\Psi}_{h,\varepsilon,\varepsilon}(X, \mathbb{H}, \mathbb{H})$ *with* $\varepsilon \ge h^{1/2-\delta}$. *Let us assume that* $T \le \rho^{-1}$ *and in* $\mathcal{K}^+_{[0,T]}(p, \mathsf{supp}(q))$ *conditions* (3.5.5), (3.5.6), $(3.5.23)_{1-3}$, $(3.5.24)_{1-3}$ *and* (3.5.25) *be fulfilled with* $\rho = h^{\delta'}$, $\delta' = \delta'(\delta) > 0$.

Further assume that conditions $(3.5.26)_1$, $(3.5.26)_2$ *are fulfilled and*

(3.5.36) $K^+_{[0,T]}(p,z)$ *consists of no more than* $C\rho^{-l}$ *non-branching billiards.*

Let $Q \in \mathbf{\Psi}_{h,\varepsilon,\varepsilon}$ *with the symbol supported in* $\mathcal{V}$ *which is* ε*-vicinity of* z *and* $Q' \in \mathbf{\Psi}_{h,\varepsilon',\varepsilon'}$ *with the symbol equal* 1 *in* $\Psi_{\bar t}(\mathcal{V}')$ *where* $\mathcal{V}'$ *is* $(1+\rho^3)\varepsilon$*-vicinity of* z. *Then*

$$|F_{t\to h^{-1}\tau}\chi_{T'}(t-\bar t(1-Q')e^{-ih^{-1}tA}Q| \le Ch^s \qquad \forall t \in [\tau_1,\tau_2]. \tag{3.5.37}$$

Proof. Since each non-branching billiard has no more than $C\rho^{-2}$ reflection points it is sufficient to prove theorem as

(3.5.38) Each of non-branching billiards comprising $K^+_{[0,T]}(p,z)$ has no more than 1 reflection point each.

Indeed, in the general case we can select points t'_j such that condition $(3.5.26)_2$ is fulfilled with t'_j instead of $\bar t$ and on interval $[t'_j, t''_j]$ each billiard has no more than 1 reflection point. Then statement of the theorem in the general case follows from it under restriction (3.5.38).

Then due to finite speed of propagation it is sufficient to prove this theorem as $t_1 < \bar t$ and $t_1, \bar t - t_1 \asymp \rho^3$. But then the rescaling brings us to the standard case. □

3.A Appendices

3.A.1 On Solutions to Linear Matrix Equations

Let $\mathbb{H}$ and $\mathbb{H}'$ be Hilbert spaces and let us consider the equation

$$AX - XB = C \tag{3.A.1}$$

with respect to $X \in \mathcal{L}(\mathbb{H}, \mathbb{H}')$ with given $A \in \mathcal{L}(\mathbb{H}, \mathbb{H})$, $B \in \mathcal{L}(\mathbb{H}', \mathbb{H}')$, $C \in \mathcal{L}(\mathbb{H}, \mathbb{H}')$.

Let us assume that the

(3.A.2) spectra of A and B are disjoint

and let $\mathbb{C} \supset \gamma$ be a closed counter-clockwise oriented curve, $\gamma = \partial G$ where G is a bounded domain in $\mathbb{C}$, $\mathsf{Spec}(A) \subset G$, $\mathsf{Spec}(B) \subset \complement \bar{G}$. Then

$$X = \frac{1}{2\pi i} \int_\gamma (z - A)^{-1} C (z - B)^{-1} dz \tag{3.A.3}$$

is the (unique) solution of equation (3.A.1).

Further, if $\mathbb{C} \supset \gamma'$ is a closed counter-clockwise oriented curve, $\gamma' = \partial G'$, G' is a bounded domain in $\mathbb{C}$, $\mathsf{Spec}(B) \subset G'$, $\mathsf{Spec}(A) \subset \complement \bar{G}'$ then

$$X = -\frac{1}{2\pi i} \int_{\gamma'} (z - A)^{-1} C (z - B)^{-1} dz. \tag{3.A.3}'$$

Moreover,

(3.A.4) formula (3.A.3) ((3.A.3)$'$ respectively) remains true also in the case when B (A respectively) is an unbounded operator.

If $A \geq \epsilon I, B \leq -\epsilon I$ with $\epsilon > 0$ then the (unique) solution of (3.A.1) is also given by the formula

$$X = \int_0^{+\infty} e^{-At} C e^{Bt} dt \tag{3.A.5}$$

and if $B \geq \epsilon I, A \leq -\epsilon I$ then

$$X = \int_{-\infty}^0 e^{-At} C e^{Bt} dt;. \tag{3.A.5}'$$

Furthermore,

(3.A.6) formula (3.A.5) ((3.A.5)$'$ respectively) remains true if $-A$ and B (A and $-B$ respectively) are unbounded operators which generate semigroups tending to 0 quickly enough as t tends to $+\infty$.

Let us note that if X is a solution of (3.A.1) then $X^\dagger$ is a solution of

$$X^\dagger A^\dagger - B^\dagger X^\dagger = C^\dagger. \tag{3.A.1}^\dagger$$

and hence if $\mathbb{H} = \mathbb{H}'$ and $A^\dagger = B$, $C^\dagger = -C$ then the uniqueness of solutions implies that $X^\dagger = X$.

Similarly, if $\mathbb{H}$ and $\mathbb{H}'$ are complexifications of the real spaces then the operation of complex conjugation is defined and if A, B, C are real then X is also real; moreover, if $\mathbb{H} = \mathbb{H}'$ and $A^T = B$, $C^T = -C$ then $X^T = X$.

Let us emphasize that the fundamental condition is (3.A.2).

3.A.2 Structure of Gliding Points

Assume that a and b are two Hermitean $m \times m$-matrices, a is non-degenerate and $a^{-1}b$ has the only eigenvalue 0 and $\operatorname{rank} a^{-1}b = m-1$. Let f_k ($k = 1, \dots, m$) be the corresponding (associated) eigenvectors:

$$bf_k = af_{k-1} \qquad k = 1, \dots, m;\ f_0 = 0. \tag{3.A.7}$$

Then

$$\langle af_k, f_j\rangle = \langle bf_{k+1}, f_j\rangle = \langle f_{k+1}, bf_j\rangle = \langle af_{k+1}, f_{j-1}\rangle$$

which implies that

$$\langle af_k, f_j\rangle = \rho_{j+k} \qquad \text{with } \rho_l = 0 \text{ as } l = 2, \dots, m; \tag{3.A.8}$$

however since a is non-degenerate $\rho = \rho_{m+1} \neq 0$. Then replacing f_j by $\lambda f_j + \sum_{k<j} \lambda_{j-k} f_k$ we can achieve $\rho_l = 0$ as $l = m+2, \dots, 2m$ and $\rho_{m+1} = \pm 1$.

Let ρ be a linear transformation mapping k-th coordinate vector to f_k; then $\rho^\dagger a \rho$ is of the form of a_0 in (3.1.15) (modulo factor $\varsigma = \pm 1$). Then (3.A.7) implies that modulo the same factor b is of the form of a_1 there.

Let now a and b be variable matrices being of form (3.1.15) only at point z^*. Then $g = \det(\eta - a^{-1}b)$ coincides with $r = b_{mm}$ modulo term vanishing with its first order derivatives at $z = z^*$ and $\eta = 0$.

Comments

There are many papers devoted to the propagation of singularities near the boundary.

Theorems 3.1.7 and 3.1.12 were first proven for wave front sets in V. Ivrii [2] in the less general form. We already explained that these theorems, in contrast to Theorem 2.1.2, are not very precise.

A certain analog of Theorem 3.3.6 for the wave equation was first proven in V. Ivrii [8], which was the first proof of this type of theorem without an explicit parametrix construction.

In order to exclude boundary bicharacteristics for a bicharacteristically concave domain one needs to prove results similar to those of Theorems 3.4.9, 3.4.10 etc. Energy estimates based proof was published first in V. Ivrii [7]. see also L. Hörmander [1]. Alternatively one can use explicit parametrix construction; see R. Melrose [1], G Eskin [1,2], M. Taylor [2,3,5]

where the theorem was proved by an explicit parametrix construction. Idea to combine two types of results (general but not excluding bicharacteristics for a bicharacteristically concave domain and one which excludes them at strictly concave points was used first by R. Melrose and J. Sjöstrand in [1].

Otherwise only analytic or Gevrey (with Gevrey index ≤ 3) singularities propagate along boundary bicharacteristics J. Sjöstrand [3–6], G. Lebeau [1, 2].

Propagation near edges and vertices was studied in G. Lebeau [3, 4] and in P. Gérard and G. Leabeau [1].

Propagation near conical points was studied in M. E. Taylor [6] and in J. Cheeger and M. E. Taylor [1, 2].

All this propagation results are of no importance for integrated spectral asymptotics and we do not cover them here.

One should be note that there is a huge literature devoted to diffraction partially predating microlocal analysis and coexisting with it. I just mention fundamental monographs with a richness of the material and references; we highly recommend V. A. Borovikov [1], V. A. Borovikov and B. Ye. Kinber [1] and especially V. M. Babich and V. S. Buldyrev [1].

In particular in V. A. Borovikov [1] was treated the propagation of the plane waves for constant coefficients near angles (in dimension 2), vertices and edges (in dimension 3). In addition to several plane waves appearing due to (possibly multiple) reflections from all illuminated sides or faces, there appear circular waves generated by vertices with amplitudes acquiring factor $h^{\frac{1}{2}}$ (in dimension 2) and cylindrical waves generated by edges with amplitudes acquiring factor $h^{\frac{1}{2}}$ and spherical waves generated by vertices with amplitudes acquiring factor h (in comparison to incidental plane waves).

Part II

Local and Microlocal Semiclassical Spectral Asymptotics in the Interior of the Domain

Chapter 4

General Theory in the Interior of the Domain

4.1 Introduction

4.1.1 Main Objects

In this chapter we begin the treatment of the *local and microlocal semiclassical spectral asymptotics*. At this stage we are interested in the *standard* spectral asymptotics, i.e., in asymptotics with respect to the Planck constant $h \to +0$ only; namely, in this chapter we are interested in the asymptotics of the *spectral function* $e(x,x,\tau)$ which is the restriction to the diagonal of the Schwartz kernel of the spectral projector, and the spatial means of $e(x,x,\tau)$. Further, we are interested in the *microlocal spectral asymptotics*; so our main object is either

$$\mathsf{N}_x := \Gamma_x\bigl(Q_{1x}\psi(x)e(.,.,\tau)\psi(y)\,{}^tQ_{2y}\bigr) \tag{4.1.1}$$

and

$$\mathsf{N} := \Gamma\bigl(Q_{1x}\psi(x)e(.,.,\tau)\psi(y)\,{}^tQ_{2y}\bigr) \tag{4.1.2}$$

where Qj are h-pseudodifferential operators with compactly supported symbols and $\psi_j(x)$ are compactly supported functions. We also consider averaging with respect to the spectral parameter i.e. we consider

$$\mathsf{N}_{x,\varphi,L} := \int \varphi\bigl(\frac{h}{L}\bigr)\,d_\tau\Gamma_x\bigl(Q_{1x}\psi(x)e(.,.,\tau)\psi(y)\,{}^tQ_{2y}\bigr) \tag{4.1.3}$$

and

$$\mathsf{N}_{\varphi,L} \coloneqq \int \varphi\big(\frac{h}{L}\big)\, d_\tau \Gamma\big(Q_{1x}\psi(x)e(.,.,\tau)\psi(y)\,{}^tQ_{2y}\big) \tag{4.1.4}$$

where φ is an *averaging function* and L is an *averaging parameter*. What kind of averaging functions we use we discuss later.

Let be clear: the most interesting case is (4.1.2) with $Q_1 = Q_2 = I$ and $\psi_1 = \psi_2 = 1$ and microlocalized expressions will be used in the future as building blocks for localized expressions i.e. with $Q_1 = Q_2 = I$ and those in turn will be used as building blocks for those with $\psi_1 = \psi_2 = 1$.

As an approximation to $e(x,y,\tau)$ to be used in expressions (4.1.1)–(4.1.4) we pick up first the *Tauberian approximation*

$$e^{\mathsf{T}}(x,y,\tau) \coloneqq h^{-1}\int_{-\infty}^{\tau} \Big(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)u(x,y,t)\Big)\, d\tau \tag{4.1.5}$$

where $u(x,y,t)$ is the corresponding propagator i.e. the Schwartz kernel of $e^{ih^{-1}tA}$, $\bar{\chi} \in \mathscr{C}_0^\infty([-1,1])$ equal to 1 on $[-\frac{1}{2},\frac{1}{2}]$ and $\bar{\chi}_T(t) = \bar{\chi}(t/T)$ and *usually* T will be a small constant here. Plugging $e^{\mathsf{T}}(x,y,\tau)$ into (4.1.1)–(4.1.4) we get corresponding *Tauberian approximations* $\mathsf{N}_*^{\mathsf{T}}$ and the difference between N_* and $\mathsf{N}_*^{\mathsf{T}}$ we denote by $\mathsf{R}_*^{\mathsf{T}}$ which will be called *Tauberian error*.

So, our first goal is to estimate $\mathsf{R}_*^{\mathsf{T}}$ either in the general case or under microhyperbolicity assumption. In the most papers devoted to sharp spectral asymptotics this step is masked but we prefer to make it explicit.

Surely, $u(x,y,t)$ and thus $e^{\mathsf{T}}(x,y,\tau)$ is no more implicit object than $e(x,y,\tau)$ was but actually we can calculate $\mathsf{N}_*^{\mathsf{T}}$ it with a high precision: in the standard papers it is made by constructing $u(x,y,t)$ as an oscillatory integral but this imposes very strong assumptions on operator we consider.

Instead here we use completely different approach: first using our propagation results we show that under microhyperbolicity condition one can replace in the Tauberian expressions $T \gtrsim 1$ by much smaller value $T = h^{1-\delta}$ with an arbitrarily small exponent $\delta > 0$.

On such small time interval we can construct $u(x,y,t)$ by the *successive approximation method* picking up as unperturbed operator $A(y, hD_x, 0)$. This construction is much more robust and universal than construction as an oscillatory integral.

Actually this construction does not use microhyperbolicity at all and we need to assume only that $T \le h^{\frac{1}{2}+\delta}$ with an arbitrarily small exponent

$\delta > 0$. However without microhyperbolicity it is useful only for $\mathsf{N}_{\varphi,L}$ and $\mathsf{N}_{x,\varphi,L}$ and only as $L \ge h^{\frac{1}{2}-\delta}$.

Plugging $u(x,y,t)$ into Tauberian approximations N_*^{T} we will be able to prove that with a certain precision

$$\mathsf{N}_x^{\mathsf{T}} \sim \mathsf{N}_x^{\mathsf{W}} := \sum_{0\le n\le m} \kappa_{nx}(\tau) h^{-d+n}, \tag{4.1.6}$$

or

$$\mathsf{N}^{\mathsf{T}} \sim \mathsf{N}_x^{\mathsf{W}} := \sum_{0\le n\le m} \kappa_{n}(\tau) h^{-d+n} \tag{4.1.7}$$

with

$$\kappa_n(\tau) = \int \kappa_{nx}(\tau)\, dx \tag{4.1.8}$$

and the corresponding equalities for $\mathsf{N}^{\mathsf{T}}_{x,\varphi,L}$ and $\mathsf{N}^{\mathsf{T}}_{\varphi,L}$.

Here κ_{nx} and κ_n depend on Q_1, Q_2, ψ_1, ψ_2; in particular

$$\kappa_{0x}(\tau) = (2\pi)^{-d} \int q_1^0(x,\xi)\psi_1(x) e(x,\xi,\tau)\psi_2(x) q_2^0(x,\xi)\, d\xi \tag{4.1.9}$$

where q_j are principal symbols of Q_j and

$$e(x,\xi,\tau) = \theta\big(\tau - a(x,\xi)\big) \tag{4.1.10}$$

is the spectral projector of the principal symbol $a(x,\xi)$ of operator A; as usual θ is the Heaviside function.

Expressions for next coefficients are much more complicated but they could be also calculated explicitly. These coefficients could be defined formally but without microhyperbolicity condition they could be rather irregular.

We call $\mathsf{N}_*^{\mathsf{W}}$ *Weyl approximation* and Weyl approximation is our final goal. We denote by $\mathsf{R}_*^{\mathsf{W}}$ the difference between N_* and $\mathsf{N}_*^{\mathsf{W}}$ and call it *Weyl error*.

So, on the second step we estimate $(\mathsf{N}_*^{T} - \mathsf{N}_*^{\mathsf{W}})$. Since $\mathsf{R}_x^{\mathsf{T}}$ and R^{T} usually are estimated by $O(h^{1-d})$ (but under assumptions of the global nature one can improve them marginally) it would be sufficient to pick up in $\mathsf{N}_x^{\mathsf{W}}$ and N^{W} usually $m = 0$ but sometimes $m = 1$. However estimates of $\mathsf{R}^{\mathsf{T}}_{x,\varphi,L}$ and $\mathsf{R}^{\mathsf{T}}_{\varphi,L}$ could be (much) better than $O(h^{1-d})$ and we need (much) more terms.

Remark 4.1.1. If we consider operators acting on sections then all the symbols and thus κ_{nx} are sections as well; then Γ and κ_n are not defined. However, $\Gamma\,\mathsf{tr}$ is defined and we should use instead

$$\kappa_n(\tau) = \int \mathsf{tr}(\kappa_{nx}(\tau))\,dx$$

in this case as well.

4.1.2 Kind of Assumptions to be Used

The only global condition we assume is that the operator A is self-adjoint.

All other conditions will be local or even microlocal: we derive asymptotics for $\mathsf{N}_{x,*}$ or N_* with x belonging and symbols of q_j supported in the ball $B(0,\frac{1}{2})$ under the assumption that the ball $B(0,1)$ is contained in the domain and all conditions are imposed in this ball.

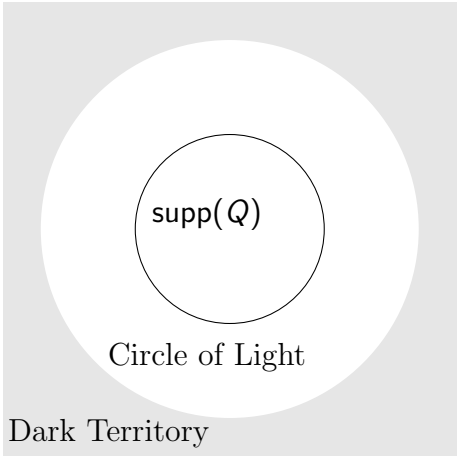


Figure 4.1: We impose condition in the "Circle of Light"; the only global condition is self-adjointness.

This is exactly true only if we want to derive asymptotics of N_x or N with an error estimate $O(h^{1-d})$; then we pick up T as a small constant.

However if we want to derive (marginally) better remainder estimates we need to pick up large T and impose some conditions at the points which we can touch by some generalized Hamiltonian trajectory of the length T starting in $B(0,\frac{1}{2})$.

Finally, there are two microhyperbolicity conditions: microhyperbolicity and ξ-microhyperbolicity:

Definition 4.1.2. We say that

(i) Symbol $a(x,\xi)$ is *microhyperbolic* at energy level τ in Ω if at each point $z\in\Omega$ symbol $(\tau - a(x,\xi))$ is microhyperbolic at z in some direction $\ell=\ell(z)$ ($|\ell(z)|\le 1$) in the sense of Definition 2.1.1;

(ii) Symbol $a(x,\xi)$ is *ξ-microhyperbolic* at energy level τ in Ω if in the framework of (i) one can select $\ell((z) = (0,\ell_\xi(z))$ at each point.

Normally microhyperbolicity is used to derive asymptotics with the spatial averaging (i.e. of N and $\mathsf{N}_{\varphi,L}$ while ξ-microhyperbolicity is used to derive asymptotics without spatial averaging (i.e. of N_x and $\mathsf{N}_{x,\varphi,L}$ but in Section 4.6 we will always need ξ-microhyperbolicity.

4.1.3 Plan of the Chapter

In Section 4.2 we introduce the basic notions, obtain the first rough estimates and under microhyperbolicity or ξ-microhyperbolicity conditions estimate

$$\Gamma\Big(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)Q_{1x}u(x,y,t)\,{}^tQ_{2y}\Big) \tag{4.1.11}$$

and

$$\Gamma_x\Big(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)Q_{1x}u(x,y,t)\,{}^tQ_{2y}\Big) \tag{4.1.12}$$

by Ch^{1-d} as $T \asymp 1$; we also derive certain estimates for larger T under non-periodicity or no-loop condition. Then using Tauberian technique we derive from (4.1.11) or (4.1.12) estimates respectively $\mathsf{R}^{\mathsf{T}} \le Ch^{1-d}$ and $\mathsf{R}^{\mathsf{T}}_x \le Ch^{1-d}$ and certain estimates for $\mathsf{R}^{\mathsf{T}}_{\varphi,L}$ and $\mathsf{R}_{x,\varphi,L}$.

Then in Section 4.3 by means of the successive approximations we construct $u(t,x,y)$ in a bounded domain in phase space for $|t| \le h^{\frac12+\delta}$ with an arbitrarily small exponent $\delta > 0$ and then we construct $(\Gamma u)(t)$ and $(\Gamma_x u)(t,x,x)$ and related functions.

Next, in Section 4.4 we prove the main general theorems. Namely, the construction of Section 4.3 immediately yields the complete asymptotic decomposition of $\mathsf{N}_{x,\varphi,L}$ as $\varphi \in \mathscr{C}_0^\infty(\mathbb{R})$ and $L \ge h^{\frac12-\delta}$. Further, under microhyperbolicity or ξ-microhyperbolicity condition respectively we prove that coefficients κ_n or κ_{nx} are smooth as $\tau \in [-\epsilon,\epsilon]$. Moreover, the same condition allows us in the Tauberian approximation (4.1.5) for N_* and $\mathsf{N}_{x,*}$ to replace small constant T by any lesser $T \ge h^{1-\delta}$ and thus we can apply construction of Section 4.3 and pass from Tauberian to Weyl approximations.

In Section 4.5 we using the long time propagation theorems increase T from small constant T_0 either to arbitrarily large constant or even to larger $T \le h^{-\delta}$ and derive even sharper remainder estimates; actually we apply these estimates not everywhere but with exception of sets of small measure (not exceeding ε). Combining with estimates there we arrive to remainder estimates

$$C_0(T^{-1}+\varepsilon)h^{1-d} + C'h^{1-d+\delta}$$

for errors R^{W} or $\mathsf{R}^{\mathsf{W}}_x$ and improved estimates with averaging.

Further, in Section 4.6 we partially generalize the results of the previous sections to operators with "rough" with respect to x symbols. As an application we derive spectral asymptotics for number of negative eigenvalues for operators with non-smooth coefficients on compact closed manifolds; here we combine results for "rough" operators with variational ideas. This is a first taste of things to come much later.

Finally, in Section 4.7 we consider operators with non-smooth ($\mathscr{C}^l$) coefficients directly by the perturbation method.

In Appendix 4.A.1 we consider the boundary values of almost-analytic functions (which are useful for the calculation of successive approximations via the Fourier transform) and in Appendix 4.A.2 we study approximations of non-smooth functions by "rough" functions.

Remark 4.1.3. (i) During the whole chapter we consider pseudodifferential operators cut-offs Q_j of different classes; we include ψ_j into them: $Q_j := Q_j\psi_j$ and thus ${}^{t}Q_j := \psi_j\, {}^{t}Q_j$.

(ii) Further, we assume by default that

$$\sup_{(x,\xi)} \|q_j^0(x,\xi)\| \le 1 \qquad \forall j \tag{4.1.13}$$

which simplifies some statements; obviously there is no loss of the generality.

4.2 Preliminary Analysis and Tauberian Method

4.2.1 General Estimates

In this chapter we consider the differential operator

$$A = \mathsf{Op}^{\mathsf{w}}(A(x,\xi,h)) = \sum_{|\alpha|+|\beta|\le m} \frac{(\alpha+\beta)!}{\alpha!\beta!}(hD)^{\alpha} a_{\alpha+\beta}(x,h)(hD)^{\beta} \tag{4.2.1}$$

with the full symbol

$$A(x,\xi,h) = \sum_{|\alpha|\le m} a_{\alpha}\xi^{\alpha} \tag{4.2.2}$$

and $h \in (0,1]$, given on the manifold X some part of which is identified with the ball $B(0,1)$ where the coefficients $a_\alpha(x,h)$ are linear (not necessarily bounded) symmetric operators in some auxiliary Hilbert space $\mathbb{H}$.

Alternatively, we consider pseudodifferential operators $A \in \boldsymbol{\Psi}_h(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ and $A \in \tilde{\boldsymbol{\Psi}}_h^{(m)}(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ and more general operators; at this moment we need only to assume that

(4.2.3) A with the given domain $\mathfrak{D}(A)$ is a self-adjoint operator in $\mathscr{L}^2(X, \mathbb{H})$

(this condition will be formulated more precisely later). Then there are Hermitean spectral projectors and an unitary propagator.

Let $E(\tau)$ be the spectral projector of A corresponding to the interval $(-\infty, \tau)$; then $E(\tau_1, \tau_2) = E(\tau_1) - E(\tau_2)$ for $\tau_1 < \tau_2$ is the spectral projector corresponding to the interval $[\tau_1, \tau_2)$. Our goal is to derive asymptotics as $h \to +0$ for various functions associated with $e(x, y, \tau_1, \tau_2)$, the Schwartz kernel of $E(\tau_1, \tau_2)$.

Our main tool is the analysis of $u(x, y, t)$, the Schwartz kernel of the propagator $U(t) := e^{ih^{-1}tA}$. Obviously,

$$u(x,y,t) = \int e^{ith^{-1}\tau} d_\tau e(x,y,\tau_1,\tau) \tag{4.2.4}$$

with an arbitrary $\tau_1 \in [-\infty, +\infty]$.

Equality (4.2.1) is understood in the sense of distributions and in the general case we cannot set $x = y$ here.

However, Proposition 1.1.11 and Theorem 1.1.12 yield that we can set $x = y$ if we first apply operators Q_{1x} and ${}^tQ_{2y}$ on the left and right respectively provided the symbols of Q_1 and Q_2 are compactly supported. Namely, since the operator norms of $U(t)$ and $E(\tau_1, \tau_2)$ do not exceed 1 Theorem 1.1.12 yields the following

Proposition 4.2.1. *Let A be an operator self-adjoint in $\mathscr{L}^2(X, \mathbb{H})$ and let functions $\psi_1, \psi_2 \in \mathscr{L}^\infty$ be supported in $B(0,1)$ and $|\psi_j| \le 1$.*

Moreover, let $q_1, q_2 \in \mathsf{S}_{h,\rho,1,K}(T^*\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ be supported in $\{|\xi| \le \rho\}$ and $Q_i = \mathsf{Op}^{\mathsf{w}}(q_i)$ $(i = 1, 2)$, with $\rho \ge h$. Then

$$\|Q_{1x}\psi_1(x) e(x,y,\tau_1,\tau_2)\psi_2(y)\, {}^tQ_{2y}\| \le C'\Big(\frac{\rho}{h}\Big)^d \tag{4.2.5}$$

$$\forall x, y \in \mathbb{R}^d \quad \forall \tau_1, \tau_2 \in \mathbb{R}$$

and

$$\|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)\big(Q_{1x}\psi_1(x)u(t,x,y)\psi_2(y)\,{}^tQ_{2y}\big)\| \le C'T\big(\frac{\rho}{h}\big)^d \qquad \forall x,y\in\mathbb{R}^d \quad \forall T\ge h,\ T'\in\mathbb{R} \tag{4.2.6}$$

where here and below $C'=C\|\!|q_1|\!|\cdot|\!|q_2|\!|$, $|\!|\!|\cdot|\!|\!|$ *means the* $\mathsf{S}_{h,\rho,1,K}$*-norm; further* $K=K(d)$, $C=C(d)$ *here,* $\chi\in\mathscr{C}_0([-1,1])$ *and* $\chi_{T,T'}(t)=\chi(\frac{t-T'}{T})$.

In the case of $\dim H=\infty$ we need to obtain a more refined assertion. Let us assume that B_1 is a (self-adjoint) operator in $\mathbb{H}$ such that

$$B_1^\dagger = B_1, \qquad B_1\ge I \tag{4.2.7}$$

where $\ge$ and $\le$ denote the standard order of symmetric operators; furthermore, assume tat the estimate

$$\|B_1^l u\|_{\mathscr{L}^2(B(0,1),\mathbb{H})} \le c\big(\|A^l u\|_{\mathscr{L}^2(X,\mathbb{H})}+\|u\|_{\mathscr{L}^2(X,\mathbb{H})}\big) \qquad \forall u\in\mathfrak{D}(A^l) \tag{4.2.8}_l$$

holds with some $l>0$.

Proposition 4.2.2. *Let the assumptions of Proposition 4.2.1 and conditions* (4.2.7) *and* $(4.2.8)_l$ *be fulfilled.*

(i) Then

$$\|Q_{1x}\psi_1(x)B_1^j e(x,y,\tau_1,\tau_2)B_1^k\psi_2(y)\,{}^tQ_{2y}\| \le C'\big(\frac{\rho}{h}\big)^d(|\tau_1|+|\tau_2|+1)^{j+k} \qquad \forall x,y\in\mathbb{R}^d \quad \forall\tau_1,\tau_2\in\mathbb{R} \quad \forall j,k\in[0,l] \tag{4.2.9}_l$$

where here and below K *and* C *also depend on* l *and* C *also depends on* c *in condition* $(4.2.8)_l$.

(ii) Moreover, if $\chi\in\mathscr{C}_0^{k+j}([-1,1])$ *and* $T\ge h$, $T'\in\mathbb{R}$ *then*

$$\|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)Q_{1x}\psi_1(x)B_1^j u(t,x,y)B_1^k\psi_2(y)\,{}^tQ_{2y}\| \le C'T|\!|\!|\chi|\!|\!|\big(\frac{\rho}{h}\big)^d(|\tau|+1)^{j+k} \qquad \forall x,y\in\mathbb{R}^d \quad \forall\tau\in\mathbb{R} \quad \forall j,k\in[0,l] \tag{4.2.10}_l$$

where $|\!|\!|\chi|\!|\!|$ *means the* $\mathscr{C}^{k+j}$*-norm of* χ.

In particular, in the framework of Proposition 4.2.1 this estimate holds for $j=k=0$.

Proof. Let us note that the interpolation theorems of J. L. Lions, and E. Magenes [1] imply that (4.2.8) holds for l replaced by arbitrary $j \in [0, l]$ and hence

$$\|B_1^j(A \pm i)^{-j}\| \le c, \qquad \|(A \mp i)^{-j} B_1^j\| \le c \tag{4.2.11}$$

where we derived the second inequality just taking an adjoint operator. Therefore

$$\|B_1^j E(\tau_1, \tau_2) B_1^k\| \le c^2 \|(A+i)^j E(\tau_1, \tau_2)(A+i)^k\| \le C(|\tau_1| + |\tau_2| + 1)^{j+k};$$

the resulting inequality combined with Theorem 1.1.12 implies $(4.2.9)_l$.

Similarly

$$\|F_{t\to h^{-1}\tau} \chi_{T,T'}(t) B_1^j U(t) B_1^k\| \le C\|F_{t\to h^{-1}\tau}(A+i)^j \chi_{T,T'}(t) U(t)(A+i)^k\| =$$
$$C\|F_{t\to h^{-1}\tau}((-hD_t + i + \tau)^{j+k} \chi_{T,T'}) U(t)\| \le CT(|\tau|+1)^{j+k} \|\!|\chi|\!\|$$

where the inequality $T \ge h$ was also used. Applying Theorem 1.1.12 again we obtain (4.2.10). □

4.2.2 Semiclassical Spectral Gaps. I

Now we are interested in the spectral gaps (of the simplest nature) appearing because some interval in $\mathbb{R}$ is free of $\mathsf{Spec}\, a(x,\xi)$ for all $(x,\xi) \in \mathsf{supp}(q)$ [1)]. To do so let us now investigate the equations

$$(hD_t - A_x)u = 0, \tag{4.2.12}$$

$$u(hD_t - {}^tA_y) = 0. \tag{4.2.12}^\dagger$$

from the elliptic point of view.

Since it is relatively simple, we do it in the general case of $\mathcal{L}(\mathbb{H}, \mathbb{H})$ valued symbols.

First Standard Set of Assumptions

Let us assume that

$$\|D_x^\beta a_\alpha(x,h) B^{-(1-\frac{|\alpha|}{m})}\| \le c \tag{4.2.13}_1$$
$$\forall x \in B(0,1) \quad \forall \beta : |\beta| \le K \quad \forall \alpha : |\alpha| \le m$$

[1)] More complicated smaller gaps of the different nature will be examined much later.

and

$$(4.2.14)_1 \quad \|a_0(x,h)w\| \ge \epsilon_0\|Bw\| - c\|w\| \qquad \forall x \in B(0,1) \quad \forall w \in \mathfrak{D}(B)$$

where

(4.2.15) B is an operator in $\mathbb{H}$ satisfying (4.2.7) [2)]

and the meaning of the subscript "1" will be clear later (see $(4.2.13)_l$, $(4.2.14)_l$ below). Moreover, let us assume that

(4.2.16) A is a self-adjoint operator in $\mathscr{L}^2(X,\mathbb{H})$ and $\mathfrak{D}(A) \supset \mathscr{C}_0^m(B(0,1),\mathfrak{D}(B))$

and

(4.2.17) $a_0(x,h)$ with $\mathfrak{D}(a_0) = \mathfrak{D}(B)$ is a self-adjoint operator in $\mathbb{H}$.

Then

(4.2.18) $a_\alpha(x,h)$ with $\mathfrak{D}(a_\alpha) = \mathfrak{D}(B)$ are symmetric operators in $\mathbb{H}$ for all $x \in B(0,1), h$ and $\alpha : |\alpha| \le m$

and

(4.2.19) $A(x,\xi,h)$ with the domain $\mathfrak{D}(B)$ is a self-adjoint operator in $\mathbb{H}$ for all $x \in B(0,1), \xi \in \mathbb{R}^d, h$

and, moreover, the conditions

$$(4.2.13)_l \quad \|D_x^\beta B^{-(1-\frac{|\alpha|}{m})(1-l)} a_\alpha(x,h) B^{-(1-\frac{|\alpha|}{m})l}\| \le c \qquad \forall x \in B(0,1) \quad \forall \beta : |\beta| \le K \quad \forall \alpha : |\alpha| \le m,$$

$$(4.2.14)_l \quad \|B^{l-1}a_0(x,h)w\| \ge \epsilon_0\|B^l w\| - c\|B^{l-1}w\| \qquad \forall x \in B(0,1) \quad \forall w \in \mathfrak{D}(B)$$

are fulfilled for all $l \in [0,1]$.

[2)] But not necessarily coinciding with B_1; in what follows $B_1 \le B$. In Remark 4.2.22 we explain why we need B_1 different from B.

In fact, (4.2.18) follows from (4.2.16), while $(4.2.13)_l$ and $(4.2.14)_l$ follow from $(4.2.13)_1$ and $(4.2.14)_1$ by means of transition to the adjoint operator, (44.2.18) and interpolation. Finally, (4.2.19) follows from $(4.2.13)_1$ with $K = 0$, $(4.2.14)_1$ and (4.2.17).

Finally, let us assume that $(4.2.20)_1$ Condition $(4.2.8)_1$ is fulfilled with $B_1 = B$.

Conditions $(4.2.13)_1$, $(4.2.14)_1$, (4.2.15)–(4.2.17) and $(4.2.20)_1$ form *the first standard set of assumptions*.

Second Standard Set of Assumptions

However, we can also deal with *the second standard set of assumptions* obtained by replacing in the previous set conditions $(4.2.13)_1$, $(4.2.14)_1$, (4.2.16) and $(4.2.20)_1$ respectively by $(4.2.13)_{\frac12}$,

$$(4.2.14)_{\pm} \qquad \pm\langle a_0(x,h)w,w\rangle \ge \epsilon_0\|B^{\frac12}w\|^2 - c\|w\|^2 \qquad \forall w \in \mathfrak{D}(B),$$

(4.2.21) $\pm A$ is a self-adjoint semibounded from below operator in $\mathscr{L}^2(X,\mathbb{H})$ and $\mathfrak{D}(A^{\frac12}) \supset \mathscr{C}_0^m(B(0,1),\mathfrak{D}(B^{\frac12}))$.

and $(4.2.20)_{\frac12}$ Condition $(4.2.8)_{\frac12}$ is fulfilled with $B_1 = B$ and preserving (4.2.15).

Obviously, then (4.2.17) and (4.2.18) hold and, moreover, (4.2.19) is replaced by $(4.2.19)_{\pm}$ $\pm A(x,\xi,h)$ is a self-adjoint semibounded from below operator in $\mathbb{H}$ with $\mathfrak{D}(A(x,\xi,h)^{\frac12}) = \mathfrak{D}(B^{\frac12})$ for all $x \in B(0,1)$, $\xi \in \mathbb{R}^d$, h; in particular, it holds for $a_0(x,h)$.

Estimates to Propagator

(i) If the first standard set of assumptions is fulfilled then multiplying the operator $(hD_t - A)$ by B^{-1} on the left or on the right we see that the parametrices G_1 and G_0 of the operators $P_1 = B^{-1}(hD_t - A)$ and $P_0 = (hD_t - A)B^{-1}$ can be constructed on $\mathsf{supp}(q) \times \mathbb{R}_t \times [\tau_1,\tau_2]$ provided

$$(4.2.22) \qquad \|(\tau - A(x,\xi,h))B^{-1}w\| \ge \epsilon_0\nu\|w\|$$

and

$$\|A^{(\alpha)}_{(\beta)}(x,\xi,h)(\tau - A(x,\xi,h))^{-1}\| \le c\rho^{-|\alpha|}\gamma^{-|\beta|} \tag{4.2.23}$$
$$\forall (x,\xi) \in \mathsf{supp}(q) \quad \forall \tau \in [\tau_1,\tau_2] \quad \forall w \in \mathbb{H} \quad \forall \alpha,\beta : |\alpha|+|\beta| \le K$$

where

$$\gamma \le 1, \quad \rho\gamma \ge h^{1-\delta}\omega^{\delta}, \quad \omega \ge 1, \quad \tau_2 - \tau_1 \ge \nu \ge \big(\frac{h}{\omega}\big)^n \tag{4.2.24}$$

with an arbitrarily small exponent $\delta > 0$ and arbitrary n (surely, K depends on n).

(ii) Similarly, if the second standard set of assumptions is fulfilled then multiplying the operator $(hD_t - A)$ by $B^{-\frac12}$ on the left and right we see that the parametrix of the operator $P_{\frac12} = B^{-\frac12}(hD_t - A)B^{-\frac12}$ can be constructed on $\mathsf{supp}(q) \times \mathbb{R}_t \times [\tau_1,\tau_2]$ provided

$$\pm\langle (A(x,\xi,h) - \tau)w, w\rangle \ge \epsilon_0\nu\|B^{\frac12}w\|^2 \qquad \forall w \in \mathfrak{D}(B) \tag{4.2.22}_{\pm}$$

and

$$\|(A(x,\xi,h) - \tau)^{-\frac12} A^{(\alpha)}_{(\beta)}(x,\xi,h)(A(x,\xi,h) - \tau)^{-\frac12}\| \le c\rho^{-|\alpha|}\gamma^{-|\beta|} \tag{4.2.23}'$$
$$\forall (x,\xi) \in \mathsf{supp}(q) \quad \forall \tau \in [\tau_1,\tau_2] \quad \forall w \in \mathbb{H} \quad \forall \alpha,\beta : |\alpha|+|\beta| \le K$$

where condition (4.2.24) is assumed to be fulfilled in this case also.

Let $q \in \mathsf{S}_{h,\rho,\gamma,K}(\mathbb{R}^d,\mathbb{H},\mathbb{H})$ and $Q = \mathsf{Op}^{\mathsf{w}}(q)$. Then in all three cases ($l = 0, 1, \frac12$) the operator norm of $Q(G_lP_l - I)$ does not exceed $C'\|\!|Q|\!\|(h\omega^{-1})^s$ where here and in what follows s is arbitrarily large and K, C depend on d, m, δ, n, s and C' also depends on c. Until further notice $l = 1$ or $l = \frac12$.

Then Theorem 1.1.12 immediately implies

Proposition 4.2.3. *(i) Let conditions* $(4.2.13)_1$, $(4.2.14)_1$, (4.2.15), (4.2.16), (4.2.17), $(4.2.20)_1$ *and* (4.2.22)–(4.2.24) *be fulfilled with either* $q = q_1$ *or* $q = q_2$ *and let us assume that*

$$|\xi| \le c\omega \qquad \forall (x,\xi) \in \mathsf{supp}(q_1) \cup \mathsf{supp}(q_2) \tag{4.2.25}$$

where $q_1, q_2 \in \mathsf{S}_{h,\rho,\gamma,K}(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ *and* $Q = \mathsf{Op}^{\mathsf{w}}(q)$ *and* $Q_i = \mathsf{Op}^{\mathsf{w}}(q_i)$. *Let* $\psi_1, \psi_2 \in \mathscr{C}_0^K(B(0, 1-\epsilon))$, $\epsilon > 0$ *be an arbitrarily small constant,*

$$T \geq h^{1-\delta}\omega^{\delta} \tag{4.2.26}$$

and $\chi \in \mathscr{C}_0^K([-1,1])$. *Then*

$$\|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)Q_{1x}B^j\psi_1(x)u(t,x,y)B^k\psi_2(y)\,{}^tQ_{2y}\| \leq C'T(|\tau|+1)^{j+k}\Big(\frac{h}{\omega}\Big)^s \tag{4.2.27}_l$$

$$\forall x, y \in \mathbb{R}^d \quad \forall \tau \in [\tau_1, \tau_2] \quad \forall j, k \in [0, l]$$

with $l = 1$ *where here* $C' = C\|\!|q_1|\!\| \cdot \|\!|q_2|\!\| \cdot \|\!|\psi_1|\!\| \cdot \|\!|\psi_2|\!\| \cdot \|\!|\chi|\!\|$, *all the norms are calculated in the spaces indicated above, and* C *also depends on* ϵ.

(ii) Let conditions $(4.2.13)_{\frac12}$, $(4.2.14)_{\pm}$, (4.2.15), (4.2.21), $(4.2.20)_{\frac12}$ *and* $(4.2.22)_{\pm}$, $(4.2.23)'$, (4.2.24), (4.2.25) *be fulfilled with either* $q = q_1$ *or* $q = q_2$ *as well as the conditions of (ii) on* $Q_1, Q_2, \psi_1, \psi_2, \chi$. *Then estimate* $(4.2.27)_l$ *holds with* $l = \frac12$.

(iii) In particular, conditions (4.2.22)–(4.2.23) *are automatically fulfilled in the* γ*-neighborhood of* $(\bar{x}, \bar{\xi})$ *with* $\gamma = h^{\frac12-\delta}$ *provided* (4.2.22) *is fulfilled at this point with* $\nu = h^{\frac12-\delta}$.

On the other hand, one can use the sharp Gårding inequality (Theorem 1.1.29) and its sharpest form for scalar operators (Theorem 1.1.31) combined with the fact that $B^{-\frac12}[A,\psi]B^{-\frac12}$ is a skew-self-adjoint operator modulo $O(h^2)$) instead of the parametrix construction and easily obtain

Proposition 4.2.4. *Let conditions* $(4.2.13)_{\frac12}$, $(4.2.14)_{\pm}$, (4.2.15), $(4.2.19)_{\pm}$ (4.2.21), *be fulfilled for* $\tau = \bar{\tau}$ *and let condition* $(4.2.22)_{\pm}$ *be fulfilled in the* ϵ*-neighborhood of* $\mathsf{supp}(q_2)$ *where* $\epsilon > 0$ *is an arbitrarily small constant. Then*

(i) Estimate $(4.2.27)_l$ *holds with* $l = \frac12$ *provided* $\nu = Ch$ *with a large enough constant* C *and* $\tau \in [\tau_1, \tau_2]$ *with* $[\tau_1, \tau_2] = [\bar{\tau}, \infty)$ *respectively.*

(ii) Further, if $a(x,\xi) = \lambda(x,\xi)$ *is a scalar-valued symbol this estimate remain true provided* $\nu = Ch^{2-\delta}$ *with a large enough constant* C.

(iii) Furthermore, reducing A *to the block-diagonal matrix* $\begin{pmatrix} A_I & 0 \\ 0 & A_{II} \end{pmatrix}$ *where* $\operatorname{Spec} A_{II}(x,\xi)$ *does not intersect* $[\tau_1-\epsilon, \tau_2+\epsilon]$ *we obtain the same assertion as in (ii) provided*

$$\operatorname{Spec} a(x,\xi) \cap [\tau_1-\epsilon, \tau_2+\epsilon] = \{\lambda(x,\xi)\}. \tag{4.2.28}$$

Finally, if $B = I$ and if in the ϵ-neighborhood of $(\bar{x}, \bar{\xi})$ either

$$\operatorname{Spec} A(x,\xi,h) \cap [\bar{\tau}-\epsilon, \bar{\tau}+\nu] = \emptyset \tag{4.2.29}_+$$

or

$$\operatorname{Spec} A(x,\xi,h) \cap [\bar{\tau}-\nu, \bar{\tau}+\epsilon] = \emptyset \tag{4.2.29}_-$$

then reducing the operator to the block structure $\begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}$ with positively defined $\pm(A_\pm - \bar{\tau})$ and applying the same arguments as above we obtain

Proposition 4.2.5. *Let condition* $(4.2.13)_1$, $(4.2.14)_1$, (4.2.15), (4.2.16) *and* (4.2.17), $(4.2.20)_1$ *be fulfilled with* $B = I$. *Furthermore, let condition* $(4.2.29)_\pm$ *be fulfilled with* $\nu = Ch$ *in the general case and with* $\nu = h^{2-\delta}$ *in the case of condition* (4.2.28).

Then for τ *residing in the indicated interval estimate* $(4.2.27)_I$ *holds with* $I = \frac{1}{2}$.

Estimates to Projector

In order to pass from estimates of $u(\cdot,\cdot,\cdot)$ to estimates of $e(\cdot,\cdot,\cdot,\cdot)$ we apply a *simplified Tauberian technique.*[3)].

Proposition 4.2.6. *(i) In the frameworks of Proposition 4.2.3(i) or (ii) the estimate*

$$\begin{aligned} \|Q_{1x}\psi_1(x) B^j e(x,y,\tau',\tau'') B^k \psi_2(y)\, {}^tQ_{2y}\| \le \\ C'\bigl(\frac{h}{\omega}\bigr)^s (|\tau'|+|\tau''|+1)^{j+k}(|\tau''-\tau'|+1) \\ \forall x,y \in \mathbb{R}^d \quad \forall \tau',\tau'' \in [\tau_1,\tau_2] \quad \forall j,k \in [0,I] \end{aligned} \tag{4.2.30}$$

holds with $I = 1$ *and* $I = \frac{1}{2}$ *respectively with the same* K *and* C *as before and with* $C' = C\|\!|q_1|\!\| \cdot \|\!|q_2|\!\| \cdot \|\!|\psi_1|\!\| \cdot \|\!|\psi_2|\!\|$.

[3)] In what follows using these words we refer to the proof of Proposition 4.2.6 below.

(ii) Moreover, the estimate holds in the frameworks of Propositions 4.2.4 and 4.2.5.

Proof. We apply Tauberian arguments. Namely, equality (4.2.4) implies the equality

$$F_{t\to h^{-1}\tau}\chi_T(t)Q_{1x}\psi_1(x)B^j u(t,x,y)B^k\psi_2(y)\,{}^tQ_{2y} = T\int \hat\chi(\frac{(\tau-\lambda)T}{h})\,d_\lambda \mathsf{N}_{x,y}(\lambda) \tag{4.2.31}$$

with

$$\mathsf{N}_{x,y}(\lambda) = Q_{1x}\psi_1(x)B^j e(x,y,\lambda_1,\lambda)B^k\psi_2(y)\,{}^tQ_{2y}. \tag{4.2.32}$$

Let us pick $j=k$, $Q_2=Q_1^*$, $\psi_2=\psi_1^\dagger$ (where † means complex conjugation) and $x=y$; then we get

$$\Gamma_x F_{t\to h^{-1}\tau}\chi_T(t)Q_{1x}\psi_1(x)B^j u(t,x,y)B^k\psi_2(y)\,{}^tQ_{2y} = T\int \hat\chi(\frac{(\tau-\lambda)T}{h})\,d_\lambda \mathsf{N}_{x}(\lambda) \tag{4.2.33}$$

with the $\mathcal{L}(\mathbb{H},\mathbb{H})$-valued function $\mathsf{N}_x(\lambda)=\Gamma_x\mathsf{N}_{x,y}(\lambda)$.

Note that in this special case $\mathsf{N}_x(\lambda)$ is monotone[4)] with respect to λ because in this case $\mathsf{N}_x(\lambda)$ is a restriction to the diagonal of the Schwartz kernel of the monotone operator-valued function $RE(\lambda_1,\lambda)R^*$ with $R=Q_1\psi_1B^j$ and $A\le B \implies \mathcal{K}_A(x,x)\le \mathcal{K}_B(x,x)$ where in the second case we talk about natural partial order in $\mathcal{L}(\mathbb{H},\mathbb{H})$ and that that operator, trace and Hilbert-Schmidt are monotone with respect to this partial order on the space of self-adjoint operators.

Let us pick a *Hörmander function* χ [5)].

Lemma 4.2.7. *(i) Let f be monotone increasing function, χ Hörmander function and*

$$\int \hat\chi(\frac{(\tau-\lambda)}{\hbar})\,d_\lambda f(\lambda)\le g(\tau) \tag{4.2.34}$$

[4)] Recall that Hermitian operators on Hilbert space are partially ordered.

[5)] That means that $\chi\in \mathscr{C}_0^\infty([-1,1])$ is even and satisfies $\chi(0)=1$, $\hat\chi(\tau)>0\ \ \forall\tau\in\mathbb{R}$ where the hat means Fourier transform. Such a function exists (see, e.g., Shubin [3]).

Then

$$|f(\tau'')-f(\tau')|\le g^*(\tau,\tau',\tau'') := C_0\hbar^{-1}\int_{\tau'}^{\tau''} g(\tau)\,d\tau + g(\tau) \qquad \forall \tau',\tau'',\tau : \tau'-\hbar\le\tau\le\tau'+\hbar, \tau'\le\tau'' \tag{4.2.35}$$

where C_0 depends only on the choice of Hörmander function.

(ii) Above assertion remains true for symmetric operator-valued functions as well.

Proof. Note first that $g(\tau)\ge 0$. Observe that the right-hand expression of (4.2.34) will not increase if we take the integral only on the interval $[\tau-\hbar,\tau+\hbar]\ni\lambda$ and replace after this $\hat\chi(\frac{(\tau-\lambda)}{\hbar})$ by $\epsilon_0=\min_{\tau\in[-1,1]}\hat\chi(\tau)>0$; then we obtain in the right-hand expression $\epsilon_0 f(\tau-\hbar,\tau+\hbar)$. Therefore

$$|f(\tau'')-f(\tau')|\le C_0 g(\tau) \qquad \forall \tau'',\tau' : |\tau'-\tau|\le\hbar, |\tau''-\tau|\le\hbar. \tag{4.2.36}$$

Then (4.2.35) is proven as $|\tau''-\tau'|\le 2\hbar$ and moreover, as $|\tau''-\tau'|=2\hbar$ we can skip the second term in the right-hand expression. This immediately yields (4.2.35) without the second term in the right-hand expression as $|\tau''-\tau'|\ge 2\hbar$. Combining with (4.2.36) we arrive to (4.2.35) without this assumption. □

This Lemma 4.2.7 with $\hbar = hT^{-1}$ and $f(\tau)=\mathsf{N}_x(\tau)$ combined with estimate $(4.2.27)_l$ yield that

$$\mathsf{N}_x(\tau-\frac{h}{T},\tau+\frac{h}{T})\le C'\big(\frac{h}{\omega}\big)^s(|\tau|+1)^{2j}I_{\mathbb{H}} \qquad \forall \tau\in[\tau_1,\tau_2]$$

where $I_{\mathbb{H}}$ is the identity operator. This inequality generalizes immediately to

$$\mathsf{N}_x(\tau',\tau'')\le C'\big(\frac{h}{\omega}\big)^s\big(\frac{|\tau''-\tau'|T}{h}+1\big)\big(|\tau'|+|\tau''|+1\big)^{2j}I_{\mathbb{H}}. \tag{4.2.37}$$

where here and in what follows $\mathsf{N}_*(\tau',\tau'')=\mathsf{N}_*(\tau'')-\mathsf{N}_*(\tau')$.

Taking an appropriate constant T we obtain (4.2.30) in the special case $Q_2=Q_1^*$, $\psi_2=\psi_1^\dagger$ and $k=j$ provided the conditions of the proposition are fulfilled with $q=q_1$.

But in this case the left-hand expression in (4.2.30) equals RR^* where for fixed x we define operator R from $\mathscr{L}^2(X,\mathbb{H})$ to $\mathbb{H}$ by

$$Rv = \big(Q_1\psi_1 B^j E(\tau',\tau'')v\big)(x) \tag{4.2.38}$$

and hence R^* acts in the opposite direction. Therefore the already proven version of (4.2.30) implies that

$$\|R\| = \|R^*\| \le C'\big(\frac{h}{\omega}\big)^s(|\tau'|+|\tau''|+1)^j(|\tau''-\tau'|+1)^{\frac12}. \tag{4.2.39}$$

On the other hand, Proposition 4.2.2 and the same arguments as before yield the estimate

$$\|S\| = \|S^*\| \le C'\big(\frac{h}{\omega}\big)^{-\frac{d}{2}}(|\tau'|+|\tau''|+1)^k(|\tau''-\tau'|+1)^{\frac12} \tag{4.2.40}$$

where S is defined in the same way as R was defined but with Q_1,ψ_1,x replaced by Q_2,ψ_2,y. Since the left-hand expression in (4.2.30) equals the norm of RS^* we obtain (4.2.30) in complete form; recall that exponent s is chosen arbitrarily. □

Let us consider explicitly the case in which the second standard set of assumptions holds. Then $(4.2.14)_\pm$ implies that if

$$\pm\tau \le -C_0\langle\xi\rangle^m \tag{4.2.41}_\pm$$

with $C_0 = C_0(d,m,c)$ [6] then conditions $(4.2.22)_\pm$ and $(4.2.23)'$ are fulfilled with $\nu=1$, $\rho = |\tau|^{\frac1m}$, $\gamma=1$ and moreover

$$\|(\tau - A(x,\xi,h))^{-1}\| \le C|\tau|^{-1}. \tag{4.2.42}$$

In fact, $(4.2.13)_{\frac12}$ implies that

$$|\langle (A(x,\xi,h)-a_0(x,h))w,w\rangle| \le C_0|\xi|^m\|w\|^2 + C_0|\xi|\cdot\|B^{\frac{m-1}{2m}}w\|^2$$

and then $(4.2.14)_\pm$ and $(4.2.41)_\pm$ yield that

$$\pm\langle (A(x,\xi,h)-\tau)w,w\rangle \ge C_0^{-1}(|\tau|\cdot\|w\|^2 + \|B^{\frac12}w\|^2)$$

[6] We use the notation C_0 instead of C in the case when in its definition c can be calculated for K replaced by $K_0 = 1,2,3$ where the definition of K_0 in every case we leave to the reader.

and this inequality yields $(4.2.22)_{\pm}$, (4.2.42) and $(4.2.23)'$.

Therefore if $q_i \in \mathsf{S}_{h,\rho,\gamma,K}(T^*\mathbb{R}^d, \mathbb{H}, \mathbb{H})$, $Q_i = \mathsf{Op}^{\mathsf{w}}(q_i)$ and (4.2.24), (4.2.25) hold and if $\pm\tau \le -C_0\omega^m$, then (4.2.30) holds for $\tau' = \tau$, $\tau'' = 2\tau$; hence the estimate

$$\|Q_{1x}\psi_1(x)B^j e(x,y,\tau,2\tau)B^k\psi_2(y)\,{}^tQ_{2y}\| \le C' h^s|\tau|^{-s} \quad \forall x,y\in\mathbb{R}^d \quad \forall j,k\in[0,l] \tag{4.2.43}$$

holds with $Q_1 = \zeta(\bar\omega^{-1}hD)$ where $\bar\omega = C_0^{-1}|\tau|^{\frac{1}{m}}$, $\zeta \in \mathscr{C}_0^K(B(0,1))$.

It easily follows that this estimate holds for $Q_1 = \mathsf{Op}^{\mathsf{w}}(q_1)$ provided $q_1 \in \mathsf{S}_{h,\rho,\gamma,K}$ is supported in $\{|\xi| \le C_0^{-1}|\tau|^{\frac{1}{m}}\}$, $\pm\tau \le -C_0$ and condition (4.2.24) is fulfilled; here $l = \frac{1}{2}$.

Moreover, if conditions $(4.2.13)_1$, $(4.2.14)_1$ and $(4.2.20)_1$ are fulfilled then one can take $l = 1$. Replacing τ by $2^p\tau$ and summing with respect to $p = 0, 1, 2, \ldots$ we obtain

Proposition 4.2.8. *Let conditions* $(4.2.13)_{\frac{1}{2}}$, $(4.2.14)_{\pm}$, (4.2.15), (4.2.21) *and* $(4.2.20)_{\frac{1}{2}}$ *be fulfilled and let* $q_i \in \mathsf{S}_{h,\rho,1,K}(T^*\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ *satisfy* (4.2.25) *and* ρ *and* $\gamma = 1$ *satisfy* (4.2.24). *Let* $\psi_{1,2}$ *be as in Proposition 4.2.3. Then*

$$\|Q_{1x}\psi_1(x)B^j e(x,y,\mp\infty,\tau)B^k\psi_2(y)\,{}^tQ_{2y}\| \le C' h^s|\tau|^{-s} \quad \forall x,y\in\mathbb{R}^d \quad \forall \tau : \pm\tau \le -C_0\omega^m \quad \forall j,k\in[0,l] \tag{4.2.44}$$

with $l = \frac{1}{2}$*; moreover, if conditions* $(4.2.13)_1$, $(4.2.14)_1$ *and* $(4.2.20)_1$ *are fulfilled then one can take* $l = 1$.

Estimates without Cut-offs

The following two assertions show how one can set $x = y$ without multiplication by cutting operators.

Proposition 4.2.9. *Let conditions* $(4.2.13)_1$ *and* $(4.2.14)_1$ *be fulfilled and let*

$$|\xi|^m\|w\| + \|Bw\| \le c(\|A(x,\xi,h)w\| + \|w\|) \quad \forall x\in B(0,1) \quad \forall \xi\in\mathbb{R}^d \quad \forall w\in\mathfrak{D}(B). \tag{4.2.45}$$

(i) Then $(4.2.20)_1$ *holds.*

Further let us assume that $q_1, q_2 \in \tilde{\mathsf{S}}^0_{h,\rho,\gamma,K}(T^*\mathbb{R}^d, \mathbb{H}, \mathbb{H})$, $Q_i = \mathsf{Op}^{\mathsf{w}}(q_i)$, (2.1.25) *holds and* ψ_1, ψ_2 *are as in Proposition 4.2.3. Then*

(ii) If either q_1 *or* q_2 *is supported in* $\{|\xi| \ge \omega\}$ *with* $\omega \ge C_0$ *then*

$$\|D_x^\alpha D_y^\beta Q_{1x}\psi_1(x)B^j e(x,y,\tau',\tau'')B^k\psi_2(y)\,{}^tQ_{2y}\| \le C'\Big(\frac{h}{\omega}\Big)^s \tag{4.2.46}$$
$$\forall x, y \in \mathbb{R}^d\ \forall \tau', \tau'' : |\tau'| \le C_0^{-1}\omega,\ |\tau''| \le C_0^{-1}\omega$$
$$\forall j, k \in [0, l]\ \forall \alpha, \beta : |\alpha| + |\beta| \le r$$

where K *and* C' *now also depend on* r, r *arbitrarily large, and the* $\tilde{\mathsf{S}}$*-norms of* $q_{1,2}$ *are now used in the definition of* C'*, and* $l = 1$ *here.*

(iii) In the general case

$$\|D_x^\alpha D_y^\beta Q_{1x}\psi_1(x)B^j e(x,y,\tau',\tau'')B^k\psi_2(y)\,{}^tQ_{2y}\| \le \tag{4.2.47}$$
$$C' h^{-d-|\alpha|-|\beta|}(|\tau'| + |\tau''| + 1)^{\frac{1}{m}(d+|\alpha|+|\beta|)+j+k}$$
$$\forall x, y \in \mathbb{R}^d \quad \forall \tau', \tau'' \quad \forall j, k \in [0, l] \quad \forall \alpha, \beta : |\alpha| + |\beta| \le r.$$

Proof. (i) We will prove our claim that $(4.2.20)_1$ is fulfilled later (see Subsection 4.2.3).

(ii) Proposition 4.2.6 implies estimate (4.2.46) under the additional assumption that q_1 and q_2 are supported in $\{|\xi| \le 4\omega\}$ and either q_1 or q_2 is supported in $\{|\xi| \ge \omega\}$ where as usual $Q_j = \mathsf{Op}(q_j)$.

Replacing ω by 2ω and summing over a partition of unity by means of pseudodifferential operators with symbols supported in $\{2^p\omega \le |\xi| \le 2^{p+2}\omega\}$, $p = 0, 1, 2, \dots$ we obtain (ii); more precisely, we can take two partitions (one to the left of Q_1 and another to the right of Q_2, and p means the largest index).

(iii) Assertion (iii) follows from (i) and Proposition 4.2.3. □

Proposition 4.2.10. *Let conditions* $(4.2.13)_{\frac12}$, $(4.2.14)_\pm$, (4.2.15), (4.2.21) *be fulfilled and let*

$$\pm\langle A(x,\xi,h)w, w\rangle \ge \epsilon_0(|\xi|^m\|w\|^2 + \|B^{\frac12}w\|^2) - c\|w\|^2 \tag{4.2.48}_\pm$$
$$\forall x \in B(0,1) \quad \forall \xi \in \mathbb{R}^d \quad \forall w \in \mathfrak{D}(B).$$

(i) Then $(4.2.20)_{\frac12}$ *holds.*

Further, let us assume that $q_1, q_2 \in \tilde{\mathsf{S}}^0_{h,\rho,\gamma,K}(T^*\mathbb{R}^d, \mathbb{H}, \mathbb{H})$, $Q_i = \mathsf{Op}^{\mathsf{w}}(q_i)$, (4.2.24) *holds and* $\psi_{1,2}$ *are as in Proposition 4.2.3. Then*

(ii) The estimate

$$(4.2.49)_{\pm} \quad \|D_x^\alpha D_y^\beta Q_{1x}\psi_1(x)B^j e(x,y,\mp\infty,\tau)B^k\psi_2(y)\,{}^tQ_{2y}\| \le C' h^s|\tau|^{-s} \qquad \forall x,y\in\mathbb{R}^d \quad \forall \pm\tau \le -C_0 \quad \forall j,k\in[0,l]$$

holds with $l = \frac12$*; moreover, if either the symbol of* Q_1 *or the symbol of* Q_2 *is supported in* $\{|\xi| \ge \omega\}$ *then this estimate remains true with the additional factor* ω^{-s} *in the right-hand expression.*

(iii) In the general case the following estimate holds:

$$(4.2.50)_{\pm} \quad \|D_x^\alpha D_y^\beta Q_{1x}\psi_1(x)B^j e(x,y,\mp\infty,\tau)B^k\psi_2(y)\,{}^tQ_{2y}\| \le C' h^{-d-|\alpha|-|\beta|}(|\tau|+1)^{\frac1m(d+|\alpha|+|\beta|)+j+k} \qquad \forall x,y\in\mathbb{R}^d \quad \forall j,k\in[0,l]$$

with $l = \frac12$.

(iv) Moreover, if conditions $(4.2.13)_1$, $(4.2.14)_1$ *are fulfilled then all these estimates hold with* $l = 1$.

Proof. The proof is similar to that of Proposition 4.2.9 but also uses Proposition 4.2.8. □

Semiclassical Spectral Gaps under Reduced Smoothness Assumptions

The following assertions follow from Propositions 4.2.6–4.2.10 under smoothness conditions which are too restrictive; here we give an independent proof.

Proposition 4.2.11. *Let conditions* $(4.2.13)_1$, $(4.2.14)_1$, (4.2.15), (4.2.16) *and* (4.2.17) *be fulfilled with* $K = \mathsf{max}(1+\frac{d}{2}, m) + r$. *Then*

(i) If condition (4.2.45) *is fulfilled then*

$$\|D_x^\alpha D_y^\beta B^j e(x,y,\tau',\tau'')B^k\| \le C' h^{-d-|\alpha|-|\beta|}(|\tau'|+|\tau''|+1)^{\frac{1}{m}(d+|\alpha|+|\beta|)+j+k} \tag{4.2.51}$$
$$\forall x,y \in B(0,1-\epsilon) \quad \forall \tau',\tau'' \quad \forall \alpha,\beta : |\alpha| \le r, |\beta| \le r \quad \forall j,k \in [0,l]$$

with $l = 1$ *where here and below* C' *equals* $C(d,m,r,c,\epsilon)$ *multiplied by norm of all auxiliary symbols and functions.*

(ii) If the following condition

$$\|(A(x,\xi,h)-\tau)w\| \ge \epsilon((\omega+|\xi|^m)\|w\| + \|Bw\|) \tag{4.2.52}$$
$$\forall x \in B(0,1) \quad \forall \xi \in \mathbb{R}^d \quad \forall \tau \in [\tau_1,\tau_2] \quad \forall w \in \mathfrak{D}(B)$$

is fulfilled with $\omega \ge 1$ *then*

$$\|D_x^\alpha D_y^\beta B^j e(x,y,\tau',\tau'')B^k\| \le C'\bigl(\frac{h}{\omega}\bigr)^s(|\tau'|+|\tau''|+1)^{j+k} \tag{4.2.53}$$
$$\forall x,y \in B(0,1-\epsilon) \quad \forall \tau',\tau'' \in [\tau_1,\tau_2] \quad \forall \alpha,\beta : |\alpha| \le r, |\beta| \le r \quad \forall j,k \in [0,l]$$

with $l = 1$ *where* s *is arbitrary and* C *depending also on* s.

Proposition 4.2.12. *Let conditions* $(4.2.13)_{\frac12}$, $(4.2.14)_\pm$, (4.2.15), (4.2.21) *and* $(4.2.48)_\pm$ *be fulfilled with* $K = \max(1+\frac{d}{2}, m) + r$. *Then*

$$\|D_x^\alpha D_y^\beta B^j e(x,y,\mp\infty,\tau)B^k\| \le \begin{cases} C' h^s(|\tau|+1)^{-s} & \forall \tau \le -C_0 \\ C' h^{-d-|\alpha|-|\beta|}(|\tau|+1)^{\frac{1}{m}(d+|\alpha|+|\beta|)+j+k} & \forall \tau \end{cases} \tag{4.2.54}$$
$$\forall x,y \in B(0,1-\epsilon) \quad \forall \alpha,\beta : |\alpha| \le r, |\beta| \le r \quad \forall j,k \in [0,l]$$

with $l = \frac12$.

Moreover, if conditions $(4.2.13)_1$ *and* $(4.2.14)_1$ *are fulfilled then this estimate holds with* $l = 1$.

Proof of propositions 4.2.11 and 4.2.12. Let us note that if (4.2.45) is fulfilled, $v \in \mathscr{C}_0^m(B(0,1), \mathfrak{D}(B))$ and

$$Av = \sum_{|\beta|\le m-l} (hD)^\beta w_\beta$$

with $k = 0, \dots, m$ then

$$\sum_{|\alpha|\le k} \|(hD)^\alpha B^{\frac{1}{m}(l-|\alpha|)} v\| \le \sum_{|\beta|\le m-k} C\|B^{\frac{1}{m}(-m+k+|\beta|)} w_\beta\| + C\|v\|. \tag{4.2.55}$$

In fact, one can easily prove this estimate in the constant coefficient case by means of the Fourier transform. In the general case let ψ_n be a partition of unity in a neighborhood of $B(0,1)$ with small enough supports and let A_n be obtained from A by freezing coefficients at the point $x^n \in B(0,1) \cap \mathsf{supp}(\psi_n)$. Then

$$A_n \psi_n v = \sum_{|\beta|\le m-k} \psi_n (hD)^\beta w_\beta + (A_n - A)\psi_n v + [A, \psi_n] v;$$

representing the last two terms in the form

$$\sum_{|\alpha|\le k, |\beta|\le m-k} (hD)^\beta b_{\beta\alpha} (hD)^\alpha v$$

with small coefficients $b_{\beta\alpha}$ for $|\alpha| = k$, $|\beta| = m-k$ and applying estimate (4.2.55) to A_n we obtain that $M_1 \le \nu M_1 + C_\nu M_2$ where M_1 and M_2 are the left-hand and right-hand expressions respectively in (4.2.55) and one can choose $\nu > 0$ arbitrarily small (before choosing the partition of unity). Hence (4.2.55) holds in the general case.

Similarly one can prove that for $k = m+1, \dots, K$ the estimate

$$\sum_{l-m\le|\alpha|\le k} \|(hD)^\alpha B^{\frac{1}{m}(k-|\alpha|)} v\| \le C \sum_{|\beta|\le m-k} C\|(hD)^\beta Av\| + C\|v\| \tag{4.2.56}$$

holds for $v \in \mathscr{C}_0^m(B(0,1), \mathfrak{D}(B))$; one should use in this deduction estimate (4.2.55).

Repeatedly substituting ψv for v where $\psi \in \mathscr{C}_0^\infty(B(0, 1-\epsilon))$ in (4.2.55) and (4.2.56) we obtain that both these estimates remain true for $v \in \mathscr{C}^p(B(0,1), \mathfrak{D}(B))$ with $p = \max(m,k)$ if in the left-hand and right-hand expressions we take the norms in the balls $B(0, 1-\epsilon)$ and $B(0, 1-\frac{\epsilon}{2})$ respectively; here C also depends on $\epsilon > 0$.

Further, iterating if necessary we obtain that for $|\alpha| \le k, j \in [0,1]$ and $\frac{1}{m}|\alpha| + j \in \mathbb{Z}$

$$\|(hD)^\alpha B^j v\|_{B(0,1-\epsilon)} \le C(\|A^{\frac{1}{m}|\alpha|+j} v\| + \|v\|).$$

Substituting $E(\tau', \tau'')v$ for v we conclude that

(4.2.57) The operator norm of the operator $(hD)^\alpha B^j E(\tau', \tau'') : \mathscr{L}^2(X, \mathbb{H}) \to \mathscr{L}^2(B(0, 1-\epsilon), \mathbb{H})$ does not exceed $C(|\tau'| + |\tau''| + 1)^{\frac{1}{m}|\alpha|+j}$;

interpolation and estimate (4.2.56) again yield that this estimate remains true without restriction $\frac{1}{m}|\alpha| + j \in \mathbb{Z}$.

Then the embedding inequalities imply that

(4.2.58) The operator norm of the operator

$$R : \mathscr{L}^2(X, \mathbb{H}) \ni v \to ((hD)^\alpha B^j E(\tau', \tau'')v)(x) \in \mathbb{H}$$

with $x \in B(0, 1-\epsilon)$ does not exceed $Ch^{-\frac{d}{2}-|\alpha|}(|\tau'| + |\tau''| + 1)^{\frac{d}{2m}+\frac{1}{m}|\alpha|+j}$.

Obviously, $D_x^\alpha(-D_y^\beta)B^j e(x, y, \tau', \tau'')B^i = RS^*$ where S is defined in the same way as R but with β, i, y instead of α, j, x. Therefore we obtain (4.2.51).

In a similar way one can prove two estimates below:

$$\|D_x^\alpha D_y^\beta F_{t\to h^{-1}\tau}\chi_T(t)B^j U(t)B^i\| \le C' T h^{-|\alpha|-|\beta|}(|\tau|+1)^{\frac{1}{m}(|\alpha|+|\beta|)+j+i} \quad \forall \tau \quad \forall \alpha, \beta : |\alpha| \le k, |\beta| \le k \quad \forall j, i \in [0,1] \tag{4.2.59}$$

where $T \ge h^{-1}$, $\chi \in \mathscr{C}_0^\infty([-1,1])$ is a fixed function, $\|\cdot\|$ means the operator $\mathcal{L}(\mathscr{L}^2(B(0,1-\epsilon), \mathbb{H}), \mathscr{L}^2(B(0,1-\epsilon), \mathbb{H}))$-norm and the operator is applied only to functions supported in $B(0, 1-\epsilon)$; and also

$$\|D_x^\alpha D_y^\beta F_{t\to h^{-1}\tau}\chi_T(t)B^j u(t,x,y)B^i\| \le C' T h^{-d-|\alpha|-|\beta|}(|\tau|+1)^{\frac{1}{m}(d+|\alpha|+|\beta|)+j+i} \quad \forall x, y \in B(0,1-\epsilon) \quad \forall \tau \quad \forall \alpha, \beta : |\alpha| \le k, |\beta| \le k \quad \forall j, i \in [0,1] \tag{4.2.60}$$

where $\|\cdot\|$ means the $\mathcal{L}(\mathbb{H}, \mathbb{H})$-norm.

In this deduction we use (4.2.56), (4.2.12), (4.2.12)† and (4.2.59) with $\alpha = \beta = 0$, $j = i = 0$ which follows from the fact that $U(t)$ is an unitary operator.

On the other hand, if condition (4.2.52) is fulfilled then the arguments used to deduce (4.2.55) and (4.2.56) yield that these estimates remain true if in them as well as in their conditions A is replaced by $A - \tau$ with $\tau \in [\tau_1, \tau_2]$ and the last term in their right-hand expressions (namely $C\|v\|$) is replaced by $'(\frac{h}{\omega})^s\|v\|$.

These modified estimates, (4.2.12), (4.2.12)† and (4.2.59) imply that for $T \ge h^{1-\delta}\omega^{-\delta}$, $\tau \in [\tau_1, \tau_2]$ the left-hand expressions in (4.2.59) and (4.2.60) do not exceed

$$C'\big(\frac{h}{\omega}\big)^s\big(|\tau|+1\big)^{\frac{1}{m}(|\alpha|+|\beta|)+j+k-1} \quad \text{and} \quad C'\big(\frac{h}{\omega}\big)^s\big(|\tau|+1\big)^{\frac{1}{m}(d+|\alpha|+|\beta|)+j+k-1}$$

respectively.

Then the Tauberian arguments used in the proof of Proposition 4.2.6 yield estimate (4.2.53). So Proposition 4.2.11 is proven.

Let us assume that $(4.2.48)_\pm$ is fulfilled. Then the arguments used to derive estimates (4.2.55) and (4.2.56) imply that

(i) If

$$Av = \sum_{|\beta|\le m-k} (hD)^\beta w_\beta,$$

$k = 0, 1, \dots, \frac{m}{2}$ (obviously m is even in this case) then

$$\sum_{|\alpha|\le k} \|(hD)^\alpha B^{\frac{1}{m}(k-|\alpha|)}v\| \le \sum_{|\beta|\le m-k} C\|B^{-\frac{1}{2}+\frac{1}{m}(-\frac{m}{2}+k+|\beta|)_+}w_\beta\| + C\|B^{\frac{1}{2}}v\| \tag{4.2.61}$$

for $v \in \mathscr{C}_0^m(B(0,1), \mathfrak{D}(B^{1/2}))$.

(ii) For $k \ge \frac{m}{2}$

$$\sum_{\frac{m}{2}\le|\alpha|\le k} \|(hD)^\alpha B^{\frac{1}{m}(k-|\alpha|)}v\| \le \sum_{|\beta|\le k-\frac{m}{2}-i} C\|B^{\frac{i}{m}-\frac{1}{2}}Av\| + C\|B^{\frac{1}{2}}v\| \tag{4.2.62}$$

where $v \in \mathscr{C}_0^p(B(0,1), \mathfrak{D}(B^{\frac{1}{2}}))$, $i = 0, 1, \dots, \min(\frac{m}{2}, k - \frac{m}{2})$, $p = \max(m, k)$ is arbitrary.

(iii) Moreover, these estimates remain true if we replace A by $A-\tau$ with $\pm\tau \le -C_0$, add

$$|\tau| \sum_{|\alpha|\le k-\frac{m}{2}-\frac{i}{2}} \|(hD)^{\alpha} B^{-\frac{1}{2}+\frac{i}{m}} v\|$$

to the left-hand expression and in the right-hand expression replace $C\|B^{\frac{1}{2}}v\|$ by $Ch^{s}|\tau|^{-s}\|B^{\frac{1}{2}}v\|$.

(iv) Finally, all these estimates remain true for $v \in \mathscr{C}^{p}(B(0,1), \mathfrak{D}(B^{\frac{1}{2}}))$ if in their left-hand and right-hand expressions we take norms on $B(0,1-\epsilon)$ and $B(0,1-\frac{\epsilon}{2})$ respectively.

Then the arguments used in the proof of Proposition 4.2.11 yield Proposition 4.2.12. □

4.2.3 Main Tauberian Estimate

Now we are going to discuss the main scheme used to derive spectral asymptotics. We will use the series of different abstract statements. Because of the fundamental importance of these statements we will discuss them in all details.

Main Tauberian Formula

Lemma 4.2.13. *Assume that f satisfies* (4.2.35). *Let $\chi \in \mathscr{C}_0^{\infty}([-1,1])$ and $\chi(0)=1$. Then*

$$|\big(f(\tau'')-f(\tau')-f^{\mathsf{T}}(\tau'',\tau')\big)| \le C_1\mathcal{R}(\tau')+C_1\mathcal{R}(\tau'') \tag{4.2.63}$$

with the Tauberian expression

$$f^{\mathsf{T}}(\tau'',\tau') := \hbar^{-1}\int_{\tau'}^{\tau''}\Big(\int \hat{\chi}\Big(\frac{(\tau-\lambda)}{\hbar}\Big)\,d_{\lambda}f(\lambda)\Big)\,d\tau, \tag{4.2.64}$$

$$\mathcal{R}(\tau) := g(\tau)+\int g(\tau+\hbar z)(1+|z|^2)^{-m}\,dz \tag{4.2.65}$$

and $C_0^{-1}C_1$ depending only on χ and arbitrarily large m.

Proof. Let us rewrite the left-hand expression in (4.2.34) as

$$\frac{d}{d\tau}\int \hat{\chi}\Big(\frac{(\tau-\lambda)}{\hbar}\Big)f(\lambda)\,d\lambda.$$

Integrating from τ' to τ'' we get that

$$f^{\mathsf{T}}(\tau'',\tau') = f^{\mathsf{T}}(\tau'') - f^{\mathsf{T}}(\tau') \tag{4.2.66}$$

where

$$f^{\mathsf{T}}(\tau) \coloneqq \hbar^{-1}\int \hat{\chi}\Big(\frac{(\tau-\lambda)}{\hbar}\Big)f(\lambda)\,d\lambda. \tag{4.2.67}$$

Then $\chi(0)=1$ implies that

$$f^{\mathsf{T}}(\tau) - f(\tau) = \hbar^{-1}\int \hat{\chi}\Big(\frac{(\tau-\lambda)}{\hbar}\Big)\big(f(\lambda)-f(\tau)\big)\,d\lambda;$$

then inequalities $|\hat{\chi}(\eta)|\le C(|\eta|+1)^{-2m-1}$ and (4.2.35) imply that

$$|f^{\mathsf{T}}(\tau)-f(\tau)| \le C_0\hbar^{-1}\int (|\tau-\lambda|\hbar^{-1}+1)^{-2m-1}\Big(\hbar^{-1}\Big|\int_\tau^\lambda g(\tau')\,d\tau'\Big| + g(\tau)\Big)\,d\lambda.$$

Opening parenthesis, changing order of integration in the double integral and calculating the simple integrals we arrive to

$$C_1\hbar^{-1}\int g(\tau')\,d\tau'(|\tau'-\lambda|\hbar^{-1}+1)^{-2m} + C_1 g(\tau);$$

this expression is obviously equal to $C_1\mathcal{R}(\tau)$. This estimate combined with (4.2.66) implies (4.2.63)–(4.2.65). □

We will call (4.2.63)–(4.2.65) *main Tauberian formula*. (with an error estimate).

Asymptotics with Mollification

We consider now mollifications with kernel $L^{-1}\varphi_L(\tau)$ where $\varphi\in \mathscr{C}_0^\infty([-1,1])$ and

$$\varepsilon \coloneqq \hbar/L \le 1. \tag{4.2.68}$$

Let us consider

$$\int L^{-1}\varphi_L(\tau-\bar\tau)\big(f(\tau)-f^{\mathsf T}(\tau)\big)\,d\tau=\int L^{-1}\tilde\varphi_L(\tau-\bar\tau)f(\tau)\,d\tau \tag{4.2.69}$$

with $\tilde\varphi := \varphi * \varepsilon^{-1}\hat\chi_\varepsilon - \varphi$. Assuming that

$$\chi(0)=1,\qquad \chi^{(k)}(0)=0\quad \forall k=1,2,... \tag{4.2.70}$$

we note that $\varepsilon^{-n}\tilde\varphi$ is an uniformly $\mathscr{C}_0^\infty$ function for all n and then repeating arguments of the proof of Lemma 4.2.13 one can prove easily

Lemma 4.2.14. *Let f satisfy* (4.2.35). *Let* (4.2.68), (4.2.70) *be fulfilled. Let $\varphi\in\mathscr{C}_0^\infty([-1,1])$.*

Then

$$\Big|\int \varphi_L(\tau-\bar\tau)\big(f(\tau)-f^{\mathsf T}(\tau)\big)\,d\tau\Big|\le C\mathcal{R}(\bar\tau)\big(\frac{\hbar}{L}\big)^s \tag{4.2.71}$$

where s is arbitrarily large and $C_0^{-1}C$ depends on χ,φ,s, and m.

Moreover

$$\Big|\int \varphi_L(\tau-\bar\tau)\big(f(\tau,\bar\tau)-f^{\mathsf T}(\tau,\bar\tau)\big)\,d\tau\Big|\le C\mathcal{R}(\bar\tau). \tag{4.2.72}$$

Formula (4.2.71) is a *mollified Tauberian formula*, and (4.2.72) is an *alternative Tauberian formula*. The advantage of the latter is that it invokes only one energy level.

Asymptotics with Averaging

Let us consider *averaging functions* with singularities[7]. Let us assume that

(4.2.73) $\vartheta\in\mathscr{C}(\mathbb{R}^+)$ is a positive monotone function on $\mathbb{R}^+$, such that

$$\vartheta(2z)\le c\vartheta(z)\qquad \forall z\in(0,1) \tag{4.2.74}$$

and $\varphi\in\mathscr{C}^\infty(\dot{\mathbb{R}}^+)$, compactly supported in $[0,\infty]$ such that its derivatives $\varphi^{(m)}$ satisfy

$$|\varphi^{(m)}(z)|\le c_m\vartheta(z)z^{-m}\qquad \forall z>0\ \forall m\ge 0. \tag{4.2.75}$$

[7] The term "mollifying" is reserved for smooth functions.

Lemma 4.2.15. *Let f satisfy* (4.2.35) *and let* (4.2.68) *and* (4.2.70) *be fulfilled. Further, assume that φ satisfies* (4.2.73)–(4.2.75).

Then

$$|\int \varphi_L(\tau-\bar\tau)\bigl(f(\tau)-f^{\mathsf T}(\tau)\bigr)\,d\tau| \le C\mathcal{R}(\bar\tau)\vartheta\bigl(\frac{\hbar}{L}\bigr) \tag{4.2.76}$$

where $C_0^{-1}C$ depends on χ,φ,ϑ, and m.

Moreover as $\varphi_i,\vartheta_i,L_i$ satisfy (4.2.73)–(4.2.75), (4.2.68) *then*

$$|\iint \varphi_{1L_1}(\tau-\bar\tau_1)\varphi_{2L_2}(\tau'-\bar\tau_2)\bigl(f(\tau,\tau')-f^{\mathsf T}(\tau,\tau')\bigr)\,d\tau d\tau'| \le C\sum_{i=1,2}\mathcal{R}(\bar\tau_i)\vartheta_i\bigl(\frac{\hbar}{L_i}\bigr) \tag{4.2.77}$$

Proof. It is sufficient to prove only estimate (4.2.76) since (4.2.77) follows from it due to the same arguments as before. Without any loss of the generality one can assume that as $L=1$; otherwise we can make a rescaling.

One can find $\psi\in\mathscr{C}_0^\infty([\frac13,1])$ such that $\sum_{n\ge1}\psi(2^{-n}z)=1$ on $(0,1)$. Then

$$\varphi(z)=\sum_{n\ge0,L_n\ge h}\vartheta(L_n)\phi_n(zL_n^{-1})+\vartheta(h)\bar\phi(z/h)$$

where $\phi_n\in\mathscr{C}_0^\infty([\frac13,1])$ uniformly and $L_n=2^{-n}$ and $\bar\phi\in\mathscr{L}^\infty$ is supported in $[0,2]$.

Then according to estimates with mollification contribution of each term $\vartheta(L_n)\phi_n(zL_n^{-1})$ to (4.2.76) does not exceed $C\mathcal{R}(\bar\tau)(hL_n^{-1})^m\vartheta(L_n)$ with arbitrarily large m.

One can see easily that $L^{-m/2}\vartheta(L)\le ch^{-m/2}\vartheta(h)$ for large enough m and therefore the total contributions of these terms does not exceed the same expression as $L_n=h$ which is exactly the right-hand expression of (4.2.76). On the other hand, estimate without mollification implies that contribution of the last term $\vartheta(h)\bar\phi(zh^{-1})$ does not exceed the same result. □

Remark 4.2.16. (i) The most prominent application are asymptotics of Riesz means with $\varphi(z)=\vartheta(z)=z_+^s/\Gamma(s+1)$.

(ii) Obviously Lemma 4.2.15 extends to functions

$$\varphi(z) = \int \phi(z - z'; z')\, df(z'), \qquad \text{with} \quad \int |d\mu(z')| \le c \tag{4.2.78}$$

where $\phi(z; z')$ satisfy (4.2.75) with respect to z uniformly with respect to z' and μ is a function of the bounded variation.

One may try to find an alternative description of functions of this type.

Operator Valued Functions

Remark 4.2.17. Obviously all these results hold when f and g are functions with the values in the space of symmetric operators. In this case each inequality of the type $|F| \le G$ is replaced by $F' \prec G$ which is a short notation for

$$|(Fw, w)| \le (Gw, w) \qquad \forall w \in \mathfrak{D}(G)$$

where G is self-adjoint operator. We say that G *dominates* F.

4.2.4 Applications

Scheme of Things

Let us consider some applications to the spectral asymptotics. From now again $\chi \in \mathscr{C}_0^\infty([-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1])$ and $\bar\chi \in \mathscr{C}_0^\infty([-1, 1])$, $\bar\chi = 1$ on $[-\frac{2}{3}, \frac{2}{3}]$.

Let $Q_k = \mathsf{Op}^{\mathsf{w}}(q_k)$ be a fixed h-pseudodifferential operators with symbols $q_k \in \mathsf{S}_{h,K}(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$, $\mathsf{supp}(q_k) \Subset \Omega$ and let us assume that A is microhyperbolic on Ω on energy level 0. This is a crucial assumption.

Then one can apply Theorem 2.1.19 estimates

$$\|F_{t\to h^{-1}\tau}\chi_{T'}(t)\Gamma(Q_{1x}u\,{}^tQ_{2y})\| \le Ch^{1-d}\bigl(\frac{h}{T}\bigr)^s \tag{2.1.58}$$

and

$$\|F_{t\to h^{-1}\tau}\chi_{T}(t)\Gamma_x(Q_{1x}u\,{}^tQ_{2y})\| \le Ch^{1-d}\bigl(\frac{h}{T}\bigr)^s \quad \forall x \in \mathbb{R}^d \tag{2.1.59}$$

hold for all $\tau : |\tau| \le \varepsilon_0$, where for the second estimate we need ξ-microhyperbolicity. Due to self-adjoint properties of A we have $M = 1$ in (2.1.43)–(2.1.45) and therefore in (2.1.58), (2.1.59). Recall that in these estimates $T \in [\bar T, T_0]$ is arbitrary where $\bar T = h$ and T_0 is a small constant.

Then making t-admissible partition on $[\bar{T}, T_0]$ we derive that

$$\|F_{t\to h^{-1}\tau}\big(\bar{\chi}_{T_0}(t) - \bar{\chi}_{\bar{T}}(t)\big)\Gamma(Q_{1x}u\,{}^tQ_{2y})\| \le Ch^{1-d} \tag{4.2.79}$$

and

$$\|F_{t\to h^{-1}\tau}\big(\bar{\chi}_{T_0}(t) - \bar{\chi}_{\bar{T}}(t)\big)\Gamma_x(Q_{1x}u\,{}^tQ_{2y})\| \le Ch^{1-d} \quad \forall x \in \mathbb{R}^d \tag{4.2.80}$$

respectively.

On the other hand, there are "dumb" estimates without any microhyperbolicity condition

$$\|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(Q_{1x}u\,{}^tQ_{2y})\| \le CTh^{-d} \tag{4.2.81}$$

and

$$\|F_{t\to h^{-1}\tau}\bar{\chi}_T\Gamma_x(Q_{1x}u\,{}^tQ_{2y})\| \le CTh^{-d} \quad \forall x \in \mathbb{R}^d. \tag{4.2.82}$$

Here $T \ge h$ is arbitrary. Plugging $T = \bar{T} = h$ we get

$$\|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(Q_{1x}u\,{}^tQ_{2y})\| \le CM_Th^{1-d} \tag{4.2.83}$$

and

$$\|F_{t\to h^{-1}\tau}\bar{\chi}_T\Gamma_x(Q_{1x}u\,{}^tQ_{2y})\| \le CM_Th^{1-d} \quad \forall x \in \mathbb{R}^d \tag{4.2.84}$$

as $T = \bar{T}$. Then *by virtue of* (4.2.79), (4.2.80) *these estimates hold with* $T = T_0$ where actually $M_T = 1$, $T = T_0$ but we do not need to make these assumptions if just assume that either (4.2.83) or (4.2.84) hold.

Note that all these estimates (2.1.58), (2.1.59), (4.2.83), (4.2.84) remain valid if we substitute $\bar{\chi}_T(t)$ by $\chi_{T_0}(t)\bar{\chi}_T(t)$. Therefore estimate (4.2.83) or (4.2.84) hold with $\chi_{T_0}(t)$ instead of $\bar{\chi}_{T_0}(t)$.

In particular, it holds with Hörmander function $\chi_T(t)$ where from now we use T instead of T_0.

Let us recall that $U(t) = \int e^{ih^{-1}\tau t}\, d_\tau E(\tau)$ and thus we are dealing with

$$f(\tau) = \langle \Gamma Q_{1x} e(.,.,\tau)\, {}^tQ_{2y} w, w\rangle \tag{4.2.85}$$

and

$$f_x(\tau) = \langle \Gamma_x Q_{1x} e(.,.,\tau)\, {}^tQ_{2y} w, w\rangle \tag{4.2.86}$$

respectively which are monotone functions as $Q_1 = Q_2^*$; here $w \in \mathbb{H}$ is arbitrary with $\|w\| = 1$.

Thus we conclude that (4.2.34) holds with

$$g(\tau) = \begin{cases} CM_T h^{1-d} & \text{as } |\tau| \le \epsilon_0 \\ Ch^{-d} T(|\tau+1|)^{2l} & \text{as } |\tau| \ge \epsilon_0 \end{cases} \tag{4.2.87}$$

Further, plugging $Q_2 = Q_1^*$ and $j = k$ we have monotone function $f(\tau)$ and due to virtue of Lemma 4.2.7 we conclude that in this case $f(\tau)$ satisfies (4.2.35).

Now it is time to drop this restriction. Since

$$Q_1 E(\tau,\tau') Q_2^* = Q_1 E(\tau,\tau') \cdot E(\tau,\tau') Q_1^* = (E(\tau,\tau') Q_1^*)^* E(\tau,\tau') Q_1^* \tag{4.2.88}$$

we conclude that

Proposition 4.2.18. *Under microhyperbolicity condition*

$$\|R(\tau,\tau')\|_{\mathsf{HS}} \le C(g^*(\tau,\tau',\tau))^{\frac12} \tag{4.2.89}$$

and under ξ-microhyperbolicity condition

$$\|R_x(\tau,\tau')\|_{\mathsf{HS}} \le C(g^*(\tau,\tau',\tau))^{\frac12} \tag{4.2.90}$$

with g^* defined by (4.2.35) *and g defined by* (4.2.87) *where*

$$R(\tau,\tau') \coloneqq R_w(\tau,\tau') : \mathscr{L}^2(X,\mathbb{H}) \ni f \to \langle Q_1 E(\tau,\tau') f, w\rangle \in \mathscr{L}^2(X,\mathbb{C})$$

and

$$R_x(\tau,\tau') \coloneqq R_{w,x}(\tau,\tau') : \mathscr{L}^2(X,\mathbb{H}) \ni f \to \langle Q_1 E(\tau,\tau') f, w\rangle(x) \in \mathbb{C}.$$

However the same is true for operator S defined as R but with Q_2, w' instead of Q_1, w; further, in the second case one may substitute x by y.

Then the trace norm (and thus the trace) of operator RS^* does not exceed the right-hand expression of (4.2.35). One can see easily that the trace of $(RS^*)(\tau',\tau'')$ is exactly

$$f(\tau,\tau') = \langle \Gamma Q_{1x} e(.,.,\tau,\tau')\, {}^t Q_{2y} w', w\rangle \tag{4.2.91}$$

and

$$f_{x,y}(\tau,\tau') = \langle Q_{1x} e(.,.,\tau)\, {}^t Q_{2y} w', w\rangle \tag{4.2.92}$$

where in the second case we have f depending not only on w, w' but on x, y as well. Let us recall that $w, w' \in \mathbb{H}$ are arbitrary with $\|w\| = \|w'\| = 1$. Thus we can junk w, w' and estimate $\mathcal{L}(\mathbb{H},\mathbb{H})$ norms of operators rather than their "bra-kets" with w, w'.

Therefore we arrive to the following

Proposition 4.2.19. *(i) Under microhyperbolicity assumption* f *satisfies* (4.2.35) *with* g *defined by* (4.2.87).

(ii) Under ξ*-microhyperbolicity assumption* $f_{x,y}$ *satisfies* (4.2.35) *with* g *defined by* (4.2.87).

So, what We Get?

Now we can apply Lemmas 4.2.13, 4.2.13 and 4.2.15; plugging $T = T_0$ and calculating g^* and $\mathcal{R}$ according to (4.2.35), (4.2.65) and (4.2.87) we get immediately

Proposition 4.2.20. *(i) Under microhyperbolicity assumption on energy levels* $\bar\tau$ *and* $\bar\tau'$ *on* $\mathsf{supp}(q_1) \cup \mathsf{supp}(q_2)$

$$\|\mathsf{N}(\bar\tau, \bar\tau') - \mathsf{N}^{\mathsf{T}}(\bar\tau, \bar\tau')\| \le Ch^{1-d} \tag{4.2.93}$$

and

$$\|\mathsf{N}_{\varphi,L}(\bar\tau) - \mathsf{N}^{\mathsf{T}}_{\varphi,L}(\bar\tau)\| \le Ch^{1-d}\vartheta\big(\frac{h}{L}\big); \tag{4.2.94}$$

(ii) Under ξ*-microhyperbolicity assumption on energy levels* $\bar\tau$ *and* $\bar\tau'$ *on* $\mathsf{supp}(q_1) \cup \mathsf{supp}(q_2)$

$$\|\mathsf{N}_{x,y}(\bar\tau, \bar\tau') - \mathsf{N}^{\mathsf{T}}_{x,y}(\bar\tau, \bar\tau')\| \le Ch^{1-d} \tag{4.2.95}$$

and

$$\|\mathsf{N}_{x,y,\varphi,L}(\bar\tau) - \mathsf{N}^{\mathsf{T}}_{x,y,\varphi,L}(\bar\tau)\| \le Ch^{1-d}\vartheta\big(\frac{h}{L}\big). \tag{4.2.96}$$

Remark 4.2.21. (i) In Statement (i) we need microhyperbolicity only on $\mathsf{supp}(q_1) \cap \mathsf{supp}(q_2)$ since this expression change negligibly as we replace Q_1 and Q_2 by $Q_1' \equiv Q_1$ and $Q_2 \equiv Q_2$ in the vicinity of $\mathsf{supp}(q_1) \cap \mathsf{supp}(q_2)$.

(ii) From Proposition 4.2.18 it follows that

$$\|\mathsf{N}_{x,y}(\tau', \tau'')w\|_{\mathscr{H}} \le Ch^{-\frac{d}{2}}\big(|\tau' - \tau''| + h\big)^{\frac{1}{2}} \tag{4.2.97}$$

as both $\tau;, \tau''$ are close to $\bar\tau$ and $w \in \mathbb{H}$, $\|w\| = 1$ where $\mathscr{H} = \mathscr{L}^2(\mathbb{R}^{2d}_{x,y}, \mathbb{H})$ under microhyperbolicity assumption and $\mathscr{H} = \mathscr{L}^2(\mathbb{R}^{d}_{y}, \mathbb{H})$ under ξ microhyperbolicity assumption. Then repeating arguments of the proofs of Lemmas 4.2.13–4.2.15 we arrive to

$$\|\big(\mathsf{N}_{x,y}(\bar\tau, \bar\tau') - \mathsf{N}^{\mathsf{T}}_{x,y}(\bar\tau, \bar\tau')\big)w\|_{\mathscr{H}} \le Ch^{-\frac{d-1}{2}} \tag{4.2.98}$$

and

$$\|\big(\mathsf{N}_{x,y,\varphi,L}(\bar\tau)-\mathsf{N}^{\mathsf{T}}_{x,y,\varphi,L}(\bar\tau)\big)w\|_{\mathscr{H}}\le Ch^{-\frac{d-1}{2}}. \tag{4.2.99}$$

(iii) We also get the same array of formulae with Q_{1x} and ${}^tQ_{2y}$ replaced by $Q_{1x}B_1^j$ and $B_1^k\,{}^tQ_{2y}$ respectively (thus Q_2 is replaced by $Q_2B_1^k$).

It looks that instead of difficult to calculate expressions N_* we got equally difficult to calculate expressions $\mathsf{N}_*^{\mathsf{T}}$. However it is not completely true. First of all in some special cases we can construct $u(x,y,t)$ as an oscillatory integral and calculate corresponding Tauberian expressions.

However for N^{T}, $\mathsf{N}^{\mathsf{T}}_{\varphi,L}$ or $\mathsf{N}^{\mathsf{T}}_{x}$, $\mathsf{N}^{\mathsf{T}}_{x,\varphi,L}$ under microhyperbolicity or ξ-microhyperbolicity condition respectively we can apply Theorem 2.1.19 again and replace $T=T_0$ by *any* $T\in[h^{1-\delta},T_0]$. So, we need to be able to calculate these expressions with really small T. Thus we need to be able to construct $u(x,y,t)$ for $|t|\le T$. We will do it in sSection 4.3 by the successive approximation method.

On the other hand, there is no Theorem 2.1.19 for expressions with $x\ne y$; it is just plain false. Still we will be able to get some less sharp results.

We would be able to derive better estimates if we increased T without significant increase M_T. Everything is left for Sections 4.4, 4.5 after calculations of Section 4.3.

4.2.5 Estimates in the Hilbert Scale

Hilbert Scale

In conclusion we return to the scale of Hilbert spaces $\mathbb{H}_j=\mathfrak{D}(B_1^j)$ with norms $\|u\|_j=\|B_1^ju\|$, $j\in[0,l]$ where B_1 satisfies $(4.2.8)_l$. Moreover, we introduce the spaces $\mathbb{H}_j=\mathbb{H}^\dagger_{-j}$ for $j\in[-l,0]$. So far we used sometimes Hilbert scale associated with B and with $l=\bar l$ where $\bar l=1$ or $\bar l=\frac12$ depending on assumptions[8)].

Let us assume that

$$\|(a_0(x,h)+i)^{-1}w\|_j\le c\|w\|_{j-1}\qquad\forall w\in\mathbb{H}_{j-1}, \tag{4.2.100}$$

[8)] We use l as notation for "new" l and adopt $\bar l$ as a value of "old" $l=1,\frac12$.

$$(4.2.101)\quad \|(a_0(x,h)+i)^{-1}D^\beta a_\alpha(x,h)w\|_j \le c\|w\|_{j-\frac{|\alpha|}{m}}$$
$$\forall w\in \mathbb{H}_{j-\frac{|\alpha|}{m}}\cap \mathfrak{D}(B^l)\quad \forall x\in B(0,1)\quad \forall \beta:|\beta|\le K\quad \forall j\in[0,l]$$

and let us also assume that

$$(4.2.102)_l\qquad \|B_1^{\bar l}w\|\le c\|B^{\bar l}w\|\qquad \forall w\in\mathfrak{D}(B^{\bar l}).$$

Remark 4.2.22. Let us explain why we need to have different operators B and B_1. Consider for example applications to asymptotics of number of eigenvalues of the elliptic operator in domain with thick cusps[9]. After rescaling cusp is replaced by a cylinder $X\times Y$ where X and Y are the base and cross-section of the cusp respectively and instead of the old spatial variable we have (x,y). We replace $\mathscr{L}^2(X\times Y)$ by $\mathscr{L}^2(X,\mathbb{H})$ with $\mathbb{H}=\mathscr{L}^2(Y)$; actually there will be weight but it is not important at the moment.

However we are interested in the genuine trace $\mathsf{Tr}(E(\tau))=\Gamma\,\mathsf{tr}(e(.,.,\tau))$ where $e(x,z,\tau)$ is a "matrix" operator (i. e. operator in $\mathscr{L}(\mathbb{H},\mathbb{H})$) and tr is a "matrix trace" (i.e. trace in $\mathscr{L}(\mathbb{H},\mathbb{H})$). In this example $B=(h^2\Delta_Y+I)^{m/2}$ with the Dirichlet boundary conditions where Δ_Y is positive Laplacian in Y, so associated Hilbert scale would be $\mathbb{H}_j=\mathscr{H}^{jm}_0(Y)$ (we consider Dirichlet boundary condition on the border of the cusp $X\times\partial Y$ [10].

There will be a problem since we need to increase $\bar l=1$ (or $\bar l=\frac12$) to l large enough to have operator of imbedding $I:\mathbb{H}_{l/2}\to\mathbb{H}_{-l/2}$ of the trace class but increasing l in the above scale would not be possible. However it will be possible in the scale $\mathbb{H}_j=\mathscr{H}^{jm}(Y)$. So as B_1 we will take operator generating this scale.

Remark 4.2.23. Note that in our estimates B_1^j and B_1^k appear inside, between Q_{1x} and $e(.,.,\tau)$ and between $e(.,.,\tau)$ and ${}^tQ_{y2}$ respectively and thus in order to estimate trace we need to move them outside. Thus we need to assume that $B_1^jQ_1B_1^{-j}$ and $B_1^{-k}Q_2B_1^k$ are (uniformly) bounded operators. However usually Q_1 and Q_2 are scalar h-pseudodifferential operators.

Further, let us assume that $K=K(d,m,s)$ is large enough. Then

(4.2.103) $u\in\mathscr{H}^s_{\mathsf{loc}}(B(0,1),\mathbb{H}_k\cap\mathfrak{D}(B^{1-\bar l}))$, $Au\in\mathscr{H}^s_{\mathsf{loc}}(B(0,1),\mathbb{H}_j)$ imply that $u\in\mathscr{H}^{s'}_{\mathsf{loc}}(B(0,1),\mathbb{H}_{j+1})$ with $k=j$, $s'=s-m$, $s\in\mathbb{R}$

[9] See in details in Section 12.2. Cusp is *thick* if the standard Weyl asymptotics either fails or does not provide good remainder estimate.

[10] Neumann Laplacian there is a completely different animal; see Section 12.7.

This deduction remains true for $k = 0$, $s' = s - (j+1)m$. Therefore

(4.2.104) $u \in \mathscr{H}^s_{\mathsf{loc}}(B(0,1), \mathfrak{D}(B^{1-\bar{l}}))$ and $A^l u \in \mathscr{H}^s_{\mathsf{loc}}(B(0,1), \mathbb{H})$, $l \in \mathbb{Z}^+$ imply $u \in \mathscr{H}^{s'}_{\mathsf{loc}}(B(0,1), \mathbb{H}_l)$ with an admissible s'.

Then these statements imply the inequality

$$\|\psi B_1^l u\|_{(s)} \le C'(\|A^l u\| + \|u\|) \tag{4.2.105}$$

where $\psi \in \mathscr{C}_0^\infty(B(0,1-\epsilon))$; here $\|\cdot\|_{(s)}$ means the Sobolev $\mathscr{H}^s(\mathbb{R}^d, \mathbb{H})$-norm, and $s = s(m,l) < 0$, $C' = C\|\psi\|_{\mathscr{C}^K}$, $C = C(d,m,l,c,\epsilon)$.

On the other hand, let us assume that inequality (4.2.105) is fulfilled for all $\psi \in \mathscr{C}_0^K(B(0,1-\frac{\epsilon}{2}))$ and that under certain conditions we have proven either estimate (4.2.61) or estimate (4.2.62) for some q_1 and q_2 and for all $\psi_1, \psi_2 \in \mathscr{C}_0^K(B(0,1-\frac{\epsilon}{2}))$ with $q_1 = l$, $q_2 = l$ in neighborhoods of $\mathsf{supp}(q_1')$, $\mathsf{supp}(q_2')$ respectively.

Then one can see easily that this estimate ((4.2.61) or (4.2.62)) will also be fulfilled for q_1', q_2', B_1^j, B_1^k with $j,k \in [0,l]$ instead of q_1, q_2, B^j, B^k with $j,k \in [0,\bar{l}]$ respectively; however, one should replace in the modified estimate T, ϵ, s by $T(1-\epsilon)$, 2ϵ, $s - K_0(m,l)$ respectively.

Estimates in the Hilbert Scale

In particular, we arrive to

Proposition 4.2.24. *Let conditions* (4.2.100)–(4.2.102)$_l$ *be fulfilled. Then one can enhance Propositions 4.2.6, 4.2.8–4.2.12, 4.2.18–4.2.20 to 4.2.6*, 4.2.8*–4.2.12*, 4.2.18*–4.2.20* where * means the following modification:*

(4.2.106) *we replace B^j, B^k with $j,k \in [0,\bar{l}]$ by B_1^j, B_1^k with $j,k \in [0,l]$. One can enhance* (4.2.97)–(4.2.99) *in the same way.*

Moreover, let us assume that

$$\|(A(x,\xi,h)+i)^{-1}w\|_j + |\xi|^m \cdot \|(A(x,\xi,h)+i)^{-1}w\|_{j-1} \le c\|w\|_{j-1} \qquad \forall w \in \mathbb{H}_{j-1} \tag{4.2.107}$$

and

$$\|(A(x,\xi,h)+i)^{-1}D^\beta a_\alpha(x)w\|_j \le c\|w\|_{j-\frac{|\alpha|}{m}} \qquad \forall w \in \mathbb{H}_{j-\frac{|\alpha|}{m}} \cap \mathfrak{D}(B^l) \quad \forall x \in B(0,1) \quad \forall \xi \in \mathbb{R}^d. \tag{4.2.108}$$

Then one can easily prove by the standard elliptic methods that for $B(0,1-\epsilon)$ estimate $(4.2.8)_l$ holds instead of (4.2.105) and then it is easy to prove by the same methods

Proposition 4.2.25. *Under conditions* (4.2.100)–$(4.2.102)_l$ *and* (4.2.107)–(4.2.108) *one can enhance propositions 4.2.9–4.2.11 to propositions 4.2.9*–4.2.11* where * means the same modification* (4.2.106)*.*

We leave trivial details to the reader. Again all final statements will come in Section 4.4. Here we are more interested in the scheme of things.

4.3 The Method of Successive Approximations for Propagator

Now our purpose is to construct propagator $u(x,y,t)$ on a rather short interval $[-T,T]$ with $T \ge h^{1-\delta}$. We will do it for $T = h^{\frac{1}{2}-\delta}$.

4.3.1 The Method of Successive Approximations in the Standard Ccase

Pilot-Construction

Let us temporarily assume that $X = \mathbb{R}^d$ and that $A \in \mathbf{\Psi}_{h,K}(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ (so we consider neither operator B nor Hilbert scale $\mathbb{H}_j$). We also assume that K is large enough (a non-smooth case will be treated separately in Section 4.6).

Moreover, let us assume that A is a self-adjoint operator in $\mathscr{L}^2(\mathbb{R}^d, \mathbb{H})$ and let $u(t,x,y)$ be the Schwartz kernel of the operator $U(t) = e^{ih^{-1}tA}$.

Then (4.2.12) (namely, $(hD_t - A_x)u = 0$) and the initial condition $u|_{t=0} = \delta(x-y)I$ imply the equations

$$(hD_t - A)u^{\pm} = \mp ih\delta(x-y)\delta(t)I \tag{4.3.1}$$

where $u^{\pm} = \theta(\pm t)u$, $\theta(.)$ is the Heaviside function, $I = I_{\mathbb{H}}$ is the identity operator in $\mathbb{H}$ and here and in what follows *all the operators without index y act on x and t.*

Let

$$\bar{A} = a(y, hD_x) \tag{4.3.2}$$ so the operator $\bar{A}$ is the principal part of operator A frozen at the point $x = y$;

then

$$A = \bar{A} + R \tag{4.3.3}$$

where

$$R = \sum_{1\le|\alpha|+p\le N} (x-y)^{\alpha} h^{j} A_{\alpha,p}, \tag{4.3.4}$$

(4.3.5) Operators $A_{\alpha,p} = A_{\alpha,p}(y, hD_x)$ for $|\alpha|+p < N$ and $A_{\alpha,p} = A_{\alpha,p}(x, y, hD_x, h)$ for $|\alpha| + p = N$ belong to $\boldsymbol{\Psi}_{h,K-N}(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$.

Then (4.3.1) yields the equality

$$(hD_t - \bar{A})u^{\pm} = \mp ih\delta(x-y)\delta(t)I + Ru^{\pm} \tag{4.3.6}$$

and hence

$$u^{\pm} = \mp ih\bar{G}^{\pm}\delta(x-y)\delta(t)I + \bar{G}^{\pm}Ru^{\pm} \tag{4.3.7}$$

where $\bar{G}^{\pm}$ and $G^{\pm}$ are parametrices of the problems

$$(hD_t - \bar{A})v = f, \qquad \mathsf{supp}(v) \subset \{\pm(t-t_0) \ge 0\}, \tag{4.3.8}_{\pm}$$

$$(hD_t - A)v = f, \qquad \mathsf{supp}(v) \subset \{\pm(t-t_0) \ge 0\} \tag{4.3.8}'_{\pm}$$

respectively with $\mathsf{supp}(f) \subset \{\pm(t-t_0) \ge 0\}$ for some $t_0 \in \mathbb{R}$.

Moreover, equation (4.3.1) yields that

$$u^{\pm} = \mp ihG^{\pm}\delta(x-y)\delta(t)I. \tag{4.3.9}$$

Iterating (4.3.7) N times and then substituting (4.3.9) we obtain the equality

$$u^{\pm} = \mp ih \sum_{0\le n\le N-1} (\bar{G}^{\pm}R)^n \bar{G}^{\pm}\delta(x-y)\delta(t) \mp ih(\bar{G}^{\pm}R)^n G^{\pm}\delta(x-y)\delta(t). \tag{4.3.10}$$

Therefore the method of the successive approximations works provided

(4.3.11) An operator norm of $\bar{G}^{\pm}R$ does not exceed $h^{\delta'}$ with an arbitrarily small exponent $\delta' > 0$;

then one needs to take $N = N(s, \delta')$ to get $O(h^{-d+s})$ error.

Observe that on the time interval $[0, T]$ the operator norms of the parametrices $\bar{G}^{\pm}$ do not exceed CTh^{-1} (due to the Duhamel principle; see below) and $R = O(|x-y|+h) = O(T)$ for $T \geq h$ due to the finite speed of propagation property and normally one cannot improve this.

Therefore the restriction to T is $T^2 \leq h^{1+\delta'}$ or

$$T \leq h^{\frac{1}{2}+\delta} \qquad \text{with } \delta > 0. \tag{4.3.12}$$

This is the heuristic idea but the actual implementation will be slightly different.

Transformation of the Successive Approximation Formula

Namely, let us consider the factors $(x_j - y_j)$ entering into all the copies of R defined by (4.3.4) and let us transfer them from the left to the right (to $\delta(x-y)$). If some factor reaches $\delta(x-y)$ then the corresponding term vanishes and so only those terms survive in which all factors $(x_j - y_j)$ are killed. There are two possibilities for killing $(x_j - y_j)$:

(i) Commuting it with a pseudodifferential operator; then there arises a pseudodifferential operator again and an additional factor h;

(ii) Commuting it with either $\bar{G}^{\pm}$ or $G^{\pm}$. Let us note that for $\bar{P} = hD_t - \bar{A}$

$$\bar{P}(x_j - y_j)v = (x_j - y_j)\bar{P}v - [\bar{A}, (x_j - y_j)]v$$

and $\operatorname{supp}((x_j - y_j)v)$ is contained in $\{\pm(t - t_0) \geq 0\}$ provided $\operatorname{supp}(v)$ is contained here; therefore

$$(x_j - y_j)v = \bar{G}^{\pm}(x_j - y_j)\bar{P}v - \bar{G}^{\pm}[\bar{A}, (x_j - y_j)]v$$

and we finally obtain the equality

$$(x_j - y_j)\bar{G}^{\pm} = \bar{G}^{\pm}(x_j - y_j) - \bar{G}^{\pm}[\bar{A}, (x_j - y_j)]\bar{G}^{\pm}; \tag{4.3.13}$$

similarly

$$(x_j - y_j)G^{\pm} = G^{\pm}(x_j - y_j) - G^{\pm}[A, (x_j - y_j)]G^{\pm}. \tag{4.3.13$'$}$$

Therefore as a result of commuting of $(x_j - y_j)$ with $\bar{G}^{\pm}$ there appear a pseudodifferential operator, a factor h and an additional factor $\bar{G}^{\pm}$; the same is also true for $G^{\pm}$.

Now our goal is to prove that

(4.3.14) All the terms in (4.3.10) containing more than $(N-1)$ factors h and $(x_j - y_j)$ are negligible in some appropriate sense; in particular, one can then drop the last term in (4.3.10) and in all the remaining terms in R (defined by (4.3.4)) we can drop all terms with $|\alpha| + p = N$; all the remaining terms we call *regular*.

Recall that we consider the case $X = \mathbb{R}^d$, $A \in \boldsymbol{\Psi}_{h,N}(X, \mathbb{H}, \mathbb{H})$; later we will justify the same procedure in other situations. Then the Duhamel equality

$$(\bar{G}^{\pm} v)(t, .) = ih^{-1} \int_{\pm t' \le \pm t} \bar{U}(t - t') v(t', .)\, dt' \tag{4.3.15}$$

with the unitary operator $\bar{U}(t) = e^{ih^{-1} t \bar{A}}$ and a similar formula for $G^{\pm}$ imply that

(4.3.16) $\mathcal{L}\big(\mathscr{L}^2([-T, T] \times X, \mathbb{H}), \mathscr{L}^2([-T, T] \times X, \mathbb{H})\big)$-norms of $\bar{G}^{\pm}$ and $G^{\pm}$ do not exceed $2Th^{-1}$.

Moreover, the equality

$$\bar{G}^{\pm}(\delta(t) v) = \pm ih^{-1} \theta(\pm t) \bar{U}(t) v \tag{4.3.17}$$

and a similar equality for $G^{\pm}$ imply that the

(4.3.18) $\mathcal{L}\big(\mathscr{L}^2([-T, T] \times X, \mathbb{H}), \mathscr{L}^2(X, \mathbb{H})\big)$-norms of the operators $v \to \bar{G}^{\pm}(\delta(t) v)$ and $v \to G^{\pm}(\delta(t) v)$ do not exceed $T^{1/2} h^{-1}$.

Hence we conclude that

(4.3.19) $\mathcal{L}\big(\mathscr{L}^2([-T, T] \times X, \mathbb{H}), \mathscr{L}^2([-T, T] \times X, \mathbb{H})\big)$-norm[11] of every term of decomposition (4.3.10) does not exceed $C' h^{r-k} T^{p+k+\frac{1}{2}}$ where r, p and $k+1$ are the original numbers of factors h, $(x_j - y_j)$, $\bar{G}^{\pm}$ (or $G^{\pm}$) respectively.

Obviously $k \le p + r$ and therefore the norm in question does not exceed $C' h^{\sigma(p+r)}$ with $\sigma = \sigma(\delta) > 0$ provided T satisfies (4.3.12) with an arbitrarily small exponent $\delta > 0$. Therefore (4.3.14) is proven.

Applying Theorem 1.1.12 we arrive to

[11] One can consider this term as the Schwartz kernel of an operator and calculate its operator norm.

Proposition 4.3.1. *Let $A \in \boldsymbol{\Psi}_{h,K}(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$, $q_1, q_2 \in \mathsf{S}_{h,K}(T^*\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ be supported in $\{|\xi| \le c\}$, $Q_i = \mathsf{Op}^{\mathsf{w}}(q_i)$, $i = 1, 2$, $\chi \in \mathscr{C}_0^K([-1, 1])$ and $T \in [h^{1-\delta}, h^{\frac{1}{2}+\delta}]$ with $\delta > 0$. Then*

$$\|F_{t\to h^{-1}\tau}\chi_T(t)Q_{1x}\big(u^{\pm}\pm ih\sum_{0\le n\le N-1}(\bar{G}^{\pm}R')^n\delta(x-y)\delta(t)\big)^tQ_{2y}\| \le C'h^s \quad \forall x, y \in \mathbb{R}^d \quad \forall \tau \in \mathbb{R} \tag{4.3.20}$$

where

$$R' = \sum_{1\le|\alpha|+p\le N-1}(x-y)^\alpha h^j A_{\alpha,p}(y, hD_x), \tag{4.3.4}'$$

$\delta > 0$ and s are arbitrary, $N = N(d, s, \delta)$ and $K = K(d, s, \delta, N)$ are large enough, $C = C(d, s, \delta, N, c)$, $C' = C\|\!|q_1|\!\| \cdot \|\!|q_2|\!\| \cdot \|\!|\chi|\!\|$ and all the norms are calculated in the corresponding spaces.

Minor Modifications

Let us make minor modifications to the standard case. Namely, let us replace condition $A \in \boldsymbol{\Psi}_{h,K}(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ by one one of the following conditions: $(4.3.21)_1$ $\zeta A \in \boldsymbol{\Psi}_{h,\rho,\gamma,K}(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ with $\rho, \gamma \in [0, 1]$, $\zeta \ge h^s$, $\rho\gamma \ge h^{1-\delta}$; $(4.3.21)_2$ $A \in \tilde{\boldsymbol{\Psi}}^m_{h,K}(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$; $(4.3.21)_3$ A is a differential operator with coefficients satisfying (4.2.13) with $B = I$. Now we consider the domain $\Omega \subset T^*X$ (or $\Omega \subset T^*B(0, 1-\epsilon_0)$ in the third case) and we assume that

(4.3.22) In Ω the symbol of A coincides with that of $A' \in \boldsymbol{\Psi}_{h,K}(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$.

Moreover, let us assume that

$$\|\nabla_{x,\xi}A(x, \xi, h)\| \le c_0 \qquad \forall (x, \xi) \in \Omega \tag{4.3.23}$$

Then Corollary 2.1.7 yields that if $q_1, q_2 \in \mathsf{S}_{h,\rho,\gamma,K}(\mathbb{R}^d, \mathbb{H}, \mathbb{H})$ with $\rho, \gamma \in [0, 1]$, $\rho\gamma \ge h^{1-\delta}$ are supported in $\Omega_{(\epsilon_0)}$ and $Q_1 = \mathsf{Op}^{\mathsf{w}}(q_1)$, $Q_2 = \mathsf{Op}^{\mathsf{w}}(q_2)$, then

(4.3.24) For $t \in [-T_0, T_0]$ with $T_0 = T_0(c_0, \epsilon_0) > 0$ $u(t,x,y)\,{}^tQ_{2y}$ [12] is negligible outside of $\Omega_{\frac{1}{2}\epsilon_0}$ where $\Omega_\varepsilon = \{(x,\xi) : \mathsf{dist}((x,\xi), \complement\Omega) \ge \varepsilon\}$ and ϵ_0 is arbitrarily small.

Hence (4.3.1) yields that

$$(4.3.1)' \quad (hD_t - A')u^{\pm}\,{}^tQ_{2y} \equiv \mp ih\delta(t)\mathcal{K}(x,y) \qquad \text{in } [-T_0, T_0]\times\mathbb{R}^d\times\mathbb{R}^d$$

modulo negligible functions where $\mathcal{K}(x,y) = \mathcal{K}_{Q_2}(x,y) = \delta(x-y)\,{}^tQ_{2y}$ is the Schwartz kernel of Q_2; hence one can replace (4.3.10) by

$$(4.3.10)' \quad u^{\pm}\,{}^tQ_{2y} \equiv \mp ih \sum_{0\le n\le N-1} (\bar G^{\pm}R)^n \bar G^{\pm}\mathcal{K}(x,y)\delta(t) \mp ih(\bar G^{\pm}R)^n G^{\pm}\mathcal{K}(x,y)\delta(t)$$

where $\bar A$, R, $\bar G^{\pm}$ and $G^{\pm}$ are constructed for A' instead of A [13].

Let us apply the previous procedure of the transfer of the factors $(x_j - y_j)$ to the right; now these factors can be also killed in the commutator with ${}^tQ_{2y}$ but it is still in consistent with the case (i) in the beginning of the previous subsubsection. So we obtain the following estimate

$$(4.3.20)' \quad \|F_{t\to h^{-1}\tau}\chi_T(t)Q_{1x}\big(u^{\pm}\,{}^tQ_{2y} \pm ih^{-1}\sum_{0\le n\le N-1} (\bar G^{\pm}R')^n\bar G^{\pm}\delta(t)(\delta(x-y)\,{}^tQ_{2y})\| \le C'h^s.$$

Remark 4.3.2. (i) This estimate differs from (4.3.20) only by position of ${}^tQ_{2y}$; let us remember that while operator A does not depend on y, operators $\bar A$ and R do;

(ii) Moreover, let us replace here ϵ_0 by $\frac{1}{3}\epsilon_0$ and q_2 by q_2' equal to 1 in $\Omega_{\frac{2}{3}\epsilon_0}$; let us multiply the function inside the norm by ${}^tQ_{2y}$; then Corollary 2.1.7 yields that the obtained function will not change modulo negligible terms if we replace the operator ${}^tQ'_{2y}$ by I.

So we have proven

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[12] In the case $(4.3.21)_3$ one should replace Q_i by $Q_i\psi_i$ with ψ_i supported in $B(0, 1-\epsilon_0)$; usually we do it without special mention.

[13] Later we will see that in the final answer this replacement is not necessary.

Proposition 4.3.3. *(i) The assertion of Proposition 4.3.1 remains true for operators A of the type $(4.3.21)_j$–(4.3.22), $j = 1, 2$, with K, C' depending on m and C' also depending on c_0, ϵ_0, $c = \|R\| + \|A\|$ where all the norms are calculated in the corresponding spaces.*

(ii) Moreover, for operators A of the type $(4.3.21)_3$–(4.3.22) this assertion remains true if we replace Q_1, Q_2 by $Q_1\psi_1, Q_2\psi_2$ respectively with $\psi_i \in \mathscr{C}_0^K(B(0, 1-\epsilon_0))$; i.e., the estimate

$$(4.3.20)'' \quad \|F_{t\to h^{-1}\tau}\chi_T(t)Q_{1x}\psi_1(x)\big(u^{\pm}\pm ih\sum_{0\le n\le N-1}(\bar{G}^{\pm}R')^n\delta(x-y)\delta(t)\big)\psi_2(y)\,{}^tQ_{2y}\| \le C'h^s \quad \forall x, y \in \mathbb{R}^d \quad \forall \tau \in \mathbb{R}$$

holds, where now $C' = C\|q_1\| \cdot \|q_2\| \cdot \|\psi_1\| \cdot \|\psi_2\|$.

4.3.2 Successive Approximations in the Hilbert Scale

Let us consider the general case of a differential operator with the coefficients which are unbounded operators with the coefficients satisfying $(4.2.13)_l$–$(4.2.14)_l$.

We will use the successive approximations method with the operator A replaced by the operator

$$A' = a_0(x, h) + \zeta(D)\psi_3(x)\big(A - a_0(x, h)\big)\psi_3(x)\zeta(D) \tag{4.3.25}$$

where $a_0(x, h)$ is a coefficient in A without derivatives extended to $\mathbb{R}^d$ preserving properties $(4.2.13)_l$–$(4.2.14)_l$ (for this coefficient) and here and in what follows $\psi_3, \psi_4, \psi_5 \in \mathscr{C}_0^K$ are real-valued functions supported in $B(0, 1-\frac{7}{9}\epsilon_0)$, $B(0, 1-\frac{5}{9}\epsilon_0)$, $B(0, 1-\frac{3}{9}\epsilon_0)$ and are equal to 1 in $B(0, 1-\frac{8}{9}\epsilon_0)$, $B(0, 1-\frac{2}{9}\epsilon_0)$, $B(0, 1-\frac{4}{9}\epsilon_0)$ respectively (recall that ψ_1 and ψ_2 are supported in $B(0, 1-\epsilon_0)$), $\zeta \in \mathscr{C}^\infty(\mathbb{R}^d)$ equal 1 as $|\xi| \le c$. Then due to finite speed of propagation $u\,{}^tQ_2$ satisfies propagation equation $(hD_t - A')u\,{}^tQ_2 \equiv 0$ modulo negligible f.

Recall that $a_0(x, hD)$ is "the most powerful" coefficient in A as operator in $\mathbb{H}$ and we do not cut it of because we want

$$(a_0 \pm i I_{\mathbb{H}})^{-1} \in \mathcal{L}\big(\mathscr{L}^2(X, \mathbb{H}_j), \mathscr{L}^2(X, \mathbb{H}_{j-1})\big)$$

uniformly.

Such an extension for "potential" $a_0(x,h)$ can be easily chosen provided in the case (i) we replace $B(0,1)$ by any ball from its ϵ-covering (if necessary).

The *principal difficulty now is that the coefficients of* A' *and thus perturbation* $R = A' - \bar{A}$, $\bar{A}' = A'(y,0,hD)$ *are unbounded operators and parametrices* $G^{\pm}$ *and* $\bar{G}^{\pm}$ *do not compensate this unboundedness in an explicit way*[14)].

To tackle this unboundedness we apply the *sandwich procedure*. Namely, let us note that

$$\bar{G}^{\pm} = -(\bar{A}'+i)^{-1} + \bar{G}^{\pm}(hD_t+i)(\bar{A}'+i)^{-1} \tag{4.3.26}$$

and that all three operators $\bar{G}^{\pm}$, $(\bar{A}'+i)^{-1}$ and (hD_t+i) commute.

Let us transform in this way all the terms in the right-hand expression in $(4.3.10)'$ and let us move all the copies of the factor (hD_t+i) to the left; in this way we can replace $\bar{G}^{\pm}$ in the original version of the right-hand expression in $(4.3.10)'$ either by $-(\bar{A}'+i)^{-1}$ or $(\bar{A}'+i)^{-2}$ or $(\bar{A}'+i)^{-1}\bar{G}^{\pm}(\bar{A}'+i)^{-1}$ (with the additional factor $(hD_t+i)^k$ with $k=0,1,2$ respectively).

One can easily see that for $k \ge 1$ the $\mathcal{L}\bigl(\mathscr{L}^2(\mathbb{R}^d,\mathbb{H}),\mathscr{L}^2(\mathbb{R}^d,\mathbb{H})\bigr)$-norms of the operators $B^{1-j}(\bar{A}'+i)^{-k}B^j$ do not exceed C where here and what follows $j\in[0,1]$ is arbitrary in the case of the first standard set of assumptions and $j=\frac{1}{2}$ in the case of the second standard set of assumptions.

Therefore

(4.3.27) $\mathcal{L}\bigl(\mathscr{L}^2([-T,T]\times\mathbb{R}^d,\mathbb{H}),\mathscr{L}^2([-T,T]\times\mathbb{R}^d,\mathbb{H})\bigr)$-norm of the operator $B^{1-j}(\bar{A}'+i)^{-1}\bar{G}^{\pm}(\bar{A}'+i)^{-1}B^j$ does not exceed CTh^{-1}.

Similarly we can deal with $G^{\pm}$ or $\hat{G}^{\pm}$. Hence we conclude that

(4.3.28) Let us consider a term (in the right-hand expression in $(4.3.10)'$ which was negligible in the context of Subsection 4.3.1. Then after multiplication from the left by $F_{t\to h^{-1}\tau}\chi_T(t)Q_{1x}\psi_1(x)$ with $Q_1=\mathsf{Op}^{\mathsf{w}}(q_1)$, q_1 supported in $\{|\xi|\le c\}$ we obtain an expression with $\mathcal{L}(\mathbb{H},\mathbb{H})$-norm not exceeding $C'h^s(|\tau|+1)^{\mu}$ with $\mu=\mu(d,m,s,\delta)$ for all $x,y\in\mathbb{R}^d$.

Moreover, this assertion remains true if we replace Q_{1x} and ${}^tQ_{2y}$ by $Q_{1x}B^j$ and $B^k\,{}^tQ_{2y}$ respectively where $j,k\in[0,1]$, $j+k\le 1$ in the case

[14)] Actually, there is unboundedness with respect to ξ and the failure of the coefficients to be $\mathcal{L}(\mathbb{H},\mathbb{H})$-operators: they are only $\mathcal{L}(\mathbb{H},B^{-1}\mathbb{H})$-operators. The unboundedness with respect to ξ will be compensated by cutting of A by $\zeta(D)$.

of the first standard set of assumptions and $j, k \in [0, \frac{1}{2}]$ in the case of the second standard set of assumptions.

However, we should remember that instead of equation $(4.3.1)'$ we now have

$(4.3.1)''$ $\quad (hD_t - R)u^{\pm}B^k\psi_2(y)\,{}^tQ_{2y} = \mp ih\delta(t)\delta(x-y)B^k\psi_2(y)\,{}^tQ_{2y} + f^{\pm}$

where

$$f^{\pm} = (A - A')u^{\pm}B^k\psi_2(y)\,{}^tQ_{2y}$$

and hence we need to examine corresponding terms which were automatically negligible in the context of Subsection 4.3.1.

Let us note that if we replace $u^{\pm}$ by its cut-offs $u_1^{\pm} = \psi_4(x)u^{\pm}$ then $(4.3.1)''$ remains true with $f^{\pm}$ given now by formula

$$f^{\pm} = \psi_4(x)(A - A')\psi_5 u^{\pm}B^k\psi_2(y)\,{}^tQ_{2y}$$

where

$$u^{\pm} = \mp ih\hat{G}^{\pm}\delta(t)\delta(x-y)I_{\mathbb{H}}$$

and $\hat{G}^{\pm}$ is a parametrix of the problem $(4.3.8)_{\pm}$ (for the operator $hD_t - A$ while $G^{\pm}$ are reserved for parametrices of $hD_t - A'$).

One can apply the sandwich procedure to the parametrices $\hat{G}^{\pm}$ as well; then we obtain terms in which $\hat{G}^{\pm}$ is sandwiched between $\psi_5(A+i)^{-1}$ on the left and $(A+i)^{-1}\psi_4$ from the right and also we obtain the terms containing no factor $\hat{G}^{\pm}$ at all.

Condition $(4.2.20)_l$ [15] yields that

(4.3.29) For $\psi \in \mathscr{C}_0^K(B(0,1-\varepsilon))$ operator $\psi(A \pm i)^{-1}$ acts from $\mathscr{H}^r(X, \mathbb{H}_{j-1})$ to $\mathscr{H}^{r-m}(X, \mathbb{H}_j)$ where $\mathscr{H}^r$ is a Sobolev space and r is arbitrary (but K depends on r)[16]. Then the same is true for the adjoint operator $(A \pm i)^{-1}\psi$ [17].

Further, let us consider the commutator of $(A \pm i)^{-1}$ with an operator of the form $\psi Q\psi$ where Q is a quantization of a compactly supported scalar symbol. Then

$$[(A \pm i)^{-1}, \psi Q\psi] = -(A \pm i)^{-1}[A, \psi Q'\psi](A \pm i)^{-1}$$

[15] Namely, either $(4.2.20)_1$ or $(4.2.20)_{\frac{1}{2}}$ in the cases of the first or second standard set of assumptions.

[16] And $j, 1-j \in [-l, l]$.

[17] Condition $(4.2.13)_l$ is assumed to be fulfilled for given l and for $1-l$ simultaneously.

and hence (4.3.29) yields that its operator norm from $\mathscr{L}^2(X,\mathbb{H}_{j-1})$ to $\mathscr{L}^2(X,\mathbb{H}_j)$ [16)] does not exceed Ch.

This assertion immediately yields that if $q_1, q_2 \in \mathsf{S}_{h,K}(T^*\mathbb{R}^d,\mathbb{H},\mathbb{H})$, q_2 is compactly supported with respect to (x,ξ) and its support does not intersect that of q_1 then the operator norms from $\mathscr{L}^2(X,\mathbb{H}_{j-1})$ to $\mathscr{L}^2(X,\mathbb{H}_j)$ [16)] of the operator $\psi Q_1\psi(A\pm i)^{-1}\psi Q_2\psi$ do not exceed $C'h^s$ and the same is true for $(1-\psi')(A\pm i)^{-1}\psi Q_2\psi$ provided $\psi'=1$ in the neighborhood of $\mathsf{supp}\,\psi$; here $Q_i=\mathsf{Op}^{\mathsf{w}}(q_i)$.

This assertion and Corollary 2.1.7 imply that

$$\|F_{t\to h^{-1}\tau}B^{-j}\chi_{2T}(t)f^{\pm}w\|\le C'h^s\|w\| \qquad \forall y\in\mathbb{R}^d \quad \forall\tau:|\tau|\le 2c \quad \forall w\in\mathbb{H} \tag{4.3.30}$$

where $\chi\in\mathscr{C}_0^K([-1,1])$ is arbitrary and we estimate the $\mathscr{L}^2(\mathbb{R}^d,\mathbb{H})$-norm with respect to x.

Therefore the extra terms arising in $(4.3.10)'$ in connection with $f^{\pm}$ after multiplication by $F_{t\to h^{-1}\tau}\chi_T(t)Q_{1x}\psi_1(x)B^j$ with $|\tau|\le c$ have $\mathcal{L}(\mathbb{H},\mathbb{H})$-norms not exceeding $C'h^s$. So we have proven

Proposition 4.3.4. *(i) Let conditions* $(4.2.13)_1$, $(4.2.14)_1$, (4.2.15), (4.2.16), (4.2.17) *and* $(4.2.20)_1$ *be fulfilled and let* $T,\chi,\psi_1,\psi_2,Q_1,Q_2,N$ *be as in Proposition 4.3.1; further, let* Q_1 *and* Q_2 *be scalar operators. Then the estimates*

$$\|F_{t\to h^{-1}\tau}\chi_T(t)Q_{1x}\psi_1(x)B^j\bigl(u^{\pm}\pm ih\sum_{0\le n\le N-1}(\bar G^{\pm}R')^n\bar G^{\pm}\delta(t)\delta(x-y)\bigr)B^k\psi_2(y)\,{}^tQ_{2y}\|\le C'h^s \qquad \forall x,y\in\mathbb{R}^d \quad \forall\tau:|\tau|\le c \quad \forall j,k\in[0,\bar l], j+k\le 1 \tag{4.3.31}$$

hold with $\bar l=1$ [8)].

(ii) Let conditions $(4.2.13)_{\frac12}$, $(4.2.14)_{\pm}$, (4.2.15), (4.2.21), (4.2.17), $(4.2.20)_{\frac12}$ *be fulfilled and let* $T,\chi,\psi_1,\psi_2,Q_1,Q_2,N$ *be as in Proposition 4.3.1. Then estimates* $(4.3.31)_{\pm}$ *hold with* $\bar l=\frac12$.

Furthermore, let us assume that conditions (4.2.100)–$(4.2.102)_l$ are fulfilled.

Consider first terms which do not contain $\hat G^{\pm}$. Then (4.3.26) yields that every copy of $\bar G^{\pm}$ can either be sandwiched between two copies of $(\bar A+i)^{-p}$

with arbitrarily large p or replaced by $(\bar{A}+i)^{-q}$ with $q = 1, \dots, 2p$, and the same is true for $G^{\pm}$. Let us perform this procedure for the leftmost and the rightmost copies of $\bar{G}^{\pm}$ or $G^{\pm}$. If after this a factor $\bar{G}^{\pm}$ or $G^{\pm}$ disappears we go to the next factor of this type. Then two final outcomes are possible:

(a) The term contains a factor $(\bar{A}+i)^{-p}$ or a factor $(A'+i)^{-p}$ with no factor $\bar{G}^{\pm}$, $G^{\pm}$ to the left of it, and the term contains a factor $(\bar{A}+i)^{-p}$ or a factor $(A'+i)^{-p}$ or a factor $(A+i)^{-p}$ with no factor $\bar{G}^{\pm}$, $G^{\pm}$ to the right of it;

(b) The term contains no factor $\bar{G}^{\pm}$, $G^{\pm}$.

In the case (a) (with large enough p) after multiplication on the left by an operator $F_{t\to h^{-1}\tau}\chi_T(t)Q_{1x}\psi_1(x)B_1^j$ and replacing B^k with $k \in [0, \bar{l}]$ by B_1^k with $k \in [0, l]$ we conclude that the $\mathcal{L}(\mathbb{H}, \mathbb{H})$-norm of the term obtained does not exceed $C'h^s$ provided this term was negligible under the assumptions of Subsection 4.3.1.

In the case (b) it is true only if $j + k \le 1$ but *if we consider* $u = u^+ + u^-$ *then two copies of this term will be annihilated* because they are equal but of the opposite sign.

Further, let us consider terms containing $\hat{G}^{\pm}$ (and hence originating from $f^{\pm}$). Let us note that

$$(A+i)^{-1} = (a_0+i)^{-1} \sum_{0\le n\le N-1} ((a_0-A)(a_0+i)^{-1})^n + ((a_0-A)(a_0+i)^{-1})^N (A+i)^{-1};$$

therefore the arguments used in the proof of Proposition 4.3.4 and conditions (4.2.100)–(4.2.102)$_{\bar{l}}$ imply that if q_2 is compactly supported with respect to (x, ξ) then the operator norm from $\mathscr{L}^2(X, \mathbb{H}_{j-1})$ to $\mathscr{L}^2(X, \mathbb{H}_j)$ [16)] of the operator $(A \pm i)^{-1}\psi Q_2 \psi$ does not exceed $C\|\!|q_1|\!\| \cdot \|\!|q_2|\!\| \cdot \|\!|\psi_1|\!\| \cdot \|\!|\psi_2|\!\|$.

Moreover, if the support of q_2 does not intersect the support of q_1 then the operator norms from $\mathscr{L}^2(X, \mathbb{H}_{j-1})$ to $\mathscr{L}^2(X, \mathbb{H}_j)$ of the operators $\psi Q_1 \psi (A \pm i)^{-1}\psi Q_2 \psi$ do not exceed $C'h^s$, and the same is true for $(1-\psi')(A\pm i)^{-1}\psi Q_2\psi$ provided $\psi' = 1$ in a neighborhood of $\operatorname{supp}\psi$; here $Q_i = \mathsf{Op}(q_i)$ and $\nu \in [0, l]$ is arbitrary.

Then applying the sandwich procedure in the recently modified form and referring in case (a) to Corollary 2.1.7 we obtain that terms of type (a) satisfy the same estimate as before and terms of type (b) annihilate when we add u^+ and u^-. So we have proven

Proposition 4.3.5. *Suppose that* either *conditions* $(4.2.13)_1$, $(4.2.14)_1$, (4.2.15), (4.2.16), (4.2.17), $(4.2.20)_1$ or *conditions* $(4.2.13)_{\frac12}$, $(4.2.14)_{\pm}$, (4.2.15), (4.2.21), (4.2.17), $(4.2.20)_{\frac12}$ *are fulfilled and let* $T, \chi, \psi_1, \psi_2, Q_1, Q_2, N$ *be as in Proposition 4.3.1.*

Moreover, let (4.2.100)–$(4.2.102)_l$ *be fulfilled (surely* N, K *and* C *now also depend on* l*). Then*

$$\|F_{t\to h^{-1}\tau}\chi_T(t)Q_{1x}\psi_1(x)B^j\big(u+ ih\sum_{\varsigma=\pm}\sum_{0\le n\le N-1}\varsigma\big(\bar G^\varsigma R'\big)^n\bar G^\varsigma\delta(t)\delta(x-y)\big)B^k\psi_2(y)\,{}^tQ_{2y}\|\le C'h^s \quad \forall x,y\in\mathbb{R}^d \quad \forall\tau:|\tau|\le c \quad \forall j,k\in[0,l]. \tag{4.3.32}$$

4.3.3 Calculations

Let us go from estimates to calculations. For a sake of simplicity we include now powers of B_1 into Q_1 and Q_2 if needed.

Since $\bar A = a(y, hD_x)$ is an operator with symbol which does not depend on x, an obvious method is a Fourier transform $x \to h^{-1}\xi$ with $\xi\in\mathbb{R}^d$ and the Fourier-Laplace transform $t\to h^{-1}\tau$ with $\tau\in\mathbb{C}_\mp\setminus\mathbb{R}$.

Then the following table of transformations is valid:

$\delta(x-y)\psi_2(y)\,{}^tQ_{2y}$	$e^{ih^{-1}\langle x-y,\xi\rangle}q_2(y,\xi,h)$
$\delta(t)$	1
$\bar G^\pm$	$(\tau-a(y,\xi))^{-1}$ as $\pm\operatorname{Im}\tau<0$
$b(y,hD_x,h)$	$b(y,\xi,h)$
(x_j-y_j)	$ih\partial_{\xi_j}$

Table 4.1: Table of transformations

Namely, $\delta(x-y)\psi_2(y)\,{}^tQ_{2y}$ is transformed into $e^{ih^{-1}\langle x-y,\xi\rangle}q_2(y,\xi,h)$ where $q_2(y,\xi,h)$ is the pq-symbol of operator Q_2 (replaced by $\psi_2 Q_2$ in the formulas and in this definition if necessary)[18], and $\delta(t)$ is transformed into 1; parametrices $\bar G^{\pm}$ are transformed into operators of multiplication by $(\tau - a(y,\xi))^{-1}$ [19]; let us recall that $a(x,\xi)=A(x,\xi,0)$ is a principal symbol of A. Finally, operators $b(y,hD_x,h)$ and (x_j-y_j) are transformed into $b(y,\xi,h)$ and $ih\partial_{\xi_j}$ respectively.

Therefore for $\tau\in\mathbb{C}_{\mp}\setminus\mathbb{R}$

$$(4.3.33)\quad F_{t\to h^{-1}\tau}\sum_{0\le n\le N-1}\bigl(\bar G^{\pm}R'\bigr)^n\bar G^{\pm}\delta(t)\delta(x-y)\bigr)\,{}^tQ_{2y}= \mp i\sum_{0\le n\le N'}(2\pi)^{-d-1}h^{-d+n}\int e^{ih^{-1}\langle x-y,\xi\rangle}F_n(y,\xi,\tau)[q_2(y,\xi,h)]\,d\xi$$

where

(4.3.34) F_n are differential operators[20] with respect to ξ of degree not exceeding n with $\mathcal{L}(\mathbb{H},\mathbb{H})$-valued coefficients which are holomorphic with respect to $\tau\in\mathbb{C}\setminus\mathbb{R}$ and $\mathscr{C}^s$ with respect (x,ξ) where s is arbitrarily large.

(4.3.35) $q_2=\mathsf{symb}_{qp}(Q_2)$ (and $q_1=\mathsf{symb}_{qp}(Q_1)$ later).

Moreover

(4.3.36) Under the assumptions of Propositions 4.3.1, 4.3.3 the $\mathcal{L}(\mathbb{H},\mathbb{H})$-norms of these coefficients[21] do not exceed $C|\operatorname{Im}\tau|^{-2n-1-|\beta|}$.

The sandwich procedure yields that

(4.3.37) Under the assumptions of Proposition 4.2.3 the $\mathcal{L}(\mathbb{H},\mathbb{H})$-norms (with the same j as in this proposition) of these coefficients[21] do not exceed $C(|\tau|+1)^{2r}|\operatorname{Im}\tau|^{-r}$ with large enough r depending on n.

Finally, the modified sandwich procedure yields that

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[18] In the previous analysis the method of quantization was not essential.

[19] With $\tau\in\mathbb{C}_{\mp}\setminus\mathbb{R}$; so the only difference here is the range of τ.

[20] Applied to q_2.

[21] And their derivatives of degree β, $|\beta|\le s$ with respect to $(x,\xi)\in\Omega$.

(4.3.38) Under the assumptions of Proposition 4.3.5 these estimates hold for Q_1, Q_2 incorporating B_1^j and B_1^k with $j, k \in [0, l]$.

Finally, $F_n(y, \xi, \tau)$ depend only on the values of $A(., ., h)$ in a neighborhood of (y, ξ) and hence the replacement of A by A' was not essential from the formal point of view.

Remark 4.3.6. The way in which we obtained formula (4.3.33) yields that its right-hand expression is approximately equal[22)] to

$$\mp i(2\pi)^{-d-1}h^{-d}\int e^{ih^{-1}\langle x-y,\xi\rangle}\,\mathsf{symb}_{pq(N')}\big((\tau-A)^{-1}\psi_2 Q_2\big)\,d\xi \tag{4.3.39}$$

where $(\tau - A)^{-1}$ means a parametrix *formally constructed in terms of the symbol calculus* (or in terms of the formal calculus of pseudodifferential operators), "symb" means symbol, the subscript pq means the type of the symbol and the subscript (N') means that we cut of the decomposition at $h^{N'}$; this subscript is not written in what follows.

In fact, for every fixed $\tau \in \mathbb{C} \setminus \mathbb{R}$ the construction of this parametrix by the method of successive approximations gives exactly the same formulas (without the factor $\pm(2\pi i)^{-1}$).

Therefore

(i) Expression (4.3.39) is the Schwartz kernel of the operator $\mp\frac{1}{2\pi i}(\tau - A)^{-1}\psi_2 Q_2$;

(ii) In particular,

$$F_0(y, \tau, \xi) = \big(\tau - a(y, \xi)\big)^{-1} \tag{4.3.40}$$

and

$$\begin{aligned} F_1(y, \tau, \xi)[q_2] = \big(\tau - a(y, \xi)\big)^{-1}\cdot \\ \big(a^s(y, \xi) + \frac{i}{2}\sum_j a^{(j)}_{(j)}(y, \xi) + \sum_j a_{(j)}(y, \xi)(i\partial_{\xi_j})\big)\big[\big(\tau - a(y, \xi)\big)^{-1} q_2\big] \end{aligned} \tag{4.3.41}$$

where a^s is the subprincipal symbol of A.

[22)] Errors will be only in terms with $n \ge s + d$.

Formula (4.3.33) yields that

$$(4.3.42)\quad F_{t\to h^{-1}\tau}\sum_{0\le n\le N-1} Q_{1x}\big(\bar G^{\pm}R')^n\bar G^{\pm}\delta(t)\delta(x-y)\big)\,{}^tQ_{2y}=$$

$$\mp i\sum_{0\le n\le N'}(2\pi)^{-d-1}h^{-d+n}\int e^{ih^{-1}\langle x-y,\xi\rangle}q_1(x,\xi,h)F_n(y,\xi,\tau)[q_2(y,\xi,h)]\,d\xi$$

where q_1 is the qp-symbol of Q_1 (replaced by $Q_1\psi_1$ in the formulas and in this definition if necessary).

On the other hand, Appendix 4.A.1 yields that one can set $\tau\in\mathbb{R}$ in formulas (4.3.33) and (4.3.42) and then one should replace the τ-analytic functions $F_n(y,\xi,\tau)$ by distributions

$$(4.3.43)\qquad F_n^{\mp}(y,\xi,\tau)=F_n(y,\xi,\tau\mp i0)\in\mathscr{S}'\big(\mathbb{R}_\tau,\mathscr{C}^s(\Omega,\mathcal{L}(\mathbb{H}_{\bar l},\mathbb{H}_{\bar l-1}))\big).$$

More precisely, the coefficients of $F_n^{\mp}$ belong to this space; recall that F_n and hence $F_n^{\mp}$ and $\mathcal{F}_n$ (see below) are differential operators.

Thus the left-hand expression of (4.3.42) equals as $\tau\in\mathbb{R}$ (in the sense of distributions) to

$$(4.3.44)\quad \mp i\sum_{0\le n\le N'}(2\pi)^{-d-1}h^{-d+n}\times$$

$$\int e^{ih^{-1}\langle x-y,\xi\rangle}q_1(x,\xi,h)F_n^{\mp}(y,\tau,\xi)[q_2(y,\xi,h)]\,d\xi$$

and therefore

$$(4.3.45)\quad F_{t\to h^{-1}\tau}\sum_{\varsigma=\pm}\sum_{0\le n\le N-1} Q_{1x}\varsigma\big(\bar G^{\varsigma}R')^n\bar G^{\varsigma}\delta(t)\delta(x-y)\big)\,{}^tQ_{2y}=$$

$$\sum_{0\le n\le N'}(2\pi)^{-d}h^{-d+n}\int e^{ih^{-1}\langle x-y,\xi\rangle}q_1(x,\xi,h)\mathcal{F}_n(y,\xi,\tau)[q_2(y,\xi,h)]\,d\xi$$

with

$$(4.3.46)\qquad \mathcal{F}_n=\mathsf{Res}_{\mathbb{R}}(F_n)=\frac{1}{2\pi i}(F_n^{-}-F_n^{+}).$$

Obviously

$$(4.3.47)\qquad \mathsf{supp}\,\mathcal{F}_n=\{(x,\xi,\tau):\ \tau\in\mathsf{Spec}\,a(x,\xi)\}.$$

Moreover,

(4.3.48) Under the assumptions of Proposition 4.3.5

$$\mathcal{F}_n = \mathsf{Res}_{\mathbb{R}}(F_n) = \frac{1}{2\pi i}(F_n^- - F_n^+) \in \mathscr{S}'\big(\mathbb{R}_\tau, \mathscr{C}^s(\Omega, \mathcal{L}(\mathbb{H}_l, \mathbb{H}_{-l}))\big). \tag{4.3.49}$$

Therefore Propositions 4.3.1–4.3.5 yield the following

Proposition 4.3.7. *Let $T \in [h^{1-\delta}, h^{\frac{1}{2}+\delta}]$ with an arbitrarily small exponent $\delta > 0$. Then*

(i) Under the assumptions of Proposition 4.3.1 the following estimates hold $\forall x, y \in \mathbb{R}^d$ $\forall \tau \in \mathbb{R}$:

$$\|F_{t\to h^{-1}\tau}\chi_T(t)Q_{1x}u^{\pm}\,{}^tQ_{2y} \pm i\sum_{0\le n\le N'}(2\pi)^{-d-1}h^{-d+n+1}\times \iint T\hat{\chi}(\frac{(\tau-\tau')T}{h})e^{ih^{-1}\langle x-y,\xi\rangle}q_1(x,\xi,h)F_n^{\mp}(y,\xi,\tau')[q_2(y,\xi,h]\,d\tau' d\xi\| \le C'h^s \tag{4.3.50}$$

and

$$\|F_{t\to h^{-1}\tau}\chi_T(t)Q_{1x}u\,{}^tQ_{2y} - \sum_{0\le n\le N'}(2\pi)^{-d}h^{-d+n+1}\times \iint T\hat{\chi}(\frac{(\tau-\tau')T}{h})e^{ih^{-1}\langle x-y,\xi\rangle}q_1(x,\xi,h)\mathcal{F}_n(y,\xi,\tau')[q_2(y,\xi,h]\,d\tau' d\xi\| \le C'h^s. \tag{4.3.51}$$

(ii) Under the assumptions of Proposition 4.3.3 this estimate also remains true; in terms of condition $(4.3.21)_3$ one should replace Q_1 and Q_2 by $Q_1\psi_1$ and $\psi_2 Q_2$ respectively.

(iii) Under the assumptions of Proposition 4.3.4 this estimate also holds with Q_1, Q_2 and q_1, q_2 replaced by $Q_1\psi_1 B^j$, $\psi_2 B^k Q_2$ and $q_1 B^j$, $B^k q_2$ respectively (with $j, k \in [0, \bar{l}]$, $j + k \le 1$) and with $\tau \in [-2c, 2c]$;

(iv) Under the assumptions of Proposition 4.3.5 the following estimate holds $\forall x, y \in \mathbb{R}^d\ \forall \tau \in [-2c, 2c]\ \forall j, k \in [0, l]$*:*

$$\|F_{t\to h^{-1}\tau}\chi_T(t)Q_{1x}\psi_1(x)B_1^j u B_1^k \psi_2(y)\,{}^tQ_{2y} - \sum_{0\le n\le N'}(2\pi)^{-d}h^{-d+n+1}\iint T\hat{\chi}\Big(\frac{(\tau-\tau')T}{h}\Big)e^{ih^{-1}\langle x-y,\xi\rangle}q_1(x,\xi,h)\times B_1^j\mathcal{F}_n(y,\tau',\xi)[B_1^k q_2(y,\xi,h)]\,d\tau' d\xi\| \le C'h^s; \tag{4.3.52}$$

further, in this case $\mathcal{F}_n$ *is* $\mathcal{L}(\mathbb{H}_l, \mathbb{H}_{-l})$*-valued distribution.*

Remark 4.3.8. (i) Surely, Remark 4.3.6 remains true with the obvious modifications for the further formulas. In particular, the sum of the left-hand expressions in (4.3.33) (with both signs) is approximately equal[22] to

$$(2\pi)^{-d}h^{-d}\int e^{ih^{-1}\langle x-y,\xi\rangle}\,\mathsf{Res}_{\mathbb{R}}\,\mathsf{symb}_{qp(N')}\big((\tau-A)^{-1}\psi_2 Q_2\big)\,d\xi \tag{4.3.53}$$

which is the Schwartz kernel of the operator $\mathsf{Res}_{\mathbb{R}}\big((\tau-A)^{-1}\psi_2 Q_2\big)$.

(ii) We also have the important formula

$$\mathcal{F}_0(y,\xi,\tau) = \delta\big(\tau - a(y,\xi)\big) = \partial_\tau\theta\big(\tau - a(y,\xi)\big) \tag{4.3.54}$$

(where $\theta(.)$ is the Heaviside function) and

$$\mathcal{F}_1[q_2] = \mathsf{Res}_{\mathbb{R}}\Big\{(\tau-a)^{-1}\Big(a^s + \frac{i}{2}\sum_j\Big(a_{(j)}(\tau-a)^{-1}a^{(j)} - a^{(j)}(\tau-a)^{-1}a_{(j)}\Big)\Big)(\tau-a)^{-1}q_2 + \frac{i}{2}\sum_j\Big((\tau-a)^{-1}a_{(j)}(\tau-a)^{-1}q_2^{(j)} - \big((\tau-a)^{-1}a_{(j)}(\tau-a)^{-1}q_2\big)^{(j)}\Big)\Big\}. \tag{4.3.55}$$

(iii) In particular, as a is scalar

$$\mathcal{F}_1[q_2] = -\delta'(\tau-a)a^s q_2 + \frac{i}{2}\sum_j\Big(-\delta'(\tau-a)a_{(j)}q_2^{(j)} + \big(\delta'(\tau-a)a_{(j)}q_2\big)^{(j)}\Big). \tag{4.3.56}$$

4.3.4 Exploiting Microhyperbolicity and Increasing T

Proposition 4.3.7 holds with any $T \in [h^{1-\delta}, h^{\frac12+\delta}]$ and without microhyperbolicity condition. Let us consider the possibility to increase T in the corollaries of estimates (4.3.51) and (4.3.52) to T which is a small constant and may be beyond; as we know it is very important for the Tauberian method. We also discuss the property of coefficients under microhyperbolicity condition.

Exploiting Microhyperbolicity

Let us set $x = y$ in (4.3.51)

$$F_{t\to h^{-1}\tau}\chi_T(t)\Gamma_x\bigl(Q_{1x}u\,{}^tQ_{2y}\bigr) \equiv \sum_{0\le n\le N'} (2\pi)^{-d}h^{-d+n+1}\times \iint T\hat\chi\Bigl(\frac{(\tau-\tau')T}{h}\Bigr)q_1(x,\xi,h)\mathcal{F}_n(y,\xi,\tau')[q_2(y,\xi,h]\,d\xi d\tau' \tag{4.3.57}$$

and integrate with respect to x

$$F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\bigl(Q_{1x}u\,{}^tQ_{2y}\bigr) \equiv \sum_{0\le n\le N'} (2\pi)^{-d}h^{-d+n+1}\times \iint T\hat\chi\Bigl(\frac{(\tau-\tau')T}{h}\Bigr)q_1(x,\xi,h)\mathcal{F}_n(y,\xi,\tau')[q_2(y,\xi,h]\,dxd\xi d\tau'; \tag{4.3.58}$$

these formulae hold modulo matrices with $\mathcal{L}(\mathbb{H},\mathbb{H})$ or even $\mathcal{L}(\mathbb{H}_l,\mathbb{H}_l)$ norms[23] bounded by Ch^s.

Proposition 4.3.9. *Let the operator A be ξ-microhyperbolic on energy level 0 on $\operatorname{supp} q_1 \cup \operatorname{supp} q_2$ where $q_1, q_2 \in \mathsf{S}_{h,K}(T^*\mathbb{R}^d,\mathbb{H},\mathbb{H})$. Then*

(i) For $T \in [h^{1-\delta}, T_0]$ and $\tau \in [\bar\tau-\epsilon, \bar\tau+\epsilon]$ with small constants $T_0 > 0$, $\epsilon > 0$ equality (4.3.57) holds modulo matrices with $\mathcal{L}(\mathbb{H},\mathbb{H})$ norms bounded by Ch^s.

(ii) Further, then

$$\kappa'_{nx}(\tau) = (2\pi)^{-d}\int q_1(x,\xi,h)\mathcal{F}_n(x,\xi,\tau)[q_2(x,\xi,h)]\,d\xi \tag{4.3.59}$$

belong to $\mathscr{C}_0^s\bigl(\mathbb{R}^d\times[\bar\tau-\epsilon,\bar\tau+\epsilon],\mathcal{L}(\mathbb{H},\mathbb{H})\bigr)$.

[23] In the framework of the corresponding statement of Proposition 4.3.7.

(iii) Moreover, in the framework of Proposition 4.3.5[23)] *equality* (4.3.57) *holds with* Q_1, Q_2 *and* q_1, q_2 *replaced by* $Q_1B_1^j$, Q_2B^k *and* $q_1B_1^j$, $q_2B_1^k$ *with* $j,k \in [0,l]$.

Proposition 4.3.10. *Let the operator* A *be microhyperbolic on energy level* 0 *on* $\mathsf{supp}(q_1) \cup \mathsf{supp}(q_2)$ *where* $q_1, q_2 \in \mathsf{S}_{h,K}(T^*\mathbb{R}^d, \mathbb{H}, \mathbb{H})$. *Then*

(i) For $T \in [h^{1-\delta}, T_0]$ *and* $\tau \in [\bar\tau - \epsilon, \bar\tau + \epsilon]$ *with small constants* $T_0 > 0$, $\epsilon > 0$ *equality* (4.3.57) *holds modulo matrices with* $\mathcal{L}(\mathbb{H},\mathbb{H})$ *norms bounded by* $C'h^s$.

(ii) Further, then

$$\kappa'_n(\tau) = (2\pi)^{-d} \int q_1(x,\xi,h)\mathcal{F}_n(x,\xi,\tau)[q_2(x,\xi,h)]\,dxd\xi \tag{4.3.60}$$

belong to $\mathscr{C}_0^s\bigl([\bar\tau - \epsilon, \bar\tau + \epsilon], \mathcal{L}(\mathbb{H},\mathbb{H})\bigr)$.

(iii) Moreover, in the framework of Proposition 4.3.5[23)] *equality* (4.3.58) *holds with* Q_1, Q_2 *and* q_1, q_2 *replaced by* $Q_1B_1^j$, Q_2B^k *and* $q_1B_1^j$, $q_2B_1^k$ *with* $j,k \in [0,l]$.

Proof of Propositions 4.3.9, 4.3.10. Let us note that under the microhyperbolicity condition the assertions on the regularity properties of κ'_{nx} and κ_n follow from Appendix 4.A.1.

As we mentioned, for $T \in [h^{1-\delta}, h^{\frac12}+\delta]$ asymptotics (4.3.57) and (4.3.58) are due to (4.3.51), (4.3.52). Let us take $\chi = \bar\chi$ where here we assume that $\bar\chi \in \mathscr{C}_0^K([-1,1])$ equals 1 on $[-\frac34, \frac34]$.

Then Corollary 2.1.17 yields that under corresponding microhyperbolicity condition in the left-hand expressions of (4.3.57) and (4.3.58) one can replace $\bar\chi_T$ by $\bar\chi_{T'}$ with arbitrary $T' \in [h^{1-\delta}, T_0]$; moreover, in the framework of Proposition 4.3.5 we also use the arguments of Subsection 4.2.4.

Further, Proposition 4.3.11 below implies that under corresponding microhyperbolicity condition in the right-hand expressions of (4.3.57) and (4.3.58) one can replace $\bar\chi_T$ by $\bar\chi_{T'}$ with arbitrary $T' \in [h^{1-\delta}, \infty]$. Therefore for $\chi = \bar\chi$ all statements are proven.

Since replacing a factor $\bar\chi$ by a factor $\chi = \chi\bar\chi$ (with χ supported in $[-1,1]$) is equivalent to convoluting the Fourier transform with $2\pi i T\hat\chi(\tau T h^{-1})$ we conclude that the proposition has also been proven for χ. □

Proposition 4.3.11. *(i) Let $\chi = \bar{\chi}$ (equal 1 on $[-\frac{3}{4}, \frac{3}{4}]$). Then under ξ-microhyperbolicity condition or microhyperbolicity condition one can replace in the right-hand expression of* (4.3.57) *or* (4.3.58) *respectively $\bar{\chi}_T$ by $\bar{\chi}_{T'}$ with arbitrary $T' \in [h^{1-\delta}, \infty]$; then $\mathcal{L}(\mathbb{H}, \mathbb{H})$-norm of error does not exceed $C' h^s$.*

(ii) Moreover, in the framework of Proposition 4.3.5 Statement (i) remains true with Q_1, Q_2 and q_1, q_2 replaced by $Q_1 B_1^j$, $Q_2 B^k$ and $q_1 B_1^j$, $q_2 B_1^k$ with $j, k \in [0, l]$.

Remark 4.3.12. As $T = \infty$ and $\chi = \bar{\chi}$ the right-hand expressions of (4.3.57) and (4.3.58) become

$$\sum_{0 \le n \le N'} h^{-d+n+1} \kappa'_{nx}(\tau), \tag{4.3.61}$$

$$\sum_{0 \le n \le N'} h^{-d+n+1} \kappa'_{n}(\tau) \tag{4.3.62}$$

respectively.

Proof of Proposition 4.3.11. Since $\hat{\chi} \in \mathcal{S}$ then in the right-hand expressions of (4.3.57) and (4.3.58) for $\tau \in [\bar{\tau} - \epsilon, \bar{\tau} + \epsilon]$ we can insert an additional factor $\phi(\tau')$ with $\phi \in \mathscr{C}_0^K([\bar{\tau} - 2\epsilon, \bar{\tau} + 2\epsilon]$ equal to 1 on $[\bar{\tau} - \frac{3}{2}\epsilon, \bar{\tau} + \frac{3}{2}\epsilon]$.

Let us note that if $\chi(t)$ is replaced by $t\chi(t)$ then $\hat{\chi}((\tau - \tau')Th^{-1})$ should be replaced by $-ihT^{-1}\partial_{\tau'}\hat{\chi}((\tau - \tau')Th^{-1})$ and integration by parts moves $\partial_{\tau'}$ from $\hat{\chi}((\tau - \tau')Th^{-1})$ to $\phi(\tau')\kappa_{nx}(\tau')$ or $\phi(\tau')\kappa_n(\tau')$.

Repeating this procedure we conclude from the regularity properties of κ_{nx} or κ_n that if we replace $\chi_T(t)$ by $t^r\chi_T(t)$ then we obtain in the right-hand expressions of (4.3.57) and (4.3.58) negligible terms. Taking $\chi(t) = t^{-r}(\bar{\chi}(t\zeta^{-1}) - \bar{\chi}(t))$ we conclude that we can replace T by ζT in the right-hand expression of (4.3.57) and (4.3.58) provided $\chi = \bar{\chi}, \zeta \in [1, 2]$.

Repeating this procedure $\lfloor \log_2(\frac{T'}{T}) \rfloor + 1$ times we conclude that we can replace T by T' provided $\chi = \bar{\chi}$; obviously, we can skip $\phi(\tau')$ now. □

While the previous arguments used microhyperbolicity indirectly through smoothness of κ'_{nx} and κ'_n with respect to τ, the following arguments use it directly through propagation:

Alternative proof of Proposition 4.3.11. Note that without any loss of the generality one can assume ξ-microhyperbolicity; otherwise we can apply metaplectic transformation (exactly as in the proof of Theorem 2.1.19). Let us remember that the right-hand expressions of (4.3.57) came from parametrices $\bar{G}^{\mp}$ and one can apply the same propagation results to them but with any $T \ge h^{1-\delta}$ (since those are parametrices of operators with symbols $A(hD)$ in $\mathbb{R}^d$ scaling brings any $T \ge T_0$ to $T = T_0$). □

Remark 4.3.13. (i) Therefore the microhyperbolicity condition works many times: it allows us to derive Tauberian formula with a sharp remainder estimate; further, it allows us to replace in the Tauberian formula T by $h^{1-\delta}$ without spoiling the remainder estimate and, finally, it implies regularity of the coefficients in the decompositions (4.3.61) or (4.3.62).

(ii) We will discuss if it is possible to increase T further in Section 4.5.

4.4 General Basic Theorems

In this section we obtain and improve local and microlocal semiclassical spectral asymptotics. We consider here the most important and sophisticated case of partial differential operators (possibly with unbounded coefficients). Moreover, the results obtained here remain true for both classes of pseudodifferential operators introduced in Section 4.3[24)] in which case a cutting function ψ_i is not necessary. We assume that

$$Q_i = \mathsf{Op}(q_i)\psi_i,\ \mathsf{supp}(q_i) \subset \Omega \subset \{|\xi| \le c\}, \tag{4.4.1}$$
$$\mathsf{dist}(\mathsf{supp}(q_i), \complement\Omega) \ge \epsilon_0, \psi_i \in \mathscr{C}_0^\infty\big(B(0, \frac{1}{2})\big)$$

while our conditions are fulfilled in $B(0,1)$.

We start from the general case in which no microhyperbolicity condition is assumed, then treat the microhyperbolic case. In both these cases we consider microlocal spectral asymptotics with no global condition (save self-adjointness of operator) assumed.

Then the most special case when conditions on Hamiltonian trajectories are assumed. These conditions are semi-global i.e. imposed where these trajectories reach.

[24)] See conditions $(4.3.21)_1$ and $(4.3.21)_2$.

4.4.1 General Theory: $T \asymp h^{\frac12-\delta}$

Let us start with the most general case. In this case we can derive only asymptotics for (smooth) spectral means:

Theorem 4.4.1. *Let either conditions* $(4.2.13)_1$, $(4.2.14)_1$, (4.2.15), (4.2.16), (4.2.17), $(4.2.20)_1$ *or* $(4.2.13)_{\frac12}$, $(4.2.14)_{\pm}$, (4.2.15), (4.2.21), (4.2.17), $(4.2.20)_{\frac12}$ *be fulfilled.*

Moreover, let conditions (4.4.1) *and* (4.2.100)–$(4.2.102)_l$ *be fulfilled. Let* $q_i \in \mathsf{S}_{h,K}(T^*\mathbb{R}^d, \mathbb{H}, \mathbb{H})$, $|\bar\tau| \le c$ *where as usual* $_*$ *means any subscript (including an empty one).*

Let $\varphi, \varphi_1, \varphi_2 \in \mathscr{C}_0^\infty(\mathbb{R})$ *and*

$$L_* \ge h^{\frac12-\delta}. \tag{4.4.2}$$

Then

(i) The following estimates hold:

$$\mathsf{R}^{\mathsf{W}}_{x,y,\varphi,L}(\bar\tau) := \int \varphi\big(\frac{\tau-\bar\tau}{L}\big)\Big(d_\tau\big(Q_{1x}B_1^j e(x,y,\tau_1,\tau)B_1^k\, {}^tQ_{2y}\big) - \sum_{0\le n\le N-1} h^{-d+n}\int \mathcal{K}'_n(x,y,\tau)\,d\tau\Big) \prec C'h^s \tag{4.4.3}$$

and

$$\mathsf{R}^{\mathsf{W}}_{x,y,\varphi_1,\varphi_2,L_1,L_2}(\bar\tau_1,\bar\tau_2) := \frac{1}{L_1L_2}\iint \Big(Q_{1x}B_1^j e(x,y,\tau',\tau'')B_1^k\, {}^tQ_{2y} - \sum_{0\le n\le N-1} h^{-d+n}\mathcal{K}_n(x,y,\tau',\tau'')\Big)\varphi_1\big(\frac{\tau'-\bar\tau_1}{L_1}\big)\varphi_2\big(\frac{\tau''-\bar\tau_2}{L_2}\big)\,d\tau' d\tau'' \prec C'h^s \qquad \forall x,y\in\mathbb{R}^d \quad \forall j,k\in[0,l] \tag{4.4.4}$$

where s *and* $\delta>0$ *are arbitrary,* $N = N(d,m,s,\delta)$, $K = K(d,m,s,N,\delta)$, $C' = C\|q_1\|\cdot\|q_2\|\cdot\|\psi_1\|\cdot\|\psi_2\|\cdot\|\varphi\|$ *or* $C' = C\|q_1\|\cdot\|q_2\|\cdot\|\psi_1\|\cdot\|\psi_2\|\cdot\|\varphi_1\|\cdot\|\varphi_2\|$,

$$\mathcal{K}'_n(x,y,\tau) = \int e^{ih^{-1}\langle x-y,\xi\rangle} q_1(x,\xi,h)B_1^j\mathcal{F}_n(y,\tau,\xi)B_1^k[q_2(y,\xi,h)]\,d\xi, \tag{4.4.5}$$

and

$$\mathcal{K}_n(.,.,\tau',\tau'') = \int_{\tau'}^{\tau''} \mathcal{K}'_n(.,.,\tau)\,d\tau; \tag{4.4.6}$$

$\mathsf{R}^{\mathsf{W}}_{\dots}$ *with various subscripts means Weyl remainder estimate* [25].

(ii) Without conditions (4.2.100)–(4.2.102)$_\prime$*estimates* (4.4.3) *and* (4.4.4) *remain true with* B^j_1, B^k_1 *replaced by* B^j, B^k, $j,k \in [0,\bar{l}]$;

(iii) Moreover, estimates (4.4.3) *and* (4.4.4) *remain true for* $|\bar{\tau}_*| \le c(L_*+1)$.

Furthermore in the case $B = I$ *one can take arbitrary* $\bar{\tau}_* \in \mathbb{R}$.

Proof. (a) Let us pick $T \in [h^{1-\delta}, h^{\frac12+\delta}]$ such that

$$TL \ge h^{1-\delta} \tag{4.4.7}$$

(which can be done due to condition (4.4.2); we decrease $\delta > 0$ if necessary). Applying (4.2.31) and (4.3.52) we obtain that estimate (4.4.3) holds with $T\hat{\chi}\big((\tau'-\tau)Th^{-1}\big)$ instead of $\varphi\big((\tau-\bar{\tau})L^{-1}\big)$ and with $\tau' \in [-c,c]$.

Let us multiply this estimate by $h^{-1}\varphi\big((\tau-\bar{\tau})L^{-1}\big)$ and integrate over $\tau \in \mathbb{R}$ [26]. Then (4.4.3) holds with the function $\varphi(\lambda)$ replaced by

$$\varphi_\varepsilon(\lambda) = \varepsilon^{-1}\int \varphi(\lambda')\hat{\chi}\big((\lambda-\lambda')\varepsilon^{-1}\big)\,d\lambda', \qquad \varepsilon = \frac{h}{LT}.$$

Let us pick $\chi = \bar{\chi}$. Recall that $\bar{\chi} = 1$ on $[-\frac12, \frac12]$. Then one can easily prove that

$$\varphi_\varepsilon(\lambda) = \varphi(\lambda) + O\big(\varepsilon^s(1+|\lambda|)^{-s}\big).$$

Therefore it is sufficient to estimate the norm of

$$\varepsilon^s \int \big(1+\frac{|\tau-\bar{\tau}|\,T}{h}\big)^{-s} d_\tau \mathsf{N}_{x,y}(\tau) \tag{4.4.8}$$

with $\mathsf{N}_{x,y}$ defined by modified (4.2.32):

$$\mathsf{N}_{x,y}(\tau) := Q_{1x}B^j_1 e(x,y,\tau_1,\tau)B^k_1\ {}^tQ_{2y}. \tag{4.4.9}$$

Dividing this integral into a sum of integrals over intervals $[n,n+1]$ with $n \in \mathbb{Z}$ and applying Proposition 4.2.1 we obtain that the norm of the n-th integral does not exceed $Ch^s(|n|+1)^{-s}$ (recall that $\varepsilon \le h^\delta$) and hence estimate (4.4.3) is proven.

---

[25] Let us recall that $\mathcal{K}'_n$ are distributions with respect to τ; so $\mathcal{K}_n$ are distributions with respect to (τ',τ).

[26] This means exactly that we apply operator $\varphi\big((hD_t-\bar{\tau})L^{-1}\big)$ to $Q_{1x}B^j_1 u B^k_1\ {}^tQ_{2y}$ and to its approximation obtained in the previous section and set $t=0$ afterwards.

(b) Let us prove (4.4.4). Integrating by parts in (4.4.3) and replacing $\partial_\tau\varphi(\tau)$ by $\varphi(\tau)$ we obtain that under the additional restriction

$$\int_{-\infty}^{\infty} \varphi(\tau)d\tau = 0 \tag{4.4.10}$$

the following estimate holds:

$$\mathsf{R}^{\mathsf{W}}_{x,y,\varphi_2,L_2}(\bar\tau_1,\bar\tau_2) \coloneqq L_2^{-1}\int \Big(Q_{1x}B_1^j e(x,y,\bar\tau_1,\tau)B_1^k\, {}^tQ_{2y}\,d\tau - \sum_{0\le n\le N-1} h^{-d+n}\mathcal{K}_n(x,y,\bar\tau_1,\tau)\Big)\varphi\big(\frac{\tau-\bar\tau_2}{L_2}\big)\,d\tau \prec C'h^s \qquad \forall x,y\in\mathbb{R}^d \quad \forall j,k\in[0,l] \tag{4.4.11}$$

where under condition (4.4.10) $\mathsf{R}^{\mathsf{W}}_{x,y,\varphi_2,L_2}$ does not depend on $\bar\tau_1$. It implies (4.4.4) as one of functions φ_1,φ_2 satisfies (4.4.10).

Let us note that the function

$$\varphi(\tau) = \varphi_2(\tau)\int_{-\infty}^{\infty}\varphi_1(\tau''')\,d\tau''' - \varphi_1(\tau)\int_{-\infty}^{\infty}\varphi_2(\tau''')\,d\tau'''$$

satisfies condition (4.4.10); substituting it into (4.4.11) we obtain the difference of two double integrals; let us replace (τ,τ''') in the first integral by (τ'',τ') and in the second one by (τ',τ''); then using the equality $e(.,.,\tau_1,\tau'') - e(.,.,\tau_1,\tau') = e(.,.,\tau',\tau'')$ and similar equalities for $\mathcal{K}_n$ we obtain (4.4.4) as $L = L_1 = L_2$ and $\bar\tau_1 = \bar\tau_2$.

Consider $L = L_1 \le L_2$ (which is not any loss of the generality); then (4.4.4) for $L_2 = L$ implies it for $L_2 \ge L$ with $\varphi_2 \coloneqq \varphi_2 * \varphi_{L/L_2}$ with $\varphi\in\mathscr{C}_0^\infty(\mathbb{R})$ and $\varphi_\varepsilon(\tau) = \varepsilon\varphi(\tau\varepsilon^{-1})$. However, picking φ such that $\int_{\mathbb{R}}\varphi(\tau)\,d\tau = 1$ we note that $\varphi_2 - \varphi_2 * \varphi_{L/L_2}$ satisfies (4.4.10) and thus φ_2 is decomposed into the sum of two functions for which (4.4.4) has been proven. So, we can drop condition $L_1 = L_2$.

Note that

$$e(x,y,\tau_1,\tau_2) = \sum_{i=1,2}(-1)^i\int e(x,y,\tau',\tau_i)\varphi_i(\tau')\,d\tau' \qquad \text{as} \quad \int_{\mathbb{R}}\varphi_i(\tau')\,d\tau' = 1. \tag{4.4.12}$$

Therefore it is sufficient to consider the case $L_2 = 1$. In this case shifting φ_2 we can remove assumption $\bar\tau_2 = \bar\tau_1$.

As $L_* \ge 1$ shifting φ_* replace assumption $|\bar\tau_*| \le c$ in (4.4.33), (4.4.34) by $|\bar\tau_*| \le cL_* + c$

Let us first note that in the case $B = I$, $L \in [h^{\frac12-\delta}, 1]$ one can choose $\bar\tau_i$ arbitrarily (because $\tau : |\tau| \ge C_0$ falls into the semiclassical spectral gap; recall that q_i are compactly supported). □

This theorem combined with Propositions 4.2.3–4.2.5 immediately yields

Theorem 4.4.2. *Let the conditions of Theorem 4.4.1 be fulfilled.*

(i) Moreover, let either conditions of Proposition 4.2.3 be fulfilled in an interval of the length $\ge h^{\frac12-\delta}$ containing $\bar\tau_1$ or let the conditions of one of Propositions 4.2.4, 4.2.5 be fulfilled with $\bar\tau$ replaced by $\bar\tau_1 \in \mathbb{R}$. Finally, let $|\bar\tau_1| \le cL + c$. Then

$$\mathsf{R}^{\mathsf{W}}_{x,y,\varphi_2,L_2}(\bar\tau_1,\bar\tau_2) \prec C'h^s; \tag{4.4.13}$$

here $\mathsf{R}^{\mathsf{W}}_{x,y,\varphi_2,L_2}(\bar\tau_1,\bar\tau_2)$ does not depend (modulo $O(h^s)$) on $\bar\tau_1$ lying in the indicated interval.

(ii) Further, let the assumptions of (i) be fulfilled for $\bar\tau = \bar\tau_1$ as well as for $\bar\tau = \bar\tau_2$. Then

$$\mathsf{R}^{\mathsf{W}}_{x,y}(\bar\tau_1,\bar\tau_2) \coloneqq \mathsf{N}_{x,y}(\bar\tau_1,\bar\tau_2) - \sum_{0\le n\le N-1} h^{-d+n}\mathcal{K}_n(x,y,\bar\tau_1,\bar\tau_2) \prec C'h^s; \tag{4.4.14}$$

here $\mathsf{R}^{\mathsf{W}}_{x,y}(\bar\tau_1,\bar\tau_2)$ does not depend (modulo $O(h^s)$) on $\bar\tau_1,\bar\tau_2$ residing in the indicated intervals.

(iii) Moreover, if conditions $(4.2.14)_\pm$, (4.2.21) *are fulfilled then in Statements (i) and (ii) one can take $\bar\tau_1 = \mp\infty$.*

(iv) If $B = I$ then in in Statements (i) and (ii) one can take $\bar\tau_1 \le -C_0$ (or $\bar\tau_1 = -\infty$ and $\bar\tau_2 \ge C_0$ (or $\bar\tau_2 = \infty$) with sufficiently large C_0.

(v) Assertions (ii) and (iii) of Theorem 4.4.1 remain true for the estimates of this theorem.

Proof. The corresponding proposition of Subsection 4.2.2 yields that in the function to be estimated one can replace τ_1 and (or) τ_2 by any other value from the neighborhood indicated above and hence by the mean value on this interval. On the other hand, for the spectral mean the corresponding estimate is obvious. □

Remark 4.4.3. One can prove easily that if an averaging function φ is not infinitely smooth but merely satisfies (4.2.73)–(4.2.75) then the statements above hold with the remainder estimate $C'h^{-d}\vartheta(h^{\frac{1}{2}-\delta}L^{-1})$.

4.4.2 General Microhyperbolic Theory: $T \asymp 1$

In the previous subsection we found the complete Weyl asymptotics of $\mathsf{N}_{x,y}(\tau_1,\tau_2)$ but with a twist: either τ_* falls into semiclassical spectral gap or it is "tamed" by mollification with $L_* \ge h^{\frac{1}{2}-\delta}$. Now we consider the main theory when at least one of energy levels τ_* neither falls into semiclassical spectral gap nor is "tamed".

Remark 4.4.4. Actually we can assume that only one τ_i is of this type; the general case is reduced to this due to (4.4.12).

Asymptotics with Mollification

Still let us explore first asymptotics with mollification. We know that under ξ-microhyperbolicity (microhyperbolicity) condition complete asymptotics of $F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma_x(Q_{1x}B_1^j uB_1^k\, {}^tQ_{2y})$ ($F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(Q_{1x}B_1^j uB_1^k\, {}^tQ_{2y})$ respectively) was holds for $T \asymp 1$ rather than $T = h^{\frac{1}{2}+\delta}$. Then in virtue of the same arguments as in the previous subsection all the results with applied Γ_x or Γ hold as condition (4.4.2) (which was $L_* \ge h^{\frac{1}{2}-\delta}$)is replaced by

$$L_* \ge h^{1-\delta}. \tag{4.4.15}$$

Theorem 4.4.5. *Let assumptions of Theorem 4.4.1 be fulfilled with* (4.4.2) *replaced by* (4.4.15).

Further, let A be ξ-microhyperbolic on $\mathsf{supp}(q_1)\cap\mathsf{supp}(q_2)$ on the energy level $\bar{\tau}_*$. Then

(i) The following estimates hold

$$\mathsf{R}^{\mathsf{W}}_{x,\varphi,L}(\bar\tau) \coloneqq \int \varphi\big(\frac{\tau-\bar\tau}{L}\big)\Big(d_\tau \Gamma_x\big(Q_{1x}B_1^j e(x,y,\tau_1,\tau)B_1^k\, {}^tQ_{2y}\big) - \sum_{0\le n\le N-1} h^{-d+n}\int \kappa'_{nx}(\tau)\,d\tau\Big) \prec C'h^s \tag{4.4.16}$$

and

$$\mathsf{R}^{\mathsf{W}}_{x,\varphi_1,\varphi_2,L_1,L_2}(\bar\tau_1,\bar\tau_2) \coloneqq \frac{1}{L_1L_2}\iint \Gamma_x\Big(Q_{1x}B_1^j e(x,y,\tau',\tau'')B_1^k\, {}^tQ_{2y} - \sum_{0\le n\le N-1} h^{-d+n}\kappa_{nx}(\tau',\tau'')\Big)\varphi_1\big(\frac{\tau'-\bar\tau}{L_1}\big)\varphi_2\big(\frac{\tau''-\bar\tau}{L_2}\big)\,d\tau' d\tau'' \prec C'h^s \qquad \forall x,y\in\mathbb{R}^d \quad \forall j,k\in[0,l] \tag{4.4.17}$$

with the same s, $\delta>0$, N, K, C' *as before.*

(ii)–(iii) Statements (ii)–(iii) of Theorem 4.4.1 remain true with (4.4.16), (4.4.17) *instead of* (4.4.3), (4.4.4).

Theorem 4.4.6. *Let assumptions of Theorem 4.4.1 be fulfilled with* (4.4.2) *replaced by* (4.4.15).

Further, let A *be microhyperbolic on* $\mathsf{supp}(q_1)\cap\mathsf{supp}(q_2)$ *on the energy level* $\bar\tau_*$. *Then all the results of Theorem 4.4.5 remain true with* Γ_x *replaced by* Γ *and* κ_{nx}, κ'_{nx} *replaced by* κ_n, κ'_n *respectively.*

One can modify Theorem 4.4.2 in the same way as well.

Remark 4.4.7. Usually in asymptotics of N_{x*} one needs to assume ξ-microhyperbolicity at $\mathsf{supp}(q_1)\cup\mathsf{supp}(q_2)$ but in this section due to the finite speed of propagation we can to assume ξ-microhyperbolicity at $\mathsf{supp}(q_1)\cap\mathsf{supp}(q_2)$ instead.

Asymptotics without Mollification

The following statement immediately follows from the Tauberian estimates of Section 4.2 and calculations of Section 4.3:

Theorem 4.4.8. *Let assumptions of Theorem 4.4.1 be fulfilled with* (4.4.2) *replaced by*

$$L_* \ge h. \tag{4.4.18}$$

Further, let A be ξ-microhyperbolic on $\mathsf{supp}(q_1) \cap \mathsf{supp}(q_2)$ on the energy level $\bar\tau_*$. Then

(i) The following estimate holds:

$$\mathsf{R}^{\mathsf{W}}_x(\bar\tau_1,\bar\tau_2) \coloneqq \Gamma_x\big(Q_{1x}B_1^j e(x,y,\bar\tau_1,\bar\tau_2)B_1^k\, {}^tQ_{2y} - \sum_{0\le n\le N-1} h^{-d+n}\kappa_{nx}(\bar\tau_1,\bar\tau_2)\big) \prec C'h^{1-d} \tag{4.4.19}$$

(ii) As φ_* satisfy (4.2.73)–(4.2.75) *the following estimates hold:*

$$\mathsf{R}^{\mathsf{W}}_{x,\varphi,L}(\bar\tau) \prec C'h^{1-d}\vartheta\big(\frac{h}{L}\big), \tag{4.4.20}$$

$$\mathsf{R}^{\mathsf{W}}_{x,\varphi_1,L_1,\varphi_2,L_2}(\bar\tau_1,\bar\tau_2) \prec C'h^{1-d}\vartheta_1\big(\frac{h}{L_1}\big) + C'h^{1-d}\vartheta_2\big(\frac{h}{L_2}\big) \tag{4.4.21}$$

with the left-hand expressions defined by (4.4.16) *and* (4.4.17).

Theorem 4.4.9. *Let assumptions of Theorem 4.4.1 be fulfilled with* (4.4.2) *replaced by* (4.4.18). *Further, let A be microhyperbolic on $\mathsf{supp}(q_1)\cap\mathsf{supp}(q_2)$ on the energy level $\bar\tau_*$. Then all the results of Theorem 4.4.8 remain true with $\mathsf{R}^{\mathsf{W}}_{x*}$ replaced by $\mathsf{R}^{\mathsf{W}}_*$ and κ_{nx}, κ'_{nx} replaced by κ_n, κ'_n respectively.*

Remark 4.4.10. (i) In Theorem 4.4.8 one can skip ξ-microhyperbolicity condition on some level $\bar\tau_i$ by making mollification with $\varphi_i \in \mathscr{C}_0^\infty(\mathbb{R})$ and $L_* \ge h^{1-\delta}$ in then (4.4.19) or just assuming that $\varphi_* \in \mathscr{C}_0^\infty(\mathbb{R})$ and $L_* \ge h^{1-\delta}$ in (4.4.21). The similar statement in the framework of Theorem 4.4.9 also holds.

(ii) In asymptotics without averaging only one term $\kappa_{0x}h^{-d}$ or $\kappa_0 h^{-d}$ is needed. In the asymptotics with averaging one can skip all the terms such that $t^{n-1} \lesssim \vartheta(t)$ for all $t \ll 1$ (obviously this is true as $L \lesssim 1$; as $L \gtrsim 1$ we apply Statement (iii) below). One can skip even more terms as $L \ll 1$.

(iii) One can prove easily that under ξ-microhyperbolicity condition

$$(\kappa'_{nx} * \varphi_L)(\bar\tau) = O(L^{-n}) \qquad \text{as} \quad L \ge 1 \tag{4.4.22}$$

and under microhyperbolicity condition this is true for κ'_n.

(iv) One can easily improve asymptotics derived in the previous subsections to $O(h^s L^{-s})$ as $L \ge h^{-M}$.

Improved Asymptotics without Mollification

Here we will improve asymptotics without mollification. Actually this improvement will be probably superficial in the current settings but it will open the door for the real improvement under conditions of the semi-global nature in the next section. Let us observe that we know now (4.1.12)- or (4.1.11)-like expressions with far better precision than before and therefore as $Q_2 = Q_1^*$ and $j = k$ we have estimates

$$\Gamma_x\Big(F_{t\to h^{-1}\tau}\chi_T(t) Q_{1x} B_1^j u(x,y,t) B_1^k\, {}^t Q_{2y}\Big) \prec C_0 J_x(q_1^0 B_1^j, \tau) h^{1-d} + C' h^{2-d} \tag{4.4.23}$$

and

$$\Gamma\Big(F_{t\to h^{-1}\tau}\chi_T(t) Q_{1x} B_1^j u(x,y,t) B_1^k\, {}^t Q_{2y}\Big) \prec C_0 J(q_1^0 B_1^j, \tau) h^{1-d} + C' h^{2-d} \tag{4.4.24}$$

as $|\tau - \bar\tau| \le \epsilon$ under ξ-microhyperbolicity and microhyperbolicity assumptions respectively where

$$J_x(q_1^0 B_1^j, \tau) \coloneqq \int q_1^0 B_1^j \, d_\xi \mu_{\tau x}\, B_1^j q_1^0 \tag{4.4.25}$$

$$J(q_1^0 B_1^j, \tau) \coloneqq \int q_1^0 B_1^j \, d_{(x,\xi)} \mu_\tau\, B_1^j q_1^0 \tag{4.4.26}$$

q_i^0 is the principal symbol of Q_i,

$$\mu_\tau = \delta\big(\tau - a(x,\xi)\big) = \partial_\tau \theta\big(\tau - a(x,\xi)\big) \tag{4.4.27}$$

is a non-negative $\mathcal{L}(\mathbb{H},\mathbb{H})$-valued measure and $\mu_{\tau x}$ is the same measure for given x.

Finally, here and below $C_0 = C_0(d, \vartheta, \varphi)$ is an absolute constant; so $C_0 = C_0(d)$ in the most important case $\vartheta = \varphi = 1$.

Indeed, the leading coefficients in the asymptotics of the left-hand expressions are $\kappa'_{0x}(\tau)$ and $\kappa'_0(\tau)$ which due to (4.3.52), (4.3.54), (4.3.60) and (4.3.59) are exactly

$$\kappa'_{0x}(\tau) \coloneqq (2\pi)^{-d} \int q_1^0 B_1^j \, \delta\big(\tau - a(x,\xi)\big)\, B_1^j q_2^0 \, d\xi \tag{4.4.28}$$

and

$$\kappa'_0(\tau) \coloneqq (2\pi)^{-d} \int q_1^0 B_1^j \, \delta\big(\tau - a(x,\xi)\big)\, B_1^j q_2^0 \, dx d\xi \tag{4.4.29}$$

respectively and coincide with J_{0x}, J_0 as $q_2^0 = q_1^{0\dagger}$ and $j = k$.

Estimates (4.4.23), (4.4.24) replace previously used estimates with $C'h^{1-d}$ right-hand expressions. Let us recall that

(4.4.30) Under ξ-microhyperbolicity condition $J_x(q_1^0 B_1^j, \tau)$ is a smooth function of (x, τ). Under microhyperbolicity condition $J(q_1^0 B_1^j, \tau)$ is a smooth function of τ.

Taking Hörmander function χ and applying Lemma 4.2.7) we derive easily from (4.4.24) that for $j = k$ and $Q_2 = Q_1^*$ under microhyperbolicity or ξ-microhyperbolicity on the energy level 0

$$\mathsf{N}(\tau', \tau'') \prec C_0 g^*(\tau', \tau''), \tag{4.4.31}$$

$$\mathsf{N}_x(\tau', \tau'') \prec C_0 g_x^*(\tau', \tau'') \tag{4.4.32}$$

with g^*, g_x^* defined by (4.2.35) where g, g_x are right-hand expressions of (4.4.24), (4.4.23) respectively as $|\tau - \bar\tau| \le \epsilon_0$; we do not change definition (4.2.87) of $g(\tau) = g_x(\tau) = C'T(|\tau| + 1)^{2l}$ as $|\tau - \bar\tau| \ge \epsilon_0$.

Now again we want to get rid of assumption $j = k$, $Q_2 = Q_1^*$. For this purpose we still under this assumption rewrite (4.4.31) or (4.4.32) respectively as

$$|(\mathsf{N}(\tau', \tau'')w, w)| \le C_0 (g^*(\tau', \tau'')w, w),$$
$$|(\mathsf{N}_x(\tau', \tau'')w, w)| \le C_0 (g_x^*(\tau', \tau'')w, w)$$

for all $w \in \mathbb{H}$.

Then arguments of the proof of Proposition 4.2.19 imply that without assumption $j = k$, $Q_2 = Q_1^*$

$$|(\mathsf{N}(\tau',\tau'')w,w)| \le C_0(g_1^*(\tau',\tau'')w,w)^{\frac12}(g_2^*(\tau',\tau'')w,w)^{\frac12}, \tag{4.4.33}$$

$$|(\mathsf{N}_x(\tau',\tau'')w,w)| \le C_0(g_{x1}^*(\tau',\tau'')w,w)^{\frac12}(g_{x2}^*(\tau',\tau'')w,w)^{\frac12} \tag{4.4.34}$$

respectively where subscript $_1$, $_2$ refers to g, g^* defined with $Q_1B_1^j$ and $Q_2B_2^k$.

Then

$$\begin{aligned}&|(\mathsf{N}(\tau',\tau'')w,w)| \le C_0(g_1^*(\tau',\tau'')w,w) + C_0(g_2^*(\tau',\tau'')w,w),\\ &|(\mathsf{N}_x(\tau',\tau'')w,w)| \le C_0(g_{x1}^*(\tau',\tau'')w,w) + C_0(g_{x2}^*(\tau',\tau'')w,w)\end{aligned}$$

respectively and we do not need w anymore.

Now we can apply Lemmas 4.2.13, 4.2.15; we need to calculate $\mathcal{R}(\bar\tau)$ according to (4.2.65). Since due to (4.4.30)

$$g(\tau) = g(\bar\tau) + O\big(h^{1-d}|\tau-\bar\tau| + h^{2-d}\big)$$

we conclude that $\mathcal{R}(\bar\tau) = g(\bar\tau) + O(T^{-1}h^{2-d} + h^{2-d})$.

Therefore we arrive to estimates with the right hand expressions defined by these calculations:

$$\mathsf{R}^{\mathsf{T}}_{\varphi_2,L_2}(\bar\tau,\bar\tau_2) \prec \frac{h^{1-d}}{T}\Big(C_0J(q_1^0B_1^j,\bar\tau) + C_0J(q_2^0B_1^k,\bar\tau) + C'\frac{h}{T} + C'h\Big) \tag{4.4.35}$$

and

$$\begin{aligned}&\mathsf{R}^{\mathsf{T}}_{\varphi,L,\varphi_2,L_2}(\bar\tau,\bar\tau_2) \prec\\ &\qquad \frac{h^{1-d}}{T}\Big(C_0J(q_1^0B_1^j,\bar\tau) + C_0J(q_2^0B_1^k,\bar\tau) + C'\frac{h}{T} + C'h\Big)\vartheta\big(\frac{h}{LT}\big)\end{aligned} \tag{4.4.36}$$

provided $\bar\tau_2 = \bar\tau$ and $\varphi_2 \in \mathscr{C}_0^\infty(\mathbb{R})$ while φ satisfies (4.2.73)–(4.2.75).

These estimates hold under microhyperbolicity condition; the similar estimates (with J_x, R^{T}_{x*}) hold under ξ-microhyperbolicity condition.

Let us recall that $T \asymp 1$ in the current settings. Also in virtue of the calculations of the Section 4.3 we can replace $\mathsf{R}^{\mathsf{T}}_*$ by $\mathsf{W}^{\mathsf{T}}_*$. Then we arrive to estimates (4.4.38) and (4.4.39) below but only as $\bar\tau_2 = \bar\tau_1$ and $\varphi_2 \in \mathscr{C}_0^\infty(\mathbb{R})$. However in virtue of (4.4.12) we can remove these restrictions arriving to the improved versions of Theorems 4.4.8 and 4.4.9:

Theorem 4.4.11. *Let assumptions of Theorem 4.4.1 be fulfilled with* (4.4.2) *replaced by* (4.4.18). *Further, let* A *be* ξ*-microhyperbolic on* $\mathsf{supp}(q_1) \cap \mathsf{supp}(q_2)$ *on the energy level* $\bar\tau_*$. *Then for* $j, k \le l$

$$\mathsf{R}^{\mathsf{W}}_x(\bar\tau_1, \bar\tau_2) \prec \sum_{*=1,2} C_0 \frac{h^{1-d}}{T_0}\Big(J_x(q_1^0 B_1^j, \bar\tau_*) + J_x(q_2^0 B_1^k, \bar\tau_*) + C'h\Big), \tag{4.4.37}$$

$$\begin{aligned}\mathsf{R}^{\mathsf{W}}_{x,\varphi_2,L_2}(\bar\tau_1, \bar\tau_2) \prec\ & C_0 \frac{h^{1-d}}{T_0}\Big(J_x(q_1^0 B_1^j, \bar\tau_1) + J_x(q_2^0 B_1^k, \bar\tau_1) + C'h\Big) + \\ & C_0 \frac{h^{1-d}}{T_0}\Big(J_x(q_1^0 B_1^j, \bar\tau_2) + J_x(q_2^0 B_1^k, \bar\tau_2) + C'h\Big)\vartheta_2\big(\frac{h}{L_2 T_0}\big)\end{aligned} \tag{4.4.38}$$

and

$$\begin{aligned}&\mathsf{R}^{\mathsf{W}}_{x,\varphi_1,L_1,\varphi_2,L_2}(\bar\tau_1, \bar\tau_2) \prec \\ &\qquad \sum_{*=1,2} C_0 \frac{h^{1-d}}{T_0}\Big(J_x(q_1^0 B_1^j, \bar\tau_*) + J_x(q_2^0 B_1^k, \bar\tau_*) + C'h\Big)\vartheta_*\big(\frac{h}{L_* T_0}\big)\end{aligned} \tag{4.4.39}$$

with the left-hand expressions defined respectively by (4.4.19), (4.4.16) *and* (4.4.17).

Theorem 4.4.12. *Let assumptions of Theorem 4.4.1 be fulfilled with* (4.4.2) *replaced by* (4.4.18). *Further, let* A *be microhyperbolic on* $\mathsf{supp}(q_1) \cap \mathsf{supp}(q_2)$ *on the energy level* $\bar\tau_*$. *Then all the results of Theorem 4.4.11 remain true with* $\mathsf{R}^{\mathsf{W}}_{x*}$ *replaced by* $\mathsf{R}^{\mathsf{W}}_*$ *and* κ_{nx}, κ'_{nx} *replaced by* κ_n, κ'_n *respectively.*

Useful Formulae

Remark 4.4.13. (i) Due to (4.4.28), (4.4.29)

$$\kappa_{0x}(\tau) := (2\pi)^{-d} \int q_1^0 B_1^j\, \theta\big(\tau - a(x,\xi)\big)\, B_1^j q_2^0\, d\xi \tag{4.4.40}$$

and

$$\kappa_0(\tau) := (2\pi)^{-d} \int q_1^0 B_1^j\, \theta\big(\tau - a(x,\xi)\big)\, B_1^j q_2^0\, dx d\xi. \tag{4.4.41}$$

(ii) Formulae for the next coefficients are rather complicated. Formulae for κ_{1x}, κ_1 are due to (4.3.52), (4.3.55), (4.3.60) and (4.3.59). In particular, in the case of scalar a (4.3.56) implies that

$$\kappa_{1x} = (2\pi)^{-d} \int \delta(\tau - a)\Big(-q_1^0 a^s q_2^0 - \frac{i}{2}\{a, q_1^0 q_2^0\}\Big)\, d\xi + (2\pi)^{-d} \int \theta(\tau - a)(q_1 q_2)^s \, d\xi \tag{4.4.42}$$

where $(q_1 q_2)^s$ is subprincipal symbol of $Q_1 Q_2$. Important is that as $Q_1 = Q_2 = I$ (we can set it only formally now)

$$\kappa_{1x} = -(2\pi)^{-d} \int \delta(\tau - a) a^s \, d\xi. \tag{4.4.43}$$

Similar formulae hold for κ_1.

(iii) Again, as $Q_1 = Q_2 = I$ but in the matrix case there is an extra term due to the second line in (4.3.55). The explicit formula and geometric interpretation of this term see in O. Chervova, R. J. Downes and D. Vassiliev [1, 2].

Problem 4.4.14. Consider (magnetic) Schrödinger operator

$$H = (-ih\nabla - \boldsymbol{A}(x))^2 + V(x)$$

and take $Q_1 = -ih\partial_j - A_j(x)$, $Q_2 = I$. Then obviously $\kappa_{0x} = 0$ and we leave to the reader to prove that $\kappa_{1x} = \mathsf{const}\, \partial_j (\tau - V(x))^{d/2}$ and calculate the constant.

Remark 4.4.15. Assume that Q_1, Q_2 are scalar operators. Then one can set $j = k = I$ in (4.4.37) and multiplying by B_1^{-I} from both sides one can rewrite it as

$$\mathsf{R}^{\mathsf{W}}_x(\bar\tau_1, \bar\tau_2) \prec h^{1-d} \sum_{*=1,2} \Big(C_0 \int (|q_1^0|^2 + |q_2^0|^2)\, d_\xi \mu_{x\bar\tau_*} + C' h B_1^{-2I} \Big) \tag{4.4.44}$$

where now the left-hand expression is defined by (4.4.19) with $j = k = 0$. Then

$$|\,\mathsf{tr}\big(\mathsf{R}^{\mathsf{W}}_x(\bar\tau_1, \bar\tau_2)\big)| \le h^{1-d} \sum_{*=1,2} \Big(C_0 \int (|q_1^0|^2 + |q_2^0|^2)\, d_\xi(\mathsf{tr}(\mu_{x\bar\tau_*})) + C' h\, \mathsf{tr}(B_1^{-2I}) \Big). \tag{4.4.45}$$

One can modify all other estimates in the same way.

4.5 Long Term Propagation and Sharper Remainder Estimates

4.5.1 Preliminary Discussion

Increasing T

In the previous section we under microhyperbolicity (or ξ-microhyperbolicity) condition increased T to a small constant T_0. Now we want to increase it further but in a such way that expression

$$F_{t\to h^{-1}\tau}\Big(\bar{\chi}_T(t)\Gamma\big(Q_{1x}u\,{}^tQ_{2y}\big)\Big) \tag{4.5.1}$$

or

$$F_{t\to h^{-1}\tau}\Big(\bar{\chi}_T(t)\Gamma_x\big(Q_{1x}u\,{}^tQ_{2y}\big)\Big) \tag{4.5.2}$$

would change negligibly for τ in the vicinity of $\bar{\tau}$. Then (almost) the same estimate for this expression with this new T would have place as it was for $T = T_0$ which would allow us to estimate the Tauberian error by

$$\frac{h^{1-d}}{T}\Big(C_0J(q_1^0B_1^j,\bar{\tau}) + C_0J(q_2^0B_1^k,\bar{\tau}) + C'\frac{h}{T} + C'h\Big)\vartheta\big(\frac{h}{LT}\big) \tag{4.5.3}$$

or

$$\frac{h^{1-d}}{T}\Big(C_0J_x(q_1^0B_1^j,\bar{\tau}) + C_0J_x(q_2^0B_1^k,\bar{\tau}) + C'\frac{h}{T} + C'h\Big)\vartheta\big(\frac{h}{LT}\big) \tag{4.5.4}$$

where $\vartheta = 1$ if there is no averaging (but then we take mollification with respect to the second energy level) where now T means "new" T.

However then the Tauberian expressions should change by

$$F_{t\to h^{-1}\tau}\Big(\big(\bar{\chi}_T(t) - \bar{\chi}_{T_0}(t)\big)\Gamma\big(Q_{1x}u\,{}^tQ_{2y}\big)\Big) \tag{4.5.5}$$

or

$$F_{t\to h^{-1}\tau}\Big(\big(\bar{\chi}_T(t) - \bar{\chi}_{T_0}(t)\big)\Gamma_x\big(Q_{1x}u\,{}^tQ_{2y}\big)\Big) \tag{4.5.6}$$

integrated from $-\infty$ to $\bar{\tau}$. We know that this operation is equivalent to the substitution

$$\big(\bar{\chi}_T(t) - \bar{\chi}_{T_0}(t)\big) \ \mapsto\ iht^{-1}\big(\bar{\chi}_T(t) - \bar{\chi}_{T_0}(t)\big) \tag{4.5.7}$$

and pinning $\tau = \bar{\tau}$. The same *no-periodicity* or *no-loop* condition which ensures respectively the negligibility of (4.5.5) or (4.5.6) also ensures the negligibility of these side-kicks.

Since we know that under microhyperbolicity or ξ-microhyperbolicity condition one can replace Tauberian expression for $T = T_0$ by Weyl expression, it is now true for new T as well. Therefore *Weyl error will be also estimated by* (4.5.3) *or* (4.5.4).

Synthesis

However it is not realistic to expect that no-periodicity or no-loop condition we use to derive such improved remainder estimates is fulfilled in every point of $\mathsf{supp}(q_1)$ or $\mathsf{supp}(q_2)$. To overcome this problem we will use different approaches in he cases of Γ- and Γ_x-related expressions.

In the former (Γ) case we replace our *original* operators Q_1 and Q_2 by $Q_{1i} = R_i Q_1$ and $Q_2 = R_i^* Q_2$ where

(4.5.8) R_i are *scalar pseudodifferential operators*

and assume the non-periodicity condition with time T_i on $\mathsf{supp}(r_i) \cap \mathsf{supp}(q_1)$ or $\mathsf{supp}(r_i) \cap \mathsf{supp}(q_2)$ for $i = 1, \dots, m$ while for $i = 0$ we pick up T_0 as a small constant. Then we estimate the error in question by

$$C_0 \frac{h^{1-d}}{T_i} \vartheta\Big(\frac{h}{LT_i}\Big)\big(J(r_i^0 q_1^0 B_1^j, \bar{\tau}) + J(r_i^0 q_2^0 B_1^k, \bar{\tau}) + C'h\Big). \tag{4.5.9}$$

Now we select R_i such that

$$\sum_{0\le i\le m} R_i^* R_i \equiv I, , \tag{4.5.10}$$

and thus

$$\sum_{0\le i\le m} |r_i^0|^2 = 1, . \tag{4.5.11}$$

Note that due to (4.5.8)

$$\Gamma\big(R_i Q_{1x} v(x,y)\, {}^t Q_{2y}\big) = \Gamma\big(Q_{1x} e(\dots)\, {}^t(R_i Q_2)_y\big). \tag{4.5.12}$$

Plugging $Q_2 \coloneqq R_i^* Q_2$ we arrive due to (4.5.10) to

$$\Gamma\big(Q_{1x} v(x,y)\, {}^t Q_{2y}\big) \equiv \sum_{0\le i\le m} \Gamma\big(R_i Q_{1x} v(x,y)\, {}^t(R_i^* Q_2)_y\big) \tag{4.5.13}$$

and also to

$$\kappa_n = \sum_{0\le i\le m} \kappa_{ni} \tag{4.5.14}$$

where κ_{ni} is coefficient κ_n calculated for pair $Q_{1i} = R_i Q_1$ and $Q_2 = R_i^* Q_2$. Then

$$\mathsf{N} = \sum_{0\le i\le m} \mathsf{N}_i,\ \mathsf{N}^{\mathsf{W}} = \sum_{0\le i\le m} \mathsf{N}_i^{\mathsf{W}}\ (\implies)\ \mathsf{R}^{\mathsf{W}} = \sum_{0\le i\le m} \mathsf{R}_i \tag{4.5.15}$$

in the obvious notations.

Therefore $\mathsf{R}_*^{\mathsf{W}}$ will be estimated by the sum of expressions (4.5.9) which is

$$C_0 \sum_{0\le i\le m} \frac{h^{1-d}}{T_i}\vartheta\big(\frac{h}{LT_i}\big)\Big(J(r_i^0 q_1 B_1^j, \bar\tau) + J(r_i^0 q_2 B_1^k, \bar\tau) + C'h\Big). \tag{4.5.16}$$

Note that due to (4.5.11), (4.5.8)

$$\sum_{0\le i\le m} J(r_i^0 q_1^0 B_1^j, \bar\tau) = J(q_1^0 B_1^j, \bar\tau) \tag{4.5.17}$$

which shows that inserting r_i^0 into $J(.,.)$ does not hurt.

On the other hand, as far as Γ_x-expressions are concerned the bulk of our arguments will not work; instead we simply decompose Q_1 and Q_2 into sums of operators. Then estimate would depend on m in a more explicit way. However in applications usually $m = 2$ in this case.

Later we will apply these improved (no double summation) versions of (4.5.9) and (4.5.10) in the following context: $T_1, \dots, T_m$ are large, T_0 is a fixed small constant.

Sharp and Sharper

So, to increase T we need to prove that (4.5.5)- or (4.5.6)-like expressions are negligible. For this purpose we will use propagation results of Chapters 2 and 3. However, there are *two* kinds of results: first there are standard results of Sections 2.1, 2.2 and 3.1–3.4 when we can take T as an arbitrarily large constant and long-term results of Sections 2.4 and 3.5 when depending on conditions we can take T up to $h^{-\delta}$ with sufficiently small exponent $\delta > 0$.

The results of the first kind allow us to acquire an arbitrarily small constant factor ε in the estimate of the contribution of the corresponding element Q_i with $i \geq 1$ [27)] to Weyl error basically going from "O"-type estimates to "o"-type estimates. However contribution of Q_0 is not improved this way and so not to lose the improvement we achieved we need to take Q_0 with an arbitrarily small but fixed support; "small" means here that $J_*(q_0^0 B_1^j, \bar\tau)$ is small (less than ε). Actually we cannot do better in the framework of the standard theory. In this case C' depends also on T. We will do it in Subsection 4.5.2.

The results of the second kind allow us to acquire a smaller factor ε in the estimate of the contribution of the corresponding element Q_i with $i \geq 1$ [27)] to Weyl error; now ε is a small but not very small parameter $\varepsilon \geq h^\delta$. Then not to lose this improvement we need to take Q_0 with a really small support; "small" means again that $J_*(q_0^0 B_1^j, \bar\tau) \leq \varepsilon$ but "really" indicates that ε is a parameter now. We can accommodate this only with Q with variable size support and in this case the theory of the previous Section 4.4 needs to be rebuilt. We will do it in Subsection 4.5.3.

Finally, Subsection 4.5.4 is devoted to simplest applications, examples and discussions.

4.5.2 Standard Sharp Theory

Analysis of Wave Front Sets

Let us recall that according to Theorem 1.3.13

$$\begin{aligned}(4.5.18)\quad &\mathsf{WF}(\Gamma_x Q_{1x} u\, {}^t Q_{2y}) \subset \big\{(x,\zeta,t,\tau):\ \exists \xi,\eta \text{ s.t. } \zeta = \xi - \eta\\ &\qquad \text{and } (x,\xi,x,-\eta,t,\tau) \in \mathsf{WF}(u),\ (x,\xi)\in \mathsf{supp}(q_1),\ (y,\eta)\in \mathsf{supp}(q_2)\big\}\end{aligned}$$

and

$$\begin{aligned}(4.5.19)\quad &\mathsf{WF}(\Gamma Q_{1x} u\, {}^t Q_{2y}) \subset \big\{(t,\tau):\ \exists x,\xi \text{ s.t.}\\ &\qquad (x,\xi,x,-\xi,t,\tau) \in \mathsf{WF}(u),\ (x,\xi)\in \mathsf{supp}(q_1)\cap \mathsf{supp}(q_2)\big\}.\end{aligned}$$

Meanwhile

$$(4.5.20)\quad (x,\xi,y,-\eta,t,\tau)\notin \mathsf{WF}(u) \implies \tau \in \mathsf{Spec}\, a(x,\xi)\cap \mathsf{Spec}\, a(y,\eta)$$

[27)] We ignore double summation and even the presence of the second factor in this discussion.

and according to Theorem 2.2.10 for bounded T

(4.5.21) $(x,\xi,y,-\eta,t,\tau)\notin \mathsf{WF}(u)$ provided *either*

(i) There exists a complete Hamiltonian flow $\Psi_{\tau,t'}(y,\eta)$ for $t'/t\in[0,1]$ which stays where symbol $A(.,.)$ is smooth and does not hit the boundary[28)] and $(x,\xi)\notin \Psi_{\tau,t}(y,\eta)$ *or*

(ii) There exists a complete Hamiltonian flow $\Psi_{\tau,-t'}(y,\eta)$ for $t'/t\in[0,1]$ which stays where symbol $A(.,.)$ is smooth and does not hit the boundary[28)] and $(y,\eta)\notin \Psi_{\tau,t}(x,\xi)$.

where $\Psi_{\tau,-t'}$ is a generalized Hamiltonian flow (possibly multivalued) on the energy level τ generated by symbol $-a$ [29)].

Then combining this claim with (4.5.18) we arrive to

(4.5.22) $(x,\zeta,t,\tau)\notin \mathsf{WF}(\Gamma_x Q_{1x}u\,{}^tQ_{2y})$ provided *either*

(i) For each $(y,\eta)\in \mathsf{supp}\, q_2$ s.t. $\tau\in a(y,\eta)$ there exists complete Hamiltonian flow $\Psi_{\tau,t'}(y,\eta)$ for $t'/t\in[0,1]$ which stays where symbol $A(.,.i)$ is smooth and does not hit the boundary[28)] and for each $(x,\xi)\in\Psi_t(y,\eta)$ either $x\neq y$ or $\zeta\neq\xi-\eta$ *or*

(ii) For each $(x,\xi)\in \mathsf{supp}\, q_1$ s.t. $\tau\in a(x,\xi)$ there exists complete Hamiltonian flow $\Psi_{\tau,-t'}(x,\xi)$ for $t'/t\in[0,1]$ which stays where symbol $A(.,.)$ is smooth and does not hit the boundary[28)] and for each $(y,\eta)\in\Psi_{-t}(x,\xi)$ either $x\neq y$ or $\zeta\neq\xi-\eta$.

Moreover, combining claim (4.5.21) with (4.5.19) we arrive to

(4.5.23) $(t,\tau)\notin \mathsf{WF}(\Gamma Q_{1x}u\,{}^tQ_{2y})$ provided *either*

(i) For each $(x,\xi)\in \mathsf{supp}(q_2)\cap\mathsf{supp}(q_1)$ s.t. $\tau\in a(x,\xi)$ there exists complete Hamiltonian flow $\Psi_{\tau,t'}(x,xi)$ for $t'/t\in[0,1]$ which stays where symbol $A(.,.i)$ is smooth and does not hit the boundary[28)] and $(x,\xi)\notin\Psi_t(x,\xi)$ *or*

(ii) Above condition (i) is fulfilled for t replaced by $-t$.

[28)] If we consider differential operators in domain with the boundary.

[29)] Where $z\in\Psi_{\tau,\pm t}(z')$ with $t>0$ simply means by definition that $(z;\pm t,\tau)\in \mathcal{K}_t^{\pm}(\tau-a,(z;0,\tau))$. We prefer to use terminology of the Hamiltonian flows.

Periodic Trajectories and Loops

Let us discuss what conditions of (4.5.22) and (4.5.23) actually mean.

Assume first that the complete Hamiltonian flow exists and is a single-valued. Then condition (4.5.22)(i) is equivalent to (4.5.22)(ii) and if we are not interested in ζ this condition simply means that

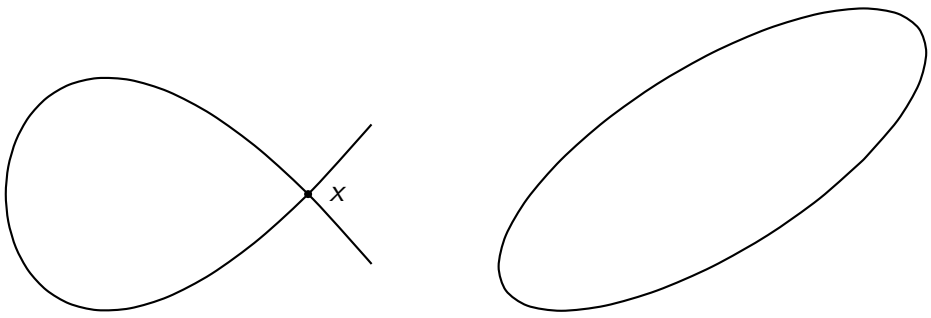
$$x \neq \pi_x \Psi_{\tau,t}(x,\xi); \tag{4.5.24}$$

simply flow does not return to point x at moment t (even from another direction). This means *absence of bicharacteristic loops*.

In the same framework condition (4.5.23)(i) is equivalent to (4.5.23)(ii) and means that

$$(x,\xi) \neq \Psi_{\tau,t}(x,\xi); \tag{4.5.25}$$

simply flow does not return to point (x,ξ) at moment t. This means *absence of periodic bicharacteristics*.



(a) Bicharacteristic loop (b) Periodic bicharacteristics

Figure 4.2: The case of a simple hamiltonian flow

Note that the looping trajectory (unless it is also a periodic trajectory) does not intersect itself, only its x projection (as shown on Figure 4.2(a)) does. Also note that x-projection of the loop could look differently from the one shown on Figure 4.2(a): it can be periodic[30] or be the same curve ran twice (in one and then in the opposite direction)[31].

[30] F.e. as $a = (\xi_1 - x_2)^2 + (\xi_2 + x_1)^2 - x_3$.

[31] F.e. as $a(x,\xi) = |\xi|^2 - x_1$.

Another difference is the *time direction* which manifests itself in more general settings. While in (4.5.23) we can choose time direction as we want, it is not the case in (4.5.22) since then we need to permute $\mathsf{supp}(q_1)$ and $\mathsf{supp}(q_2)$ as well. Moreover, to employ Tauberian technique we need a symmetric interval $[-T, T]$ and to extend $[-T_0, T_0]$ to it we need to verify our conditions for $t \in [-T, -T_0] \cup [T_0, T]$. In the case of (4.5.23) we should do it either for $t \in [-T, -T_0]$ or for $t \in [T_0, T]$ but in the case of (4.5.22) we should do it for $t \in [-T, -T_0]$ and for $t \in [T_0, T]$. For example, let (x, ξ) and (x, η) be such that $\Psi_{\tau,-t}(x, \xi)$ and $\Psi_{\tau,t}(x, \eta)$ hit "Dark Territory" as $t > 0$ and $t \asymp T_1$ and $(x, \xi) \in \mathsf{supp}(q_1)$, $(x, \eta) \in \mathsf{supp}(q_2)$; then in (4.5.22) we cannot go beyond time interval $[-T_1, T_1]$ (and we need to check no-loop condition).

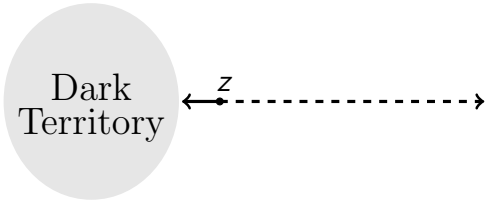


Figure 4.3: In (4.5.22) analysis we must follow both directions; in (4.5.23) analysis we can chose direction and we choose the dashed one.

On the other hand, if $\Psi_{\tau,-t}(x, \xi)$ neither hits "Dark Territory" nor comes back as $-T_2 < t < 0$ (for all $(x, \xi) \in \mathsf{supp}(q_1) \cap \mathsf{supp}(q_2)$) then (4.5.23) we can extend a time interval to $[-T_2, T_2]$.

In the current settings a "Dark Territory" could mean even a smooth boundary. However applying results of Chapter 3 we would not consider a smooth transversally met boundary as a "Dark Territory" but as a possible source of looping. For example, for a geodesic billiard flow the trajectory which hits the boundary normally goes back the same way and we need either to take $T_1 < 2\mathsf{dist}(x, \partial X)$ which is not an improvement over $T_1 < \mathsf{dist}(x, \partial X)$ or to analyze the contribution of this loop explicitly getting a boundary layer type correction to $e(x, x, .)$; this would have no affect on $\int e(x, x, .)\psi(x)\,dx$ as $\mathsf{supp}(\psi) \cap \partial X = \emptyset$. For details see Section 8.1.

For branching flow one should replace (4.5.24) and (4.5.25) by

$$x \notin \pi_x \Psi_{\tau,t}(x, \xi), \tag{4.5.26}$$

$$(x, \xi) \notin \Psi_{\tau,t}(x, \xi) \tag{4.5.27}$$

respectively.

Results

Remark 4.5.1. (i) Instead of Theorem 2.2.10 which does not allow bicharacteristic trajectory to hit the boundary, we can refer to Theorem 3.2.4 or geometric versions of Theorems 3.4.6 and 3.4.14.

item We definitely need to use the quantitate conditions: instead "trajectory does not hit the boundary" we need to assume that

$$\mathsf{dist}\big(\Psi_{t,\tau}(x,\xi),\partial X\big)\ge \gamma \tag{4.5.28}$$

and instead of (4.5.26), (4.5.27) we need to assume respectively that

$$\mathsf{dist}\big(x,\pi_x\Psi_{\tau,t}(x,\xi)\big)\ge \begin{cases}\epsilon|t| & \text{as } |t|\le T_0,\\ \gamma & \text{as } T_0\le |t|\le T,\end{cases}' \tag{4.5.29}$$

$$\mathsf{dist}\big((x,\xi),\Psi_{\tau,t}(x,\xi)\big)\ge \begin{cases}\epsilon|t| & \text{as } |t|\le T_0,\\ \gamma & \text{as } T_0\le |t|\le T,\end{cases} \tag{4.5.30}$$

where γ is a small constant. Conditions in this form are fulfilled automatically for $|t|\le T_0$ due to microhyperbolicity and ξ-microhyperbolicity respectively. Assuming that these conditions are fulfilled for all t (may be of a selected sign) with $|t|\le T$ completely removes implicit T_0 from the final estimates.

So, our analysis leads us to the following results:

Theorem 4.5.2. *Let assumptions of Theorem 4.4.1 be fulfilled with* (4.4.2) *replaced by* (4.4.18).

(i) Let A be microhyperbolic on $\mathsf{supp}(q_1)\cup\mathsf{supp}(q_2)$ on the energy level $\bar\tau$. Consider a generalized Hamiltonian flow $\Psi_{t,\bar\tau}$ issued from γ-vicinity $\mathcal{U}$ of $\mathsf{supp}(q_1)\cap\mathsf{supp}(q_2)$. Assume that as $\varsigma t>0$ [32] *and $|t|\le T$ conditions* (4.5.28), (4.5.30) *are fulfilled and also*

(4.5.31) *In γ-vicinity of $\pi_x\Psi_{t,\bar\tau}(\mathcal{U})$ assumptions of Theorem 4.4.1 are fulfilled.*

Then formula (4.3.58) *and all its derivative formulae remain valid for indicated T (rather than $T=T_0$) modulo terms not exceeding $C'h^s$ where here and in what follows constant C' depends also on T and γ.*

[32] Where we can select $\varsigma=\pm1$.

(ii) Let A be ξ-microhyperbolic on $\mathsf{supp}(q_1) \cup \mathsf{supp}(q_2)$. Consider a generalized Hamiltonian flow $\Psi_{t,\bar\tau}$ issued from $\mathsf{supp}(q_1) \cup \mathsf{supp}(q_2)$. Assume that as $|t| \le T$ conditions (4.5.28), (4.5.29) *and* (4.5.31) *are fulfilled. Then formula* (4.3.57) *and all its derivative formulae remain valid for indicated T (rather than $T = T_0$) modulo terms not exceeding $C'h^s$.*

Remark 4.5.3. Under additional assumptions of Proposition 4.3.5 statements of Theorem 4.5.2 hold with Q_1, Q_2 and q_1, q_2 replaced by $Q_1B_1^j$, $Q_2B_1^k$ and $q_1B_1^j, q_2B_1^k$ respectively with $j, k \in [0, l]$.

Now combining with arguments of Subsubsection *4.4.2.2 Improved Asymptotics without Mollification* we arrive to the sharp[33)] Theorems 4.4.12 and 4.4.11. Before doing it just note

Theorem 4.5.4. *Let assumptions of Theorem 4.4.1 be fulfilled with* (4.4.2) *replaced by* (4.4.18). *Then*

(i) In the framework of Theorem 4.5.2(i) and Remark 4.5.3 for both energy levels $\bar\tau_1$ and $\bar\tau_2$ sharp[3)] estimates (4.4.37)–(4.4.39) *hold (in "without $_x$" version).*

(ii) In the framework of Theorem 4.5.2(ii) and Remark 4.5.3 for both energy levels $\bar\tau_1$ and $\bar\tau_2$ sharp[33)] estimates (4.4.37)–(4.4.39) *hold.*

Now we are ready to the synthesis as described in Subsubsection *4.5.1.2 Synthesis*.

Theorem 4.5.5. *Let assumptions of Theorem 4.4.1 be fulfilled with* (4.4.2) *replaced by* (4.4.18). *Let A be microhyperbolic on $\mathsf{supp}(q_1) \cup \mathsf{supp}(q_2)$ on energy levels $\bar\tau_*$, $* = 1, 2$. Let $r_i^0 \in \mathsf{S}_{h,\gamma,\gamma,K}(T^*X)$ [34)] be scalar symbols satisfying* (4.5.11) .

Consider generalized Hamiltonian flow $\Psi_{t,\bar\tau_*}$ issued from γ-vicinity $\mathcal{U}_i$ of $\mathsf{supp}(r_i^0) \cap \mathsf{supp}(q_1) \cap \mathsf{supp}(q_2)$. Assume that as $\varsigma t > 0$ with $\varsigma := \varsigma_{*i}$ [32)] *and $|t| \le T_{*i}$ conditions* (4.5.28), (4.5.30) *and* (4.5.31) *are fulfilled with $\mathcal{U} := \mathcal{U}_i$.*

[33)] Where "sharp" means "with T instead of T_0" and with C' depending on γ, T.

[34)] With an arbitrarily small constant γ.

Then

$$\mathsf{R}^{\mathsf{W}}(\bar\tau_1,\bar\tau_2) \prec \sum_{*=1,2}\ \sum_{0\le i\le m} C_0\frac{h^{1-d}}{T_{*i}}\Big(J(r_i^0q_1^0B_1^j,\bar\tau_*)+J(r_i^0q_2^0B_1^k,\bar\tau_*)+C'h\Big), \tag{4.5.32}$$

$$\mathsf{R}^{\mathsf{W}}_{\varphi_2,L_2}(\bar\tau_1,\bar\tau_2) \prec \sum_{0\le i\le m} C_0\frac{h^{1-d}}{T_{*i}}\Big\{\Big(J(r_i^0q_1^0B_1^j,\bar\tau_1)+J(r_i^0q_2^0B_1^k,\bar\tau_1)+C'h\Big)+ \Big(J(r_i^0q_1^0B_1^j,\bar\tau_2)+J(r_i^0q_2^0B_1^k,\bar\tau_2)+C'h\Big)\vartheta_2\big(\frac{h}{L_2T_{*i}}\big)\Big\} \tag{4.5.33}$$

and

$$\mathsf{R}^{\mathsf{W}}_{\varphi_1,L_1,\varphi_2,L_2}(\bar\tau_1,\bar\tau_2) \prec \sum_{*=1,2}\ \sum_{0\le i\le m} C_0\frac{h^{1-d}}{T_{*i}}\Big(J(r_i^0q_1^0B_1^j,\bar\tau_*)+J(r_i^0q_2^0B_1^k,\bar\tau_*)+C'h\Big)\vartheta_*\big(\frac{h}{L_*T_{*i}}\big) \tag{4.5.34}$$

with the left-hand expressions defined respectively by (4.4.19), (4.4.16) *and* (4.4.17) *(without subscript "$_x$") with* $J(q^0B^j,\bar\tau)$ *defined by* (4.4.26). *Here* C' *depends on* γ, m, $\max_i T_{*i}$.

Theorem 4.5.6. *Let assumptions of Theorem 4.4.1 be fulfilled without* (4.4.2). *Let* A *be* ξ*-microhyperbolic on* $\operatorname{supp}(q_1)\cup\operatorname{supp}(q_2)$ *on energy levels* $\bar\tau_*$, $*=1,2$. *Let* $r_i^0\in \mathsf{S}_{h,\gamma,\gamma,K}(T^*X)$ [34] *be scalar symbols satisfying*

$$\sum r_i^0 = 1. \tag{4.5.35}$$

Consider generalized Hamiltonian flow $\Psi_{t,\bar\tau_*}$ *issued from* γ*-vicinity* $\mathcal{U}_i$ *of* $\operatorname{supp}(r_i^0)\cap(\operatorname{supp}(q_1)\cup\operatorname{supp}(q_2))$. *Assume that as* $|t|\le T_{*i}$ *conditions* (4.5.28), (4.5.29) *and* (4.5.31) *are fulfilled with* $\mathcal{U}:=\mathcal{U}_i$. *Then*

$$\mathsf{R}^{\mathsf{W}}_x(\bar\tau_1,\bar\tau_2) \prec \sum_{*=1,2}\ \sum_{0\le i\le m} C_0(m+1)\frac{h^{1-d}}{T_{*i}}\Big(J_x(r_i^0q_1^0B_1^j,\bar\tau_*)+J_x(r_i^0q_2^0B_1^k,\bar\tau_*)+C'h\Big), \tag{4.5.36}$$

(4.5.37) $\mathsf{R}^{\mathsf{W}}_{x,\varphi_2,L_2}(\bar\tau_1,\bar\tau_2) \prec$
$$\sum_{0\le i\le m} C_0(m+1)\frac{h^{1-d}}{T_{*i}}\Big\{\Big(J_x(r_i^0 q_1^0 B_1^j,\bar\tau_1)+J_x(r_i^0 q_2^0 B_1^k,\bar\tau_1)+C'h\Big)+ \Big(J_x(r_i^0 q_1^0 B_1^j,\bar\tau_2)+J_x(r_i^0 q_2^0 B_1^k,\bar\tau_2)+C'h\Big)\vartheta_2\big(\frac{h}{L_2 T_{*i}}\big)\Big\}$$

and

(4.5.38) $\mathsf{R}^{\mathsf{W}}_{x,\varphi_1,L_1,\varphi_2,L_2}(\bar\tau_1,\bar\tau_2) \prec$
$$\sum_{*=1,2} C_0(m+1)\sum_{0\le i\le m}\frac{h^{1-d}}{T_{*i}}\Big(J_x(r_i^0 q_1^0 B_1^j,\bar\tau_*)+J_x(r_i^0 q_2^0 B_1^k,\bar\tau_*)+C'h\Big)\vartheta_*\big(\frac{h}{L_* T_{*i}}\big)$$

with left-hand expressions defined respectively by (4.4.19), (4.4.16) *and* (4.4.17) *with* $J_x(q^0,\bar\tau)$ *defined by* (4.4.25).

Proof. Consider scalar operators R_i with principal symbols r_i^0 such that $\sum_i R_i = I$. Then plugging $Q_1 := R_{i_1}Q_1$, $Q_2 := Q_2 R_{i_2}^*$ and $T = \min(T_{i_1}, T_{i_2})$ we see that now we need to analyze not $\mathsf{N}_{xi}(\tau',\tau'')$ but $\mathsf{N}_{xi_1i_2}(\tau',\tau'')$ and the problem is that we have now T_{i_1} and T_{i_2}. We do not indicate which of two energy levels is considered because due to (4.4.12) they are treated independently.

Still as $Q_1 = Q_2^*$, $j = k$ and $i_1 = i_2 = i$ we using Lemma 4.2.7 estimate $\mathsf{N}_{xii}(\tau',\tau'')$ as before:

$$\mathsf{N}_{xii}(\tau',\tau'') \prec C_0 J_x(r_i^0 q_1^0 B_1^j,\bar\tau)\cdot\big(|\tau'-\tau''|+\frac{h}{T_i}\big)h^{-d}+\dots$$

where dots denote

$$C'\big(|\tau'-\tau''|+h\big)\cdot\big(|\tau'-\bar\tau|+|\tau''-\bar\tau|+h\big)$$

and therefore in the general case

$$\mathsf{N}_{xi_1i_2}(\tau',\tau'') \prec C_0 J_x(r_{i_1}^0 q_1^0 B_1^j,\bar\tau)\cdot\big(|\tau'-\tau''|+\frac{h}{T_{i_1}}\big)h^{-d}+ C_0 J_x(r_{i_2}^0 q_2^0 B_1^k,\bar\tau)\cdot\big(|\tau'-\tau''|+\frac{h}{T_{i_2}}\big)h^{-d}+\dots$$

which again leads to Tauberian and Weyl errors

$$C_0\frac{h^{1-d}}{T_{i_1}}\vartheta\big(\frac{h}{LT_{i_1}}\big)J(r^0_{i_1}q^0_1B^j_1,\bar\tau)+C_0\frac{h^{1-d}}{T_{i_2}}\vartheta\big(\frac{h}{LT_{i_2}}\big)J(r^0_{i_2}q^0_2B^k_1,\bar\tau)+C'h^{2-d}.$$

Now summation with respect i_1, i_2 delivers announced estimates. □

Note that if $r^0=\big(\sum_i|r^0_i|^2\big)^{\frac12}\le 1$ (which is easy to satisfy in the framework of (4.5.35)) then

$$\sum_i J_x(r^0_iq_0,\bar\tau)=J_x(r^0q_0,\bar\tau)\le J_x(q_0,\bar\tau). \tag{4.5.39}$$

4.5.3 Sharper Remainder Estimates

To establish truly sharper remainder estimates we need to work simultaneously in two directions: to increase T up to $h^{-\delta}$ (depending on assumptions) and to decrease γ. While it is sufficient to decrease γ to h^δ we will go here further, up to $\gamma=h^{\frac12-\delta'}$ where δ' and δ are related.

Analysis as $|t|\le T_0$

We need to reexamine first results which are due to propagation.

Proposition 4.5.7. *Let A satisfy all the assumptions of Theorem 4.4.1. Let $q_1,q_2\in \mathsf{S}_{h,\rho,\gamma,K}(T^*X,\mathbb{H},bH)$ with $\rho\gamma\ge h^{1-\delta}$. Then*

(i) If A is ξ-microhyperbolic on $\mathsf{supp}(q_1)$ on the energy level τ then

$$F_{t\to h^{-1}\tau}\chi_T(t)\Gamma_x\big(Q_{1x}B^j_1uB^k_1{}^tQ_{2y}\big)\equiv 0 \tag{4.5.40}$$

as

$$h^{1-\delta}\rho^{-1}\le T\le T_0 \tag{4.5.41}$$

where as usual $\chi\in \mathscr{C}_0^\infty([-1,-\frac13]\cup[\frac13,1])$ and T_0 is a small constant, $j,k\in[0,l]$.

(ii) If $\rho=\gamma$ and A is microhyperbolic on $\mathsf{supp}(q_1)$ on the energy level τ then instead

$$F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\big(Q_{1x}B^j_1uB^k_1{}^tQ_{2y}\big)\equiv 0 \tag{4.5.42}$$

as

$$h^{1-\delta}\rho^{-1}\le T\le T_0. \tag{4.5.43}$$

Proof. Let A be ξ-microhyperbolic on $\mathsf{supp}\,\bar{q}_1$ and $\bar{q}_1, \bar{q}_2 \in \mathsf{S}_{h,1,1,K}(T^*X, \mathbb{H}.bH)$. Let $\psi \in \mathscr{C}_0^\infty(\mathbb{R}^d)$, supported in $\{|x-y| \le \epsilon\}$ with small enough $\epsilon > 0$. Then ξ-microhyperbolicity and the standard propagation results imply that

$$F_{t\to h^{-1}\tau}\chi_T(t)\big(\bar{Q}_{1x}B_1^j u\psi_T(x-y)B_1^k\,{}^t\bar{Q}_{2y}\big) \equiv 0$$

as $T \ge h^{1-\delta}$.

Applying Q_{1x} and ${}^tQ_{2y}$ from the left and right-respectively we derive (4.5.40) under (4.5.41) immediately. Statement (i) is proven.

Finally, (i) follows from (i). □

Successive Approximations

Let us look at successive approximation method of Section 4.3 with $q_2 \in \mathsf{S}_{h,\rho,\gamma,K}(T^*X, \mathbb{H}, bH)$. The only difference appears as $(x_j - y_j)$ commutes with Q_2. In this case factor $[x_j - y_j, Q_2]$ of magnitude $\rho^{-1}h$ appears (instead of h) and since $(x_j - y_j)$ may be accompanied by a single parametrix $G^\pm$ or $\bar{G}^\pm$ with the norm $\lesssim Th^{-1}$, we gain a factor $Ch\rho^{-1}\cdot Th^{-1} = CT\rho^{-1}$ in the estimate of the norm. Then for the method of successive approximations to work we need in addition to condition $T \le h^{\frac12+\delta}$ also condition $T \le \rho h^\delta$; so we need

$$T \le \min\big(h^{\frac12+\delta}, \rho h^\delta\big). \tag{4.5.44}$$

Under this condition we get asymptotic expansions

$$F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma_x\big(Q_{1x}B_1^j uB_1^k\,{}^tQ_{2y}\big) \sim \sum_{n\ge 0}\kappa'_{nx}(\tau)h^{-d+n+1} \tag{4.5.45}$$

and

$$F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma\big(Q_{1x}B_1^j uB_1^k\,{}^tQ_{2y}\big) \sim \sum_{n\ge 0}\kappa'_{n}(\tau)h^{-d+n+1} \tag{4.5.46}$$

Actually we get a similar asymptotic expansion (with $\mathcal{K}'_n(x,y,\tau)$) even without Γ_x. However, we want to increase T and we can do it only if we apply Γ_x. In this case we get expansion for $T = T_0$ (where T_0 is again a small constant) provided upper bound (4.5.44) to T is larger than the lower bound (4.5.41); this condition is equivalent to

$$\rho \ge h^{\frac12-\delta} \tag{4.5.47}$$

where we adjust an arbitrarily small exponent $\delta > 0$ if needed.

Thus we arrive to

Proposition 4.5.8. *(i) In the framework of Proposition 4.5.7(i) and* (4.5.47) *asymptotic expansion* (4.5.45) *holds with* $T = T_0$.

(ii) In the framework of Proposition 4.5.7(ii) asymptotic expansion (4.5.46) *holds with* $T = T_0$.

Further, one can prove easily

Proposition 4.5.9. *In the framework of Proposition 4.5.8(i) and (ii) respectively*

$$\|\partial_\tau^p \kappa_{nx}\| \le C_{np}\rho^{-n-p}\gamma^{-n} \tag{4.5.48}$$

and

$$\|\partial_\tau^p \kappa_n\| \le C_{np}\gamma^{-2n-p} \tag{4.5.49}$$

Increasing T

Now the long-range propagation results of Sections 2.4 and 3.5 allow us to increase T with negligible error in (4.5.45), (4.5.46) under no-loop and no-periodicity conditions (4.5.29), (4.5.30) respectively where $\gamma \ge h^{\frac{1}{2}-\delta}$ now is a small parameter. Those are assumptions of Theorem 2.4.17 if there is no boundary or Theorem 3.5.3 otherwise. Let us mention that due to the nature of these theorems we *must* assume that $\rho = \gamma$ even if we consider asymptotics without spatial averaging.

Tauberian Estimates

Now due to Proposition 4.5.9 estimates (4.4.23), (4.5.25) are replaced by the same estimates but with $C'h^{2-d}$ replaced by $C'h^{2-d}\gamma^{-2}$. Then the right-hand expression of (4.5.32) is replaced by

$$C_0\Big(J(q_1^0 B_1^j, \tau) + C'h\gamma^{-2}\Big)\frac{h^{1-d}}{T}$$

which due to Proposition 4.5.9 does not exceed

$$C_0\Big(J(q_1^0 B_1^j, \bar\tau) + C'\gamma^{-1}|\tau-\bar\tau| + C'h\gamma^{-2}\Big)\frac{h^{1-d}}{T}.$$

Then the right-hand expression of (4.5.33) is replaced by

$$C_0\Big(J(q_1^0 B_1^j, \bar\tau) + C'\gamma^{-1}|\tau'-\bar\tau| + C'\gamma^{-1}|\tau''-\bar\tau| + C'h\gamma^{-2}\Big)\cdot\big(|\tau'-\tau''| + \frac{h}{T}\big)$$

as $Q_2 = Q_1^*$, $j = k$ which yields the similar estimate

$$C_0\Big(J(q_1^0 B_1^j, \bar\tau) + J(q_2^0 B_1^k, \bar\tau) + C'\gamma^{-1}|\tau' - \bar\tau| + C'\gamma^{-1}|\tau'' - \bar\tau| + C'h\gamma^{-2}\Big)\times \Big(|\tau' - \tau''| + \frac{h}{T}\Big)$$

in the general case.

Therefore in the end of the day we arrive to the remainder estimate

$$C_0\frac{h^{1-d}}{T}\vartheta\big(\frac{h}{LT}\big)\Big(J(q_1^0 B_1^j, \bar\tau) + J(q_2^0 B_1^k, \bar\tau) + C'h\gamma^{-2}\Big) \tag{4.5.50}$$

which is obtained from the previous one by transformation $|\tau' - \bar\tau| \mapsto hT^{-1}$, $|\tau'' - \bar\tau| \mapsto hT^{-1}$.

First we get estimate (4.5.50) for Tauberian and then for Weyl errors. Similar estimates holds without spatial averaging.

So, *the only difference in comparison with the earlier estimates is the replacement of the term $C'h$ by $C'h\gamma^{-2}$ where C' does not depend on γ, T.*

The synthesis stage repeats one for $\gamma \asymp 1$.

Remark 4.5.10. (i) As $q_1, q_2 \in \mathsf{S}_{h,\gamma,\gamma,K}$ we may need to take more terms in the Weyl expansion than we used to have because junior terms are larger; however on the synthesis stage we return to our original $q_1, q_2 \in \mathsf{S}_{h,K}$ and in the Weyl expansions for them we can drop all these extra terms.

(ii) Probably one can improve term $C'h\gamma^{-2}$ to $C'h\gamma^{-1}$ but this term seems to be rather superficial.

Finally, Results

Theorem 4.5.11. *Let assumptions of Theorem 4.4.1 be fulfilled with* (4.4.2) *replaced by* (4.4.18). *Let A be microhyperbolic on $\mathsf{supp}(q_1) \cup \mathsf{supp}(q_2)$ on the energy levels $\bar\tau_*$, $* = 1, 2$. Let $r_i^0 \in \mathsf{S}_{h,\gamma,\gamma,K}(T^*X)$* [35] *be scalar symbols satisfying* (4.5.11).

Consider generalized Hamiltonian flow $\Psi_{t,\bar\tau_*}$ issued from γ-vicinity $\mathcal{U}_i$ of $\mathsf{supp}(r_i^0) \cap \mathsf{supp}(q_1) \cap \mathsf{supp}(q_2)$. Assume that as $\varsigma t > 0$ with $\varsigma := \varsigma_{*i}$ [32] *and $|t| \le T_{*i}$ conditions* (4.5.28), (4.5.30) *and* (4.5.31) *are fulfilled and moreover*

[35] With $h^{\frac12-\delta} \le \gamma \le 1$.

as $i = 1, \dots, m$ *along* $\Psi_{t,\tau}(\mathcal{U})$ *conditions of Theorem 2.4.17 or Theorem 3.5.3 are fulfilled with* $\mathcal{U} := \mathcal{U}_i$.

Then estimates (4.5.32)–(4.5.34) *hold with term* $C'h$ *in the factor replaced by* $C'h\gamma^{-2}$ *with the left-hand expressions defined respectively by* (4.4.19), (4.4.16) *and* (4.4.17) *(without* $_x$*) with* $J(q^0 B^j, \bar\tau)$ *defined by* (4.4.26). *Here* C' *does not depend on* γ, T_{*i}.

Theorem 4.5.12. *Let all assumptions of Theorem 4.4.1 be fulfilled except may be* (4.4.2). *Let* A *be* ξ*-microhyperbolic on* $\operatorname{supp}(q_1) \cup \operatorname{supp}(q_2)$ *on the energy levels* $\bar\tau_*$, $* = 1, 2$. *Let* $r_i^0 \in \mathsf{S}_{h,\gamma,\gamma,K}(T^*X)$ *be scalar symbols satisfying* (4.5.35).

Consider generalized Hamiltonian flow $\Psi_{t,\bar\tau_*}$ *issued from* γ*-vicinity* $\mathcal{U}_i$ *of* $\operatorname{supp}(r_i^0) \cap (\operatorname{supp}(q_1) \cup \operatorname{supp}(q_2))$. *Assume that as* $|t| \le T_{*i}$ *conditions* (4.5.28), (4.5.29) *and* (4.5.31) *are fulfilled and moreover as* $i = 1, \dots, m$ *along* $\Psi_{t,\tau}(\mathcal{U})$ *conditions of Theorem 2.4.17 or Theorem 3.5.3 are fulfilled with* $\mathcal{U} := \mathcal{U}_i$.

Then estimates (4.5.36)–(4.5.38) *hold with term* $C'h$ *in the factor replaced by* $C'h\gamma^{-2}$ *with the left-hand expressions defined respectively by* (4.4.19), (4.4.16) *and* (4.4.17) *(without* $_x$*) with* $J(q^0 B^j, \bar\tau)$ *defined by* (4.4.26). *Here* C' *does not depend on* γ, T_{*i}.

4.5.4 Discussion

Local Results

We already achieved rather general results as far as $O(h^{1-d})$ remainder estimate is concerned.

Let $A(x, D)$ be an elliptic operator on the closed manifold and we are interested in its spectral asymptotics for a large spectral parameter $\lambda \gg 1$. So, we are looking at its spectrum in $(0, \lambda)$ [36]. Alternatively we can consider operator $\lambda^{-1}A(x, D)$ and look at its spectrum at $(0, 1)$.

Further, we can rewrite the latter operator as $a(x, hD, h)$ where $h = \lambda^{-1/m}$, m is an order of $A(x, D)$. Obviously the principal symbol $a(x, \xi, 0)$ of h-differential operator $a(x, hD, h)$ and the principal symbol $A_m(x, \xi)$ of the differential operator $A(x, D)$ coincide. One can see easily that 0 is a

[36] Negative eigenvalues are covered in the same manner.

lacunar value unless we are looking at $\xi = 0$ [37]. All the details are left for Sections 11.2 and 11.3.

On the other hand, in this case $a(x, hD, 0)$ is microhyperbolic on the energy level 1 in the direction $\ell = (\ell_\xi, 0) = (\xi, 0)$. Therefore asymptotics

$$e(x, x, 0, \lambda) = \kappa_0(x)\lambda^{d/m} + O(\lambda^{(d-1)/m}) \qquad \text{as} \quad \lambda \to +\infty \tag{4.5.51}$$

holds where we used the trick described in [37]. Also we get a host of asymptotics for spectral mollifications. These results hold for matrix operators and for operators acting on the Hermitean (or even Hilbert) sections of the bundles. All details are left for Chapter 11 and for elliptic operators on manifolds with boundaries as long as x is disjoint from the boundary.

Simple Improvements

However the completely local nature of the above results (where the only global condition is the self-adjointness of A) is lost if we want marginally improve the remainder estimate. Let us start from $o(\lambda^{(d-1)/m})$. Then we need to consider Hamiltonian dynamics Ψ_t on $\Sigma_\tau = \{\tau \in \operatorname{Spec} a(x, \xi)\}$; the concrete value of the energy level > 0 here is of no importance due to homogeneity.

As $A_m(x, \xi)$ is the scalar symbol this is the ordinary Hamiltonian dynamics of it; further, if $\operatorname{Spec} A_m(x, \xi)$ consists of eigenvalues $\rho_1(x, \xi), \dots, \rho_r(x, \xi)$ of the constant multiplicities we get a bunch of dynamics generated by these eigenvalues; finally, in the general case this is the generalized Hamiltonian dynamics as described in Chapter 2.

Then the result depends strongly on what we are looking for: if we want remainder estimate $o(\lambda^{(d-1)/m})$ in (4.5.51) upgraded to

$$e(x, x, 0, \lambda) = \kappa_0(x)\lambda^{d/m} + \kappa_1(x)\lambda^{(d-1)/m} + o(\lambda^{(d-1)/m}) \qquad \text{as} \quad \lambda \to +\infty \tag{4.5.52}$$

we need to fight bicharacteristic loops as on Figure 4.2(a), if we want this remainder estimate in

[37] To avoid this little problem we can consider spectrum of $A(x, D)$ in $(\frac{1}{2}\lambda, \lambda)$ and thus the spectrum of $a(x, hD, h)$ in $(\frac{1}{2}, 1)$ and then plug $2^{-n}\lambda$ instead of λ and finally to sum what we got with respect to $n = 0, \dots, \lfloor \log_2 \lambda \rfloor$.

$$\int e(x,x,0,\lambda)\psi(x)\,dx = \int \Big(\kappa_0(x)\lambda^{d/m} + \kappa_1(x)\lambda^{(d-1)/m}\Big)\psi(x)\,dx + o\big(\lambda^{(d-1)/m}\big) \qquad \text{as } \lambda \to +\infty \tag{4.5.53}$$

we need to fight periodic bicharacteristics as on Figure 4.2(b). Recall that the bicharacteristic loops and periodic bicharacteristics are defined by (4.5.24), (4.5.25).

Let Π denote the set of all periodic points (x,ξ) and Π_T denote the set of periodic points with periods $t \le T$. The latter set is closed, the former is the union of the sequence of closed sets. For a sake of simplicity we assume that $\dim \mathbb{H} < \infty$.

Assume that

$$\mu_\tau(\Pi) = 0. \tag{4.5.54}$$

Then $\mu_\tau(\Pi_T) = 0$ for any T. However Π_T is a closed set[38]. Then if we consider its ε-vicinity $\Pi_{T,\varepsilon}$

$$\mu_\tau(\Pi_{T,\varepsilon}) \to +0 \qquad \text{as } \varepsilon \to +0. \tag{4.5.55}$$

Applying Theorem 4.5.5 with $\mathcal{U}_0 = \Pi_{T,\varepsilon}$ we arrive to asymptotics (4.5.53) with the remainder

$$C\big(T^{-1} + \mu_\tau(\Pi_{T,\varepsilon})\big)\lambda^{(d-1)/m} + C'_{T,\varepsilon}\lambda^{(d-2)/m} \tag{4.5.56}$$

with arbitrary T, ε and thus (4.5.56) is actually $o\big(\lambda^{(d-2)/m}\big)$ due to (4.5.55). Therefore

(4.5.57) Asymptotics (4.5.53) holds provided assumption (4.5.54) is fulfilled.

On the other hand, without assumption (4.5.54) asymptotics (4.5.53) could fail as for Laplace-Beltrami operator on $\mathbb{S}^d$ or on other manifolds with all periodic bicharacteristics. We will consider such operators in Section 6.2.

Meanwhile the same arguments and Theorem 4.5.6 show that asymptotics (4.5.52) holds provided

$$\mu_{\tau,x}(\Lambda) = 0. \tag{4.5.58}$$

[38] Therefore due to $\mu_\tau(\Pi_T) = 0$ it is of measure 0 and thus Π is of the first Baire category.

where Λ is the set of loop points (x,ξ).

For example, on the surface of ellipsoid in $\mathbb{R}^3$ with the semi-axis (a_1, a_2, a_3) condition (4.5.58) is fulfilled if x is not an umbilic point[39].

Similarly, considering an ellipse we arrive to asymptotic (4.5.52) in all interior points which are not focal (we need to consider billiards but Chapter 3 provides all the necessary tools).

The further analysis can go in two directions: sharper reminder estimates and more general operators.

Sharper Remainder Estimates

In this case we need to apply the long-term propagation which involves an analysis how fast $D\Psi_t$ grows as $t\to\infty$ and how fast $\mu_\tau(\Pi_{T,\varepsilon})$ tends to 0 as $\varepsilon\to+0$. There are two cases:

Example 4.5.13 (Exponential case). It is the case when one can prove that

$$|D\Psi_t|\le ce^{c|t|} \tag{4.5.59}$$

and

$$\mu_\tau(\Pi_{T,\varepsilon})\le CT^{-1}\qquad \text{as } \varepsilon\le c^{-1}e^{-cT}. \tag{4.5.60}$$

Then taking $T=\epsilon\log\lambda$ one can derive (4.5.53) with the remainder estimate $O\bigl(\lambda^{(d-1)/m}(\log\lambda)^{-1}\bigr)$.

This happens f.e. on the manifolds on of the negative sectional curvature or on 2-dimensional manifolds without conjugate points P. Bérard [1].

Similarly, under condition (4.5.60) replaced by

$$\mu_{\tau,x}(\Lambda_{T,\varepsilon})\le CT^{-1}\qquad \text{as } \varepsilon\le c^{-1}e^{-cT}. \tag{4.5.61}$$

one can derive (4.5.52) with the remainder estimate $O\bigl(\lambda^{(d-1)/m}(\log\lambda)^{-1}\bigr)$.

Example 4.5.14 (Power case). It is the case when one can prove that

$$|D\Psi_t|\le c(|t|+1)^c \tag{4.5.62}$$

and

$$\mu_\tau(\Pi_{T,\varepsilon})\le CT^{-\delta}\qquad \text{as } \varepsilon\le c^{-1}T^{-c}. \tag{4.5.63}$$

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[39] There are four of such points located at $x_2=0$ as $a_1>a_2>a_3$ and there are two of them at $(0,0,\pm a_3)$ if $a_1=a_2\ne a_3$.

Then taking $T = \epsilon\lambda^{\delta'}$ one can derive (4.5.53) with the remainder estimate $O\bigl(\lambda^{(d-1-\delta'')/m}\bigr)$ with unspecified small exponent $\delta'' > 0$.

This happens for example in some Euclidean domains (ellipse, ball, domain between two confocal ellipses or between two spheres with the same center), on the surface of 3-dimensional ellipsoids which are not spheres. In this examples dynamics is completely integrable.

It happens also in Euclidean polyhedra which is not completely integrable case.

Similarly, under condition (4.5.63) replaced by

$$\mu_{\tau,x}(\Lambda_{T,\varepsilon}) \le CT^{-1} \qquad \text{as } \varepsilon \le c^{-1}T^{-c}. \tag{4.5.64}$$

one can derive (4.5.52) with the remainder estimate $O\bigl(\lambda^{(d-1-\delta'')/m}\bigr)$.

This happens for example in some Euclidean domains (ellipse, ball, domain between two confocal ellipses or between two spheres with the same center), on the surface of 3-dimensional ellipsoids which are not spheres, as long as we keep away from some special points:

$$\operatorname{dist}(x, \mathcal{L}) \ge \lambda^{-\delta''} \tag{4.5.65}$$

where $\mathcal{L}$ is the set of umbilical points for the surface of the ellipsoid, $\mathcal{L}$ is the set of foci for elliptical domain, $\mathcal{L}$ is the center for a ball.

All details are left for Chapter 11.

Remark 4.5.15. Bringing polygons and polyhedra into consideration brings also the question what to do with billiards which hit the "bad" points which are in this case corners or edges. One can decide that they branch here[40)] or simply dynamics breaks.

In the second case we define dynamics not on the whole Σ_τ but on $\Sigma_\tau \setminus \Xi$ where Ξ is the set of points for which global dynamics does not exists. Then not Π_T or Λ_T are not necessarily closed sets, but but Ξ_T, $\Pi_T \cup \Xi_T$ and $\Lambda_T \cup \Xi_T$ are where Ξ_T is the set of points for which dynamics breaks for time $t, |t| \le T$. The second approach leaves us with the simple dynamics and in all above arguments we replace Π_T or Λ_T by their unions with Ξ_T.

This is the case for billiards even in the smooth domains due to appearance of the pathological billiards which either touch the boundary or make an infinite number of jumps in the finite time.

[40)] F.e. if we consider an incoming ray close to one which hits a vertex of the polygon the direction of the outgoing ray is defined by which side was encountered first, so for the ray which hits the vertex there will be two outgoing rays. For polyhedra there will be many rays leaving vertex.

Matrix Case and Branching Bicharacteristics

There we are again with the case described in Remark 4.5.15: the billiards are branching when they hit the sets $\Theta_{jk} = \{(x,\xi) : \rho_j = \rho_k\}$ $(j \neq k)$ where ρ_j are eigenvalues of $A_m(x,\xi)$ and we do not distinguish eigenvalues which coincide everywhere. However now the set Ξ_{jk} (points starting from which dynamics eventually hits the point of Θ_{jk}) could be rather large. F.e. if λ_j are smooth and $\{\lambda_j, \lambda_k\} \neq 0$ then the set Ξ_{jk} has a positive measure (unless $\Theta_{jk} = \emptyset$). Does it mean that we are doomed to consider branching bicharacteristics?

Not necessarily. In the example just considered we know that the dynamics of ρ_j changes to those of $\rho_k = 0$ the singularity loses order by $\frac{1}{2}$ (see Proposition 2.2.16). Consider more general case assuming that ρ_j are analytic. Then any given trajectory of ρ_j either belongs to Θ_{jk} completely or hits Θ_{jk} only in some locally finite number of points. However again due to analyticity $\mu_\tau(\Theta_{jk}) = 0$ and the first case could be ignored.

Along trajectories of the second type the order of the singularity also decays by $\delta > 0$ as the dynamics of $\rho_j = 0$ changes to those of ρ_k. This should allow us to ignore branching.

Further, if only $A_m(x,\xi)$ is analytic then there is a set $\hat{\Theta}_j$ consisting of points in vicinity of which ρ_j fails to be analytic but this should be very small and the set of points $\hat{\Xi}_j$ such that the generalized Hamiltonian dynamics originated from them hits $\hat{\Theta}_j$ should have measure 0.

Conjecture 4.5.16. *(i) As $A_m(x,\xi)$ is analytic and there is no boundary, it is sufficient to consider only points of $\{\rho_j = \tau\}$ such that the Hamiltonian dynamics of ρ_j originated in these points remains in the domain where ρ_j is analytic and ignore branching.*

(ii) As there is a boundary only branching at the boundary needs to be addressed.

We will discuss the affect of the branching on the boundary later.

4.6 Operators with Rough Coefficients

In this section we establish results similar to those we obtained earlier, for operators with symbols satisfying (2.3.4)–(2.3.6):

$$\|\partial_\xi^\alpha \partial_x^\beta \partial_{x,\xi}^\gamma A_m(x,\xi)\| \le c(m!|\alpha|!|\beta|!)^\sigma \boldsymbol{\rho}^{-\alpha}\boldsymbol{\gamma}^{-\beta}\varepsilon^{-m} \quad \forall m,\alpha,\beta : |\alpha|+|\beta|+2m \le N\ \forall \gamma : |\gamma| \le 1 \tag{4.6.1}$$

with $N = \epsilon\hbar^{-1}$, $\hbar = h/\varepsilon$ and

$$\min_j \rho_j\gamma_j \ge \varepsilon = Ch|\log h|^\sigma \qquad \sigma \ge 1. \tag{4.6.2}$$

Actually we need $\sigma = 1$.

The "pedestrian way" is to consider operators satisfying

$$\|\partial_\xi^\alpha \partial_x^\beta \partial_{x,\xi}^\gamma A_m(x,\xi)\| \le c_{m\alpha\beta}\boldsymbol{\rho}^{-\alpha}\boldsymbol{\gamma}^{-\beta}\varepsilon^{-m} \quad \forall m,\alpha,\beta : |\alpha|+|\beta|+2m \le N\ \forall \gamma : |\gamma| \le 1 \tag{4.6.3}$$

with $\boldsymbol{\rho},\boldsymbol{\gamma}$ satisfying (2.3.2):

$$\min_j \rho_j\gamma_j \ge h^{1-\delta}. \tag{4.6.4}$$

We will be mainly interested in the case of $\rho_j = 1$ and $\gamma_j = \gamma \ll 1$ briefly mentioning other cases too.

4.6.1 Discussion: Rationale

Actually we are interested in the operators with irregular coefficients. F.e. let us consider elliptic self-adjoint semiclassical (matrix) operator in the divergent form

$$A(x,hD) = \sum_{\alpha,\beta:|\alpha|\le m,|\beta|\le m} (hD)^\alpha a_{\alpha\beta}(x)(hD)^\beta \tag{4.6.5}$$

on compact closed manifold X, defined by quadratic form (then it must be of even order $2m$ and semibounded). We can also consider first order (matrix) operator

$$A(x,hD) = \frac{1}{2}\sum_j \bigl(hD_j a_j(x) + a_j(x) hD_j\bigr) + a_0(x). \tag{4.6.6}$$

We assume that the coefficients belong to $\mathscr{C}_\nu^l(X,.)$ where

(4.6.7) $\nu \in \mathscr{C}(0,\infty)$ is a positive as $t \ge 0$ function, $\nu(t)t^{-l}$ and $\nu(t)t^{-l-1}$ are respectively non-decreasing and non-increasing functions on $(0,\infty)$, $l \in \mathbb{Z}^+$

and

Definition 4.6.1. $\mathscr{C}^l_\nu(X)$ is a subspace of $\mathscr{C}^l(X)$ consisting of functions with l-th order derivatives *continuous with continuity modulus*

$$|\partial^\alpha f(x) - \partial^\alpha a(y)| \le C\nu(|x-y|)\cdot |x-y|^{-l}, \qquad \text{as } \ |\alpha| = l. \tag{4.6.8}$$

Remark 4.6.2. Surely this definition would hold for vector- or matrix- valued functions and one can generalize it easily to sections of vector bundles.

So we are looking for the dimension $\mathsf{N}_A(\tau_1,\tau_2)$ of the spectral projector $E_A(\tau_1,\tau_2)$ of operator A corresponding to interval $[\tau_1,\tau_2)$ with $\tau_1<\tau_2$. Since we cannot apply microlocal analysis to such operators we approximate them by operators with symbols satisfying (4.6.1)–(4.6.2). The most obvious approach is to *bracket* such operators. Namely in the case of operator A semibounded from below we can assume without any loss of the generality that $\tau_1=-\infty$ and $\tau_2=0$ and we are interested in $\mathsf{N}^-_A := \mathsf{N}_A(-\infty,0)$. Let us bracket A between A_- and A_+:

$$A_- \le A \le A_+. \tag{4.6.9}$$

Then

$$\mathsf{N}^-_{A^+} \le \mathsf{N}^-_A \le N^-_{A^-}. \tag{4.6.10}$$

In the case of non-semibounded A we can either modify this method or just to notice that

$$\mathsf{N}_A(\tau_1,\tau_2) = \mathsf{N}^-\bigl(\bigl(A-\frac{1}{2}(\tau_1+\tau_2)\bigr)^2 - \frac{1}{4}(\tau_1-\tau_2)^2\bigr). \tag{4.6.11}$$

We want $A_\pm$ satisfy (4.6.1)–(4.6.4). If we manage to derive remainder estimates for such operators

$$|\mathsf{N}^-_{A_-} - \mathsf{N}^{\mathsf{W}\,-}_{A_-}| \le R_-, \tag{4.6.12}_-$$

$$|\mathsf{N}^-_{A_+} - \mathsf{N}^{\mathsf{W}\,-}_{A_+}| \le R_+ \tag{4.6.12}_+$$

and

$$|\mathsf{N}^{\mathsf{W}\,-}_{A_-} - \mathsf{N}^{\mathsf{W}\,-}_{A}| \le R'_-, \tag{4.6.13}_-$$

$$|\mathsf{N}^{\mathsf{W}\,-}_{A_+} - \mathsf{N}^{\mathsf{W}\,-}_{A}| \le R'_+, \tag{4.6.13}_+$$

where $\mathsf{N}_A^{\mathsf{W}-}$ are Weyl approximations for N_A^- etc, then

$$|\mathsf{N}_A^- - \mathsf{N}_A^{\mathsf{W}-}| \le \max(R_- + R'_-, R_+ + R'_+). \tag{4.6.14}$$

So, remainder estimate for operator with irregular coefficients consists of two parts: remainder estimates $R_\pm$ for approximate operators and *approximation error* $R'_\pm$. While $R_\pm$ is the subject of this section, $R'_\pm$ is much more simple object. Let us approximate coefficients a_α by $a_{\alpha\varepsilon}$ according to Proposition 4.A.2. As A is defined by (4.6.5) or (4.6.6) let A_ε be defined in the same way but with a_α replaced by $a_{\alpha\varepsilon}$.

Remark 4.6.3. From the heuristic point of view would like to take $A_\pm := A_\varepsilon \pm C\nu(\varepsilon)$ but it is not exactly possible since $A - A_\varepsilon$ is an unbounded operator. However we can fix it.

Namely, assume that A given by (4.6.5) is a semi-bounded from below elliptic operator $A \ge -C_0 + 1$ and $\tau_1 = -\infty$. Then the following sequence

$$A \mapsto A + C_0 \mapsto \bigl(A_\varepsilon \pm C\nu(\varepsilon)\bigr)\cdot\bigl(1 \pm C_1\nu(\varepsilon)\bigr) \mapsto A_\pm := \bigl(A_\varepsilon \pm +C\nu(\varepsilon)\bigr)\cdot\bigl(1 \pm C_1\nu(\varepsilon)\bigr) - C_0$$

takes care of this issue.

One can prove then easily that

Lemma 4.6.4. *Let A be ξ-microhyperbolic on energy levels 0. Then as $\varepsilon \ge h$ an approximation error does not exceed $Ch^{-d}\nu(\varepsilon)$.*

So we are interested to take ε as small as possible. Since our symbol is analytic with respect to ξ we can take the smallest ε satisfying logarithmic uncertainty principle:

$$\varepsilon = Ch|\log h|. \tag{4.6.15}$$

So, in this case an approximation error is $Ch^{-d}\nu(Ch|\log h|)$ and it is $O(h^{1-d})$ or $o(h^{1-d})$ provided

$$\nu(t) = O(t|\log t|^{-1}), \tag{4.6.16}$$

$$\nu(t) = o(t|\log t|^{-1}) \tag{4.6.17}$$

respectively.

We will prove in this section that under assumption (4.6.16) the remainder estimate is $O(h^{1-d})$. Moreover under appropriate condition to the dynamic system we will prove in this section that under assumption (4.6.17) remainder estimate is $o(h^{1-d})$.

Further we will prove that under assumption $\nu(t) \ge t|\log t|^{-1}$ for all $t \in (0,1)$ the remainder estimate is $O\big(h^{-d}\nu(Ch|\log h|)\big)$.

Thus spectral asymptotics for operators with symbols satisfying (4.6.1)–(4.6.4) are instrumental for spectral asymptotics for operators with $\mathscr{C}^{l}_{\nu}$-coefficients.

Remark 4.6.5. (i) We will assume that $\rho_1 = ... = \rho_d = \rho$ and $\gamma_1 = ... = \gamma_d = \gamma$.

(ii) We can see that the most important case is

$$\rho = 1, \qquad \gamma = \varepsilon. \tag{4.6.18}$$

Actually one can reduce more general case $\rho \ge \gamma$ to it by rescaling. So we assume that (4.6.18) holds.

(iii) In Section 4.7 we replace the bracketing method by perturbation method which allows us to improve remainder estimates in asymptotics for Riesz means N^{-}_{φ} (as ϑ is sufficiently large) and consider asymptotics of $\mathsf{N}_x(\tau_1,\tau_2)$ and $\mathsf{N}^{-}_{x,\varphi}$.

4.6.2 Estimates

The simplest thing to try would be to scale $x \mapsto x\varepsilon^{-1}$, $t \mapsto t\varepsilon^{-1}$, $\xi \mapsto \xi\varepsilon^{-1}$ and $h \mapsto h\varepsilon^{-1}$. However then scaling back we have estimate

$$\mathsf{R} \prec C\big(\frac{h}{\varepsilon}\big)^{1-d}\varepsilon^{-d} \asymp C\varepsilon^{-1}h^{1-d} \tag{4.6.19}$$

where R denotes both Tauberian R^{T} and Weyl R^{W} errors. Thus we got an extra factor ε^{-1} which is really bad. For errors in averaged expressions we would get even worse factor ε^{-1-s} where an extra factor ε^{-s} comes from $\vartheta\big(h/L\varepsilon\big) : \vartheta\big(h/L\big)$.

These bad factors are no surprise: actually we use usual Tauberian estimate but with time $T \asymp \varepsilon$ instead of $T \asymp 1$. Fortunately we have Theorem 2.3.5(vi) which implies that

Proposition 4.6.6. *Let $\varepsilon \ge Ch|\log h|$. Then under ξ-microhyperbolicity condition on the energy level 0 estimate* (2.1.59) *remains true as $T \in [h, T_0]$ with small constant T_0 and then*

$$\|F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma_x\big(Q_{1x}u\,{}^tQ_{2y}\big)\| \le Ch^{1-d} \qquad \forall \tau : |\tau| \le \epsilon_0 \tag{4.6.20}$$

Then our standard Tauberian technique implies the standard estimate

$$\mathsf{R}^{\mathsf{T}}_{\varphi} \prec Ch^{1-d}\vartheta\big(\frac{h}{L}\big) \tag{4.6.21}$$

where ϑ is a continuity modulus of φ.

4.6.3 Successive Approximations and Calculations

Now we need to pass from Tauberian expression

$$\mathsf{N}^{\mathsf{T}}_x = \int_{-\infty}^{\tau} \Big(F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma_x\big(Q_{1x}u(x,y,t)\,{}^tQ_{2y}\big)\Big)\,d\tau \tag{4.6.22}$$

to Weyl expression.

As in Sections 4.3–4.5 Q_1 and Q_2 are fixed operators, $\bar\chi$ is a fixed $\mathscr{C}_0^\infty([-1,1])$ function, $\bar\chi = 1$ on $[-\frac{2}{3}, \frac{2}{3}]$ and T is a small enough constant.

Remark 4.6.7. In order to deal with operators of (4.6.1)–(4.6.4) type we assume that $Q_j \in \boldsymbol{\Psi}_{h,\mathsf{an}}$.

Since we know from (2.1.59) that $F_{t\to h^{-1}\tau}\chi_T(t)\Gamma_x Q_{1x}u\,{}^tQ_{2y}$ is negligible as $T \in [h^{1-\delta}, T_0]$ we conclude that

(4.6.23) In the Tauberian expression we can take any $T \in [h^{1-\delta}, T_0]$.

Remark 4.6.8. (i) Minor improvement of the propagation technique implies that actually one can take any $T \in [Ch|\log h|, T_0]$; we will not need this improvement for a very long time and we will make it much later.

(ii) Improvements connected with our ability to chose T_0 as an arbitrarily large constant will be discussed later.

We will apply successive approximation method. Almost as in Section 4.3 we introduce operators $\bar{A} = A(y, hD_x)$ and $R = A - \bar{A}$ and arrive to the same formula (4.3.10)′:

$$(4.3.10)' \quad u^{\pm}\, {}^{t}Q_{2y} \equiv \mp ih \sum_{0\le n\le N-1} (\bar{G}^{\pm}R)^n \bar{G}^{\pm}\mathcal{K}(x,y)\delta(t) \mp ih(\bar{G}^{\pm}R)^n G^{\pm}\mathcal{K}(x,y)\delta(t).$$

Plugging it into (4.6.23) we conclude that

(4.6.24) n-th term (with $n = 0, 1, ...$) in the resulting formula does not exceed

$$(4.6.25) \qquad Ch^{-d}T\big(\frac{T^2}{h}\big)^n$$

where we can take $T = h^{1-\delta}$ with arbitrarily small exponent $\delta > 0$ [41].

(4.6.26) In particular, as we are looking for R^{W} it is sufficient to take 2 terms and as we are looking for $\mathsf{R}^{\mathsf{W}}_{\varphi}$ with $L \le 1$ it is sufficient to take $2 + \lfloor s \rfloor$ terms provided $\vartheta(z) \ge \epsilon_0 z^s \quad \forall z \in (0,1)$.

Let us prove

Proposition 4.6.9. *Under ξ-microhyperbolicity condition n-th term does not exceed* (4.6.24) *with $T = h$, i.e. Ch^{1-d+n}.*

Proof. It is sufficient to consider n-th term with $\bar{\chi}_T(t)$ replaced by $\chi_T(t)$ with "new" T running from h to "old" T and to estimate this term by (4.6.25) multiplied by $(hT^{-1})^M$ with sufficiently large exponent M; here and below $h \le T \le \epsilon_0$.

Scaling $x \mapsto (x-y)T^{-1}$, $t \mapsto tT^{-1}$, $h \mapsto hT^{-1}$ we conclude that it is sufficient to consider $T = \epsilon_0$ only. Let us recall that n-th term could be rewritten as a "sandwich" u_n with operators $\bar{G}^{\pm}$ interlaced with h-pseudodifferential operators; the latter now are of $\boldsymbol{\Psi}_{\varepsilon T^{-1},1,hT^{-1},\mathsf{an}}$ class.

[41] Actually according to Remark 4.6.7 we can take $T = Ch|\log h|$.

However Theorem 2.3.2 and ξ-microhyperbolicity condition on the energy level 0 which is preserved under rescaling imply that

$$\mathsf{WF}(Q_{1x}u_n^{\pm}\,{}^tQ_{2y})\cap\{|\tau|\le\epsilon\}\cap\{\epsilon\le\pm t\le\epsilon_0\}\subset \{\pm\langle\ell_\xi, y-x\rangle\ge\pm\epsilon_0 t\} \tag{4.6.27}$$

where $(0,\ell_\xi)$ is the direction of microhyperbolicity and therefore

$$\mathsf{WF}\big(\Gamma_x(Q_{1x}u_n^{\pm}\,{}^tQ_{2y})\big)\cap\{|\tau|\le\epsilon\}\cap\{\epsilon\le\pm t\le\epsilon_0\}=\emptyset \tag{4.6.28}$$

and the required estimate is proven. □

Thus, if we are interested just in $O(h^{1-d})$ estimate i.e. we are looking for $s=0$, we are done: we need to consider just one term, i.e. we can take $N=1$ in (4.3.10)′; to be able to take $A_0(y,hD_x)$ instead of $A(y,hD_x)$ we will need to make assumption that their difference is $O(h)$ (see assumption (4.6.33) below with $l=1$).

If we are interested in estimate better than $O(h^{1-d})$ but not better than $O(h^{2-d})$ we should take $N=2$ and consider a new term

$$\mp ih\bar{G}^{\pm}R\bar{G}^{\pm}\mathcal{K}(x,y)\delta(t), \tag{4.6.29}$$

plug here

$$\bar{R}=\sum_{1\le|\alpha|+k\le l}\frac{1}{\alpha!}(x-y)^{\alpha}A_{k(\alpha)}(y,hD_x)h^k \tag{4.6.30}$$

with $l=1$ instead of R, getting again the standard term

$$\mp ih\bar{G}^{\pm}\bar{R}\bar{G}^{\pm}\mathcal{K}(x,y)\delta(t), \tag{4.6.31}$$

(and plugging it to expression we need to calculate) and find estimate for the error due to

$$\mp ih\bar{G}^{\pm}R'\bar{G}^{\pm}\mathcal{K}(x,y)\delta(t), \tag{4.6.32}$$

with $R'=R-\bar{R}$. For remainder estimate better than $O(h^{1-d})$ we will need more regularity assumptions than (4.6.1)–(2.3.6).

Further, if we are interested in the remainder estimate $O(h^{l-d})$ with $l>1$ we need to assume more regularity conditions: namely that (4.6.1) holds for all γ with $|\gamma|\le m$ and modify (4.6.2) in the similar way to keep $A_{k(\alpha)}$

with $k+|\alpha|\le l$ bounded. Then expression due to (4.6.32) will be obviously $O(h^{l-d})$. We need to consider higher order successive approximation terms in (4.3.10)$'$, replace R by $\bar{R}$ in each of them and estimate an error. However these terms will be easier and we will be able even to shorten $\bar{R}$ in them.

Finally, for remainder estimates between $O(h^{l-d})$ and $O(h^{l+1-d})$ we will need intermediate regularity conditions. So, let us assume that

$$\|A^{(\alpha)}_{i+k(\beta+\gamma)}(x,\xi)\| \le c(i!|\alpha|!|\beta|!)\varepsilon^{-|\beta|-i} \qquad \forall i,\alpha,\beta : |\alpha|+|\beta|+2i\le N\ \forall \gamma, k : |\gamma|+k\le l \tag{4.6.33}$$

and

$$\|A^{(\alpha)}_{i+k(\beta+\gamma)}(x,\xi) - A^{(\alpha)}_{i+k(\beta+\gamma)}(y,\xi)\| \le c(i!|\alpha|!|\beta|!)\varepsilon^{-|\beta|-i}\nu(|x-y|)|x-y|^{-l} \qquad \forall i,\alpha,\beta : |\alpha|+|\beta|+2i\le N\ \forall \gamma, k : |\gamma|+k=m \tag{4.6.34}$$

with $N=\epsilon\hbar^{-1}$, $\hbar=h/\varepsilon$ where ν satisfies (4.6.7).

Proposition 4.6.10. *Let ξ-microhyperbolicity condition,* (4.6.33) *and* (4.6.34) *be fulfilled with ν satisfying* (4.6.7). *Then replacing in* (4.3.10)$'$ R *by* $\bar{R}$ *defined by* (4.6.30) *leads to* $O\big(h^{-d}\nu(h)\big)$ *error.*

Proof. So, we need to consider sandwiches

$$\bar{G}^{\pm}R_1\bar{G}^{\pm}R_2\bar{G}^{\pm}\cdots R_{n-1}\bar{G}^{\pm}\mathcal{K}(x,y)\delta(t)$$

with n factors $\bar{G}^{\pm}$ and $(n-1)$ factors R_k where each R_k is either $\bar{R}$ or $(R-\bar{R})$ and there is at least one latter factor. Then acting as before we can estimate the contribution of each sandwich by $Ch^i(h/T)^{-M}$ with $i=n_1+ln_2$ and $M\le Ci$ where n_1, n_2 is the number of factors of each type, $n_1+n_2=n$ and $n_2\ge 1$.

Then the contribution of each sandwich does not exceed $Ch^{-d+i-\delta}$ while $\nu(h)\ge h^{l+1}$. Obviously $i\ge l+2$ unless $n\le 3$ and the contribution of the sandwich is properly estimated.

Further, we can apply the same procedure as in the proof of Proposition 4.6.9: let us pick up $T=h$ and then the contribution of sandwich does not exceed Ch^{-d+l}; then let us plug $\chi_T(t)$ instead of $\bar{\chi}_T(t)$ with "new" T running from h to "old" T and basing on Theorem 2.3.2 estimate this

contribution by $Ch^{-d+l}(h/\mathsf{T})^{M'-M}$ where we can take M' arbitrarily large, say $M' = M+1$ and then summation with respect to partition results in Ch^{-d+l}. Obviously $i \ge l+1$ unless $n=2$ and the contribution of the sandwich is properly estimated.

So, only case $n=2$ needs a special consideration and only in the case when $\nu(h) \asymp h^l$ fails. Then we have "a simple sandwich"

$$\bar{G}^{\pm}(R-\bar{R})\bar{G}^{\pm}\mathcal{K}(x,y)\delta(t);$$

let us insert factor ψ_r in the middle:

$$\bar{G}^{\pm}\psi_r(R-\bar{R})\bar{G}^{\pm}\mathcal{K}(x,y)\delta(t);$$

where $\psi_r = \psi\big((x-y)r^{-1}\big)$ and we consider three cases:

(i) $r = CT$, $\psi \in \mathsf{S}^0_{h,\mathsf{an}}$ is supported in $\{|x-y| \le 1\}$, $T = h$. Then since norm of $\psi_r(R-\bar{R})$ does not exceed $C\nu(r)$ [42] and and as $T \asymp h$ we estimate the contribution of such modified sandwich by $Ch^{-d}\nu(r)$ which is $O\big(h^{-d}\nu(h)\big)$.

(ii) $r = CT$, $\psi \in \mathsf{S}^0_{h,\mathsf{an}}$ is supported in $\{|x-y| \le 1, |\xi| \le c\}$, $T \ge h$ and $\bar{\chi}$ is replace by χ. Then we estimate the contribution of such modified sandwich by

$$Ch^{-d}\nu(r)\big(\frac{h}{T}\big)^{M-M'} \lesssim Ch^{-d}\nu(T)\big(\frac{h}{T}\big)^{M'-M} \lesssim Ch^{-d}\nu(h)\big(\frac{h}{T}\big)^{M-M'-l-1}$$

and summation with respect to t-partition results in $Ch^{-d}\nu(h)$ since M' is arbitrarily large. I remind that we apply Theorem 2.3.2 after rescaling $x \mapsto (x-y)T^{-1}$, $t \mapsto tT^{-1}$ and $h \mapsto hT^{-1}$. We again used estimate of $\psi_r(R-\bar{R})$ with footnote [42].

(iii) $r \ge CT$, $\psi \in \mathsf{S}^0_{h,\mathsf{an}}$ is supported in $\{|x-y| \le 1, |\xi| \le c\}$, $T \ge h$. Then we estimate the contribution of such modified sandwich by

$$Ch^{-d}\nu(r)\big(\frac{h}{T}\big)^{M-M'} \lesssim Ch^{-d}\nu(r)\big(\frac{h}{T}\big)^{-M}\big(\frac{h}{R}\big)^{M'} \lesssim Ch^{-d}\nu(h)\big(\frac{h}{r}\big)^{M-M'-l-1}$$

where we use the same procedure as before but we apply Theorem 2.3.2 after rescaling $x \mapsto (x-y)r^{-1}$, $t \mapsto tr^{-1}$ and $h \mapsto hr^{-1}$. We again used estimate of $\psi_r(R-\bar{R})$ with footnote [42]. Again summation with respect to r results in $Ch^{-d}\nu(h)$ since M' is arbitrarily large. □

[42] Actually it may be true only if we multiply this operator by $\bar{Q} = \bar{q}(hD)$ with compactly supported symbol but we can insert such factor with the negligible error.

Thus we constructed "Weyl approximation" with $O\big(h^{-d}\nu(h)\big)$ error for Tauberian expressions (4.6.23) and also for

$$\mathsf{N}_{x,\varphi,L} = \int \varphi_L(\tau)\Big(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma_x\big(Q_{1x}u(x,y,t)\,{}^tQ_{2y}\big)\Big)\,d\tau \tag{4.6.35}$$

(with small constant T). Let us recall that according to Tauberian arguments $\mathsf{N}_x^{\mathsf{T}}(0)$ defined by (4.6.22) and $\mathsf{N}_{x,\varphi,L}(0)$ defined by (4.6.35) approximate $\mathsf{N}_x(0)$ and

$$\mathsf{N}_{x,\varphi,L}(0) = \int \varphi_L(\tau) d_\tau\Big(\Gamma_x\big(Q_{1x}e(.,.,0)\,{}^tQ_{2y}\big)\Big) \tag{4.6.36}$$

with $O(h^{1-d})$ and $O(h^{1-d}\vartheta(h/L))$ errors.

So, we arrive to

Corollary 4.6.11. *(i) For N_x Weyl approximation is as good as the Tauberian one provided $\nu(t) = O(t)$ as $t \to +0$ (which we assumed).*

(ii) For $\mathsf{N}_{x,\varphi,L}$ Weyl approximation is as good as the Tauberian one as $L \le 1$ and $\nu(t) = O\big(t\vartheta(t)\big)$ as $t \to +0$.

(iii) In both cases the difference between Weyl and Tauberian approximations is "o" in comparison with the remainder estimate provided $\nu(t) = o(t)$ as $t \to +0$ or $\nu(t) = o\big(t\vartheta(t)\big)$ as $t \to +0$ respectively.

Proof. These statements are clearly true for N_x and for $\mathsf{N}_{x,\varphi,L}$ with $L \lesssim 1$. Then for $\mathsf{N}_{x,\varphi,L}$ with $L \gtrsim 1$ we can apply arguments of Remark 4.4.10. □

4.6.4 Main Theorems

Now we arrive immediately to the following

Theorem 4.6.12. *Let A be a self-adjoint operator with the symbol satisfying (4.6.34) where ν satisfies (4.6.7) and $l \ge 1$. Assume that ξ-microhyperbolicity condition on the energy level 0 at each point of $\pi_x\big(\mathsf{supp}(q_1) \cup \mathsf{supp}(q_2)\big)$.*

Then

$$\mathsf{R}^{\mathsf{W}}_{x,\varphi,L} \prec Ch^{1-d}\Big(\vartheta\big(\frac{h}{L}\big) + \nu(h)h^{-1}\Big); \tag{4.6.37}$$

in particular

$$\mathsf{R}^{\mathsf{W}}_{x} \prec Ch^{1-d}\Big(1 + \nu(h)h^{-1}\Big). \tag{4.6.38}$$

Corollary 4.6.13. *The same estimates hold for* $\mathsf{R}^{\mathsf{W}}_{\varphi,L}$, R^{W}.

Now let us slightly improve remainder estimates using long term propagation.

Theorem 4.6.14. *Let conditions of Theorem 4.6.12 be fulfilled. Let* $r_i^0 \in \mathsf{S}_{h,\gamma,\gamma,K}(T^*X)$ [34] *be scalar symbols satisfying* (4.5.11).

Consider generalized Hamiltonian flow $\Psi_{t,0}$ *issued from* γ*-vicinity* $\mathcal{U}_i$ *of* $\mathsf{supp}(r_i^0) \cap \mathsf{supp}(q_1) \cap \mathsf{supp}(q_2)$. *Assume that as* $\varsigma t > 0$ *with* $\varsigma := \varsigma_{*i}$ [32] *and* $|t| \le T_i$ *conditions* (4.5.28), (4.5.30) *and* (4.5.31) *are fulfilled with* $\mathcal{U} := \mathcal{U}_i$.

(i) Then

$$\mathsf{R}^{\mathsf{T}}(0) \prec \sum_{0\le i\le m} C_0 \frac{h^{1-d}}{T_i}\Big(J(r_i^0 q_1^0, 0) + J(r_i^0 q_2^0, 0) + C'h\Big), \tag{4.6.39}$$

$$\mathsf{R}^{\mathsf{T}}_{\varphi,L} \prec \sum_{0\le i\le m} C_0 \frac{h^{1-d}}{T_i}\Big(J(r_i^0 q_1^0, 0) + J(r_i^0 q_2^0, 0) + C'h\Big)\vartheta\big(\frac{h}{LT_i}\big). \tag{4.6.40}$$

Here C' *depends on* γ, m, $\max_i T_i$.

(ii) Further, the same estimates with an extra term $Ch^{-d}\nu(h)$ *in the right-hand expressions hold for* $\mathsf{R}^{\mathsf{W}}(0)$ *and* $\mathsf{R}^{\mathsf{W}}_{\varphi,L}$.

Theorem 4.6.15. *Let conditions of Theorem 4.6.12 be fulfilled. Consider generalized Hamiltonian flow* $\Psi_{t,\bar\tau_*}$ *issued from* γ*-vicinity* $\mathcal{U}_i$ *of* $\mathsf{supp}(r_i^0) \cap (\mathsf{supp}(q_1) \cup \mathsf{supp}(q_2))$. *Assume that as* $|t| \le T_i$ *conditions* (4.5.28), (4.5.29) *and* (4.5.31) *are fulfilled with* $\mathcal{U} := \mathcal{U}_i$.

(i) Then

$$\mathsf{R}^{\mathsf{T}}_x(0) \prec \sum_{0\le i\le m} C_1 \frac{h^{1-d}}{T_i}\Big(J_x(r_i^0 q_1^0, 0) + J_x(r_i^0 q_2^0, 0) + C'h\Big), \tag{4.6.41}$$

$$\mathsf{R}^{\mathsf{T}}_{x,\varphi,L} \prec \sum_{0\le i\le m} C_1 \frac{h^{1-d}}{T_i}\Big(J_x(r_i^0 q_1^0, 0) + J_x(r_i^0 q_2^0, 0) + C'h\Big)\vartheta\big(\frac{h}{LT_i}\big). \tag{4.6.42}$$

Here $C_1 = C_0(l+1)$ *and* C' *depends on* γ, m, $\max_i T_i$.

(ii) Further, the same estimates with an extra term $Ch^{-d}\nu(h)$ *in the right-hand expressions hold for* $\mathsf{R}^{\mathsf{W}}(0)$ *and* $\mathsf{R}^{\mathsf{W}}_{\varphi,L}$.

4.6.5 Applications: Bracketing Method

At this moment we want to demonstrate technique rather than achieve the most general results. In particular we want to use only the simplest variational arguments. Therefore we assume that operator A defined by consider operator (4.6.5) is elliptic with the positively defined senior symbol:

$$\langle a^0(x,\xi)w,\ w\rangle \ge \epsilon_0|\xi|^{2m}\|w\|^2 \qquad \forall \xi \in \mathbb{R}^d\ \forall w \tag{4.6.43}$$

with

$$a^0(x,\xi) \coloneqq \sum_{\alpha:|\alpha|+|\beta|=2m} .a_{\alpha\beta}(x)\xi^{\alpha+\beta} \tag{4.6.44}$$

Then provided $a_{\alpha\beta} \in \mathscr{C}$ (as $|\alpha| = |\beta| = m$)

$$(Au, u) \ge \epsilon_0 h^{2m}\|u\|_{\mathscr{H}^m} - C_0\|u\|^2 \qquad \forall u \in \mathscr{H}^m \tag{4.6.45}$$

First of all, applying Theorem 4.6.12 and Corollary 4.6.13 we immediately arrive to

Theorem 4.6.16. *Let A be an partial differential operator in the divergent form* (4.6.5) *on compact closed manifold with the coefficients belonging to $\mathscr{C}^l_\nu$ with ν satisfying* (4.6.7) *and $l \ge 1$. Let* (4.6.43) *be fulfilled. Assume that ξ-microhyperbolicity condition on energy level 0 is fulfilled at each point. Let $Q_1 = Q_2 = I$, $\psi_1 = \psi_2 = 1$.*

Then

$$|\operatorname{tr}(\mathsf{R}^{\mathsf{W}}_{\varphi,L})| \le Ch^{1-d}\Big(\vartheta(\frac{h}{L}) + \nu(\varepsilon)h^{-1}\Big)\operatorname{dim}\mathbb{H} \tag{4.6.46}$$

with $\varepsilon = h|\log h|$; in particular

$$|\operatorname{tr}(\mathsf{R}^{\mathsf{W}})| \le Ch^{1-d}\Big(1 + \nu(\varepsilon)h^{-1}\Big)\operatorname{dim}\mathbb{H}. \tag{4.6.47}$$

Proof. Indeed, we can bracket A between two operators with the rough coefficients, different from those of A by $O(\nu(\varepsilon))$. We leave easy details to the reader. □

Corollary 4.6.17. *In the framework of Theorem 4.6.16 with fixed $\operatorname{dim}\mathbb{H} < \infty$ the usual remainder estimate $|\operatorname{tr}(\mathsf{R}^{\mathsf{W}}_{\varphi,L})| \le Ch^{1-d}\vartheta(h/L)$ holds as $L \le 1$ provided continuity modulus ν satisfies*

$$\nu(\varepsilon) \lesssim \vartheta(\varepsilon|\log\varepsilon|^{-1})\cdot\varepsilon|\log\varepsilon|^{-1}; \tag{4.6.48}$$

in particular, the usual remainder estimate $|\mathsf{tr}(\mathsf{R}^{\mathsf{W}})| \le Ch^{1-d}$ *holds provided*

$$\nu(\varepsilon) \lesssim \varepsilon |\log \varepsilon|^{-1}. \tag{4.6.49}$$

Now let us combine Theorem 4.6.14 with the approximation arguments. We will get an approximation error $O(h^{-d}\nu(\varepsilon))$ but there is a problem: in this theorem non-periodicity condition is supposed to be fulfilled with the generalized Hamiltonian flow generated by a_ε while it is natural to consider generalized Hamiltonian flow generated by the original symbol a. The good news are that

(4.6.50) As $\varepsilon \to +0$ the generalized bicharacteristics of $(\tau - a_\varepsilon)$ converge to those of $(\tau - a)$

and therefore

$$\Psi_{t,a}(\mathcal{U}) \supset \lim_{\varepsilon \to +0} \Psi_{t,a_\varepsilon}(\mathcal{U}). \tag{4.6.51}$$

Thus non-periodicity condition (4.5.30) for $\Psi_{t,a}$ implies the same condition for Ψ_{t,a_ε} with γ replaced by $\frac{1}{2}\gamma$ and Theorem 4.6.14 implies

Theorem 4.6.18. *Let conditions of Theorem 4.6.16 be fulfilled.*

Consider generalized Hamiltonian flow $\Psi_{t,0}$ *issued from* γ*-vicinity* $\mathcal{U}_i$ *of* $\mathsf{supp}(r_i^0) \cap \mathsf{supp}(q_1) \cap \mathsf{supp}(q_2)$*. Assume that as* $\varsigma t > 0$ *with* $\varsigma := \varsigma_{*i}$ [32] *and* $|t| \le T_i$ *conditions* (4.5.28), (4.5.30) *and* (4.5.31) *are fulfilled with* $\mathcal{U} := \mathcal{U}_i$*.*

Then

$$\mathsf{R}^{\mathsf{W}}(0) \prec \sum_{0\le i\le m} C_0 \frac{h^{1-d}}{T_i}\Big(J(r_i^0 q_1^0, 0) + J(r_i^0 q_2^0, 0) + C'h\Big) + C_0 h^{-d}\nu(\varepsilon), \tag{4.6.52}$$

$$\mathsf{R}^{\mathsf{T}}_{\varphi,L} \prec \sum_{0\le i\le m} C_0 \frac{h^{1-d}}{T_i}\Big(J(r_i^0 q_1^0, 0) + J(r_i^0 q_2^0, 0) + C'h\Big)\vartheta\big(\frac{h}{LT_i}\big) + C_0 h^{-d}\nu(\varepsilon) \tag{4.6.53}$$

hold with $\varepsilon = Ch|\log h|$.

Corollary 4.6.19. *Assume that one can select T_i as arbitrarily large constants for $i \ge 1$ and simultaneously $\mu_0(\mathcal{U}_0)$ to be arbitrarily small then*

$$\mathsf{R}^{\mathsf{W}} = o\big(h^{1-d}\big), \tag{4.6.54}$$

$$\mathsf{R}^{\mathsf{W}}_{\varphi,L} = o\big(h^{1-d}\vartheta\big(\frac{h}{L}\big)\big), \tag{4.6.55}$$

provided $\nu(\varepsilon) = o\big(h^{1-d}\big)$ and $\nu(\varepsilon) = o\big(h^{1-d}\vartheta(h/L)\big)$ respectively (with $\varepsilon = h|\log h|$) i.e.

$$\nu(\varepsilon) = o\big(\varepsilon|\log\varepsilon|^{-1}\big), \tag{4.6.56}$$

$$\nu(\varepsilon) = o\big(\vartheta(\varepsilon|\log\varepsilon|^{-1})\cdot\varepsilon|\log\varepsilon|^{-1}\big). \tag{4.6.57}$$

Remark 4.6.20. (i) It is not clear if sets in both sides of (4.6.51) coincide; so may be non-periodicity assumption for a is more restrictive than for a_ε.

(ii) Even in the scalar case when equation of the generalized bicharacteristics becomes

$$\frac{dz}{dt} = -\nabla^{\#}a(z)$$

those could branch unless the right-hand expression satisfies Lipshitz condition i.e. $l \ge 2$ and $\nu(\varepsilon) = O(\varepsilon^2)$.

Finally, let us consider less regular case of $l = 0$. Let us apply the same approximation as before. However we can apply Theorem 4.6.12 only after rescaling $x \mapsto x/T$, $t \mapsto t/T$ with

$$T = T(\varepsilon) = \varepsilon\nu(\varepsilon)^{-1} \tag{4.6.58}$$

which leads to the remainder estimates

$$\mathsf{R}^{\mathsf{T}}_x \prec Ch^{1-d}\frac{1}{T(\varepsilon)} = Ch^{1-d}\varepsilon^{-1}\nu(\varepsilon). \tag{4.6.59}$$

One can see easily that this is less then the approximation error $Ch^{-d}\nu(\varepsilon)$ and therefore

Theorem 4.6.21. *Let conditions of Theorem 4.6.16 be fulfilled but with $l = 0$. Then*

$$\mathsf{R}^{\mathsf{W}}_x \prec C\nu(\varepsilon)h^{-d}. \tag{4.6.60}$$

Since approximation error dominates, mollification with respect to the spectral parameter would not improve this estimate.

4.6.6 Final Remarks

The following problem may be easy, or not:

Problem 4.6.22. (i) Replace assumption (4.6.43) by

$$\|a_0(x,\xi)w\| \ge \epsilon_0|\xi|^2 m\|w\| \qquad \forall \xi \in \mathbb{R}^d\ \forall w. \tag{4.6.61}$$

Then as $a_{\alpha\beta} \in \mathscr{C}$ (as $|\alpha| = |\beta| = m$)

$$\|(1+h^2|D|^2)^{-m/2}Au\| \ge \epsilon_0\|(1+h^2|D|^2)^{m/2}u\| - C_0\|u\| \qquad \forall u \in \mathscr{H}^m. \tag{4.6.62}$$

(ii) Instead of operators (4.6.5) consider

$$A(x,hD) = \sum_{\substack{\alpha,\beta:|\alpha|\le m+1,\,|\beta|\le m+1,\\ |\alpha|+|\beta|\le 2m+1}} (hD)^\alpha a_{\alpha\beta}(x)(hD)^\beta \tag{4.6.63}$$

under assumption (4.6.61) with

$$a^0(x,\xi) \coloneqq \sum_{\alpha:|\alpha|+|\beta|=2m+1} a_{\alpha\beta}\xi^{\alpha+\beta}. \tag{4.6.64}$$

Then as $a_{\alpha\beta} \in \mathscr{C}^1$ as $|\alpha|+|\beta| = 2m+1$

$$\|(1+h^2|D|^2)^{-\bar m/2}Au\| \ge \epsilon_0\|(1+h^2|D|^2)^{\bar m/2}u\| - C_0\|u\| \qquad \forall u \in \mathscr{H}^{\bar m} \tag{4.6.65}$$

where here and below $\bar m = m$ for for operators (4.6.5) and $\bar m = m + \frac{1}{2}$ for operators (4.6.63).

Remark 4.6.23. (i) We did not consider in this section operators with unbounded coefficients. Most likely that this two generalizations of the standard case do not mix well together.

(ii) We did not consider in this section sharper remainder estimates. While they are possible the conditions of theorems (especially in applications) become too exotic.

Remark 4.6.24. Can we replace ξ-microhyperbolicity by microhyperbolicity retaining our results for $\mathsf{N}^{\mathsf{T}}, \mathsf{N}^{\mathsf{T}}_{\varphi,L}, \mathsf{N}^{\mathsf{W}}, \mathsf{N}^{\mathsf{W}}_{\varphi,L}$? The answer is positive but with a lot of caveats.

(i) First of all, in Subsection 4.6.2 we need to replace $\bar{T} = h^{1-\delta}$ by $\bar{T} = h^{1-\delta}\varepsilon^{-1}$. Really, as we go to ξ-microhyperbolic case applying metaplectic transformation, we get operator belonging to $\mathbf{\Psi}_{h,\varepsilon,\varepsilon,\mathsf{an}}$. Then shift $\asymp T$ is observable as microlocal uncertainty principle $T\cdot\varepsilon \ge h^{1-\delta}$ is fulfilled; improvements as we apply logarithmic uncertainty principle are superficial.

It looks really bad because then we should estimate expression (4.1.11)

$$F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma\big(Q_{1x}u\,{}^tQ_{2y}\big)$$

by $Ch^{1-d}\varepsilon^{-1}$ which would lead to remainder estimates with an extra factor ε^{-1} but actually we can repair this after calculations.

(ii) Still the method of the successive approximations requires $T \le h^{\frac12+\delta}$ and $\bar{T}$ satisfies this if and only if

$$\varepsilon \ge h^{\frac12-\delta} \tag{4.6.66}$$

where we change an arbitrarily small exponent $\delta > 0$ if needed. Under this condition and microhyperbolicity one can prove easily that

$$|\kappa_n^{(i)}| \le C\,\mathsf{max}\big(\varepsilon^{-n-i}\nu(\varepsilon),1\big). \tag{4.6.67}$$

Then as $l \ge 1$ and ε satisfies (4.6.66) we can estimate (4.1.11) by Ch^{1-d} and therefore no factor ε^{-1} actually appears. Note that we must assume that $l \ge 1$; otherwise we cannot differentiate with respect to x in applications and microhyperbolicity does not make sense (while ξ-microhyperbolicity does not require such differentiation).

(iii) Much larger ε given by (4.6.66) increases an approximation error and to achieve the proper remainder estimates we must assume that $h^{-d}\nu(\varepsilon) \le h^{1-d}\vartheta(h)$ (as $L = 1$) which means now

$$\nu(\varepsilon) = O\big(\vartheta\big(\varepsilon^{2+2\delta}\big)\varepsilon^{2+2\delta}\big); \tag{4.6.68}$$

in particular, as $\vartheta = 1$ we get

$$\nu(\varepsilon) = O\big(\varepsilon^{2+2\delta}\big). \tag{4.6.69}$$

Further to recover sharp remainder estimates we need to replace in these conditions "O" by "o".

4.7 Operators with Irregular Coefficients

4.7.1 Preliminary Analysis

Our goal is to derive spectral asymptotics for operators (4.6.5) and (4.6.63) satisfying ellipticity condition (4.6.61).

For brevity of statements we consider

(4.7.1) operator A of the type either (4.6.5) or (4.6.63), with $a_{\alpha\beta} \in \mathscr{C}^l$ for all α, β and with $a_{\alpha\beta} \in \mathscr{C}^{l+1}$ as $|\alpha| + |\beta| = 2m + 1$. Let $\bar{m} = \frac{1}{2}\mathsf{ord}A$.

Proposition 4.7.1. *Let us consider operator A of the type* (4.7.1).

(i) Then the following estimate holds

$$\|(1+h^2|D|^2)^{(-\bar{m}+s)/2}Au\| \le C_0\|(1+h^2|D|^2)^{(\bar{m}+s)/2}u\| \qquad \forall u \in \mathscr{H}^{\bar{m}}\ \forall s : |s| \le l. \tag{4.7.2}$$

(ii) If ellipticity condition (4.6.61) *is fulfilled then*

$$\|(1+h^2|D|^2)^{(-\bar{m}+s)/2}Au\| \ge \epsilon_0\|(1+h^2|D|^2)^{(\bar{m}+s)/2}u\| - C_0\|(1+h^2|D|^2)^{s}u\| \qquad \forall u \in \mathscr{H}^{\bar{m}+s}\ \forall s : |s| \le l. \tag{4.7.3}$$

Proof. Easy proof is left to the reader. □

Then in virtue of embedding theorem we immediately arrive to

Corollary 4.7.2. *In the framework of Proposition 4.7.1(ii) with*

$$m^* := l + m' - d/2 > 0 \tag{4.7.4}$$

the following estimate holds

$$|(hD_x)^\alpha (hD_y)^\beta e(x,y,\tau_1,\tau_2)| \le Ch^{-d} \tag{4.7.5}$$

for all $\alpha : |\alpha| < m^*$, $\beta : |\beta| < m^*$, for $|\tau_1| \le c$, $|\tau_2| \le c$.

4.7.2 Tauberian Estimates

Now we can unleash the full power of microlocal analysis but we need to extend it to our framework.

Proposition 4.7.3. *For a commutator of a pseudodifferential operator with a smooth compactly supported symbol and $\mathscr{C}^l$-function $b(x)$ a usual commutator formula holds modulo $O(h^l \|\partial b\|_{l-1})$ for any $l > 0$ where*

$$\|b\|_l := \begin{cases} \sum_{\alpha:|\alpha|=l} \sup_x |\partial^\alpha f(x)| & l \in \mathbb{Z}^+, \\ \sum_{\alpha:|\alpha|=\lfloor l \rfloor} \sup_{x\neq y} |x-y|^{\lfloor l \rfloor - l} \cdot |\partial^\alpha f(x) - \partial^\alpha f(y)| & l \notin \mathbb{Z}^+. \end{cases} \tag{4.7.6}$$

Proof. Easy proof is left to the reader. □

For operator A of type (4.7.1) let

$$\|A\|_l := \sum_{\alpha,\beta} \|a_{\alpha\beta}\|_l + \sum_{\alpha,\beta:|\alpha|+|\beta|=2m+1} \|a_{\alpha\beta}\|_{l+1}. \tag{4.7.7}$$

Proposition 4.7.4. *Let us consider operator A of the type* (4.7.1) *satisfying ellipticity condition* (4.6.61). *Let $u(x,y,t)$ be the Schwartz kernel of $e^{ih^{-1}tH_{A,V}}$. Then for $T \asymp 1$*

(i) Estimate

$$\|F_{t\to h^{-1}\tau}\chi_T(t)(hD_x)^\alpha (hD_y)^\beta \psi_1(x)\psi_2(y)u\| \le Ch^s \tag{4.7.8}$$

for all $\alpha : |\alpha| < \bar{m} + l$, $\beta : |\beta| < \bar{m} + l$, s and all $\psi_1, \psi_2 \in \mathscr{C}_0^\infty(B(0,1))$, such that $\operatorname{dist}(\operatorname{supp}(\psi_1), \operatorname{supp}(\psi_2)) \ge C_0 T$, $\chi \in \mathscr{C}_0^\infty([-1,1])$ and $\tau : |\tau| \le c_0$; in this Proposition $\|.\|$ means an operator norm from $\mathscr{L}^2$ to $\mathscr{L}^2$.

(ii) As $l \ge 1$ estimate

$$\|F_{t\to h^{-1}\tau}\chi_T(t)(hD_x)^\alpha (hD_y)^\beta \varphi_1(hD_x)\varphi_2(hD_y)u\| \le Ch^s + Ch^{l-1}\|\partial A\|_{l-1} \tag{4.7.9}$$

holds for all $\alpha : |\alpha| < \bar{m} + l$, $\beta : |\beta| < \bar{m} + l$, s and all $\varphi_1, \varphi_2 \in \mathscr{C}_0^\infty$, such that $\operatorname{dist}(\operatorname{supp}(\varphi_1), \operatorname{supp}(\varphi_2)) \ge C_0 T$, $\chi \in \mathscr{C}_0^\infty([-1,1])$ and $\tau : |\tau| \le c_0$.

(iii) If also $l > 1$ *and in* $B(0,2)$ *operator* A *is* ξ*-microhyperbolic on the energy level* $\tau : |\tau| \le c$ *then then for a small constant* $T = \epsilon$ *estimate* (4.7.9) *holds for all* $\alpha : |\alpha| \le \bar{m} + l$, $\beta : |\beta| \le \bar{m} + l$, s *and all* $\psi_1, \psi_2 \in \mathscr{C}_0^\infty(B(0,1))$, *such that* $\operatorname{diam}(\operatorname{supp}(\psi_1) \cup \operatorname{supp}(\psi_2)) \le \epsilon_0 T$ *and* $\chi \in \mathscr{C}_0^\infty([-1,-\frac{1}{2}] \cup [\frac{1}{2}, 1])$.

Proof. Let $v = e^{ith^{-1}H} f$ with arbitrary f.

(i) Statement (i)[43] is easily proven by the same arguments as in the proof of Theorem 2.1.2: we consider just usual function $\phi(x)$ and operators of multiplication like $\chi(\phi(x))$ so there are no "bad" commutators due to non-smoothness of the coefficients.

(ii) Statement (ii)[44] is also proven by the same arguments; however in this case $\phi = \phi(x,\xi)$ so we need to involve "bad" commutators but their contributions are bounded by

$$C\|Q_1 u\| \cdot \Bigl(h^l \|\!| A |\!\|_l + \|u\| + h^{1+\delta} \|Q' u\| \Bigr)$$

in the right-hand expression while the left-hand expression is $\epsilon h \|Q_1 u\|^2$ where Q, Q_1, and Q' are operators with symbols $\chi(\phi(x,\xi))$, $\chi_1(\phi(x,\xi))$, and $\chi_1(\phi(x,\xi) - \eta)$ respectively, $\eta > 0$ is an arbitrarily small constant (so the latter symbol has a bit larger support than the former one), $\delta > 0$ is a small exponent, $\chi_1(t) = (-\chi'(t))^{\frac{1}{2}}$, and f.e. $\chi(t) = e^{-|t|^{-1}}$ as $t < 0$, $\chi(t) = 0$ as $t \ge 0$.

Therefore we conclude that

$$\|Qu\| \le Ch^{l-1} \|\!| A |\!\|_l + \|u\| + Ch^\delta \|Q' u\|$$

and similarly we can estimate $\|Q'u\|$ with $\|Q''u\|$ in the right-hand expression etc and thus we conclude that

$$\|Qu\| \le Ch^{l-1} \|\!| A |\!\|_l + Ch^s \|u\|$$

which is what we need.

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[43] Finite speed of propagation with respect to x.

[44] Finite speed of propagation with respect to ξ.

(iii) Statement (iii) is easily proven also by the same arguments: we consider just usual function $\phi(x)$ and operators of multiplication like $\chi(\phi(x))$ so there are no "bad" commutators due to non-smoothness of the coefficients. However we need to consider a contribution of u which is not confined to the small vicinity of (y,η) and we need Statement (ii) for this so the last term in the right-hand expression of (4.7.9) is inherited. We leave easy details to the reader. □

Proposition 4.7.5. *In the framework of Proposition 4.7.4(iii) with however $T : h \le T \le \epsilon$ estimates*

$$|F_{t\to h^{-1}\tau}\chi_T(t)\big((hD_x)^\alpha (hD_y)^\beta u(x,y,t)\big)\big|_{x=y}| \le Ch^{1-d+s}T^{-s} + Ch^{l-1}T^{2-d-l}\Big(\sum_{1\le k\le \lfloor l\rfloor} T^k \|\partial A\|_{k-1} + T^l \|\partial A\|_{l-1}\Big) \tag{4.7.10}$$

and as $l+d>2$

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)\big((hD_x)^\alpha (hD_y)^\beta u(x,y,t)\big)\big|_{x=y}| \le Ch^{1-d} \tag{4.7.11}$$

hold for all $\alpha : |\alpha| < \bar m + l - d/2$, $\beta : |\beta| < \bar m + l - d/2$, s as $T=\epsilon$ and $|\tau| \le \epsilon$ where as usual $\chi \in \mathscr{C}_0^\infty([-1,-\frac12]\cup[\frac12,1])$, $\bar\chi \in \mathscr{C}_0^\infty([-1,1])$.

Proof. Observe first that estimate (4.7.10) holds as $T \asymp 1$ in virtue of embedding theorem.

Let us consider $T \in [C_0 h, \epsilon]$; then we apply the standard rescaling $t \mapsto tT^{-1}$, $x \mapsto xT^{-1}$, $h \mapsto hT^{-1}$. Then we need to add factor $T^{-d}\times T$ to the right-hand expression where the first factor appears since $u(x,y,t)$ is a density with respect to y and the second one appears since there is an integral with respect to t in the left-hand expression.

Observation how $\|.\|_l$ scales concludes the proof of (4.7.10).

To prove (4.7.11) observe first that it holds for $T=2h$. To prove it for $h\le T\le\epsilon$ we use t-admissible partition on $[-1,1]\setminus[-h,h]$ and apply (4.7.10) for each element of the partition. The result is Ch^{1-d} unless $l=d=1$ when the result is $O(|\log h|)$. □

Then for $\mathsf{R}^{\mathsf{T}}_x(\tau_1,\tau_2)$ and its averages the standard estimates hold: $\mathsf{R}^{\mathsf{T}}_x \prec Ch^{1-d}$ etc.

4.7.3 Weyl Asymptotics

Now let us pass from Tauberian expression $\mathsf{N}_x^{\mathsf{T}}(\tau_1,\tau_2)$ to Weyl expression $\mathsf{N}_x^{\mathsf{W}}(\tau_1,\tau_2)$.

Theorem 4.7.6. *Let A be operator* (4.7.1) *with $l \ge 1$ satisfying ellipticity condition* (4.6.61) *and* (4.7.4). *Let on the energy levels τ_1 and τ_2 ξ-microhyperbolicity condition is fulfilled.*

(i) Then

$$\Gamma_x(hD_x)^\alpha(hD_y)^\beta e(x,y;\tau_1,\tau_2) - \mathsf{N}_x^{\mathsf{W}}(x;\tau_1,\tau_2) \prec C\bigl(h^{1-d}+h^{-d}\varsigma^{1/2}\bigr) \tag{4.7.12}$$

as $|\alpha| < m^* = m + l - d/2$, $|\beta| < m^*$ and $\varsigma = h^l$ where $\mathsf{N}_x^{\mathsf{W}}(x,\tau)$ means corresponding Weyl expression.

(ii) Further, for φ supported in the small vicinity of $\bar\tau_i$ such that $\partial^\alpha\varphi$ is continuous with the modulus of continuity $\tau^{r-\lceil r\rceil+1}$

$$\int \varphi(\tau)\,d_\tau\Bigl(\Gamma_x(hD_x)^\alpha(hD_y)^\beta e(x,y;\tau,\tau_i) - \mathsf{N}_x^{\mathsf{W}}(x;\tau,\tau_i)\Bigr) \prec C\bigl(h^{1+r-d}+h^{-d}\varsigma^{(r+1)/2}+\delta_{r1}h^{-d}\varsigma|\log\varsigma|\bigr) \tag{4.7.13}$$

as $|\alpha| < m^*$, $|\beta| < m^*$ and $\varsigma = h^l$ where $\mathsf{N}_x^{\mathsf{W}}(x,\tau)$ means corresponding Weyl expression including the second term (if needed).

Remark 4.7.7. It is definitely not as good as $C\bigl(h^{1-d}+h^{-d}\varsigma\bigr)$ estimate we got in the previous section but it is a different object (pointwise vs global), we consider a larger class of operators and it really does not matter provided $\nu(\varepsilon) = O(h^2)$ which is the case as $l \ge 2$ (at least if we set $\varepsilon = h$).

Proof of Theorem 4.7.6. (a) We will use the method of successive approximations but in two steps and the first step will be different. As unperturbed operator we consider A_ε with the coefficients $a_{\alpha\beta\varepsilon} = a_{\alpha\beta} * \phi_\varepsilon$ where $\phi_\varepsilon(x) = \varepsilon^{-d}\phi(\varepsilon^{-1}x)$. Then $|a_{\alpha\beta\varepsilon} - a_{\alpha\beta}| \le \varsigma := \nu(\varepsilon)$ where $\nu(\varepsilon) = C_0\varepsilon^l$ under assumption that coefficients belong to $\mathscr{C}^l$.

Therefore on the energy level $\tau : |\tau| \le C$ the error norm does not exceed $C_0\varsigma$ and then according to calculations of Section 4.4 Tauberian expressions

on intervals $[-T,-T/2]\cup[T/2,T]$ for A and A_ε differ by $Ch^{-d-1}\varsigma T$ and then this is true for Tauberian expressions on interval $[-T,T]$: in the obvious notations

$$\mathsf{N}^{\mathsf{T}}_x(\tau_1,\tau_2;T;A)-\mathsf{N}^{\mathsf{T}}_x(\tau_1,\tau_2;T;A_\varepsilon)\prec Ch^{-d-1}\varsigma T \tag{4.7.14}$$

while Tauberian errors do not exceed $Ch^{1-d}T^{-1}$.

Let us set $\varepsilon = C_0h|\log h|$ [45]. Then according to previous Section 4.6

$$\mathsf{N}^{\mathsf{T}}_x(\tau_1,\tau_2;T;A_\varepsilon)-\mathsf{N}^{\mathsf{W}}_x(\tau_1,\tau_2;A_\varepsilon)\prec Ch^{1-d} \tag{4.7.15}$$

and due to microhyperbolicity one can prove easily that

$$\mathsf{N}^{\mathsf{W}}_x(\tau_1,\tau_2;A_\varepsilon)-\mathsf{N}^{\mathsf{W}}_x(\tau_1,\tau_2;A)\prec Ch^{-d}\varsigma. \tag{4.7.16}$$

Combining, we arrive to

$$\mathsf{R}^{\mathsf{W}}_x(\tau_1,\tau_2;A)\prec C\bigl(h^{1-d}T^{-1}+h^{-d-1}\varsigma T\bigr). \tag{4.7.17}$$

Minimizing by $T\in[C_0h,\epsilon]$ we get

$$\mathsf{R}^{\mathsf{W}}_x(\tau_1,\tau_2;A)\prec C\bigl(h^{1-d}+h^{-d}\varsigma^{\frac12}\bigr). \tag{4.7.18}$$

(b) Similarly, let us consider $\mathsf{N}^{\mathsf{W}}_{x,\varphi}$, with φ supported in the small vicinity of τ_i such that $\partial^\alpha\varphi$ is continuous with the modulus of continuity $\tau^{r-\lceil r\rceil+1}$ as $|\alpha|\le\lceil r\rceil-1$ as $r>0$. Then the Tauberian error is $Ch^{1+r-d}T^{-1-r}$, Tauberian expressions on intervals $[-T,-T/2]\cup[T/2,T]$ for A and A_ε differ by $Ch^{-d-1}\varsigma T\times(h/T)^r$ and therefore this is also true for Tauberian expressions on interval $[-T,T]$ provided $r<1$ while for $r=1$ we obtain $Ch^{-d}\varsigma(1+|\log T/h|)$:

$$\mathsf{N}^{\mathsf{T}}_{x,\varphi}(T;A)-\mathsf{N}^{\mathsf{T}}_{x,\varphi}(T;A_\varepsilon)\prec Ch^{-d-1+r}\varsigma\bigl(T^{1-r}+C\delta_{r1}|\log T/h|\bigr). \tag{4.7.14}'$$

Similarly, the right-hand expression in (4.7.15) becomes Ch^{1+r-d} but one needs to include in Weyl expressions the second term here and below (unless it is 0), the right-hand expression in (4.7.15) remains and combining we get $Ch^{-d}\varsigma$:

$$\mathsf{R}^{\mathsf{W}}_{x,\varphi}(A)\prec C\Bigl(h^{1+r-d}T^{-r-1}+Ch^{-d-1+r}\varsigma\bigl(T^{1-r}+C\delta_{r1}|\log T/h|\bigr)\Bigr). \tag{4.7.17}'$$

[45] Later we will set $\varepsilon=C_0h$.

Minimizing by $T \in [C_0 h, \epsilon]$ we get

$$\mathsf{R}^{\mathsf{W}}_{x,\varphi}(A) \prec C\big(h^{1+r-d} + h^{-d}\varsigma^{(r+1)/2} + \delta_{r1} h^{-d}\varsigma |\log \varsigma|\big). \tag{4.7.18}'$$

(c) We need to prove that we can set $\varepsilon = C_0 h$. Let us apply the previous arguments to $A_{2\varepsilon}$ and A_ε instead of A_ε and A respectively, and set $\varsigma = \nu(\varepsilon)$, $\varepsilon \ge C_0 h$.

One can observe easily that for operator A_ε rather than A all our previous results could be improved. Namely, in Proposition 4.7.3 a usual commutator formula holds modulo $O\big(h^l(h/\varepsilon)^s \|\partial b\|_{l-1}\big)$ with arbitrarily large exponent s and therefore this factor $(h/\varepsilon)^s$ pops-up in some other estimates in the correct places: in Proposition 4.7.4 it appears alongside factor h^{l-1} in the last term of the right-hand expression in (4.7.9).

Consider now (4.7.14)′. Interval $[-T, -T/2] \cup [T/2, T]$ contributes now $Ch^{-d-1}(h/T)^r \varsigma T\big((h/\varepsilon)^s + (h/T)^s\big]$. Indeed, it follows from the fact that rescaling $x \mapsto x/T$ we have not only $h \mapsto h/T$ but also $\varepsilon \mapsto \min(\varepsilon/T, 1)$. Then summation with respect to partition with respect to tT results in $Ch^{-d-1}(h/T)^r \varsigma T(h/\varepsilon)^s + Ch^{-d}\varsigma$ which is now the new right-hand expression in (4.7.14)′. Plugging $\varepsilon = \varepsilon_n := 2^n h$ and taking a sum with respect to $n = 0, 1, \dots \bar{n} = \lceil \log_2(\varepsilon/h)\rceil$ we conclude that

$$\mathsf{N}_{x,\varphi}(A) - \mathsf{N}^{\mathsf{T}}_{x,\varphi}(T, A_\varepsilon) \prec Ch^{-d}(h/T)^r + Ch^{-d-1}(h/T)^r \varsigma(h) T + Ch^{-d}\varsigma(\varepsilon) \tag{4.7.19}$$

where the first term estimates the Tauberian error, the second term estimates an error when we replace A by A_h in the Tauberian expression, and it and the third term estimate together an error when we replace A_h by A_ε. Here $0 \le r \le 1$ where for $r = 0$ we consider instead N_x and $\mathsf{N}^{\mathsf{T}}_x$ and for $r = 1$ we need to replace the second term by $Ch^{-d}\varsigma(h)|\log h|$.

Next in virtue of Section 4.6 we can replace here $\mathsf{N}^{\mathsf{T}}_{x,\varphi}(T, A_\varepsilon)$ by $\mathsf{N}^{\mathsf{W}}_{x,\varphi}(T, A_\varepsilon)$ as long as $\varepsilon \ge C_0 h|\log h|$ and, finally, we can replace $\mathsf{N}^{\mathsf{W}}_{x,\varphi}(T, A_\varepsilon)$ by $\mathsf{N}^{\mathsf{W}}_{x,\varphi}(T, A)$.

Minimizing with respect to $T \in [C_0 h, \epsilon]$ we conclude that the left-hand expression in (4.7.19) is estimated by the right-hand expression of (4.7.18)′ with $\varsigma = \varsigma(h)$ plus $Ch^{-d}\varsigma(\varepsilon)$. One can see easily that the latter term is smaller than the right-hand expression of (4.7.18)′ as long as $\varepsilon \le h^{1-\delta}$ for $r < 1$ and $\varepsilon \le C_0 h|\log h|$ for $r = 1$. This concludes the proof. □

Theorem 4.7.8. *In the framework of Theorem 4.7.6(i) and (ii) respectively*

(i) Estimate

$$|\Gamma\,\mathsf{tr}\big(b(x)(hD_x)^\alpha(hD_y)^\beta e(x,y;\tau_1,\tau_2)\big) - \mathsf{N}^{\mathsf{W}}(\tau_1,\tau_2)| \le C\big(h^{1-d}+h^{-d}\varsigma^{2/3}\big)\,\mathsf{dim}\,\mathbb{H} \tag{4.7.20}$$

holds where $\mathsf{N}^{\mathsf{W}}(\tau_1,\tau_2)$ *means corresponding Weyl expression and* $b\in \mathscr{C}_0^\infty$ *is a matrix-valued function.*

(ii) Estimate

$$|\int \varphi(\tau)\,d_\tau\Big(\Gamma\,\mathsf{tr}\big(b(x)(hD_x)^\alpha(hD_y)^\beta e(x,y;\tau,\tau_i)\big) - \mathsf{N}^{\mathsf{W}}(\tau,\tau_i)\Big)| \le C\big(h^{1+r-d}+h^{-d}\varsigma^{2(r+1)/3}+\delta_{r2}h^{-d}\varsigma|\log\varsigma|\big)\,\mathsf{dim}\,\mathbb{H} \tag{4.7.21}$$

holds where $\mathsf{N}^{\mathsf{W}}(\tau,\tau_i)$ *means corresponding Weyl expression including the second term.*

Proof. (a) Let us apply a two term approximation rather than one term. Then the error will be the third rather than the second term; then the error estimate will be $Ch^{-d-2}\varsigma^2T^2$ rather than $Ch^{-d-1}\varsigma T$ and we get expression

$$C\Big(h^{1+r-d}T^{-r-1}+h^{-d-2+r}\varsigma\big(T^{2-r}+\delta_{r2}|\log T/h|\big)\Big) \tag{4.7.22}$$

instead of the right-hand expression in (4.7.17)′. Minimizing by $T\in[C_0h,\epsilon]$ we get the right hand expression in (4.7.21) but without $\mathsf{dim}\,\mathbb{H}$ which appears after we take trace. Here $\varsigma=\nu(\varepsilon)$, $\varepsilon=C_0h|\log h|$ at the moment.

Consider the second term; it appears from

$$\mathsf{Tr}\big(BG_\varepsilon^{\pm}(A-A_\varepsilon)G_\varepsilon^{\pm}\big)=\mathsf{Tr}\big((A-A_\varepsilon)G_\varepsilon^{\pm}BG_\varepsilon^{\pm}\big) \tag{4.7.23}$$

with $B=(-hD)^\beta b(hD)^\alpha$ and since in $G_\varepsilon^{\pm}BG_\varepsilon^{\pm}$ operator B has smooth coefficients and $\varepsilon=C_0h|\log h|$ we conclude that due to propagation of singularities results contribution of time intervals $[-T,-\bar T]\cup[\bar T,T]$ with $\bar T=Ch|\log h|$ is negligible.

Meanwhile on time interval $[-\bar T,\bar T]$ we apply our standard successive approximation method with freezing symbol at point y etc resulting in

$$\mathsf{tr}\big((a-a_\varepsilon)(\tau-a_\varepsilon)^{-1}b(\tau-a_\varepsilon)^{-1}\big)=\mathsf{tr}\big(b(\tau-a_\varepsilon)^{-1}(a-a_\varepsilon)(\tau-a_\varepsilon)^{-1}\big)$$

calculated at point (y,ξ).

To this expression one needs to apply $\mathsf{Res}_{\mathbb{R}}$, multiply by $(2\pi h)^{-d}$, integrate by $(y,\xi)\in\mathbb{R}^{2d}$ and by $\tau\in[\tau_1,\tau_2]$ [46]. Due to microhyperbolicity of a on the energy level s τ_1,τ_2 and estimate $|a-a_\varepsilon|\le C_0\varsigma$ the result is equal to the result of the same procedure applied to

$$\mathsf{tr}\big(b(\tau-a_\varepsilon)^{-1}-b(\tau-a)^{-1}\big)$$

with an error, not exceeding $Ch^{-d}\varsigma^2$. However the latter result is equal to the difference between $\mathsf{N}^{\mathsf{W}}(\tau_1,\tau_2;A)$ and $\mathsf{N}^{\mathsf{W}}(\tau_1,\tau_2;A_\varepsilon)$ and therefore Statement (i) is proven albeit with $\varsigma=\nu(\varepsilon)$.

(b) To improve this statement to one with $\varsigma=\nu(h)$ we apply the same procedure as in Part (c) of the proof of Theorem 4.7.6. Then estimate of the third (error) term in the successive approximation is trivial and is left to the reader, let us consider the second term which now is due to the following modification of (4.7.23):

$$\sum_{0\le n\le \bar n}\mathsf{Tr}\Big((A_{\varepsilon_n}-A_{\varepsilon_{n+1}})G^{\pm}_{\varepsilon_{n+1}}BG^{\pm}_{\varepsilon_{n+1}}\Big) \tag{4.7.24}$$

where $h\le\varepsilon_n\le\varepsilon=C_0h|\log h|$. We want to convert both factors $G^{\pm}_{\varepsilon_{n+1}}$ to $G^{\pm}_{\varepsilon}$. To do this properly observe that $G^{\pm}_{\varepsilon_m}BG^{\pm}_{\varepsilon_m}-G^{\pm}_{\varepsilon_{m+1}}BG^{\pm}_{\varepsilon_{m+1}}$ contains one factor $(A_{\varepsilon_m}-A_{\varepsilon_{m+1}})$ and three factors $G^{\pm}_{\varepsilon_{m+1}}$ and $G^{\pm}_{\varepsilon_m}$ (together). Then applying arguments Part (c) of the proof of Theorem 4.7.6 we estimate the error in (4.7.24) by

$$T^3h^{-3}\sum_{0\le n\le m\le\bar n}\nu(\varepsilon_n)\nu(\varepsilon_m)h^{l-1}\big[(h/\varepsilon_m)^s+(h/T)^s\big]$$

and we estimate the resulting error in the second term properly.

So, instead of (4.7.24) we have $\mathsf{Tr}\Big((A-A_\varepsilon)G^{\pm}_{\varepsilon}BG^{\pm}_{\varepsilon}\Big)$ and we are be able to pass to Weyl expression as in Part (a). □

Remark 4.7.9. (i) As $r\ge\frac12$ and $1\le l\le 1+r$ we have a better estimate than one delivered by the bracketing.

(ii) One can generalize our arguments for more general ν, in paricular, for $\nu(s)=s^l(\log|s|)^\sigma$.

[46] If we consider Statement (i); Statement (ii) is proven the same way.

4.7.4 Sharper Asymptotics

Observe that under assumption

$$\nu(s) = o(s^{\frac{2}{3}}) \qquad \text{as } s \to +0. \tag{4.7.25}$$

the second terms in the right-hand expressions of (4.7.20) and (4.7.21)are "o" in comparison with the first terms.

Theorem 4.7.10. *In the framework of Theorem 4.7.8 let* (4.7.25) *be fulfilled.*

Further, let the standard non-periodicity condition

$$z \notin \Phi_{(}z) \qquad \forall t \neq 0\, \forall z \tag{4.7.26}$$

(where Φ_t means the generalized (multivalued) Hamiltonian flow) be fulfilled. Then estimates (4.7.20) *and* (4.7.21) *hold with the right-hand expressions $o(h^{1-d})$ and $o(h^{1-d+r})$ respectively.*

Proof. Under assumption (4.7.26) for A_ε one can take T arbitrarily large (so in all estimates $Ch^{1-d+r}T^{-1-r}$ with $h \le T \le \epsilon$ is replaced by $Ch^{1-d+r}T^{-1-r} + o_T(h^{1-d+r})$ for any $T \ge \epsilon$.

Then under assumption (4.7.26) proof of Theorem 4.7.8 delivers improved estimates. □

4.A Appendices

4.A.1 On the Traces of Almost Analytic Functions

Let $\Omega \subset \mathbb{R}^d$ be an open domain, Γ an open convex domain with a vertex at 0 and let $\tilde\Omega = \Omega + i\Gamma z \subset \mathbb{C}^d$. Let $\mathbb{B}$ be a Banach space. A function $F(.) \in \mathscr{C}^1(\tilde\Omega, B)$ is called *r-almost analytic* if

$$\|F(z)\| \le M|\operatorname{Im} z|^{-l+\delta} \qquad \forall z \in \tilde\Omega, \tag{4.A.1}$$

and

$$\|\bar\partial_{z_j} F(z)\| \le M|\operatorname{Im} z|^{r-1+\delta} \qquad \forall z \in \tilde\Omega \quad \forall j = 1, \dots, d \tag{4.A.2}$$

where $\bar\partial_{z_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$ and $\delta > 0$, $l \in \mathbb{N}$.

Proposition 4.A.1. *Let conditions* (4.A.1) *and* (4.A.2) *with* $r = 0$ *be fulfilled. Then*

For every $y \in \Gamma$ *with* $|y| = 1$ *there exists a limit*

$$f(x) = F(x + i0y) = \lim_{t \to +0} F(x + ity) \tag{4.A.3}$$

in the space of $\mathbb{B}$*-valued distributions* $\mathscr{D}'(\Omega, \mathbb{B})$*;*

Moreover $f = \sum_{0 \le k \le l} \mathcal{M}^k f_k$ *where* $f_k \in \mathscr{C}(\Omega, \mathbb{B})$ *and* $\|f_k\| \le CM$ *with* $C = C(l, \delta)$, $\mathcal{M} = \langle y, \bar\partial_x \rangle$.

$f(x)$ *does not depend on the choice of* y.

Proof. For $l = 0$ this assertion is obvious. Let us note that

$$\partial_t F(x + ity) = -\mathcal{M} F(x + ity) + 2\langle y, \bar\partial_z \rangle F(x + ity)$$

and hence

$$F(x + ity) = F(x + iy) + 2\int_1^t \langle y, \bar\partial_z \rangle F(x + it'y) dt' - \int_1^t \mathcal{M} F(x + it'y)\, dt'. \tag{4.A.4}$$

Then (4.A.1) and (4.A.1) yield that the first and the second terms in the right hand expression belong to $\mathscr{C}(\Omega_x \times J_t \times \Gamma_y, \mathbb{B})$, $J = [0, 1]$; the same is also true for the "shortened" third term $\int_1^t F(x + it'y) dt'$ provided $l = 1$. On the other hand, for $l \ge 2$ we replace $F(x + it'y) dt'$ in (4.A.4) by the right-hand expression of (4.A.4) with t, t' replaced by t', t'' respectively. Repeating this procedure $l - 1$ times we obtain that

$$F(x + ity) = \sum_{0 \le k \le l-1} \mathcal{M}^k F_k(x) + \frac{1}{(l-1)!}(-\mathcal{M})^l \int_1^t (t - t')^{l-1} F(x + it'y)\, dt' \tag{4.A.5}$$

where F_k with $k = 0, \dots, l - 1$ as well as

$$F_l(x) = \frac{1}{(l-1)!}(-\mathcal{M})^l \int_1^t (t - t')^{l-1} F(x + it'y)\, dt'$$

obviously belong to $\mathscr{C}(\Omega_x \times J_t \times \Gamma_y, B)$. Therefore the limit $F(x+i0y)$ exists in the distribution sense and is the sum of $\mathcal{M}^k f_k$ with $f_k(x) = F_k(x+i0y)$. So Statements 4.A.1 and 4.A.1, are proven.

In order to prove 4.A.1 let us note that

$$\partial_s F(x+iy^0-is\omega) = \mathcal{M}'F(x+iy^0-is\omega) - 2\langle\omega,\bar\partial_z\rangle F(x+iy^0-is\omega)$$

with $\mathcal{M}' = -\langle\omega,\partial_x\rangle$; setting $y, y^0 \in \Gamma$, $s, t \in [0,1]$ and $\omega = y^0 - ty$ we obtain a formula similar to (4.A.4) with $F(x+ity)$, $\mathcal{M}$, $\langle y,\bar\partial_z\rangle$, $t, t', 1$ replaced by $F(x+iy^0-is\omega)$, $-\mathcal{M}'$, $-\langle\omega,\bar\partial_z\rangle$, $s, s', 0$ respectively. Then formula (4.A.5) also holds modified in the same way:

$$F(x+ity) = \sum_{0\le k\le l-1} (-\mathcal{M}')^k F_k(x,y^0,\omega,t) + \frac{1}{(l-1)!}\mathcal{M}'^l \int_0^1 (1-s')^{l-1} F(x+iy^0-is'\omega)\,ds';$$

moreover, in this sum we can take the limit as $t \to +0$ (and $\omega \to y^0$, $\mathcal{M}' \to -\mathcal{M}$), and this limit obviously does not depend on y. □

Let us consider a symbol $a \in \mathscr{C}^k(\Omega, \mathbb{H}, \mathbb{H})$ (where $\mathbb{H}$ is an auxiliary Hilbert space) such that

(4.A.6) $a^\dagger(x) = a(x) \qquad$ in Ω,

and

(4.A.7) $\langle(\ell a)(x)w, w\rangle \ge \epsilon_0\|w\|^2 - c\|(a(x)-\tau)w\|^2 \qquad \forall x\in\Omega \quad \forall w \in \mathbb{H}$

where we write now x instead of (x,ξ) and $\ell \in T\Omega$. Let $\tilde a(x+iy)$ be the q-analytic extension of $a(x)$:

$$\tilde a(x+iy) = \sum_{|\alpha|\le q} \frac{1}{\alpha!}(\partial_x^\alpha a)(x)(iy)^\alpha. \tag{4.A.8}$$

We assert that for small enough $\epsilon \ge 0$, $\nu \ge 0$

$$\|\tilde P w\| \ge \epsilon_1(\varepsilon+\nu)\|w\| \tag{4.A.9}$$

where here and below $\tilde P = \tilde a(x+i\varepsilon\ell) - \tau + i\nu I$ and $P = a(x) - \tau$. In fact,

$$\tilde P(x)w = P(x)w + i(\varepsilon(\ell a)(x) + \nu)w + O(\varepsilon^2\|w\|)$$

and hence (4.A.6) and (4.A.7) yield that

$$\|\tilde{P}(x)w\| \cdot \|w\| \ge \mathsf{Im}\langle \tilde{P}(x)w, w\rangle \ge \epsilon_0(\varepsilon+\nu)\|w\|^2 - C_0\varepsilon^2\|w\|^2 - c\|P(x)w\|^2 \ge (\frac{1}{2}(\varepsilon+\nu) - C\varepsilon^2 - C\nu^2)\|w\|^2 - c\|\tilde{P}(x)w\|^2,$$

and (4.A.9) trivially follows from this inequality.

Therefore $F(z) = (\tilde{a}(z') - z_0)^{-1}$ satisfies (4.A.1) and (4.A.2) with $I = 2$, $\delta = 1$, $r = q-2$, $z = (z_0, z')$ and $\Gamma = \{y_0 \le 0, y' = t\ell, t \ge 0, y_0 - t < 0\}$. Moreover,

$$\big(a(x) - \tau \pm i0\big)^{-1} = \big(\tilde{a}(x \pm i0\ell) - \tau\big)^{-1} \tag{4.A.10}$$

(we have analyzed the "+" case; the "−" case can be analyzed in the same way) and the same is true for all products of the factors $(\tilde{a}(z') - z_0)^{-1}$ (the number of which should not exceed j) and regular[47] q-almost analytic factors, $q = q(j)$.

We will use these properties in the forthcoming sections and chapters.

4.A.2 Smoothing of $\mathscr{C}^m_\vartheta$ Functions

Proposition 4.A.2. *Let $f \in \mathscr{C}^m_\vartheta(X, .)$ where ϑ satisfies* (4.6.7).

Then for every $\varepsilon > 0$ there exits f_ε such that

$$|\partial^\alpha f_\varepsilon(x) - \partial^\alpha f(x)| \le C\vartheta(\varepsilon)\varepsilon^{-|\alpha|} \qquad \forall \alpha : |\alpha| \le m, \tag{4.A.11}$$

$$|\partial^\alpha f_\varepsilon(x)| \le C\vartheta(\varepsilon)\varepsilon^{-|\alpha|} \qquad \forall \alpha : |\alpha| > m; \tag{4.A.12}$$

here C does not depend on α.

Proof. Let ϕ be a function such that

$$\hat{\phi} \in \mathscr{C}_0^\infty\big(B(0, \tfrac{1}{2})\big),\ \partial^\alpha\hat{\phi}(0) = \delta_{\alpha 0} \text{ where } \hat{\phi} \text{ denotes Fourier transform of } \phi. \tag{4.A.13}$$

Then $|\partial^\alpha\phi| \le C_n(1+|x|)^{-n}$ for any fixed n and every α. Let $f_\varepsilon := f * \phi_\varepsilon$ with $\phi_\varepsilon = \varepsilon^{-d}\phi(x/\varepsilon)$.

[47] I.e., of the class C^q up to $\Omega \subset \mathbb{R}^d$.

Then for $|\alpha| \le m$

$$\partial^\alpha f_\varepsilon(x) - \partial^\alpha f(x) = \int \phi(y)\big(\partial^\alpha f(x-\varepsilon y) - \partial^\alpha f(x)\big)\,dy.$$

Taking Taylor expansion

$$\partial^\alpha f(x-\varepsilon y) = \sum_{\beta:|\beta|\le m-|\alpha|} \frac{1}{\beta!}(-\varepsilon y)^\beta \partial^{\alpha+\beta} f(x) + R_\alpha$$

with

$$R_\alpha \le C(\varepsilon|y|)^{-|\alpha|}\vartheta(\varepsilon|y|)$$

we see that the left hand expansion in (4.A.11) does not exceed

$$C\varepsilon^{-|\alpha|}\int |y|^{-|\alpha|}\vartheta(\varepsilon|y|)(1+|y|)^{-n}\,dy$$

and the integrand does not exceed $\vartheta(\varepsilon)(1+|y|)^{1-n}$ for both $|y| \le 1$ and $|y| \ge 1$ due to assumption $f \in \mathscr{C}^m_\vartheta$; so we get (4.A.11).

For $|\alpha| > m$ we have

$$\partial^\alpha f_\varepsilon = \varepsilon^{m-|\alpha|}\int \partial^\beta\phi(y)(\partial^{\alpha-\beta}f)(x-\varepsilon y)\,dy$$

with $|\beta| = |\alpha| - m$; replacing $(\partial^{\alpha-\beta}f)(x-\varepsilon y)$ by $(\partial^{\alpha-\beta}f)(x)$ we get 0 and the difference does not exceed

$$C\varepsilon^{-|\alpha|}\int \vartheta(\varepsilon|y|)(|y|+1)^{-n}\,dy$$

which leads to (4.A.12). □

Comments

The number of papers devoted to spectral asymptotics is huge. Apart of the Tauberian approach developed by T. Carleman in [1, 2] and used here, and variational approach we will use later to cover singular zones, we should mention a method of almost spectral projector due to M. Shubin and V. Tulovsky [1].

Depending which function $f(t, A)$ is used and what Tauberian theorem is applied we talk about resolvent method ($f = (z - A)^{-1}$), complex power method ($f = A^z$), heat kernel method ($f = e^{-tA}$) or hyperbolic operator method ($f = e^{itA}$, $f = \cos(t\sqrt{A})$ or $f = \sin(t\sqrt{A})$ applied here.

The absolute majority of the papers using the latter use explicit construction of the hyperbolic kernel and we refer to Yu. Safarov and D. Vassiliev [3]. Here we comment only on the constructions of this Chapter.

The whole construction of Sections 4.3–4.4 appeared first in paper of V. Ivrii [8] in the standard rather than semi-classical situation: first propagation results and rescaling were used to prove that singularity at $t = 0$ of $\sigma(t) = \Gamma(Qu)$ was of the *normal type* i.e. $(tD_t)^n\sigma(t)$ had the same order of singularity in the vicinity of 0:

$$t^n D_t^n \sigma(t) \in \mathscr{H}^{-M}[-T_0, T_0] \qquad \forall n \in \mathbb{Z}^+.$$

Then the successive approximation was used and each next term contained an extra factor of $t^2 D_t$ type:

$$\sigma_m(t) \in t^{2m}\mathscr{H}^{-M-m}[-T_0, T_0] \qquad \forall m \in \mathbb{Z}^+.$$

Then combining these two statements it was proven that actually

$$\sigma_m(t) \in \mathscr{H}^{M-m}[-T_0, T_0] \qquad \forall m \in \mathbb{Z}^+.$$

Long term propagation to prove a "sharp" remainder estimate was first used by J. J. Duistermaat and V. Guillemin [1] and without Fourier integral operators again by V. Ivrii [8]. These allowed to prove Weyl conjecture: two-term asymptotics of eigenvalue counting function under assumption that measure of the set of all periodic billiards is 0.

These arguments were repeated in V. Ivrii [10] to study matrix operators with characteristic roots of variable multiplicity.

"Sharper" remainder estimates were first derived by P. Bérard [1] and then by A. Volovoy [1] and later repeated by V. Ivrii [23] in the general settings.

The sharp remainder estimates without smoothness conditions appeared first in L. Zielinski [1–4], followed by V. Ivrii [26] and M. Bronstein and V.Ivrii [1]. While L. Zielinski was able to develop microlocal analysis as $\varepsilon = h^{\frac{1}{2}}$ and in V.Ivrii [26] it was done as $\varepsilon = h^{1-\delta}$ with arbitrarily small $\delta > 0$; then it was improved to $\varepsilon = h|\log h|^{1+\delta}$ (unpublished) and in M. Bronstein–V.Ivrii [1] envelope was pushed further to the limit $\varepsilon = Ch|\log h|$; this

improvement leads to smaller approximation errors as regularity is fixed and to weaker regularity condition as approximation error is fixed.

Direct approach to operators with non-smooth coefficients was developed first by V. Ivrii while researching for Chapter 27 of this book.

Chapter 5

Scalar Operators in the Interior of the Domain. Rescaling Technique

5.1 Introduction

5.1.1 Pilot: Schrödinger Operator

In this chapter we consider scalar and similar operators and weaken microhyperbolicity conditions while trying to retain sharp remainder estimates. Note that for such operators we can rewrite ellipticity, microhyperbolicity and ξ-microhyperbolicity on energy level τ conditions respectively as

$$|a(x,\xi) - \bar{\tau}| \ge \epsilon_0, \tag{5.1.1}$$

$$|a(x,\xi) - \bar{\tau}| + |\nabla_{x,\xi} a(x,\xi)| \ge \epsilon_0, \tag{5.1.2}$$

$$|a(x,\xi) - \bar{\tau}| + |\nabla_{\xi} a(x,\xi)| \ge \epsilon_0. \tag{5.1.3}$$

To illustrate, for Schrödinger operator

$$h^2|D|^2 + V(x) \tag{5.1.4}$$

and $Q_1 = Q_2 = I$ these conditions become

$$V \le \bar{\tau} - \epsilon_0, \tag{5.1.5}$$

$$|V(x)| + |\nabla V(x)| \ge \epsilon_0, \tag{5.1.6}$$

$$|V(x)| \ge \epsilon_0 \tag{5.1.7}$$

respectively. These conditions are assumed to be fulfilled in $B(0,1)$ and then respectively

$$|e(x,y,\bar\tau)| \le C'h^s, \qquad e^{\mathsf{W}}(x,y,\bar\tau) = 0 \tag{5.1.8}$$

with arbitrarily large s,

$$|\int \mathsf{R}^{\mathsf{W}}_x(\bar\tau)\psi(x)\,dx| \le Ch^{1-d} \tag{5.1.9}$$

$$|R^{\mathsf{W}}_x(\bar\tau)| \le Ch^{1-d}. \tag{5.1.10}$$

as $x, y \in B(0,\frac12)$ or $\psi \in \mathscr{C}_0^\infty(B(0,\frac12))$

To weaken these conditions we apply *rescaling with variable scale parameters*.

First, however, consider constant parameters ρ, γ such that

$$\rho\gamma \ge h \tag{5.1.11}$$

and that V is $\lesssim \rho^2$ in an appropriate sense and our playground is $B(0,\gamma)$. Then we scale $x \mapsto x\gamma^{-1}$, divide operator by ρ^2 [1)] and arrive to operator (5.1.4) but with $h \mapsto \hbar = h\rho^{-1}\gamma^{-1}$ and potential $V' = \rho^{-2}V(x\gamma)$ now satisfying all the assumptions of Chapter 4.

In order our new operator to be elliptic, microhyperbolic or ξ-microhyperbolic on the energy level 0 we must assume that *originally*

$$V \le -\epsilon_0\rho^2, \tag{5.1.12}$$

$$|V(x)| + \gamma|\nabla V(x)| \ge \epsilon_0\rho^2, \tag{5.1.13}$$

$$|V(x)| \ge \epsilon_0\rho^2 \tag{5.1.14}$$

respectively and then we get estimates

$$|e(x,y,0)| \le C'h^l\rho^{-l}\gamma^{-l-d}, \quad e^{\mathsf{W}}(x,y,\bar\tau) = 0, \tag{5.1.15}$$

$$\gamma^{-d}|\int \mathsf{R}^{\mathsf{W}}_x(0)\psi(x)\,dx| \le Ch^{1-d}\rho^{d-1}\gamma^{-1} \tag{5.1.16}$$

$$|R^{\mathsf{W}}_x(0)| \le Ch^{1-d}\rho^{d-1}\gamma^{-1}. \tag{5.1.17}$$

[1)] Without any loss of the generality we can assume that $\bar\tau = 0$.

Now, if we want to get rid of assumptions (5.1.6) or (5.1.7) in the original standard problem we need to introduce *scaling functions*

$$\gamma(x) = \epsilon\big(|V(x)| + |\nabla V(x)|^2\big)^{\frac{1}{2}} + \bar{\gamma}, \qquad \rho(x) = \gamma(x), \tag{5.1.18}$$

or

$$\gamma(x) = \epsilon|V(x)| + \bar{\gamma}, \qquad \rho(x) = \gamma(x)^{\frac{1}{2}} \tag{5.1.19}$$

where $\bar{\gamma} = h^{\frac{1}{2}}$, $\bar{\gamma} = h^{\frac{2}{3}}$ respectively in order to fulfill (5.1.11).

Then $|\nabla\gamma| \le \frac{1}{2}$ and therefore both γ and ρ have constant magnitudes in balls $B(x,\gamma(x))$. Further, as $\gamma \ge 2\bar{\gamma}$ in $B(x,\gamma(x))$ condition (5.1.13) (or (5.1.14)) is fulfilled. On the other hand, as $\gamma \asymp \bar{\gamma}$ these conditions are of no importance since $\hbar \asymp 1$ then.

So, we arrive to estimates

$$|\int \mathsf{R}^{\mathsf{W}}_x(0)\psi(x)\,dx| \le Ch^{-d}\int \rho(x)^{d-1}\gamma^{-1}\,dx \tag{5.1.20}$$

or

$$|\mathsf{R}^{\mathsf{W}}_x(0)| \le Ch^{-d}\rho(x)^{d-1}\gamma^{-1}. \tag{5.1.21}$$

Plugging $\rho = \gamma$ or $\rho = \gamma^{\frac{1}{2}}$ we arrive to

$$|\int \mathsf{R}^{\mathsf{W}}_x(0)\psi(x)\,dx| \le Ch^{-d}\int \gamma(x)^{d-2}\,dx \tag{5.1.22}$$

or

$$|\mathsf{R}^{\mathsf{W}}_x(0)| \le Ch^{-d}\rho(x)^{d-1}\gamma^{(d-3)/2}. \tag{5.1.23}$$

Observe that the right-hand expression in (5.1.22) is $O(h^{1-d})$ as $d \ge 2$ and $O(h^{-\frac{1}{2}})$ as $d = 1$ while the right-hand expression in (5.1.23) is $O(h^{1-d})$ as $d \ge 2$, $O(h^{-\frac{4}{3}})$ as $d = 2$ and $O(h^{-\frac{2}{3}})$ as $d = 1$.

Therefore we can remove in some cases microhyperbolicity or ξ-microhyperbolicity condition either without any punishment at all or with some punishment.

5.1.2 Plan of the Chapter

The goal of this Chapter is to develop this approach further.

In Section 5.2 we consider operators with scalar principal symbols as well as operators with eigenvalues of the principal symbols having constant multiplicities. First, we prove results similar to those for the Schrödinger

operator we discussed above; the role of the dimension is played by $\mathsf{rank}\,\mathsf{Hess}\,a$ in stationary points. Then for asymptotics with spatial mollification we step by step weaken these non-degeneracy conditions until the become really weak. We also explore sharp and sharper asymptotics.

In Section 5.3 we apply these results to the Schrödinger and Dirac operators. For the Schrödinger operator the conditions of these theorems are fulfilled automatically for any dimension d (if we are interested in spatial means) but for the Dirac operator some additional treatment is necessary because we want asymptotics uniform with respect to the mass which can be not only finite but also very large or very small parameter; in the latter case eigenvalues of the principal symbols are not disjoint but it happens only at one point. We also derive improved asymptotics without spatial mollification for 1- and 2-dimensional Schrödinger operator as operator is not ξ-microhyperbolic; then asymptotics contains a correction term due to short bicharacteristic loops.

In Section 5.4 we generalize some of the results of the previous two sections to non-smooth operators. While we start from ξ-microhyperbolicity condition, at some moment we weaken it and microhyperbolicity condition also plays a role and we observe that results with or without spatial mollification become different. For example, if $d = 1, 2$ and we consider Schrödinger operator then in the framework of Subsection 5.1.1 remainder estimate is

$$|\mathsf{R}^{\mathsf{W}}| = |\int R^{\mathsf{W}}_x \psi(x)\,dx| \le Ch^{1-d}\int \gamma^{(d-3)/2}(x)\,dx; \tag{5.1.24}$$

and under microhyperbolicity condition

$$\gamma(x) + |\nabla\gamma(x)| \ge \epsilon_0. \tag{5.1.25}$$

Then as $d = 2, 1$ we arrive to the remainder estimates $|\mathsf{R}^{\mathsf{W}}| = O(h^{-1})$ and $|\mathsf{R}^{\mathsf{W}}| = O(|\log h|)$ respectively and in the latter case additional arguments lead to the remainder estimate $|\mathsf{R}^{\mathsf{W}}| = O(1)$. Thus we replaced ξ-microhyperbolicity by microhyperbolicity.

Rescaling again, we arrive to certain estimates without microhyperbolicity condition.

Furthermore, in some proofs we use not only rescaling but also long-term propagation (where long-term refers to time scale after rescaling) and ability to choose time direction arbitrarily is very handy (but this is possible only for asymptotics with spatial mollification).

Finally, in Appendix 5.A.1 we calculate spectral kernels for Schrödinger operator with potential $V(x) = x_1$ and in Appendix 5.A.2 we list some properties of the Pauli matrices.

5.2 Spectral Asymptotics for Scalar Operators

In this Section we consider operators with scalar principal symbols[2)] for which we weaken the restrictions on averaging parameter L of Subsection 4.4.1 and the microhyperbolicity condition we had in Subsection 4.4.2. Our main tool is rescaling applied to the results of the previous Chapter 4.

This theory is applied to more general operators as well. Namely, all the results remain true for operators such that

$$\mathsf{Spec}\, A^0(x,\xi) \cap [-\epsilon_0, \epsilon_0] = \{a(x,\xi)\} \qquad \text{in } \Omega \tag{5.2.1}$$

where here $A^0(x,\xi)$ is the principal symbol of A and $a(x,\xi)$ is the scalar function. Then in a fixed neighborhood of $\mathsf{supp}(q_1) \cup \mathsf{supp}(q_2)$ the operator A can be reduced[3)] to the block-diagonal form $A \equiv \begin{pmatrix} A_I & 0 \\ 0 & A_{II} \end{pmatrix}$ where the principal symbol of A_I equals $a(x,\xi)$ and $\mathsf{Spec}\, a_{II} \cap [-\epsilon_0, \epsilon_0] = \emptyset$; let Ω be this neighborhood.

If $u = \begin{pmatrix} u_{I,I} & u_{I,II} \\ u_{II,I} & u_{II,II} \end{pmatrix}$ corresponds to this reduction then the standard ellipticity arguments yield that

(5.2.2) All the components of u apart from $u_{I,I}$ are negligible in $\Omega \times \Omega^T \times [-\epsilon_0, \epsilon_0] \ni (x, y, \tau)$;

in what follows we will consider only $\tau \in [-\epsilon_0, \epsilon_0]$. Thus we can assume without any loss of the generality that

(5.2.3) $B = I$ and A is an operator with the scalar principal symbol a.

[2)] Later when we consider more degenerate operators we will need to include more terms in the analysis and assume that these terms are scalar.

[3)] By means of multiplication on the right and left by G^* and G respectively where G is a pseudodifferential operator with $G^*G \equiv GG^* \equiv I$.

For operators satisfying assumption (5.2.3) the ellipticity, microhyperbolicity and ξ-microhyperbolicity conditions on the energy level $\bar\tau$ are exactly (5.1.1), (5.1.2) and (5.1.3) meaning respectively that $\bar\tau$ is not a value of $a(x,\xi)$ at all, and $\bar\tau$ is not a *critical value of* a, $\bar\tau$ is not a critical value as $a(x,\xi)$ is considered to be function of ξ only.

One can easily see that

$$d\mu_\tau = I_{\mathbb{H}}\, dxd\xi : da\big|_{\Sigma_\tau} \tag{5.2.4}$$

in the sense of exterior forms.

5.2.1 From Condition "No Critical Point" to "$\operatorname{rank}\operatorname{Hess} a \ge 2$ at Critical Points." I

Standard Asymptotics

Let us introduce the *scaling functions*

$$\gamma(x,\xi) = \epsilon_1\big(|a(x,\xi)-\tau| + |\nabla_{(x,\xi)}a|^2\big)^{\frac12} + \bar\gamma, \qquad \rho(x,\xi) = \gamma(x,\xi) \tag{5.2.5}$$

with $\bar\gamma = h^{\frac12}$; this choice of $\bar\gamma$ is due to the fact that at this step γ will be scale with respect x and ξ.

One can easily check that γ is an *admissible scaling function*

$$|\nabla_{(x,\xi)}\gamma(x,\xi)| \le \frac12 \tag{5.2.6}$$

provided $\epsilon_1 > 0$ is a small enough constant.

Let us consider a γ-admissible partition of unity by pseudodifferential operators Q'_l [4)] Let $Q''_l = \operatorname{Op}(q''_l)$ where q''_l is the symbol of the same type as q'_l. Then one can easily see that

$$\rho_l^{-2} Q'_l A_l \in \Psi_{h,\gamma_l,\gamma_l,K}(T^*\mathbb{R}^d,\mathbb{H},\mathbb{H}) \text{ uniformly with respect to } l. \tag{5.2.7}$$

Then one needs to consider the separate elements of the partition and to treat two cases:

[4)] That means that $Q'_l = \operatorname{Op}(q'_l)$ where $q_l \in \mathsf{S}_{h,\gamma(z^l),\gamma(z^l),K}(T^*\mathbb{R}^d)$ are supported in the balls $B(z^l, \frac12\gamma_l)$ with $\gamma_l = \gamma(z^l)$ and the multiplicity of the covering by balls $B(z^l,\gamma_l)$ does not exceed $K_0 = K_0(d)$. Such a partition exists (see Hörmander [1], for example).

(i) *Regular case* when $\gamma := \gamma_l \ge c_0\bar{\gamma}$; then for this element all the conditions including the microhyperbolicity condition will be fulfilled after the rescaling

$$x \mapsto x_{\text{new}} = (x - x^l)\gamma^{-1},\ \xi \mapsto \xi_{\text{new}} = (\xi - \xi^l)\gamma^{-1}, \tag{5.2.8}$$
$$h \mapsto h_{\text{new}} = h\gamma^{-1}\rho^{-1},\ A \mapsto A_{\text{new}} = \rho^{-2}A_l$$

and the gauge transformation $A_l \mapsto e^{ih^{-1}\langle x,\xi_l\rangle}A_l e^{-ih^{-1}\langle x,\xi_l\rangle}$.

So, there are (apart from inessential shifts with respect to (x,ξ)) rescalings with respect to (x,ξ) and the rescaling with respect to the spectral parameter τ as well

$$\tau \mapsto \tau_{\text{new}} = \tau\varrho^{-1} \tag{5.2.9}$$

with $\varrho = \rho^2$ here. We can assume that $\bar{\tau} = 0$ to keep this energy level unaffected by rescaling. We selected $\bar{\gamma} = h^{\frac{1}{2}}$ to keep $h_{\text{new}} \le 1$ (because $\rho = \gamma$ at this moment).

Therefore one can apply all the results of Subsection 4.4.2.

Thus, let us apply all the results of Subsubsection 4.4.2.3 *Improved Asymptotics without Mollification* and then return back to the original variables. At this moment we consider *only* estimates of $\mathsf{R}^{\mathsf{W}}(\tau)$ and $\mathsf{R}^{\mathsf{W}}_{\varphi,L}(\tau)$.

We need to plug there $L_{\text{new}} = L\varrho^{-1}$. Moreover we can apply the results of Section 4.5 as well noticing that *on this step* no rescaling on t is necessary because $h_{\text{new}}^{-1}A_{\text{new}} = h^{-1}A$ modulo an inessential shift and thus $T_{\text{new}} = T$ here.

Note that

$$h_{\text{new}}L_{\text{new}}^{-1} = hL^{-1} \ \text{ and } \ h_{\text{new}}(L_{\text{new}}T)^{-1} = h(LT)^{-1} \tag{5.2.10}$$

and therefore

$$\frac{h_{\text{new}}^{1-d}}{T_{\text{new}}}\int_{(\mathcal{U}\cap\Sigma_\tau)} dx_{\text{new}}d\xi_{\text{new}} : da_{\text{new}} = \frac{h^{1-d}}{T}\int_{\mathcal{U}\cap\Sigma_\tau} dxd\xi : da \tag{5.2.11}$$

(surely in the left-hand expression not $(x_{\text{new}},\xi_{\text{new}})$ but (x,ξ) belongs to $\mathcal{U}\cap\Sigma_\tau$) and

$$h_{\text{new}}^{1-d+\delta}\vartheta\Big(\frac{h_{\text{new}}}{L_{\text{new}}T}\Big) = \Big(\frac{h}{\gamma^2}\Big)^{1-d+\delta}\vartheta\Big(\frac{h}{L}\Big). \tag{5.2.12}$$

We obtain that

(5.2.13) For this element of the partition all the estimates of Subsubsection 4.4.2.3 *Improved Asymptotics without Mollification* and Section 4.5 remain true with an additional term

$$C' h^{1-d+\delta} \vartheta\big(\frac{h}{L}\big) \int_{\mathcal{U}} \gamma^{-2-2\delta}\, dxd\xi \tag{5.2.14}$$

in the right-hand expression where $\mathcal{U} = B(z', \gamma_l)$ now.

(ii) *Irregular case* when $\gamma := \gamma_l \le c_0\bar{\gamma}$; then for this element all the conditions apart from the microhyperbolicity condition will be fulfilled after the same change of variables as before. However, the microhyperbolicity condition provides no advantage because $h_{\mathsf{new}} \asymp 1$ now.

Therefore one can apply Proposition 4.2.1 and Theorem 4.4.12. Returning to the original variables and operator we see that the contribution of this element to the remainder does not exceed

$$C_0 h^{1-d}\Big(\int_{\mathcal{U}'\cap\Sigma_\tau} dxd\xi : da + C' h^{\delta} \int_{\mathcal{U}'} \gamma^{-2-2\delta}\, dxd\xi\Big)\vartheta\big(\frac{h}{L}\big) \tag{5.2.15}$$

where $\mathcal{U}' = B(z', \gamma_l)$ now and

$$\int_{\mathcal{U}'\cap\Sigma_\tau} dxd\xi : da = O\Big(\int \gamma^{-2}\, dxd\xi\Big); \tag{5.2.16}$$

moreover one can replace "$= O$" by "$\asymp$" if in $B(z', \frac{1}{2}\gamma_l)$ ellipticity $a \asymp \gamma^2$ fails.

Note that

(5.2.17) Contribution[5)] of each irregular element does not exceed $C_0\vartheta(h/L)$

(5.2.18) Contribution[5)] of each regular elliptic element does not exceed $C_k h^{-d+k}\gamma^{-2d-2k}\vartheta(h/L)$ with an arbitrarily large exponent k.

[5)] Both to the remainder and to the main term.

Now let us sum the contributions of all the elements of this partition:

$$|\mathsf{R}^{\mathsf{W}}_{\varphi,L}(\tau)| \le C_0 h^{1-d}\vartheta\Big(\frac{h}{L}\Big)\Big(\int_{\mathcal{U}\cap\Sigma_\tau\cap\{\gamma\ge c_0\bar\gamma\}} dxd\xi : da + C' h^{\delta}\int_{\mathcal{U}\cap\{|a-\tau|\le\epsilon\gamma^2\}} \gamma^{-2-2\delta}\, dxd\xi + C' h^{k}\int_{\mathcal{U}} \gamma^{-2-2k}\, dxd\xi\Big) \tag{5.2.19}$$

where $\mathcal{U}$ is a fixed vicinity of $\mathsf{supp}(q)$. Here the last term in the right-hand expression is the contribution of all elliptic and of all irregular elements.

We need to consider only one level asymptotics and $\mathsf{R}^{\mathsf{W}}(\tau)$ is also covered since we can always set the second energy level to $-C$ since $B = I$.

First of all we obtain immediately

Theorem 5.2.1. $\mathsf{R}^{\mathsf{W}}(\tau) = O(h^s)$ *with arbitrarily large* s *provided* $L \ge h^{1-\delta}$ *and* $\varphi \in \mathscr{C}_0^\infty(\mathbb{R})$.

To cover more interesting case we need to impose some non-degeneracy condition; namely let us assume that (5.2.20)$_r$ $|a(x,\xi)-\tau| + |\nabla_{x,\xi} a(x,\xi)| \le \epsilon_0 \implies \mathsf{Hess}\, a(x,\xi)$ has r eigenvalues (counting multiplicities) $f_1, \dots, f_r$ such that $|f_1| \ge \epsilon_0, \dots, |f_r| \ge \epsilon_0$ [6].

Then in the vicinity of every point $z = (x,\xi)$ with

$$|a(z)-\tau| + |\nabla_z a(z)| \le \epsilon_0 \tag{5.2.21}$$

one can choose a coordinate system[7] $z = (z'; z'') = (z_1, \dots, z_r; z_{r+1}, \dots, z_{2d})$ such that

$$a(z) = \sum_{1\le j\le r} \varsigma_j z_j^2 + V(z'') \qquad \text{with } \varsigma_j = \pm 1. \tag{5.2.22}$$

Then in a smaller vicinity

$$\gamma(z) \asymp |z'| + \gamma'(z''), \quad \gamma'(z'') \coloneqq \epsilon_2\big(|V(z'')-\tau| + |\nabla V(z'')|^2\big)^{\frac12} + \bar\gamma. \tag{5.2.23}$$

One can see easily that then

$$\int_{\Sigma_\tau\cap\mathcal{U}\cap\{\varepsilon\le\gamma(x,\xi)\le 2\varepsilon\}} dxd\xi : da \le C\varepsilon^{r-2}; \tag{5.2.24}$$

[6] The other eigenvalues are arbitrary.

[7] This system is not assumed to be symplectic.

then for $r \ge 3$

$$\int_{\Sigma_\tau \cap \Omega} dxd\xi : da < \infty, \tag{5.2.25}$$

and

$$\int_{\Sigma_\tau \cap \Omega \cap \{\gamma(x,\xi)<\varepsilon\}} dxd\xi : da = O(\varepsilon^{(r-2)}) \tag{5.2.26}$$

and, moreover, all the additional terms are $O\big(h^{1-d+\delta'}\vartheta(h/L)\big)$ with a small enough exponent $\delta' > 0$.

Meanwhile for $r = 2$ we get a logarithmic divergence in (5.2.25).

On the other hand, assume instead that $(5.2.20)_2^+$ $|a(x,\xi) - \tau| + |\nabla_{x,\xi} a(x,\xi)| \le \epsilon_0 \implies \mathsf{Hess}\, a(x,\xi)$ has 2 eigenvalues (counting multiplicities) f_1, f_2 of the same sign such that $|f_1| \ge \epsilon_0, |f_2| \ge \epsilon_0$ [6].

Then (5.2.25), (5.2.26) hold. Really, in this case for each z'' on Σ_τ magnitude of $|z'|$ is defined which prevents logarithmic divergence. So, in this case $\int_{\Sigma_\tau} dxd\xi : da$ is uniformly bounded[8].

Finally, under condition $(5.2.20)_2$ the additional terms are $O\big(h^{1-d}\vartheta(h/L)\big)$.

Thus we arrive to the remainder estimate $O\big(h^{1-d}\vartheta(h/L)\big)$ under condition $(5.2.20)_2^+$.

Now look what happens to the coefficients. They are given by the "regularized" formulae in which Q_2 is replaced by $Q_2(I - \zeta(x, hD))$ where

$$\mathsf{supp}(\zeta) \subset \{\gamma(x,\xi) \le c_0\bar\gamma\}, \qquad \zeta = 1 \quad \text{as} \quad \gamma(x,\xi) \ge \bar\gamma, \tag{5.2.27}$$
$$|D^\alpha \zeta| \le c_\alpha \gamma^{-|\alpha|} \qquad \forall \alpha;$$

so these regularized coefficients are given by

$$\kappa'_{n\,\mathsf{reg}}(\tau, h) = \tag{5.2.28}$$
$$(2\pi)^{-d} \int q_1(x,\xi,h)\mathcal{F}_n(x,\tau,\xi)[q_2(x,\xi,h)]\big(1 - \zeta(x,\xi)\big)\, dxd\xi$$

instead of (4.3.60); exact choice of ζ is not important. Meanwhile integral in (4.3.60) may lead to distribution with respect to τ because there is no microhyperbolicity anymore.

So, we arrive immediately to

[8] However, as example $a = z_1^2 + z_2^2$ shows, this integral is not necessarily uniformly convergent.

Theorem 5.2.2. *(i) Under condition* $(5.2.20)_3$

$$|\mathsf{R}^{\mathsf{W}}_{\varphi,L,\mathsf{reg}}(\bar\tau)| \le C_0 h^{1-d}\vartheta\big(\frac{h}{L}\big)\Big(\int_{\Sigma_{\bar\tau}\cap\mathcal{U}} dxd\xi : da + C'h^{\delta'}\Big) \tag{5.2.29}$$

where

$$\mathsf{R}^{\mathsf{W}}_{\varphi,L,\mathsf{reg}}(\bar\tau) \coloneqq \int \varphi\big(\frac{\tau-\bar\tau}{L}\big)\Big(d_\tau\big(\Gamma Q_{1x} Be(.,.,\tau',\tau)\,{}^t Q_{2y}\big) - \sum_{0\le n\le N'-1} h^{-d+n}\kappa'_{n,\mathsf{reg}}(\tau,h)\,d\tau\Big) \tag{5.2.30}$$

with regularized coefficients $\kappa'_{n,\mathsf{reg}}$.

(ii) Under condition $(5.2.20)^+_2$

$$|\mathsf{R}^{\mathsf{W}}_{\varphi,L,\mathsf{reg}}(\bar\tau)| \le Ch^{1-d}\vartheta\big(\frac{h}{L}\big). \tag{5.2.31}$$

(iii) Under condition $(5.2.20)_2$

$$|\mathsf{R}^{\mathsf{W}}_{\varphi,L,\mathsf{reg}}(\bar\tau)| \le Ch^{1-d}\vartheta\big(\frac{h}{L}\big)\cdot\big(|\log h|+1\big). \tag{5.2.32}$$

(iv) Under condition $(5.2.20)_1$

$$|\mathsf{R}^{\mathsf{W}}_{\varphi,L,\mathsf{reg}}(\bar\tau)| \le Ch^{\frac12-d}\vartheta\big(\frac{h}{L}\big). \tag{5.2.33}$$

Proof. We proved Statements (i) and (ii) already. Proof of Statements (iii) and (iv) follows the same scheme but as we integrate with respect to z' we note that this integral becomes

$$\int |\log\big(|V(z'')-\bar\tau|+h\big)|\,dz'' \tag{5.2.34}$$

and

$$\int \big(|V(z'')-\bar\tau|+h\big)^{-\frac12}\,dz'' \tag{5.2.35}$$

as $r=2$, $a=z_1^2-z_2^2+V(z'')$ and as $r=1$, $a=\pm z_1^2+V(z'')$ respectively. In the worst case $V(z'')=0$ these expressions become $|\log h|$ and $h^{-\frac12}$ factors.

We will use improved remainder estimates later. □

Remark 5.2.3. (i) We do not need to include terms such that $\vartheta(h) \ge C^{-1}h^{n-1}$ for all $h \in (0, h_0)$;

(ii) Under condition $(5.2.20)^+_r$ we do not need to regularize coefficients in the terms such that $\vartheta(L)L^{-n}$ is a monotone non-decreasing function on $(0, h_0)$.

Sharp and Sharper Asymptotics

Let us leave analysis of Statements (iii) and (iv) of Theorem 5.2.2 for coming subsections and consider improvements of (i) and (ii) under condition to long-term dynamics.

If we assume that a is scalar globally we have just Hamiltonian flow (not the generalized Hamiltonian flow) and we do not need to consider sharp asymptotics separately; however we do not make such assumption.

We will use ν instead of γ of Section 4.5 because γ is in use.

Then we can apply Theorems 4.5.5 and 4.5.12 in the following settings. Let $\mathcal{U}_i$ denote $\mathsf{supp}(r_i^0)$ and Ω denote $\mathsf{supp}(q_1) \cap \mathsf{supp}(q_2)$ (from these theorems). Assume that

$$\mathcal{U}_i \subset \Omega^+_\varepsilon := \{z \in \Omega : \gamma(z) \ge \varepsilon\} \qquad \forall i = 1, \dots, m \tag{5.2.36}$$

with ν either a small constant (for sharp asymptotics) or a small parameter $\nu \in [h^\delta\epsilon, \epsilon]$ (for sharper asymptotics). Then in the zone Ω^+_ε we can use Theorems 4.5.5 and 4.5.12 with condition (4.5.30) replaced by

$$\mathsf{dist}\big((x,\xi), \Psi_{\tau,t}(x,\xi)\big) \ge \begin{cases} \epsilon\gamma|t| & \text{as } |t| \le T_0, \\ \gamma\nu & \text{as } T_0 \le |t| \le T. \end{cases} \tag{5.2.37}$$

In particular, we arrive to

Proposition 5.2.4. *Let condition* $(5.2.20)^+_2$ *be fulfilled. Further, assume that* $\mu_\tau(\Pi \cap \Omega^+) = 0$ *where* Π *is the set of all periodic trajectories and* $\Omega^+ := \{z \in \Omega : \gamma(z) > 0\}$. *Then for any* $\varepsilon > 0$ *contribution of* Ω^+_ε *to the remainder estimate is* $o\big(h^{1-d}\vartheta(h/L)\big)$ *uniformly with respect to* $h \ge L$.

We leave to the reader to formulate theorems. However it would be completely futile unless we can properly estimate contribution to the remainder of *near-critical zone*

$$\Omega_\varepsilon := \{z \in \Omega : \gamma(x,\xi) < \varepsilon\}. \tag{5.2.38}$$

Near-Critical Zone

Observe first that due to (5.2.26) as $r \ge 3$ contribution of Ω_ε to the remainder does no exceed $C\varepsilon h^{1-d}$ and thus is small. In particular, Proposition 5.2.4 immediately implies the final Theorem 5.2.9(i).

However under condition $(5.2.20)_2^+$ situation is more complicated. Due to (5.2.23)

$$\mu_\tau(\Omega_\varepsilon) \lesssim \mathsf{mes}(\Omega''_\varepsilon), \quad \Omega''_\varepsilon := \{z'' \in \Omega'' : \gamma'(z'') < \varepsilon\} \tag{5.2.39}$$

where we assigned $\mu_\tau(\Omega_0) = 0$ because μ_τ is not defined on $\Omega_0 = \{z \in: \gamma(z) = 0\}$, and $\Omega_\varepsilon = B(0,\varepsilon) \times \Omega''$.

Proof of Theorem 5.2.2(ii) immediately implies

Proposition 5.2.5. *Let condition* $(5.2.20)_2^+$ *be fulfilled. Then contribution of* Ω_ε *to the remainder estimate does not exceed* $Ch^{1-d}\vartheta(h/L)\,\mathsf{mes}(\Omega''_\varepsilon)$.

In particular, Proposition 5.2.4 immediately implies the final Theorem 5.2.9(ii) under assumption $\mathsf{mes}(\{V = \bar\tau\}) = 0$. However we can do better than this. Let us introduce the set

$$\Sigma'_\tau = \{(x,\xi) : \ \tau \text{ is an eigenvalue of } \ a(x,\xi) + ha^s(x,\xi)\} \tag{5.2.40}$$

and the $\mathcal{L}(\mathbb{H},\mathbb{H})$-valued[9] measure

$$d\mu'_\tau = \partial_\tau \theta\big(\tau - a(x,\xi) - a^s(x,\xi)h\big) \tag{5.2.41}$$

supported on Σ'_τ.

Proposition 5.2.6. *Let condition* $(5.2.20)_2^+$ *be fulfilled. Then* $\mathsf{R}^{\mathsf{T}}_{\varphi,L}(\tau)$ *is dominated by*

$$\Big(Ch^{1-d}\int_{\Sigma'_\tau\cap\Omega\cap\{\gamma\ge\varepsilon\}} d\mu'_\tau + Ch^{1-d}\int_{\Sigma'_\tau\cap\Omega\cap\{\gamma\le\varepsilon\}} |\{z_1,z_2\}|\, d\mu'_\tau + C''h^{1-d+\delta'}\Big)\vartheta\big(\frac{h}{L}\big) \tag{5.2.42}$$

where $\varepsilon = h^\delta$ *with arbitrarily small* $\delta > 0$.

[9] We do not assume here that the subprincipal symbol a^s is scalar.

Proof. It is sufficient to prove (5.2.42) as

$$\mathsf{supp}(q_1) \cap \mathsf{supp}(q_2) \subset \Omega_\varepsilon. \tag{5.2.43}$$

Without any loss of the generality we can assume that two disjoint from 0 eigenvalues of $\mathsf{Hess}\, a$ are positive. Introducing appropriate local coordinates $z = (z'; z'') = (z_1, z_2; z'')$ in $\mathbb{R}^{2d}$ we obtain that $a - \tau = (z_1^2 + z_2^2 + V(z''))$ and then (5.2.23) holds for γ defined by (5.2.5); let us take $\bar\gamma = h^{\frac12-\delta}$ and consider a γ-admissible partition of unity. Then the arguments of the proof of Theorem 5.2.2 provide us with an appropriate estimate in the zone $\{\gamma \ge c_0\varepsilon\}$ and we need to consider only the zone $\{|z'| \le \varepsilon\} \times \Omega''$.

Let us cover Ω'' with balls of radius $\gamma_1 = h^{\delta_1}$ with $\delta_1 < \delta$ and consider balls of different types:

(i) Let $|\mathsf{Hess}\, V| \ge c_0\gamma_1$ at some point $z'' \in \Omega$. Then this remains true in $B(z'', \gamma_1)$ (with c_0 replaced by $\frac12 c_0$) and

$$\mathsf{mes}\big(\Omega''_\varepsilon \cap B(z'', \gamma_1)\big) \le \mathsf{mes}\big(B(z'', \gamma_1)\big)h^{\delta'}, \quad \delta' > 0. \tag{5.2.44}$$

Then the total contribution of these balls to the remainder does not exceed $C'h^{1-d+\delta'}\vartheta\big(h/L\big)$; thus we need to consider only balls with $|\mathsf{Hess}\, V| \le \gamma_1$.

Moreover, the same arguments combined with the induction with respect to m yield that one need only treat balls with $\sum_{2\le|\alpha|\le m} |D^\alpha V| \le c_0\gamma_1$ with arbitrarily large m.

Now as V is almost constant in the balls in question we need to consider $\{z_1, z_2\}$ which is either rather small (less than ω) or has a constant magnitude ω in balls in question; then in both cases classical dynamics could be periodic but its period is $\gtrsim \omega^{-1}$. We exploit this in a bit different way.

(ii) Let us consider a ball with $|\{z_1, z_2\}| \le c\omega$ where $\omega \ge h^{\delta_0}$. This inequality holds at all points of this ball simultaneously. Then in the appropriate *symplectic* coordinates

$$(a - \bar\tau) = \xi_1^2 + (\xi_2 + b_1(x,\xi))^2 + b_2(x,\xi) \tag{5.2.45}$$

where $|\nabla b_1| \le c\omega$ and $|\mathsf{Hess}\, b| \le c\omega^2$ in this ball.

Let us redefine the scaling functions γ and ρ in the ball $B(z'', \gamma_1)$. Namely, pick

$$\begin{aligned} &\rho := \epsilon\Big(|\xi_1| + |\xi_2 + b_1| + \big(|b_2 - \tau| + \omega^{-2}|\nabla b_2|^2\big)^{\frac12}\Big) + \bar\rho', \\ &\gamma := \rho\omega^{-1}, \qquad \bar\rho' = h^{\frac12}\omega^{\frac12}. \end{aligned} \tag{5.2.46}$$

Then $|\nabla\rho| \le \frac{1}{2}$ (due to the condition on the Hessian) and $\rho\gamma \ge h$.

Moreover, one can check that

$$\mathsf{dist}_x(z',z'') \le \gamma(z''),\ \mathsf{dist}_\xi(z,z'') \le \rho(z'') \implies \rho(z') \asymp \rho(z''),\ \gamma(z') \asymp \rho(z''). \tag{5.2.47}$$

Let us rescale $x_{\mathsf{new}} = x\gamma^{-1}$, $\xi_{\mathsf{new}} = \xi\rho^{-1}$. Then we should replace h by $h_{\mathsf{new}} = h\rho^{-1}\gamma^{-1}$ and L by $L_{\mathsf{new}} = L\rho^{-2}$ because we divide A by ρ^2 now. Obviously,

$$\frac{h_{\mathsf{new}}}{L_{\mathsf{new}}} = \frac{h}{L}\omega. \tag{5.2.48}$$

One can easily check that after this rescaling the derivatives (up to order K) of the symbol of the operator are bounded.

Surely, the symbol itself is not necessary bounded because of the term $a^s h\rho^{-2}$ but in this case one can introduce operator $B = a^s(\bar z)h\rho^{-2}$ where $\bar z$ is a fixed point in the original ball of radius γ_1.

For a scalar symbol a^s one can avoid this difficulty simply by replacing a with $a + ha^s$ in the definitions of γ and ρ.

Moreover, the microhyperbolicity condition is fulfilled for $\rho \ge c_0\bar\rho'$ [10)].

Then Theorem 4.5.12 provides all the necessary estimates in this zone and its total contribution to $\mathsf{R}_{\varphi,L}$ surely does not exceed the right-hand expression of estimate (5.2.41) with ω instead of $|\{z_1,z_2\}|$.

Further, Theorem 5.2.2 [11)] applies in the domain $\rho \le \bar\rho'$, the total contribution to $\mathsf{R}^{\mathsf{W}}_{\varphi,L}$ of which does not exceed

$$C'\bar\rho'^{2-d}\bar\gamma'^{-d}\vartheta\Big(\frac{h}{L}\Big) \le C'h^{1-d}\omega\vartheta\Big(\frac{h}{L}\Big).$$

Therefore the contribution of balls with $\omega \le h^{\delta_0}$ is estimated properly.

[10)] According to our procedures, one should check this condition for the symbol $a + a^s h$ after rescaling, but the derivatives of $a^s h$ are negligible, so one may check the condition for the scalar symbol a instead.

[11)] With obvious modifications for matrix a^s.

(iii) Finally, let us consider balls of radius $\gamma_2 = h^{\frac{1}{2}-\delta_2}$ where $|\mathsf{Hess}\, V| \le c_0\gamma_1$ and $|\{z_1, z_2\}| \asymp \omega \ge h^{\delta_0}$. In these balls the variation of b_2 is less than $h^{1+\delta_3}$ and we simply include $h^{-1}b_2$ in a^s. Without any loss of the generality one can assume that in the ball, $h^{-1}b_2 + a^s - \tau = \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix}$ where $B_+ \ge \nu I$, $B_- < (1-\epsilon)\nu I$ and $\epsilon''\omega \le \nu \le \epsilon'\omega$ with small enough positive constants $\epsilon, \epsilon', \epsilon''$. Then one can easily see that

$$\int_{\Sigma'_\tau \cap B(z,\gamma_2)} d\mu'_\tau \succ \pi_- \gamma_2^{2-2d}$$

where $\pi_- = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$.

Let us consider $A_+ = Z_1^2 + Z_2^2 + B_+$. Then Gårding inequality yields that

$$\begin{aligned}(A_+u, u) \ge \|Z_1u\|^2 + \|Z_2u\|^2 - C_0h^{1+2\delta_0}\|u\|^2 - C\|\mathcal{Q}u\|^2 \ge \\ \frac{1}{2}(\|Z_1u\|^2 + \|Z_2u\|^2 + |([Z_1, Z_2]u, u)|) - C_0h^{1+2\delta_0}\|u\|^2 - C\|\mathcal{Q}u\|^2 \ge \\ \epsilon(\|Z_1u\|^2 + \|Z_2u\|^2 + \omega h\|u\|^2) - C\|\mathcal{Q}u\|^2\end{aligned}$$

where $\mathcal{Q}$ is negligible in the ball in question. Therefore for $\tau' \le \tau$ the operator $A_+ - \tau'$ is invertible in this ball and the norm of the inverse does not exceed $Ch^{-1}\omega^{-1}$.

Therefore $F_{t\to h^{-1}\tau}\chi_\tau(t)(Q_1u^{++})$ is negligible for $\tau \le \epsilon\omega h$, $T \ge h^{-1-\delta'''}\omega^{-1}$ with an arbitrarily small exponent $\delta''' > 0$ and simplified Tauberian arguments yield that $Q_1E(\tau', \tau)$ is negligible for $\tau', \tau'' \le \tau + \epsilon\omega h$. This and estimates from below of the right-hand expressions of (5.2.42) yield that (5.2.42) holds. □

From the proof above immediately follows

Theorem 5.2.7. *Let condition* $(5.2.20)_2^+$ *be fulfilled. Then* $\mathsf{R}^{\mathsf{W}}_{\varphi,L,\mathsf{reg}}$ *is dominated by* (5.2.42) *where now regularization also means that* $a^s h$ *is included in the principal symbol while calculating coefficients.*

Remark 5.2.8. (i) Obviously $\mu_\tau(U)$ may be not continuous function; in particular, it has no-zero jump at $\bar\tau$ as $a = z_1^2 + z_2^2 + \bar\tau$; however in the case (iii) which is the only case when this really matters the bottom of the spectrum is actually pushed up and it helps; otherwise we would need to replace in the right-hand expression τ by $\tau + C_h$.

(ii) Theorem 5.2.7 clearly provides improvement for example for 2-dimensional Schrödinger operator without magnetic field but it provides improvement neither for 2-dimensional magnetic Schrödinger operator

$$h^2D_1^2 + h^2(D_2 - x_1)^2 \tag{5.2.49}$$

nor for 1-dimensional Schrödinger operator

$$h^2D_1^2 + h^2x_1^2; \tag{5.2.50}$$

these operators are actually equivalent.

In particular, Proposition 5.2.4 immediately implies the final Theorem 5.2.9(ii) in the full generality.

Sharp and Sharper Asymptotics (End)

So, we can apply Theorem 5.2.7 in the near-critical zone. In particular we get

Theorem 5.2.9. *Assume that* $\mu_{\bar\tau}(\Pi \cap \Omega^+) = 0$. *Then*

(i) Under condition $(5.2.20)_3$

$$\mathsf{R}^{\mathsf{W}}_{\varphi,L,\mathsf{reg}}(\bar\tau) = o\Big(h^{1-d}\vartheta\big(\frac{h}{L}\big)\Big). \tag{5.2.51}$$

(ii) Let condition $(5.2.20)_2^+$ *be fulfilled. Then as as*

$$\mathsf{mes}(\{V = \bar\tau,\ \{z_1, z_2\} \neq 0\}) = 0 \tag{5.2.52}$$

estimate (5.2.51) *holds.*

5.2.2 From Condition "No Critical Point" to "rank Hess $a \geq 2$ at cCritical Points. " II

Now we begin treating the more complicated case in which condition $(5.2.20)_2$ is fulfilled but condition $(5.2.20)_2^+$ fails. We want to obtain remainder estimate $O(h^{1-d}\vartheta(h/L))$ in this case. Without any loss of the generality one can assume that $(5.2.20)_2^-$ $|a(x,\xi) - \tau| + |\nabla_{x,\xi}a(x,\xi)| \leq \epsilon_0 \implies \mathsf{Hess}\, a(x,\xi)$

has two eigenvalues (counting multiplicities) f_1, f_2 such that $f_1 \ge \epsilon_0, -f_2 \ge \epsilon_0$ [6]. Then in the appropriate coordinates locally

$$a - \tau = z_1 z_2 + V(z'') \tag{5.2.53}$$

with $z'' = (z_3, \dots, z_{2d})$.

The goal of this Subsection is

Theorem 5.2.10. *Let condition and* $(5.2.20)_2^-$ *be fulfilled. Let* $\operatorname{diam} \Omega \le c$. *Then* $\mathsf{R}^{\mathsf{W}}_{\varphi,L,\mathsf{reg}}$ *is dominated by* $Ch^{1-d}\vartheta(h/L)$.

Proof. Let us observe first that for fixed z'' hyperbole $\Sigma_{\tau,z''} = \{(z_1, z_2) : (z_1, z_2, z'') \in \Sigma_\tau\}$ is parametrized by $z_1 \pm z_2$ as $\pm V(z'') < 0$ and

$$\int_{\Sigma_{\tau,z''} \cap \{|z'| \ge h^{\frac12}\}} d\mu_{\tau,z''} \lesssim |\log\bigl(|V(0)| + h\bigr)|; \tag{5.2.54}$$

after integration with respect to z'' this may lead to the factor $|\log h|$ in the final estimate and our purpose is to fight it.

(a) First we consider zone where V is "variable enough".

Let us introduce the scaling functions

$$\zeta_l(z'') = \epsilon\Bigl(\sum_{|\alpha|\le l} |D^\alpha V|^{\frac{l+1}{l+1-|\alpha|}}\Bigr)^{\frac{1}{l+1}} + \bar\zeta_l, \qquad \bar\zeta_l = h^{\frac{2}{l+3}} \tag{5.2.55}$$

with $l \ge 1$. Then

$$|\nabla \zeta_l| \le \frac12 \tag{5.2.56}$$

as $\epsilon > 0$ is small enough constant and

$$\gamma(z) \asymp |z_1| + |z_2| + \zeta_1(z'') \tag{5.2.57}$$

and it trivially follows from the arguments of the previous Subsection 5.2.1 [12] that

(5.2.58) The contribution of the zone $\Omega \cap \{\gamma(z) \le C_0 h^{\frac12}\}$ to the remainder does not exceed $Ch^{1-d} \operatorname{mes}\bigl(\{z'' \in \Omega'' : \zeta_1(z'') \le C_1 h^{\frac12}\}\bigr)\vartheta(h/L)$

[12] One needs to consider balls in $\mathbb{R}^{2d}$ of radius $\zeta_1(z)$ for each z.

and

(5.2.59) The contribution of the zone $\Omega\cap\{\gamma(z) \le C_0\zeta_1(z'')\}$ to the remainder estimate does not exceed $Ch^{1-d}\vartheta(h/L)$

where $\Omega = \{(z', z'') : |z'| \le \epsilon, z'' \in \Omega''\}$ and domain Ω'' is fixed.

Moreover, the same arguments yield that

(5.2.60) The contribution of the zone $\Omega \cap \{\gamma(z) \le \varrho(z'')\}$ to the remainder estimate does not exceed

$$Ch^{1-d}\vartheta\big(\frac{h}{L}\big) \int_\Omega \big(\log \frac{\varrho(z'')}{\zeta_1(z'')} + 1\big)\, dz'' \tag{5.2.61}$$

provided $\varrho(z'')$ satisfies conditions $\varrho(z'') \ge \zeta_1(z'')$ and $|\nabla\varrho| \le 1$.

Denote by J_l the integral in (5.2.61) with

$$\varrho_l := \sum_{1\le k\le l} \zeta_k^{p_k}, \qquad p_k = 2^{k-1}. \tag{5.2.62}$$

Let us prove by induction with respect to $l \ge 1$ that

(5.2.63) J_l does not exceed C_l.

For $l = 1$ this is obvious. Let us assume that it is true for some l; to make an induction in (5.2.63) we should prove that

$$J'_l := \int \log\big(\frac{\varrho_l(z'')}{\varrho_{l-1}(z'')}\big)\, dz'' \tag{5.2.64}$$

is bounded.

Let us consider a ζ-admissible covering of Ω'' (with $\zeta = \zeta_l$) and let us consider the contribution of each element to J'_l. Surely, one needs to consider only elements $B\big(z'', \zeta(z'')\big)$ with $\varrho_l \ge C_2\varrho_{l-1}$ with arbitrarily large constant C_2. The latter inequality means that $\zeta_k \le \epsilon\zeta^{p_l/p_k}$ i. e.

$$|D^\alpha V| \le \epsilon_2\zeta^{2^{l-k}(k-|\alpha|+1)} \qquad \forall k : |\alpha| \le k \le l-1 \tag{5.2.65}$$

with arbitrarily small constant $\epsilon_2 > 0$. On the other hand,

$$|D^\alpha V| \le c\zeta^{l-|\alpha|+1} \qquad \forall \alpha : |\alpha| \le l \tag{5.2.66}$$

and for some $\alpha = \bar\alpha$ with $|\alpha| \le l$ the opposite inequality holds

$$|D^\alpha V| \ge \epsilon_0 \zeta^{l-|\alpha|+1} \qquad \text{as} \quad \alpha = \bar\alpha. \tag{5.2.67}$$

Picking in (5.2.65) $k = l-1$ we get inequality

$$|D^\alpha V| \le \epsilon_2 \zeta^{2(l-|\alpha|)} \qquad \forall \alpha : |\alpha| \le l-1$$

which is stronger than (5.2.66) and incompatible with (5.2.67) since ϵ_0 is fixed and ϵ_2 is arbitrarily small; thus (5.2.67) holds with some $\alpha = \bar\alpha : |\bar\alpha| = l$.

Meanwhile (5.2.64) can only increase if we replace ϱ_{l-1} by the lesser value $\zeta_{l-1}^{p_{l-1}}$ where ζ_{l-1} in turn is greater than

$$\eta \coloneqq \sum_{\alpha:|\alpha|=l-1} |D^\alpha V| \tag{5.2.68}$$

while $\varrho_l \varrho_{l-1} \asymp 1 + \zeta^{p_l} \varrho_{l-1}^{-1}$.

Thus contribution of $B(z', \zeta(z'))$ to (5.2.64) does not exceed

$$C_0 \int_{B(z',\zeta(z'))} \log\bigl(\frac{\zeta^l}{\eta}\bigr)\, dz'. \tag{5.2.69}$$

Rescaling $y = z'/\zeta$ and introducing function $W(y) = V(\zeta y)\zeta^{-l}$ we see that (5.2.69) does not exceed $C\zeta^{2d-2} \int |\log \phi(y)|\, dy$ with $\phi(y) = \sum_{\alpha:|\alpha|=l-1} |D_y^\alpha W|$ and

(5.2.70) For some β with $|\beta| = l$, $|D_y^\beta W| \ge \epsilon_0$.

Therefore it is sufficient to prove that $\int |\log \phi(y)|\, dy \le C$ but this is obvious because in the appropriate coordinate system $\phi(y) \ge |y_1|$ (due to (5.2.70)). Thus, inequality $J_l' \le C$ is proven, the induction step is complete and (5.2.63) is also proven.

As a corollary we obtain that

(5.2.71) The contribution of the zone

$$\{z=(z',z''):\, z''\in\Omega, \gamma(z)\le \varrho_l(z')=\sum_{1\le k\le l}\zeta_k^{p_k},\quad p_k=2^{k-1}\}$$

to the remainder does not exceed $Ch^{1-d}\vartheta(h/L)$.

Thus, we have estimated properly the contribution of the zone $\{\varrho_l\ge\epsilon\gamma\}$; so far the non-covered zone is $\{\zeta_k\le\gamma^{1/p_k}\ \forall k\le l\}$ or equivalently

$$\{|D^\alpha V|\le \gamma^{(k-|\alpha|+1)/2^{k-1}}\quad \forall\alpha,k:|\alpha|\le k\le l\}.$$

Picking the optimal value $k=|\alpha|$ we get this zone

$$\{(z_1,z_2,z''):|D^\alpha V|\le c\gamma^{2^{1-|\alpha|}}\ \forall\alpha:|\alpha|\le l,\ \gamma\asymp |z_1|+|z_2|\} \tag{5.2.72}$$

with arbitrarily large l.

(b) Now we want to estimate the contribution of the larger (than (5.2.72)) zone defined by

$$|D^\alpha V|\le c\gamma^\delta\quad \forall\alpha:|\alpha|=2,\qquad \gamma\asymp |z_1|+|z_2| \tag{5.2.73}$$

with arbitrarily small $\delta>0$ (as $l=2$ we need to do it for $\delta=\frac12$).

First of all we consider the part of this zone with

$$|\nabla V|\ge \gamma^{1+\delta_1} \tag{5.2.74}$$

with $\delta_1<\delta$. Let us consider a point $\bar z''$ satisfying both inequalities (5.2.73) and (5.2.74). Without any loss of the generality one can assume that all the components of $\nabla V(\bar z'')$ vanish excluding $\sigma=\partial_{z_3}V$. Then in the ν-vicinity $\mathcal{U}$ of $(0,\bar z'')$

$$c^{-1}\le|\nabla V|\le c\sigma,\quad V(z)=V(\bar z'')+\sigma(z_3-\bar z_3)+O(\nu^2\gamma^\delta) \tag{5.2.75}$$

provided $\nu\le\min(\sigma\gamma^{-\delta},\gamma^\delta)$.

Let $\mathcal{V}_1=\mathcal{V}\cap\{z'':|V(z'')|\le c\gamma^2\}$. One can see easily that due to (5.2.75)

$$\mathsf{mes}(\mathcal{V}_1)\nu^{2-2d}\le\gamma^2\sigma^{-1}\nu^{-1}\le\gamma^{\delta_2} \tag{5.2.76}$$

with $\delta_2>0$ provided $\nu\sigma\ge\gamma^{2+\delta_2}$, $\nu\ge\gamma^{1-\delta_2}$. Obviously conditions to ν are not contradictory for small enough $\delta_2>0$, $\delta_1>0$.

Therefore

(5.2.77) For any ε the contribution of the zone

$$\{z : \frac{1}{2}\varepsilon \le \gamma(z) \le 2\varepsilon,\ |\nabla^2 V| \le \gamma^{\delta},\ |\nabla V| \ge \gamma^{1+\delta'}\}$$

to the remainder does not exceed $Ch^{1-d}\varepsilon^{\delta_2}\vartheta(h/L)$ and therefore the total contribution of the zone in question does not exceed $Ch^{1-d}\vartheta(h/L)$.

So, we should consider only zone where inequalities (5.2.73) and

$$|\nabla V| \le \gamma^{1+\delta} \tag{5.2.78}$$

hold.

In this zone we apply Theorem 4.5.12 after rescaling: so we use the long term propagation of singularities.

(c) In this part of the proof we establish that

(5.2.79) Contribution of γ-admissible element on which

$$\omega := |\{z_1, z_2\}| \le \gamma^{\delta_1} \tag{5.2.80}$$

to $\mathsf{R}^{\mathsf{W}}_{\varphi,L,\mathsf{reg}}$ does not exceed

$$Ch^{1-d}\gamma^{2d-2}\Big(\gamma^{\delta_2} + h^{\delta_2}\gamma^{-2\delta_2}\Big)\vartheta\big(\frac{h}{L}\big). \tag{5.2.81}$$

To prove (5.2.79) let us consider $\mathcal{V}$ which is $\epsilon_0\gamma$-vicinity with respect to z' and $\gamma^{1-\delta_3}$-vicinity with respect to z''.

Due to (5.2.77) we can assume that both (5.2.78) and (5.2.80) hold. Then in $\mathcal{V}$ the propagation speed with respect to z' does not exceed $c\omega\gamma + c\gamma^{1+\delta} \le c\gamma^{1+\delta_1}$ while the propagation speed with respect to z'' does not exceed $c\gamma$. Therefore for time $T = \gamma^{-\delta_2}$ Hamiltonian trajectories remain in $\mathcal{V}$.

Moreover, for infinitesimal variations we obtain the inequality $|\frac{d}{dt}\mathfrak{d}z| \le C\gamma^{\delta}|\mathfrak{d}z|$ and therefore $|\mathfrak{d}z(t)| \le C|\mathfrak{d}z(0)|$. Hence, $|D\Psi_t| \le C$.

Finally, let us check the non-periodicity condition for $|t| \le T$. One should consider only initial points with $|\bar{z}_1| \ge \gamma$ (or with the same inequality for $\bar{z}_2$). Obviously there exists a linear function ψ such that $\{\psi, z_1\} = \{\psi, V\} = 0$

and $\{\psi, z_2\} = 1$ at $\bar{z}$ in which case $\frac{d}{dt}\psi = z_1 + O(\gamma^{1+\delta})$ and therefore $\psi(z(t)) = \psi(z(0)) + z_1(0)t + O(\gamma^{1+\delta_1})$ and the non-periodicity condition is fulfilled as well.

Thus (5.2.79) is proven. It immediately implies that the total contribution of zone defined by (5.2.80) and (5.2.78) to the remainder does not exceed $Ch^{1-d}\vartheta(h/L)$ and we arrive to

(5.2.82) The total contribution of zone defined by (5.2.80) to the remainder does not exceed $Ch^{1-d}\vartheta(h/L)$.

(d) So, we should now consider only points satisfying (5.2.72), (5.2.73) and

$$\omega = |\{z_1, z_2\}| \ge \gamma^{\delta_2} \tag{5.2.83}$$

where one can choose $\delta_2 > 0$ arbitrarily.

Consider $\epsilon\omega(0, z'') + h^{\frac{1}{2}}$ as a scaling function and a corresponding partition. Now we need to prove that

(5.2.84) As $\hat{z} = (0, \hat{z}'')$ contribution to $\mathsf{R}^{\mathsf{W}}_{\varphi,L}$ of η admissible element $B(\hat{z}, \eta)$ does not exceed $Ch^{1-d}\eta^{2d-2}\vartheta(h/L)$ with $\eta := \epsilon\omega(\hat{z})$.

Let us again consider the Hamiltonian flow (which will be hyperbolic here); to be definite we assume that $\{z_1, z_2\} > 0$. We consider point $\bar{z}$ and trace the trajectory $z(t)$ with $z(0) = \bar{z}$ as long as it remains in $\mathcal{V}$ which is η vicinity of $\hat{z}$ with respect to z'' and $\epsilon_1\eta^2$ vicinity with respect to z'; due to (5.2.83) $\bar{z} \in \mathcal{V}$.

Recall that one can choose the time direction arbitrarily. Therefore without any loss of the generality one can assume that $|\bar{z}_2| \asymp \gamma \gg \bar{z}_1$ because $|z_1| \asymp |z_2|$ implies that $\gamma \asymp \zeta_1$ of Part (a). Note that due to (5.2.83) $\gamma \le \omega^M$ with arbitrarily large M and that $|\nabla V| \le \gamma^{1+\delta} \le \omega\gamma^{1+\delta_4}$.

So, consider Hamiltonian equation; then under extra assumptions just imposed, for time

$$T := \epsilon\omega^{-1}|\log\gamma| \ge C\omega^{-1}|\log\omega| \tag{5.2.85}$$

this trajectory remains in $\mathcal{V}$; and for positive time $T^* \le C\omega^{-1}|\log\gamma|$ it leaves $\mathcal{V}$ for sure through $|z_2| \asymp \omega^2$.

Let us try to look at this trajectory in the positive time direction; then considering equation to variations we conclude that $|\frac{d}{dt}\mathfrak{d}z| \le c\omega|\mathfrak{d}z|$ and therefore $|\mathfrak{d}z(t)| \le Ce^{c\omega|t|}|\mathfrak{d}z(0)|$. Hence, $|D\Psi_t| \le Ce^{c\omega|t|}$.

Finally, since $|z_2(t) - z_2(0)| \ge \epsilon_1\omega te^{\epsilon_1\omega t}$ we conclude that the non-periodicity condition is fulfilled as well. Thus we can take T defined by (5.2.85). Then the total contribution of the element in question to the remainder does not exceed $Ch^{1-d}J\omega^{2d-1}\vartheta(h/L)$ with

$$J = \int \gamma^{-1}|\log\gamma|^{-1}\,d\gamma; \tag{5.2.86}$$

this integral has loglog divergence and we can avoid it considering only $\gamma \le h^{\delta'}$ with arbitrarily small $\delta' > 0$; to be on the safe side we also assume that $\gamma \ge h^{\frac12-\delta'}$. So we arrive to

(5.2.87) Contribution of subzone described in (5.2.84) with extra condition $\{h^{\frac12-\delta'} \le \gamma \le h^{\delta'}\}$ does not exceed $Ch^{1-d}\omega^{2d-1}\vartheta(h/L)$.

Thus we need to reconsider two subzones: *outer subzone* $\{h^{\delta'} \le \gamma \le \omega^M\}$ and *inner core* $\{h^{\frac12} \le \gamma \le h^{\frac12-\delta'}\}$.

(e) To tackle the former subzone let us consider trajectories issued from $\bar z$ in the negative time direction (while before we considered positive time direction) as long as it remains in $\mathcal{V}$ but capping T by $\epsilon|\log h|$. Then it remains in the zone $\{|z_2| \ge h^{2\delta'}\}$ and $|d\Psi_t| \le h^{-3\delta'}$ and non-periodicity condition is fulfilled as well. Then the contribution of subzone in question does not exceed

$$\Big(Ch^{1-d}\int_{\Sigma_\tau\cap\mathcal{U}}\big(T_-(x,\xi)^{-1} + |\log h|^{-1}\big)\,dxd\xi : da + C''h^{1-d+\delta''}\omega^{2d-2}\Big)\vartheta\big(\frac{h}{L}\big) \tag{5.2.88}$$

where $\mathcal{U}$ is an element in question and we do not cap $T_-(z)$ anymore so $T_-(z) = \infty$ if the trajectory in question does not leave $\mathcal{V}$ for time $\ge \epsilon|\log h|$. Obviously $|\log h|^{-1}\int dxd\xi : da| \le C$ and we are left with

$$\int_{\Sigma_\tau} T_-(x,\xi)^{-1}dxd\xi : da; \tag{5.2.89}$$

and trajectories counted can leave $\mathcal{V}$ only through $|z_1| \asymp \omega^2$.

Since the measure $dxd\xi : da$ is invariant we conclude that integral (5.2.89) is of the same magnitude as if it was restricted to $\{\frac{1}{2}\omega^2 \le \gamma \le \omega^2\}$ and T_- was replaced by the time T' spent by the trajectory in this zone; this removes logarithmic divergency. So we arrive to

(5.2.90) Contribution of subzone described in (5.2.84) with extra condition $\{h^{\delta'} \le \gamma \le \omega^M\}$ does not exceed $Ch^{1-d}\omega^{2d-1}\vartheta(h/L)$.

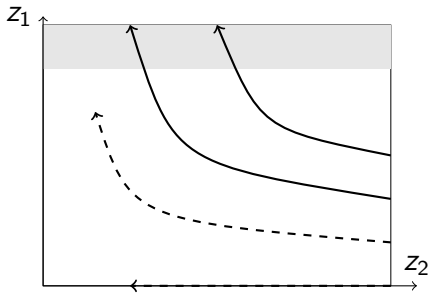


Figure 5.1: Solid trajectories leave $\mathcal{V}$ for a capped time; dashed trajectories do not and are not considered in (5.2.89); the shadow indicate zone $\gamma \asymp \omega^2$.

(f) In order to treat the remaining zone let us apply the routine rescaling $x_{\mathsf{new}} = (x - \bar{x})\bar{\gamma}^{-1}$ and $h_{\mathsf{new}} = \hbar := h\bar{\gamma}^{-2}$ with $\bar{\gamma} = h^{\frac{1}{2}-\delta'}$. We get an operator of the same type but with complete symbol $a = z_1z_2 + \bar{V} + ...$ where $\{z_1, z_2\} = \omega + ...$ where the dots denote symbols all the derivatives of which do not exceed $\hbar^M$ with arbitrarily large M and are therefore negligible, $\bar{V}$ and ω are constant and $\bar{V}$ is not necessarily a scalar; here $\omega \ge \hbar^{\delta_1}$ (otherwise we can apply rescaling and arrive into case already considered in Part (c) and only the contribution of the zone $h^{\frac{1}{2}}\gamma \le \hbar^{\frac{1}{2}-\delta''}$ needs to be treated (with $\gamma = \gamma_{\mathsf{new}} = \gamma_{\mathsf{old}}/\bar{\gamma}$ as well and δ_1 and δ_2 are new but still arbitrarily small). Now $\delta_0 = 0$.

Without any loss of the generality one can assume that $z_1 = hD_1$ and either $z_2 = \omega x_1$ or $z_2 = \omega x_1 + x_2$ (which can be arranged otherwise by an appropriate symplectic change of variables). Moreover, changing coordinates $x_{1\mathsf{new}} = x_1 + \omega^{-1}x_2$, $x'_{\mathsf{new}} = x'$ if necessary and multiplying the operator by ω^{-1} we get exactly the operator $x_1\hbar D_1 + \frac{i\hbar}{2} + \bar{V}_{\mathsf{new}}$ (modulo negligible) with $\bar{V}_{\mathsf{new}} = \omega^{-1}\bar{V}$.

All that remains to do is to consider points $(\bar{x}, \bar{\xi})$ with $\bar{\gamma} = |\bar{x}_1| \ge |\bar{\xi}_1|$ (points with the opposite inequality are treated similarly with the negative time direction) and to prove that one can take $T = \epsilon|\log\hbar|$. But for this very specific operator the proof is direct: one can consider operators of multiplication; then $e^{-it\hbar^{-1}A}\phi e^{it\hbar^{-1}A} \equiv \phi_t$ with $\phi_t(x) = \phi(e^{-t}x)$ and for the indicated $t \ge \epsilon$ we obtain that $\phi_t\phi = 0$ provided $\operatorname{supp}\phi$ is small enough.

This proves that the contribution of the element of the partition in question to the remainder estimate is $O\big(h^{1-d}\omega\gamma^{2d-2}|\log h|^{-1}\vartheta(h/L)\big)$ and the total contribution of such elements is $O\big(h^{1-d}\omega^{2d-1}\vartheta(h/L)\big)$ as well because factor $|\log h|^{-1}$ compensate for logarithmic divergence of the integral. Thus we arrive to

(5.2.91) Contribution to the remainder of the subzone described in (5.2.84) with extra condition $\{h^{\frac12} \le \gamma \le h^{1/2-\delta''}\}$ does not exceed $Ch^{1-d}\omega^{2d-1}\vartheta(h/L)$.

Now combining (5.2.87), (5.2.90) and (5.2.91) we obtain (5.2.84) which implies the proper estimate $Ch^{1-d}\vartheta(h/L)$ of the remaining zone.

Finally, Theorem 5.2.10 has been proven. □

Remark 5.2.11. (i) Thus in the inner zone $\{\gamma \le h^{\delta'}\}$ we considered trajectories moving from degeneration while in the outer zone we considered trajectories moving to degeneration.

(ii) We did not exploit that actually instead of factor $\vartheta(h/L)$ we had factor $\vartheta(h/LT)$.

(iii) Now regularization means that $a^s h$ must be included in the principal part; otherwise generally remainder estimate would contain logarithmic factor.

(iv) Assuming that $V = 0$ identically, one can use the canonical form of such operators. Namely, it was proven (for similar proofs see, e. g., Section 13.3) that in the framework of the current analysis with $\{z_1, z_2\} \asymp 1$

$$\mathcal{F}^*A\mathcal{F} \equiv \sum_{m,n:m+n\ge 1} a_{mn}(x', hD')(x_1 hD_1 - \frac{1}{2}ih)^m h^n \tag{5.2.92}$$

for an appropriate Fourier integral operator $\mathcal{F}$ such that in the vicinity of the point in question $\mathcal{F}^*\mathcal{F} \equiv I$.

(v) The most reasonable example is 1-dimensional Schrödinger operator

$$h^2D_1^2 - h^2x_1^2. \tag{5.2.93}$$

Actually we proved even more precise

Theorem 5.2.12. *Let condition* $(5.2.20)_2^-$ *be fulfilled. Then contribution of* Ω_ε *to the remainder does not exceed*

$$Ch^{1-d}\Big(\eta + \int_{\{\zeta_m<\eta\}} |\{z_1,z_2\}|\,dz'' + C'h^\delta\Big)\vartheta\big(\frac{h}{L}\big) \tag{5.2.94}$$

with $\eta = O(\varepsilon^\delta)$ *as* $\varepsilon \to +0$*, arbitrarily large* m *and* $\delta = \delta_m > 0$*.*

Proof. Consider ζ_m-admissible covering. Arguments of the proof of Theorem 5.2.10 imply that the total contribution of zone $\{\zeta_m \ge \varepsilon_1\}$ does not exceed $\big(\eta h^{1-d} + C''h^{1-d+\delta''}\big)\vartheta(h/L)$; therefore only zone $\{\zeta_m \le \varepsilon_1\}$ needs to be reexamined. Here ε_1 is an arbitrarily small constant.

Further, reexaming this proof one can notice that then only zone where $\{\omega > \varepsilon_1\}$ needs to be reconsidered and contribution of the latter zone does not exceed (5.2.94). □

This opens the door to more precise remainder estimates. In particular, we have

Theorem 5.2.13. *Assume that*

$$\mu_{\bar\tau}(\Pi \cap \Omega^+) = 0 \tag{5.2.95}$$

and

$$\mathsf{mes}(\{V = \bar\tau, \{z_1,z_2\} \ne 0\}) = 0. \tag{5.2.96}$$

Then estimate (5.2.51)

$$\mathsf{R}^{\mathsf{W}}_{\varphi,L,\mathsf{reg}}(\bar\tau) = o\Big(h^{1-d}\vartheta\big(\frac{h}{L}\big)\Big)$$

holds.

So, despite very essential differences between $(5.2.20)_2^+$ and $(5.2.20)_2^-$ cases, both estimates $O\big(h^{-d}\vartheta(h/L)\big)$ and $o\big(h^{-d}\vartheta(h/L)\big)$ are proven in the both of them; the latter under assumptions (5.2.95) and (5.2.96).

Our analysis under assumption $(5.2.20)_2$ is complete.

5.2.3 From Condition "rank Hess $a \geq 2$ at Critical Points" to "rank Hess $a \geq 1$ at Critical Points"

Now we will consider operators with stronger degenerations and we consider only scalar operators. We will indicate later what order of matrix symbol can be added. So, let us assume that

(5.2.97) A is a scalar operator[13].

In this subsection we assume that condition $(5.2.20)_1$ is fulfilled. Then locally in the appropriate coordinate system

$$a = \pm\bigl(z_1^2 + V(z')\bigr), \qquad z' = (z_2, \dots, z_{2d}) \tag{5.2.98}$$

where here and below a means the complete symbol of A [13]. However we need to pass to symplectic coordinates with $z_1 = \xi_1$.

Without any loss of the generality we can assume that $|\partial_{\xi_1} z_1| \geq \epsilon_0$ (locally) and then by Malgrange preparation theorem (Malgrange [1]) we rewrite (5.2.98) as

$$a = b(x,\xi)\bigl(\bigl(\xi_1 - \alpha(x,\xi')\bigr)^2 + V(x,\xi')\bigr).$$

Then after the symplectic transformation

$$\xi_{1\mathsf{new}} = \xi_1 - \alpha(x,\xi'),\ (x,\xi')_{\mathsf{new}} = \psi(x,\xi')$$

we obtain the same formula with $\alpha = 0$. Thus, without any loss of the generality one can assume that

$$a(x,\xi) = b(x,\xi)\bigl(\xi_1^2 + V(x,\xi')\bigr), \qquad b(x,\xi) \geq \epsilon_0 \tag{5.2.99}$$

where we changed the sign if needed. In what follows $z' = (x,\xi')$.

Theorem 5.2.14. *Let conditions* (5.2.98) *and* $(5.2.20)_1$ *be fulfilled. Then*

(i) If

$$\sum_{|\alpha|\leq m} |D^\alpha V| \geq \epsilon \tag{5.2.100}_m$$

with some $m \geq 2$ *then* $\mathsf{R}^{\mathsf{W}}_{\varphi,L}$ *is dominated by* $Ch^{1-d}\vartheta\bigl(h/L\bigr)$.

[13] In fact, in this subsection one needs to assume that this condition is fulfilled only modulo $O(h^2)$.

(ii) If

(5.2.101) $\vartheta(\tau)\tau^{-s}$ *is a monotone increasing function with some* $s>0$

then $\mathsf{R}^{\mathsf{W}}_{\varphi,L}$ *is dominated by* $Ch^{1-d}\vartheta\big(h/L\big)$ *as well.*

(iii) In the general case estimate

$$\mathsf{R}^{\mathsf{W}}_{\varphi,L}\le Ch^{1-d-\delta}\vartheta\big(\frac{h}{L}\big) \tag{5.2.102}$$

holds for arbitrarily small $\delta>0$.

In the smoothness conditions K *depends on* m, ϑ *and* $\delta>0$.

Proof, Part I. First of all, this theorem is proven under condition $(5.2.100)_m$ for $m=2$ (see Theorems 5.2.2 and 5.2.10). Let us assume that Statements (i), (ii) are proven for given $m=\bar m-1\ge 2$ and let us prove it for $m=\bar m$.

Let us introduce the function $\gamma=\zeta_{m-1}$ by formula (5.2.55):

$$\gamma\coloneqq\epsilon\Big(\sum_{|\alpha|\le m-1}|D^\alpha V|^{\frac{l}{m-|\alpha|}}\Big)^{\frac{1}{l+1}}+\bar\gamma_m,\qquad \bar\gamma_m=h^{\frac{2}{m+2}} \tag{5.2.103}$$

then $|\nabla\gamma|\le\frac12$ (so γ is a scaling function) and

$$|D^\alpha V|\le C\rho^2\gamma^{-|\alpha|}\qquad\forall\alpha:|\alpha|\le K \tag{5.2.104}$$

with

$$\rho=\gamma^{\frac{m}{2}}. \tag{5.2.105}$$

Let us consider the point $(\bar x,\bar\xi')$. Let $\mathcal{U}'$ be its γ-vicinity (with respect to (x,ξ')) with $\gamma=\gamma(\bar x,\bar\xi')\ge h^{2/(m+2)}$.

(a) Let us consider zone $\{|\xi_1|\ge C_0\rho\}$ with ρ defined by (5.2.105) and let

$$\mathcal{U}=\{(x,\xi):(x,\xi')\in\mathcal{U}',|\xi_1-\bar\xi_1|\le\frac12|\bar\xi_1|\}$$

with $|\bar\xi_1|\ge C_O\rho$.

Let us rescale $x_{\mathsf{new}} = (x - \bar{x})\gamma^{-1}$, multiply the operator by ρ^{-2} and introduce $h_{\mathsf{new}} = h\rho^{-1}\gamma^{-1}$ with redefined temporarily $\rho := |\bar{\xi}_1|$. Let us note that the ellipticity condition is fulfilled and

$$\hbar \le 1, \qquad h\gamma^{-2} \le h^{1-\frac{2}{m}} \tag{5.2.106}$$

due to our choice of $\bar{\gamma}$. Therefore contribution of $\mathcal{U}$ to the remainder does not exceed

$$R(\gamma,\rho,h) := C\vartheta\Big(\frac{h\rho}{L\gamma}\Big)\Big(\frac{h}{\rho\gamma}\Big)^{r-d+1}\Big(\frac{\gamma}{\rho}\Big)^{d-1} = Ch^{1-d}\vartheta\Big(\frac{h\rho}{L\gamma}\Big)\Big(\frac{h}{\rho\gamma}\Big)^{r}\gamma^{2d-2} \tag{5.2.107}$$

with arbitrarily large r; here factor $(\gamma/\rho)^{d-1}$ appears since $\mathcal{U}'$ is covered by this number of balls of radius ρ with ξ' and we plugged $L_{\mathsf{new}} = L\rho^{-2}$ and $h_{\mathsf{new}}/L_{\mathsf{new}} = h\rho/L\gamma$ into $\vartheta(.)$.

Observe that $R(\gamma,\rho,h)\rho^{r/2}$ is decaying function of ρ provided r is large enough. Therefore after summation on a $|\xi_1|$-admissible partition of unity in $\{(x,\xi) : (x,\xi') \in \mathcal{U}', |\xi_1| \ge c\rho\}$ we obtain that the contribution of this zone to the remainder estimate does not exceed CR as well with ρ restored to its (5.2.105) definition. Note that

$$\vartheta\Big(\frac{h\rho}{L\gamma}\Big) \le \vartheta\Big(\frac{h}{L}\Big)\Big(\frac{\rho}{\gamma}\Big)^{s} \tag{5.2.108}$$

with $s > 0$ under assumption (5.2.101) and $s = 0$ otherwise. Using (5.2.108) and (5.2.105) we arrive to

$$R \le Ch^{1-d}\vartheta\Big(\frac{h}{L}\Big)\Big(\frac{h}{\gamma^{\frac{1}{2}(m+2)}}\Big)^{r}\gamma^{\frac{1}{2}(m-2)s+2d-2}. \tag{5.2.109}$$

Summation with respect to $(2d-1)$-dimensional γ-admissible partition results in

$$Ch^{1-d}\vartheta\Big(\frac{h}{L}\Big)\int \Big(\frac{h}{\gamma^{\frac{1}{2}(m+2)}}\Big)^{r}\gamma^{\frac{1}{2}(m-2)s-1}\,dz'. \tag{5.2.110}$$

Note that under condition $(5.2.100)_m$ this integral is converging as $r > 0$.

(b) Now let us consider zone

$$\mathcal{U} = \{(x,\xi) : (x,\xi') \in \mathcal{U}', |\xi_1| \le c\rho|\}.$$

Let us rescale and multiply the operator in the same way as above. Let us note that if $\gamma \ge c\bar{\gamma}$ then condition $(5.2.100)_{m-1}$ is fulfilled[14] and the contribution of $\mathcal{U}$ to the remainder does not exceed (5.2.107) with $r = 0$; this we estimate by (5.2.109) with $r = 0$ and summation with respect to partition results in (5.2.110) with $r = 0$, and this integral is bounded under condition $(5.2.100)_m$ as $s > 0$.

(c) proof-5-2-14-c If $\gamma \le c\bar{\gamma}$ then $h_{\mathsf{new}} \asymp 1$ and one does not need to check condition $(5.2.100)_{m-1}$ and therefore the same arguments hold.

(d) So, under assumption $s > 0$ we can make the step of induction and $\mathsf{R}^{\mathsf{W}}_{\varphi,L}$ is properly estimated under condition $(5.2.100)_m$. Further, as $(m-2)s > 2$ integral (5.2.109) is bounded even without condition $(5.2.100)_m$ and therefore Statement (ii) of theorem is proven.

On the other hand, if $s = 0$ with each step of induction factor $|\log h|$ appears and we arrive to estimate (5.2.102) but with an extra factor $|\log h|^{m-3}$. Further, on the step without condition $(5.2.100)_m$ with large enough m factor γ^{-1} appears with $\gamma = h^{2/(m+2)}$. Therefore statement (iii) of theorem is proven as well. □

Proof, Part II. Now, our goal is to get rid of logarithmic factor for $s = 0$ under condition $(5.2.100)_m$. We will follow the proof of Theorem 5.2.10 with some modifications; to make comparison more clear we restart numeration of the parts of the proof.

(a) First of all, without any loss of the generality one can assume that

$$|\partial_{z_2}^m V| \ge \epsilon_0 \qquad (5.2.111)_m$$

[14] More precisely, this condition is fulfilled if $\gamma' \ge \epsilon\gamma$ where γ' is introduced by the same formula (5.2.103) with differentiation only on x. However before rescaling one can apply the symplectic transformation $(x_j,\xi_j) \to (\xi_j,-x_j)$ with some $j = 2,\dots,d$ if necessary and fulfill this condition.

(condition $(5.2.100)_m$ yields this condition after ϵ-partition and appropriate choice of coordinate system). Then Malgrange preparation theorem yields that

$$V = \beta(z')\big(z_2^m + W_1(z'')z_2^{m-1} + ... + W_m(z'')\big) \tag{5.2.112}$$

with $|\beta(z'')| \ge \epsilon$, $z'' = (z_3, ..., z_{2d})$. Without any loss of the generality one can assume that $W_1 = 0$; really we can achieve it by $z_2 \mapsto z_2 + \frac{1}{m}W_1$. Then on the induction step $\gamma_{m-1} = \epsilon_1|z_2| + \gamma'_{m-1}(z'')$ where

$$\gamma'_{m-1} = \epsilon\Big(\sum_{2\le k\le m}\ \sum_{|\alpha|\le(k-1)} |D^\alpha W_k|^{\frac{m}{k-|\alpha|}}\Big)^{\frac{1}{m}} \tag{5.2.113}_m$$

with $|\nabla\gamma'_m| \le \frac{1}{2}$ and one can see easily that due to the induction assumption

(5.2.114) Contribution of zone $\{|z_2| \le C_0\gamma'_m\}$ to $\mathcal{R}_{2\vartheta}$ does not exceed $Ch^{1-d}\vartheta(h/L)$.

Thus we need to consider only zone $\{|z_2| \ge C_0\gamma'_m\}$ which was the source of the logarithmic factor.

Now let us introduce

$$\zeta_{m-1,j} = \epsilon\Big(\sum_{2\le k\le m}\ \sum_{|\alpha|\le j(k-1)/m} |D^\alpha W_k|^{\frac{mj}{kj-|\alpha|m}}\Big)^{\frac{1}{j}} \tag{5.2.115}_{m,l}$$

with $j \ge m$; it coincides with $(5.2.113)_m$ as $j = m$. Here $j \in \mathbb{R}$, $j > 0$. Note that

(5.2.116) Conditions

$$\zeta_{m-1,j} \ge \epsilon \tag{5.2.117}_{m,j}$$

are essentially different for given m only for a monotone sequence $j_i \in \mathbb{Q}$ (which denotes the set of rational numbers), $j_i \to \infty$, $j_{i+1} - j_i \le 1$ $\forall i$.

Introducing ϱ_j in the same way as in the proof of Theorem 5.2.10, Part (a) and repeating relevant arguments we arrive to the proper estimate of zone $\{|z_2| \le C\rho_{m,l}\}$ with redefined

$$\gamma \coloneqq \big(|z_2|^m + |z_1|^2\big)^{1/m} \ge ch^{2/(m+2)} \tag{5.2.118}$$

and in particular of zone defined by

$$\sum_{|\alpha|+k=m} |D^\alpha W_k| \ge \epsilon\gamma^\delta$$

with some $\delta > 0$. Easy details are left to the reader.

(b) So, we need to consider zone where opposite inequality

$$\sum_{|\alpha|+k=m} |D^\alpha W_k| \le \epsilon\gamma^\delta \tag{5.2.119}$$

holds; this inequality is a new incarnation of (5.2.73) and repeating arguments of the proof of Theorem 5.2.10, Part (b) we reduce it to zone defined by

$$|D^\alpha W_k| \le C\gamma^{m+\delta-k-|\alpha|} \qquad \forall \alpha, k : |\alpha| + k \le m; \tag{5.2.120}$$

this is a new incarnation of (5.2.78); we use induction with respect to j to prove that if (5.2.120) is fulfilled for all α, k with $|\alpha| + k > m - j$ and some $\delta > 0$ and if (5.2.120) fails for some α, k with $|\alpha| + k = m - j$ and any $\delta > 0$ then (5.2.76) holds with $\mathcal{V}_1 = \mathcal{V} \cap \{z'' : |V(z'')| \le c\gamma^{(m+2)/2}\}$. So, $\delta = \delta_j$ is decreasing but we take the smallest one in the end.

(c) So, we need to consider zone where (5.2.120) and (5.2.80) hold. Recall that

$$a = \pm\big(z_1^2 + \beta(z')z_2^m + W(z')\big) \tag{5.2.121}$$

where

$$|D^\alpha W| \le c\gamma^{m+\delta-|\alpha|}. \tag{5.2.122}$$

As $\beta > 0$ and m is even proof is no different from one under condition $(5.2.20)_2^+$. As m is odd this is true for subzone where $\beta z_2^m > 0$.

So we need to consider the remaining cases: <u>either</u> even m and $\beta < 0$ <u>or</u> odd m, $\beta < 0$ and $z_2 > 0$ [15)]. Consider

$$w_1 = z_1 - (-\beta)^{1/2} z_2^{m/2}, \ w_1 = z_2 + (-\beta)^{1/2} z_2^{m/2} \tag{5.2.123}$$

[15)] One can reduce the case of odd m, $\beta > 0$ and $z_2 < 0$ to this one replacing z_2, β by $-z_2$, $-\beta$.

which will play role of z_1 and z_2. Then

$$|w_1| + |w_2| \asymp \gamma^{m/2}, \tag{5.2.124}$$

$$\{w_1, w_2\} = 2\omega(-\beta)^{1/2} z_2^{(m-2)/2} + O(z_2^{m/2}), \qquad \omega = \{z_1, z_2\} \tag{5.2.125}$$

and we need to rescale time $t = t_{\mathsf{new}} = t_{\mathsf{old}}\gamma^{(2-m)/2}$.

Then analysis in zone $\{|\omega| \le \gamma^{\delta_2}\}$ is exactly as it was in the proof of Theorem 5.2.10(c); as before $\omega = \{z_1, z_2\}$.

(d) Finally analysis in zone where $|\omega| \ge \gamma^{\delta_2}$ repeats arguments of the proof of Theorem 5.2.10 (d)–(f) but with a twist: we always consider trajectory launched in the direction of decreasing $\gamma(z(t))$.

Indeed, because propagation slows down near $w = 0$ faster than before we can cap time by $T = \epsilon(\gamma/\bar\gamma)^{\delta_2}$ with $\delta_2 > 0$ rather than by $T = \epsilon|\log\gamma/\bar\gamma|$ as we did then; now $\int_{\Sigma_\tau} T^{-1}dxd\xi$ is bounded (instead of possible loglog divergence) and the same arguments of that proof Part (e) work in the whole zone $\{\bar\gamma \le \gamma \le \epsilon\}$. □

In the case $V = \mathsf{const}$ (treated as a parameter) one can divide the operator by $\rho^2 = \max(|V|, h^2)$ and introduce $h_{\mathsf{new}} = h\rho^{-1}$ (no rescaling with respect to x is necessary). Then the remainder estimate $O\big(h^{1-d}\vartheta(h/L)\big)$ holds. Therefore we arrive to

Conjecture 5.2.15. *For operators with analytic symbols the remainder estimate $O\big(h^{1-d}\vartheta(h/L)\big)$ holds under condition $(5.2.20)_1$ even without condition (5.2.101).*

5.2.4 From Condition "rank Hess $a \ge 1$ at Critical Points" to a Weak Non-Degeneracy cCondition

Now even condition "$\mathsf{rank}\,\mathsf{Hess}\, a \ge 1$" seems to be too restrictive for us and we are going to get rid of it.

Theorem 5.2.16. *Let A be a scalar operator. Then*

(i) Under assumption

$$\sum_{|\alpha|\le n} |D^\alpha a| \ge \epsilon_0 \tag{5.2.126}_n$$

$\mathsf{R}^{\mathsf{W}}_{\varphi,L}$ *is dominated by $C\vartheta(h/L)h^{1-d-\delta}$ with arbitrarily small $\delta > 0$.*

(ii) Under condition (5.2.101) *with* $s > 0$ $\mathsf{R}^{\mathsf{W}}_{\varphi,L}$ *is dominated by* $C\vartheta(h/L)h^{1-d}$.

Proof. To prove both Statements (i) and (ii) we apply induction on n. Note that for $n = 2$ theorem is already proven.

(a) So suppose that under condition $(5.2.126)_{n-1}$ both Statements (i) and (ii) are already proven and let us consider some point $\bar{z}$ at which condition $(5.2.126)_n$ holds but $(5.2.126)_{n-1}$ fails.

Then in the appropriate symplectic coordinates $|\partial^n_{\xi_1} a(\bar{z})| \ge \epsilon_0$ and then according to Malgrange preparatory theorem in the vicinity of $\bar{z}$

$$a(x,\xi) = b(x,\xi)\Big(\sum_{0\le j\le n} \xi_1^{n-j} V_j(x,\xi')\Big) \qquad \text{with} \quad |b(x,\xi)| \ge \epsilon_0 \tag{5.2.127}$$

and with

$$V_0 = 1, \qquad V_1 = 0. \tag{5.2.128}$$

Now instead of a single function V we have several functions V_j and we want to prove remainder estimate (5.2.126) under an extra condition

$$\sum_{2\le j\le n,\ |\alpha|\le lj/n} |D^\alpha V_j| \ge \epsilon_0 \tag{5.2.129}_{n,l}$$

with $l \ge n$; later we will drop it for large ν.

First let us treat the case $l = n$. Let us introduce the functions

$$\gamma := \gamma_n = \epsilon\Big(\sum_{2\le j\le n,\ |\alpha|\le j-1} |D^\alpha V_j|^{\frac{n}{j-|\alpha|}}\Big)^{\frac{1}{n}} + \bar{\gamma} \qquad \text{with} \quad \bar{\gamma} = h^{\frac{1}{2}} \tag{5.2.130}_n$$

and $\rho = \gamma$. Then $|\nabla\gamma| \le \frac{1}{2}$ for small enough $\epsilon > 0$ (so γ is a scaling function) and

$$|D^\beta V_j| \le c\rho^j \gamma^{-|\beta|} \qquad \forall j, \beta. \tag{5.2.131}$$

Let us rescale as usual $x_{\mathsf{new}} = (x - \bar{x})\gamma^{-1}$, multiply the operator by ρ^{-n} and introduce $h_{\mathsf{new}} = h\rho^{-1}\gamma^{-1}$. Then for $\gamma \ge c_0\bar{\gamma}$ after this procedure the smoothness conditions and condition $(5.2.126)_{n-1}$ are fulfilled and hence the contribution of the zone $\{z :\ z' \in B\big(\bar{z}', \gamma_n(\bar{z}')\big), |\xi_1| \le c\rho\}$ to the remainder does not exceed

$$C\Big(\frac{h}{\rho\gamma}\Big)^{-d+1-\delta} \vartheta\Big(\frac{h\rho^{n-1}}{L\gamma}\Big) \tag{5.2.132}$$

where $\delta = 0$ under condition (5.2.101) with $s > 0$ and $\delta = 0$ otherwise; later ρ will be different from γ. We used the fact that $L_{\mathsf{new}} = L\rho^{-n}$ due to multiplication of operator and therefore

$$\frac{h_{\mathsf{new}}}{L_{\mathsf{new}}} = \frac{h\rho^{n-1}}{L\gamma} \tag{5.2.133}$$

Therefore the total contribution of the zone $\{z : |\xi_1| \le c\rho, \rho\gamma \ge h\}$ to the remainder does not exceed

$$Ch^{1-d-\delta} \int \gamma^{-1+\delta} \vartheta\big(\frac{h\rho^{n-1}}{L\gamma}\big)\, dz'. \tag{5.2.134}$$

Meanwhile for $\rho\gamma \asymp h, |\xi_1| \le c\rho$ (i. e. for $\rho \asymp \bar\rho$ and $\gamma \asymp \bar\gamma$) we obtain $h_{\mathsf{new}} = 1$ and the contribution of the corresponding element of the partition to the remainder estimate does not exceed $C\vartheta(h\rho^{n-1}/L\gamma)$ and the total contribution of zone $\{\rho\gamma \asymp h, |z_1| \le c\rho\}$ to the remainder does not exceed (5.2.134) with integral restricted to his zone.

Finally, in the zone $\{|\xi_1| \ge c\rho\}$ we apply an $\epsilon|\xi_1|$-admissible partition and standard elliptic arguments and then the contribution of each element does not exceed $C(h/\xi_1^2)^r \vartheta(h\xi_1^{n-2}/L)$ with arbitrarily large r and therefore the total contribution does not exceed (5.2.134).

(b) Let us notice that under condition $(5.2.129)_{n,n}$ in the appropriate coordinate system $\gamma \ge \epsilon'|z_2|$ and then (5.2.134) with $\rho = \gamma$ does not exceed $Ch^{1-d-\delta}\vartheta(h/L)$. This is trivial as $\delta > 0$; as $\delta = 0$ and (5.2.101) is fulfilled we use estimate

$$\vartheta\big(\frac{h\rho^{n-1}}{L\gamma}\big) \le \vartheta\big(\frac{h}{L}\big) \cdot \big(\frac{\rho^{n-1}}{\gamma}\big)^s \tag{5.2.135}$$

with $s > 0$.

Therefore we proved that

(5.2.136) Assuming that under condition $(5.2.126)_{n-1}$ a proper remainder estimate is proven, we prove it also under conditions conditions $(5.2.126)_n$ and $(5.2.129)_{n,n}$.

It is not a complete induction; in the next step we apply induction with respect to l as well.

Moreover, let us notice that

(5.2.137) Under condition (5.2.101) we obtain a proper remainder estimate even without condition $(5.2.129)_{n,n}$ for $(n-2)s \ge 1$.

So, under condition (5.2.101) the estimate will be proven for any n if it was proven for $n \le 2 + s^{-1}$.

(c) Let us assume that Statements (i) and (ii) are proven under condition $(5.2.126)_{n-1}$ and also under conditions $(5.2.126)_n$ and $(5.2.129)_{n,l}$. Now our goal is to prove them under conditions $(5.2.126)_n$ and $(5.2.129)_{n,l'}$ with some $l' > 0$ and to get rid of the latter condition for l' large enough.

Obviously statement similar to (5.2.116) holds.

Let us introduce the scaling function

$$\gamma \coloneqq \gamma_{n,l}(z') = \epsilon\Big(\sum_{2\le j\le n,\ |\alpha|\le jl/n} |D^\alpha V_j|^{\frac{nl}{jl-|\alpha|n}}\Big)^{\frac{1}{nl}} + \bar\gamma_{nl}, \quad \bar\gamma_{nl} = h^{\frac{n}{n+l}} \tag{5.2.138}$$

with small enough $\epsilon > 0$. Further, let $\rho = \gamma^{l/n}$. One can easily see that $|\nabla\gamma| \le \frac12$ and that (5.2.131) holds.

Moreover, for some j and $\alpha : |\alpha| < jl/n$ the opposite inequality holds (with $\epsilon_1 > 0$ instead of c) provided $\gamma \ge c\bar\gamma$.

Therefore, after the routine rescaling procedure with $x_{\mathsf{new}} = (x - \bar x)\gamma^{-1}$, $h_{\mathsf{new}} = h\rho^{-1}\gamma^{-1}$ and with multiplication operator ρ^{-n} we obtain the routine smoothness conditions and either condition $(5.2.129)_{n,l}$ or $h_{\mathsf{new}} \asymp 1$; we consider now the zone $\{|\xi_1| \le c\rho\}$ and if necessary apply a linear symplectic transformation for (x', ξ'). In both cases the contributions of the corresponding balls was described above and the contributions of the corresponding zones are given by (5.2.134).

In the zone $\{|\xi_1| \ge c\rho\}$ elliptic arguments are applicable and taking into account that $\gamma^2 \ge h^{1-\sigma}$ for $l > n$ we easily obtain that the contribution of this zone does not exceed the (5.2.134) as well. So, the total remainder estimate is given by (5.2.134). Recall that this deduction holds under the assumption that for given $l \ge 1$ condition $(5.2.129)_{n,l}$ yields the proper remainder estimate.

(d) Let us assume now that condition $(5.2.129)_{n,l'}$ is fulfilled. Then there exist α and j with $|\alpha| \le l'j/n$ such that

$$|D^\alpha V_j| \ge \epsilon.$$

Then in the appropriate coordinates

$$|D^\beta V_j| \ge \epsilon |z_2|$$

for some $\beta : |\beta| = |\alpha| - 1$ and picking $l = (|\alpha| - 1)j/n$ we obtain that $\gamma \ge \epsilon |z_2|$. Then integral in (5.2.134) is bounded.

Therefore remainder estimate (5.2.126) under condition $(5.2.129)_{n,l'}$ holds provided it holds under condition $(5.2.129)_{n,l}$. We can restrict ourselves to numbers $l = m/(n-1)!$ with integers $m \ge n$ and apply induction on m.

Moreover, under condition (5.2.101) with $s > 0$ and $l > l(n,s)$ none of the final estimates linked with z_2 and condition $(5.2.129)_{n,l}$ are necessary. These estimates can also be avoided in the case $s = 0$ as well (for $m \ge m_{n,\delta}$) by doubling δ. Therefore the induction step from $n-1$ to n is complete.

So, Statements (i) and (ii) are proven under condition $(5.2.126)_n$ with arbitrarily large n.

(e) Finally we need to get rid of condition $(5.2.126)_n$ in Statement (ii). Let us introduce the scaling functions

$$\gamma_n = \epsilon \Big(\sum_{\alpha : |\alpha| \le n-1} |D^\alpha a|^{\frac{n}{n-|\alpha|}} \Big)^{\frac{1}{n}} + \bar\gamma_n, \qquad \bar\gamma_n = h^{\frac12}. \tag{5.2.139}$$

Then $|\nabla \gamma| \le \frac12$ and

$$|D^\alpha a| \le c\gamma^{n-|\alpha|} \qquad \forall \alpha : |\alpha| < n$$

and either $\gamma \le ch^{\frac12}$ or the opposite estimate holds for some α with $|\alpha| < n$.

Then applying the rescaling procedure $x_{\mathsf{new}} = (x - \bar x)\gamma^{-1}$, $h_{\mathsf{new}} = h\gamma^{-2}$ and multiplying the operator by γ^{-n} we get into the situation of $(5.2.126)_{n-1}$ or $h \asymp 1$ respectively and applying an estimate $Ch^{1-d}\vartheta(h/L)$ we obtain that the contribution of γ-element to the remainder does not exceed

$$Ch^{1-d}\gamma^{2d-2}\vartheta\Big(\frac{h\gamma^{n-2}}{L}\Big) \le Ch^{1-d}\gamma^{2d-2+(n-2)s}\vartheta\Big(\frac{h}{L}\Big) \le Ch^{1-d}\gamma^{2d}\vartheta\Big(\frac{h}{L}\Big)$$

as $(n-2)s \ge 2$.

In this case summation with respect to partition yields the required remainder estimate. □

Remark 5.2.17. (i) Without condition (5.2.101) with $s > 0$ we obtain the unpleasant factor $h^{-\delta}$ only when we get rid of condition $(5.2.129)_{n,l}$. All other steps of our analysis give only additional factors $|\log h|^M$. However, we cannot go from conditions $(5.2.126)_{n-1}$, $(5.2.129)_{n-1,l}$ to $(5.2.126)_n$, $(5.2.129)_{n,n}$ without dropping condition $(5.2.129)_{n-1,l}$ first unless we assume that the condition

$$(5.2.140)_r \qquad \sum_{2\le j\le n-r,\ |\alpha|\le j} |D^\alpha V_j| \ge \epsilon$$

holds with $r = n-2$.

Indeed, this jump invokes γ_n defined by $(5.2.130)_n$ and we need to check that for some $k \le n-1$ and β

$$|D^\beta V_k| \asymp \gamma_n^{(k-1-|\beta|(n-1)/l)} \asymp \sum_{2\le j\le n, |\alpha|\le j-1} |D^\alpha V_j|^{\frac{k-1-|\beta|(n-1)/l}{j-|\alpha|}};$$

one can rewrite this condition as

$$\sum_{2\le j\le n-1,\ \alpha\le j} |D^\alpha V_j|^{1/(j-1-|\alpha|(n-1)/l)} \asymp \sum_{2\le j\le n,\ |\alpha|\le j-1} |D^\alpha V_j|^{1/(j-|\alpha|)};$$

and this is not guaranteed unless $|D^\alpha V_j| \asymp 1$ for some $\alpha : |\alpha| = 1$ and $j = 2$; let us recall that $V_1 = 0$.

So in order to provide the remainder estimate $O(h^{1-d}|\log h|^M)$ one should assume that condition $(5.2.140)_r$ is fulfilled for $r = n-2$; under this condition one can prove the estimate easily.

(ii) On the other hand, if we assume that $V_2, \dots, V_{n-r-1}$ vanish identically while condition $(5.2.140)_r$ holds, we observe that the problem is reduced (after a few steps of the induction described above) to the problem with $a = b\xi_1^r$ with $r = n-r-1$ and non-vanishing symbol b. Then the same partition arguments yield the remainder estimate $O(h^{1-d})$ because the zone $\{|\xi_1| \ge h\}$ is elliptic. Therefore a proper remainder estimate remains true but we do not know whether it is uniform with respect to the spectral parameter.

(iii) In certain cases (under some additional conditions) propagation of singularities arguments provide the remainder estimate $O(h^{1-d})$ but I never investigated it.

The remark above leads us to

Conjecture 5.2.18. *For operators with analytic symbols $R^{\mathsf{W}}_{\varphi,L}$ is dominated by $O\big(h^{1-d}\vartheta(h/L)\big)$.*

5.2.5 From Microlocal to Local Spectral Asymptotics

Let us go from microlocal to local spectral asymptotics. Then the results of the previous subsections combined with the results of Section 4.2 imply

Theorem 5.2.19. *(i) Let condition* (4.2.45) *be fulfilled. Then the assertions of all the above statements remain true for $Q_1 = Q_2 = I$ provided their conditions are fulfilled for $\Omega = T^*B(0,1)$* [16].

(ii) Moreover if condition $(4.2.48)_{\pm}$ *is fulfilled then in every estimate without mollification with respect to τ_1 one can take $\tau_1 = \mp\infty$ and in every estimate with mollification one can drop the mollification and take $\tau_1 = \mp\infty$.*

5.3 Local and Pointwise Spectral Asymptotics for the Schrödinger and Dirac Operators

In this section we treat the Schrödinger operator

$$A = \sum_{j,k} P_j g^{jk} P_k + V, \qquad P_j = hD_j - V_j \tag{5.3.1}$$

and Dirac operator

$$A = \sum_{j,l} \frac{1}{2}\sigma_l(\omega^{jl}P_j + P_j\omega^{jl}) + \sigma_0 m + V\cdot I \tag{5.3.2}$$

where *the components g^{jk} of the inverse metric tensor, coefficients ω^{jl}, the components of the vector-potential V_j* and *the scalar potential V* are real-valued functions, σ_l are *Pauli matrices* (see Subsection 5.3.2 and Appendix 5.A.2), $m = \mathsf{const} \ge 0$ is a *mass parameter*, and $h \in (0,1)$. Constants in conditions and in estimates depend neither on h nor m.

[16] In fact, it is necessary to check them only for $\Omega = B(0,1)\times B(0,C_0)$ with a large enough C_0.

Remark 5.3.1. As $d \ge 2$ one can consider also the Schrödinger-Pauli operator

$$A = \Big(\sum_{j,l} \frac{1}{2}\sigma_l(\omega^{jl}P_j + P_j\omega^{jl})\Big)^2 + V \tag{5.3.3}$$

which differs from (5.3.1) only by matrix lower order terms, but in the current framework (without strong magnetic field as in Volumes III and IV) it does not make any difference.

We assume that

(5.3.4) A is self-adjoint operator in $\mathscr{L}^2(X, \mathbb{H})$, $\mathbb{H} = \mathbb{C}^{\mathsf{D}}$, $\mathsf{D} = 1$ for the Schrödinger operator.

5.3.1 Schrödinger Operator

Main Assumptions

Let us start with the Schrödinger operator assuming that

$$B(0,1) \subset X, \tag{5.3.5}$$

$$|D^\alpha g^{jk}| \le c, \quad |D^\alpha V_j| \le c, \quad |D^\alpha V| \le c \quad \forall \alpha : |\alpha| \le K, \tag{5.3.6}_{1-3}$$

and

$$\epsilon_0 \le \sum_{j,k} g^{jk}(x)\xi_j\xi_k|\xi|^{-2} \le c \quad \forall x \in B(0,1) \quad \forall \xi \in \mathbb{R}^d \setminus 0. \tag{5.3.7}$$

Remark 5.3.2. A (unitary) *gauge transformation* $u \to e^{ih^{-1}\varphi(x)}u$ with a real-valued function φ can be applied which for the Schrödinger and Dirac operators is equivalent to the transformation $V_j \mapsto V_j + \partial_j\varphi$. Hence one can weaken condition $(5.3.6)_2$, replacing it with

$$|D^\alpha F_{jk}| \le c \quad \forall \alpha : |\alpha| \le K \tag{5.3.6}'_2$$

where

$$F_{jk} = \partial_k V_j - \partial_j V_k \tag{5.3.8}$$

are *components of the tensor magnetic intensity.*

Here the principal symbol of A equals

$$a(x,\xi) = \sum_{j,k} g^{jk}(\xi_j - V_j)(\xi_k - V_k) + V \tag{5.3.9}$$

and the subprincipal symbol vanishes and hence the coefficients in the spectral asymptotics are

$$\kappa_0(x,\tau) = (2\pi)^{-d}\varpi_d(\tau - V)_+^{d/2}\sqrt{g} \tag{5.3.10}$$

and

$$\kappa_1(x,\tau) = 0 \tag{5.3.11}$$

where $\tau_1 = -\infty$, $\tau_2 = \tau$, ϖ_d is the volume of the unit ball in $\mathbb{R}^d$ and $g = \det(g^{jk})^{-1}$. We prefer to use notation $\kappa_n(x,\tau)$ rather than $\kappa_{xn}(\tau)$ [17].

Asymptotics with Spatial Mollification

Then the results of the previous Section 5.2 immediately yield the following assertions:

Theorem 5.3.3. *Let conditions* (5.3.1), (5.3.4), (5.3.5), $(5.3.6)_{1-3}$ *and* (5.3.7) *be fulfilled and* $d \geq 2$. *Let* φ *satisfy* (4.2.73)–(4.2.75). *Then as* $Q_1 = I$, $Q_2 = \psi(x)$

(i) The following estimate holds:

$$|\mathsf{R}^{\mathsf{W}}_{\varphi,L}| \leq \bigl(Ch^{1-d} + C'h^{1-d+\delta''}\bigr)\vartheta\bigl(\frac{h}{L}\bigr) \tag{5.3.12}$$

where here and below $C = C(d,c)$, $K = K(d)$, $\delta'' = \delta''(d) > 0$ *is small enough and* $\sup_x |\psi(x)| \leq 1$, $C' = C\|\psi\|_{\mathscr{C}^K}$.

(ii) Let $\Lambda_1, \dots, \Lambda_n$ *be closed subsets of* $\Sigma_\tau = \{(x,\xi) \in T^*B(0,1), a(x,\xi) = \tau\}$ *and let* $\Lambda_j \subset \{\gamma^2 = |\tau - V| + |\nabla_x V|^2 \geq h^{1-\delta}\}$ *with* $\delta > 0$.

Further, let us assume that through each point of Λ_j *there passes a Hamiltonian trajectory of* $a(x,\xi)$ *with* $t \in \mathcal{J}_j$, $\mathcal{J}_j = [0, T_j]$ *or* $\mathcal{J}_j = [-T_j, 0]$, $T_j \in [1, h^{-\delta}]$ *along which*

$$|D^\alpha g^{jk}| \leq ch^{-\delta'}, \quad |D^\alpha V_j| \leq ch^{-\delta'}, \quad |D^\alpha V| \leq ch^{-\delta'} \quad \forall \alpha : |\alpha| \leq K' \tag{5.3.13}$$

[17] For Schrödinger-Pauli operator (5.3.3) κ_0 acquires factor D and $\kappa_1 = 0$ because $\mathsf{tr}(a^s) = 0$.

and

$$\mathsf{dist}\big(x(t), \partial X\big) \ge \epsilon h^{\delta'}, \tag{5.3.14}$$

$$\mathsf{dist}\big((x(t), \xi(t)), (x, \xi)\big) \ge \epsilon |t| h^{\frac{1}{2}-\delta} \tag{5.3.15}$$

and condition (2.4.29) *holds with* $\omega = h^{-\delta'}$ *with small enough exponent* $\delta' = \delta'(d, \delta) > 0$.

Furthermore, let (5.2.101) *be fulfilled with* $s \ge 0$. *Then*

$$|\mathsf{R}^{\mathsf{W}}_{\varphi,L}| \le C\Big(\sum_{0\le j\le n} \frac{h^{1-d}}{T_j^{1+s}} \int_{\Lambda_j} dx d\xi : da + C'' h^{1-d+\delta''}\Big)\vartheta\big(\frac{h}{L}\big) \tag{5.3.16}$$

where $T_0 = 1$, $\Lambda_0 = \Sigma_\tau \setminus (\Lambda_1 \cup \dots \cup \Lambda_n)$, $K' = K'(d, s, \delta)$, $C'' = \|\psi\|_{\mathscr{C}^{K'}}$.

Proof. Indeed, condition $(5.2.20)^+_d$ is fulfilled and therefore Statement (i) as $d \ge 2$ and (ii) as $d \ge 3$ follow automatically.

Further, as $d = 2$ for Statement (ii) we need to apply Proposition 5.2.6 but since $a^s = 0$ we have $\mu'_\tau = \mu_\tau$. Note that in this case condition $\mathsf{mes}(\{V = \bar\tau, \{z_1, z_2\} \ne 0\}) = 0$ of Theorem 5.2.9(ii) translates into

$$\mathsf{mes}(\{V = \bar\tau, F_{12} \ne 0\}) = 0 \tag{5.3.17}$$

with F_{jk} defined by (5.3.8). □

Alternative proof. Alternatively, one can repeat more rigorously arguments of Subsection 5.1.1. Let us start from estimate (5.3.12), (5.3.16) which hold under microhyperbolicity condition

$$|V| + |\nabla V| \ge \epsilon_0 \qquad \text{in } B(0,1), \tag{5.3.18}$$

estimate

$$|e(x, x, 0)| \le C h^r \qquad \text{in } B(0, \frac{1}{2}) \tag{5.3.19}$$

which holds with arbitrarily large exponent r under assumption

$$V(x) \ge \epsilon_0 \qquad \text{in } B(0,1) \tag{5.3.20}$$

and generic estimate

$$|e(x, x, 0)| \le C h^{-d} \qquad \text{in } B(0, \frac{1}{2}) \tag{5.3.21}$$

which holds with no extra condition.

Rescaling them $x \mapsto x\gamma^{-1}$, $hD \to \hbar D$ and $h \to \hbar = h\rho^{-1}\gamma^{-1}$ we get estimate

$$|\mathsf{R}^{\mathsf{W}}| \le Ch^{1-d}\rho^{d-1}\gamma^{-1} \tag{5.3.22}$$

under condition

$$|V| + \gamma|\nabla V| \ge \epsilon_0\rho^2 \qquad \text{in } B(0,\gamma), \tag{5.3.18)*}$$

estimate

$$|e(x,x,0)| \le Ch^r\rho^{-s}\gamma^{-d-r} \qquad \text{in } B(0,\frac{1}{2}\gamma) \tag{5.3.19)*}$$

which holds with arbitrarily large exponent r under assumption

$$V(x) \ge \epsilon_0\rho^2 \qquad \text{in } B(0,\gamma) \tag{5.3.20)*}$$

and generic estimate

$$|e(x,x,0)| \le Ch^{-d}\rho^d \qquad \text{in } B(0,\frac{1}{2}) \tag{5.3.21)*}$$

which holds with no extra condition; here we assume that

$$\rho\gamma \ge h \tag{5.3.23}$$

and replace $(5.3.6)_{1-3}$ by

$$|D^\alpha g^{jk}| \le c\gamma^{-|\alpha|}, \quad |D^\alpha V_j| \le c\rho\gamma^{-|\alpha|}, \quad |D^\alpha V| \le c\rho^2\gamma^{-|\alpha|} \quad \forall x \in B(0,\gamma),\ \forall \alpha : |\alpha| \le K \tag{5.3.6)$^*_{1-3}$}$$

and (5.3.5) by

$$B(0,\gamma) \subset X. \tag{5.3.5)*}$$

Further, assuming (5.2.101) with $s \ge 0$ and also $\rho \le \gamma$ we get estimate

$$|\mathsf{R}^{\mathsf{W}}_{\varphi,L}| \le Ch^{1-d}\rho^{d-1+s}\gamma^{-s}\vartheta\big(\frac{h}{L}\big). \tag{5.3.24}$$

Let us introduce scaling functions

$$\gamma(x) = \epsilon\big(|V| + |\nabla V|^2\big)^{\frac{1}{2}} + \frac{1}{2}\bar\gamma, \qquad \rho = \gamma \qquad \text{with } \bar\gamma = h^{\frac{1}{2}} \tag{5.3.25}$$

with small enough constant ϵ and observe that conditions $(5.3.6)^*_{1-3}$ are fulfilled[18], as $\gamma \ge h^{\frac12}$ condition $(5.3.18)^*$ is fulfilled and as $\gamma \asymp h^{\frac12}$ all estimates are equally good (actually generic estimate even does not need spatial mollification).

After summation over γ-admissible partition we arrive to Statement (i).

Proof of Statement (ii) follows the same scheme but with the use of long time propagation results. □

Remark 5.3.4. Statement (i) as $d \ge 2$ and Statement (ii) as $d \ge 3$ remain true for Schrödinger-Pauli operator as well. As $d = 2$ (5.3.16) should be replaced by while for $d = 2$

$$|\mathcal{R}^{\mathsf{W}}_{\varphi,L}| \le C_0 h^{-1}\Big(\sum_{0\le j\le n} \frac{1}{T_j^{1+s}} \int_{\Lambda_j^{\pm}} dx d\xi : da + \int_{\{\varrho\le h^{\delta}\}} |F_{12}|\, dx + C'' h^{\delta''}\Big)\vartheta\big(\frac{h}{L}\big) \tag{5.3.26}$$

where we assume that

$$\Lambda_j^{\pm} \subset \big\{\varrho^2 \coloneqq |V| + |\nabla V|^2 \ge h^{1-\delta}\big\}.$$

For the proof we refer to the proof of the Statement (ii) of Theorem 5.3.25 below where it is done for Dirac operator.

Theorem 5.3.5. *Let conditions* (5.3.1), (5.3.4)–(5.3.7) *be fulfilled and* $d = 1$. *Then*

(i) The following estimate holds

$$|\mathsf{R}^{\mathsf{W}}| \le C h^{-\delta}. \tag{5.3.27}$$

(ii) Moreover, if either

$$\sum_{|\alpha|\le n} |D^{\alpha} V| \ge \epsilon_0 \tag{5.3.28}$$

or (5.2.101) *holds with* $s > 0$ *then Statements (i) and (ii) of Theorem 5.3.3 remain true. However, for* $s \in \mathbb{N}$ *one should regularize the coefficient* κ_{1+s} *and take it depending on* h *(and it is* $O(|\log h|^M)$ *in virtue of the analysis of the previous Section 5.2).*

[18] Actually condition to V_j may fail as $\alpha = 0$ but we need to check it only as $|\alpha| \ge 1$ due to Remark 5.3.2.

Proof. Proof follows arguments to the previous section but with essential simplification. We leave details to the reader. □

Remark 5.3.6. Conditions $(5.2.20)^{\pm}_2$ in this case mean exactly that V has non-degenerate minimum (maximum) in the point in question.

Asymptotics without Spatial Mollification

Let us consider spectral asymptotics without spatial mollification. First of all, let us check ξ-microhyperbolicity condition; it is

$$|V| \ge \epsilon_0 \qquad \forall x \in B(0,1) \tag{5.3.29}$$

and we immediately arrive to

Theorem 5.3.7. *Let conditions* (5.3.1), (5.3.4), (5.3.5), (5.3.6), (5.3.7) *and* (5.3.29) *be fulfilled. Then*

$$|\mathsf{R}^{\mathsf{W}}_{x,\varphi,L}| \le C_0 h^{1-d}\vartheta\Big(\frac{h}{L}\Big) \qquad \forall x \in B(0,\frac{1}{2}). \tag{5.3.30}$$

Now let us use the same rescaling scheme as in the alternative proof of Theorem 5.3.3. Then we should replace $(5.3.18)^*$ by

$$|V| \ge \epsilon_0\rho^2 \qquad \forall x \in B(0,\gamma) \tag{5.3.29}^*$$

and therefore we need to pick up the scaling functions

$$\gamma = \epsilon|V| + \bar\gamma, \qquad \rho = \gamma^{\frac12} \qquad \text{with } \bar\gamma = \frac{1}{2}h^{\frac23} \text{ and } \bar\rho = h^{\frac13}; \tag{5.3.31}$$

we select the $\bar\gamma$ equal $h^{\frac23}$ to keep $\rho\gamma \ge h$. Obviously $|\nabla_x\gamma| \le \frac12$.

We immediately arrive to the estimate

$$|\mathsf{R}^{\mathsf{W}}_{x,\varphi,L}| \le Ch^{1-d}\rho^{d-1}\gamma^{-1}\vartheta\Big(\frac{h\rho}{L\gamma}\Big) \asymeq \begin{cases} Ch^{1-d}|V|^{(d-3)/2}\vartheta\Big(\frac{h}{L}|V|^{-\frac12}\Big) & \text{as } |V| \ge h^{\frac23},\\ Ch^{-\frac23 d}\vartheta\Big(\frac{h^{2/3}}{L}\Big) & \text{as } |V| \le h^{\frac23}. \end{cases} \tag{5.3.32}$$

So, we proved

Theorem 5.3.8. *Let conditions* (5.3.1), (5.3.4), (5.3.5), (5.3.6) *and* (5.3.7) *be fulfilled. Then estimate* (5.3.32) *holds.*

Corollary 5.3.9. *Let conditions of Theorem 5.3.8 be fulfilled. Let us assume that*

(5.3.33) $\vartheta(\tau)\tau^{-s}$ *is a monotone decreasing function*[19] *with some* $s \ge 0$.

Then as $d-3 \ge s$ *estimate* (5.3.30) *holds and as* $d-3 < s$ *estimate*

$$|\mathsf{R}^{\mathsf{W}}_{x,\varphi,L}| \le C_0 h^{-\frac{2}{3}(d+s)} \vartheta\Big(\frac{h}{L}\Big) \qquad \forall x \in B(0,\frac{1}{2}) \tag{5.3.34}$$

holds.

The reader can reformulate Theorem 5.2.14 for the Schrödinger operator and improved asymptotics with the "no loop" condition. Finally the results of Section 5.2 yield

Theorem 5.3.10. *Let conditions* (5.3.1), (5.3.4), (5.3.5), (5.3.6) *and* (5.3.7) *be fulfilled. Then for* $\tau \le V_* = \inf_{B(0,1)} V$ *the following estimate holds*

$$|e(x,y,\tau)| \le C' h^{-d}\big(1+\frac{|V_*-\tau|}{h}\big)^{-l} \quad \forall x,y \in B(0,1-\epsilon) \tag{5.3.35}$$

where $C' = C'(d,c,l,\epsilon)$ *and* $l,\epsilon > 0$ *are arbitrary.*

Here we use that $|\nabla V| \le C|V-V_*|^{\frac{1}{2}}$.

Asymptotics without Spatial Mollification and Short Loops

In this subsubsection we consider very special case $\vartheta = 1$ and $d = 1,2$ when estimate $|\mathsf{R}^{\mathsf{W}}_x| \le Ch^{1-d}$ fails replaced by $|\mathsf{R}^{\mathsf{W}}_x| \le Ch^{-\frac{2}{3}}$ and $\mathcal{R}_x \le Ch^{-\frac{4}{3}}$ as $d = 1,2$. More precisely, as $|V| \ge h^{\frac{2}{3}}$ estimates $|\mathsf{R}^{\mathsf{W}}_x|| \le Ch^{1-d}|V|^{(d-3)/2}$ holds.

Our purpose is to improve these estimates, possibly adding a *correction term* associated with the short loop. As $|\nabla_x V|$ is our foe rather than our friend here, we assume that condition (5.3.18) holds. Later we will get rid of it by scaling.

Our goal is to prove

[19] Before we assumed that it is increasing function.

Theorem 5.3.11. *Let conditions* (5.3.1), (5.3.4), (5.3.5), (5.3.6), (5.3.7) *and* (5.3.18) *be fulfilled. Then*

$$|e(x,x,0) - \kappa_0(x)h^{-d} - h^{-\frac{2}{3}d}|\nabla V(x)|_g^{\frac{1}{3}d}\mathcal{Q}\big(W(x)h^{-\frac{2}{3}}\big)g| \le Ch^{1-d}\gamma(x)^{\frac{1}{2}(d-2)} \tag{5.3.36}$$

where in the correction term $W(x) \asymp V(x)$ *and* $\mathcal{Q}$ *will be defined by* (5.3.62), *and* (5.A.11) *respectively and*

$$\mathcal{Q}(\lambda) = O(\lambda^{-\frac{1}{4}(d+3)}) \quad \text{as} \quad \lambda \to \infty \tag{5.3.37}$$

(see (5.A.12) *for its asymptotics as* $\lambda \to +\infty$*) and* $\gamma(x) \asymp |W(x)| + h^{\frac{2}{3}}$,

$$|\nabla V|_g = \big(\sum_{j,k} g^{jk}\partial_j V \cdot \partial_k V\big)^{\frac{1}{2}}. \tag{5.3.38}$$

The crucial step in the proof is

Proposition 5.3.12. *Let* $u(x,y,t)$ *be the Schwartz kernel of* $e^{ih^{-1}A}$. *In the framework of Theorem 5.3.11*

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma_x u| \le Ch^{1-d}\gamma(x)^{\frac{1}{2}(d-2)} \qquad \forall \tau : |\tau| \le \epsilon\gamma(x) \tag{5.3.39}$$

where T *is the small constant.*

Note that the rescaling arguments yield (5.3.39) with $T = \epsilon_0\gamma(x)^{\frac{1}{2}}$ and this would yield estimates $O\big(h^{1-d}\gamma^{(d-3)/2}\big)$. Generalizing it to $T = \epsilon_0$ leads to remainder estimates $O\big(h^{1-d}\gamma^{(d-2)/2}\big)$ which are announced in Theorem 5.3.11, but with Tauberian main part. This Tauberian expression would have two contributors: one equal $h^{-d}\kappa_0(x)$ which is the main term from $t = 0$ and another equal to the correction term which from the loop and also collects all other terms from $t = 0$.

Proof of Proposition 5.3.12. Consider first the left-hand expression of (5.3.39) with $\bar{\chi}_T(t)$ replaced with $\chi_T(t)$ supported in $\frac{1}{2}T \le |t| \le T$ with $T = T_0$ which is a small constant and with $\gamma(x) \le \epsilon_1 T$; then it would be less than Ch^s.

Rescaling $x \mapsto x_{\mathsf{new}} = xT^{-2}$, $t \mapsto t_{\mathsf{new}} = tT^{-1}$, $h \mapsto h_{\mathsf{new}} = hT^{-3}$ and multiplying operator by T^{-2} we reduce the general case of $T \ge \gamma(x)^{\frac{1}{2}}$ to the

previous one; so now the left-hand expression of (5.3.39) with $\chi_T(t)$ instead of $\bar{\chi}_T(t)$ does not exceed

$$CT^{-2d}\Big(\frac{h}{T^3}\Big)^s$$

where $T^{1-2d} = T \times T^{-2d}$ and the first factor T appears from Fourier transform while T^{-2d} appears because we have a density.

Summation with respect to $T \in [C_0\gamma^{\frac12}, T_0]$ results in the same expression as $T = \gamma^{\frac12}$ i. e.

$$C\gamma^{-\frac12-d}\Big(\frac{h}{\gamma^{\frac32}}\Big)^s$$

which does not exceed the right-hand expression of (5.3.39).

Therefore we need to prove (5.3.39) with $T = C_0\gamma^{\frac12}$. Rescaling $x \mapsto x_{\text{new}} = x\gamma^{-1}$, $t \mapsto t_{\text{new}} = t\gamma^{-\frac12}$, $h \mapsto h_{\text{new}} = h\gamma^{-\frac32}$ and multiplying operator by γ^{-1} we reduce (5.3.39) to the case $\gamma \asymp 1$.

However, if originally $\gamma(x) \ge C_0h^{\frac23}$ then condition (5.3.29) is fulfilled after rescaling this estimate follows from (4.2.82). On the other hand, if originally $\gamma \asymp h^{\frac23}$ then after rescaling $h_{\text{new}} \asymp 1$ and this estimate holds as well. □

Therefore due to Tauberian theorem we arrive to

Corollary 5.3.13. *In the framework of Theorem 5.3.11*

$$|\mathsf{R}^{\mathsf{T}}_{x,\varphi,L}| \le Ch^{1-d}\gamma(x)^{\frac12(d-2)}\vartheta\Big(\frac{h}{L}\Big) \tag{5.3.40}$$

as $\mathsf{N}^{\mathsf{T}}_{x,\varphi,L}$ *is defined with* $T = T_0$ *which is a small constant.*

Now our goal is to calculate $\mathsf{N}^{\mathsf{T}}_x$ with the indicated error. Due to the same arguments as in the proof of Proposition 5.3.12

(5.3.41) Estimate (5.3.40) remains true as $\mathsf{N}^{\mathsf{T}}_{x,\varphi,L}$ is defined with $T = \max\big(C\gamma^{\frac12}, h^{\frac13-\delta}\big)$.

Now let us calculate. We will do first the general calculations in the case of $\gamma(x) \asymp 1$ and then we rescale. To do this we need to prove

Theorem 5.3.14. *Let* $\bar{x}$, $\bar{y}$ *be fixed points and let* $t_1 < t_2$ *be of the same sign and* $t_1 \asymp t_2 \asymp (t_2 - t_1) \asymp 1$. *Assume that*

(5.3.42) *There exist only one Hamiltonian trajectory* $(x(t),\xi(t))$ *such that* $x(0)=\bar y$, $x(t)=\bar x$ *and* $t\in[t_1,t_2]$; *let it happen as* $t=\bar t$, $\xi(0)=\bar\eta$ *and* $(\bar t-t_1)\asymp(t_2-\bar t)\asymp 1$,

$$\frac{dx}{dt}\Big|_{t=\bar t}\asymp \frac{dx}{dt}\Big|_{t=\bar t}\asymp 1 \tag{5.3.43}$$

and

(5.3.44) *Map* $\Sigma_0\cap T^*_{\bar y}X\ni\eta\to\pi_x\Psi_t(y,\eta)$ *is nondegenerate in* $(\bar y,\bar\eta)$.

Then as $t_1\le\bar t-\epsilon$, $t_2\ge\bar t+\epsilon$, $x=\bar x$, $y=\bar y$

$$h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\bar\chi_\epsilon(t-T^*)u\Big)\,d\tau\equiv e^{ih^{-1}\phi(x,y)}\sum_{n\ge0}b_n(x,y)h^{\frac{1-d}{2}+n} \tag{5.3.45}$$

where

$$\phi(x,y)=\int_0^{\bar t}\ell(x(t),\xi(t))\,dt,\qquad \ell(x,\xi)\coloneqq\langle\partial_\xi a,\xi\rangle. \tag{5.3.46}$$

Proof. First, let us rewrite the left-hand expression of (5.3.45) as

$$i\Big(F_{t\to h^{-1}\tau}t^{-1}\bar\chi_\epsilon(t-\bar t)\Big)\Big|_{\tau=0}. \tag{5.3.47}$$

On the other hand we know that

$$u(x,y,t)\equiv(2\pi h)^{-\frac12(d+m)}\int e^{ih^{-1}\varphi(x,y,\theta,t)}\sum_{n\ge0}b'_n(x,y,t,\theta)\,d\theta \tag{5.3.48}$$

where θ is m-dimensional variable and $\varphi(x,y,t,\vartheta)$ is defined in the corresponding way. For example, one can take $m=d$, $\theta=\eta$,

$$\varphi(x,y,t,\eta)=-\langle y,\eta\rangle+\psi(x,t,\eta) \tag{5.3.49}$$

where

$$\partial_t\psi=-a(x,\partial_x\psi),\qquad \psi(x,0,\eta)=\langle x,\eta\rangle. \tag{5.3.50}$$

Let us apply stationary phase method with respect to θ,t; condition (5.3.42) and $\partial_t\psi=-a(x,\xi)$ imply that there is only one stationary point and it is $(\bar\eta,\bar t)$. Further, conditions (5.3.43) and (5.3.44) imply that this is nondegenerate point. Thus we gain a factor $h^{\frac12(d+1)}$ and (5.3.45)–(5.3.46) are proven. □

Proof of Theorem 5.3.11. To apply Theorem 5.3.14 to our case we need just rescale $x \mapsto x\gamma^{-1}$, $h \mapsto \hbar = h\gamma^{-\frac{3}{2}}$; then as $\hbar \le h^{\delta}$ i. e.

$$\gamma \ge h^{\frac{2}{3}(1-\delta)} \tag{5.3.51}$$

we conclude that

$$\mathsf{N}_{x,\mathsf{corr}} := \mathsf{N}_x^{\mathsf{T}} - \mathsf{N}_x^{\mathsf{W}} \equiv e^{i\phi(x,x)} \sum_{n\ge 0} b_n(x,x)\gamma^{\frac{1}{4}(-d-3-6n)} h^{\frac{1-d}{2}+n} \tag{5.3.52}$$

where we also multiplied by γ^{-d} since we are dealing with densities.

Thus

(5.3.53) Correction term $\mathsf{N}_{x,\mathsf{corr}}$ is of magnitude $h^{\frac{1}{2}(1-d)}\gamma^{-\frac{1}{4}(d+3)}$.

This implies a drastic difference between $d = 1$ when the correction term is below remainder estimate $h^{1-d}\gamma(x)^{\frac{1}{2}(d-2)}$ of (5.3.40) only as $\gamma \asymp 1$ and $d \ge 2$ it is always so as

$$\gamma \ge \bar{\gamma}_1 := h^{(2d-2)/(3d-1)}. \tag{5.3.54}$$

Let us introduce

$$X^- = \{x :\ V(x) < 0\}, \qquad X^0 = \partial X^- = \{x :\ V(x) = 0\}, \tag{5.3.55}$$

Without any loss of the generality one can assume that $X^- = \{x_1 < 0\}$.

Case $d = 1$. We can assume without any loss of the generality that $a(x,\xi) = \beta(x)\big(\xi_1^2 - V_0(x)\big)$ (we can always get rid of V_1 by gradient transform); then

$$W(x_1) = \Big(\frac{3}{2}\int_0^{x_1} V_0(y_1)^{\frac{1}{2}}\, dy_1\Big)^{\frac{2}{3}} \tag{5.3.56}$$

is a travel time from $x_1 \in X^-$ to X^0 (on energy level 0, if we replace β by 1) and one can see easily that then

(5.3.57) Under assumption (5.3.18) $W(x)/V_0(x)$ is a smooth and disjoint from 0 function on $X^- \cup X^0$.

We redefine $x_1 = W(x)$ and then $a(x,\xi)$ will be in the same form as before but with different β and with $W(x_1) = x_1$:

$$a(x,\xi) = \beta(x)\big(\xi_1^2 - x_1\big) \tag{5.3.58}$$

Let us prove that *in calculations one can replace* $\beta(x)$ *by* $\beta(x) = 1$ *and assume that*

$$A = h^2 D_1^2 - x_1 \qquad \text{on } \mathbb{R}. \tag{5.3.59}$$

Really, let $\bar{x}$ be a point where calculations are done, while x be a "running" point. Without any loss of the generality one can assume that $\beta(\bar{x}) = 1$. Let us rescale as before. Then we can assume that we are at he point with $\gamma(x)$ but on $[-c, c]$

$$|D_1^j \beta(x)| \le c_j \varepsilon^j \tag{5.3.60}$$

where ε is an original $\gamma(\bar{x})$.

Let us apply Theorem 5.3.14. Note that phase functions for the original operator and for the model operator coincide identically while amplitudes differ by $O(\varepsilon)$ (where $\varepsilon = \gamma$); so an error is $O(h^{1-d}\varepsilon)$ and scaling back we get an error estimate $Ch^{1-d}\gamma^{\frac{1}{2}(d-3)}\varepsilon = Ch^{1-d}\gamma^{\frac{1}{2}(d-1)}$ which is actually better by factor $\gamma^{\frac{1}{2}}$ than we need. Calculations for model operator (5.3.59) are produced in Appendix 5.A.1. Theorem 5.3.11 is proven as $d = 1$.

Case $d \ge 2$. Again without any loss of the generally one can assume that $V(x) = -k(x)x_1$ with $k(x) > 0$ disjoint from 0.

Let

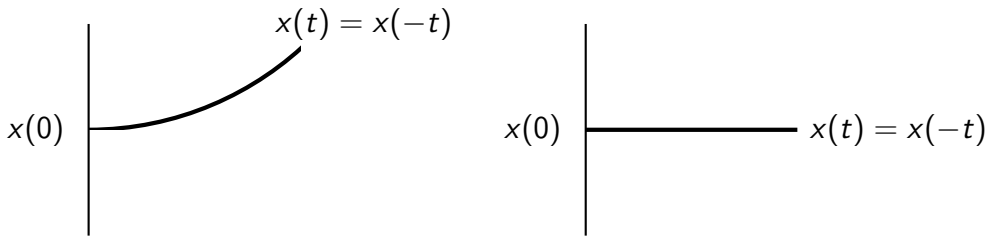
$$\Theta = \{(x,\xi) : V(x) = 0\} \cap \Sigma_0 = \{(x,\xi) : \ x_1 = 0, \xi_j = V_j(x)\} \tag{5.3.61}$$

parametrized by $x' = (x_2, \dots, x_d)$.

Consider Hamiltonian trajectory passing through $(0, x') \in \Theta$. One can select ξ_1 as a natural parameter along this trajectory, so we get a d-dimensional manifold $\Lambda \subset \Sigma_0$; since Σ_0 is $(2d-1)$-dimensional, there exists $(d-1)$-dimensional variable η: $\Lambda = \Sigma_0 \cap \{\eta = 0\}$.

Assume first that

(5.3.62) There is no magnetic field, i. e. $V_1 = V_2 = \dots = V_d = 0$.



(a) Original coordinates (b) Straighten coordinates

Figure 5.2: : Short loops

Then trajectory passing through Θ as $t = 0$ is symmetric: $x(-t) = x(t)$ and $\xi(-t) = \xi(t)$. Then the set of loops coincides with Λ and x-projection of the loop is exactly as on the left picture below

If we introduce new coordinates replacing x' by $x'(0)$ we get picture as on the right. Note that due to the same arguments as for $d = 1$ one can assume that

$$a(x,\xi) = \beta(x)\Big(\xi_1^2 - x_1 + \sum_{j,k\geq 2} g'^{jk}(x)\big(\xi_j - \alpha_j(x)\xi_1\big)\big(\xi_k - \alpha_k(x)\xi_1\big)\Big) \tag{5.3.63}$$

with positive definite matrix (g'^{jk}). Then $\eta = 0$ if and only if $\xi_j - \alpha_j(x)\xi_1 = 0$ $\forall j = 2, \dots, d$. Also note that on Σ_0 we must have $\eta = 0 \implies \{a, \eta\} = 0$ i. e. $\{\xi_1^2 - x_1, \xi_j - \alpha_j(x)\xi_1\} = 0$ which easily yields that $\alpha_j(x) = 0$ $\forall j = 2, \dots, d$ i. e.

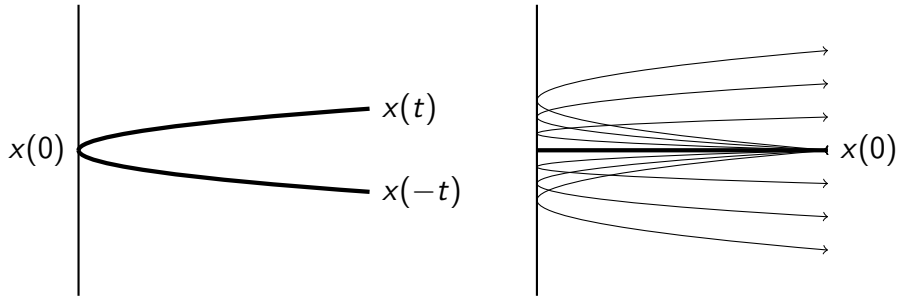
$$a(x,\xi) = \beta(x)\Big(\xi_1^2 - x_1 + \sum_{j,k\geq 2} g'^{jk}(x)\xi_j\xi_k\Big) \tag{5.3.64}$$

and $\eta = \xi'$.

One can see easily that in the special case $g'^{jk}(x) = \delta_{jk}$ trajectories are parabolas

$$\begin{aligned} &\xi_1 = t, && x_1 = t^2,\\ &\xi' = \mathrm{const}, && x' = \mathrm{const} + \xi' t \ (j \geq 2) \end{aligned}$$

and similarly looking in more general case.



(a) (b)

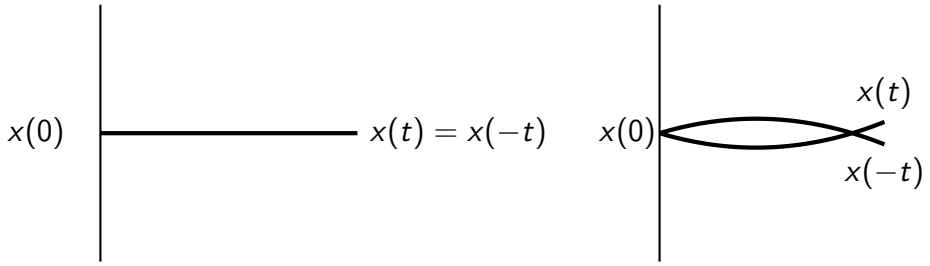
Figure 5.3: Selecting $\xi' \neq 0$ destroys the loop

Again, as $\gamma \asymp 1$ we can estimate contribution of the loop by $Ch^{1-d} \times h^{\frac{1}{2}(d-1)} = h^{-\frac{1}{2}(d-1)}$ where extra factor $h^{\frac{1}{2}(d-1)}$ is due to Theorem 5.3.14. Scaling procedure leads to estimate (5.3.53).

As $g'^{jk} = \delta_{jk}$ Appendix 5.A.1 implies that estimate (5.3.36) holds. Thus it holds for $g'^{jk} = \mathsf{const}$. For variable g'^{jk} we get instead of phase $|\xi'|^2$ another phase $|\xi'|^2 + \omega(x, \xi')$ with $\omega = O(\gamma|\xi'|^2)$ which leads to an error with an extra factor γ in the error estimate which becomes $h^{-\frac{1}{2}(d-1)}\gamma^{-\frac{1}{4}(d-1)}$ and is less than the right hand expression of (5.3.36).

Therefore *under assumption (5.3.62) Theorem 5.3.11 is proven*.

We can consider operator (5.3.64) and add magnetic field to it.



(a) No magnetic field case (b) Magnetic field case

Figure 5.4: Short loop without and with magnetic field

In this case equalities $\xi(-t) = -\xi(t)$ and $x(-t) = x(t)$ along trajectories passing through Θ fail, Λ loses is value and picture on Figure 5.2(b) is replaced by the picture on the left where actually the arcs are symmetric only asymptotically. However as before for each x (with $0 < x_1 < \epsilon$) there

exists $\xi = \xi(x)$ such that $(x,\xi) \in \Sigma_0$ and Hamiltonian trajectory passing through (x,ξ) comes back to x. So, instead of few looping trajectories and each such trajectory being a loop for each of its points, we have now many looping trajectories but each of them serves only one point.

Further, one can prove that then the phase is changed by $O(\varepsilon)$ where $\varepsilon = \gamma$ and scaling back we get an extra term $O(h^{-1}\gamma^{\frac{3}{2}} \times \varepsilon) = O(h^{-1}\gamma^{\frac{5}{2}})$ in the exponent which leads to the error not exceeding $Ch^{-\frac{1}{2}(d+1)}\gamma^{-\frac{1}{4}(d-7)}$ which does not exceed the right-hand expression of (5.3.36) as $h^{\frac{2}{3}} \le \gamma \le \bar{\gamma}_1$.

Theorem 5.3.11 proven completely. □

Remark 5.3.15. Obviously

$$W(x) = \mathsf{const}\, \varrho^{\frac{4}{3}}(x), \qquad \mathsf{const} = \big(\frac{3}{2}\big)^{\frac{2}{3}} \tag{5.3.65}$$

where $\varrho(x)$ is the distance in the metrics $g^{jk}V(x)^{-1}$ from x to X^0. Thus one can rewrite the correction term as

$$h^{-\frac{2}{3}d}|\nabla V(x)|^{\frac{d}{3}}\mathcal{Q}\Big(\mathsf{const}\, h^{-\frac{2}{3}}\varrho^{\frac{4}{3}}(x)\Big). \tag{5.3.66}$$

Finally, let us get rid of condition $|\nabla V(x)| \ge \epsilon$. To do this let us rewrite first the right-hand expression of (5.3.36) as

$$Ch^{-\frac{1}{2}(d-1)}\big(\max(|V|, h^{\frac{2}{3}})\big)^{\frac{1}{2}(d-2)} \tag{5.3.67}$$

thus releasing notation γ.

Let us introduce a standard scale $\gamma = \big(|V| + |\nabla V|^2\big)^{\frac{1}{2}} + h^{\frac{1}{2}}$ but exclusively at point $\bar{x}$ in question. Then applying the standard rescaling $x \to x\gamma^{-1}$, $h \to \hbar = h\gamma^{-2}$, $V \to V\gamma^{-2}$, $\nabla V \to \nabla V\gamma^{-1}$ we get instead of (5.3.67) expression

$$Ch^{1-d}\gamma^{d-2}\big(\max(|V|\gamma^{-2}, h^{\frac{2}{3}}\gamma^{-\frac{4}{3}})\big)^{\frac{1}{2}(d-2)} = Ch^{1-d}\big(\max(|V|, h^{\frac{2}{3}}\gamma^{\frac{2}{3}})\big)^{\frac{1}{2}(d-2)}.$$

As $d \ge 2$ the latter expression can only increase as γ is replaced by 1. However as $d = 1$ situation is different and we arrive to

$$C\min(|V|^{-\frac{1}{2}}, h^{-\frac{1}{3}}\gamma^{-\frac{1}{3}})\big). \tag{5.3.68}$$

Thus we proved

Theorem 5.3.16. *Let conditions* (5.3.1), (5.3.4), (5.3.5), (5.3.6) *and* (5.3.7) *be fulfilled. Then*

(i) As $d = 2$ *asymptotics* (5.3.36) *holds; furthermore one can skip the correction term without penalty as* $h^{d-1}|\nabla V|^{d+1} \le |V|^{3d-1}$.

(ii) As $d = 1$ *the left-hand expression of* (5.3.36) *does not exceed*

$$C\begin{cases} |V(x)|^{-\frac12} & \text{as } \ h^{\frac23}|\nabla V(x)|^{\frac23} \le |V(x)| \le |\nabla V(x)|^2 \\ & \text{or as } \ |V(x)| \ge \max\bigl(|\nabla V(x)|^2, h\bigr), \\ h^{-\frac13}|\nabla V(x)|^{-\frac13} & \text{as } \ |V(x)| \le h^{\frac23}|\nabla V(x)|^{\frac23},\ |\nabla V(x)| \ge h^{\frac12}, \\ h^{-\frac12} & \text{as } \ |V(x)| \le h, |\nabla V(x)| \le h^{\frac12}; \end{cases} \tag{5.3.69}$$

further, as $|V(x)| \ge \max\bigl(|\nabla V(x)|^2, h\bigr)$ *one can skip the correction term without penalty.*

Problem 5.3.17. Consider averaged with respect to the spectral parameter correction term with $s > 0$ and, respectively, as $d > s + 3$ and generalize results we derived as $s = 0$, $d = 1, 2$.

5.3.2 Dirac Operator

Assumptions

So, we consider Dirac operator (5.3.2)

$$A = \sum_{j,l} \frac{1}{2}\sigma_l(\omega^{jl}P_j + P_j\omega^{jl}) + \sigma_0 m + V \cdot I$$

where coefficients ω^{jl}, the components of the vector-potential V_j and the scalar potential V are real-valued functions, σ_l are $\mathsf{D} \times \mathsf{D}$-dimensional Pauli matrices (see Appendix 5.A.2), $\mathsf{D} \ge 2$, $m = \mathsf{const} \ge 0$ is a mass parameter, and $h \in (0, 1)$.

We can also consider *massless Dirac operator*

$$A = \sum_{j,l} \frac{1}{2}\sigma_l(\omega^{jl}P_j + P_j\omega^{jl}) + V \cdot I \tag{5.3.70}$$

which is not necessarily a special case of (5.3.2) because for a given dimension D there may be only d rather than $(d + 1)$ of $\mathsf{D} \times \mathsf{D}$-dimensional Pauli

matrices[20]; however unless otherwise specified we consider it under $m = 0$ blanket as well.

Moreover, for the future applications we also consider the *generalized Dirac operator*

$$\boldsymbol{A} = J^{-\frac{1}{2}}(A + W\sigma_0)J^{-\frac{1}{2}} \tag{5.3.71}$$

with

$$J = \frac{1}{2}J_+(I + \sigma_0) + \frac{1}{2}J_-(I - \sigma_0) \tag{5.3.72}$$

where $J_\pm$ are positive-valued functions.

Let us discuss operator (5.3.2) in the simple case of $\omega^{jl} = \delta_{jl}$, $V_j = V = 0$:

$$a(x,\xi) = \sum_j \sigma_j \xi_j + m\sigma_0 + V(x). \tag{5.3.73}$$

Then $a(x,\xi)$ has eigenvalues

$$\lambda_\pm = \pm(m^2 + |\xi|^2)^{\frac{1}{2}} + V(x) \tag{5.3.74}$$

and observe that

(5.3.75) For given x there is a spectral gap $(-m + V(x), m + V(x))$ which is empty as $m = 0$.

As $m \lesssim 1$ we get rather standard case of the matrix operator and actually we will improve results getting rid of the microhyperbolicity condition.

On the other hand, as $m \gg 1$ we get a huge spectral gap and to avoid it we need to take energy level close to its end; alternatively we can pick up energy level 0 but replace $V(x)$ by $\mp m + V_0(x)$. Then the "interesting" eigenvalue of $a(x,\xi)$ becomes

$$\lambda_\pm = \pm(m^2 + |\xi|^2)^{\frac{1}{2}} \mp m + V(x) \approx \pm \frac{1}{2m}|\xi|^2 + V(x) = \pm\frac{1}{2m}\Big(|\xi|^2 \pm 2mV(x)\Big) \tag{5.3.76}$$

as $|\xi| \lesssim 1$ and we arrive basically to the Schrödinger operator with the potential $\pm 2mV(x)$ and we want it to be of the type we considered before.

[20] This can happen only for odd d.

The following array of assumptions codifies these two different situations for the generalized Dirac operator (5.3.71)–(5.3.72). Let us assume that

$$(5.3.77)_{1-2} \qquad |D^\alpha \omega^{jl}| \le c, \quad |D^\alpha V_j| \le c \qquad \forall \alpha : |\alpha| \le K,$$

$$(5.3.77)_3 \qquad |D^\alpha V| \le c \min(1, \frac{1}{m}) \qquad \forall \alpha : 1 \le |\alpha| \le K,$$

$$(5.3.77)_4^- \qquad (\tau_j - m - V)_- \le c \min(1, \frac{1}{m}),$$

$$(5.3.77)_4^+ \qquad (\tau_j + m - V)_+ \le c \min(1, \frac{1}{m}),$$

$$(5.3.77)_5 \qquad |D^\alpha W| \le c \min(1, \frac{1}{m}) \qquad \forall \alpha : |\alpha| \le K$$

and

$$(5.3.77)_6 \qquad |D^\alpha J_\varsigma| \le c \quad \forall \alpha : |\alpha| \le K, \quad \varsigma = \pm,$$

$$(5.3.77)_7 \qquad \epsilon_0 \le J_\varsigma \le c$$

where $\tau_1 < \tau_2$ are the energy levels in question.

Moreover let us assume that

(5.3.78) The metric tensor

$$(5.3.79) \qquad g^{jk} = \sum_{l,r} \omega^{jl}\omega^{kr}\delta_{lr}$$

satisfies $(5.3.6)_1$ and (5.3.7).

Let us recall that

$$(5.3.80) \qquad \sigma_l^\dagger = \sigma_l,$$

$$(5.3.81) \qquad \sigma_l\sigma_k + \sigma_k\sigma_l = 2\delta_{kl}I \qquad (k, l = 0, \dots, d),$$

i. e., that σ_l are Hermitian, unitary and anti-commuting. The properties of Pauli matrices are presented more explicitly in Appendix 5.A.2.

Calculations

Lemma 5.3.18. *For $d \ge 1$ the principal symbol (of the Dirac operator)*

$$(5.3.82) \qquad a(x, \xi) = \sum_{j,l} \sigma_l \omega^{jl}(\xi_j - V_j) + \sigma_0 m + V \cdot I$$

has two eigenvalues

$$\lambda_{\mp} = V \mp (g^{jk}(\xi_j - V_j)(\xi_k - V_k) + m^2)^{\frac{1}{2}} \tag{5.3.83}$$

of multiplicity $\frac{1}{2}\mathsf{D}$.

Proof. Obviously it is sufficient to treat the case $V = V_1 = ... = V_d = 0$. Then (5.3.81) implies that $a(x,\xi)^2 = g^{jk}\xi_j\xi_k + m^2$ and hence there is no eigenvalue of a different from $\lambda_{\pm}$.

Since $\lambda_{\pm}$ coincide only if $\xi = 0, m = 0$ then in the calculation of multiplicities one can assume that $\xi = 0, m > 0$. But in this case the multiplicities of $\lambda_{\pm}$ are equal to $\frac{1}{2}\mathsf{D}$ since $\sigma_1^{-1}\sigma_0\sigma_1 = -\sigma_0$; for the massless Dirac operator we can consider instead $\xi = (1, 0, ..., 0)$. □

Remark 5.3.19. (i) Let $d = 1$; this is possible only for massless Dirac operator $A = \omega(D_1 - V_1) + V$ and the conclusion fails; however it is still valid for $d \geq 2$ and

$$a(x,\xi) = \sum_{j,l} \sigma_l \omega^{jl}(\xi_j - V_j) + V \cdot I. \tag{5.3.84}$$

(ii) Without any loss of the generality one can assume that $\mathsf{D} = \bar{\mathsf{D}} := 2^{\lfloor \frac{d+1}{2} \rfloor}$ because $\sigma_k = q^{\dagger}(\bar{\sigma}_k \otimes I_{\mathsf{D}'})q$ with the unitary matrix q, $\bar{\mathsf{D}} \times \bar{\mathsf{D}}$ matrices $\bar{\sigma}_k$ and $\mathsf{D}' = \mathsf{D}\bar{\mathsf{D}}^{-1}$. In particular, for $d = 1, 2$ we get $\bar{\mathsf{D}} = 2$ and the eigenvalues of $a(x,\xi)$ are simple.

(iii) Similarly for massless Dirac operator without any loss of the generality one can assume that $\mathsf{D} = \bar{\mathsf{D}} := 2^{\lfloor \frac{d}{2} \rfloor}$. In particular, for $d = 3$ we get $\bar{\mathsf{D}} = 2$ and the eigenvalues of $a(x,\xi)$ are simple.

As a corollary we obtain that for the Dirac operator

$$\mathsf{tr}(\kappa_0(x,\tau_1,\tau_2)) = \frac{\mathsf{D}}{2}(2\pi)^{-d} \sum_{\varepsilon=\pm} \int_{\tau_1<\lambda_\varepsilon<\tau_2} d\xi \tag{5.3.85}$$

and for the generalized Dirac operator

$$\mathsf{tr}(\kappa_0(x,\tau_1,\tau_2)) = (2\pi)^{-d} \int \boldsymbol{n}(x,\xi,\tau_1,\tau_2) d\xi \tag{5.3.86}$$

where $\boldsymbol{n}(x,\xi,\tau_1,\tau_2)$ is the eigenvalue counting function for the symbol

$$\boldsymbol{a}(x,\xi) = J(x)^{-\frac{1}{2}}\big(a(x,\xi)+\sigma_0 W(x)\big)J(x)^{-\frac{1}{2}} \tag{5.3.87}$$

and $\boldsymbol{a}$ is defined by (5.3.82).

Let us calculate $\kappa_1'(y,\tau)$. The arguments of Section 4.3 imply that $\kappa_1'(y,\tau)$ depends only on the values of ω^{jl}, V_j, V, $m+W$, J_+, J_- and their first-degree derivatives at the point y and the dependence on the derivatives is linear.

Since the gauge transformation is equivalent to multiplying the spectral function $e(x,y,\tau_1,\tau_2)$ by $\exp ih^{-1}(\varphi(x)-\varphi(y))$ and hence it does not change $e(y,y,\tau_1,\tau_2)$ then it does not change $\kappa_1'(y,\tau)$ either. Then $\kappa_1'(y,\tau)$ depends on V_j only through $F_{jk}(y)$. Therefore it is sufficient to be able to calculate $\kappa_1'(y,\tau)$ in two cases:

(a) $V_j=0$ and ω^{jl}, V, W, J_+ and J_- are linear functions;

(b) V_j are linear functions and ω^{jl}, V, W, J_1 and J_2 are constants.

and then add the results.

But in the case (a) the substitution $A\mapsto \sigma_0 A\sigma_0$, $x_j\mapsto 2y_j-x_j$ (which obviously does not change $\mathsf{tr}(e(y,y,\tau_1,\tau_2))$) is equivalent to reversing the signs of the coefficients at (x_j-y_j) in ω^{kl}, V, W, J_+ and J_- and therefore $\mathsf{tr}(e(y,y,\tau_1,\tau_2))$ does not depend on the derivatives of these functions.

In the case (b) we reach $g^{jk}=\delta_{jk}$ after an appropriate linear change of variables; then one can assume that $\omega^{jl}=\delta_{jl}$ (otherwise we obtain these equalities by replacing σ_l with $\sigma_l'=\sum_k \omega^{lk}\sigma_k$ as $l=1,\dots,d$ and $\sigma_0'=\sigma_0$; one can check that these new matrices also satisfy (5.3.80) and (5.3.81).

Let us now apply an explicit calculation based on (4.3.39). Taking into account that $\boldsymbol{a}^s=0$ and J is constant we obtain that

$$\mathsf{tr}(\kappa_1'(y,\tau)) = \frac{1}{2}(2\pi)^{-d-1}\int \mathsf{Res}_{\mathbb{R}}\,\mathcal{F}_1\,d\xi \tag{5.3.88}$$

with

$$\mathcal{F}_1 = \sum_j \mathsf{tr}\big(J(J\tau-\boldsymbol{a})^{-1}\boldsymbol{a}_{(j)}(J\tau-\boldsymbol{a})^{-1}\boldsymbol{a}^{(j)}(J\tau-\boldsymbol{a})^{-1}\big). \tag{5.3.89}$$

Including for the sake of simplicity $-\frac{1}{2}(J_++J_-)\tau$ in V and $\frac{1}{2}(J_--J_+)\tau+W$ in m; then taking into account that $\boldsymbol{a}^s=0$ we obtain that

$$\mathcal{F}_1 = \mathsf{tr}\big(J(\tau-\boldsymbol{a})^{-1}\boldsymbol{a}_{(j)}(\tau-\boldsymbol{a})^{-1}\boldsymbol{a}^{(j)}(\tau-\boldsymbol{a})^{-1}\big) \tag{5.3.90}$$

where $a_{(j)}$ and $a^{(j)}$ are the derivatives of a with respect to x_j and ξ_j respectively and after calculations we need to take $\tau = 0$ and finally substitute $V \mapsto V - \frac{1}{2}(J_+ + J_-)\tau$ and $m \mapsto m + \frac{1}{2}(J_- - J_+)\tau + W$.

Let us note that

$$\bigl(\alpha I + \sum_j \alpha_j \sigma_j\bigr)^{-1} = \bigl(\alpha^2 - \sum_k \alpha_k^2\bigr)^{-1}\bigl(\alpha I - \sum_j \alpha_j \sigma_j\bigr)$$

for $\alpha, \alpha^1, \dots, \alpha^d \in \mathbb{C}$; therefore

$$\mathcal{F}_1 = -\Delta^{-3}\,\mathrm{tr}\Bigl(J(\tau + a' - V)\sum_{j,k}\sigma_k V_{k(j)}(\tau + a' - V)\sigma_j \cdot (\tau + a' - V)\Bigr)$$

where

$$\Delta = (\tau - V)^2 - |\xi|^2 - m^2, \qquad a' = \sum_l \sigma_l \xi_l + \sigma_0 m, \qquad V_{k(j)} = \partial_j V_k$$

and we have taken into account that $a^{(j)} = \sigma_j$, $a_{(j)} = -\sigma_k V_{k(j)}$ because of simplifying assumptions. Since

$$(\tau + a' - V)\sigma_k = \sigma_k(\tau + a' - V) + 2\xi_k I$$

and $(\tau + a' - V)(\tau - a' - V) = \Delta I$ and these two factors commute we conclude that

$$\begin{aligned}(5.3.91)\quad \mathcal{F}_1 = -\Delta^{-2}\,\mathrm{tr}\Bigl(\sum_{j,k} J\sigma_k\sigma_j(\tau + a' - V)V_{k(j)}\Bigr) - \\ 2\Delta^{-2}\,\mathrm{tr}\Bigl(\sum_{j,k} J\sigma_j\xi_k V_{k(j)}\Bigr) - 2\Delta^{-2}\,\mathrm{tr}\Bigl(\sum_{j,k} J(\tau + a' - V)V_{k(j)}\xi_k\xi_j\Bigr).\end{aligned}$$

The properties of Pauli matrices (see Appendix 5.A.2) yield that the second term vanishes. Without any loss of the generality one can assume that $V_{k(j)} = -V_{j(k)} = -\frac{1}{2}F_{kj}$; we can always arrange this by a gauge transformation. Then the third term also vanishes and (5.3.91) becomes

$$\mathcal{F}_1 = \frac{1}{2}\Delta^{-2}\,\mathrm{tr}\Bigl(\sum_{j,k}\sigma_k\sigma_j F_{kj} J(\tau + a' - V)\Bigr).$$

Then the properties of Pauli matrices yield that $\mathcal{F}_1$ vanishes unless $d=2$, in which case only terms of type $f\sigma_0$ with scalar f in the product $J(\tau+a'-V)$ are essential and hence

$$\mathcal{F}_1 = \frac{1}{2}\Delta^{-2}\operatorname{tr}\Big((\sigma_1\sigma_2\sigma_0)F_{12}\big((J_+ + J_-)m + (J_+ - J_-)(\tau - V)\big)\Big). \tag{5.3.92}$$

Explicit calculation in polar coordinates in $\mathbb{R}^2_\xi$ yields that

$$\int \Delta^{-2} d\xi = \pi\big((\tau - V)^2 - m^2\big)^{-1}$$

and the real residue of this function is equal to

$$\frac{\pi^2 i}{m}\big(\delta(\tau - V - m) - \delta(\tau - V + m)\big)$$

and therefore

$$\operatorname{tr}\kappa_1'(x,\tau) = \frac{1}{8\pi}F_{12}\operatorname{tr}(i\sigma_1\sigma_2\sigma_0)\big(\delta(\tau - V - m) - \delta(\tau - V + m)\big) \tag{5.3.93}$$

which implies

$$\operatorname{tr}(\kappa_1(x,\tau_1,\tau_2)) = \frac{1}{8\pi}F_{12}\operatorname{tr}(i\sigma_1\sigma_2\sigma_0)\Big(\theta(\tau - V - m) - \theta(\tau - V + m)\Big)\Big|_{\tau=\tau_1}^{\tau=\tau_2}. \tag{5.3.94}$$

Recalling that some terms were "hidden" in V and m we conclude that

$$\operatorname{tr}(\kappa_1'(x,\tau)) = \frac{1}{8\pi}F_{12}\operatorname{tr}(i\sigma_1\sigma_2\sigma_0)\sum_{\varsigma=\pm}\varsigma J_\varsigma \delta\big(J_\varsigma\tau - \varsigma(m+W) - V\big) \tag{5.3.95}$$

and thus finally

$$\operatorname{tr}(\kappa_1(x,\tau_1,\tau_2)) = \frac{1}{8\pi}F_{12}\operatorname{tr}(i\sigma_1\sigma_2\sigma_0)\sum_{\varsigma=\pm}\varsigma\theta\big(J_\varsigma\tau - \varsigma(m+W) - V\big)\big|_{\tau=\tau_1}^{\tau=\tau_2}. \tag{5.3.96}$$

These arguments with necessary modifications work for the massless Dirac operator as well resulting the answer which is 0. So we have proven

Lemma 5.3.20. *(i) For the generalized Dirac operator* (5.3.71)–(5.3.72) $\operatorname{tr}(\kappa_1(x,\tau_1,\tau_2))$ *vanishes for* $d\neq 2$ *and* $\operatorname{tr}(\kappa_1(x,\tau_1,\tau_2))$ *is given by* (5.3.96) *for* $d=2$*; this formula becomes* (5.3.94) *for Dirac operator.*

(ii) For the massless Dirac operator $\operatorname{tr}(\kappa_1(x,\tau_1,\tau_2))=0$.

All calculations are modulo gradient.

Asymptotics for $m \ge c_0$

Let us pass from calculations to asymptotics. We assume that $\tau_2 = 0$, which is not a restriction for Dirac operator (5.3.2) but is a restriction for the more general operator (5.3.71). In this subsubsection we consider the case

$$m \ge c_0. \tag{5.3.97}$$

Then at least one of the two conditions

$$W + m \mp V \ge \epsilon_0 m \tag{5.3.98}_{\pm}$$

is fulfilled. Moreover let us assume that

$$m + W \ge |V - \tau_1| + \epsilon_0 m; \tag{5.3.99}$$

under assumption (5.3.97) we can always satisfy this condition by an appropriate choice of τ_1 but we cannot guarantee that $\tau_1 < \tau_2$. So, we select τ_1 belonging to the semiclassical spectral gap which is here for sure under assumption (5.3.97).

Without any loss of generality one can assume that

$$\sigma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{5.3.100}$$

(see Appendix 5.A.2). Separating the $\mathbb{C}^{\mathsf{D}}$-valued functions into two components $\begin{pmatrix} v_+ \\ v_- \end{pmatrix}$ and the $\mathbb{C}^{\mathsf{D}} \times \mathbb{C}^{\mathsf{D}}$-valued functions into four blocks $\begin{pmatrix} u_{++} & u_{+-} \\ u_{-+} & u_{--} \end{pmatrix}$ we obtain that the non-stationary generalized Dirac equation

$$(JhD_t - A)J^{-\frac{1}{2}}u = 0 \tag{5.3.101}$$

is equivalent to two equations

$$(J_+ hD_t - m - W - V)u^+ - \mathcal{M}u^- = 0, \tag{5.3.102}_+$$

and

$$-\mathcal{M}^\dagger u^+ + (J_- hD_t + m + W - V)u^- = 0 \tag{5.3.102}_-$$

where

$$\mathcal{M} = \frac{1}{2}\sum_{j,l} \sigma'_l(\omega^{jl}P_j + P_j\omega^{jl}), \tag{5.3.103}$$

$$\mathcal{M}^\dagger = \frac{1}{2}\sum_{j,l} \sigma'^\dagger_l(\omega^{jl}P_j + P_j\omega^{jl}) \tag{5.3.103}^\dagger$$

and due to (5.3.100) $\sigma_l = \begin{pmatrix} 0 & \sigma_l' \\ \sigma_l'^{\dagger} & 0 \end{pmatrix}$.

In this case if condition $(5.3.98)_{\pm}$ is fulfilled then the second (the first respectively) of these equations is elliptic for $|\tau| \le \epsilon_0 m$ and hence we can exclude the corresponding component. Let us assume that condition $(5.3.98)_{-}$ is fulfilled. Then plugging

$$u^- \equiv (J_- hD_t + m + W - V)^{-1} \mathcal{M}^{\dagger} u^+ \tag{5.3.104}$$

into $(5.3.101)_{+}$ we obtain that

$$m\Big((J_+ hD_t - m - W - V) - \mathcal{M}(J_- hD_t + m + W - V)^{-1}\mathcal{M}^{\dagger}\Big)u^+ \equiv 0 \tag{5.3.105}$$

where $(J_- hD_t + m + W - V)^{-1}$ is a parametrix of $(J_- hD_t + m + W - V)$ in the zone $\{|\tau| \le \epsilon_0 m\}$ and the norms of the errors in (5.3.104) and (5.3.105) do not exceed $C' m^{-1} h^s$.

Let us consider the operator

$$m\mathcal{M}(J_- hD_t + m + W - V)^{-1}\mathcal{M}^{\dagger}; \tag{5.3.106}$$

the properties of Pauli matrices yield that

(5.3.107) The principal symbol of operator (5.3.106) is equal to

$$m(J_-\tau + m + W - V)^{-1} \sum_{j,k} g^{jk}(\xi_j - V_j)(\xi_k - V_k).$$

Rescaling the time $t_{\mathsf{new}} = m^{-1}t$ we rewrite (5.3.105) as

$$(J_+ hD_t - \mathcal{A})u^+ \equiv 0 \tag{5.3.108}$$

with

$$\mathcal{A} = m(m + W + V) + m\mathcal{M}(J_- hD_t + m + W - V)^{-1}\mathcal{M}^{\dagger} \tag{5.3.109}$$

which due to (5.3.107) would be Schrödinger operator (modulo lower order terms) if not hD_t in the parametrix:

(5.3.110) The principal symbol of $\mathcal{A}$ is equal to

$$m(m + W - V)^{-1} b + O(\tau) \tag{5.3.111}$$

with

$$b \coloneqq \sum_{j,k} g^{jk}(\xi_j - V_j)(\xi_k - V_k) + (m + W)^2 - V^2. \tag{5.3.112}$$

Note that due to $(5.3.77)_{3-5}$ the potential

$$V^* := (m+W)^2 - V^2 \tag{5.3.113}$$

satisfies $(5.3.6)_3$.

One can prove easily the following

Proposition 5.3.21. *(Uniform) ellipticity, ξ-microhyperbolicity or microhyperbolicity conditions for operator $\boldsymbol{A}$ on the energy level 0 are equivalent to those for the Schrödinger operator with potential* (5.3.113) *also on the energy level 0 i. e. respectively to*

$$(m+W)^2 - V^2 \ge \epsilon_0, \tag{5.3.114}$$

$$|(m+W)^2 - V^2| \ge \epsilon_0, \tag{5.3.115}$$

$$|(m+W)^2 - V^2 + |\nabla\big((m+W)^2 - V^2\big)| \ge \epsilon_0. \tag{5.3.116}$$

All the arguments for operators scalar in principal and all the arguments for the Schrödinger operator remain true excluding arguments linked with the subprincipal symbol. First of all we have the standard elliptic statement[21)]

Proposition 5.3.22. *(i) Let conditions* (5.3.71), (5.3.71), $(5.3.77)_{1-7}$, (5.3.7), (5.3.97) *and* $(5.3.98)_{\pm}$ *be fulfilled and let*

$$m + W \pm V \ge \epsilon \frac{r}{m} \qquad \forall x \in B(0,1), \tag{5.3.117}_{\pm}$$

with $\epsilon > 0$, $m^2 \ge r \ge 1$. Then

$$|F_{t\to h^{-1}\tau}\chi_T(t)u| \le C'\big(\frac{h}{r}\big)^s T \tag{5.3.118}$$
$$\forall x, y \in B(0, 1-\epsilon) \quad \forall \tau \in [-\epsilon' r, \epsilon' r]$$

where $\epsilon' = \epsilon'(d, c, \epsilon_0, \epsilon) > 0$ is a small constant and $C' = C'(d, s, c, \epsilon_0, \epsilon)$.

(ii) Moreover, the same estimate with an additional factor m^{-1}, m^{-1}, m^{-2} in the right-hand expression holds for the blocks $u^{\mp,\cdot}(.,.,.)$, $u^{\cdot,\mp}(.,.,.)$ and $u^{\mp,\mp}(.,.,.)$ respectively.

[21)] Under assumptions already made condition $(5.3.117)_{\pm}$ below is equivalent to (5.3.114) with the right-hand expression $\epsilon_0 r$.

Then the standard Tauberian arguments yield

Corollary 5.3.23. *(i) In the framework of Proposition 5.3.22*

$$|e(x,y,\tau',\tau'')| \le C'\bigl(\frac{h}{r}\bigr)^s \qquad \forall x,y \in B(0,1-\epsilon)] \quad \forall \tau',\tau'' \in [-\epsilon' r, \epsilon' r]. \tag{5.3.119}$$

(ii) Moreover, the same estimate with a factor m^{-1}, m^{-1}, m^{-2} in the right-hand expression holds for the blocks $e^{\mp,\cdot}(x,y,\tau',\tau'')$, $e^{\cdot,\mp}(x,y,\tau',\tau'')$ and $e^{\mp,\mp}(x,y,\tau',\tau'')$ respectively.

Remark 5.3.24. Corollary 5.3.23 yields that we can assume in what follows that

$$m + W \pm V \le \frac{c}{m}. \tag{5.3.120}_{\pm}$$

Then the following assertions similar to those for the Schrödinger operator are proven easily:

Theorem 5.3.25 [22]**.** *Let assumptions* (5.3.71)–(5.3.71), $(5.3.77)_{1-7}$, (5.3.78), (5.3.97) *and* $(5.3.98)_{\pm}$, $(5.3.120)_{\pm}$ *be fulfilled and let* $d \ge 2$.

Let $\psi \in \mathscr{C}_0^K(B(0,\frac{1}{2}))$, $|\psi| \le 1$ *and let* (4.2.73)–(4.2.75) *be fulfilled and* $L \ge m^{-1}h$. *Then*

(i) The following estimate holds

$$|\mathcal{R}^{\mathsf{W}}_{\varphi,L}| \le \bigl(Ch^{1-d} + C'h^{1-d+\delta'}\bigr)\vartheta\bigl(\frac{h}{mL}\bigr). \tag{5.3.121}$$

(ii) Moreover, let $\Lambda_1^{\pm},\dots,\Lambda_n^{\pm}$ be closed subsets of $\Sigma^{\pm} = \{(x,\xi) : \lambda_{\pm} = 0\}$ where $\lambda_{\pm}$ are given by (5.3.83),

$$\Lambda_j^{\pm} \subset \bigl\{\varrho_{\pm}^2 := m| \pm V + m + W| + m^2|\nabla(V \pm W)|^2 \ge h^{1-\delta}\bigr\}$$

and suppose that through each point of the $h^{\frac{1}{2}-\delta}$-neighborhood of $\Lambda_j^{\pm}$ there passes a Hamiltonian trajectory generated by $mJ_{\pm}(x)^{-1}\lambda_{\pm}(x,\xi)$ with $t \in \mathcal{J}_j = [0,T_j]$ or $\mathcal{J}_j = [-T_j,0]$, $T_j \in [1,h^{-\delta'}]$, along which

[22] Cf. Theorem 5.3.3.

$$(5.3.122)_{1,2} \quad |D^{\alpha}\omega^{jl}| \le c_1 h^{-\delta'}, \quad |D^{\alpha}V_j| \le c_1 h^{-\delta'},$$

$$(5.3.122)_3 \quad |D^{\alpha}V| \le c_1 h^{-\delta'} \min(1, \frac{1}{m}) \quad (|\alpha| \ge 1)$$

$$(5.3.122)_4 \quad |D^{\alpha}W| \le c_1 h^{-\delta'} \min(1, \frac{1}{m}), \quad |D^{\alpha}J_{\varepsilon}| \le c_1 h^{-\delta'} \quad \forall \alpha : |\alpha| \le K',$$

$$(5.3.122)_5 \quad \epsilon_1 h^{\delta'} \le J_{\varepsilon} \le c_1 h^{-\delta'}$$

and conditions (2.4.29), (5.3.14), (5.3.15) *are fulfilled. Further, let* (5.2.101) *with* $s \ge 0$ *be fulfilled.*

Then for $d \ge 3$, $d = 2$ *respectively estimates* (5.3.16) *and* (5.3.26) *hold with* $\vartheta(h/L)$ *replaced by* $\vartheta(h/Lm)$ *and* $a = \lambda_{\pm}$ *and* $\gamma = \varrho_{\pm}$ *in the right-hand expressions.*

Proof. In this modification only the proof of Statement (ii) with $d = 2$ must be changed, only the part at which we consider a ball $B(z', \gamma_1)$ with $|\operatorname{Hess} V^*| \le \gamma_1$, $|\{z_1, z_2\}| \asymp \gamma_3$ where $V^* = (m + W \pm V)m$, $\gamma_3 \ge h^{\delta_3}$ with arbitrarily small $\delta_3 > 0$ and we refer to the proof of Theorem 5.2.7. The difference between magnetic Schrödinger and Schrödinger-Pauli or Dirac operators will be explored later.

Let us consider Dirac operator in the ball of radius $\gamma_4 = h^{\frac{1}{2}-\delta_4}$ with respect to x with a small enough exponent $\delta_4 > 0$. Let us introduce new coordinates $x_{\mathsf{new}} = x\gamma_4^{-1}$, divide the Dirac operator by $\gamma_4\gamma_3$ and replace h, m by $h_{\mathsf{new}} = h/(\gamma_4^2\gamma_3) < h^{\delta_5}$, $m_{\mathsf{new}} = m/(\gamma_3\gamma_4)$. One can easily check that we remain consistent with all the assumptions of this subsection provided $V^* \ge -c_0\gamma_3^2\gamma_4^2$ in the ball in question and in this case the contribution of this ball to the remainder estimate does not exceed

$$C_0 h_{\mathsf{new}}^{-1} + C' h_{\mathsf{new}}^{\delta'-1} \le C_0 h^{-1}\gamma_4^2 |F_{12}| + C' h_{\mathsf{new}}^{\delta'-1}$$

where exponents and constants are changed if necessary; after summation with respect to all the balls of this type we obtain the second and the fourth terms of the right-hand expression in (5.2.26).

On the other hand, balls with $V^* \le -c_0\gamma_3^2\gamma_4^2$ are consistent with the proof of Theorem 5.2.7: after rescaling we divide by $(-V^*)^{\frac{1}{2}}$ and obtain the case in which the microhyperbolicity condition is fulfilled; so the total contribution of the balls of this type does not exceed the sum of the first and the fourth terms of the right-hand expression in (5.3.26).

The similar proof works also for Schrödinger-Pauli operators as mentioned in Remark 5.3.4. □

Also one can prove easily

Theorem 5.3.26 [23]**.** *In the framework of Theorem 5.3.25 but with* $d = 1$ *estimate* (5.3.27) *holds.*

Moreover, if either V^* *satisfies* (5.3.28) *or* (5.2.101) *holds with* $s > 0$ *then Statements (i) and (ii) of Theorem 5.3.25 remain true.*

Also one can explore asymptotics without spatial mollification:

Problem 5.3.27. (i) Prove statements similar to those of Theorems 5.3.7, 5.3.8, 5.3.10, Corollary 5.3.9, with the only difference that not L but Lm should be greater than h and $\vartheta(h/L)$ needs to be replaced by $\vartheta(h/Lm)$ in the right-hand expressions of estimates.

(ii) Prove the statements similar to those of Theorems 5.3.11, 5.3.16.

We believe that the complete proofs of these statements are worth to be published.

Asymptotics for $m \le c_0$

Let us now consider the opposite case

$$m \le c_0. \tag{5.3.123}$$

Remark 5.3.28. Actually in the previous Subsubsection 5.3.2.4 *Asymptotics for* $m \le c_0$ we used only ellipticity of one of the equations $(5.3.102)_\pm$ and one can achieve this by ϵ'-partition provided

$$m + W \ge \epsilon_0 \tag{5.3.124}$$

with the only difference that in sharp or sharper asymptotics both signs $\pm$ may contribute to the remainder estimate; so in (5.3.26) one needs to sum with respect to sign $\pm$. So our main interest is when $m + W \ll 1$. In this case the eigenvalues $\lambda_\pm$ of $a(x,\xi)$ are not necessarily disjoint.

[23] Cf. Theorem 5.3.5.

One can prove easily

Proposition 5.3.29. *(i) Operator* (5.3.71)–(5.3.72) *is microhyperbolic at energy level* 0 *in* ξ*-directions if and only if*

$$|m + W \pm V| \ge \epsilon_0 \tag{5.3.125}$$

for both signs;

(ii) Operator (5.3.71)–(5.3.72) *is microhyperbolic at energy level* 0 *if and only if*

$$|m + W \pm V| + |\nabla(W \pm V)| \ge \epsilon_0 \tag{5.3.126}$$

for both signs and also

$$|m + W| \le \epsilon_0 \implies \exists \ell : \ |\ell_x| = 1,\ \ell_x(V \pm W) \ge \epsilon_0 \tag{5.3.127}$$

for both signs.

Then the general results of Section 4.4 imply

Corollary 5.3.30. *Let conditions* (5.3.71)–(5.3.71), $(5.3.77)_{1-7}$, (5.3.78), *and* (5.3.123), (4.2.73)–(4.2.75) *be fulfilled.*

(i) Further, let (5.3.126)–(5.3.127) *be fulfilled. Then*

$$|\mathsf{R}^{\mathsf{W}}_{\varphi,L}| \le Ch^{1-d}\vartheta\big(\frac{h}{L}\big).$$

(ii) On the other hand, let (5.3.125) *be fulfilled. Then*

$$|\mathsf{R}^{\mathsf{W}}_{x,\varphi,L}| \le Ch^{1-d}\vartheta\big(\frac{h}{L}\big).$$

We will discuss sharp and sharper estimates later. If we want to rescale our main obstacle is condition (5.2.128). The good news is that in our applications we can assume that either (5.2.135) holds or

$$W = 0 \tag{5.3.128}$$

and since the first case is already covered we assume that (5.2.129) holds. Then condition (5.2.128) is equivalent to (5.2.127).

As long we remain in the framework of $m \le \epsilon_0$ there are two modes of rescaling:

(a) If we are interested in estimates with the spatial averaging and magnetic field is 0, i.e.

$$V_1 = ... = V_d = 0 \tag{5.3.129}$$

we can take

$$\gamma = \epsilon\big(|V| + |\nabla V|^2\big)^{\frac{1}{2}} + \bar{\gamma}, \qquad \rho = \gamma^2, \qquad \bar{\gamma} = \max\big(h^{\frac{1}{3}}, m^{\frac{1}{2}}\big) \tag{5.3.130}$$

(b) If we are interested in estimates without the spatial averaging or magnetic field is 0 we should take

$$\gamma = \epsilon|V| + \bar{\gamma}, \qquad \rho = \gamma, \qquad \bar{\gamma} = \max\big(h^{\frac{1}{2}}, m\big). \tag{5.3.131}$$

Then rescaling $x \mapsto x\gamma^{-1}$ and dividing by ρ we get to $h \mapsto h\rho^{-1}\gamma^{-1}$, $m \mapsto m\rho^{-1}$, $L \mapsto L\rho^{-1}$; here we want $m_{\mathsf{new}} = m\rho^{-1} \le 1$ i. e. $\gamma \gtrsim m^{\frac{1}{2}}$ in the case (a) and $\gamma \gtrsim m$ in the case (b).

We also want $L_{\mathsf{new}} = L\rho^{-1} \ge h_{\mathsf{new}} = h\rho^{-1}\gamma^{-1}$ i. e. $\gamma \ge hL^{-1}$ but we will ignore it since estimates hold without this condition as well.

As a result the right-hand expressions with or without spatial averaging become

$$Ch^{1-d} \int \rho^{d-1} \vartheta\big(\frac{h}{L\gamma}\big)\, dx, \tag{5.3.132}$$

where dimmed elements are present only if there is a spatial averaging. Under assumption (5.3.33) we can estimate (5.3.132) by

$$Ch^{1-d}\vartheta\big(\frac{h}{L}\big) \int \rho^{d-1}\gamma^{-s}\, dx, \tag{5.3.133}$$

and it is where all our cases finally separate:

Proposition 5.3.31. *Let conditions* (5.3.71)–(5.3.71), $(5.3.77)_{1-7}$, (5.3.78), *and* (5.3.123), (5.3.128), (4.2.73)–(4.2.75) *be fulfilled. Then*

(i) If condition (5.3.129) *is fulfilled*[24] *the contribution of zone* $\{\gamma \ge 2\bar{\gamma}\}$ *with* γ *and* $\bar{\gamma}$ *defined by* (5.3.130) *to* $\mathsf{R}^{\mathsf{W}}_{\varphi,L}$ *does not exceed*

$$Ch^{1-d}\big(\frac{h}{L}\big) \cdot \begin{cases} 1 & \text{if either } 2d-3-s \ge 0 \\ & \quad \text{or } 2d-2-s+r > 0, \\ |\log \bar{\gamma}| & \text{if } 2d-3-s+r = 0, \\ \bar{\gamma}^{2d-3-s+r} & \text{if } 2d-3-s+r < 0. \end{cases} \tag{5.3.134}$$

[24] Then we plug $\rho = \gamma^2$.

Here and below r is defined by assumption

(5.3.135)$_r$ $\mathsf{Hess}\, V$ *has at least r eigenvalues disjoint from 0 as* $|\nabla V| \le \epsilon$.

(ii) Otherwise[25] *the contribution of zone* $\{\gamma \ge 2\bar\gamma\}$ *with* γ *and* $\bar\gamma$ *defined by* (5.3.131) *to* $\mathsf{R}^{\mathsf{W}}_{\varphi,L}$ *does not exceed*

$$Ch^{1-d}\bigl(\frac{h}{L}\bigr)\cdot \begin{cases} 1 & \text{if either } d-2-s\ge 0 \\ & \quad \text{or } d-2-s+r>0, \\ |\log\bar\gamma| & \text{if } d-2-s+r=0, \\ \bar\gamma^{d-2-s+r} & \text{if } d-2-s+r<0. \end{cases} \tag{5.3.136}$$

(iii) If $|V(x)^2-m^2|\ge \bar\gamma^2$ *with* $\bar\gamma$ *defined by* (5.3.131) *then*

$$|\mathsf{R}^{\mathsf{W}}_{x,\varphi,L}| \le Ch^{1-d}\vartheta\bigl(\frac{h}{L}\bigr)|V(x)|^{d-1-s}. \tag{5.3.137}$$

Now we need to consider zone $\{2\bar\gamma_0 \le \gamma \le 2\bar\gamma\}$ and either calculate its contribution to $\mathsf{R}^{\mathsf{W}}_{\varphi,L}$ or to $\mathsf{R}^{\mathsf{W}}_{x,\varphi,L}$ as $\bar\gamma_0 \le |V(x)| \le \bar\gamma$. There we need to distinguish between $\bar\gamma$ defined by h or by m. In the former case $h_{\mathsf{new}} \asymp 1$ and we skip this step completely and define $\bar\gamma_0 := \bar\gamma$.

In the latter case after rescaling we have $m_{\mathsf{new}} \asymp 1$ and $h\le 1$. It means that we can apply directly results of the previous Subsubsection 5.3.2.3 *Asymptotics for* $m\ge c_0$. However let us explain what this actually means. If we consider asymptotics with the spatial averaging we need to pick up here

$$\gamma = \epsilon\bigl(|V^*| + |\nabla V^*|^2\bigr)^{\frac12} + \bar\gamma_0, \qquad \rho=\gamma, \qquad \bar\gamma_0 = h^{\frac12}$$

with $V^* = V^2-m^2$ and get

$$h_{\mathsf{new}} = h\rho^{-1}\gamma^{-1}, \qquad L_{\mathsf{new}} = L\rho^{-2}.$$

But this would be true after original rescaling; so if we want to make proper definitions in the original settings we need to replace in these equations γ, ρ, V, m, $\bar\gamma_0$, L and h by $\gamma\bar\gamma^{-1}$, $\rho\bar\rho^{-1}$, $V\bar\rho^{-1}$, $m\bar\rho^{-1}$, $\bar\gamma_0\bar\gamma^{-1}$, $L\bar\rho^{-1}$ and $h\bar\gamma^{-1}\bar\rho^{-1}$ respectively thus arriving to

$$\gamma = \epsilon\bar\gamma\bigl(\bar\rho^{-2}|V^*| + \bar\rho^{-4}\bar\gamma^4|\nabla V^*|^2\bigr)^{\frac12} + \bar\gamma_0, \qquad \rho=\bar\rho\bar\gamma^{-1}\gamma, \tag{5.3.138}$$

$$\bar\gamma_0 = h^{\frac12}\bar\gamma^{\frac12}\bar\rho^{-\frac12}, \qquad h_{\mathsf{new}} = h\bar\gamma\bar\rho^{-1}\gamma^{-2}, \qquad L_{\mathsf{new}} = L\bar\rho^{-1}\bar\gamma^2\gamma^{-2}.$$

[25] Then we plug $\rho=\gamma$.

(i) Let us recall that in the case (a) we need to check if $m \ge h^{\frac{2}{3}}$ and plug $\bar\gamma = m^{\frac12}$, $\bar\rho = m$ thus arriving to

$$\gamma = \epsilon\big(m^{-1}|V^2 - m^2| + m|\nabla V|^2\big)^{\frac12} + \bar\gamma_0, \qquad \rho = m^{\frac12}\gamma, \tag{5.3.139}$$
$$\bar\gamma_0 = h^{\frac12} m^{-\frac14}, \qquad h_{\mathsf{new}} = hm^{-\frac12}\gamma^{-2}, \qquad L_{\mathsf{new}} = L\gamma^{-2}$$

where we actually slightly modified γ (without changing its magnitude).

Therefore contribution of the zone $\{2\bar\gamma_0 \le \gamma \le 2\bar\gamma\}$ to $\mathsf{R}^{\mathsf{W}}_{\varphi,L}$ does not exceed

$$Ch^{1-d}m^{d-1-\frac12 s}\vartheta\big(\frac{h}{L}\big)\int_{\{\gamma(x)\le 2\bar\gamma\}} \gamma^{d-2}\,dx \le Ch^{1-d}m^{\frac12(2d-3+r-s)}\vartheta\big(\frac{h}{L}\big) \tag{5.3.140}$$

provided $d \ge 2$ [26) 27)].

(ii) Meanwhile in the case (b) we need check if $m \ge h^{\frac12}$ and to plug into (5.3.138) $\bar\gamma = \bar\rho = m$ thus replacing (5.3.139) by

$$\gamma = \epsilon\big(|V^2 - m^2| + m^4|\nabla V|^2\big)^{\frac12} + \bar\gamma_0, \qquad \rho = \gamma, \tag{5.3.141}$$
$$\bar\gamma_0 = h^{\frac12}, \qquad h_{\mathsf{new}} = h\gamma^{-2}, \qquad L_{\mathsf{new}} = Lm\gamma^{-2}.$$

and (5.3.140) by

$$Ch^{1-d}\vartheta\big(\frac{h}{Lm}\big)\int_{\{\gamma(x)\le 2\bar\gamma\}} \gamma^{d-2}\,dx \le Ch^{1-d}m^{d-2+r-s}\vartheta\big(\frac{h}{L}\big) \tag{5.3.142}$$

as $d \ge 2$ [26)].

(iii) Finally, in the asymptotics without spatial mollification we need to check again if $m \ge h^{\frac12}$ and then take

$$\gamma = \epsilon|V^*| + \bar\gamma_0, \qquad \rho = \gamma^{\frac12}, \qquad \bar\gamma_0 = h^{\frac23}, \qquad h_{\mathsf{new}} = h\gamma^{-\frac32}, \qquad L_{\mathsf{new}} = L\gamma^{-1}$$

after the first rescaling; to derive these relations in the original coordinates one needs the same replacement as in (ii) thus arriving to

$$\gamma = \epsilon m^{-1}|V^2 - m^2| + \bar\gamma_0, \qquad \rho = \gamma^{\frac12}m^{\frac12}, \qquad \bar\gamma_0 = h^{\frac23}m^{-\frac13}, \tag{5.3.143}$$
$$h_{\mathsf{new}} = h\gamma^{-\frac32}m^{-\frac12}, \qquad L_{\mathsf{new}} = L\gamma^{-1}$$

[26)] It was proven that for $d = 2$ for Schrödinger operator there is no logarithmic factor.

[27)] Case $d = 1$ needs to be reexamined anyway.

and deriving estimate

$$|\mathsf{R}^{\mathsf{W}}_{x,\varphi,L}| \le Ch^{1-d}\gamma^{\frac{1}{2}(d-3)}\vartheta\Big(\frac{h}{L\gamma^{\frac{1}{2}}m^{\frac{1}{2}}}\Big) \le Ch^{1-d}|V^2-m^2|^{\frac{1}{2}(d-3-s)}m^{-\frac{1}{2}(d-3)}\vartheta\Big(\frac{h}{L}\Big). \tag{5.3.144}$$

Let us recall that there is also zone $\{\gamma \le 2\bar\gamma_0\}$ but in this zone $h_{\mathsf{new}} \asymp 1$ and one can prove easily that contribution of this zone to the remainder does not exceed the estimate we derived for the first zone as $\bar\gamma_0 = \bar\gamma$ and for the second zone as $\bar\gamma_0 \le \bar\gamma$ (the second statement was actually proven as $m \ge \epsilon_0$.

Thus we arrive to

Theorem 5.3.32. *Let conditions* (5.3.71)–(5.3.71), $(5.3.77)_{1-7}$, (5.3.78), (5.3.123), (5.3.128) *and* (4.2.73)–(4.2.75) *be fulfilled.*

(i) Moreover let $d \ge 2$ *and* (5.3.129) *be fulfilled. Then* $|\mathsf{R}^{\mathsf{W}}_{\varphi,L}|$ *does not exceed* (5.3.134) *with* $\bar\gamma = \mathsf{max}(m^{\frac{1}{2}}, h^{\frac{1}{3}})$.

(ii) In the general case $d \ge 2$ $|\mathsf{R}^{\mathsf{W}}_{\varphi,L}|$ *does not exceed* (5.3.136) *with* $\bar\gamma = \mathsf{max}(m, h^{\frac{1}{2}})$. *In particular,* $|\mathsf{R}^{\mathsf{W}}| \le Ch^{1-d}$ *as* $d \ge 2$.

(iii) $\mathsf{R}^{\mathsf{W}}_{x,\varphi,L}$ *satisfies* (5.3.137) *as* $|V(x)^2-m^2| \ge \bar\gamma^2$ *and* (5.3.144) *as* $m \ge h^{\frac{1}{2}}$, $m^2 \ge |V(x)^2-m^2| \ge h^{\frac{2}{3}}m^{\frac{2}{3}}$. *Furthermore*

$$|\mathsf{R}^{\mathsf{W}}_{x,\varphi,L}| \le Ch^{\frac{1}{2}(1-d-s)}\vartheta\Big(\frac{h}{L}\Big) \tag{5.3.145}$$

as $m + |V(x)| \le h^{\frac{1}{2}}$, *and*

$$|\mathsf{R}^{\mathsf{W}}_{x,\varphi,L}| \le Ch^{\frac{1}{3}(-2d-s)}m^{-\frac{1}{6}(d-3+2s)}\vartheta\Big(\frac{h}{L}\Big) \tag{5.3.146}$$

as $m \ge h^{\frac{1}{2}}$, $|V(x)^2-m^2| \le h^{\frac{2}{3}}m^{\frac{2}{3}}$.

Remark 5.3.33. We did not formulate Theorem 5.3.32(i) as $d = 1$ only because as we know contribution of zone $\{\gamma(x) \le cm^{\frac{1}{2}}\}$ is not necessarily estimated by $C\vartheta(h/L)$.

Further Improvements

Now our goal is to improve Theorem 5.3.32(i) especially as $d = 1$, $s = 0$.

Theorem 5.3.34. *Let conditions* (5.3.71)–(5.3.71), $(5.3.77)_{1-7}$, (5.3.78), *and* (5.3.123), (5.3.128), (5.3.129) *and* (4.2.73)–(4.2.75) *be fulfilled. Then*

(i) Estimate $|\mathsf{R}^{\mathsf{W}}_{\varphi,L}| \le Ch^{1-d}\vartheta(h/L)$ *holds as* $2d > s+2$.

(ii) Estimate

$$|\mathsf{R}^{\mathsf{W}}_{\varphi,L}| \le Ch^{1-d}\vartheta\Big(\frac{h}{L}\Big)\,|\log(h+m)| \tag{5.3.147}$$

holds as $2d = s+2$, $d \ge 2$.

(iii) Estimate

$$|\mathsf{R}^{\mathsf{W}}_{\varphi,L}| \le C\vartheta\Big(\frac{h}{L}\Big)\,|\log(h+m)|^{n-1} \tag{5.3.148}$$

holds as $d = 1$, $s = 0$ *and and both functions* $V \pm m$ *satisfy non-degeneracy condition* $(5.2.100)_n$.

(iv) Estimate

$$|\mathsf{R}^{\mathsf{W}}_{\varphi,L}| \le C\vartheta\Big(\frac{h}{L}\Big)h^{-\delta} \tag{5.3.149}$$

holds as $d = 1$, $s = 0$.

(v) Estimate

$$|\mathsf{R}^{\mathsf{W}}_{\varphi,L}| \le Ch^{1-d}\vartheta\Big(\frac{h}{L}\Big)\big(\max(h^2, m^3)\big)^{\frac{1}{6}(2d-2-s)} \tag{5.3.150}$$

holds as $2d < s+2$.

Proof. Let us introduce scaling functions

$$\gamma_n(x) = \epsilon\Big(\sum_{\alpha:|\alpha|\le n-1} |\nabla^\alpha V|^{\frac{n}{(n-|\alpha|)}}\Big)^{\frac{1}{n}} + \bar\gamma_n, \qquad \rho_n = \gamma_n^n, \tag{5.3.151}$$

$$\bar\gamma_n = \max\big(h^{\frac{1}{n+1}}, m^{\frac{1}{n}}\big).$$

(i) Assume that under assumption $\gamma_n \asymp 1$ Statement (i) is proven. This is the case as $n = 2$ because then the microhyperbolicity condition holds as $m \le \epsilon$ and as $m \ge \epsilon$ we are in the framework of the previous Subsubsection 5.3.2.4 *Asymptotics for* $m \le c_0$.

Let us consider ball $B(x, \gamma_n(x))$ with $\gamma_n(x) \ge C_0\bar{\gamma}_n$. Rescaling it to $B(0,1)$ and simultaneously multiplying operator by ρ_n^{-1} we find ourselves in the framework of $\gamma_n \asymp 1$ with $h \mapsto h\rho_n^{-1}\gamma_n^{-1}$, $L \mapsto L\rho_n^{-1}$ and therefore the contribution of such ball to $\mathsf{R}^{\mathsf{W}}_{\varphi,L}$ does not exceed

$$C\big(\frac{h}{\rho_n\gamma_n}\big)^{1-d}\vartheta\big(\frac{h}{L\gamma_n}\big) \le Ch^{1-d}\rho_n^{d-1}\gamma_n^{d-1-s}\vartheta\big(\frac{h}{L}\big);$$

then the total contribution of all such balls does not exceed

$$Ch^{1-d}\vartheta\big(\frac{h}{L}\big)\int \rho_n^{d-1}\gamma_n^{-1-s}\,dx = Ch^{1-d}\vartheta\big(\frac{h}{L}\big)\int \gamma_n^{(d-1)n-1-s}\,dx. \tag{5.3.152}$$

Meanwhile as $m \ge h^{n/(n+1)}$ consider $B(x, \gamma_n(x))$ with $\gamma_n(x) \asymp \bar{\gamma}_n \asymp m^{1/n}$; rescaling as before we arrive to $m \asymp 1$ and $h \mapsto h\bar{\rho}_n^{-1}\bar{\gamma}_n^{-1}$, $L \mapsto L\bar{\rho}_n^{-1}$. Applying results of the previous Subsubsection we conclude that the contribution of such ball to the remainder does not exceed the same expression as before and again the contribution of all such balls does not exceed

$$Ch^{1-d}\bar{\gamma}_n^{(d-1)n-s-1}\vartheta\big(\frac{h}{L}\big)\,\mathsf{mes}(\{\gamma(x) \le \bar{\gamma}_n\}). \tag{5.3.153}$$

Finally, as $m \le h^{n/(n+1)}$ consider $B(x, \gamma_n(x))$ with $\gamma_n(x) \asymp \bar{\gamma}_n \asymp h^{1/(n+1)}$; rescaling as before we arrive to $m \le 1$ and $h \mapsto 1$ and again estimate (5.3.153) is achieved.

Note that as $(d-1)n > s$ under condition that $\gamma_{n+1} \asymp 1$ an integral in the right-hand expression of (5.3.152) converges and we estimate the right-hand expression of (5.3.152) by $Ch^{1-d}\vartheta(h/L)$. In this case $\mathsf{mes}(\{\gamma(x) \le \bar{\gamma}_n\}) \lesssim \bar{\gamma}_n$ and we estimate (5.3.153) by $Ch^{1-d}\vartheta(h/L)$ as well.

So, we made the step of induction. This inequality $(d-1)n > s$ is fulfilled for $n \ge 3$ provided it is fulfilled for $n = 2$ and $d \ge 2$. So, everything works well and as $(d-1)n \ge s+1$ integrand in (5.3.152) is bounded and we do not even need assumption $\gamma_{n+1} \asymp 1$ to get the proper estimate.

Statement (i) is proven.

(ii)–(iv) As $d = s + 1$ integral in (5.3.152) is logarithmically diverging for $n = 2$ only as $d \ge 2$ and for every n as $d = 1$, $s = 0$; furthermore as $d \ge 2$ for large enough n this integral converges even without condition $\gamma_{n+1} \asymp 1$ and thus Statements (ii) and (iii) are also proven.

On the other hand, as $d = 1$ dropping condition $\gamma_{n+1} \asymp 1$ leads to an extra factor $\bar{\gamma}_n^{-1} \le h^{-\delta}$ with $\delta = 1/(n+1)$. This proves Statement (iv).

Surely, one needs to consider also contribution of the zone $\{\gamma_n \asymp \bar{\gamma}_n\}$; as $m \ge h^{n/(n+1)}$ we get (5.3.153) as $d \ge 2$, or as $d = 1$ and non-degeneracy condition is fulfilled and we get

$$C\bar{\gamma}_n^{-1}\bigl(\frac{h}{\bar{\gamma}_n^{n+1}}\bigr)^{-\delta}\vartheta\bigl(\frac{h}{L}\bigr)$$

as $d = 1$ and we assume no non-degeneracy condition.

Finally, as $m \le h^{n/(n+1)}$ the analysis of zone $\{\gamma_n \asymp \bar{\gamma}_n\}$ is also trivial. So, Statements (ii)–(iv) are proven completely.

(v) As $d < s + 1$, $n = 2$ expression (5.3.152) has the same magnitude as (5.3.153) which is exactly the right-hand expression of (5.3.150). So, as $\gamma_3 \asymp 1$ estimate (5.3.150) is proven even for $d = 1$. Let us scale the right-hand expression of (5.3.150): i. e. plug $m \mapsto m\gamma^{-n}$, $h \mapsto h\gamma^{-(n+1)}$ and multiply by γ^{-d}; we get

$$C\bigl(h\gamma^{-n-1}\bigr)^{1-d}\gamma^{-d}\vartheta\bigl(\frac{h}{L\gamma}\bigr)\bigl(\max\bigl(h^2\gamma^{-2(n+1)}, m^3\gamma^{-3n}\bigr)\bigr)^{\frac{1}{6}(2d-2-s)} \le$$
$$Ch^{1-d}\gamma^{-1+\frac{1}{3}(d+s-1)(n-2)}\vartheta\bigl(\frac{h}{L}\bigr)\bigl(\max\bigl(h^2, m^3\gamma^{-n-2}\bigr)\bigr)^{\frac{1}{6}(2d-2-s)}$$

and then integrating with respect to dx we get the same right-hand expression of (5.3.150) provided either $\gamma_{n+1} \asymp 1$ or n is large enough. This takes care of Statement (v). □

Let us improve Statement (iii) of Theorem 5.3.34:

Theorem 5.3.35. *Let $d = 1$ and conditions* (5.3.71)–(5.3.71)$'$, $(5.3.77)_{1-7}$, (5.3.78)$'$, *and* (5.3.123), (5.3.128) (4.2.73)–(4.2.75) *be fulfilled.*

Furthermore, let both $V \pm m$ satisfy non-degeneracy condition $(5.2.100)_n$. *Then estimate $|\mathsf{R}^{\mathsf{W}}| \le C$ holds.*

Proof. Let us repeat the proof of the previous theorem and treat the behavior of the Hamiltonian trajectories in the zone $\{\rho \ge C_0 m\}$ (the only zone which yields the logarithmic factor)[28)].

Note that on each step of induction $\gamma(x) \asymp \varepsilon + |x_1 - \hat{x}_1|$ with $\varepsilon \ge 0$ where without any loss of the generality one can assume that $\hat{x}_1 = 0$. Then on each step of the induction a logarithmic factor appears only due to the zone $\{|x_1| \ge C_0 \bar{\gamma}_n\}$ with $\bar{\gamma}_n$ increased to

$$\bar{\gamma}_n := \mathsf{max}\big(\varepsilon, m^{\frac{1}{n}}, h^{\frac{1}{n+1}}\big) \tag{5.3.154}$$

and an arbitrarily large constant C_0. But in this zone one should not apply the induction assumption but only the microhyperbolic theorem; so the power of logarithm in the right-hand expression of (5.3.148) is always 1.

Moreover, let us note that after rescaling $t_{\mathsf{new}} = t\gamma^{-1}$ and therefore the only term to be treated is

$$\int_{\{|x_1|\ge C_0\bar{\gamma}_n\}} \frac{1}{T(x_1)}\, dx_1 \tag{5.3.155}$$

and a logarithmic factor appears if we take $T = \epsilon_0\gamma$. Here on the energy level 0

$$\xi_1 = \pm\big(V^2 - m^2\big)^{\frac{1}{2}}. \tag{5.3.156}$$

Let us consider the trajectory of the Hamiltonian system (after ξ_1 is excluded)

$$\frac{dx_1}{dt} = \pm\frac{\big(V^2 - m^2\big)^{\frac{1}{2}}}{V} \tag{5.3.157}$$

starting from the point $\bar{x}_1$ in the time direction in which $|x_1|$ increases.

Then the time until x_1 remains in $\{|x_1| \le \epsilon_0\}$ is disjoint from 0 and one can see easily that in this time interval $|x_1(t)| \asymp |x_1(0)| + |t|$ because as $|x_1|$ increases $|V(x)|$ also increases and stays larger than $2m$.

Then non-periodicity condition is fulfilled trivially. Further, one can prove easily that

$$|\mathfrak{d}x_1(t)| \le C|\mathfrak{d}x_1(0)| + C(|x_1(0)| + |t|)^r |x_1(0)|^{-r-n+1}|\mathfrak{d}\tau|$$

with large enough r and so in order to fulfill condition on the Jacobi matrix we should take $T(x_1) = \epsilon_1 \min(1, |x_1|^{1+\delta}\bar{\gamma}_n^{-\delta})$ with sufficiently small exponent $\delta > 0$; we replaced here $x_1(0)$ by x_1. But then integral (5.3.155) is bounded.□

[28)] Surely, these trajectories coincide geometrically with the trajectories of the Schrödinger operator (depending on the energy level) but dynamically there is an essential difference.

Sharp Remainder Estimates

Now our goal is to derive sharp spectral asymptotics. Let us first formulate them in microhyperbolic case; the following statement is due to Section 4.4:

Theorem 5.3.36. *Let conditions of Corollary 5.3.30 be fulfilled.*

Further, let $\Lambda_j^{\pm}$ *be closed subsets of* $\Sigma^{\pm}$ *and suppose that through every point of* $\Lambda_j^{\pm}$ *there passes a Hamiltonian trajectory generated by* $J_{\pm}^{-1}\lambda_{\pm}$ *with* $t \in \mathcal{J}_j$, $\mathcal{J}_j = [0, T_j]$ *or* $\mathcal{J}_j = [-T_j, 0]$, $T_j \in [1, h^{-\delta'}]$ *in the* $h^{\frac{1}{2}-\delta}$*-neighborhood for which conditions* $(5.3.122)_{1-7}$, (2.4.29), (5.3.14) *are fulfilled and*

$$(m+W)^2 + g^{jk}(\xi_j - V_j)(\xi_k - V_k) \ge \epsilon_1 h^{2\delta'}. \tag{5.3.158}$$

(i) Moreover, let microhyperbolicity condition (5.3.126)-(5.3.127) *be fulfilled and along trajectories non-periodicity condition* (5.3.15) *be fulfilled.*

Then estimate

$$|\operatorname{tr}(\mathsf{R}^{\mathsf{W}}_{\varphi,L})| \le C\Big(\sum_{0\le j\le n} \frac{h^{1-d}}{T_j^{1+s}} \int_{\Lambda_j} d\mu + C' h^{1-d+\delta''}\Big)\vartheta\big(\frac{h}{L}\big) \tag{5.3.159}$$

holds with $\Lambda_j = \Lambda_j^+ \cup \Lambda_j^-$, $d\mu := J dx d\xi : d\lambda_{\pm}$ *on* $\Sigma^{\pm}$.

(ii) Moreover, let microhyperbolicity condition (5.3.125) *be fulfilled and along trajectories non-looping condition*

$$\operatorname{dist}\big(x(t), x\big) \ge \epsilon |t| h^{\frac{1}{2}-\delta} \tag{5.3.160}$$

be fulfilled.

Then estimate (5.3.159) *holds with* $\mathsf{R}^{\mathsf{W}}_{\varphi,L}$, Λ_j *and* $d\mu$ *replaced by* $\mathsf{R}^{\mathsf{W}}_{x,\varphi,L}$, Λ_{xj}, $d\mu_x = J d\xi : d\lambda_{\pm}$ *respectively.*

Remark 5.3.37. (i) Here condition (5.3.158) prevents eigenvalues $\lambda_{\pm}(x,\xi)$ from meeting one another. One can see easily that with $\mu_x = dx : d\lambda$

$$\mu_x(\{|\xi - \boldsymbol{V}(x)|_g \le \varepsilon\}) \le C(m + W(x) + \varepsilon)\varepsilon^{d-2} \tag{5.3.161}$$

where $|.|_g$ is associated with matrix g^{jk} and $\boldsymbol{V} = (V_1, \dots, V_d)$.

(ii) Further, $J_+ \asymp J_-$ and we can skip index here.

(iii) We took the trace to avoid dealing with matrices.

Proposition 5.3.38. *Let $W = 0$ and $d \geq 2$ and suppose the condition*

$$\sum_{j,k} g^{jk}\eta_j\eta_k \geq h^{\delta''}|\eta|^2 \qquad \forall \eta \in \mathbb{R}^d \tag{5.3.162}$$

holds along trajectories. Then in Theorem 5.3.36(i) one can drop condition (5.3.158).

Proof. It is sufficient to prove that $\mu(\mathcal{Z}) \leq Ch^{\delta''}$ with small exponent $\delta'' > 0$. Here $\mathcal{Z}$ is the set of all points $(x,\xi) \in \Lambda$ (we skip index j for the sake of simplicity) such that (5.3.158) is violated somewhere along trajectory.

Let us first note that because of (5.3.161) this estimate holds for the measure of the set of all points of $\Sigma^{\pm} \subset T^*B(0,1)$ at which $|V(x)| \leq \rho_1 := h^{\delta''}$. Hence we should consider points in which $|V(x)| \geq h^{\delta_1}$ and therefore condition (5.3.158) holds but is violated somewhere along the trajectory. Let $\mathcal{Z}'$ be the set of these points. Without any loss of the generality we can assume that $T \leq h^{-\delta_1}$ and all inequalities along trajectories are fulfilled with δ_1 instead of δ'; so δ' which is arbitrarily small but much larger than δ_j is invoked only in definition of $\mathcal{Z}$.

Let X' be the union of x-projections of all the Hamiltonian trajectories of length not exceeding T passing through $\mathcal{Z}'$. Then every such trajectory intersects $\mathcal{Y}_1$ where

$$\mathcal{Y}_\nu = \{(x,\xi) \in \Sigma, 2^{-\nu}\rho \leq |V| \leq 2^{\nu}\rho\}, \quad \rho = h^{\delta'}$$

and one can assume that $m \leq \frac{1}{8}\rho$. Along these trajectories

$$|\frac{dz}{dt}| \leq c_1 h^{-\delta_2}; \tag{5.3.163}$$

hence $\mathcal{X}' \subset B(0, h^{-\delta_3})$ and the trajectory intersecting $\mathcal{Y}_1$ belongs to $\mathcal{Y}_2$ in a time interval of length $t_0 = h^{\delta_4}\rho$. Hence due to ability to select invariant measure equivalent to μ

$$\mu(\mathcal{Z}') \leq \frac{T}{t_0}\mu(\mathcal{Y}_2) \leq C'h^{-\delta_5}\rho^{d-2}\,\mathsf{mes}(\{x \in X', |V| \leq 4\rho\}) \leq C'h^{-\delta_6}\rho^{d-2}$$

because $\mu(\mathcal{Y}_2) \leq C\rho^{d-1}h^{-\delta_5}$ due to (5.3.161), (5.3.162) and $T \leq h^{-\delta_1}$. Then for $d \geq 3$ we obtain the estimate in question as $\delta' > 2\delta_6$.

For $d = 2$ a more sophisticated analysis is necessary. Let

$$\mathcal{Y}'_\nu = \{z \in \mathcal{Y}_\nu, |\nabla V| \le 2^\nu \rho_1\}, \qquad \mathcal{Y}''_\nu = \{z \in \mathcal{Y}_\nu, |\nabla V| \ge 2^{-\nu}\rho_1\}.$$

with small enough $\delta_1 > 0$.

Then the previous arguments are applicable to $\mathcal{Y}'_\nu$, $t'_0 = h^{\delta_4}\rho\rho_1^{-1}$, $\mathcal{Y}''_\nu$ and $t''_0 = h^{\delta_4}\rho$ and hence

$$\mu(\mathcal{Z}') \le C'h^{-\delta_5}\rho^{d-2}\Big(\rho_1 \operatorname{mes}(\{x \in X', |V| \le 4\rho\}) + \operatorname{mes}(\{x \in X', |V| \le 4\rho, |\nabla V| \ge \rho_1\})\Big) \le C'h^{-\delta_6}\rho^{d-\frac{3}{2}}$$

where we estimated the second measure by $Ch^{-\delta_5}\rho\rho_1^{-1}$ and picked up $\rho_1 = \rho^{\frac{1}{2}}$.□

Theorem 5.3.39. *Let conditions of Corollary 5.3.30 be fulfilled.*

Moreover, let $\Lambda_1^\pm, \dots, \Lambda_n^\pm$ *be closed subsets of* $\Sigma^\pm = \{(x,\xi) : \lambda_\pm = 0\}$ *where* $\lambda_\pm$ *are given by* (5.3.83)*,*

$$\Lambda_j^\pm \subset \{\gamma = |\nabla V| + \min\big(|V|, \frac{|V|^2}{m}\big) \ge h^{\frac{1}{2}-\delta}\}$$

and through every point of $\Lambda_j^\pm$ *there passes a Hamiltonian trajectory generated by* $J_\pm^{-1}\lambda_\pm$ *with* $t \in \mathcal{J}_j$, $\mathcal{J}_= [0, T_j]$ *or* $\mathcal{J}_= [-T_j, 0]$, $T_j \in [1, h^{-\delta'}]$ *in the* $h^{\frac{1}{2}-\delta}$*-neighborhood for which conditions* $(5.3.122)_{1-5}$ *and conditions* (2.4.29), (5.3.14), (5.3.15) *are fulfilled. Further, let* (5.2.101) *with* $s \ge 0$ *be fulfilled.*

(i) Let <u>*either*</u> $d > s + 2$ <u>*or*</u> $V_j = 0$ *and* $2d > s + 2$*. Then estimate* (5.3.16) *with* $\Lambda_j = \Lambda_j^+ \cup \Lambda_j^-$, $da = d\lambda_\pm$ *at* $\Sigma^\pm$ *remains true.*

(ii) Let $d = 2$, $s = 0$*. Then estimate* (5.3.26) *holds where the notation of Theorem 5.3.3 is used.*

Proof. Proof of Statement (i) is trivial. Let us prove Statement (ii). We need to consider only balls of radius $\gamma_1 = h^{\frac{1}{2}-\delta}$ in which

$$(5.3.164)_{1,2,3} \qquad |V| \le c_0 h^{\frac{1}{2}-\delta'}, \quad |\nabla V| \le c_0 h^{\frac{1}{2}-\delta'}, \quad m \le c_0 h^{\frac{1}{2}-\frac{1}{4}\delta'};$$

the last condition follows from the fact that $\gamma \ge \epsilon_0 m$ in this proof. Let us consider various cases:

(a) If $|F_{12}(x')| = \mu \ge h^{\frac{1}{4}\delta'}$ and $v = |V(x')| \le \mu\gamma_1$ then $|F_{12}| \asymp \mu$ and $|V| \le c\mu\gamma_1$ in the ball $B(x', \gamma_1)$. Let us change variables $x_{\mathsf{new}} = x\gamma_1^{-1}$, divide A by $\mu\gamma_1$ and set $h_{\mathsf{new}} = h(\mu\gamma_1^2)^{-1}$, $m_{\mathsf{new}} = m(\mu\gamma_1)^{-1}$; then we remain in the framework of our theorem and the same arguments as in the proof of Theorem 5.3.25 yield that the contribution of this ball to the remainder does not exceed

$$C\gamma_1^2\mu h^{-1} + C''\gamma_1^2 h^{\delta''-1} + C''h^r.$$

(b) Let $|F_{12}(x')| = \mu \ge h^{\frac{1}{4}\delta'}$ and $v = |V(x')| \ge \mu\gamma_1$. Let us change variables $x_{\mathsf{new}} = x\gamma_1^{-1}$, divide A by v and set $h_{\mathsf{new}} = h(v\gamma_1)^{-1}$, $m_{\mathsf{new}} = mv^{-1}$; then we remain in the framework of this theorem and the microhyperbolicity condition is fulfilled; applying Theorem 5.3.25 we obtain that the contribution of this ball to the remainder does not exceed

$$C\gamma_1 vh^{-1} + C'\gamma_1^2 h^{\delta''-1} + C''h^r.$$

(c) If $|F_{12}(x')| \le \mu = h^{\frac{1}{2}\delta'}$ then the analysis of Parts (a) and (b) is applicable for $v = |V(x')| \le \mu\gamma_1$ and $v = |V(x')| \ge \mu\gamma_1$ respectively and we obtain that the estimate of (b) remains true.

Calculating the total contribution of all balls of this type we obtain (5.3.26). □

Remark 5.3.40. Thus again we can easily derive asymptotics with the remainder estimate $o\big(h^{1-d}\vartheta(h/L)\big)$ under assumption that the set of all periodic trajectories has measure 0; as $m = 0$ we should consider only trajectories which do not hit

$$\Theta_\tau = \{(x,\xi) :\ \xi_1 - V_1(x) = ... = \xi_d - V_d(x) = 0, V(x) = \tau\}.$$

As $d = 2$ we need an extra condition

$$\mathsf{mes}(\{x : V(x) = \pm m, |F(x)| > 0\}) = 0. \tag{5.3.165}$$

Remark 5.3.41. (i) Similar results could be proven for asymptotics without spatial mollification; however in this case we cannot get rid of condition (5.3.158) along trajectories but one could check that actual assumption should be with δ' replaced by $\frac{1}{2} - \delta'$.

(ii) It would be rather interesting to improve asymptotics without averaging as $V_1 = ... = V_d = 0$ with addition of correction term, especially in the case of $m = 0$. However condition (5.3.158) seem to be an obstacle.

5.4 Operators with Irregular Coefficients

5.4.1 Preliminary Remarks

Now we consider operators with irregular coefficients (as in Sections 4.6 and 4.7) and for such scalar operators we are going to extend to a certain degree results of the previous sections of this chapter. We consider only asymptotics with the spatial averaging due to the following reason:

Analysis of asymptotics without spatial averaging is much more straightforward and thus simpler. Really, our base points for rescaling procedure used to be results of Sections 4.4 and 4.5 for microhyperbolic and ξ-microhyperbolic operators respectively as we consider asymptotics with or without spatial averaging. However the results Sections 4.6 and 4.7 require ξ-microhyperbolicity and only in the case of ridiculously large roughness parameter[29)] we could derive them under microhyperbolicity condition. So our first goal is to recover these results under microhyperbolicity condition (plus some extra assumptions).

On the other hand, as analysis of Section 5.3 shows, for a Schrödinger operator $h^2|D|^2+V(x)$ exactly points with $V=0$ and $|\nabla V| \asymp 1$ are the main obstacle for pointwise spectral asymptotics. Still, it is not that simple: as this analysis shows the remainder estimate obtained just by rescaling is not necessary optimal even if we consider only asymptotics without correction term. Therefore we formulate the following problem which we consider really challenging, especially its second part:

Problem 5.4.1. (i) Extend results of Subsection 5.3.1 (i.e. only for Schrödinger operator) partially (i.e. without correction term and respectively with the remainder at least of the same magnitude as a correction term) to operators with irregular coefficients.

(ii) Extend results of Subsection 5.3.1 completely (i.e. with correction term) to operators with irregular coefficients.

Recall that our goal are operators with irregular coefficients rather than the "rough" operators which are our main objects of study and we either bracket irregular operators between rough ones but this bracketing has implications only for asymptotics of the number of eigenvalues which will

[29)] $\varepsilon = C(h|\log h|)^{\frac{1}{2}}$ as opposed to $\varepsilon = Ch|\log h|$.

be assembled from (micro)local asymptotics with the spatial averaging or use the method of perturbation as in Sections 4.6 and 4.7 respectively.

Let us discuss in more detail bracketing method. Recall that applying our rescaling technique we have two scaling functions $\gamma(x)$ and $\rho(x)$. However, applying results of Section 4.6 we need to remember about *roughness parameter* ε which becomes a *roughness function* which is (much) smaller than $\gamma(x)$ and usually is defined to satisfy logarithmic uncertainty principle (see (5.4.14)).

Applications: we construct approximations which provide a rather small error in Weyl expressions and also rather small remainder estimate. Further, for semi-bounded operators we can select two approximations of this type $A_\varepsilon^{\pm}$ [30] such that $A_\varepsilon^- \le A \le A_\varepsilon^+$ and thus numbers of negative eigenvalues have opposite order: $\mathsf{N}^-(A_\varepsilon^-) \ge \mathsf{N}^-(A) \ge \mathsf{N}^-(A_\varepsilon^+)$.

For non-semibounded operators we are looking for number of eigenvalues between τ_1 and τ_2 and thus for number of negative eigenvalues of

$$B = \bigl(A - \frac{1}{2}(\tau_2+\tau_1)\bigr)^2 - \frac{2}{4}(\tau_2-\tau_1)^2 \tag{5.4.1}$$

and one can construct two operators $A_\varepsilon^{\pm}$ such that $B_\varepsilon^- \le A \le B_\varepsilon^+$. We will do it when we consider applications (so much later).

5.4.2 Schrödinger Operator

Reminder

We start from the Schrödinger operator as a toy-model but we consider the general one (5.3.1). Our first goal is to replace ξ-microhyperbolicity condition

$$|V(x)| \ge \epsilon_0, \tag{5.4.2}$$

by microhyperbolicity ocondition

$$|V(x)| + |\nabla V(x)| \ge \epsilon_0, \tag{5.4.3}$$

where $V(x)$ is the potential.

Since such condition makes sense only as original operator had at least $\mathscr{C}^1$-coefficients we assume this. In fact we assume that

[30] Where subscript ε means not the actual selection of the parameter function but simply fact that we are talking about approximations

(5.4.4) g^{jk}, V_j, V are $\mathscr{C}^l_\nu$-functions with $t^l \le \nu(t) \le t^{l+1}$, $l = 1, 2, ...$ (see Definition 4.6.1).

Consider operator A_ε with g^{jk}, V_j, V replaced by g^{jk}_ε, $V_{j\varepsilon}$, V_ε which are their ε-mollifications.

Then under condition (5.4.2) we derived asymptotics

$$|\mathsf{R}^{\mathsf{W}}_{x,\varphi,L,\varepsilon}| \le Ch^{1-d}\vartheta\Big(\frac{h}{L}\Big) \tag{5.4.5}$$

and exactly the same estimate for $\mathsf{R}^{\mathsf{W}}_{\varphi,L,\varepsilon}$. Let us recall that approximation error was $Ch^{-d}\nu(\varepsilon)$.

Let us recall that we could take ε pretty small:

$$\varepsilon = Ch|\log h|. \tag{5.4.6}$$

Getting Rid of Condition (5.4.2)

First we consider the case of low smoothness:

(5.4.7) Let $\nu(t)$ be monotone increasing function such that $\nu(0) = 0$

and

$$\nu(t) \ge t, \qquad \frac{d}{dt}(\nu(t)t^{-1}) \le 0 \tag{5.4.8}$$

and also

$$\nu(C_1 t) \ge c_1\nu(t) \quad \forall t \in (0,1) \qquad \text{with} \quad c_1 > 1,\ C_1 > 1. \tag{5.4.9}$$

Here if (5.4.9) holds for some $c_1 > 1$ and $C_1 > 1$ then it holds for every $c_1 > 1$ and $C_1 = C_1(c_1)$.

Then $\gamma(x)$ defined from

$$\nu(\gamma(x)) = \max\big(\epsilon|V(x)|, \nu(\bar\gamma)\big) \tag{5.4.10}$$

satisfies

$$|x-y| \le \epsilon\gamma(x) \implies C^{-1}\gamma(x) \le \gamma(y) \le C\gamma(x) \tag{5.4.11}$$

and thus is a proper scaling function.

Really, $|x-y| \le \epsilon\gamma$ implies $|V(x)-V(y)| \le \nu(\epsilon\gamma)$. Then $|x-y| \le \epsilon\gamma(x)$ implies that $|V(y)-V(x)| \le \epsilon_1 \max(|V(x)|, \nu(\bar\gamma)$ which due to (5.4.9)–(5.4.10) implies that $|\gamma(x)-\gamma(y)| \le \epsilon_2\gamma(x)$.

Further, let

$$\rho(x) = \nu(\gamma(x))^{\frac{1}{2}} \tag{5.4.12}$$

and therefore we should define $\bar{\gamma}$, $\bar{\rho}$ from

$$\bar{\gamma}\nu(\bar{\gamma})^{\frac{1}{2}} = 2h, \qquad \bar{\rho} = \nu(\bar{\gamma})^{\frac{1}{2}}. \tag{5.4.13}$$

Let us consider an element $B(y,\gamma)$ with $\gamma = \gamma(y)$ etc and let us scale $x \mapsto x\gamma^{-1}$, $\xi \mapsto \xi\rho^{-1}$, $h \mapsto h\rho^{-1}\gamma^{-1}$. This also scales $\varepsilon \mapsto \varepsilon\gamma^{-1}$ and we need to satisfy $\varepsilon = Ch|\log h|$ after rescaling; thus we need to pick up

$$\varepsilon = C\frac{h}{\rho}|\log \frac{h}{\rho\gamma}| \tag{5.4.14}$$

Now let us notice that $\nu(t)$ scales as well:

$$\nu_{\mathsf{new}}(t) = \rho^{-2}\nu(\gamma t) = \nu(\gamma)^{-1}\nu(\gamma t) \tag{5.4.15}$$

which satisfies (5.4.7).

We know (see Theorem 4.6.21) that under condition (5.4.2)

$$|\mathsf{R}^{\mathsf{W}}_{x,\varepsilon}| \le Ch^{-d}\nu(\varepsilon); \tag{5.4.16}$$

there $\varepsilon = Ch|\log h|$.

Then (5.4.16) scales to

$$|\mathsf{R}^{\mathsf{W}}_{x,\varepsilon}| \le Ch^{-d}\rho^{d-2}\nu\big(\frac{h}{\rho}|\log \frac{h}{\rho\gamma}|\big)\big|_{\rho=\nu(\gamma)^{\frac{1}{2}}}. \tag{5.4.17}$$

Let us consider the right-hand expression of (5.4.17) as a function of ρ (so $\gamma = \gamma(\rho)$ is defined from $\nu(\gamma) = \rho^2$). As $d \le 2$ it is clearly monotone decreasing and therefore it does not exceed its value as $\rho = \bar{\rho}$ which is

$$|\mathsf{R}^{\mathsf{W}}_{x,\varepsilon}| \le Ch^{-d}\nu\big(\bar{\gamma}\big)^{\frac{1}{2}d} \asymp C\bar{\gamma}^{-d}. \tag{5.4.18}$$

On the other hand, the right-hand expression of (5.4.17) does not exceed $Ch^{-d}\rho^{d-2}\nu\big(h\rho^{-1}|\log h|\big)$ which for $d \ge 3$ is a monotone increasing function of ρ due to (5.4.8) and therefore the right-hand expression of (5.4.17) does not exceed its value as $\rho = 1$.

Thus we arrive to

Theorem 5.4.2. *Let* (5.4.7)–(5.4.9) *be fulfilled and let us drop condition* (5.4.2). *Then for described above operator* A_ε *estimate* (5.4.17) *holds.*

In particular, as $d \ge 3$ *estimate* $|\mathsf{R}^{\mathsf{W}}_{x,\varepsilon}| \le Ch^{-d}\nu(h|\log h|)$ *still holds and as* $d = 1, 2$ *estimate* $|\mathsf{R}^{\mathsf{W}}_{x,\varepsilon}| \le C\bar\gamma^{-d}$ *holds with* $\bar\gamma$ *defined from* (5.4.13).

Remark 5.4.3. We conclude immediately that in the framework of Theorem 5.4.2 for ψ with compact support[31)].

(i) As $d \ge 3$ estimate (5.4.16) holds for $\mathsf{R}^{\mathsf{W}}_\varepsilon$ with $\varepsilon = Ch|\log h|$;

(ii) As $d = 1, 2$ estimate

$$|\mathsf{R}^{\mathsf{W}}_\varepsilon| \le c\bar\gamma^{-d} \tag{5.4.19}$$

holds with $\bar\gamma$ defined by (5.4.13).

(iii) One can easily improve (5.4.19) under assumption $\mathsf{mes}(Z(t)) \le \sigma(t)$ where $Z(t) = \{x : |V(x)| \le t\}$;

(iv) So far the same estimates hold for bracketing-approximation errors (actually estimates for $|\mathsf{R}^{\mathsf{W}}_\varepsilon|$, $|\mathsf{R}^{\mathsf{W}}_{x,\varepsilon}|$ are marginally better). Indeed, we take $A^\pm_\varepsilon = A_\varepsilon + C_0\nu(\varepsilon)$. Thus we got estimates for the combined error.

(v) Due to low smoothness we did not apply results of Section 4.7.

Example 5.4.4. Let $\nu(t) = t^r|\log t|^\sigma$ where

$$r \in [0, 1], \qquad r = 0 \implies \sigma < 0, \qquad r = 1 \implies \sigma \ge 0. \tag{5.4.20}$$

Then

(i) If either (5.4.2) is fulfilled or $d \ge 3$ then

$$|\mathsf{R}^{\mathsf{W}}_\varepsilon| \le Ch^{-d+r}|\log h|^{r+\sigma}; \tag{5.4.21}$$

(ii) If $d = 1, 2$ and (5.4.2) fails then

$$|\mathsf{R}^{\mathsf{W}}_\varepsilon| \le Ch^{-\frac{2d}{r+2}}|\log h|^{\frac{\sigma d}{r+2}}. \tag{5.4.22}$$

We leave to the reader

Problem 5.4.5. Generalize these results to the spectral means averages.

[31)] One can easily get rid of this assumption and take $\psi = 1$ in $\mathbb{R}^d$ provided $\{x : V(x) \le \epsilon_0\}$ is compact.

From Condition (5.4.2) to Condition (5.4.3)

Now we change our assumptions: in addition to (5.4.7) we assume that and

$$\nu(t)\le t,\qquad \frac{d}{dt}(\nu(t)t^{-1})\ge 0,\qquad \nu(t)t^{-1}\to 0 \quad \text{as} \quad t\to +0. \tag{5.4.23}$$

Let us recall that then under condition (5.4.2) $|\mathsf{R}^{\mathsf{W}}_{x,\varepsilon}|\le Ch^{1-d}$ and we are not talking about approximation error now.

Then we can introduce scaling functions as they were introduced in the smooth case when we wanted to get rid of condition (5.4.2):

$$\gamma(x):=\epsilon\,\mathsf{max}(|V(x)|,\bar\gamma),\qquad \rho(x):=\gamma(x)^{\frac12},\qquad \bar\gamma=C_1h^{\frac23} \tag{5.4.24}$$

(which coincides with the definition of the previous subsubsection as $\nu(t)=t$) and rescale as we used to; then $h\mapsto h_{\mathsf{new}}=h\rho^{-1}\gamma^{-1}$ and $\varepsilon\mapsto\varepsilon\gamma^{-1}$; we define ε by (5.4.14). Then after rescaling $\nu_{\mathsf{new}}(t)$ satisfies $\nu_{\mathsf{new}}(t)\le t$.

Then remainder estimate scales to

$$|\mathsf{R}^{\mathsf{W}}_{x,s,\varepsilon}|\le C(h/\rho\gamma)^{1-d+s}\gamma^{-d}\times\rho^{2s}=Ch^{1-d+s}\rho^{d-1+s}\gamma^{-1-s}= Ch^{1-d+s}\gamma^{\frac12(d-s-3)} \tag{5.4.25}$$

because $\rho^2=\gamma$ and $\mathsf{R}^{\mathsf{W}}_{x,s,\varepsilon}$ is a spectral averaging with $\varphi(\tau)=\tau_+^s$, $s\ge 0$ (in particular, $\mathsf{R}^{\mathsf{W}}_{x,0,\varepsilon}=\mathsf{R}^{\mathsf{W}}_{x,\varepsilon}$). So far all as it used to be in the smooth case when we study local spectral asymptotics without averaging.

Therefore

$$|\mathsf{R}^{\mathsf{W}}_{x,s,\varepsilon}|\le Ch^{1-d+s}\int \gamma^{\frac12(d-s-3)}\,dx \tag{5.4.26}$$

and now condition (5.4.3) changes the rules of the game:

Proposition 5.4.6. *For Schrödinger operator under conditions* (5.4.3), (5.4.7) *and* (5.4.23)

$$|\mathsf{R}^{\mathsf{W}}_{x,s,\varepsilon}|\le C\begin{cases} h^{1-d+s} & \text{as}\quad d>s+1,\\ h^{1-d+s}|\log h| & \text{as}\quad d=s+1,\\ h^{\frac23(-d+s+1)} & \text{as}\quad d<s+1.\end{cases} \tag{5.4.27}$$

So, a proper remainder estimate is recovered for $d > s+1$; while for $d = s+1$ logarithmic factor appears. Meanwhile due to (5.4.3) bracketing-approximation error does not exceed

$$Ch^{-d}\int \rho^{d-2+2s}\nu(\varepsilon(\rho))\,dx \asymp Ch^{-d}\int \rho^{d-1+2s}\nu(\varepsilon(\rho))\,d\rho. \tag{5.4.28}$$

Therefore

Theorem 5.4.7. *Let conditions* (5.4.3), (5.4.7) *and* (5.4.23) *be fulfilled. Then for appropriate bracketing operators* $A_\varepsilon^{\pm}$ [30] ($A_\varepsilon^+ \ge A \ge A_\varepsilon^-$) *the combined Weyl and bracketing-approximation error does not exceed* (5.4.27) + (5.4.28) [32] *with* $\varepsilon(\rho) = C\rho^{-1}h|\log(h\rho^{-3})$ *and* ρ *running from* $\bar\rho = Ch^{\frac13}$ *to a small constant.*

Remark 5.4.8. (i) (5.4.28) is $O(h^{1-d+s})$ provided <u>either</u> $d+s \ge 2$ and $\nu(t) = t^{s+1}|\log t|^{-s-1}$ <u>or</u> $d=1$, $s=0$ and $\nu(t) = t|\log t|^{-2-\delta}$.

(ii) As $d \le s+1$ we are rather unhappy with Weyl error because in arguments leading to Proposition 5.4.6 no smoothness requirement can produce such remainder $O(h^{1-d+s})$.

Case $d \le s+1$ Reloaded

Now our goal is to improve arguments leading to Theorem 5.4.7 and to prove $|\mathsf{R}^{\mathsf{W}}_{s,\varepsilon}| \le Ch^{1-d}$ as $d \le s+1$ without significantly increasing ε. We need to better estimate the contribution of $B\big(y,\gamma(y)\big)$ to the remainder.

To do so we need to study a long term propagation of wave front sets. For simplicity of notations we assume that $\boldsymbol{V}=0$, $g^{jk}=\delta_{jk}$. Let us introduce

$$\gamma(y) = \epsilon_0|V(y)| + C_0(h|\log h|)^{\frac23}, \qquad \rho(y) = C_0\gamma(x)^{\frac12}, \qquad \varepsilon(y) = Ch|\log h|\rho^{-1}. \tag{5.4.29}$$

Let us consider point y and operator Q with the symbol supported in

$$\{(x,\xi):\ |x-y|\le \epsilon\gamma,\ \pm\langle\xi,\nabla V(y)\rangle \ge -\epsilon\rho\}$$

with γ,ρ calculated at x and sufficiently small constant ϵ.

[32] Here and in what follows (5.4.28) denotes the right-hand expression only.

Then

$$\mathsf{WF}(u_\varepsilon\, {}^{t}Q)\cap\{|\tau|\le \epsilon'\rho^2\}\cap\{C_0\epsilon\gamma^{\frac12}\le \mp t\le \epsilon_0\gamma^{\frac12}\}\subset \tag{5.4.30}$$
$$\{(x,\xi):\ |x-y|\le C\epsilon_0\gamma,\ \pm\langle\xi,\nabla V(y)\rangle\ge C\epsilon\rho,\ \langle x-y,\nabla V(y)\rangle\ge C_0\epsilon^2\gamma\}.$$

Indeed, it follows after rescaling due from standard propagation results. Here t is an "absolute time", after rescaling it is divided by $\gamma^{\frac12}$. Here and below u_ε is the Schwartz kernel of $U_\varepsilon(t)=e^{ih^{-1}tA_\varepsilon}$.

Then one can extend it for larger t:

$$\mathsf{WF}(u_\varepsilon\, {}^{t}Q)\cap\{|\tau|\le \epsilon'\rho^2\}\cap\{T\le \mp t\le 2T\}\subset \tag{5.4.31}$$
$$\{(x,\xi):\ |x-y|\le C\epsilon_0 T^2,\ \pm\langle\xi,\nabla V(y)\rangle\ge \epsilon_1 T,\ \langle x-y,\nabla V(y)\rangle\ge \epsilon_1^2\gamma\}$$

as $\epsilon_0\gamma^{\frac12}\le T\le\epsilon_0$. Indeed, one can prove it easily by recursion $T\mapsto 2T$.

Then, using property of Γ we conclude

(5.4.32) Let ψ be supported in $\{|V(x)|\le\gamma\}$. Then $F_{t\mapsto h^{-1}\tau}\chi_T(t)\Gamma\big(u_\varepsilon\psi\big)$ is negligible for $T_*\le T\le T^*$ with $T_*=\epsilon\gamma^{\frac12}$ and $T^*=\epsilon$, as $|\tau|\le\epsilon'\gamma$.

As usual $\chi\in\mathscr{C}_0^\infty([-1,-\frac12]\cup[\frac12,1])$ and therefore in the Tauberian estimates we can replace $T\asymp\gamma^{\frac12}$ by $T\asymp 1$. This works not only as $\gamma\ge 2C_0(h|\log h|)^{\frac12}$ where we use ξ-microhyperbolicity but also as $\gamma\le 2C_0(h|\log h|)^{\frac12}$ where we use microhyperbolicity (5.4.3) which is possible because there $\varepsilon\asymp\gamma$ and after rescaling $h\mapsto\hbar=C_0^{-1}|\log h|^{-1}$.

So, contribution of $B(x,\gamma(x))$ to $\mathsf{R}^{\mathsf{T}}_{s,\varepsilon}$ does not exceed

$$C\big(\frac{h}{\rho\gamma}\big)^{1-d+s}\times\big(\frac{T_*}{T^*}\big)^{1+s}\times\rho^{2s}\asymp Ch^{1-d+s}\rho^{d-2}\gamma^d \tag{5.4.33}$$

which sums to $Ch^{1-d+s}\int\rho^{d-2}\,dx$ which is does not exceed Ch^{1-d+s} even as $d=1$ in virtue of (5.4.3).

So, in the current framework $|\mathsf{R}^{\mathsf{T}}_{s,\varepsilon}|\le Ch^{1-d+s}$.

Remark 5.4.9. Now we need to pass from Tauberian to Weyl expression and there is a twist: we need to take sufficiently many terms.

(i) If $d=s+1$ there was only logarithmic divergence and we need to take only terms with h^{-d+n} with $0\le n\le s+1$ (the term where logarithmic divergence was because "out of the box" this term does not exceed $C\int\gamma^{-1}\,dx\asymp C|\log h|$ and remainder estimate should be $O(1)$; next term does not exceed $Ch\int\gamma^{-\frac52}\,dx=o(1)$.

(ii) If $d < s+1$ there was a power divergence and we need to take many terms, and only if

$$(5.4.29)^* \quad \gamma(y) = \epsilon_0 |V(y)| + C_0 h^{\frac{2}{3}-\delta}, \qquad \rho(y) = C_0 \gamma(x)^{\frac{1}{2}}, \qquad \varepsilon(y) = Ch|\log h|\rho^{-1}.$$

Recall that the bracketig error is delivered by (5.4.28). Thus we arrive to

Theorem 5.4.10. *Let $d \le s+1$ and conditions* (5.4.3), (5.4.7) *and* (5.4.23) *be fulfilled. Then for appropriate bracketing operators $A_\varepsilon^{\pm}$ [30] ($A_\varepsilon^+ \ge A \ge A_\varepsilon^-$) the combined Weyl and bracketing-approximation error does not exceed*

(i) Ch^{-d+s+1} + (5.4.28) [32] *with ρ running from $\bar\rho = (h|\log h|)^{\frac{1}{3}}$ to a small constant as $d = s+1$.*

(ii) Ch^{-d+s+1} + (5.4.28) + (5.4.34) *with the third term*

$$(5.4.34) \qquad Ch^{-d}\bar\rho^{d+2s}\nu(\bar\varepsilon)$$

and ρ running from $\bar\rho = h^{\frac{1}{3}-\delta}$ to a small constant and and $\bar\varepsilon = \bar\rho^2$ as $d < s+1$.

Example 5.4.11. In the framework of Theorem 5.4.10 let $\nu(t) = t^l|\log t|^{-\sigma}$.

(i) Then the combined error is $O(h^{-d+s+1})$ provided $l = s+1$ and either $d+s > 1$, $\sigma = s+1$ or $= 1$, $s = 0$, $\sigma = 2$. Recall that $l = s+1$, $\sigma = -1$ needed for this remainder estimate to work under condition (5.4.2).

(ii) Other cases are left to the reader as well as $\nu(t) = t^l|\log t|^{\sigma}$.

The following problem seems to be challenging, at least Part (ii):

Problem 5.4.12. (i) Prove that in the framework of Theorem 5.4.10 one can preserve in the Weyl expression $\mathsf{N}^{\mathsf{W}}_{s,\varepsilon}$ only terms containing h to the power lesser than $-d+s+1$, and then one can replace Weyl expression for approximation $\mathsf{N}^{\mathsf{W}}_{s,\varepsilon}$ by Weyl expression for the operator itself $\mathsf{N}^{\mathsf{W}}_{s}$. Recall that Weyl coefficients appear first as distributions with respect to τ.

(ii) Prove that Statement (i) of Theorem 5.4.10 holds even as $d < s+1$.

Dropping Condition (5.4.3)

Recall that as $d \ge s+3$ we simply dropped condition (4.5.2) but as $d < s+3$ we traded it to (4.5.3). Now we want to get rid of the latter. So, let $d < s+3$. Let $\nu(t)$ satisfy (5.4.7), (5.4.23) as before and let us replace (5.4.9) by a stronger assumption

$$\nu(C_1 t) \ge C_1 c_1 \nu(t) \quad \forall t \in (0,1) \qquad \text{with } \ c_1 > 1,\ C_1 > 1 \tag{5.4.35}$$

which means exactly that $\nu(t)t^{-1}$ also satisfies (5.4.9). We also assume that

$$t^{-2}\nu(t) \text{ is a monotone decreasing function} \tag{5.4.36}$$

so that we do not assume more smoothness than $\mathscr{C}^2$. Let us introduce γ, ρ by the following formula similar to (5.4.10)

$$\gamma = \max(\gamma', \bar\gamma), \quad \gamma' = \epsilon \min\{t : |V| \le \nu(t), |\nabla V| \le t^{-1}\nu(t)\}, \tag{5.4.37}$$
$$\text{with } \bar\gamma \ \text{ defined from } \ \bar\gamma \nu(\bar\gamma)^{\frac12} = h$$

Then γ is a scaling function i.e. (5.4.11) holds. Indeed, if $|x-y| \le \epsilon\gamma$ then

$$|\nabla V(x) - \nabla V(y)| \le \nu(|x-y|)|x-y|^{-1} \le \epsilon_1 \nu(\gamma)\gamma^{-1}$$

and therefore $|\nabla V(y)| \le 2\nu(\gamma)\gamma^{-1}$. Then

$$|V(x) - V(y)| \le 2\nu(\gamma)\gamma^{-1}|x-y| + \nu(|x-y|) \le \epsilon_1 \nu(\gamma).$$

Combining these two inequalities and using (5.4.9), (5.4.35) we conclude that $\gamma(x)$ satisfies (5.4.11).

Let us define $\rho, \bar\gamma$ by (5.4.12), (5.4.13). Then after our usual rescaling, we obtain an operator satisfying (5.4.3) as $\gamma \ge 2\bar\gamma$. Then h^{-d+s+1} is replaced by $Ch^{-d+s+1}\int \nu(\gamma)^{\frac12(d+s-1)}\gamma^{-s-1}\,dx$ which in the current framework we cannot estimate by anything better than

$$Ch^{-d+s+1} \sup_{\gamma:\gamma\nu(\gamma)^{1/2}\ge h} \nu(\gamma)^{\frac12(d+s-1)}\gamma^{-s-1}. \tag{5.4.38}$$

One can observe easily that in this scaling procedure $h \mapsto h/(\rho\gamma)$, $\nu(\varepsilon) \mapsto \nu(\varepsilon\gamma)/\rho^2$ with $\rho = \nu(\gamma)^{1/2}$ and terms (5.4.28) and (5.4.34) (in the framework of Theorem 5.4.10(ii)) do not increase.

Also one can observe that the contribution of zone $\{\gamma\nu(\gamma)^{1/2} \le C_0 h\}$ does not exceed (5.4.28).

Therefore we arrive to

Theorem 5.4.13. *Let* $d \le s+3$ *and conditions* (5.4.7) *and* (5.4.23), (5.4.35), (5.4.36) *be fulfilled. Then for appropriate bracketing operators* $A_\varepsilon^{\pm}$ [30] ($A_\varepsilon^+ \ge A \ge A_\varepsilon^-$) *the combined Weyl and bracketing-approximation error does not exceed*

(i) (5.4.38) + (5.4.28) [32] *with* ρ *running from* $\bar\rho = (h|\log h|)^{\frac13}$ *to a small constant as* $d = s+1$.

(ii) (5.4.38) + (5.4.28) + (5.4.34) *with* ρ *running from* $\bar\rho = h^{\frac13-\delta}$ *to a small constant and and* $\bar\varepsilon = \bar\rho^2$ *as* $d < s+1$.

Remark 5.4.14. (i) Let $d \ge 2$ and $\nu(\gamma) \le \gamma^l$ with $l = l(s) = 2(s+1)/(d+s-1)$. Then expression (5.4.38) is $\asymeq Ch^{-d+s+1}$). Here $l(s) = 2$ as $d = 2$ and $l(s) < 2$ as $d \ge 3$.

(ii) Let $d = 1$, $l = 2$ expression (5.4.38) is $\asymeq Ch^{-d+s+\frac12}$). Assuming that $\nu(t) = o(t^l)$ and $\nu(t) \ge t^{l+1}$ with $l = 2, 3, \dots$, one can introduce a scaling function

$$(5.4.37)' \quad \gamma = \max(\gamma', \bar\gamma), \quad \gamma' = \epsilon \min\{t : |\nabla^j V| \le t^{-j}\nu(t)\ \forall j = 1, \dots, l\}$$
$$\text{with } \bar\gamma \text{ defined from } \bar\gamma\nu(\bar\gamma)^{\frac12} = h$$

and as $l = 2$ recover remainder estimate $O(1)$ or $O(|\log h|)$ provided V has a single non-degenerate minimum or maximum. One can even recover remainder estimate $O(1)$ if either V has a single non-degenerate minimum or $s > 0$, $\nu(t) = O(t^2|\log t|^{-\sigma}$, $\sigma > 2$. The one can continue as in Subsection 5.2.3.

Sharp Remainder Estimates

So far the best possible combined remainder estimate we derived is $O(h^{-d+1+s})$:

Theorem 5.4.15. *(a) In the framework of Theorem 5.4.7 assume that* $d > s+1$ *and expression* (5.4.28) *is* $O(h^{1-d+s})$.

(b) In the framework of Theorem 5.4.10(i) assume that expression (5.4.28) *is* $O(h^{1-d+s})$.

(c) In the framework of Theorem 5.4.10(ii) assume that expressions (5.4.28) *and* (5.4.34) *are* $O(h^{s+1})$.

(d) In the framework of Theorem 5.4.13(i) assume that expressions (5.4.28) *and* (5.4.38) *are* $O(h^{1-d+s})$.

(e) In the framework of Theorem 5.4.13(ii) assume that expressions (5.4.28), (5.4.34) *and* (5.4.38) *are* $O(h^{1-d+s})$.

Here all terms are calculated in the frameworks of the corresponding theorems.

Then the combined error is $O(h^{-d+1})$

Could we do better than this? Observe that all terms in the estimates we derived which contain ν estimate approximating-bracketing errors while Ch^{1-d+s} estimates semiclassical error[33)]. Only the latter could be improved. Therefore

Theorem 5.4.16. *(a) In the framework of Theorem 5.4.7 assume that* $d > s+1$ *and expression* (5.4.28) *is* $o(h^{1-d+s})$.

(b) In the framework of Theorem 5.4.10(i) assume that expression (5.4.28) *is* $o(h^{1-d+s})$.

(c) In the framework of Theorem 5.4.10(ii) assume that expression (5.4.28) *is* $o(h^{s+1})$ *and*

$$\sup_{\gamma<t:\gamma\nu(\gamma)^{1/2}\ge h} \nu(\gamma)^{\frac{1}{2}(d+s-1)}\gamma^{-s-1} = o(1) \qquad \text{as } h\to+0, t\to+0. \tag{5.4.39}$$

(d) In the framework of Theorem 5.4.13(i) assume that expressions (5.4.28) *and* (5.4.38) *are* $o(h^{1-d+s})$.

(e) In the framework of Theorem 5.4.13(ii) assume that expressions (5.4.28), (5.4.34) *are* $o(h^{1-d+s})$ *and* (5.4.39) *holds.*

Here all terms are calculated in the frameworks of the corresponding theorems.

Further, let $d\ge 2$. *Assume that the measure of the set of all dead-end*[34)] *or periodic points*[35)] *is* 0.

Then the combined error is $o(h^{-d+1})$.

[33)] The only exception is (5.4.38) which we discuss later.

[34)] Let us recall we call point z dead-end if dynamics $\Psi_t(z)$ leaves our domain for some $t : \pm t > 0$.

[35)] Let us recall we call point z periodic if $z\in\Psi_t(z)$ for some $t : \pm t > 0$. Let us recall that to dismiss point as dead-end or periodic we are free to chose (the same) sign $\pm$.

Proof. The easy proof is left to the reader. □

Applying Results of Section 4.7

We know from Section 4.7 that under assumption (5.4.2)

$$|\mathsf{R}^{\mathsf{W}}_{x,s}| \le Ch^{1-d+s} + Ch^{-d}\nu(\varepsilon)^{\frac{1}{2}(s+1)} \tag{5.4.40}$$

and

$$|\mathsf{R}^{\mathsf{W}}_{s}| \le Ch^{1-d+s} + Ch^{-d}\nu(\varepsilon)^{\frac{2}{3}(s+1)} \tag{5.4.41}$$

with $\varepsilon = h$. Then applying the arguments of the proof of Theorem 5.4.7 we arrive to

Therefore

Theorem 5.4.17. *Let conditions* (5.4.3), (5.4.7) *and* (5.4.23) *be fulfilled. Then* $|\mathsf{R}^{\mathsf{W}}_{s}|$ *does not exceed* (5.4.27) + (5.4.42) *with the second term*

$$Ch^{-d}\int \rho^{d-\frac{1}{3}+\frac{2}{3}s}\nu(\varepsilon(\rho))^{\frac{2}{3}(s+1)}\,d\rho, \tag{5.4.42}$$

$\varepsilon(\rho) = C\rho^{-1}h$ *and* ρ *running from* $\bar\rho = Ch^{\frac{1}{3}}$ *to a small constant.*

Applying Results of Section 4.7. II

To improve (5.4.27) as $d \le s+1$ we can apply arguments of the proof of Theorem 5.4.10 but we need to recall how estimate (5.4.41) was derived to understand how term (5.4.42) should be modified. However this improvement would not make any sense if $T := (h\nu(\varepsilon(\rho))^{-\frac{2}{3}} < \rho$, i.e. $\nu(\varepsilon) \ge \varepsilon^{\frac{3}{2}}$.

So we assume that

$$\nu(\varepsilon) = \varepsilon^{l}, \qquad l > \frac{3}{2}. \tag{5.4.43}$$

The trouble is that we consider propagation for time larger than ρ and then the magnitude $\varepsilon(\rho)$ (and $\nu(\varepsilon(\rho))$ changes) and it may increase or decrease. So

$$Ch^{-d}\rho^{d-s-1}\gamma^{d-s-1}\times\rho^{2s}\times\rho^{-2s-1}T^3\nu(\varepsilon(\rho))^2 \asymp Ch^{-d}\rho^{d-s-3}\gamma^{d-s-1}\times T^3\nu(\varepsilon(\rho))^2 \tag{5.4.44}$$

is not necessarily the correct estimate of the third term in the successive approximation (on the given γ-element, $\rho^2 \asymp \gamma$) an it seems that we need to replace here $\varepsilon(\rho)$ by $\bar{\varepsilon} = h^{\frac{2}{3}}|\log h|$ but it would be too bad in the end. While we cannot completely cure it, we can actually do significantly better.

Recall that for each h-pseudodifferential partition element Q we need to consider $\mathsf{Tr}(G^{\pm}Q)$ which for the third term translates into $\mathsf{Tr}(G^{\pm}A'_{\varepsilon}G_{\varepsilon}^{\pm}A'_{\varepsilon}G_{\varepsilon}^{\pm}Q)$ where $G^{\pm}$ and $G_{\varepsilon}^{\pm}$ are parametrices for perturbed (original) and unperturbed (mollified) operators and A'_{ε} is a perturbation.

If for given Q propagator $G_{\varepsilon}^{\pm}$ increases ρ and thus decreases $\varepsilon(\rho)$ we can indeed preserve one copy of $\nu(\varepsilon(\rho))$. Furthermore, since we actually have $\int \nu(\varepsilon(\rho(t)))\,dt$ in virtue of (5.4.43) we can replace here T by $\rho < T$. Application of A'_{ε} forces propagator to "lose direction". However in front of A'_{ε} we can insert $Q_1^{\pm}$ of the same nature as $Q^{\pm}$ and in the case "$Q^{\pm}$ meets $Q_1^{\pm}$" we can do the same, but in the case "$Q^{\pm}$ meets $Q_1^{\mp}$" we need to take $\bar{\rho}\nu(\bar{\varepsilon})$ instead.

Now we need to understand what to do in the opposite case, $Q = Q^{\mp}$. Observe that in successive approximation we can put $G^{\pm}$ to the left as we did or to the right as we will do now and the choice is ours. But because of the trace we can move $Q^{\mp}$ to the left. And finally, because the result is real we can complex conjugate the result, thus take adjoint operator under the trace which restores this term to its original form but with a twist: while $Q^{\pm}$ and A'_{ε} are preserved, $G^{\pm}$ and $G_{\varepsilon}^{\pm}$ are replaced by $G^{\mp}$ and $G_{\varepsilon}^{\mp}$ and are now in the accordance with $Q^{\mp}$ and the same estimate as before holds. Further, instead of three term approximation we can make many term approximation, each time gaining factor $h^{-1}\bar{\rho}\nu(\bar{\varepsilon}) \le h^{\delta}$ due to (5.4.3).

Therefore we can consider only "regular terms" with only $G_{\varepsilon}^{\pm}$ and therefore the last copy of T could be replaced by ρ since we need to return to ρ^2-vicinity of $\mathsf{supp}(\psi)$.

Thus we estimate the total error on the element Q by

$$Ch^{-d+s+1}\rho^{d-2}\gamma^d + Ch^{-d-2}\rho^{d+2s}\gamma^d(\rho\nu(\varepsilon(\rho)))(\bar{\rho}\nu(\bar{\varepsilon})) \tag{5.4.45}$$

and summation over partition in $\rho \le r\}$ returns

$$Ch^{-d+s+1} + Ch^{-d+\frac{5}{3}(l-1)-\delta}r^{d+2s+3-l} \tag{5.4.46}$$

because to avoid troubles with Weyl terms we may need to take $\bar{\varepsilon} = h^{\frac{2}{3}-\delta}$. Here the first term is actually smaller but it would not make any difference.

However taking $r = 1$ we get rather poor estimate as the last term usually would exceed both the first term and $Ch^{-d+\frac{2}{3}(s+1)l}$ whicjh is the second term in (5.4.41). Therefore we apply arguments of the previous subsubsection in $\{r \le \rho \lesssim 1\}$ and estimate its contribution by the second line in the following expression

$$Ch^{-d+\frac{5}{3}(l-1)-\delta} r^{d+2s+3-l} + Ch^{-d+s+1} r^{d-s-1} + Ch^{-d+\frac{2}{3}(s+1)l} r^{d-\frac{2}{3}l+\frac{2}{3}s-2sl} \tag{5.4.47}$$

and optimizing by $r : h^{\frac{1}{3}} \le r \le 1$ we arrive to

Theorem 5.4.18. *Let $d \le s+1$ and conditions* (5.4.3), (5.4.7) *and* (5.4.43) *be fulfilled. Then the total error does not exceed espression* (5.4.47) *minimized over $r : h^{\frac{1}{3}} \le r \le 1$.*

We leave to the reader

Problem 5.4.19. (i) Get rid of condition (5.4.3) like in Subsubsection 5.4.2.5.

(ii) Analyze separately $d = 1$ as in this dimension there $\{V = 0\}$ is a point and the movement could be either to it or from it which probably allows to replace one factor ρ by a factor $\rho^{-2}h|\log h|$.

(iii) Consider case of more regular V and irregular magnetic potential (the case we will deal in Chapters 27 and 28 and use the fact that a factor ρ pops-up in $\nu(\varepsilon(\rho))$.

5.4.3 Dirac Operator

Let us consider Dirac operator now. We assume that $m \le 1$; otherwise it brings nothing new in comparison with Schrödinger operator. As $|V(x)| \ge 2m$ we can apply the same approach as before but with the obvious modifications described below.

Getting Rid of Condition (5.4.2)

Assume that (5.4.7)–(5.4.9) hold and define $\gamma(x)$ by (5.4.10). Then (5.4.11) holds but now (5.4.12)–(5.4.13) are replaced respectively by

$$\rho(x) = \nu(\gamma(x)) \tag{5.4.48}$$

and

$$\bar{\gamma}\nu(\bar{\gamma}) = h. \tag{5.4.49}$$

This is a correct definition as long as $\nu(\bar{\gamma}) \ge 2m$ which is equivalent to

$$\nu\big(\frac{h}{m}\big) \ge 2m. \tag{5.4.50}$$

Let us define ε by (5.4.14).

Let us recall that the combined error does not exceed $Ch^{-d}\nu(h|\log h|)$ under condition (5.4.2). Let us scale $x \mapsto x\gamma^{-1}$, $\xi \mapsto \xi\rho^{-1}$, $h \mapsto h\rho^{-1}\gamma^{-1}$, $\varepsilon \mapsto \varepsilon\gamma^{-1}$ and $\nu(t) \mapsto \nu_{\mathsf{new}}(t) \coloneqq \rho^{-1}\nu(\gamma t)$. Then we conclude that

$$|R_x^{\mathsf{W}}| \le Ch^{-d}\rho^{d-1}\nu\big(\frac{h}{\rho}|\log\frac{h}{\rho\gamma}|\big) \tag{5.4.51}$$

and this expression reaches its maximum as $\rho \asymp 1$ as $d \ge 2$ and as $\rho = \bar{\rho}$ as $d = 1$. Thus we arrive to

Theorem 5.4.20. *Let* (5.4.7)–(5.4.9) *and* (5.4.50) *be fulfilled and let us drop condition* (5.4.2). *Then*

(i) For $d \ge 2$ the combined error still does not exceed $Ch^{-d}\nu(h|\log h|)$.

(ii) For $d = 1$ estimate $|\mathsf{R}_x^{\mathsf{W}}| \le C\bar{\gamma}^{-1}$ holds with $\bar{\gamma}$ defined from (5.4.49).

On the other hand if (5.4.50) fails, i. e. if

$$\nu\big(\frac{h}{m}\big) \le 2m \tag{5.4.52}$$

then in the zone $\{\gamma(x) \le \bar{\gamma}_1\}$ we find ourselves in the framework of the Schrödinger operator with $V(x)$ replaced by $V(x)/2m$ and therefore with $\nu(t)$ replaced by $m^{-1}\nu(\bar{\gamma}_1 t)$; here $\bar{\gamma}_1$ is defined defined by

$$\nu(\bar{\gamma}_1) = 2m. \tag{5.4.53}$$

Then as $d \ge 3$ we have still the same estimates rescaled; in particular, $|\mathsf{R}_x^{\mathsf{W}}| \le Ch^{-d}\nu(h|\log h|)$.

However for $d = 1, 2$ we need to rescale estimate $C|\mathsf{R}_x^{\mathsf{W}}| \le \bar{t}^{-d}$ with $\bar{t}$ defined after rescaling by (5.4.13); the latter equation after rescaling is

$\bar{t}\big(m^{-1}\nu(\bar{\gamma}_1\bar{t})\big)^{\frac{1}{2}} = h(m\bar{\gamma}_1)^{-1}$. Thus we arrive to estimate $|\mathsf{R}^{\mathsf{W}}_x| \le C\bar{\gamma}_1^{-d}\bar{t}^{-d}$. Denoting $\bar{\gamma}_1\bar{t}$ by $\bar{\gamma}$ we arrive to $|\mathsf{R}^{\mathsf{W}}_x| \le C\bar{\gamma}^{-d}$ with $\bar{\gamma}$ defined by

$$\bar{\gamma}\nu(\bar{\gamma})^{\frac{1}{2}} = hm^{-\frac{1}{2}}. \tag{5.4.54}$$

Thus we proved

Theorem 5.4.21. *Let* (5.4.7)–(5.4.9) *and* (5.4.52) *be fulfilled and let us drop condition* (5.4.2). *Then*

(i) For $d \ge 3$ *the combined error still does not exceed* $Ch^{-d}\nu(h|\log h|)$;

(ii) For $d \ge 1, 2$ *estimate* $|\mathsf{R}^{\mathsf{W}}_x| \le C\bar{\gamma}^{-d}$ *holds with* $\bar{\gamma}$ *defined from* (5.4.54).

From Condition (5.4.2) to Condition (5.4.3)

Let (5.4.7), (5.4.23) be fulfilled. Define $\gamma(x)$ by (5.4.37). Then we need to define $\bar{\gamma}_1 = m$ and replace (5.4.50), (5.4.52) by $m \le h^{\frac{1}{2}}$ and $m \ge h^{\frac{1}{2}}$ respectively. However in both cases we arrive to

Theorem 5.4.22. *Let* (5.4.7), (5.4.23) *be fulfilled and let us drop condition* (5.4.2). *Let* <u>*either*</u> $d \ge 3$ <u>*or*</u> $d = 2$ *and* $m \le h^{\frac{1}{2}}$ <u>*or*</u> *condition* (5.4.3) *be fulfilled. Then* $|\mathsf{R}^{\mathsf{W}}| \le Ch^{1-d}$ *while an approximation error does not exceed* $Ch^{-d}\nu(h|\log h|)$ *as* $d \ge 2$ *and* (5.4.27) *as* $d = 1$ *(and* $s = 0$*).*

Dropping Condition (5.4.3)

We are left with the case $d = 1, 2$ and $m \ge h^{\frac{1}{2}}$ only. Otherwise we arrive to estimate Ch^{1-d} for the combined error with a single exception of $d = 1$ when this estimate is replaced by $C\bar{\gamma}^{-1}$ with $\bar{\gamma}$ defined by (5.4.49).

Using scaling defined by (5.4.37) we arrive to

Theorem 5.4.23. *Let* $d = 1, 2$*; let* (5.4.7), (5.4.23), (5.4.35) *and* (5.4.36) *be fulfilled and let us drop condition* (5.4.3). *Let* <u>*either*</u> $d = 2$ *and* $m \ge h^{\frac{1}{2}}$ <u>*or*</u> $d = 1$.

Then the combined error does not exceed $C\bar{\gamma}^{-d}$ *where* $\bar{\gamma}$ *is defined by* (5.4.54) *as* $m \ge h^{\frac{1}{2}}$ *and by* (5.4.49) *otherwise (which happens only as* $d = 1$*).*

Sharp Remainder Estimate

Obviously we arrive

Theorem 5.4.24. *Let* (5.4.7) *and* (5.4.23) *be fulfilled. Further, let* $d \geq 3$.

Alternatively let $d = 2$, $\mathsf{mes}(Z(0)) = 0$ *and either* (5.4.3) *be fulfilled or* $m \leq h^{\frac{1}{2}}$ *or* $\nu(t) = t^2$.

Assume that the measure of the set of all dead-end[34] *or periodic points*[35] *is* 0.

Then $|\mathsf{R}^{\mathsf{W}}| = o(h^{1-d})$ *and the same is true for approximation error provided* (5.4.28) *holds.*

We leave details to the reader. Let us observe that if $m \to \bar{m}$ we need to investigate Hamiltonian system corresponding to $\bar{m}$ only.

We leave to the reader

Problem 5.4.25. (i) Consider $\mathsf{R}^{\mathsf{W}}_s$ (see also Subsubsection 5.3.2.4).

(ii) Consider Riesz means using also methods and results of Section 4.7.

5.4.4 General oOperators

In this subsection we consider general differential operators with irregular coefficients. We approximate them by operators with the "rough" coefficients and estimate both Weyl remainder estimate and the approximation error in Weyl expression. The applications to asymptotics will come later.

Dropping Condition $|\nabla_\xi a| \neq 0$

Again, instead of the microhyperbolicity condition

$$|a| + |\nabla_{x,\xi} a| \geq \epsilon_0 \tag{5.4.55}$$

as in the smooth case our "reference theorem" requires a stronger ξ-microhyperbolicity condition

$$|a| + |\nabla_{\xi} a| \geq \epsilon_0 \tag{5.4.56}$$

which also works for asymptotics without spatial mollification.

Assume first that $\nu(t)$ satisfies (5.4.7)–(5.4.9) and define scaling functions from equations

$$\nu(\gamma) = \epsilon \max\bigl(|a| + |\nabla_\xi a|^2, \nu(\bar\gamma)\bigr), \qquad \rho = \nu(\gamma)^{\frac12}, \qquad h_{\text{new}} = \frac{h}{\rho\gamma} \tag{5.4.57}$$

with $\bar\gamma$ defined from equation (5.4.13) and a roughness function

$$\varepsilon = C\frac{h}{\rho}|\log\frac{h}{\rho\gamma}|. \tag{5.4.58}$$

Then instead of (5.4.11) we have

$$|x-y| \le \epsilon\gamma(x), \qquad |\xi-\eta| \le \epsilon\rho \implies \tag{5.4.59}$$
$$C^{-1}\gamma(x,\xi) \le \gamma(y,\eta) \le C\gamma(x,\xi), \qquad C^{-1}\rho(x,\xi) \le \rho(y,\eta) \le C\rho(x,\xi).$$

Indeed,

$$|a_\eta(y,\eta) - a_\xi(x,\xi)| \le |a_\eta(y,\eta) - a_\eta(x,\eta)| + |a_\eta(x,\eta) - a_\xi(x,\xi)| \le \\ C_0\nu(|x-y|) + C_0|\xi-\eta|$$

(where $a_\xi = \nabla_\xi a$ etc) which implies that

$$|a_\eta(y,\eta) - a_\xi(x,\xi)| \le \epsilon\rho$$

and also

$$|\eta(y,\eta) - a(x,\xi)| \le |a(y,\eta) - a(x,\eta)| + |a(x,\eta) - a(x,\xi)| \le \\ C_0\nu(|x-y|) + C_0|a_\xi(x,\xi)|\cdot|\xi-\eta| + C_0|\xi-\eta|^2 \le \epsilon\rho^2.$$

These two inequalities imply (5.4.59). So, (γ,ρ) is a scaling vector-function.

Consider first asymptotics *without* spatial mollification. Scaling estimate $Ch^{-d}\nu(h|\log h|)$ of the combined error we estimate the contribution of (γ,ρ) element by

$$Ch^{-d}\rho^{d-2}\nu\bigl(\frac{h}{\rho}|\log\frac{h}{\rho\gamma}|\bigr)$$

(let us recall that $\nu_{\text{new}}(t) = \rho^{-2}\nu(\gamma t)$)) and the total combined error by

$$Ch^{-d}\int \rho^{-2}\nu\bigl(\frac{h}{\rho}|\log\frac{h}{\rho\gamma}|\bigr)\,d\xi. \tag{5.4.60}$$

Let us assume that one of two following conditions holds: $(5.4.61)^+_r$ $a_{\xi\xi}$ has r eigenvalues of the same sign with absolute values larger than ϵ_1; $(5.4.61)_r$ $a_{\xi\xi}$ has r eigenvalues with absolute values larger than ϵ_1 where here and below $a_{\xi\xi}$, $a_{\xi x}$, $a_{x\xi}$ and a_{xx} denote the matrices of the second derivatives (we don't use the last one since it does not necessarily exist).

Observe that under assumption $(5.4.61)_r$

$$a = Q(z;\xi' - \mu(z)) + W(z), \qquad \xi' = (\xi_1, \dots, \xi_r),\ z = (x, \xi''),\quad \xi'' = (x_{r+1}, \dots, \xi_d) \tag{5.4.62}$$

where is Q a non-degenerate quadratic form with respect to $\xi' - \mu(z)$; further, under condition $(5.4.61)^+_r$ this form is either positive or negative definite.

Then $\rho \gtrsim |\xi' - \mu(z)| + |V(z)|$ and (5.4.60) equals

$$Ch^{-d} \int \Big(\int \rho^{-2} \nu\big(\frac{h}{\rho} |\log \frac{h}{\rho\gamma}\big)\, d\xi' \Big)\, d\xi'' \tag{5.4.63}$$

where the integrand does not exceed $C\rho^{-3}\nu(h|\log h|)$.

Then the inner integral in expression (5.4.63) and thus the whole expression (5.4.63) do not exceed $Ch^{-d}\nu(h|\log h|)$ as either $(5.4.61)_4$ or $(5.4.61)^+_3$ is fulfilled. The same is true under condition $(5.4.61)_3$ provided $\nu(r)r^{-l}$ is monotone decreasing with some $l < 1$.

Really, the inner integral does not exceed $\int \rho^{-l-2}\, d\rho^r$ with integral generally taken over $\bar\rho \le \rho \le c$ and with $l = 1$; hence we need $r = 4$ to have this integral bounded; $l < 1$ relaxes it to $r = 3$; finally, under condition $(5.4.61)^+_r$ integration is taken over set $|\xi' - \mu(z)| \asymp |W(z)|^{\frac12}$ and therefore as $r = 3$ this integral is also bounded.

On the other hand, under weaker assumptions we get weaker estimates:

Theorem 5.4.26. *Let ν satisfy* (5.4.7)–(5.4.9).

(i) Let <u>*either*</u> $(5.4.61)_4$ <u>*or*</u> $(5.4.61)^+_3$ *be fulfilled* <u>*or*</u> $(5.4.61)_3$ *be fulfilled and $\nu(r)r^{-l}$ be monotone decreasing with some $l < 1$. Then the combined error (for both N and N_x) does not exceed $Ch^{-d}\nu(h|\log h|)$.*

(ii) Let $(5.4.61)_3$ be fulfilled. Then the combined error (for both N and N_x) does not exceed $Ch^{-d}\nu(h|\log h|)$.

(iii) Let condition $(5.4.61)_r$ with $r = 1, 2$ be fulfilled. Then the combined error (for both N and N_x) does not exceed $Ch^{-d}\nu(h|\log h|)$ does not exceed $Ch^{-d}\bar\rho^r$ with $\bar\rho$, $\bar\gamma$ defined by (5.4.13).

Case $\nu(t)=o(t)$

Assuming instead that $\nu(t)$ satisfies (5.4.7), (5.4.23) and using scaling (5.4.57) associated with $\nu(t)=t$, i. e.

$$\gamma=\epsilon\max\bigl(|a|+|\nabla_\xi a|^2,\bar\gamma\bigr),\qquad \rho=\gamma^{\frac12},\qquad h_{\mathsf{new}}=h\gamma^{-\frac32} \tag{5.4.64}$$

one arrives instantly to the following

Theorem 5.4.27. *Let ν satisfy* (5.4.7), (5.4.23).

(i) Let <u>*either*</u> $(5.4.61)_3^+$ <u>*or*</u> $(5.4.61)_4$ *be fulfilled. Then the combined error (for both* N *and* N_x*) does not exceed* $Ch^{-d}\bigl(\nu(h|\log h|)+h\bigr)$*;*

(ii) Let $(5.4.61)_3$ *be fulfilled. Then the combined error (for both* N *and* N_x*) does not exceed* $Ch^{-d}\bigl(\nu(h|\log h|)+h|\log h|\bigr)$*;*

(iii) Let (5.4.55) *and* $(5.4.61)_2$ *be fulfilled. Then the combined error (for* N *only) does not exceed* $Ch^{-d}\bigl(\nu(h|\log h|)+h|\log h|\bigr)$*;*

(iv) Let (5.4.55) *and* $(5.4.61)_1$ *be fulfilled. Then the combined error (for* N *only) does not exceed* $Ch^{-d}\bigl(\tilde\nu(h|\log h|)+h^{\frac13}|\log h|\bigr)$ *where*

$$\tilde\nu=\int_{h^{1/3}}^1 \nu(\rho^{-1}|\log\rho^{-3}h|)\rho^{-1}\,d\rho. \tag{5.4.65}$$

Problem 5.4.28. Following the proof of Theorem 5.4.10 one can prove that in the framework of Statement (iv) estimate is actually $Ch^{-d}\bigl(\tilde\nu(h|\log h|)+h\bigr)$.

Now we encounter strange new possibilities not available for Schrödinger operator. First, it may happen that (5.4.55) holds but (5.4.56) and $(5.4.61)_1$ fail.

Second, it may happen that (5.4.55) also fails and $\mathsf{rank}(a_{\xi\xi})$ is not large enough, but $\mathsf{rank}(a_{\xi\xi}\ a_{\xi x})$ helps. Therefore let us assume that

$(5.4.66)_m$ The following $(d+1)\times 2d$-matrix has rank at least m [36]:

$$\begin{pmatrix} da \\ da_{\xi_1} \\ \vdots \\ da_{\xi_d} \end{pmatrix} = \begin{pmatrix} a_{\xi_1} & \dots & a_{\xi_d} & a_{x_1} & \dots & a_{x_d} \\ a_{\xi_1\xi_1} & \dots & a_{\xi_1\xi_d} & a_{\xi_1 x_1} & \dots & a_{\xi_1 x_d} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{\xi_d\xi_1} & \dots & a_{\xi_d\xi_d} & a_{\xi_d x_1} & \dots & a_{\xi_d x_d} \end{pmatrix} \tag{5.4.67}$$

Then we can prove easily

Theorem 5.4.29. *Let ν satisfy* (5.4.7) *and* (5.4.23).

(i) Let <u>either</u> (5.4.55) *and* $(5.4.66)_2$ *<u>or</u>* $(5.4.66)_4$ *be fulfilled; then the combined error does not exceed* $Ch^{-d}\big(\nu(h|\log h|+h\big)$.

(ii) Let <u>either</u> (5.4.55) *and* $(5.4.66)_1$ *<u>or</u>* $(5.4.66)_3$ *be fulfilled; then the combined error does not exceed* $Ch^{-d}\big(\nu(h|\log h|+h\big)|\log h|$.

(iii) Let <u>either</u> (5.4.55) *<u>or</u>* $(5.4.66)_2$ *be fulfilled; then the combined error does not exceed* $Ch^{-d+\frac{2}{3}}$.

(iv) Let $(5.4.66)_1$ *be fulfilled; then the combined error does not exceed* $Ch^{-d+\frac{1}{3}}$.

Proof. Let us introduce γ and ρ by (5.4.64) again (only in the vicinity of $\{\nabla a_x = 0\}$). Then the total combined error does not exceed again

$$Ch^{-d}\int \Big(\gamma^{-1}\nu\big(\frac{h}{\gamma^{\frac{1}{2}}}|\log\frac{h}{\gamma^{\frac{3}{2}}}|\big) + Ch\gamma^{-\frac{3}{2}}\Big)\,dxd\xi \tag{5.4.68}$$

where we divided the previous expression by $\rho^d\gamma^d$ and plugged $\rho=\gamma^{\frac{1}{2}}$.

Note that under condition $\big((5.4.66)\big)_m$ in an appropriate non-symplectic coordinates

$$\gamma \gtrsim \sum_{1\le j\le m} z_j^2 + \bar\gamma$$

and under conditions (5.4.55) and $\big((5.4.66)\big)_m$

$$\gamma \gtrsim |z_1| + \sum_{2\le j\le m} z_j^2 + \bar\gamma$$

which leads to the corresponding estimates of integral (5.4.68). □

[36] Which means that there exists m-minor with an absolute value not less than ϵ_1.

Case $\nu(t)=o(t)$. II

Basically we are left with the cases when (5.4.55) or $(5.4.66)_2$ or $(5.4.66)_1$ is fulfilled and we hope to improve remainder estimate as $\nu(t)\le t^l$ with $l\in(1,2]$.

Problem 5.4.30. Following the proof of Theorem 5.2.14 prove that under condition (5.4.55) the combined error does not exceed $Ch^{-d}\bigl(\tilde\nu(h^{1-\delta})+h^{1-\delta}\bigr)$ with arbitrarily small $\delta>0$.

Now we need to cover case when only $(5.4.66)_m$ with $m=1,2$ is fulfilled. Assuming that (5.4.35), (5.4.36) are fulfilled we can introduce γ, ρ by the following formula similar to (5.4.37)

$$\gamma=\mathsf{max}(\gamma',\bar\gamma),\quad \gamma'=\epsilon\,\mathsf{min}\{t:|a|+|a_\xi|^2\le\nu(t),|a_x|\le t^{-1}\nu(t)\}. \tag{5.4.69}$$

with ρ, $\bar\gamma$ defined by (5.4.12) and (5.4.13).

One can prove easily that (5.4.59) holds.

Theorem 5.4.31. *Let continuity modulus ν satisfy* (5.4.7), (5.4.23), (5.4.35), (5.4.36) *and* $(5.4.66)_m$ *be fulfilled with $m=1,2$. Then the combined error is $O(h^{-d}\bar\rho^m+h^{1-\delta-d})$ with $\bar\gamma$, $\bar\rho$ defined by* (5.4.13).

Proof. Let us consider (ρ,γ)-element.

(a) As $\gamma\ge c_0\bar\gamma$ and $|a|+|\nabla_\xi a|^2\asymp\rho^2$ there its contribution to the remainder does not exceed $Ch^{1-d}\rho^{d-1}\gamma^{d-1}$ and contribution of all these elements does not exceed

$$Ch^{1-d}\int\rho^{-1}\gamma^{-1}\,dxd\xi.$$

(b) As $\gamma\ge c_0\bar\gamma$ and $|\nabla_x a|\asymp\nu(\gamma)\gamma^{-1}$ there its contribution to the remainder does not exceed due to problem 5.4.30 $Ch^{1-\delta-d}\rho^{d-1+\delta}\gamma^{d-1+\delta}$ and contribution of all these elements does not exceed

$$Ch^{1-\delta-d}\int\rho^{\delta-1}\gamma^{\delta-1}\,dxd\xi.$$

Thus, we got the second integral anyway. Then due to condition $(5.4.66)_m$ $\mathsf{mes}(\{z:\rho(z)\asymp\rho\})\lesssim\rho^m$ and this integral does not exceed

$$Ch^{1-\delta-d}\int\rho^{m+\delta-2}\gamma(\rho)^{\delta-1}\,d\rho \tag{5.4.70}$$

and if

(5.4.71) $M(\rho) = \rho^{m+\delta-1}\gamma(\rho)^{\delta-1}$ satisfies $M(k\rho) \ge k^{-\sigma}M(\rho)$ with $\sigma > 0$ as $k \le 1$

we conclude that this integral does not exceed $Ch^{1-\delta-d}\bar\rho^{m+\delta-1}\bar\gamma^{\delta-1} = Ch^{-d}\bar\rho^{m}$.

Obviously (5.4.71) holds as $m = 1$, $\delta = \frac{1}{3}$ for sure, we do not need even Problem 5.4.30 but can refer to Theorem 5.4.29(ii). If $m = 2$ situation becomes more complicated and slightly varying δ we can achieve either (5.4.71) or $M(k\rho) \le k^{\sigma}M(\rho)$; in the latter case (5.4.70) does not exceed $Ch^{1-d+\delta}$. □

Problem 5.4.32. (i) Assuming that $\nu(t) = o(t^2)$ introduce a_{xx} into consideration and replace assumption $(5.4.66)_m$ by a weaker assumption

$(5.4.72)_m$ The following $(2d+1)\times 2d$-matrix has rank at least m[36]:

$$(5.4.73)\qquad \begin{pmatrix} da \\ da_{\xi_1} \\ \vdots \\ da_{x_d}\end{pmatrix} = \begin{pmatrix} a_{\xi_1} & \dots & a_{\xi_d} & a_{x_1} & \dots & a_{x_d} \\ a_{\xi_1\xi_1} & \dots & a_{\xi_1\xi_d} & a_{\xi_1 x_1} & \dots & a_{\xi_1 x_d} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{x_d\xi_1} & \dots & a_{x_d\xi_d} & a_{x_d x_1} & \dots & a_{x_d x_d}\end{pmatrix}.$$

(ii) Alternatively introduce $\varepsilon = C(h|\log h|)^{\frac{1}{2}}$, derive the combined estimate $Ch^{-d}\big(\nu((h|\log h|)^{\frac{1}{2}}) + h\big)$ under microhyperbolicity condition and scale it.

Problem 5.4.33. (i) Consider $\mathsf{R}^{\mathsf{W}}_s$ (see also Subsubsection 5.3.2.4).

(ii) Consider Riesz means using also methods and results of Section 4.7.

Sharp Remainder Estimates

As long as the contribution of $\{z : \gamma(z) \le \eta\}$ to the remainder estimate does not exceed $\big(\mu(\eta) + o_\eta(1)\big)h^{1-d}$ as $h \to +0$ with $\mu(\eta) = o(1)$ as $\eta \to +0$, under standard non-periodicity condition one can derive remainder estimate $\mathsf{R}^{\mathsf{W}} = o(h^{1-d})$. Moreover if an approximation error is $o(h^{1-d})$ then the combined error is $o(h^{1-d})$ as well.

5.A Appendices

5.A.1 Spectral Kernel Calculations for Some Model Operators

Case $d = 1$

Consider operator (5.3.59).

Note first that

$$B = -x_1 \implies \partial_\tau e_B(x, y, \tau) = \delta(x_1 - y_1)\delta(x_1 + \tau). \tag{5.A.1}$$

Then making unitary h-Fourier transform we conclude that for

$$B = hD_{\xi_1} \implies \partial_\tau e_B(\xi_1, \eta_1, \tau) = (2\pi h)^{-1} \exp\bigl(ih^{-1}(\xi_1 - \eta_1)\tau\bigr). \tag{5.A.2}$$

Therefore since $\mathcal{T}^* hD_{\xi_1}\mathcal{T} = hD_{\xi_1} + \xi_1^2$ for $\mathcal{T} = \exp(ih^{-1}\frac{1}{3}\xi_1^3)$ we conclude that

$$\begin{aligned} &B = hD_{\xi_1} + \xi_1^2 \implies \\ &\qquad \partial_\tau e_B(\xi_1, \eta_1, \tau) = (2\pi h)^{-1} \exp\bigl(ih^{-1}\bigl[(\xi_1 - \eta_1)\tau - \frac{1}{3}(\xi_1^3 - \eta_1^3)\bigr]\bigr). \end{aligned} \tag{5.A.3}$$

Finally making inverse Fourier transform we conclude that for A defined by (5.3.59)

$$\begin{aligned} &\partial_\tau e(x_1, y_1, \tau) = \\ &\qquad (2\pi h)^{-2} \iint \exp\bigl(ih^{-1}\bigl[(\xi_1 - \eta_1)\tau + x_1\xi_1 - y_1\eta_1 - \frac{1}{3}(\xi_1^3 - \eta_1^3)\bigr]\bigr)\, d\xi_1\, d\eta_1 \end{aligned} \tag{5.A.4}$$

and therefore

$$\begin{aligned} &\partial_\tau e(x_1, x_1, \tau) = \\ &\qquad (2\pi h)^{-2} \iint \exp\bigl(ih^{-1}\bigl[(\xi_1 - \eta_1)(\tau + x_1) - \frac{1}{3}(\xi_1^3 - \eta_1^3)\bigr]\bigr)\, d\xi_1\, d\eta_1. \end{aligned} \tag{5.A.5}$$

One can rewrite it as

$$e(x_1, x_1, \tau) = h^{-\frac{2}{3}} \int_{-\infty}^{(x_1+\tau)h^{-\frac{2}{3}}} F(\tau)\, d\tau, \tag{5.A.6}$$

with

$$F(t) = (2\pi)^{-2} \iint \exp\bigl(i\bigl[(\xi_1 - \eta_1)t - \frac{1}{3}(\xi_1^3 - \eta_1^3)\bigr]\bigr)\, d\xi_1\, d\eta_1 = \tag{5.A.7}$$
$$2(2\pi)^{-2} \iint \exp\bigl(i\bigl[2\beta t - \frac{2}{3}\beta^3 - 2\beta\alpha^2\bigr]\bigr)\, d\alpha\, d\beta =$$
$$(2\pi)^{-\frac{3}{2}} \int |\beta|^{-\frac{1}{2}} \exp\bigl(-i\frac{\pi}{4}\operatorname{sgn}\beta + i\bigl[2\beta t - \frac{2}{3}\beta^3\bigr]\bigr)\, d\beta$$

where we first substituted $\xi_1 = \alpha + \beta$, $\eta_1 = \alpha - \beta$ and then integrated with respect to α.

The remaining integral has stationary points $\beta = \pm t^{\frac{1}{2}}$ and a singular point $\beta = 0$. Decomposing for $t > 0$ $F(t) = F_1(t) + F_2(t)$,

$$F_1(t) = (2\pi)^{-\frac{3}{2}} \int |\beta|^{-\frac{1}{2}} \exp\bigl(-i\frac{\pi}{4}\operatorname{sgn}\beta + 2i\beta t\bigr)\, d\beta = \frac{1}{2}\kappa t^{-\frac{1}{2}} \tag{5.A.8}$$

with

$$\kappa = 2(2\pi)^{-\frac{3}{2}} \int_0^\infty \beta^{-\frac{1}{2}} \cos(2\beta - \frac{\pi}{4})\, d\beta, \tag{5.A.9}$$

and

$$F_2(t) = \tag{5.A.10}$$
$$(2\pi)^{-\frac{3}{2}} \int |\beta|^{-\frac{1}{2}} \exp\bigl(-i\frac{\pi}{4}\operatorname{sgn}\beta + 2i\beta t\bigr)\Bigl(\exp\bigl(-\frac{2}{3}i\beta^3\bigr) - 1\Bigr) d\beta.$$

Then primitives of $F_1(t)$ and $F_2(t)$ produce exactly main term and correction term in (5.3.36):

$$\mathcal{Q}(t) = \int_\infty^t F_2(t')\, dt' = \tag{5.A.11}$$
$$-\,2^{-2}(2\pi)^{-\frac{3}{2}} \int |\beta|^{-\frac{3}{2}} \bigl(i\frac{\pi}{4}\operatorname{sgn}\beta + 2i\beta t\bigr)\Bigl(\exp\bigl(-\frac{2}{3}i\beta^3\bigr) - 1\Bigr)\, d\beta.$$

In this integral singularity at $\beta = 0$ gives a relatively small contribution (one can prove it is $O(t^{-\frac{3}{2}})$ as $t \to \infty$); the main contribution comes from the stationary points $\beta = \pm t^{\frac{1}{2}}$ and modulo $O(t^{-\frac{3}{2}})$

$$\mathcal{Q}(t) \equiv -2^{-1}(2\pi)^{-1} \sum_{\beta = \pm t^{\frac{1}{2}}} |\beta|^{-2} \exp\bigl(i\frac{\pi}{2}\operatorname{sgn}\beta + \frac{4}{3}i\beta t\bigr) = \tag{5.A.12}$$
$$(2\pi)^{-1} t^{-1} \sin\bigl(\frac{4}{3}t^{\frac{3}{2}}\bigr).$$

Case $d = 2$

Now we consider operator

$$A = h^2 D_{x_1}^2 + h^2 D_{x_2}^2 - x_1$$

leading to $B = \xi_1^2 + \xi_2^2 + hD_{\xi_1}$ and then instead of (5.A.5) we have

(5.A.13) $\partial_\tau e(x_1, x_1, \tau) =$

$$(2\pi h)^{-3} \iint \exp\bigl(ih^{-1}\bigl[(\xi_1 - \eta_1)(\tau + x_1 - \xi_2^2) - \frac{1}{3}(\xi_1^3 - \eta_1^3)\bigr]\bigr)\, d\xi_1 d\xi_2 d\eta_1.$$

and all previous formulae are adjusting accordingly. Then (5.A.6)–(5.A.7) are replaced by

(5.A.14) $e(x_1, x_1, \tau) = h^{-\frac{4}{3}} \int_{-\infty}^{(x_1+\tau)h^{-\frac{2}{3}}} F(\tau)\, d\tau,$

with

(5.A.15) $F(t) = 2(2\pi)^{-3} \iint \exp\bigl(i\bigl[2\beta(t + \gamma^2) - \frac{2}{3}\beta^3 - 2\beta\alpha^2\bigr]\bigr) d\alpha d\beta d\gamma =$

$$(2\pi)^{-2} \int |\beta|^{-1} \exp\bigl(i\bigl[2\beta t - \frac{2}{3}\beta^3\bigr]\bigr)\, d\beta$$

and (5.A.8)–(5.A.10) are replaced by

(5.A.16) $$F_1(t) = (2\pi)^{-2} \int |\beta|^{-1} \exp\bigl(2i\beta t\bigr)\, d\beta = \kappa + \kappa_1 t^{-3}$$

with

(5.A.17) $$\kappa = 2(2\pi)^{-\frac{3}{2}} \int_0^\infty \beta^{-\frac{1}{2}} \cos(2\beta - \frac{\pi}{4})\, d\beta,$$

where integral was understood in the sense of distributions and

(5.A.18) $$F_2(t) = (2\pi)^{-2} i^{-1} \int |\beta|^{-1} \exp\bigl(2i\beta t\bigr)\Bigl(\exp\bigl(-\frac{2}{3}i\beta^3\bigr) - 1\Bigr) d\beta.$$

Then (5.A.11) is replaced by

(5.A.19) $\mathcal{Q}(t) = \int_\infty^t F_2(t')\, dt' =$

$$-\, 2^{-1}(2\pi)^{-2} \int |\beta|^{-2} \operatorname{sgn}\beta \exp\bigl(2i\beta t\bigr)\Bigl(\exp\bigl(-\frac{2}{3}i\beta^3\bigr) - 1\Bigr)\, d\beta.$$

In this integral singularity at $\beta = 0$ gives a small contribution; really, decomposing $\exp\bigl(-\frac{2}{3}i\beta^3\bigr) - 1$ into powers of β^3 we get that he leading term is $O(t^{-2})$. The main contribution comes from the stationary points $\beta = \pm t^{\frac{1}{2}}$ and modulo $O(t^{-\frac{3}{2}})$

$$\mathcal{Q}(t) \equiv (2\pi)^{-1} t^{-\frac{5}{4}} \sin\bigl(\frac{4}{3} t^{\frac{3}{2}}\bigr). \tag{5.A.20}$$

5.A.2 On Pauli Matrices

The results of this appendix are important for Section 4.6 and for later applications.

Definition 5.A.1. *Pauli matrices*[37] are $\mathsf{D} \times \mathsf{D}$ matrices $\sigma_0, \dots, \sigma_d$ such that

$$\sigma_j^\dagger = \sigma_j, \tag{5.A.21}$$

and

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk} I \qquad j, k = 0, \dots, d. \tag{5.A.22}$$

We have the key statement

Theorem 5.A.2. *(i) A set $\sigma_0, \dots, \sigma_d$ of $\mathsf{D} \times \mathsf{D}$ Pauli matrices exists if and only if $\mathsf{D} \in \bar{\mathsf{D}}\mathbb{Z}$ where $\bar{\mathsf{D}} = 2^{\lfloor \frac{d+1}{2} \rfloor}$.*

(ii) Moreover, if $\mathsf{D} = \bar{\mathsf{D}}$ and d is odd then all sets of Pauli matrices are unitarily equivalent[38] *On the other hand, for $\mathsf{D} = \bar{\mathsf{D}}$ and even d there exist precisely two sets of Pauli matrices which are not unitarily equivalent.*

(iii) If $\mathsf{D} = \bar{\mathsf{D}}$ and d is even then

$$\sigma_0 \cdots \sigma_d = \varsigma i^{\frac{d}{2}} I \quad \text{with } \varsigma = \pm 1 \tag{5.A.23}$$

and two sets are equivalent if and only if $\varsigma = \varsigma'$. If $\mathsf{D} = \bar{\mathsf{D}}$ and d is odd then $\sigma_0, \dots, \sigma_d, \sigma_{d+1}$ with

$$\sigma_{d+1} = \varsigma(-i)^{\frac{d+1}{2}} \tag{5.A.24}$$

are also Pauli matrices.

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[37] As $\mathsf{D} = 4$ they are also called Dirac matrices.

[38] The sets $\sigma_0, \dots, \sigma_d$ and $\sigma'_0, \dots, \sigma'_d$ are unitarily equivalent if $\sigma'_j = q^\dagger \sigma'_j q \quad \forall j = 0, \dots, d$ with appropriate unitary matrix q. Obviously the transition to a unitarily equivalent set preserves properties (5.A.21) and (5.A.22).

(iv) If $p = \mathsf{D}/\bar{\mathsf{D}} \in \mathbb{Z}$ *and* d *is odd then every set of Pauli matrices is unitarily equivalent to the set* $\sigma_j = \bar{\sigma}_j \otimes I_p$ $j = 0, \dots, d$ *where* $\{\bar{\sigma}_j \quad j = 0, \dots, d\}$ *is a standard set in dimension* $\bar{\mathsf{D}}$*.*

Further, if $p = \mathsf{D}/\bar{\mathsf{D}} \in \mathbb{Z}$ *and* d *is even then every set of Pauli matrices is unitarily equivalent to the set* $\sigma_j = \bar{\sigma}_j \otimes J_{p,q}$ $j = 0, \dots, d$ *where* $J_{p,q} = \mathsf{diag}(1, \dots, 1, -1, \dots, -1)$ *is a diagonal* $p \times p$ *matrix,* $q = 0, \dots, p$*. Here sets with different* q *are not unitarily equivalent.*

So a set of Pauli matrices is irreducible if and only if $\mathsf{D} = \bar{\mathsf{D}}$*;*

(v) The property "Every matrix commuting with all the Pauli matrices is scalar" is fulfilled if and only if $\mathsf{D} = \bar{\mathsf{D}}$*.*

Proof. Property (5.A.22) yields that $\sigma_j^2 = I$ and hence $\mathsf{Spec}\,\sigma_j \subset \{-1, 1\}$. Let us reduce σ_0 to the diagonal form; then for $d = 0$ a set of Pauli matrices is unitarily equivalent to the set consisting of one matrix

$$\sigma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{5.A.25}$$

and hence for $d = 0$ we obtain all the statements of the theorem and $\bar{\mathsf{D}} = 1$ and for $\mathsf{D} = 1$ $\sigma_0 = \varepsilon = \pm 1$.

Thus without any loss of the generality one can assume that (5.A.25) holds and for $d \geq 1$ (5.A.21) and (5.A.22) yield that $-\sigma_0 = \sigma_1^{-1}\sigma_0\sigma_1$ and hence the multiplicities of the eigenvalues ± 1 are equal to $\frac{1}{2}\mathsf{D}$.

Moreover, (5.A.21) and (5.A.22) yield that

$$\sigma_j = \begin{pmatrix} 0 & \alpha_j \\ \alpha_j^\dagger & 0 \end{pmatrix} \qquad \text{for} \quad j = 1, \dots, d \tag{5.A.26}$$

where α_j are unitary. Let $q = \begin{pmatrix} I & 0 \\ 0 & q' \end{pmatrix}$ where q' is a unitary $\frac{1}{2}\mathsf{D} \times \frac{1}{2}\mathsf{D}$ matrix; then $q^\dagger \sigma_0 q = \sigma_0$ and $\sigma'_j = q^\dagger \sigma_j q$ $(j \geq 1)$ have the same form as σ_j but with α_j replaced by $\alpha'_j = \alpha_j q$. Hence without loss of generality one can assume that $\alpha_1 = iI$, i. e. , that

$$\sigma_1 = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}. \tag{5.A.27}$$

So we obtain Statements (i) and (ii) for $d = 1$ with $\bar{\mathsf{D}} = 2$.

Moreover, (5.A.21) and (5.A.22) yield that if σ_0 and σ_1 have the indicated form then the matrices α_j with $j = 2, \dots, d$ also satisfy (5.A.21) and (5.A.22). The proof is complete. □

Lemma 5.A.3. *If $d \ge 2$ and conditions* (5.A.21) *and* (5.A.22) *are fulfilled then by a unitary transformation the set $\sigma_0, \dots, \sigma_d$ can be reduced to the form*

$$\sigma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}, \quad \sigma_j = \begin{pmatrix} 0 & \alpha_j \\ \alpha_j & 0 \end{pmatrix} \tag{5.A.28}$$

$(j = 2, \dots, d)$ *where $\alpha_2, \dots, \alpha_d$ are $\frac{1}{2}\mathsf{D} \times \frac{1}{2}\mathsf{D}$ matrices satisfying* (5.A.21) *and* (5.A.22) *also.*

Conversely, if $\alpha_2, \dots, \alpha_d$ are $\frac{1}{2}\mathsf{D} \times \frac{1}{2}\mathsf{D}$ matrices satisfying (5.A.21) *and* (5.A.22) *then the $\mathsf{D} \times \mathsf{D}$ matrices $\sigma_0, \dots, \sigma_d$ introduced by* (5.A.28) *way also satisfy* (5.A.21) *and* (5.A.22).

Moreover if $\sigma_0, \dots, \sigma_d$ satisfy (5.A.23) *if and only if $\alpha_2, \dots, \alpha_d$ do with the same ς and with d replaced by $d - 2$.*

Theorem 5.A.2 can be easily proven by induction with respect to d with step 2.

We have obtained in particular that for $d = 2$ $\bar{\mathsf{D}} = 2$ and

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_j = \varsigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.A.29}$$

modulo unitary equivalence; we have obtained the *Pauli matrices;* moreover,

$$[\sigma_j, \sigma_k] = 2i\varsigma \bar{\varepsilon}^{jkl} \sigma_l \tag{5.A.30}$$

where $\bar{\varepsilon}^{jkl}$ is an absolutely skew-symmetric tensor with $\bar{\varepsilon}^{123} = 1$ (we do not discuss the change of coordinates) and we write σ_3 instead of σ_0 here. We have also obtained that for $d = 3, 4$ $\bar{\mathsf{D}} = 4$ and we have a method to construct the corresponding sets. In particular, for $d = 3$ one can assume that

$$\sigma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \sigma_j = \begin{pmatrix} 0 & \alpha_j \\ \alpha_j & 0 \end{pmatrix} \quad j = 1, 2, 3 \tag{5.A.31}$$

where $\alpha_3 = \alpha_0, \alpha_1, \alpha_2$ are Pauli matrices.

We do not prove the two following simple assertions here:

Proposition 5.A.4. *Let $\sigma_0, \dots, \sigma_d$ be Pauli matrices with $d \geq 2$. Let $\mathbb{K}^{\pm} = \mathsf{Ker}(\sigma_1 \pm i\sigma_2)$. Then*

(i) $\dim \mathbb{K}^{\pm} = \frac{1}{2}\mathsf{D}$, *$\mathbb{K}^+$ and $\mathbb{K}^-$ are orthogonal, and $\mathbb{K}^+ \oplus \mathbb{K}^- = \mathbb{H}$;*

(ii) $\sigma_j \mathbb{K}^{\pm} = \mathbb{K}^{\pm}$ *for $j \neq 1, 2$ and $\sigma_j \mathbb{K}^{\pm} = \mathbb{K}^{\mp}$ for $j = 1, 2$.*

(iii) If (5.A.23) *is fulfilled then*

$$\sigma_0 \cdots \sigma_d|_{\mathbb{K}^{\pm}} = \pm\varsigma i^{\frac{d-2}{2}} I. \tag{5.A.32}$$

Proposition 5.A.5. *Let $\sigma_0, \dots, \sigma_d$ be Pauli matrices and $(\omega^{jk})_{j,k=0,\dots,d}$ be a real orthogonal matrix. Then*

$$\sigma'_j = \sum_{0 \leq k \leq d} \omega^{jk} \sigma_k \quad j = 0, \dots, d$$

are also Pauli matrices. Moreover, (5.A.23) *for $\sigma_0, \dots, \sigma_d$ yields the same equality for $\sigma'_0, \dots, \sigma'_d$ with $\varsigma' = \varsigma \operatorname{sgn} \det(\omega^{jk})$.*

Moreover the following assertion is useful:

Proposition 5.A.6. *Let $\sigma_0, \dots, \sigma_d$ be Pauli matrices. Let us introduce Hermitian matrices $\beta_j = \frac{1}{2i}[\sigma_{2j-2}, \sigma_{2j-1}]$ with $j = 1, \dots, r \leq q = \lfloor \frac{d+1}{2} \rfloor$. Then*

(i) $\beta_j^2 = I$, $\quad \beta_1, \dots, \beta_r$ *commute.*

(ii) For every set $(\varepsilon_1, \dots, \varepsilon_r) \in \{-1, 1\}^r$

$$\dim\Big(\bigcap_{1 \leq j \leq r} \mathsf{Ker}(\beta_j - \varepsilon_j) \Big) = 2^{-r}\mathsf{D}. \tag{5.A.33}$$

Proof. Statement (i) is obvious; Statement (ii) is easily proven by induction with respect to r with a constant $d - 2r$ where the inductive construction of the proof of Theorem 5.A.2 is used. □

Propositions 5.A.5 and 5.A.5 yield

Corollary 5.A.7. *Let $\sigma_0, \dots, \sigma_d$ be Dirac matrices, $(F_{jk})_{j,k=0,\dots,d}$ be a skew-symmetric real matrix and $\pm i f_j$, $j = 1, \dots, q$ be its non-zero eigenvalues (we take multiplicities into account), $f_j > 0$. Then*

$$\mathsf{Spec}\Big(\sum_{j,k} F_{jk}[\sigma_j, \sigma_k]\Big) = \Big\{\nu_\varepsilon = \sum_{1\le j\le r} \varepsilon_j f_j, \quad \varepsilon \in \{-1,1\}^r\Big\}. \tag{5.A.34}$$

Further, the multiplicity of each eigenvalue on the left coincides with the multiplicity of each number on the right multiplied by $2^{-r}\mathsf{D}$, assuming that if ν_ε and $\nu_{\varepsilon'}$ coincide then their multiplicities are combined.

Chapter 6

Operators in the Interior of Domain. Esoteric Theory

This chapter is devoted to even more specialized results than the previous one.

In Section 6.1 we extend some of the above results to conjoint spectral asymptotics for families of commuting operators (general or scalar).

Then in Section 6.2 we consider scalar operators with periodic Hamiltonian flow of the principal symbol (or some perturbations of such operators) and under certain conditions we obtain spectral asymptotics with highly accurate remainder estimates; these asymptotics are not necessarily completely Weyl-type. In the case of the cluster character of the spectrum we get the asymptotic distribution of the eigenvalues inside the clusters. Our main tools here are very long term propagation of singularities (the main result is that singularities propagate along periodic trajectories which drift along the Hamiltonian field of some auxiliary symbol) and conjoint spectral asymptotics of two commuting operators (namely, the operator in question can be represented as the sum of an operator A_0 with spectrum $\{2\pi h T_0^{-1} n, n \in \mathbb{Z}\}$ and a perturbing operator μB commuting with A_0, where μ is a small parameter).

Next, in Section 6.4 we consider asymptotics of

$$\int \omega(x,y)|e(x,y,0)|^2\,dxdy$$

with $\omega(x,y)$ singular at the diagonal $x=y$.

Finally, in Section 6.5 we consider mollification with respect to non-spectral parameters and establish that under corresponding microhyperbolicity condition it would lead to the same effect as mollification with respect to spectral parameter; we will need it much later.

As usual we have few appendices.

6.1 Spectral Asymptotics for Families of Commuting Operators

In this section we generalize the results of the previous sections to the case of families of commuting operators.

6.1.1 Introduction

Let $A_j = A_j(x, hD, h)$ ($j = 1, \dots, I$) be h-pseudodifferential operators on a manifold X. We assume that

(6.1.1) Either $X = \mathbb{R}^d$ or X is a compact closed $\mathscr{C}^K$ manifold or X is a compact $\mathscr{C}^K$ manifold with boundary $\partial X \in \mathscr{C}^K$ with large enough $K = K(d)$, $d = \mathsf{dim}\, X$. In the third case A_j are differential operators and their original domains are given by some boundary value conditions.

Moreover, let us assume that

(6.1.2) The operators A_j are self-adjoint operators in $\mathscr{L}^2(X, \mathbb{H})$ where $\mathbb{H}$ is a Hermitian space[1)] and commute.

Let us recall that

Definition 6.1.1. Unbounded self-adjoint operators $A_1, \dots, A_I$ commute if[2)]

(i) Their spectral projectors $E_1(\tau_1), \dots, E_I(\tau_I)$ commute;

(ii) Their propagators $U_1(t_1), \dots, U_I(t_I)$ commute;

[1)] Surely, one can consider operators acting in the Hermitian bundles.

[2)] These conditions are equivalent.

(iii) Their resolvents $(z_1 - A_1)^{-1}, \dots, (z_l - A_l)^{-1}$ commute.

Then

$$A_j = \int_{\mathbb{R}^l} \tau_j \, d_\tau E(\boldsymbol{\tau}) \tag{6.1.3}$$

where $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_l) \in \mathbb{R}^l$, $E(\tau) = E_1(\tau_1) \dots E_l(\tau_l)$ is the *spectral projector of the family of commuting operators* $\{A_1, \dots, A_l\}$ and $E_j(\tau_j)$ is the spectral projector of A_j.

The standard question is how to get the semiclassical asymptotics of the number of conjoint eigenvalues τ lying in the (compact) domain $\Lambda \subset \mathbb{R}^l$. Usually, however, the case of a domain with the smooth boundary is treated when $\Lambda = \{\boldsymbol{\tau}, \phi(\boldsymbol{\tau}) \le 0\}$ with $\phi \in \mathscr{C}^K$ and $|\phi| + |\nabla\phi|$ disjoint from 0. But results of this type are due to the standard theory applied to the operator $A = \phi(A_1, \dots, A_l)$.

The real problem arises when $\partial\Lambda$ is not smooth. These settings are too general to get a meaningful answer. We assume that

$$\Lambda = [\lambda_1, \tau_1] \times [\lambda_2, \tau_2] \times \cdots \times [\lambda_l, \tau_l] \qquad \text{with} \quad \tau'_j < \tau_j. \tag{6.1.4}$$

Remark 6.1.2. Then we immediately will be able to generalize to the case when $\Lambda = \{\boldsymbol{\tau} : \phi(\boldsymbol{\tau}) \in \Lambda'\}$ where Λ' rather than λ is defined by (6.1.4) with l replaced by m and $\phi : \mathbb{R}^l \to \mathbb{R}^m \supset \Lambda'$ is a smooth map (dimensions of Λ' and Λ do not necessarily coincide. Really, if $\phi = (\phi_1, \dots, \phi_m)$ one can consider family $(A'_1, \dots, A'_m) = \phi_1(A_1, \dots, A_l), \dots, \phi_m(A_1, \dots, A_l)$.

However conditions we will use may be not preserved when we pass from family $(A_1, \dots, A_l)$ to $(A'_1, \dots, A'_m)$ unless $m = l$ and ϕ is a diffeomorphism.

Therefore we assume that (6.1.4) is fulfilled i.e. that Λ is the box. In this case one can consider the dimensions of this box as parameters as well. Thus we are going to treat the semiclassical asymptotics of $\mathsf{Tr}(E(\boldsymbol{\tau}'; \boldsymbol{\tau}))Q$ where

$$E(\boldsymbol{\lambda}; \boldsymbol{\tau}) = \big(E_1(\tau_1) - E_1(\lambda_1)\big) \cdots \big(E_l(\tau_l) - E_l(\lambda_l)\big)$$

and Q is an appropriate h-pseudodifferential operator.

To give a more precise description of the operators in question we assume that

(6.1.5) Either A_j are h-differential operators of orders m_j or $A_j \in \tilde{\boldsymbol{\Psi}}^{(m_j)}_{h,\rho,\gamma,h}$ where the classes of h-pseudodifferential operators $\boldsymbol{\Psi}_{h,\rho,\gamma}$ and $\tilde{\boldsymbol{\Psi}}^{(m)}_{h,\rho,\gamma}$ were introduced in Subsection 6.4.2 for $\rho, \gamma \in (0,1]$, $\rho\gamma \ge h^{1-\delta}$, $\delta > 0$ is arbitrarily small.

To attack this problem let us consider the *propagator of the family of commuting operators* $\{A_1, \dots, A_l\}$

$$U(\boldsymbol{t}) = \exp ih^{-1}(t_1A_1 + \cdots + t_lA_l) = \exp(ih^{-1}t_1A_1)\cdots\exp(ih^{-1}t_lA_l) \tag{6.1.6}$$

with $\boldsymbol{t} = (t_1, \dots, t_l) \in \mathbb{R}^l$. Then

$$U(\boldsymbol{t}) = \int_{\mathbb{R}^l} e^{ih^{-1}\langle \boldsymbol{t},\boldsymbol{\tau}\rangle}\, d_{\boldsymbol{\tau}}E(\boldsymbol{\tau}) \tag{6.1.7}$$

and its Schwartz kernel $u(x, y, \boldsymbol{t})$ satisfies the "Cauchy" problem

$$(hD_{t_j} - A_j)u = 0 \quad (j = 1, \dots, l), \qquad u|_{\boldsymbol{t}=0} = \delta(x-y)I. \tag{6.1.8}$$

Since A_j commute we conclude that

(6.1.9) The principal symbols of A_j commute as well: $[a_j, a_k] = 0$.

6.1.2 General Theory

Elliptic Arguments

Applying routine elliptic arguments one can easily prove the following

Proposition 6.1.3. *Let $(A_1, \dots, A_l)$ be a system of commuting operators of* (6.1.5) *type and let the ellipticity condition*

$$\sum_{1\le j\le l} |a_j(z)v| \ge 2\varrho|v| \qquad \forall v \in \mathbb{H} \tag{6.1.10}$$

be fulfilled in the γ-neighborhood $\mathcal{U}$ of $\bar{z} \in T^*\mathbb{R}^d$ with $\gamma \ge h^{\frac12-\delta}$, $\varrho \ge \gamma^2$ where a_j are the principal symbols of A_j. Let $Q_k \in \boldsymbol{\Psi}_{h,\gamma,\gamma}$ and their symbols be supported in $\mathcal{U}$. Then

$$|F_{\boldsymbol{t}\to h^{-1}\boldsymbol{\tau}}\chi_T(\boldsymbol{t})Q_{1x}u\,{}^tQ_{2y}| \le Ch^s T_1\cdots T_l \tag{6.1.11}$$
$$\forall \boldsymbol{\tau} : |\boldsymbol{\tau}| \le \varrho \quad \forall x, y \in \mathbb{R}^d$$

provided $T_j\varrho \ge h^{1-\delta}$ $\forall j$ where $\chi \in \mathscr{C}_0^K(B_l(0,1))$ is any fixed function, subscript l means dimension here, $T = \operatorname{diag}(T_1, \dots, T_l)$, $\chi_T(\boldsymbol{t}) = \chi(T^{-1}\boldsymbol{t})$.

Successive Approximations

As in Section 4.3 let us first construct

$$F_{t\to h^{-1}\tau}\chi_T(\boldsymbol{t})\,\mathsf{Tr}(U(\boldsymbol{t})Q) \tag{6.1.12}$$

with $\chi_T(\boldsymbol{t}) = \chi_{1T_1}(t_1)\cdots\chi_{lT_l}(t_l)$ and $h^{1-\delta} \le T \le h^{\frac12+\delta}$ by the successive approximation method. To do it we do not need to repeat the construction of that section. An easy induction on l with Proposition 4.3.7 at each step gives us the following

Proposition 6.1.4. *Let $(A_1,\dots,A_l)$ be a system of commuting operators of* (6.1.5) *type and let $\chi_j \in \mathscr{C}_0^K[-1,1]$ and*

$$T_j \in [h^{1-\delta}, h^{\frac12+\delta}] \qquad \forall j. \tag{6.1.13}$$

Let $Q_k \in \boldsymbol{\Psi}_h(\mathbb{R}^d)$ be operators with compactly supported symbols[3)]*. Then*

$$\|F_{\boldsymbol{t}\to h^{-1}\boldsymbol{\tau}}\chi_T(\boldsymbol{t})Q_{1x}u(x,y,\boldsymbol{t})\,{}^tQ_y - h^{-d}T_1\cdots T_l\iint e^{ih^{-1}\langle x-y,\xi\rangle}\hat\chi(h^{-1}T(\boldsymbol{\tau}-\boldsymbol{\tau}'))\mathcal{F}(y,\xi,\boldsymbol{\tau}',h)\,d\xi d\tau'\| \le Ch^s \tag{6.1.14}$$

where

$$\mathcal{F}(y,\xi,\tau',h) = \mathsf{Res}_{\mathbb{R}}\,F(y,\xi,\tau',h) \tag{6.1.15}$$

with l-dimensional operation $\mathsf{Res}_{\mathbb{R}}$ and

(6.1.16) *F is the symbol of the operator $(2\pi)^{-d}Q_1(\tau_1-A_1)^{-1}\cdots(\tau_l-A_l)^{-1}Q_2$ constructed in the framework of the formal symbol calculus*

and s is arbitrarily large.

Therefore

$$F = \sum_{0\le n\le M} F_n(y,\xi,\tau) \tag{6.1.17}$$

where $M = M(s)$ and F_n are sums of terms of the sandwich type

$$b_{j_0}(y,\xi)\big(\tau_{j_1}-a_{j_1}(y,\xi)\big)^{-1}b_{j_1}(y,\xi)\big(\tau_{j_2}-a_{j_2}(y,\xi)\big)^{-1}\cdots b_{j_{r-1}}(y,\xi)\big(\tau_{j_r}-a_{j_r}(y,\xi)\big)^{-1}b_{j_r}(y,\xi) \tag{6.1.18}$$

[3)] If $X\ne\mathbb{R}^d$ we replace Q_k by $Q_k\psi_k$ with $\psi\in\mathscr{C}_0^K(B(\bar x,\frac12))$ where $B(\bar x,1)\subset X$.

and b_* *are pseudodifferential symbols supported in* $\mathsf{supp}(q_1) \cup \mathsf{supp}(q_2)$ *as* $l+1 \le r \le l+2n$ *for* $n \ge 1$.

In particular,

$$F_0 = q_1^0(\tau_1 - a_1)^{-1} \cdots (\tau_l - a_l)^{-1} q_2^0 \tag{6.1.19}$$

and

$$\mathcal{F}_0 = (2\pi)^{-d} q_1^0 \delta(\tau_1 - a_1) \cdots \delta(\tau_l - a_l) q_2^0 = \partial_{\tau_1} \cdots \partial_{\tau_l} (2\pi)^{-d} q_1^0 \theta(\tau_1 - a_1) \cdots \theta(\tau_l - a_l) q_2^0 \tag{6.1.20}$$

where q_k *are symbols of* Q_k.

Basic Results (General Case)

Let us start from the really rough estimates:

Theorem 6.1.5. *Let* $(A_1, \dots, A_l)$ *be a system of commuting operators of* (6.1.5) *type and let* Q_k *be operators with compactly supported symbols. Then*

(i) The following estimate holds:

$$|(Q_{1x} e\, {}^t Q_{2y})(x, y, \boldsymbol{\tau})| \le C h^{-d}. \tag{6.1.21}$$

(ii) In the framework of Proposition 6.1.3 (i.e. if system $(A_1, \dots, A_l)$ *is elliptic on either* $\mathsf{supp}(q_1)$ *or* $\mathsf{supp}(q_2)$*) the following estimate holds:*

$$|(Q_{1x} e\, {}^t Q_{2y})(x, y, \boldsymbol{\tau}', \boldsymbol{\tau})| \le C h^s \qquad \forall \tau', \tau : |\tau| \le \rho, |\tau'| \le \rho. \tag{6.1.22}$$

Proof. Let us recall that $e(x, y, \boldsymbol{\lambda}, \boldsymbol{\tau})$ is a Schwartz kernel of operator $E(\boldsymbol{\lambda}, \boldsymbol{\tau}) = \prod_j (E(\tau_j) - E(\lambda_j))$ (which is a projector as $\boldsymbol{\tau} \ge \boldsymbol{\lambda}$ where $\boldsymbol{\tau} \ge \boldsymbol{\lambda}$ means exactly that $\tau_j \ge \lambda_j$ for $j = 1, \dots, l$.

Then estimates (6.1.21), (6.1.22) follow from the similar estimates for $\Gamma_x (Q_{1x} e\, {}^t Q_{2y})(\boldsymbol{\tau}, \boldsymbol{\lambda})$ replaced by $F_{\boldsymbol{t} \to h^{-1}\boldsymbol{\tau}} \phi_T(\boldsymbol{t}) \Gamma_x (Q_{1x} u\, {}^t Q_{2y})$ (where the first esxtimate is obvious and the second one is due to Proposition 6.1.3) as we pick $T \asymp 1$ and l*-dimensional Hörmander function* ϕ (which can be taken as a product of l ordinary Hörmander functions: $\phi(\boldsymbol{\tau}) = \prod_j \phi(\tau_j)$).

We get rid of Γ_x and assumption that $Q_1 = Q_2^*$ following the same scheme as in Section 4.2. □

Theorem 6.1.6. *Let* $(A_1, \dots, A_l)$ *be a system of commuting operators of* (6.1.5) *type and let* Q_k *be operators with compactly supported symbols. Let* $\varphi \in \mathscr{C}_0^K(\mathbb{R})$. *Then estimate*

$$\Big|\int \varphi(hL^{-1}(\bar{\tau}-\tau))\big(d_\tau \int (Q_{1x}e^{\,t}Q_{2y})(x,x,\tau)dx - h^{-d}\iint \mathcal{F}(x,\xi,\tau)dxd\xi d\tau\big)\Big| \le Ch^s \tag{6.1.23}$$

holds provided $L = \mathsf{diag}(L_1, \dots, L_l)$ *with* $h^{\frac{1}{2}-\delta} \le L_j \le h^{-n} \quad \forall j$.

Proof. We just apply $\varphi(L^{-1}(hD_{\boldsymbol{t}} - \bar{\tau}))$ to (6.1.14) (inside absolute value) and then take $\boldsymbol{t} = 0$. □

The Microhyperbolicity Condition

Let us replace our standard microhyperbolicity condition by the similar condition for a system $(a_1, \dots, a_l)$:

Definition 6.1.7. We call system $(a_1, \dots, a_l)$ microhyperbolic (or ξ-microhyperbolic) in Ω if for each $\zeta \in \mathbb{S}^l$ symbol $a_\zeta = \sum_{1\le j\le l} \zeta_j a_j(x,\xi)$ is microhyperbolic (or ξ-microhyperbolic respectively) in Ω.

Note that in this definition direction of (ξ-)microhyperbolicity of a_ζ may depend on ζ.

Let us repeat arguments of Section 2.1.5. The routine propagation of singularities arguments (combined with elliptic arguments) yield

Proposition 6.1.8. *Let* $(A_1, \dots, A_l)$ *be a system of commuting operators of* (6.1.5) *type.*

(i) Let symbol a_ζ *be* ξ*-microhyperbolic at the point* $\bar{z}$ *as* $\zeta = \bar{\zeta}$ *and let* $\chi \in \mathscr{C}_0^K$ *be supported in the* ϵ*-vicinity of* $\bar{\zeta}$ *with sufficiently small* $\epsilon > 0$. *Then*

$$\|F_{\boldsymbol{t}\to h^{-1}\tau}\chi_T(\boldsymbol{t})\Gamma_x(Q_{1x}u\,{}^tQ_{2y})\| \le Ch^s \qquad \forall x \quad \forall \boldsymbol{\tau} : |\boldsymbol{\tau}| \le \epsilon \tag{6.1.24}$$

with any scalar $T \in [\epsilon_1, T_0]$ *and with* $Q_j \in \boldsymbol{\Psi}_h$ *with symbols supported in the* ϵ*-neighborhood of* $\bar{z}$ *where* $\epsilon_1 > 0$ *is arbitrarily small, and* T_0 *is a sufficiently small constant which does not depend on* ϵ_1 *or the choice of* χ.

(ii) Let $J \subset \{1, \dots, l\}$ *and let system* $(a_j)_{j\in J}$ *be* ξ*-microhyperbolic at point* $\bar{z}$*. Let* $\chi \in \mathscr{C}_0^K\big(B_r(0,1) \setminus B_r(0, \frac{1}{2})\big)$ *where* $r = \#J$*,* $T \in [\epsilon_1, T_0]$*. Further, let* $\phi \in \mathscr{C}_0^K\big(B_{l-r}(0,1)\big)$*,* $T'' \in [h^{1-\delta}, \epsilon_0 T]$*.*

Then (6.1.24) *holds with* $\chi_T(t)$ *replaced by* $\chi_T(t')\phi_{T''}(t'')$*,* $t' := (t_j)_{j\in J}$*,* $t'' := (t_j)_{j\notin J}$*.*

Then we immediately arrive to

Corollary 6.1.9. *Let* $(A_1, \dots, A_l)$ *be a system of commuting operators of* (6.1.5) *type. Let* $J \subset \{1, \dots, l\}$ *and let system* $(a_j)_{j\in J}$ *be* ξ*-microhyperbolic at point* $\bar{z}$*. Let* Q_k *be operators with symbols supported in vicinity of* $\bar{z}$*. Then*

(i) Estimate (6.1.14) *holds provided*

$$T_j \in [h^{1-\delta}, T_0] \quad \forall j \in J, \qquad T_j \in [h^{1-\delta}, h^{\frac{1}{2}+\delta}] \quad \forall j \notin J. \tag{6.1.25}$$

(ii) Estimate (6.1.23) *holds provided*

$$L_j \in [h^{1-\delta}, h^{-n}] \quad \forall j \in J, \qquad L_j \in [h^{\frac{1}{2}-\delta}, h^{-n}] \quad \forall j \notin J. \tag{6.1.26}$$

Exploiting Microhyperbolicity

Repeating rescaling arguments of Theorem 2.1.19 and remembering that in scaling $t \to T^{-1}t$ brings factor T^l rather than T as it was then (when $l = 1$) we arrive to

Proposition 6.1.10. *In the framework of Proposition 6.1.8(ii)*

$$\|F_{t\to h^{-1}\tau}\chi_T(t')\phi_{T''}(t'')\Gamma_x(Q_{1x}u\,{}^tQ_{2y})\| \le Ch^{l-d}\big(\frac{h}{T}\big)^s \qquad \forall x \quad \forall \tau : |\tau| \le \epsilon \quad \forall T \in [h, T_0]. \tag{6.1.27}$$

Proposition 6.1.11. *Let* $(A_1, \dots, A_l)$ *be a system of commuting operators of* (6.1.5) *type. Let* $(a_1, \dots, a_l)$ *be* ξ*-microhyperbolic at point* $\bar{z}$ *and let* Q_k *be operators with symbols supported in vicinity of* $\bar{z}$*. Then*

(i) Estimate

$$\|F_{t\to h^{-1}\tau}\phi_T(t)\Gamma_x(Q_{1x}u\,{}^tQ_{2y})\| \le Ch^{l-d}\prod_{1\le j\le l}\big(\frac{T_j}{T_0}+1\big) \qquad \forall x \quad \forall \tau : |\tau| \le \epsilon \quad \forall T_j \ge h \tag{6.1.28}$$

holds as $\phi \in \mathscr{C}_0^K(B(0,1))$*.*

(ii) the following estimate holds:

$$\|\Gamma_x(Q_{1x}e^{\,t}Q_{2y})(\boldsymbol{\tau}',\boldsymbol{\tau})\| \le Ch^{l-d}T_0^{-l}\prod_{1\le j\le l}\bigl(1+\frac{T_0}{h}|\tau_j-\lambda_j|\bigr) \tag{6.1.29}$$
$$\forall \boldsymbol{\tau}',\boldsymbol{\tau} : |\boldsymbol{\tau}'|\le \epsilon_0, |\boldsymbol{\tau}|\le \epsilon_0.$$

Proof. (a) Statement (i) for $T_j \in [h, T_0]$ follows from Proposition 6.1.10 after we pick up an appropriate partition of unity with respect to $\boldsymbol{\tau}$.

(b) Let us pick up $T = T_0$ and multidimensional Hörmander function ϕ. Then we derive immediately from Proposition 6.1.10 the following estimate

$$\|\Gamma_x(e^{\,t}Q_y)(\boldsymbol{\tau}',\boldsymbol{\tau})\| \le Ch^{l-d}T_0^{-l} \qquad \forall \boldsymbol{\tau}',\boldsymbol{\tau} : |\boldsymbol{\tau}|\le \epsilon_0,\ |\boldsymbol{\tau}-\boldsymbol{\lambda}|\le C\frac{h}{T_0}$$

which immediately implies (6.1.29). We can get rid of Γ_x and assumption that $Q_1 = Q_2^*$ following the same scheme as in Section 4.2.

(c) After (6.1.29) is proven we can get prove (6.1.28) in the full generality as it was done in Section 4.2. □

Remark 6.1.12. (i) Let us recall that $T_0 \asymp 1$ but in certain settings this could change.

(ii) One can replace ξ-microhyperbolicity condition by microhyperbolicity condition and Γ_x by Γ.

(iii) Finally, if we know only that $(a_j)_{j\in J}$ is ξ-microhyperbolic where $J \subset \{1,\dots,l\}$, $\#J = r$ then we conclude that

$$\|F_{\boldsymbol{t}\to h^{-1}\boldsymbol{\tau}}\phi_T(\boldsymbol{t})\Gamma_x(Q_{1x}u\,{}^tQ_{2y}\| \le Ch^{r-d}\prod_{j\in J}\bigl(\frac{T_j}{T_0}+1\bigr)\cdot\prod_{j\notin J}(T_j+1) \tag{6.1.30}$$
$$\forall x \quad \forall \boldsymbol{\tau} : |\boldsymbol{\tau}|\le \epsilon \quad \forall T_j \ge h$$

and

$$\|\Gamma_x(Q_{1x}e^{\,t}Q_{2y})(\boldsymbol{\tau}',\boldsymbol{\tau})\| \le Ch^{r-d}T_0^{-r}\prod_{j\in J}\bigl(1+\frac{T_0}{h}|\tau_j-\lambda_j|\bigr) \tag{6.1.31}$$
$$\forall \boldsymbol{\tau}',\boldsymbol{\tau} : |\boldsymbol{\tau}'|\le \epsilon_0, |\boldsymbol{\tau}|\le \epsilon_0.$$

Tauberian Arguments

Let

$$\mathsf{N}_{x,y}(\boldsymbol{\lambda},\boldsymbol{\tau}) \coloneqq Q_{1x}e(x,y,\boldsymbol{\lambda},\boldsymbol{\tau})\,{}^{t}Q_{2y} \tag{6.1.32}$$

and

$$\mathsf{N}_{x,y,\varphi_1,\dots,\varphi_l;L_1,\dots,L_l}(\boldsymbol{\tau}) \coloneqq \int \prod_{1\le j\le l} \varphi_j(L_j^{-1}(\tau_j-\lambda_j))d_{\boldsymbol{\lambda}}\mathsf{N}_{x,y}(\boldsymbol{\lambda}) \tag{6.1.33}$$

(where $\mathsf{N}_{x,y}(\boldsymbol{\tau}) = \mathsf{N}_{x,y}(-\boldsymbol{\infty},\boldsymbol{\tau})$ etc).

Now we want to approximate $e(x,y,\boldsymbol{\tau})$ by

$$e^{\mathsf{T}}(x,y,\boldsymbol{\tau}) \coloneqq h^{-l}\int_{\boldsymbol{\tau}+\mathbb{R}^{-,l}} \Big(F_{\boldsymbol{t}\to h^{-1}\boldsymbol{\lambda}}\bar{\chi}(T^{-1}\boldsymbol{t})u(x,y,\boldsymbol{t})\Big)d\boldsymbol{\lambda} \tag{6.1.34}$$

with matrix $T = \mathsf{diag}(T_1,\dots,T_l)$. Namely,

(6.1.35) Let $\mathsf{N}^{\mathsf{T}}_{x,y}(\boldsymbol{\lambda},\boldsymbol{\tau})$ and $\mathsf{N}^{\mathsf{T}}_{x,y,\varphi_1,\dots,\varphi_l;L_1,\dots,L_l}$ be defined by (6.1.32) and (6.1.33) with $e(x,y,\boldsymbol{\tau})$ replaced by $e^{\mathsf{T}}(x,y,\boldsymbol{\tau})$.

Let $\mathsf{R}_* \coloneqq \mathsf{N}_* - \mathsf{N}^{\mathsf{T}}_*$ denote the Tauberian error.

Theorem 6.1.13. *Let $(A_1,\dots,A_l)$ be a system of commuting operators of* (6.1.5) *type.*

(i) Assume that the following modification of condition (6.1.29)

$$\|(Q_{1x}e\,{}^{t}Q_{2y})(\boldsymbol{\tau}',\boldsymbol{\tau})\| \le Ch^{l-d}\prod_{1\le j\le l}\big(\frac{1}{T_j}+\frac{1}{h}|\tau_j-\lambda_j|\big) \tag{6.1.36}$$
$$\forall \boldsymbol{\tau}',\boldsymbol{\tau} : |\boldsymbol{\tau}'|\le\epsilon_0, |\boldsymbol{\tau}|\le\epsilon_0$$

holds where $T_j \ge h$.

Then

$$\mathsf{R}^{T}_{x,y}(\boldsymbol{\lambda},\boldsymbol{\tau}) \prec Ch^{-d}\sum_{1\le k\le l}\frac{h}{T_k}\prod_{1\le j\le k: j\ne k}\big(|\tau_j-\lambda_j|+\frac{h}{T_j}\big) \tag{6.1.37}$$

and if φ_k satisfy (4.2.73)–(4.2.75) *and $L_jT_j \ge h$ for all j*

$$\mathsf{R}_{x,y,\varphi_1,\dots,\varphi_l;L_1,\dots,L_l} \prec Ch^{-d}L_1\cdots L_l\sum_{1\le k\le l}\frac{h}{L_kT_k}\vartheta_k\big(\frac{h}{L_kT_k}\big) \tag{6.1.38}$$

(ii) Further, if (6.1.36) *holds (only) after application of* Γ_x *or* Γ *then the same is true for* (6.1.37) *and* (6.1.38).

Note that (6.1.37) is a special case of (6.1.38) with φ_k which are characteristic functions of $[0,1]$.

Proof. Proof is trivial: we just apply l times Tauberian Lemma 4.2.13 with respect to variable τ_j with j running from 1 to l. □

Principal Results

Now assuming that system $(a_1,\dots,a_l)$ is ξ-microhyperbolic we can apply Theorem 6.1.13 with $T_j \asymp 1$. However due to ξ-microhyperbolicity with the negligible error we can replace in the Tauberian expressions $\mathsf{N}^{\mathsf{T}}_{x,x,*}$ (as $x=y$) $T_j \asymp 1$ by any $T_j \ge h^{1-\delta}$ thus arriving to Weyl expression

$$e^{\mathsf{W}}(x,y,\boldsymbol{\tau}) = (2\pi)^{-d}\int\int_{\boldsymbol{\tau}+\mathbb{R}^{-l}} \mathcal{F}(x,y,\boldsymbol{\lambda}) e^{i\langle x-y,\xi\rangle}\, d\boldsymbol{\lambda} d\xi. \tag{6.1.39}$$

Thus we proved

Theorem 6.1.14. *Let* $(A_1,\dots,A_l)$ *be a system of commuting operators of* (6.1.5) *type and let* Q_1, Q_2 *be operators with symbols compactly supported in* Ω.

(i) Let system $(a_1,\dots,a_l)$ *be* ξ*-microhyperbolic in* Ω. *Then*

$$\begin{aligned}\mathsf{R}^{\mathsf{W}}_x(\boldsymbol{\tau},\boldsymbol{\lambda}) \coloneqq \mathsf{N}_x(\boldsymbol{\tau},\boldsymbol{\lambda}) - {}& \\ (2\pi h)^{-d}\int q_1^0(x,\xi)\mathsf{e}(\boldsymbol{\tau},\boldsymbol{\lambda},x,\xi) q_2^0(x,\xi)\, d\xi \prec {}& \\ Ch^{1-d}\sum_{1\le k\le l}\ \prod_{1\le j\le l: j\ne k} \bigl(|\tau_j-\lambda_j|+h\bigr)&\end{aligned} \tag{6.1.40}$$

where

$$\mathsf{e}(\boldsymbol{\tau},\boldsymbol{\lambda},x,\xi) = \bigl(\theta(\tau_1 - a_1(x,\xi))\cdots(\tau_l - a_l(x,\xi)\bigr) \tag{6.1.41}$$

and skipped subscript y *as usual means that* $y=x$.

Further, assume that φ_k *satisfy* (4.2.73)–(4.2.75) *and* $L_k \ge h$*; then*

$$\mathsf{R}^{\mathsf{W}}_{x,\varphi_1,L_1,\dots,\varphi_l,L_l} \prec Ch^{-d}L_1\cdots L_l \sum_{1\le k\le l} \frac{h}{L_k}\vartheta_k\bigl(\frac{h}{L_k}\bigr) \tag{6.1.42}$$

where the left-hand expression is the difference between $\mathsf{N}_{x,\varphi_1,L_1,\dots,\varphi_l,L_l}$ *and* $\mathsf{N}^{\mathsf{W}}_{x,\varphi_1,L_1,\dots,\varphi_l,L_l}$ *obtained by plugging* $e^{\mathsf{W}}(x,y,\boldsymbol{\tau})$ *instead of* $e(x,y,\boldsymbol{\tau})$ *in the definition of* $\mathsf{N}_{x,\varphi_1,L_1,\dots,\varphi_l,L_l}$*.*

(ii) Let system $(a_1,\dots,a_l)$ *be microhyperbolic in* Ω*. Then all above estimates hold after integration with respect to* x*. In particular*

$$\mathsf{R}^{\mathsf{W}}(\boldsymbol{\tau},\boldsymbol{\lambda}) \coloneqq \mathsf{N}(\boldsymbol{\tau},\boldsymbol{\lambda}) - (2\pi h)^{-d}\int q_1^0(x,\xi)e(\boldsymbol{\tau},\boldsymbol{\lambda},x,\xi)q_2^0(x,\xi)\,dxd\xi \prec Ch^{1-d}\sum_{1\le k\le l}\ \prod_{1\le j\le k: j\ne k}\bigl(|\tau_j-\lambda_j|+h\bigr). \tag{6.1.43}$$

Possible Generalizations, Remarks

One can easily generalize Theorem 6.1.14 to the case when only subsystem is (ξ-)microhyperbolic:

Theorem 6.1.15. *Let* $(A_1,\dots,A_l)$ *be a system of commuting operators of* (6.1.5) *type and let* Q_1, Q_2 *be operators with symbols compactly supported in* Ω*. Let* $\varphi_j \in \mathscr{C}_0^K$*,* $L_j \ge h^{\frac12-\delta}$ *for all* $j\notin J$ *where* $J\subset\{1,\dots,l\}$*.*

(i) Let system $(a_j)_{j\in J}$ *be* ξ*-microhyperbolic in* Ω*. Then*

$$\mathsf{R}^{\mathsf{W}}_{x,(\varphi_j,L_j)_{j\notin J}}(\boldsymbol{\tau}',\boldsymbol{\lambda}') \coloneqq \int \prod_{j\notin J}\varphi_{jL_j}(\tau_j)\,d_{\boldsymbol{\tau}''}\mathsf{R}^{\mathsf{W}}_x(\boldsymbol{\tau},\boldsymbol{\lambda}) \prec Ch^{1-d}\sum_{k\in J}\ \prod_{j\in J: j\ne k}\bigl(|\tau_j-\lambda_j|+h\bigr) + Ch^s \tag{6.1.44}$$

with arbitrarily large exponent s*.*

Further, assume that φ_k *satisfy* (4.2.73)–(4.2.75) *and* $L_k \ge h$ *for all* $k\in J$*; then*

$$\mathsf{R}^{\mathsf{W}}_{x,\varphi_1,L_1,\dots,\varphi_l,L_l} \prec Ch^{-d}\prod_{k\in J}L_j\sum_{k\in J}\frac{h}{L_k}\vartheta_k\bigl(\frac{h}{L_k}\bigr) + Ch^s. \tag{6.1.45}$$

(ii) Let system $(a_j)_{j\in J}$ be microhyperbolic in Ω. Then all above estimates hold after integration with respect to x. In particular,

$$\mathsf{R}^{\mathsf{W}}_{(\varphi_j,L_j)_{j\notin J}}(\boldsymbol{\tau}',\boldsymbol{\lambda}') \coloneqq \int \prod_{j\notin J} \varphi_{jL_j}(\tau_j)\, d_{\tau''}\mathsf{R}^{\mathsf{W}}(\boldsymbol{\tau},\boldsymbol{\lambda}) \prec Ch^{1-d}\sum_{k\in J}\ \prod_{j\in J: j\neq k} \bigl(|\tau_j-\lambda_j|+h\bigr) + Ch^s. \tag{6.1.46}$$

Problem 6.1.16. Get sharp remainder estimates (namely with right-hand expressions (6.1.37) or (6.1.38) with $T_j \geq 1$. To do this one needs to introduce *multitime generalized Hamiltonian flow* exactly as we introduced in Section 2.2 generalized Hamiltonian flow and then to define loop or periodic points for such flow. However since $a_1,\dots,a_l$ commute one can introduce such flow as a product of generalized Hamiltonian flows:

$$\Psi_{\boldsymbol{t},a_1,\dots,a_l} = \Psi_{t_1,a_1}\circ\Psi_{t_2,a_2}\circ\cdots\circ\Psi_{t_l,a_l} \tag{6.1.47}$$

(but one needs to show that these flows commute).

We will do it only in the case of scalar symbols $a_1,\dots,a_l$ in the next subsection. The general construction is left to the reader and we believe that full exposition deserves to be published.

Remark 6.1.17. We do not need large smoothness; in particular we can assume that operators $A_1,\dots,A_l$ satisfy regularity conditions of Section 4.6 (but then we need ξ-microhyperbolicity). However our main target are operators with irregular coefficients rather than rough operators. So if we have commuting irregular operators $A_1,\dots,A_l$ one should be able to approximate them by rough operators $A_{1\varepsilon},\dots,A_{l\varepsilon}$ in such way that these operators also commute. While in some special cases it is possible but there is no general theory and it even does not seem to be possible.

Remark 6.1.18. Let $a_1,\dots,a_m$, $m\leq l$ be scalar symbols. Then $(a_1,\dots,a_l)$ is microhyperbolic if and only if for all $J\subset\{1,\dots,m\}$

(a) $(da_j)_{j\in J}$ are linearly independent at $(a_j)_{j\in J}=0$ and therefore $\Sigma_J \coloneqq \{(a_j)_{j\in J}=0\}$ is a smooth manifold in T^*X and

(b) as $m<l$ $(a_{m+1},\dots,a_l)$ restricted to Σ_m is microhyperbolic.

One can describe ξ-microhyperbolicity as microhyperbolicity for fixed x and therefore the above description works (with fixed x) for ξ-microhyperbolicity.

6.1.3 Operators with Scalar Symbols

Preliminary Remarks

Let us assume now that

(6.1.48) All the symbols $a_1, \dots, a_l$ are scalar.

Then since $(A_1, \dots, A_l)$ is a system of commuting operators

$$\{a_j, a_k\} = 0 \qquad \forall j, k = 1, \dots, l \tag{6.1.49}$$

and assuming that at $\bar{z}$ a subsystem $(a_1, \dots, a_r)$ is microhyperbolic and no its subsystem is elliptic one can introduce symplectic coordinates in the vicinity of $\bar{z}$ such that $a_j = \xi_j$ for $j = 1, \dots, m$ there.

Furthermore one can prove easily

Proposition 6.1.19. *Let $(A_1, \dots, A_l)$ be a system of commuting operators. Assuming that at $\bar{z}$ a subsystem $(a_1, \dots, a_m)$ is microhyperbolic and no its subsystem is elliptic one can find a h-Fourier integral operator $\mathcal{F}$ such that $\mathcal{F}\mathcal{F}^* \equiv I$ in vicinity of $\bar{z}$ and*

$$\mathcal{F}^*\mathcal{F} \equiv I, \qquad \mathcal{F}^* A_j \mathcal{F} \equiv hD_j \quad \forall j = 1, \dots, m \qquad \text{in vicinity of } 0. \tag{6.1.50}$$

Furthermore

$$\mathcal{F}^* A_j \mathcal{F} \equiv A_j'(x', hD, h) \quad \forall j = m+1, \dots, l \qquad \text{in vicinity of } 0 \tag{6.1.51}$$

with $(x'; x'') = (x_1, \dots, x_m; x_{m+1}, \dots, x_l)$ etc.

Asymptotics with Mollifications

Our first task is to replace condition $L_j \ge h^{\frac{1}{2}-\delta}$ by the condition $L_j \ge h^{1-\delta}$ in Theorems 6.1.6 and 6.1.15.

Theorem 6.1.20. *Theorems 6.1.6 (with mollification with respect to x) and 6.1.15(ii) hold under extra condition* (6.1.48) *if we relax conditions to L_j to $L_j \ge h^{1-\delta}$ for all $j = 1, \dots, l$ (in Theorem 6.1.6) and or all $j = m+1, \dots, l$ (in Theorem 6.1.15) respectively.*

Proof. Let us apply induction with respect to (r,m) where r is the number of $j=1,\dots,l$ such that $L_j \le h^{1-\delta}$. Note that as $r=0$ condition $L_j \asymp 1$ is fulfilled for all j and we can apply Theorems 6.1.6 and 6.1.15; meanwhile as $m=l$ system $(a_1,\dots,a_l)$ is microhyperbolic and we can apply Theorem 6.1.15.

Due to Proposition 6.1.19 one can assume without any loss of the generality that $\bar{z}=0$ and in its vicinity $A_j = hD_j$ for $j=1,\dots,m$ and $A_j = A_j(x'', hD', hD'')$ for $j=m+1,\dots,l$.

Let us introduce the scaling function

$$\gamma = \epsilon\Big(\sum_{m+1\le j\le l} (|b_j| + |\nabla b_j|^2) + |\xi'|\Big)^{\frac{1}{2}} + \bar{\gamma} \tag{6.1.52}$$

with $b_j = a_j(x'',0,\xi'')$ and

$$\bar{\gamma} = |L|^{\frac{1}{2}}, \qquad |L| = \max_{1\le j\le l} L_j \tag{6.1.53}$$

and a small constant $\epsilon>0$. Let us recall that $|L| \ge h^{\frac{1}{2}-\delta'}$ unless $m=l$.

Then $|\nabla\gamma| \le \frac{1}{2}$. Let us make a $(\epsilon,\gamma,\gamma,\gamma^2)$-admissible partition with respect to (x',x'',ξ',ξ'') and consider the contribution of one element of this partition. Obviously, we need to rescale $x'' \mapsto x''_{\mathsf{new}} = x''\gamma^{-1}$ and introduce $h_{\mathsf{new}} = h\gamma^{-2}$. This yields the rescaling $\xi' \mapsto \xi'_{\mathsf{new}} = \xi'\gamma^{-2}$, $\xi'' \mapsto \xi''_{\mathsf{new}} = \xi''\gamma^{-1}$. Moreover, let us divide the operators A_j by γ^2 for all $j=1,\dots,l$; so, $L_j \mapsto L_{j,\mathsf{new}} = L_j\gamma^{-2}$ and h/L_j are preserved.

First, let us consider elements of the partition with $|\xi'| \ge \epsilon_0\gamma^2$ and $\gamma \ge C_1\bar{\gamma}$.

Then system is elliptic near 0 and Theorem 6.1.5(ii) yields that the contribution of such elements to the principal part is 0 and the contribution to the remainder is $O(h^s)$ provided C_1 is large enough.

Now, if $|L| \ge \epsilon_0\gamma^2$ then after this rescaling procedure we are in the case $(r-1,m)$ [4] and due to induction assumption we obtain that the contribution of such an element does not exceed the rescaled right-hand expression of estimate (6.1.45) which is

$$C\gamma^{2d-2m}\Big(h^{-d}\prod_{k=1,\dots,m} L_j \sum_{k=1,\dots,m} \frac{h}{L_k}\vartheta_k\big(\frac{h}{L_k}\big) + h^{s'}\Big) \tag{6.1.54}$$

[4] Really, $L_{j\mathsf{new}} \asymp 1$ for at least one number j.

where s' is another but still arbitrarily large exponent.

On the remaining elements with $\gamma \ge C\bar{\gamma}$ microhyperbolicity condition is fulfilled for a subsystem with $(m+1)$ elements after rescaling and permutation of A_j with $j = m+1, \dots, l$. Therefore we can also apply the induction hypothesis. We obtain that the contribution of such element to the remainder also does not exceed the rescaled right-hand expression of estimate (6.1.45) which is (6.1.54).

Now let us restrict γ to $\Lambda := \{x' = \xi' = 0\}$ and consider contribution of $B(z'', \gamma(z'')) \times \mathbb{R}^{2m} \cap \Omega$ where $z'' = (x'', \xi'')$; due to the previous arguments it does not exceed (6.1.54) and therefore the total contribution of such balls does not exceed (6.1.54) multiplied by γ^{-2d+2m} (since $\dim \Lambda = 2d - 2m$) and integrated over Λ; this is exactly (6.1.45). □

Principal Results

Now we can get rid of the mollification.

Theorem 6.1.21. *Let $(A_1, \dots, A_l)$ be a system of commuting operators of* (6.1.5) *type and condition* (6.1.48) *be fulfilled. Moreover, let $(a_1, \dots, a_{l-1})$ be microhyperbolic in the vicinity of $\bar{z}$ (in which case $\Sigma := \{a_1 = \dots = a_{l-1} = 0\}$ is a smooth manifold). Assume that φ_j satisfy* (4.2.73)–(4.2.75) *and $L_j \ge h$.*

(i) Let $b = a_l|_\Sigma$ satisfy condition $(5.2.20)_r$

$(6.1.55)_r$ $|b(x'', \xi'') - \tau_l| + |\nabla_{x'',\xi''} b(x'', \xi'')| \le \epsilon_0 \implies \operatorname{Hess} b(x'', \xi'')$ *has r eigenvalues (counting multiplicities) $f_1, \dots, f_r$ such that $|f_1| \ge \epsilon_0, \dots, |f_r| \ge \epsilon_0$* [5] *with $r \ge 3$ or*

$(6.1.55)_2^+$ $|b(x'', \xi'') - \tau_l| + |\nabla_{x'',\xi''} b(x'', \xi'')| \le \epsilon_0 \implies \operatorname{Hess} b(x'', \xi'')$ *has 2 eigenvalues (counting multiplicities) f_1, f_2 of the same sign such that $|f_1| \ge \epsilon_0, |f_2| \ge \epsilon_0$* [5].

Then estimates (6.1.43) *and*

$$\mathsf{R}^{\mathsf{W}}_{\varphi_1, L_1, \dots, \varphi_l, L_l} \prec Ch^{-d} L_1 \cdots L_l \sum_{1 \le k \le l} \frac{h}{L_k} \vartheta_k\Big(\frac{h}{L_k}\Big) \tag{6.1.56}$$

hold.

[5] The other eigenvalues are arbitrary.

(ii) Let b *satisfy condition* $(6.1.55)_2$. *Then estimates*

$$\mathsf{R}^{\mathsf{W}} \prec Ch^{1-d} \sum_{1\le k\le l} \prod_{1\le j\le k: j\ne k} \bigl(|\tau_j-\lambda_j|+h\bigr)\,|\log h| \tag{6.1.57}$$

and

$$\mathsf{R}^{\mathsf{W}}_{\varphi_1,L_1,\dots,\varphi_l,L_l} \prec Ch^{-d}L_1\cdots L_l \sum_{1\le k\le l} \frac{h}{L_k}\vartheta_k\bigl(\frac{h}{L_k}\bigr)\,|\log h| \tag{6.1.58}$$

hold.

(iii) Let b *satisfy condition* $(6.1.55)_1$. *Then estimates*

$$\mathsf{R}^{\mathsf{W}} \prec Ch^{\frac12-d} \sum_{1\le k\le l} \prod_{1\le j\le k: j\ne k} \bigl(|\tau_j-\lambda_j|+h\bigr) \tag{6.1.59}$$

and

$$\mathsf{R}^{\mathsf{W}}_{\varphi_1,L_1,\dots,\varphi_l,L_l} \prec Ch^{-\frac12-d}L_1\cdots L_l \sum_{1\le k\le l} \frac{h}{L_k}\vartheta_k\bigl(\frac{h}{L_k}\bigr) \tag{6.1.60}$$

hold.

Proof. Let us introduce scaling function $\gamma(z)$ by (6.1.52) with $\bar\gamma = h^{\frac12}$. Let us rescale as in the proof of Theorem 6.1.20. Then the right-hand expressions of (6.1.43) and (6.1.57) scale to the same expressions but with an extra factor γ^{2d-2l} and therefore the contribution of $B(z',\gamma(z'))\times\mathbb{R}^{2m}\cap\Omega$ to the remainder does not exceed $\gamma^{2d-2l}R$ where R denotes the original expression, $z''\in\Sigma$, $\dim\Sigma = 2d-2l+2$ and we again restricted γ to Σ.

Then the total contribution to the remainder does not exceed $\int \gamma^{-2}\,dz''\times R$ and we arrive to all estimates; under condition $(6.1.55)_2^+$ we introduce coordinates $z''=(z_1,z_2;z''')$ on Σ such that $\gamma(z'')\gtrsim |z_1|+|z_2|$ and $|z_1|+|z_2|$ has a constant magnitude as z''' is constant and ellipticity is violated (as we did in the proof of Theorem 5.2.2). □

Problem 6.1.22. (i) It would be very interesting to carry out the analysis of Section 5.2 but it requires large t_l which is impossible due to terms with $|\alpha|=1$ in the decomposition $b\sim\sum b_\alpha(x'',\xi'')\xi'^\alpha$. Really, these terms mean the propagation with respect to x' with the speed $b_\alpha(x,\xi'')$.

On the other hand, if $b_\alpha=\mathsf{const}$ we would be able to replace A_l by $A_l' := A_l - \sum_\alpha b_\alpha A^\alpha$ where $\alpha\in\mathbb{Z}^{+\,l-1}$ and $A^\alpha := A_1^{\alpha_1}\cdot A_{l-1}A^{\alpha_{l-1}}$ and analysis of this new system $(A_1,\dots,A_{l-1},A_l)$ would lead us to the desired results. However b_α are variable and this is a difficulty we could not tackle.

(ii) It would be even more challenging to consider few degenerating symbols $a_1, \dots, a_l$.

We strongly believe that solution of any of these two problems deserves to be published.

Example 6.1.23. We leave to the reader the hosts of examples related to spherically symmetric operators: e. g. $A_1 = h^2\Delta$ on $\mathbb{S}^d$ ($d \geq 3$) or $A_1 = h^2\Delta + V(|x|)$ in $\mathbb{R}^d$ while $A_2 = x_2 hD_1 - x_1 hD_2$ is a component of the angular momentum.

6.2 Scalar Operators with Periodic Hamiltonian Flow

In this section we treat scalar operators with the periodic Hamiltonian flow and weak perturbations of such operators. We consider very long-time propagation of the singularities. Then under some restrictions we prove a microlocal version of a theorem concerning spectral gaps (i.e. lacunas in the spectrum), and under other restrictions we obtain highly accurate semiclassical spectral asymptotics which may contain a non-Weyl term. Thus periodic trajectories which were so far the worst enemy of sharp spectral asymptotics are tamed and become their best friend.

6.2.1 Introduction

Let us assume that

(6.2.1) $A \in \boldsymbol{\Psi}_h(X, \mathbb{K})$ is a self-adjoint operator with principal symbol $a(x,\xi)$.

Assume for a while that a is a scalar symbol; later we abandon this condition but then we will return to it. Let $\bar\tau$ be a fixed energy level (usually for a sake of simplicity $\bar\tau = 0$), and let Ω be a connected domain in $T^*\mathbb{R}^d$. Let us assume that

(6.2.2) a is microhyperbolic in Ω on the energy level $\bar\tau$

which for scalar symbol means that

$$|\nabla_{(x,\xi)} a(x,\xi)| + |a(x,\xi) - \bar\tau| \geq \epsilon_0 \quad \text{in } \Omega, \tag{6.2.3}$$

and that all Hamiltonian trajectories are periodic in Ω i.e.

$$\Psi_{a,T}(x,\xi) = (x,\xi) \quad \forall (x,\xi) \in \Omega \qquad \text{for} \quad T = T(x,\xi) \tag{6.2.4}$$

where $\Psi_{a,t}$ is a Hamiltonian flow generated by a. Then one can replace Ω by

$$\bigcup_{(x,\xi)\in\Omega} \quad \bigcup_{t\in[0,T(x,\xi)]} \Psi_{a,t}(x,\xi)$$

and hence without any loss of the generality we can assume that

$$\Psi_{a,t}(\Omega) = \Omega \qquad \forall t. \tag{6.2.5}$$

It is well known that (6.2.4) then holds with $T(x,\xi) = T(a(x,\xi))$ (i.e., the period depends only on the energy level) with a smooth function $T(\tau)$ as $\tau \in [\tau_1, \tau_2]$ provided this interval is contained in the set of values of $a(x,\xi)$ at Ω [6)].

Remark 6.2.1. (i) There can be *subperiodic points* (x,ξ) with $T(x,\xi) = \frac{1}{m}T(a(x,\xi))$, $m \in \mathbb{Z}^+$ but condition (6.2.2) forbids stationary points of $\Psi_{a,t}$. Further, the set of subperiodic points is always a symplectic submanifold (see Proposition 6.3.1).

(ii) This conclusion would be wrong if we assume only that all the trajectories on the given energy level $\bar\tau$ are periodic. Really, let $\bar\tau = 0$; then replacing

[6)] More precisely, if we can select $T = T(z)$ as $\mathscr{C}^1$ function then $T(z)$ is constant on each connected component of $\{a(z) = \tau\}$ provided $\nabla a(z)$ is disjoint from 0 here.

Indeed, let us consider $\iint dH(z) \wedge dt(z)$ over

$$\Sigma = \{z = \Phi_t(x(s)), 0 \le s \le 1, 0 \le t \le t(x(s))\}$$

where $x(s)$ $(0 \le s \le 1)$ is a $\mathscr{C}^1$-curve $\{a(z) = E\}$. Then with an arbitrary constant c

$$0 = \iint da(z) \wedge dT(z) = \int (a+c)dT|_{s=0}^{s=1} = (E+c)\big(T(x(1)) - T(x(0))\big) \implies$$
$$T(x(1)) = T(x(0)).$$

Obviously, if $da \ne 0$ and $T(z)$ is bounded in the neighbourhood of some point $\bar z$, then one can select T such that $\Phi_{T(z)}(z) = z$; however at some points $T(z)$ could be the multiple of the shortest period.

$a(x,\xi)$ by $a'(x,\xi) = b(x,\xi)a(x,\xi)$ with $b \neq 0$ we do not change trajectories on $\{a(x,\xi) = 0\}$, so they remain periodic but their periods are now

$$T'(x,\xi) = \int_0^{T(x,\xi)} b^{-1}(\Psi_{a,t}(x,\xi))\, dt$$

which is not necessarily constant on Σ_τ.

Further, it is well known that the quantity

$$\frac{1}{T}\kappa_1 = \frac{1}{T}\int_0^T (-\xi dx + a dt) \tag{6.2.6}$$

(where the integral is taken along the Hamiltonian trajectory $(x(t),\xi(t)) = \Psi_{a,t}(x,\xi)$) is a constant. Here κ_1 is a (classical) *action*.

Furthermore, we assume that

$$T(\tau) = T_0, \quad \text{i.e.} \quad \Psi_{a,T_0} = I; \quad \text{in } \Omega \tag{6.2.7}$$

then κ_1 will also is a constant.

Remark 6.2.2. Starting from (6.2.4) one can always reach (6.2.7) with $T_0 = 1$ replacing A by $f(A)$ with

$$f(\tau) = \int^\tau T(\tau')\, d\tau'. \tag{6.2.8}$$

However we will formulate main theorems with general T_0.

So, we assumed that *the classical dynamics is periodic* (with period T_0). What can we say about quantum dynamics? We know that $e^{ih^{-1}tA}$ is an h-Fourier integral operator (uniformly with respect to $t : |t| \le c$). The corresponding symplectomorphism is $\Psi_{a,-t}$ which is an identity as $t = T_0$ due to (6.2.7). Therefore we conclude that it is actually h-pseudodifferential operator (at least in Ω). More precisely

Theorem 6.2.3. *(i) Let conditions* (6.2.1), (6.2.2) *and* (6.2.7) *be fulfilled. Then*

$$e^{ih^{-1}T_0A} \equiv e^{ih^{-1}\kappa} G \qquad \textit{in } \Omega \tag{6.2.9}$$

where $G \in \mathbf{\Psi}_h(X, \mathbb{K})$ *is an operator with the principal symbol* $V(x, \xi, T_0)$ *where* $V(x, \xi, t) \in S(X, \mathbb{K})$ *is a solution to Cauchy problem*

$$\frac{d}{dt} V(z, t) = i a^s \circ \Psi_{a,-t}(z) V(z), \qquad V(z, 0) = I, \tag{6.2.10}$$

a^s *is subprincipal symbol of* A *and* $\kappa = \kappa_1 - \frac{1}{2}\pi\kappa_0 h$ *is the* Maslov constant*,* κ_0 *is the* Maslov index *of the periodic trajectory* $\Psi_{t,a}(x, \xi)$ *(which is also a constant and* κ_1 *is an action.*

(ii) In particular, if a^s *is also a scalar symbol*

$$e^{ih^{-1}T_0 A} \equiv e^{ih^{-1}\kappa + iB} \tag{6.2.11}$$

where B *is an operator with the principal symbol*

$$b := \int_0^{T_0} a^s \circ \Psi_{a,-t}(z)\, dt. \tag{6.2.12}$$

Proof. Proof easily follows from the construction of Subsection 1.2.2. Also one can find the proof at Y. Colin de Verdiere [2]. □

Now for the purpose of introduction assume that $\Omega = T^*X$.

First of all, assume that $G = e^{i\kappa_2}$ and (6.2.9) holds precisely rather than modulo negligible operator

$$e^{ih^{-1}T_0 A} = e^{i\kappa'}, \qquad \kappa' = h^{-1}\kappa_1 + \kappa_2 - \frac{1}{2}\pi\kappa_0, \quad \kappa' \in \mathbb{R}; \tag{6.2.13}$$

then we conclude that $\mathsf{Spec}(h^{-1}T_0 A) \subset 2\pi\mathbb{Z} + \kappa'$ and equivalently

$$\mathsf{Spec}(A) \subset \frac{2\pi h}{T_0}\mathbb{Z} + \frac{h}{T_0}\kappa'; \tag{6.2.14}$$

let us recall that T_0 is a period. This ratio hT_0^{-1} plays important role in all statements.

In this case we get pure point spectrum; if X is not compact manifold it could be of the infinite multiplicity. So, there are *spectral gaps* of the length $2\pi h T_0^{-1}$.

Further, the results of Chapter 4 imply that if X is a compact manifold and then for "typical" $n \in \mathbb{Z}$ such that

$$\tau_n = \frac{2\pi h}{T_0} n + \frac{h}{T_0}\kappa' \tag{6.2.15}$$

belongs to $[-\epsilon_0, \epsilon_0]$ multiplicity of τ_n as an eigenvalue of A is $\asymp h^{1-d}$ (provided a is not elliptic).

This in turn implies that we cannot reasonably expect asymptotics with the remainder estimate better than $O(h^{1-d})$, however we can reformulate the problem in this case in the opposite way: *Calculate precisely multiplicity* N_n *of* τ_n. This problem we will be able to solve.

If T_0 is not a minimal period then replacing T_0 by mT_0 we conclude that for $n \in \mathbb{Z} \setminus m\mathbb{Z}$ formula (6.2.15) gives us fake possible eigenvalues. So it is crucial to assume during this section that

(6.2.16) T_0 is a minimal period of the Hamiltonian flow $\Psi_{a,t}$;

then we will prove that every τ_n given by (6.2.15) is actually an eigenvalue and its multiplicity is actually $\asymp h^{1-d}$.

It would not preclude from subperiodic points but their contribution to τ_n would be of magnitude h^{1-d+2r} where $2r$ is a codimension of the manifold of subperiodic points. Their effect would be basically to shift some eigenvalues from τ_n with $n \notin m\mathbb{Z}$ to τ_n with $n \in m\mathbb{Z}$.

More generally, instead of (6.2.13) let us assume that (6.2.11) holds with the bounded Hermitean operator B; then instead of (6.2.14) we conclude that

$$\mathsf{Spec}(A) \subset \frac{2\pi h}{T_0}\mathbb{Z} + \frac{h}{T_0}\kappa' + \frac{h}{T_0}\,\mathsf{Spec}(B). \tag{6.2.17}$$

in particular if $\mathsf{Spec}\, B \subset [-\eta\pi, \eta\pi]$ with $\eta < 1$ we conclude that $\mathsf{Spec}(A)$ is consisting in *spectral clusters* contained in η-vicinities of τ_n defined by (6.2.15).

So, a single eigenvalue τ_n of large multiplicity N_n would be split into eigenvalues of the total multiplicity of N_n and our calculation of N_n will be still useful. However the new and more challenging question arrises: *How eigenvalues are distributed inside the cluster?* which boils down to asymptotics of $\mathsf{N}(\tau, h)$ again; under certain assumptions we will be able to give an answer with the remainder estimate $O(h^{2-d})$. In this case η may be a constant or a small parameter ($h' \le \eta < 1$), in the latter case the spectral clusters will be narrow.

The task to find sharp asymptotics of $\mathsf{N}(\tau, h)$ is not restricted to the case of $\eta < 1$ even if otherwise the spectral gaps disappear and clusters overlap.

We will add some twist replacing A by $A+\varepsilon A'$ with $\varepsilon \ll 1$ we will arrive to (6.2.17) with B replaced by $\varepsilon' h^{-1}B$ where typically $\varepsilon' = \varepsilon$.

Note that $\varepsilon' h^{-1}$ could be a large parameter. In this case under certain assumptions we will be able to give an answer with the remainder estimate $O(\varepsilon h^{1-d})$.

Let us describe the method. Let us replace (6.2.11) by

$$e^{ih^{-1}T_0 A} = e^{i\varepsilon h^{-1}B} \tag{6.2.18}$$

where

(6.2.19) A and B commute.

We reached $\kappa = 0$ by replacing A by $A - hT_0^{-1}\kappa$ and included εh^{-1} in B explicitly. We will justify these assumptions later (modulo negligible). Then

$$e^{ih^{-1}nT_0 A} = e^{inh^{-1}\varepsilon B} \qquad n\in\mathbb{Z},\ |n|\le h^{-l'}$$

and therefore

$$e^{ih^{-1}tA} = e^{ih^{-1}t'A}e^{ih^{-1}nT_0 A} = e^{ih^{-1}t'A}e^{ih^{-1}\varepsilon nB} = e^{ih^{-1}(t'A+t''B)} \qquad \text{as}\ \ t = nT_0 + t',\ t'' = n\varepsilon. \tag{6.2.20}$$

Here the right-hand expression is h-Fourier integral operator uniformly with respect to $|t'|\le c$, $|t''|\le c$; picking up $n = \lfloor tT_0^{-1}\rfloor$ we rewrite this assumption as

$$|t| \le T := c\varepsilon^{-1}. \tag{6.2.21}$$

Therefore instead of one-parameter long time evolution we get two-parameter normal time evolution described by Hamiltonian flow $\Psi_{a,b,t',t''} := \Psi_{a,t'}\Psi_{b,t''}$ provided B has a scalar principal symbol b; in the general case $\Psi_{b,t''}$ would be generalized Hamiltonian flow. Instead of one time t we have a *normal time* t' and a *slow time* t''. Our classical dynamics contains periodic movement and a *drift* controlled by b.

So, if the drift breaks periodicity instantly we get Weyl asymptotics with the remainder estimate $CT^{-1}h^{1-d} \asymp C\varepsilon h^{1-d}$ where T is defined by (6.2.21) with c replaced by ϵ_0. Periodicity is broken instantly in the classical dynamics provided (a,b) satisfies microhyperbolicity condition (see Definition 6.1.7). We will deal with a more degenerate scenario of Subsection 6.1.3 later.

However, situation is more complicated for a quantum dynamics: the drift for time t'' shifts trajectory by $\asymp |t''|$ and therefore the shift is observable only as $|t''| \ge Ch|\log h|$ (due to logarithmic uncertainty principle) or as $|n|\varepsilon \ge Ch|\log h|$.

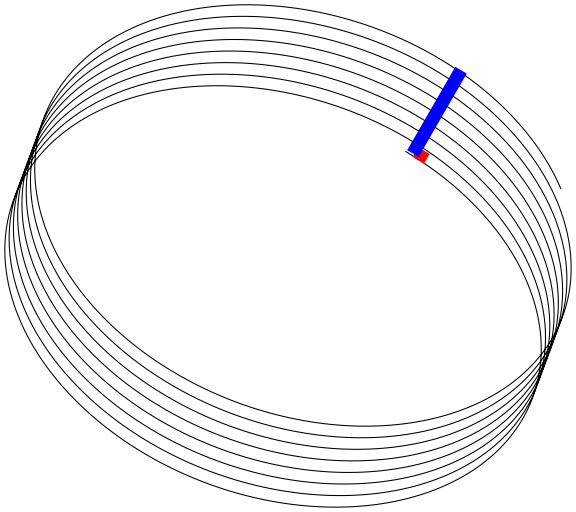


Figure 6.1: Periodicity classical and quantum: the shift between two consecutive windings may be quantum observable only after many windings.

So, the shift is observable instantly (i.e. after the first period) only as $\varepsilon \ge Ch|\log h|$ (or $\varepsilon \ge C|\log h|$) and our conclusion stays only then. Otherwise we need to wait time $T' := Ch\varepsilon^{-1}|\log h|$ and then

$$\|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(u\,{}^tQ)| \le CT'h^{1-d} = C\varepsilon^{-1}h^{2-d}|\log h|. \tag{6.2.22}$$

In fact, a more accurate analysis allows us to prove a better estimate

$$\|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(u\,{}^tQ)| \le C(\varepsilon^{-1}+h^{-1})h^{2-d} \tag{6.2.23}$$

valid for $\varepsilon \le 1$; so as $\varepsilon \le h$ effectively $T' \asymp \varepsilon^{-1}h$ which corresponds to the standard uncertainty principle.

Then (6.2.23) leads to the Tauberian estimate

$$\mathsf{R}^{\mathsf{T}} \prec CT^{-1}(\varepsilon^{-1}+h^{-1})h^{2-d} \asymp C(\varepsilon+h)h^{1-d}. \tag{6.2.24}$$

Remark 6.2.4. (i) We can apply logarithmic uncertainty principle only under analyticity assumption; in the general case one needs to replace $|\log h|$ by $h^{-\delta}$ but we are getting rid of both factors anyway.

(ii) However instant transition from Tauberian asymptotics to the Weyl one is possible only as $\varepsilon \ge Ch|\log h|$ (or $\varepsilon \ge h^{1-\delta}$); in the general case one needs to calculate the correction term which is due to contribution of the windings with $1 \le |n| \le \varepsilon^{-1}T'$.

6.2.2 Operator B

Let us analyze the *crucial assumption* (6.2.18)–(6.2.19). In fact, it is the only assumption we made so far which we really need. We do not need even assumptions that a is scalar and Hamiltonian flow periodic as long as (6.2.18)–(6.2.19) hold. However we need to discover how realistic this assumption is.

So, let us assume that a is a scalar symbol and it generates the Hamiltonian flow which is 1-periodic in Ω.

Scalar Case

Let a^s be a scalar symbol as well. Then (6.2.11) holds (at least modulo $O(h)$) with operator B with the scalar principal symbol b defined in Ω by (6.2.12); this symbol obviously is $\Psi_{a,t}$-invariant and therefore $\{a,b\} = 0$ there. We can always extend it to the smooth real-valued symbol.

Consider $B = \mathsf{Op}^{\mathrm{w}}(b)$; this operator commutes with A modulo $O(h)$ and (6.2.11) holds modulo $O(h)$. However we can fix it due to the following procedure described in more general settings.

Assume first that (6.2.18) holds. Let $R \in \mathbf{\Psi}_h(X,\mathbb{K})$. Consider averaging of R over period equal 1:

$$\bar{R} := \int_0^1 e^{-ih^{-1}t'A} R e^{ih^{-1}t'A}\, dt'. \tag{6.2.25}$$

Consider $\bar{R}_t := e^{-ih^{-1}tA} R e^{ih^{-1}tA}$; it is also given by (6.2.25) but with an integral taken from t to $1+t$ and thus

$$\begin{aligned}\bar{R}_t - \bar{R} = {}& \\ & \int_0^t \Big(e^{-ih^{-1}(t'+1)A} R e^{ih^{-1}(t'+1)A} - e^{-ih^{-1}tA} R e^{ih^{-1}tA}\Big)\, dt' = \\ & \qquad \int_0^t e^{-ih^{-1}t'A}\Big(e^{-ih^{-1}A} R e^{ih^{-1}A} - R\Big) e^{ih^{-1}t'A}\, dt'\end{aligned} \tag{6.2.26}$$

and the latter expression vanishes as $R = B$. So, $\bar{B}$ commutes with A modulo an operator negligible in Ω.

Let us compare B and $\bar{B}$. Due to Theorem 6.2.3 their principal symbols coincide and $B - \bar{B} \in h\boldsymbol{\Psi}_h(X, \mathbb{K})$. Then in Ω

$$e^{ih^{-1}A} \equiv e^{iB} \equiv e^{i\bar{B}+i(B-\bar{B})} \equiv e^{i\bar{B}} + ie^{i\bar{B}}(B-\bar{B}) + R, \qquad R \in h^2\boldsymbol{\Psi}_h(X, \mathbb{K}).$$

Since $e^{ih^{-1}A}$ and $\bar{B}$ commute with A, after averaging according to (6.2.26) we arrive to

$$e^{ih^{-1}A} \equiv e^{i\bar{B}} + \bar{R}, \qquad \bar{R} \in h^k\boldsymbol{\Psi}_h(X, \mathbb{K}) \tag{6.2.27}$$

with $k = 2$. Let us assume that (6.2.27) holds with some $\bar{B}$ and $\bar{R}$ which commute with A (modulo negligible in Ω). Then replacing $\bar{B}$ by $\bar{B}' := \bar{B} + e^{-i\bar{B}}\bar{R}$ we get the same (6.2.27) but with k replaced by $k+1$. Here again all operators commute with A.

Iterating and adjusting $\bar{B}$ we can achieve (6.2.18)–(6.2.19). Therefore we arrive to

Proposition 6.2.5. *Let conditions* (6.2.1)–(6.2.4) *and* (6.2.7) *be fulfilled and let a^s be scalar symbol. Then one can find operator $B \in \boldsymbol{\Psi}_h(X, \mathbb{K})$ with the scalar principal symbol b defined by* (6.2.12) *and satisfying* (6.2.18)–(6.2.19)*.*

Then we conclude that

Corollary 6.2.6. *In the framework of Proposition 6.2.5 operator $A_0 := A - hB$ commutes (modulo negligible) with A and B and*

$$e^{ih^{-1}T_0A_0} \equiv I \tag{6.2.28}$$

modulo an operator which is negligible in Ω.

So A could be considered as perturbation of A_0 by εB with $\varepsilon = h$. Let us forget about (6.2.18)–(6.2.19), consider scalar *unperturbed operator* $A_0 \in \boldsymbol{\Psi}_h(X)$ satisfying (6.2.28) and perturb it by operator εP where $P \in \boldsymbol{\Psi}_h(X, \mathbb{K})$ and it does not necessarily commute with A_0 anymore.

Proposition 6.2.7. *Let conditions* (6.2.1)–(6.2.4) *and* (6.2.7) *be fulfilled and let* $A_0 \in \mathbf{\Psi}_h(X)$ *be an operator with the principal symbol* a *satisfying* (6.2.28). *Let* $P \in \mathbf{\Psi}_h(X)$ *and*

$$A(\varepsilon) = A_0 + \varepsilon P, \qquad \varepsilon \in [h^l, h^\delta]. \tag{6.2.29}$$

Then there exists an operator

$$B(\varepsilon) \sim \sum_{k\ge 0} B_k \varepsilon^k \tag{6.2.30}$$

commuting with $A(\varepsilon)$ *and satisfying*

$$e^{ih^{-1}T_0 A(\varepsilon)} \equiv e^{ih^{-1}\varepsilon B(\varepsilon)} \tag{6.2.31}$$

in Ω. *Furthermore the principal symbol of* B_0 *is defined by*

$$b_0' = (x,\xi) = \int_0^{T_0} b\circ \Psi_{a,-t}(z)\,dt. \tag{6.2.32}$$

Proof. Let $U_0(t) = e^{ith^{-1}A_0}$ and $U(t) = e^{ith^{-1}A}$; so in fact objects we study depend on ε. Let us consider operator $V(t) = U(t)U_0(-t)$; then

$$D_t V(t) = h^{-1}U(t)PU_0(-t) = \varepsilon h^{-1} V(t)L(t), \qquad V(0) = I \tag{6.2.33}$$

with

$$L(t) \coloneqq U_0(t)PU_0(-t). \tag{6.2.34}$$

Our purpose is to construct $V(t)$.

Let us construct a self-adjoint operator $R(t) \in \mathbf{\Psi}_h(X)$ such that

$$V(t) \equiv \exp\bigl(i\varepsilon h^{-1}\int_0^t R(t')\,dt'\bigr) \qquad \text{in } \Omega. \tag{6.2.35}$$

To construct $R(t)$ let us consider

$$W(\sigma,t) \coloneqq \exp\bigl(i\varepsilon h^{-1}\sigma\int_0^t R(t')\,dt'\bigr). \tag{6.2.36}$$

Then

$$\partial_\sigma W - i\varepsilon h^{-1}\int_0^t R(t')dt'\cdot W = 0$$

and differentiating with respect to t we obtain that

$$\Big(\partial_\sigma - i\varepsilon h^{-1}\int_0^t R(t')\,dt'\Big)(\partial_t W) = i\varepsilon h^{-1}R(t)W.$$

Then

$$\partial_t W(\sigma,t) = i\varepsilon h^{-1}\int_0^\sigma W(\sigma-\sigma',t)R(t)W(\sigma',t)\,d\sigma' = i\varepsilon h^{-1}R'(\sigma,t)W(\sigma,t)$$

due to definition (6.2.36) and equality $\partial_t W = 0$ as $\sigma = 0$; here we define

$$R'(\sigma,t) = \int_0^\sigma W(\sigma',t)R(t)W(-\sigma',t)\,d\sigma'.$$

Setting $\sigma = 1$ we obtain that

$$\partial_t W_1(t) = i\varepsilon h^{-1}R'(1,t)W_1(t) \tag{6.2.37}$$

for $W_1(t) = W(1,t)$ and

$$R'(1,t) = \int_0^1 W(\sigma,t)R(t)W(-\sigma,t)d\sigma. \tag{6.2.38}$$

Comparing (6.2.35) and (6.2.37)–(6.2.38) and recalling that $V(0) = W_1(0) = I$ we conclude that (6.2.35) is fulfilled provided $R'(1,t) \equiv L(t)\ \forall t \in [-T_0, T_0]$ modulo negligible operators in a fixed neighborhood of $\bar\Omega$. Thus we need to construct $R(t)$ such that

$$R'(t) \coloneqq \int_0^1 W(\sigma,t)R(t)W(-\sigma,t)\,d\sigma \equiv L(t) \qquad \text{in } \Omega \quad \forall t \in [-T_0, T_0] \tag{6.2.39}$$

where we denote this neighborhood by Ω again.

Theorem 2.4.10 yields that R' and L belong to $\boldsymbol{\Psi}_h(X)$ provided R and P belong to this class; moreover $R'(t) \equiv R(t) \mod \varepsilon\boldsymbol{\Psi}_h(X)$.

Further, (6.2.39) is equivalent to

$$\sum_{m\ge 0}\frac{1}{(m+1)!}\big(i\varepsilon h^{-1}\operatorname{Ad}_{\int_0^t R(t)\,dt}\big)^m R(t) \equiv L(t) \qquad \text{in } \Omega \quad \forall t \in [-T_0, T_0]. \tag{6.2.40}$$

Really, the left-hand expressions of (6.2.39) and (6.2.40) coincide as $\varepsilon \le h^{\delta}$. This allows to fulfill (6.2.40) and in the end of the day (6.2.35) recurrently by

$$R(t,\varepsilon) = \sum_{n\ge 1} R_n(t)\varepsilon^{n-1}. \tag{6.2.41}$$

Namely, as $R_1(t), \dots, R_{n-1}(t)$ are defined we can calculate the corresponding partial sum $R_{(n-1)}(t,\varepsilon)$ of (6.2.41), plug it instead of $R(t)$ in any term with $m \ge 1$ in (6.2.40) and drop all the terms with power of ε greater than $(n-1)$; finally we plug $R(t) = R_{(n-1)}(t,\varepsilon) + R_n(t)\varepsilon^{n-1}$ to the term with $m = 0$. Then terms with the power of ε less than $(n-1)$ will be reduced and we get

$$R_n(t) = \mathfrak{R}\big[R_1(t), \dots, \mathcal{R}_{n-1}(t)\big] + \delta_{n1} L(t)$$

where $\mathfrak{R}$ is non-linear operator, differential with respect to (x,ξ) but containing integrations with respect to t. One can see easily that all operators $R_k(t)$ are self-adjoint.

In particular

$$R_1(t) = L(t) = U_0(t) P U_0(-t), \tag{6.2.42}$$

$$R_2(t) = -\frac{1}{2} i h^{-1} \int_0^t [U_0(t') P U_0(-t'), U_0(t) P U_0(-t)]\, dt'. \tag{6.2.43}$$

Finally, setting $t = T_0$ in (6.2.35) we arrive to $U(t)U_0(-1) = V(T_0) \equiv \exp(i\varepsilon h^{-1} B)$ and therefore since $U_0(-T_0) \equiv I$ in Ω we arrive to (6.2.31) with

$$B(\varepsilon) = \sum_{n\ge 0} B_n \varepsilon^{n-1}, \tag{6.2.44}$$

$$B_n = \int_0^{T_0} R_n(t)\, dt. \tag{6.2.45}$$

In particular, (6.2.42), (6.2.43) yield

$$B_1 = \int_0^{T_0} U_0(t) P U_0(-t)\, dt, \tag{6.2.46}$$

$$B_2 = -\frac{1}{2} i h^{-1} \int_{0\le t'\le t\le T_0} \big[U_0(t') P U_0(-t'), U_0(t) P U_0(-t)\big]\, dt' dt \tag{6.2.47}$$

with the principal symbols

$$b_1' = \int_0^{T_0} b \circ \Psi_{a,t}\, dt \tag{6.2.48}$$

$$b_2' = -\frac{1}{2} \int_{0\le t'\le t\le T_0} \bigl\{ b \circ \Psi_{a,t'}, b \circ \Psi_{a,t} \bigr\}\, dt' dt. \tag{6.2.49}$$

Proposition 6.2.8. *In the framework of Proposition 6.2.7*

$$[A, B] \equiv 0 \quad \textit{in } \Omega. \tag{6.2.50}$$

Proof. Let us assume that $C = ih^{-1}[A, B] \in h^p \mathbf{\Psi}_h(X)$ in Ω, $p \ge 0$. Let us apply formula

$$e^{-i\varepsilon h^{-1}tB} A e^{i\varepsilon h^{-1}tB} = A + i\varepsilon h^{-1} \int_0^t e^{-i\varepsilon h^{-1}t'B} [A, B] e^{i\varepsilon h^{-1}t'B}\, dt' \tag{6.2.51}$$

to C_1 instead of A. We then obtain that (in Ω)

$$e^{-i\varepsilon h^{-1}tB} C_1 e^{i\varepsilon h^{-1}tB} \equiv C_1 + \varepsilon t C_2$$

with $C_2 = ih^{-1}[C_1, B] \in h^p \mathbf{\Psi}_h(X)$.

Applying (6.2.51) to A we then obtain that

$$e^{-i\varepsilon h^{-1}tB} A e^{i\varepsilon h^{-1}tB} = \equiv A + \varepsilon t C_1 \mod \varepsilon^2 \mathbf{\Psi}_h(X).$$

Taking $t = T_0$ and replacing $e^{\pm i\varepsilon h^{-1}B}$ by $e^{\pm ih^{-1}T_0 A}$ we obtain that $C_1 \equiv 0 \mod \varepsilon \mathbf{\Psi}_h(X)$ and hence p can be replaced by $p+\delta$ because $\varepsilon \le h^\delta$; meanwhile division by ε is justified by $\varepsilon \ge h^l$. Induction with respect to p concludes this analysis. □

Remark 6.2.9. (i) Surely if the right-hand expression in (6.2.48) vanishes then $B(\varepsilon) = \max(\varepsilon, h) B'$ with $B' \mathbf{\Psi}_h(X)$ and therefore effectively ε is redefined as $\varepsilon_{\mathsf{new}} = \varepsilon \max(\varepsilon, h)$ etc.

(ii) Most likely Propositions 6.2.7–6.2.8 hold for $\varepsilon \in [h^\delta, \varepsilon_0]$ with small constant ε_0 and the proof would be very interesting.

(iii) However we will investigate the case $\varepsilon \in [h^\delta, \varepsilon_0]$ by using results of Section 4.5.

Matrix Case

The case of matrix operator P is more complicated. However in this case we need to assume that $\varepsilon \le ch$ because otherwise the nature of operator $e^{ih^{-1}(A_0+\varepsilon P)}$ is not clear unless we assume that b has eigenvalues of constant multiplicities.

Unfortunately we were unable to prove existence of $B(\varepsilon)$ with desired properties unless a bit stronger assumption is made:

Proposition 6.2.10. *Let conditions* (6.2.1)–(6.2.4) *and* (6.2.7) *be fulfilled and let* $A_0 \in \boldsymbol{\Psi}_h(X)$ *be an operator with the principal symbol* a *satisfying* (6.2.28). *Let* $P \in \boldsymbol{\Psi}_h(X, \mathbb{K})$ *and*

$$A(\varepsilon) = A_0 + \varepsilon P, \qquad \varepsilon \in [h', \epsilon_0 h] \tag{6.2.52}$$

Then there exists an operator

$$B(\varepsilon) \sim \sum_{k\ge 0} B_k (h^{-1}\varepsilon)^k \tag{6.2.53}$$

commuting with $A(\varepsilon)$ *and satisfying* (6.2.31) *in* Ω.

Proof. Consider operator $e^{ih^{-1}t(A_0+\eta hP)}$. According to Subsection 1.2.2 its parametrix is a h-Fourier integral operator with a phase function independent on η and a symbol analytic with respect to η. Therefore $F(\eta) := e^{ih^{-1}t(A_0+\eta hP)}$ is (in Ω) unitary h-pseudodifferential operator with the symbol analytic with respect to η. Also it is I as $\eta = 0$. Then as $\|F(\eta) - I\| \le 1 - \epsilon$ (which is fulfilled as $\eta \le \epsilon_0$) one can define

$$\log F(\eta) = \sum_{n\ge 1} \frac{1}{n}(-1)^{n-1}\big(F(\eta) - I\big)^n$$

analytic with respect to η; further this function is skew-symmetric, vanishes as $\eta = 0$ (again, modulo negligible in Ω) and thus $B := -i\eta^{-1}F(\eta)$ is a required operator. □

Problem 6.2.11. Extend the statement above to $\epsilon_0 h \le \epsilon \le ch$.

Example 6.2.12. (i) Operator

$$A = \sum_{1\le i<j\le} \sigma_{ij}(x_i hD_j - x_j hD_i) + m\sigma_0 \tag{6.2.54}$$

on $\mathbb{R}^d$ where $\{\sigma_0, \sigma_{ij}, 1 \le i < j \le d\}$ is a set of Pauli matrices; one can skip $m\sigma_0$. This operator satisfies (6.2.7). It could be considered also on the positively homogeneous of degree 0 functions. One can "fix" it to operator satisfying (6.2.28).

(ii) Let a be as above and $A_k = A(S_k x, S_k^{*-1} hD)$ where S_k are nondegenerate matrices. Then $a_1 \oplus a_2 \oplus \cdots \oplus a_m$ may have characteristics of variable multiplicity and satisfies (6.2.7) if we ignore branching of the generalized Hamiltonian flow; it satisfies (6.2.28). So case of truly matrix a in the current framework is reasonable.

6.2.3 Long Term Evolution

At this step we need assume neither that a, b are scalar symbols nor that $\varepsilon \le h^\delta$ ($\varepsilon \le \epsilon$ is enough). However some of the statements below require such assumptions; then these assumptions are mentioned explicitly.

Main Theorem

Now we consider the long-range evolution:

Proposition 6.2.13. *Let conditions* (6.2.1), (6.2.4), (6.2.7), (6.2.18) *and* (6.2.19) *be fulfilled in* Ω. *Further let* $\Omega_0 \subset \Omega$ *also satisfy* (6.2.4) *and*

$$\Psi_{b,t}\Omega_0 \subset \Omega \qquad \forall t : |t| \le T'' \tag{6.2.55}$$

where T'' *is a constant. Let*

$$\varepsilon \in [h', \epsilon_0]. \tag{6.2.56}$$

Then for all t, $|t| \le T^* = \varepsilon^{-1} T'' T_0$ *the uniform equality*

$$e^{ih^{-1}tA} \equiv e^{it''B} e^{ih^{-1}t'A} \qquad \text{in } \Omega_0 \tag{6.2.57}$$

holds with $t'' = \varepsilon n$, $t' = t - n$ *provided* $n \in \mathbb{Z}$, $|n - tT_0^{-1}| \le c$.

Proof. Let us first note that $e^{i\varepsilon h^{-1} nB} = e^{ih^{-1}t''B}$ with $t'' = \varepsilon n$ is a standard propagator for operator B as $|t''| \le \mathsf{const}$. Then the standard theory of propagation of singularities (see Theorem 2.2.10) yields that due to condition (6.2.55) $(I - Q')e^{i\varepsilon nB} Q \equiv 0$ as $|t''| \le T''$ (i.e. $|n| \le \varepsilon^{-1} T''$, and $Q = \mathsf{Op}(q)$, $Q' = \mathsf{Op}(q')$, $\mathsf{supp}(q) \subset \Omega_0$, $q' = 1$ in Ω.

Then as $0 \le \pm n \le \varepsilon^{-1}T''$

$$e^{i\varepsilon h^{-1}nB}Q \equiv e^{\pm ih^{-1}T_0A}e^{i\varepsilon h^{-1}(n\mp 1)B}Q \equiv e^{ih^{-1}mT_0A}e^{i\varepsilon h^{-1}(n\mp m)B}Q \equiv e^{ih^{-1}nT_0A}Q \tag{6.2.58}$$

modulo negligible operator; we used the fact that the number of transitions is $O(h^{-l})$ and therefore accumulated error is $O(h^{-l+K})$ as $O(h^K)$ error is made each time. Then

$$e^{ih^{-1}tA}Q \equiv e^{ih^{-1}(nT_0+t')A}Q \equiv e^{ih^{-1}nT_0A}e^{ih^{-1}t'A}Q \equiv e^{ih^{-1}\varepsilon nB}e^{ih^{-1}t'A}Q \tag{6.2.59}$$

because we can use identity (6.2.58) with Q replaced by $e^{ih^{-1}t'A}Qe^{-ih^{-1}t'A}$ which is h-pseudodifferential operator with the symbol supported in Ω_0. □

This proposition combined with different versions of propagation results (Theorems 2.2.10, 2.3.6, 2.4.17 etc) yields the following "metastatement":

Corollary 6.2.14. *In the framework of Proposition 6.2.13 singularities of $e^{ih^{-1}tA}$ with $|t| \le \varepsilon^{-1}T''T_0$ propagate along the (periodic) flow $\Psi_{a,t'}$ drifting along the flow $\Psi_{b,t''}$ where $t'' = t - nT_0$ and $t'' = \varepsilon n$ are quick and slow times respectively.*

Corollaries

Proposition 6.2.13 allows us to apply results of successive approximations of Subsubsection *6.1.2.2 Successive Approximations* on the series of isolated time intervals

$$\mathcal{J}_n := (n-\nu, n+\nu), \qquad |n| \le \varepsilon^{-1}T \tag{6.2.60}$$

where *currently* $\nu = h^{\frac{1}{2}+\delta}$ and $T = \varepsilon h^{\frac{1}{2}+\delta}$ but later we will increase them. Indeed, according to successive approximations

$$U(x,y,t',t'')\,{}^tQ_{2y} \equiv \sum_{m\ge 0} h^{-d+m}\iiint \mathcal{F}_m(y,\xi,\tau',\tau'')e^{ih^{-1}(\langle x-y,\xi\rangle+t'\tau'+t''\tau'')}\,d\xi\,d\tau'd\tau''$$

where $U(x,y,t',t'')$ is the Schwartz kernel of $e^{ih^{-1}(t'A+t''B)}$. Plugging $t''=\varepsilon n$, $t'=t-nT_0$ we arrive to

$$u(x,y,t)\,{}^tQ_{2y}\equiv \sum_{m\ge 0} h^{-d+m}\iiint \mathcal{F}_m(y,\xi,\tau',\tau'')e^{ih^{-1}(\langle x-y,\xi\rangle+t\tau'+n(\varepsilon\tau''-T_0\tau'))}\,d\xi\,d\tau' d\tau''$$

as $|t'|\le h^{\frac12+\delta}$, $|t''|\le h^{\frac12+\delta}$. Then, as $\phi\in \mathscr{C}_0^\infty([-1,1])$, multiplying by $\phi_\nu((t-nT_0))$, making h-Fourier transform and applying Γ we arrive to

$$\Gamma_{t\to h^{-1}\tau}\phi_\nu(t-nT_0)\Gamma\big(Q_{1x}u(x,y,t)\,{}^tQ_{2y}\big)\equiv \tag{6.2.61}$$
$$\sum_{m\ge 0} h^{1-d+m}\iiint \nu\hat\phi\big(\frac{(\tau-\tau')\nu}{h}\big)\mathcal{F}_m(y,\xi,\tau',\tau'')e^{ih^{-1}n(\varepsilon\tau''-T_0\tau)}\,dyd\xi\,d\tau' d\tau''.$$

In particular, as $\mathcal{F}_0=(2\pi)^{-d}q_1^0\delta(\tau'-a)\delta(\tau''-b)q_2^0$ the main part of the right-hand expression is a convolution of

$$(2\pi)^{-d}h^{1-d}\iint q_1 e^{ih^{-1}n(\varepsilon b-T_0 a)}q_2\,dyd\xi \tag{6.2.62}$$

with $\nu h^{-1}\hat\phi(\nu h^{-1}\tau)$.

Proposition 6.2.15. *Let conditions* (6.2.1), (6.2.4), (6.2.7), (6.2.18) *and* (6.2.19) *be fulfilled in* Ω. *Further, let* a *be microhyperbolic in* Ω *on energy level* $\bar\tau$ *and let* (6.2.55) *be fulfilled. Let* $Q_1,Q_2\in \boldsymbol{\Psi}_{h,\gamma,\gamma,K}(x,\mathbb{K})$ *be supported in* Ω_0, $\gamma\ge h^{\frac12-\delta}$.

Then (6.2.61) *holds with a small constant* ν.

Proof. We constructed successive approximations as $|t'|\le \epsilon^{-1}T'$ and $|t''|\le \epsilon^{-1}T'$ where $T'=h^{\frac12+\delta}$ and ϵ is an arbitrarily small constant. However if a is microhyperbolic, so is $a+\alpha b$ as $|\alpha|\le\epsilon$ and therefore due to arguments of Subsubsection *6.1.2.4 The Microhyperbolicity Condition* intervals with $|t''|\le T'$, $\epsilon^{-1}T'\le |t'|\le \epsilon$ provide negligible contribution to the trace. □

So, as far as trace is concerned we slightly extended interval system (6.2.55). These intervals still do not overlap and cannot overlap due to possible subperiodic trajectories. We ban them temporarily.

Let $\mathcal{U}$ be an open subset in Ω such that

$$\mathsf{dist}\big(\Psi_{a,t}(z), z\big) \ge \epsilon_0\gamma \sin\frac{\pi t}{T_0} \qquad \forall t \in [0, T_0] \quad \forall z \in \mathcal{U} \quad \text{with} \ \ \gamma \ge h^{\frac{1}{2}-\delta}. \tag{6.2.63}$$

This condition excludes subperiodic trajectories, starting from $\mathcal{U}$. Later we will modify our arguments in order to permit them. However, the latter makes sense only as a is scalar. In the matrix case we must assume that $\gamma = \epsilon'$ and we will not be able to allow subperiodic trajectories.

If (6.2.63) is fulfilled for $\mathcal{U}$, then it is also fulfilled for $\epsilon_1\gamma$-neighborhood of $\mathcal{U}$.

Proposition 6.2.16. *Let conditions of Proposition 6.2.15 be fulfilled. Further, let* (6.2.63) *be fulfilled where* $\gamma \ge h^{\frac{1}{2}-\delta}$ *as* a *is scalar and* $\gamma \asymp 1$ *in the general case. Let* $Q_1, Q_2 \in \mathbf{\Psi}_{h,\gamma,\gamma,K}(X, \mathbb{K})$ *be supported in the* $\epsilon_1\gamma$*-neighborhood of* $\mathcal{U}$*. Then*

(i) (6.2.61) *holds with any constant* $\nu < T_0$*.*

(ii) Moreover, as $\chi \in \mathscr{C}_0^K([-1,1])$ *and* $T \in [\frac{1}{2}T_0, \varepsilon^{-1}h^{\frac{1}{2}+2\delta}]$, $N = \lfloor TT_0^{-1}\rfloor$

$$F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\big(Q_{1x}u(x,y,t)\,{}^tQ_{2y}\big) \equiv \sum_{m\ge 0}\sum_{n:|n|\le T} h^{1-d+m}\iiiint T\hat{\chi}\big(\frac{(\tau-\tau')T}{h}\big)\mathcal{F}_m(y,\xi,\tau',\tau'')e^{ih^{-1}n(\varepsilon\tau''-T_0\tau')}\times dyd\xi\, d\tau' d\tau''. \tag{6.2.64}$$

Proof. (i) Statement (i) follows from the simple fact that under condition (6.2.63) the contribution of time intervals $\big((n-1)T_0+\epsilon, nT_0-\epsilon\big)$ is negligible.

Really, consider first $\gamma \asymp 1$. Then as $|t'| \ge \epsilon$ the right-hand expression of (6.2.63) is disjoint from 0; this is not broken by perturbation of $t'A$ by $t''B$ with $|t''| \le h^{\frac{1}{2}+\delta}$. Applying results of Section 2.3 we arrive to the required conclusion.

Consider the case of $\gamma \ge h^{\frac{1}{2}-\delta}$ and scalar a. In this case we can assume with no loss of the generality that $\mathsf{diam}(\mathsf{supp}(Q_1)\cup\mathsf{supp}(Q_2)) \le \epsilon''\gamma$. Then

$$Q_1 e^{ih^{-1}t''B}e^{ih^{-1}t'A}Q_2 \equiv Q_1 e^{ih^{-1}t''B}Q_{2t'}e^{ih^{-1}t'A} \equiv 0$$

where $Q_{2t'} = e^{ih^{-1}t'A}Q_2e^{-ih^{-1}t'A} \in \mathbf{\Psi}_{h,\gamma,\gamma}(X,\mathbb{K})$ and $\mathsf{dist}(\mathsf{supp}(Q_1),\mathsf{supp}(Q_{2t'})) \ge \epsilon_2\gamma$ due to (6.2.63). Due to the finite speed of propagation for B and $|t''| \le h^{\frac12+\delta} \le \epsilon_2\gamma$ we conclude that $Q_1e^{ih^{-1}t''B}Q_{2t'} \equiv 0$ which proves the second identity above.

(ii) Then representing $\chi_T(t) = \chi_T(t)\sum_n \phi(t-nT_0)$ with some ϕ and applying Statement (i) we arrive to (6.2.64). One should notice that τ in the exponent in (6.2.61) is replaced by τ' in (6.2.64) because ϕ_ν is shifted by nT_0 in the first term while χ_T is not. □

Let $\chi = 1$ at $[-\frac12 T_0, \frac12 T_0]$ and $\varphi \in \mathscr{C}_0^K([-c,c])$; let $L \ge T^{-1}h^{1-\delta}$. In this case we know that modulo $O(h^s)$ with arbitrarily large s

$$\int \varphi\Big(\frac{\tau}{L}\Big)\, d_\tau \Gamma Q_{1x} e(.,.,\tau_1,\tau)\, {}^tQ_{2y} - h^{-1}\int \varphi\Big(\frac{\tau}{L}\Big)\big(F_{t\to h^{-1}\tau}\chi_T \Gamma Q_{1x}u(.,.,t)\, {}^tQ_{2y}\big)\, d\tau \equiv 0. \tag{6.2.65}$$

Combining with (6.2.64) we arrive to

$$\int \varphi\Big(\frac{\tau}{L}\Big)\Big(d_\tau \Gamma Q_{1x} e(.,.,\tau_1,\tau)\, {}^tQ_{2y} - \sum_{m\ge 0}\sum_{n:|n|\le T} h^{-d+m}\iiint \mathcal{F}_m(y,\xi,\tau,\tau'') e^{ih^{-1}n(\varepsilon\tau''-T_0\tau)} \times dyd\xi\, d\tau'' d\tau\Big) \equiv 0 \tag{6.2.66}$$

where we have applied estimate

$$|\varphi\Big(\frac{\tau'}{L}\Big) - \frac{T}{h}\int \hat\chi\Big(\frac{(\tau-\tau')T}{h}\Big)\varphi\Big(\frac{\tau'}{L}\Big)\, d\tau| \le C'h^s \tag{6.2.67}$$

and then after τ was removed we renamed τ' by τ.

In fact, replacing $\varphi(\tau L^{-1})$ by

$$\sum_{0\le k\le K-1} \varphi^{(k)}\Big(\frac{\tau'}{L}\Big)\frac{1}{k!}(\tau-\tau')^k L^{-k} + O\big((\tau'-\tau)^K L^{-K}\big),$$

estimating $\hat\chi(t)$ by $C(|t|+1)^{-2K}$ and recalling that $\chi(0)=1$, $\chi'(0)=\chi''(0)=\ldots=0$ and that $LT \ge h^{1-\delta}$ we obtain inequality (6.2.67).

Note that actually in (6.2.66) we can extend summation over $n \in \mathbb{Z}$. Indeed, the norm of m-th term of successive approximations on $[-T,T]$

does not exceed $CT^{l(m)}h^{-l(m)}$ and multiplying by $\phi_T(t)$ with $\phi \in \mathscr{C}_0^\infty(\mathbb{R})$, then applying $\varphi(hL^{-1}D_t)$ with $LT \ge h^{1-\delta}$ and finally setting $t=0$ makes it $O(h^s)$.

Then using formula

$$\sum_{n\in\mathbb{Z}} e^{-int} = 2\pi \sum_{p\in\mathbb{Z}} \delta(t-2\pi p) = \partial_t \Upsilon(t) \tag{6.2.68}$$

with

$$\Upsilon(t) \coloneqq 2\pi \sum_{p\in\mathbb{Z}} \theta(t-2\pi p) = 2\pi \lceil \frac{t}{2\pi} \rceil \tag{6.2.69}$$

we conclude that

$$\sum_{n\in\mathbb{Z}} e^{inh^{-1}(\varepsilon\tau''-T_0\tau)} = hT_0^{-1}\partial_\tau \Upsilon\big(h^{-1}(T_0\tau-\varepsilon\tau'')\big)$$

thus rewriting (6.2.66) as

$$\int \varphi\big(\frac{\tau}{L}\big)\Big(d_\tau \Gamma Q_{1x} e(.,.,\tau_1,\tau)\, {}^tQ_{2y} - \sum_{m\ge 0} h^{1-d+m} T_0^{-1} \iiint \mathcal{F}_m(y,\xi,\tau,\tau'') d_\tau \Upsilon\big(h^{-1}(T_0\tau-\varepsilon\tau'')\big)\, dy d\xi\, d\tau''\Big) \equiv 0. \tag{6.2.70}$$

In particular, the main part is

$$(2\pi)^{-d} h^{1-d} T_0^{-1} \int_{\Sigma_\tau} q_1^0 \Upsilon\big(h^{-1}(T_0 a - \varepsilon b)\big)\, d\mu_\tau q_2^0 \tag{6.2.71}$$

(multiplied by $-\partial_\tau \varphi(\tau/L)$ and integrated over τ) where

(6.2.72) $\Sigma_\tau = \{(x,\xi) : \tau \in \mathsf{Spec}(a(x,\xi))\}$ and $\mu_\tau = \delta(\tau - a)$ is a measure there as usual.

So, we arrive to

Theorem 6.2.17. *Let conditions* (6.2.1), (6.2.4), (6.2.7), (6.2.18) *and* (6.2.19) *be fulfilled in* Ω. *Further, let* a *be microhyperbolic in* Ω *on energy level* $\bar\tau$ *and let* (6.2.55) *be fulfilled.*

Furthermore, let (6.2.63) *be fulfilled where* $\gamma \ge h^{\frac12-\delta}$ *as* a *is scalar and* $\gamma \asymp 1$ *in the general case. Let* $Q_1, Q_2 \in \mathbf{\Psi}_{h,\gamma,\gamma,K}(X, \mathbb{K})$ *be supported in the* $\epsilon_1\gamma$*-neighborhood of* $\mathcal{U}$*. Then as*

$$L \ge \varepsilon h^{\frac12-\delta} \tag{6.2.73}$$

formula (6.2.70) *holds.*

Here we took $T = \varepsilon^{-1}h^{\frac12-\delta}$, defined bound to L from $LT \ge h^{1-\delta}$ and denoted 2δ by δ.

Spectral Gaps (the First Attempt)

We conclude that if $\Omega = T^*X$ and (6.2.28) holds precisely rather than modulo negligible then $\mathsf{Spec}(A_0) \subset 2\pi h T_0^{-1}\mathbb{Z}$; otherwise it is contained in h^s-vicinity of $2\pi h T_0^{-1}\mathbb{Z}$.

Further, if A_0 and B commute precisely rather than modulo negligible then $\mathsf{Spec}(A_0 + \varepsilon B) \subset \mathsf{Spec}(A_0) + \varepsilon\,\mathsf{Spec}(B)$; otherwise it is contained in h^s-vicinity of the latter set.

Therefore as $\varepsilon \le \epsilon_0 h$ spectrum is split into *spectral bands* of the width $\asymp \varepsilon$ (there may be degenerations) separated by *spectral gaps*. As $\varepsilon \le \mathsf{ch}$ there may be spectral gaps (but as ε grows from $\epsilon_0 h$ to ch spectral bands start overlap and spectral gaps start disappear (barring degenerations), and as $\varepsilon \ge \mathsf{ch}$ one should not expect them at all.

Criteria of τ belonging to the spectral gap (in Ω) is

$$\begin{aligned} &|\tau - 2\pi p\frac{h}{T_0} - \varepsilon\lambda| \ge L \\ &\qquad \forall p \in \mathbb{Z} \quad \forall \lambda \in \mathsf{Spec}(b(x,\xi)) \quad \forall (x,\xi) \in \Omega : |a(x,\xi) - \tau| \le \mathsf{ch} \end{aligned} \tag{6.2.74}$$

where *currently* L is defined by (6.2.73). Later in the dedicated Subsection 6.4 we using elliptic arguments weaken condition to L.

We say that

(6.2.75) τ' and τ'' belong to the same spectral gap if (6.2.74) holds for all $\tau \in [\mathsf{min}(\tau', \tau''), \mathsf{max}(\tau', \tau'')]$.

Corollary 6.2.18. *Let conditions of Theorem 6.2.17 be fulfilled. Then*

(i) If τ_1, τ_2 belong to spectral gaps in Ω then estimate (6.2.70) *holds without mollification:*

$$\Gamma Q_{1x} e(.,.,\tau_1,\tau_2)\, {}^t Q_{2y} \equiv \sum_{m\ge 0} h^{-d+m} T_0 \iiint \int_{\tau_1}^{\tau_2} \mathcal{F}_m(y,\xi,\tau,\tau'') d_\tau \Upsilon\big(h^{-1}(T_0\tau - \varepsilon\tau'')\big)\, dy d\xi\, d\tau''. \tag{6.2.76}$$

(ii) In particular, if τ_1, τ_2 belong to the same spectral gap in Ω then

$$\Gamma Q_{1x} e(.,.,\tau_1,\tau_2)\, {}^t Q_{2y} \equiv 0. \tag{6.2.77}$$

Proof. (a) Note first that since due to Section 6.1

$$\operatorname{supp}\big(\mathcal{F}_m(x,\xi,\tau,\lambda)\big) \subset \{\tau \in \operatorname{Spec}(a(x,\xi)),\ \lambda \in \operatorname{Spec}(b(x,\xi))\} \tag{6.2.78}$$

we conclude that

$$\operatorname{supp}\big(\mathcal{F}_m(x,\xi,\tau,\lambda)\delta(h^{-1}(T_0\tau - \varepsilon\lambda - 2\pi p h))\big) \subset \{\tau \in \operatorname{Spec}(a(x,\xi)),\ \lambda \in \operatorname{Spec}(b(x,\xi)), \tau = 2\pi p \frac{h}{T_0} + \lambda\}. \tag{6.2.79}$$

Therefore if $\operatorname{supp}(\varphi) \subset (-1,1)$ and $[-L, L]$ belongs to the spectral gap we conclude that

$$\int \varphi(\tau/L) d_\tau \Gamma Q_{1x} e(.,.,\tau_1,\tau_2)\, {}^t Q_{2y} \equiv 0.$$

Taking $\varphi \ge 0$ and disjoint from 0 on $(-\frac12, \frac12)$ and $Q_2 = Q_1^*$ we arrive to (6.2.77) for $\tau_1, \tau_2 \in (-\frac12 L, \frac12 L)$; then we can get rid of condition $Q_2 = Q_1^*$. Shifting with respect to spectral parameter we get Statement (ii) in the full generality. Moreover Statement (i) is proven as τ_1, τ_2 belong to the same spectral gap.

(b) Let τ_1, τ_2 belong to different spectral gaps; we can assume without any loss of the generality that $\tau_2 \ge \tau_1 + L$. We can apply (6.2.70) with $\varphi(\tau/L)$ replaced by $\varphi_j((\tau - \tau_j')/L)$ and thus with $\varphi(\tau/L)$ replaced by their sum $\omega(\tau)$ containing no more than h^{-l} terms. However we can always make $\omega(\tau)$ supported in $[\tau_1, \tau_2]$ and equal 1 on $[\tau_1 + \frac12 L, \tau_2 - \frac12]$. Applying Statement (i) to both spectral gaps we can replace $\int \omega(\tau)(\dots)$ by $\int_{\tau_1}^{\tau_2}(\dots)$ with the negligible error and therefore Statement (i) is proven in the full generality. □

6.2.4 Exploiting Microhyperbolicity

In this subsection we assume that

(6.2.80) System (A, B) is microhyperbolic in Ω on energy level $(\bar{\tau}, \bar{\lambda})$ (see Definition 6.1.7)

and derive asymptotics without mollification and with mollification with a smaller parameter L; as usual $\bar{\tau} = 0$.

Extending Time Interval

Let us recall that the successive approximation delivers $U(x, y, t', t'')$ as $|t'| \le T' = h^{\frac{1}{2}+\delta}$ and $|t''| \le T'' = h^{\frac{1}{2}+\delta}$. Then microhyperbolicity of A allowed us to increase T' to ϵ without affecting T'' and then (6.2.63) allowed us to increase T' so time intervals for $t = t' + nT_0$ began overlap provided $\gamma \ge CT'' = C\varepsilon T$ which was automatically fulfilled under assumptions there.

On the other hand, microhyperbolicity of (A, B) combined with the results of Subsubsection *6.1.2.5 Exploiting Microhyperbolicity* allow to increase T', T'' to $T' = T'' = \epsilon$ then (6.2.63) allowed us to increase T' so time intervals for $t = t' + nT_0$ began overlap provided $\gamma \ge CT'' = C\varepsilon T$. This condition is automatically fulfilled as $\gamma \asymp 1$ but may be a problem otherwise. We need to allow γ small if we want to approach subperiodic points.

Therefore we immediately arrive to

Theorem 6.2.19. *Let condition* (6.2.80) *be fulfilled. Then if we replace* $e(., y, \tau)$ *by* $\psi(B)_x e(., y, \tau)$ *with* $\psi \in \mathscr{C}_0^\infty([\bar{\lambda} - \epsilon, \bar{\lambda} + \epsilon])$ *with small enough* ϵ

(i) Proposition 6.2.15 extends to $n : |n| \le TT_0^{-1} \le \epsilon\varepsilon^{-1}$.

(ii) Proposition 6.2.16 extends to n, T*:* $|n| \le TT_0^{-1} = \epsilon\varepsilon^{-1}\gamma$.

(iii) Theorem 6.2.17 extends to L *satisfying*

$$L \ge \varepsilon\gamma^{-1}h^{1-\delta}. \tag{6.2.81}$$

Tauberian Arguments

To apply Tauberian arguments we need to estimate

$$M(\phi) := F_{t\to h^{-1}\tau}\phi_T(t)\Gamma\big(\psi(B)_x Q_{1x} u(., ., t)\, {}^t Q_{2y}\big). \tag{6.2.82}$$

Theorem 6.2.20. *Let conditions* (6.2.1), (6.2.4), (6.2.7), (6.2.18), (6.2.19) *and* (6.2.80) *be fulfilled in* Ω. *Further, let* (6.2.55) *be fulfilled.*

Furthermore, let (6.2.63) *be fulfilled with* $\gamma \asymp 1$. *Let* $Q_1, Q_2 \in \mathbf{\Psi}_{h,\gamma,\gamma,K}(X, \mathbb{K})$ *be supported in the* $\epsilon_1\gamma$*-neighborhood of* $\mathcal{U}$. *Let* $\chi \in \mathscr{C}_0^\infty([-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1])$, $\bar{\chi} \in \mathscr{C}_0^\infty([-1, 1])$. *Then with the arbitrarily large exponent* I

(i) As $|n| \le T^* T_0^{-1} := \epsilon\varepsilon^{-1}$, $\phi(t) = \bar{\chi}(t - nT_0)$

$$|M(\phi)| \le Ch^{1-d}(1 + \varepsilon h^{-1}|n|)^{-I}. \tag{6.2.83}$$

(ii) As $1 \le T \le T^*$

$$|M(\chi_T)| \le Ch^{1-d} T(1 + \varepsilon h^{-1} T)^{-I} \tag{6.2.84}$$

and

$$|M(\bar{\chi}_T)| \le Ch^{1-d}(\varepsilon^{-1}h + 1). \tag{6.2.85}$$

We took $\gamma \asymp 1$ because we will study subperiodic points later.

Proof. We need a following statement:

(6.2.86) Consider system $(A_1, \dots, A_m)$ microhyperbolic at $(\bar{\tau}_1, \dots, \bar{\tau}_m)$. Let $\varphi_j \in \mathscr{C}_0^\infty([\bar{\tau}_j - \epsilon, \bar{\tau}_j + \epsilon])$. Then

$$|\Gamma\big(Q_{1x}\varphi_1(A_1)_x\varphi_2(A_2)_x \cdots \varphi_m(A_m)_x U(., ., \boldsymbol{t})\, {}^tQ_{2y}\big)| \le Ch^{-d}\big(1 + h^{-1}|\boldsymbol{t}|\big)^{-I} \qquad \text{as} \quad |\boldsymbol{t}| \le \epsilon. \tag{6.2.87}$$

Really, due to propagation theory the left-hand expression is $O(h^s)$ as $\frac{1}{2}\epsilon \le |\boldsymbol{t}| \le \epsilon$. Scaling as usual we arrive to (6.2.87) as $h \le |\boldsymbol{t}| \le \epsilon$. Finally, combining with the trivial estimate $O(h^{-d})$ we get (6.2.87) as $|\boldsymbol{t}| \le h$. Here $\phi_j(A_j)$ replace $\phi_j(\tau_j)$.

This estimate immediately implies that

$$|F_{\boldsymbol{t}' \to h^{-1}\boldsymbol{\tau}'}\phi_1(t_1) \cdots \phi_{m'}(t_{m'})\Gamma\big(Q_{1x}\varphi_{m'+1}(A_{m'+1})_x \cdots \varphi_m(A_m)_x \times U(., ., \boldsymbol{t})\, {}^tQ_{2y}\big)| \le Ch^{-d+m'}\big(1 + h^{-1}|\boldsymbol{t}|\big)^{-I} \qquad \text{as} \quad |\boldsymbol{t}'| \le \epsilon, \tag{6.2.88}$$

$\mathsf{supp}(\phi_j) \subset [-\epsilon, \epsilon]$ and $\boldsymbol{t}' = (t_1, \dots, t_{m'})$.

Applying (6.2.88) with $m'=1$ to microhyperbolic system (A,B) and picking up $|t_2|=\varepsilon n$ we derive (6.2.83) as $\mathsf{supp}(\phi)\subset [nT_0-\epsilon, nT_0+\epsilon]$.

However assumption (6.2.63) with $\gamma\asymp 1$ implies that the contribution of time intervals $[(n-1)T_0+\epsilon, nT_0-\epsilon]$ is negligible and we arrive to (6.2.83) in its full generality.

It immediately implies (6.2.84), which in turn implies (6.2.85). □

Now Tauberian theorem implies the Tauberian estimate

$$|\mathsf{R}^{\mathsf{T}}_{\psi}|\le Ch^{1-d}T^{-1}(\varepsilon^{-1}h+1)=Ch^{1-d}(\varepsilon+h) \tag{6.2.89}$$

and more generally

$$|\mathsf{R}^{\mathsf{T}}_{\psi,\varphi,L}|\le Ch^{1-d}(\varepsilon+h)\vartheta\Big(\frac{\varepsilon h}{L}\Big)\qquad \text{as}\quad L\ge \varepsilon h \tag{6.2.90}$$

where we plugged $T=\epsilon\varepsilon^{-1}$ and subscript ψ means that we replaced $e(.,.,\tau)$ by $\psi(B)_x e(.,.,\tau)$.

So, we have proven

Theorem 6.2.21. *Let conditions of Theorem 6.2.20 be fulfilled and let φ satisfy* (4.2.73)–(4.2.75). *Then estimates* (6.2.89), (6.2.90) *hold as $\mathsf{R}^{\mathsf{T}}_{\psi}$ is the difference between $\mathsf{N}^{\mathsf{T}}_{\psi}$ and Tauberian expression, which is the right-hand expression of* (6.2.64) *or equivalently* (6.2.76), *and $\mathsf{N}^{\mathsf{T}}_{\psi,\varphi,L}$ is the same expression mollified.*

Let us analyze the contribution of the different terms in (6.2.64) into the final answer. Here we need to distinguish terms with $n=0$ which deliver Weyl expression (probably with many terms) and terms with $n\ne 0$ which deliver *correction terms*.

Then Theorem 6.2.20 yields that

(6.2.91) The contribution of term with $n\ne 0$ to $\mathsf{R}^{\mathsf{T}}_{\psi,\varphi,L}$ does not exceed

$$Ch^{1-d+m}|n|^{-1}\big(1+h^{-1}\varepsilon|n|\big)^{-l}\vartheta\Big(\frac{h}{L|n|}\Big). \tag{6.2.92}$$

In particular we arrive to

Theorem 6.2.22. *In the framework of Theorem 6.2.21 as* $\varepsilon \ge h^{1-\delta}$ *the same estimates* (6.2.89), (6.2.90) *hold for* $\mathsf{R}^{\mathsf{W}}_{\psi}$ *and* $\mathsf{R}^{\mathsf{W}}_{\psi,\varphi,L}$.

Further trivially we have

Proposition 6.2.23. *In the framework of Theorem 6.2.21 the total contribution of all terms with* $m \ge 1$, $n \ne 0$ *to the Tauberian expression* $\mathsf{N}^{\mathsf{T}}_{\psi,\varphi,L}$ *does not exceed*

$$Ch^{2-d}\bigl(\frac{h}{\varepsilon}\bigr)^{l}\vartheta\bigl(\frac{h}{\max(h,L)}\bigr) \tag{6.2.93}$$

as $\varepsilon \ge h$ *and*

$$Ch^{2-d}\sum_{1\le n\le \varepsilon^{-1}h} n^{-1}\vartheta\bigl(\frac{h}{\max(h,Ln)}\bigr) \tag{6.2.94}$$

as $\varepsilon \ge h$. *In particular, this contribution to the Tauberian expression* N^{T} *does not exceed* Ch^{2-d} *and* $Ch^{2-d}\bigl(|\log \varepsilon^{-1}h|+1\bigr)$ *respectively.*

Consider now terms with $m=0$, $n\ne 0$. According to (6.2.71) their contribution is

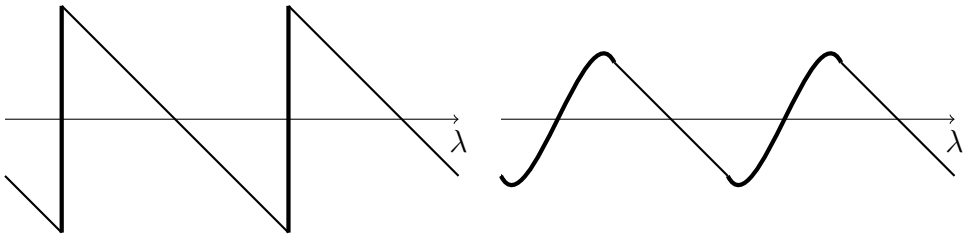
$$\mathsf{N}_{\mathsf{corr},\psi} := (2\pi)^{-d}h^{1-d}T_0^{-1}\int_{\Sigma_\tau} q_1^0\psi(x)\Upsilon_1\bigl(h^{-1}(aT_0-\varepsilon b)\bigr)\,d\mu_\tau q_2^0 \tag{6.2.95}$$

where $\Upsilon_1(t)=\Upsilon(t)-t+\frac{1}{2}$. So, we have proven

Theorem 6.2.24. *In the framework of Theorem 6.2.21 with an error not exceeding* (6.2.93) *as* $\varepsilon \ge h$ *and* (6.2.94) *as* $\varepsilon \le h$ *one can replace* $\mathsf{N}^{\mathsf{T}}_{\psi}$ *or* $\mathsf{N}^{\mathsf{T}}_{\psi,\varphi,L}$ *in Theorem 6.2.21 by* $\mathsf{N}^{\mathsf{W}}_{\psi}+\mathsf{N}_{\mathsf{corr},\psi}$ *or* $\mathsf{N}^{\mathsf{W}}_{\psi,\varphi,L}+\mathsf{N}_{\mathsf{corr},\psi,\varphi,L}$ *respectively.*

Using approach of Section 6.3.1 in the scalar case one can eliminate logarithm.

Problem 6.2.25. Prove that in the case of operators belonging to $\boldsymbol{\Psi}_{h,\mathsf{an}}(X,\mathbb{K})$ one can replace in the estimates above $(1+\varepsilon h^{-1}|n|)^{-l}$ by $\exp(-\epsilon h^{-1}|n|)$; in particular, prove that $\mathsf{N}^{\mathsf{T}}_{\psi}\equiv \mathsf{N}^{\mathsf{W}}_{\psi}$ as $\varepsilon \ge Ch|\log h|$.



(a) $\Upsilon_1(t)$ (b) $\int \Upsilon_1(t-\varepsilon b(x,\xi))d\mu_\tau$

Figure 6.2: Mollified with respect to $d\mu_\tau$ "a sharp saw" becomes a "dull saw"; bold curved parts correspond to the spectral bands.

Removing Cut-off ψ

Remark 6.2.26. Theorems 6.2.19, 6.2.21, 6.2.22, 6.2.24 and Proposition 6.2.23 hold with $\psi(B) = I$ provided

(i) Either $\varepsilon \gg h$ and microhyperbolicity condition (6.2.80) holds for every $\bar\lambda$ (and we cannot do better in this case).

(ii) Or $\varepsilon \lesssim h$, we interested in asymptotics at $\tau : |\tau - \bar\tau| \le \epsilon\varepsilon$ and microhyperbolicity condition (6.2.80) holds for all $\bar\lambda_p = \varepsilon^{-1}(\bar\tau - 2\pi p h)$.

(iii) As $\varepsilon \ll h$ and $\bar\tau$ does not belong to spectral gap microhyperbolicity condition should be checked exactly for one p (such that $|\bar\tau - 2p\pi h T_0^{-1}| \ll h$.

Proof is trivial because values in the spectral gaps are already eliminated.

Remark 6.2.27. Obviously wit is sufficient to fulfill microhyperbolicity of spectral gap condition on $\mathsf{supp}\, Q_1 \cap \mathsf{supp}\, Q_2$ only.

6.3 Scalar Operators with Periodic Hamiltonian Flow (End)

6.3.1 Scalar a and Subperiodic Trajectories

Structure of Subperiodic Trajectories

Let a be a scalar symbol. Let Π be the set of all the subperiodic points. Due to Colin de Verdiere [2], Proposition 1.3, we have the following

Proposition 6.3.1. *(i) The set of all subperiodic points* Π *is a union of a finite number of symplectic submanifolds* $\Pi_{p,r}$ *of codimension* $2r$ *consisting of* $p^{-1}T_0$*-periodic points*

$$\Pi = \bigcup_{(p,r)\in\mathcal{J}} \Pi_{p,r} \tag{6.3.1}$$

$\mathcal{J}$ *is a finite subset of* $\{2,3,...\}\times\{1,...,(d-1)\}$.

(ii) Moreover $\Psi_{a,t}(\Pi_{p,r}) = \Pi_{p,r}$ *and for all* t *and*

$$z\in\Omega_0,\ z'\in\Psi_{a,t}(z) \implies \operatorname{dist}(z',\Pi_{p,r}) \asymp \operatorname{dist}(z',\Pi_{p,r}) \qquad \forall t. \tag{6.3.2}$$

(iii) As $z\in\Pi_{p,r}$

$$T_z\Pi_{p,r} = \operatorname{Ker}(D\Psi_{a,p^{-1}T_0}(z) - I) \tag{6.3.3}$$

and

$$(D\Psi_{a,p^{-1}T_0}(z))^p = I; \tag{6.3.4}$$

hence

$$\operatorname{Spec}(D\Psi_{a,p^{-1}T_0}(z)) \subset \{e^{2\pi i\frac{q}{p}}, q = 0,...,p-1\} \tag{6.3.5}$$

and all the Jordan cells of $D\Psi_{a,p^{-1}T_0}(z)$ *are* 1*-dimensional.*

(iv) Further, $\Pi_{p,r}$ *and* $\Pi_{p,r'}$ *are disjoint for* $r\neq r'$, $\Pi_{p'}\subset\Pi_p$ *if and only if* p *divides* p' *and* $\Pi_{p'}\cap\Pi_{p''} = \Pi_p$ *where* p *is the greatest common divisor of* p' *and* p''*. Moreover, this intersection is clean*[7)]*.*

(v) Furthermore, the intersection of $\Pi_{p,r}$ *and* Σ_τ *is clean and* $\operatorname{codim}\Pi_{p,r}\cap\Sigma_\tau = 2r+1$.

Proof. Note first that (6.3.4) follows from $\Psi^p_{a,p^{-1}T_0} = I$; (6.3.5) and the lack of adjoint eigenvectors follow from this. The other statements follow obviously. □

Further, due to $\{a,b\}=0$ (recall that a is scalar) $\Psi_{a,t'}$ and $\Psi_{b,t''}$ commute (an easy proof is left to the reader),

$$\Psi_{a,t}\Psi_{b,t''}(\Pi_{p,r}\cap\Omega_0)\subset\Pi_{p,r} \tag{6.3.6}$$

[7)] See J. J. Duistermaat & V. Guillemin [1] for the definition.

and

$$(6.3.7)\quad z \in \Omega_0,\ z' \in \Psi_{a,t}\Psi_{b,t''}(z) \implies \mathsf{dist}(z', \Pi_{p,r}) \asymp \mathsf{dist}(z', \Pi_{p,r}) \qquad \forall |t''| \le T'',\ \forall t$$

where T'' is the same constant as in condition (6.2.55).

We currently assume neither that b is scalar no microhyperbolicity of (a, b).

Propagation of Singularities near Π

In the previous analysis we using condition (6.2.63) were able to reduce the analysis of $\chi_T(t)\Gamma(Q_1 e^{ih^{-1}At}Q_2)$ to the analysis of $\phi(t - nT_0)\Gamma(Q_1 e^{ih^{-1}At}Q_2)$ with $|n| \le T$ and ϕ supported in $[-\epsilon, \epsilon]$. Now we have condition (6.2.63) as $\mathsf{dist}(z, \Pi) \ge \gamma$ and condition

$$(6.3.8)\quad \mathsf{dist}\big(\Psi_{a,t}(z), z\big) \ge \epsilon_0\big(|\sin \frac{\pi p t}{T_0}| + \gamma |\sin \frac{\pi t}{T_0}|\big) \qquad \forall t \in [0, T_0] \quad \forall z \in \mathcal{U}$$

as z is near Π_p, but disjoint from $\Pi_{p'}$ with p' not multiple of p, $\gamma \asymp \mathsf{dist}(z, \Pi_p)$.

Anyway we need to consider additionally $\phi(t - nT_0 - t_{p,q})\Gamma(Q_1 e^{ih^{-1}At}Q_2)$ where $t_{p,q} = qp^{-1}T_0$, $q = 1, \dots, p-1$ and p, q are coprimes.

The following statement will be needed only if (a, b) is microhyperbolic but does not require microhyperbolicity by itself:

Proposition 6.3.2. *Let $Q_1, Q_2 \in \mathbf{\Psi}_{h,\gamma,\gamma}(X, K)$ be supported respectively in $\{\mathsf{dist}(z, \Pi_p) \ge C_0\gamma\}$ and $\{\mathsf{dist}(z, \Pi_p) \le \gamma\} \cap \Omega_0$ with large enough constant C_0 and $\gamma \ge h^{\frac{1}{2}-\delta}$.*

Then $Q_1 e^{ih^{-1}tB}Q_2 \equiv 0$ as $|t| \le T''$ where T'' is a constant from (6.2.55).

Proof. If b is scalar this statement follows immediately from Section 2.4. Let us prove it in the general case. Obviously it is sufficient to provide proof for small constant T''; then without any loss of the generality one can assume that

(6.3.9) $\Pi_p = \{f_1 = \dots = f_m = 0\}$ with smooth real-valued functions $f_1, \dots, f_m$ such that $df_1, \dots, df_m$ are linearly independent on Π_p.

Actually one can even assume that $f_j = x_j$ for $j = 1, \dots, r$ and $f_j = \xi_{j-r}$ for $j = r+1, \dots, 2r = m$ because Π_p is symplectic but this is immaterial now. We need however the following consequence of (6.3.7):

$$\{b, f_j\} = \sum_k b_{jk} f_k \tag{6.3.10}$$

Then $\pm\{\tau - b, e^{\pm Ct}|f|^2\}e^{\pm Ct} \ge |f|^2$ and one can apply the energy estimate method with

$$Q_\alpha = \mathsf{Op}^{\mathsf{w}}\big(\chi(\gamma^{-2}e^{\pm Ct}|f - \bar f|^2 - \alpha)\big) \tag{6.3.11}$$

with $\bar f = 0$ here where χ is a function as in the proof of Theorem 2.1.2 (i.e. described by (2.1.11)) and to prove the following

Proposition 6.3.3. *Assume that v is temperate and*

$$\|Q_\alpha v|_{t=0}\| \le Ch^s, \tag{6.3.12}$$

$$\|Q_\alpha(hD_t - B)v^\pm\| \le Ch^{s+1}, \tag{6.3.13}$$

then for $\beta < \alpha$

$$\|Q_\beta v^\pm\| \le Ch^s \tag{6.3.14}$$

where $\beta < \alpha$ are arbitrary constants.

First (6.3.14) is proved under additional assumption $\|Q_\alpha v^\pm\| \le Ch^{s-\delta}$ with arbitrarily small exponent $\delta > 0$ which can be easily dropped.

An easy proof of Proposition 6.3.3 following exactly the same scheme as in Theorem 2.1.2 is left to the reader. This completes the proof of Proposition 6.3.2. □

Proposition 6.3.4. *Let $Q_1, Q_2 \in \mathbf{\Psi}_{h,\gamma,\gamma}(X, K)$ be supported in $\{f \in \mathcal{V}_1\}$ and $\{f \in \mathcal{V}_2\} \cap \Omega_0$ respectively where $f = (f_1, \dots, f_m)$. Let T'' be a constant and $\mathcal{V}_j \subset \{|f| \le \gamma\}$ for $j = 1, 2$, $\mathsf{dist}(\mathcal{V}_1, \mathcal{V}_2) \ge C_0 T''\gamma$.*

Then $Q_1 e^{ih^{-1}tB} Q_2 \equiv 0$ as $|t| \le T''$. Here C_0 does not depend on T''.

Proof. Without any loss of the generality one can assume that $\mathcal{V}_1 = \{|f - \bar f| \le \epsilon\gamma\}$, $\mathcal{V}_2 = \{|f - \bar f| \ge (2\epsilon + CT'')\gamma\}$ with $|\bar f| \le \gamma$.

An easy proof following exactly the same scheme as in Theorem 2.1.2 with Q_α defined by (6.3.11). It is left to the reader. □

Corollary 6.3.5. *Assume that* $Q_1, Q_2 \in \mathbf{\Psi}_{h,\gamma,\gamma}(X, K)$ *are supported in* $\{z\colon \mathsf{dist}(z, \Pi_p) \asymp \gamma\} \cap \Omega_0$ *with* $\gamma \ge h^{\frac{1}{2}-\delta}$.

Then $\Gamma Q_1 e^{ih^{-1}tA} Q_2 \equiv 0$ *as* $|t - t_{p,q} - nT_0| \le \epsilon$ *with* $q = 1, \dots, p-1$.

Proof. Without any loss of the generality one can assume that Q_1, Q_2 are supported in $\{|f - \bar{f}| \le \epsilon'\gamma\}$ with the same $\bar{f}$ and arbitrarily small constant ϵ', $|\bar{f}| \asymp \gamma$. Note that

$$e^{ih^{-1}tA} Q_2 \equiv e^{ih^{-1}(t-nT_0)A} e^{ih^{-1}\varepsilon nB} Q_2 \equiv e^{ih^{-1}(t-nT_0)A_0} e^{ih^{-1}\varepsilon tB} Q_2. \tag{6.3.15}$$

Due to Proposition 6.3.4

$$e^{ih^{-1}\varepsilon tB} Q_2 \equiv Q_3 e^{ih^{-1}\varepsilon tB} Q_2$$

where planted operator $Q_3 \in \mathbf{\Psi}_{h,\gamma,\gamma}(X, \mathbb{K})$ is supported in $\{|f - \bar{f}| \le 3\epsilon'\gamma\} \cap \Omega$ and $Q_3 \equiv I$ in $\{|f - \bar{f}| \le 3\epsilon'\gamma\}$. However then $Q_1 e^{ih^{-1}(t-n)A_0} Q_3 \equiv 0$ as $|t - t_{p,q} - n| \le \epsilon$ due to (6.3.8) and the theory of Section 2.4. □

Therefore in what follows we need to consider only

$$\phi(t - nT_0 - t_{p,q}) \Gamma \big(Q_{1x} u(., ., t) Q^{t}{}_{2y}$$

with Q_j supported in $\{z\colon \mathsf{dist}(z, \Pi_p) \le \gamma = h^{\frac{1}{2}-\delta}\}$.

Successive Approximations

At this moment we do not assume that (a, b) is microhyperbolic; therefore we need to take $T = \varepsilon h^{\frac{1}{2}+2\delta}$ and $T'' = h^{\frac{1}{2}+2\delta}$.

Since Π_p is symplectic manifold we can assume without any loss of the generality that $\Pi_p = \{x'' = \xi'' = 0\}$ with $x = (x'; x'') = (x_1, \dots, x_{d-r}; x_{d-r+1}, \dots, x_d)$, $\xi = (\xi'; \xi'') = (\xi_1, \dots, \xi_{d-r}; \xi_{d-r+1}, \dots, \xi_d)$.

Let us run successive approximations to construct $e^{it''h^{-1}B} Q_2$ only; and let us freeze not $x = y$ but $x = (y', 0)$ and also replace hD'' by 0; so if $B = B(x', x'', hD_{x'}, hD_{x''}, h)$ we pick up $\bar{B} = b(y', 0, hD_{x'}, 0)$ where $b(x, \xi)$ is the principal symbol of B. We can do it because x'' and ξ'' are confined to $\{|x''| \le \gamma, |\xi''| \le \gamma\}$ with $\gamma = h^{\frac{1}{2}-\delta}$ since due to the previous subsubsection we should consider Q_j supported in γ-vicinity of Π_p. Note that we selected exponents in such way that $T''\gamma = h^{1+\delta}$.

Then the Schwartz kernel of $e^{it''h^{-1}B} Q_2$ is given by

$$\sum_{m\ge 0} h^{-d+m} \iint \mathcal{G}_m(y,\xi,\lambda) e^{ih^{-1}(t''\lambda+\langle x-y,\xi\rangle)}\, d\xi d\lambda \tag{6.3.16}$$

where $\mathcal{G}_m$ are functions described in Section 4.3 as $\mathcal{F}_m$ but they are constructed with $(\lambda - b(x',\xi'))^{-1}$ and they are polynomials with respect to (x'',ξ''). Note that since $\{b,x''\} = \{b,\xi''\} = 0$ as $x'' = \xi'' = 0$ each perturbation of this new type brings factor $\gamma^2 T'' h^{-1}$ so actually we need only $T'' \le \min\bigl(h^{\frac12+\delta}, h^{1+\delta}\gamma^{-2}\bigr)$.

Meanwhile $e^{ih^{-1}t'A_0}$ is h-Fourier integral operator corresponding to symplectomorphism $\Psi_{a,t}$ and one can write it as

$$\sum_{m\ge 0} h^{-\frac12(d+N)+k} \int f_k(x,y,t',\theta) e^{ih^{-1}S(x,y,t',\theta)}\, d\theta \tag{6.3.17}$$

where $\theta \in \mathbb{R}^N$, $S(x,y,t',\theta)$ and $f_k(x,y,t',\theta)$ are phase functions and amplitudes respectively. Therefore

$$\begin{aligned}\Gamma(Q_1 e^{ih^{-1}t'A_0} e^{it''h^{-1}B} Q_2) \equiv \sum_{m\ge 0} h^{-\frac12(3d+N)+m}\times \\ \iint\iiint \mathcal{G}_m(y,\xi,\lambda,t',\theta) e^{ih^{-1}(t''\lambda+\langle x-y,\xi\rangle+S(x,y,t',\theta))}\, d\theta d\xi d\lambda\, dxdy\end{aligned} \tag{6.3.18}$$

where we collected together equal powers of h and functions $\mathcal{G}_m(y,\xi,\lambda,\theta)$ are polynomials with respect to x'', ξ'' and are smooth with respect to t',θ (but there are also cut-off factors like $q(x''\gamma^{-1},\xi''\gamma^{-1})$ if we assume that symbols of Q_j are products of $q_j'(x',\xi')$ and such symbols.)

Consider an oscillatory integral

$$\begin{aligned}h^{-\frac12(3d+N)+m}\times \\ \iiiint \mathcal{G}_m(y,\xi,\lambda,t',\theta) e^{ih^{-1}(t''\lambda+\langle x-y,\xi\rangle+S(x,y,t',\theta))}\, d\theta dxdy''d\xi''.\end{aligned} \tag{6.3.19}$$

One can prove easily based on (6.3.7) that

(6.3.20) As x',y',λ,ξ',t' are fixed with $|t'-t_{p,q}| \le \epsilon$ the phase function $\langle x-y,\xi\rangle + S(x,y,t',\theta)$ has just one stationary point defined by

$$\{x''=y''=\xi''=0,\ \xi' = -\partial_{x'}S_0(x',y',t',\theta),\ \partial_\theta S_0(x',y',t',\theta)=0\}$$

with $S_0(x',y',t',\theta) \coloneqq S(x',0,y',0,t',\theta)$ and this point is non-degenerate.

Then due to the stationary phase principle one can eliminate integration with respect to (x'', y'', ξ'') (without any interference of $q(x''\gamma^{-1}, \xi''\gamma^{-1})$ factors if $q = 1$ in the vicinity of $\{x'' = \xi'' = 0\}$) resulting in (6.3.19) becoming the sum of

$$h^{-\frac{1}{2}(3d-3r+N)+m+k}\times \iint \mathcal{G}_{m,k}(y', \xi', \lambda, t', \theta) e^{ih^{-1}(t''\lambda+\langle x'-y', \xi\rangle+S_0(x', y', t', \theta))}\, d\theta dx'. \tag{6.3.21}$$

Let us recall that $S_0(x', y', t', \theta)$ corresponds to symplectomorphism $\Psi_{a,t}|_{\Pi_p}$ which is I as $t = t_{p,q}$; further, without any loss of the generality one can assume that locally $a = \xi_1$. Therefore without any loss of the generality one can assume that $N = d - r$ and $S(x', y', t', \theta) = -\langle x' - y', \theta\rangle + (t' - t_{p,q})\theta_1$ and (6.3.18) becomes

$$\sum_{m\ge 0} h^{-2(d-r)+m}\times \iint \iiint \mathcal{G}_m(y', \xi', \lambda, \theta) e^{ih^{-1}(t''\lambda+\langle x'-y', \xi'-\theta\rangle+(t'-t_{p,q})\theta_1)}\, d\theta d\xi' d\lambda\, dx' dy'. \tag{6.3.22}$$

This is exactly (but with the different functions) would look like

$$\Gamma\big(Q_{1x} e^{ih^{-1}(t'-t'_{p,q})A_0} e^{-ih^{-1}t''B_0} Q_2\big)$$

with $(d-r)$ dimensional operators $A_0 = a(x', 0, hD', 0)$ and $B_0 = b(x', 0, hD', 0)$. It invokes $\big(\tau' - a(x', \xi')\big)^{-1}$ and $\big(\lambda - b(x', \xi')\big)^{-1}$. Therefore in view (6.3.15) formula (6.2.64) should be corrected to

$$\begin{aligned} &F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\big(Q_{1x} u(x, y, t)\, {}^tQ_{2y}\big) \equiv \\ &\sum_{m\ge 0}\sum_{n:|n|\le T} h^{1-d+m} \iiiint T\hat{\chi}\big(\tfrac{(\tau-\tau')T}{h}\big)\mathcal{F}_m(y, \xi, \tau', \tau'') e^{ih^{-1}n(\varepsilon\tau''-T_0\tau')}\times \\ &\qquad dyd\xi\, d\tau' d\tau'' + \\ &\sum_{p,r}\sum_{\Lambda}\sum_{m\ge 0}\sum_{n:|n|\le T} h^{1-d+r+m} \iiiint T\hat{\chi}\big(\tfrac{(\tau-\tau')T}{h}\big)\times \\ &\qquad \mathcal{F}_{p,r,m}(y', \xi', \tau', \tau'') e^{ih^{-1}np^{-1}(\varepsilon\tau''-T_0\tau'+2\pi q(\Lambda))}\, dy' d\xi'\, d\tau' d\tau''. \end{aligned} \tag{6.3.23}$$

where we sum with respect to connected components Λ of $\Pi_{p,r}$, use $dy'd\xi'$ to denote natural measure on $\Pi_{p,r}$ and we also replaced T_0 and ε by $T_0' = T_0 p^{-1}$ and $\varepsilon' = \varepsilon p^{-1}$.

Since we can only be sure that $e^{ih^{-1}T_0'A_0'} = e^{2\pi p^{-1}qi}$ with $q(\Lambda) = 0, \dots, p-1$ rather than with $q(\Lambda) = 0$, we get the last term in the exponent in the second part of (6.3.23).

Similarly (6.2.70) should be corrected to

$$\begin{aligned}(6.3.24)\quad &\int \varphi\big(\frac{\tau}{L}\big)\Big(d_\tau \Gamma Q_{1x} e(.,.,\tau_1,\tau)\,{}^tQ_{2y} - \\ &\sum_{m\ge 0} h^{1-d+m}\iiint \mathcal{F}_m(y,\xi,\tau,\tau'')d_\tau \Upsilon\big(h^{-1}(T_0\tau - \varepsilon\tau'')\big)\,dyd\xi\,d\tau'' - \\ &\sum_{p,r} p \sum_{\Lambda}\sum_{m\ge 0} h^{1-d+r+m}\iiint \mathcal{F}_{p,r,m}(y',\xi',\tau,\tau'')\times \\ &\qquad d_\tau \Upsilon\big(h^{-1}p^{-1}(T_0\tau - \varepsilon\tau'' - q(\Lambda))\big)\,dy'd\xi'\,d\tau''\Big) \equiv 0\end{aligned}$$

and similarly (6.2.76) should be corrected to

$$\begin{aligned}(6.3.25)\quad &\Gamma Q_{1x} e(.,.,\tau_1,\tau_2)\,{}^tQ_{2y} \equiv \\ &\sum_{m\ge 0} h^{-d+m}\iiint \int_{\tau_1}^{\tau_2} \mathcal{F}_m(y,\xi,\tau,\tau'')d_\tau \Upsilon\big(h^{-1}(T_0\tau - \varepsilon\tau'')\big)\,dyd\xi\,d\tau'' + \\ &\qquad \sum_{p,r} p \sum_{m\ge 0} h^{1-d+r+m}\iiint \int_{\tau_1}^{\tau_2} \mathcal{F}_{p,r,m}(y',\xi',\tau,\tau'')\times \\ &\qquad\qquad d_\tau \Upsilon\big(h^{-1}p^{-1}(T_0\tau - \varepsilon\tau'' - q(\Lambda))\big)\,dyd\xi\,d\tau''.\end{aligned}$$

Thus we arrive to the following statement

Theorem 6.3.6. *Let a be scalar symbol. Let conditions* (6.2.1), (6.2.4), (6.2.7), (6.2.18) *and* (6.2.19) *be fulfilled in Ω. Further, let a be microhyperbolic in Ω on the energy level $\bar\tau$ and let* (6.2.55) *be fulfilled.*

Then

(i) As $L \ge \varepsilon h^{\frac12-\delta}$ formula (6.3.24) *holds (cf. Theorem 6.2.17).*

(ii) If τ_1, τ_2 belong to spectral gaps in Ω then formula (6.3.25) *holds (cf. Corollary 6.2.18).*

Exploiting Microhyperbolicity

Now we assume that (a,b) is microhyperbolic on the level $(\bar\tau,\bar\lambda)$ and discuss the consequences of this assumption. First of all we prove that we can take $T\asymp \varepsilon^{-1}$. To do this we should consider

$$\Gamma\big(Q_1\psi(B)\varphi(A_0)e^{ih^{-1}t''B}e^{ih^{-1}t'A_0}Q_2\big) \tag{6.3.26}$$

with φ,ψ supported in ϵ-vicinities of $\bar\tau$ and $\bar\lambda$ but this time as $|t'-t_{p,q}|\le\epsilon$, $|t''|\le T''=\epsilon$.

The key to extend constriction of the previous Subsubsection 6.3.1.4 to increased time interval is

Proposition 6.3.7. *Let a be scalar symbol and (a,b) be microhyperbolic on level $(\bar\tau,\bar\lambda)$. Then expression* (6.3.26) *is negligible as $|t'-t_{p,q}|+|t''|\asymp T$ with $\bar T:=h^{1-\delta}\le T\le\epsilon$ (this is a temporary notation).*

Proof. Consider instead

$$\Gamma\big(Q_1\psi(B)\varphi(A_0)e^{ih^{-1}t''B}e^{ih^{-1}t'A_0}e^{ih^{-1}t_{p,q}A_0}Q_2\big) \tag{6.3.27}$$

but with $|t'|+|t''|\asymp T$. Without any loss of the generality we can assume that near $\mathsf{supp}(Q_j)$ system (a,b) is ξ'-microhyperbolic on the level $(\bar\tau,\bar\lambda)$ where as before we need to consider only γ-vicinity of $\Xi_{p,r}=\{x''=\xi''=0\}$ and

$$\{a,x''\}=\{a,\xi''\}=\{b,x''\}=\{b,\xi''\}=0 \qquad \text{as } x''=\xi''=0. \tag{6.3.28}$$

Let us scale $x'\mapsto x'T^{-1}$, $x''\mapsto x''T^{-\frac12}$, leading to $\xi'\mapsto\xi'$, $\xi''\mapsto\xi''T^{-\frac12}$, $h\mapsto h_{\mathsf{new}}:=hT^{-1}$, $\gamma\mapsto\gamma_{\mathsf{new}}:=\gamma T^{-\frac12}$. Then again $\gamma\asymp h^{\frac12-\delta'}$ leads to $\gamma_{\mathsf{new}}\asymp h_{\mathsf{new}}^{\frac12-\delta''}$ but there is a new question: how scales $v(x,y)$ which is the Schwartz kernel of $e^{ih^{-1}t_{p,q}A_0}Q_2$? In the definition of $v(x,y)$ parameters h, $t_{p,q}$ are not scaled.

However $e^{ih^{-1}t_{p,q}A_0}$ is h-Fourier integral operator and with the scaling of x, y it becomes h_{new}-Fourier integral operator with

$$\mathsf{WF}(v)\subset\{|x'-y'|\lesssim(|x''|+|\xi''|)\asymp(|y''|+|\eta''|)\}.$$

Now we apply cut-off Q_2 (rescaled) and conclude that

$$\mathsf{WF}(v)\subset\{x'=y',x''=y''=\xi''=\eta''=0\}$$

and due to propagation of singularities and ξ'-microhyperbolicity we conclude that

$$\mathsf{WF}(U) \subset \{|x'-y'| \ge \epsilon, x''=y''=\xi''=\eta''=0\}$$

where $U(x,y,t',t'')$ is a Schwartz kernel of $\psi(B)\varphi(A_0)e^{ih^{-1}t''B}e^{ih^{-1}t'A_0}v$ after this rescaling. Then $\Gamma(U)\equiv 0$ as $|t'|+|t''|\asymp 1$ and (6.3.27) is negligible as $|t'|+|t''|\asymp T$ and therefore (6.3.26) is negligible as $|t'-t_{p,q}|+|t''|\asymp T$.□

Proposition 6.3.7 immediately implies

Theorem 6.3.8. *(cf. Theorem 6.2.19) Let a be scalar symbol and microhyperbolicity condition* (6.2.80) *be fulfilled. Then*

(i) Expression (6.2.15) *is negligible outside of zone where it was constructed by the method of successive approximations.*

(ii) Let us replace $\mathsf{e}(.,y,\tau)$ by $\psi(B)_x\mathsf{e}(.,y,\tau)$ with $\psi\in \mathscr{C}_0^\infty([\bar\lambda-\epsilon,\bar\lambda+\epsilon])$ with small enough ϵ; then the statement of Theorem 6.3.6 extends to L satisfying $L\ge \varepsilon h^{1-\delta}$.

Tauberian Estimates

Analysis of subsubsections *6.3.1.2 Propagation of Singularities near* Π and *6.3.1.4 Exploiting Microhyperbolicity* immediately implies that

(6.3.29) Contribution of $\Pi_{p,r}$ and including its subperiodical points to $F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma\big(Q_1\psi(B)e^{ih^{-1}tA}Q_2\big)$ does not exceed

$$Ch^{-d}\gamma^{2r}\bar T(\varepsilon^{-1}\bar T+1)\le Ch^{1-d+r-\delta'}(\varepsilon^{-1}h+1) \tag{6.3.30}$$

where factor γ^{2r} is a volume of γ-vicinity of $\Pi_{p,r}$, factor $\bar T$ comes as a length of the time interval on one period which contributes non-negligibly (recall that $|t'-t_{p,q}|\le \bar T$ then) and factor $(\varepsilon^{-1}\bar T+1)$ is a number of periods to be counted (since as $|t''|\ge \bar T$ contribution is negligible).

Probably this statement holds with $\delta'=0$ but we do not need this improvement since it is less anyway than estimate $Ch^{1-d}(\varepsilon^{-1}h+1)$ we got earlier and therefore Tauberian estimates (6.2.89) and (6.2.90) do not change.

Consider corrected Weyl expressions. One can see easily that the corrections look exactly as main oscillatory terms but with integration with respect to Λ and with an extra factor h^r where $2r=\mathsf{codim}\,\Lambda$. Therefore we arrive to two statements below:

Theorem 6.3.9 [8]**.** *In the framework of Theorem 6.2.24 with an error not exceeding* (6.2.93) *as* $\varepsilon \ge h$ *and* (6.2.94) *as* $\varepsilon \le h$ *one can replace* $\mathsf{N}^{\mathsf{T}}_{\psi}$ *or* $\mathsf{N}^{\mathsf{T}}_{\psi,\varphi,L}$ *in Theorem 6.2.21 by* $\mathsf{N}^{\mathsf{W}}_{\psi} + \mathsf{N}_{\mathsf{corr},\psi}$ *or* $\mathsf{N}^{\mathsf{W}}_{\psi,\varphi,L} + \mathsf{N}_{\mathsf{corr},\psi,\varphi,L}$ *respectively where correction term now contains also contributions of subperiodic trajectories.*

Proposition 6.3.10 [9]**.** *Contributions of* $\Pi_{p,r}$ *to the Tauberian expression* $\mathsf{N}^{\mathsf{T}}_{\psi,\varphi,L}$ *does not exceed expression* (6.2.93) *multiplied by* h^{r-1} *as* $\varepsilon \ge h$ *and expression* (6.2.94) *multiplied by* h^{r-1} *as* $\varepsilon \ge h$.

In particular, this contribution to the Tauberian expression N^{T} *does not exceed* Ch^{2-d} *and* $Ch^{2-d}\bigl(|\log \varepsilon^{-1}h| + 1\bigr)$ *as* $\varepsilon \ge h$ *and* $\varepsilon \le h$ *respectively.*

6.3.2 The case of Degenerate Scalar Operator B

Now we assume that not only a but also b is scalar symbol. In this case microhyperbolicity condition of (a, b) means exactly that

$$|\nabla_{\Sigma} b| + |b - \bar\lambda| \ge \epsilon_0 \qquad \text{on } \ \Sigma := \{a = \bar\tau\}. \tag{6.3.31}$$

Now we want to tackle degenerating symbol b. Without any loss of the generality one can assume that locally $a = \xi_1$, $\bar\tau = 0$.

Asymptotics with mMollification

Let introduce scaling function ρ by

$$\rho = \epsilon_1\bigl(|\nabla_{x',\xi'} b|^2 + |b - \bar\lambda|\bigr)^{\frac12} + \bar\rho \tag{6.3.32}$$

with

$$\bar\rho = h^{\frac12 - \delta'} \tag{6.3.33}$$

here; $(x', \xi') = (x_2, \dots, x_d; \xi_2, \dots, \xi_d)$. Then this is an admissible function on Σ. Consider $\rho = \bar\rho$ vicinity of point $\bar z$ with respect to (x', ξ') and $(\epsilon, \epsilon\rho^2)$ vicinity with respect to (x_1, ξ_1). Then in this vicinity

$$b = \bar\lambda + \rho^2 b' + \mu a + O(\rho^2), \quad b' = \rho^{-2}\bigl(b(x', 0, \xi') - \bar\lambda\bigr) \tag{6.3.34}$$

[8] Cf. Theorem 6.2.24.

[9] Cf. Theorem 6.2.23.

with , $\mu = \mathsf{const}$. Then for $Q \in \mathbf{\Psi}_{h,\boldsymbol{\rho},\boldsymbol{\gamma}}(X,\mathbb{K})$ supported in this vicinity, $\boldsymbol{\rho} = (\rho^2, \rho, \dots, \rho)$, $\boldsymbol{\gamma} = (1, \rho, \dots, \rho)$

$$e^{ih^{-1}T_0 A} Q \equiv e^{i\varepsilon h^{-1} B} Q \equiv e^{i\varepsilon h^{-1}(\bar{\lambda}+\rho^2 B'+\mu A_0)} Q \equiv e^{i\varepsilon h^{-1}(\bar{\lambda}+\rho^2 B'+\mu A_0)} Q \tag{6.3.35}$$

where B' is an operator with the principal symbol b'.

Then after rescaling $x' \mapsto x'\rho^{-1}$, $\xi' \mapsto \xi'\rho^{-1}$, $x_1 \mapsto x_1$, $\xi_1 \mapsto \xi_1\rho^{-2}$, $h \mapsto h_{\mathsf{new}} = h\rho^{-2}$, $\varepsilon \mapsto \varepsilon_{\mathsf{new}} := \varepsilon\rho^2$ we are in the framework of the previous theory (at least if there are no subperiodic trajectories). Further, as $\rho \ge c\bar{\rho}$ we are in the framework of the microhyperbolic theory and are allowed to use mollification parameter $L = \varepsilon_{\mathsf{new}} h_{\mathsf{new}}^{1-\delta'} \le \varepsilon h^{1-\delta'}$. On the other hand, as $\rho \le c\bar{\rho}$ we are allowed to use mollification parameter $L = \varepsilon_{\mathsf{new}} h_{\mathsf{new}}^{\frac{1}{2}-\delta'} \le \varepsilon h^{1-2\delta}$.

Consider subperiodic trajectories. Then also nothing changes because cases $\mathsf{dist}(z,\Pi) \ge h^{\frac{1}{2}-\delta}$ and $\mathsf{dist}(z,\Pi) \le h^{\frac{1}{2}-\delta}$ are transformed to the similar cases after rescaling.

Therefore we arrive to

Theorem 6.3.11 [10]**.** *Let a and b be scalar symbols. Let conditions* (6.2.1), (6.2.4), (6.2.7), (6.2.18) *and* (6.2.19) *be fulfilled in Ω. Further, let a be microhyperbolic in Ω on the energy level $\bar{\tau}$ and let* (6.2.55) *be fulfilled.*

Then

(i) As $L \ge \varepsilon h^{1-\delta}$ formula (6.3.24) *holds (cf. Theorem 6.2.17).*

(ii) If τ_1, τ_2 belong to spectral gaps in Ω then estimate (6.3.25) *holds (cf. Theorem 6.2.18). Here we use the same definition* (6.2.74) *of the spectral gap but with $L = h^{1-\delta}$ rather than $L = h^{\frac{1}{2}-\delta}$.*

Estimates without Mollification

Let us consider the same partition. Note that one should take $T \asymp \varepsilon^{-1}$ again (since evolution remains in zone $\{\rho \asymp \rho(\bar{z})\}$ for this time) but estimate for $M(\pi)$ defined by (6.2.82) would be different from (6.2.83):

$$|M(\phi)| \le Ch^{1-d}(1 + \varepsilon\rho^2 h^{-1}|n|)^{-l}\rho^{2d-2} \tag{6.3.36}$$

and therefore

$$|M(\bar{\chi}_T)| \le Ch^{1-d}\big(1 + \varepsilon^{-1}\rho^{-2}h\big)\rho^{2d} \tag{6.3.37}$$

[10] Cf. Theorem 6.3.6.

and therefore contribution of this element to the remainder estimate does not exceed

$$Ch^{1-d}\bigl(\varepsilon+\rho^{-2}h\bigr)\rho^{2d-2}\vartheta\bigl(\frac{\varepsilon h}{L}\bigr) \tag{6.3.38}$$

and thus summation over partition elements with $\rho \ge c\bar\rho$ results in

$$Ch^{1-d}\vartheta\bigl(\frac{\varepsilon h}{L}\bigr)\int_{\Sigma_{\bar\tau}\cap\{\rho\ge\bar\rho\}}\bigl(\varepsilon+\rho^{-2}h\bigr)\,dx'd\xi' \le Ch^{1-d}\Bigl(\varepsilon+h\int_{\Sigma_{\bar\tau}\cap\{\rho\ge\bar\rho\}}\rho^{-2}\,dx'd\xi'\Bigr)\vartheta\bigl(\frac{\varepsilon h}{L}\bigr). \tag{6.3.39}$$

where we again consider locally $a=\xi_1$.

Meanwhile contribution of the balls with $\rho\asymp\bar\rho$ does not exceed

$$Ch^{1-d}\Bigl(\int_{\Sigma_{\bar\tau}\cap\{\rho\le\bar\rho\}}dx'd\xi'\Bigr)\vartheta\bigl(\frac{\varepsilon h}{L}\bigr). \tag{6.3.40}$$

Really, we apply here the Tauberian estimates with $T\asymp\varepsilon^{-1}$ but here we have

$$M(\bar\chi_T)\le Ch^{1-d}\varepsilon^{-1}\rho^{2d-2} \tag{6.3.41}$$

since number of windings is $O(\varepsilon^{-1})$.

Combining estimates (6.3.39) and (6.3.40) we arrive to

Theorem 6.3.12 [11]**.** *Let a and b be scalar symbols. Let conditions* (6.2.1), (6.2.4), (6.2.7), (6.2.18) *and* (6.2.19) *be fulfilled in Ω. Further, let a be microhyperbolic in Ω on energy level $\bar\tau$ and let* (6.2.55) *be fulfilled.*

(i) Then

$$|\mathsf{R}^{\mathsf{T}}_{\varphi,\psi,L}|\le Ch^{1-d}\Bigl(\varepsilon+h\int_{\Sigma_{\bar\tau}\cap\{\rho\ge\bar\rho\}}\rho^{-2}\,dx'd\xi'+\int_{\Sigma_{\bar\tau}\cap\{\rho\le\bar\rho\}}dx'd\xi'\Bigr)\vartheta\bigl(\frac{\varepsilon h}{L}\bigr). \tag{6.3.42}$$

[11] Cf. Theorem 6.3.6.

(ii) In particular, if

$(6.3.43)_k$ $|a-\bar\tau|+|b-\bar\lambda|+|\nabla_{x',\xi'}b| \le \epsilon$ *implies that* $\mathsf{Hess}_{x',\xi'}\, b$ *has* k *eigenvalues disjoint from* 0

then as $k \ge 3$ *estimate*

$$|\mathsf{R}^{\mathsf{T}}_{\varphi,\psi,L}| \le Ch^{1-d}(\varepsilon + h)\vartheta\big(\varepsilon\frac{h}{L}\big) \tag{6.3.44}$$

holds as if there was no degeneration, as $k = 2$ *estimate*

$$|\mathsf{R}^{\mathsf{T}}_{\varphi,\psi,L}| \le Ch^{1-d-\delta}(\varepsilon + h)\big(\frac{\varepsilon h}{L}\big) \tag{6.3.45}$$

holds and as $k = 1$ *estimate*

$$|\mathsf{R}^{\mathsf{T}}_{\varphi,\psi,L}| \le Ch^{\frac12-d}(\varepsilon + h^{1-\delta})\big(\frac{\varepsilon h}{L}\big) \tag{6.3.46}$$

holds.

This result is not completely satisfactory as $k = 1, 2$; in this case we need to be able to take $\bar\rho = h^{\frac12}$. It is possible despite the fact that condition $\varepsilon_{\mathsf{new}} \ge h^l_{\mathsf{new}}$ breaks because we need condition $\varepsilon \ge h^l$ only because error is $O(h^r)$ in the original equations. However in fact error is this with the original h and we need only "$\varepsilon_{\mathsf{new}} \ge h^l$" rather than "$\varepsilon_{\mathsf{new}} \ge h^l_{\mathsf{new}}$". Therefore we arrive to

Theorem 6.3.13. *(i) In the framework of Theorem 6.3.12 estimate* (6.3.42) *holds with* $\bar\rho = h^{\frac12}$.

(ii) In particular, if condition

$(6.3.43)^+_2$ $|a-\bar\tau| + |b-\bar\lambda| + |\nabla_{x',\xi'}b| \le \epsilon$ *implies that* $\mathsf{Hess}_{x',\xi'}\, b$ *has* 2 *eigenvalues disjoint from* 0 *and both of the same sign*

then estimate (6.3.44) *holds as if there was no degeneration.*

(iii) Under condition $(6.3.43)_2$ *estimate*

$$|\mathsf{R}^{\mathsf{T}}_{\varphi,\psi,L}| \le Ch^{1-d-\delta}(\varepsilon + h|\log h|)\big(\frac{\varepsilon h}{L}\big) \tag{6.3.47}$$

holds.

(iv) Under condition $(6.3.43)_1$ *estimate*

$$|\mathsf{R}^{\mathsf{T}}_{\varphi,\psi,L}| \le Ch^{\frac{1}{2}-d}(\varepsilon + h^{\frac{1}{2}})(\frac{\varepsilon h}{L}) \tag{6.3.48}$$

holds.

6.3.3 Some Remarks

Remark 6.3.14. (i) The following question arises naturally: can we improve estimate (6.3.44), surely including if necessary the next terms into the left-hand expression. The answer will be affirmative if we manage to increase T. It is not a difficult task and the additional arguments are completely in terms of the Tauberian method provided the following conditions are fulfilled:

$$\Psi_{b,t''}(\Omega') \subset \Omega \qquad \forall t'' \in [-T'', T''], \tag{6.3.49}$$

$$|D_{x,\xi}\Psi_{b,t''}| \le h^{-\delta'} \qquad \forall (x,\xi) \in \Omega' \quad \forall t'' \in [-T'', T''] \tag{6.3.50}$$

and

$$\mathsf{dist}(\Psi_{a,t} \circ \Psi_{b,t'}(x,\xi), (x,\xi) \ge \epsilon_0 \gamma |t'| t(1-t) \tag{6.3.51}$$
$$\forall (x,\xi) \in \mathcal{U} \quad \forall t \in (0,1) \quad \forall t'' \in [-T'', T'']$$

where Ω' contains the ϵ_1-neighborhood of Ω_0, $\Psi_{a,t}(\Omega_0) = \Omega_0$ and $1 \le T'' \le h^{-\delta'}$, $\delta' > 0$ is a small enough exponent. Then in the corresponding estimates one can replace T by $\varepsilon^{-1}T''$ and insert factor T''^{-1} in the right hand expression of (6.3.44) In this case $d \ge 3$ automatically.

(ii) In particular if one can take T'' as an arbitrarily large constant, then one can insert factor $o(1)$.

(iii) On the other hand, one can investigate the case when the flow $\Psi_{b,t''}$ on $\tilde{\Omega} = \Omega/\sim$ is T_1-periodic where $\sim$ is the equivalence relation given by $z \sim z'$ if $\Psi_{a,t}(z) = z'$ for some t, and if $\tilde{\Omega}$ contains the ϵ-neighborhood of $\tilde{\Omega}_0 = \Omega_0/\sim$; in this case the arguments of this section work; moreover this chain could be prolonged but these conditions seem rather insane (but possible to fulfill).

Let $\varepsilon \ll h$ and $\varepsilon' \ll h$ where now $B = B_0 + \varepsilon' B'$ and quantum evolution for B_0 is periodic i.e. $e^{ih^{-1}T_1 B_0} \equiv I$. Then spectral bands of the width $\asymp \varepsilon$

separated by spectral gaps of the width h in turn consist of spectral subbands of the width $\asymp \varepsilon\varepsilon'$ separated by spectral gaps of the width εh (we assume that perturbations are generic).

Remark 6.3.15. (i) Let us recall that in the case of one operator formally[12)]

$$\partial_\tau E(\tau) \approx (2\pi h)^{-d} \operatorname{Res}_{\mathbb{R}} (\tau - A)^{-1} \tag{6.3.52}$$

Similarly in the case of I commuting operators

$$\partial_{\tau_1}\cdots\partial_{\tau_I} E_1(\tau_1)\cdots E_I(\tau_I) \approx (2\pi h)^{-d} \operatorname{Res}_{\mathbb{R}} (\tau_1 - A_1)^{-1}\cdots \operatorname{Res}_{\mathbb{R}} (\tau_I - A_I)^{-1} \tag{6.3.53}$$

we have now

$$\partial_\tau E(\tau) \approx (2\pi h)^{-d} \sum_{p\in\mathbb{Z}} \operatorname{Res}_{\mathbb{R}} (\tau' - A_0)^{-1}\big|_{\tau'=2\pi ph T_0^{-1}} \theta(\lambda - B)\big|_{\lambda=\varepsilon^{-1}(\tau-2\pi ph T_0^{-1})}. \tag{6.3.54}$$

In particular, as τ_1 and τ_2 are two gap values we have

$$Q_1 E(\tau) Q_2 \equiv (2\pi h)^{-d} \sum_{p\in\mathbb{Z}} Q_1 \theta(\tau' - A_0)\big|_{\tau'=2\pi ph} \theta(\lambda - B)\big|_{\lambda=\varepsilon^{-1}(\tau-2\pi ph)} Q_2 \Big|_{\tau=\tau_1}^{\tau=\tau_2} \tag{6.3.55}$$

with here $\theta(\tau - A)$ etc is understood as $\int^\tau \operatorname{Res}_{\mathbb{R}} (\tau - A)^{-1}\, d\tau$ with $(z - A)^{-1}$ constructed as formal pseudodifferential operators.

(ii) We know that as $\varepsilon \ll h$ there are spectral gaps of the width $\asymp h$; however as b is matrix-valued there could be spectral gaps of the width $\asymp \varepsilon$. Let us ignore them. Then as $\tau_1 < \tau_2$ are adjacent gap values; define $p \in \mathbb{Z}$ by

$$\varepsilon \operatorname{Spec}(b) \subset (\tau_2 - \frac{2\pi(p+1)h}{T_0}, \tau_2 - \frac{2\pi ph}{T_0}) \tag{6.3.56}$$

then

$$Q_1 E(\tau) Q_2 \equiv (2\pi h)^{-d} Q_1 \theta(\tau' - A_0)^{-1}\big|_{\tau'=\frac{2\pi ph}{T_0}} \tag{6.3.57}$$

expression which does not depend on ε (as long as ε is small enough not to destroy spectral gaps).

[12)] So, it makes sense after substitution in the various "taming" expressions.

6.3.4 Spectral Gaps Reloaded

Let us try a bit different approach. Assume that a is a scalar symbol and consider operator

$$G \equiv e^{ih^{-1}T_0A} \tag{6.3.58}$$

This is h-pseudodifferential operator. Without any loss of the generality one can assume that

$$G = \mathsf{const}\cdot I + \eta G', \quad G' \in \boldsymbol{\Psi}_{h,K}(\mathbb{R}^d, \mathbb{H}), \quad \eta \in [h^r, h]; \tag{6.3.59}$$

for $r = 1$ this condition can be checked effectively because we know G modulo $h\boldsymbol{\Psi}$.

Let us consider the self-adjoint operators $G_1 = \frac{1}{2}(G + G^*)$ and $G_2 = -\frac{1}{2}i(G - G^*)$. Then (6.3.58) remains true for $\exp(ih^{-1}T_0A)$ and G replaced by either $\cos(ih^{-1}T_0A)$ and G_1 or $\sin(ih^{-1}T_0A)$ and G_2 respectively.

Proposition 6.3.16. *(i) If for $j = 1$ or $j = 2$*

$$|G_j(x,\xi,h) - \mu_j| \ge \zeta \qquad \forall (x,\xi) \in \Omega, \tag{6.3.60}$$

where $\zeta \in [C_0h\eta, 1]$ and $G_j(x,\xi,h)$ and $G(x,\xi,h)$ are Weyl symbols of G_j G respectively, then

$$E_j'(\mu_j - \frac{1}{2}\eta\zeta, \mu_j + \frac{1}{2}\eta\zeta) \equiv 0 \qquad \textit{in } \Omega \tag{6.3.61}$$

where E_1' and E_2' are the spectral projectors of $\cos(h^{-1}A)$ and $\sin(h^{-1}A)$ respectively.

(ii) Let

$$|G(x,\xi,h) - \exp(ih^{-1}T_0\tau)| \ge \eta\zeta \qquad \forall (x,\xi) \in \Omega \quad \forall \tau \in [\tau_1, \tau_2] \tag{6.3.62}$$

where $\zeta \in [C_0h\eta, 1]$. Then

$$E(\tau_1, \tau_2) \equiv 0 \qquad \textit{in } \Omega. \tag{6.3.63}$$

(iii) For scalar G one can assume that $\zeta \in [C_0h^2\eta, 1]$.

Proof. (i) An easy proof is based on sharp Gårding inequality combined with the simplified Tauberian arguments.

(ii) Taking into account the link between these spectral projectors and the spectral projector E of A we obtain that

$$E(\tau - \frac{1}{2}h\zeta, \tau + \frac{1}{2}h\zeta) \equiv 0 \quad \text{in} \quad \Omega$$

provided $\cos(h^{-1}T_0\tau) = \mu_1$ or $\sin(h^{-1}T_0\tau) = \mu_2$.

(iii) To prove Statement (iii) one should apply Melin inequality instead of sharp Gårding inequality.

Easy details are left to the reader. □

We call intervals $[\tau_1, \tau_2]$ satisfying (6.3.62) *(semiclassical) spectral gaps*. We already obtained complete asymptotics in the case when the spectral parameters belong to different spectral gaps. However now we just defined border of such gaps with a better precision.

6.3.5 Families of Commuting Operators Approach

Operators with Completely Periodic Evolution

Now let us consider a bit different approach. However, the first step of the new approach is based on the previous ideas. Let us consider in this subsubsection operator $A = A_0$ satisfying (6.2.28) (i.e. $e^{ih^{-1}T_0A_0} \equiv I$).

Proposition 6.3.16 immediately yields that the spectral zones in Ω lie in very narrow vicinities of the points $2\pi mhT_0^{-1}$ with $m \in \mathbb{Z}$.

Let $\bar\tau = 2\pi mhT_0^{-1}$; further, let $\tau_1 \in (2\pi(m-1)\pi hT_0^{-1}, 2\pi mhT_0^{-1})$ and $\tau_2 \in (2\pi mhT_0^{-1}, 2\pi(m+1)hT_0^{-1})$ be lacunar values, and let $\phi \in \mathscr{C}_0^K([-1,1])$ equal 1 on $[-\frac{1}{3}, \frac{1}{3}]$. Then

$$QE(\tau_1, \tau_2) \equiv \int \varphi\bigl(\frac{\tau - \bar\tau}{L}\bigr) Q\, d_\tau E(*, \tau) \equiv \varphi\bigl(\frac{1}{L}(hD_t - \bar\tau)\bigr)\bar\chi_T QU|_{t=0} \tag{6.3.64}$$

for $L = \frac{1}{2}h$, φ and $\bar\chi$ the same type of function as ϕ, and $LT \ge h^{1-\delta}$.

One can replace $\bar\chi_T(t)$ here by $\sum_{|n|\le T} \bar\chi(t - nT_0)$ for $\bar\chi \in \mathscr{C}_0^K([-\frac{2}{3}, \frac{2}{3}])$ such that the last sum with n running over $\mathbb{Z}$ equals 1. Then applying (6.2.28) we obtain that the right-hand expression of (6.2.52) is equal to

$$\int_{\mathbb{R}} \varphi\bigl(\frac{\tau - \bar\tau}{L}\bigr) \sum_{|n|\le T} e^{ih^{-1}T_0n\tau} F_{t\to h^{-1}\tau}\bar\chi(t) QU\, d\tau \tag{6.3.65}$$

provided $T \le h^{-l}$ with arbitrarily large l.

Recall that QU is a Fourier integral operator. Let us write its Schwartz kernel in the form of a Lagrangian distribution of the class $I_{h,M}$ [13] where temporarily

$$M := \{(x,\xi,y,\eta,t,\tau) : (x,\xi) = \Psi_{a,t}(y,-\eta), \tau = a(x,\xi)\}.$$

Then one can easily see that the Schwartz kernel of $F_{t\to h^{-1}\tau}\bar{\chi}(t)QU$ is a Lagrangian distribution of the class $I_{h,\Lambda}$ with the Lagrangian manifold

$$\Lambda_\tau = \{(x,\xi;y,\eta),\ a(x,\xi)=\tau,\ \exists t : (y,-\eta) = \Psi_{a,t}(x,\xi)\} \tag{6.3.66}$$

depending smoothly on $h^{-1}\tau$ provided there are no subperiodic trajectories[14].

Then one can take the sum with respect to n running over $\mathbb{Z}$ in (6.3.65); the error will be negligible. Recall that the sum of the exponents then equals $\sum_m \delta(\tau - 2\pi m h T_0^{-1}) h T_0^{-1}$.

Moreover, one can take here only $m = 0$ (other terms vanish because of the choice of φ and L). Then the right-hand expression of (6.3.65) is a Lagrangian distribution of the class hI_{h,Λ_τ} depending smoothly on τ with Lagrangian manifold Λ_τ [15].

So, the following statement is proven:

Theorem 6.3.17. *Let A be an operator with a scalar symbol a and let conditions* (6.2.1), (6.2.2), (6.2.5), (6.2.7) *and* (6.2.28) *be fulfilled. Then*

(i) Let there be no subperiodic trajectories. Then for all $\tau = 2\pi m h T_0^{-1}$ with $|\tau| \le \epsilon$ the Schwartz kernel of $E_A(\tau-\zeta;\tau+\zeta)$ is a Lagrangian distribution with Lagrangian manifold Λ_τ; $\zeta = C_0 h^s$ here [16].

(ii) In the general case Λ_τ is (locally) the union of a finite number of Lagrangian manifolds and the distribution in question is microlocally the sum of the corresponding Lagrangian distributions.

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[13] Recall that this means that it is defined by $h^{-m}\int b(x,y,t,\theta)e^{ih^{-1}\varphi(x,y,t,\theta)}\,d\theta$ with m-dimensional variable θ and phase function φ satisfying standard conditions.

[14] If there are subperiodic trajectories then Λ_τ is a locally finite union of Lagrangian manifolds and we obtain a sum of the corresponding Lagrangian distributions.

[15] By definition $\tau = \bar{\tau}$ which runs over a lattice; nevertheless we can obviously extend the phase functions and amplitudes to the interval.

[16] However, $E(\tau-\zeta;\tau+\zeta)$ surely is not a Fourier integral operator because $\dim \operatorname{Ker} d\pi_X = \dim \operatorname{Ker} d\pi_Y = 1 (\ne 0)$ for projectors $\pi_X : \Lambda \to T^*X$, $\pi_Y : \Lambda \to T^*Y$, $Y = X$ here.

Now let us consider $\mathsf{Tr}\, QE(\tau_1,\tau_2)$. The same techniques are applicable (it seems to be more convenient to go from $F_{t\to h^{-1}\tau}(\chi_T(t)QU)$ than from QE).

First of all, let us assume that there are no subperiodic trajectories. Then one can take $\bar\chi$ supported in a small vicinity of 0. Then in virtue of the calculations of Subsection 4.3.3 we immediately obtain that modulo $O(h^s)$ the answer is

$$\sum_{m\ge 0} h^{-d+m+1}\, T_0^{-1} \iiint \mathcal{F}_m(y,\xi,\tau)\, d\mu_\tau \tag{6.3.67}$$

for $\tau=\bar\tau$ where $d\mu_\tau$ is a density on Σ_τ.

On the other hand, let us assume now that there are subperiodic trajectories with the (minimal) subperiod $T=p^{-1}$, $p=2,3,...$ (then one should also take into account subperiods np^{-1}, $n\in\mathbb{Z}$).

Therefore in order to take into account subperiodic trajectories one should in addition treat

$$\int \varphi\big(\frac{\tau-\bar\tau}{L}\big) F_{t\to h^{-1}\tau}\chi_T(t-T')\,\mathsf{Tr}\big(QU(t)\big)\, d\tau \tag{6.3.68}$$

with $T'=(k+qp^{-1})T_0$, $k\in\mathbb{Z}$, $|k|\le N$ and coprimes p and q, $1\le q\le p-1$. The easy calculations of oscillatory integrals give us an answer of the form

$$\int \varphi\big(\frac{\tau-\bar\tau}{L}\big) h^{-d+r} \int_{\Pi_{p,r}\cap\Sigma_\tau} F_{p,q}(x,\xi,\tau,h) e^{ih^{-1}T'\tau}\, d\mu_{p,r,\tau} d\tau \tag{6.3.69}$$

with smooth functions F_* where $d\mu=d\mu_\tau$ is a smooth density on $\Pi_{p,r}\cap\Sigma_\tau$.

One can see easily that the error will be negligible if we sum over $k\in\mathbb{Z}$ instead of $|k|\le N$ with $N=h^{-K}$ with large enough K. After summation with respect to $k\in\mathbb{Z}$ we get a sum of $hT_0^{-1}\delta(\tau-2\pi pkhT_0^{-1})$ and only $k=m$ gives a contribution to the final answer which at this step is equal to

$$h^{1-d+r} \int_{\Pi_{p,r}\cap\Sigma_\tau} F_{p,q}(x,\xi,\tau,h) e^{2\pi i h^{-1}T_0 qp^{-1}}\, d\mu_{p,r,\tau}. \tag{6.3.70}$$

So, the following statement has been proven:

Theorem 6.3.18. *Let A be an operator with a scalar symbol a and let conditions* (6.2.1), (6.2.2), (6.2.5), (6.2.7) *and* (6.2.28) *be fulfilled.*

Then $\mathsf{Tr}\big(QE(2\pi mh-\zeta, 2\pi mh+\zeta)\big)$ *is the sum of the terms* (6.3.67) *and* (6.3.70) *where* (p,r) *runs over the set of the admissible indices and* q *runs over* $(0),\dots,p-1$ [17].

Operators with the Global dDecomposition

In this subsubsection we treat operators of the form

(6.3.71) $A = A_0 + \varepsilon B$ where $e^{ih^{-1}A_0} = I$ and $B \in \mathbf{\Psi}_h$ and A_0 commute.

Then (6.2.18) holds with the same operator B. Later we will show that the general case can be reduced to this one. Therefore

$$E_A\big(\tau',\tau''\big) = \sum_m \boldsymbol{\pi}_n E_B\big(\varepsilon^{-1}(\tau' - 2\pi m\frac{h}{T_0}), \varepsilon^{-1}(\tau'' - 2\pi m\frac{h}{T_0})\big) \tag{6.3.72}$$

where

$$\boldsymbol{\pi}_m = E_{A_0}(2\pi m\frac{h}{T_0} - 0, 2\pi m\frac{h}{T_0} + 0) \tag{6.3.73}$$

is the spectral projector of A_0 to the eigenspace corresponding to eigenvalue $2\pi mhT_0^{-1}$.

We are interested in the spectrum of A close to $\bar\tau = 2\pi nhT_0^{-1}$. Note that for $\varepsilon \le \epsilon_2 h$ only the term with $m = n$ is essential in the right-hand expression of (6.3.72). Without any loss of the generality one can assume that microlocally $a = \xi_1$.

Let us consider a $(\rho,\rho,\varepsilon,\rho^2)$-admissible partition of unity with respect to $(x',\xi';x_1,\xi_1)$ with $\rho = \rho(x',\xi')$ given by (6.3.32) with $\bar\rho = h^{\frac12}$. For $\rho^2\varepsilon \le \epsilon_2 h$ only the term with $m = n$ is essential in the right-hand expression of (6.3.71) after application of the operator Q with the corresponding support. We treat only this case below. One can treat the opposite case by the same method but the results of the previous subsections are sufficient for our needs in this case.

Let us consider the two-parameter group $\boldsymbol{U}(t',t'') = e^{ih^{-1}(t'A+t''B)}$. This group is 1-periodic with respect to t'. The following statement is due to the results of Section 6.1:

---

[17] One can easily see that the character of the answer does not change if we include $q = 0$ and (or) q with reducible ratio qp^{-1}.

Proposition 6.3.19. *In the previous assumptions for small enough positive constants T', T''*

$$F_{(t',t'')\to h^{-1}(\tau',\tau'')}\bigl(\bar\chi_{T'}(t')\bar\phi_{T''}(t'')\,\mathsf{Tr}\,QU\bigr) \tag{6.3.74}$$

is given modulo $O(\hbar^s)$ by the successive approximations method where $\hbar = \rho^{-2}h$ and the absolute value of this expression does not exceed Ch^{-d}.

Then the same answer with an additional factor $e^{inh^{-1}T_0\tau}$ holds for this expression with $\bar\chi_T(t-nT_0)$ instead of $\bar\chi$. Therefore in absence of subperiodic trajectories

$$\int \varphi\bigl(\frac{\tau-\bar\tau}{L}\bigr)F_{(t,t')\to h^{-1}(\tau,\tau')}\bigl(\bar\chi_T(t)\,\mathsf{Tr}(QU)\bigr)\,d\tau \tag{6.3.75}$$

is given modulo $O(\hbar^s)$ by the successive approximations method and further calculations: it is equal to

$$\int \mathsf{Res}_{\mathbb{R}^2}\,\mathsf{Tr}\bigl(Q(\tau-A)^{-1}(\tau'-B)^{-1}\bigr)\varphi\bigl(\frac{\tau-\bar\tau}{L}\bigr)\,d\Upsilon(h^{-1}\tau) \tag{6.3.76}$$

provided $TL \ge h^{1-\delta}$; here $\mathsf{Res}_{\mathbb{R}^2}$ is taken with respect to (τ,τ').

For $L \le h$ the absolute value of the last expression does not exceed Ch^{2-d}. Applying the Tauberian method we obtain then that for a function φ satisfying (4.2.73)–(4.2.75)

$$\int \varphi\bigl(\frac{\tau'}{L}\bigr)\,d_{\tau'}QE_B(*,\tau')\boldsymbol{\pi}_n \equiv \int \varphi\bigl(\frac{\tau'}{L}\bigr)\,\mathsf{Res}_{\mathbb{R}^2}\,\mathsf{Tr}\bigl(Q(\bar\tau-A)^{-1}(\tau'-B)^{-1}\bigr)\,d\tau'$$

modulo $O(\hbar^{2-d}\vartheta(\varepsilon h/L)$.

This immediately yields (after summation with respect to the partition) all statements of Theorem 6.3.12 (under assumptions made).

Now let us consider the contribution of the subperiodic trajectories. Direct calculation based on Proposition 6.3.1 yields that the contribution of such trajectories to $F_{t\to h^{-1}\tau}\chi_T(t)QU$ does not exceed $Ch^{2-d}T$ (which does not exceed the standard estimate of this Fourier transform) and therefore their contribution to the remainder does not exceed $Ch^{2-d}\vartheta(\varepsilon h/L)$ which does not exceed the final remainder estimate. Furthermore, their contributions to the principal parts are also known and we again recover all statements of Theorem 6.3.12 (under assumptions made).

General oOperators

Let us consider the general scalar operator A satisfying conditions (6.2.1), (6.2.2), (6.2.5) and (6.2.7). Then condition (6.2.18) is also fulfilled and without any loss of generality one can assume that B is a self-adjoint operator. Let $A_0 = A - \varepsilon B$. Then

$$e^{ih^{-1}T_0A_0} \equiv I, \qquad [A_0, B] \equiv 0 \qquad \text{in } \Omega_1 \tag{6.3.77}$$

where Ω_1 is the intermediate domain between Ω and Ω_0. Then for the operator A_0 the results of Subsubsection *6.2.3.3 Spectral Gaps (the First Attempt)* hold. Let us set

$$\bar{A}_0 = \sum_n \tau_n E(\tau_n - \zeta, \tau_n + \zeta) \tag{6.3.78}$$

with $\epsilon h \ge \zeta \ge h^l$ and $\tau_n = 2\pi nh T_0^{-1}$, thus replacing eigenvalues of A_0 close to τ_n by τ_n. Then

$$\bar{A}_0 \equiv A_0 \qquad \text{in } \Omega_1, \qquad e^{ih^{-1}T_0\bar{A}_0} = I. \tag{6.3.79}$$

Furthermore, let us set

$$\bar{B} = \frac{1}{T_0}\int_0^{T_0} e^{-ih^{-1}t\bar{A}_0} B e^{ih^{-1}t\bar{A}_0}\, dt. \tag{6.3.80}$$

Then

$$\bar{B} \equiv B \qquad \text{in } \Omega_1, \qquad [\bar{A}_0, \bar{B}] = 0 \tag{6.3.81}$$

and all the operators are self-adjoint.

Then the operator $\bar{A} = \bar{A}_0 + \bar{B}$ satisfies the assumptions of the previous subsubsection and we can apply those results to it.

On the other hand, $A \equiv \bar{A}$ in Ω_1. Then propagation of singularities results of Subsection 6.2.3 yield that $e^{ith^{-1}tA}Q \equiv e^{ith^{-1}t\bar{A}}Q$ for $|t| \le T = \epsilon\varepsilon^{-1}$ and therefore

$$\int \hat{\chi}\big(\frac{(\tau-\lambda)T}{h}\big)\, d_\lambda \nu(\tau') \equiv \int \hat{\chi}\big(\frac{(\tau-\tau')T}{h}\big)\, d_{\tau'}\bar{\nu}(\tau') \tag{6.3.82}$$

with $\nu(\tau) = \mathsf{Tr}\big(QE_A(-\infty,\tau)\big)$ and $\bar{\nu}(\tau) = \mathsf{Tr}\big(QE_{\bar{A}}(-\infty,\tau)\big)$. Picking up a Hörmander function for χ we obtain by the Tauberian method that

$$|\nu(\tau + \frac{h}{T}) - \nu(\tau)| \le C \sup_\tau |\bar{\nu}(\tau + \frac{h}{T}) - \bar{\nu}(\tau)| + Ch^s \tag{6.3.83}$$

and one can easily estimate the right-hand expression.

Then applying Tauberian arguments in the full one can estimate

$$|\int \varphi\big(\frac{\tau-\bar{\tau}}{L}\big)\, d_\tau\big(\nu(\tau)-\bar{\tau}(\tau)\big)|$$

and recover Theorem 6.3.12 in the current assumptions.

6.3.6 Strong Perturbations

Consider now the case of

$$\varepsilon \in [h^\delta, 1] \tag{6.3.84}$$

with arbitrarily small exponent $\delta > 0$. For a sake of simplicity we assume that $T_0 = 1$; it will not enter into our final results anyway.

We assume that A_0 and B are scalar operators. The problem here is that our construction of $\bar{B}$ such that

$$e^{ih^{-1}(A_0+\varepsilon B)} \equiv e^{ih^{-1}\varepsilon \bar{B}}, \qquad [A_0, \bar{B}] \equiv 0$$

leads to these equalities modulo $O(\varepsilon^K)$ with arbitrarily large K; however in view of (6.3.84) this is not negligible. Thus arguments of the current section fail. Instead we apply results of Section 4.5 and here (6.3.84) is our ally rather than foe. Still we can assume without any loss of the generality that above relations are *almost fulfilled for* b *in its classical variant*[18)], namely

$$\exp(\nabla^{\#}(a+\varepsilon b)) \equiv \exp(\varepsilon \nabla^{\#} b), \qquad \{a, b\} \equiv 0 \qquad \mathsf{mod}\ O(\varepsilon^{K+1}) \tag{6.3.85}$$

with large $K = K(I, d)$ where I is defined below. Here the first identity obviously follows from the second one and from $\exp(\nabla^{\#} a) = I$.

Therefore we can follow classical dynamics until $T \asymp \varepsilon^{-1}$. Indeed,

$$\exp(t\nabla^{\#}(a+\varepsilon b)) \equiv \exp(t''\nabla^{\#} b)\cdot \exp(t', \nabla^{\#}(a+\varepsilon b)) \qquad \text{with}\ \ t' = t-n,\ t'' = \varepsilon n,\ |n-t| \le c. \tag{6.3.86}$$

Now let us consider $\mathsf{dist}\big(z, \exp(t\nabla^{\#}(a+\varepsilon b))z\big)$. First of all, one can see easily that

[18)] I.e. for Hamiltonian fields.

(6.3.87) If $\epsilon_1 \le |t| \le T = \epsilon_0\varepsilon^{-1}$ and $\rho := |\nabla_{\Sigma} b(z)| \ge C_0\varepsilon$ with small enough constants ϵ_0 and C_0^{-1} then

$$\operatorname{dist}\bigl(z, \exp(t\nabla^{\#}(a+\varepsilon b))z\bigr) \ge \epsilon_2\varepsilon\rho.$$

We recall that locally $a = \xi_1$ and then $b \equiv b(x', \xi) \mod O(\varepsilon^K)$; then we consider derivatives of b with respect to (x', ξ').

Then due to results of Section 4.5 we arrive to the following

Proposition 6.3.20. *(i) Contribution of the zone* $\{|\nabla_{\Sigma} b| \ge C_0\varepsilon\}$ *to* $\mathsf{R}^{\mathsf{W}}_{\varphi,L}$ *does not exceed* $C\varepsilon h^{1-d}\vartheta(\varepsilon h/L)$*; in particular, its contribution to* R^{W} *does not exceed* $C\varepsilon h^{1-d}$*.*

(ii) Contribution of the zone $\{|\nabla_{\Sigma} b| \le C_0\varepsilon\}$ *to* $\mathsf{R}^{\mathsf{W}}_{\varphi,L}$ *does not exceed*

$$C\varepsilon h^{1-d}\vartheta\bigl(\frac{h}{L}\bigr)\mu\bigl(\{|\nabla_{\Sigma} b| \le C_0\varepsilon\}\bigr); \tag{6.3.88}$$

in particular, its contribution to R^{W} *does not exceed* $C\varepsilon h^{1-d}\mu\bigl(\{|\nabla_{\Sigma} b| \le C_0\varepsilon\}\bigr)$*.*

(iii) In particular, under condition $(6.3.43)_1$ $\mathsf{R}^{\mathsf{W}} \le C\varepsilon h^{1-d}$*.*

Problem 6.3.21. Prove that (6.3.88) holds with $\vartheta(\varepsilon h/L)$ rather than $\vartheta(h/L)$.

Instead we consider R^{W} and weaken condition $(6.3.43)_1$.

Theorem 6.3.22. *Let* A, B *be scalar self-adjoint operators,* a *microhyperbolic and* $\exp(\nabla^{\#}) = I$*. Assume that conditions* (6.3.84) *and* (6.3.85) *are fulfilled and*

$$\sum_{\alpha:|\alpha|\le I} |\nabla_{\Sigma}^{\alpha} b| \ge \epsilon_0. \tag{6.3.89}_I$$

Then $\mathsf{R}^{\mathsf{W}} \le C\varepsilon h^{1-d}$ *provided* $\delta = \delta(I, d)$*.*

Proof. We leave to the reader a standard proof based on induction with respect to I and scaling $(x_1, x'; \xi_1, \xi') \mapsto (x_1, \rho^{-1}x'; \rho^{-1}\xi_1, \rho^{-2}\xi')$, $h \mapsto \rho^{-2}h$ with the scaling function

$$\rho_I := \Bigl(\sum_{\alpha:|\alpha|\le I-1} |\nabla_{\Sigma}^{\alpha} b|^{\frac{I}{(I-|\alpha)}}\Bigr)^{\frac{1}{I}} + \bar{\rho} \tag{6.3.90}_I$$

with $\bar{\rho} = h^{\delta'}$. □

6.3.7 Operators with Massive set of Periodic tTrajectories

Let us reproduce some results of Yu. Safarov (the monograph Yu. Safarov & D. Vassiliev [2] and series of articles) concerning the case when Hamiltonian flow is not periodic but $\mu_\tau(\Pi) > 0$ where Π is the set of periodic trajectories. Here we cannot expect any remainder estimate better than $o(h^{1-d})$. So, let A be an operator with a scalar principal symbol a microhyperbolic at energy level τ. For the simplicity of statements (but not arguments) we assume that symbols are infinitely smooth.

Let $z \in \Sigma = \Sigma_\tau$ be periodic point with the minimum period $T(z)$. Consider some surface $\Theta \ni z$, $\Sigma \supset \Theta$, transversal to the Hamiltonian flow at z, and measure μ'_τ generated by μ_τ on Θ. Consider map $\Phi : \Theta \to \Theta$, mapping point $\Theta \ni w \to \Psi_{t(w)}(w)$ where $t(w)$ is a time when Ψ_t hits Θ the next time.

Definition 6.3.23. Let locally $a = \xi_1$, $\theta = \{x_1 = 0\}$ and $x' = (x_2, \dots, x_d)$, $\xi' = (\xi_2, \dots, \xi_d)$.

Periodic point z is called *absolutely periodic* if $\nabla_{(x'',\xi'')}\big(\Phi - I\big)(x,\xi) = 0$ $\forall \alpha$ for some $t \neq 0$.

Obviously,

(6.3.91) Definition of the absolute periodicity does not depend on the choice of $\Theta \ni z$.

Let Π_T be a set of all periodic points with periods not exceeding T. Let Π^∞ be the set of all absolutely periodic points and Π^∞_T be the set of all absolutely periodic points with periods not exceeding T.

Obviously

(6.3.92) Π_T and Π^∞_T are closed sets and $\Pi = \bigcup_T \Pi_T$, $\Pi^\infty = \bigcup_T \Pi^\infty_T$;

(6.3.93) Π, Π^∞, Π_T and Π^∞_T are $\exp\big(t\nabla^\# a\big)$-invariant;

(6.3.94) If $z \in \Sigma_\tau$ but $z \notin \Pi^\infty$ then for each $T > 0$ there exists $\varepsilon > 0$ such that $\mu_\tau\big(\Pi_T \cap B(z,\varepsilon)\big) = 0$.

Also obviously we arrive to

Proposition 6.3.24. *(i) Let $z \in \Pi_T^\infty \cap \Sigma_{\bar\tau}$ with $\bar\tau = 0$ and the minimal period in z be exactly $T_0 := T_0(z) \le T$. Let $\gamma = h^\delta$ with arbitrarily small $\delta > 0$. Let us extend $t(w)$ from $\Theta \ni z$ to vicinity of z so that it remains constant along Ψ_t* [19] *Let $t^{\mathsf w}$ denote operator with t as a principal symbol, we can construct it locally so that $[A, t^{\mathsf w}] = 0$.*

Then in $B(z,\gamma)$

$$e^{ih^{-1}t^{\mathsf w}A} \equiv G + \mathcal{F}(A - \bar\tau) \tag{6.3.95}$$

with $G \in \mathbf{\Psi}_{h,\gamma,\gamma,\infty}(X, \mathbb{K})$ such that $G^*G \equiv GG^* \equiv I$ and 0-th order h-Fourier integral operator $\mathcal{F}$.

(ii) Principal symbol of G is uniquely defined at z.

(iii) In particular, if a^s is scalar then the principal symbol of G (at z) is given by (6.2.12).

Then modulo $o_T(h^{1-d})$

$$F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\big(Q_{1,x}u\,{}^tQ_{2y}\big) \equiv \tag{6.3.96}$$
$$(2\pi h)^{-d}\int T\hat\chi\big((\tau-\tau')Th^{-1}\big)\sum_n q_1(x,\xi)G(x,\xi)^n e^{ih^{-1}nT_0(x,\xi)\tau'}q_2(x,\xi)\,d\mu_{\tau'}$$
$$\text{as}\quad |\tau-\bar\tau| = o(1)$$

where sum is actually taken over $n : |n| \le TT_0(x,\xi)^{-1}$ and we use the same symbol G for operator and its principal symbol. Here we do not need to assume that $\bar\tau = 0$ anymore: we can get rid of this assumption in the standard way. We used here that

$$\liminf_{z'\to z} T_0(z') = T_0(z). \tag{6.3.97}$$

Further,

(6.3.98) With the same error one can replace $\mu_{\tau'}$ by μ_τ in (6.3.96).

Then obviously

[19] We cannot extend it globaly this way unless all trajectories are periodic.

(6.3.99) The absolute value of the right-hand expression of (6.3.96) does not exceed

$$\Big(C_\varepsilon + CTJ(T',\varepsilon) + CTT'^{-1} + o_{T,T',\varepsilon}(1)\Big)h^{1-d} \tag{6.3.100}$$

with

$$J(T',\varepsilon) \coloneqq \int_{\{\mathsf{dist}(\mathsf{Spec}(G(z)e^{ih^{-1}T_0(z)\tau}),1)<\varepsilon,\ T_0(z)\le T'\}} T_0^{-1}\,d\mu_\tau, \tag{6.3.101}$$

arbitrarily small $\varepsilon>0$, arbitrarily large $T'\le T$ and $o_{T,T',\varepsilon}(1)$ with respect to $h\to 0$ but not uniform with respect to T, T', ε.

Multiplying by $h^{d-1}T^{-1}$ and tending $h\to +0$ and *only then* $T\to\infty$ we get $CJ(T',\varepsilon)+CT'^{-1}$. After this tending $\varepsilon\to +0$ and *only then* $T'\to\infty$ we get $C\mu_\tau\big(\Pi^\infty\cap\Sigma_\tau\cap\{1\in\mathsf{Spec}(G(z)e^{ih^{-1}T_0(z)\tau})\}\big)$.

Applying Tauberian theorem we arrive to

$$\mathsf{R}^T_{\varphi,L;\tau} = o\big(h^{1-d}\vartheta(hL^{-1})\big) \tag{6.3.102}$$

provided

$$\mu_\tau\Big(\Pi^\infty\cap\Sigma_\tau\cap\big\{e^{-ih^{-1}T_0(z)\tau}\in\mathsf{Spec}\big(G(z)\big)\big\}\Big)=0; \tag{6.3.103}$$

Denoting by $e^{i\lambda_j(z)}$ eigenvalues of $G(z)$ one can rewrite this condition as

$$\mu_\tau\Big(\Pi^\infty\cap\Sigma_\tau\cap\big\{\tau\in -\frac{h}{T_0(z)}\lambda_j(z)+\frac{2\pi h}{T_0(z)}\mathbb{Z}\big\}\Big)=0\qquad \forall j. \tag{6.3.104}$$

In the condition (6.3.104) choice of τ is important. In the Tauberian expression we need to take $T\to\infty$ slowly.

Theorem 6.3.25. *Let $A\in\boldsymbol{\Psi}_h$ be a self-adjoint operator with a scalar principal symbol a microhyperbolic at energy level τ. Let condition* (6.3.104) *be fulfilled with $G(z)$ introduced on $\Pi^\infty\cap\Sigma_\tau$ by Proposition 6.3.24. Let φ satisfy* (4.2.73)–(4.2.75).

Then

$$\mathsf{N}_{\varphi,L}(\tau) = \mathsf{N}^{\mathsf{W}}_{\varphi,L}(\tau) + \mathsf{N}_{\mathsf{corr},\varphi,L} + o\big(h^{1-d}\vartheta(hL^{-1})\big) \tag{6.3.105}$$

with

$$\mathsf{N}_{\mathsf{corr}} \coloneqq (2\pi)^{-d}h^{1-d}\int_{\Pi^\infty\cap\Sigma_\tau} q_1^0\,T_0^{-1}\Upsilon_1\big(T_0h^{-1}(a-hb)\big)q_2^0\,d\mu_\tau \tag{6.3.106}$$

$b\coloneqq -iT_0^{-1}\log G$ *and* $\Upsilon_1(t)=\Upsilon(t)-t+\frac12$, Υ *defined by* (6.2.69).

In particular as conditions are fulfilled on both levels τ_1 and τ_2 we recover asymptotics of $\mathsf{N}(\tau_1,\tau_2)$ with an error estimate $o(h^{1-d})$.

Let us recall that here T_0 depends on the point of Π^∞.

6.3.8 Examples

Laplacian

As the first example let us consider Laplace-Beltrami operators on the manifolds with all closed geodesics. One can find the theory of such manifolds in A. Besse [1]. Instead of Hamiltonian flow one can consider geodesic flow on $\mathbb{S}^*X$ and there could be subperiods. Let us recall that for spectral asymptotics of such operators one should multiply it by h^{-2} with $h := \lambda^{-\frac{1}{2}}$ and consider energy level 1.

Then the general theory claims that $\mathsf{N} = \mathsf{N}^{\mathsf{W}} + O(\lambda^{\frac{1}{2}(d-1)})$ with $\mathsf{N}^{\mathsf{W}} = \kappa_0 \lambda^{\frac{1}{2}d}$. Now with the generic lower order terms $\varepsilon = h$ and therefore correction is of the type $\lambda^{\frac{1}{2}(d-1)}\mathcal{Q}(\lambda^{\frac{1}{2}})$ with bounded but oscillating function $\mathcal{Q}$. Later we consider even the case when such manifolds have isolated singularities (see Example 11.3.28).

Symbol b is defined on $\mathbb{S}^*X/\sim$ where "$\sim$" means equivalence along geodesic flow.

In particular,

(6.3.107) (see Appendix 6.A.3) As $X = \mathbb{S}^d$ eigenvalues of Laplacian are

$$\lambda_n = n\big(n+d-1\big) = \big(n+\frac{1}{2}(d-1)\big)^2 - \frac{1}{4}(d-1)^2 \tag{6.3.108}$$

of multiplicity

$$N_n = \frac{(n+d-2)!(2n+d-1)}{(d-1)!n!}, \qquad n = 0,1,... \tag{6.3.109}$$

General perturbations of Laplacian are of the form $A'(x,D)$ where A' is a first order operator and then $\varepsilon = h$. In this case the width of the spectral cluster is generically $\asymp \lambda\varepsilon \asymp \lambda^{\frac{1}{2}}$ while the distance between eigenvalues is $\asymp \lambda h \asymp \lambda^{\frac{1}{2}}$ (actually $\lambda_n - \lambda_{n-1} = 2n + \frac{1}{2}(d-3)$); so spectral gaps could disappear.

There is an exceptional case when the principal symbol of A' is odd: $A'(x,\xi) = -A'(-x,-\xi)$. Then the the principal symbol of b which is an average of a' along Hamiltonian trajectories is identically 0. Really these trajectories are circles and $\exp(\frac{1}{2}\pi\nabla^{\#}a) = -I$ so antipodal points contribute

equally. Therefore in this case we can take $\varepsilon = h^2 = \lambda^{-\frac{1}{2}}$. In this case the width of the eigenvalue cluster is generically $\asymp \lambda\varepsilon \asymp 1$.

More generally, if perturbation is an operator of order k, $k \le 1$ then generically $\varepsilon = h^{2-k} \asymp \lambda^{\frac{k}{2}-1}$ and the width of the spectral cluster is generically $\asymp \lambda\varepsilon \asymp \lambda^{\frac{k}{2}}$. However for odd perturbations $\varepsilon = h^{2-2k} \asymp \lambda^{-k}$.

The most interesting example in this case is "Laplacian plus potential" with $k = 0$.

Harmonic Oscillator

Consider harmonic oscillator $\Delta + |x|^2$ on $\mathbb{R}^d$.

(6.3.110) Eigenvalues of harmonic oscillator $\Delta + |x|^2$ on $\mathbb{R}^d$

$$\lambda_n = 2n + d \tag{6.3.111}$$

of multiplicity

$$N_n = \frac{(n+d-1)!}{(d-1)!n!}, \qquad n = 0, 1, ... \tag{6.3.112}$$

which follows from the similar fact as $d = 1$.

Dividing by λ and rescaling $x \mapsto x\lambda^{-1}$ we get into framework of our theory with $h = \lambda^{-1}$; the domain is basically $X = \{|x| \le 2\}$ since $\{|x| > 1\}$ is classically forbidden. So standard theory yields $\mathsf{N} = \mathsf{N}^{\mathsf{W}} + O(\lambda^{d-1})$ with $\mathsf{N}^{\mathsf{W}} = \kappa_0\lambda^d$. This is completely consistent with the theory of global operators (see Subsubsection *1.1.4.4 Global Operators*) in which harmonic oscillator has degree 1 while operators x_j and D_j have degree $\frac{1}{2}$.

Note that x-projections of Hamiltonian trajectories are ellipses with center at the origin. Let us perturb such operator by a weak constant magnetic field:

$$A = \sum_k \big(D_k - \mu F_k(x)\big)^2, \qquad F_k = \sum_j f_{kj} x_j D_k \tag{6.3.113}$$

with $\mu \ll 1$. Then

$$b(x,\xi) = -\frac{2}{\pi}\mu \int_0^\pi \sum_{j,k} f_{kj} x_j \xi_k \, dt = -\frac{1}{\pi}\mu \int_\ell \sum_{j,k} f_{kj} x_j dx_k = \\ -\frac{1}{\pi}\mu \int_{\mathcal{D}(\ell)} \sum_{j,k} f_{kj} dx_j \wedge dx_k \tag{6.3.114}$$

where $\mathcal{D}(\ell)$ is a planar elliptical domain "inside" ellipse ℓ.

Thus $\varepsilon = \mu$ and as $\mu\lambda \le \epsilon_0$ spectral gaps separate clusters.

Problem 6.3.26. It should be easy to prove that b has only non-degenerate critical points on $\Sigma/\sim$ where $\Sigma = \{|\xi|^2 + |x|^2 = 1\} = \mathbb{S}^{2d-1}$ and "$\sim$" means equivalence along Hamiltonian flow.

Remark 6.3.27. A related, but a special and separate example is a Schrödinger operator on $\mathbb{R}^2$ with a constant magnetic and 0 electric field (see Volumes III and IV).

Coulomb Potential

Consider Schrödinger operator with Coulomb potential $\Delta - |x|^{-1}$ on $\mathbb{R}^d$. Surely it is singular in origin; so we consider more formal theory. The justification will be given much later in Sections 11.2, 11.3 and 11.4.

(6.3.115) (see Appendix 6.A.3) Eigenvalues of such operator

$$-\lambda_n = -(2n + d - 1)^{-2} \tag{6.3.116}$$

of multiplicity

$$\nu_n = \sum_{0\le m\le n} \frac{(m + d - 3)!(2m + d - 2)}{(d - 2)!m!}, \qquad n = 0, 1, \dots \tag{6.3.117}$$

Dividing by λ and rescaling $x \mapsto x\lambda$ we get into framework of our theory with $h = \lambda^{\frac12}$; the domain is basically $X = \{|x| \le 2\}$ since $\{|x| > 1\}$ is classically forbidden. However part with $|x| \le \epsilon$ would be a problem.

So the standard theory yields $\mathsf{N} = \mathsf{N}^{\mathsf{W}} + O(\lambda^{-\frac{d-1}{2}})$ with $\mathsf{N}^{\mathsf{W}} = \kappa_0\lambda^{-\frac{d}{2}}$.

Note that x-projections of Hamiltonian trajectories are Kepler ellipses with focus at the origin. Let us perturb such operator by a weak constant magnetic field:

$$A = \sum_k \big(D_k - \mu F_k(x)\big)^2 - |x|^{-1}, \qquad F_k = \sum_j f_{kj}x_jD_k \tag{6.3.118}$$

with $\mu \ll \lambda^{\frac32}$. Then (6.3.114) holds.

Thus $\varepsilon = \mu\lambda^{-\frac32}$ and as $\mu\lambda \le \epsilon_0\lambda^2$ spectral gaps may be separating clusters.

Problem 6.3.28. Prove that b has only non-degenerate critical points on $\Sigma/\sim$ where $\Sigma = \{|\xi|^2 - |x|^{-1} = -1\}$ and "$\sim$" means equivalence along Hamiltonian flow.

6.4 Asymptotics for Dirac Energy

6.4.1 Introduction

In *Part XI Application to Multiparticle Quantum Theory* devoted to the *sharp asymptotics of the ground state energy of heavy atoms and molecules* it was needed to calculate*Dirac correction term*[20)] which in that approximation was equal to

$$\mathsf{I} := \iint |x-y|^{-1} \cdot |e(x,y,\tau)|^2 \, dxdy \tag{6.4.1}$$

where $e(x,y,\tau)$ is the Schwartz kernel of the spectral projector $E(\tau)$ of the (magnetic) Schrödinger operator

$$A = \frac{1}{2}\Big(\sum_{j,k} P_j g^{jk}(x) P_k - V\Big), \qquad P_j = hD_j - \mu V_j \tag{6.4.2}$$

$\tau \approx 0$ and $h \to +0$ (while either $\mu \to +\infty$ or remains constant). Actually the corresponding parts of Chapters 25–28 was originally more complicated but it was reduced to the problem above.

Then $\mathsf{I} \asymp h^{-d-1}$ (where $d = 3$ in the above papers) and it is needed to prove that $\mathsf{I} = \mathsf{I}^{\mathsf{W}} + O(h^{-d-1+\delta})$ with I^{W} defined by the same formula but with $e(x,y,\tau)$ replaced by

$$\mathsf{N}^{\mathsf{W}}_y(x,y,\tau) := (2\pi h)^{-d} \int_{g(y,\xi)\le V(y)+2\tau} e^{ih^{-1}\langle x-y,\xi\rangle} \, d\xi \tag{6.4.3}$$

and with a small exponent $\delta > 0$; for Magnetic Schrödinger it was needed to prove as binding parameter $\mu \le h^{-\delta}$ only. Expression (6.4.3) is a *Weyl expression* for $e(x,y,\tau)$ for operator with coefficients frozen at point y.

[20)] Representing Coulomb interaction of electrons with themselves which should not to be counted in the energy calculation and should be subtracted from the Thomas-Fermi expression.

However we believe that the asymptotics of expression (6.4.1) or more general ones are interesting by themselves and that there are sharp asymptotics. Still our attempts to derive it were not very successful. Let us bring some degree of the order to this matter.

Let us consider a matrix h-differential operator $A(x, hD)$ and find asymptotics of

$$\mathsf{I} := \iint \omega(x,y) e(x,y,\tau)\psi_2(x) e(y,x,\tau)\psi_1(y)\,dx\,dy \tag{6.4.4}$$

with a matrix-valued function $\omega(x,y)$ such that

(6.4.5) $\omega(x,y) := \Omega(x,y;x-y)$ where function Ω is smooth in $B(0,1)\times B(0,1)\times B(\mathbb{R}^d\setminus 0)$ and positively homogeneous of degree $-\kappa$ ($0<\kappa<d$) with respect to its third argument[21)]

and with smooth cut-off functions ψ_1, ψ_2.

The main part of asymptotics should have a magnitude of $h^{-d-\kappa}$ and our goal is to get a remainder estimate $O(h^{1-d-\kappa})$.

One can also consider a more general expression

$$\begin{aligned}\mathsf{I}_m := \iint \omega(x^1,\dots,x^m)\times \\ e(x^1,x^2,\tau)\psi_2(x^1)e(x^2,x^3,\tau)\cdots e(x^m,x^1,\tau)\psi_{m+1}(x^0)\,dx^1\cdots dx^m\end{aligned} \tag{6.4.6}$$

with $x^{m+1}=x^1$, $\psi_{m+1}:=\psi_1$ etc and

(6.4.7) $\omega(x^1,\dots,x^m) := \Omega(x^1,\dots,x^m;\{x^j-x^{j+1}\}_{1\le j\le m})$ where function Ω is smooth in $B(0,1)^m\times B(\mathbb{R}^d\setminus 0)^m$ and homogeneous of degree $-(m-1)\kappa$ with respect to $\{x^j-x^k\}_{1\le j<k\le m}$. Moreover,

$$|D_z^\beta D_x^\alpha \Omega| \le C_{\beta,\alpha}|z^1|^{-\kappa-|\beta^1|}\cdots|z^m|^{-\kappa-|\beta^m|} \qquad \text{as} \quad \sum_k |z^k|^2=1, \sum_k z^k=0$$

where $\boldsymbol{x}=(x^1,\dots,x^m)$, $\boldsymbol{z}=(z^1,\dots,z^m)$, etc.

[21)] In other words it is Michlin-Calderon-Zygmund kernel.

However we are able to implement in this case only some of our arguments.

The main part of asymptotics should have a magnitude of $h^{-d-(m-1)\kappa}$ (see Theorem 6.4.7) and we would like to get a remainder estimate $O(h^{1-d-(m-1)\kappa})$.

We are leaving for Chapter 16 book the similar but much more delicate and difficult analysis for a 2- and 3-dimensional Magnetic Schrödinger operator(6.4.2) with the trajectories having many loops.

Remark 6.4.1. (i) To avoid the necessity to cut-off with respect to hD one needs to assume that its symbol satisfies

$$\|a(x,\xi)^{-1}\| \le C|\xi|^{-m} \qquad \text{as } |\xi| \ge C_0 \tag{6.4.8}$$

as $a \in \mathbf{\Psi}^m_h$ (one can weaken this condition but we leave it to the reader).

(ii) One needs to assume that a is semibounded from below which under assumption (6.4.8) is equivalent to

$$\langle a(x,\xi)v, v\rangle \ge c^{-1}\|v\|^2 \qquad \text{as } |\xi| \ge C_0; \tag{6.4.9}$$

otherwise instead of $E(\tau)$ one should consider $E(\tau_1,\tau_2) \coloneqq E(\tau_2) - E(\tau_1)$; we leave it to the reader as well.

This section consist of four more subsections: in Subsection 6.4.2 we derive asymptotics with the sharp remainder estimate but with the implicit Tauberian approximation for $e(x,y,0)$. In Subsections 6.4.4 and 6.4.3 we replace it by expression (6.4.3) without deteriorating remainder estimate for scalar operators and for certain matrix operators and with some not sharp remainder estimates for other matrix operators. Finally, in Subsection 6.4.5 we apply a rescaling technique to the Schrödinger operator.

Let us mention that for larger κ we need less restrictive conditions to operator.

6.4.2 Estimates

Special Case

Let us assume first that $\omega = 1$ but relax conditions to $\psi_1,\dots,\psi_m$, assuming only that $\psi_1,\dots,\psi_m \in \mathscr{L}^\infty$. This is definitely not the case we are interested in but one needs to make few clarifications first. Then

$$\mathsf{I}_m \coloneqq \Gamma\big(E(\tau)\psi_2 E(\tau)\psi_3 E(\tau)\psi_4 \cdots \psi_m E(\tau)\psi_{m+1}\big) \tag{6.4.10}$$

containing m factors $E(\tau)$.

Under condition (6.4.9) we know (see f.e. Proposition 4.2.1) that if $\mathscr{L}^\infty$ norms and the diameters of supports ψ, ψ_1 are bounded, then

$$\|\psi E(\tau)\psi_1\|_1 \le Ch^{-d} \qquad \text{as } |\tau| \le c \tag{6.4.11}$$

where $\|.\|_\infty$ and $\|.\|_1$ denote operator and trace norms respectively. Then since an operator norm of $E(\tau)$ does not exceed 1 we conclude that $|\mathsf{I}_m| \le Ch^{-d}$. Therefore

(6.4.12) If $\psi_j \in \mathscr{L}^\infty$ and I_m is given by (6.4.10) then $|\mathsf{I}_m| \le Ch^{-d}$.

Further, let us assume that

(6.4.13) $a(x,\xi)$ is microhyperbolic on energy level 0.

Then as (6.4.13) is fulfilled on supports of ψ, ψ_1 it is known (see Section 4.2) that

$$\|\psi\big(E(\tau)-E(\lambda)\big)\psi_1\| \le C(|\tau-\lambda|+hT^{-1})h^{-d} \qquad \text{as } |\tau|\le\epsilon_1,\ |\lambda|\le c. \tag{6.4.14}$$

Here and for a while $T \asymp 1$ but we want to keep a track of it.

Since this property holds under wider assumptions than microhyperbolicity, we will assume so far only that (6.4.14) holds. On the other hand, if we want to be able to take $T \ll 1$ microhyperbolicity is not enough; in this case we assume that

(6.4.15) $a(x,\xi)$ is ξ-microhyperbolic on energy level 0.

Anyway, under condition (6.4.14)

$$|\Gamma\Big(\big(E(\tau)-E(\lambda)\big)\psi_2E(\tau_2)\psi_3E(\tau_3)\psi_4\cdots\psi_mE(\tau_m)\psi_{m+1}\Big)| \tag{6.4.16}$$

also does not exceed the right hand expression of (6.4.14) as $|\tau| \le \epsilon_1$ and therefore due to the standard Tauberian arguments (second part, see again Section 4.2) the following inequality holds:

$$|\Gamma\Big(\Big(E(0)-h^{-1}\int_{-\infty}^{0} F_{t\to h^{-1}\tau}\big(\bar\chi_T(t)U(t)\big)\,d\tau\Big)\times \psi_2E(\tau_2)\psi_3E(\tau_3)\psi_4\cdots\psi_mE(\tau_m)\psi_{m+1}\Big)| \le CT^{-1}h^{1-d} \tag{6.4.17}$$

where we use our standard notations $\bar{\chi}$ and χ in the future and $\bar{\chi}_T(t) = \bar{\chi}(t/T)$ and as usual Γ is the "scalar trace" of the operator, and does not include taking matrix trace tr.

Here and below $U(t) = e^{ih^{-1}tA}$ is the propagator of A and $u(x,y,t)$ is its Schwartz kernel.

So with $O(T^{-1}h^{1-d})$ error we can replace one copy of $E(0)$ in I_m by its standard implicit Tauberian approximation

$$h^{-1}\int_{-\infty}^{0} F_{t\to h^{-1}\tau}\bigl(\bar{\chi}_T(t)U(t)\bigr)\,d\tau \tag{6.4.18}$$

and in by the virtue of the same arguments we can do it with another copy of $E(0)$. Therefore

Proposition 6.4.2. *Under conditions* (6.4.14) *with an error* $O(T^{-1}h^{1-d})$ I_m *is equal to*

(6.4.19)
$$h^{-m}\Gamma\int_{\boldsymbol{\tau}\in\mathbb{R}^{-,m}} F_{\boldsymbol{t}\to h^{-1}\boldsymbol{\tau}}\Bigl(\bar{\chi}_T(t_1)U(t_1)\psi_2\bar{\chi}_T(t_2)U(t_2)\psi_3\cdots\psi_m U(t_m)\psi_{m+1})\Bigr)\,d\boldsymbol{\tau}$$

with $\boldsymbol{t} = (t_1,\dots,t_m)$, $\boldsymbol{\tau} = (\tau_1,\dots,\tau_m)$.

Note that here one can take any $T\in[h^{1-\delta},c]$ (but then an error depends on T). Further, note that as $\mathsf{dist}(\mathsf{supp}\,\psi_j,\mathsf{supp}\,\psi_{j+1})\ge(c_0+\epsilon)T$ where c_0 here and below is the upper bound of the propagation speed on energy level 0 and $x^{m+1}:=x^1$, expression (6.4.19) as $m=2$ or the similar expression as $m\ge 3$ become negligible and we arrive to

Corollary 6.4.3. *If in framework of Proposition 6.4.2*

$$\mathsf{dist}(\mathsf{supp}(\psi_j),\mathsf{supp}(\psi_{j+1}))\ge(c_0+\epsilon)T$$

for some $j=1,\dots,m$ *then* $|\mathsf{I}_m|$ *does not exceed* $CT^{-1}h^{1-d}$.

Smooth Case

The next step is to assume that ω is a general smooth function rather than the product as before. Without any loss of the generality one can assume

that ω is also compactly supported (since ψ, ψ_1 are). Then from

$$\omega(x^1,\dots,x^m) = \int \omega(y^1,\dots,y^m)\cdot \delta(y^1-x^1)\cdots\delta(y^m-x^m)\,dy = \int \omega'(y^1,\dots,y^m)\cdot \theta(y^1-x^1)\cdots\theta(y^m-x^m)\,dy^1\cdots dy^m \tag{6.4.20}$$

one arrives to

$$\mathsf{I}_m = \int \omega'(y^1,\dots,y^m)\cdot J_m(y^1,\dots,y^m)\,dy^1\cdots dy^m \tag{6.4.21}$$

with $J_m(y^1,\dots,y^m)$ defined by (6.4.6) with $\omega = 1$ and $\psi_j(x)$ redefined as $\psi_j(x)\theta(y^j - x)$ where

$$\omega'(y^1,\dots,y^m) = (-1)^{md} \prod_{1\le j\le m}\prod_{1\le k\le d} \partial_{y_k^j}\omega(y^1,\dots,y^m)$$

and here and below $\theta(x) = \theta(x_1)\cdots\theta(x_d)$. Then we immediately arrive to

Proposition 6.4.4. *Let ω and $\psi_1,\dots,\psi_m$ be smooth functions and let condition* (6.4.9) *be fulfilled. Then $|\mathsf{I}_m| \le Ch^{-d}$.*

Problem 6.4.5. As $m = 2$ and $\omega, \psi_1, \psi_2 \in \mathscr{L}^\infty$ $|\mathsf{I}_2| \le Ch^{-d}$ obviously (it follows from the estimate $\|\psi E\psi\|_2 \le Ch^{-d/2}$ where $\|.\|_2$ is the Hilbert-Schmidt norm). Is it possible prove the similar result for $m \ge 3$?

Proposition 6.4.6. *Let ω and $\psi_1,\dots,\psi_m$ be smooth functions and let conditions* (6.4.9) *and* (6.4.14) *be fulfilled. Then*

(i) With an error $O(T^{-1}h^{1-d})$ I_m is equal to

$$\mathsf{I}_m^{\mathsf{T}} = h^{-m}\int\int_{\tau\in\mathbb{R}^{-,m}} \omega(x^1,\dots,x^m)\times F_{t\to h^{-1}\tau}\Big(\bar\chi_T(t_1)u(x^1,x^2,t_1)\psi_2(x^2)\bar\chi_T(t_2)u(x^2,x^3,t_2)\psi_3(x^3)\cdots \psi_m(x^m)u(x^m,x^{m+1},t_m)\psi_{m+1}(x^{m+1})\Big)\,d\tau\,dx^1\cdots dx^m \tag{6.4.22}$$

with $x^{m+1} := x^1$.

(ii) Further, if $\operatorname{dist}(\operatorname{supp}(\psi_j),\operatorname{supp}(\psi_{j+1})) \ge (c_0+\epsilon)T$ for some $j = 1,\dots,m$ then $|\mathsf{I}_m|$ does not exceed $CT^{-1}h^{1-d}$ where so far $T \asymp 1$.

Proof. Proof follows immediately from Proposition 6.4.2 and representation (6.4.20)–(6.4.21). □

Singular homogeneous case

Theorem 6.4.7. *Let conditions* (6.4.9) *and* (6.4.7) *be fulfilled.*
Then $|\mathsf{I}_m| \le Ch^{-d-(m-1)\kappa}$.

Proof. Let us replace $\Omega(x,z)$ by $\Omega(x,z)\beta(z^1\gamma_1^{-1})\cdots\beta(z^m\gamma_m^{-1})$ where $\gamma_j \ge h$ and $\beta, \bar\beta$ are functions (on $\mathbb{R}^d$) similar to $\chi, \bar\chi$ respectively i.e. $\beta, \bar\beta \in \mathscr{C}_0^\infty(B(0,1))$ and $\beta = 0$, $\bar\beta = 1$ in $B(0,\frac12)$.

Then similarly to the analysis of the smooth case one can estimate the contribution of such partition element to I_m by

$$Ch^{-d}\bigl(\gamma_1\cdots\gamma_m\bigr)^{-1}(\gamma_1+\cdots+\gamma_m)^{1-\kappa}; \tag{6.4.23}$$

then summation with respect to all such elements with $\gamma_j \ge \bar\gamma = h$ results in the value of this expression as $\gamma_j = \bar\gamma$ which is exactly $Ch^{-d-(m-1)\kappa}$.

However one needs to consider the other partition elements when some (or all) of $\beta(z^j\gamma_j^{-1})$ are replaced by $\bar\beta(z^j\bar\gamma^{-1})$. So we get "sandwiches" consisting of the factors

$$e(x^k, x^{k+1}, \tau)\beta(z^{k+1}\gamma_{k+1}^{-1})\cdots e(x^{j-1}, x^j, \tau)\beta(z^j\gamma_j^{-1})e(x^j, x^{j+1}, \tau)$$

with $j \ge k$ and in between them factors $\bar\beta(z^k\bar\gamma^{-1})$.

Let J be the set of indices appearing in $\bar\beta(z^k\bar\gamma^{-1})$ (for a given type of a "sandwich"). One can see easily that the contribution of each "sandwich" to I_m does not exceed

$$Ch^{-dr}\prod_{j\notin J}\gamma_j^{-\kappa}\times\Bigl(\int_{\{|z|\le\bar\gamma\}}|z|^{-\kappa}\,dz\Bigr)^{r-1} \asymp Ch^{-dr}\prod_{j\notin J}\gamma_j^{-\kappa}\times\bar\gamma^{(d-\kappa)(r-1)}$$

where $r = \#J$. Then after summation with respect to $\gamma_j \ge \bar\gamma$ one gets the same expression with $\gamma_j = \bar\gamma$ i.e.

$$Ch^{-dr}\bar\gamma^{\kappa(m-r)+(d-\kappa)(r-1)} = Ch^{-dr}\bar\gamma^{-\kappa(m-1)+d(r-1)}$$

which is exactly what we want as $\bar\gamma \asymp h$. □

The following statement immediately follows from the proof:

Proposition 6.4.8. *Let conditions* (6.4.9) *and* (6.4.7) *be fulfilled. Then replacing* $\Omega(x,z)$ *by* $\Omega(x,z)\bar\beta(z^1\gamma^{-1})\cdots\bar\beta(z^m\gamma^{-1})$ *with* $\gamma \ge h$ *results in the error not exceeding*

$$Ch^{-d-(m-2)\kappa}\gamma^{-\kappa}. \tag{6.4.24}$$

Singular Homogeneous Case: Using ξ-Microhyperbolicity

Now let assume instead of condition (6.4.14) that ξ-microhyperbolicity condition (6.4.15) is fulfilled. Then we can improve estimate (6.4.24):

Proposition 6.4.9. *Let conditions* (6.4.9), (6.4.7) *and* (6.4.15) *be fulfilled. Then replacing* $\Omega(x,z)$ *by* $\Omega(x,z)\bar{\beta}(z^1\gamma^{-1})\cdots\bar{\beta}(z^m\gamma^{-1})$ *with* $\gamma\ge h$ *results in the error not exceeding*

$$Ch^{1-d-(m-2)\kappa}\gamma^{-1-\kappa}. \tag{6.4.25}$$

This is equivalent to taking $T\asymp\gamma$ *in* (6.4.18) *and plugging Schwartz kernel of it instead of* $e(x,y,0)$ *in the definition of* I_m.

Proof. Proof follows from the combined arguments of the proofs of Theorem 6.4.7 and Proposition 6.4.2; in this case one needs to consider only "sandwiches" containing at least one factor $\beta(x^j\gamma_j^{-1})$ with $\gamma_j\ge\gamma$ which accounts for an extra factor $h\gamma_j^{-1}$ and summation with respect to partition results in an extra factor $h\gamma^{-1}$. □

So one needs to study expression (6.4.22) with some $T=T^*$; let us recall that the remainder estimate contains factor T^{*-1}. One can decompose $\bar{\chi}_{T^*}(t)$ into the sum of $\bar{\chi}_{\bar{T}}(t)$ and $\chi_T(t)$ with T running between $\bar{T}$ and T^* and also one can take $\bar{T}=Ch$. Then expression (6.4.22) becomes the sum of the similar expressions with $\bar{\chi}_T(t)$ (with $T=T^*$) replaced by $\phi_{jT_j}(t)$ where <u>either</u> $\phi_j=\chi_j$ and $\bar{T}\le T_j\le T^*$ <u>or</u> $\phi_j=\bar{\chi}$ and $T_j=\bar{T}$.

In this expression as $\phi_j=\chi_j$ one can replace $\int_{-\infty}^0(\dots)\,d\tau_j$ by $(\dots)|_{\tau_j=0}$ simultaneously replacing $h^{-1}\chi_{j,T_j}(t_j)$ by $it^{-1}\chi_{j,T_j}(t)=T_j^{-1}\phi_{j,T_j}(t)$ with redefined $\phi_j(t)=it^{-1}\chi_j(t)$.

So we get a modified expression (6.4.22) with

(a) r factors $\bar{\chi}_{\bar{T}}(t_j)$ and integration over $\mathbb{R}^{-r}\ni\tau_J$, $r=\#J$ and

(b) $(m-r)$ factors $\phi_{k,T_k}(t_k)$ and with τ_k snapped to 0 (instead of integration) as $k\notin J$

(c) furthermore, factor h^{-m} is replaced by $h^{-r}\prod_{k\notin J}T_k^{-1}$.

Finally we will need

Proposition 6.4.10. *Let conditions* (6.4.9) *and* (6.4.15) *be fulfilled and let* ω *be a smooth function,*

$$\omega = O\big((|x^1 - x^2| + \cdots + |x^m - x^1|)^K\big). \tag{6.4.26}$$

Then $\mathsf{I}_m = O(h^{1-d})$ *as* $K > 1$ *and* $\mathsf{I}_m = O(h^{1-d}|\log h|)$ *as* $K = 1$.

Proof. Proof follows from the combined arguments of the proofs of Theorem 6.4.7 and Proposition 6.4.2 exactly as in the proof of Proposition 6.4.9.

Here however the main contribution (as $K \geq 1$) is delivered by zone $\{|x^1 - x^2| + \cdots + |x^m - x^1| \asymp 1\}$. □

One can consider certain generalizations but we will do them later.

6.4.3 Calculations

Now our purpose is to go from implicit Tauberian expression (6.4.22) to more explicit one.

Constant Coefficients Case

Let us first consider case $A(x,\xi) = A(\xi)$. In this case

$$e(x,y,\tau) = (2\pi h)^{-d} \int e^{ih^{-1}\langle x-y,\xi\rangle} E(\xi)\, d\xi \tag{6.4.27}$$

where $E(\xi,\tau)$ is the matrix projector corresponding to $A(\xi)$. Then

$$\mathsf{I}_m = (2\pi h)^{-dm} \int\int \omega(x^1,\dots,x^m) E(\xi^1,0)\cdots E(\xi^m,0)\times \tag{6.4.28}$$
$$e^{ih^{-1}\left(\langle x^1-x^2,\xi^1\rangle + \langle x^2-x^3,\xi^2\rangle + \cdots + \langle x^m-x^1,\xi^m\rangle\right)}\, dx^1\cdots dx^m\, d\xi^1\cdots d\xi^m.$$

(6.4.29) From now and until the end of the paper we are assuming that $m = 2$.

Without any loss of the generality one can assume that either $\omega(x,y)$ is of the form

$$\omega(x,y) = \Omega\big(\frac{1}{2}(x+y), x-y\big). \tag{6.4.30}$$

or it is of the same singular type as before but multiplied by $(x_k - y_k)$. However in the latter case (under microhyperbolicity condition) one can apply a Tauberian approximation for $e(x, y, \tau)$ equal 0 with the remainder estimate $O(h^{1-d}|x-y|^{-1})$ (in the same trace class as before) which leads to $\mathsf{I} \approx 0$ with the sought remainder estimate $O(h^{1-d-\kappa})$.

In the former case (6.4.30) we get

$$\mathsf{I} := \mathsf{I}_2 = \int \mathcal{J}(x)\, dx, \tag{6.4.31}$$

where

$$\mathcal{J}(x) = 2(2\pi h)^{-2d} \iiint \Omega(x,z) E(\xi,0) E(\eta,0) e^{ih^{-1}\langle z, \xi-\eta\rangle}\, dz d\xi d\eta = G(x) h^{-d-\kappa}, \tag{6.4.32}$$

with

$$G(x) = \iint \widehat{\Omega}(x, \xi-\eta) E(\xi,0) E(\eta,0)\, d\xi d\eta, \tag{6.4.33}$$

and

$$\widehat{\Omega}(x,\zeta) = 2(2\pi)^{-2d} \int \Omega(x,z) e^{i\langle z,\zeta\rangle}\, dz. \tag{6.4.34}$$

One always can take Ω having a compact support with respect to x (since we had originally cutoffs $\psi_1(x^1), \dots, \psi_m(x^m)$.

Remark 6.4.11. (i) One can easily generalize (6.4.31)–(6.4.34) to $m > 2$.

(ii) Integral (6.4.34) converges as $|z| \le 1$ since $\kappa < d$. On the other hand it defines a distribution with respect to ζ which is positively homogeneous of degree $\kappa - d$ and also is smooth as $\zeta \ne 0$; thus $\widehat{\Omega} \in \mathscr{L}^1_{\mathsf{loc}}$ and (6.4.33) is well-defined. However generalization to $m > 2$ is not that easy.

General ξ-Microhyperbolic Case

Note first that due to the ξ-microhyperbolicity condition (6.4.15) one should take $T \asymp \gamma$ as $m = 2$ [22]. Otherwise as $T \in [h^{1-\delta}, T^*]$, T^* is the small constant, the contribution of $[-\frac{1}{2}T, -T] \cup [T, \frac{1}{2}T]$ would be negligible.

[22] And $T_j \asymp |x^j - x^{j+1}|$ in the general case.

To calculate u let us apply the successive approximation method on the time interval $[-T, T]$ with $h^{1-\delta} \le T$. Then plugging the successive approximation into any copy on that interval we arrive to an error in u in the trace norm equal to $O\big(h^{-d}(T^2h^{-1})^n\big)$ where n is the number of the first dropped term (starting from 0). This leads to the error in I $O\big(h^{-d-\kappa}(T^2h^{-1})^n\gamma^{-\kappa}\big)$ as $T \ge \gamma$. Since under ξ-microhyperbolicity assumption (6.4.15) we need to consider only $T \asymp \gamma$, the error does not exceed

$$Ch^{-d}(T^2h^{-1})^n T^{-\kappa}. \tag{6.4.35}$$

On the other hand if we just take $u = 0$ then we get an error not exceeding

$$Ch^{1-d}T^{-1-\kappa}. \tag{6.4.36}$$

Let us minimize the sum of (6.4.35) and (6.4.36). Finding T from the equation $h^{-d}(T^2h^{-1})^n = h^{1-d}T^{-1}$ we get

$$T = h^{(n+1)/(2n+1)} \tag{6.4.37}$$

(which is greater than $h^{1-\delta}$ with $\delta > 0$) and this leads to an error not exceeding

$$Ch^{1-d-(n+1)(\kappa+1)/(2n+1)}. \tag{6.4.38}$$

Proposition 6.4.12. *Let conditions* (6.4.9), (6.4.7) *and* (6.4.15) *be fulfilled. Then*

(i) Using successive approximation as $|t| \le T$ *given by* (6.4.28) *and taking* $u = 0$ *otherwise we get* I *with an error given by* (6.4.38).

(ii) In particular, this is the sharp remainder estimate $O(h^{1-d-\kappa})$ *as*

$$\kappa \ge (n+1)/n; \tag{6.4.39}$$

in particular, as $\kappa \ge 2$ *one can skip all perturbation terms and get the same answer* (6.4.30)–(6.4.33).

On the other hand, if we cannot skip some term, then this is given by the same formulae (6.4.30)–(6.4.33) as before but with the factor $h^{-d-\kappa+s}$ instead of $h^{-d-\kappa}$ and with Ω replaced by Ω_s positively homogeneous of degree $-\kappa+s$ [23]. Then as long as $s < \kappa$ one can see that these terms are less than the remainder estimate and we arrive to

[23] Provided these formulae have sense!

Proposition 6.4.13. *Let conditions* (6.4.9), (6.4.7) *and* (6.4.15) *be fulfilled. Then*

(i) As $\kappa > 1$ *formulae* (6.4.30)–(6.4.33) *provide an answer with the remainder estimate* $O(h^{1-d-\kappa})$.

(ii) As $\kappa \le 1$ *formulae* (6.4.30)–(6.4.33) *provide an answer with the remainder estimate* $O\bigl(h^{\frac{1}{2}(1+\kappa)-d-\kappa-\delta}\bigr)$ *with arbitrarily small exponent* $\delta > 0$.

6.4.4 Scalar Case

Let us completely analyze the case of the scalar ξ-microhyperbolic operator.

Smooth case. I

Assume first that $\omega = 1$ and ψ_1, ψ_2 are smooth functions. Then one can rewrite (6.4.19) with $m = 2$

$$h^{-2}\Gamma \int_{(\tau_1,\tau_2)\in\mathbb{R}^{-,2}} F_{t_1\to h^{-1}\tau_1, t_2\to h^{-1}\tau_2}\times \Bigl(\bar{\chi}_T(t_1)\bar{\chi}_T(t_2)\psi_1 U(t_1)\psi_2 U(t_2)\Bigr)\, d\tau \tag{6.4.40}$$

with $T = T^*$ which is the largest value for which remainder estimate $O(T^{-1}h^{1-d})$ for the standard asymptotics was derived; here $T^* \asymp 1$.

Let us recall that if we replace some copies of $\bar{\chi}_T(t_k)$ by $\chi_{T_k}(t_k)$ with $Ch \le T_k \le T^*$ then one can replace also operator $h^{-1}\int_{-\infty}^0 (\dots)\, d\tau_k$ by $T^{-1}(\dots)|_{\tau_k=0}$ and χ by $it^{-1}\chi$.

If we do it with both $k = 1, 2$ then we get a term $O(h^{-d})$ (the better estimate is actually possible) and the summation with respect to all partitions with respect to T_1, T_2 results in $O(h^{-d}|\log h|^2)$ which differs from the proper estimate by $|\log h|^2$ factor.

If we replace some copies of $\bar{\chi}_T(t_k)$ by $\bar{\chi}_h(t_k)$ then we do not make a transformation with respect to these factors but we gain a factor h due to the size of the support. So after summation with respect to partition we arrive to estimate $O(h^{-d}|\log h|^{2-r})$ for I where r is the number of $\bar{\chi}_h(t_k)$ factors.

On the other hand, expression (6.4.40) is equal to

$$h^{-2}\Gamma \int_{(\tau_1,\tau_2)\in\mathbb{R}^{-,2}} F_{t_1\to h^{-1}\tau_1, t_2\to h^{-1}\tau_2}\times \Big(\bar{\chi}_T(t_1)\bar{\chi}_T(t_2)\psi_1\psi_{2,t_1}U(t_1+t_2))\Big)\, d\tau \tag{6.4.41}$$

with $\psi_t = U(t)\psi U(-t)$.

Applying our standard approach we arrive to

$$\mathsf{I}^{\mathsf{T}} \sim \sum_{n\ge 0} \varkappa_n h^{-d+n} \tag{6.4.42}$$

where $\mathsf{I}^{\mathsf{T}} = \mathsf{I}^{\mathsf{T}}_2$ is defined by (6.4.22).

Let us replace in (6.4.41) ψ_{2,t_1} by ψ_2. Plugging $t_{1,2} = \frac{1}{2}t \pm z$, $\tau_{1,2} = \tau \pm \lambda$ we arrive to

$$h^{-1}\Gamma \int_{\infty}^{0} \Big(\int_{\mathbb{R}} \rho_T(t,\tau)\psi_1\psi_2 U(t) e^{-ih^{-1}t\tau}\, dt\Big)\, d\tau \tag{6.4.43}$$

where $\rho_T(t,\tau) = \rho(tT^{-1},\tau)$, $\tau < 0$

$$\rho(t,\tau) = -\pi^{-1}h^{-1}\int_{\mathbb{R}} \bar{\chi}_T(\frac{1}{2}t+z)\bar{\chi}_T(\frac{1}{2}t-z)z^{-1}\sin(h^{-1}Tz\tau)\, dz \tag{6.4.44}$$

belongs to $\mathscr{C}_0^{\infty}([-2,2])$ and one can prove easily that

$$|\partial_t^n(\rho(t,\tau) \mp \bar{\chi}^2(\frac{1}{2}t))| \le C_{nm}(1+|\tau|\,Th^{-1})^{-m} \qquad \forall m,n\ \forall \tau \leqslant 0. \tag{6.4.45}$$

Then due to (6.4.45) only zone $\{|\tau| \le h^{1-\delta}\}$ gives a non-negligible contribution to this error and due to the microhyperbolicity condition there $|\Gamma\psi_1\psi_2 U(t)| \le Ch^{-d}(1+|t|h^{-1})^{-m}$ which together with (6.4.45) implies

(6.4.46) Under microhyperbolicity condition (6.4.13) expression (6.4.43) is equal modulo $O(h^{1-d})$ to the same expression with ρ replaced by $\bar{\chi}^2(\frac{1}{2}t)$.

On the other hand, if we replace ψ_{2,t_1} by $\psi_{2,t_1} - \psi_2 = t_1\psi'_{2,t_1}$ then we can apply the same transformation as before just getting rid of one factor h^{-1} and integration with respect to τ_1, which simply snaps to 0, resulting in the expression, similar to (6.4.43) but with $\rho\psi_2$ replaced by

$$\rho'(t,\tau,x) = (2\pi)^{-1} i \int_{\mathbb{R}} \bar{\chi}_T(z)\bar{\chi}_T(t-z)e^{ih^{-1}\tau z}\psi'_{2,z}\, dz \tag{6.4.47}$$

which satisfies inequality similar to (6.4.45)

$$|\partial_t^n \rho(t,\tau)| \le C_{nm}(1+|\tau|\,Th^{-1})^{-m} \qquad \forall m,n\ \forall \tau \lessgtr 0. \tag{6.4.48}$$

and therefore we arrive to

(6.4.49) Under microhyperbolicity condition (6.4.13) this new (6.4.43)-type expression is $O(h^{1-d})$.

So, we are left with expression (6.4.43) with $\rho(t) = \bar{\chi}^2(\frac{1}{2}t)$ but due to the standard theory we get modulo $O(h^{1-d})$ expression

$$\Gamma\psi_1\psi_2 E(0) \equiv (2\pi h)^{-d} \iint_{\{a(x,\xi)<0\}} \psi_1\psi_2\, dx\, d\xi. \tag{6.4.50}$$

So, I^{T} is given by (6.4.50) modulo $O(h^{1-d})$ and therefore

$$\varkappa_0 = (2\pi)^{-d} \int \iint_{\{a(x,\xi)<0\}} \psi_1\psi_2\, dx\, d\xi \qquad \text{in (6.4.42)}. \tag{6.4.51}$$

Smooth case. II

Then in the general smooth case we get

Proposition 6.4.14. *Let ω and $\psi_1,\dots,\psi_m$ be smooth functions and let* (6.4.9) *and microhyperbolicity condition* (6.4.13) *be fulfilled. Then with an error $O(T^{-1}h^{1-d})$ where $T \asymp 1$ here decomposition* (6.4.42) *holds with*

$$\varkappa_0 = (2\pi)^{-d} \iint_{\{a(x,\xi)<0\}} \omega(x,x)\psi_1(x)\psi_2(x)\, dx\, d\xi. \tag{6.4.52}$$

Proof. Follows from the standard decomposition (6.4.20)–(6.4.21). □

Singular Homogeneous Case

Consider now the case of singular homogeneous ω. However, let us consider first $\mathsf{I}^{\mathsf{T}}_\gamma$ defined by (6.4.22) with $\omega = 1$ and ψ_1,ψ_2 replaced by $\psi_{1,\gamma},\psi_{2,\gamma}$ which are some smooth functions scaled at some point z with the scaling parameter $\gamma \in (h^{1-\delta}, h^{\delta})$. To have microhyperbolicity condition sustain

scaling we replace it by ξ-microhyperbolicity condition (6.4.15). Then (6.4.42) implies

$$\mathsf{I}^{\mathsf{T}}_{\gamma} \sim \sum_{n,m\ge 0} \varkappa_{nm} h^{-d+n}\gamma^{m-n+d} \tag{6.4.53}$$

and obviously

$$(2\pi)^{-d}\iint_{\{a(x,\xi)<0\}} \psi_{1,\gamma}(x)\psi_{2,\gamma}(x)\,dx\,d\xi \sim \sum_{m\ge 0} \varkappa'_m \gamma^{m+d}. \tag{6.4.54}$$

One can see easily that in (6.4.53) terms with $m=0$ would be the same for operator $A^0_z = a_0(z, hD)$ where $a_0(x,\xi)$ is the principal symbol of A; this z is not necessarily the original one, but distance between them should not exceed $c\gamma$; similarly in (6.4.54) term with $m=0$ coincides with the left-hand expression with $a(x,\xi)$ replaced by $a(z,\xi)$.

What is more, under ξ-microhyperbolicity condition (6.4.15) integration with respect to x is not needed, so all these results would hold (without factor γ^d in the decomposition and estimates) without it; thus one can take $z=x$ (or y, does not matter).

Thus we arrive to

Proposition 6.4.15. *Let $\mathsf{I}^{\mathsf{T}}_{\gamma}$ be defined by* (6.4.22) *with $\omega=1$ and ψ_1,ψ_2 replaced by $\psi_{1,\gamma},\psi_{2,\gamma}$ which are the same smooth functions scaled at some point z with the parameter $\gamma\in(h^{1-\delta},h^{\delta})$.*

Further, let $\mathsf{I}^{\mathsf{T}\,0}_{\gamma}$ be defined the same way but with $U(t)$ replaced by $U^0(t) = e^{ih^{-1}tA^0}$ where $A^0 = a(z,hD)$ and later z is set to x. Then

$$\mathsf{I}^{\mathsf{T}}_{\gamma} - \mathsf{I}^{\mathsf{T}\,0}_{\gamma} = O(h^{1-d}\gamma^d).$$

Now we can calculate I^{T} in the scalar case:

Proposition 6.4.16. *In the framework of Proposition 6.4.15 as ω satisfies* (6.4.17) *and $\kappa>0$*

$$\mathsf{I}^{\mathsf{T}} - \mathsf{I}^{\mathsf{T}\,0} = O(h^{1-d-\kappa}) \tag{6.4.55}$$

where $\mathsf{I}^{\mathsf{T}\,0}$ is defined for constant-coefficient operator obtained by freezing coefficients of A at point x (or y, does not matter).

Proof. Consider three zones: $\{|x-y| \gtrsim \gamma_1\}$ with $\gamma_1 \asymp h^{\delta}$, $\{\gamma \lesssim |x-y| \lesssim \gamma_1\}$ with $\gamma_0 \asymp h^{1-\delta}$ and $\{|x-y| \lesssim \gamma\}$.

Then the contribution of the first zone to the reminder for I^{T} and $\mathsf{I}^{\mathsf{T}\,0}$ does not exceed $Ch^{1-d}\gamma_1^{-1-\kappa} = O(h^{1-d-\kappa})$ (while main parts are 0); in virtue of proposition 6.4.15 and decomposition (6.4.20)–(6.4.21) the contribution of the second zone to (6.4.55) does not exceed $O(h^{1-d}\gamma^{-\kappa}) = o(h^{1-d-\kappa})$.

In the third zone one can apply the method of successive approximations resulting in

$$\mathsf{I}^{\mathsf{T}} - \mathsf{I}^{\mathsf{T}\,0} \sim h^{-d} \sum_{m+n+k\ge 1} \varkappa''_{mnk} h^{-d+n-m+k-\kappa}\gamma^{2m-n}.$$

However since the final answer does not depend on γ only terms with $2m = n$ are posed to survive just resulting in $\bigl(\varkappa + o(1)\bigr)h^{-d+1-\kappa}$. □

Summarizing results of Subsection 6.4.2, Proposition 6.4.16 and formulae (6.4.31)–(6.4.34) we arrive to

Theorem 6.4.17. *Let A be a scalar operator satisfying conditions* (6.4.9) *and* (6.4.15). *Then*

$$\mathsf{I} = \int \mathsf{J}(x)\psi_1(x)\psi_2(x)\,dx + O(h^{1-d-\kappa}), \tag{6.4.56}$$

where

$$\begin{aligned}\mathsf{J}(x) = G(x)h^{-d-\kappa} = {}& \\ 2(2\pi h)^{-2d} \iiint E(x,\xi,0)\Omega(x,z)E(x,\eta,0)e^{ih^{-1}\langle z,\xi-\eta\rangle}\,dzd\xi d\eta = {}& \\ 2(2\pi h)^{-2d} \iiint_{\{a(x,\xi)<0,\ a(x,\eta)<0\}} \Omega(x,z)e^{ih^{-1}\langle z,\xi-\eta\rangle}\,dzd\xi d\eta & \end{aligned} \tag{6.4.57}$$

with

$$\begin{aligned} G(x) = \iint E(x,\xi,0)\widehat{\Omega}(x,\xi-\eta)E(x,\eta,0)\,d\xi & \\ \iint_{\{a(x,\xi)<0,\ a(x,\eta)<0\}} \hat{\Omega}(x,\xi-\eta)\,d\xi d\eta, & \end{aligned} \tag{6.4.58}$$

and $\hat{\Omega}$ is defined by (6.4.34).

Remark 6.4.18. (i) Alternatively one can prove this theorem using oscillatory integral representation of $u(x,y,t)$ as $|t| \le T = \epsilon$.

(ii) Alternatively one can replace one or both copies of x in $E(x,.,.)$ or in $a(x,0)$ by y.

Definition 6.4.19. We will refer to formulae (6.4.56)–(6.4.58), (6.4.34) as to *standard Weyl expression* even in the matrix case. However in this case the the last parts of (6.4.56), (6.4.57) containing $a < 0$ should be skipped.

6.4.5 Rescaling Technique

Now our goal is to weaken and eventually to get rid of (ξ-)microhyperbolicity condition for the Schrödinger operator.

Schrödinger Operator

Let us recall that for a Schrödinger operator condition of ξ-microhyperbolicity (6.4.15) means that

$$V \ge \epsilon_0. \tag{6.4.59}$$

If this condition is violated let us introduce scaling functions $\rho(x)$, $\gamma(x)$ in the usual way $\gamma = \epsilon|V|$ and $\rho = \gamma^{\frac12}$.

Then, as $\rho\gamma \ge h$, due to ξ-microhyperbolicity of the rescaled operator, the contribution of $B(\bar{x},\gamma(\bar{x})) \times B(\bar{x},\gamma(\bar{x}))$ to the remainder does not exceed

$$C\big(\frac{h}{\rho\gamma}\big)^{1-d-\kappa}\gamma^{-\kappa} \asymp Ch^{1-d-\kappa}\rho^{d-1-\kappa}\gamma^{d-1} \tag{6.4.60}$$

with $\rho = \rho(\bar{x})$ and $\gamma = \gamma(\bar{x})$. Therefore the total contribution of the zone

$$\big\{(x,y) : |x-y| \le \epsilon\gamma(x)\big\} \tag{6.4.61}$$

(where automatically $\gamma(x) \asymp \gamma(y)$) to the remainder does not exceed

$$Ch^{1-d-\kappa}\int \rho^{d-1+\kappa}\gamma^{-1}\,dx \tag{6.4.62}$$

and with $\rho = \gamma^{\frac12}$ here it becomes $Ch^{1-d-\kappa}M$ with

$$M := \int \gamma^{\frac12(d-3+\kappa)}\,dx + 1; \tag{6.4.63}$$

obviously, $M = O(1)$ provided *either* $d + \kappa \ge 3$ *or*

$$|V| + |\nabla V| \ge \epsilon_0. \tag{6.4.64}$$

and $d + \kappa > 1$ (which is surely the case).

Remark 6.4.20. (i) Note that (6.4.64) is microhyperbolicity condition (6.4.13).

(ii) As usual, one should take $\rho\gamma \ge Ch$ and thus to add $Ch^{\frac{1}{3}}$ and $Ch^{\frac{2}{3}}$ to ρ, γ respectively. However it does not affect our conclusion due to the standard fact that if $\rho\gamma \asymp h$ then $h_{\mathsf{eff}} \asymp 1$ and condition (6.4.64) is not needed.

Consider now the complement of the zone (6.4.61). Let us consider γ-admissible partitions $\phi_\alpha(x)$ and $\phi_\beta(y)$. We say that elements are adjacent $\alpha \sim \beta$ if $|x-y| \le \max(\gamma(x), \gamma(y))$ as $x \in \operatorname{supp}\phi_\alpha$, $y \in \operatorname{supp}\phi_\beta$; then $\gamma(x) \asymp \gamma(y)$ and we already calculated the contribution of all pairs of adjacent elements. Now we need to consider contribution of all pairs of not adjacent elements with $|x-y| \asymp \nu$.

Due to our previous analysis the contribution of such group does not exceed

$$C\nu^{-\kappa} \| \sum_{\alpha \not\sim \beta} \phi_\alpha E(0) \phi_\beta \|_1 \tag{6.4.65}$$

where summation is taken over elements belonging to the given group.

Due to the standard theory of Subsubsection *5.3.1.3 Asymptotics without Spatial Mollification* for such elements $e^{\mathsf{W}}(x,y,0) = 0$ and therefore

$$|e(x,y,0)| \le R_x^{\frac{1}{2}} R_y^{\frac{1}{2}} \le Ch^{1-d} \big(\rho(x)^{d-1}\gamma(x)^{-1}\big)^{\frac{1}{2}} \big(\rho(y)^{d-1}\gamma(y)^{-1}\big)^{\frac{1}{2}}. \tag{6.4.66}$$

There are two cases: $d \ge 3$ and $d = 1, 2$. In the former case we do not need ξ-microhyperbolicity condition at all, so we can increase γ, ρ to $\nu^{\frac{1}{2}}$, ν respectively. Then the norm in (6.4.66) does not exceed $Ch^{1-d}\nu^{\frac{1}{2}(d-3)} \times \nu^d$; the contribution of each ν-group does not exceed $Ch^{1-d}\nu^{\frac{1}{2}(3d-3)-\kappa}$ and contribution of all ν-groups does not exceed $Ch^{1-d}\nu^{\frac{1}{2}(d-3)-\kappa}$.

Finally, summation with respect to ν results in Ch^{1-d} as $d-3 > 2\kappa$, $Ch^{1-d}|\log h|$ as $d-3 = 2\kappa$ and $Ch^{1-d} \times h^{\frac{2}{3}(\frac{1}{2}(d-3)-\kappa)}$ as $d-3 < 2\kappa$ (since the least value of ν is $h^{\frac{2}{3}}$; each of these expressions does not exceed $Ch^{1-d-\kappa}$).

Consider now case $d = 1, 2, 3$ (so $d = 3$ fits both analysis). Observe that

$$\sup \big(\rho(x)^{d-1}\gamma(x)^{-1}\big)^{\frac{1}{2}} \le Ch^{\frac{1}{3}(d-3)}$$

and therefore

$$\|\sum_{\alpha\gamma\beta}\phi_\alpha E(0)\phi_\beta\|_1 \le Ch^{-\frac{2}{3}d}\int \bigl(\rho(x)^{d-1}\gamma(x)^{-1}\bigr)^{\frac{1}{2}}\,dx$$

and then contribution of each ν-group does not exceed

$$Ch^{-\frac{2}{3}d}\nu^{-\kappa}\int \bigl(\rho(x)^{d-1}\gamma(x)^{-1}\bigr)^{\frac{1}{2}}\,dx$$

with integral over $B(z,\nu)$. Then the total contribution of all ν-groups does not exceed the same expression with integral taken over $\{\gamma(x)\le\nu\}$. Finally summation with respect ν results in

$$Ch^{-\frac{2}{3}d}\int \gamma(x)^{\frac{1}{4}(d-3)-4\kappa}\,dx$$

which is less than $Ch^{1-d-\kappa}M$ because $\gamma\ge h^{\frac{2}{3}}$.

So, we arrive to

Proposition 6.4.21. *Consider Schrödinger operator. Then the the standard Weyl asymptotics* (6.4.56)–(6.4.58), (6.4.34) *holds with the remainder estimate* $Ch^{1-d-\kappa}$ *provided* <u>*either*</u> $d+\kappa>3$ <u>*or*</u> *condition* (6.4.64) *is fulfilled.*

This completely covers the case $d\ge 3$. Furthermore, after Proposition 6.4.21 is proven, we can introduce scaling functions

$$\gamma=\rho=\epsilon\bigl(|V|+|\nabla V|^2\bigr)^{\frac{1}{2}}+Ch^{\frac{1}{2}} \tag{6.4.67}$$

and then applying the same arguments we arrive to

Proposition 6.4.22. *Consider Schrödinger operator. Then the the standard Weyl asymptotics* (6.4.56)–(6.4.58), (6.4.34) *holds with the remainder estimate* $Ch^{1-d-\kappa}$ *provided* <u>*either*</u> $d+\kappa>2$ <u>*or*</u> *condition*

$$|V|+|\nabla V|+|\nabla^2 V|\ge\epsilon \tag{6.4.68}$$

is fulfilled.

This completely covers the case $d=2$. As $d=1$ we get the required remainder estimate under condition (6.4.68).

Now, combining this with the arguments of the proof of Theorem 5.2.14 we arrive to

Proposition 6.4.23. *Consider Schrödinger operator with* $d = 1$, $\kappa > 0$. *Then the standard Weyl asymptotics* (6.4.56)–(6.4.58), (6.4.34) *holds with the remainder estimate* $O(h^{1-d-\kappa})$.

We are leaving easy details of both proofs to the reader; see also the proofs of the next subsection.

Remark 6.4.24. Actually all above results hold as $\kappa = 0$ as well with the singular exception of $d = 1$ when the remainder estimate $O(1)$ is recovered under condition

$$\sum_{|\beta|\le K} |\nabla_x^\beta V| \ge \epsilon; \tag{6.4.69}$$

without it the remainder estimate is $O(h^{-\delta})$ with arbitrarily small $\delta > 0$.

6.5 Mollification with Respect to Non-Spectral Parameters

6.5.1 Linear Dependence

Let operator $A = A_\zeta$ depend on parameter $\zeta \in (-1, 1)$ smoothly and microhyperbolically, i.e.,

$$\operatorname{Re} \langle \partial_\zeta a v, v\rangle \ge \epsilon_0 \|v\|^2 - c\|av\|^2 \qquad \forall \zeta \quad \forall (x,\xi) \in \Omega \quad \forall v \in \mathbb{H} \tag{6.5.1}$$

where a is the principal symbol of A.

Then the propagation of singularities theory yields that $u_\zeta(x, y, t)$ is negligible in zones $\{|t| \le T_0, \pm t > 0, \pm z \le \pm\epsilon_1 t - T, |\tau| \le \epsilon_2\}$ with appropriate constants $\epsilon_j > 0$ and $T_0 > 0$ where z is dual to the ζ variable and $T > h^{1-\delta}$ is a parameter, $\delta > 0$.

In particular, $u_\zeta(x, y, t)$ is negligible for $t = \pm T_0$, $\pm z \le \epsilon_3$. Moreover, let us assume that

(6.5.2) A_ζ is a linear function of ζ,

$$\operatorname{Re} \langle \partial_\zeta A_\zeta u, u\rangle \ge 0 \quad \forall u. \tag{6.5.3}$$

Then using the standard energy estimates technique with $\psi = \psi(z)$ one can prove easily that U is negligible in zones $\{T_0 \le \pm t \le h^{-l} = T_1, \pm z \le \frac{1}{2}\epsilon_3\}$ with an arbitrary exponent l.

In the z-representation, let us apply the operator $\phi_1(hL^{-1}D_z)$ where $\phi_1 \in \mathscr{C}_0^K([-\frac{1}{2}, \frac{1}{2}])$ and set $z = 0$; in the ζ-representation this procedure means multiplication by $\phi_1(\zeta L^{-1})$ and integration with respect to $\zeta \in \mathbb{R}$. The resulting function is negligible for $T_1 \ge |t| \ge T, \tau \in [-\epsilon_2, \epsilon_2]$ and for $|t| \le h^{\frac{1}{2}+\delta}$ this function was constructed by a successive approximation method. Thus, for $\tau, \tau' \in [-\frac{1}{2}\epsilon_2, \frac{1}{2}\epsilon_2]$ we obtain the complete asymptotic expansion for

$$\frac{1}{L}\int \phi_1\bigl(\frac{\zeta}{L}\bigr)(\Gamma Q_x'^{\,t} Q_y'' e)(\tau', \tau; \zeta)\, d\zeta \tag{6.5.4}$$

and therefore for $\tau \in [-\frac{1}{2}\epsilon_2, \frac{1}{2}\epsilon_2]$ we obtain the complete asymptotic expansion for

$$\frac{1}{LL_1}\iint \phi\bigl(\frac{\tau'}{L_1}\bigr)\phi_1\bigl(\frac{\zeta}{L}\bigr)(\Gamma Q_x'^{\,t} Q_y'' e)(\tau', \tau; \zeta)\, d\zeta d\tau'. \tag{6.5.5}$$

We assume in these expansions that L and L_1 are such that $1 \ge L \ge h^{\frac{1}{2}-\delta}$ (because the only condition on L is $LT \ge h^{1-\delta}$ and we can take $T = h^{\frac{1}{2}+\delta}$) and $1 \ge L_1 \ge 0$.

Moreover, if a has only one eigenvalue in the interval $[-h^{\delta'}, h^{\delta'}]$ with a small enough exponent $\delta' > 0$ (in which case this eigenvalue is of constant multiplicity) then the same arguments as for mollification on the spectral parameter in Theorem 5.2.1 yield that we can weaken the restrictions on L and obtain the condition $1 \ge L \ge h^{1-\delta}$.

Furthermore, later for the two-dimensional Schrödinger and Dirac operators with strong magnetic field we will be able to weaken this condition also, replacing it by the same condition as for mollification with respect to spectral parameter. So the following statement has been proven:

Proposition 6.5.1. *(i) Under conditions* (6.5.1)–(6.5.3) *all the theorems and complete asymptotics for mollification with respect to spectral parameter remain true if we replace mollification with respect to spectral parameter by mollification with respect to ζ with the same restrictions on L; here we assume that $\tau', \tau \in [-\epsilon, \epsilon]$ with a small constant $\epsilon > 0$.*

(ii) Moreover, these theorems and asymptotics remain true if we also include mollification with respect to the spectral parameter τ' with $L_1 \in [0,1]$ where we take $e(\lambda,\tau,\zeta)\,d\lambda d\zeta$ instead of $d_\tau(.,\tau)$ in the original mollification.

On the other hand, if we drop conditions (6.5.2) and (6.5.3) then we are forced to take $T_1 = T_0$ [24]. Then the Tauberian method provides the following

Proposition 6.5.2. *Under condition* (6.5.1) *in the theorems on asymptotics without mollification we can drop the microhyperbolicity condition provided we mollify with respect to ζ and take $\tau', \tau \in [-\epsilon,\epsilon]$ (we can mollify also with respect to ζ, τ' and take $\tau \in [-\epsilon,\epsilon]$). Here we have the same restrictions on L and L_1 as before (recall that in the scalar and similar cases these restrictions are weaker than in the general case).*

Let us observe that applying the same energy estimates technique as before in the framework of (6.5.1) and (6.5.2) we can reach $T_1 = \epsilon h^{-2}$ and so we can improve the remainder estimate in Proposition 6.5.2 under the additional condition (6.5.2).

6.5.2 Microhyperbolic Case

Let us treat the general case and assume that

(6.5.6) At level 0 the operator A is microhyperbolic with respect to (x,ξ) in Ω for every $\zeta \in (-1,1)$.

Then taking into account that under this condition for $T = T_0$ and $\tau_1, \tau_2 \in [-\epsilon,\epsilon]$ with $|\tau_2 - \tau_1| \le hT^{-1}$ the trace norm of the operator $Q'E(\tau_1,\tau_2)Q''$ does not exceed Ch^{1-d} and that this norm does not exceed $Ch^{-m}(|\tau_1|+|\tau_2|+1)^m$ for general $\tau_1, \tau_2 \in \mathbb{R}$ with appropriate m, we obtain from the formula

$$F_{t\to h^{-1}\tau}\chi_T(t)u = \int \hat{\chi}_T((\tau-\tau')h)\, d_{\tau'}e(\tau_1,\tau')$$

that (after multiplication by Q'_x and ${}^tQ''_y$) the trace norm of the left-hand expression does not exceed Ch^{1-d} for $T = T_0$ and $\tau \in [-\frac{1}{2}\epsilon, \frac{1}{2}\epsilon]$.

[24] But we can increase T_1 under standard conditions on the Hamiltonian flow or under the conditions of Section 6.2.

This and the previous arguments easily yield that for $|T| \ge 2T_0$ and $|\tau| \le \frac{1}{2}\epsilon$ the trace norm of

$$F_{t\to h^{-1}\tau}\chi_{T_0,T}(t)MQ'_x u\, {}^tQ''_y$$

does not exceed $CTh^{1-d}(hL^{-1})^{3s}$ where $L \in [h,1]$ is arbitrary, $M = M_L$: $Mv = L^{-1}\int \phi(\zeta L^{-1})v_\zeta\, d\zeta$. Moreover, for $|T| \le 2T_0$ this norm does not exceed Ch^{1-d}. But then the trace norm of

$$F_{t\to h^{-1}\tau}\chi_{T}(t)MQ'_x u\, {}^tQ''_y$$

does not exceed $Ch^{1-d} + CT^2h^{1-d}(hL^{-1})^{3s}$ and therefore the standard Tauberian arguments with $T_1 = (\frac{h}{L})^{-s}$ yield that

$$|M\Gamma Q'_x\Big(e - \frac{1}{h}\int_{\tau_1}^{\tau_2} F_{t\to h^{-1}\tau}\bar\chi_{T_1}(t)u\, {}^tQ''_y\, d\tau\Big)| \tag{6.5.7}$$

does not exceed $Ch^{1-d}(hL^{-1})^s$.

Let us now consider

$$|M\Gamma Q'_x\frac{1}{h}\int_{\tau_1}^{\tau_2} F_{t\to h^{-1}\tau}\big(\bar\chi_{T_1}(t) - \bar\chi_{T_0}(t)\big)u\, {}^tQ''_y\, d\tau| = |M\Gamma Q'_x F_{t\to h^{-1}\tau}\frac{1}{t}\big(\bar\chi_{T_1}(t) - \bar\chi_{T_0}(t)\big)u\, {}^tQ''_y\Big|_{\tau=\tau_1}^{\tau=\tau_2}|.$$

The previous analysis yields that the right-hand expression of this equality does not exceed $CT_1^2h^{1-d}(hL^{-1})^{3s} = Ch^{1-d}(hL^{-1})^s$ and therefore we can replace T_1 by T_0 in (6.5.7) without increasing the error. However, under condition (6.5.6) for $T = T_0$ we have complete asymptotics for $\Gamma Q'_x F_{t\to h^{-1}\tau}\bar\chi_T(t)u\, {}^tQ''_y$ and hence we have obtained *complete asymptotics for* $M\Gamma Q'_x e\, {}^tQ''_y$ *with remainder estimate* $Ch^{1-d}(hL^{-1})^s$.

Remark 6.5.3. (i) Without condition (6.5.6) we would obtain remainder estimate $Ch^{-d}(hL^{-1})^s$.

(ii) Let us assume that condition (6.5.1) is fulfilled for all $\zeta \in (-1,1)$. Using a partition of unity and the arguments of Section 5.2 we obtain that under conditions $(5.2.20)_1$ the remainder estimate is $O\big(h^{\frac{1}{2}-d}(hL^{-1})^s\big)$; under condition $(5.2.20)_2^+$ the remainder estimate is $O\big(h^{1-d}(hL^{-1})^s\big)$ and under condition $(5.2.20)_2$ the remainder estimate is $O\big(h^{1-d}(hL^{-1})^s|\log h|\big)$.

Moreover, the more advanced arguments of Section 5.2 yield that under condition $(5.2.20)_2$ the remainder estimate is $O\big(h^{1-d}(hL^{-1})^s\big)$ as well and, moreover, we get the same remainder estimate under conditions $(5.2.20)_1$ and $(5.2.100)_m$.

Furthermore, under condition $(5.2.20)_1$ only the remainder estimate is $O\big(h^{1-d-\delta}(hL^{-1})^s\big)$ with an arbitrarily small exponent $\delta > 0$ and the same estimate holds under condition (5.2.126).

(iii) Similarly, under conditions of a global nature linked to the Hamiltonian trajectories the remainder estimate can be improved; now in the remainder estimates $T_1 = (hL^{-1})^{-s}T_2$ where T_2 is the time controlled by the Hamiltonian trajectories.

(iv) For Schrödinger and Dirac operators with strong magnetic field we will be able to obtain similar results.

6.5.3 Averaging

Starting from the previous results and applying a partition of unity with respect to ζ we obtain that if condition $\varphi \in \mathscr{C}_0^K([-\frac{1}{2},\frac{1}{2}])$ is replaced by the condition (4.2.73)–(4.2.75) then in general under condition (6.5.6) the remainder estimate is $O\big(h^{1-d}\vartheta(hL^{-1})\big)$, etc.: in all cases the remainder estimates are as good as those we obtained for similar averaging with respect to τ.

Therefore under conditions (6.5.1)–(6.5.3) the estimates for averaging remain true if they are taken with respect to ζ (instead of the spectral parameter).

6.A Appendices

6.A.1 Hamiltonian Flows of Commuting Operators

Proposition 6.A.1. *Let a be scalar and*

$$\{a,b\} = 0 \tag{6.A.1}$$

Then as long as we do not leave domain where (6.A.1) *holds*

$$\Psi_{b,t}(\{a=\tau\}) \subset \{a=\tau\}; \tag{6.A.2}$$

$$\Psi_{b,t}\circ\Psi_{a,t'} = \Psi_{a,t'}\circ\Psi_{b,t} = \Psi_{at'+bt,1} \tag{6.A.3}$$

Proof. Since a is scalar $\Psi_{a,t} = e^{t\nabla^{\#}a}$ is single-valued. Due to (6.A.1) $\nabla^{\#}a(b) = 0$ and $\Psi_{a,t'}$ preserves b: $b \circ \Psi_{a,t'} = b$. Then $\Psi_{a,t'}$ also preserves $K_b(z)$ and $K_b^{\sigma}(z)$ and therefore $\Psi_{a,t'}$ also preserves $\Psi_{b,t}$. The first of two equalities (6.A.3) is proven.

Then

$$\Psi_{a,t'} \circ \Psi_{b,t} = \bigl(\Psi_{a,n^{-1}t'}\bigr)^n \circ \bigl(\Psi_{b,n^{-1}t}\bigr)^n = \bigl(\Psi_{a,n^{-1}t'}\Psi_{b,n^{-1}t}\bigr)^n$$

and due to the fact that $\Psi_{a,t'}$ preserves $\Psi_{b,t}$ the latter expression is

$$\bigl(\Psi_{n^{-1}(at'+bt),1} + O(n^{-2})\bigr)^n$$

which tends to $\Psi_{at'+bt,1}$ as $n \to \infty$.

Finally, by definition

$$K_b(z) = \{\ell : \langle \ell(b)v, v\rangle > 0 \ \forall v \in \operatorname{Ker} b\}$$

and thus $\ell \in K_b(z)$ implies that $\ell + \alpha\nabla^{\#}a(z) \in K_b(z)$ for any $\alpha \in \mathbb{R}$. Then $\zeta \in K_z^{\sigma}(b)$ implies that $\sigma(\zeta, \nabla^{\#}a) = 0$ i.e. $\zeta(a) = 0$. Therefore $K_z^{\sigma}(b) \subset T\{a = \tau\}$ as $a(z) = \tau$. This implies (6.A.2). □

Problem 6.A.2. Does (6.A.3) hold as both a and b are matrix symbols of two commuting operators so $[a, b] = 0$ and

$$-i\{a, b\} + [a', b] + [a, b'] = 0$$

with

$$\{a, b\} \coloneqq \frac{1}{2}\sum_j \bigl(a^{(j)}b_{(j)} + b_{(j)}a^{(j)} - a_{(j)}b^{(j)} - b^{(j)}a_{(j)}\bigr),$$

a', b' are subprincipal symbols (so we actually have a standard formula for Poisson brackets but including symmetrized products of $a^{(j)}$ and $b_{(j)}$ and $b^{(j)}$ and $a_{(j)}$?

6.A.2 One Property of Real Residues

Proposition 6.A.3. *Let $F(z)$ be analytic in $\mathbb{C} \setminus [\tau_1, \tau_2]$ with $[\tau_1, \tau_2] \subset \mathbb{R}$ and $-\infty < \tau_1 \le \tau_2 < \infty$. Then*

$$\int_{\tau_1'}^{\tau_2'} (\operatorname{Res}_{\mathbb{R}} F)(\tau)\, d\tau = 2\pi i \operatorname{Res}_{\infty} F \tag{6.A.4}$$

as $-\infty < \tau_1' < \tau_1$, $\tau_2 < \tau_2' < \infty$. In particular,

$$\operatorname{Res}_{\mathbb{R}} F(\tau) = \partial_\tau\bigl(k\tau + g(\tau)\bigr), \quad k = 2\pi i \operatorname{Res}_{\infty} F, \quad \operatorname{supp}(g) \subset [\tau_1, \tau_2]. \tag{6.A.5}$$

Proof. Obviously the left hand expression in (6.A.4) equals to $-\int_\gamma F(z)\,dz$ where γ is any contour winding once in the counter-clock-wise direction around $[\tau_1,\tau_2]$ and in turn equals to the right-hand expression. □

6.A.3 Eigenvalues of Some Model Operators

Proof of (6.3.105). Indeed, eigenfunctions of Laplacian on the sphere $\mathbb{S}^d$ are closely connected to homogeneous harmonic polynomials in $\mathbb{R}^{d+1} \ni x$: $P(x) = |x|^n\phi(x/|x|)$ and denoting $|x| = r$, $x/|x| = y$ we arrive to

$$0 = \Delta_x P(x) = \bigl(-\partial_r^2 - dr^{-1}\partial_r + r^{-2}\Delta_y\bigr)\bigl(r^n\phi(y)\bigr) = r^{n-2}\bigl(-n(n-1) - nd + \Delta_y\bigr)\phi(y)$$

which results in (6.3.106) in the statement "N_n is equal to the dimensional of the space of all harmonic polynomials of degree n in $\mathbb{R}^{d+1}$". However such polynomial $P(x)$ is uniquely defined by $P(0,x')$ and $\partial_{x_1}P(0,x')$ which are arbitrary homogeneous polynomials of degrees n and $(n-1)$ respectively on $\mathbb{R}^d \ni x' = (x_2,\dots,x_{d+1})$. Since dimension of the space of homogeneous polynomials of degree n polynomials on $\mathbb{R}^d$ is equal to $\frac{(n+d-1)!}{n!(d-1)!}$ we get (6.3.107). □

Proof of (6.3.113). Let $A = \Delta - r^{-1}$; let us rewrite it as

$$A = -\partial_r^2 - (d-1)r^{-1} + r^{-2}\Lambda - r^{-1} \tag{6.A.6}$$

where Λ is Laplacian on $\mathbb{S}^{d-1}$. Then plugging $m(m+d-2)$ as its eigenvalues we get for eigenvalues $-\lambda$ of A

$$\Bigl(\partial_r^2 + (d-1)r^{-1}\partial_r - m(m+d-2)r^{-2} + r^{-1} - \lambda\Bigr)u = 0;$$

plugging $u = r^m v$ we get

$$\Bigl(\partial_r^2 + (d-1+2m)r^{-1}\partial_r + r^{-1} - \lambda\Bigr)v = 0.$$

Plugging $v = e^{-\sqrt{\lambda}r}w$ to compensate the last term we get after multiplication by r

$$\Bigl(r\partial_r^2 - 2\sqrt{\lambda}r\partial_r + (d-1+2m)\partial_r + \bigl(1-(d-1+2m)\sqrt{\lambda}\bigr)\Bigr)w = 0.$$

This equation scales to

$$\Big(r\partial_r^2 - 2\sqrt{\lambda}\beta r\partial_r + (d-1+2m)\partial_r + \beta\big(1-(d-1+2m)\sqrt{\lambda}\big)\Big)w = 0.$$

and picking up $\beta = 1/2\sqrt{\lambda}$ we arrive to

$$\Big(r\partial_r^2 - r\partial_r + (d-1+2m)\partial_r + \big(1-(d-1+2m)\sqrt{\lambda}\big)/2\sqrt{\lambda}\Big)w = 0 \tag{6.A.7}$$

which is the equation defining associated Laguerre polynomial $L_k^{(\alpha)}$ as

$$\begin{aligned}(d-1+2m) &= \alpha+1\\ \big(1-(d-1+2m)\sqrt{\lambda}\big)/2\sqrt{\lambda} &= k\end{aligned} \tag{6.A.8}$$

with $\alpha = 0, \dots, k$. The second equation here is

$$1/\sqrt{\lambda} = d-1+2m+2k$$

or

$$\lambda = \lambda_n := (2n+d-1)^{-2}, \qquad n = m+k. \tag{6.A.9}$$

This proves (6.3.114). To calculate ν_n we need to sum (6.3.107) with d replaced by $(d-1)$ and n by m from $m=0$ to $m=n$. This proves (6.3.115).□

Part III

Local and Microlocal Semiclassical Spectral Asymptotics near the Boundary

Chapter 7

Standard Local Semiclassical Spectral Asymptotics near the Boundary

In this chapter we continue the treatment of local and microlocal semiclassical spectral asymptotics. Now with the singular exception of Section 7.5 we consider the case when our (unit) "Ball of Light" intersects the boundary of the domain in a part of the boundary which is smooth. This chapter consists of five sections mainly matching to the corresponding sections of Chapter 4.

Namely, Section 7.1 is devoted to the preliminary analysis; here we obtain rough estimates of the spectral function. More precisely, we find that there are three different cases; two of them (when the spectral function on a fixed interval has order h^{-d} and $O(h^{\infty})$ respectively) we have met before; in the third case the operator A is elliptic (on the given energy level) but the boundary value problem $(A, \eth B)$ is not, and the spectral function has order h^{1-d}–exactly as in the case of elliptic operator on the closed manifold of dimension $(d-1)$. This is not a coincidence.

In Section 7.2 we find successive approximations, which are now essentially more complicated because we need to consider the boundary value problem. The good news is that if the operator A is not elliptic we have already constructed free-space propagator and we need to consider only the part of propagator generated by the boundary; if the operator A is elliptic the "free space" propagator is negligible.

In Sections 7.3 and 7.4 we prove the basic theorems of this chapter. Again

in the former we derive asymptotics with the remainder estimate $O(h^{1-d})$ (if $(A, \partial B)$ is microhyperbolic and A is not elliptic on the energy level in question) or $O(h^{2-d})$ (if $(A, \partial B)$ is microhyperbolic but not elliptic and A is elliptic on the energy level in question). In the latter section we improve estimates $O(h^{1-d})$ to $o(h^{1-d})$ or even slightly better under non-periodicity condition of the global nature.

In Section 7.5 as in Section 4.6 we study operators with the irregular coefficients and reimplement method of R. Seeley; in conjugation with our own technique it works extremely nicely. In particular the microhyperbolicity condition for $(A, \partial B)$ disappears leaving us with the microhyperbolicity condition for A only which is a nice development even in the smooth case.

Finally, in Appendix 7.A we prove that the set of "bad" energy levels where the microhyperbolicity property is violated has measure 0; this result is reasonable even if there is no boundary.

7.1 Preliminary Analysis

7.1.1 General Case

In this chapter we treat the operator

$$A = \sum_{|\alpha|\le m} a_\alpha(x,h)(hD)^\alpha \tag{7.1.1}$$

in the domain $X \subset \mathbb{R}^d$ assuming that

$$X \cap B(0,1) = \mathbb{R}^d_+ \cap B(0,1) \tag{7.1.2}$$

where $\mathbb{R}^d_+ = \mathbb{R}_+ \times \mathbb{R}^{d-1}$ [1)] the slightly more general case

$$X \cap B(0,1) = \{x_1 \ge \phi(x')\} \cap B(0,1), \quad |D^\beta \phi| \le c \quad \forall \beta, |\beta| \le K \tag{7.1.3}$$

is reduced to the former after a regular change of coordinates which preserves (in particular) the smoothness condition

$$\|D^\beta a_\alpha\| \le c \qquad \forall \alpha : |\alpha| \le m, \quad \forall \beta : |\beta| \le K; \tag{7.1.4}_1$$

we assume that

[1)] Except the last Section 7.5 describing irregular operators and domains with irregular boundary.

(7.1.5) $\mathbb{H} = \mathbb{C}^D$ and so a_α are $D \times D$ matrices.

We also assume that the boundary operator B on $Y = \partial X$ is given; more precisely we assume that

(7.1.6) $u \in \mathscr{C}_0^m(B(0,1), \mathbb{H})$ belongs to $\mathfrak{D}(A)$ if and only if

$$B_k u|_Y = 0 \qquad k = 1, \dots, l \tag{7.1.7}$$

where

$$B_k = \sum_{|\alpha| \le m_k} b_{k\alpha}(x', h)(hD)^\alpha \tag{7.1.8}$$

where $0 \le m_1 < m_2 < \dots < m_l \le m-1$, $b_{k\alpha}$ are $\mathsf{D}_k \times \mathsf{D}$ matrices and

$$\|D^\beta b_{k\alpha}\| \le c \qquad \forall \alpha, |\alpha| \le m_k, \quad \forall \beta, |\beta| \le K \quad \forall k = 1, \dots, l. \tag{7.1.4)$_2$}$$

Moreover, we assume that the boundary is not characteristic, i.e., we assume that

$$\|a_{(m,0,\dots,0)}(x)^{-1}\| \le c \qquad \forall x \in B(0,1). \tag{7.1.9}$$

Finally we assume that

(7.1.10) A is a self-adjoint operator in $\mathscr{L}^2(X, \mathbb{H})$ where $\mathfrak{D}(A)$ satisfies condition (7.1.6).

One can then prove that then

$$\mathsf{D}_1 + \dots + \mathsf{D}_l = \frac{m\mathsf{D}}{2}. \tag{7.1.11}$$

Let us introduce a condition similar to (3.1.9): there exist symbols $\beta_j(z)$ with $j = 1, \dots, m$ and $\beta = (\beta_1, \dots, \beta_m)$ such that

$$T(z) + (\beta^\dagger b)(z) + (b^\dagger \beta)(z) = 0 \qquad \forall z \in T^*(Y \cap B(0,1)) \tag{7.1.12}$$

where $T(z)$ is written in the form (3.1.7) after A has been written in the form (3.1.1) and operators A_k have been defined, with principal symbols a_k,

and b is the principal symbol of the boundary operator $B = (B_{(1)}, \dots, B_{(m)})$ where the boundary conditions (7.1.7) have been rewritten in the form

$$Bu = \sum_{1 \le j \le m} B_{(j)} (hD_1)^{j-1} u|_Y = 0. \tag{7.1.13}$$

Condition (7.1.12) follows from the equality

$$(Au, v)_X + \sum_{1 \le k \le l} (B_k u, B'_k v)_Y = (u, Av)_X + \sum_{1 \le k \le l} (B'_k u, B_k v)_Y \qquad \forall u, v \in \mathscr{C}_0^m(B(0,1), \mathbb{H}) \tag{7.1.14}$$

where B'_k are h-differential operators of orders $(m - m_k - 1)$ with the coefficients satisfying condition $(7.1.4)_2$. Let us note that conditions (7.1.14), (7.1.9) and (7.1.11) yield that

$$\operatorname{rank} b(z) = \frac{m\mathsf{D}}{2} \qquad \forall z \in T^*(Y \cap B(0,1)) \tag{7.1.15}$$

and

$$\operatorname{rank} b_{k(m_k,0,\dots,0)}(x) = \mathsf{D}_k \qquad \forall x \in Y \cap B(0,1) \tag{7.1.16}$$

where these conditions are satisfied uniformly (i.e., there exits a minor of the corresponding dimension the absolute value of which is not less than ϵ_0).

Let $e(x, y, \tau)$ be the Schwartz kernel of the spectral projector of A and let $e(x, y, \tau_1, \tau_2) = e(x, y, \tau)|_{\tau=\tau_1}^{\tau=\tau_2}$.

We start from a statement similar to Proposition 4.2.1:

Theorem 7.1.1. *Let conditions* (7.1.1), (7.1.2), $(7.1.4)_1$, (7.1.9) *and* (7.1.10) *be fulfilled. Let* $\psi_1, \psi_2 \in \mathscr{C}_0^K(B(0, 1-\epsilon))$, $\epsilon > 0$ *and let* $q_1, q_2 \in \mathsf{S}'_{h,K}$ *be supported in* $\{|\xi'| \le c_1\}$, $Q_i = \mathsf{Op}(q_i)$. *Then*

$$\|Q_{1x}\psi_1(x)e(.,.,\tau_1,\tau_2)\psi_2(y)^t Q_{2y}\| \le C' h^{-d} \big(1 + |\tau_1| + |\tau_2|\big)^{\frac{1}{m}} \qquad \forall x, y \in \mathbb{R}^d \quad \forall \tau_1, \tau_2 \in \mathbb{R} \tag{7.1.17}$$

with $K = K(d, m)$, $C' = C'(d, m, c, c_1, \epsilon)$ *defined in the same way as in Proposition 4.2.1 (with the corresponding norms of* q_i*).*

Proof. Let us note first of all that the conditions of this theorem imply the very rough estimate

$$\|D_1^j \psi u\|_{(-n)} \le C' h^{-n}(\|Au\| + \|u\|) \qquad \forall j = 0, \dots, m-1 \tag{7.1.18}$$

where in the left-hand expression the $\mathscr{L}^2$-norm with respect to x_1 and the $\mathscr{H}^{(-n)}$-norm with respect to x' is calculated, $\psi \in \mathscr{C}_0^K(B(0, 1-\epsilon'))$, C' also depends on $\epsilon' > 0$ and ψ, and $n = n(d, m)$.

In fact, (7.1.9) yields that

$$(hD_1)^m \psi u = \psi Au + \sum_{|\alpha| \le m, \alpha_1 \le m-1} (hD)^\alpha a'_\alpha u \tag{7.1.19}$$

where $a'_\alpha \in \mathscr{C}_0^K(B(0, 1-\epsilon'))$ and hence

$$\begin{aligned}(7.1.20)_j \quad (hD_1)^j \psi u = \mathcal{K}^{m-j} \psi Au + \\ \sum_{|\alpha| \le m, m-j \le \alpha_1 \le m-1} (hD')^{\alpha'} (hD_1)^{\alpha_1 - m + j} a'_\alpha u + \\ \sum_{|\alpha| \le m, \alpha_1 \le m-j} \mathcal{K}^{m-j-\alpha_1} (hD')^{\alpha'} a'_\alpha u\end{aligned}$$

where $(\mathcal{K}u)(x) = ih^{-1} \int_\infty^{x_1} u(t, x')dt$.

Substituting $(7.1.20)_1$ into $(7.1.20)_2$ and continuing this process we remove from the right-hand expression of $(7.1.20)_m$ every degree of hD_1; in this expression there will only be multiple compositions of operators of multiplication by smooth compactly supported functions, hD' and $\mathcal{K}$; this representation yields (7.1.18) with $n = n(m)$.

Let $u \in \mathscr{C}_0^m(B(0, 1), \mathbb{H})$. Then (7.1.9) yields that

$$\begin{aligned}\|(hD_1)^m u\| \le C\|Au\| + C \sum_{|\alpha| \le m,\, \alpha_1 \le m-1} \|(hD)^\alpha u\| \le \\ C\|Au\| + C \sum_{|\alpha| \le m,\, \alpha_1 = 0} \|(hD)^\alpha u\|\end{aligned}$$

where we have applied the inequality

$$\sum_{|\alpha| \le m,\, \alpha_1 \le m-1} \|(hD)^\alpha u\| \le \epsilon \|(hD_1)^m u\| + C_\epsilon \sum_{|\alpha| \le m,\, \alpha_1 = 0} \|(hD)^\alpha u\|$$

with arbitrarily small $\epsilon > 0$. Then substituting $u = Q_1\psi_1 v$ we again obtain by means of (7.1.18) that

$$\sum_{j\le m} \|(hD_1)^j Q_1\psi_1 v\| \le C(\|Av\| + \|v\|) + \sum_{j\le m-1} Ch\|(hD_1)^j Q_1'\psi_1' v\|$$

where Q_1' and ψ_1' are an operator and function of the same type as Q_1 and ψ_1 but with c_1 and ϵ replaced by $2c_1$ and $\frac{1}{2}\epsilon$. Applying the same arguments to Q_1' and ψ_1' repeatedly we finally obtain estimate

$$\sum_{j\le m} \|(hD_1)^j Q_1\psi_1 v\| \le C(\|Av\| + \|v\|). \tag{7.1.21}$$

Next, applying inequality

$$\|v(x_1,.)\|' \le Ch^{-\frac{1}{2}}\|v\|^{\frac{2m-1}{2m}} \sum_{j\le m} \|(hD_1)^j v\|^{\frac{1}{2m}} \tag{7.1.22}$$

with the $\mathscr{L}^2$-norm with respect to x' in the left-hand expression we conclude that

$$\|(Q_1\psi_1 v)(x_1,.)\|' \le Ch^{-\frac{1}{2}}\big(\|v\| + \|Av\|^{\frac{1}{2m}}\|v\|^{\frac{2m-1}{2m}}\big).$$

Observing that instead of Q_1 we could take in this analysis $(hD)^\alpha Q_1$ with any fixed α we obtain from the Sobolev embedding theorems that

$$\|(Q_1\psi_1 v)(x)\| \le Ch^{-\frac{d}{2}}\Big(\|v\| + \|Au\|^{\frac{1}{2m}}\|v\|^{1-\frac{1}{2m}}\Big). \tag{7.1.23}$$

Substituting $v = Q_1\psi_1 E(\tau_1,\tau_2)w$ we conclude that

(7.1.24) The operator norm of operator $M = Q_1\psi_1 E(\tau_1,\tau_2) : \mathscr{L}^2(X,\mathbb{H}) \to \mathbb{H}$ does not exceed $Ch^{-\frac{d}{2}}(1+|\tau_1|+|\tau_2|)^{\frac{1}{2m}}$,

and since $Q_{1x}\psi_1(x)e(.,.,\tau_1,\tau_2)\psi_2^t Q_2 = MN^*$ where $N = Q_2^*\bar\psi_2 E(\tau_1,\tau_2)$ is an operator of the same type as M we obtain (7.1.17). □

Remark 7.1.2. Our proof implies the estimate

$$\|D_{x_1}^j D_{y_1}^k Q_{1x}\psi_1(x)e(.,.,\tau_1,\tau_2)\psi_2(y)^t Q_{2y}\| \le C'h^{-d-j-k}\big(1+|\tau_1|+|\tau_2|\big)^{\frac{j+k+1}{m}} \tag{7.1.17}'$$

for $j \le m-1, k \le m-1$. If we apply our analysis to the powers of the operator A instead of A itself we obtain this estimate for arbitrary j,k but now K and C' also depend on j,k.

7.1.2 Elliptic Case

Our next step is to treat the zone where A is elliptic:

Theorem 7.1.3. *Let conditions* (7.1.1), (7.1.2), $(7.1.4)_1$, (7.1.9) *and* (7.1.10) *be fulfilled and let* Q_i *and* ψ_i *be the same as in Theorem 7.1.1*[2)]. *Let us assume that*

$$\|(\tau - a(x,\xi))^{-1}\| \le c \qquad \forall (x,\xi') \in \operatorname{supp}(q_1) \cup \operatorname{supp}(q_2) \quad \forall \xi_1 \in \mathbb{R} \quad \forall \tau \in [\tau_1, \tau_2]. \tag{7.1.25}$$

Then for $\tau_1', \tau_2' \in [\tau_1, \tau_2]$ *the following estimates hold:*

$$\|D_{x_1}^j D_{y_1}^k Q_{1x} \psi_1(x) e(.,.,\tau_1',\tau_2') \psi_2(y)^t Q_{2y}\| \le C' h^{-d-j-k} (1 + |\tau_1'| + |\tau_2'|)^{\frac{j+k+1}{m}} \big(1 + \frac{1}{h}(x_1 + y_1)\big)^{-s} \tag{7.1.26}$$

and

$$\|\Gamma'' Q_{1x} \psi_1(x) e(.,.,\tau_1',\tau_2') \psi_2(y)^t Q_{2y}\| \le C' h^{1-d} (1 + |\tau_1'| + |\tau_2'|)^{\frac{j+k+1}{m}} \tag{7.1.27}$$

where Γ'' *is introduced by* (3.3.43).

Proof. Let $\chi \in \mathscr{C}_0^K([-1,1]$ be fixed. Then partially repeating arguments of the proof of Theorem 7.1.1 (see also the proof of Proposition 4.2.2) we obtain that

$$\|F_{t \to h^{-1}\tau} \chi_{T,T'}(t)_x D_{x_1}^j Q_1' \psi_1' u\| \le C' T h^{-\frac{1}{2}-j} (1 + |\tau|)^{\frac{1}{m}(j+\frac{1}{2})} \|v\| \tag{7.1.28}$$

where $v \in \mathscr{L}^2(X, \mathbb{H})$ is arbitrary, $u = \exp(ih^{-1}tA)v$, $T \ge h$, $T' \in \mathbb{R}$ and $\tau \in \mathbb{R}$ are arbitrary, Q_i' and ψ_i' are operators and functions of the same type as Q_i and ψ_i, equal to 1 in the neighborhoods of the supports of Q_i and ψ_i respectively, and $\mathfrak{d}$ is the operator of restriction to $Y = \partial X$.

Now let $\mathfrak{C}$ be the operator realizing the reduction of Subsubsection *3.1.2.1 Reduction to a Special Case* in a neighborhood of $\operatorname{supp} Q_1' \times [\tau_1, \tau_2]$; then estimate (7.1.28) also holds for u, v replaced by $\tilde{u} = \mathfrak{C}u, \tilde{v} = \mathfrak{C}v$.

Moreover, $\tilde{u}$ is divided into two blocks $(\tilde{u}', \tilde{u}'')$ and Subsubsection *3.1.2.2 Elliptic Theory* yields that one of them $(\tilde{u}'')$ is negligible while the second

[2)] Including condition $\operatorname{supp}(q_i) \subset \{|\xi'| \le c_1\}$.

satisfies the estimate

$$\|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)D^j_{x_1}Q_1\psi_1\tilde{u}'\|' \le C'Th^{-\frac12-j}(1+|\tau|)^{\frac1m(j+\frac12)}\|v\|\big(1+\frac{x_1}{h}\big)^{-s} \quad \forall\tau\in[\tau_1,\tau_2] \tag{7.1.29}$$

where in the left-hand expression the $\mathscr{L}^2$-norm with respect to x' is calculated; then the same estimate holds also for $\tilde{u}'$ replaced either by $\tilde{u}=\mathfrak{C}v$ or by u itself.

Further, here we can replace Q_1 by $(hD')^\alpha Q_1$ and hence the embedding theorems yield that

$$\|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)Q_1\psi_1 u\| \le C'Th^{-\frac d2}(1+|\tau|)^{\frac{1}{2m}}\big(1+\frac{x_1}{h}\big)^{-s}\|v\| \quad \forall x\in\bar{\mathbb{R}}^+\times\mathbb{R}^{d-1} \quad \forall\tau\in[\tau_1,\tau_2].$$

Taking into account the connection between u, v and propagator $U = U(x,y,t)$ we conclude that

$$\|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)Q_{1x}\psi_1(x)f^\dagger U\| \le C'T^{-\frac d2}(1+|\tau|)^{\frac{1}{2m}}(1+\frac{x_1}{h})^{-s} \quad \forall x\in\bar{\mathbb{R}}^+\times\mathbb{R}^{d-1} \quad \forall\tau\in[\tau_1,\tau_2] \quad \forall f\in\mathbb{H}$$

where in the left-hand expression the $\mathscr{L}^2$-norm with respect to y is calculated. Repeating the same arguments with x replaced by y we obtain from this estimate and from the rough estimate with $s=0$ (which holds for all $\tau\in\mathbb{R}$) estimate

$$\|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)Q_{1x}\psi_1(x)U\psi_2(y)^tQ_{2y}\| \le C'T^{-d}(1+|\tau|)^{\frac12}\big(1+\frac{1}{h}(x_1+y_1)\big)^{-s} \quad \forall x,y\in\bar{\mathbb{R}}^+\times\mathbb{R}^{d-1} \quad \forall\tau\in[\tau_1,\tau_2].$$

Obviously, one can assume that $|\tau_2-\tau_1|\ge 1$. Applying the standard simplified Tauberian arguments we obtain estimate (7.1.26) from which estimate (7.1.27) follows. □

Let us now treat the completely elliptic zone:

Theorem 7.1.4. *Let conditions* (7.1.1), (7.1.2), $(7.1.4)_{1,2}$, (7.1.9), (7.1.10) *and* (7.1.12) *be fulfilled. Let Q_i and ψ_i be the same as in Theorem 7.1.1.*

Let us assume that condition (7.1.18) *is fulfilled and that boundary problem is elliptic at the domain in question:*

$$\|\eth b(x', D_{x_1}, \xi')v\| \ge \epsilon_0\|v\|_+ \quad \forall v \in \mathscr{S}(\bar{\mathbb{R}}^+): \quad \big(a(0, x', D_{x_1}, \xi')v - \tau\big) = 0 \quad \forall (x', \xi') \in (\mathrm{supp}(q_1) \cup \mathrm{supp}(q_2)) \cap Y \quad \forall \tau \in [\tau_1, \tau_2]. \tag{7.1.30}$$

Then the following estimate holds for $\tau_1', \tau_2' \in [\tau_1, \tau_2]$*:*

$$\|D_{x_1}^j D_{y_1}^k Q_{1x}\psi_1(x)e(., ., \tau_1', \tau_2')\psi_2(y)^t Q_{2y}\| \le C' h^s \big(1 + |\tau_1'| + |\tau_2'|\big)^{\frac{j+k+1}{m}}. \tag{7.1.31}$$

Proof. The proof is similar to the proof of Proposition 4.2.6 but also uses Theorem 3.1.2 (see also the end of the proof of the next Theorem 7.1.5). We leave easy details to the reader. □

Theorem 7.1.5. *Let conditions* (7.1.1), (7.1.2), $(7.1.4)_{1,2}$, (7.1.9), (7.1.10) *and* (7.1.12) *be fulfilled. Let us assume that*

$$\langle a(x, \xi_1, \xi')v, v\rangle \ge (\epsilon_0|\xi|^m - c)\|v\|^2 \quad \forall x \in B(0,1) \cap X \quad \forall \xi_1 \in \mathbb{R}, \xi' = 0 \quad \forall v \in \mathbb{H} \tag{7.1.32}$$

and that

$$\sum_{1\le j\le m} |\tau|^{-\frac{1}{m}(m_j+\frac{1}{2})}\|\eth b_j(x', D_1, \xi')v\| \ge \epsilon_0\|v\|_+^2 \quad \forall x' \in B(0,1) \cap Y,\ \xi' = 0 \quad \forall \tau \le -c \quad \forall v \in \mathscr{S}(\bar{\mathbb{R}}^+, H):\ (a(0, x', D_1, \xi') - \tau)v = 0. \tag{7.1.33}$$

Assume that Q_i *and* ψ_i *are the same as in Theorem 7.1.1. Then for* $\tau \le -C_1 = C_1(d, c, c_1)$ *the following estimate holds:*

$$\|D_{x_1}^j D_{y_1}^k Q_{1x}\psi_1(x)e(., ., -\infty, \tau)\psi_2(y)^t Q_{2y}\| \le C' h^s |\tau|^{-s}. \tag{7.1.34}$$

Proof. Obviously (7.1.32) and (7.1.33) imply the same conditions for symbols a and b_j from which all the terms not containing ξ_1^m and $\xi_1^{m_j}$ respectively are excluded. Then for these symbols the following estimate holds:

$$\sum_j |\tau|^{\frac{1}{m}(2m-2m_j-1)}\|\bar{\eth} b_j(x', D_1, \xi')v\|^2 + \|(a(0, x', D_1, \xi') - \tau)v\|_+^2 \geq \epsilon_0 \sum_{0\leq k\leq m} |\tau|^{\frac{2(m-k)}{m}}\|D_1^k v\|_+^2 - c|\xi'|^m\|v\|_+^2 \tag{7.1.35}$$

and hence the same estimate holds also for the original symbols a and b_j where now $\xi' \in \mathbb{R}^d$, $\tau \leq -C_1$ and $v \in \mathscr{S}(\bar{\mathbb{R}}^+, \mathbb{H})$ are arbitrary and c, C_1 are increased if necessary. Therefore h-Fourier transform with respect to x' and rescaling with respect to x_1 yield that if a and b_j do not depend on x and $A = a(hD)$, $B = b(hD)$ then

$$\sum_j |\tau|^{\frac{2m-2m_j-1}{m}}\|\bar{\eth} B_j u\|_Y^2 + \|(A-\tau)u\|^2 \geq \epsilon_0 \sum_{0\leq k\leq m} |\tau|^{\frac{2(m-k)}{m}}\|(hD_1)^k u\|_+^2 - c\sum_{|\alpha|\leq m}\|(hD')^\alpha u\|^2 \tag{7.1.36}$$

for $u \in \mathscr{C}_0^K(B(0,1), \mathbb{H})$.

This estimate is obviously preserved when we perturb operators A and B_j by arbitrary bounded lower order terms; moreover, it is preserved when we perturb these operators by operators of the same orders with small coefficients (i.e., with small maxima of the $\mathcal{L}(\mathbb{H}, \mathbb{H})$-norms of the coefficients).

Hence for general operators A and B_j estimate (7.1.36) remains true provided we replace u by $\psi_\nu u$ where $\psi_\nu \in \mathscr{C}_0^K(B(y_\nu, \epsilon))$, $y_\nu \in B(0,1)$ and $\epsilon > 0$ is a small constant. Taking real-valued functions ψ_ν such that $\sum_\nu \psi_\nu^2 = 1$ in $B(0, 1+\frac{\epsilon}{2})$ and summing estimates (7.1.36) for $\psi_\nu u$ with respect to ν we obtain inequality (7.1.36) for u in the general case.

Let us substitute

$$u = \phi\big(\frac{\tau}{L}\big) F_{t\to h^{-1}\tau}\chi_T(t)\bar{\psi}_1 Q_1 \psi_1 v$$

into (7.1.36) where $v = \exp(ih^{-1}tA)w$, $\chi \in \mathscr{C}_0^K([-1,1])$, $\phi \in \mathscr{C}_0^K([-c, -\epsilon_1,])$, $Q_1 = \mathsf{Op}(q_1)$, $q_1 \in \mathsf{S}_{h,\rho,1,K}(\mathbb{R}^{d-1}, \mathbb{C})$ is supported in $\{|\xi'| \leq c_2^{-1}\rho\}$, and $\rho = L^{\frac{1}{M}}$ with an appropriate constant c_2.

Let $\bar{\psi}_1$, $\bar{q}_1$, $\bar{\phi}$ and $\bar{\chi}$ be functions of the same type as ψ_1, q_1, ϕ and χ, equal to 1 in the neighborhoods of the respective supports of the latter functions. Then (7.1.36) yields that for $L \le -C_1$, $TL \ge h$

$$\sum_{0\le k\le m} \|(hD_1)^k \phi\big(\frac{h}{L}D_t\big)\chi_T(t)Q_1\psi_1 v\| \le$$
$$C'hL^{-1}\sum_{0\le k\le m} \|(hD_1)^k \bar{\phi}\big(\frac{h}{L}D_t\big)\bar{\chi}_T(t)\bar{Q}_1\bar{\psi}_1 v\| +$$
$$C'h^sL^{-s}\sum_{0\le k\le m-1} \|(hD_1)^k \bar{\phi}\big(\frac{h}{L}D_t\big)\bar{\chi}_T(t)v\|_{(-m)}$$

where we calculate the $\mathscr{L}^2$-norm with respect to (x,t) everywhere but the last term in which we calculate the $\mathscr{L}^2$-norm with respect to (x_1,t) and the $\mathscr{H}^{(-m)}$-norm with respect to x', which obviously does not exceed

$$C'h^sL^{-s}\Big(\|\bar{\phi}\big(\frac{h}{L}D_t\big)\chi_T(t)v\| + \|A\bar{\phi}\big(\frac{h}{L}D_t\big)\bar{\chi}_T(t)v\|\Big);$$

replacing Av by hD_tv and permuting hD_t with $\bar{\chi}_T(t)$ we obtain that none of these expressions exceeds $C'h^sL^{-s}\|v_0\|$. Thus,

$$\sum_{0\le k\le m} \|(hD_1)^k \phi\big(\frac{h}{L}D_t\big)\chi_T(t)Q_1\psi_1 v\| \le$$
$$C'hL^{-1}\sum_{0\le k\le m} \|(hD_1)^k \bar{\phi}\big(\frac{h}{L}D_t\big)\bar{\chi}_T(t)\bar{Q}_1\bar{\psi}_1 v\| + C'h^sL^{-s}\|w\|.$$

Then a repeated argument yields that

$$\|(hD_1)^j \phi\big(\frac{h}{L}D_t\big)\chi_T(t)Q_1\psi_1 v\| \le C'h^sL^{-s}\|w\|$$

for $j = 0,\dots,m-1$. Then the non-characteristic property of $\partial X = \{x_1 = 0\}$ yields that this estimate holds for all $j \in \mathbb{Z}^+$. Therefore the embedding theorems yield that

$$\|(hD_1)^j \phi\big(\frac{h}{L}D_t\big)\chi_T(t)Q_1\psi_1 v\| \le C'h^sL^{-s}\|w\| \qquad \forall x \in B(0,1).$$

Further, this inequality and the relation between v, w and propagator U imply that

$$\|(hD_{x_1})^j\phi\big(\frac{h}{L}D_t\big)\chi_T(t)Q_{1x}\psi_1(x)f^\dagger U\| \le C'h^sL^{-s}\|f\| \qquad \forall x\in B(0,1)\quad \forall f\in\mathbb{H}$$

where in the left-hand expression the $\mathscr{L}^2$-norm with respect to (y,t) is calculated.

Repeating all these arguments for y instead of x we easily obtain inequality

$$\|(hD_{x_1})^j(hD_{y_2})^k\phi\big(\frac{h}{L}D_t\big)\chi_T(t)Q_{1x}\psi_1(x)f^\dagger U\psi_2(y)^tQ_{2y}\| \le C'h^sL^{-s}\|f\| \qquad \forall x,y\in B(0,1)\quad \forall f\in\mathbb{H}.$$

where in the left-hand expression the $\mathscr{L}^2$-norm with respect to t is calculated. Therefore we arrive to estimate

$$\|F_{t\to h^{-1}\tau}(hD_{x_1})^j(hD_{y_2})^k\chi_T(t)Q_{1x}\psi_1(x)U\psi_2(y)^tQ_{2y}\| \le C'h^s|\tau|^{-s} \qquad \forall x,y\in B(0,1)\quad \forall \tau\le -C_1.$$

The rest of the proof coincides with the end of the proof of Proposition 4.2.8. □

Remark 7.1.6. Note that we have proven this theorem under the restriction $\mathsf{supp}(q_i)\subset\{|\xi'|\le C_0^{-1}|\tau|^{\frac{1}{m}}\}$ which is weaker than $\mathsf{supp}(q_i)\subset\{|\xi'|\le c_1\}$.

The following two statements are similar to Propositions 4.2.9 and 4.2.10 and are proven in the same way (taking into account the proof of Theorem 7.1.5).

Theorem 7.1.7. *Let conditions* (7.1.1), (7.1.2), $(7.1.4)_{1,2}$, (7.1.9), (7.1.10) *and* (7.1.12) *be fulfilled and let*

$$|\xi|^m\|v\| \le c(\|a(x,\xi)v\|+\|v\|) \qquad \forall x\in B(0,1)\cap X\quad \forall \xi\in\mathbb{R}^d\quad \forall v\in\mathbb{H} \tag{7.1.37}$$

and

$$|\xi'|^m\|v\|_+ \le c(\|a(0,x',D_1,\xi')v\|_+ + \sum_j |\xi'|^{m-m_j-\frac12}\|\mathring{\partial} b_j(x',D_1,\xi')v\| + \|v\|_+) \quad \forall x' \in B(0,1)\cap Y \quad \forall \xi' \in \mathbb{R}^{d-1} \quad \forall v \in \mathscr{S}(\bar{\mathbb{R}}^+,\mathbb{H}). \tag{7.1.38}$$

Let us assume that $Q_i = \mathsf{Op}(q_i)$, $q_i \in \tilde{\mathsf{S}}^{0\prime}_{h,\rho,\gamma,K}$,

$$\rho\gamma \ge h^{1-\delta}\omega^\delta, \qquad \omega \ge 1, \quad \gamma \le 1 \tag{7.1.39}$$

where $\delta > 0$ *is arbitrarily small; let* ψ_1 *and* ψ_2 *be the same as above. Then*

(i) If the symbol q_i *is supported in* $\{|\xi'| \ge \omega\}$ *for some* $i = 1,2$ *then for* $\omega \ge C_0$, $|\tau_1'| + |\tau_2'| \le \epsilon_2\omega^m$, $x, y \in \mathbb{R}^+ \times \mathbb{R}^{d-1}$,

$$\|D^j_{x_1} D^k_{y_1} Q_{1x}\psi_1(x) e(.,.,\tau_1',\tau_2')\psi_2(y)^t Q_{2y}\| \le C' h^s \omega^{-s} \tag{7.1.40}$$

where C *and* C' *include products of the norms of* ψ_i, q_i *(*$i = 1,2$*) in the corresponding spaces and in the case of* C' *these spaces and* C' *itself depend on* s.

(ii) In the general case

$$\|D^j_{x_1} D^k_{y_1} Q_{1x}\psi_1(x) e(.,.,\tau_1',\tau_2')\psi_2(y)^t Q_{2y}\| \le C' h^{-d-j-k}\big(|\tau_1'| + |\tau_2'| + 1\big)^{\frac{d+j+k}{m}}. \tag{7.1.41}$$

(iii) If condition (7.1.32) *is fulfilled then*

$$\|D^j_{x_1} D^k_{y_1} Q_{1x}\psi_1(x) e(.,.,\tau_1',\tau_2')\psi_2(y)^t Q_{2y}\| \le Ch^{1-d-j-k}\big(|\tau_1'| + |\tau_2'| + 1\big)^{\frac{d+j+k-1}{m}} \quad \forall \tau_1' \le -C_0, \tau_2' \le -C_0. \tag{7.1.42}$$

Remark 7.1.8. Obviously, under condition (7.1.37) it is sufficient to check condition (7.1.38) only for $v \in \mathscr{S}(\bar{\mathbb{R}}^+,\mathbb{H})$ such that $a(0,x'D_1,\xi')v = 0$.

Theorem 7.1.9. *Let conditions* (7.1.1), (7.1.2), $(7.1.4)_{1,2}$, (7.1.9), (7.1.10), (7.1.12), (7.1.37) *and* (7.1.38) *be fulfilled.*

Further let conditions (7.1.32) *and* (7.1.33) *be fulfilled for all* $\xi \in \mathbb{R}^d$ *and* $\xi' \in \mathbb{R}^{d-1}$ *respectively*[3]. *Furthermore, let* Q_i *and* ψ_i *be the same as in Theorem 7.1.7. Then*

(i) For $\tau \le -C_0$ *estimate* (7.1.28) *holds.*

(ii) Moreover, if q_i *is supported in* $\{|\xi'| \ge \omega\}$ *for some* $i = 1, 2$ *then estimate* (7.1.28) *remains true with an additional factor* ω^{-s} *in the right-hand expression.*

(iii) In the general case

$$\|D^j_{x_1} D^k_{y_1} Q_{1x}\psi_1(x) e(.,.,-\infty,\tau)\psi_2(y)^t Q_{2y}\| \le Ch^{d-j-k}(|\tau|+1)^{\frac{d+j+k}{m}}. \tag{7.1.43}$$

Remark 7.1.10. (i) Condition (7.1.37) means exactly that the senior symbol[4] of a is elliptic and condition (7.1.38) means exactly that the senior symbols of a, $b_1, \dots, b_l$ satisfy the Shapiro-Lopatinskii condition.

(ii) If (7.1.38) holds then condition (7.1.33) for all ξ' means exactly that the bottom of the spectrum of the operator $a(0, x', D_1, \xi')$ on $\mathscr{L}^2(\mathbb{R}^+, \mathbb{H})$ with the boundary condition $b(x', D_1, \xi')v = 0$ is not less than $\bigl(\epsilon_0^{-1}|\xi'|^m - C_0\bigr)$.

7.1.3 Operators with Minimal Smoothness Conditions

On the following step of our analysis we will prove assertions similar to Propositions 4.2.11 and 4.2.12, i.e., obtain estimates under minimal restrictions[5] on smoothness:

Theorem 7.1.11. *Assume that conditions* (7.1.1), (7.1.2), $(7.1.4)_{1,2}$ *with* $K = \max(\frac{d}{2}+1, m) + n$, (7.1.9), (7.1.10), (7.1.14), (7.1.37) *and* (7.1.38) *are fulfilled. Then*

[3] That means exactly that the operator $a(0, x', D_1, \xi')$ in $\mathscr{L}^2(\bar{\mathbb{R}}^+, \mathbb{H})$ with the boundary condition $\eth b(x', D_1, \xi')v = 0$ is semibounded from below uniformly with respect to (x', ξ').

[4] I.e., the higher degree homogeneous symbol.

[5] Later in Section 7.5 we will apply different approaches (partly similar to those of Section 4.7) to consider even far less regular case.

(i) The following estimate holds:

$$\|D_x^\alpha D_y^\beta e(x,y,\tau_1',\tau_2')\| \le C'' h^{-d-|\alpha|-|\beta|}\big(1+|\tau_1'|+|\tau_2'|\big)^{\frac{1}{m}(d+|\alpha|+|\beta|)} \quad \forall x,y\in B(0,1-\epsilon) \quad \forall \alpha,\beta : |\alpha|\le n, |\beta|\le n. \tag{7.1.44}$$

(ii) Moreover, if condition

$$\|(a(x,\xi)-\tau)v\| \ge \epsilon_0\big(|\xi|^m+\omega\big)\|v\| \quad \forall x\in B(0,1)\cap X \quad \forall \xi\in\mathbb{R}^d \quad \forall v\in\mathbb{H} \quad \forall \tau\in[\tau_1,\tau_2] \tag{7.1.45}$$

is fulfilled with $\omega = |\tau|+1$ then estimate

$$\|D_x^\alpha D_y^\beta e(x,y,\tau_1',\tau_2')\| \le C'' h^{-d-|\alpha|-|\beta|}\big(1+|\tau_1'|+|\tau_2'|\big)^{\frac{1}{m}(d+|\alpha|+|\beta|-1)}\big(1+\frac{1}{h}(x_1+y_1)\big)^{-s} \tag{7.1.46}$$

holds with the same x, y, α, β as before and with $\tau_1',\tau_2'\in[\tau_1,\tau_2]$;

(iii) Moreover, if condition (7.1.45) *is fulfilled and*

$$\|(a(0,x',D_1,\xi')-\tau)v\|_+ + \sum_j \big(|\xi'|+\omega^{\frac{1}{m}}\big)^{m-m_j-\frac{1}{2}}\|\mathfrak{d}b_j(x',D_1,\xi')v\| \ge \epsilon_0(|\xi'|^m+\omega)\|v\|_+ \quad \forall x'\in B(0,1)\cap Y \quad \forall \xi'\in\mathbb{R}^{d-1} \quad \forall v\in\mathscr{S}(\bar{\mathbb{R}}^+,\mathbb{H}) \quad \forall \tau\in[\tau_1,\tau_2] \tag{7.1.47}$$

then estimate

$$\|D_x^\alpha D_y^\beta e(x,y,\tau_1',\tau_2')\| \le C' h^s \omega^{-s} \tag{7.1.48}$$

holds with the same x, y, α, β as before and with $\tau_1',\tau_2'\in[\tau_1,\tau_2]$.

Theorem 7.1.12. *Assume that conditions* (7.1.1), (7.1.2), $(7.1.4)_{1,2}$ *with $K=\max(\frac{d}{2}+1,m)+n$,* (7.1.9), (7.1.10) *and* (7.1.14) *are fulfilled. Further, let the following conditions be fulfilled:*

$$\epsilon_0|\xi|^m\|v\|^2 \le \big(\langle a(x,\xi)v,v\rangle + c\|v\|^2\big) \quad \forall x\in B(0,1)\cap X \quad \forall \xi\in\mathbb{R}^d \quad \forall v\in\mathbb{H} \tag{7.1.49}$$

and

$$\epsilon_0|\xi'|^m\|v\|^2_+ \le \big(\langle a(0,x',D_1\xi')v,v\rangle_+ + c\|v\|^2_+\big) \tag{7.1.50}$$
$$\forall x'\in B(0,1)\cap Y \quad \forall \xi'\in\mathbb{R}^{d-1} \quad \forall v\in \mathscr{S}(\bar{\mathbb{R}}^+,\mathbb{H})\colon\ \eth b(x'D_1,\xi')v=0.$$

Then estimate

$$\|D_x^\alpha D_y^\beta e(x,y,-\infty,\tau)\| \le \begin{cases} C''h^s(1+|\tau|)^{-s} & \forall \tau\le -C_0\\ C''h^{-d-|\alpha|-|\beta|}(1+|\tau|)^{\frac{1}{m}(d+|\alpha|+|\beta|)} \end{cases} \tag{7.1.51}$$

holds with the same x, y, α, β as before.

Proof of Theorems 7.1.11 and 7.1.12. Let us note first of all that in the framework of Theorem 7.1.11(i)

$$\sum_{|\alpha|\le n}\|(hD)^\alpha v\| \le \tag{7.1.52}$$
$$C\Big(\sum_{|\beta|\le n-m}\|(hD)^\beta Av\| + \|v\| + \sum_j \sum_{|\beta|\le n-m_j-1} h^{\frac12}\|(hD')^\beta B_j v\|_{Y,\frac12}\Big)$$
$$\forall v\in \mathscr{C}_0^K(B(0,1),\mathbb{H})$$

where

$$\|v\|_{Y,\frac12} \coloneqq \inf\{\sum_{1\le k\le d}\|hD_k w\| :\quad \eth w = v\} \tag{7.1.53}$$

and $m\le n\le K$; this estimate is similar to (4.2.54).

Estimate (7.1.52) can be proven in the constant coefficient case by means of the Fourier transform with respect to x' and then it could be extended to the general case by means of a partition of unity. Then we easily obtain (see the proof of Proposition 4.2.11) that estimate (7.1.52) remains true for $v\in \mathscr{C}^n(B(0,1),\mathbb{H})$ provided that in the left and right-hand expressions the norms on $B(0,1-\epsilon)\cap X$ and $B(0,1-\frac{\epsilon}{2})\cap X$, $B(0,1-\frac{\epsilon}{2})\cap Y$ are calculated.

Iterating if necessary we obtain for $n\in m\mathbb{Z}^+$, $n\le K$ estimate

$$\sum_{|\alpha|\le n}\|(hD)^\alpha v\|_{B(0,1-\epsilon)} \le C\big(\|A^{\frac{n}{m}}v\|_{B(0,1-\epsilon')} + \|v\|_{B(0,1-\epsilon')}\big) \tag{7.1.54}$$
$$\forall v\in \mathfrak{D}(A^{\frac{n}{m}}).$$

On the other hand, for $0 \le n \le m$, (7.1.52) yields the inequality

$$\sum_{|\alpha|\le n} \|(hD)^\alpha v\|_{B(0,1-\epsilon)} \le \tag{7.1.55}$$
$$C\|v\|^{1-\frac{n}{m}}_{B(0,1-\frac{\epsilon}{2})} \cdot \|Av\|^{\frac{n}{m}}_{B(0,1-\frac{\epsilon}{2})} + C\|v\|_{B(0,1-\frac{\epsilon}{2})} \qquad \forall v \in \mathfrak{D}(A).$$

Substituting $E(\tau_1', \tau_2')v$ for v we obtain from (7.1.54) and (7.1.55) that the norm of the operator

$$(hD)^\alpha E(\tau_1', \tau_2') : \mathscr{L}^2(X, \mathbb{H}) \to \mathscr{L}^2(B(0, 1-\epsilon) \cap X, \mathbb{H})$$

does not exceed $C(|\tau_1'| + |\tau_2'| + 1)^{\frac{|\alpha|}{m}}$ for all $\alpha : |\alpha| \le K$. The rest of proof of Theorem 7.1.11(i) repeats the end of the proof of Proposition 4.2.11.

In a similar way the estimate

$$\|(hD)^\alpha F_{t\to h^{-1}\tau}\chi_T(t)v\|_{B(0,1-\epsilon)} \le C''T(|\tau|+1)^{\frac{|\alpha|}{m}}\|w\| \tag{7.1.56}$$

can be proven for $v = e^{ih^{-1}tA}w$, $w \in \mathfrak{D}(A^{\frac{n}{m}})$, $|\alpha| \le n$ with $n \in m\mathbb{Z}^+$, $n \le K$ where $T \ge h^{-1}$ and $\chi \in \mathscr{C}_0^\infty([-1,1])$ is fixed.

Starting from this estimate one can prove Theorems 7.1.11(iii) and 7.1.12 in the same way as we proved Propositio 4.2.11(ii). Namely, in estimates (7.1.52) and (7.1.54) for $\tau \in [\tau_1, \tau_2]$, the term $\|v\|$ in the right hand expression is replaced by $(1+|\tau|)^{-s}h^s\|v\|$ and A is replaced by $A - \tau$; all the other arguments repeat those of the proof of Proposition 4.2.11(ii).

Let us now prove Theorem 7.1.11(ii). Let us first note that without any loss of the generality we can assume that $|\tau_1| + |\tau_2| \le c$. In fact, for $d \ge 2$ it is sufficient to prove estimate (7.1.45) under this restriction or the restriction $c\epsilon_0 \le \tau_1/\tau_2 \le c$, $|\tau_j| \ge 1$; the general case can be reduced to these special cases by division of the interval $[\tau_1, \tau_2]$ into intervals $[-2^l, -2^{l-1}]$ and two other intervals, the first of which satisfies the first restriction and the second of which satisfies the second restriction.

However if $\epsilon_0\omega \le \pm\tau_j \le c\omega \quad \forall j = 1, 2$ with $\omega \ge 1$ then dividing A by ω and replacing τ_j and h by $\tau_j\omega^{-1}$ and $h\omega^{-\frac{1}{m}}$ respectively we reduce the second case to the first one. On the other hand, one can easily see that if $d = 1$, $|\tau| \ge C_0$ and (7.1.45) holds then (7.1.47) also holds and hence it is sufficient for us to treat the case $|\tau_1| + |\tau_2| \le c$.

But if condition (7.1.45) is fulfilled then

$$\sum_{|\alpha|\le n} \|(hD)^{\alpha} u\| \le C \sum_{|\beta|\le n-m} \|(hD)^{\beta}(A-\tau)u\| + Ch^{s}\|u\| \qquad \forall \tau \in [\tau_1,\tau_2] \quad \forall u \in \mathscr{C}_0^{n}\big(B(0,1)\cap\{x_1 \ge -\epsilon\},\mathbb{H}\big) \tag{7.1.57}$$

where here (and only here) we calculate the $\mathscr{L}^2$-norms on $B(0,1)$ instead of its intersection with X. Let us substitute here u by $\phi(x_1\nu^{-1})u$ where $\phi \in \mathscr{C}^{\infty}(\mathbb{R})$, $\phi(t)=1$ for $t \ge 1$, $\phi(t)=0$ for $t \le 1-\epsilon$ and $\nu \in [h,1]$ is arbitrary; we then obtain inequality

$$\sum_{|\alpha|\le n} \|(hD)^{\alpha} u\|_{X_\nu} \le C \sum_{|\beta|\le n-m} \|(hD)^{\beta}(A-\tau)u\|_{X_{\nu(1-\epsilon)}} + Ch\nu^{-1} \sum_{|\beta|\le n-1} \|(hD)^{\beta} u\|_{X_{\nu(1-\epsilon)}}$$

where $X_\eta = \{x, x_1 \ge \eta\}$. Since $\epsilon > 0$ is arbitrary, then iterating this inequality we obtain another one:

$$\sum_{|\alpha|\le n} \|(hD)^{\alpha} u\|_{X_\nu} \le C \sum_{|\beta|\le n-m} \|(hD)^{\beta}(A-\tau)u\|_{X_{\nu(1-\epsilon)}} + Ch^{s}\nu^{-s} \sum_{|\beta|\le n-1} \|(hD)^{\beta} u\|_{X_{\nu(1-\epsilon)}}; \tag{7.1.58}$$

moreover, standard arguments yield that we can take $u \in \mathscr{C}^{n}((B(0,1),\mathbb{H})$ but in this case the norms in the left and right-hand expressions should be calculated on $B(0,1-\epsilon)\cap X_\nu$ and $B(0,1-\frac{\epsilon}{2})\cap X_{\nu(1-\epsilon)}$ respectively.

Further, let us substitute into this inequality $u = F_{t\to h^{-1}\tau}\chi_T(t)v$ with $v = e^{ih^{-1}t}Aw$ and let us note that $(A-\tau)u = ihT^{-1}u_1$ with $u_1 = F_{t\to h^{-1}\tau}\chi_T'(t)v$ where χ' is the derivative of χ. Applying this inequality to u_1, etc., we finally obtain that

$$\sum_{|\alpha|\le n} \|(hD)^{\alpha} F_{t\to h^{-1}\tau}\chi_T(t)v\|_{B(0,1-\epsilon)\cap X_\nu} \le Ch^{s}\nu^{-s} \sum_{|\beta|\le n-1} \|(hD)^{\beta} F_{t\to h^{-1}\tau}\bar{\chi}_T(t)v\|_{B(0,1-\frac{\epsilon}{2})\cap X_{\frac{\nu}{2}}} \le Ch^{s}\nu^{-s}T\|w\|$$

where $\bar{\chi} \in \mathscr{C}_0^\infty([-2,2])$ is equal to 1 on $[-\frac{3}{2}, \frac{3}{2}]$ and we used (7.1.57). Then embedding theorems imply that

$$\|(hD)^\alpha F_{t\to h^{-1}\tau}\chi_T(t)v\| \le Ch^{-\frac{d}{2}+s}T\nu^{-s}\|w\| \\ \forall x \in B(0,1-\epsilon) : x_1 \ge \nu \quad \forall \tau \in [\tau_1,\tau_2] \quad \forall \alpha : |\alpha| \le n \quad \forall T \ge \nu$$

where in the left-hand expression the norms in $\mathbb{H}$ are calculated. Since w is arbitrary we obtain the following estimate

$$\|(hD_x)^\alpha F_{t\to h^{-1}\tau}\chi_T(t)f^\dagger U(x,y,t)\| \le Ch^{-\frac{d}{2}+s}T\nu^{-s}\|f\| \\ \forall x \in B(0,1-\epsilon) : x_1 \ge \nu \\ \forall \tau \in [\tau_1,\tau_2] \quad \forall \alpha : |\alpha| \le n \quad \forall T \ge \nu \quad \forall f \in \mathbb{H} \tag{7.1.59}$$

where in the left-hand expression the $\mathscr{L}^2$-norm with respect to y is calculated.

Starting with estimates (7.1.59) and

$$\sum_{|\alpha|\le n, |\beta|\le k} \|(hD_x)^\alpha (hD_y)^\beta F_{t\to h^{-1}\tau}\chi_T(t)f^\dagger U(x,y,t)\| \le \\ Ch^{-\frac{d}{2}}(|\tau|+1)^{\frac{1}{m}(d+|\alpha|+|\beta|)}\|f\| \\ \forall x \in B(0,1-\epsilon) : x_1 \ge \nu \quad \forall \tau \in \mathbb{R} \quad \forall T \ge \nu \quad \forall f \in \mathbb{H} \tag{7.1.60}$$

(where again in the left-hand expression the $\mathscr{L}^2$-norm with respect to y is calculated) and repeating the same arguments with y instead of x we obtain that

$$\|(hD_x)^\alpha (hD_y)^\beta F_{t\to h^{-1}\tau}\chi_T(t)U(x,y,t)\| \le Ch^{-d+s}T\nu^{-s} \\ \forall x,y \in B(0,1-\epsilon) : x_1 \ge \nu, y_1 \ge \nu \quad \forall \tau \in [\tau_1,\tau_2] \quad \forall T \ge \nu \\ \forall \alpha,\beta : |\alpha| \le n, |\beta| \le n.$$

Restricting to $x = y$ and $\alpha = \beta$ and applying the standard Tauberian arguments we obtain estimate (7.1.46) in this case; then the estimate also holds without the restriction due to the arguments used in the proof of estimate (4.2.53). □

7.1.4 Estimates for the Schrödinger Operator

One of the most important assertions of Subsection 7.1.2 from the point of view of applications is Theorem 7.1.7 which permits cutting with respect to

ξ' (for $|\tau| \le c$) taking into account only the zone $\{|\xi'| \le C_0\}$. For this we were forced to assume that the operator and the boundary value problem are elliptic in the classical sense. Let us demonstrate that in a certain special case one can weaken this restriction. Let us treat the Schrödinger operator (5.3.1) satisfying $(5.3.6)_{1-3}$ and (5.3.7).

Let us choose a coordinate system in a neighborhood of ∂X in which x_1 is the Riemannian distance in X from x to Y corresponding to the Riemannian metric g^{jk}; in this system $g^{1k} = \delta_{1k}$. Using gradient transform we can reduce the general case to $V_1 = 0$. So we treat operator

$$A = (hD_1)^2 + A'(x', hD', h) = (hD_1)^2 + \sum_{2\le j,k\le d} (hD_j - V_j) g^{jk} (hD_k - V_k) + V; \tag{7.1.61}$$

and assumption (5.3.7) becomes

$$\sum_{2\le j,k\le d} g^{jk}\xi'_j\xi'_k \ge \epsilon_0 |\xi'|^2. \tag{7.1.62}$$

Let us consider the boundary operator

$$B = -i\alpha(x')hD_1 + M(x', hD') + \beta(x') + h\beta'(x'), \qquad M(x',\xi') = \sum_{2\le j\le d} m^j(x')(\xi_j - V_j). \tag{7.1.63}$$

One can see easily that the self-adjointness of A (with the boundary condition $Bu = 0$) yields that functions $\alpha^{-1}m^j$ and $\alpha^{-1}\beta$ are real-valued (at points where $\alpha \ne 0$). Let us assume that

(7.1.64) α, β, m^j are real-valued functions.

Proposition 3.1.11 yields that if $(A-\tau)v = 0$ and $Q = \mathsf{Op}(q)$, $q \in \mathsf{S}'_{h,\rho,1,K}$ with $\rho \ge C_0$ and if q is supported in $\{\epsilon_0\rho \le |\xi'| + |\tau|^{\frac12} \le c\rho\}$ and if *either* $|\xi'| \ge C_0|\tau|^{\frac12}$ on $\mathsf{supp}(q)$ *or* $\tau \le 0$ then

$$\|Q\eth(ihD_1 + L)v\| \le C'\rho^{-s}h^s\|v\|$$

where $L = \mathsf{Op}(\ell)$, $\ell \in \tilde{\mathsf{S}}^{1\prime}_{h,K}$ does not depend on Q and its principal symbol equals

$$\ell_0 = (a'(x',\xi') - \tau)^{\frac12}, \qquad a'(x',\xi') = \sum_{2\le j,k\le d} g^{jk}(\xi_j - V_j)(\xi_k - V_k). \tag{7.1.65}$$

Therefore in the zone $\Omega = \{|\xi'| \ge C_0(|\tau|+1)\} \cup \{\tau \le -C_0\}$ the boundary condition $\mathfrak{d}Bv = 0$ is equivalent to the condition $\tilde{B}\mathfrak{d}v = 0$ where

$$\tilde{B} = \alpha L + M + \beta + h\beta'. \tag{7.1.66}$$

It is easy to see that for $d \ge 3$ the Shapiro-Lopatinskii condition means exactly that

$$|\alpha| \ge \Big(\sum_{2\le j,k\le d} g_{jk} m^j m^k\Big)^{\frac{1}{2}} + \epsilon_0 \tag{7.1.67}$$

where (g_{jk}) is the tensor inverse to (g^{jk}).

However in the case $d = 2$ this condition is stronger than the Shapiro-Lopatinskii condition which in this case means exactly that

$$|\alpha \pm \sqrt{g}_{22} m^2| \ge \epsilon_0 \tag{7.1.68}$$

for both signs $\pm$; moreover, speaking about this condition we assume that $|\alpha| \ge \epsilon_0$. We replace condition (7.1.67) or (7.1.68) by the pair of conditions

$$\alpha \ge (1+\epsilon_0)\Big(\sum_{2\le j,k\le 2} g_{jk} m^j m^k\Big)^{\frac{1}{2}}, \tag{7.1.69}_1$$

and

$$\beta \ge \epsilon_0. \tag{7.1.69}_2$$

Let us note that $(7.1.69)_2$ and (7.1.64) provide (7.1.11). Moreover, $(7.1.69)_{1,2}$ and (7.1.64) imply that in Ω

$$\tilde{B}_0(x',\xi',\tau) \ge \epsilon_1 \alpha(x')\big(|\xi'| + |\tau|^{\frac{1}{2}}\big) + \epsilon_1$$

with $\epsilon_1 = \epsilon_1(d, c, \epsilon_0) > 0$.

We need the following assertion:

Lemma 7.1.13. *Let conditions* (7.1.61)–(7.1.64), $(7.1.69)_{1,2}$ *and* $(7.1.4)_{1,2}$ *be fulfilled. Let* $Q = q(hD')$ *with* $q \in \mathsf{S}'_{h,\rho,1,K}$ *supported in*

$$\{\epsilon_0\rho \le |\xi'| + |\tau|^{\frac{1}{2}} \le c\rho\} \cap \Omega;$$

moreover, let Q' *and* q' *be an operator and symbol of the same nature, with* $q' = 1$ *in the* $\epsilon\rho$*-neighborhood of* $\mathsf{supp}(q)$*. Finally let* $\psi, \psi' \in \mathscr{C}_0^K(B(0,1))$ *and assume* ψ' *equals* 1 *in the* ϵ*-neighborhood of* $\mathsf{supp}\,\psi$*. Then*

$$\|Q\psi v\|_Y \le C(\|Q'\tilde{B}\psi' v\|_Y + \rho^{-s}h^s\|\psi' v\|). \tag{7.1.70}$$

where $Y = \partial X$.

Proof. Obviously it is sufficient to prove estimate

$$\|Q\psi v\|_Y \le C\big(\|Q'\tilde{B}\psi' v\|_Y + \rho^{-\frac12}h^{\frac12}\|Q'\psi' v\|_Y + \rho^{-s}h^s\|\psi' v\|\big). \tag{7.1.71}$$

Applying this estimate to Q', ψ' instead of Q, ψ and repeating this process we obtain (7.1.70). We can assume without loss of generality that ψ, ψ', Q, Q' are real-valued.

Then (7.1.64), $(7.1.69)_{1,2}$ and the sharp Gårding inequality yield that

$$\mathsf{Re}(\tilde{B}Q\psi v, Q\psi v)_Y \ge \epsilon_0\|Q\psi v\|_Y^2 - C\rho^{-s}h^s\|\psi' v\|_Y^2$$

and we need to consider

$$\begin{aligned}\mathsf{Re}(\tilde{B}Q\psi v, Q\psi v)_Y = \mathsf{Re}(Q\psi\tilde{B}\psi' v, Q\psi v)_Y + \\ \mathsf{Re}([\tilde{B}, Q]\psi v, Q\psi v)_Y + \mathsf{Re}(Q[\tilde{B}, \psi]\psi' v, Q\psi v)_Y.\end{aligned} \tag{7.1.72}$$

The absolute value of the first term in the right-hand expression does not exceed

$$\|Q\psi v\|_Y \cdot \|Q'\tilde{B}\psi' v\|_Y + C\rho^{-s}h^s\|\psi' v\|_Y^2.$$

One can rewrite the second term in the right-hand expression of (7.1.72) in the form $\frac12\,\mathsf{Re}(\mathcal{K}_1\psi v, \psi v)_Y$ with

$$\mathcal{K}_1 = [Q, [\tilde{B}, Q]] - [\tilde{B}^* - B, Q]Q \in \rho^{-1}h\boldsymbol{\Psi}'_{h,\rho,1,K}$$

uniformly with respect to ρ and hence the absolute value of this term does not exceed

$$C\rho^{-1}h\|Q'\psi' v\|_Y^2 + \rho^{-s}h^s\|\psi' v\|_Y^2. \tag{7.1.73}$$

Finally, the third term in the right-hand expression of (7.1.72) is equal to $\mathsf{Re}(\mathcal{K}_2\psi' v, \psi' v)_Y$ where

$$\mathcal{K}_2 = \psi Q^2[B, \psi] = Q\psi[\tilde{B}, \psi]Q + [\psi, Q][\tilde{B}, \psi]Q + Q\psi[Q, [\tilde{B}, \psi]];$$

here the second and the third terms obviously belong to $\rho^{-1}h\boldsymbol{\Psi}'_{h,\rho,1,K}$ and hence modulo an expression with absolute value not exceeding (7.1.73) the third term in the right-hand expression of (7.1.72) is equal to

$$\mathsf{Re}(\psi[\tilde{B}, \psi]Q\psi' v, Q\psi' v)_Y = \frac12\,\mathsf{Re}(\mathcal{K}_3\psi' v, \psi' v)_Y$$

where

$$\mathcal{K}_3 = [\psi, [\tilde{B}, \psi]] - [\tilde{B}^* - \tilde{B}, \psi] \in \rho^{-1} h \mathbf{\Psi}'_{h,\rho,1,K}.$$

Therefore the absolute value of the third term in the right-hand expression of (7.1.72) also does not exceed (7.1.73). These estimates clearly yield (7.1.71). □

This Lemma 7.1.13, Proposition 3.1.11 and the standard Tauberian technique obviously yield

Theorem 7.1.14. *Let conditions* (7.1.61)–(7.1.64), $((7.1.69)_1)_{1,2}$, $(7.1.4)_{1,2}$ *and* (7.1.5) *be fulfilled. Then statements (i)–(iii) of Theorem 7.1.9 remain true with* $\omega = \rho$.

7.2 Method of Successive Approximations

The goal of this section is the same as the goal of Section 4.3. There is an additional difficulty arising from the fact that there is no mollification with respect to x_1 and y_1 because there is no cutting with respect to ξ_1 and η_1 (so smoothness with respect to x_1 and y_1 should be treated carefully).

7.2.1 Approximating the Free Space Term

It will be convenient for us to represent propagator $U = U(x, y, t)$ as a sum $U = U^0 + U^1$ as was done in Section 3.3; to construct $U^0 = U^0(x, y, t)$ we continue operator A into the domain $\{x_1 \geq -\epsilon\} \cap B(0, 1 - \epsilon)$ with a small enough constant $\epsilon > 0$. Then for U^0 the construction of Subsection 4.3.1 is applicable with the modifications indicated in Subsection 4.3.2 (but the coefficients are bounded operators now) and with the calculations of Subsection 4.3.3. Namely, for U^0 a slightly modified form of Proposition 4.3.7 holds:

Standard Calculations Revised

Proposition 7.2.1. *Let conditions* (7.1.1), (7.1.2), $(7.1.4)_1$, (7.1.9) *and* (7.1.10) *be fulfilled. Let* $Q_i = \mathsf{Op}(q_i)$, $q_i \in \mathsf{S}'_{h,K}$ *be supported in* $\{|\xi'| \leq c\}$ *for*

$i = 1, 2$, *and let* $\psi_i \in \mathscr{C}_0^K(B(0, 1-\epsilon))$. *Then*

$$\mathcal{R} := \|D_x^\alpha D_y^\beta \Big(F_{t\to h^{-1}\tau}\chi_T(t) Q'_{1x}\psi_1(x) U^0 \psi_2(y)^t Q'_{2y} - \sum_{0\le n\le N'} h^{-d+n} \iint T\hat{\chi}(\frac{(\tau-\tau')T}{h}) e^{ih^{-1}\times\langle x-y,\xi\rangle} q'_1(x,\xi')\times \mathcal{F}^0_n(y,\tau',\xi)[q'_2(y,\xi')]d\tau' d\xi' \Big)\| \le C' h^s \tag{7.2.1}$$

$$\forall x, y \in \mathbb{R}^d \quad \forall \tau : |\tau| \le c \quad \forall \alpha, \beta : |\alpha| \le r, |\beta| \le r$$

where we recall that

$$\mathcal{F}^0_n(y,\tau',\xi) = \mathsf{Res}_{\mathbb{R}}\, F^0_n(y,\tau',\xi), \tag{7.2.2}$$

$T \in [h^{1-\delta}, h^{\frac12+\delta}]$, $\delta > 0$ *is an arbitrarily small exponent,* r *is arbitrary, and* K *and the constants depend on* r *also.*

Proof. Proposition 4.3.7(ii) yields this estimate for Q'_{1x} and Q'_{2y} replaced by $Q_{1x} = Q'_{1x}Q''_{1x}$ and $Q_{2y} = Q'_{2y}Q''_{2y}$ respectively where $Q''_{1x} = q''_1(hD_{x_1})$, $Q''_{2y} = q''_2(hD_{y_1})$ with $q''_1, q''_2 \in \mathsf{S}_{h,\rho,1,K}(\mathbb{R})$, q'_1 supported in $\{|\xi_1| \le c_1\}$ and with similar q''_2; the constant c_1 will be chosen later. Surely $q'_1(x,\xi')$, $q'_2(x,\xi')$ should be replaced by $q_1(x,\xi)$, $q_2(y,\xi)$ respectively.

On the other hand, the standard elliptic arguments yield that

$$\|D_x^\alpha D_y^\beta \big(F_{t\to h^{-1}\tau}\chi_T(t) Q_{1x}\psi_1(x) U^0 \psi_2(y)^t Q_{2y}\big)\| \le C' h^s \rho^{-s}$$

provided Q_1 and Q_2 have the same structure as before and $q''_i \in \mathsf{S}_{h,K}(\mathbb{R})$ are supported in $\{|\xi_1| \le c\rho\}$ for $i = 1, 2$ and $q''_i \in \mathsf{S}_{h,\rho,1,K}(T^*\mathbb{R})$ is supported in $\{|\xi_1| \ge \epsilon_0\rho\}$ for some $i = 1, 2$ where $\rho \ge c_1$ with an appropriate constant c_1.

Moreover, the regularity property of F^0_n for $|\tau| \le c$, $|\xi_1| \ge c_1\epsilon_0$ and $|\xi'| \le c$ yields that all the other terms in $\mathcal{R}$ vanish.

Summing with respect to an admissible partition of unity on $\mathbb{R}_{\xi_1}$, we obtain estimate (7.2.1). □

Calculating Restriction to the Boundary

Now our goal is to obtain a decomposition for U^1. The first step is to obtain a decomposition for $\eth_x D^j_{x_1} U^{0\pm}$ for $j = 0, \dots, m-1$. For this purpose let us use Proposition 4.3.7(ii) again. However, now we modify the construction of Subsection 4.3.1.

First of all, since we are looking at restriction as $x_1 = 0$ let us freeze the symbol of A at the point $(0, y')$ instead of y as we did before. Then the factor $(x_1 - y_1)$ will be replaced by the factor x_1 which we now move to the left (to $\bar{\partial}_x$) instead of moving it to the right (to $\delta(x-y)$) as was done before with the factor $(x_1 - y_1)$. However the final result remains the same: if the factor x_1 reaches $\bar{\partial}_x$ then the corresponding term vanishes.

Further, now we cut only with respect to ξ', we do not cut with respect to ξ_1. In order to control the smoothness with respect to x_1 and y_1 we apply the construction of Subsection 4.3.2 with $B^{\frac{1}{m}}$ replaced by hD_1 (therefore we need to assume that the extended operator A' satisfies condition (7.1.9) at every point). All other details of the construction are preserved without any change and then the calculations of Subsection 4.3.3 yield the following estimate

$$\begin{aligned}(7.2.3)\quad \mathcal{R}'_{\pm} = \|\bar{\partial}_x D_x^{\alpha} D_y^{\beta}\Big(F_{t\to h^{-1}\tau}\chi_T(t)Q'_{1x}\psi_1(x)U^0\psi_2(y)^t Q'_{2y}\pm \\ \sum_{0\le n\le N'} h^{-d+n}\iint T\hat{\chi}(\frac{\tau' T}{h})e^{ih^{-1}\langle x-y,\xi\rangle}q'_1(x,\xi')\times \\ F_n^1(y,\tau-\tau',\xi)[q'_2(y,\xi')]d\tau' d\xi'\Big)\| \le C' h^s(1+|\tau|)^N\end{aligned}$$

$$\forall x', y' \in \mathbb{R}^{d-1} \quad \forall \tau : \pm \operatorname{Im}\tau < 0 \quad \forall \alpha, \beta : |\alpha| \le r, |\beta| \le r, \alpha_1 + \beta_1 \le m-1$$

where F_n^1 are $2n$-th order differential operators with respect to ξ' with coefficients analytic in $\mathbb{C}_{\mp} \setminus \mathbb{R}$ and $F_0^1 = F_0^0$; the norm in the left-hand expression is the operator norm from $\mathscr{L}^2(\mathbb{R}_{y_1}, \mathbb{H})$ to $\mathbb{H}$ and we assume that $Q'_i = \mathsf{Op}(q'_i)$ with $q'_i \in \mathsf{S}'_{h,K}$ supported in $\{|\xi| \le c\}$ for $i = 1, 2$.

Proposition 7.2.2. *Let conditions* (7.1.1), (7.1.2), $(7.1.4)_1$, (7.1.9) *and* (7.1.10) *be fulfilled and let* Q_i, ψ_i *be the same as in Proposition 7.2.1.*

Then estimate (7.2.3) *remains true under the restriction* $\alpha_1 \le m-1$ *instead of the previous restriction* $\alpha_1 + \beta_1 \le m-1$ *provided* $y_1 > 0$ *and in the left-hand expression the operator norm from* $\mathbb{H}$ *to* $\mathbb{H}$ *is calculated.*

Proof. Obviously it is sufficient to prove that this estimate holds for the $\mathcal{L}(\mathbb{H}, \mathscr{L}^2(\mathbb{R}^+, \mathbb{H}))$-norm with the condition $\alpha_1 + \beta_1 \le m-1$ replaced by $\alpha_1 \le m-1$.

If we perform the sandwich procedure of Subsection 4.3.2 then we see that the only terms preventing us from obtaining this estimate are terms which

contain no factor $\bar{G}^{\pm}$, $G^{\pm}$, $\hat{G}^{\pm}$ which are parametrices of $(4.3.8)_{\pm}$-problems for $(hD_t - \bar{A})$, $(hD_t - A')$, $(hD_t - A)$ respectively.

Therefore each of these terms may contain only factors $(\bar{A}+i)^{-1}$, $(A'+i)^{-1}$, $(A+i)^{-p}$ and it must contain at least one copy of the factor $(\bar{A}+i)^{-1}$ and no more than one copy of the factor $(\bar{A}+i)^{-p}$. Further, every factor $(\bar{A}+i)^{-1}$ is to the left of every factor $(A'+i)^{-1}$ and $(A+i)^{-p}$ is to the right of every factor $(A'+i)^{-1}$. Furthermore, these terms contain h-pseudodifferential operators with compact supports with respect to (x,ξ'), functions with compact supports with respect to x and operators hD_t and hD_{x_1}, hD_{y_1}.

One can see easily that all these terms vanish if we sum U^{0+} and U^{0-}. However we need to consider U^{0+} and U^{0-} separately.

Let us replace all copies of the operator D_{y_1} applied to $\delta(x-y)$ by $-D_{x_1}$.

(i) Let us first treat all terms which do not contain $(A+i)^{-1}$. Let us move all the copies of hD_{x_1} from the left to the right by applying the commutation formula

$$[(A'+i)^{-1}, T] = -(A'+i)^{-1}[A', T](A'+i)^{-1};$$

we know that $(\bar{A}+i)^{-1}$ and hD_1 commute. Moreover, if there is a copy of $(hD_1)^k$ with $k \ge m$ near either $(A'+i)^{-1}$ or $(\bar{A}+i)^{-1}$ then (7.1.9) yields that this term equals a sum of terms one of which does not contain a copy of $(A'+i)^{-1}$ or $(\bar{A}+i)^{-1}$ while the other contains hD_1 to a degree not exceeding $m-1$. Therefore we obtain terms of two types as the result of these transformations:

(a) Terms which are products of factors $(hD_1)^j(\bar{A}+i)^{-1}$, $(hD_1)^j(A'+i)^{-1}$ with $j \le m-1$, operators belonging to $\boldsymbol{\Psi}'_{h,K}$, smooth functions and hD_t;

(b) Terms which contain neither $(\bar{A}+i)^{-1}$ nor $(A'+i)^{-1}$.

Furthermore, terms originating from terms which we threw away before contain a factor h^L with arbitrary large L. Then the norms of *these* terms of type (a) do not exceed $C'h^s(1+|\tau|)^N$ because of (7.1.9) and terms of type (b) will vanish for $x_1 \ne y_1$ because they are differential operators with respect to x_1, y_1.

(ii) Let us now treat terms containing $(A+i)^{-p}$; these terms necessarily contain a factor $(A-A')$ as well. Let us recall that there is operator $\psi(x)Q_x\psi(x)$ applied to $\delta(x-y)$; the symbol of this operator is supported

in the vicinity of $(\bar{y}, \bar{\xi}')$. Let us drag this operator to the left; if it meets $(A - A')$ whose symbol is 0 in the vicinity of this point, we get a high degree of h. Thus applying the commutation formula for $(A + i)^{-1}$ and $\psi Q \psi$ we can reduce our analysis to the case in which there is no factor $(A - A')$ but there is a high degree of h. Really, we need a lot of commutations since on each step we get h-pseudodifferential operators (with respect to (x', t)) with symbols supported in vicinity of $(\bar{y}, \bar{\xi}')$.

Let us recall that there are also factors hD_1, hD_t, operators belonging to $\boldsymbol{\Psi}'_{h,K}$ with compactly supported symbols and compactly supported functions. If we consider the commutators of $(A + i)^{-1}$ with ψD_1^q and apply division of ψD_1^q by $(A + i)$ with remainder (we have $\psi \in \mathscr{C}_0^K(B(0, 1 - \epsilon))$ decreasing $\epsilon > 0$ if necessary) we obtain as before three types of terms:

(a) Terms which are products of factors $(hD_1)^j(\bar{A} + i)^{-1}$, $(hD_1)^j(A' + i)^{-1}$, $\psi(hD_1)^j(A + i)^{-1}$ with $j \le m - 1$, operators belonging to $\boldsymbol{\Psi}'_{h,K}$, smooth functions and hD_t;

(b) Terms which are products of factors $(hD_1)^j(\bar{A} + i)^{-1}$, $(hD_1)^j(A' + i)^{-1}$ with $j \le m - 1$, $(A + i)^{-p}$ (only one copy), operators belonging to $\boldsymbol{\Psi}'_{h,K}$, smooth functions and hD_t;

(c) Terms which contain neither $(\bar{A} + i)^{-1}$, nor $(A' + i)^{-1}$, nor $(A + i)^{-1}$.

All these terms contain factor h^L with large L. Let us note that for $0 \le j \le m$ operators $(hD_1)^j(\bar{A} + i)^{-1}$, $(hD_1)^j(A' + i)^{-1}$ and $\psi(hD_1)^j(A + i)^{-1}$ act from $\mathscr{L}^2(\mathbb{R}^d, \mathbb{H})$ ($\mathscr{L}^2(B(0, 1), \mathbb{H})$ in the third case) into $\mathscr{L}^2(\mathbb{R}^d, \mathbb{H})$. Then we can estimate terms of types (a) and (b) while terms of type (c) vanish. □

Remark 7.2.3. Surely one could refer to the theory of h-pseudodifferential operators with the transmission property and prove that $\psi(A + i)^{-p}\psi$ is an operator of this type, but one should first develop this theory!

Corollary 7.2.4. *Let conditions* (7.1.1), (7.1.2), $(7.1.4)_1$, (7.1.9) *and* (7.1.10) *be fulfilled; let* Q'_i, ψ'_i *be the same as in Proposition 7.2.1. Then*

$$\mathcal{R}''_{\pm} := \|D_x^{\alpha} D_y^{\beta}\Big(F_{t\to h^{-1}\tau}\chi_T(t)Q'_{1x}\psi_1(x)V^{\pm}\psi_2(y)^t Q'_{2y} - \sum_{0\le n\le N'} h^{-d+n}\iint T\hat{\chi}(\frac{\tau\tau' T}{h})e^{ih^{-1}\langle x-y,\xi\rangle}q'_1(x,\xi')\times F_n^2(y,\tau-\tau',\xi)[q'_2(y,\xi')]\,d\tau' d\xi'\Big)\| \le C' h^s(|\tau|+1)^{N''} \tag{7.2.4}$$

$$\forall x', y' \in \mathbb{R}^{d-1} \quad \forall y_1 > 0 \quad \forall \tau : \pm\tau < 0 \quad \forall \alpha, \beta : |\alpha| \le r, |\beta| \le r$$

where $T \in [h^{1-\delta}, h^{\frac{1}{2}+\delta}]$ *as before,* $V^{\pm} = -\mathfrak{d}_x B(x, hD_x)U^{0\pm}$, *and* F_n^k *are functions of the same type as before; here they are sums of terms*

$$\Big(\prod_{1\le j\le l_n}\big(\tau - a(0,y',\xi)\big)^{-1} p_j(y',\xi)\Big)\cdot\big(\tau - a(0,y',\xi)\big)^{-1} \tag{7.2.5}$$

where p_j *are polynomials with respect to* ξ *the degrees of which with respect to* ξ_1 *do not exceed* $m-1$.

7.2.2 Approximations for the Reflected Wave

Let us calculate $U^{1\pm}$ by the method of successive approximations. For this purpose we write the problem (3.3.26), (3.3.32)–(3.3.33) in the previous notations:

$$(hD_t - A)U^{1\pm} = 0, \tag{7.2.6}$$

$$\mathfrak{d}BU^{1\pm} = -\mathfrak{d}BU^{0\pm}, \tag{7.2.7}$$

$$U^{1\pm}|_{\pm t<0} = 0; \tag{7.2.8}$$

hence

$$(hD_t - A')U^{1\pm} = F^{\pm}, \tag{7.2.9}$$

$$\mathfrak{d}B'U^{1\pm} = -\mathfrak{d}B'U^{0\pm} + f^{\pm} \tag{7.2.10}$$

with

$$F^{\pm} = (A - A')U^{1\pm}, \tag{7.2.11}$$

$$f^{\pm} = -\mathfrak{d}(B - B')(U^{0\pm} + U^{1\pm}). \tag{7.2.12}$$

Therefore

$$(hD_t - \bar{A})U^{1\pm} = F^{\pm} + RU^{1\pm}, \tag{7.2.13}$$

$$\eth\bar{B}U^{1\pm} = -V^{\pm} + \eth R_1 U^{1\pm} + f^{\pm} \tag{7.2.14}$$

where exactly as in Subsubsection *7.2.1.2 Calculating Restriction to the Boundary* $R = A' - \bar{A}$, $R_1 = B' - \bar{B}$, $\bar{A}$ and $\bar{B}$ are operators A' and B' with symbols frozen at the point $(0, y')$, A' and B' are obtained from A and B by cutting outside of a neighborhood of $\mathsf{supp}(q_2')$, and all the operators act with respect to x.

Then (7.2.13), (7.2.14) and (7.2.8) yield that

$$U^{1\pm} = \bigl(\bar{G}^{\pm}R + \bar{G}'^{\pm}\eth R_1\bigr)U^{1\pm} + \bar{G}^{\pm}F^{\pm} + \bar{G}'^{\pm}\bigl(-V^{\pm} + f^{\pm}\bigr). \tag{7.2.15}$$

Furthermore, (7.2.9), (7.2.10), (7.2.8) yield that

$$U^{1\pm} = G^{\pm}F^{\pm} + G'^{\pm}(-V^{\pm} + f^{\pm}) \tag{7.2.16}$$

where $\bar{G}^{\pm}$ and $\bar{G}'^{\pm}$ are parametrices of the problems

$$(hD_t - \bar{A})u = f, \quad \mathsf{supp}\, f \subset \{\pm(t - t_0) \ge 0\}, \tag{7.2.17}$$

$$\eth\bar{B}u = 0, \tag{7.2.18}$$

$$u|_{\pm(t-t_0)<0} = 0 \tag{7.2.19}$$

and

$$(hD_t - \bar{A})u = 0, \tag{7.2.17}$$

$$\eth\bar{B}u = f, \quad \mathsf{supp}\, f \subset \{\pm(t - t_0) \ge 0\}, \tag{7.2.18}$$

$$u|_{\pm(t-t_0)<0} = 0 \tag{7.2.19}$$

respectively, $G^{\pm}$ and $G'^{\pm}$ are defined in the same way for A' and B' instead of $\bar{A}$ and $\bar{B}$ and in what follows we will use $\hat{G}^{\pm}$ and $\hat{G}'^{\pm}$ defined in the same way for A and B.

Iterating (7.2.15) and applying (7.2.16) we obtain that

$$\begin{aligned} U^{1\pm} = -\sum_{0\le n\le N'-1} \bigl(\bar{G}^{\pm}R + \bar{G}'^{\pm}\eth R_1\bigr)^n \bar{G}'^{\pm}\eth B' U^{0\pm} - \\ \bigl(\bar{G}^{\pm}R + \bar{G}'^{\pm}\eth R_1\bigr)^{N'} G'^{\pm}\eth U^{0\pm} + \\ \sum_{0\le n\le N'-1} \bigl(\bar{G}^{\pm}R + \bar{G}'^{\pm}\eth R_1\bigr)^n \bigl(\bar{G}^{\pm}F^{\pm} + \bar{G}'^{\pm}f^{\pm}\bigr) + \\ \bigl(\bar{G}^{\pm}R + \bar{G}'^{\pm}\eth R_1\bigr)^{N'} \bigl(G^{\pm} + G'^{\pm}f^{\pm}\bigr). \end{aligned} \tag{7.2.20}$$

Our goal is to prove that as $|t| \le h^{\frac12+\delta}$ only the first line matters for the final answer.

Let us first treat the first and second lines (the third and fourth lines will be called "extra terms").

For this task we need the commutation relations for $\bar G^{\pm}$ and $\bar G'^{\pm}$ with $(x_j - y_j)$ $(j = 2, \dots, d)$ and x_1 and similar relations for $G^{\pm}$, $G'^{\pm}$. Let us take u supported in $\{t : \pm(t - t_0) \ge 0\}$ and let us consider the equalities

$$\bar P x_j u = x_j \bar P u - [\bar A, x_j]u,$$
$$\eth \bar B x_j u = \eth [\bar B, x_j]u + x_j \eth \bar B u$$

where $\bar P = hD_t - \bar A$ and for $j = 1$ the last term should be skipped; then

$$x_j u = \bar G^{\pm} x_j \bar P u - \bar G [\bar A, x_j]u + \bar P'^{\pm}[\bar B, x_j]u + \bar G'^{\pm} x_j \eth \bar B u$$

where for $j = 1$ the last term should be skipped.

Substituting $u = \bar G^{\pm} v$ or $u = \bar G'^{\pm} w$ with arbitrary v, w supported in $\{t : \pm(t - t_0) \ge 0\}$ we obtain that

$$x_j \bar G^{\pm} = \bar G^{\pm} x_j - \bar G^{\pm}[\bar A, x_j]\bar G^{\pm} + \bar G'^{\pm}\eth[\bar B, x_j]\bar G^{\pm}, \tag{7.2.21}$$

$$x_j \bar G'^{\pm} = -\bar G^{\pm}[\bar A, x_j]\bar G'^{\pm} + \bar G'^{\pm}\eth[\bar B, x_j]\bar G'^{\pm} + \bar G'^{\pm} x_j \tag{7.2.22}$$

where for $j = 2$ one can replace x_j by $x_j - y_j$ and for for $j = 1$ the last term in (7.2.22) should be skipped:

$$x_1 \bar G'^{\pm} = -\bar G^{\pm}[\bar A, x_1]\bar G'^{\pm} + \bar G'^{\pm}\eth[\bar B, x_1]\bar G'^{\pm}. \tag{7.2.23}$$

The same relations hold for $G^{\pm}$ and $G'^{\pm}$.

Let us recall that

$$\begin{aligned}\eth B' U^{0\pm} = \mp i \sum_{0\le n\le N'-1} & \eth B' (\bar G_0^{\pm} R)^n \bar G_0^{\pm} \delta(t)\delta(x-y) \mp \\ & ih \eth B' (\bar G_0^{\pm} R)^{N'} G_0^{\pm} \delta(t)\delta(x-y) + \\ & \sum_{0\le n\le N'-1} \eth B' (\bar G_0^{\pm} R)^n \bar G_0^{\pm} F_0^{\pm} + \eth B' (\bar G_0^{\pm} R)^{N'} \bar G_0^{\pm} F_0^{\pm}\end{aligned} \tag{7.2.24}$$

where $\bar G_0^{\mp}$ and $G_0^{\mp}$ are parametrices of the Cauchy problems (i.e., of problems $(4.3.8)_{\pm}$, $(4.3.8)'_{\pm}$) and $F_0^{\pm} = (A - A')U^{0\pm}$.

Further, let us recall that for $\bar{G}_0^{\mp}$ and $G_0^{\mp}$ the commutation relations (4.3.13) and (4.3.13)$'$ hold. Let us apply commutation relations (4.3.13), (4.3.13)$'$, (7.2.21) and (7.2.22) to the first and second lines in the right-hand expression of (7.2.20); we transfer $(x_j - y_j)$ with $j \ge 2$ and x_1 to the right, i.e. to $B'U^{0\pm}$. When x_1 reaches it the corresponding term vanishes (which is equivalent to the absence of the last term in in (7.2.22) for $j = 1$).

Meanwhile in (7.2.24) we transfer x_1 to the left (i.e., to $\eth$); when x_1 reaches it the corresponding term vanishes; in (7.2.20) and (7.2.24) we transfer $(x_j - y_j)$ with $j \ge 2$ to the right (i.e., to $\delta(x - y)$) with the same result.

Therefore each such factor should be killed this way, and then a factor h arises.

Let us substitute (7.2.24) into (7.2.20) and let us omit all terms containing $F^{\pm}$, $f^{\pm}$, $F_0^{\pm}$ which will be analyzed later (so we include the terms generated by the third line in (7.2.24) in "extra terms").

Let us transfer all factors $(x_j - y_j)$ with $j \ge 2$ and x_1 (or large number of them) in the same way as described above. In addition, let us apply sandwiching procedure to all operators $G^{\pm}$, $\bar{G}^{\pm}$, $G_0^{\pm}$, $\bar{G}_0^{\pm}$; for $G^{\pm}$ and $\bar{G}^{\pm}$ the sandwiching procedure is the same as for $G_0^{\pm}$ and $\bar{G}_0^{\pm}$. Moreover, we can "sandwich" the operators $\bar{G}'^{\pm}$ and $G'^{\pm}$ on their left. In fact, equalities

$$(\bar{A} + i)\bar{G}'^{\pm} = (-hD_t + \bar{A})\bar{G}'^{\pm} + (hD_t + i)\bar{G}'^{\pm} = (hD_t + i)\bar{G}'^{\pm}$$

and $\eth\bar{B}\bar{G}'^{\pm} = I$ yield that

$$\bar{G}'^{\pm} = (hD_t + i)(\bar{A} + i)^{-1}\bar{G}'^{\pm} + \bar{E}' \tag{7.2.25}$$

where $\bar{E}'$ is a parametrix of the problem

$$(\bar{A} + i)u = 0, \qquad \eth\bar{B}u = f$$

and $(\bar{A} + i)^{-1}$ means a parametrix of the problem

$$(\bar{A} + i)u = f, \qquad \eth\bar{B}u = 0$$

as it was in the sandwiching procedure for $\bar{G}^{\pm}$. The same method is also applicable to $G'^{\pm}$.

Furthermore, we also have the procedure of division by $\bar{A}$ (or A) with the remainder which we apply to $(hD_1)^k$ with $k \ge m$. Namely, if we have a factor $(hD_1)^k\bar{G}_0^{\pm}$ then we can transform this factor into a sum of factors of

the same type with $k \le m-1$ and an operator which is h-differential with respect to x_1 and h-pseudodifferential with respect to x'. The same is true for $\bar{G}^{\pm}$, $\bar{G}_0^{\pm}$, $G^{\pm}$ and for operators $\bar{G}_0^{\pm}$, $G_0^{\pm}$ staying to the left of $(hD_1)^k$.

Moreover, we can transform operator $(hD_1)^k \bar{G}'^{\pm}$ into a sum of operators of the same type with $k \le m-1$ multiplied on the left by operators belonging to $\boldsymbol{\Psi}'_{h,K}$.

Let us note that if between the rightmost copy of $\eth$ and $\delta(x-y)$ there is no parametrix[6)] then the corresponding term vanishes for $y_1 > 0$. Therefore treating the commutators of $(hD_1)^k$ with $(A'+i)_0^{-1}$ in the same way as before we conclude that to the right of the rightmost copy of $\eth$ we can obtain a product of operators $(hD_1)^k(A'+i)_0^{-1}$, $(hD_1)^k(\bar{A}+i)_0^{-1}$ with $k \le m-1$, $G_0^{\pm}$, $\bar{G}_0^{\pm}$, h-pseudodifferential operators with respect to x' with symbols compactly supported with respect to (x,ξ'), and compactly supported functions (even if we have applied $(hD_{y_1})^k$ with arbitrarily large k before this procedure).

Similarly, if we apply $(hD_{x_1})^k$ with arbitrarily large k we can provide that to the left of the rightmost copy of $\eth$ there is a product of operators $(hD_1)^k(A'+i)^{-1}$, $(hD_1)^k(\bar{A}+i)^{-1}$ with $k \le m-1$, $h^{\frac12}\eth(hD_1)^k(A'+i)^{-1}$, $h^{\frac12}\eth(hD_1)^k(\bar{A}+i)^{-1}$, $h^{-\frac12}(hD_1)^k E'$, $h^{-\frac12}(hD_1)^k \bar{E}'$, $\eth(hD_1)^k \bar{E}'$, $\eth(hD_1)^k E'$ with $k \le m-1$, $\bar{G}^{\pm}$, $G^{\pm}$, hD_t, h-pseudodifferential operators with respect to x' with symbols compactly supported with respect to (x,ξ') and compactly supported functions.

Further, the total quantity of copies of $\bar{G}^{\pm}$, $G^{\pm}$, $\bar{G}_0^{\pm}$ and $G_0^{\pm}$ in every product does not exceed $(2\nu+1)$ where ν is the degree of h gained in the commutation process. All these operators, excluding hD_t, $\bar{G}^{\pm}$, $G^{\pm}$, $\bar{G}_0^{\pm}$ and $G_0^{\pm}$, have operator norms from $\mathscr{L}^2(Z,.)$ to $\mathscr{L}^2(Z',.)$ not exceeding C' where Z and Z' are corresponding domains from the following list: $\mathbb{R}^d$, $\mathbb{R}^{d-1}$, $\bar{\mathbb{R}}^+ \times \mathbb{R}^{d-1}$.

Moreover, the operator norms of $\bar{G}^{\pm}$, $G^{\pm}$, $\bar{G}_0^{\pm}$ and $G_0^{\pm}$ in $\mathscr{L}^2([-T,T]\times Z)$ do not exceed $C'h^{-1}T$. Therefore the norm of the term with gained h^{ν} does not exceed $Ch^{-d+\nu}(h^{-1}T)^{2\nu+1}$ and in particular

(7.2.26) All terms which contain neither $F_0^{\pm}$ nor $F^{\pm}$ nor $f^{\pm}$ and contain either $G^{\pm}$ or $G'^{\pm}$ or terms of the decomposition of $\bar{A}-A'$ or $\bar{B}-B'$ with $x_1^{\alpha_1}(x'-y')^{\alpha'}h^{\nu}$ with $|\alpha|+\nu \ge N$ are negligible as $T \in [h^{1-\delta}, h^{-\frac12}]$;

[6)] I.e., no operator $G_0^{\pm}$, $\bar{G}^{\pm}$, $(A'+i)_0^{-1}$, $(\bar{A}+i)_0^{-1}$; there can be no other parametrix at this place and the index "0" means that the parametrix is constructed in $\mathbb{R}^d$.

we say that *a term is negligible* if after applying $Q'_{1x}\psi_1(x)(hD_{x_1})^j$ and $Q'_{2y}\psi_2(y)(hD_{y_1})^j$ to it with $j,k\le s$ the corresponding norms will be less than $C'h^s$ where s is arbitrarily large and $N'=N'(d,m,s)$.

Let us consider terms originating from $F_0^{\pm}$, $F^{\pm}$ and $f^{\pm}$ (we called them "extra terms" and skipped so far). However, we change the first step of our construction slightly. Namely, we write the problems for $U^{0\pm}$ and $U^{1\pm}$ in the form

$$(hD_t-A')\psi_4U^{0\prime\pm}=\mp ih\delta(t)\delta(x-y)\psi_4(x)+F_0^{\pm},$$
$$(hD_t-A')\psi_4U^{1\prime\pm}=F^{\pm},$$
$$\eth B'\psi_4U^{1\prime\pm}=-\eth B'\psi_4U^{0\prime\pm}+f^{\pm}$$

where now

$$F_0^{\pm}=\psi_4(A-A')\psi_5U^{0\pm}-[A,\psi_4]\psi_5U^{0\pm},$$
$$F^{\pm}=\psi_4(A-A')\psi_5U^{1\pm}-[A,\psi_4]\psi_5U^{1\pm},$$
$$f^{\pm}=\eth\psi_4(B-B')\psi_5(U^{0\pm}+U^{1\pm})+\eth[B,\psi_4](U^{0\pm}+U^{1\pm}).$$

Then when we commute h-pseudodifferential operators with $\bar G^{\pm}$ and $G^{\pm}$, sandwich and divide with the remainder operators $(hD_1)^k$ with $k\ge m$ by A' (or $\bar A$) we obtain[7] from $F_0^{\pm}$, $F^{\pm}$ and $f^{\pm}$ terms which after multiplication by $Q'_{1x}\psi_1(x)$ [8] are of one of the following forms:

(a) $(hD_t)^{N'''}\mathcal{K}^{\pm}(hD_1)^j\psi_5U^{1\pm}$ with $j\le m-1$;

(b) $(hD_t)^{N'''}\mathcal{K}'^{\pm}\eth(hD_1)^j\psi_5U^{i\pm}$ with $i=0,1$ and $j\le m-1$

where the operator norms of the operators

$$\left.\begin{aligned}&\mathcal{K}^{\pm}:\mathscr{L}^2\bigl(\mathbb{R}^+\times[-T,T],\mathscr{H}^{-\mu}(\mathbb{R}^{d-1},.)\bigr)\\ &\mathcal{K}'^{\pm}:\qquad\mathscr{L}^2\bigl([-T,T],\mathscr{H}^{-\mu}(\mathbb{R}^{d-1},.)\bigr)\end{aligned}\right\}\to\mathscr{L}^2\bigl(\mathbb{R}^+\times[-T,T],\mathscr{H}^{\mu}(\mathbb{R}^{d-1},.)\bigr)$$

will be less than $C'h^{\mu}$ where μ is arbitrary and does not depend on N' (but now K and N''' depend on N' and μ).

On the other hand, the following lemma is a consequence of the proof of Theorem 7.1.1.

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[7] Even if we apply $(hD_{x_1})^k$ with arbitrarily large k before this procedure.

[8] Let us recall that $Q'_1=\mathsf{Op}(q'_1)$ where $\mathsf{supp}\,q'_1$ does not intersect the support of $(A'-A)$.

Lemma 7.2.5. *Let conditions* (7.1.1), (7.1.2), $(7.1.4)_1$, (7.1.9) *and* (7.1.10) *be fulfilled and let ψ_i be the same as before. Then*

$$\|\psi_1(x)(hD_{x_1})^j(hD_{y_1})^k\psi_2(y)U^{1\pm}\| \le C'h^{-L},$$
$$\|\eth_x\psi_1(x)(hD_{x_1})^j(hD_{y_1})^k\psi_2(y)U^{0\pm}\| \le C'h^{-L},$$

where j and k are arbitrary (less than L), in the left-hand expression the operator norms

$$\mathscr{L}^2(\mathbb{R}^+\times[-T,T],\mathscr{H}^L(\mathbb{R}^{d-1},.)) \to \begin{cases}\mathscr{L}^2\big(\mathbb{R}^+\times[-T,T],H_{-L}(\mathbb{R}^{d-1},.)\big)\\ \qquad\mathscr{L}^2\big([-T,T],\mathscr{H}^{\mu}(\mathbb{R}^{d-1},.)\big)\end{cases}$$

respectively are calculated, and $L = L(d,m,j,k)$.

Proof. The proof is very simple and is based on the equalities

$$(hD_t - A_x)U^{i\pm} = 0, \quad U^{i\pm t}(hD_t - A_y) = 0$$

(provided $x_1 \ne y_1$ for $i = 0$). □

Therefore

(7.2.27) All terms originating from $F_0^{\pm}$, $F^{\pm}$ and $f^{\pm}$ are negligible.

Combining this with (7.2.26) we arrive to

Proposition 7.2.6. *Let conditions* (7.1.1), (7.1.2), $(7.1.4)_{1,2}$–(7.1.10) *be fulfilled and let the operators Q_i and functions ψ_i be the same as before. Finally, let $\chi \in \mathscr{C}_0^K([-1,1])$ and $T \in [h^{1-\delta}, h^{\frac12+\delta}]$. Then*

$$(7.2.28)\quad \|(hD_x)^{\alpha}(hD_y)^{\beta}Q'_{1x}\psi_1(x)F_{t\to h^{-1}\tau}\chi_T(t)\times \Big(U^{1\pm} - \sum_{n\le N'-1}\big(\bar G^{\pm}R' + \bar G'^{\pm}\eth R'_1\big)^n\bar G'^{\pm}V^{\pm}\Big)\psi_2(y)^tQ'_{2y}\| \le C'h^s(1+|\tau|)^{N'''}$$

with

$$(7.2.29)\quad V^{\pm} = \mp ih\eth(\bar B + R'_1)\sum_{k\le N'-1}(\bar G_0^{\pm}R')^k\delta(t)\delta(x-y)$$
$$\forall x,y\in\bar{\mathbb{R}}^+\times\mathbb{R}^{d-1}\quad \forall\tau : \pm\operatorname{Im}\tau<0\quad \forall\alpha,\beta : |\alpha|\le s, |\beta|\le s$$

where $N' \ge N'(d,m,s,\delta)$, $K' = K = K(d,m,s,\delta,N')$, and R', R'_1 are obtained from R, R_1 by dropping terms containing $x_1^{\alpha_1}(x'-y')^{\alpha'}h^{\nu}$ with $|\alpha|+\nu \ge N'$; all operators without the index "y" act with respect to x.

7.2.3 Calculations

Starting Point

Let us go from estimates to calculations. Let us recall that according to Corollary 7.2.4

$$F_{(t,x')\to h^{-1}(\tau,\xi')}V^{\pm} = \pm\sum h^{-d+k}\int_{-\infty}^{\infty} F_k^2(y',\eta_1,\xi',\tau)e^{-ih^{-1}y_1\eta_1}\,d\eta_1 \tag{7.2.30}$$

where F_k^* are functions smooth with respect to (y',ξ') and holomorphic with respect to (η_1,τ) in the domain containing $\mathbb{R}\times\mathbb{C}_{\mp}$; these functions tend to 0 in this domain as $|\eta_1|\to\infty$.

One Dimensional Parametrices

Let us construct parametrices $\bar{G}'^{\pm}$ and $\bar{G}^{\pm}$. Let us note that

$$F_{(t,x')\to h^{-1}(\tau,\xi')}\bar{G}'^{\pm} = \mathcal{G}'(z,h^{-1}x_1,\tau)F_{(t,x')\to h^{-1}(\tau,\xi')} \qquad (\pm\operatorname{Im}\tau<0) \tag{7.2.31}$$

where $z=(y',\xi')$ and $\mathcal{G}'$ is a solution of the problem

$$\bigl(\tau-\bar{A}(z,D_1)\bigr)\mathcal{G}'=0, \tag{7.2.32}$$

$$\eth\bar{B}(z,D_1)\mathcal{G}'=0, \tag{7.2.33}$$

$$\mathcal{G}'\in\mathscr{S}(\mathbb{R}^+,.) \tag{7.2.34}$$

and for $z\in\operatorname{supp}q_2$ (as will be assumed without being specially mentioned) we can replace $\bar{A}$ and $\bar{B}$ by a and b respectively.

Using $\bar{G}'^{\pm}$ and parametrices in the whole space $\mathbb{R}^d$ for the same operator $(\tau-\bar{A})$ we can construct the parametrices $\bar{G}^{\pm}$. Namely, let $\mathcal{G}(z,\tau)$ be the one-dimensional (on $\mathbb{R}^+$) integral operator with Schwartz kernel

$$\begin{aligned}\mathcal{K}_{\mathcal{G}}(x_1,y_1;z,\tau) = {}& (2\pi)^{-1}\text{v.p.}\int_{-\infty}^{\infty}\bigl(e^{i(x_1-y_1)\xi_1}-\mathcal{G}'(z,x_1,\tau)\bar{B}(z,\xi_1)e^{-iy_1\xi_1}\bigr)\times\\ &\bigl(\tau-\bar{A}(z,x_1,\xi_1)\bigr)^{-1}\,d\xi_1\end{aligned} \tag{7.2.35}$$

where the principal value of the integral is calculated; then

$$F_{(t,x')\to h^{-1}(\tau,\xi')}\mathcal{L}\bar{G}^{\pm} = \mathcal{G}(z,\tau)\mathcal{L}F_{(t,x')\to h^{-1}(\tau,\xi')} \qquad (\pm\operatorname{Im}\tau<0) \tag{7.2.36}$$

where $(\mathcal{L}v)(x_1) = v(x_1 h)$.

It is easy to see that for $\eta_1 \in \mathbb{R}$

$$\mathcal{G}(e^{ix_1\eta_1}v) = \bigl(e^{ix_1\eta_1} - \mathcal{G}(z,x_1,\tau)\bar{B}(z,\eta_1)\bigr)\bigl(\tau - \bar{A}(z,\eta_1)\bigr)^{-1}v. \tag{7.2.37}$$

We need to establish certain properties of $\mathcal{G}'$ and $\mathcal{G}$ for $z \in \mathsf{supp}(q_2)$.

Lemma 7.2.7. *(i) Functions* $W_j(z,\tau) = \eth D_1^j \mathcal{G}'(z,x_1,\tau)$ *are regular on* $\mathbb{R}^{2d-2} \times (\mathbb{C}\setminus\mathbb{R}) \ni (z,\tau)$ *and satisfy inequalities*

$$\|D_z^\alpha D_\tau^\beta W_j\| \le C|\operatorname{Im}\tau|^{-N}(1+|\tau|)^N \qquad \forall \alpha,\beta,j : |\alpha|+\beta+j \le s \tag{7.2.38}$$

where $N = N(d,m,s)$ *and* s *is arbitrary.*

(ii) Further, $\eth_y (D_{y_1})^j \mathcal{K}_{\mathcal{G}} = \mathcal{G}' W_j'$ *where the functions* $W_j'(z,\tau)$ *have the same properties as* W_j*.*

Proof. Since $\mathcal{G}$ is a parametrix for $\bigl(\tau - a(z,D_1)\bigr)$ with the boundary condition $\eth b(z,D_1)u = 0$ and since the operator $a(z,D_1)$ with this condition is self-adjoint we conclude that the operator norm in $\mathscr{L}^2(\mathbb{R}^+,\mathbb{H})$ of the operator $\mathcal{G}$ does not exceed $|\operatorname{Im}\tau|^{-1}$. On the other hand, (7.1.9) and the definition of $\mathcal{G}$ as a parametrix yield that

$$\|D_1^m \mathcal{G}\| \le C\Bigl(\sum_{0\le j\le m-1} \|D_1^j \mathcal{G}\| + |\tau|\cdot\|\mathcal{G}\| + 1\Bigr);$$

this inequality, the inequality $\|\mathcal{G}\| \le C|\operatorname{Im}\tau|^{-1}$ and interpolation yield that

$$\|D_1^j \mathcal{G}\| \le C|\operatorname{Im}\tau|^{-1}(|\tau|+1)^{\frac{j}{m}} \qquad (j = 0,\dots,m) \tag{7.2.39}$$

which implies

$$\|\eth_y D_{y_1}^j \mathcal{K}_{\mathcal{G}}\| \le C|\operatorname{Im}\tau|^{-1}(|\tau|+1)^{\frac{j+1}{m}} \qquad (j = 0,\dots,m-1) \tag{7.2.40}$$

where in the left-hand expression the operator norm from $\mathbb{H}$ to $\mathscr{L}^2(\mathbb{R}^+,\mathbb{H})$ is calculated. Then the equality $\bigl(\tau - a(z,D_{y_1})\bigr)\mathcal{K}_{\mathcal{G}}^\dagger = 0$ for $y_1 \ne x_1$ yields that

(7.2.41) Estimate (7.2.40) holds for all j with constant $C = C_j$.

Starting from (7.2.40)–(7.2.41) and treating equation

$$\big(\tau - a(z, D_{x_1})\big)\eth_y D_{y_1}^j \mathcal{K}_{\mathcal{G}} = 0 \tag{7.2.42}$$

for $x_1 > 0$ we obtain that

$$\|D_{x_1}^k \eth_y D_{y_1}^j \mathcal{K}_{\mathcal{G}}\| \le C_{jk} |\operatorname{Im}\tau|^{-1} (|\tau| + 1)^{\frac{j+k+1}{m}} \tag{7.2.43}$$

and

$$\|\eth_x D_{x_1}^k \eth_y D_{y_1}^j \mathcal{K}_{\mathcal{G}}\| \le C_{jk} |\operatorname{Im}\tau|^{-1} (|\tau| + 1)^{\frac{j+k+2}{m}} \tag{7.2.44}$$

for all j, k.

Moreover, equation (7.2.42) yields the equality $\eth_y D_{y_1}^j \mathcal{K}_{\mathcal{G}} = \mathcal{G}' W_j'$ with $W_j' = \eth_x b(z, D_{x_1}) \eth_y D_{y_1}^j \mathcal{K}_{\mathcal{G}}$; therefore Statement (ii) holds with (7.2.38) proven only for $\alpha = \beta = 0$ so far.

Furthermore, equality

$$\mathcal{G}'(z, \tau) = \mathcal{R} - \mathcal{G}(z, \tau)\big(\tau - a(z, D_1)\big)\mathcal{R}$$

holds for every operator $\mathcal{R}$ from $\mathbb{H}'$ to $\mathscr{S}(\bar{\mathbb{R}}^+, \mathbb{H})$ such that $\eth b(z, D_1)\mathcal{R} = I$ (e.g., for $\mathcal{R} = \mathcal{G}'(z, i)$). Therefore Statement (i) also holds holds (but with (7.2.38) proven only for $\alpha = \beta = 0$ so far).

For arbitrary α and β (7.2.38) follows from the following formulas similar to (7.2.21) and (7.2.22):

$$\partial \mathcal{G} = -\mathcal{G}(\partial(\tau - a))\mathcal{G} - \mathcal{G}' \eth(\partial b)\mathcal{G},$$
$$\partial \mathcal{G}' = -\mathcal{G}(\partial(\tau - a))\mathcal{G}' - \mathcal{G}' \eth(\partial b)\mathcal{G}'$$

where $\partial = \partial_{x_j}, \partial_{\xi_j}$ with $j = 2, \dots, d$ or $\partial = \partial_\tau$; in the last case the last terms in these equalities vanish. □

One Dimensional Parametrices (End)

Let us investigate regularity of $\mathcal{G}'(z, \tau)$ and $\mathcal{G}(z, \tau)$ with respect to (z, τ). Let us start from the obvious

Remark 7.2.8. Operator $\mathcal{G}$ is bounded in a neighborhood of (z^*, τ^*) if and only if the problem $\big(\tau - a(z, D_1), \eth b(z, D_1)\big)$ is elliptic, i.e., if $z^* \notin \Sigma_{\tau^*, b}$. In this case the functions W_j and W_j' are regular in a neighborhood of (z^*, τ^*).

Let us assume that the problem $\bigl(\tau - a(z, D_1), \eth b(z, D_1)\bigr)$ is microhyperbolic at (z^*, τ^*) in the multidirection $\mathcal{T} = (\ell', \nu_1, \dots, \nu_n)$; in particular, in this case (3.1.25) is fulfilled. Without any loss of the generality one can assume that either

(a) $\ell = \partial_{z_1}$ or

(b) $\ell' = 0$

(otherwise we can arrange this by a uniformly non-degenerate change of coordinates in $\mathbb{R}^{2d-2}$).

In the former case let us treat the almost analytic (with respect to z_1) extensions of $a(z, D_1)$ and $b(z, D_1)$ to the neighborhood $\tilde{\Omega} \subset \mathbb{C} \times \mathbb{R}^{2d-3} \times \mathbb{C}$ of the point (z^*, τ^*). Let us make the reduction of Subsection 3.1.2 but now the coefficients at powers of hD_1 are functions of (z, τ) rather than h-pseudodifferential operators. Let $\mathfrak{C} = (\mathfrak{C}_j) = (\mathfrak{C}_j(z, D_1, \tau))_{j=0,\dots,n}$ be the operator realizing this reduction.

Proposition 7.2.9. *Let the problem $(\tau - a, \eth b)$ be microhyperbolic at (z^*, τ^*) in the multidirection $\mathcal{T} = (\ell', \nu_1, \dots, \nu_n)$. Then*

(i) If $\ell' = 0$ then W_j and W_j' are regular in a neighborhood of (z^*, τ^*), i.e., (7.2.38) *holds with $N = 0$;*

(ii) If $\ell' = \partial_{z_1}$ then there are almost analytic (with respect to z_1) extensions of W_j and W_j' in $\tilde{\Omega}_\pm = \tilde{\Omega} \cap \{\pm \operatorname{Im} \tau < 0, \mp \operatorname{Im} z_1 < 0\}$ and here

$$\|D_z^\alpha D_\tau^\beta W_j\| \le C(|\operatorname{Im} \tau| + |\operatorname{Im} z_1|)^{-N} \qquad \forall \alpha, \beta, j \tag{7.2.45}$$

with $C = C_{\alpha\beta j}$, $N = N_{\alpha\beta j}$.

Proof of proposition 7.2.9, Part I. Let us note that case (b) is reduced to (a) by introducing an additional variable z_1 (on which symbols a and b do not depend); therefore let us treat case (a).

Let us consider almost analytic extension of $\mathcal{G}'(z, \tau)$; it should satisfy (7.2.32) and (7.2.33) but (7.2.34) does not necessarily remain true.

Let us recall that we reduce operator $p(z, D_1, \tau) = \bigl(\tau - a(z, D_1)\bigr)$ to operator $\underline{p}(z, D_1, \tau)$ in the block-diagonal form as in Subsubsection *3.2.1.1 Notations and Definitions*.

Let us note that the microhyperbolicity condition of $p(z, \tau, D_1)$ at (z, τ, ν_l) yields that

(7.2.46) $g_l(z,\xi_1,\tau)=\det \underline{p}_l(z,\xi_1,\tau)$ has no root ξ_1 in the strip

$$\{|\operatorname{Im}\xi_1-\nu_l\operatorname{Im}z_1|\le \epsilon(|\operatorname{Im}\tau|+|\operatorname{Im}z_1|)\}$$

provided $(z,\tau)\in\tilde{\Omega}_{\pm}$.

Therefore we can replace (7.2.34) by condition

$$e^{\nu_l\sigma x_1}\boldsymbol{\pi}_l\mathcal{G}'\in \mathscr{S}(\mathbb{R}^+,.)\qquad \forall l=0,\dots,n \tag{7.2.47}$$

where $\sigma=\operatorname{Im}z_1$, $\nu_0=0$; here and below $\boldsymbol{\pi}_l$ means projection to l-th block and $l=0$ means an elliptic block.

Introducing $u=(u_0,\dots,u_n)$ with $u_l=e^{-\nu_j z_1 x_1}\boldsymbol{\pi}_l\mathcal{G}'$ we obtain the problem

$$\underline{p}_l(z,D_1-\nu_l z_1,\tau)u_l=0, \tag{7.2.48}$$

$$\eth\underline{b}(z,\tau)u=I, \tag{7.2.49}$$

$$u\in\mathscr{S}(\mathbb{R}^+,.); \tag{7.2.50}$$

let us recall that for $z_1\in\mathbb{R}$, $\operatorname{Im}\tau\ne 0$ this problem is solvable. □

Let us prove

Lemma 7.2.10. *Problem* (7.2.48)–(7.2.50) *is solvable for* $(z,\tau)\in\tilde{\Omega}_{\pm}$*; its solution is almost analytic with respect to* z_1 *and satisfies estimates*

$$\|D_1^k u\|_+\le C_k\big(|\operatorname{Im}\tau|+|\operatorname{Im}z_1|\big)^{-1}\qquad \forall k \tag{7.2.51}$$

Proof. Since this problem is solvable for $\operatorname{Im}z_1=0$, $\operatorname{Im}\tau\ne 0$, its solubility in the general case follows from (7.2.51). The deduction of the inequality in the proof of Theorem 3.1.12 implies that for $\omega:=(z,\tau)\in\mathbb{R}^{2d-2}\times\mathbb{R}$

$$\begin{aligned} M:=&-\operatorname{Re}\big(\langle \ell'\underline{p}(\omega,D_1)\underline{v},\underline{v}\rangle_+\\ &+\sum_{1\le l\le n}\nu_l\langle \ell'\underline{p}'_l(\omega,D_1)\underline{v}_l,\underline{v}_l\rangle_+ - i\langle \ell'\underline{b}(\omega)\eth\underline{v},\underline{b}'(\omega)\eth\underline{v}\rangle_b\big)\ge\\ &C^{-1}\|\underline{v}\|_+^2-C\|\underline{p}(\omega,D_1)\underline{v}\|_{(r)+}-C\|\underline{b}(\omega)\eth\underline{v}\|^2\\ &\forall \underline{v}\in\mathscr{S}(\bar{\mathbb{R}}^+,\mathbb{H}^m) \end{aligned} \tag{7.2.52}$$

with some r and C where $\underline{p}'_l=\partial_{\xi_1}\underline{p}_l$, $\underline{v}_l$ are the components of $\underline{v}$ after the reduction and

$$\|w\|_{(r)+}=\sum_{0\le k\le r}\|D_1^k w\|_+.$$

Let us take into account that for $\omega \in \mathbb{R}^{2d-2} \times \mathbb{R}$ expression

$$\langle \underline{p}(\omega, D_1)\underline{v}, \underline{v}\rangle_+ - i\langle \underline{b}(\omega)\eth\underline{v}, \underline{b}'(\omega)\eth\underline{v}\rangle_b$$

is real-valued due to (7.1.12). Further, let us take into account the following inequality

$$|\operatorname{Im}\big(\langle \underline{p}_\sigma(\omega, D_1)\underline{v}, \underline{v}\rangle_+ - i\langle \underline{b}_\sigma(\omega)\eth\underline{v}, \underline{b}'_\sigma(\omega)\eth\underline{v}\rangle_b\big) - M\sigma - M_0 \operatorname{Im}\tau| \le$$
$$C|\sigma| \cdot \|\underline{b}_\sigma(\omega)\eth\underline{v}\|_b \cdot \|\eth\underline{v}\|_b + C\big(|\sigma| + |\operatorname{Im}\tau|\big)^2\|\underline{v}\|^2_{(r)+}$$

where $\omega \in \mathbb{R}^{2d-2} \times \mathbb{C}$,

$$\underline{p}_\sigma(\omega, D_1) = \big(\underline{p}_{jl}(\omega + i\sigma\ell', D_1 - i\sigma\nu_l)\big)_{l=0,\dots,n},$$
$$\underline{b}'_\sigma(\omega) = \underline{b}'(\omega + i\sigma\ell'),$$

M_0 is defined as M by (7.2.52) with ℓ' and ν_l replaced by ∂_τ and 0 respectively; these inequality is due to microhyperbolicity of $(\underline{p}, \eth\underline{b})$ in multidirection $\mathcal{T}$. It yields an estimate

$$\big(|\sigma| + |\operatorname{Im}\tau|\big)\|\underline{v}\|_+ \le C\big(\epsilon|\sigma|\big)^{-1}\big(\|\underline{p}_\sigma(\omega, D_1)\underline{v}\|^2_{(r)+} + \|\underline{b}_\sigma(\omega)\eth\underline{v}\|^2_b\big) +$$
$$C\big((|\sigma| + |\operatorname{Im}\tau|)^2 + \epsilon(|\sigma| + |\operatorname{Im}\tau|)\big)\|\underline{v}\|_{(r+1)+}$$

with an arbitrarily small constant $\epsilon > 0$. Note that in the left-hand expression of this estimate one can replace $\|\underline{v}\|_+$ by $\|\underline{v}\|_{(r+1)+}$ because the coefficient of hD_1 in $\underline{p}_\sigma(z, D_1)$ is non-degenerate and so for small $|\sigma| + |\operatorname{Im}\tau|$ one can drop the last term in the right-hand expression. Then we arrive to

$$\|\underline{v}\|_{(r+1)+} \le \big(|\sigma| + |\operatorname{Im}\tau|\big)^{-1}\big(\|\underline{p}_\sigma(\omega, D_1)\underline{v}\|_{(r)+1} + \|\underline{b}_\sigma(z)\eth\underline{v}\|_b\big). \tag{7.2.53}$$

This inequality yields (7.2.51). Moreover, (7.2.53) and the equalities

$$(\partial\underline{p}_\sigma)u + \underline{p}_\sigma(\partial u) = 0,$$
$$\eth(\partial\underline{b}_\sigma)u + \eth\underline{b}_\sigma(\partial u) = 0$$

for every vector field ∂ yield that u is almost analytic with respect to τ and z_1 and that the estimate

$$\|D_z^\alpha D_\tau^\beta D_1^j u\| \le C\big(|\operatorname{Im} z_1| + |\operatorname{Im}\tau|\big)^{-N} \tag{7.2.54}$$

holds for $|\alpha| + \beta = 1$. If we make a few differentiations we can easily obtain the same estimate arbitrary α, β with $C = C_{\alpha\beta}$, $N = N_{\alpha\beta}$. □

Proof of proposition 7.2.9 (end). Lemma 7.2.10, estimate (7.2.54) and the definition of W_j immediately imply that for W_j of Proposition 7.2.9 holds.

Let us prove Proposition 7.2.9 for W_j'. Formula (7.2.35) yields that for real z, $\operatorname{Im}\tau \neq 0$ and $x_1 + y_1 > 0$ the following formula holds:

$$\mathcal{K}_{\mathcal{G}}(x_1, y_1; z, \tau) = (2\pi)^{-1} \int_{\gamma_+} e^{i(x_1-y_1)\xi_1} \big(\tau - a(z,\xi_1)\big)^{-1} d\xi_1 + \mathcal{G}'(z, x_1, \tau) \times (2\pi)^{-1} \int_{\gamma_-} b(z,\xi_1) e^{-iy_1\xi_1} \big(\tau - a(z,\xi_1)\big)^{-1} d\xi_1 \tag{7.2.55}$$

where

(7.2.56) $\gamma_\pm$ are closed contours in $\mathbb{C}$ passing once around the poles of $\big(\tau - a(z,\xi_1)\big)^{-1}$ lying in $\mathbb{C}_\pm$ and not passing around the poles lying in $\mathbb{C}_\mp$; for real z and $\operatorname{Im}\tau \neq 0$ there is no real pole.

Note that the factor to the right of $\mathcal{G}'$ in the second term in the right-hand expression is a sum of terms of the form

$$\int_{\gamma_-} \xi_1^k e^{-iy_1\xi_1} \big(\tau - a(z,\xi_1)\big)^{-1} d\xi_1$$

with regular coefficients and with $k \leq m-1$. Differentiating j times with respect to y_1 and setting $y_1 = 0$ we obtain terms of the form

$$W_k''(z,\tau) = \int_{\gamma_-} \xi_1^k \big(\tau - a(z,\xi_1)\big)^{-1} d\xi_1$$

with coefficients regular with respect to z, τ. In order to treat $\operatorname{Im} z_1 \neq 0$ let us replace the contours $\gamma_\pm$ by homotopic contours $\gamma_\pm^\sigma$ such that for $0 \leq \pm\sigma' \leq \pm\sigma$ there is no pole of $(\tau - a(z,\xi_1))^{-1}$ in $\gamma_\pm^\sigma$ where $\sigma = \operatorname{Im}\zeta$.

Property (7.2.46) implies that one can assume without any loss of the generality that in the intersection with the ϵ-neighborhood of ζ_j the contour $\gamma_\pm^\sigma$ coincides with $\{\operatorname{Im}\xi_1 = \nu_j \operatorname{Im} z_1\}$; then the microhyperbolicity condition yields that on $\gamma_\pm^\sigma$

$$\big\|\big(\tau - a(z,\xi_1)\big)^{-1}\big\| \leq C(|\operatorname{Im}\tau| + |\operatorname{Im} z_1|)^{-1}$$

and hence W_k'' also satisfies the statement of Proposition 7.2.9. Hence we need to treat only the first term in $\mathcal{K}_{\mathcal{G}}$. Applying $\partial D_{y_1}^j$ we obtain a sum of terms of the form

$$v_j := (2\pi)^{-1} \int_{\gamma_+} e^{ix_1\xi_1} \xi_1^j \big(\tau - a(z,\xi_1)\big)^{-1} d\xi_1.$$

These terms satisfy equation $\bigl(\tau - a(z,D_1)\bigr)v = 0$ for $x_1 > 0$ and hence $v_j = \mathcal{G}'(z,x_1,\tau)\eth b(z,D_1)v_j$. Repeating the same arguments as before we see that $W_k''' = \eth b(z,D_1)v$ satisfy statement of Proposition 7.2.9; therefore this statement holds for W_k' also. □

Let us represent $\mathcal{G}'$ as an oscillatory integral:

Lemma 7.2.11. *(i) The following formula holds:*

$$\mathcal{G}'(z,x_1,\tau) = \mathsf{v.p.}\int_{-\infty}^{\infty} \sum_{0\le j\le m-1} \xi_1^{m-j-1}\bigl(\tau - a(z,\xi_1)\bigr)^{-1}\,d\xi_1 \cdot W_j''(z,\tau) \tag{7.2.57}$$

where W_j'' satisfy (7.2.38).

(ii) In this formula one can replace integration along $\mathbb{R}$ by integration along γ_+.

(iii) Hence the same representation holds for $D_1^k\mathcal{G}'(z,x_1,\tau)$ for all $k \ge 0$.

(iv) Under the microhyperbolicity condition for W_j'' the statement of Proposition 7.2.9 holds.

Proof. Starting from the equality

$$\bigl(\tau - a(z,D_1)\bigr)\bigl(\mathcal{G}'\theta(x_1)\bigr) = \sum_{j+k\le m-1} \delta^{(k)}(x_1)a_{kj}(z)\eth D_1^j\mathcal{G}'$$

with regular coefficients a_{kj} and applying the Fourier transform with respect to x_1, multiplication by $(\tau - a(z,\xi_1))^{-1}$, and the inverse Fourier transform we obtain (7.2.56); estimate (7.2.38) follows from Lemma 7.2.7.

Further, Statement (iv) follows from Proposition 7.2.9 while Statement (iii) is obvious. Applying D_1^k to (7.2.56) with γ_+ instead of $\mathbb{R}$ and dividing with the remainder by $(\tau - a(z,\xi_1))$ we obtain the same representation for $D_1^k\mathcal{G}'$, and at the last moment we can replace γ_+ by $\mathbb{R}$ again; so, Statement (ii) is proven. □

Now we can prove

Proposition 7.2.12. *(i) For $\mathsf{Im}\,\tau \ne 0$ one-dimensional operator*

$$\mathcal{A}(z,D_1) = c_1(z)D_1^{j_1}\mathcal{G}\cdot c_2(z)D_1^{j_2}\mathcal{G}\cdots c_{\mu-1}(z)D_1^{j_{\mu-1}}\mathcal{G}\cdot c_\mu(z)D_1^{j_\mu}\mathcal{G}'$$

containing $(\mu-1)$ *factors* $\mathcal{G}$ *is a sum of terms of the form*

$$\int_{\gamma_+} e^{ix_1\xi_1}\xi_1^k F_{kl}(z,\xi_1,\tau)d\xi_1 \cdot W_{lk}(z,\tau) \tag{7.2.58}$$

where

$$F_{kl} = c'_{kl1}(z,\tau)\big(\tau - a(z,\xi_1)\big)^{-1}\cdots c'_{kll}(z,\tau)\big(\tau - a(z,\xi_1)\big)^{-1}$$

$k=0,\dots,m-1$, $l\le\mu$, *and* W_{lk} *satisfy* (7.2.38) *(there could be many terms with the same* (k,l) *but different coefficients* c_{lk*}*).*

(ii) Moreover, $\mathfrak{d}\mathcal{A} = W(z,\tau)$ *satisfies* (7.2.38)*.*

(iii) Under the microhyperbolicity condition W_{lk} *and* W *satisfy the statement of Proposition 7.2.9.*

Proof. Let us prove by induction with respect to μ. For $\mu=1$ this proposition follows from Lemma 7.2.10 and Proposition 7.2.9. Let us rewrite (7.2.58) as the principal value of an integral along $\mathbb{R}$ and apply (7.2.37); we then obtain terms of the same form with F_{lk} replaced by $(\tau-a(z,\xi_1))^{-1}F_{lk}$ and terms of the form $\mathcal{G}(z,\xi_1,\tau)W'_{lk}$ with

$$W'_{lk} = \mathsf{v.p.}\int_{-\infty}^{\infty} b(z,\xi_1)\xi_1^k\big(\tau-a(z,\xi_1)\big)^{-1}F_{lk}(z,\xi_1,\tau)\,d\xi_1.$$

In this integral we can take integration over the contour γ_+ coinciding with the real axis in the ϵ-neighborhood of ζ_j and therefore W'_{lk} satisfy (7.2.38); moreover, under the microhyperbolicity condition taking $\mathsf{Im}\, z_1\ne 0$ we can deform this contour into γ_+^σ (see the proof of Proposition 7.2.9) and obtain that W'_{lk} satisfy the assertion of Proposition 7.2.9.

We rewrite $\mathcal{G}'$ in the form (7.2.56), pass to integration along γ_+, apply the operator $c_1(z)D_1^{j_1}$ and divide by $(\tau-a(z,\xi_1))$ with the remainder any factor ξ_1^k with k exceeding $(m-1)$. Since the quotient contains no pole its integral is 0, so only the remainder matters and we finish the induction step.

Then Statement (i) and the first part of Statement (iii) are proven (in fact, Statement (iii) consists of two parts corresponding to the two parts of Proposition 7.2.9).

Applying operator $\mathfrak{d}$ to (7.2.58) and estimating $\int_{\gamma_+}\xi_1^k F_{kl}(z,\xi_1,\tau)d\xi_1$ (and its derivatives) in a manner similar to the previous arguments (and with $\mathsf{Im}\, z_1\ne 0$ under the microhyperbolicity condition) we prove Statement (ii) and the second part of Statement (iii). □

Calculations: what We Got

Now we can finish our calculations. Starting from Proposition 7.2.6 we can eliminate all the degrees of x_1 and $(x'-y')$ by commutation; we can use formulae (7.2.31) and (7.2.36) to pass from operators $\bar{G}^{\pm}$ and $\bar{G}'^{\pm}$ to $\mathcal{G}$ and $\mathcal{G}'$ in $\mathbb{C}_{\mp}\setminus\mathbb{R}$ and Proposition 7.2.12; we can also use formulae for $\bar{G}_0^{\pm}$. Thus we arrive to

Theorem 7.2.13. *Let conditions* (7.1.1), (7.1.2), $(7.1.4)_{1,2}$, (7.1.9) *and* (7.1.10) *be fulfilled.*

Further, let $Q_i'=\mathsf{Op}(q_i')$ *with symbols* $q_i'\in \mathsf{S}'_{h,K}$ *supported in* $\{|\xi'|\le c\}$ *and* $\psi_i\in \mathscr{C}_0^K(B(0,1-\epsilon))$. *Finally, let* $\chi\in\mathscr{C}_0^K([-1,1])$ *and* $T\in[h^{1-\delta},h^{\frac12+\delta}]$.

(i) Then

$$
\begin{aligned}
(7.2.59)\quad \|D_x^{\alpha}D_y^{\beta}\Big(F_{t\to h^{-1}\tau}\chi_T(t)Q'_{1x}\psi_1(x)U^{1\pm}\psi_2(y)^{t}Q'_{2y}\mp\\
\sum_{0\le n\le N'-1}h^{-d+n}\int T\hat{\chi}(\frac{\tau' T}{h})d\tau'\int d\xi'\int_{\gamma_+}d\xi_1\int_{\gamma_-}d\eta_1\times\\
e^{ih^{-1}(x_1\xi_1-y_1\eta_1+\langle x'-y',\xi'\rangle)}q_1'(x,\xi')F_n(y',\xi',\xi_1,\eta_1,\tau-\tau')[q_2'(y,\xi')]\Big)\|\le\\
C'h^{s}(|\tau|+1)^{N''}\\
\forall\alpha,\beta:|\alpha|\le s,|\beta|\le s\quad\forall\tau\in\mathbb{C}_{\mp}\quad\forall x,y\in\bar{\mathbb{R}}^{+}\times\mathbb{R}^{d-1}
\end{aligned}
$$

provided $N'=N'(d,m,s,\delta)$, $N''=N''(d,m,s,\delta,N')$, $K=K(d,m,s,\delta,N')$ *where* F_n *are sums of terms of the form*

$$
\begin{aligned}
(7.2.60)\quad \xi_1^{j}\eta_1^{k}\prod_{1\le\mu\le M}\big(\tau-a(y',\xi_1,\xi')\big)^{-1}c_{\mu}(y',\xi')\times\\
W(y',\xi',\tau)\prod_{1\le\nu\le M'}c'_{\nu}(y',\xi')\big(\tau-a(y',\eta_1,\xi')\big)^{-1}
\end{aligned}
$$

with $j,k\le m-1$, $M+M'\le M_0(n)=2n+2$, *and functions* W *satisfying* (7.2.38)*; here* q_1' *and* q_2' *are symbols of* $Q_1'\psi_1$ *and* $Q_2'\psi_2$ *respectively and there are no more than* $M_0(n)$ *differentiations with respect to* (ξ',x_1).

(ii) Moreover, under the microhyperbolicity condition the functions W *satisfy the assertion of Proposition 7.2.9.*

Remark 7.2.14. (i) In formula (7.2.59) we have replaced integration with respect to ξ_1 along $\mathbb{R}$ by integration along γ_-.

(ii) Multiplication of all the terms of decomposition (7.2.59) by x_1 or y_1 is equivalent (after integration by parts) to the application to F_n of the operator $ih\partial_{\xi_1}$ or $-ih\partial_{\eta_1}$ respectively.

(iii) Further, note that application of ∂_{ξ_1} or ∂_{η_1} increases the singularity of F_n at $\xi_1, \eta_1 \in \mathbb{R}$ and $\tau \to \mathbb{R} \pm i0$ by no more than an extra factor $(\operatorname{Im}\tau)^{-1}$ and hence the singularity of the distribution $F_n(y', \xi', \xi_1, \eta_1, \tau \pm i0)$ increases by no more than by 1. Hence when we estimate the corresponding integral we should replace the $\mathscr{C}^N$-norm of $\hat{\chi}(\tau' Th^{-1})$ by its $\mathscr{C}^{N+1}$-norm, i.e., there arises an extra factor Th^{-1}. Also multiplication by x_1 or y_1 essentially multiplies the corresponding integral by T. Therefore for $(x_1 + y_1) \geq Th^{-\delta}$ with $\delta > 0$ all terms in the decomposition are negligible.

Calculations: Taking Traces

Let us set $x_1 = y_1$ and integrate over x_1 along $\mathbb{R}^+$; let us take the Taylor decomposition of q_i' with respect to x_1 and y_1 and apply Remark 7.2.14(ii),7.2.14-iii; then we can drop all the remainder terms and eliminate factors x_1 and y_1 by means of integration by parts. As a result there arises inside the exterior integrals with respect to τ' and ξ' a term

$$W_n'(z, \tau - \tau') = ih \int_{\gamma_+} \int_{\gamma_-} F_n(y', \xi', \xi_1, \eta_1, \tau - \tau')(\xi_1 - \eta_1)^{-1} d\xi_1 d\eta_1$$

where we have taken non-intersecting contours γ_+ and γ_- and we do not write factors q_1' and q_2' and $z = (x', \xi')$. Moreover, we can choose γ_+ and γ_- (depending on τ) such that

$$\operatorname{dist}(\gamma_+, \gamma_-) \geq \epsilon |\operatorname{Im}\tau| \tag{7.2.61}$$

and at these contours

$$|\big(\tau - a(y', \xi', \xi_1)\big)^{-1}| \leq C |\operatorname{Im}\tau|^{-1}. \tag{7.2.62}$$

Hence functions W_n' satisfy (7.2.38).

Furthermore, under microhyperbolicity condition we can pass to non-real variable z_1 and choose γ_+ and γ_- such that in the preceding inequalities

(7.2.61), (7.2.62) $|\operatorname{Im}\tau|$ will be replaced by $(|\operatorname{Im}\tau| + |\operatorname{Im} z_1|)$ (let us recall the that signs of $\operatorname{Im}\tau$ and $\operatorname{Im} z_1$ should be opposite). But then for W'_n the statement of Proposition 7.2.9 holds. So we have proven

Theorem 7.2.15. *Let conditions* (7.1.1), (7.1.2), $(7.1.4)_{1,2}$, (7.1.9) *and* (7.1.10) *be fulfilled.*

Further, let $Q'_i = \mathsf{Op}(q'_i)$ *with symbols* $q'_i \in \mathsf{S}'_{h,K}$ *supported in* $\{|\xi'| \le c\}$ *and* $\psi_i \in \mathscr{C}_0^K(B(0,1-\epsilon))$. *Finally, let* $\chi \in \mathscr{C}_0^K([-1,1])$ *and* $T \in [h^{1-\delta}, h^{\frac12+\delta}]$.

(i) Then

$$
\begin{aligned}
(7.2.63)\quad \|D^\alpha_{x'} D^\beta_{y'} \Big(F_{t\to h^{-1}\tau}\chi_T(t) \int_0^\infty \big(Q'_{1x}\psi_1(x) U^{1\pm}\psi_2(y)^t Q'_{2y}\big)\big|_{x_1=y_1} dy_1 \mp \\
\sum_{0\le n\le N'-1} h^{-d+n+1} \int T\hat\chi\big(\frac{\tau' T}{h}\big)\, d\tau' \int d\xi' \times \\
e^{ih^{-1}\langle x'-y',\xi'\rangle} q'_1(x,\xi') W'_n(y',\xi',\tau-\tau')[q'_2(y,\xi')]\Big)\| \le C' h^s (|\tau|+1)^{N''} \\
\forall \alpha,\beta : |\alpha|\le s, |\beta|\le s \quad \forall \tau \in \mathbb{C}_\mp \quad \forall x', y' \in \mathbb{R}^{d-1}
\end{aligned}
$$

with the same N', N'' *and* K *as before and with* W'_n *satisfying* (7.2.38).

(ii) Moreover, under microhyperbolicity condition for W'_n *the statement of Proposition 7.2.9 holds.*

(iii) Furthermore, the same Statements (i), (ii) remain true if we replace ψ_1 *and (or)* ψ_2 *by* $\psi_1\psi'_1$ *and (or)* $\psi_2\psi'_2$ *with* $\psi'_i(x_1) = \psi''_i(x_1 h^{-\delta} T^{-1})$, $\psi''_i \in \mathscr{C}_0^K([-1,1])$ *equal to* 1 *on* $[-\frac12, \frac12]$*; moreover by virtue of Remark 7.2.14 doing this we change none of the terms in the decomposition.*

Calculations: Mollification with Respect to τ

On the other hand, applying the operator $\phi(hL^{-1}D_t)$ and taking $t = 0$ we obtain

Theorem 7.2.16. *Let conditions* (7.1.1), (7.1.2), $(7.1.4)_{1,2}$, (7.1.9) *and* (7.1.10) *be fulfilled.*

Further, let $Q'_i = \mathsf{Op}(q'_i)$ *with symbols* $q'_i \in \mathsf{S}'_{h,K}$ *supported in* $\{|\xi'| \le c\}$ *and* $\psi_i \in \mathscr{C}_0^K(B(0,1-\epsilon))$. *Finally, let* $\chi \in \mathscr{C}_0^K([-1,1])$, $\chi = 1$ *at* $[-\frac12, \frac12]$,

$T \in [h^{1-\delta}, h^{\frac12+\delta}]$ *and* $L \ge h^{1-\delta'}T^{-1}$. *Then*

$$\|D_x^\alpha D_y^\beta \Big(\phi(L^{-1}hD_t)\chi_T(t)\big(Q'_{1x}\psi_1(x)U^{1\pm}\psi_2(y)^t Q'_{2y}\big)\big|_{t=0}\mp \sum_{0\le n\le N'-1} h^{-d+n+1}\int \phi\big(\frac{\tau'}{L}\big)\,d\tau' \int d\xi' \int_{\gamma_+} d\xi_1 \int_{\gamma_-} d\eta_1 \times e^{ih^{-1}(x_1\xi_1-y_1\eta_1+\langle x'-y',\xi'\rangle}q'_1(x,\xi')F_n(y',\xi',\xi_1,\eta_1,\tau'\mp i0)[q'_2(y,\xi')]\Big)\| \le C'h^s \quad \forall \alpha,\beta : |\alpha|\le s, |\beta|\le s \quad \forall x,y \in \bar R^+ \times \mathbb{R}^{d-1} \tag{7.2.64}$$

and

$$\|D_{x'}^\alpha D_{y'}^\beta \Big(\phi(L^{-1}hD_t)\chi_T(t)\int_0^\infty \big(Q'_{1x}U^{1\pm}(y)^t Q'_{2y})|_{x_1=y_1}\big)\,dy_1\big|_{t=0}\mp \sum_{0\le n\le N'-1} h^{-d+n+2}\int \phi\big(\frac{\tau'}{L}\big)\,d\tau' \int d\xi' \times e^{ih^{-1}\langle x'-y',\xi'\rangle}q'_1(x,\xi')W'_n(y',\xi',\tau'\mp i0)[q'_2(y,\xi')]\Big)\| \le C'h^s \quad \forall \alpha,\beta : |\alpha|\le s, |\beta|\le s \quad \forall x',y' \in \mathbb{R}^{d-1} \tag{7.2.65}$$

with the same N', N'', K, δ, δ' *as before*[9)].

7.2.4 Exploiting Microhyperbolicity

Now let us apply the microhyperbolicity condition and try to prove that inequalities (7.2.59) and (7.2.63) remain true (under some assumptions) for $T \in [h^{1-\delta}, T_0]$ with a small enough constant $T_0 > 0$; then the same will be true for (7.2.64) and (7.2.65) as well.

Theorem 7.2.17. *Let conditions* (7.1.1), (7.1.2), $(7.1.4)_{1,2}$, (7.1.9) *and* (7.1.10) *be fulfilled.*

Further, let $Q'_i = \mathsf{Op}(q'_i)$ *with symbols* $q'_i \in \mathsf{S}'_{h,K}$ *supported in* $\{|\xi'| \le c\}$ *and* $\psi_i \in \mathscr{C}_0^K(B(0,1-\epsilon))$.

Furthermore, let us assume that the problem $(\tau - A, \eth B)$ *is microhyperbolic on* $\mathsf{supp}(q'_2)$ *in the multidirection* $\mathcal{T} = (\ell', \nu_1, \dots, \nu_n)$ *for every* $\tau \in [\tau_1, \tau_2]$.

Finally, let $\chi \in \mathscr{C}_0^K([-1,1])$ *and* $T \in [h^{1-\delta}, T_0]$, $T_0 = T_0(d, m, D, c, \epsilon)$. *Then for* $\tau \in [\tau_1, \tau_2]$, $|\alpha| \le s$, $|\beta| \le s$, *the following estimates hold:*

[9)] We assume for the sake of simplicity that Q'_i and ψ'_i do not depend on x_1 for $x_1 \le Th^{-\delta'}$.

(i) If $\ell'_x = 0$, $\nu_1 = ... = \nu_n = \nu$ *then*

$$\|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma_x\big(D_x^\alpha D_y^\beta Q'_{1x}\psi_1(x)U^{1\pm}\psi_2(y)^t Q'_{2y}\big)\big|_{t=0}\mp \sum_{0\le n\le N'-1} h^{-d+n-|\alpha|-|\beta|}\int T\hat\chi\big(\frac{\tau' T}{h}\big)d\tau'\int_{\gamma_+}d\xi_1\int_{\gamma_-}d\eta_1\times e^{ih^{-1}y_1(\xi_1-\eta_1)}F^1_{n,\alpha,\beta}(y',\xi_1,\eta_1,\tau-\tau'\mp i0)\| \le C'h^s \quad \forall y\in \bar{\mathbb{R}}^+\times\mathbb{R}^{d-1}, \tag{7.2.66}$$

where (as $\tau\in[\tau_1,\tau_2]$*)* $F^1_{n,\alpha,\beta}(y',\xi_1,\eta_1,\tau\mp i0)$ *are regular functions with respect to all variables and analytic with respect to* ξ_1 *and* η_1 *such that*

$$F^1_{n,\alpha,\beta}(y',\xi_1,\eta_1,\tau\mp i0)\sim\sum_{0\le k} c_{nk\alpha\beta}(y',\eta_1,\tau\mp i0)\xi_1^{-k-1} \tag{7.2.67}$$

as $|\xi_1|\to\infty$ *(and a similar decomposition holds as* $|\eta_1|\to\infty$*), coefficients* $c_{nk\alpha\beta}$ *are regular and for them similar decompositions hold as* $|\xi_1|\to\infty$ *or* $|\eta_1|\to\infty$.

Moreover, functions $F^1_{n,\alpha,\beta}(y',\xi_1,\eta_1,\tau+i0)$ *and* $F^1_{n,\alpha,\beta}(y',\xi_1,\eta_1,\tau-i0)$ *coincide provided* $(y',\xi')\notin\Sigma_{\tau,b}\cap\operatorname{supp}(q'_1)\cap\operatorname{supp}(q'_2)$ *for every* $\xi'\in\mathbb{R}^{d-1}$.

(ii) If $\ell'_x = 0$ *then*

$$\|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma''\big(D_x^\alpha D_y^\beta Q'_{1x}\psi_1(x)U^{1\pm}\psi_2(y)^t Q'_{2y}\big)\big|_{t=0}\mp \sum_{0\le n\le N'-1} h^{-d+n-|\alpha|-|\beta|+1}\int T\hat\chi\big(\frac{\tau' T}{h}\big)\,d\tau'\times W_{n,\alpha,\beta}(y',\tau-\tau'\mp i0)\| \le C'h^s \tag{7.2.68}$$

and

$$\|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma'_{x'}\big(\eth_x\eth_y D_x^\alpha D_y^\beta Q'_{1x}\psi_1(x)U^{1\pm}\psi_2(y)^t Q'_{2y}\big)\big|_{t=0}\mp \sum_{0\le n\le N'-1} h^{-d+n-|\alpha|-|\beta|}\int T\hat\chi\big(\frac{\tau' T}{h}\big)\,d\tau'\times W'_{n,\alpha,\beta}(y',\tau-\tau'\mp i0)\| \le C'h^s \quad \forall y'\in\mathbb{R}^{d-1} \tag{7.2.69}$$

where (as $\tau\in[\tau_1,\tau_2]$*)* $W_{n,\alpha,\beta}(y',\tau\mp i0)$ *and* $W'_{n,\alpha,\beta}(y',\tau\mp i0)$ *are regular functions which*[10] *coincide provided* $(y',\xi')\notin\Sigma_{\tau,b}\cap\operatorname{supp}(q'_1)\cap\operatorname{supp}(q'_2)$ *for every* $\xi'\in\mathbb{R}^{d-1}$.

[10] With the signs "+" and "−".

(iii) If $\nu_1 = ... = \nu_n = \nu$, *then*

$$\|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma'''\big(D_x^\alpha D_y^\beta Q'_{1x}\psi_1(x)U^{1\pm}\psi_2(y)^t Q'_{2y}\big)\big|_{t=0}\mp \sum_{0\le n\le N'-1} h^{-d+n-|\alpha|-|\beta|}\int T\hat\chi\big(\frac{\tau' T}{h}\big)d\tau'\int_{\gamma_+} d\xi_1\int_{\gamma_-} d\eta_1\times e^{ih^{-1}y_1(\xi_1-\eta_1)}F^2_{n,\alpha,\beta}(\xi_1,\eta_1,\tau-\tau'\mp i0)\| \le C'h^s \quad \forall y_1\in\bar{\mathbb{R}}^+ \tag{7.2.70}$$

where $F^2_{n,\alpha,\beta}(\xi_1,\eta_1,\tau-\tau'\mp i0)$ *are functions of the same type as in (i) which*[10] *coincide provided* $(y',\xi')\notin \Sigma_{\tau,b}\cap\mathrm{supp}(q'_1)\cap\mathrm{supp}(q'_2)$ *for every* $(y',\xi')\in T^*Y$.

(iv) If $\ell'=0$ *then*

$$\|F_{t\to h^{-1}\tau}\chi_T(t)\big(\eth_x\eth_y D_x^\alpha D_y^\beta Q'_{1x}\psi_1(x)U^{1\pm}\psi_2(y)^t Q'_{2y}\big)\big|_{t=0}\mp \sum_{0\le n\le N'-1} h^{-d+n-|\alpha|-|\beta|}\int T\hat\chi\big(\frac{\tau' T}{h}\big)\,d\tau'\times e^{ih^{-1}\langle x'-y',\xi'\rangle}W''_{n,\alpha,\beta}(y',\tau-\tau'\mp i0)d\xi'\| \le C'h^s \quad \forall x',y'\in\mathbb{R}^{d-1} \tag{7.2.71}$$

where $W''_{n,\alpha,\beta}(y',\tau\mp i0)$ *are functions of the same type as in (ii) which*[10] *coincide provided* $(y',\xi')\notin\Sigma_{\tau,b}\cap\mathrm{supp}(q'_1)\cap\mathrm{supp}(q'_2)$.

(v) In the general case

$$\|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\big(D_x^\alpha D_y^\beta Q'_{1x}\psi_1(x)U^{1\pm}\psi_2(y)^t Q'_{2y}\big)\big|_{t=0}\mp \sum_{0\le n\le N'-1} h^{-d+n-|\alpha|-|\beta|+1}\int T\hat\chi\big(\frac{\tau' T}{h}\big)d\tau'\times e^{ih^{-1}\langle x'-y',\xi'\rangle}W'''_{n,\alpha,\beta}(\tau-\tau'\mp i0)\,d\xi'\| \le C'h^s \tag{7.2.72}$$

and

$$\|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma'\big(\eth_x\eth_y D_x^\alpha D_y^\beta Q'_{1x}\psi_1(x)U^{1\pm}\psi_2(y)^t Q'_{2y}\big)\big|_{t=0}\mp \sum_{0\le n\le N'-1} h^{-d+n-|\alpha|-|\beta|}\int T\hat\chi\big(\frac{\tau' T}{h}\big)\,d\tau'\times e^{ih^{-1}\langle x'-y',\xi'\rangle}W^{IV}_{n,\alpha,\beta}(y',\tau-\tau'\mp i0)\,d\xi'\| \le C'h^s \tag{7.2.73}$$

where $W'''_{n,\alpha,\beta}(\tau \mp i0)$ *and* $W^{IV}_{n,\alpha,\beta}(\tau \mp i0)$ *are functions of the same type as in (ii) which*[10] *coincide provided* $(y',\xi') \notin \Sigma_{\tau,b} \cap \mathsf{supp}(q'_1) \cap \mathsf{supp}(q'_2)$ *for every* $(y',\xi') \in T^*Y$.

(vi) Moreover, for $n = 0$ *all the functions* $F^{\cdots}_{\cdots}$, $W^{\cdots}_{\cdots}$ *contain* q^0_1, q^0_2, ψ_1 *and* ψ_2 *as factors.*

(vii) Finally, in (ii), (iv) and (v) one can replace ψ_1 *and* ψ_2 *by* $\psi_1\psi'_1(x_1 h^{\delta'} T^{-1})$ *and* $\psi_2\psi'_2(y_1 h^{\delta'} T^{-1})$ *respectively.*

Proof. The proof evidently follows from Theorems 7.2.13, 7.2.13 and 3.3.5 if we prove that for χ supported in $[-1,-\epsilon'] \cup [\epsilon',1]$ with arbitrarily small $\epsilon' > 0$ all the terms of the decomposition are $O(h^s)$ and also prove that the functions $F^{\cdots}_{\cdots}$, $W^{\cdots}_{\cdots}$ are as regular as is announced in the corresponding statements. Repeating the arguments of the proof of Proposition 4.3.9 we obtain that it is sufficient to prove only the statements concerning regularity.

Further, all the statements concerning regularity of the functions $W^{\cdots}_{\cdots}$ follow from the corresponding assertions of Theorems 7.2.13 and 7.2.15 and from Appendix 4.A.1. Therefore it is sufficient to prove regularity properties of $F^{\cdots}_{\cdots}$ in the cases (i) and (iii). We treat case (i) and case (iii) which is treated in the same way is left to the reader.

Let us replace the contour γ_+ by the union of contours $\gamma_{+,j}$ with $\gamma_{+,j}$ located near $\xi_1 = \zeta_j$ for $j \ge 1$ and $\gamma_{+,0}$ located in $\mathsf{Im}\,\xi_1 \ge \epsilon''$; similarly we replace γ_- where now $\gamma_{-,0}$ is located in $\mathsf{Im}\,\xi_1 \le -\epsilon$; as a result we obtain the sum of terms

$$\int_{\gamma_{+,j}} d\xi_1 \int_{\gamma_{-,k}} d\eta_1 \times \xi_1^p \eta_1^r e^{ix_1(\xi_1-\eta_1)} f_1(\xi_1,\xi',\tau) W(\xi',\tau) f_2(\eta_1,\xi',\tau) \coloneqq \Phi(\xi',\tau)$$

where f_1 is a product of $(\tau - a(\xi_1,\xi'))^{-1}$ separated by regular functions of ξ' and f_2 has a similar form, $p \le m-1$, $r \le m-1$ and for the sake of simplicity we do not indicate here dependence on y' and do not write the exterior integral with respect to ξ'.

Without any loss of the generality one can assume that $\ell' = \partial_{\xi_2}$ (if $\ell' = 0$ we can introduce an additional variable). Then in W the replacement of $\tau \in \mathbb{R}$ by $\tau \mp i0$ is equivalent to the replacement of $\xi_2 \in \mathbb{R}$ by $\xi_2 \pm i0$ and in f_1 and f_2 the replacement of $\tau \in \mathbb{R}$ by $\tau \mp i0$ is equivalent to the replacement

$$\xi_2 \to \xi_2 \pm i0, \quad \xi_1 \to \xi_1 + i\nu\,\mathsf{Im}\,\xi_2, \quad \eta_1 \to \eta_1 + i\nu\,\mathsf{Im}\,\xi_2$$

(let us recall that $\nu_1 = ... = \nu_n = \nu$ in the case (i) and ν_0 can be chosen arbitrarily). Therefore the replacement of $\tau \in \mathbb{R}$ by $\tau \mp i0$ is equivalent to the replacement of $\xi_2 \in \mathbb{R}$ by $\xi_2 \pm i0$ with the deformation of contours $\gamma_\pm \to \gamma_\pm + i\nu \operatorname{Im} \xi_2$ and without any loss of the generality one can assume that $\gamma_\pm \subset \mathbb{C}_\pm \cup \mathbb{R}$; then the absolute value of the exponent does not exceed 1 and on the deformed contours $\|(\tau - a(\xi', \xi_1))^{-1}\| \le C|\operatorname{Im} \xi_2|^{-1}$.

Therefore we obtain that there exists an almost analytic (with respect to ξ_2) extension of $\Phi(y', \xi', \tau \mp i0)$ to the domain $\{\pm \operatorname{Im} \xi_2 > 0\}$ such that

$$\|D_{y'}^\alpha D_\tau^\beta \Phi n(y', \xi', \tau \mp i0)\| \le C'|\operatorname{Im} \xi_2|^{-N} \qquad \forall \alpha, \beta : |\alpha| + \beta \le s.$$

Therefore after multiplication by functions compactly supported with respect to ξ' and integration with respect to ξ_2 we obtain functions $F_{...}^{...}(y', \tau \mp i0)$ which are smooth due to Appendix 4.A.2. □

7.2.5 Conclusion

We leave to the reader to write down corollaries of Theorems 7.2.13 and 7.2.15 in which $U^{1\pm}$ are replaced by their sum $U^1 = U^{1+} + U^{1-}$ and $\mp F_{...}^{...}(\tau)$, $\mp W_{...}^{...}(\tau)$ with $\tau \in \mathbb{C}_\mp$ are replaced by

$$\begin{aligned} \mathcal{F}_{...}^{...}(\tau) &= F_{...}^{...}(\tau - i0) - F_{...}^{...}(\tau + i0), \\ \mathcal{W}_{...}^{...}(\tau) &= W_{...}^{...}(\tau - i0) - W_{...}^{...}(\tau + i0) \end{aligned}$$

with $\tau \in \mathbb{R}$ where we skip all other arguments.

Similarly, in Theorems 7.2.16 and 7.2.17 one can do the similar modifications except here we replace $\tau - \tau' \mp i0$ in the arguments just by $\tau - \tau'$.

Finally, we leave to the reader to write down decompositions when U^1 (or $U^{1\pm}$) is replaced by $U = U^0 + U^1$ (or $U^\pm = U^{0\pm} + U^{1\pm}$) and decompositions of this section are added to decomposition of Section 4.3.

We will use all these decompositions in the next section as needed.

7.3 General Basic Theorems

In this section we obtain the principal microlocal and local asymptotics. Proofs which are completely similar to those of Section 4.4 are usually left to the reader.

7.3.1 Asymptotics with Mollification

We begin with microlocal asymptotics with mollification with respect to the spectral parameter.

Theorem 7.3.1. [11] *Let conditions* (7.1.1), (7.1.2), $(7.1.4)_{1,2}$, (7.1.9) *and* (7.1.10) *be fulfilled and let* $Q_i = \mathsf{Op}(q_i)$, $q_i \in \mathsf{S}_{h,K}$ *be supported in* $\{|\xi'| \le c\}$, $\psi_i \in \mathscr{C}_0^K(B(0,1-\epsilon))$*; let* $\phi \in \mathscr{C}_0^K([-c,c])$ *and* $\bar\tau \in [-c,c]$. *Then*

(i) For $L \in [h^{\frac12-\delta}, 1]$ *the following estimates hold:*

$$\frac{1}{L}\Big\|\int \phi\big(\frac{\tau-\bar\tau}{L}\big)\Big(d_\tau Q'_{1x}\psi_1(x)e(x,y,\tau_1,\tau)\psi_2(y)^t Q'_{2y} - \sum_{0\le n\le N'-1} h^{-d+n}\mathcal{K}'_n(x,y,\tau)\,d\tau\Big)\Big\| \le C'h^s \quad \forall x,y \in \bar{\mathbb{R}}^+\times\mathbb{R}^{d-1}, \tag{7.3.1}$$

$$\frac{1}{L}\Big\|\int \phi\big(\frac{\tau-\bar\tau}{L}\big)\Big(d_\tau \Gamma_x Q'_{1x}\psi_1(x)e(x,y,\tau_1,\tau)\psi_2(y)^t Q'_{2y} - \sum_{0\le n\le N'-1} h^{-d+n}\kappa'_n(x,\tau)\,d\tau\Big)\Big\| \le C'h^s \quad \forall x \in \bar{\mathbb{R}}^+\times\mathbb{R}^{d-1}, \tag{7.3.2}$$

$$\frac{1}{L}\Big\|\int \phi\big(\frac{\tau-\bar\tau}{L}\big)\Big(d_\tau \Gamma Q'_{1x}\psi_1(x)e(x,y,\tau_1,\tau)\psi_2(y)^t Q'_{2y} - \sum_{0\le n\le N'-1} h^{-d+n}\varkappa'_n(x,y,\tau)\,d\tau\Big)\Big\| \le C'h^s \tag{7.3.3}$$

where $\mathcal{K}'_n = \mathcal{K}^{0\prime}_n + \mathcal{K}^{1\prime}_n$, $\mathcal{K}^{0\prime}_n$ *and* $\mathcal{K}^0_n$ *are defined by* (4.4.5) *and* (4.4.6) *respectively,* $\kappa'_n = \kappa^{0\prime}_n + \kappa^{1\prime}_n$ *and* $\kappa^{\cdots}_{\cdots} = \mathcal{K}^{\cdots}_{\cdots}|_{x=y}$, $\varkappa'_n = \varkappa^{0\prime}_n + \varkappa^{1\prime}_n$ *and* $\varkappa^{\cdots}_{\cdots} = \int \kappa^{\cdots}_{\cdots}\,dx$ *while*

$$\mathcal{K}^{1\prime}_n(x,y,\tau) = -2\pi i \int d\xi' \int_{\gamma_+} d\xi_1 \int_{\gamma_-} d\eta_1 \times e^{ih^{-1}(x_1\xi_1 - y_1\eta_1 + \langle x'-y',\xi'\rangle)}\,\mathsf{Res}_{\mathbb{R}}\, F^1_{n,0,0}(x',\xi_1,\eta_1,\xi',\tau), \tag{7.3.4}$$

[11] Compare with Theorem 4.4.1.

$$\kappa_n^{1\prime}(x,\tau)=\Gamma_x\mathcal{K}_n^{1\prime}(x,y,\tau)=-2\pi i\int d\xi'\int_{\gamma_+}d\xi_1\int_{\gamma_-}d\eta_1\times e^{ih^{-1}x_1(\xi_1-\eta_1)}\operatorname{Res}_{\mathbb{R}}F^1_{n,0,0}(x',\xi_1,\eta_1,\xi',\tau), \tag{7.3.5}$$

$$\varkappa_n^{1\prime}(\tau)=\Gamma\mathcal{K}_n^{1\prime}(x,y,\tau)=-2\pi i\operatorname{Res}_{\mathbb{R}}W_{n-1,0,0}(\tau),\quad \varkappa_0^{1\prime}(\tau)=0, \tag{7.3.6}$$

and functions $F^1_{n,0,0}(x',\xi_1,\eta_1,\xi',\tau)$ *and* $W_{n,0,0}$ *are defined in Theorem 7.2.17.*

(ii) Assume that the operator $(\tau-A)$ *is microhyperbolic for* $\tau=\bar\tau$ *at* $\operatorname{supp}(q_1')\cap\operatorname{supp}(q_2')\cap T^*(X\cap B(0,1))$ *in some direction* ℓ *and that the problem* $(\tau-A,\eth B)$ *is microhyperbolic for* $\tau=\bar\tau$ *at* $\operatorname{supp}(q_1')\cap\operatorname{supp}(q_2')\cap T^*(Y\cap B(0,1))$ *in some multidirection* $\mathcal{T}=(\ell',\nu_1,\dots,\nu_N)$*. Then* (7.3.3) *remains true for* $L\in[h^{1-\delta},1]$*.*

Moreover, if $\ell_x=\ell'_x=0$ *then the same is true for asymptotics* (7.3.2)*.*

7.3.2 Asymptotics under Microhyperbolicity Condition

Assuming that the problem is microhyperbolic let us now consider non-mollified asymptotics and asymptotics with smaller L.

Theorem 7.3.2. [12] *Let conditions* (7.1.1), (7.1.2), $(7.1.4)_{1,2}$, (7.1.9) *and* (7.1.10) *be fulfilled and let* $Q_i=\operatorname{Op}(q_i)$, $q_i\in\mathsf{S}_{h,K}$ *be supported in* $\{|\xi'|\le c\}$, $\psi_i\in\mathscr{C}_0^K(B(0,\frac12))$.

Let us assume that the problem $(\tau-A,\eth B)$ *is microhyperbolic for* $\tau=\bar\tau$ *at* $(\operatorname{supp}(q_1')\cup\operatorname{supp}(q_2'))\cap T^*(Y\cap B(0,1))$ *in some multidirection* $\mathcal{T}$*. Then*

(i) Estimates

$$\mathsf{R}^{\mathsf{W}}=\|\Gamma\big(Q'_{1x}\psi_1(x)e(x,y,\tau_1,\tau_2)\psi_2(y)^tQ'_{2y}\big)-h^{-d}\varkappa_0(\tau_1,\tau_2)-h^{1-d}\varkappa_1(\tau_1,\tau_2)\|\le C\big(M_0(\tau_1)+M_0(\tau_2)\big)h^{1-d}+Ch^{2-d}\quad\forall\tau_1,\tau_2\in[\bar\tau-\epsilon_0,\bar\tau+\epsilon_0], \tag{7.3.7}$$

[12] Compare with Theorems 4.4.5, 4.4.6, 4.4.8, 4.4.11 and 4.4.12.

$$\mathsf{R}_1^{\mathsf{W}} = \frac{1}{L}\| \int \varphi\big(\frac{\tau'-\bar\tau}{L}\big)\Big(\Gamma\big(Q'_{1x}\psi_1(x)e(x,y,\tau',\tau)\psi_2(y)^t Q'_{2y}\big) - h^{-d}\varkappa_0(\tau',\tau) - h^{1-d}\varkappa_1(\tau',\tau)\Big)d\tau'\| \le M_0(\tau)h^{1-d} + Ch^{2-d} \quad \forall \tau \in [\bar\tau-\epsilon_0, \bar\tau+\epsilon_0] \tag{7.3.8}$$

hold where $\epsilon_0 = \epsilon_0(d, m, D, c) > 0$ *is small enough,* $\varphi \in \mathscr{C}_0^K([-c, c])$, $L \in [h^{1-\delta}, 1]$ *is arbitrary,*

$$M_0(\tau) = \|\varkappa'_{0,q_1}(\tau)\| + \|\varkappa'_{0,q_2}(\tau)\| \tag{7.3.9}$$

and

$$\varkappa_n(\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} \varkappa'_n(\tau)d\tau \tag{7.3.10}$$

and other notations of Section 4.4 are used.

(ii) Assume instead that φ *satisfies* (4.2.73)–(4.2.75). *Then*

$$\mathsf{R}^{\mathsf{W}}_{\varphi,L} = \| \int \varphi\big(\frac{\tau'-\bar\tau}{L}\big)d_\tau\Big(\Gamma\big(Q'_{1x}\psi_1(x)e(x,y,\tau_1,\tau)\psi_2(y)^t Q'_{2y}\big) - h^{-d}\varkappa_0(\tau_1,\tau) - h^{1-d}\varkappa_1(\tau_1,\tau)\Big)\| \le C\big(M_0(\bar\tau)h^{1-d} + Ch^{2-d}\big)\vartheta\big(\frac{h}{L}\big). \tag{7.3.11}$$

Remark 7.3.3. (i) Obviously, the list of statements could be extended in the correspondence with the statements of Theorem 7.2.17.

(ii) In particular, in the same assumptions as above one can replace Γ by $\Gamma'\eth_x\eth_y$ i.e. with integration over ∂X after restriction to $x = y \in \partial X$; In this case $\varkappa_n$, $\varkappa'_n$ are replaced by $\int \eth\kappa_n dx'$ and $\int \eth\kappa'_n dx'$ both in asymptotics (7.3.7), (7.3.8), (7.3.11) and in definition (7.3.9) of M_0; note that zone where $(\tau - a)$ is elliptic may contribute to thus redefined M_0.

(iii) Further, if $\ell'_x = 0$ both in Theorem 7.3.2 and in the Statement (ii) above we can skip integration with respect to x' (but not with respect to x_1).

(iv) On the other hand, if $\nu_1 = ... = \nu_m = \nu$ in Theorem 7.3.2 we can skip integration with respect to x_1.

(v) Finally, if both $\ell'_x = 0$ and $\nu_1 = ... = \nu_m = \nu$ in Theorem 7.3.2 we can skip integration with respect to x.

7.3.3 Asymptotics in the Elliptic Zone

Let us now treat the case in which A is elliptic for $\tau \in [\tau_1, \tau_2]$ on $\mathsf{supp}(q_2')$. In this case $\varkappa_0(\tau', \tau'') = 0$ for $\tau', \tau'' \in [\tau_1, \tau_2]$.

Remark 7.3.4. Ellipticity of the operator yields that $U^0\, {}^tQ_{2y} \equiv 0$; however it is not true for $U^{0\pm}\, {}^tQ_{2y}$ which are given by

$$U^{0\pm}\, {}^tQ_{2y} \equiv \mp iG\delta(t)\delta(x-y)\, {}^tQ_{2y} \tag{7.3.12}$$

where G is a parametrix of elliptic in this zone operator $(hD_t - A)$ (and due to ellipticity we do have separate $G^{\pm}$.

Let us calculate $\varkappa_1'(\tau)$, $\varkappa_1(\tau', \tau'')$ and $\mathcal{K}_0'(x, y, \tau)$, $\mathcal{K}_0(x, y, \tau', \tau'')$ for these τ, τ', τ''. Here we skip superscript ^{1} since U^0 does not contribute into these functions.

Construction of these functions yields that in our case it is sufficient to do the special case $A = a(hD_1)$, $B = b(hD_1)$; this easily yields

Lemma 7.3.5. *Let the conditions of Theorem 7.3.1 be fulfilled and let A be elliptic for $\tau \in [\tau_1, \tau_2]$ on $\mathsf{supp}(q_2')$. Then for $\tau', \tau'' \in [\tau_1, \tau_2]$*

$$\mathcal{K}_0(x, y, \tau', \tau'') = (2\pi)^{1-d} \int e^{ih^{-1}\langle x'-y', \xi'\rangle} q_1^0(x', \xi') \times \mathsf{e}(h^{-1}x_1, h^{-1}y_1; y', \xi', \tau', \tau'') q_2^0(x', \xi')\, d\xi' \tag{7.3.13}$$

and

$$\mathcal{K}_0'(x, y, \tau) = (2\pi)^{1-d} \int e^{ih^{-1}\langle x'-y', \xi'\rangle} q_1^0(x', \xi') \times \partial_\tau \mathsf{e}(h^{-1}x_1, h^{-1}y_1; y', \xi', \tau, \tau'') q_2^0(x', \xi')\, d\xi' \tag{7.3.14}$$

where τ', τ'' are spectral parameters, $\mathsf{e}(x_1, y_1; y', \xi', \tau', \tau'')$ is the Schwartz kernel of the spectral projector of one-dimensional operator

$$\mathsf{a} := a(0, y', D_1, \xi'), \tag{7.3.15}$$

on $\mathscr{L}^2(\mathbb{R}^+, \mathbb{H})$ with the domain

$$\mathfrak{D}(\mathsf{a}) = \{v \in \mathscr{H}^1(\mathbb{R}^+, \mathbb{H}) : \eth b(y', D_1, \xi')v = 0\} \tag{7.3.16}$$

Proof. Easy calculations are left to the reader. □

Corollary 7.3.6. *Therefore in the framework of Lemma 7.3.5*

$$\varkappa_1 := \Gamma\mathcal{K}_0(.,,,\tau',\tau'') = (2\pi)^{1-d}\int q_1^0(x',\xi')\upsilon(x',\xi',\tau',\tau'')q_2^0(x',\xi')\,dx'd\xi' \tag{7.3.17}$$

where

$$\upsilon = \int_0^\infty \mathsf{e}(x_1,x',x_1,x')\,dx_1. \tag{7.3.18}$$

Theorem 7.3.7. *Let conditions* (7.1.1), (7.1.2), $(7.1.4)_{1,2}$, (7.1.9) *and* (7.1.10) *be fulfilled and let* $Q_i = \mathsf{Op}(q_i)$, $q_i \in \mathsf{S}_{h,K}$ *be supported in* $\{|\xi'| \le c\}$, $\psi_i \in \mathscr{C}_0^K(B(0,\frac12))$.

Further, let us assume that the operator $(\tau - A)$ *is elliptic for* $\tau = \bar\tau$ *at* $(\mathsf{supp}(q_1') \cup \mathsf{supp}(q_2')) \cap T^*(X \cap B(0,1))$ *and that the problem* $(\tau - A, \eth B)$ *is microhyperbolic for* $\tau = \bar\tau$ *at* $(\mathsf{supp}(q_1') \cup \mathsf{supp}(q_2')) \cap T^*(Y \cap B(0,1))$ *in some multidirection* $\mathcal{T}$ [13].

(i) Then

$$\mathsf{R}_b^{\mathsf{W}} := \|\Gamma\big(Q_{1x}'\psi_1(x)e(x,y,\tau_1,\tau_2)\psi_2(y)\,{}^tQ_{2y}'\big) - h^{1-d}\varkappa_1(\tau_1,\tau_2) - h^{2-d}\varkappa_2(\tau_1,\tau_2)\| \le C\big(M_1(\tau_1)+M_1(\tau_2)\big)h^{2-d} + Ch^{3-d} \quad \forall \tau_1,\tau_2 \in [\bar\tau-\epsilon_0,\bar\tau+\epsilon_0], \tag{7.3.19}$$

and

$$\mathsf{R}_{b,1}^{\mathsf{W}} := \frac{1}{L}\|\int \phi\big(\frac{\tau'-\bar\tau}{L}\big)\Big(\Gamma\big(Q_{1x}'\psi_1(x)e(x,y,\tau',\tau)\psi_2(y)^tQ_{2y}'\big) - h^{1-d}\varkappa_1(\tau',\tau) - h^{2-d}\varkappa_2(\tau',\tau)\Big)\,d\tau'\| \le CM_1(\tau)h^{2-d} + Ch^{3-d} \quad \forall,\tau \in [\tau-\epsilon_0,\bar\tau+\epsilon_0]; \tag{7.3.20}$$

where ϵ_0, ϕ, L *are the same as in Theorem 7.3.2(i) and*

$$M_1(\tau) = \|\varkappa_{1q_1}'(\tau)\| + \|\varkappa_{1q_2}'(\tau)\|. \tag{7.3.21}$$

[13] Due to ellipticity of $(\tau - A)$ $\mathcal{T} = \ell' \in T^*Y$.

(ii) Assume instead that φ satisfies (4.2.73)–(4.2.75). *Then*

$$\mathcal{R}^{\mathsf{W}}_{\varphi,L} \le C\big(M_1(\bar\tau)+h\big)h^{2-d}\vartheta\big(\frac{h}{L}\big). \tag{7.3.22}$$

(iii) If $\ell'_x = 0$ then

$$\begin{aligned}\mathsf{R}^{\mathsf{W}}_x = \|\Gamma_{x'}\big(Q'_{1x}\psi_1(x)e(x,y,\tau_1,\tau_2)\psi_2(y)\,{}^{t}Q'_{2y}\big)- \\ h^{-d}\mathcal{K}_0(x,x,\tau_1,\tau_2)\| \le C'h^{1-d}\big(1+\frac{(x_1+y_1)}{h}\big)^{-s} \\ \forall x_1,y_1\in[0,1]\quad \forall x'\in\mathbb{R}^{d-1}\quad \forall \tau_1,\tau_2\in[\bar\tau-\epsilon_0,\bar\tau+\epsilon_0].\end{aligned} \tag{7.3.23}$$

Proof. Proof follows from arguments of the proof of Theorem 4.4.2. Microhyperbolicity of the boundary value problem provides smoothness with respect to τ all expressions involved (after corresponding integrations).

In the proof of Statement (iii) we also use the estimate

$$\|\Gamma_{x'}\mathcal{K}_n(x,y,\tau)\| \le C'\big(1+\frac{(x_1+y_1)}{h}\big)^{-s} \qquad \forall n\le s \tag{7.3.24}$$

for $\tau\in[\bar\tau-\epsilon_0,\bar\tau+\epsilon_0]$. This latter estimate follows from the fact that contours $\gamma_\pm$ can be chosen (due to the ellipticity condition) lying in the zone $\{\pm\operatorname{Im}\xi_1\ge\epsilon>0\}$. □

Remark 7.3.8. In all estimates of this section, involving two energy levels one can take one of them equal to $-\infty$ provided the following conditions are fulfilled:

$$\begin{aligned}&\langle a(x,\xi)v,v\rangle \ge -c\|v\|^2 \qquad \forall v\in\mathbb{H}\\ &\forall (x,\xi')\in(\operatorname{supp}(q'_1)\cup\operatorname{supp}(q'_2))\cap(X\cap B(0,1))\times\mathbb{R}^{d-1}\quad\forall\xi_1\in\mathbb{R}\end{aligned} \tag{7.3.25}$$

and

$$\begin{aligned}&\langle a(0,x',D_1,\xi')v,v\rangle_+ \ge -c\|v\|^2_+\\ &\qquad\forall v\in\mathscr{S}(\bar{\mathbb{R}}^+,\mathbb{H}):\mathfrak{d}b(x',D_1,\xi')v=0\\ &\forall (x',\xi')\in(\operatorname{supp}(q'_1)\cup\operatorname{supp}(q'_2))\cap(Y\cap B(0,1))\times\mathbb{R}^{d-1};\end{aligned} \tag{7.3.26}$$

moreover, these conditions imply the same inequalities (7.3.25) and (7.3.26) with the additional terms $\epsilon_0|\xi_1|^m\|v\|^2$ and $\epsilon_0\|D_1^{\frac{m}{2}}v\|_+^2$ respectively in their right-hand expressions (because $a(0,x',D_1,\xi')$ is elliptic as a one-dimensional differential operator; we assume that either (7.1.12) or the conditions of Lemma 7.1.13 are fulfilled).

Furthermore, in all the other estimates mollification with respect to τ' can be replaced by setting $\tau'=-\infty$ in this situation.

Example 7.3.9. Consider Schrödinger operator with the principal symbol

$$a=\sum_{j,k} g^{jk}(x)\bigl(\xi_j-V_j(x)\bigr)\bigl(\xi_k-V_k(x)\bigr)+V(x) \quad \text{with} \quad g^{1j}=\delta_{1j} \tag{7.3.27}$$

with the boundary condition

$$\mathfrak{d}\bigl(hD_1-i\beta(x,hD')\bigr)u=0. \tag{7.3.28}$$

We consider elliptic energy level

$$\tau<a'(x',\xi') \coloneqq \sum_{2\le j,k\le d} g^{jk}(x')\bigl(\xi_j-V_j(x')\bigr)\bigl(\xi_k-V_k(x')\bigr)+V(x') \tag{7.3.29}$$

as $x=(0,x')$ and then

$$\partial_\tau e(x_1,y_1,x',\xi',\tau,-\infty)= 2\beta(x',\xi')^{\frac12}e^{-\beta(x',\xi')(x_1+y_1)}\delta\bigl(\tau-a'(x',\xi')+\beta(x',\xi')^2\bigr)\theta\bigl(\beta(x',\xi')\bigr); \tag{7.3.30}$$

and therefore as τ satisfies (7.3.29)

$$\varkappa_1(\tau_2,\tau_1)=(2\pi)^{1-d}\iint \theta\bigl(\tau-a'(x',\xi')+\beta(x',\xi')^2\bigr)\Big|_{\tau=\tau_1}^{\tau=\tau_2}\times \theta\bigl(\beta(x',\xi')\bigr)q_1^0(x',\xi')q_2^0(x',\xi')\,dx'd\xi' \tag{7.3.31}$$

and microhyperbolicity condition is the one for $\bigl(\tau-a'(x',\xi')+\beta(x',\xi')^2\bigr)$ as $\beta(x',\xi')>0$.

7.3.4 Local Asymptotics

Let us proceed from microlocal asymptotics derived above to the local asymptotics. This transition is based on the results of Section 7.1 that in the zones $\{|\xi'| \ge C_0\}$ problem is elliptic and therefore contributions of these zones into asymptotics are negligible. The easy details are left to the reader.

Theorem 7.3.10. *Let conditions* (7.1.1), (7.1.2), $(7.1.4)_{1,2}$, (7.1.9), (7.1.10) *and* (7.1.37), (7.1.38) *be fulfilled and let* $\phi \in \mathscr{C}_0^K([-c,c])$, $\bar\tau \in [-c,c]$ *and* $L \in [h^{\frac12-\delta},1]$. *Then*

(i) The following estimate holds:

$$\frac{1}{L}\Big\| \int \phi\big(\frac{\tau-\bar\tau}{L}\big)\Big(d_\tau e(x,y,\tau_1,\tau)- \sum_{0\le n\le N'-1} h^{-d+n}\mathcal{K}'_n(x,y,\tau)\,d\tau\Big)\Big\| \le C'h^s \qquad \forall x,y \in B(0,1-\epsilon) \tag{7.3.32}$$

where $\mathcal{K}'_n$, $\mathcal{K}_n$ *are defined above for* $q_1 = q_2 = \psi_1 = \psi_2 = 1$.

(ii) Moreover, if symbol $a(x,\xi)$ *is semi-bounded from below uniformly with respect to* $(x,\xi) \in T^*(X \cap B(0,1))$ *and if operator* $a(0,x',D_1,\xi')$ *is semi-bounded from below uniformly with respect to* $(x',\xi') \in T^*(Y \cap B(0,1))$ *then*

$$\frac{1}{L}\Big\| \int \big(\phi\big(\frac{\tau-\bar\tau}{L}\big)\Big(e(x,y,-\infty,\tau)- \sum_{0\le n\le N'-1} h^{-d+n}\mathcal{K}_n(x,y,-\infty,\tau)\Big)\,d\tau\Big\| \le C'h^s \qquad \forall x,y \in B(0,1-\epsilon). \tag{7.3.33}$$

Theorem 7.3.11. *Let the conditions of the previous Theorem 7.3.10 be fulfilled and let us assume that the operator* $(\tau - A)$ *is microhyperbolic for* $\tau = \bar\tau$ *at* $T^*(X \cap B(0,1))$ *and that the problem* $(\tau - A, \eth B)$ *is microhyperbolic for* $\tau = \bar\tau$ *at* $T^*(Y \cap B(0,1))$. *Then*

(i) The following estimate holds:

$$\|\Gamma\big(\psi_1(x)e(.,.,\tau_1,\tau_2)\big) - h^{-d}\varkappa_0(\tau_1,\tau_2) - h^{1-d}\varkappa_1(\tau_1,\tau_2)\| \le C\big(M_0(\tau_1)+M_0(\tau_2)\big)h^{1-d} + Ch^{2-d} \quad \forall \tau_1,\tau_2 \in [\bar\tau-\epsilon_0,\bar\tau+\epsilon_0] \tag{7.3.34}$$

and

$$\frac{1}{L}\|\int \phi\big(\frac{\tau-\bar\tau}{L}\big)\Big(\Gamma\big(\psi_1(x)e(.,.,\tau',\tau)\big) - h^{-d}\varkappa_0(\tau',\tau) - h^{1-d}\varkappa_1(\tau',\tau)\Big)\,d\tau'\| \le M_0(\tau)h^{1-d} + CMh^{2-d} \quad \forall \tau \in [\bar\tau-\epsilon_0,\bar\tau+\epsilon_0] \tag{7.3.35}$$

where $\epsilon_0 = \epsilon_0(d,m,D,c) > 0$ is small enough, $\phi \in \mathscr{C}_0^K([-c,c])$ is fixed, $L \in [h^{1-\delta},1]$ is arbitrary, $M_0(\tau)$ and $\varkappa_n$ were defined above for $q_1 = q_2 = 1$; moreover, estimates for $\mathsf{R}^{\mathsf{W}}_{\varphi,L}$ defined by (7.3.11) *with proper modifications remain true as well.*

(ii) If the operator $(\tau - A)$ is elliptic for $\tau = \bar\tau$ at $T^*(X \cap B(0,1))$ then

$$\|\Gamma\big(\psi_1(x)e(.,.,\tau_1,\tau_2)\big) - h^{1-d}\varkappa_1(\tau_1,\tau_2) - h^{2-d}\varkappa_2(\tau_1,\tau_2)\| \le C\big(M_1(\tau_1)+M_1(\tau_2)\big)h^{2-d} + Ch^{3-d} \quad \forall \tau_1,\tau_2 \in [\bar\tau-\epsilon_0,\bar\tau+\epsilon_0] \tag{7.3.36}$$

and

$$\frac{1}{L}\|\int \phi\big(\frac{\tau-\bar\tau}{L}\big)\Big(\Gamma\big(\psi_1(x)e(.,.,\tau',\tau)\big) - h^{1-d}\varkappa_1(\tau',\tau) - h^{2-d}\varkappa_2(\tau',\tau)\Big)\,d\tau'\| \le M_1(\tau)h^{2-d} + Ch^{3-d} \quad \forall \tau \in [\bar\tau-\epsilon_0,\bar\tau+\epsilon_0] \tag{7.3.37}$$

where $M_1(\tau)$ is defined above for $q_1 = q_2 = 1$; moreover, estimates for $\mathsf{R}^{\mathsf{W}}_{\varphi,L}$ with proper modifications remain true as well.

(iii) In the framework of Statement (ii) of the previous Theorem 7.3.10 estimates (7.3.34) *and* (7.2.37) *remain true for $\tau_1 = -\infty$.*

(iv) Statements (i) and (ii) of the previous Theorem 7.3.10 remain true for $L \in [h^{1-\delta}, 1]$.

Remark 7.3.12. If we assume that the conditions of some theorem (or some part of a theorem) are fulfilled for $\tau = \bar{\tau}_i$, $i = 1, 2$, then the corresponding estimates (without mollification) remain for τ_i belonging to the ϵ_0-neighborhoods of $\bar{\tau}_i$.

7.4 Sharper Asymptotics

7.4.1 Discussion

The purpose of this section is similar to the purpose of Section 4.5: using long term propagation we want to make sharper asymptotics. However the answer strongly depends what asymptotics we are talking about:

(i) sect-7-4-1-i If our goal are asymptotics of

$$\int e(x, x, \tau)\psi(x)\, dx \tag{7.4.1}$$

(and the similar) then it follows from the previous Section 7.3 that (under microhyperbolicity assumption) contribution of the boundary strip $Y_\varepsilon := \{x \in X, \mathsf{dist}(x, \partial X) \le \varepsilon\}$ to the remainder does not exceed $C(\varepsilon + h)h^{1-d}$ and therefore we need to impose conditions of the global nature only at inner points; so basically all results of Section 4.5 remains valid but now generalized Hamiltonian flow is defined as in Chapter 3 rather than Chapter 2.

Thus even in the simplest case of the Schrödinger operator we should consider Hamiltonian billiards rather than just Hamiltonian trajectories (we will prove that the generalized Hamiltonian billiards which are not Hamiltonian billiards form set of measure zero here). Still checking non-periodicity conditions for billiards is an extremely difficult task[14)].

Situations becomes even more complicated if there is a branching on the boundary which will be the case for example for few Schrödinger operators intertwined through boundary conditions, in particular in the case when X is a "layered" domain so in each layer we have its own Schrödinger

[14)] We are talking about "classical" rather than "quantum" dynamics here.

operator which are tied through conditions on the smooth "inner boundaries"; then branching happens when incoming ray generates two outgoing rays: reflected and refracted. Another meaningful example comes from elasticity (in isotropic media) when we have compression waves and sheer waves and at reflection incoming wave of one type may generate outgoing waves of both types. Analysis of branching billiards is an incredibly difficult task and there is very little known[14]. This revives the question "what is a periodic branching billiard"?

(ii) Situation is rather different if we are discussing either

$$\int_{\partial X} e(x,x,\tau)\psi(x)\,dx' \tag{7.4.2}$$

or we are still discussing asymptotics of (7.4.1) but operator $(\tau - A)$ is elliptic (but the boundary value problem is microhyperbolic). In the former case the contribution of the inner points is missing and in the latter it is negligible. So in both cases we have a measure concentrated on $\Sigma_{\tau,b}$. One can easily generalize the results of Section 4.5 to these cases–with the obvious modifications.

(iii) And analysis of $e(x,x,\tau)$ just changes the game completely. At this moment here the previous section just does not provide complete results because the corresponding microhyperbolicity condition requires $\nu_1 = ... = \nu_N$ and cannot be satisfied at some points: for Schrödinger operator it fails near points $(x',\xi') \in T^*\partial X$ such that the reflected ray has the direction opposite to direction of the incoming ray (and repeats it if there is no magnetic field). Situation is similar to those considered in Subsubsection *5.3.1.4 Asymptotics without Spatial Mollification and Short Loops*. Namely: we have short loops (of the length $\asymp \gamma(x) := \mathsf{dist}(x,\partial X)$) and the standard rescaling technique gives us that their contribution is $O(\gamma^{-1}h^{-d})$.

However under certain assumptions we can construct $U(x,x,t)$ using oscillatory integrals and we arrive to the terms of the boundary layer type $h^{-d}\Upsilon(x',h^{-1}\gamma(x))$ with $\Upsilon(x,s) = O(s^{-\frac{1}{2}(d+1)})$. While results are similar to those for degenerating Schrödinger operator there is a difference: this term is of magnitude $O(h^{-d})$ at the boundary no matter how large dimension is. We will do it in Section 8.1.

After this difficulty is tackled (or simply avoided assuming that x is disjoint from ∂X) there are no other obstacles to get sharper remainder

estimates except that we should assume no loop condition which is very difficult to check[14)].

7.4.2 General Theory

Sharp Asymptotics

The proof of the following statements is trivial in view of the results of the previous section and propagation results of Section 3.2:

Theorem 7.4.1. *Let conditions of Theorem 7.3.2 be fulfilled. Assume that there exists set $\Xi_\tau \subset \Sigma_\tau$ such that*

$$(x,\xi) \notin \Psi_t(x,\xi) \qquad \forall (x,\xi) \in \Sigma_\tau \Xi_\tau \quad \forall t : \varsigma t > 0 \tag{7.4.3}$$

where $\Psi_t(z)$ is a generalized Hamiltonian billiard flow in the sense of Definitions 3.2.2–3.2.3[15)] *where $\varsigma = \pm$ may depend on (x,ξ).*

Further, assume that $Q_1, Q_2 \in \mathbf{\Psi}_{h,K}(X \setminus \partial X, \mathbb{H})$ have symbols q_1, q_2 with the compact supports

$$\mu_\tau\big(\Xi_\tau \cap \operatorname{supp}(q_1^0) \cap \operatorname{supp}(q_2^0)\big) = 0 \tag{7.4.4}$$

as $\tau = \tau_1$, $\tau = \tau_2$ and $\tau = \bar\tau$ where $\mu_\tau = \delta(\tau - a)$ is the same measure as in Section 4.5. Then

(i) The left-hand expressions of (7.3.7) *and* (7.3.8) *are $o(h^{1-d})$.*

(ii) Moreover, in the framework of Theorem 7.3.2(ii) the left-hand expression of (7.3.11) *is $o\big(h^{1-d}\vartheta(hL^{-1})\big)$.*

Corollary 7.4.2. *Let conditions of Theorem 7.3.11(i) be also fulfilled. Then Theorem 7.4.1 remains true as $Q_j = \psi_j(x) \in \mathscr{C}_0^K(X \setminus \partial X, \mathbb{H})$.*

Remark 7.4.3. If (7.4.3) is fulfilled for measure μ'_τ which is the restriction of μ_τ to ∂X then all the statements above remain true for $Q_1, Q_2 \in \mathbf{\Psi}_{h,K}(\partial X, \mathbb{H})$, $\psi_j(x) \in \mathscr{C}_0^K(\partial X, \mathbb{H})$ and Γ replaced by $\Gamma' \eth_x \eth_y$ (see Remark 7.3.3(ii)).

[15)] I.e. it is an intersection of $\mathcal{K}^\varsigma(z, \tau - a, b)$ with $\{x_0 = t\}$.

Theorem 7.4.4. *Let conditions of Theorem 7.3.7 be fulfilled. Assume that there exists set* $\Xi_{\tau,b} \subset \Sigma_{\tau,b}$ *such that* (7.4.3) *is fulfilled.*

Further, assume that $Q_1, Q_2 \in \mathbf{\Psi}_{h,K}(X, \mathbb{H})$ *have symbols* q_1, q_2 *with the compact supports and condition* (7.4.3) *is fulfilled as* $\tau = \tau_1$, $\tau = \tau_2$ *and* $\tau = \bar{\tau}$ *with* $\mu_{\tau,b} = \delta(\tau - a)$ *which is the similar measure but now it is concentrated on* ∂X *and* $\mathbb{H}$ *is replaced by* $\mathscr{S}(\mathbb{R}^+, \mathbb{H})$. *Then*

(i) The left-hand expressions of (7.3.19) *and* (7.3.20) *are* $o(h^{2-d})$.

(ii) Moreover, in the framework of Theorem 7.3.7(ii) the left-hand expression of (7.3.22) *is* $o\big(h^{2-d}\vartheta(hL^{-1})\big)$.

Corollary 7.4.5. *Let conditions of Theorem 7.3.11(ii) be fulfilled. Then Theorem 7.4.4 remains true as* $Q_j = \psi_j(x) \in \mathscr{C}_0^K(\partial X, \mathbb{H})$.

Sharper Asymptotics

Our goal is to establish either remainder estimate $O(h^{1-d-\delta})$ with unspecified exponent $\delta > 0$ or $O(h^{1-d}|\log h|^{-1})$.

Theorem 7.4.6 [16]**.** *Let assumptions of Theorem 7.3.2 be fulfilled. Let* $r_i^0 \in \mathsf{S}_{h,\gamma,\gamma,K}(T^*X)$ [17] *be scalar symbols satisfying* (4.5.11).

Consider generalized Hamiltonian flow $\Psi_{t,\bar{\tau}_*}$ *issued from* γ*-vicinity* $\mathcal{U}_i$ *of* $\operatorname{supp}(r_i^0) \cap \operatorname{supp}(q_1 \cap) \operatorname{supp}(q_2)$. *Assume that as* $\varsigma t > 0$ *with* $\varsigma := \varsigma_{*i}$ [32] *and* $|t| \le T_{*i}$ *conditions* (4.5.28), (4.5.30) *and* (4.5.31) *are fulfilled and, moreover, as* $i = 1, \dots, m$ *along* $\Psi_{t,\tau}(\mathcal{U})$ *conditions of either Theorem 2.4.17 or Theorem 3.5.3 are fulfilled with* $\mathcal{U} := \mathcal{U}_i$.

Then estimates (4.5.32)–(4.5.34) *hold with term* $C'h$ *in the factor replaced by* $C'h\gamma^{-2}$ *with the left-hand expressions defined respectively by* (4.4.19), (4.4.16) *and* (4.4.17) *(without* $_x$*) with* $J(q^0 B^j, \bar{\tau})$ *defined by* (4.4.26). *Here* C' *does not depend on* γ, T_{*i}.

Theorem 7.4.7 [18]**.** *Let assumptions of Theorem 7.3.2 be fulfilled. Let* A *be* ξ*-microhyperbolic on* $\operatorname{supp}(q_1) \cup \operatorname{supp}(q_2)$ *on the energy levels* $\bar{\tau}_*$, $* = 1, 2$. *Let* $r_i^0 \in \mathsf{S}_{h,\gamma,\gamma,K}(T^*X)$ *be scalar symbols satisfying* (4.5.35).

---

[16] Cf. Theorem 4.5.11.

[17] With $h^{\frac{1}{2}-\delta} \le \gamma \le 1$.

[18] Cf. Theorem 4.5.12.

Consider generalized Hamiltonian flow $\Psi_{t,\bar\tau_*}$ *issued from* γ*-vicinity* $\mathcal{U}_i$ *of* $\mathsf{supp}(r_i^0) \cap (\mathsf{supp}(q_1) \cup \mathsf{supp}(q_2))$. *Assume that as* $|t| \le T_{*i}$ *conditions* (4.5.28), (4.5.29) *and* (4.5.31) *are fulfilled and, moreover, as* $i = 1, \dots, m$ *along* $\Psi_{t,\tau}(\mathcal{U})$ *conditions of either Theorem 2.4.17 or Theorem 3.5.3 are fulfilled with* $\mathcal{U} := \mathcal{U}_i$.

Then estimates (4.5.36)–(4.5.38) *hold with term* $C'h$ *in the factor replaced by* $C'h\gamma^{-2}$ *with the left-hand expressions defined respectively by* (4.4.19), (4.4.16) *and* (4.4.17) *(without* $_x$*) with* $J(q^0 B^j, \bar\tau)$ *defined by* (4.4.26). *Here* C' *does not depend on* γ, T_{*i}.

7.4.3 Billiard Flow for Schrödinger Operator

We need to tackle Hamiltonian billiard flow in different domains and manifolds with the boundary.

Definition of the Flow

Consider Schrödinger operator with the symbol

$$a = \sum_{j,k} g^{jk}(x)\bigl(\xi_j - V_j(x)\bigr)\bigl(\xi_k - V_k(x)\bigr) + V(x) \tag{7.4.5}$$

and instead of generalized Hamiltonian billiards we consider ordinary Hamiltonian billiards composed of Hamiltonian trajectories

$$\frac{dz}{dt} = \nabla^{\#} a(z) \tag{7.4.6}$$

which must hit boundary transversally:

$$|\phi| + |\{a, \phi\}| > 0 \tag{7.4.7}$$

where $X = \{\phi > 0\}$ and $|\phi| + |\nabla\phi| \ge \epsilon_0$.

Then Hamiltonian billiard flow is not necessarily defined on Σ_τ but on the set of the full measure:

Theorem 7.4.8. *Let* X *be a smooth manifold with the boundary and let* a *be microhyperbolic on level* τ. *Assume that* g^{jk} *and* V *are uniformly bounded and* (g^{jk}) *is uniformly elliptic.*

Then there exists Ξ_τ *the set of measure* 0 *and of the first Baire category on* Σ_τ *such that through each point of* $\Sigma_\tau \setminus \Xi_\tau$ *there passes an (unique)*

infinitely long in both directions Hamiltonian billiard and thus a (single-valued) Hamiltonian billiard flow Ψ_t *is defined on* $\Sigma_\tau \setminus \Xi_\tau$*; this set is* Ψ_t*-invariant.*

This flow conserves the natural measure $dxd\xi : da$ *on* Σ*.*

Proof. The proof coincides with the arguments of Chapter 6 in the book of Cornfeld, Sinai, Fomin [1]. Consider set $\Xi_{\tau,T} \subset$ of points on Σ_τ such that Ψ_t cannot be defined on interval $[0, T]$. Then as T_1, T_2 have the same sign

$$\Xi_{\tau,T_1+T_2} \subset \Xi_{\tau,T_1} \cup \Psi_{-T_1}(\Xi_{\tau,T_2}).$$

Therefore since Ψ_t preserves measure $d\mu = dxd\xi : da$ on Σ we conclude that

$$\mu_\tau(\Xi_{\tau,T_1+T_2}) \le \mu_\tau(\Xi_{\tau,T_1}) + \mu_\tau(\Xi_{\tau,T_2})$$

as $T > 0$, $\Delta > 0$; thus $\mu_\tau(\Xi_{\tau,T}) \le T\Delta^{-1}\mu_\tau(\Xi_{\tau,\Delta})$. However

$$\Xi_{\tau,\Delta} \subset \{\phi \le c\Delta, |\{H,\phi\}| \le \varepsilon(\Delta)\}$$

with $\varepsilon(\Delta) = o(1)$ as $\Delta \to 0$. Hence $\mu_\tau(\Xi_{\tau,\Delta}) \le C\varepsilon(\Delta)\Delta$ and thus $\mu_\tau(\Xi_{\tau,T}) \le C\varepsilon(\Delta)$; therefore $\mu_\tau(\Xi_{\tau,T}) = 0$ for any $T > 0$ and thus $\mu_\tau(\Xi_\tau) = 0$.

The same arguments work in domain with corners, edges and vertices (see Chapter 11). □

Remark 7.4.9. If we are interested only in trajectories of length not exceeding T then in the statement of this lemma one can replace Ξ by a closed nowhere dense subset Ξ_T. However then we have no invariancy.

Growth of the Billiard Flow. I

We need to study how this flow grows.

Let $J_t(x,\xi) = D\Psi_t(x,\xi)$ be the Jacobi matrix of Ψ_t calculated at the point (x,ξ); then $\det J_t(x,\xi) = 1$. We need the following statements concerning the Hamiltonian flows on various manifolds:

Let us treat different classes of strongly convex compact Riemannian manifolds with boundary.

We assume that uniformly $\phi \in \mathscr{C}^3$,

$$\phi > 0 \quad \text{on} \quad X, \qquad \phi = 0 \text{on} \quad \partial X \tag{7.4.8}$$

and

$$|\nabla\phi| + |\phi| \ge \epsilon. \tag{7.4.9}$$

Further, we assume that X is uniformly *strongly convex* i.e. we assume the uniform version of

$$\phi(x) = 0,\ \{a,\phi\}(x,\xi) = 0 \implies \{a,\{a,\phi\}\}(x,\xi) < 0 \qquad \forall (x,\xi) \in \Sigma_\tau. \tag{7.4.10}$$

Thus we assume

$$\phi(x) \le \epsilon,\ |a(x,\xi) - \tau| \le \epsilon,\ |\{a,\phi\}(x,\xi)| \le \epsilon \implies \{a,\{a,\phi\}\}(x,\xi) \le -\epsilon. \tag{7.4.11}$$

Recall that $\{.,.\}$ are Poisson brackets. In this case

$$\Xi_\tau = \big\{(x,\xi)\colon a = \tau, \phi = \{a,\phi\} = 0\big\} \tag{7.4.12}$$

is a manifold of codimension 2 in Σ_τ. Define

$$\rho(x,\xi) = \frac{1}{2}\{a,\phi\}^2 - \{a,\{a,\phi\}\}\phi \tag{7.4.13}$$

in a small enough neighborhood of Ξ_τ (on Σ_τ).

Then due to strict convexity $\rho(x,\xi) = 0 \iff (x,\xi) \in \Xi_\tau$ and $\rho \asymp 1$ outside of this neighborhood in Σ_τ. Moreover,

$$\rho \asymp \{a,\phi\}^2 + \phi \qquad \text{on } \Sigma_\tau. \tag{7.4.14}$$

Theorem 7.4.10. *Let X be a smooth manifold with the boundary. Let $g^{jk}, V_j, V \in \mathscr{C}^3$ uniformly. Under the assumptions made above (most notably strong convexity) the following estimates hold:*

$$C^{-1}e^{-C|t|} \le \frac{\rho(\Psi_t(x,\xi))}{\rho(x,\xi)} \le Ce^{C|t|}, \tag{7.4.15}$$

$$|J_t(x,\xi)| \le Ce^{C|t|}\rho(x,\xi)^{-M} \tag{7.4.16}$$

for all $(x,\xi) \in \Sigma_\tau \setminus \Xi_\tau$, $t \in \mathbb{R}$ where C does not depend on (x,ξ,t).

Proof. Note that due to (7.4.14) $|\{a,\rho\}| \le C_0\rho$. Therefore along Hamiltonian trajectories $|\frac{d}{dt}\rho| \le C_0\rho$. Note that

(7.4.17) ρ does not change at the moment of reflection at ∂X.

This yields (7.4.15) with $C = C_0$. Let $\Sigma' = T^*X|_{\partial X}$. Important: we use coordinates (x', ξ) rather than (x', ξ', τ) on Σ'.

Let us prove (7.4.16). Let us consider the map $\Phi : \Sigma' \to \Sigma'$ transforming the point $z' \in \Sigma'$ to the point $\varsigma\Psi_{t_1(z')}(z')$ where $t_1(z') > 0$ is the time of the first return of the trajectory passing through z' to ∂X and $\varsigma : \Sigma' \to \Sigma'$ is the reflection map; if a trajectory never hits ∂X again then $t_1(z') = +\infty$ and $\Psi(z')$ is not defined.

One can see easily that

(7.4.18) Map Φ preserves measure ν_τ on $\Sigma'_\tau = \Sigma_\tau|_Y$ with the density

$$d\nu_\tau = |\eta| dx d\xi : (da \wedge d\phi) \tag{7.4.19}$$

with $\eta = \{a, \phi\} = \pm(2\rho)^{\frac{1}{2}}$ on Σ'; obviously ν does not depend on the choice of ϕ.

Further, one can easily see that (7.4.16) holds if the trajectory $\Psi_{t'}(z)$ does not hit ∂X for $0 \le t' \le t$ or $0 \ge t' \ge t$. On the other hand, one can easily see that

$$t_1(z') \asymp |\eta(z')|, \tag{7.4.20}$$

and

$$D\Phi(z') = I + O\big(t_1(z')\big) \tag{7.4.21}$$

in the local coordinates provided $\rho(z') \le \epsilon_1$ (where $\epsilon_1 > 0$ is small enough) and therefore

$$|D\Phi^n(z')| \le e^{C_1 t_n(z')} \tag{7.4.22}$$

for $n = 1$ under the same assumption. On the other hand, if $\rho(z') \ge \epsilon_1$ we conclude that (7.4.22) for $n = 1$ holds as well. So, (7.4.22) for $n = 1$ always holds.

Let us consider the map Φ^n. Then (7.4.15), (7.4.20) and (7.4.21) yield that

$$n \le C_1 e^{C|t_n|} |\eta(z')|^{-1} \tag{7.4.23}$$

with an appropriate constant C_1 and that (7.4.22) holds for arbitrary $n \in \mathbb{Z}^+$. Here $t_n(z') > 0$ is the time of the n-th return of the billiard Hamiltonian trajectory to ∂X (it is possible that $t_n(z') = +\infty$ in which case $\Phi^n(z')$ is not defined). Inverting time in our arguments we obtain that (7.4.22) holds for arbitrary $n \in \mathbb{Z}^+$.

Now we should finish the proof of (7.4.16). Let us consider an arbitrary point $z \in T^*X$ and let us rewrite $\Psi_t(z)$ in the form

$$\Psi_t(z) = \Psi_{t-t_{n+1}(z)} \circ \Phi^n \circ \Psi_{t_1(z)}(z) \tag{7.4.24}$$

where $t_j(z) > 0$ is the time of the j-th return of the billiard Hamiltonian trajectory to ∂X and we assume that $t_{n+1}(z) \le t < t_{n+2}(z)$ and $n \ge 0$.

Our purpose now is to estimate the differential of this product. Consider $\Psi_{t_1(z)}(z)$ mapping T^*X to $T^*X|_{\partial X}$. Note that

$$e^{-C_0|t_k|}\rho^{\frac{1}{2}}(z) \le |\eta(\Psi_{t_k(z)}(z))| \le e^{C_0|t_k|}\rho^{\frac{1}{2}}(z) \tag{7.4.25}$$

and

$$|dt_k(z)| + |d\Psi_{t_k(z)}(z)| \le C_0 e^{C_0 t_1(z)}\rho(z)^{-\frac{1}{2}} \tag{7.4.26}$$

hold as $k = 1$ while

$$|D\Psi_{t'}(z')| \le C_0 e^{C_0 t'} \qquad \text{as } \ 0 \le t' \le t_1(z'),\ z' \in T^*X|_Y. \tag{7.4.27}$$

So we need to understand how t_1, t_{n+1} depend on z. One can easily see that for $z' \in \Sigma'$ the inequality $|dt_1(z')| \le C_1 e^{C_1 t_1(z')}|dz'|$ holds and therefore

$$|dt_k(z')| \le |dt_{k-1}(z')| + C_1 e^{C_1|t_{k+1}-t_k|}|d\Phi^{n-1}(z')| \le |dt_{k-1}(z')| + C_2 e^{C_2|t_k|}|dz'|$$

and finally we obtain by induction by k (using the above inequalities) that

$$|dt_k(z')| \le Ce^{C|t_k|(z')}|\eta|^{-1} \cdot |dz'|. \tag{7.4.28}$$

Combining with (7.4.26) for $k = 1$ we arrive to (7.4.25) and (7.4.26) for arbitrary $k \in \mathbb{Z}^+$; combining with (7.4.27) we arrive to (7.4.16) as $t > 0$. Reverting time we arrive to (7.4.16) as $t < 0$. □

Growth of the Billiard Flow. II

Now let us consider cases with a power growth.

Theorem 7.4.11 (Polyhedra). *Let X be a compact polyhedral domain in $\mathbb{R}^d$ with $g^{jk} = \delta_{jk}$. Let $V_j = V = 0$. Then*

$$|J_t(x,\xi)| \le C(|t|+1) \tag{7.4.29}$$

uniformly with respect to $(x,\xi) \in S^*\bar{X} \setminus \Xi$.

Proof. One can prove easily that the measure of rays hitting the corners and perishing there is 0. We will prove it later anyway.

Note that Ψ_t consists of linear movements $(x,\xi) \to (x + t_j\xi, \xi)$ and reflections from various faces:

(7.4.30) The reflection from the face Z at point $y_Z \in Z$ is

$$(x,\xi) \to (\varsigma_Z(x - y_Z) + y_Z, \varsigma_Z\xi)$$

with $\varsigma_Z \in \mathcal{G}$ where $\mathcal{G}$ is a group of reflections in $\mathbb{R}^d$ (i.e., linear orthogonal maps ς such that $\varsigma^* = \varsigma$ and $\varsigma^2 = I$).

Therefore

(7.4.31) If $(\bar{x},\bar{\xi}) \notin \xi_{\bar{t}}$ and $(\bar{x},\bar{\xi}), \Psi_{\bar{t}}(\bar{x},\bar{\xi}) \notin S^*\bar{X}|_{\partial X}$ then in a neighborhood of $(\bar{x},\bar{\xi},\bar{t})$

$$\Psi_t(x,\xi) = (\bar{\varsigma}(x + \xi t) + \bar{y}, \bar{\varsigma}\xi)$$

with $\bar{y} \in \mathbb{R}^d$ and $\bar{\varsigma} \in \mathcal{G}$ depending on $(\bar{x},\bar{\xi},\bar{t})$.

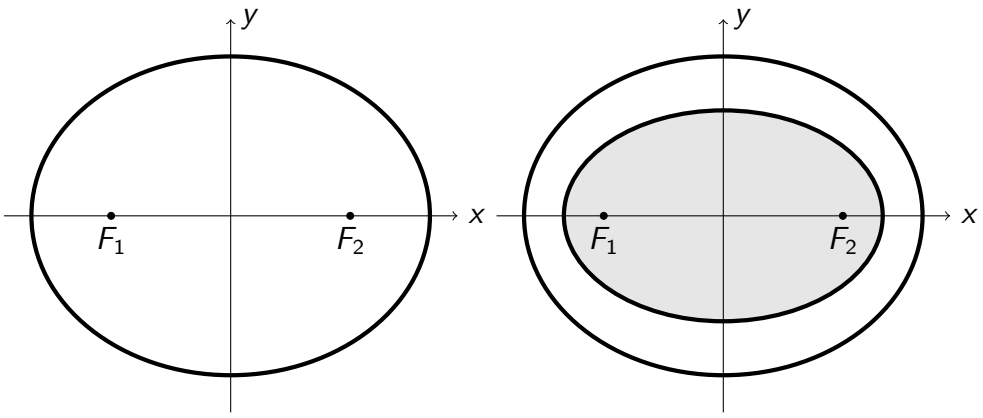
This immediately yields (7.4.29). □

Theorem 7.4.12 (Elliptical domains). *(i) Let X be the interior of an ellipse in $\mathbb{R}^2$ or, more generally, an elliptic annulus, i.e. domain lying between two confocal ellipses where the standard Euclidean metric is given on $\mathbb{R}^2$.*

Then there exists a continuous non-negative function $\rho(z)$ on $\Sigma = S^*X$ such that

$$\rho(\Psi_t(z)) = \rho(z), \tag{7.4.32}$$

$$|D\Psi_t(z)| \le C\rho^{-L}(z)(|t|+1)^{L'} \qquad \forall z \in \Sigma \setminus \Xi \quad \forall t \in \mathbb{R} \tag{7.4.33}$$



(a) Ellipse (b) Elliptical annulus

Figure 7.1: Elliptical domains

with $\Xi = \{z, \rho(z) = 0\}$ *and*

$$\mathsf{mes}_\Sigma(\{z : \rho(z) < h\}) = O(h^\sigma) \qquad \text{as } h \to +0 \tag{7.4.34}$$

with $\sigma > 0$ *and* $L' = 1$. *Here* mes_Σ *is the standard measure on* Σ.

(ii) The same is true for a ball in $\mathbb{R}^d$ *and for a domain lying between two spheres with the same center (where the standard Euclidean metric is given on* $\mathbb{R}^d$*).*

Proof. (i) Let us consider first a more difficult case of an elliptical domain or a domain lying between two ellipses with the same foci $F_1 \neq F_2$; let Y_2 be an outer and Y_1 an inner ellipses.

In the first case ($Y_1 = \emptyset$) let us define $\Xi = S^* Y$. In the second case let us introduce $\Xi_2 = S^* Y_2$ and $\Xi'_1 = S^* Y_1$ for Y_2 and Y_1 respectively and let us set

$$\Xi_1 = \big\{(x + t\xi, \xi) \colon (x, \xi) \in \Xi'_1, t \in \mathbb{R}, x + t\xi \in \bar{X}\big\} \tag{7.4.35}$$

(thus, the definitions of Ξ_1 and Ξ_2 and are different; moreover, $\mathsf{dim}\, \Xi_1 = 2$, $\mathsf{dim}\, \Xi_2 = 1$). Then let $\Xi = \Xi_2 \cup \Xi_1$. Note that $\Xi_2 \cap \Xi_1 = \emptyset$.

In the domains of this type every billiard belongs to one of the following three classes:

(a) All the straight lines containing at least one segment of this billiard do not pass between foci and are tangent to an ellipse Y which has the same foci F_1 and F_2 and is smaller than Y_2 (although not necessarily larger than Y_1).

(b) All the straight lines containing at least one segment of this billiard pass between foci and are tangent to the same hyperbola Y which has the same foci F_1 and F_2.

(c) All the straight lines containing at least one segment of this billiard pass through the foci (either by turns or simultaneously). Then Y is the straight segment between foci.

For elliptical domains it is well known[19]. However this property (preserving Y) holds for an internal reflection from an ellipse Y_k then it holds also for an external reflection from it since in this case extensions of rays reflect internally from Y_k (see Figure 7.2(d)). So Y is preserved both at reflections from Y_2 and Y_1.

Then

(7.4.36) $\beta := +b^2, -b^2, 0$ in the cases (a), (b) and (c) respectively is an invariant of the Hamiltonian flow. Here a_k and b_k are semiaxes of Y_k ($k = 1, 2$) directed along the axes x_1 and x_2 respectively, $a_k > b_k$ and a and b are are semiaxes of Y; $a_k^2 - b_k^2 = a^2 - y^2 = c^2$ where $c = \frac{1}{2}\mathsf{dist}(F_1, F_2)$.

Direct calculation shows that[20]

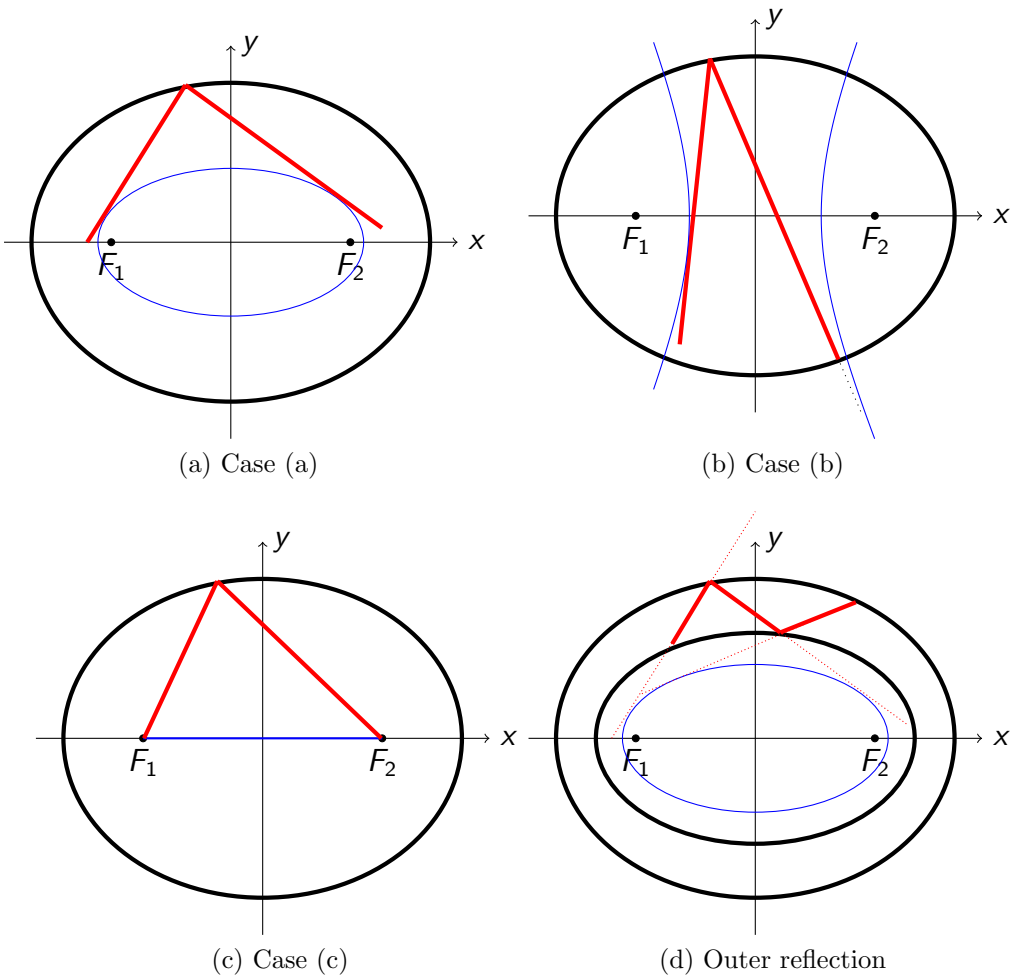
$$\beta = b_k^2 - a_k^2 b_k^2 \Big(\frac{x_1\xi_1}{a_k^2} + \frac{x_2\xi_2}{b_k^2}\Big)^2 = (x_1\xi_2 - x_2\xi_1)^2 - c^2\xi_2^2 \qquad \text{for } x \in Y_k,\ |\xi| = 1. \tag{7.4.37}$$

Let us introduce $\Sigma_2' = \Sigma|_{Y_2} = B^* Y_2$ and the map $\Phi : \Sigma_2' \to \Sigma_2'$: we launch ray from $\iota^{-1}z \cap S^*X$ and then $\Phi(z)$ and $t_1(z)$ are the point and the time of its first return to Σ_2' (maybe after one reflection from Y_1); so

$$\Phi(z) = \iota \Psi_{t_1(z)}(\iota^{-1}z \cap S^*X). \tag{7.4.38}$$

[19] See, e.g., V. M. Babich & V. S. Buldyrev [1], section 4.5 and the beautiful pictures there. See also Figure 7.2(a)–(c)

[20] One can see easily that the middle and the right-hand expression coincide, that the latter is invariant along rays while $(x_1 x_1 a_1^2 + x_2\xi_2 a_2^2)$ acquires factor (-1) at reflections. Then it is sufficient to calculate β at point of the tangency with Y.



(a) Case (a) (b) Case (b)

(c) Case (c) (d) Outer reflection

Figure 7.2: Cases (a), (b), (c) in ellipse and (d) in annulus. Red lines are rays, dotted lines their extensions, Y is drawn in blue; cases (b), (c) in an annulus a similar

Then

$$t_1(z') \asymp \rho_2(z')^{\frac{1}{2}} = \Big|\frac{x_1\xi_1}{a_2^2} + \frac{x_2\xi_2}{b_2^2}\Big|. \tag{7.4.39}$$

Here the right-hand expression is an invariant of the map Φ and therefore in what follows

$$n \le C(|t_n(z')| + 1)\rho_2(z')^{-\frac{1}{2}} \tag{7.4.40}$$

with the obvious definition of $t_n(z')$. Let us introduce the function

$$\eta = \frac{x_1\xi_1}{a_2^2} + \frac{x_2\xi_2}{b_2^2} \tag{7.4.41}$$

on Σ_2'. Then

$$|\nabla_{\Sigma_2'}\eta| \asymp \rho_3(z') := (|x_1| + |\xi_1|)\cdot(|x_2| + |\xi_2|). \tag{7.4.42}$$

One can prove it easily introducing coordinates $(u, v) : x_1 = a_2 \cos u$, $x_2 = b_2 \sin u$, $\xi_1 = \cos v$, $\xi_2 = \sin v$ on Σ_2'; in what follows $z' = (u, v)$.

Let us calculate the density $d\nu : d\eta$ on the lines $\{\eta = \mathsf{const}\}$ by different methods where $d\nu$ is the density on Σ_2' corresponding to

$$\phi = \frac{1}{2}\big(1 - \frac{x_1^2}{a_2^2} - \frac{x_2^2}{b_2^2}\big).$$

Then under constrain $\eta = \mathsf{const}$ we obtain equality

$$|\frac{du}{U}| = |\frac{dv}{V}| \quad \text{with} \quad U = U(z') = -\frac{x_1\xi_2}{a_2^2} + \frac{x_2\xi_1}{b_2^2},\ V = V(z') = -\frac{x_2\xi_1}{a_2^2} + \frac{x_1\xi_2}{b_2^2},$$

and obviously[21)]

$$|U| + |V| \asymp \rho_3 \tag{7.4.43}$$

Taking into account that $d\nu$ and $|\eta|$ are invariants of Φ we see that $|d\nu : d\eta|$ is also an invariant and therefore the middle equality in this chain holds:

$$|\frac{du}{U}| = |\frac{dv}{V}| = |\frac{du'}{U'}| = |\frac{dv'}{V'}|$$

under the constrain $d\eta = 0$ where $(u', v') = \Phi(u, v)$ and $U' = U(u', v')$, $V' = V(u', v')$. One can easily see that $\rho_3(u, v) \asymp \rho_3(u', v')$.

Starting with this chain of equalities one can easily prove that without the constrain the following equivalence chain holds:

$$|\frac{du}{U}| \equiv |\frac{dv}{V}| \equiv |\frac{du'}{U'}| \equiv |\frac{dv'}{V'}|$$

[21)] As $a_2 \neq b_2$; as $a_2 \approx b_2$ $\rho_3 \gtrsim |U| + |V| \gtrsim (a_2 - b_2)/a_2 b_2$.

modulo terms with absolute values no greater than $C\bigl(\rho_3^{-L}+\rho_1^{-L}|d\eta|\bigr)$ where

$$\rho_1(z')=\mathsf{dist}(z',\Xi)\asymp \min\bigl(|\beta-b_1^2|,|\beta-b_2^2|\bigr).$$

Here ρ_3 is not Φ-invariant but we know that

$$\rho_3\ge \epsilon\min\bigl(|\beta|,|\beta+a_2^2-b_2^2|\bigr)$$

and β is invariant. Therefore for (u',v') the previous chain of equivalences holds modulo terms with absolute values not exceeding $Cn\rho^{-L}|d\eta|$ with

$$\rho=\min\bigl(|\beta|,|\beta+a_2^2-b_2^2|,|\beta-b_1^2|,|\beta-b_2|^2\bigr).$$

Increasing L if necessary we obtain the inequality $|D\Phi^n|\le Cn\rho^{-L}$. The remainder of the proof repeats the final arguments of the proof of the Theorem 7.4.10 with (7.4.16) replaced by (7.4.40). This yields (7.4.33). The proof of (7.4.34) is trivial.

(ii) If $a_2=b_2$ and $a_1=b_1$ then the proof is trivial, Φ is just a rotation of (x,ξ) by an angle 2β which is constant along billiards.

In this case from the beginning we use $\rho(z)$ instead of $\rho_3(z)$. The case of a ball or of a spherical shell can be reduced to case $d=2$ because every billiard lies in some two-dimensional plane passing through the center and the Hamiltonian flow commutes with the map $(x,\xi)\to(Gx,G\xi)$ for every orthogonal matrix G. Therefore one can consider only billiards and their variations in the plane (x_1,x_2). Singularities related to this reduction near the manifold $\{(x,\xi),x=t\xi,t\in\mathbb{R}\}=\{(x,\xi),\beta=0\}$ are taken into account because $|\beta|\ge\epsilon\rho$. □

Remark 7.4.13. For elliptic annulus we were lucky that if ray is tangent to Y_1 it remains tangent all the time and thus Ξ_1 cuts Σ into two connected components. For more general domains (f.e. disk with a round hole, but with shifted center) $\Xi_1=\bigcup_{n\in\mathbb{Z}}\Xi_{1(n)}$ where $\Xi_{1(n)}$ is a set of rays tangent to Y_1 on its n-th segment; then Ξ_1 cuts Σ into infinite number of components.

Non-Periodicity Condition and Results. I

Let us discuss first the "sharp" asymptotics. Then results of Subsubsection *7.4.2.1 Sharp Asymptotics* imply immediately

Theorem 7.4.14. *Consider Schrödinger operator and assume that τ is microhyperbolicity level*[22]*. Assume that*

$$\mu_\tau(\Pi_\tau) = 0 \tag{7.4.44}$$

where Π_τ denotes the set of points on Σ_τ periodic with respect to billiard Hamiltonian flow. Then Weyl asymptotics holds with the remainder estimate $o(h^{1-d})$.

Remark 7.4.15. (i) Theorem 7.4.6 immediately implies that $\mu'_\tau(\iota(\Xi_\tau|_{\partial X})) = 0$ where $\iota : T^*X|_Y \to T^*Y$ is a standard map and μ'_τ is a measure on T^*Y; first we define $dx'd\xi : da$ on $\Sigma_\tau|_Y$ and then ι pushes it to T^*Y, where as usual $Y = \partial X$. This measure is absolutely continuous with respect to $dx'd\xi'$ and concentrated where $\tau - a' > 0$.

(ii) Further, condition (7.4.44) implies that

$$\mu'_\tau(\iota\Pi_\tau)_Y) = 0 \tag{7.4.45}$$

Then (modulo condition in the elliptic zone, i.e. zone where $\tau - a' < 0$) asymptotics with integration only over ∂X holds with the remainder estimate $o(h^{1-d})$.

While (7.4.44) fails say for hemisphere everybody believes in so-called *Ivrii's conjecture* (1980):

Conjecture 7.4.16. *For standard Euclidean billiards in Euclidean domain* (7.4.44) *holds.*

Considered to be an easy game in 1980 it turned to be really hard even in the case of $d = 2$ (see f.e. E. Gutkin [1]); the sketch of the proof of E. Gutkin [2] contained unpatchable holes. For *generic* domains in $\mathbb{R}^d$ conjecture 7.4.16 was proven by V. Petkov, L. Stoyanov in [2] (see also their book V. Petkov, L. Stoyanov [4]).

The following theorem is due to D. Vassiliev

Theorem 7.4.17. *Consider manifold and operator described in Theorem 7.4.10. Further, assume that g^{jk}, V_j, V and ϕ are analytic functions. Then* (7.4.45) *and*

$$\mu_\tau(\Pi_\tau \setminus \Pi_\tau^0) = 0 \tag{7.4.46}$$

[22] Currently for both a and $(A, \eth B)$; see Section 7.5 for generalizations.

hold where Π^0_τ is the set of periodic points but with trajectories never hitting Y.

Proof. All we need to prove is that $\mu'(\Pi_{n,\tau}) = 0$ for all $n \in \mathbb{Z}^+$, $n \neq 0$ where $\Pi_{n,\tau} = \{z' \in \Sigma'_\tau, \Phi^n(z) = z\}$. Due to analyticity Φ is also analytic and it is sufficient to prove that $\Phi^n \neq I$ on Σ_τ. However this is true since due to microhyperbolicity $|\{a, \phi\}| \asymp \varepsilon \le \epsilon n^{-1}$ $\mathsf{dist}(\Phi^n(z), z) \asymp \varepsilon n$. □

Remark 7.4.18. Note that for domains in $\mathbb{R}^d$, with $g^{jk} = \delta_{jk}$, $V_j = V = 0$ $\Pi^0_\tau = \emptyset$.

Non-Periodicity Condition and Results. II

Theorem 7.4.19. *Consider operator $A = h^2\Delta - 1$ in X which is an ellipse, elliptic annulus, ball or spherical annulus. Then Weyl asymptotic holds with the remainder estimate $O(h^{1-d+\delta})$ with (unspecified) $\delta > 0$.*

Proof. (i) Let us apply Theorems 7.4.6 and 7.4.7; we know already that $D\Psi_t$ has a power growth but we need to check that

$$\mathsf{mes}(\Pi_{T,\varepsilon}) \le h^\delta \tag{7.4.47}$$

where

$$\Pi_{T,\varepsilon} := \{z : \exists t : \epsilon_0 < t \le T : \ \mathsf{dist}(z, \Psi_t(z)) \le \epsilon\} \tag{7.4.48}$$

as we pick up $T = h^{-\delta_1}$, $\varepsilon = h^{\delta_2}$ and consider billiard flow on $\Sigma_0 = S^*X$ and μ_0 is the usual measure. Really, due to Theorem 7.4.12 we estimate $|D\Psi_t|$ by $CT^{-1}\varepsilon^{-N}$ as $|t| \le T$ everywhere with the exception of $\Lambda = \{\rho_3 \le \varepsilon\}$. Therefore taking exponents δ_1 and δ_2 small enough we conclude that the contributions of $\Lambda_\varepsilon \cup \Pi_{T,\varepsilon}$ and $\Sigma_0 \setminus (\Lambda_\varepsilon \cup \Pi_{T,\varepsilon})$ to the remainder estimate do not exceed $C\,\mathsf{mes}(\Lambda_\varepsilon \cup \Pi_{T,\varepsilon})h^{1-d}$ and $CT^{-1}h^{1-d}$ respectively. However one can see easily that $\mathsf{mes}(\Lambda_\varepsilon) \le \varepsilon^{\delta''}$ and (7.4.47) is all we need to check. We will use notations of the proof of Theorem 7.4.12.

Obviously (7.4.47) is equivalent to

$$\mathsf{mes}\big\{z' \in \Sigma'_2 : \exists k = 1, \dots, n : \mathsf{dist}(z', \Phi^k z') \le \varepsilon = n^{-M}\big\} = O(n^{-\sigma}) \quad \text{as } n \to +\infty \tag{7.4.49}$$

for appropriate M and $\sigma > 0$. Let us recall that $\Sigma'_2 = B^*Y_2$ and $\Phi : \Sigma'_2 \to \Sigma'_2$ is a map.

(ii) Let us first consider the zone $\{(x,\xi),\beta > 0\}$. Let us consider the lines $\{\eta = \mathsf{const}\}$. These lines are analytically parameterized by a cyclic parameter $u \in \mathbb{R}/2\pi\mathbb{Z}$ [23].

Let $w \in \mathbb{R}/\mathbb{Z}$ be a new cyclic coordinate: $w = w_\eta(u) = \nu_\eta((0,u))m_\eta^{-1}$ for $u \in [0,2\pi]$ and then by periodicity. Here $m_\eta = \nu_\eta([0,2\pi])$. One can easily see that $m_\eta \asymp \log|\beta|$ as $\beta \to +0$ for $a_2 > b_2$ and $m_\eta \asymp \beta^{-1}$ as $\beta \to +0$ for $a_2 = b_2$.

For every fixed η the map Φ_η of this line is analytic and preserves the measure $dw_\eta = m_\eta^{-1} d\nu_\eta$ and the orientation[24] and therefore Φ_η is a shift $\Phi_\eta : w_\eta \to w_\eta + f(\eta)$. Then to prove (7.4.49) for $\Sigma_2' \cap \{\beta > 0\}$ one needs and suffices to prove that

$$\mathsf{C}n\,\mathsf{mes}\big(\{\eta : \exists k, l = 1,\dots,n : |f(\eta) - \frac{k}{l}| \le \varepsilon = n^{-L}\}\big) = O(n^{-\sigma}) \quad \text{as } n \to +\infty \tag{7.4.50}$$

for admissible L and $\sigma > 0$; here mes means Lebesgue measure.

Obviously, $f(\eta)$ is a continuous function. One can easily prove that

$$\epsilon\beta^p \le |\frac{df}{d\eta}| \le c\beta^{-p} \qquad \text{for } |\eta| \le a_2 - \epsilon_0, \tag{7.4.51}$$

i.e., for $\beta \in (0,\beta_0)$ with small enough $\epsilon_0 > 0, \beta_0 > 0$. Let us recall that $\beta = b_2^2 - a_2^2 b_2^2 \eta^2$. Therefore (7.4.50) holds for values of η indicated here.

On the other hand, Φ is an analytic map for $\beta \ne b_1^2$ (the case when rays are tangent to the inner ellipse Y_1). Therefore the function $f(\eta)$ is analytic for $|\eta| < \eta_1 = a_2^{-1}b_2^{-1}(b_2^2 - b_1^2)^{\frac12}$ and for $\eta_1 < |\eta| \le a_2^{-1}$. Obviously, $f(\eta)$ is not constant in each of these three intervals. One can easily see that $f_\pm(\zeta) = f(\bar\eta \pm \zeta^2)$ is an analytic function for $|\zeta| \le \epsilon_0$ and $\bar\eta = +\eta_1, -\eta_1$.

But one can easily see that

$$\mathsf{mes}(\{\eta : |f(\eta)| \le \varepsilon\}) \le C\varepsilon^\delta \tag{7.4.52}$$

[23] More precisely, this line consists of two connected components. Φ transforms every component into itself and the x-projection is an analytic diffeomorphism of this component to the ellipse Y_2. Every component is parameterized by u. Permutation of these components is equivalent to replacing Φ by Φ^{-1}.

[24] Moreover, it is well known from Cornfeld, Sinai, Fomin [1] that for generic η, Φ_η is an ergodic map.

for every $\mathscr{C}^K$ function on the closed segment provided

$$\sum_{1\le j\le K-1} |f^{(j)}| \ge \epsilon_0. \tag{7.4.53}$$

Therefore (7.4.50) holds.

(iii) Let us treat the zone $\{(x,\xi) : 0 > \beta > b_2^2 - a_2^2\}$. In this case every line $\{\eta = \mathsf{const}\}$ consists of two connected components and every component is equivalent to $\mathbb{S}^1$ and the x-projection maps every component to one of two segments on the ellipse Y_2 describing by the inequality $x_2^2 \ge -\beta b_2^2(a_2^2 - b_2^2)^{-1}$. The inverse map is two-valued. Let us introduce on this connected component the cyclic coordinate u' which can be done in the following way: let us consider the confocal hyperbola described by the equation

$$\frac{x_1^2}{a'^2} - \frac{x_2^2}{b'^2} = 1$$

with $a'^2 + b'^2 = a_2^2 - b_2^2$. This hyperbola intersects Y_2 in four points $(\bar{x}_1, \bar{x}_2)$ with

$$\bar{x}_1^2 = a_2^2(a_2^2 - b_2^2 + \beta)(a_2^2 - b_2^2)^{-1},\ \bar{x}_2^2 = -b_2^2\beta(a_2^2 - b_2^2)^{-1}.$$

Let us introduce the parameter $u' \in \mathbb{S}^1$ on this hyperbola: $x_1 = a' \sec u'$, $x_2 = b' \tan u'$. Through every point of the hyperbola corresponding to this parameter there passes a tangent straight line hitting the corresponding segment of the ellipse. Then (x_1, x_2) is an intersection point and (ξ_1, ξ_2) is a unit vector along this tangent line, directed inside and outside for $\eta < 0$ and $\eta > 0$ respectively.

Further, let us introduce the parameter $w' \in \mathbb{R}/\mathbb{Z}$ in the same way as before. Again we take $dw' = m_\eta^{-1} d\nu_\eta$ with $m_\eta \asymp \log|\beta|$ as $\beta \to -0$ and $m_\eta \asymp 1$ as $\beta \to b_2^2 - a_2^2$. Moreover, in the latter case m_η is an analytic function of $\zeta' = (\beta + a_2^2 - b_2^2)^{\frac{1}{2}}$.

One can easily see that if there is no inner ellipse Y_2 then Φ transforms one connected component of every line $\{\eta = \mathsf{const}\}$ to another and therefore for $\beta < -n^{-\sigma}$ the inequality $\mathsf{dist}(z', \Psi^k(z')) < \epsilon$ is impossible for odd k and the measure of the set $\{|\beta| \le n^{-\sigma}\}$ is $O(n^{-\delta})$.

On the other hand, if there is an inner ellipse Y_1 then the map Φ preserves both connected components of $\eta = \mathsf{const}$ but changes their orientations, i.e., the map is of the form $w_\eta \to \frac{1}{2} f(\eta) - w_\eta$ and therefore for odd k the

$d\nu_\eta$-measure of points with $\mathsf{dist}(z', \Phi^k z') \le \varepsilon$ is $O(n^p\varepsilon)$. Therefore *in both cases only even* k *should be treated*.

Furthermore, on every connected component Φ^2 is of the form $w_\eta \to w_\eta + f(\eta)$ where the function f is again continuous and (7.4.51) holds this time for $|\eta| \ge (a_2^{-1} + \epsilon_0)$, i.e., $\beta \in (-\beta_0, 0)$ with small enough ϵ_0 and β_0. This function is analytic for $a_2^{-1} < |\eta| < b_2^{-1}$, i.e., for $\beta \in (0, b_2^2 - a_2^2)$, and the functions $f(\pm(b_2^{-1} - \zeta^2))$ are analytic with respect to ζ with $|\zeta| \le \epsilon_0$. Therefore we obtain that (7.4.53) and (7.4.50) remain true in this case. □

Non-Periodicity Condition for Branching Billiards

Let us consider the case when several Schrödinger operators are intertwined on the boundary through boundary conditions; in this case Hamiltonian billiard flow (in contrast to generalized Hamiltonian billiard flow) may not even be defined on the set of the full measure and non-periodicity is a very long shot. However we consider a very special case.

Case 1. Let us consider first Hamiltonian billiard flow in the ball $X = B(0,1)$ assuming that

(7.4.54) The metrics (g^{jk}) and potential V are spherically symmetric and $V_j = 0$.

Then each billiard stays in the same plane and we can assume without any loss of the generality that $d = 2$. Introducing polar coordinates $(r, \theta) \in (0,1) \times \mathbb{R}/2\pi\mathbb{Z}$ we can rewrite our Hamiltonian as

$$a(r, \xi_r, \xi_\theta) = \frac{1}{2}\lambda(r)^2\Big(\xi_r^2 + \mu(r)^2 r^{-2}\xi_\theta - V(r)\Big) \tag{7.4.55}$$

with $\epsilon \le \lambda, \mu \le c$. Then $\xi_\theta = \eta$ is a movement integral. Obviously

$$\Sigma_0 := \big\{(r, \xi_r, \eta) \in (0,1] \times \mathbb{R}^2 :\ a(r, \xi_r, \eta) = 0\big\} = \big\{(r, \xi_r, \eta) :\ \xi_r = \big(V(r) - \mu(r)^2 r^{-2}\eta^2\big)^{\frac{1}{2}}\big\}. \tag{7.4.56}$$

We are interested only in level 0 trajectories which hit the boundary $\partial X = Y = \{r = 1\}$. Let us denote by $\mathcal{X} \subset \Sigma_0$ the set of points such that

trajectories passing through them hit Y; then

$$\mathcal{X} = \big\{(r,\xi_r,\eta)\in \Sigma_0,\ \eta : |\eta|\le \eta_0 := (V(1)/\mu(1))^{-\frac{1}{2}}, \ r_1(\eta)\le r\le 1\big\} \tag{7.4.57}$$

where $r = r_1(\eta) < 1$ is the largest root of the equation $V(r)-\mu(r)^2r^{-2}\eta^2 = 0$.

Studying trajectories one can assume without any loss of the generality that $\lambda(r) = 1$; then along trajectories

$$\frac{dr}{dt} = \xi_r = \big(V(r)-\mu(r)^2r^{-2}\eta^2\big)^{\frac{1}{2}}, \qquad \frac{d\theta}{dt} = \mu(r)^2\eta \tag{7.4.58}$$

and therefore increment of θ between two consecutive hits is equal to

$$f(\eta) = 2\int_{r_1(\eta)}^1 \big(V(r)-\mu(r)^2r^{-2}\eta^2\big)^{-\frac{1}{2}}\mu(r)^2r^{-2}\eta\, dr \tag{7.4.59}$$

One can see easily that Ψ_t satisfies power-growth condition

$$|D\Psi_t| \le C(|t|+1)\rho^{-M}, \qquad \rho = \eta_0^2-\eta^2 \tag{7.4.60}$$

and the strong non-periodicity condition is due to (7.4.52) which in turn follows from condition (7.4.53) i.e.

$$\sum_{1\le k\le K-1} |f^{(k)}(\eta)| \ge \epsilon_0 \qquad \forall \eta : 0<\eta\le\eta_0. \tag{7.4.61}$$

Thus we arrive to

Proposition 7.4.20. *Let problem $(A,\eth B)$ be self-adjoint and microhyperbolic on the energy level 0 where A is an operator with the principal symbol* (7.4.55)*; let conditions* (7.4.54) *and* (7.4.60) *be fulfilled. Then the contribution of the small vicinity of $\mathcal{X}$ to the remainder in the Weyl asymptotics is $O(h^{1-d+\delta})$.*

Case 2. Consider now several operators of the same type (albeit with its own s and potential) linked through boundary conditions; so $A = \mathsf{diag}(A_1,\dots,A_m)$. We consider each operator A_j in its own copy X_j of ball $B(0,1)$ which are glued along Y.

Theorem 7.4.21. *Let problem* $(A, \eth B)$ *be self-adjoint and microhyperbolic on the energy level* 0 *where* $A = \mathsf{diag}(A_1, \dots, A_m)$ *and each* A_j *is an operator with the principal symbol* (7.4.55) *with* $\lambda = \lambda_j$, $\mu = \mu_j$, $V = V_j$ *satisfying* (7.4.54).

Assume that <u>*either*</u>

$$\sum_{0\le k\le K-1} |\sum_j \bigl(n_j f_j(\eta) - 2\pi q\bigr)| \ge q^{-K} \qquad \forall n = (n_1, \dots, n_m) \in \mathbb{Z}^{+\,m} \tag{7.4.62}$$

$$\text{with} \quad n_j = 0 \quad \text{as} \quad \eta_j \ge \eta_{j0} \quad \text{and} \quad q \in \mathbb{Z}^+ \setminus 0$$

provided $N \le \varepsilon^{-\delta}$ <u>*or*</u> *for all* j *functions* f_j *satisfy* (7.4.61) *and also*

$$\pm \frac{d}{d\eta} f_j(\eta) \le 0 \qquad \forall \eta : 0 < \eta \le \eta_{0j} \tag{7.4.63}_{\pm}$$

(with the same sign). Then the contribution of small vicinity of $\bigcup_j \mathcal{X}_j$ *to the remainder in the Weyl asymptotics is* $O(h^{1-d+\delta})$.

Proof. We need to prove that

$$\mathsf{mes}(\Pi_{\varepsilon,N}) = O(\varepsilon^{\delta}) \qquad \text{as} \quad \varepsilon \to +0 \tag{7.4.64}$$

where

$$\Pi_{\varepsilon,N}\bigl(\{\eta > 0,\ \exists n = (n_j, \dots, n_m) \in \mathbb{Z}^{+\,m} \ \text{ with } \ n_j = 0 \ \text{ as } \ \eta_j \ge \eta_{j0},$$
$$\text{and } \ q = 1, \dots, N \ \text{ such that } \ |\sum_{1\le j\le m} n_j f_j(\eta) - 2\pi q| \le \varepsilon\} \tag{7.4.65}$$

provided $N \le \varepsilon^{-\delta}$.

However (7.4.61), (7.4.63) imply (7.4.64). □

Example 7.4.22. (i) (flat disk) As $\mu = \mathsf{const}$, $V = \alpha^2 = \mathsf{const}$ we see that $r_1(\eta) = \alpha^{-1}\mu\eta$ and

$$f(\eta) = 2\int_{\alpha^{-1}\mu\eta}^{1} \bigl(\alpha^2 - \mu^2 r^{-2}\eta^2\bigr)^{-\frac{1}{2}} \mu^2 r^{-2}\eta \, dr = \mu\pi - 2\mu \arcsin(\alpha^{-1}\mu\eta)$$

satisfies (7.4.61) and $(7.4.63)_+$.

(ii) (spherical cut) As $\mu(r) = \beta r \sin^{-1}(\beta r)$, $V = \alpha^2 = \mathsf{const}$ we see that $r_1 = \beta^{-1} \arcsin(\alpha^{-1}\beta\eta)$ and

$$f(\eta) = 2\int_{\beta^{-1}\arcsin(\alpha^{-1}\beta\eta)}^{1} \bigl(\alpha^2 - \beta^2 \sin^{-2}(\beta r)\eta^2\bigr)^{-\frac{1}{2}} \beta^2 \sin^{-2}(\beta r)\eta \, dr = \pi - 2\arcsin\Bigl(\beta\eta(\alpha^2 - \beta^2\eta^2)^{-\frac{1}{2}} \cot\beta\Bigr)$$

satisfies $(7.4.63)_+$ as $\beta \in (0, \frac{\pi}{2})$ (less than semi-sphere) and $(7.4.63)_-$ as $\beta \in (0, \frac{\pi}{2})$ (more than semi-sphere).

Case 3. Alternatively consider a cylinder $X = (-1, 1) \times \mathbb{R}/2\pi\mathbb{Z} \ni (z, \theta)$ with

$$\mathsf{a}(r, \xi_r, \xi_\theta) = \frac{1}{2}\lambda(z)^2\Bigl(\xi_z^2 + \mu(z)^2\xi_\theta^2 - V(z)\Bigr) \tag{7.4.66}$$

with $\mu = \mathsf{const}$, $\alpha = \mathsf{const}$. Then $\partial X = Y_- \cup Y_+$, $Y_\pm = \{z = \pm 1\}$ and increment of θ between consecutive hits of Y_j and Y_{-j} is

$$f(\eta) = 2\int_0^1 \bigl(V(z) - \mu^2(z)\eta^2\bigr)^{-\frac{1}{2}} \mu^2(z)\eta \, dz \tag{7.4.67}$$

provided $|\eta| \le \inf_z \mu^{-1}(z)V(z)^{\frac{1}{2}}$; otherwise trajectory returns to Y_j and

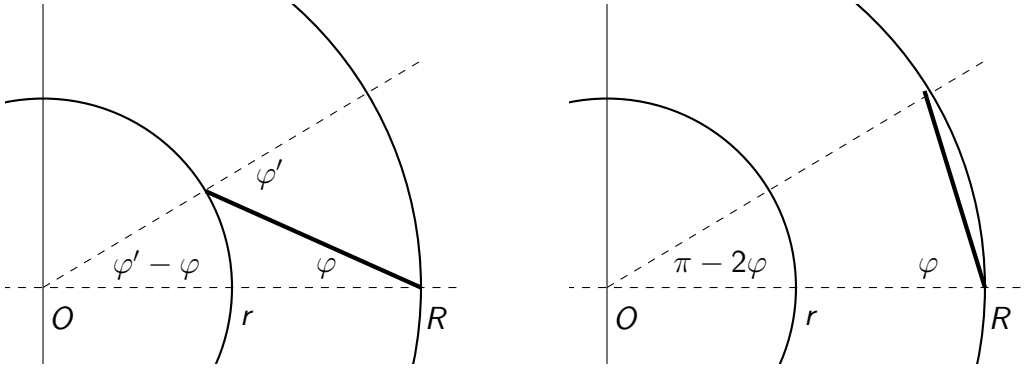
$$f(\eta) = 2\int_{z_1(\eta)}^1 \bigl(V(z) - \mu^2(z)\eta^2\bigr)^{-\frac{1}{2}} \mu^2(z)\eta \, dz. \tag{7.4.68}$$

Example 7.4.23. As $\mu(z) = \mathsf{const}$, $V = \alpha^2 = \mathsf{const}$, $f(\eta) = 2\mu^2\eta(1-\mu^2\eta^2)^{-\frac{1}{2}}$ satisfies $(7.4.63)_-$.

Remark 7.4.24. (i) Theorem 7.4.21 holds also in the case of $X_1, \dots, X_{m_1}$ balls with the common boundary Y_-, $X_{m_1+1}, \dots, X_{m_1+m_2}$ balls with the common boundary Y_+, and $X_{m_1+m_2+1}, \dots X_m$ cylinders with the common boundary $Y_- \cup Y_+$ (so $X_1, \dots, X_{m_1}$ and $X_{m_1+m_2+1}, \dots X_m$ are glued together along Y_- and $X_{m_1+1}, \dots, X_{m_1+m_2}$ and $X_{m_1+m_2+1}, \dots X_m$ are glued together along Y_+) provided f_j satisfy either (7.4.62) or $(7.4.63)_\pm$ with the same sign.

(ii) We avoided to discuss boundary conditions; we need to assume that for almost all billiards boundary bicharacteristics do not appear (see assumption (8.3.34) below).

Example 7.4.25. Consider $X_1, \dots, X_m$ circular annuli with the outer and inner radii R_k and r_k and if $R_k = r_j$ or $R_k = R_k$ or $R_j = r_j$ we consider the corresponding boundaries to be identified and operators to be intertwined there.



(a) Ray hits inner boundary (b) Ray does not hit inner boundary

Figure 7.3: Circular annulus: dashed lines are normals to the boundary

Then if φ_k is an incidence angle[25)] of the ray at the outer boundary and φ'_k is an incidence angle of the same ray at the inner boundary then as an easy geometrical exercise shows $r_k \sin \phi'_k = R_k \sin \phi_k$. Furthermore if $R_k = R_j$, $R_k = r_j$, $r_k = R_j$, $r_k = r_j$, then according to the Snell's law $c_k \sin \varphi_k = c_j \sin \varphi_j$, $c_k \sin \varphi_k = c_j \sin \varphi'_j$, $c_k \sin \varphi'_k = c_j \sin \varphi_j$, $c_k \sin \varphi'_k = c_j \sin \varphi'_j$ respectively, and therefore $R_k c_k \sin \varphi_k$ and $r_k c_k \sin \varphi'_k$ are preserved along rays and at reflections and they are equal (if the ray hits inner boundary). Let us denote this value by η.

Further, if the ray hits an inner boundary, the increment of the central angle along the ray segment (between hitting the outer and then inner or inner and then outer boundaries) is $\varphi'_k - \varphi_k$. On the other hand, if the ray does not hit the inner boundary the increment between two consecutive hits of the outer boundary is $\pi - 2\varphi_k$.

Therefore the total increment of the central angle is

$$F(\eta, n_1, \dots, n_m) = \sum_{k:|\eta|\le c_k r_k} n_k \Big(\arcsin\big(\frac{\eta}{r_k c_k}\big) - \arcsin\big(\frac{\eta}{R_k c_k}\big) \Big) + \sum_{k:c_k r_k < |\eta| \le c_k R_k} n_k \Big(\pi - 2 \arcsin\big(\frac{\eta}{R_k c_k}\big) \Big)$$

[25)] Incidence angle is an angle between the ray and the normal to the boundary.

where n_k denotes the number of segments of the billiard residing in X_k.

Then

$$\partial_\eta F(\eta, n_1, \dots, n_m) = \sum_{k:|\eta|\le c_k r_k} n_k\Big(\big(r_k^2c_k^2-\eta^2\big)^{-\frac12}-\big(R_k^2c_k^2-\eta^2\big)^{-\frac12}\Big)-\sum_{k:c_kr_k<|\eta|\le c_kR_k} n_k\big(R_k^2c_k^2-\eta^2\big)^{-\frac12}$$

and one can justify strong non-periodicity provided

$$c_kR_k \ne c_jr_j \qquad \forall j,k. \tag{7.4.69}$$

Really, $F(\eta, n_1, \dots, n_m)$ is analytic outside of points $\{\pm\eta_k, \eta_k = c_kR_k\}$ and $\{\pm\eta'_k, \eta'_k = c_kr_k\}$ and condition (7.4.69) ensures that $\partial_\eta F(\eta, n_1, \dots, n_m)$ tends to infinity as $\eta \to \eta_k - 0$ (and $\eta \to -\eta_k + 0$) which implies necessary diophantine condition.

7.5 Operators with Irregular Coefficients. Seeley's Method

In this section we consider operators with irregular coefficients in the domain with the irregular boundary. As usual we want smoothness slightly greater than $\mathscr{C}^1$. Observe, that we need to reject an idea to straighten the boundary we heavily used before. Indeed only if $\partial X \in \mathscr{C}^2$ we will get after an operator with $\mathscr{C}^1$-coefficients (assuming that original coefficients were smooth enough).

To avoid this difficulty we apply method originally due to R.Seeley [1] which with the minor modifications could be used for operators with the smooth coefficients but in the domains with the irregular boundary.

To accomodate also operators with irregular coefficients we need to modify it more and to accomodate matrix operators we need to modify it even more. We cover differential operators considered in Section 4.6 assuming that they have semi-bounded and elliptic principal symbols. If the principal symbol is not semi-bounded we will just need to consider two energy levels and only obvious modification is required (except boundary strip).

Let us consider ball $B(0,1)$ with $B(0,1) \cap \partial X \ne \emptyset$; otherwise either $B(0,1) \subset X$ and one can apply results of Chapters 4–5 or $B(0,1) \cap X = \emptyset$.

7.5.1 Discussion

Rescaling Technique

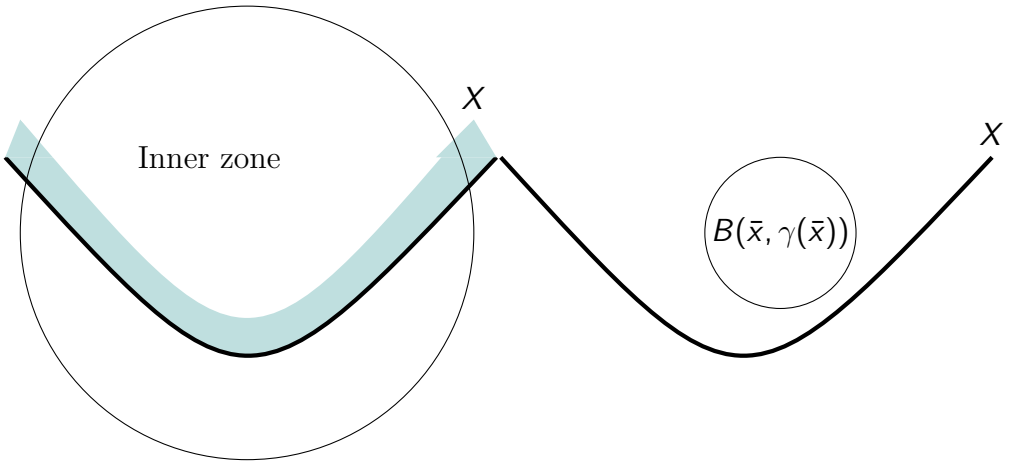
We consider contribution of the *inner zone*

$$\{x \in X \cap B(0,1) : \ \mathsf{dist}(x, \partial X) \geq \bar{\gamma}\} \tag{7.5.1}$$

to asymptotics and to the remainder estimate; we estimate contribution of the *boundary strip*

$$\{x \in X \cap B(0,1) : \ \mathsf{dist}(x, \partial X) \leq \bar{\gamma}\} \tag{7.5.2}$$

later. Usually $\bar{\gamma} \asymp h$ but we commit only to an assumption that $\bar{\gamma} \geq h$.



(a) Inner zone and boundary strip (b) $B(\bar{x}, \gamma(\bar{x}))$ to be rescaled

Figure 7.4: Rescaling settings

Let us introduce

$$\gamma(x) = \frac{1}{2}\mathsf{dist}(x, \partial X) \tag{7.5.3}$$

which is obviously scaling function and consider $B(\bar{x}, \gamma)) \subset X \cap B(0,1)$ with $\gamma = \gamma(\bar{x})$ here.

We rescale it $x \mapsto x\gamma^{-1}$, $h \mapsto \hbar = h\gamma^{-1}$. We want that

(7.5.4) Contribution of $B(\bar{x}, \gamma(\bar{x}))$ to the remainder does not exceed $\hbar^{1-d}$.

This happens for example if operator is ξ-microhyperbolic or scalar and satisfies certain non-degeneracy assumptions (which should survive rescaling; we will discuss them later in Section 8.1). Due to Sections 4.6 and 5.4 we do not need restrictive smoothness conditions.

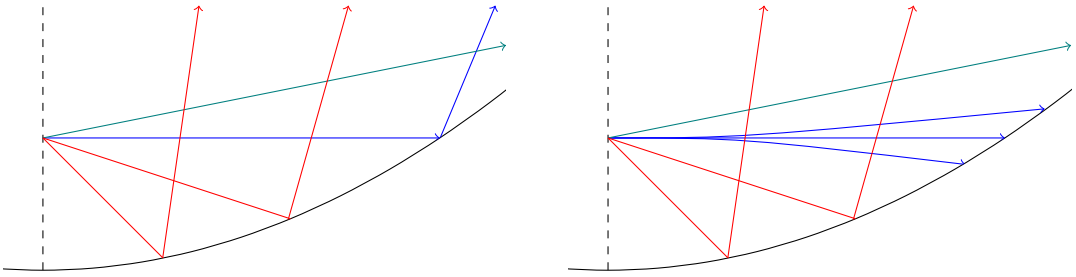
Then the total contribution of inner zone (7.5.1) to the remainder does not exceed

$$Ch^{1-d}\int \gamma(x)^{-1}\,dx. \tag{7.5.5}$$

Really, contribution of each ball does not exceed $C\hbar^{1-d} = Ch^{1-d}\gamma^{-1}\times\gamma^{d}$.

The bad news are that even in the smooth case integral in (7.5.5) is unbounded; the good news are that this divergence is only logarithmic.

Propagation



(a) Geometry of rays (b) Approximate geometry of rays

Figure 7.5: Green rays are outgoing, red are incoming and blue are almost parallel to the boundary

The idea of R. Seeley was to consider rays (only wave equation in the domain with the smooth boundary was analyzed then). In the smooth case almost parallel rays (with an angle no more than $\hbar^{\delta}$) do not meet the boundary for a time $T \asymp \min\bigl(\gamma^{\frac{1}{2}}, \hbar^{-\delta}\gamma\bigr)$, while outgoing rays require even larger time. Incoming rays meet boundary once during this time, they are not tangent to the boundary and one can use oscillatory integrals to calculate their contribution. However here we are interested only in the asymptotics of $e(x,x,0)$ integrated with respect to x and therefore we can select time direction along each ray bundle as we want; so we do not need to consider incoming rays at all.

Thus at least heuristically we can replace $T(x) \asymp \gamma(x)$ by larger value which in the R. Seeley settings was

$$T(x) \asymp \min\bigl(\gamma(x)^{\frac{1}{2}}, \hbar^{-\delta}\gamma(x)\bigr). \tag{7.5.6}$$

Then the contribution of $B(\bar{x}, \gamma)$ to the remainder does not exceed

$$CT^{-1}h^{1-d} \tag{7.5.7}$$

and the total contribution of zone (7.5.1) does not exceed

$$Ch^{1-d}\int T(x)^{-1}\,dx \asymp Ch^{1-d}\int \bigl(\gamma^{-\frac{1}{2}} + h^{\delta}\gamma^{-1-\delta}\bigr)\,dx. \tag{7.5.8}$$

Remark 7.5.1. (i) One can see easily that in the smooth settings integral in (7.5.8) is bounded. What is more: obviously it work for less smooth boundaries as well: if $\partial X \in \mathscr{C}_{\nu}$ with the continuity modulus $\nu(t) = o(t)$ all we need is to replace $\gamma^{\frac{1}{2}}$ in (7.5.6) by $\check{\nu}(\gamma)$ where $\check{\nu}$ is an inverse function. In particular, integral in (7.5.8) is bounded provided $\nu(t) = O(t|\log t|^{1+\delta})$ with $\delta > 0$.

(ii) Contribution of zone (7.5.2) is easy to estimate: in the current (smooth boundary) settings $e(x, x, 0) \le Ch^{-d}$ and the total measure of this zone is $O(\bar{\gamma})$; so the total contribution of this zone does not exceed $Ch^{-d}\bar{\gamma} \asymp Ch^{1-d}$. In the case of irregular boundary we will discuss this later in the different settings (Dirichlet boundary conditions being the most favourable).

What Then?

However we want to deal not only with irregular boundary but also with irregular coefficients. In this section we mainly

(i) Generalize this analysis (and justify propagation) to general ξ-microhyperbolic scalar operators (including rough ones);

(ii) Generalize this analysis (and justify propagation) to general ξ-microhyperbolic matrix operators (including rough ones);

(iii) Using rescaling consider degenerating scalar operators (including rough ones);

(iv) Improve the remainder estimate to $o(h^{1-d})$ under conditions of the global nature;

(v) Describe construction of mollification for irregular operators.

Mollification

To recover $O(h^{1-d})$ and $o(h^{1-d})$ remainder estimate in the case of irregular operators we will select smoothing parameter ε in a special way (assuming ξ-microhyperbolicity): again as in Section 5.4 we can take *variable smoothing parameter*. Namely we take

$$\varepsilon(x) = \eta\gamma(x)^{\delta_0-1} \tag{7.5.9}$$

with $\gamma(x) = \frac{1}{2}\mathsf{dist}(x, \partial X)$ and arbitrarily small exponent $\delta_0 > 0$; here η denotes the *original smoothing parameter* which was

$$\eta = Ch|\log h|, \quad \eta = C'(h)h|\log h| \tag{7.5.10}$$

in the proof of remainder estimates $O(h^{1-d})$ and $o(h^{1-d})$ respectively where C is a large enough constant and $C'(h) \to \infty$ slowly enough: $C'(h)\nu(\eta)\eta^{-1} \to 0$ where $\nu(t)t^{-1}$ is the continuity modulus of the first derivatives of the coefficients.

However, we define $\varepsilon(x)$ by (7.5.9) only as $\gamma(x) \geq C\varepsilon(x)$ i.e only in the interior domain

$$X_{\hat\gamma} := \{x \in X, \gamma(x) \geq \hat\gamma\} \qquad \text{with } \hat\gamma_1 = C(h|\log h|)^{1/(2-\delta_0)} \tag{7.5.11}$$

while in

$$Y_{\hat\gamma} := \{\gamma(x) \leq \hat\gamma\} \tag{7.5.12}$$

we pick

$$\varepsilon(x) = \epsilon_0\gamma(x) \tag{7.5.13}$$

with an appropriate constant ϵ_0; then in this zone

$$\hat\gamma \leq \varepsilon(x) \leq \epsilon\gamma(x). \tag{7.5.14}$$

We mollify operator in question; namely we replace it's coefficients $a_*(x)$ by

$$\tilde a_*(x) = \int a_*\big(x + \varepsilon(x)y\big)\phi(y)dy$$

where ϕ is a mollifying function of the same type as in Section 5.4.

Remark 7.5.2. (i) Again *in what follows we skip "˜" over mollified symbol a and over its level surfaces "Σ_τ"* .

(ii) Note that near the boundary we favor mollification over precision (and we will see that there is a good reason for this); however one can see easily that the differences for the Weyl expressions for mollified and original operators are $O(h^{1-d})$, $o(h^{1-d})$ as soon as operator is ξ-microhyperbolic, domain satisfies condition (7.5.28) below and also

(7.5.15) First derivatives of the coefficients are continuous with the continuity modulus $O\big(\nu(t)t^{-1}\big)$ in the vicinity of $\bar{X}$ with $\nu(t) = O\big(t|\log t|^{-1}\big)$ or $\nu(t) = o\big(t|\log t|^{-1}\big)$ respectively.

7.5.2 $O(h^{1-d})$ Remainder Estimate for Scalar Operators

Let us consider a point $\bar{x}$ and γ-vicinity of it with $\gamma = \gamma(\bar{x})$. Rescaling it to $B(0,1)$ by $x \mapsto (x-\bar{x})\gamma^{-1}$ we get the operator of the same type with $\hbar = h\gamma^{-1}$ instead of h.

Inner Zone

Now let us start our real arguments. Namely, let us consider a point $\bar{x} \in Y_\mu$ and a $\frac{1}{4}\zeta$-vicinity of it with parameter $\zeta \le \gamma := \gamma(\bar{x})$ defined later and an (arbitrarily) small constant μ. We consider also $\bar{\xi}$ and $\frac{1}{4}\rho$-vicinity of it with parameter ρ defined later.

Let us consider generalized Hamiltonian trajectories of $a(x,\xi)$ with $|t| \le T$,

$$T \le T_0(\gamma) := \gamma^{1-\delta_1}. \tag{7.5.16}$$

These trajectories remain in $(\frac{1}{2}\rho, \frac{1}{2}\zeta)$-vicinity of $(y(t), \bar{\xi})$ with $y(t) := \bar{x} + \ell t$, $\ell = \nabla_\xi a(\bar{x}, \bar{\xi})$ as long as they do not hit the boundary provided

$$\rho \ge CT, \qquad \zeta \ge C\rho T. \tag{7.5.17}_{1,2}$$

Let us recall that $|\nabla_{x,\xi}\nabla_\xi a| \le c$. Therefore

(7.5.18) Assumption $\gamma(y(t)) \ge \zeta$ for all $t : 0 \le \pm t \le T$ implies that $\gamma(x(t)) \ge \frac{1}{2}\zeta$ at every trajectory of this "beam" (of the trajectories).

To pass from the "classical" to "quantum" dynamics we need to request the logarithmic uncertainty principle $\zeta\rho \ge Ch|\log h|$ and $\varepsilon(x(t))\rho \ge Ch|\log h|$ at each point of the trajectory. So let us assume that

$$\zeta\rho \ge Ch|\log h|, \qquad \rho \ge CT^{1-\delta_0} \tag{7.5.19}_{1,2}$$

where the second condition assures the second of the uncertainty principles due to observation $\gamma(x(t)) \le CT$ and definition of $\varepsilon(x)$.

Note, that the only condition bounding ρ from above is $(7.5.17)_2$; plugging $\rho = \epsilon\zeta T(\gamma)^{-1}$ into all other conditions and selecting appropriate exponents δ_0, δ_1 we observe that we can satisfy all of them if and only if $\gamma \ge \bar\gamma_1 := h^{1-\delta_2}$ and $\zeta = \max\bigl(\gamma^{\frac12(1-\delta)}h^{\frac12}, \gamma^{2-\delta}\bigr)$ with arbitrarily small exponents $\delta > 0$ and $\delta_2 = \delta(1+\delta)^{-1}$.

Remark 7.5.3. Observe that we could take $\varepsilon(x) = Ch|\log h|\gamma^{-1}|\log\gamma|^{-1-\delta_0}$ and $T_0(\gamma) = \gamma|\log\gamma|^{1+\delta_0}$ and push $\bar\gamma_0$ to $h|\log h|^\sigma$ with large enough σ.

These assumptions enable us to prove

Proposition 7.5.4. *The singularity of $v = e^{ih^{-1}tA}v_0$ contained at $t = 0$ in $B(\bar x, \frac14\zeta)\times B(\bar\xi, \frac14\rho)$ is contained in $B(y(t), \frac12\zeta)\times B(\bar\xi, \frac12\rho)$ as long as*

$$0 \le \pm t \le T^*_\pm(x,\xi) = \min\bigl(T_\pm(x,\xi), T_0(\gamma)\bigr) \tag{7.5.20}$$

where

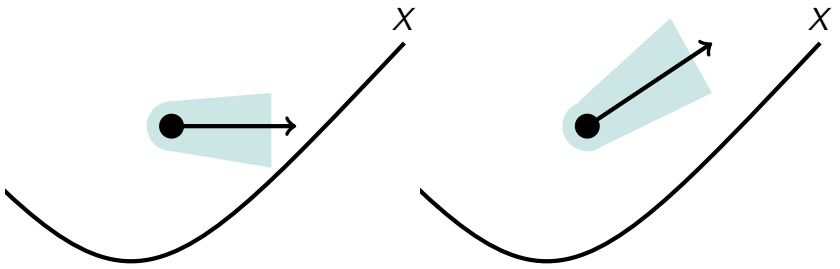
$$T_\pm(x,\xi) := \max\{T : x(t) \in X_\zeta \quad \forall t : 0 \le \pm t \le T\}. \tag{7.5.21}$$

See Figure 7.6 below for illustration of the assertion of this proposition.

Remark 7.5.5. (i) So $T^*_\pm(x,\xi)$ is $T_\pm(x,\xi)$ *capped* by $T_0(x)$.

(ii) Let us recall that $x(t)$ could be a multi-valued function because the Hamiltonian flow could be multi-valued due to the irregularity of the right-hand expressions in the Hamiltonian equations. Note however that defining $T_\pm$ we can take *either* multivalued Hamiltonian trajectories *or* single-valued Hamiltonian trajectories of the mollified symbol *or* just a straight lines $y(t)$ tangent to $x(t)$ at $t = 0$.

Proof of Proposition 7.5.4. Without any loss of the generality one can assume that $(\bar x, \bar\xi) = (0,0)$. Rescaling $t \mapsto t_{\text{new}} = tT^{-1}$, $x \mapsto x_{\text{new}} = (x - \ell t)\zeta^{-1}$, $\xi \mapsto \xi_{\text{new}} = \xi\rho^{-1}$ we find ourselves in the framework of Theorem 2.3.2 with $\Omega = [0,1]\times B(0,1)\times B(0,1)$, $h_{\text{new}} = h(\rho\zeta)^{-1} \le h^\delta$ and $\varepsilon_{\text{new}} = \varepsilon_0 T^{\delta_0-1}\zeta^{-1}$ which satisfy condition $\varepsilon_{\text{new}} \ge Ch_{\text{new}}|\log h_{\text{new}}|$. □



(a) (b)

Figure 7.6: Singularity initially confined in black disk with initial direction indicated by arrow remains in the colored zone for time T.

Without any loss of the generality one can assume that

(7.5.22) operator is ξ_1-microhyperbolic (i.e. microhyperbolic in the direction $\ell_\xi = (1, 0, \dots, 0)$);

then we can take operators $Q_j = Q_j(x, hD')$ with $\xi' = (\xi_2, \dots, \xi_d)$.

Then

$$F_{t\to h^{-1}\tau}\bigl(\bar\chi_T(t) - \bar\chi_{T'}(t)\bigr)\Gamma\bigl(Q_{1x}u\,{}^tQ_{2y}\bigr) \equiv 0 \tag{7.5.23}$$

as Q_2 is supported in $B(\bar x, \frac{1}{4}\zeta)\times B(\bar\xi', \frac{1}{4}\rho)$ and $T' = h^{1-\delta'}$ with arbitrarily small exponent $\delta' > 0$.

Therefore

(7.5.24) the contribution of $\Omega'(\bar x, \bar\xi') := B(\bar x, \frac{1}{4}\zeta)\times B(\bar\xi', \frac{1}{4}\rho)$ to the remainder does not exceed $Ch^{1-d}\zeta^d T^{*-1}$ with $T^* = \max(T^*_+, T^*_-)$.

Really, if $T_-(x,\xi) > T_+(x,\xi)$ one can always revert time.

Plugging $T^*_\pm$ defined by (7.5.20) we can rewrite this expression as

$$Ch^{1-d}\int_{\Omega'(\bar x,\bar\xi)\cap\Sigma_\tau} T(x,\xi)^{-1}dx\,d\xi : da + Ch^{1-d}\zeta^d T_0(\gamma) \tag{7.5.25}$$

where

$$T(x,\xi) = T_+(x,\xi) + T_-(x,\xi) \tag{7.5.26}$$

and $\Sigma_\tau = \{(x,\xi) : a(x,\xi) = \tau\}$.

Then the contribution of the strip $\mathcal{X}_{(\beta)} = \{x \in X : \beta \le \gamma(x) \le 2\beta\}$ with $\beta \in (\bar\gamma_1, \mu)$ to the remainder does not exceed

$$Ch^{1-d} \int_{\mathcal{X}_{(\beta)}\times\mathbb{R}^d\cap\Sigma_\tau} T(x,\xi)^{-1} dx\, d\xi : da + Ch^{1-d} T_0(\beta)^{-1} \cdot \mathsf{mes}(\mathcal{X}_{(\beta)}). \tag{7.5.27}$$

The second term does not exceed $C\beta^\delta h^{1-d}$ provided

$$\mathsf{mes}(\mathcal{X}_{(\beta)}) \le c\beta \tag{7.5.28}$$

which is a very weak regularity condition to ∂X (intersected with $B(0,1)$); we will deal with even with less regular boundaries in Subsection 7.5.5.

Then estimate (7.5.27) for contribution of $\mathcal{X}_{(\beta)}$ implies that the contribution of the strip

$$Y_\beta \cap X_{\beta'} = \{x \in X : \beta' \le \gamma(x) \le \beta\} \tag{7.5.29}$$

to the remainder does not exceed

$$Ch^{1-d} \int_{(Y_\beta\cap X_{\beta'})\times\mathbb{R}^d\cap\Sigma_\tau} T(x,\xi)^{-1} dx\, d\xi : da + Ch^{1-d}\beta^\delta \tag{7.5.30}$$

provided $\frac{1}{2}\beta > \beta' > \bar\gamma_1$.

Let us recall definition of $T(x,\xi)$: according to (7.5.21) and (7.5.26)

(7.5.31) $T(x,\xi)$ is the maximal time such that any trajectory passing through (x,ξ) remains in X_β: i.e. if $x(t) \in X_\beta$ as $t \in [-T_-(x,\xi), T_+(x,\xi)]$, then $T(x,\xi) = T_+(x,\xi) + T_-(x,\xi)$.

Observe that increasing ζ, we decrease $T(x,\xi)$ and thus increase the first term in (7.5.30); so we are interested to pick up the smallest possible ζ. Further, note that $(7.5.17)_{1,2}$ is the only restriction to ρ from above; selecting $\rho = \epsilon\zeta T^{-1}$ and plugging into $(7.5.17)_{1,2}$ and $(7.5.19)_{1,2}$ we get

$$\zeta \ge C\max\bigl((h|\log h|\,T)^{\frac{1}{2}}, T^{2-\delta_0}\bigr).$$

Recalling that $T_0 = \gamma^{1-\delta_1}$ and taking $T = T_0$, we pick up

$$\zeta = \max\bigl(h^{\frac{1}{2}-\delta}\beta^{\frac{1}{2}+\delta}, \beta^{2-\delta}\bigr) \tag{7.5.32}$$

with (arbitrarily) small $\delta > 0$ (with all previous exponents δ_j depending on it). Furthermore, for any given β let us define $\beta' = \zeta$ by (7.5.32). Let us recall that $\zeta \le \gamma$ and thus β' is the lowest value of γ we can afford for given ζ.

So, let us take β and define $\beta' = \zeta(\beta) \le \beta$ by (7.5.32) and then pick up this ζ in the whole strip (7.5.29). Therefore the contribution of this strip (7.5.29) to the remainder is given by (7.5.30) where now $T(x,\xi)$ is a time for which Hamiltonian trajectory of the mollified symbol $a(x,\xi)$ remains in (7.5.29).

Since the corresponding Hamiltonian flow preserves both Σ_τ and measure $dx\,d\xi : da$ on it, we conclude that thus interpreted (7.5.30) does not exceed

$$Ch^{1-d}\int_{(\mathcal{X}_{(\beta')}\cup\mathcal{X}_{(\beta')})\times\mathbb{R}^d\cap\Sigma_\tau} t(x,\xi)^{-1}dx\,d\xi : da + Ch^{1-d}\beta^{\delta} \tag{7.5.33}$$

where $t(x,\xi)$ is the time spent by trajectory in $\mathcal{X}_{(\gamma)}$ as $x \in \mathcal{X}_{(\gamma)}$; however $t(x,\xi) \ge \epsilon\gamma(x,\xi)$ and the first term in (7.5.33) does not exceed $Ch^{1-d}\gamma^{-1}\operatorname{vol}\mathcal{X}_{(\gamma)}$ and in virtue of condition (7.5.28) it does not exceed Ch^{1-d}. Therefore

(7.5.34) Contribution of the strip (7.5.29) with $\beta \ge h^{1-\delta}$ and $\beta' = \zeta$ defined by (7.5.32) to the remainder does not exceed Ch^{1-d}.

Since zones

$$\mathcal{X}'' := \{x : h^{1-\delta} \le \gamma(x) \le h^{\delta}\} \tag{7.5.35}$$

and

$$\mathcal{X}' := \{x : h^{\delta} \le \gamma(x) \le \mu\} \tag{7.5.36}$$

can be covered by no more than C_δ and $C\log|\log h|$ respectively of such stripes, we have proven

Proposition 7.5.6. *Let coefficients be $\mathscr{C}^1(\bar{X})$ and mollification parameter is selected as above. Let condition* (7.5.28) *be fulfilled. Then*

(i) The contribution of zone $\mathcal{X}''$ defined by (7.5.35) *to the remainder does not exceed $C_\delta h^{1-d}$.*

(ii) The contribution of zone $\mathcal{X}'$ defined by (7.5.36) *to the remainder does not exceed*

$$Ch^{1-d}\int_{(X_{h^\delta}\cap Y_\mu)\times\mathbb{R}^d\cap\Sigma_\tau} T(x,\xi)^{-1}dx\,d\xi : da + Ch^{1-d}\mu^\delta \tag{7.5.37}$$

with $T(x,\xi)$ defined by (7.5.31) *with*

$$\zeta = \beta^{2-\delta}. \tag{7.5.38}$$

(iii) In particular, it does not exceed $Ch^{1-d}\log|\log h|$.

(iv) If

$$\int_{Y_1} T(x,\xi)^{-1}dx\,d\xi : da < \infty \tag{7.5.39}$$

then the contributions of both zones $\mathcal{X}'$, $\mathcal{X}''$ to the remainder do not exceed $C_\mu h^{1-d}$ with $C_\mu = o(1)$ as $\mu \to +0$.

Moving Closer to the Boundary

Let us consider zone

$$\mathcal{X}''' := \{x : \mu^{-1}h \le \gamma(x) \le h^{1-\delta}\} \tag{7.5.40}$$

with $\mu \le 1$.

We can repeat our analysis leading to Proposition 7.5.6 but we must replace conditions $(7.5.19)_{1,2}$ by a single condition

$$\zeta\rho \ge Ch|\log\hbar| \qquad \text{with} \quad \hbar = \frac{h}{\gamma}; \tag{7.5.19}'_1$$

we do not need the second condition since trajectory is contained in the zone where $\varepsilon(x) = \gamma(x)$ and we need only $|\log\hbar|$ because $\hbar$ is our effective semiclassical parameter. Therefore we can define ζ by (7.5.38) and take $T_0(\gamma) = \gamma^{1+\delta'}h^{-\delta'}$ with small exponents $\delta, \delta' > 0$.

Then the contribution of the strip $\mathcal{X}_{(\beta)}$ to the remainder does not exceed (7.5.27) and repeating the rest of arguments leading to Proposition 7.5.6 we arrive to

Proposition 7.5.7. *(i) Contribution of zone $\mathcal{X}'''$ with to the remainder does not exceed*

$$Ch^{1-d}\int_{(X_{\mu^{-1}h}\bigcap Y_{h^{1-\delta}})\times\mathbb{R}^d\cap\Sigma_\tau} T(x,\xi)^{-1}dx\,d\xi : da + Ch^{1-d}\mu^{\delta''}; \tag{7.5.41}$$

(ii) In particular, it does not exceed $Ch^{1-d}\log|\log h|$.

(iii) Under condition (7.5.39) *it does not exceed $C_\mu h^{1-d}$ with $C_\mu = o(1)$ as $\mu\to+0$.*

Boundary Strip

Simple rescaling $x\mapsto x\gamma^{-1}$ with $\gamma = h$ implies

Proposition 7.5.8. *Under extra condition*

$$X\cap B(0,1) = \{x_1 > \phi(x')\} \qquad \text{with} \qquad |D^\beta\phi|\le ch^{1-|\alpha|} \qquad \forall \alpha : 1\le|\beta|\le K \tag{7.5.42}$$

contributions of zone Y_β with $\beta\ge h$ to the main part of the asymptotics and to the remainder do not exceed $Ch^{-d}\beta$.

Currently we do not have good tools to estimate contribution of zone Y_β as $\beta\asymp h$ and ∂X is very irregular. So, instead of inventing a special proof (which is not very difficult) we refer to the results of Chapter 9 which imply

Remark 7.5.9. (i) With the Dirichlet boundary conditions contributions of zone Y_β with $\beta\ge h$ to the main part of the asymptotics and to the remainder do not exceed $Ch^{-d}\operatorname{vol} Y_\beta$; in particular, under condition (7.5.28) these contributions do not exceed $Ch^{-d}\beta$.

(ii) We will discuss other boundary conditions as well.

Synthesis

Thus after summation of contribution of these four zones ($\mathcal{X}', \mathcal{X}'', \mathcal{X}'''$ and Y_h) and adding the contribution of the interior zone X_μ with (arbitrarily) small constant $\mu > 0$ which is $O(h^{1-d})$ due to results of Section 4.6, we arrive to

Theorem 7.5.10. *Let A be a scalar h-differential operator of* (4.6.5) *type with coefficients belonging to $\mathscr{C}^{l}_{\nu}(\bar{X})$ with $l = 1$ and $\nu(t) = t|\log t|^{-1}$ (see Definition 4.6.1).*

Assume that the principal symbol of A is elliptic and semi-bounded from below and the Dirichlet boundary conditions are imposed on ∂X. Further, assume that condition (7.5.28) *is fulfilled. Let X be a bounded domain. Then under ξ-microhyperbolicity condition on level τ*

(i) The following remainder estimate holds:

$$|\mathsf{N}(\tau) - \mathsf{N}^{\mathsf{W}}(\tau)| \le Ch^{1-d} \int_{(\mathcal{X}'\cup\mathcal{X}''')\times\mathbb{R}^d\cap\Sigma_\tau} T(x,\xi)^{-1} dx\, d\xi : da + Ch^{1-d} \tag{7.5.43}$$

with $\mathcal{X}'$ and $\mathcal{X}'''$ defined by (7.5.36) *and* (7.5.40) *respectively and $T(x,\xi)$ defined by* (7.5.31), (7.5.23) *with ζ given by* (7.5.38) *(namely, $\zeta(x) = \gamma(x)^{2-\delta}$).*

(ii) Moreover, if also condition (7.5.39) *is fulfilled then*

$$|\mathsf{N}(\tau) - \mathsf{N}^{\mathsf{W}}| \le Ch^{1-d}. \tag{7.5.44}$$

Remark 7.5.11. If domain X is unbounded then the statement remains true after we replace A by its regularization with ε given by (7.5.13) and include cut-off with $\psi \in \mathscr{C}^1_0(B(0,\frac{1}{2}))$.

Corollary 7.5.12. *In the framework of Theorem 7.5.10*

(i) Remainder (i.e. the left hand expression in (7.5.43)*) does not exceed $Ch^{1-d}\log|\log h|$.*

(ii) If ∂X is more regular, namely, if

$$\partial X = \bigcup_k \{x_k = \phi_k(x_{\hat{k}})\} \qquad \text{with} \quad \phi_k \in \mathscr{C}^1_{\nu_1} \tag{7.5.45}$$

then the remainder does not exceed

$$Ch^{1-d} + Ch^{1-d} \int_J \nu_1(t)\, \frac{dt}{t^2} dt \tag{7.5.46}$$

with $J = [h, h^{1-\delta}] \cup [h^{\delta}, 1]$*; in particular, it does not exceed* Ch^{1-d} *provided*

$$\int_0^1 \nu_1(t)\,\frac{dt}{t^2} < \infty \tag{7.5.47}$$

which happens f.e. as $\nu_1(t) = t|\log t|^{-1-\delta}$ *with* $\delta > 0$.

(iii) For the Schrödinger operator the remainder does not exceed

$$Ch^{1-d} + Ch^{1-d}\int_J \nu_1(t)^2\,\frac{dt}{t^3}; \tag{7.5.48}$$

in particular, it does not exceed Ch^{1-d} *provided*

$$\int_0^1 \nu_1(t)^2\,\frac{dt}{t^3} < \infty. \tag{7.5.49}$$

(iv) Under assumption (7.5.39)[26] *contribution of zone* $\{x, \mu^{-1} \le \gamma(x) \le \mu\}$ *to the remainder does not exceed* $C_\mu h^{1-d}$ *with* $C_\mu = o(1)$ *as* $\mu \to +0$.

Proof. (i) Proof of Statement (i) is trivial.

(ii) Under condition (7.5.45) one can see easily that $T(x,\xi) \ge \epsilon_0 \tilde{\nu}_1(\gamma(x,\xi))$ and $\tilde{\nu}_1$ is the corresponding inverse function; we will improve these arguments later to cover more general case. Then

$$h^{1-d}\int T^{-1}(x,\xi)\,dxd\xi : da \le Ch^{1-d}\int_J \tilde{\nu}_1(\gamma)^{-1}d\gamma$$

which does not exceed (7.5.46).

(iii) Further, note that for the Schrödinger operator under condition (7.5.45) $T(x,\xi) \ge \epsilon_0 \max\bigl(\tilde{\nu}_1(\gamma), \tilde{\nu}_0(\varrho)\bigr)$ where ϱ is a slope (an angle between trajectory and the boundary), $\nu_0(t) = \nu_1(t)t^{-1}$ and $\tilde{\nu}_0$ is the corresponding inverse function. Then

$$h^{1-d}\int T^{-1}(x,\xi)\,dxd\xi : da \le$$
$$Ch^{1-d}\int_J \frac{1}{t}d_t\,\mathsf{mes}\bigl(\{(x,\xi) : \gamma(x) \le \nu_1(t), \varrho(x,\xi) \le \nu_0(t)\}\bigr) \le$$
$$Ch^{1-d}\int \frac{1}{t}\,d_t\bigl(t^{-1}\nu_1^2(t)\bigr)$$

which does not exceed (7.5.48).

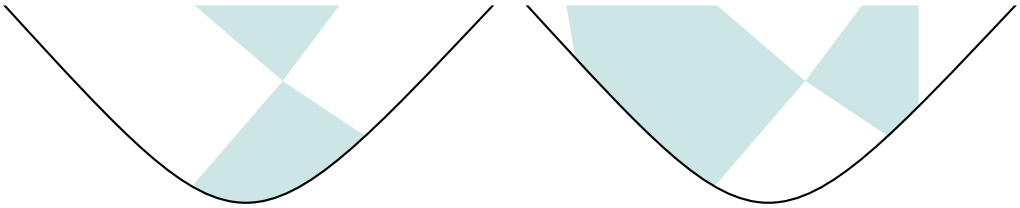
[26] In particular, in the framework either of Statement (ii) and (7.5.47) or Statement (iii) and (7.5.49).

(iv) Finally, Statement (iv) trivially follows from the proofs of Statements (ii) and (iii). □

Remark 7.5.13. If the first derivatives of the coefficients of a are Hölder continuous (i.e. $\nu(t) = t^{1+\delta}$), then one can pick up $\varepsilon = h^{1-\delta}$ as $\gamma > h^{\delta}$ with small exponent $\delta > 0$ and with an approximation error $o(h^{1-d})$; then for $\gamma > h^{\delta}$ we can pick up $\zeta = h^{\delta}$ and then contribution of $\mathcal{X}'$ to the remainder would not exceed $C_{\mu}h^{1-d}$ with $C_{\mu} = o(1)$ as $\mu \to +0$; then in estimate (7.5.43) we can skip $\mathcal{X}'$ and therefore estimates (7.5.46), (7.5.48) hold with $J = [h, h^{1-\delta}]$.

7.5.3 $O(h^{1-d})$ Remainder Estimate for Matrix Operators

Let us consider the general matrix case; we define $\gamma(x)$, η, $\varepsilon(x)$, $\bar{\gamma}_1$ in the same way as before.



(a) (b)

Figure 7.7: Matrix case: massive propagation

We have a major problem now: in the general matrix case some singularities propagate along "massive" propagation cones rather than along "thin" Hamiltonian trajectories (even if Hamiltonian flow is multivalued due to non-smoothness the corresponding cones were thin enough in the previous subsection). And we have issue not with the case of incoming and outgoing cones as on the Subfigure (a), but with what is now instead of parallels rays as on Subfigure (b).

However we will show that arguments of the previous Subsection 7.5.2 remain true because such massive propagation happens in rather small number of points. Heuristically if $\dim \ker a(\bar{x}, \bar{\xi}) = l$ at some point, then one can assume that a is $l \times l$-matrix and two cases are possible: *either*

$\mathsf{dim}\,\mathsf{span}_{j,k}\,\nabla_\xi a_{jk} = 1$ and then $\partial_{\xi_i} a_{jk}(\bar{x},\bar{\xi}) = b_{jk}\ell_i$ for all j,k,i and therefore the x-projection of the propagation direction at $(\bar{x},\bar{\xi})$ is proportional to ℓ and arguments of the previous subsection work *or* $\mathsf{dim}\,\mathsf{span}_{j,k}\,\nabla_\xi a_{jk} \ge 2$ and then in majority of points $(x,\xi)\in\Sigma$ in the vicinity of $(\bar{x},\bar{\xi})$ $\mathsf{dim}\,\mathsf{ker}\,a(x,\xi) < l$; let us recall that $\Sigma = \{(x,\xi) :\ \mathsf{ker}\,a(x,\xi)\neq\{0\}\}$. Let us formalize this idea.

As in the previous subsection we should consider zones $\mathcal{X}'\cup\mathcal{X}''$ defined by (7.5.36), (7.5.35) and $\mathcal{X}'''$ defined by (7.5.40) separately; boundary strip Y_β does not require any modification in comparison with the scalar case.

Let us introduce small parameter

(7.5.50) $\eta=\gamma$ and $\eta = h\gamma^{-1}$ in $\mathcal{X}'\cup\mathcal{X}''$ and $\mathcal{X}'''$ respectively.

Covering

Let $(\bar{x},\bar{\xi})\in\Omega\subset\bigl(\mathcal{X}'\cup\mathcal{X}''\cup\mathcal{X}'''\bigr)\times\mathbb{R}^d$. As in the scalar case we want to analyze our operator in $\gamma\eta^{-\delta}$-vicinity of $(\bar{x},\bar{\xi})$ because for time t with $|t|\le c_0^{-1}\gamma\eta^{-\delta}$ we do not expect singularities to leave it. However, with respect to ξ we will take ρ-vicinity with $\rho=\eta^\delta$ for a sake of uncertainty principle.

More precisely, as before we assume that a is ξ_1-microhyperbolic in the whole domain in question and we consider ρ-vicinity with respect to ξ' and thus use operators $Q_j = Q_j(x,hD')$.

Further, without any loss of the generality one can assume that

$$\|a(\bar{x},\bar{\xi})\|\le\epsilon\eta^{\delta_1} \tag{7.5.51}$$

with some small exponent $\delta_1>0$.

Indeed, otherwise in η^{δ_1}-vicinity of $(\bar{x},\bar{\xi})$ we can transform our operator A [27] into the direct sum of an operator, satisfying this condition, and an elliptic operator with the following explanations:

(a) The transformation and resulting operators are h-pseudodifferential operators which is fine as long as we stay away from the boundary. So actually we can take only $B(\bar{x},\gamma)\times B(\bar{\xi}',\epsilon\eta^{\delta_1})$ but then after we consider propagation we can take the similar vicinity of another point and so on as long as propagation keeps away from ∂X in the same way as it was in the scalar case. Then propagation is reduced to propagation of operator A_0-type;

[27] We use the same notation for operator with mollified coefficients.

(b) Actually the resulting operator is block-diagonal $\mathsf{diag}(A_0; A_1, \dots, A_k)$ where A_0 satisfies (7.5.51) and $\varrho_j^{-1} A_j$ are regular elliptic operators with $\nu^{\delta_1} < \varrho_1 < \dots < \varrho_k$.

After rescaling $x \mapsto x\eta^{-\delta_1}$, $\xi \mapsto \xi\eta^{-\delta_1}$, $t \mapsto t\eta^{-\delta_1}$, $h \mapsto h\eta^{-2\delta_1}$ this operator would satisfy the proper regularity conditions.

Note that due to assumption of ξ_1-microhyperbolicity $b := a_{\xi_1}(\bar{x}, \bar{\xi})$ is a positive matrix. Then for $\xi \in B(\bar{\xi}, \rho)$

$$a'(\xi) := b^{-\frac{1}{2}} a(\bar{x}, \xi) b^{-\frac{1}{2}} = (\xi_1 - \bar{\xi}_1) + a''(\xi' - \bar{\xi}') + O(\rho^2) \qquad (7.5.52)$$

$$\text{with} \quad a''(\zeta') = \sum_{2 \le k \le d} a'_{\xi_k}(\bar{x}, \bar{\xi}) \zeta_k,$$

$$a'_{\xi_k}(\bar{x}, \xi) = a_{\xi_k}(\bar{x}, \xi) + O(\rho). \qquad (7.5.53)$$

Now let us consider for $|\zeta'| = 1$ the top $\lambda_{\mathsf{max}}(\zeta')$ and the bottom $\lambda_{\mathsf{min}}(\zeta')$ eigenvalues of the matrix $a''(\zeta')$. Two cases are possible

$$\forall \zeta' : |\zeta'| = 1 \qquad \lambda_{\mathsf{max}}(\zeta') - \lambda_{\mathsf{min}}(\zeta') \le \eta^{\delta_2}, \qquad (7.5.54)$$

$$\exists \zeta' : |\zeta'| = 1 \qquad \lambda_{\mathsf{max}}(\zeta') - \lambda_{\mathsf{min}}(\zeta') \ge \eta^{\delta_2} \qquad (7.5.55)$$

with some exponent $\delta_2 \in (0, \frac{1}{8}\delta_1)$.

One can prove easily that in the former case $a''(\xi') = \sum_k \lambda_k \xi_k + O(\eta^{\delta_2})$ for all $\xi' : |\xi'| = 1$ and therefore

$$a'(\xi) = \sum_{1 \le k \le d} \lambda_k \xi_k + O(\rho\eta^{\delta_2}) \qquad (7.5.56)$$

and

$$a'_{\xi_k}(\xi) = \lambda_k + O(\eta^{\delta_2}) \qquad \forall \xi \in B(\bar{\xi}, \rho) \qquad (7.5.57)$$

with $\lambda_1 = 1$; for our needs it will be effectively scalar operator with the principal symbol $g = \mathsf{tr}(a_0)$ (alternatively with $g = \sum_k \lambda_k(\xi_k - \bar{\xi}_k)$).

In the latter case (7.5.55) one can find the covering of the sphere $\mathbb{S}^{d-2}(0,1)$ by the union of balls $\bigcup_{j \in \mathcal{I}' \cup \mathcal{I}''} B(\zeta'_j, \eta^{\delta_3})$ such that

$$\lambda_{\mathsf{max}}(\zeta') - \lambda_{\mathsf{min}}(\zeta') \ge \eta^{2\delta_2} \qquad \forall \xi' \in B(\zeta'_j, \eta^{\delta_3}), \quad j \in \mathcal{I}',$$

$$\mathsf{mes}'\Big(\bigcup_{j \in I''} B(\zeta'_j, \eta^{\delta_3}) \cap \mathbb{S}^{d-2}(0,1) \Big) \le \eta^{\delta_4}$$

where mes' is a standard measure on $\mathbb{S}^{d-2}(0,1)$. Here $\delta_3>0$ and $\delta_4>0$ are small exponents.

Then we will be able to cover $B(\bar\xi', \eta^{\delta_1-3\delta_2})$ by by the union of balls $\bigcup_{j\in\mathcal{J}'\cup\mathcal{J}''} B(\xi'_j, \eta^{\delta_5})$ such that

$$\lambda_{\mathsf{max}}(\xi') - \lambda_{\mathsf{min}}(\xi') \ge \eta^{\delta_1-\delta_6} \qquad \forall \xi' \in B(\xi'_j, 2\eta^{\delta_5}), \quad j\in\mathcal{J}', \tag{7.5.58}$$

$$\mathsf{mes}\Big(\bigcup_{j\in\mathcal{J}''} B(\xi'_j,\eta^{\delta_5})\Big) \le \eta^{\delta_6}\,\mathsf{mes}\, B(\bar\xi, \eta^{\delta_1-3\delta_2}). \tag{7.5.59}$$

Thus, there will be "not very many" balls of the second type (with $j\in\mathcal{J}''$) and in each ball of the first type (with $j\in\mathcal{J}'$) one can reduce our operator to direct sum of two operators of lesser (matrix) dimension. Continuing this process with the balls of the first type we arrive to

Proposition 7.5.14. *One can cover $B(\bar\xi,\rho)$ with $\rho=\eta^\delta$ by the union of balls $\bigcup_{j\in\mathcal{J}'\cup\mathcal{J}''} B(\xi_j,\rho_1)$ with $\rho_1=\rho\eta^{\delta'}$ such that*

(i) For each $\bar x$ and each $j\in\mathcal{J}'$ in the domain

$$\big(B(\bar x, 2\gamma_1)\cap\{\frac{1}{2}\gamma\eta^\delta \le \mathsf{dist}(x,\partial X)\le 2\gamma\eta^{-\delta}\}\big)\times B(\xi_j, 2\rho_1)$$

with $\gamma_1=\gamma\eta^{-\delta}$, $j\in\mathcal{J}'$ and ξ_j, $\mathcal{J}'$ depending on $\bar x$ one can transform operator A into block-diagonal matrix operator $\mathsf{diag}(A_1,\dots,A_p)$ with $p=p(\bar x, j)$ with symbols a_l of blocks such that

$$|\partial_x^\beta\partial_\xi^\alpha a_l(x,\xi)| \le N^{|\alpha|+|\beta|-1}\eta^{-2(|\alpha|+|\beta|-1)\delta'}\varepsilon^{-|\beta|} \qquad \forall \alpha,\beta: 1\le|\alpha|+|\beta|\le N,$$

and the same estimates hold for a transforming matrix; furthermore, for each block

$$|\partial_{\xi_k} a_l - \lambda_{l,k}(j) b_l| \le C\eta^{\delta''}$$

with some numbers $\lambda_{l,k}(j)$, $\lambda_{l,1}=1$, and some exponent $\delta''>0$.

(ii) prop-7-5-14-ii The total $(d-1)$-dimensional measure of $\pi\big(\bigcup_{j\in\mathcal{J}''} B(\xi_j,\rho')\big)$ does not exceed $C\rho^{d-1}\eta^{\delta''}$ where π denotes projection $(\xi_1,\xi')\to\xi'$.

One can choose δ, δ' and δ'' to be arbitrarily small.

This proposition allows us to estimate properly contribution to the remainder of the domains of each type.

Estimates

Indeed, in each domain

$$\bigl(B(\bar{x},\gamma_1)\cap\{\gamma\eta^{\delta}\le \mathsf{dist}(x,\partial X)\le \gamma\eta^{-\delta}\}\bigr)\times B(\xi_j,\rho_1)$$

of the first type (with $j\in\mathcal{J}'$) we can apply arguments of Subsection 7.5.2 and get the same estimate of their contribution to the remainder. Namely, the total contribution of these domains to the remainder does not exceed

$$Ch^{1-d}\int_{(\mathcal{X}'\bigcup\mathcal{X}''')\times\mathbb{R}^d\bigcap\Sigma_0\bigcap\Xi'} T(x,\xi)^{-1}dx\,d\xi : dg + Ch^{1-d} \tag{7.5.60}$$

where $T(x,\xi)$ is defined by Hamiltonians flows with appropriate scalar Hamiltonians[28)].

Further, applying arguments of Remark 7.5.13 we can skip $\mathcal{X}'$ in (7.5.60) if the first derivatives of coefficients are Hölder continuous.

On the other hand, the contribution to the remainder of the union Ξ'' of the boxes of the second type ($j\in\mathcal{J}''$) does not exceed $Ch^{1-d}\int \eta(x)^{\delta''}\gamma(x)^{-1}dx$ which in turn does not exceed Ch^{1-d} due to condition (7.5.28) and definition (7.5.50).

Furthermore, the contributions to the remainder of X_1 and Y_h do not exceed Ch^{1-d}. Thus we had proven

Theorem 7.5.15. *Let A be a h-differential operator of* (4.6.5) *type with coefficients belonging to $\mathscr{C}^l_\eta(\bar{X})$ with $l=1$ and $\eta(t)=t|\log t|^{-1}$ (see Definition 4.6.1) and assume that the principal symbol of A is elliptic and semi-bounded from below and the Dirichlet boundary conditions are imposed on ∂Y. Assume that* (7.5.28) *is fulfilled. Let X be a bounded domain. Then under ξ-microhyperbolicity condition on level τ*

(i) The remainder for $\tau=0$ does not exceed (7.5.60).

(ii) Further, under condition (7.5.39) *the remainder is $O(h^{1-d})$.*

(iii) Furthermore, Remark 7.5.13 also remains true.

Similarly, one can prove

[28)] One can pick up these Hamiltonians in different ways leading to the same result: as affine with respect to ξ functions or as $g(x,\xi)$ where $\{g(x,\xi)=0\}$ coincides in these boxes with Σ_0 and $c^{-1}\le|\nabla g|\le c$.

Corollary 7.5.16. *(i) Statements (i), (ii) and (iv) of Corollary 7.5.12 remain true in the matrix case.*

(ii) Statement (iii) of Corollary 7.5.12 also remains true in the matrix case for operators such that near boundary $\det a(x,\xi) = \prod_{1\le k\le m} q_k(x,\xi)$ *with elliptic quadratic forms (with respect to* ξ*)* $q_k(x,\xi)$.

In the proof of Statement (ii) one should apply the same arguments as in the proof of Proposition 7.5.14: either $\{(x,\xi') : \exists \xi_1 : q_k(x,\xi) = q_l(x,\xi) = 0\}$ is a "thin set" or q_k and q_l do not differ much and one can ignore q_l in the propagation arguments.

7.5.4 Sharp Remainder Estimate

If we want to derive a sharp remainder estimate $o(h^{1-d})$ we need first to improve handling of two zones: *boundary strip* $\{x\colon \gamma(x) \le \mu^{-1}h\}$ and *inner zone* $\{x\colon \gamma(x) \ge \mu\}$ where μ should be an arbitrarily small constant; we already proved that under appropriate conditions (ensuring remainder estimate $O(h^{1-d})$) the contribution of the *intermediate strip* $\{x : \mu^{-1}h \le \gamma(x) \le \mu\}$ to the remainder does not exceed $(C_\mu + o(1))h^{1-d}$ with $C_\mu = o(1)$ as $\mu \to +0$.

Near Boundary Zone

The better handling of the boundary strip $\{x : \gamma(x) \le \mu^{-1}h\}$ does not require assumptions of the global nature but slightly improved smoothness condition (7.5.28). The reason is that in our assumptions[29] the contribution of this zone to N does not exceed $C\mu h^{1-d}$. We will assume that the boundary is rather regular with the exception of the thinner set:

(7.5.61) There exists a closed set $Z \subset \partial X$ such that

$$\mathsf{vol}(\mathcal{X}_{(\beta)} \cap Z_\alpha) \le f_\alpha \beta \qquad \text{with} \quad f_\alpha = o(1) \quad \text{as} \quad \alpha \to +0 \tag{7.5.62}$$

where $Z_\alpha = \{x, \mathsf{dist}(x,Z) < \alpha\}$ and for each $\alpha > 0$

$$\partial X \setminus Z_\alpha = \bigcup_k \{x_k = \phi_k(x_{\hat{k}})\} \qquad \text{with} \quad \phi_k \in \mathscr{C}^1.$$

[29] Including Dirichlet boundary conditions.

Let us apply ε_1-mollification of ∂X. Then there exist two domains $X^{\pm}(\varepsilon_1)$ with mollified boundaries, such that

$$X^{+}(\varepsilon_1) \supset X \supset X^{-}(\varepsilon_1) \tag{7.5.63}$$

and

$$\mathsf{vol}\big(X^{+}(\varepsilon_1) \setminus X^{-}(\varepsilon_1)\big) = o(\varepsilon_1) \ \text{ as } \ \varepsilon_1 \to 0. \tag{7.5.64}$$

Then we can select $\varepsilon_1 = \varepsilon_1(h)$ such that $h\varepsilon_1^{-1} = o(1)$ and the approximation error in N^{W} due to the mollification of the boundary is $o(h^{1-d})$.

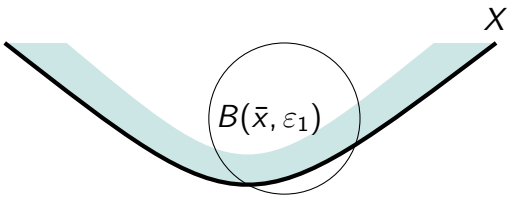


Figure 7.8: Boundary strip; $\varepsilon_1 \gg \mu^{-1}h$

Let us consider ball $B(x, \varepsilon_1)$ with $x \in Y_{\mu^{-1}h}$ and rescaling it to $B(0,1)$ we have that the problem is uniformly smooth and microhyperbolic in an appropriate multidirection provided condition (7.5.65) below is fulfilled and we can apply smooth theory to calculate contribution of $Y_{\mu^{-1}h} \cap B(x, \varepsilon_1)$; namely, its contribution to the remainder does not exceed $o(1)(h\varepsilon_1^{-1})^{1-d}$. We assume that

(7.5.65) Boundary value problem is microhyperbolic on energy level τ in an appropriate multidirection $(\ell_\xi, 0, \nu_1, \dots, \nu_M)$ at each point.

taking sum over all the balls covering $Y_{\mu^{-1}h} \setminus Z_\alpha$ we get a total estimate

$$o(1)(h\varepsilon_1^{-1})^{1-d} \times \varepsilon_1^{1-d} = o(h^{1-d})$$

because due to (7.5.28) we need to consider $M(\varepsilon_1)$ of such balls with $M(\varepsilon_1) = C\,\mathsf{mes}(\{x\colon \gamma(x) \le \varepsilon_1\})\varepsilon_1^{-d} = O(\varepsilon_1^{1-d})$.

On the other hand, the total contribution of all the balls with non-empty intersection with Z_α does not exceed $Cf_\alpha h^{1-d}$. So, we have proven

Proposition 7.5.17. *Under additional conditions* (7.5.61)–(7.5.62), (7.5.65) *contribution of the strip $Y_{\mu^{-1}h}$ to the principal part of asymptotics is*

$$\int_{Y_{\mu^{-1}h}} \mathsf{N}^{\mathsf{W}}(x)\psi(x)dx + \kappa' h^{1-d} \tag{7.5.66}$$

where $\psi(x) = 1$ in $Y_{\mu^{-1}h}$ and supported in $Y_{2\mu^{-1}h}$ and its contribution to the remainder is $o(h^{1-d})$ for each small fixed $\mu > 0$.

Remark 7.5.18. (i) With more general than Dirichlet boundary condition we do not have a monotone dependence of N on the domain and thus we cannot approximate ∂X at all; this leaves us with no other option but to flatten the boundary by changing a coordinate $x_{k,\mathsf{new}} = x_k - \phi_k(x_{\hat{k}})$. This looks rather unappealing since then the principal symbol includes the first derivatives of ϕ_k but it is not that bad since (in contrast to Remark 7.5.1) we do it only in a very marrow strip and here we we can take ε_2-approximation of the coefficients and since first derivatives of ϕ_k are uniformly bounded an approximation error does not exceed $o(\mu^{-1}h^{1-d})$ as $\varepsilon_2 \to +0$.

(ii) Further, since the only difference is that between $O(h^{1-d})$ and $o(h^{1-d})$ and we rescale $x \mapsto x\mu h^{-1}$ one can very easily weaken condition (7.5.65) to

(7.5.67) Boundary value problem is (non-uniformly) microhyperbolic in an appropriate multidirection $(\ell_\xi, 0, \nu_1, \dots, \nu_M)$ on the energy level τ at each point $(x', \xi) \in T^*Y_k \setminus \mathcal{Z}_k$ and $\mathsf{mes}_{T^*Y_k}(\mathcal{Z}_k) = 0$, where $Y_k = \{x_k = \phi_k(x_{\hat{k}})$ and $\mathsf{mes}_{T^*Y_k}$ is a natural measure there.

Inner Zone

To get remainder estimate $o(h^{1-d})$ we need also to improve the remainder estimate for $\int \psi(x)e(x,x,\tau)dx$ with ψ supported in X_μ with any fixed $\mu > 0$. This involves the analysis of the propagation of singularities starting from X_μ.

To do this we need to assume in condition (7.5.61)–(7.5.62) that $\phi_k \in \mathscr{C}^1_{\nu_1}$ with $\nu_1(t) = o(t|\log t|^{-1})$. Then taking ε_1-mollification of the boundary we get approximation error in N^{W} as $o(h^{1-d})$ provided $\varepsilon_1(h|\log h|)^{-1} \to +\infty$ slowly enough as $h \to +0$.

Theorem 7.5.19. *Let A be a Schrödinger operator with $\mathscr{C}^1_\nu$ coefficients, $\nu(t) = o\big(t|\log t|^{-1}\big)$. Let condition* (7.5.39) *be fulfilled and conditions* (7.5.61)–(7.5.62) *hold with $\phi_k \in \mathscr{C}^1_{\nu_1}$, $\nu_1(t) = o(t|\log t|^{-1})$. Let us consider mollified domain and mollified operator.*

Further, let $(\bar{x}, \bar{\xi}) \in T^*\overset{\circ}{X} := X \setminus \partial X$. Let us consider $\Psi_t(\bar{x}, \bar{\xi})$, the (multivalued) Hamiltonian flow with reflections from $\partial X = \{x_k = \phi_k(x_{\hat{k}})\}$ (where k depends on position of point near $\partial X \setminus Z$. Assume that

(7.5.68) *For all* $t \in [0, T]$ *and for all* $(x(t), \xi(t)) \in \Psi_t(\bar{x}, \bar{\xi})$

$$x(t) \in \overset{\circ}{X} \cup \partial X \setminus Z\,, \tag{7.5.69}$$

$$|\phi(x(t))| + |\frac{d}{dt}\phi(x(t))| \geq \epsilon. \tag{7.5.70}$$

(i) Furthermore, let $(\bar{x}', \bar{\xi}') \in T^*\overset{\circ}{X} \setminus \Psi_T(\bar{x}, \bar{\xi})$. *Then for pseudodifferential operators* Q_1 *and* Q_2 *with symbols supported in the small vicinities of* $(\bar{x}, \bar{\xi})$ *and* $(\bar{x}', \bar{\xi}')$ *respectively we have that* $Q_2 e^{ih^{-1}tA} Q_1 \equiv 0$ *for* t *in the small vicinity of* T.

(ii) On the other hand, let $\bar{x}' \in \bar{X}$ *and* $\bar{x}' \notin \pi_x \Psi_T(\bar{x}, \bar{\xi})$ *where* π_x *is an* x*-projection. Then for a pseudodifferential operator* Q_1 *with symbol supported in the small vicinity of* $(\bar{x}, \bar{\xi})$ *and function* ψ *supported in the small vicinity of* $\bar{x}'$ *we have that* $\psi e^{ih^{-1}tA} Q_1 \equiv 0$ *for* t *in the small vicinity of* T.

Proof. Note first, that due to Section 2.3 Statement (i) holds in the case of the free propagation

$$\pi_x \Psi_t(\bar{x}, \bar{\xi}) \cap \partial X = \emptyset \qquad \forall t \in [0, t]. \tag{7.5.71}$$

Then it is sufficient to prove theorem assuming that

(7.5.72) $\bar{x}, \bar{x}' \in Y_\beta$ and $T \leq c\beta$ with (arbitrarily) small constant $\beta > 0$ and large enough constant c which does not depend on β.

Really, let us apply induction with respect to maximal number n trajectories from $\Psi_T(\bar{x}, \bar{\xi})$ hit the boundary. Then for each n we employ induction with respect to $m = \lfloor CT/\beta \rfloor$ with large constant C, also β-independent. But then if either $\bar{x} \notin Y_\beta$ or $\bar{x}' \notin Y_\beta$ one can reduce the problem to one with lesser m due to free propagation. Now, if $\bar{x} \in Y_\beta$ and as we assumed along trajectory (7.5.70) holds (with $\epsilon \gg \beta$) and also $\pi_x \Phi_t(\bar{x}, \bar{\xi}) \in \partial X$ for some $t = c\beta$ then we will be able to reduce problem to one with lesser m, n unless $n = 1$ and $T \leq c\beta$.

Under assumption (7.5.72) $\xi(t)$ are concentrated in the ball of radius not exceeding $C\beta$ and $x(t)$ are concentrated in the ball of radius not exceeding $C\beta^2$ for all Hamiltonian trajectories $(x(t), \xi(t))$ issued from $(\bar{x}, \bar{\xi})$ as long as $|t| \leq c\beta$. Then it is sufficient to prove theorem assuming also that $\gamma(\bar{x}) = \beta$ and $\gamma(\bar{x}') = \beta$ in Statement (i).

From now on we will consider β as a small parameter rather than as a constant; $1 \gg \beta \ge h^{1-\delta}$. Further, instead of small constant vicinity of $(\bar{x}, \bar{\xi})$ we will consider (ζ, ρ)-vicinity of it with ζ given by (7.5.30) and $\rho = c\beta^{-1}\zeta$.

Then applying arguments of the proof of Proposition 7.5.4 with $T \asymp \beta$ we can reduce this problem to the problem with β replaced by $\beta' = \zeta$ and correspondingly replaced ρ. We continue this process as long as $\beta \ge h^{1-\delta}$. There will be no more than $L = C \log |\log h|$ steps. On some of these steps ρ decays and on each step we should go to the partition containing however no more than

$$\bigl(1 + \rho_n \rho_{n+1}^{-1}\bigr)^d \bigl(1 + \beta_n \beta_{n+1}^{-1}\bigr)^d$$

elements ending in the end with no more than $L(\rho_0 \rho_n^{-1})^d \le h^{-2d}$ elements but an error on each element is $O(h^s)$ and the total error is $O(h^{s-d})$ with arbitrarily large s.

So, the proof of our theorem is reduced to the case $\beta = h^{1-\delta}$.

However, then $\varepsilon \asymp \beta$ and since we used analytic mollifications of the coefficients we find ourselves in the framework of (uniformly) analytic theory of free propagation after we rescale[30]. In this theory propagation error is $O(e^{-\epsilon \hbar^{-1}})$ where $\hbar = h\beta^{-1}$ is effective semiclassical parameter. Then we can apply the same arguments as above as long as $\beta \ge Ch|\log h|$ and therefore the proof of our theorem is reduced to the case $\beta = Ch|\log h|$.

But now rescaling we find that the boundary is (uniformly) analytic as well and we just apply analytic theory of the propagation of singularities near the boundary when rays are transversal, see f.e. S. Wakabayashi [1], J. Sjostrand [1,3] to the boundary.

So our theorem has been proven. □

Immediately we have

Theorem 7.5.20. *Let A be a Schrödinger operator with $\mathscr{C}^1_\nu$ coefficients, $\nu(t) = o\bigl(t|\log t|^{-1}\bigr)$. Let condition* (7.5.39) *be fulfilled and conditions* (7.5.61)–(7.5.62) *hold with $\phi_k \in \mathscr{C}^1_{\nu_1}$, $\nu_1(t) = o(|\log t|^{-1})$. Let*

$$|a(x,\xi) - \tau| + |\nabla_\xi a(x,\xi)| \ne 0. \tag{7.5.73}$$

Let us assume that

[30] Actually because mollification scale is near boundary so close to 1 we find ourselves in the framework of small analytic perturbation of the constant coefficient operator.

(7.5.74) $dxd\xi : da$*-measure of the set of "bad" points* $z \in T^*X$ *is 0; we call* z *a "bad" point if for some* t *either* $z \in \Psi_t(z)$ *(periodicity) or* $\Psi_t(z) \cap \bigl(\pi^{-1}(Z) \cup \bigcup_k \iota_k^{-1}(\bigcup_k \mathcal{Z})\bigr) \neq \emptyset$ *(hitting bad or non-hyperbolic point on the boundary, here* $\pi : T^*X \ni (x,\xi) \to x$ *and* $\iota_k : T^*X|_{Y_k} \ni (x,\xi) \to (x',\xi') \in T^*Y_k$*) or* $\phi(z') = \frac{d}{dt}\phi(z'(t)) = 0$ *for some* $z' \in \Psi_t(z)$ *(tangency).*

Then the following asymptotics holds:

$$\mathsf{N}(\tau) = \mathsf{N}^{\mathsf{W}} + \kappa_1' h^{1-d} + o(h^{1-d}) \qquad \text{as} \quad h \to 0. \tag{7.5.75}$$

Problem 7.5.21. We leave to the reader operators more general than the Schrödinger operator.

7.5.5 Less Sharp Remainder Estimates

There are two sources of abnormally large remainder estimate

(i) Irregularity of the coefficients. Let us assume that either $\nu(t) \geq t|\log t|^{-1}$ i.e. either $\nu(t) \leq t$ and coefficients belong to $\mathscr{C}^1_\nu$ or $\nu(t) \geq t$ and coefficients belong to $\mathscr{C}^0_\nu$. In this case even if there is no boundary the remainder estimate is $R_1 = Ch^{-d}\nu\bigl(h|\log \nu(h)|\bigr)$ which is $\asymp h^{-d}\nu(h|\log h|)$ provided $\nu(h) \leq h^\delta$ with $\delta > 0$ (see Theorem 4.6.16).

(ii) Irregularity of the boundary which leads to the remainder estimate at least $R_2 = Ch^{-d}\nu(h)$ with $\nu(t) = \mathsf{mes}(Y_t)$ even for a constant coefficients.

Our goal is to recover in the general case the remainder estimate $R_1 + R_2$ or close to it. We will assume that

(7.5.76) $\nu(t)t^{-1}$ is monotonically decreasing function of t.

Then using constant scale ε-approximation with $\varepsilon \geq \eta = Ch|\log h|$ we get that the total approximation error is $h^{-d}\vartheta(\varepsilon)$ and the contribution of $B\bigl(x,\gamma(x)\bigr)$ to the semiclassical error after rescaling $x \mapsto x\varrho^{-1}$,

$$\varrho = \mathsf{min}\bigl(\gamma(x), \frac{\varepsilon}{\nu(\varepsilon)}\bigr) \tag{7.5.77}$$

(see Subsection 4.6.5 for analysis inside domain) does not exceed

$$Ch^{1-d}\gamma^d\bigl(\varepsilon^{-1}\nu(\varepsilon) + \gamma^{-1}\bigr) \tag{7.5.78}$$

and then the contribution of X_h to the semiclassical error does not exceed

$$Ch^{1-d}\varepsilon^{-1}\nu(\varepsilon) + Ch^{1-d}\int_h^1 t^{-1}d_t\nu(t) \tag{7.5.79}$$

while the total contribution of the strip Y_h does not exceed $R_2 = Ch^{-d}\nu(h)$ according to Chapter 9.

The first term in (7.5.79) is less than R_1 as $\varepsilon = \eta$ and only the second term could be a problem. However, if the boundary is quite irregular, namely $\nu(t) \ge t^\sigma$ with $\sigma < 1$, then

$$\int_s^1 t^{-1}d_t\nu(t) \le Cs^{-1}\nu(s)$$

and then this term does not exceed R_2. Therefore we have proven

Theorem 7.5.22. *(i) Remainder does not exceed $R_1 + R_2$ unless*

$$\nu(t) \coloneqq \mathsf{mes}(Y_t) \le C_\delta t^{1-\delta} \tag{7.5.80}$$

with some exponent $\delta > 0$.

(ii) If (7.5.80) *holds, then the remainder does not exceed $R_1 + R_2 + R_2'$ where R_2' is the second term in* (7.5.79)*.*

Therefore we need to reconsider only case when (7.5.80) holds and we need to assume that

(7.5.81) ν satisfies condition (7.5.80) as well: $\nu(t) \le Ct^{1-\delta}$;

otherwise R_1 will be larger than R_2' and then we pick up $\varepsilon = \eta$ again.

We will apply then our more sophisticated variable scale ε-approximation with $\varepsilon = \mathsf{min}\bigl(\eta\gamma^{-\sigma}, \hat{\gamma}\bigr)$ as in Subsections 7.5.1–7.5.3. In this case approximation error does not exceed

$$Ch^{-d}\int \nu(\eta_0 t^{-\sigma})\, d_t\nu(t) \le Ch^{-d}\int \nu(\eta)t^{-\delta}\, d_t\nu(t) = R_1\int t^{-\sigma'}\, d_t\nu(t)$$

provided

(7.5.82) $\nu(\varepsilon)\varepsilon^{-1-\sigma_1}$ is decreasing function of ε

and $\sigma' = \sigma(1+\sigma_1)$. Assuming that exponents σ and σ_1 are small enough, we see that this integral does not exceed

$$C\nu(s)s^{-\sigma'} + C\int \nu(t)t^{-\sigma'-1}dt \le C$$

and so an approximation error does not exceed R_1.

To estimate a semiclassical error we apply the same arguments as in the proof of Proposition 7.5.6 and find that the contribution of the strip $\mathcal{X}'_{(\beta)} := \{x : \beta \le \gamma(x) \le \beta^{1-\delta'}\}$ to the semiclassical error does not exceed

$$Ch^{1-d}\int_{\mathcal{X}'_{(\beta)}} \gamma^{\delta-1}\,dx + Ch^{1-d}\int_{\mathcal{X}'_{(\beta)}\times\mathbb{R}^d \bigcap \Sigma_\tau} d\mu_\tau$$

with the standard measure μ_τ on Σ_τ. One can see easily that the second integral does not exceed $d\nu(\beta)/d\beta$ and then the total semiclassical error does not exceed

$$Ch^{1-d}\int_{X_h} \gamma^{\delta-1}dx + Ch^{1-d}\sum_{k:0\le k\le C\log|\log h|} \frac{d\nu(\beta)}{d\beta}\Big|_{\beta=e^{-\epsilon e^{\epsilon k}}}. \tag{7.5.83}$$

Replacing this sum by an integral and changing variable in the integral we arrive to

$$Ch^{1-d}\int_h^1 \frac{1}{t|\log t|}\,d_t\nu(t)$$

which is larger than the first term in (7.5.83). Therefore we proved

Theorem 7.5.23. *Let ν, ν satisfy* (7.5.49)*. Then the remainder does not exceed $R_1 + R_2 + R_2''$ with*

$$R_2'' = Ch^{1-d}\int_{X_h} \gamma^{-1}|\log\gamma|^{-1}\,dx. \tag{7.5.84}$$

7.A Appendices

7.A.1 Sets Associated with the Microhyperbolicity Condition

In this section we study the structure of certain sets associated with the microhyperbolicity condition; namely let

$$\sigma(x,\xi) = \mathsf{Spec}\big(J^{-1}(x)a(x,\xi)\big) \tag{7.A.1}$$

be the set of all the eigenvalues of $J^{-1}(x)a(x,\xi)$,

$$\Sigma_\tau = \{(x,\xi) \in T^*\bar{X}, \tau \in \sigma(x,\xi)\}, \tag{7.A.2}$$

$$\Sigma'_{\tau b} = \iota\Sigma_\tau \tag{7.A.3}$$

where $\iota : T^*\bar{X}|_Y \to T^*Y$ is a natural map. Let

$$\Xi_0 = \overline{\bigcup_{T^*\bar{X}} \sigma(x,\xi)}. \tag{7.A.4}$$

Further, let $(x',\xi') \in T^*Y$ and let $\sigma_d(x',\xi')$ be the discrete spectrum of the problem

$$(a(x',D_1,\xi') - \tau J(x'))v = 0,) \tag{7.A.5}$$

$$\mathfrak{d}b(x',D_1,\xi')v = 0, \tag{7.A.6}$$

$$v = o(1) \qquad \text{as } x_1 \to +\infty; \tag{7.A.7}$$

and let

$$\Sigma''_{\tau b} = \{(x',\xi') \in T^*Y, \tau \in \sigma_d(x',\xi')\}, \qquad \Xi_1 = \overline{\bigcup_{T^*Y} \sigma_d(x,\xi)} \tag{7.A.8}$$

where we skip $x_1 = 0$ in the notation for a and J. Let $\Sigma_{\tau b} = \Sigma'_{\tau b} \cup \Sigma''_{\tau b}$.

Moreover, let us introduce the sets

(7.A.9) $\mathcal{Y}_0 = \big\{\tau \in \mathbb{R}, \Sigma_\tau$ contains at least one point at which the symbol $\tau J(x) - a(x,\xi)$ is microhyperbolic in no direction $\ell \in T\big(T^*\bar{X}\big)\big\}$,

and

(7.A.10) $\mathcal{Y}_1 = \big\{\tau \in \mathbb{R}$ and $\Sigma_{\tau b}$ contains at least one point at which the problem $\big(\tau J(x') - a(x',D_1,\xi'), \mathfrak{d}b(x',D_1,\xi')\big)$ is microhyperbolic in no multidirection$\big\}$.

Finally, let

(7.A.11) $\mathcal{Z}_0$ be the set of the points $(x,\tau) \in \overset{\circ}{X} \times \mathbb{R}$ such that for some $\xi \in T^*_x\overset{\circ}{X}$ the symbol $\tau J(x) - a(x,\xi)$ is microhyperbolic at (x,ξ) in no direction ℓ with $\ell_x = 0$;

and

(7.A.12) $\mathcal{Z}_1$ be the set of points $(x',\tau)\in Y\times\mathbb{R}$ such that for some $\xi'\in T^*_{x'}Y$ the problem $\bigl(\tau J(x')-a(x',D_1,\xi'),\eth b(x',D_1,\xi')\bigr)$ is microhyperbolic at (x',ξ') in no multidirection $\mathcal{T}=(\ell',\nu_1,\dots,\nu_M)$ with $\ell'_x=0$.

We assume that

(7.A.13) $J(x)$ and $a(x,\xi)$ are Hermitian (matrix-valued) symbols and $J(x)$ is non-degenerate,

(7.A.14) Either $J(x)$ or $a(x,\xi)$ is positive definite,

(7.A.15) Operator $a(x',D_1,\xi')$ with the boundary condition $\eth b(x',D_1,\xi')v=0$ is self-adjoint on $\mathscr{L}^2(\mathbb{R}^+,\mathbb{H})$,

(7.A.16) Either $J(x')$ or operator $a(x',D_1,\xi')$ (with the indicated boundary condition) is positive definite,

and

(7.A.17) The coefficient of D_1^m in $a(x',D_1,\xi')$ is non-degenerate.

Here and in what follows $\overset{\circ}{X}$, Y and $\bar{X}$ mean the the previous domain, its boundary and its closure respectively intersected with $\overline{B(0,1)}$.

Obviously,

(7.A.18) Ξ_0 and Ξ_1 are closed sets.

Let us recall that

(7.A.19) If a symbol is microhyperbolic at some point in some direction then it is microhyperbolic in a neighborhood of this point in all the directions close to original one.

Similarly

(7.A.20) If the problem is microhyperbolic at some point in some multidirection then it is microhyperbolic in a neighborhood of this point in all multidirections close to original one (see Proposition 3.1.5).

Hence

(7.A.21) $\mathcal{Y}_0$, $\mathcal{Y}_1$ are closed subsets of $\mathbb{R}$, $\mathcal{Z}_0$ is a closed subset of $\overset{\circ}{X} \times \mathbb{R}$, and $\mathcal{Z}_0 \cap (Y \times \mathbb{R})$, $\mathcal{Z}_1$ are closed subsets of $Y \times \mathbb{R}$; obviously $\mathcal{Z}_0 \cap (Y \times \mathbb{R}) \subset \mathcal{Z}_1$.

Our main goal is

Theorem 7.A.1. *Let conditions* (7.A.13)–(7.A.17) *be fulfilled and let* $K = K(d, m, \mathsf{D})$ *in the smoothness conditions where* D *is the dimension of the matrices and* m *is an order of symbols with respect to* ξ_1. *Then*

(i) The Lebesgue measures of the sets $\mathcal{Y}_0$ *and* $\mathcal{Y}_1$ *equal* 0.

(ii) For every $x \in \overset{\circ}{X}$, $y \in Y$ *the Lebesgue measure of the set* $\{\tau, (x, \tau) \in \mathcal{Z}_0\}$ *equals* 0.

(iii) The Lebesgue measures of the sets $\mathcal{Z}_0$ *and* $\mathcal{Z}_1$ [31] *equal* 0

(iv) The Lebesgue measure of the set $\Sigma''_{\tau b}$ *(in* T^*Y*) equals* 0 *provided* $\tau \notin \mathcal{Y} = \mathcal{Y}_0 \cup \mathcal{Y}_1$.

Proof. We will prove Statement (i) here; Statement (ii) can be proven similarly.

Obviously $\mathcal{Z}_0$ and $\mathcal{Z}_1$ are measurable sets and hence Statement (iii) follows from Statement (ii). In the proof of Statement (iv) one can assume that $(\partial_{x_2}, \nu_1, \dots, \nu_M)$ is the multidirection of microhyperbolicity; then it follows from the proof below that for every $x'' = (x_3, \dots, x_d)$ and ξ' the Lebesgue measure of the set $\{x_2, (x_2, x'', \xi') \in S''_{\tau b}\}$ equals 0; this fact and the obvious fact that $\Sigma''_{\tau b}$ is measurable imply Statement (iv).

So let us prove that $\mathsf{mes}(\mathcal{Y}_1) = 0$ (the equality $\mathsf{mes}(\mathcal{Y}_0) = 0$ can be proven in the same way but without any preliminary analysis; this equality follows directly from Lemma 7.A.2).

Let $\varepsilon > 0$ and let Ω be an open subset in $T^*Y \times \mathbb{R}$. Let us introduce set $\mathcal{Y}_1(\Omega, \varepsilon)$ as we defined $\mathcal{Y}_1$ but with $(z, \tau) \in \Omega$, $z = (x', \xi')$ and with condition (7.A.7) in the problem (7.A.5)–(7.A.7) replaced by

$$u = O(e^{-\varepsilon x_1}) \quad \text{as } x_1 \to +\infty; \tag{7.A.22}$$

then it is sufficient to prove that

[31] In $\overset{\circ}{X} \times \mathbb{R}$ and $Y \times \mathbb{R}$ respectively.

(7.A.23) Every point (z^*, τ^*) has a neighborhood $\Omega = \Omega(\varepsilon)$ such that $\mathsf{mes}\big(\mathcal{Y}_1(\Omega, \varepsilon)\big) = 0$.

Without any loss of the generality one can assume that the polynomial $g(z, \zeta, \tau)$ has no root with the imaginary part equal to ε for $(z, \tau) \in \Omega$.

Let us reduce problem (7.A.5), (7.A.6) to a first-order problem and let us replace $\underline{u}$ by $\underline{u}_{\mathsf{new}} = Q\underline{u}$ as we did in Subsection 3.1.2; then we obtain problem $(\underline{p}, \eth\underline{b})$ where

$$\underline{p} = \begin{pmatrix} \underline{p}_I & 0 & 0 \\ 0 & 0 & \underline{p}_{II} \\ 0 & \underline{p}^*_{II} & 0 \end{pmatrix}, \quad \underline{b} = (\underline{b}_I, \underline{b}_{II}, \underline{p}_{III}) \tag{7.A.24}$$

and all the roots of the polynomials $\det \underline{p}_I(z, \zeta, \tau)$ and $\det \underline{p}_{II}(z, \zeta, \tau)$ lie in the domains $\{|\operatorname{Im}\zeta| \le \varepsilon\}$, $\{|\operatorname{Im}\zeta| \ge \varepsilon\}$ of $\mathbb{C}$ respectively. Moreover, let $\underline{b}' = (\underline{b}'_I, \underline{b}'_{II}, \underline{b}'_{III})$ be defined by the equality

$$\begin{aligned} \langle \underline{p}(z, D_1, \tau)U, V\rangle_+ - \langle U, \underline{p}(z, D_1, \tau)V\rangle_+ = \\ \langle \underline{b}'(z, \tau)U(0), \underline{b}(z, \tau)V(0)\rangle - \langle \underline{b}(z, \tau)U(0), \underline{b}'(z, \tau)V(0)\rangle; \end{aligned} \tag{7.A.25}$$

then

(7.A.26) $R = \underline{b}'^*_{II}\underline{b}_{II}$ is a Hermitian matrix.

The microhyperbolicity of $\big(\tau J(x') - a(x', D_1, \xi'), \eth b(x', D_1, \xi')\big)$ means exactly that

$$\begin{aligned} \langle (\ell + \nu_k\partial_\zeta)\underline{p}_I(z, \zeta, \tau)V, V\rangle < 0 \\ \forall V \in \operatorname{Ker} \underline{p}_I(z, \zeta_k, \tau) \setminus 0 \quad \forall k = 1, \dots, M \end{aligned} \tag{7.A.27}$$

and

$$\langle \ell R(z, \tau)V, V\rangle \ < 0 \qquad \forall V \in \operatorname{Ker} \underline{b}(z, \tau) \setminus 0. \tag{7.A.28}$$

Moreover,

(7.A.29) Conditions (7.A.27) and (7.A.28) are fulfilled for $\ell' = \partial_\tau$ and $\nu_1 = \dots = \nu_M = 0$.

Let us introduce

$$S(z,\zeta_1,\dots,\zeta_M,\tau)=\begin{pmatrix}\underline{p}_I(z,\zeta_1,\tau) & \dots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & \underline{p}_I(z,\zeta_M,\tau) & 0\\ 0 & \dots & 0 & \underline{b}_{II}(z,\tau)\end{pmatrix}, \tag{7.A.30}$$

$$T(z,\tau)=\begin{pmatrix}I & \dots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & I & 0\\ 0 & \dots & 0 & \underline{b}'^{*}_{II}(z,\tau)\end{pmatrix}; \tag{7.A.31}$$

then the equality $\mathsf{mes}\big(\mathcal{Y}_1(\Omega(\varepsilon),\varepsilon)\big)=0$ follows from Lemma 7.A.2 below.□

Lemma 7.A.2. *Let $S(z,\tau)$ and $T(z,\tau)$ be $\mathscr{C}^K$ matrices of dimensions $N\times D$ and $D\times N$ respectively for $(z,\tau)\in\Omega$ where $\Omega\subset\Omega_x\times\mathbb{R}$ and Ω_x are open domains and $K=K(d,N,D)$. Let us assume that*

(7.A.32) *ST is a Hermitian matrix*

and that for $\ell=\partial_\tau$ the following condition holds

$$\langle \ell(ST(z,\tau)v,v\rangle<0 \qquad \forall v\in \mathsf{Ker}\,S(z,\tau)\setminus 0 \quad \forall (z,\tau)\in\Omega. \tag{7.A.33}$$

Let

(7.A.34) $\mathcal{N}=\{(z,\tau)$: *there exists no vector field $\ell=\ell_z\in T_x\Omega_x$ satisfying condition* (7.A.33)$\}$.

Then

$$\mathsf{mes}\big(\{\tau,\exists z:(z,\tau)\in\mathcal{N}\}\big)=0. \tag{7.A.35}$$

Proof. Let $\mathcal{X}_s=\{(z,\tau):\mathsf{rank}\,S(z,\tau)\le s\}$. Then it is sufficient to prove that

$$\mathsf{mes}\big(\{\tau,\exists z:(z,\tau)\in\mathcal{N}\cap(\mathcal{X}_s\setminus\mathcal{X}_{s-1}\}\big)=0. \tag{7.A.36}$$

Moreover, it is sufficient to treat the case $s=0$. In fact, in the proof of (7.A.36) we can assume without any loss of the generality that $S=\begin{pmatrix}S' & 0\\ 0 & S''\end{pmatrix}$

where S'' is an invertible $s \times s$ matrix. This case can obviously be reduced to the case $s = 0$.

Let $t(z,\tau) = \mathsf{tr}(TS)(z,\tau)$; then condition (7.A.33) with $\ell = \partial_\tau$ yields that

$$t(z,\tau) = 0, \quad \partial_\tau t(z,\tau) > 0 \qquad \text{for } (z,\tau) \in \mathcal{X}_0.$$

Then without any loss of the generality one can assume that $\tau = \lambda(z)$ for $(z,\tau) \in \mathcal{X}_0$ where λ is a real-valued $\mathscr{C}^K$ function.

Let $\alpha_1(z), \dots, \alpha_p(z)$ be elements of matrix $R(z) = (TS)(z,\lambda(z))$ arbitrarily enumerated. Then

$$\mathcal{X}_0 = \{(z,\tau) : \tau = \lambda(z),\ \alpha_1(z) = \dots = \alpha_p(z) = 0\}.$$

Let us note that

(7.A.37) Let $(z,\tau) \in \mathcal{N} \cap \mathcal{X}_0$. Then for arbitrary dz, $d\lambda$ is a linear combination of $d\alpha_1, \dots, d\alpha_p$.

In fact, if that is not the case then there exists a direction $\ell \in T_x\Omega_x$ such that $\ell\lambda < 0$ and $\ell\alpha_1 = \dots = \ell\alpha_p = 0$. Then

$$\ell(TS)(z,\tau) = -\partial_\tau(TS)(z,\tau) \cdot \ell\lambda(z)$$

and hence (7.A.33) holds for this direction because it holds for ∂_τ.

Therefore $\mathcal{N} \cap \mathcal{X}_0$ is contained in the union of sets $\mathcal{N}_\mathcal{K}$ for $\mathcal{K} \subset \{1, \dots, p\}$ where

(7.A.38) $\mathcal{N}_\mathcal{K} = \{\tau = \lambda(z), \alpha_j(z) = 0 \quad \forall j \in \mathcal{K},\ d\alpha_j\ (j \in \mathcal{K})$ are linearly independent and $d\lambda$ is their linear combination$\}$.

Then Sard's theorem yields that

$$\mathsf{mes}\big(\{\tau, \exists z, (z,\tau) \in \mathcal{N}_\mathcal{K}\}\big) = 0$$

and this equality yields (7.A.36) with $s = 0$. □

The following assertion also seems to be of interest:

Proposition 7.A.3. *Let conditions* (7.A.13)–(7.A.17) *be fulfilled. Then*

(i) If $\tau^* \in \partial\Xi_0$ and $z^* \in \Sigma_{\tau^*}$ then the symbol $(\tau^* J - a)$ is microhyperbolic in no direction at (z^*, τ^*).

(ii) If $\tau^* \in (\partial\Xi_1)\backslash\Xi_0$ *and* $z^* \in \Sigma''_{\tau^* b}$ *then the problem* $\big(\tau^* J - a(x', D_1, \xi'), \eth b\big)$ *is microhyperbolic in no multidirection at* (z^*, τ^*).

Proof. In the proof of Statement (i) or (ii) let $\tau_1, \dots, \tau_M$ be continuous eigenvalues of the matrix $J^{-1}(x)a(x,\xi)$ (or those of the problem (7.A.5)–(7.A.7)) depending on $z \in T^*X$ or $z \in T^*Y$ equal to τ^* for $z = z^*$. Let $\Pi = \Pi(z)$ be the corresponding spectral projector in $\mathbb{H}$ or in the weighted space $\mathscr{L}^2(\mathbb{R}^+, \mathbb{H}, Jdx)$ respectively. Then both statements will be proven if we prove Statement (i) under the additional restriction $\tau^* J(z^*) - a(z^*) = 0$.

Conditions (7.A.5)–(7.A.13) yield that τ_j are one time differentiable at z^* in every direction (K. O. Friedrichs [1], e.g.) and hence it is sufficient to prove that if ℓ is a microhyperbolicity direction then $\ell\tau_j(z^*) \neq 0$ and therefore Ξ_0 contains a neighborhood of τ^*. But the equalities

$$\ell(\tau_1 + \dots + \tau_M)(z^*) = \operatorname{tr} \ell(J^{-1}a)(z^*) = -\operatorname{tr} J^{-1}\big(\ell(\tau^* J - a)\big)(z^*)$$

yield that if the matrix $\big(\ell(\tau^* J - a)\big)(z^*)$ is either positive or negative definite then $\ell(\tau_1 + \dots + \tau_M) \neq 0$ provided J is positive definite; if a is positive definite we can permute a and J and use the same arguments. □

Comments

The results of Section 7.1 are trivial enough, but the proofs given here were developed specially for this book.

The construction of Section 7.2 (the method of successive approximations) in its present form was suggested first in V. Ivrii [8, 10] for differential and pseudodifferential operators (in fact in [8] the more complicated case of a domain with boundary was treated and namely the presence of the boundary was the reason that the traditional method did not work properly); the more sophisticated and general form which was used here was suggested in V. Ivrii [11]; for h-(pseudo)-differential operators this construction and its justification is more transparent.

The results of Section 7.3 were well known in certain special cases: for scalar operators for which $h = 1$ and the large spectral parameter τ is considered (then this problem can be reduced to a semiclassical one with $h = |\tau|^{-1/m}$ where m is the degree of the operator), asymptotics with the remainder estimate $O(|\tau|^{(d-1)/m})$ are due to R. Seeley [1, 2]; later by other methods they were obtained by V. Ivrii [8, 11]; for scalar operators with $m > 2$ and for matrix operators with constant multiplicities of the eigenvalues of the principal symbol these asymptotics were first obtained by

D. Vassiliev [1–3] and in the general case by V. Ivrii [11]; we do not discuss asymptotics with less precise remainder estimate because they are outside of the domain of the method of hyperbolic operators and we only mention that in the paper of V. M. Babich & B. M. Levitan [2] this estimate was obtained under the very restrictive condition that the domain is strictly concave in the corresponding metrics (which is fulfilled for no domain in $\mathbb{R}^d$ for the standard Laplacian) before it was obtained by R. Seeley in the general case.

More precise asymptotics of Section 7.4 with the remainder estimate $o(|\tau|^{(d-1)/m})$ were first obtained (under conditions on the Hamiltonian flow) by V. Ivrii [8, 11] and in all the papers mentioned above excluding those of R. Seeley such asymptotics were also obtained; this estimate was obtained by R. Melrose R. Melrose [8] in the case $m = 2$ simultaneously with V. Ivrii [8] but under the condition that the domain is strictly concave in the corresponding metric.

Results of Section 7.5 were first published in V. Ivrii [27], I should mention less precise results of L. Zielinski [1–4].

One can find a lot of beautiful billiard pictures in many places, in particular on Wolfram MathWorld http://mathworld.wolfram.com/Billiards.html

I am aware that these comments are very poor and inadequate. I send the reader to the excellent (but not very modern) survey of G. Rozenblioum, M. Z. Solomyak & M. Shubin [1] for more information. It is a real shame that nobody came with more modern exposition.

Chapter 8

Standard Local Semiclassical Spectral Asymptotics near the Boundary. Miscellaneous

In this chapter we continue the treatment of local and microlocal semiclassical spectral asymptotics near the boundary.

In Section 8.1 we consider asymptotics of $e(x,x,\tau)$ near boundary and derive it with a remainder estimate $O(h^{1-d})$ and a boundary layer correction term $h^{-d}\Upsilon(x', h^{-1}x_1)$.

In Section 8.2 we consider rescaling technique near the boundary; here we have two scales: one comes from the principal symbol of operator and another is the distance to the boundary.

Section 8.3 is devoted to operators with (partially) periodic Hamiltonian flow: for example we consider two manifolds with the common boundary and Hamiltonian flow is periodic only in one of them. Then asymptotics with the remainder estimate $o(h^{1-d})$ and a correction term $h^{1-d}\Upsilon(\tau h^{-1})$ appears not only due to "interference" when phase shifted differently along different trajectories, but also "dissipation" when at reflection some part of energy goes to manifold with non-periodic Hamiltonian flow and does not return to the original point. We consider also the case when Hamiltonian flows in both manifolds are periodic.

Furthermore, in Section 8.4 we consider asymptotics when operator A is restricted to its invariant subspace, projector to which is either a pseudodifferential operator (away from the boundary), or sum of a pseudodifferential operator with the transmission property and a singular Green operator.

8.1 Pointwise Spectral Asymptotics

In this section we consider boundary layer type term in $e(x,x,\tau)$ appearing near boundary. Let us recall that due to rescaling technique

$$e(x,x,\tau) = \mathsf{N}^{\mathsf{w}}_x + O\big(h^{1-d}\gamma^{-1}(x)\big) \tag{8.1.1}$$

where $\gamma(x) = \mathsf{dist}(x,\partial X)$ and this remainder estimate generated after integration $O(h^{1-d}|\log h|)$ and basically the whole Chapter 7 we spent to eliminate logarithmic factor.

Now we would like to consider $e(x,x,\tau)$ without integration with respect to x.

8.1.1 Preliminary Analysis

Discussion

Let us recall that in the previous chapter we derived asymptotics of

$$\mathsf{N} := \Gamma\big(Q_{1x}e(.,.,\tau)\,{}^tQ_{2y}\big) \tag{8.1.2}$$

and

$$\mathsf{N}' := \Gamma'\eth_x\eth_y\big(Q_{1x}e(.,.,\tau)\,{}^tQ_{2y}\big) \tag{8.1.3}$$

with the remainder $O(h^{1-d})$ provided problem $(A,\eth B)$ is microhyperbolic in multidirection $\mathcal{T} = (\ell',\nu_1,\dots,\nu_M)$; further, we derived asymptotics of

$$\mathsf{N}'_x := \Gamma_x\eth_x\eth_y\big(Q_{1x}e(.,.,\tau)\,{}^tQ_{2y}\big) \tag{8.1.4}$$

and

$$\mathsf{N}_x := \Gamma_x\big(Q_{1x}e(.,.,\tau)\,{}^tQ_{2y}\big) \tag{8.1.5}$$

with the remainder $O(h^{1-d})$ provided $\ell' = (\ell'_\xi,0)$ and for (8.1.5) we also need to assume that $\nu_1 = \dots = \nu_M$ in the microhyperbolicity condition; this asymptotics had a boundary layer type term $h^{1-d}\Upsilon(x',x_1h^{-1})$ with $D^\alpha\Upsilon(x',r) = O(r^{-\infty})$ as $r\to+\infty$.

Let us discuss this microhyperbolicity condition. Consider first the case of the Laplace operator; without any loss of the generality one can assume that $a(x,\xi) = a'(x,\xi') + \xi_1^2$ where $a(x,\xi')$ is a positive definite quadratic form. Then microhyperbolicity means that

$$\langle \ell'_\xi, \nabla'_\xi a'\rangle + 2\nu_\pm\xi_1 > 0 \qquad \text{as} \quad \xi_1 = \pm\big(\tau - a(x,\xi')\big)^{\frac12} \tag{8.1.6}$$

and we can always take $\ell'_\xi = \xi'$, $\nu_+ = \nu_- = 0$ as long as $a'(x,\xi') > 0$ (i.e. $\xi' \neq 0$).

So, with the exception of the case $\xi' = 0$ both microhyperbolicity conditions hold. In this exceptional case however $\nabla'_\xi a'(x,\xi') = 0$ and the first term in (8.1.6) is 0 no matter what ℓ'_ξ we pick up; so (8.1.6) boils up to

$$\pm\nu_\pm\big(\tau - a(x,\xi')\big)^{\frac{1}{2}} > 0$$

and for microhyperbolicity we just take $\nu_\pm = \pm 1$ provided $\tau > 0$; this is the standard ξ'-microhyperbolicity condition (we deliberately take $\ell' = (\ell'_\xi, 0)$).

However it is not the case as we assume that $\nu_+ = \nu_-$; then condition is impossible to satisfy. So, the previous section fails to find asymptotics of $e(x,x,\tau)$ but not because of the rays tangent to the boundary but on the contrary, because of the rays orthogonal to it (in the corresponding metrics).

This is a pleasant surprise because then we can find the solution of non-stationary problem in the form of the standard oscillatory integrals as we did in Subsubsection *5.3.1.4 Asymptotics without Spatial Mollification and Short Loops*. The analogy does not stop here: the exceptional rays hit the boundary orthogonally, reflect and follow the same path as before forming the short loops of the length $2\mathsf{dist}(x,\partial X)$ where distance is measured in the corresponding Riemannian metrics.

Remark 8.1.1. (i) However there is a difference: in the former case loop appeared because trajectory went "uphill", lost velocity and rolled back "downhill' repeating the path while now it reflects without losing velocity.

(ii) Another difference is that in the former case the leading term in the asymptotics of $e(x,x,\tau)$ at points where $|V(x) - \tau| \ll 1$ was smaller, of magnitude $(\tau - V(x))_+^{d/2}$, than in the generic points and therefore the remainder estimate without correction term in higher dimensions ($d \geq 3$) could be taken uniform. Now, however the leading terms is everywhere of the same magnitude and the correction term is needed in any dimension, however in the higher dimensions it matters in the smaller vicinities of the boundary (under appropriate assumptions) because the measure of "observable short loops" is smaller (see Remark 8.1.2).

Toy-Model

Consider $X = \mathbb{R}^+ \times \mathbb{R}^{d-1}$, $a(\xi) = |\xi|^2$ and either Dirichlet or Neumann boundary conditions. Then propagator is defined by

$$U(x,y,t) = U^0(x,y,t) + U^1(x,y,t) \tag{8.1.7}$$

with free space solution

$$U^0(x,y,t) = (2\pi h)^{-d} \int e^{ih^{-1}(|\xi|^2 t + \langle x-y,\xi\rangle)}\, d\xi \tag{8.1.8}$$

and reflected wave

$$U^1(x,y,t) = \varsigma(2\pi h)^{-d} \int e^{ih^{-1}(|\xi|^2 t + \langle x-\tilde{y},\xi\rangle)}\, d\xi \tag{8.1.9}$$

where $\tilde{y} = (-y_1, y_2, \dots, y_d)$ and $\varsigma = \mp 1$ for Dirichlet or Neumann boundary conditions respectively.

Since U^1 is responsible to the difference $e^1(x,y,\tau)$ between $e(x,y,\tau)$ and its free space counterpart $e^0(x,y,\tau)$ we manipulate only with it. Taking $y = x$ we get $\langle x - \tilde{x}, \xi\rangle = 2x_1\xi_1$ while $F_{t\to h^{-1}\tau}$ replaces $e^{ih^{-1}\tau}$ by $h\delta\big(\tau - |\xi|^2\big)$; so

$$F_{t\to h^{-1}\tau}\Gamma_x U^1 = \varsigma(2\pi h)^{1-d} \int \delta\big(\tau - |\xi|^2\big) e^{2ih^{-1}x_1\xi_1}\, d\xi = \tag{8.1.10}$$

$$\mathsf{const}\cdot \varsigma h^{1-d}\tau^{\frac{d}{2}-1}\cdot \begin{cases} \displaystyle\int_0^\pi \cos\big(2h^{-1}\tau^{\frac12}x_1\cos\phi\big)\sin^{d-2}\phi\, d\phi & d\ge 2,\\ \cos\big(2h^{-1}\tau^{\frac12}x_1\big) & d=1 \end{cases}$$

where we introduced spherical coordinates $(\phi, \zeta\sin\phi)$ on $\mathbb{S}^{d-1}$, $\phi \in (0,\pi)$, $\zeta \in \mathbb{S}^{d-2}$.

It leads to

$$e^1(x,x,1) \;=\; h^{-1}\int_{-\infty}^1 \Big(F_{t\to h^{-1}\tau}\Gamma_x U^1\Big)\, d\tau \;=\; h^{-d}\Upsilon_\varsigma(h^{-1}x_1) \tag{8.1.11}$$

with

$$\Upsilon_\varsigma(r) = \tag{8.1.12}$$

$$\mathsf{const}\cdot\varsigma\begin{cases} \displaystyle\int_0^1 z^{d-1}\int_0^\pi \cos\big(2zr\cos\phi\big)\sin^{d-2}\phi\, d\phi dz & d\ge 2,\\ \displaystyle\int_0^1 \cos\big(2zr\big) & d=1 \end{cases}$$

where we plugged $\tau = z^2$.

Here we took $\tau = 1$ since any other value could be reduced to this one by rescaling $h \mapsto h\tau^{-\frac{1}{2}}$.

One can find the constant so that for Dirichlet/Neumann problem $e^1(x,x,\tau) = \varsigma e^0(x,x,\tau)$ as $x_1 = 0$.

Stationary phase method implies that

$$\Upsilon(r) = \mathsf{const}\cdot r^{-(d+1)/2}\cos(2r) + O(r^{-(d+3)/2}) \qquad \text{as} \quad r \to +\infty. \tag{8.1.13}$$

Therefore

(8.1.14) With an error not exceeding

$$Ch^{-d}(x_1h^{-1}+1)^{-\frac{1}{2}(d+1)} + Ch^{1-d} \tag{8.1.15}$$

the standard Weyl expression for $e(x,x,\tau)$ holds;

this error is better than $O\big(h^{-d}(x_1h^{-1}+1)^{-1}\big)$ which was due to rescaling but it is only because critical points of $a(\xi') = |\xi'|^2$ are non-degenerate.

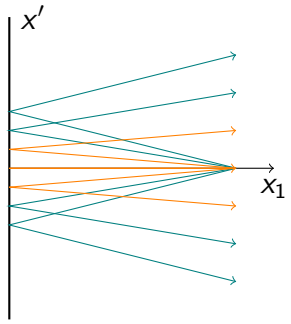


Figure 8.1: Reflection and short loops.

Remark 8.1.2. We can interpret (8.1.13) in the following way: as $|\xi'| \asymp \rho$ the point after reflection deviates from the original point (on the same distance from the boundary) by $|x'| \asymp \rho x_1$ (see Figure 8.1) and therefore this difference is quantum observable only if $\rho^2 x_1 \gg h$; then "short loops" are formed by $\{\xi' \colon |\xi'| \lesssim x_1^{-1/2}h^{1/2}\}$ and their measure is $\asymp x_1^{-(d-1)/2}h^{(d-1)/2}$ and their contribution to the asymptotics is of magnitude $h^{1-d}x_1^{-1}\cdot x_1^{-(d-1)/2}h^{(d-1)/2}$ which is exactly the first term in (8.1.13).

For second order operators this works automatically but in the general case it requires condition $(8.1.49)_r$ with $r = d-1$.

8.1.2 Schrödinger Operator

Let us consider Schrödinger operator with the symbol

$$a(x,\xi)=\sum_{j,k} g^{jk}(\xi_j-V_j)(\xi_k-V_k)+V \tag{8.1.16}$$

assuming that $\tau - V \asymp 1$ and positive. We want to study trajectories on the energy level τ only and we can assume without any loss of the generality that

$$\tau=0,\qquad V(x)\le -\epsilon,\qquad V(x)=-1 \tag{8.1.17}$$

where the last assumption is achieved by division $a(x,\xi)$ by $-V(x)$ which does not affect trajectories on energy level 0.

General Theory

We can assume without any loss of the generality that $V_1 = 0$. Let us introduce coordinates $x_1 = \mathsf{dist}(x, \partial X)$ in the metrics (g^{jk}) [1)] and x' which are constant along trajectories of $a(x,\theta)$ on level 0 which are orthogonal to the boundary, i.e. such that

$$g^{j1}=\delta_{j1}. \tag{8.1.18}$$

We are interested in the construction of parametrix in

$$\mathcal{U}=\{|\xi_j-V_j|\le\epsilon \qquad j=2,\dots,d\}. \tag{8.1.19}$$

It is be much simpler to consider this problem with the "time" x_1 and with t as one of spatial variables. Then as $l = 0$

$$u^l(x,y,t)\equiv \int \sum_{\varsigma=\pm} e^{ih^{-1}(\phi^0_\varsigma(x,y,\eta)+t\eta_0)}\sum_{n\ge 0} f^l_{\varsigma n}(x,y,\eta)h^{n-d}\,d\eta, \tag{8.1.20}$$

where $\eta=(\eta',\eta_0)=(\eta_2,\dots,\eta_d,\eta_0)$ and phase functions ϕ^0_ς are defined from

$$\partial_{x_1}\phi^l_\varsigma=\lambda_\varsigma(x,\nabla_{x'}\phi_\varsigma,\eta_0), \tag{8.1.21}$$

$$\phi^0_\varsigma(x,y,\eta,\eta_0)|_{x_1=y_1}=\langle x'-y',\eta'\rangle, \tag{8.1.22}$$

$$\lambda_\varsigma(x,\xi',\eta_0)=\varsigma\big(\eta_0-a(x,\xi')\big)^{\frac12}. \tag{8.1.23}$$

[1)] I.e. in the final run in $(g^{jk}(-V(x))^{-1})$.

Here amplitudes $f^0_{\varsigma n}(x,y,\eta)$ are smooth, satisfy transport equations and some initial conditions as $x_1 = y_1$ but we do not need to calculate them.

Then we can calculate u^0, hD_1u^0 as $x_1 = 0$ and define $u^1(x,y,t)$ in the same way (8.1.20) but with ϕ^1_ς satisfying (8.1.21) and

$$\phi^1_\varsigma = -\phi^0_{-\varsigma} \qquad \text{as} \quad x_1 = 0. \tag{8.1.24}$$

Here amplitudes $f^1_{\varsigma n}(x,y,\eta)$ are smooth, satisfy transport equations and some initial conditions as $x_1 = 0$ but we do not need to calculate them.

Obviously

$$\phi^1_\varsigma(y,y,\eta) = y_1\psi_\varsigma(y,\eta) \qquad \text{as} \quad x_1 = y_1, \tag{8.1.25}$$

$$\psi_\varsigma(y,\eta) = \lambda_\varsigma(0,y',\eta) - \lambda_{-\varsigma}(0,y',\eta) + O(y_1). \tag{8.1.26}$$

Therefore (as $|t| \le T$)

$$\Gamma_y\big(Q_{1x}u^1(x,y,t)\,{}^tQ_{2y}\big) \equiv \int \sum_{\varsigma=\pm} e^{ih^{-1}(y_1\psi_\varsigma(y,\eta)+t\eta_0)} \sum_{n\ge 0} f^2_{\varsigma n}(y,\eta)h^{n-d}\,d\eta, \tag{8.1.27}$$

where $Q_j = Q(x, hD', hD_t)$ are cut-offs with respect to ξ', τ and finally

$$F_{t\to h^{-1}\tau}\bar\chi_T(t)\Big(\Gamma_y\big(Q_{1x}u^1(x,y,t)\,{}^tQ_{2y}\big)\Big) \equiv \sum_{\varsigma=\pm}\int e^{ih^{-1}y_1\psi_\varsigma(y,\eta',\tau)} \sum_{n\ge 0} f^3_{\varsigma n}(y,\eta',\tau)h^{1+n-d}\,d\eta'. \tag{8.1.28}$$

Obviously, representation (8.1.28) implies that

(8.1.29) For small constant T expression (8.1.28) does not exceed Ch^{1-d}.

Since (8.1.29) is true for u^0 under ξ-microhyperbolicity condition

$$|\tau - V(y)| \ge \epsilon_0 \tag{8.1.30}$$

due to free space results, this statement is true also for u and therefore due to Tauberian arguments

Proposition 8.1.3. *Let* (8.1.30) *be fulfilled and* $Q_1 = Q_1(x, hD')$, $Q_2(x, hD')$ *be operators with symbols supported in* $\mathcal{U}$. *Then* $\Gamma_y\big(Q_{1x}e^1(x,y,\tau)\,{}^tQ_{2y}\big)$ *equals modulo* $O(h^{1-d})$ *Tauberian expression*

$$h^{-1}\int_{-\infty}^{\tau}\Big(F_{t\to h^{-1}\tau'}\bar\chi_T(t)\big(\Gamma_y\big(Q_{1x}u^1(x,y,t)\,{}^tQ_{2y}\big)\big)\Big)\,d\tau'. \tag{8.1.31}$$

Due to results of Subsection 7.3.1 we need to calculate only (8.1.31) with an extra factor $\chi(\tau-\tau')$ with χ supported in the small vicinity of 0:

$$\int_{-\infty}^{\tau}\chi(\tau-\tau')\sum_{\varsigma=\pm}\int e^{ih^{-1}y_1\psi_\varsigma(y,\eta',\tau')}\sum_{n\ge 0}f^3_{\varsigma n}(y,\eta',\tau')h^{n-d}\,d\eta'd\tau'. \tag{8.1.32}$$

Obviously modulo $O(h^{1-d})$ we can skip in the right-hand expression of (8.1.32) all the terms with $n\ge 1$; so we get $h^{-d}J$ with obviously defined J:

$$J := \sum_{\varsigma=\pm}\int_{-\infty}^{\tau}\chi(\tau-\tau')\int e^{i\hbar^{-1}\psi_\varsigma(y,\eta',\tau')}f^3_{\varsigma 0}(y,\eta',\tau')\,d\eta'd\tau' \tag{8.1.33}$$

with $\hbar = hy_1^{-1}$.

Further, for $y_1\ge h^{-1}$ one can consider J as an oscillatory integral with a semiclassical parameter $\hbar$. One can see easily that due to (8.1.25)–(8.1.26) in this oscillatory integral the phase function $\psi_\varsigma(y,\eta,\tau')$ satisfies $|\partial_{\tau'}\psi_\varsigma(y,\eta',\tau')|\ge\epsilon_1$ and has non-degenerate critical points with respect to η'; we denote these points by $\eta'_\varsigma(y,\tau')$. Therefore

$$J_\hbar \sim \sum_{\varsigma=\pm}\sum_{k\ge 0}e^{i\hbar^{-1}\varphi_\varsigma(y,\tau)}f^4_{\varsigma 0}(y,\tau)\hbar^{\frac12(1+d)+k}\qquad \text{as}\quad \hbar\ll 1 \tag{8.1.34}$$

where

$$\varphi_\varsigma(y,\tau)=\psi_\varsigma(y,\eta'(y,\tau),\tau). \tag{8.1.35}$$

Then

(8.1.36) $J_\hbar = J^0_\hbar + O(h)$ where $J^0_\hbar$ is also given by (8.1.33) but with y replaced by $(0,y')$.

Also the fact that f^l_0 are defined through transport equations along trajectories implies then

Proposition 8.1.4. *Modulo* $O(h^{1-d})$ *term* (8.1.31) *coincides with the same term but calculated for* $(\bar A,\eth\bar B) = (a(0,y',hD_x),\eth b(0,y',hD_x))$.

Calculations and Main Theorem

Now in virtue of Propositions 8.1.3, 8.1.4 we need only to recalculate $\Upsilon(r)$ in the case of more general boundary conditions than Dirichlet and Neumann, and then make change of variables.

Repeating arguments of Subsubsection *8.1.1.2 Toy-Model* one can prove easily that

Proposition 8.1.5. *Consider Schrödinger operator with $g^{jk} = \delta_{jk}$, $V = -1$, $\tau = 0$ and the boundary condition*

$$\big(B_0(x, hD')hD_1 + iB_1(x, hD')\big)u|_{x_1=0} = 0 \tag{8.1.37}$$

with

$$|b_0(x', \xi') + |b_1(x', \xi')| \geq \epsilon. \tag{8.1.38}$$

Assume that $\mathsf{supp}(q_j) \subset \mathcal{U}$ defined by (8.1.19)*. Then*

(i) The boundary layer term is $h^{-d}\Upsilon\big(h^{-1}x_1\big)$ with

$$\Upsilon(r) = \tag{8.1.39}$$
$$(2\pi)^{-d} \int \Big(\int_{\lambda_-(\xi',\tau)}^{\lambda_+(\xi',\tau)} \big(\xi_1 + i\beta(\xi')\big)\big(\xi_1 - i\beta(\xi')\big)^{-1} e^{2ir\xi_1}\, d\xi_1 \Big) q_1(\xi') q_2(\xi')\, d\xi'$$

where b_j are principal symbols of B_j and due to self-adjointness assumption $\beta = -ib_1 : b_0$ is real-valued.

(ii) Here function $\Upsilon(r)$ admits decomposition

$$\Upsilon(r) \sim \sum_{\varsigma=\pm} e^{2ir\varsigma} \sum_{n\geq 0} \varkappa_{\varsigma n} r^{-\frac{1}{2}(d+1)-n} \qquad \text{as} \quad r \to +\infty. \tag{8.1.40}$$

Therefore changing variables and using asymptotics in the zone where Theorems 7.3.10 and 7.3.11 work we arrive to

Theorem 8.1.6. *Let us consider Schrödinger operator with self-adjoint boundary condition* (8.1.37) *and let condition $V(x) \leq -\epsilon$, be fulfilled. Further, let ellipticity condition*

$$|b_0(x', \xi')\lambda(x', \xi') + b_1(x', \xi')| \neq 0 \qquad \text{on} \quad \mathsf{supp}(q_1) \cup \mathsf{supp}(q_2) \tag{8.1.41}$$

be fulfilled where $\lambda(x',\xi')$ *is a root of* $a(x,\xi_1,\xi')=0$ *with* $\operatorname{Im}\lambda>0$ [2)].

Then

$$\Gamma_x\big(Q_{1x}e(x,y,0)\,{}^tQ_{2y}\big)-\kappa_0(x)h^{-d}-h^{-d}\Upsilon(x',h^{-1}x_1)\sqrt{g}|\le Ch^{1-d} \tag{8.1.42}$$

with Weyl coefficient κ_0 *and with* $g=\det(g^{jk})^{-1}$ *where* x_1 *is the distance from* x *to* ∂X *in the metrics* $g^{jk}V^{-1}$ *and* $\Upsilon(r)$ *is defined by* (8.1.38) *and satisfies* (8.1.40).

Remark 8.1.7. (i) Due to Theorems 8.1.6 and 8.1.8 below the above theorem but with $\Upsilon(r)$ replaced by $\Upsilon(r)+\Upsilon_b(r)$ remains true if ellipticity condition (8.1.41) is replaced by microhyperbolicity condition

$$\begin{aligned}&|b_0(x',\xi')\lambda(x',\xi')+b_1(x',\xi')|+\\ &\qquad|\nabla_{\xi'}\big(b_0(x',\xi')\lambda(x',\xi')+b_1(x',\xi')\big)|\ne 0 \qquad \text{on}\quad \operatorname{supp}(q_1)\cup\operatorname{supp}(q_2)\end{aligned} \tag{8.1.43}$$

where $\Upsilon_b(r)\in\mathscr{S}(\mathbb{R}^+)$.

(ii) Standard Weyl asymptotics holds with the remainder estimate $O(h^{1-d})$ for $e(x,x,\tau)$ as $x_1\ge h^{(d-1)/(d+1)}$.

8.1.3 Generalizations

Now we need to consider the generalizations to the systems. We consider only Q_j with symbols supported in $\mathcal{U}$. The same approach leads to the following

Theorem 8.1.8. *Let* $(A,\eth B)$ *be self-adjoint and let* $\mathcal{U}\subset T^*\partial X$. *Assume that*

(8.1.44) *As* $(x,\xi',\tau)\in[0,\epsilon]\times\mathcal{U}\times[-\epsilon,\epsilon]$ *all the roots* ξ_1 *of equation*

$$\det\big(\tau-a(x,\xi)\big)=0 \tag{8.1.45}$$

are $\lambda_j^{\pm}(x,\xi',\tau)$ *with* $j=1,\dots,M^{\pm}$ *which are real-valued, smooth and of constant multiplicities and*

$$\pm\partial_\tau\lambda_j^{\pm}>0. \tag{8.1.46}$$

[2)] So, this is the condition only in the complement of $\pi\Sigma|_{\partial X}$; recall that $\Sigma=\{(x,\xi)\in T^*X, a(x,\xi)=0\}$.

Then

(i) Reflected solution has decomposition

$$(8.1.47)\quad F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Big(\Gamma_x\big(Q_{1x}u^1(x,y,t)\,{}^tQ_{2y}\big)\Big)\equiv \sum_{1\le j\le M^+,1\le k\le M^-}\int e^{ih^{-1}x_1\psi_{jk}(x,\eta',\tau)}\sum_{n\ge 0} f_{jkn}(x,\eta',\tau)h^{1+n-d}\,d\eta'.$$

with real phase functions ψ_{jk} and does not exceed $Ch^{1-d}(1+h^{-1}y_1)^{-\frac12(r-1)}$.

(ii) Asymptotics (8.1.42) *holds with $h^{-d}\Upsilon(x,h^{-1}x_1)$ constructed from* (8.1.47) *in the usual Tauberian way; thus*

$$(8.1.48)\quad \Upsilon(x,h^{-1}x_1)\equiv \sum_{1\le j\le M^+,1\le k\le M^-}\int^\tau\int e^{ih^{-1}x_1\psi_{jk}(y,\eta',\tau')}\sum_{n\ge 0} f_{jkn}(y,\eta',\tau')\,d\eta' d\tau'.$$

(iii) Under assumption

$$(8.1.49)_r\quad \nabla'_\xi\big(\lambda_j^+(x,\xi',\tau)-\lambda_k^-(x,\xi',\tau)\big)=0 \implies \operatorname{rank}\operatorname{Hess}_{\xi'}\big(\lambda_j^+(x,\xi',\tau)-\lambda_k^-(x,\xi',\tau)\big)\ge r$$

expressions (8.1.47) *and* (8.1.48) *do not exceed $Ch^{1-d}(1+h^{-1}x_1)^{-\frac12(r-1)}$ and $C(1+h^{-1}x_1)^{-\frac12(r+1)}$ respectively.*

(iv) Under condition $(8.1.49)_3$ *one can replace x by x' in the phase functions and amplitudes in* (8.1.48)*; then $\Upsilon(y',h^{-1}y_1)$ is constructed for the problem $\big(a(0,y',hD_x),\eth b(y',hD_x)\big)$.*

(v) As $r=d-1$ $h^{-d}\Upsilon(h^{-1}x_1)$ admits stationary phase decompositions

$$(8.1.50)\quad \sum_{j,k} e^{ix_1h^{-1}\mu_{jk}(x',\tau)}\sum_n f'_{jkn}(x',\tau)(x_1h^{-1})^{-\frac12(d+1)-n}\qquad \operatorname{mod} O(h^{1-d}).$$

Remark 8.1.9. (i) An easy proof is left to the reader.

(ii) In contrast to the Schrödinger operator we do not define x_1 as the distance to the boundary; eliminating x_1 from the phases ψ_{jk} without deteriorating estimate is possible as $r\ge 3$ while eliminating x_1 from amplitudes is possible as $r\ge 1$.

8.2 Scalar operators and Rescaling

Now our purpose is to consider scalar operators and apply to them rescaling technique. The bad news is that we do not have the all freedom of transformations we enjoyed in the smooth case since we need to preserve a boundary and also we need to look how boundary operator scales. The good news is that this restriction applies only in the boundary partition elements while in the inner partition elements we can apply the results of Chapter 5. Further, let us recall that in Section 5.4 we studied operators with irregular coefficients and did not have a freedom of transformations by means of Fourier integral operators at all.

The important question is what we study:

(a) $\Gamma\bigl(Q_{1x}e(.,.,\tau)\,{}^tQ_{2y}\bigr)$,

(b) $\Gamma\bigl(Q_{1x}e(.,.,\tau)\,{}^tQ_{2y}\bigr)$ as $\tau < \inf V$ or $\Gamma'\bigl(Q_{1x}e(.,.,\tau)\,{}^tQ_{2y}\bigr)$,

(c) $\Gamma_x\bigl(Q_{1x}e(.,.,\tau)\,{}^tQ_{2y}\bigr)$.

In the first case R. Seeley's method of Section 7.5 is our very powerful instrument to tackle balls close to the boundary. In two latter cases this method does not work very well. We consider only Schrödinger operator.

8.2.1 Schrödinger Operator. I

General Theory

As usual let us start from the Schrödinger operator with the principal symbol

$$a = \sum_{j,k} g^{jk}(\xi_j - V_j)(\xi_k - V_k) + V \tag{8.2.1}$$

(or any matrix operator with the principal symbol aI) looking for asymptotics of $\Gamma\bigl(Q_{1x}e(.,.,\tau)\,{}^tQ_{2y}\bigr)$ (where Q_j are h-differential operators, usually just cut-off functions).

We consider boundary problem

$$\Bigl(\sum_j hb^j(hD_k - V_k) + b_0\Bigr)u|_{\partial X} = 0; \tag{8.2.2}$$

we assume that it is either Dirichlet boundary problem (i. e. $b^1 = ... = b^d = 0$, $b_0 = 1$) or

$$\sum_{1\le j\le d} |b^j| \ge \epsilon. \tag{8.2.3}$$

Without any loss of the generality one can assume[3)] that $X = \{x_1 > 0\}$ locally and

$$g^{jk} = \delta_{jk} \tag{8.2.4}$$

in which case self-adjointness of the problem implies that (modulo non-vanishing factor)

(8.2.5) ib^1, b^2,...,b^d and b_0 are real-valued functions

and the standard ellipticity of the problem means exactly that

$$|ib^1\big(\sum_{2\le j,k\le d} g^{jk}\xi_j\xi_k\big)^{\frac{1}{2}} + \sum_{2\le d} b^j\xi_j| \asymp |\xi'| \qquad \forall \xi' \in \mathbb{R}^d. \tag{8.2.6}$$

We know that with appropriate boundary conditions remainder estimate $O(h^{1-d})$ is achieved under ξ-microhyperbolicity condition

$$|V| \ge \epsilon_0. \tag{8.2.7}$$

To get rid of this condition (as coefficients belong to $\mathscr{C}^1$) we applied scaling functions

$$\gamma(x) = \epsilon|V(x)| + h^{\frac{2}{3}}, \qquad \rho(x) = \gamma(x)^{\frac{1}{2}}. \tag{8.2.8}$$

Let us do the same; then as $\gamma(x) \le \ell(x)$,

$$\ell(x) = \frac{1}{2}\mathsf{dist}(x, \partial X) \tag{8.2.9}$$

everything works as before in Section 5.4 and the total contribution of these balls to the remainder is $O(h^{1-d})$ as *either* $d \ge 3$ *or* $d = 2$ and

$$|V| + |\nabla V| \ge \epsilon_0; \tag{8.2.10}$$

[3)] Surely it may affect smoothness but we need this assumption only to write down conditions to B in more compact form

as $d = 1$ under condition (8.2.10) we concluded that the total contribution of these balls to the remainder is $O(|\log h|)$; we needed to examine a propagation of singularities to fix this and we will do it later.

On the other hand, as $\ell(x) \le \gamma(x)$ we have two choices: first, we can snap $\gamma(x)$ to $\ell(x)$ but leave $\rho(x) = |V(x)|^{\frac{1}{2}}$ (as it was before for $\gamma(x) \ge 2h^{\frac{2}{3}}$):

$$\gamma(x) = \ell(x), \quad \rho(x) = |V(x)|^{\frac{1}{2}} + h\ell(x)^{-1} \quad \text{as} \quad \epsilon|V(x)| \ge \ell(x) \ge h; \tag{8.2.11}$$

then the total contribution of such balls does not exceed

$$Ch^{1-d} \int \rho(x)^{d-1}\ell(x)^{-1} \le Ch^{1-d}|\log h|$$

and we cannot generally improve this while contribution of the zone $\{x\colon \ell(x) \le Ch\}$ to the remainder is $O(h^{1-d})$. So we get remainder estimate $O(h^{1-d}|\log h|)$ under condition (8.2.10) and $d \ge 1$; then rescaling arguments of Chapter 5 continue to work and we get remainder estimate $O(h^{1-d}|\log h|)$ as *either* $d \ge 2$ *or* as $d = 1$ and non-degeneracy condition $(5.2.100)_m$ is fulfilled and the remainder estimate $O(h^{-\delta})$ as $d = 1$ without non-degeneracy condition (in the latter case we should be happy as we got the same result as in the case without boundary).

Alternatively we should not replace $\gamma(x)$ by $\ell(x)$ but then we need to analyze boundary ball $B(\bar{x}, \gamma(\bar{x}))$ in which case we need to look how boundary condition scales. As before we can assume that $V_j(\bar{x}) = 0$ as $d \ge 2$ or $V_1 = 0$ as $d = 1$.

Obviously the Dirichlet boundary condition $u|_{\partial X} = 0$ scales well. Condition (8.2.2) also scales well as $b_0 = 0$ identically otherwise assumptions (8.2.3), (8.2.6) may be in trouble. In fact, this condition scales to

$$\Big(\sum_j \hbar b^j(hD_k - V_k) + \beta b_0'\Big)u|_{\partial X} = 0. \tag{8.2.12}$$

where $\beta = 1$, $b_0' = \rho^{-1}b_0$ as $|b_0(x)| \le C\rho$ at some and thus in any point of partition element and $\beta = \rho^{-1}b_0(\bar{x})$, $b_0' = b_0 b_0(\bar{x})^{-1}$ if $|b_0(x)| \ge C\rho$ at some and thus in any point of the partition element. So, we get a system with an additional large parameter β. However we use Seeley's method of Section 7.5 and we need only estimate here (in the boundary strip $\{x_1' \le \hbar\}$ after rescaling). Therefore we can allow operators with a parameter as long

the boundary condition remains elliptic with a parameter i.e.

$$|ib_0\big(\sum_{2\le j,k\le d} g^{jk}\xi_j\xi_k\big)^{\frac{1}{2}} + \sum_{2\le j\le d} b^j\xi_j + b_0\beta| \asymp |\xi'| + \beta \qquad \forall \xi' \in \mathbb{R}^d,\ \forall \beta \ge 0. \tag{8.2.13}$$

This condition scales well.

So we assume that

(8.2.14) One of the following condition is fulfilled:

(a) $b^1 = ... = b^d = 0$, $b_0 = 1$,

(b) $b_0 = 0$ and (8.2.6) is fulfilled,

(c) (8.2.13) is fulfilled.

Thus, the contribution of ball $B(\bar{x}, \gamma(\bar{x}))$ to the remainder is $O(\hbar^{1-d}) = O(h^{1-d}\gamma^{d-1}\rho^{d-1})$. Then the contribution of all such balls to the remainder does not exceed $Ch^{1-d}\int \rho^{d-1}(x)\,dx' = O(h^{1-d})$ without any extra assumption.

Therefore we get the total remainder estimate $O(h^{1-d})$ as *either* $d \ge 3$ *or* as $d = 2$ and condition (8.2.10) is fulfilled. Then using scaling with

$$\gamma(x) = \rho(x) = \epsilon(|V(x)| + |\nabla V(x)|^2)^{\frac{1}{2}} + h^{\frac{1}{2}} \tag{8.2.15}$$

and applying the same arguments we recover remainder estimate $O(h^{-1})$ as $d = 2$ even without assumption (8.2.10).

Thus we arrive to

Theorem 8.2.1. *Consider Schrödinger operator. Assume that the boundary problem is self-adjoint and condition* (8.2.14) *is fulfilled.*

Let either $d \ge 3$ and coefficients belong to $\mathscr{C}^{1,1}$ or $d = 2$ and coefficients belong to $\mathscr{C}^2$. Then

(i) The remainder estimate is $O(h^{1-d})$.

(ii) Moreover, the contribution of the zone $\{x : \mathsf{dist}(x, \partial X) \le \varepsilon\}$ with $\varepsilon \ge h$ to the remainder does not exceed $C\varepsilon h^{1-d}$.

Sharp Remainder Estimates

We can combine Theorem 8.2.1(ii) with Theorem 5.3.3. Note that for Schrödinger operator the problem $(A-\tau, \eth B)$ is automatically microhyperbolic at points where billiards hit the boundary.

Details are left to the reader. Just to remind that as $d=2$ the remainder estimate includes the term

$$Ch^{-1}\int_{\{|V-\tau|+|\nabla V|^2\le h^{1-\delta}\}} |F_{12}|\,dx$$

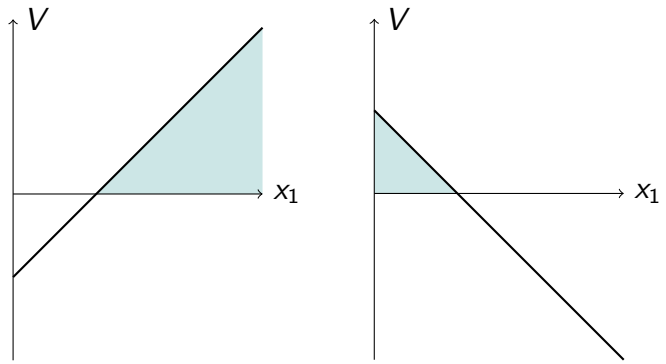
where $F_{12}=(\partial_{x_1}V_2-\partial_{x_2}V_1)$.

Special Case $d=1$

As $d=1$ we still need to get rid of $|\log h|$. Under assumption (8.2.10) it is easy. Without any loss of the generality one can assume that $|V(0)|\le\epsilon$ and then consider two cases:

(a) $\partial_{x_1}V(0)\ge\epsilon_0$;

(b) $-\partial_{x_1}V(0)\le-\epsilon_0$:



(a) $V'>0$ (b) $V'<0$

Figure 8.2: Case $d=1$ under condition (8.2.10); teal!20 gray covers classically forbidden zones.

In the former case the slope goes to the boundary and *after rescaling* $x_1\mapsto x_1\gamma^{-1}$, $\xi_1\mapsto\xi_1\gamma^{-\frac12}$, $h\mapsto\hbar=h\gamma^{-\frac32}$ with $\gamma=\frac12 x_1\ge\bar\gamma:=\big(|V(0)|+h^{\frac23}\big)$

the zone $\{x\colon x_{1\mathsf{new}} \ge C\}$ is classically forbidden and its contribution to the remainder is $O(\hbar^s) = O\big(h^s\gamma^{-\frac{2}{3}s}\big)$ and the total contribution of the zone $\{x\colon x_1 \ge C\bar\gamma\}$ does not exceed $C\int h^s\gamma^{-\frac{2}{3}s}\gamma^{-1}\,d\gamma \asymp 1$ exactly as in the proof of Theorem 5.2.7.

In the latter case the slope goes away from the boundary and therefore launching trajectory in this direction we extend $T \asymp \gamma(x)\rho(x)^{-1} \asymp x_1^{\frac12}$ to $T \asymp 1$. Then the contribution of interval $x_1 \asymp \gamma$ to the remainder improves from $O(1)$ to $O(\gamma^{\frac12})$. Therefore the total contribution of the zone $\{x\colon |x_1| \ge C(|V(0)| + h^{\frac23})\}$ improves from $\int \gamma^{-1}\,d\gamma \le |\log h|$ to $\int \gamma^{\frac12}\times\gamma^{-1}\,d\gamma \asymp 1$ exactly as in the proof of Theorem 5.2.10.

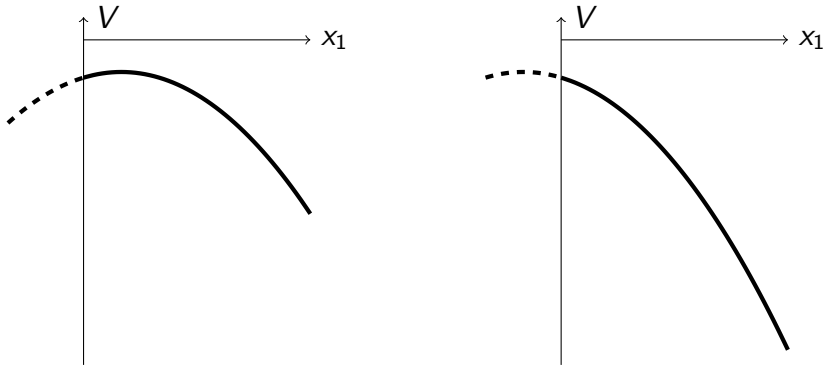
In both cases we can estimate contribution to the remainder of the zone $\{x\colon x_1 \le \bar\gamma\}$ by $O(1)$. Therefore in both cases under condition (8.2.10) we reached the remainder estimate $O(1)$.

Now we can use $\gamma = \epsilon\big(|V| + |\partial_{x_1}V|^2\big)^{\frac12} + h^{\frac12}$ and exactly as before we need to consider the case $|V(0)| + |\partial_{x_1}V(0)| \le \epsilon$ and then two cases:

(a) $\partial_{x_1}^2 V(0) > \epsilon_0$;

(b) $-\partial_{x_1}^2 V(0) < -\epsilon_0$.

Again in the former case $\int_{\Sigma_\tau} d\mu_\tau \le C$ leading us to the same remainder estimate $O(1)$; so only case (b) needs to be addressed. Again, there are two subcases: $V'(0) > 0$ and $V'(0) < 0$.



(a) $V'(0) > 0$ (b) $V'(0) < 0$

Figure 8.3: Case $d = 1$ under condition (b); sign $V(x_1^*)$ does not really matter.

Let us introduce

$$\bar{\gamma} = |V'(0)| + \bigl(|V(0)| + h\bigr)^{\frac{1}{2}}. \tag{8.2.16}$$

Then the zone $\{x\colon x_1 < C\bar{\gamma}\}$ is rescaled $x_1 \mapsto x_1\bar{\gamma}^{-1}$, $\xi_1 \mapsto \xi_1\bar{\gamma}^{-1}$, $h \mapsto \hbar = h\bar{\gamma}^{-2}$ and either $\bar{\gamma} \ge 2h^{\frac{1}{2}}$ in which case (8.2.10) is fulfilled after rescaling, or $\bar{\gamma} \asymp h^{\frac{1}{2}}$ in which case $\hbar \asymp 1$; in both cases contribution of this zone to the remainder is $O(1)$.

Assume that function $V(x)$ extended to negative x_1 reaches its maximum on $\{x\colon |x_1| \le \bar{\gamma}\}$ at x_1^*. Then in virtue of the above arguments we need to consider contribution of the zone $\{x\colon x_1 \ge C\bar{\gamma}\}$. Let us recall the proof of Theorem 5.2.10; adapting it to our case one can see easily that we considered there classical trajectories going "downhill" away from x_1^* unless $x_1 - x_1^* > h^\delta$ in which case we considered classical trajectories going "uphill" towards x_1^*. The boundary $x_1 = 0$ does not affect our arguments in the former case and affects our arguments only in the latter case and only if $x_1^* < 0$ and $V(0) < 0$; sign $V(x_1^*)$ does not matter.

However we can repeat the same arguments as in the proof of Theorem 5.2.10 but with $\bar{\gamma}$ defined by (8.2.16) rather than with $\bar{\gamma} = (|V(x_1^*)| + h)^{\frac{1}{2}}$ as we used there.

Now we can go from condition $(5.2.100)_2$ to $(5.2.100)_m$ by induction with respect to m. Note that in the proof of Theorem 5.2.14 in this case we always are moving "uphill" towards point x^* where $\gamma(x)$ reaches its minimum and

$$T \asymp \max\bigl(x_1^\delta \bar{\gamma}^{-\delta}, 1\bigr) \tag{8.2.17}$$

with $\bar{\gamma}$ introduced there by (5.2.103) as γ reaches its minimum at $x_1^* > 0$ and with $\bar{\gamma}$ replaced by $\gamma(0)$ as γ reaches its minimum at $x_1^* < 0$.

This leads us to

Theorem 8.2.2. *Consider Schrödinger operator in dimension $d = 1$ with sufficiently smooth coefficients. Assume that boundary problem is self-adjoint and satisfies condition* (8.2.14) [4]*. Then the remainder estimate is $O(1)$ under condition* $(5.2.100)_m$ *and $O(h^{-\delta})$ without it.*

Generalizations

Consider any scalar operator assuming that the classically principal part of it is (uniformly) elliptic and also classically principal part of boundary

[4] Which as $d = 1$ boils down to $|b_0| + |b^1| \asymp 1$

value problem is (uniformly) elliptic. Then in the smooth case we can apply arguments of Section 5.2 and in irregular case arguments of Section 5.4; in the latter case we either assume that the boundary is smooth or just satisfies regularity conditions of Section 7.5 and in the latter case we assume that there are Dirichlet boundary conditions.

Then under these assumptions we recover in full results of Section 5.2 or Section 5.4 respectively.

8.2.2 Schrödinger Operator. II

Preliminaries

Now we consider asymptotics either of $\Gamma\big(Q_{1x}e(.,.,\tau)\,{}^tQ_{2y}\big)$ as $\tau < \inf V$ or $\Gamma'\big(Q_{1x}e(.,.,\tau)\,{}^tQ_{2y}\big)$. In this case we cannot get rid of the microhyperbolicity assumption of $(A-\tau, \eth B)$ as easily as we did before. Also we assume that coefficients are smooth enough.

Remark 8.2.3. (i) Ellipticity condition[5)] (8.2.6) has different implications for different dimensions:

(i)′ As $d \ge 3$ it is equivalent to

$$g'(b') := \sum_{2\le j,k\le d} g_{jk} b^j b^k \le (1-\epsilon)|b^1|^2 \tag{8.2.18}$$

where $(g_{jk}) = (g^{jk})^{-1}$;

(i)″ As $d = 2$ it is equivalent to *either* (8.2.7) *or*

$$g_{22}|b^2|^2 \ge (1+\epsilon)|b^1|^2; \tag{8.2.19}$$

(ii) Under condition (8.2.18) operator A with boundary condition $\eth Bu = 0$ is semi-bounded from below and under condition (8.2.19) it is not.

(iii) Condition (8.2.13) is (8.2.6) in dimension dv rather than $(d-1)$ with $\xi_0 = \beta$ and $g^{0j} = \delta_{j0}$.

[5)] As $g^{jk} = \delta_{jk}$.

Microhyperbolicity Conditions

Let us recall that the microhyperbolicity condition at the interior point (x,ξ) is

$$|\tau - V(x)| - \sum_{1\le j\le d} |\xi_j - V_j(x)| + |\nabla_x V(x)| \ge \epsilon_0; \tag{8.2.20}$$

the microhyperbolicity condition for $(A-\tau)$ at the boundary point (x,ξ') is

$$|\tau - V(x)| - \sum_{2\le j\le d} |\xi_j - V_j(x)| + |\nabla_{x'} V(x)| \ge \epsilon_0 \quad (x_1 = 0); \tag{8.2.21}$$

moreover, ξ-microhyperbolicity in the interior and boundary point is

$$|\tau - V(x)| - \sum_{1\le j\le d} |\xi_j - V_j(x)| \ge \epsilon_0 \tag{8.2.22}$$

and

$$|\tau - V(x)| - \sum_{2\le j\le d} |\xi_j - V_j(x)| \ge \epsilon_0 \quad (x_1 = 0); \tag{8.2.23}$$

respectively.

Furthermore, if $(A-\tau)$ is ξ-microhyperbolic in the boundary point, problem $(\tau - A, \eth B)$ is ξ-microhyperbolic if and only if one of the following four conditions is fulfilled:

(8.2.24) The point (x',ξ') is either hyperbolic or "almost hyperbolic", i.e.

$$\tau \ge \sum_{2\le j,k\le d} g^{jk}(x)(\xi_j - V_j)(\xi_k - V_k) + V - \epsilon'$$

with $\epsilon' = \epsilon'(d, c, \epsilon_0) > 0$;

$$ib_1 b_0 \ge -\epsilon', \tag{8.2.25}$$

$$|\tau - W(x)| \ge \epsilon_0 \tag{8.2.26}$$

where

$$W = V - \beta^2(|b^1|^2 - g'(b')^{-1} \tag{8.2.27}$$

with $g'(b')$ defined by (8.2.18),

$$\sum_{2\le j\le d} \bigl|\xi_j - V_j + (\tau - V)\frac{b_j}{b_0}\bigr| \ge \epsilon_0, \quad b_j = \sum_{2\le k\le d} g_{jk}b^k; \tag{8.2.28}$$

in cases (8.2.25)–(8.2.27) the operator $\tilde{B}$ with the principal symbol

$$ib^1\Bigl(\sum_{2\le j,k\le d} g^{jk}\xi_j\xi_k\Bigr)^{\frac12} + \sum_{2\le d} b^j\xi_j \tag{8.2.29}$$

and the problem $(A-\tau, \eth B)$ are both either elliptic at this point or microhyperbolic in the multidirection $(0, \xi_j - V_j)_{j=2,\dots d})$; we assume that condition (8.2.24) is violated and hence operator $A-\tau$ is elliptic at this point.

Finally, if all four conditions (8.2.24)–(8.2.27) are violated then the microhyperbolicity condition is fulfilled if and only if

$$|\nabla_{x'} W| \ge \epsilon_0; \tag{8.2.30}$$

however $\ell'_x \ne 0$ in this case.

Analysis in the "Elliptic" Zone

Let us now consider the elliptic zone of the operator $(A-\tau)$. Here it is sufficient to consider only $\Gamma'\bigl(Q_{1x}e(.,.,\tau)\,{}^tQ_{2y}\bigr)$ as $\Gamma\bigl(Q_{1x}e(.,.,\tau)\,{}^tQ_{2y}\bigr)$ in this zone can be reduced to it (with modified operators Q_{1x} and Q_{2y} and) with an extra factor h.

In the zone in question case the ellipticity or microhyperbolicity of problem $(A-\tau, \eth B)$ is equivalent to the same property of the symbol

$$\tilde{b} = ib^1(x')\bigl(a'(x',\xi') - \tau\bigr)^{\frac12} + b'(x',\xi') + b_0(x') \tag{8.2.31}$$

where

$$a'(x',\xi') = \sum_{j,k} g^{jk}(\xi_j - V_j)(\xi_k - V_k) + V, \tag{8.2.32}$$

$b' = \sum_j b^j\xi_j$, we sum with respect to $j,k = 2,\dots,d$ and as before we assume that $g^{1k} = \delta_{1k}$; let us recall that the elliptic zone is given by the inequality

$$a'(x',\xi') \ge \tau + \epsilon \tag{8.2.33}$$

with a constant $\epsilon > 0$.

In turn symbol $\tilde{b}$ is elliptic in zone defined by

$$ib^1(x')^{-1}\big(b'(x',\xi')+b_0(x')\big)\ge 0 \tag{8.2.34}$$

and in zone

$$ib^1(x')^{-1}\big(b'(x',\xi')+b_0(x')\big)< 0 \tag{8.2.35}$$

the ellipticity or microhyperbolicity of $\tilde{b}$ is equivalent to those of

$$p=-\big(b^1(x')\big)^2\big(a'(x',\xi')-\tau\big)+\big(b'(x',\xi')+b_0(x')\big)^2, \tag{8.2.36}$$

Due to assumption (8.2.18) or (8.2.19) quadratic form

$$-\big(ib^1(x')\big)^2\sum_{j,k} g^{jk}(x')\xi_j\xi_k+\big(\sum_j b^j\xi_j\big)^2 \tag{8.2.37}$$

is non-degenerate and thus we basically arrive to the Schrödinger operator on the boundary; here we can assume that ib^1 is disjoint from 0; otherwise $\sum_j |b^j|^2$ is disjoint from 0 and as ib^1 is sufficiently small, the symbol p is ξ-microhyperbolic.

For this Schrödinger operator we can invoke the full power of Subsection 5.3.1 and arrive to

Theorem 8.2.4. *Consider Schrödinger operator in dimension $d\ge 2$. Assume that boundary problem is self-adjoint and satisfies condition* (8.2.14). *Let $Q_j=q_j(x,hD'_x)$ be operators with symbols supported in the zone defined by* (8.2.33). *Then*

(i) As $d\ge 3$ the following estimates hold

$$|\Gamma'\big(Q_{1x}e(.,.,\tau)\,{}^tQ_{2y}\big)-\kappa_0 h^{-d}|\le Ch^{1-d} \tag{8.2.38}$$

and

$$|\Gamma\big(Q_{1x}e(.,.,\tau)\,{}^tQ_{2y}\big)-\kappa_1 h^{1-d}|\le Ch^{2-d}. \tag{8.2.39}$$

(ii) As $d\ge 2$ and nondegeneracy condition $(5.2.100)_m$ is fulfilled for symbol

$$W(x')=-\big(ib^1(x')\big)^2\big(V(x')-\tau\big)+\big(b_0(x')\big)^2, \tag{8.2.40}$$

estimates (8.2.38) *and* (8.2.39) *hold as well.*

(iii) As $d = 2$

$$|\Gamma'\big(Q_{1x} e(.,.,\tau)\, {}^t Q_{2y}\big) - \kappa_0 h^{-2}| \le C h^{-1-\delta} \tag{8.2.41}$$

and

$$|\Gamma\big(Q_{1x} e(.,.,\tau)\, {}^t Q_{2y}\big) - \kappa_1 h^{1-d}| \le C h^{-\delta} \tag{8.2.42}$$

hold with arbitrarily small exponent $\delta > 0$.

Remark 8.2.5. (i) Under some conditions on the Hamiltonian trajectories of p one can obtain two-term asymptotics; however one should keep in mind that these trajectories can leave the elliptic zone, so it is necessary either to forbid this or to use the complete analysis of propagation of singularities in the hyperbolic zone.

(ii) Surely one can also consider averaging with respect to spectral parameter.

Asymptotics as $V \approx 0$

To cover this zone we introduce

$$\gamma = \epsilon |V| + h^{\frac{2}{3}}, \qquad \rho = \gamma^{\frac{1}{2}} \tag{8.2.43}$$

and after rescaling $V \approx 1$ or $\hbar \asymp 1$. We do not need to care then about microhyperbolicity condition in the elliptic zone (unless $d = 2$). Then we arrive immediately to

Theorem 8.2.6. *Consider Schrödinger operator in dimension* $d \ge 2$. *Assume that boundary problem is self-adjoint and satisfies condition* (8.2.14). *Then*

(i) Asymptotics (8.2.38) *holds as* $d \ge 3$.

(ii) Asymptotics

$$|\Gamma'\big(Q_{1x} e(.,.,\tau)\, {}^t Q_{2y}\big) - \kappa_0 h^{-d}| \le C h^{-1} \int \gamma^{-1}\, dx' \tag{8.2.44}$$

holds as $d = 2$ *and the boundary condition is either Dirichlet or Neumann or non-degeneracy condition* $(5.2.100)_m$ *is fulfilled for symbol* $W(x')$ *defined by* (8.2.40).

(iii) Asymptotics

$$|\Gamma'\big(Q_{1x}e(.,.,\tau)\,{}^tQ_{2y}\big) - \kappa_0 h^{-d}| \le Ch^{-1-\delta}\int \gamma^{-1+\frac{3}{2}\delta}\,dx' \tag{8.2.45}$$

holds as $d = 2$.

Proof. Really, in the frameworks of Statements (i) or (ii) the rescaling leads to the remainder estimate $Ch^{1-d}\int \rho^{d-1}\gamma^{-1}\,dx'$ which implies (8.2.38) and (8.2.44) respectively.

In the framework of Statement (iii) we get the remainder estimate

$$Ch^{1-d}\int \rho\gamma^{-1}\big(|h\gamma^{-1}\rho^{-1}|\big)^{-\delta}\,dx'$$

which leads to (8.2.45). □

Corollary 8.2.7. *Let* $d = 2$. *Then*

(i) Asymptotics with remainder estimate $O(h^{-1-\delta})$ *holds under microhyperbolicity condition*

$$|V-\tau| + |\nabla_{x'}V| + |\nabla^2_{x'}V| \ge \epsilon. \tag{8.2.46}$$

(ii) Moreover, for Dirichlet or Neumann boundary condition asymptotics with remainder estimate $O(h^{-1}|\log h|)$ *holds under condition* (8.2.46).

(iii) Asymptotics with remainder estimate $O(h^{-\frac{4}{3}})$ *in the general case.*

Remark 8.2.8. We cannot control $\partial_{x_1}V$ so we can neither scale microhyperbolicity condition $|V-\tau| + |\nabla_{x'}V| \ge \epsilon$ properly. This prevents us from getting better results as $d = 2$.

Problem 8.2.9. (i) Derive uniform with respect to $\tau \approx 0$ remainder estimates (the best possible) for toy-model operators

$$h^2\big(D_1^2 + D_2^2\big) \pm x_1 \tag{8.2.47}$$

under Dirichlet or Neumann boundary condition.

(ii) Generalize these estimates.

8.2.3 Schrödinger Operator. III

Consider now asymptotics of $\Gamma_x\bigl(Q_{1x}e(.,.,\tau)\,{}^tQ_{2y}\bigr)$. Assume for simplicity that either Dirichlet or Neumann boundary condition is imposed. Then the same scaling as before brings the remainder estimate $O(h^{1-d})$ as $d\ge 3$ and the remainder estimate $O(h^{-1}\gamma^{-\frac12})$ as $d=2$; in particular as $d=2$ the remainder estimate $O(h^{-\frac43})$ holds.

Consider case $d=2$ in more details. As $\ell(x)=\mathsf{dist}(x,\partial X)\ge\gamma(x)$ we consider a ball $B(x,\ell(x))$ and in this ball we replace $\gamma(x)$ and $\rho(x)$ by $\ell(x)$ and $\ell(x)^{\frac12}$ respectively; we can do it because we have "free space asymptotics" without assumption that $V(x)$ is disjoint from 0. This asymptotics was derived in Subsubsection *5.3.1.4 Asymptotics without Spatial Mollification and Short Loops*. Let us recall that this asymptotics had remainder estimate $O(h^{-1})$ and contained correction term described in (5.3.36) and (5.3.62) and concentrated near $V=0$.

Scaling this ball leads us to the remainder estimate $O(h^{-1}\rho^{\frac12}\gamma^{-1})$ where we plug $\gamma=\ell$ and $\rho=\ell^{\frac12}$ and arrive to the remainder estimate $O(h^{-1}\ell(x)^{-\frac12})$.

Therefore in both cases the remainder does not exceed

$$Ch^{-1}\gamma_1(x)^{-\frac12},\qquad \gamma_1(x)=\mathsf{max}\bigl(|V(x)|,\ell(x),h^{\frac23}\bigr) \tag{8.2.48}$$

but the correction term is different depending on which of $\ell(x),|V(x)$ is larger:

Theorem 8.2.10. *Consider Schrödinger operator in dimension $d=2$ under Dirichlet or Neumann boundary condition. Then asymptotics of $\Gamma_x\bigl(Q_{1x}e(.,.,\tau)\,{}^tQ_{2y}\bigr)$ holds with the remainder estimate $O\bigl(h^{-1}\gamma_1(x)^{-\frac12}\bigr)$ with the Weyl principal part and correction term $h^{-2}\Upsilon(h^{-1}x_1)$ with function Υ constructed in Section 8.1 as $|V(x)|\ge\mathsf{max}(\ell(x),h^{\frac23})$ and with the correction term described by Theorem 5.3.16 as $\ell(x)\ge\mathsf{max}(|V(x)|,h^{\frac23})$.*

Problem 8.2.11. In dimension $d=2$ assuming that $|\nabla_{x'}V(x)|\ge\epsilon$ derive asymptotics with the remainder $O(h^{-1})$ or at least which is $o(h^{-1}\gamma_1^{-\frac13})$ as $\gamma_1\ll 1$. This correction term is expected to contain a "circular wave" term corresponding to the "vertex" $\{x\in\partial X, V(x)=0\}$.

Related is the following

Problem 8.2.12. (i) In dimension $d \geq 2$ consider non-degenerate Schrödinger operator in domain with edges and derive asymptotics with the remainder estimate $O(h^{1-d})$ or at least with $o(h^{1-d}\gamma_1^{-1}(x))$ as $\gamma_1(x) \ll 1$ where γ_1 is the distance to the nearest edge.

(ii) Derive similar results in domains with edges, vertices of any dimension.

(iii) Derive similar results in domains with conical points.

Those are difficult problems solution of which clearly would deserve publication.

Remark 8.2.13. Surely we can restore results of Section 5.2 for spectral mollifications.

8.3 Operators with Periodic Hamiltonian Flows

In Section 6.2 we considered the case when there was no boundary and the Hamiltonian flow was periodic and derived very sharp spectral asymptotics. Now we want to achieve similar results when there is a boundary. We do not expect asymptotics to be that sharp, but better than $O(h^{1-d})$ and have non-Weyl correction term. However the presence of the boundary, more precisely, branching of the rays on the boundary brings the new possibilities. For simplicity we consider only Schrödinger operators.

8.3.1 Discussion and Plan

We start our analysis in Subsection 8.3.2 from the case when there is no branching and the Hamiltonian flow (with reflections) is periodic. Then if period is T_0, $\Psi_{T_0} = I$ we have the same equality

$$e^{ih^{-1}T_0A}Q = e^{iT_0L}Q \tag{8.3.1}$$

as before where $\mathsf{supp}(q)$ is disjoint from the boundary and only transversal to the boundary trajectories originate at $\mathsf{supp}(q)$ and L is h-pseudodifferential operator. Then

$$F_{t\to h^{-1}\tau}\bar{\chi}\Big(\frac{t}{T_0} - n\Big)\Gamma\big(Q_{1x}u\,{}^tQ_{2y}\big) = (2\pi h)^{1-d}J(n) + O(h^{1-d+\delta}) \tag{8.3.2}$$

with

$$J(n) = \int_{\Sigma_\tau} e^{in\ell} q_1 q_2 \, d\mu_\tau \tag{8.3.3}$$

as $\bar{\chi}$ is supported in $(-\frac{2}{3}, \frac{2}{3})$ and equals 1 in $(-\frac{1}{3}, \frac{1}{3})$ and $n \in \mathbb{Z}$, $|n| \le h^{-\delta}$. Here as before $\Sigma_\tau = \{a(x,\xi) = \tau\}$ and $\mu_\tau = dxd\xi : da$ is the measure there, ℓ is the principal symbol of L.

If ℓ is "variable enough" on Σ_τ so the right-hand expression is decaying as $n \to \infty$ we can improve remainder estimate $O(h^{1-d})$ (to what degree depends on the rate of the decay). Recall that in Section 6.2 essentially ℓ was defined by integral of the subprincipal symbol along periodic trajectories. Now however ℓ can pick up extra terms at the moment of the reflection: for example, for Schrödinger operator increment of ℓ is 0 and π for Neumann and Dirichlet boundary condition respectively; however more general boundary condition brings increment which depends on the point $(x', \xi') \in T^*Y$ of the reflection.

Then in Subsection 8.3.3 we consider a more complicated case when two operators are intertwined through the boundary conditions and all Hamiltonian trajectories of one of them are periodic but of another are not and, moreover, if the generic Hamiltonian billiard is periodic, it is in fact some Hamiltonian billiard of the first operator. Then (8.3.1) fails but (8.3.2)–(8.3.3) is replaced by (8.3.2), (8.3.4)

$$J(n) = \int_{\Sigma_{1,\tau}} e^{in\ell - |n|\ell'} q_1 q_2 \, d\mu_{1,\tau} \tag{8.3.4}$$

where $\Sigma_{1,\tau}$ and $\mu_{1,\tau}$ correspond to the first operator and $\ell' \ge 0$ depends on the portion of the energy which was whisked away along trajectory of the second operator at the reflection point. Then if

$$\mu_{1,\tau}\big(\{(x,\xi) \in \Sigma_{1,\tau}, \ell' = 0, \nabla\ell = 0\}\big) = 0 \tag{8.3.5}$$

we conclude that $J(n) = o(1)$ as $n \to \infty$ and we will be able to improve remainder estimate $O(h^{1-d})$ to $o(h^{1-d})$ and again there will be a non-Weyl correction term but it will depend also on both ℓ and ℓ'.

Finally, in Subsection 8.3.3 we consider even more complicated case when two operators are intertwined through boundary condition and the Hamiltonian trajectories of both of them are periodic and, moreover, after a while all trajectories issued from some point at T^*Y assemble there. Then we can construct a matrix symbol and its eigenvalues play a role formerly played by ℓ.

8.3.2 Simple Hamiltonian Flow

Inner Asymptotics

Let $\mathcal{U}$ be an open connected subset in T^*X disjoint from ∂X. Consider Hamiltonian billiards issued from $\mathcal{U}$, let Φ_t denote the generalized Hamiltonian flow. Assume that

$$|\nabla_{x,\xi} a(x,\xi)| \ge \epsilon \qquad \forall t \ \ \forall (x,\xi) \in \Phi_t(\mathcal{U}), \tag{8.3.6}$$

$$|\phi(x)| + |\{a,\phi\}(x,\xi)| \ge \epsilon \qquad \forall t \ \ \forall (x,\xi) \in \Phi_t(\mathcal{U}), \tag{8.3.7}$$

$$\Phi_t(x,\xi) = (x,\xi) \qquad \text{with} \ \ t = t(x,\xi) > 0 \ \ \forall (x,\xi) \in \mathcal{U}, \tag{8.3.8}$$

where $\phi \in \mathscr{C}^\infty(\bar{X})$, $\phi > 0$ in X and $\phi = 0$ on ∂X and

(8.3.9) Along $\Phi_t(\mathcal{U})$ reflections are simple without branching, which means that $\iota^{-1}\iota(y,\eta) \cap \{a(y,\eta) = \tau\}$ consists of two (disjoint) points $(y,\eta^{\pm})$ such that $\pm\{a,\phi\} > 0$ as $(y,\eta) \in \Phi_t(\mathcal{U})$, $y \in \partial X$ and $\tau = a(y,\eta)$.

Let us recall that $\iota : T^*X|_{\partial X} \to T^*\partial X$ is a natural map.

Then exactly as in the case without boundary

(8.3.10) One can select $t(x,\xi) = T(a(x,\xi))$ in $\mathcal{U}$ with $T \in \mathscr{C}^\infty$ (so period depends on the energy level only[6),7)]

and

(8.3.11) As $(x,\xi) \in \mathcal{U}$ and $0 < t < T(x,\xi)$ $\Phi_t(x,\xi)$ meets ∂X exactly $(N-1)$ times with $N = \mathsf{const}$; so as $N \ge 1$ billiard consists of N segments with the ends on the boundary.

We can consider instead of the operator A with the domain

$$\mathfrak{D}(A) = \{u \in \mathscr{H}^m(X), \eth Bu = 0\} \tag{8.3.12}$$

an operator $f(A)$ with

$$f(\tau) = \int T^{-1}(\tau)\, d\tau \tag{8.3.13}$$

[6)] See clarification in footnote [6)] in Chapter 6.

[7)] But there could be subperiodic trajectories with periods $n^{-1}T(a(x,\xi))$, $n = 2,3,\ldots$. The structure of subperiodic trajectories described in Subsection 6.3.1.

and introduce the corresponding propagator $e^{ih^{-1}tf(A)}$ so that the corresponding Hamiltonian flow Ψ_t of $f(A)$ is periodic with period 1.

However now structure of $f(A)$ near ∂X is a bit murky; but it is not really serious obstacle since we can consider problem

$$\bigl(\check{f}(hD_t)-A\bigr)e^{ih^{-1}tf(A)}=0, \tag{8.3.14}$$

$$\mathring{\partial} B e^{ih^{-1}tf(A)}=0 \tag{8.3.15}$$

where $\check{f}$ is an inverse function. Therefore its Schwartz kernel $U(x,y,t)$ satisfies problem (8.3.14)–(8.3.15) with respect to x and transposed problem with respect to y.

Note that due to assumptions (8.3.6)–(8.3.7) $e^{ih^{-1}tf(A)}Q$ is a Fourier integral operator with the symplectomorphism Ψ_t provided Q is h-pseudodifferential operator with the symbol supported in $\mathcal{U}$. Further, due to (8.3.8), (8.3.13) $\Psi_1=I$ and therefore $e^{ih^{-1}f(A)}Q$ is h-pseudodifferential operator. Therefore, we arrive to

Proposition 8.3.1. *Let A be a Schrödinger operator with the scalar principal symbol $a(x,\xi)$ satisfying* (8.3.6)–(8.3.8) *and* (8.3.13)*. Then*

$$e^{ih^{-1}f(A)-i\kappa}\equiv e^{ih^{-1}\varepsilon L} \qquad \text{in } \mathcal{U} \tag{8.3.16}$$

with $\varepsilon=h$ and h-pseudodifferential operator L; here κ is Maslov's constant and the principal symbol ℓ of L is defined by

$$e^{i\ell}=\prod_{1\le k\le N} e^{i\ell_k}e^{i\ell'_k} \tag{8.3.17}$$

where ℓ_k corresponds to k-th segment (see Theorem 6.2.3) and ℓ'_k corresponds to the k-th reflection.

Example 8.3.2. Consider Schrödinger operator. Assume that at the reflection point $a(x,\xi)=\xi_1^2+a(x',\xi')$, $b=\xi_1+i\beta(\xi')$. Then

$$e^{i\ell'_k}=-\bigl((\tau-a')^{\frac12}-i\beta\bigr)^{-1}\bigl((\tau-a')^{\frac12}+\imath\beta\bigr) \tag{8.3.18}$$

with the right-hand expression calculated in the corresponding reflection point.

Taking $\beta=0$ or $\beta=\infty$ (formally) we get $\ell'_k=0$ and $\ell'_k=\pi$ for Neumann and Dirichlet boundary conditions respectively.

Then we can use all the local results of Section 6.2, assuming that (8.3.16) holds with $h^l \le \varepsilon \le h$ and derive local spectral asymptotics with the same precision (up to $O(h^{2-d})$) as there.

Corollary 8.3.3. *(i) Let conditions of Proposition 8.3.1 be fulfilled. Then asymptotics with the remainder $o(h^{1-d})$ holds provided*

$$\mu_\tau\bigl(\{(x,\xi)\in \Sigma_\tau, \nabla_{\Sigma_\tau}\ell(x,\xi)=0\}\bigr)=0 \tag{8.3.19}$$

where ∇_{Σ_τ} means a gradient along Σ_τ.

(ii) Asymptotics with the remainder $O(h^{1-d+\delta})$ with small exponent $\delta>0$ holds provided

$$\mu_\tau\bigl(\{(x,\xi)\in \Sigma_\tau, |\nabla'\ell(x,\xi)|\le \varepsilon\}\bigr)=o(\varepsilon^{\delta'}) \tag{8.3.20}$$

with the small exponent $\delta'>0$.

(iii) Furthermore asymptotics with the remainder $o(h^{1-d})$ holds as (8.3.6)–(8.3.7) *are replaced by their non-uniform versions*

$$|\phi(x)|+|\{a,\phi\}(x,\xi)|>0 \qquad \forall t\ \ \forall (x,\xi)\in \Phi_t(\mathcal{U}\cap\Sigma_\tau\setminus\Lambda_\tau), \tag{8.3.7}'$$

where $\mu_\tau(\Lambda_\tau)=0$.

We leave to the reader to formulate statement similar to Statement (iii) but with the remainder estimate $O(h^{1-d+\delta})$.

Usually our purpose is the asymptotics with $Q_{1x}=\psi_1(x)$, $Q_{2y}=\psi_2(y)$ where ψ_j are smooth functions but not vanishing near the ∂X.

Corollary 8.3.4. *Asymptotics with the remainder estimate $o(h^{1-d})$ holds provided conditions* (8.3.6), (8.3.7)′, (8.3.8) *and* (8.3.19) *are fulfilled in $\mathcal{U}=T^*X$, $Q_j=\psi_j\in \mathscr{C}^\infty(\bar X)$ compactly supported.*

Proof. Due to corollary 8.3.3(iii) a contribution of the zone $\{x\colon x_1>\varepsilon\}$ to the remainder is $o_\varepsilon(h^{1-d})$ while a contribution of the zone $\{x\colon x_1\le\varepsilon\}$ to the remainder does not exceed $O(\varepsilon h^{1-d})$ as $\varepsilon\ge h$. □

These arguments could be improved to $\varepsilon=h^\delta$ and the remainder estimate could be improved to $O(h^{1-d+\delta})$. We leave both precise statements and precise arguments to the reader. Our goal is to derive even more sharp remainder estimate.

Asymptotics near Boundary

Let us consider $X = \{x\colon x_1 > 0\}$ and $\mathcal{U}$ open subset in $T^*\mathbb{R}^{d-1}_{x'} \times [0,\varepsilon]_{x_1} \times \mathbb{R}_\tau$ where $\varepsilon > 0$ is a very small constant. We are interested in

$$F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\,\mathsf{Tr}\big(e^{ih^{-1}tf(A)}Q\big) \tag{8.3.21}$$

with $Q = Q(x, hD'_x, hD_t)$ with symbol supported in $\mathcal{U}$; let us rewrite it as

$$F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\,\mathsf{Tr}\big(e^{ih^{-1}(t-\bar{t})f(A)}Q'e^{ih^{-1}\bar{t}f(A)}\big) = F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\,\mathsf{Tr}\big(e^{ih^{-1}tf(A)}Q'\big)$$

with $Q' = e^{ih^{-1}\bar{t}f(A)}Qe^{-ih^{-1}\bar{t}f(A)}$.

Note that if (8.3.6), (8.3.7) are fulfilled, $\varepsilon > 0$ and $\bar{t} \ge C\varepsilon$ are small enough then $Q' = Q'(x, hD_x, hD_t)$ is h-pseudodifferential operator with the symbol supported in $\{\epsilon'\bar{t} \le x_1 \le C'\bar{t}\}$ and we can rewrite

$$Q' = Q''(x, hD_x) + Q'''(x, hD_x, hD_t)\big(hD_t - f(A)\big);$$

therefore we can rewrite (8.3.21) in the same form but with Q replaced by Q''.

Now we can apply all the arguments of the previous Subsubsection *8.3.2.2 Asymptotics near Boundary* and of Section 6.2 and derive asymptotics with the remainder estimate as sharp as $O(h^{2-d})$ provided conditions (8.3.6)–(8.3.8) are fulfilled in the full measure as well as corresponding conditions of Section 6.2 to ℓ. Here as usual we lift the points of $(y', \eta', \tau) \in T^*\partial X$ to $(y', \eta^\pm) \in \Sigma_\tau$.

So, we need to analyze only a near tangent zone $\{x\colon x_1 \le \varepsilon, |\{a, x_1\}| \le \varepsilon\}$.

Let us assume that conditions (8.3.6), (8.3.7)$'$ and (8.3.8) are fulfilled for all $(x,\xi) \in T^*X : \tau_1 \le a(x,\xi) \le \tau_2$ with $\tau_1 < \tau_2$. In all examples we know (see Subsubsection *8.2.2.3 Analysis in the "Elliptic" Zone* below ∂X is *bicharacteristically flat*

$$x_1 = 0, \tau_1 \le a'(x,\xi') \le \tau_2 \implies a'_{x_1}(x,\xi) = 0; \tag{8.3.22}$$

so we assume in this section that this is the case.

Then (8.3.8) and (8.3.22) imply that

(8.3.23) All solutions of equation

$$z'' + b(\Psi_t(x',\xi'))z = 0 \qquad \text{with} \quad b(x',\xi) = \frac{1}{2}a'_{x_1x_1}(0,x',\xi') \tag{8.3.24}$$

are T_0-periodic for any $(x',\xi') \in T^*\partial X$, $\tau_1 \le a'(x,\xi') \le \tau_2$.

Therefore (8.3.8) and (8.3.22) imply that

$$\rho(x,\xi)\circ \Psi_t \asymp \rho \quad \text{with} \quad \rho = x_1 + |\{a, x_1\}| \quad \text{as} \quad \tau_1 \le a(x,\xi) \le \tau_2. \tag{8.3.25}$$

Consider zone $\mathcal{U}_\varepsilon = \{(x,\xi)\colon \rho(x,\xi) \asymp \varepsilon\}$. Blowing up $(x_1,\xi_1) \to \varepsilon^{-1}(x_1,\xi_1)$ we can prove easily that after this

$$\Psi_t(x,\xi) = \bigl(\Psi'_{t,\varepsilon}(x,\xi),\ \Psi''_{(x',\xi',t),\varepsilon}(x_1,\xi_1)\bigr), \tag{8.3.26}$$

with

$$\Psi'_{t,\varepsilon}(x,\xi) \sim \Psi'_{t,0}(x',\xi') + \sum_{n\ge 2} \varepsilon^n \Psi'_{t,(n)}(x,\xi) \tag{8.3.27}$$

and

$$\Psi''_{(x',\xi',t),\varepsilon}(x_1,\xi_1) \sim \sum_{n\ge 1} \varepsilon^n \Psi''_{(x',\xi',t),(n)}(x_1,\xi_1) \tag{8.3.28}$$

where $\Psi'_{t,0}$ is the Hamiltonian flow on $T^*\partial X$ and $\Psi''_{t,(1)}$ is the linearized with respect to (x_1,ξ_1) billiard flow (described by (8.3.24)). One can prove easily that $\Psi'_{t,(n)}$ and $\Psi''_{(x',\xi',t),(n)}$ are uniformly smooth.

Then in the blown-up coordinates (8.3.16) holds provided $\varepsilon \ge h^{\frac12-\delta}$ because $\hbar = \varepsilon^{-2}h$ and then we can apply all the previous arguments and derive asymptotics; the remainder estimate is as sharp as $O(h^{2-d})$ provided (in the original coordinates)

$$|\nabla_{(x,\xi)}\ell(x,\xi)| \ge \epsilon_0 \qquad \text{as} \quad x_1 \asymp \rho(x,\xi) \le \epsilon_1; \tag{8.3.29}$$

$x_1 \asymp \rho(x,\xi)$ means exactly that $x_1 \ge \epsilon|\xi_1|$.

Therefore we arrive to

Proposition 8.3.5. *Let A be a Schrödinger operator with the principal symbol $a(x,\xi)$ satisfying* (8.3.6)–(8.3.8) *and* (8.3.22)–(8.3.23). *Let the subprincipal symbol and the boundary condition be such that* (8.3.29) *is fulfilled. All these conditions are supposed to be fulfilled in $\{(x,\xi)\colon \rho(x,\xi)\le \epsilon_0\}$.*

Then the contribution of the zone $\{(x,\xi)\colon h^{\frac12-\delta} \le \rho(x,\xi) \le \epsilon_2\}$ to the remainder is $O(h^{2-d})$.

Since the contribution of the zone $\{(x,\xi)\colon \rho(x,\xi)\le\varepsilon\}$ to the remainder does not exceed $C\varepsilon^2 h^{1-d}$ and we can take $\varepsilon = h^{\frac12-\delta}$ we conclude that the contribution of the zone $\{(x,\xi)\colon \rho(x,\xi)\le\epsilon_2\}$ to the remainder is $O(h^{2-d-\delta})$. One can also estimate this way the contribution of the subperiodic trajectories. Thus we arrive (details are left to the reader) to

Theorem 8.3.6. *Let A be a Schrödinger operator with the principal symbol $a(x,\xi)$ satisfying* (8.3.6)–(8.3.8) *and* (8.3.22)–(8.3.23)*. Let the subprincipal symbol and the boundary condition be such that* (8.3.29) *is fulfilled. All these conditions are supposed to be fulfilled in the zone $\{(x,\xi)\colon \rho(x,\xi)\le \epsilon_0\}$.*

Then contribution of the zone $\{(x,\xi)\colon \rho(x,\xi)\le \epsilon_2\}$ to the remainder is $O(h^{2-d-\delta})$ while its contribution to the principal part of the asymptotics is given by the standard two term Weyl formula plus a correction term constructed according to Section 6.3 for symbol ℓ.

Problem 8.3.7. Recover $O(h^{2-d})$ remainder estimate. This should not be extremely hard especially if there are no subperiodic trajectories but definitely worth of publishing.

Example 8.3.8. Consider Laplacian $h^2\Delta$ on the standard hemisphere $\mathbb{S}^d_+ := \{x\colon |x|=1, x_1>0\}$ in $\mathbb{R}^{d+1}$ or harmonic oscillator $h^2\Delta+|x|^2$ on the standard half-space $\mathbb{R}^d_+ = \{x\colon x_1>0\}$.

(i) Consider a boundary operator of Example 8.3.2. Then $N=2$ and $\ell(z,\tau)=\ell'(z,\tau)+\ell'(\varrho(z),\tau)$ where $z=(x',\xi')$, $\varrho(z)$ is the *antipodal point* and ℓ' is defined by (8.3.18). Then one can express conditions to ℓ as conditions to β; we leave exact statements to the reader.

(ii) Consider Dirichlet or Neumann boundary conditions. Then if a perturbation is hx_1 one can calculate easily that $\ell=\kappa\rho+O(\rho^2)$ where ρ is the incidence angle of trajectory and $\kappa>0$. Then condition (8.3.29) is fulfilled.

Discussion

Unfortunately almost nothing is known about manifolds with the boundaries and all billiards closed; we are discussing geodesic billiards but one can ask the same questions about Hamiltonian billiards associated with the Schrödinger operator; in the last case the phase space S^*X is replaced by the portion of the phase space $\{\tau_1\le a(x,\xi)\le\tau_2\}$. We assume that the phase space is connected.

Example 8.3.9. Consider the standard Laplacian $h^2\Delta$ on one of the following manifolds:

(i) The standard hemisphere $\mathbb{S}^d_+ = \{x\in\mathbb{S}^d\,|x|=1, x_1>0\}$ is the most obvious example.

(ii) Starting from this let $\mathcal{X}$ be a connected closed manifold with all geodesics periodic and let G be a finite group of isometries of X. Then $\{x\colon j(x)=x\}$ are surfaces in $\mathcal{X}$ for any $j\in G$ (except $j=I$); all this surfaces together break $\mathcal{X}$ into several cells. Let one of them be X. Unless $\#G=2$ ∂X can have edges and other singularities but dealing with them is easy; see Sections 11.2 and 11.3.

(iii) In particular, consider Zoll-Tannery manifolds with the rotational symmetry $\mathcal{X}:=\{x=(x_1,x')\in [-L_1,L_2]\times \mathbb{S}^{d-1}\}$ with $g^{11}=1$, $g^{jk}=f(x_1)^2\delta_{jk}$ for $j=1,\dots,d$, $k=2,\dots,d$ and function f such that all geodesics are periodic[8)].

Then we can use $j(x)=(x_1,j'(x'))$ where $j'\in G'$ where G' is a finite group of isometries of $\mathbb{S}^{d-1}$.

(iv) Further, in the framework of (iii) if f is an even function[9)], we can also use $j(x)=(-x_1,x')$ alone or in conjugation with G'.

Example 8.3.10. Consider Schrödinger operator $h^2\Delta+V(|x|)$ where

(i) We consider harmonic oscillator $V(r)=r^2$.

(ii) We consider Coulomb Hamiltonian $V(r)=-r^{-1}$ on the negative energy levels.

Let G be a finite group of the isometries of $\mathbb{S}^{d-1}$. These isometries are extended to rotational isometries of $\mathbb{R}^d$ and $\{x\colon j(x)=x\}$ are surfaces in $\mathbb{R}^d$ for any $j\in\Gamma$ (except $j=I$); all this surfaces together break $\mathbb{R}^d$ into several cells. Let one of them be X. Unless $\#G=2$ ∂X can have edges and other singularities but dealing with them is easy; see Sections 11.2 and 11.3.

Problem 8.3.11. Either prove that if all billiards are closed then ∂X must be bicharacteristically flat or to construct counter-example

(i) For Laplacians only.

[8)] These manifolds are isometric (up to a factor) to a sphere $\mathbb{S}^d$ with the metrics $(r+h(\cos(s)))^2\,ds^2+\sin^2(s)\,d\theta^2$ with $s\in[0,\pi]$, $r=p/q$, $p,q\in\mathbb{N}$, $\gcd(p,q)=1$ and odd function $h:(-1,1)\to(-r,r)$. Then the length of equator is 2π, the length of meridian $2\pi r$ and the length of general geodesics $2\pi p$. See A. Besse [1], Theorem 4.13.

[9)] Which happens only as $h=0$ in footnote [8)].

(ii) For general Schrödinger operators without magnetic field.

(iii) For general Schrödinger operators with magnetic field.

8.3.3 Branching Hamiltonian Flow with "Scattering"

Let us consider two manifolds X_1 and X_2 glued along the connected component Y of the boundary. We consider Schrödinger operators A_j on X_j and these operators are intertwined through the boundary conditions. We assume that Hamiltonian flows have simple (no branching) reflections on each of them, so branching comes from "reflection-refraction". Further we are interested in the case when Hamiltonian flow on X_1 is periodic while on X_2 majority of trajectories are not periodic and, moreover, when majority of the branching billiards in $X_1 \cup X_2$ containing trajectories in X_2 are not periodic.

Analysis in X_2

Since we want to consider examples in which we can indeed analyze the long-term behavior of billiards rather than just postulate it, we will make rather restrictive assumptions about billiards. First we assume that billiard flow on X_1 satisfy all assumptions of the previous Subsection 8.3.2; in particular

$$\Phi_{1,\tau}(z)^N = I \text{ [10]}. \tag{8.3.30}$$

We also assume that $\Phi_{1,\tau}$ and $\Phi_{2,\tau}$ commute

$$\Phi_{1,\tau} \circ \Phi_{2,\tau} = \Phi_{2,\tau} \circ \Phi_{1,\tau}. \tag{8.3.31}$$

Then for points of $\Sigma_{2,\tau}$ periodicity properties of the branching Hamiltonian flow with reflections Ψ_t coincide with those of $\Psi_{2,t}$,[11].

Now we need to impose condition to the boundary operator in addition to $\{A, B\}$ being self-adjoint. Namely, consider a point $z = (x', \xi') \in T^*Y$,

[10] We define $\Phi_{j,\tau}$ in the following way: first, we lift $z \in T^*Y$ to $z \in T^*X_j|_Y \cap \Sigma_{j,\tau}$ with $\{a_j, x_1\}(z) > 0$, then launch a trajectory forward until it hits the boundary at $(y', \eta) \in T^*X_j|_Y$ and, finally, project it back to T^*Y.

[11] Indeed, it is sufficient to analyze such properties of $\Phi_{2,\tau}$ and a corresponding branching map Φ_τ which is $\Phi_{2,\tau}^k \Phi_{2,\tau}^l$ due to (8.3.30), (8.3.31). But if $\Phi_{2,\tau}^k \Phi_{2,\tau}^l(z) = z$ then $\Phi_{2,\tau}^{rl}(z) = \Phi_{2,\tau}^{rk} \Phi_{2,\tau}^{rl}(z) = z$.

and consider an auxiliary problem

$$a_j(x', D_1, \xi')u_j = 0 \qquad j = 1, 2, \tag{8.3.32}$$

$$\eth b(x', D_1, \xi')u = 0 \tag{8.3.33}$$

(where $u = (u_1, u_2)$) and consider solutions which are combinations of the exponents $e^{i\lambda_j x_1}$ where for each j either $\operatorname{Im}\lambda_j > 0$ or $\operatorname{Im}\lambda_j = 0$ and $\{a_j, x_1\}(x', \lambda_j, \xi') > 0$ [12]. Denote by Λ the set of points for which such non-trivial solutions exist. We assume that

$$\operatorname{mes}_{T^*Y}(\Lambda) = 0. \tag{8.3.34}$$

Then we arrive to the following

Proposition 8.3.12. *Let in the described setup conditions* (8.3.30), (8.3.31) *and* (8.3.34) *be fulfilled.*

Let Q_1, Q_2 be h-pseudodifferential operators with the symbols supported in T^*X_2 and

$$\mu_{2,\tau}(\Pi_{2,\tau} \cap \operatorname{supp}(q_1) \cap \operatorname{supp}(q_2)) = 0 \tag{8.3.35}$$

where $\Pi_{2,\tau}$ is the set of point $(y, \eta) \in \Sigma_{2,\tau}$ periodic with respect to $\Psi_{2,t}$.

Then for $\Gamma(Q_{1x}e(., ., \tau)\,{}^tQ_{1y})$ the standard Weyl two-term asymptotics holds with the remainder estimate $o(h^{1-d})$.

Remark 8.3.13. (i) Note that

$$\mu_{j,\tau}(\Lambda_{j,\tau}) = 0 \tag{8.3.36}$$

where $\Lambda_{j,\tau} = \Lambda'_{j,\tau} \cup \Lambda''_{j,\tau}$, $\Lambda'_{j,\tau}$ is the set of dead-end points of billiard (not generalized billiard) flow and $\Lambda''_{j,\tau}$ is the set of points $(y, \eta) \in \Sigma_{j,\tau}$ such that $\iota_j \Psi_{j,t}(y, \eta) \in \Xi_{3-j,\tau}$ where $\Xi_{k,\tau} = \iota_k\{(x, \xi) \in \Sigma_{k,\tau}, \{a_k, x_1\} = 0\}$.

(ii) Note that the grows properties of the branching Hamiltonian flow with reflections Ψ_t coincide with those of $\Psi_{2,t}$ as long as we ensure that along billiards we do not approach to glancing rays on X_1:

$$\{a_1, x_1\}(x, \xi) \ge h^{\delta_1} \qquad \forall (x, \xi) \in \iota_1^{-1}\iota_2 \Psi_{2,t}(y, \eta) \cap \Sigma_{1,\tau} \tag{8.3.37}$$

$$\forall (y, \eta) \in \operatorname{supp}(q_2) \ \ \forall t : \pm t \in [0, T] \ \ \Psi_{2,t}(y, \eta) \in T^*X_2|_Y$$

[12] As we assume that $x_1 > 0$ on both X_1 and X_2 and $x_1 = 0$ on Y, it means that we consider solutions which either are of boundary-layer type or outgoing.

where $T = T(h)$. Then in the framework of Section 7.4 one can recover similar results: namely that two-term Weyl asymptotics holds with the remainder estimate $O(T(h)^{-1}h^{1-d})$ which is usually $O(h^{1-d}|\log h|^{-1})$ or $O(h^{1-d+\delta})$ (depending on the growth and non-periodicity conditions to $\Psi_{2,t}$. Exact statement and proof are left to the reader (which is a relatively easy problem).

(iii) In Statement (ii) we need also to assume a more sharp version of (8.3.34): namely if we consider approximate solutions to (8.3.32)–(8.3.33) with precision ε we get set $\Lambda_{\tau,\varepsilon}$ and we need to impose either condition $\mathsf{mes}(\Lambda_{\tau,\varepsilon}) = o(\varepsilon^{\delta})$ or condition $\mathsf{mes}(\Lambda_{\tau,\varepsilon}) = o(|\log\varepsilon|^{-1})$ as $\varepsilon \to +0$.

(iv) In Subsubsection *7.4.3.6 Non-Periodicity Condition for Branching Billiards* we did not consider points $z \in T^*Y$ where some of operators a_j are elliptic on $\iota^{-1}z$ rather than hyperbolic. If we want to consider such points we need to assume condition (8.3.34) or more sharp version of it described in Statement (iii).

(v) Property (8.3.37) is guaranteed if either a_2 is cylindrically symmetric on T^*X_2 or if there no complete internal reflection for billiards from X_2 at all:

$$|\{a_1, x_1\}(x,\xi)| \ge \epsilon|\{a_2, x_1\}(x,\eta)|' \tag{8.3.38}$$
$$\forall (x,\xi) \in T^*X_1|_Y \cap \Sigma_{1,\tau}\ \forall (x,\eta) \in T^*X_2|_Y \cap \Sigma_{2,\tau} : \iota_1(x,\xi) = \iota_2(x,\eta).$$

Analysis in X_1

However if operators Q_1 and Q_2 have symbols supported in X_1 situation is very different: trajectories of a_1 are periodic but the typical closed branching billiard is the one which does not contain segments of Hamiltonian trajectories of a_2. More precisely, due to conditions

$$\mu_{2,\tau}(\Pi_{2,\tau}) = 0 \tag{8.3.39}$$

and (8.3.36) we conclude that

$$\mu_{1,\tau}(\Pi'_{1,\tau}) = 0 \tag{8.3.40}$$

where $\Pi'_{1,\tau}$ is the set of points $z \in \Sigma_{1,\tau}$ such that there exists a billiard trajectory of Ψ_t starting and ending in z and which is not entirely billiard of $\Psi_{1,t}$.

Then for arbitrarily large $T > 0$ and arbitrarily small $\varepsilon > 0$ there exists a set $\mathcal{U} = \mathcal{U}_{T,\varepsilon} \subset T^*X$ such that

$$\mu_{1,\tau}(\Sigma_{1,\tau} \setminus \mathcal{U}) \le \varepsilon, \tag{8.3.41}$$

$$\operatorname{dist}(\mathcal{U}, \Pi'_{1,\tau,T} \cup \Lambda_{1,\tau,T}) \ge \gamma, \tag{8.3.42}$$

$$\operatorname{dist}(Y, \pi_x \mathcal{U}) \ge \gamma \tag{8.3.43}$$

with $\gamma = \gamma(T, \varepsilon) > 0$ were $\Pi'_{1,\tau,T}$ s the set of points $z \in \Sigma_{1,\tau}$ such that there exists a billiard trajectory of Ψ_t of the length not exceeding T, starting and ending in z and which is not entirely billiard of $\Psi_{1,t}$ and $\Lambda_{1,\tau,T}$ is the set of points $z \in \Sigma_{1,\tau}$ such that billiard trajectory of a_1 hits the boundary at point of $\Lambda_{2,T,\tau}$. As usual, these sets increase as T increases, their unions (with respect to T) coincide with $\Pi'_{1,\tau}$ and $\Lambda_{1,\tau}$ respectively and $\Lambda_{1,\tau,T}$ and $\Lambda_{1,\tau,T} \cap \Pi'_{1,\tau,T}$ are closed sets.

Therefore there exists operator $Q = Q_{T,\varepsilon}$ such that $\operatorname{supp}(q) \subset \mathcal{U}$ and

$$|\Gamma\big(Q_{1x} e(.,.,\tau)\, {}^t((I-Q)Q_2)_y\big) - \kappa'_0 h^{-d} - \kappa'_1 h^{1-d}| \le C_0 \varepsilon h^{1-d} \tag{8.3.44}$$

with $\kappa'_j = \kappa'_{j,Q_1,Q_2,Q}(\tau)$.

This estimate holds for any fixed τ satisfying (8.3.39). Therefore we need to consider $\Gamma\big(Q_{1x} e(.,.,\tau)\, {}^t Q_{2y}\big)$ with $\operatorname{supp}(q_2) \subset \mathcal{U}$.

We can assume without any loss of the generality that (8.3.42), (8.3.43) are fulfilled for all τ. Then for $\epsilon \le |t| \le T$

$$\Gamma\big(Q_{1x} U(.,.,t)\, {}^t Q_{2y}\big) \equiv \Gamma\big(Q_{1x} Z(.,.,t)\, {}^t Q_{2y}\big) \tag{8.3.45}$$

where as $\pm t > 0$ $Z(x,y,t)$ is the Schwartz kernel of "the approximate propagation with the scattering semigroup" $\boldsymbol{Z}(t)$ constructed in the following way (for a sake of simplicity in notations we consider only $t > 0$:

(i) We represent $Q_2 = Q_{2,1} + ... + Q_{2,N}$ where symbols $Q_{2,\nu}$ have small supports;

(ii) For each ν we select $0 = t_{0,\nu} < ... < t_{M,\nu} = t$ in such a way that for any $z \in \operatorname{supp}(q_{2,\nu})$ $\Psi_{t'}(z)$ is disjoint from Y as $t' = t_{j,\nu}$ and $\Psi_{t'}(z)$ hits Y no more than once as $t_{j,\nu} \le t' \le t_{j+1,\nu}$;

(iii) We set

$$\boldsymbol{Z}(t) Q_{2,\nu} = \Big(\prod_{0 \le j \le M-1} \psi e^{ih^{-1}(t_{j+1,\nu} - t_{j,\nu}) f(A)} \Big) \cdot Q_{2,\nu} \tag{8.3.46}$$

where $\psi \in \mathscr{C}^\infty$ is supported in $\{x \in X_1, \mathsf{dist}(x, Y) \ge \gamma'\}$ and equals 1 in $\{x \in X_1, \mathsf{dist}(x, Y) \ge 2\gamma'\}$.

Then

$$Z(t_1 + t_2)Q_2 \equiv Z(t_2)Z(t_1)Q_2 \qquad \text{as} \quad \pm t_1 > 0, \pm t_2 > 0 \tag{8.3.47}$$

and

$$Z(t)^* \equiv Z(-t) \tag{8.3.48}$$

and due to periodicity of $\Psi_{1,t}$ with period T_0 $Z(\pm T_0)$ are h-pseudodifferential operators.

Remark 8.3.14. In contrast to what we had before they are not necessarily unitary because *reflection coefficients* are not unitary anymore. More precisely,

(i) If $\Psi_{1,t}$ hits $T^*X|_Y$ at points (x', ξ) with $(x', \xi') \notin \iota_2 \Sigma_{2,\tau}$ [13)] the reflection coefficient $\varkappa_{11}(x', \xi', \tau)$ still satisfies $|\varkappa_{11}(x', \xi', \tau)| = 1$.

(ii) However if $\Psi_{1,t}$ hits $T^*X|_Y$ at points (x', ξ) with $(x', \xi') \in \iota_2 \Sigma_{2,\tau}$ then the reflection-refraction matrix $\varkappa(x', \xi', \tau) = \big(\varkappa_{jk}(x', \xi', \tau)\big)_{j,k=1,2}$ is unitary but the reflection coefficient $\varkappa_{11}(x', \xi', \tau)$ satisfies only $|\varkappa_{11}(x', \xi', \tau)| \le 1$ and the equality means exactly that $\varkappa_{12}(x', \xi', \tau) = \varkappa_{21}(x', \xi', \tau) = 0$ (then $|\varkappa_{22}(x', \xi', \tau)| = 1$ as well).

So the principal symbol of $Z(T_0)$ is the product of $e^{i\ell_0(x,\xi)}$ where $\ell_0(x, \xi)$ is given by the usual formula for manifolds without boundary (but with trajectories replaced by billiards) and also of reflection coefficients coefficients in the point of reflection. If the reflection coefficients do not vanish we can rewrite the answer in the form

$$Z(T_0) \equiv e^{i\varepsilon h^{-1} L} \tag{8.3.49}$$

with h-pseudodifferential operator L, $L + L^* \le 0$ and $\varepsilon = h$ (but in calculations below we are not assuming this).

We will use this formula even if the coefficients vanish: in this case we simply formally allow $\mathsf{Im}\, \ell = +\infty$.

[13)] I.e. at the point of the complete internal reflection.

Then as $\bar{\chi} \in \mathscr{C}_0^\infty[\epsilon - 1, 1 - \epsilon]$

$$F_{t\to h^{-1}\tau}\bar{\chi}_{T_0}(t - nT_0)\Gamma\big(Q_{1x}Z(.,.,t)\,{}^tQ_{2y}\big) = h^{1-d}J(h,\varepsilon,n,\tau)\cdot\widehat{\bar{\chi}}\big(\frac{\tau}{h}\big) + O\big(h^{2-d}\big), \tag{8.3.50}$$

with

$$J(h,\varepsilon,n,\tau) = (2\pi)^{1-d}\int_{\Sigma_{1,\tau}} e^{ih^{-1}((\varepsilon\,\mathrm{Re}\,\ell-\tau)n+\varepsilon\,\mathrm{Im}\,\ell|n|)}\,d\mu_{1,\tau}. \tag{8.3.51}$$

We can see easily that

(8.3.52) $J(h,\varepsilon,n,\tau) = o(1)$ as $n\to\infty$ provided

$$\mu_\tau\big(\big\{(x,\xi)\in\Sigma_\tau:\ \mathrm{Im}\,\ell(x,\xi) = \nabla\,\mathrm{Re}\,\ell(x,\xi) = 0\big\}\big) = 0. \tag{8.3.53}$$

On the other hand, note that

$$\sum_{n\in\mathbb{Z}\setminus 0} e^{ih^{-1}(n\alpha-|n|\beta)} = \frac{2e^{-\beta}\cos\alpha - 2e^{-2\beta}}{1 - 2e^{-\beta}\cos\alpha + e^{-2\beta}} \qquad \text{as}\ \ \beta > 0 \tag{8.3.54}$$

and formally we can extend this for $\beta = 0$ as well.

Then after summation with respect to n for $\beta = 0$ we get exactly $\partial_\alpha\Upsilon(\alpha)$ with Υ defined by (6.2.68) and for $\beta > 0$ we get that the expression (8.3.54) is equal to $\partial_\beta\Upsilon *_\alpha (2\pi)^{-1}G(\alpha,\beta)$ where $G(\alpha,\beta) = 2\beta\big(\alpha^2+\beta^2\big)^{-1}$ and as usual $*_\alpha$ means a convolution with respect to α.

Note that

(8.3.55) Function

$$\Upsilon(\alpha,\beta) \coloneqq \Upsilon *_\alpha G(\alpha,\beta) \tag{8.3.56}$$

is harmonic function as $\beta > 0$, coincides with $\Upsilon(\alpha)$ as $\beta = 0$, is $o(1)$ as $\beta\to+\infty$ and is periodic with respect to α.

Therefore taking in account results of this subsection we arrive to the following

Theorem 8.3.15. *Let in the described setup conditions* (8.3.30), (8.3.31), (8.3.34), (8.3.39) *and* (8.3.53) *be fulfilled.*

Then asymptotics

$$\mathsf{\Gamma e}(.,.,\tau) = \kappa_0(\tau)h^{-d}+ \Big(\kappa_1(\tau) + \int_{\Sigma_\tau} \Upsilon\big(h^{-1}(\varepsilon\,\mathsf{Re}\,\ell-\tau), -h^{-1}\varepsilon\,\mathsf{Im}\,\ell\big)\,d\mu_\tau\Big)\,d\mu_\tau\Big) + o(h^{1-d}) \tag{8.3.57}$$

holds.

Problem 8.3.16. Calculate all coefficients $\varkappa_{jk}$ for two Schrödinger operators A_1 and A_2.

We will solve this problem in the very special case:

Example 8.3.17. Assume that (at the given point $z \in T^*Y$) $a_j = c_j^2|\xi|^2$ and the boundary condition is

$$u_2 = \alpha u_1, \tag{8.3.58}$$

$$D_1 u_2 = -\beta D_1 u_1 \tag{8.3.59}$$

(recall that $x_1 > 0$ in both X_1, X_2; in more standard notations of $x_1 < 0$ in X_2 one needs to skip "$-$"). Then $\{A, B\}$ to be self-adjoint requires $\alpha\beta^\dagger = c_1^2 c_2^{-2}$. We do not consider gliding points since their measure is 0.

(i) Consider point ξ' with $c_1^2|\xi'|^2 < \tau < c_2^2|\xi'|^2$. One can prove easily that

$$\varkappa_{11}(\xi',\tau) = e^{2i\varphi} \tag{8.3.60}$$

with

$$\varphi(\xi',\tau) = \mathsf{arctan}\Big(|\beta|^{-2}c_2^{-2}c_1^2\big(c_2^2|\xi'|^2-\tau\big)^{\frac{1}{2}}\big(\tau - c_1^2|\xi'|\big)^{-\frac{1}{2}}\Big) \tag{8.3.61}$$

and therefore assumption (8.3.53) is fulfilled provided $c_1 \neq c_2$, c_j, α, β are symmetric (i.e. $c_j\varrho = c_j$) and the subprincipal symbol is 0. If $c_1 > c_2$ this case does not appear.

(ii) Consider point ξ' with $c_j^2|\xi'|^2 < \tau$ for both $j = 1, 2$. One can prove easily that

$$\varkappa_{11}(\xi',\tau) = \big(\omega-1\big)\big(\omega+1\big)^{-1} \tag{8.3.62}$$

with

$$\omega(\xi',\tau) = |\beta|^{-2}c_2^{-2}c_1^2\big(\tau - c_2^2|\xi'|^2\big)^{\frac{1}{2}}\big(\tau - c_1^2|\xi'|\big)^{-\frac{1}{2}} \tag{8.3.63}$$

and $|\varkappa_{11}| < 1$ so assumption (8.3.53) is fulfilled.

The following problems seem to be rather straightforward but not easy and worth publishing:

Problem 8.3.18. Under stronger conditions to $\Psi_{2,t}$ and ℓ (see (8.3.20)) prove remainder estimate $O(h^{1-d+\delta})$.

Problem 8.3.19. Prove the same results as Hamiltonian billiards of a_1 are assumed to be periodic only on one energy level τ rather than on neighboring levels.

8.3.4 Two Periodic Flows

Now we consider the case of both flows Ψ_{1t} and Ψ_{2t} generated by $f(a_1)$ and $f(a_2)$ both satisfying (8.3.31) and (8.3.30) which becomes

(8.3.64) $\Phi_{j,\tau}^{N_j} = I$ for $j = 1, 2$ where N_1 and N_2 are not necessarily equal.

Remark 8.3.20. (i) Here as in the previous Subsection 8.3.3 we can consider several manifolds attached to one another: X_j is a connected manifold, $\partial X_j = Y_j \cup Y_j'$ with disjoint $Y_j \cup Y_j' = \emptyset$ and Y_j' and Y_{j+1} are glued together ($j = 1, \dots, m$) while Y_1 and Y_m' may be or may be not or may be only partially glued together.

(ii) Further, we can consider cases when along (some part of) Y more than two manifolds are glued together etc

We leave such generalization to the reader.

Examples and Discussion

Let us start from examples:

Example 8.3.21. (i) In the framework of Example 8.3.9(ii) we again consider partition of $\mathcal{X}$ into cells and on each cell X_j we define operator as either Laplacian $h^2 c_j^2 \Delta$ or more generally as a Schrödinger operator $h^2 c_j^2 \Delta - E_j$.

(ii) In the framework of Example 8.3.9(iv) we consider $X_1 = \{(x_1, x_2) \in (-1, 0) \times (0, 2\pi)\}$ and $X_1 = \{(x_1, x_2) \in (0, 1) \times (0, 2\pi)\}$ and we glue them along $x_1 = 0$ identifying $(+0, x_2)$ with $(-0, x_2)$. Observe that $N_j = p_j$.

(iii) In Example 8.3.9(iv) with $p_1 = 1$ we have $N_1 = 1$ and we can glue it with anything else as (8.3.31) is fulfilled automatically then.

Example 8.3.22. Consider two Schrödinger operators in $\mathbb{R}^d_+ = \mathbb{R}^+ \times \mathbb{R}^{d-1}$

$$A_j = \frac{1}{2}\omega_j^2\big(h^2\Delta + |x|^2 - E_j\big) \tag{8.3.65}$$

intertwined through boundary conditions.

Our idea was to make all flows periodic with the common period. It works with Example 8.3.22 where $T_j = 2\pi\omega_j^{-2}$. Indeed, we want to find asymptotics of $\mathsf{N}_h^-(0)$ which is the maximal dimension of the *negative subspace* of the operator A which does not change if we replace A by $J^{-\frac{1}{2}}AJ^{-\frac{1}{2}}$ with the positive self-adjoint operator J. Picking up $J = J(x)$ equal ω_j^2 in X_j we arrive to the case of $\omega_j = 1$ and $T_j = 2\pi$. Therefore in this example one can make periods equal $T_1 = T_2$.

However this approach does not seem work with the Example 8.3.21 (unless some unnatural conditions to ω_j, E_j are imposed). The main obstacle here is that multiplying operator by $J^{-\frac{1}{2}}$ from both sides and calculating $f(A)$ do not play well together.

Reduction to the Boundary

First of all observe that the contribution to the remainder of the domain $\{z, \mathsf{dist}(z, T^*Y) \le \varepsilon_0\}$ does not exceed $C_0\epsilon_0^2 h^{1-d}$ and therefore one needs to consider the contribution of $\{z, \mathsf{dist}(z, T^*Y) \ge \varepsilon_0\}$ with arbitrarily small but fixed ε_0 [14)].

In this zone one can consider $U(x, y, t)$ as the solution of the Cauchy problem with respect to x_1 (or y_1) with the data at $\{x_1 = 0\}$ (or $\{y_1 = 0\}$). Rewriting therefore $U = R_x U\, {}^tR_y$ where R is an operator resolving this Cauchy problem is given by an oscillatory integral we can rewrite

$$\Gamma\big(U(.,.,t)\, {}^tQ_y\big) \equiv \Gamma'\big(\mathcal{Q}(x', hD'_x, hD_t, h)V\big) \tag{8.3.66}$$

with h-pseudodifferential operator R and $V = \eth_{x,m-1}\eth_{y,m-1}U$ the Cauchy data for U (assuming that m is an order of A).

One can see easily that as Q is an operator with the symbol supported in T^*X_j, the principal symbol of $\mathcal{Q}$ at (x', ξ, τ) is defined as an averaging of Q along $\Psi_{jt}(z_j)$ with $t \in [0, t_j(z_j)]$ where $z_j = \iota^{-1}(x', \xi') \cap \Sigma_{j\tau}$ and $t_j(z_j)$ is the time of the next hit of the boundary.

[14)] If our target is $o(h^{1-d})$ remainder estimate.

Then instead of continuous family of Fourier integral operators $e^{ih^{-1}tA}$ we can consider a discrete family $\mathcal{F}^n$ defined by the following way: consider $v_j = \mathfrak{d}_{m-1}u_j = (v_j^-, v_j^+)$ with $v_j^{\mp}$ corresponding to incoming and outgoing solutions.

We have two types of trajectories: those in $\Sigma_{j\tau}$ which hit $T^*X|_Y$ at points elliptic for $(a_{3-j} - \tau)$ where $j = 1, 2$ and those which hit $T^*X|_Y$ at points hyperbolic for $(a_k - \tau)$ for both $k = 1, 2$. The analysis of the former is of no different from what we have seen in the previous subsection: we just consider Ψ_{jt} and construct the (real-valued) symbol ℓ as long as we assume that $(x', \xi) \notin \Lambda$ where Λ is defined in the paragraph preceding (8.3.34).

The analysis of the latter is more interesting. Here as before we have a matrix of reflection-refraction $(\varkappa_{jk})_{j,k=1,2}$ which is a symbol defined on $\mathcal{U} \times (\tau - \varepsilon', \tau + \varepsilon')$ where $\mathcal{U}$ is a a zone described above.

We need to construct a transformation along trajectories matrix L but it will be of the different type. Consider generic point $z \in T^*Y$, and collection of $\{\Phi_{1,\tau}^j \Phi_{2,\tau}^k\}_{i=0,\dots,N_1-1;k=0,\dots N_2-1}$. Not all points here are necessarily distinct; let us numerate distinct points by greek indices: z_α with $\alpha \in \{1, \dots, N\}$ [15]. Let $\phi_j : \{1, \dots, N\} \to \{1, \dots, N\}$ be permutations: $\beta = \phi_j(\alpha) \iff z_\beta = \Phi_{j,\tau}(z_\alpha)$.

We consider $2N$-vectors $v_{j\alpha}$ with norms

$$\|v\| = \Big(\sum_{\alpha=1,\dots,N} \sum_{j=1,2} |v_{\alpha j}|^2 \cdot |\{a_j, x_1\}(z_\alpha)| \Big)^{\frac{1}{2}}. \tag{8.3.67}$$

Then we introduce a sparse matrix $2N \times 2N$-matrix L in the following way:

$$L_{\beta k, \alpha j} = |\{a_j, x_1\}(z_\beta)|^{-\frac{1}{2}} |\{a_j, x_1\}(z_\alpha)|^{\frac{1}{2}} e^{i\ell_{\alpha j}} \tag{8.3.68}$$

if $k = j$ and $\beta = \phi_j(\alpha)$ and 0 otherwise; here $\ell_{\alpha j}$ is an integral of the subprincipal symbol along trajectory of Ψ_{jt} from z_α to z_β. Obviously matrix L is unitary[16].

Finally let us consider a matrix $M = \tilde{\varkappa} L$ where $\tilde{\varkappa}$ is a matrix κ raised to $2N \times 2N$-matrix: $\tilde{\varkappa}_{\beta k, \alpha j} = \kappa_{jk}(z_\alpha)$ as $\beta = \alpha$ and 0 otherwise. Matrix M shows how data on Y is transformed after a single "step" (propagation followed by reflection-refraction). One can see easily that

(8.3.69) Matrices $\tilde{\varkappa}$ and M are unitary[16].

[15] Then N divides $N_1 N_2$ and is divisible by N_1 and N_2.

[16] In the norm (8.3.67).

Analysis of the Evolution

One can prove easily that modulo $O_T(h^{2-d})$

$$(8.3.70)\quad F_{t\to h^{-1}\tau}\chi_T(t)\Gamma'\big(\mathcal{Q}(x', hD'_x, hD_t, h)V\big) \equiv$$
$$(2\pi h)^{1-d}\chi_T(T_{j_1}+\dots+T_{j_K})\iint \sum_{K\neq 1}\ \sum_{(j_1,\dots,j_K)\in\{1,2\}^K}\ \sum_{(\alpha_1,\dots,\alpha_K)\in\{1,\dots,N\}^K}$$
$$M_{\alpha_1 j_1,\alpha_2 j_2}M_{\alpha_2 j_2,\alpha_3 j_3}\cdots M_{\alpha_{K-1}j_{K-1},\alpha_K j_K}e^{-ih^{-1}\tau(T_{j_1}+\dots+T_{j_N})}\mathcal{Q}_{\alpha_K j_K,\alpha_1 j_1}\,dx'd\xi'$$

where $\chi\in\mathscr{C}_0^\infty$ is supported in $[\frac12,1]$ and symbols M_{jk} and $\mathcal{Q}$ and T_j are calculated at (x',ξ',τ) [17] and since we consider only trajectories which are periodic on the energy levels (close to) τ in this sum $\mathcal{Q}_{\alpha_K j_K,\alpha_1 j_1}=0$ unless $\alpha_K=\alpha_1$.

One can rewrite terms with equal K as

$$(2\pi h)^{1-d}\int \hat\chi(\lambda)e^{iT^{-1}\lambda(T_{j_1}+\dots+T_{j_K})}\iint \sum_{j=(j_1,\dots,j_K)\in\{1,2\}^K}\ \sum_{(\alpha_1,\dots,\alpha_K)\in\{1,\dots,N\}^K}$$
$$M_{\alpha_1 j_1,\alpha_2 j_2}M_{\alpha_2 j_2,\alpha_3 j_3}\cdots M_{\alpha_{K-1}j_{K-1},\alpha_K j_K}e^{-ih^{-1}\tau(T_{j_1}+\dots+T_{j_K})}\mathcal{Q}_{\alpha_K j_K,\alpha_1 j_1}\,dx'd\xi'\,d\lambda$$

which in turn equals

$$(8.3.71)\qquad (2\pi h)^{1-d}\int\hat\chi(\lambda)\iint \mathsf{tr}\big(S(\lambda,\tau,x',\xi')^K\mathcal{Q}\big)\,dx'd\xi'\,d\lambda$$

where S is $2N\times 2N$ unitary [16] matrix

$$(8.3.72)\qquad S(\lambda T^{-1},\tau,x',\xi')=\big(e^{i(\lambda T^{-1}+\tau h^{-1})T_j}M_{\beta k,\alpha_j}\big)_{\beta k,\alpha j}.$$

Consider eigenvalues of this matrix; as $T\to\infty$ they tend to eigenvalues of the same matrix with $\lambda=0$:

$$(8.3.73)\qquad S(0,\tau,x',\xi')=\big(e^{i\tau h^{-1}T_j}M_{jk}\big)_{j,k=0,\dots,L-1}.$$

Therefore if

$$(8.3.74)\qquad \mathsf{mes}\big(\big\{(x',\xi'):\rho\in\mathsf{Spec}\big(S(0,0,x',\xi')\big)\big\}\big)=0)\qquad \forall\rho\in\mathbb{R}$$

[17] Where $T_j=T_j^*(\tau)/N_j$ is the time to run one leg and T_j^* is a full period.

then this term (8.3.73) is $o(h^{1-d})$ and therefore we estimated expression (8.3.70) by $o(Th^{1-d})$ (as $\tau = o(1)$) which in the end of the day returns remainder estimate $o(h^{1-d})$ with the Tauberian main part.

Consider correction to the main part; so we integrate (8.3.70) with $\chi = 1$ from $\tau = -\infty$ to $\tau = \tau$ resulting in (after multiplication by h^{-1} and modulo $o(h^{1-d})$) after taking trace we get

$$\begin{aligned}(8.3.75)\quad \Omega(\tau) \coloneqq (2\pi h)^{1-d} \iint \sum_{K\neq 1} \sum_{j=(j_1,\dots,j_K)\in\{1,2\}^K} \sum_{(\alpha_1,\dots,\alpha_K)\in\{1,\dots,N\}^K} \\ (T_{j_1}+\dots+T_{j_K})^{-1} e^{-ih^{-1}\tau(T_{j_1}+\dots+T_{j_K})} \\ \times \mathsf{tr}\big(M_{\alpha_1 j_1,\alpha_2 j_2} M_{\alpha_2 j_2,\alpha_3 j_3}\cdots M_{\alpha_{K-1} j_{K-1},\alpha_K j_K}\mathcal{Q}_{\alpha_K j_K,\alpha_1 j_1}\, dx'd\xi'\big).\end{aligned}$$

Therefore we have proven

Theorem 8.3.23. *Consider two Schrödinger operators A_1 and A_2 with all periodic trajectories on levels close to τ, satisfying* (8.3.30) *and* (8.3.31). *Further, let* (8.3.34) *and* (8.3.74) *be fulfilled. Then asymptotics*

$$(8.3.76)\qquad \mathsf{N}_h(\tau) = \kappa_0(\tau)h^{-d} + \big(\kappa_1(\tau)+\Omega(\tau)\big)h^{1-d} + o(h^{1-d})$$

holds as $h\to +0$ and $\tau = o(1)$ with the standard coefficients κ_0, κ_1 and $\Omega(\tau)$ defined by (8.3.75).

Example 8.3.24. Consider Example 8.3.17 in the new conditions to Hamiltonian flows. We can use results of that example immediately to treat billiards with complete internal reflections (thus non-branching).

To treat branching billiards we need to calculate eigenvalues of the matrix M. If $\Phi_{1,\tau} = \Phi_{2,\tau}$ and therefore $N_1 = N_2$ it is easy as $M = C\otimes \varkappa$ where C is a cyclic matrix of order M and condition (8.3.74) for M is equivalent to the same condition for $\varkappa$. It follows from calculations of Example 8.3.17 that $\varkappa_{22} = -\varkappa_{11}$; thus we arrive to

$$(8.3.77)\qquad \lambda = e^{\pm i\varphi},\qquad \varphi = \arccos \varkappa_{11} = \arccos\big((\omega-1)(\omega+1)^{-1}\big);$$

then obviously (8.3.74) is fulfilled and asymptotics (8.3.76) holds.

Problem 8.3.25. (i) In the framework of Example 8.3.24 derive formula for Ω through function Υ defined by (6.2.68).

(ii) In the framework of Example 8.3.24 consider the case when $\Phi_{1,\tau} = \Phi_{2,\tau}$ is not fulfilled. In this case calculation of eigenvalues of M could be really challenging.

Remark 8.3.26. On the contrary, assume that condition (8.3.74) is not fulfilled. Then there will be eigenvalues of high multiplicity or clusters of eigenvalues located in $o(h)$-vicinities of solutions τ of equation $\det(M-1) = 0$. Assuming that $\Phi_{1,\tau} = \Phi_{2,\tau}$ one can rewrite this equation as

$$\cos\bigl(-\varphi + \frac{1}{2}\tau h^{-1}(T_1 + T_2)\bigr) = \cos\alpha \cdot \cos\bigl(-\psi + \frac{1}{2}\tau h^{-1}(T_1 - T_2)\bigr) \tag{8.3.78}$$

as $\varkappa = \begin{pmatrix} e^{i(\phi+\psi)}\cos\alpha & e^{i(\phi+\chi)}\sin\alpha \\ -e^{i(\phi-\chi)}\sin\alpha & e^{i(\phi-\psi)}\cos\alpha \end{pmatrix}$ which is the general form of the unitary matrix.

The following problems seem to be a difficult one and worth of publication:

Problem 8.3.27. Prove the same results as flows $\Psi_{j,t}$ are assumed to be periodic only on one energy level τ rather than on neighboring levels.

Problem 8.3.28. Consider two manifolds satisfying the same assumptions as before albeit with the measure 0 of periodic billiards on each of them. Assume that $\Phi_{1,\tau}^p \Phi_{2,\tau}^q = I$ for some $p, q \in \mathbb{N}$ (recall that we assume that $\Phi_{1,\tau}$ and $\Phi_{2,\tau}$ commute.

Results should be similar to those of Subsection 8.3.3.

Example 8.3.29. Consider manifolds X_1 and X_2 with metrics $r_i^2\, ds^2 + \sin^2(s)\, d\theta^2$ and $s \in [o, \pi/2]$ and $s \in [\pi/2, pi]$ respectively albeit with r_1, r_2 which are irrational but such that $q_1 r_1 + q_2 r_2 = p$ with $q_1, q_2, p \in \mathbb{N}$.

On the contrary, the following problem seems to be relatively easy:

Problem 8.3.30. (i) Prove the same results as before when there are more than two manifolds X_j $j = 1, \dots, m'$ with the all billiards periodic.

(ii) Incorporate in this scheme Problem 8.3.28.

(iii) Extend these results to the case when there are also manifolds X_j $j = m'+1, \dots, m$ with almost all billiards non-periodic.

8.4 Spectral Asymptotics on Subspaces

So far we have considered operators with domain which is dense in $\mathscr{L}^2(X,\mathbb{H})$. We need to consider also spectral problems when eigenfunctions are restricted to a certain subspace $\mathcal{H}\subset \mathscr{L}^2(X,\mathbb{H})$ such that orthogonal projector Π to $\mathcal{H}$ is either a pseudodifferential operator (inside the domain), or a singular Green operator or sum of pseudodifferential operator with transmission property and singular Green operator.

8.4.1 Examples and Discussion

Let X be a connected Riemannian manifold (possibly) with the boundary ∂X, $\Lambda^{(p)}_x$ the space of p-forms at point x, $\Lambda^{(p)}=\Lambda^{(p)}(X)$ be the corresponding bundle over X.

Let $\mathsf{d}:\mathscr{C}^\infty(X,\Lambda^{(p)})\to\mathscr{C}^\infty(X,\Lambda^{(p+1)})$ be an operator of external differentiation, $\mathsf{d}^*:=*\mathsf{d}*:\mathscr{C}^\infty(X,\Lambda^{(p)})\to\mathscr{C}^\infty(X,\Lambda^{(p-1)})$ an operator formally adjoint to it.

Indeed, due to the Riemannian structure we have an antilinear operator $*:\mathscr{C}^\infty(X,\Lambda^{(p)})\ni u\to *u\in\mathscr{C}^\infty(X,\Lambda^{(d-p)})$ which provides an inner product $(u,v)=\varsigma\int_X u\wedge *v$ and a norm $\|u\|=\bigl(\int_X u\wedge *u\bigr)^{\frac{1}{2}}$ as $**=1$ [18].

Now we can consider operator $\Delta=\mathsf{d}\mathsf{d}^*+\mathsf{d}^*\mathsf{d}$ with the domain $\mathscr{H}^2(X,\Lambda^{(p)})$; at this moment we do not specify boundary conditions.

Example 8.4.1. (i) Consider subspace

$$\mathcal{H}:=\{u\in\mathscr{L}^2(X,\Lambda^{(p)}):\mathsf{d}^*u=0\} \tag{8.4.1}$$

with $p<d$ [19]. On this subspace consider quadratic form

$$\mathcal{Q}(u)=(\mathsf{d}u,\mathsf{d}u)+(\mathsf{d}^*u,\mathsf{d}^*u) \tag{8.4.2}$$

with domain

$$\mathfrak{D}(\mathcal{Q})=\{u\in\mathscr{L}^2(X,\Lambda^{(p)}):\ \mathsf{d}u\in\mathscr{L}^2(X,\Lambda^{(p+1)}),\ \mathsf{d}^*u\in\mathscr{L}^2(X,\Lambda^{(p-1)})\} \tag{8.4.3}$$

[18] In contrast to most of textbooks where $**=(-1)^{p(d-p-1)}$.

[19] Then $\dim\mathcal{H}=\infty$; as $d=p$ $\dim\mathcal{H}=1$.

and an operator $A = \Delta$ generated by this form; it is given by a desired differential expression

$$\Delta = \mathsf{d}^*\mathsf{d} = \mathsf{d}^*\mathsf{d} + \mathsf{d}\mathsf{d}^* \tag{8.4.4}$$

with the domain

$$\mathfrak{D}(A) = \{u \in \mathcal{H} \cap \mathscr{H}^2(X, \Lambda^{(p)}),\ \eth_n \mathsf{d}u = 0\} \tag{8.4.5}$$

where $\eth_n = \eth - \eth_t$, $\eth_t$ is an operator of the restriction to ∂X; so if $w = w_0 + w_1 \wedge dx_1$ where u_k are $(p-k)$-forms with respect to x' depending on x_1, then

$$\eth w = w_0|_{x_1=0} + w_1|_{x_1=0} \wedge dx_1, \tag{8.4.6}$$

$$\eth_t w = w_0|_{x_1=0}, \quad \eth_n w = w_1|_{x_1=0} \wedge dx_1; \tag{8.4.7}$$

We call $\eth_t w$ and $\eth_n w$ *tangential* and *normal components* of $\eth w$ respectively.

However unless either $p = 0$ or $\partial X = \emptyset$ the quadratic form and operator have infinite-dimensional kernel

$$\mathcal{K} = \{u \in \mathscr{L}^2(X, \Lambda^{(p)}) :\ \mathsf{d}u = \mathsf{d}^*u = 0\} \tag{8.4.8}$$

and we need to restrict both form and operator to

$$\mathcal{H}_0 = \mathcal{H} \ominus \mathcal{K} = \{u \in \mathscr{L}^2(X, \Lambda^{(p)}) :\ \mathsf{d}u = 0,\ \eth_n u = 0\}. \tag{8.4.9}$$

(ii) We can add to (8.4.3) condition

$$\eth_t u = 0 \tag{8.4.10}$$

and get operator $A = \Delta$ with domain

$$\mathfrak{D}(A) = \{u \in \mathcal{H} \cap \mathscr{H}^2(X, \Lambda^{(p)}),\ \eth_t u = 0\} \tag{8.4.11}$$

(iii) Using duality operator $*$ we can simultaneously permute d and d^*, $\eth_t$ and $\eth_n$, $p = 0$ and $p = d$.

(iv) Let $0 < p < d$. Then we can consider quadratic form (8.4.2) on $\mathcal{H}_1 := \mathscr{L}^2(X, \Lambda^{(p)}) \ominus \mathcal{K}$ and get an operator $A = \Delta$ with the domain

$$\mathfrak{D}(A) = \{u \in \mathcal{H} \cap \mathscr{H}^2(X, \Lambda^{(p)}),\ \eth_t \mathsf{d}u = \eth_n d^*u = 0\} \tag{8.4.12}$$

Remark 8.4.2. (i) Note that the boundary conditions for A are equivalent to equality

$$(\mathsf{d}u, \mathsf{d}v) = (\mathsf{d}^*\mathsf{d}u, v) \qquad \forall v \in \mathfrak{D}(\mathcal{Q}). \tag{8.4.13}$$

(ii) Also note that in Example 8.4.1(i),(ii) there is a "hidden" boundary condition

$$\eth_t \mathsf{d}^* u = 0 \tag{8.4.14}$$

due to $\mathsf{d}^* u = 0$; with the boundary conditions (8.4.5), (8.4.14) or (8.4.10), (8.4.14) operator $A = \Delta$ given by differential equation (8.4.4) is self-adjoint;

(iii) One can see easily that

(8.4.15) A_B and Π commute[20)].

We leave to the reader

Problem 8.4.3. Describe operators Π, Π_0 and Π_1 (projectors onto $\mathcal{H}$, $\mathcal{H}_0$ and $\mathcal{H}_1$ in example 8.4.1;

Example 8.4.4. Consider operator in $\mathcal{L}^2(X, \Lambda^{(p)}) \oplus \mathcal{L}^2(X, \Lambda^{(p-1)}) \ni \begin{pmatrix} u \\ v \end{pmatrix}$

$$A = \begin{pmatrix} 0 & \alpha \mathsf{d} \beta \\ \beta^\dagger \mathsf{d}^* \alpha^\dagger & 0 \end{pmatrix} \tag{8.4.16}$$

restricted to kernel $\mathcal{H}$ of operator

$$R = \begin{pmatrix} \mathsf{d}\alpha^{-1} \\ \mathsf{d}^* \beta^{\dagger -1} \end{pmatrix} \tag{8.4.17}$$

under boundary condition

$$\eth_t \beta v = 0 \tag{8.4.18}$$

or, alternatively,

$$\eth_n \alpha^\dagger u = 0 \tag{8.4.19}$$

[20)] Which means that spectral projectors of A_B and Π commute.

where $\alpha : \Lambda_x^{(p)} \to \Lambda_x^{(p)}$, $\beta : \Lambda_x^{(p-1)} \to \Lambda_x^{(p-1)}$ are non-degenerate matrices smoothly depending on x.

Then we can make the same conclusions as before with the corresponding subspace $\mathcal{H}$.

Maxwell equations in anisotropic media (crystals) fit the example above as $\alpha = \varepsilon^{-\frac{1}{2}}$, $\beta = \mu^{-\frac{1}{2}}$ where ε, μ are positive Hermitian matrices; for scalar such matrices we get Maxwell equations in isotropic media.

Example 8.4.5. (i) Consider $\mathcal{H} = \{u : \Delta u = 0\} \subset \mathscr{L}^2(X, \Lambda^{(p)})$ and operator generated by the quadratic form $\mathcal{Q}(u) = \|\nabla u\|^2$. Expressing $u = Rv$ with $v = \eth u$ we get quadratic form $\mathcal{Q}'(v) = \mathcal{Q}(Rv)$ on $\mathscr{H}^{-\frac{1}{2}}(Y, \mathcal{E})$ where $\mathcal{E} = \Lambda^{(p)}(Y) \oplus \Lambda^{(p-1)}(Y)$, $Y = \partial X$.

Then

$$\mathcal{Q}'(v) = (B\Delta_Y^{-\frac{1}{2}} v, \Delta_Y^{-\frac{1}{2}} v)_Y, \qquad \|u\|^2 = \|\Delta_Y^{-\frac{1}{2}}\|_Y^2 \tag{8.4.20}$$

with the pseudodifferential operator B on ∂X, acting on $\mathscr{C}^\infty(Y, \mathcal{E})$ and self-adjoint in $\mathcal{H}' = \mathscr{L}^2(Y, \mathcal{E})$. Thus the problem is reduced to study of spectral properties of B.

(ii) Consider $\mathcal{H} = \{u : \mathsf{d}u = \mathsf{d}^* u = 0\} \subset \mathscr{L}^2(X, \Lambda^{(p)})$ and operator generated by quadratic form $\mathcal{Q}(u)$. Expressing $u = Rv$ with $v = \eth u$ and the same operator R as before, we get

$$\mathcal{H}' = \{w \in \mathscr{L}^2(Y, \mathcal{E}) : \mathsf{d}R\Delta_Y^{\frac{1}{2}} w = \mathsf{d}^* R\Delta_Y^{\frac{1}{2}} w = 0\}. \tag{8.4.21}$$

Thus the problem is reduced to study of spectral properties of B' defined by a quadratic form $\mathcal{Q}'(v)$ on $\mathcal{H}'$.

(iii) In (i) and (ii) we can also impose certain conditions to v thus further reducing the space $\mathcal{H}'$.

We leave to the reader the following

Problem 8.4.6. (i) Describe projector Π in $\mathscr{L}^2(X, \Lambda^{(p)})$ which is a singular Green operator and operator B in Example 8.4.5(i).

(ii) Describe projector Π in $\mathscr{L}^2(X, \Lambda^{(p)})$ which is a singular Green operator, projector $\Pi' = \Delta_Y^{\frac{1}{2}} R^* \Pi R \Delta_Y^{\frac{1}{2}}$ which is a pseudodifferential operator on Y and B' in Example 8.4.5(ii).

8.4.2 Analysis away from the Boundary

If X is a closed manifold or at least $B(0,1) \subset X$ there is an already developed theory: projector on $\mathcal{H}$ we are interested in is (locally) a pseudodifferential operator and asymptotics of $\Gamma\big(\Pi_x e(.,.,\tau)\big)$ we already defined. However there is a twist:

Theorem 8.4.7. *Let Π be an h-pseudodifferential projector commuting with A. Then all results of Chapters 4–6 remain true if we redefine microhyperbolicity according to Definition 8.4.8 below (with $r = I - \pi$) and for sharp results consider only trajectories on manifold*

$$\Sigma_\tau := \{\mathsf{Ker}\, a(x,\xi) \cap \mathsf{Ker}(1-\pi(x,\xi)) \neq \{0\}\} \tag{8.4.22}$$

where π is the principal symbol of Π.

Definition 8.4.8. We say that operator A is microhyperbolic at point z in direction ℓ at energy level τ under restriction $Ru = 0$ if

$$\langle((\ell a(z))v, v\rangle \geq \epsilon\|v\|^2 - C\|(a(z)-\tau)v\|^2 - C\|r(z)v\|^2 \qquad \forall v \tag{8.4.23}$$

with small enough constant $\epsilon > 0$; here R is a pseudodifferential operator (not necessary a projector) with the principal symbol $r(x,\xi)$ such that $\mathsf{Ker}\, r(x,\xi)$ is invariant with respect to $a(x,\xi)$.

Proof of Theorem 8.4.7. The easiest way to recover all local results is to note that one can find pseudodifferential $\mathsf{D} \times \mathsf{D}'$-matrix operator E (where $\mathsf{D}' = \mathsf{rank}\,\pi$ and A is $\mathsf{D} \times \mathsf{D}$-matrix operator) such that $(I-\Pi)u \equiv 0$ near $z \in T^*X$ if and only if there exists v such that $u \equiv Ev$ near z. Then the problem locally becomes $\mathsf{D}' \times \mathsf{D}'$-matrix and microhyperbolic in the ordinary sense. Here D' is constant on each connected component of modified Σ_τ. The easy details are left to the reader.

Sharper results propagation are also easy: we just remember that condition $(I-\Pi)u = 0$ makes some points of old Σ_τ elliptic and thus removes them from propagation. Since so far we did not discuss in details propagation near boundary we simply announce that at this moment all propagation near boundary remains the same as without restriction (but at moment when the boundary is not considered Σ_τ is "trimmed"). We will modify this in the next subsections.

Finally, results for operators with the periodic Hamiltonian flows and based on equality $e^{ih^{-1}A} \equiv e^{i\varepsilon h^{-1}B}$ are now based on

$$e^{ih^{-1}A}\Pi \equiv e^{i\varepsilon h^{-1}B}\Pi, \tag{8.4.24}$$

$$(I-\Pi)e^{ih^{-1}A}\Pi \equiv (I-\Pi)e^{i\varepsilon h^{-1}B}\Pi \equiv 0, \tag{8.4.25}$$

$$\Pi e^{ih^{-1}A}(I-\Pi) \equiv \Pi e^{i\varepsilon h^{-1}B}(I-\Pi) \equiv 0. \tag{8.4.26}$$

Again, the easy details are left to the reader. □

Remark 8.4.9. (i) If Π does not commute with A defining quadratic form. situation here does not change drastically: this quadratic form defines operator $\Pi A = \Pi A \Pi$ on $\mathcal{H} = \mathsf{Ran}\,\Pi$ and all we need is to replace in the definitions (8.4.22) of Σ_τ and (8.4.23) of microhyperbolicity principal symbol a of A by the principal symbol $a' = \pi a \pi$ of A' where π is the principal symbol of Π.

(ii) In particular, if a and π commute, (8.4.22) and (8.4.23) do not need to be modified.

(iii) This would not be the case near the border since then ΠA would not be operator of the type we have studied.

8.4.3 Analysis near the Boundary. I

This theory is much more delicate partially because it involves not only pseudodifferential operators but also Boutet-de-Monvel operators and we do not have semi-classical theory here and development of this theory is rather tedious. Luckily we do not need a full theory but a limited one and it is rather obvious:

Problem 8.4.10. Develop semi-classical theory parallel to one of Section 1.4. In particular prove that operators described there–pseudodifferential operators with transmission property (see Definition 1.4.3), boundary, trace and singular Green operators (see Definitions 1.4.10–1.4.12) and also h-pseudodifferential operators on the boundary have the property that if symbols of two such operators R_1 and R_2 of different but "matching" types have disjoint supports then R_1R_2 is a negligible operator.

We assume that (8.4.15) holds (i.e. A_B and Π commute). Then we need to study $U_\Pi(x,y,t)$ the Schwartz kernel of $e^{ih^{-1}tA_B}\Pi$ and thus we can apply the theory of Chapter 7 with the only difference that now $Q_2=\Pi$ is the sum of pseudodifferential operator with the transmission property and a singular Green operator (with respect to (x,t)) while before it was of the type

$$Q_j(x,t,hD_x,hD_t)=\sum_{0\le k\le l} Q_{j,k}(x,t,hD'_x,D'_t)(hD_{x_1})^k. \tag{8.4.27}$$

Remark 8.4.11. One can rewrite operator $Q_j=Q_j(x,t,hD_x,hD_t)$ in the form

$$Q_j=Q_j''(x,t,hD_x,hD_t)(hD_t-A)+Q_j'+hP\eth_{\min(l,m)}+G \tag{8.4.28}$$

where $\eth_l u=\eth(u,hD_1u,\dots,(hD_1)^{l-1}u)$, Q_j' is an operator of (8.4.27) type, P, G are Poisson and singular Green operators respectively. But (hD_t-A) annihilates U, so one needs to consider only $Q_j=Q_j'+P\eth_{\min(l,m)}+G$.

Repeating arguments of Sections 3.1–3.3 we get immediately that

Proposition 8.4.12. *All statements of Theorem 7.2.17 remain true as Q_1,Q_2 are sums of h-pseudodifferential operators with the transmission property and h-singular Green operators.*

Proof. The only not completely trivial part of the proof is the Proposition 3.3.4 reinvented: then additionally appear terms of the same type $\Gamma''\bigl(Q_{1x}U_{jk}(.,.,t)\,{}^tQ_{2y}\bigr)$ but in virtue of Remark 8.4.11 with at least one of operators Q_1 and Q_2 being the sum of $P\eth_l$ and G.

As Q_1 is operator of this type, it effectively confines x_1 to $\{x_1\le h^{1-\delta'}\}$ and then there is no difference if we take $\nu_j x_1$ or $\nu_k x_1$ in the application of the propagation theorem, so we get terms of the type we deal in another parts of the proof.

Details and the rest of arguments are left to the reader. □

Now we immediately arrive to

Theorem 8.4.13. *All statements of Theorems 7.3.1 and 7.3.2 remain true as Q_1,Q_2 are sums of h-pseudodifferential operators with the transmission property and h-singular Green operators.*

Remark 8.4.14. The leading term in the asymptotics is $\kappa_0(\tau,\tau')h^{-d}$. Consider $Q_2 = Q_1^*$ and $\tau > \tau'$. Then $\kappa_0 = 0$ if and only if

$$q_1\big(\theta(a-\tau') - \theta(a-\tau)\big)q_2 = 0; \tag{8.4.29}$$

recall that $I - \theta(a-\tau)$ is a spectral projector of symbol a.

As $q_2 = \pi$ is the projector-valued symbol the latter condition becomes

$$\big(\theta(a-\tau') - \theta(a-\tau')\big)\big|_{\operatorname{Ran}\pi} = 0. \tag{8.4.30}$$

One can also prove easily the following

Theorem 8.4.15. *All statements of Theorem 7.3.7 remain true as Q_1, Q_2 are sums of h-pseudodifferential operators with the transmission property and h-singular Green operators and principal symbol of Q_1 is $q_2 = \pi$, the projector-valued symbol satisfying* (8.4.30).

Problem 8.4.16. Investigate if theorems 7.3.10, 7.3.11 also remain true.

Then one can prove rather easily

Theorem 8.4.17. *All statements of Theorems 7.4.1, 7.4.4, 7.4.6, 7.4.7, 7.4.8 and 7.4.21 remain true as Q_1, Q_2 are sums of h-pseudodifferential operators with the transmission property and h-singular Green operators and principal symbol of Q_1 is $q_2 = \pi$, the projector-valued symbol satisfying* (8.4.30).

Problem 8.4.18. (i) Investigate if statements Theorems 8.1.6 and 8.1.8 also remain true.

(ii) Investigate if statements of Theorems 8.2.1, 8.2.2, 8.2.4 and 8.2.6 also remain true.

(iii) Investigate if statements of Theorems 8.3.6, 8.3.15, 8.3.23 and 8.2.6 also remain true.

Much more difficult problem would be to weaken notion of "microhyperbolic in multidirection" and still recover the same results:

Definition 8.4.19. Let conditions (3.1.1)–(3.1.6), (3.1.23), (3.1.9) be fulfilled. Let Π be a sum of the h-pseudodifferential operator with the transmission property and singular Green operator.

Then the problem (P,B) is *microhyperbolic at the point* $z \in \Omega' \subset T^*\partial X$ *in the multidirection* $\mathcal{T} = (\ell', \nu_1, \dots, \nu_n)$ under constrain $(I-\Pi)u = 0$ if the following two conditions are fulfilled:

(i) For every $k = 1, \dots, n$ the operator P is microhyperbolic at the point $z_k = (z, \eta_k)$ in the direction $\ell_k = (\ell, \nu_k, 0) \in T_{z_k}(T^*X|_{\partial X}) \subset (T_{z_k}T^*X)|_{\partial X}$ under restrain π_0 where π_0 is the principal symbol of the pseudodifferential part of Π;

(ii) The following estimate holds

$$
\begin{aligned}
&(8.4.31)\quad -\operatorname{Re}\langle(\mathcal{T}'p)(z,D_1)v, v\rangle_+ + \operatorname{Re} i\langle(\mathcal{T}'b)(z)\eth v, \beta(z)\eth v\rangle_b \geq \\
&\epsilon\|v\|^2_+ - c_1(c_1\|b(z)\eth v\|'^2 - \langle T(z)\eth v, \eth v\rangle_m) \qquad \forall v \in \mathbb{H}'(z) \cap \operatorname{Ker}(I-\pi(z))
\end{aligned}
$$

where $\pi(z) = \pi_0(z, hD_1) + \pi'(z)$ and $\pi'(z)$ is a principal symbol of singular Green part of Π; $\pi(z)$ is a projector in $\mathscr{S}(\mathbb{R}^+, \mathbb{C}^{\mathsf{D}})$.

We leave to the reader

Problem 8.4.20. Correspondingly redefine Σ and Σ_b, modify definitions 3.1.1, 3.1.3, 3.1.4, 3.2.2, 3.2.3, 3.2.7.

Here as we weaken notion of microhyperbolicity, we increase the cone of microhyperbolic multidirections and reduce the dual cones and the propagation "conoids".

The really challenging task would be to

Problem 8.4.21. (i) Prove Theorems 3.1.2, 3.1.7 3.1.12 3.1.13, 3.2.4, 3.3.1, 3.3.2, 3.3.3, 3.3.5, 3.3.6 and 3.3.7 under new definition of microhyperbolicity as we consider only solutions satisfying $(I-\Pi)u = 0$ where Π and A_B commute.

(ii) Prove Theorems 8.4.13, 8.4.15 and 8.4.17 under new definition of microhyperbolicity as we consider only solutions satisfying $(I-\Pi)u = 0$ where Π and A_B commute.

8.4.4 Analysis near the Boundary. II. Seeley's Approach

Seeley's approach works especially well (as long as (8.4.30) is violates). In this approach we analyze zone without actually considering boundary. Let us recall this approach in the simplified version as ∂X is smooth.

(a) We consider first zone $X' = \{x : \ \gamma(x) := \mathsf{dist}(x, \partial X) \ge h^{1-\delta}\}$, take $T(x) = \gamma(x)^{1-\delta'}$ which confines propagation (in appropriate direction) to

$$Z(x) := \{y : \ (\frac{h}{\gamma(x)})^{1+\delta''} \le (\frac{h}{\gamma(y)}) \le (\frac{h}{\gamma(x)})^{1-\delta''}\}. \tag{8.4.32}$$

However in this zone A is just a regular h-pseudodifferential operator with the principal symbol a and all we need is its microhyperbolicity with the restriction $(I-\pi)u = 0$. Further, if π and a commute we need to check microhyperbolicity of a with the restriction $(I-\pi)u = 0$.

So, the same arguments as before bring us that the contribution of the strip $\{x : \gamma(x) \asymp \gamma\}$ to the remainder is $O(h^{1-d}\gamma^{\delta'})$ and the total contribution of X' is $O(h^{1-d})$.

(b) Then we consider zone $X'' = \{x : \ h \le \gamma(x) \le h^{1-\delta}\}$ and rescale it $x \mapsto x/\gamma_1$, $h \mapsto \hbar = h/\gamma_1$, $\gamma \mapsto \gamma' = \gamma/\gamma_1$ selecting $\gamma_1 = (\gamma/h)^s h$ so that $\gamma' = \hbar^{1-\delta}$; then $\gamma' = (h/\gamma)^{s-1}$.

Therefore in virtue of (a) contribution of the strip $\{x : \ \gamma(x) \asymp \gamma\}$ to the remainder does not exceed $C\hbar^{1-d}\gamma'^{\delta'}\gamma_1^{1-d} \le Ch^{1-d}(h/\gamma)^{\delta'})$ and the total contribution of this zone to the remainder is $O(h^{1-d})$. This part of arguments does not require any modification.

(c) Furthermore, we consider zone $x'' = \{x : \gamma(x) \le h\}$ and we use the rough estimate similar to those of Section 7.1 we estimate contribution of this zone also by $O(h^{1-d})$. Easy details are left to the reader.

(d) Further, these arguments prove that the contribution of the zone $\{x : \ \mu^{-1}h \le \gamma(x) \le \mu\}$ to the remainder does not exceed $\varepsilon(\mu)h^{1-d}$ with $\varepsilon(\mu) = o(1)$ as $\mu \to 0$. Improved analysis in zone $\{x : \ \gamma(x) \le \mu^{-1}h\}$ (see Section 7.5) proves that its contribution to the remainder is $o(h^{1-d})$ for any fixed $\mu > 0$. Finally, propagation arguments prove that under non-periodicity billiard conditions contribution of the zone $\{x : \ \gamma(x) \le \mu\}$ to the remainder is $o(h^{1-d})$. So we arrive to Theorem below:

Theorem 8.4.22. *Let ∂X be smooth. Then all statements of Theorems 7.5.10, 7.5.15, 7.5.19 and 7.5.20 remain true as Q_1, Q_2 are sums of h-pseudodifferential operators with the transmission property and singular Green operators.*

Example 8.4.23. Consider $\mathcal{H}$ defined by (8.4.1) and operator $\mathcal{A}$ with boundary condition $\mathcal{B}$ defined by arbitrary elliptic m-th order quadratic form $\mathcal{Q}(u)$.

Then

$$\mathcal{A} = A + G + hP\eth_{2m}, \tag{8.4.33}$$

with the domain

$$\mathfrak{D}(\mathcal{A}_{\mathcal{B}}) = \{u \in \mathscr{H}^2(X, \Lambda^{(p)}),\ \mathcal{B}u = 0\}, \tag{8.4.34}$$

$$\mathcal{B} = B\eth_{2m} + T \tag{8.4.35}$$

where $\eth_m u = (\eth u, \eth hD_1 u, \dots, \eth(hD_1)^{m-1}u)$, m is an order of $\mathcal{Q}$ and A is a pseudodifferential operator of order $2m$ with transmission property, B is a pseudodifferential operator on ∂X, G, P and T are singular Green, Poisson and trace operators respectively.

We leave to the reader the following

Problem 8.4.24. Describe operators $\mathcal{A}$, $\mathcal{B}$.

Repeating arguments above[21)] we arrive to

Theorem 8.4.25. *Let ∂X be smooth. Then*

(i) Statements of Theorems 7.5.10 and 7.5.15 remain true as Q_1, Q_2 are sums of h-pseudodifferential operators with the transmission property and singular Green operators and $\mathcal{A}_{\mathcal{B}}$ is of the type described above.

(ii) As operator $\mathcal{A}$ and generalized boundary operator $\mathcal{B}$ differ from differential operator A and operator B by lower order terms only, statement of Theorems 7.5.19 and 7.5.20 also remain true.

The following problem seems to be extremely challenging:

Problem 8.4.26. To derive asymptotics with remainder estimate $o(h^{1-d})$ for general operators of (8.4.33)–(8.4.35)-type.

Here treatment of both zones $\{x\colon \gamma(x) \le \mu^{-1}h\}$ and $\{x\colon \gamma(x) \ge \mu\}$ seems to be difficult since propagation near boundary is not clear.

[21)] In part (c) correct scaling of all terms of the operator is essential.

Comments

The systematic theory outlined in Section 8.4 even without Problem 8.4.26 would be definitely worth publications.

Microlocal Analysis, Sharp Spectral Asymptotics and Applications. II

Functional Methods and Eigenvalue Asymptotics

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Part IV

Estimates of the Spectrum

Chapter 9

Estimates of the Negative Spectrum

This short chapter plays a crucial role in the whole book bridging Parts II, III and Part V. Namely here we combine the results of Parts II and III devoted to local semiclassical spectral asymptotics with the variational estimates of Rozenblioum type (upper estimates for $\mathsf{N}^-(A)$) in order to derive upper and lower estimates for $\mathsf{N}^-(A) = \mathsf{N}(-\infty, A)$ where $\mathsf{N}^-(A)$ is the number of negative eigenvalues of the self-adjoint operator A (counting their multiplicities) provided $(-\infty, 0)$ does not intersect with the essential spectrum of A and $\mathsf{N}^-(A) = \infty$ otherwise.

More precisely, the idea is that the geometric domain X consists of two overlapping zones: the *regular zone* where local semiclassical spectral asymptotics are applied and the *singular zone* where formally variational estimates are applied. We use only the regular zone and semiclassical asymptotics there to derive a lower estimate of $\mathsf{N}^-(A)$. However we use both the regular zone with its semiclassical asymptotics and the singular zone with variational estimates to derive upper estimates of $\mathsf{N}^-(A)$. Both these estimate have the same main part derived somewhere in Parts II and III.

On the *inner boundary* of the singular zone we can introduce boundary conditions of our choice and Dirichlet boundary conditions are usually the best. In the justification of this procedure (decoupling of asymptotics) the results of Chapters 2 and 3 concerning the finite speed of propagation of singularities in the intersection of these zones play an important role.

In Section 9.1 we formulate the main statements, discuss and prove them. Actually there is just one main Theorem 9.1.7 and its sidekick Theorem 9.1.13

for spectral Riesz means.

Section 9.2 is devoted to the series of initial examples; each of them could be formulated as a theorem.

In Section 9.3 we transfer the results of Section 9.2 to quadratic forms.

In Appendix 9.A we survey variational estimates and discuss some their consequences.

Later (in Part 4) we treat the case when the operator A depends on parameter(s) and we derive asymptotics of $\mathsf{N}^-(A)$ with respect to the parameters. These parameters may be of different nature but even if one of them is a semiclassical parameter h it is not the same as used here effective semiclassical parameter h which depends on partition element in the regular zone.

Surely, the results of Part II and III provide some of the asymptotics of Part V but without the additional analysis of this part we cannot cover many important cases.

9.1 Main Theorem

9.1.1 Discussion

As a preliminary example let us consider operator $\Delta - \lambda$ in the domain having singularities of different types in the domain on Figure 9.1. This domain has singularities of the type "angle" or "spike" and of the type "cusp". Later we consider many other examples of singularities of the boundary (including the case when the whole boundary itself is a singularity) or/and of the operator.

Consider *spatial scaling function* $\gamma(x)$ such that ∂X intersected with $B(\bar{x}, \gamma(\bar{x}))$ after scaling $x \mapsto (x - \bar{x})\gamma(\bar{x})^{-1}$ is uniformly smooth. After this scaling and division by λ we have Schrödinger operator $h^2\Delta - 1$ with $h = \lambda^{-\frac{1}{2}}\gamma(\bar{x})^{-1}$.

Therefore contribution of $B(\bar{x}, \gamma(\bar{x}))$ to the remainder does not exceed $Ch^{1-d} = C\lambda^{\frac{1}{2}(d-1)}\gamma(\bar{x})^{d-1}$ as long as $h \le 1$. So we can introduce a *regular zone* $X_{\mathsf{reg}} = \{x : \ \gamma(x) \ge \lambda^{-\frac{1}{2}}\}$ and its contribution to the remainder does not exceed

$$C\lambda^{\frac{1}{2}(d-1)} \int_{X_{\mathsf{reg}}} \gamma(x)^{-1}\, dx \tag{9.1.1}$$

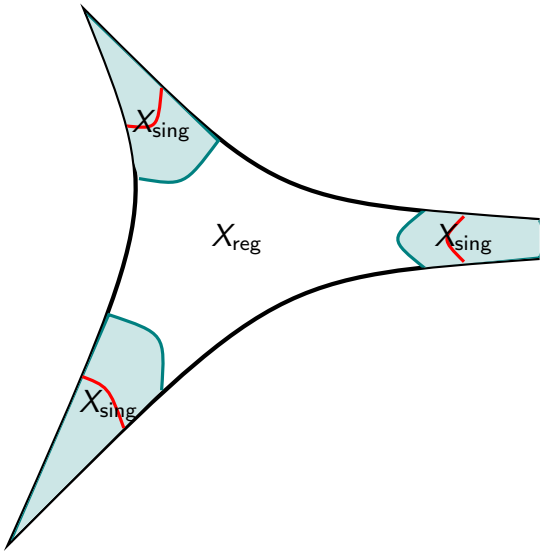


Figure 9.1: Black lines bound domain X, regular zone X_{reg} is bounded by red and black lines, singular zone X_{sing} is colored (both zones overlap).

while the principal part of asymptotics is given by the Weyl formula (with the integration over X_{reg})

$$\text{const} \cdot \lambda^{\frac{1}{2}d} \int_{X_{\text{reg}}} dx \tag{9.1.2}$$

with the standard constant.

Therefore we need to consider zone $X'_{\text{sing}} = \{x : \ \gamma(x) \le 2\lambda^{-\frac{1}{2}}\}$ which is part of the *singular zone* $X_{\text{sing}} = \{x : \ \gamma(x) \le 3\lambda^{-\frac{1}{2}}\}$. These zones shrink as $\lambda \to \infty$.

Assume that operator Δ was defined with the Dirichlet boundary conditions on ∂X. Let us formally add Dirichlet boundary conditions on the *inner boundary of singular zone* $\partial X_{\text{sing}} \setminus \partial X$ (i.e. thick lines on Figure 9.1). In this case contribution of X_{sing} and X'_{sing} to asymptotics does not exceed as $d \ge 3$

$$C\lambda^{\frac{1}{2}d} \int_{X_{\text{sing}}} dx. \tag{9.1.3}$$

So,

(9.1.4) The main part of asymptotics is (9.1.2) while the remainder estimate is given by the sum of (9.1.1) and (9.1.3).

Remark 9.1.1. (i) Obviously, the are no Dirichlet boundary condition on part of $\partial X_{\mathsf{sing}}$ and therefore this claim so far is justified only in the very special case when X is the union of infinitely many connected components, some of them completely belonging to X_{reg} and the rest to X_{sing}. However as we mentioned we will justify it.

(ii) The main part of the asymptotics is given by cut Weyl formula (9.1.2). Sure, we can always complete it, making integral over X since integral over X_{sing} is part of the error estimate. However, in the certain cases (f.e. in domains with cusps of the infinite volume) we will be able to get better estimate than (9.1.3).

(iii) Under some assumptions we are able to use sharp asymptotics[1)] in X_{reg} thus replacing (9.1.1) by a smaller expression; however in certain general cases we do not have even $O(h^{1-d})$ estimate thus replacing (9.1.1) by a larger expression.

(iv) Neumann boundary problem could be treated under certain assumptions as well but those are really restrictive assumptions due to very special nature of the Neumann boundary condition.

(v) In the discussed case the meaning of the *reference theorem* was obvious; in the general case we can use different reference theorems in the different parts of the regular zone.

In what follows by the *reference theorem* we mean the local semiclassical asymptotics we used.

9.1.2 Main Theorem Setup

Scaling Functions

Let us cover our domain X into two overlapping subdomains X_{reg} and X_{sing} in which specific conditions are assumed to be fulfilled.

Let us assume that

$$X \subset \mathbb{R}^d/(\bar{\gamma}_1\mathbb{Z} \times \bar{\gamma}_2\mathbb{Z} \times ... \times \bar{\gamma}_d\mathbb{Z}), \qquad 0 < \bar{\gamma}_j \le \infty \quad (j = 1, ..., d) \tag{9.1.5}$$

[1)] With remainder estimate $o(h^{1-d})$ or better.

and $X = X_{\mathsf{reg}} \cup X_{\mathsf{sing}}$.

Remark 9.1.2. Surely one can treat more general manifolds and in specific cases we will do it.

Let us assume that on X_{reg} a function $\gamma = \gamma(x)$ is given such that

$$0 < \gamma(x) \le \min_j \bar{\gamma}_j, \quad |\nabla\gamma(x)| \le \frac{1}{2}. \tag{9.1.6)$_1$}$$

We say that $\gamma(x)$ is a *spatial scaling function*. We also assume that on X_{reg} a *momentum scaling function* $\rho(x)$ [2)] is given such that[3)]

$$x \in \mathcal{B}(y, \gamma(y), X) \implies \rho_k(x) > 0, \quad \epsilon_0 \le \frac{\rho_k(x)}{\rho_k(y)} \le c, \quad |\nabla\rho_k(x)| \le c\rho_k(y)\gamma^{-1}(y) \tag{9.1.6)$_2$}$$

and also scaling functions ϱ and ϱ_0 are given satisfying

$$x \in \mathcal{B}(y, \gamma(y), X) \implies \varrho_j(x) > 0, \quad \epsilon_0 \le \frac{\varrho_j(x)}{\varrho_j(y)} \le c, \quad |\nabla\varrho_j(x)| \le c\varrho_j(y)\gamma^{-1}(y) \tag{9.1.6)$_3$}$$

where here and below

(9.1.7) $\mathcal{B}(y, \gamma(y), X)$ denotes a connected component of $B(y, \gamma(y)) \cap \bar{X}$ containing y.

Remark 9.1.3. Obviously

$$\mathcal{B}(y, \gamma(y), X) \supset \{x \in \bar{X} : \mathsf{dist}(x, y) < \gamma(y)\}$$

where $\mathsf{dist}(x, y) = \mathsf{dist}_X(x, y)$ is a *path-distance* in X.

Further, let us assume that

$$\rho(y)\gamma(y) \ge 1 \qquad \forall y \in X_{\mathsf{reg}}. \tag{9.1.6)$_4$}$$

Finally, X_{reg} and X_{sing} be properly overlapping: let $X_{\mathsf{reg},0} \subset X_{\mathsf{reg}}$ and

$$X = X_{\mathsf{sing}} \cup X_{\mathsf{reg},0}; \quad y \in X_{\mathsf{reg},0} \implies \mathcal{B}(y, \gamma(y), X) \subset X_{\mathsf{reg}}. \tag{9.1.8}$$

[2)] In principle one can take this function also depending on ξ.

[3)] Later other scaling functions ρ_k and ϱ_j may be introduced which should all satisfy condition (9.1.6)$_2$.

Definition 9.1.4. In what follows X_{reg} is the *zone of semiclassical asymptotics* or the *regular zone* and X_{sing} is the *zone of variational estimates* or the *singular zone*.

Operators

Let us consider the operator A defined on X_{reg} by the formula

$$A = \sum_{\alpha:|\alpha|\le m} a_\alpha(x) D^\alpha \tag{9.1.9}$$

with boundary conditions

$$B_j u = \sum_{\alpha:|\alpha|\le m_j} b_{j\alpha}(x) D^\alpha u = 0 \quad (j = 1, \dots, \mu) \qquad \text{on} \quad \partial X \cap \bar{X}_{\mathsf{reg}} \tag{9.1.10}$$

where a_α, $b_{j\alpha}$ are $\mathsf{D} \times \mathsf{D}$ matrices and $m_j \le m - 1$.

That means that A is an operator in $\mathscr{L}^2(X, \mathbb{H})$, $\mathbb{H} = \mathbb{C}^{\mathsf{D}}$ [4)] with domain $\mathfrak{D}(A)$ but $Au(x)$ is given by (9.1.9) for $u \in \mathfrak{D}(A)$, $x \in X_{\mathsf{reg}}$. Moreover, it means that if $u \in \mathscr{H}^m(X, \mathbb{H})$ is compactly supported in X_{reg} then $u \in \mathfrak{D}(A)$ if and only if $Bu|_{\partial X} = 0$.

In our analysis we replace the operator A by the operator

$$\boldsymbol{A} = J^{-1} A \tag{9.1.11}$$

in $\mathcal{H}$ where

(9.1.12) J is a Hermitian positive definite matrix depending on x

and here and below $\mathcal{H} = \mathcal{H}_J$ is a Hilbert space with inner product $(u, v)_{\mathcal{H}} = (Ju, v)$ and $(.,.)$ is the standard $\mathscr{L}^2(X, \mathbb{H})$-inner product.

We assume that

(9.1.13) $\boldsymbol{A}$ is a self-adjoint operator in $\mathcal{H}$.

Our goal is to derive upper and lower estimates for

$$\mathsf{N}^-(\boldsymbol{A}) = \int \mathsf{tr}\big(e(x, x, -\infty, 0)\big)\, dx \tag{9.1.14}$$

where $e(.,.,.,.)$ is the Schwartz kernel of the spectral projector $E(\tau)$ of $\boldsymbol{A}$.

[4)] One can also treat the case when $\mathbb{H}$ is an infinite dimensional auxiliary Hilbert space.

Remark 9.1.5. Let us note that

(i) $\boldsymbol{A}$ is unitarily equivalent to the operator $J^{-\frac{1}{2}}AJ^{-\frac{1}{2}}$ in $\mathcal{H} = \mathscr{L}^2(X, \mathbb{H})$. In what follows we use the notation $\boldsymbol{A}$ for the operator $J^{-\frac{1}{2}}AJ^{-\frac{1}{2}}$ as well. We hope that there will be no confusion.

(ii) Usually $\mathsf{N}^-(\boldsymbol{A}) = \mathsf{N}^-(A)$. It is the case, f.e. if smooth functions u satisfying boundary conditions and such that $\|J\|$ and $\|J^{-1}\|$ are bounded by $C(u)$ on $\mathsf{supp}(u)$ are dense in $\mathcal{H}$. We will discuss this with more details in Section 9.3.

(iii) Equality $\mathsf{N}^-(A) = \mathsf{N}^-(\boldsymbol{A})$ is an example of the *Birman-Schwinger* principle (see Subsection 10.1.2 for more details).

Let us describe assumptions to A, B and J.

Regular Zone

First of all, let us describe assumptions in regular zone X_{reg}. Namely, assume that uniformly with respect to $y \in X_{\mathsf{reg}}$ assumptions (9.1.15)–(9.1.18) are fulfilled:

(9.1.15) If $y \in X_{\mathsf{reg}}$ then either $B(y, \gamma(y)) \subset X$ or there exist a number $k = k(y) \in \{1, \dots, d\}$, $\varsigma = \pm 1$ and a real-valued function $\phi(x_{\hat{k}})$ such that $|\nabla\phi| \le c$ and

$$\mathcal{B}(y, \gamma(y), X) = \{\varsigma(x_k - \phi(x_{\hat{k}})) > 0\} \cap B(y, \gamma(y)) \tag{9.1.16}$$

where $x_{\hat{k}} = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_d)$;

(9.1.17) $X_{\mathsf{reg}} = \bigcup_l X_{\mathsf{reg}(l)}$ and if $y \in X_{\mathsf{reg}(l)}$ then rescaling[5)] $\mathcal{B}(y, \gamma(y), X)$ by $x \mapsto (x-y)\gamma^{-1}$, $\xi \mapsto \xi\rho^{-1}$, $\tau \mapsto \tau\varrho^{-1}\varrho_0$ and multiplying $\boldsymbol{A}$ by $\varrho^{-1}\varrho_0$ we find ourselves in the framework of one of the theorems of Chapters 4–8 which providing estimate

$$\mathsf{R}_{h,\psi} := \Big| \int \mathsf{tr}\big(e(x,x,0)\big)\psi(x)\,dx - \mathcal{N}_{h,\psi}\Big| \le h^{-d}\vartheta_l(h) \tag{9.1.18}$$

[5)] All scaling functions are calculated at y.

where $\mathcal{N}_\psi$ denotes main part of semiclassical asymptotics, $h = (\rho\gamma)^{-1}$ is an *effective semiclassical parameter* and $\psi \in \mathscr{C}^\infty(\bar{X})$ supported in $\mathcal{B}(y, \frac{1}{2}\gamma(y))$ such that

$$|D^\beta \psi| \le c_\beta \gamma^{-|\beta|} \qquad \forall \beta : |\beta| \le K \tag{9.1.19}$$

and in (9.1.19) we already scaled back to original coordinates.

Later we list the most important cases. One can prove easily

(9.1.20) There exist real-valued functions $\psi_\nu \in \mathscr{C}^\infty(\bar{X})$, supported in $\mathcal{B}(y_\nu, \frac{1}{2}\gamma(y_\nu), X)$ such that $\psi_{\mathsf{reg}} := \sum_\nu \psi_\nu^2$ equals 1 in $X_{\mathsf{reg},\frac{7}{16}}$ and no more than $L(d)$ of different balls $\mathcal{B}(y_\nu, \frac{3}{4}\gamma(y_\nu), X)$ can have non-empty intersection.

Here and below γ'-vicinity of set M is a set $\bigcup_{y\in M} \mathcal{B}(y, \gamma'(y), X)$ and $X_{\mathsf{reg},\sigma}$ denotes $\sigma\gamma$-vicinity of $X_{\mathsf{reg},0}$, $0 \le \sigma \le 1$. Condition (9.1.8) means that $X_{\mathsf{reg}} \supset X_{\mathsf{reg},1}$.

Then we instantly arrive to

Proposition 9.1.6. *Let all the above assumptions be fulfilled and let ψ satisfy* (9.1.19) *and be supported in $X_{\mathsf{reg},\frac{1}{2}}$. Then*

$$\mathsf{R}_\psi \le R_1 := C_0 \sum_l \int_{X_{\mathsf{reg}(l)}} \rho^d \vartheta_l\big((\rho\gamma)^{-1}\big)\, dx \tag{9.1.21}$$

with $C_0 = C_0(d, c, m)$.

Singular Zone

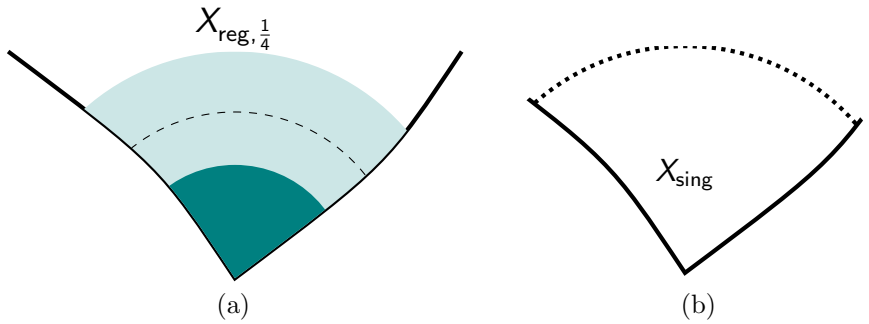
To estimate $\mathsf{N}^-(\boldsymbol{A})$ from above we need to consider $\psi = \psi_{\mathsf{sing}}$ satisfying (9.1.19) and supported in $\bar{X} \setminus \bar{X}_{\mathsf{reg},\frac{6}{16}}$.

Let us assume that

(9.1.22) There exist an operator $\boldsymbol{A}'$ in $\mathscr{L}^2(X_{\mathsf{sing}}, \mathbb{H})$ and boundary operators B_j' coinciding with $\boldsymbol{A}$ and B_j respectively on functions supported in $\bar{X}_{\mathsf{sing}} \setminus \bar{X}_{\mathsf{reg},\frac{1}{4}}$ such that under the boundary conditions $B_j' u|_{\partial X_{\mathsf{sing}}} = 0$ $(j = 1, \dots, \mu)$ this operator $\boldsymbol{A}'$ is self-adjoint and the estimate

$$\mathsf{N}^-(\boldsymbol{A}' - t) \le R_0(t+1)^p \qquad \forall t \ge 0 \tag{9.1.23}$$

holds.



(a) (b)

Figure 9.2: On (a) dark colored zone denotes $\mathsf{supp}(\psi_{\mathsf{sing}})$, together with light colored zone making X_{sing}; dashed line is the border of $X_{\mathsf{reg},\frac14}$. On (b) X_{sing} is considered alone; dotted line is $\partial X_{\mathsf{sing}} \setminus \partial X$.

Finally, let us assume that

$$\varrho_0\rho\gamma \ge \epsilon_0\varrho \quad \text{in} \quad \bigl(X_{\mathsf{reg},\frac14} \setminus X_{\mathsf{reg,ell}}\bigr) \cap X_{\mathsf{sing}} \cap \{\rho\gamma \ge c\}, \tag{9.1.24}$$

$$\epsilon_0\varrho(\rho\gamma)^{-p} \le \varrho_0 \le c\varrho(\rho\gamma)^{p} \quad \text{in} \quad X_{\mathsf{reg}} \cap X_{\mathsf{sing}} \tag{9.1.25}$$

where $X_{\mathsf{reg,ell}}$ denotes a part of X_{reg} where $\{A, B\}$ is elliptic in the semiclassical sense (after described rescaling).

The first assumption is assumed to provide the following statement:

(9.1.26) As $0 \le \sigma_1 < \sigma \le 1$ singularities starting at $X \setminus X_{\mathsf{reg},\sigma}$ reach $X_{\mathsf{reg},\sigma_1}$ only after a time no less than $\epsilon(\sigma - \sigma_1)$.

In fact, as $\rho\gamma \le c$ the propagation of singularities is absent because $h = (\rho\gamma)^{-1} \asymp 1$ and hence the notion of the singularity is empty. Further, there is no propagation in the elliptic zone either.

On the other hand, the results of Chapters 2 and 3 obviously yield that the spatial velocity of propagation of singularities for the operator $D_t - A$ in $X_{\mathsf{reg},\frac14} \cap X_{\mathsf{sing}}$ does not exceed $C_0\varrho\varrho_0^{-1}\rho^{-1}$ while the distance between $y \in X \setminus X_{\mathsf{reg},\sigma}$ and $x \in X_{\mathsf{reg},\sigma_1}$ is not less than $(\sigma - \sigma_1)\gamma(x)$. Therefore the time is not less than $\epsilon_2(\sigma - \sigma_1)\varrho^{-1}\varrho_0\rho\gamma$.

Meanwhile assumption (9.1.25) guarantees that $\varrho\varrho_0^{-1}$ is neither too large nor too small in this zone.

Finally, assume that

(9.1.27) In $X_{\mathsf{reg}} \cap X_{\mathsf{sing}}$ after rescaling $\mathcal{B}(y, \gamma(y), X)$ by $x \mapsto (x-y)\gamma^{-1}$, $\xi \mapsto \xi\rho^{-1}$, $\tau \mapsto \tau\varrho^{-1}\varrho_0$ and multiplying $\boldsymbol{A}$ by $\varrho^{-1}\varrho_0$ we find that problem $\{A, B\}$ is uniformly elliptic and positive in the classical sense[6)].

Our major result is the following

Theorem 9.1.7. *Let conditions* (9.1.5)–(9.1.13) *and* (9.1.15)–(9.1.18) *be fulfilled.*

(i) Then estimate

$$\mathsf{N}^-(\boldsymbol{A}) \ge \mathcal{N} - C_0 R_1 \tag{9.1.28}$$

holds with $\mathcal{N} = \mathcal{N}_{1,\psi}$ *defined as a sum of main parts of asymptotics in* (9.1.17)–(9.1.18) *and* R_1 *defined by* (9.1.21) *as a sum of remainder estimates there.*

(ii) Moreover, if conditions (9.1.22)–(9.1.25) *and* (9.1.27) *are fulfilled then estimate*

$$\mathsf{N}^-(\boldsymbol{A}) \le \mathcal{N} + C_0 R_1 + C_1 R_0 + C R_2 \tag{9.1.29}$$

holds with R_0 *appearing in* (9.1.23) *and*

$$R_2 = \int_{X_{\mathsf{reg},\frac{1}{4}} \cap X_{\mathsf{sing}}} \rho^{d-s}\gamma^{-s}\,dx, \tag{9.1.30}$$

$C_0 = C_0(d, c)$, $C_1 = C_1(d, p, c)$, $C = C(d, \mathsf{D}, c, s)$.

9.1.3 Proof of the Main Theorem

Let us note first that the lower estimate (9.1.23) for $\mathsf{N}^-(\boldsymbol{A})$ follows immediately from Proposition 9.1.6 since $\mathsf{N}^-(\boldsymbol{A}) \ge \mathsf{Tr}\big(E(0)\psi_{\mathsf{reg}}\big)$ as $0 \le \psi_{\mathsf{reg}} \le 1$. Further note that the upper estimate for $\mathsf{N}^-(\boldsymbol{A})$ also follows from Proposition 9.1.6 in the special case when $X_{\mathsf{sing}} = \emptyset$. Meanwhile in the general case we obtained an upper estimate for $\mathsf{Tr}\big(E(0)\psi_{\mathsf{reg}}\big)$ provided

[6)] Which means that selecting only homogeneous main parts of A, B_j with coefficients fixed in y we get a problem in the whole or half-space which is elliptic uniformly with respect to y and non-negative.

(9.1.31) $\mathsf{supp}(\psi_{\mathsf{reg}}) \subset X_{\mathsf{reg},\frac{1}{4}}$, $\psi_{\mathsf{reg}} = 1$ in $\mathsf{supp}(\psi) \subset X_{\mathsf{reg},\frac{1}{3}}$ and ψ_{reg} satisfies (9.1.19).

To derive an upper estimate for $\mathsf{N}^-(\boldsymbol{A})$ we need an upper estimate for $\mathsf{Tr}(E(0)(1-\psi_{\mathsf{reg}}))$. Let $U(x,y,t)$ be the Schwartz kernel of the operator $\exp(it\boldsymbol{A})$; recall that $\boldsymbol{A} = J^{-\frac{1}{2}} A J^{-\frac{1}{2}}$. We need some additional analysis in the zone $X_{\mathsf{reg},\mathsf{ell}} \cap X_{\mathsf{sing}} \cap \{\rho\gamma \ge c\}$.

Proposition 9.1.8. *Let conditions* (9.1.5)–(9.1.13), (9.1.15)–(9.1.18), (9.1.24)–(9.1.25) *and* (9.1.27) *be fulfilled.*

Let $\epsilon_0 > 0$ and $T_0 = T_0(d,c,\epsilon_0) > 0$ be small enough constants. Then the following estimates hold uniformly with respect to $x \in X_{\mathsf{reg},\frac{2}{7}}$:

$$\|D_x^\alpha F_{t\to h^{-1}\tau}\chi_T(t)U^{\pm}\psi'(y)\| \le C_1' h^s (|\tau|+1)^{-s}\gamma(x)^{-\frac{d}{2}-|\alpha|} \qquad \forall \tau \le c \qquad \forall \alpha : |\alpha| \le s, \tag{9.1.32}$$

and

$$\|D_x^\alpha F_{t\to h^{-1}\tau}\chi_T(t)U^{\pm}\psi'(y)\| \le C_1' h^{-M} (|\tau|+1)^{M}\gamma(x)^{-\frac{d}{2}-|\alpha|} \qquad \forall \tau \ge c \qquad \forall \alpha : |\alpha| \le s \tag{9.1.33}$$

where $\mathscr{L}^2$-norms with respect to y are calculated in the left-hand expression and here and below $\psi' \in \mathscr{L}^\infty$ is an arbitrary function such that

$$|\psi'| \le 1,\ \mathsf{supp}(\psi') \cap \mathcal{B}(x, \epsilon_0\gamma(x), X) = \emptyset \tag{9.1.34}$$

with $M = M(d,m,s)$, $C_1' = C_1'(d,m,s,c)$ and $h := \big(\rho(x)\gamma(x)\big)^{-1}$.

Proof. (a) Let us consider point $\bar{x}$ instead of x and operator $P = hD_t - \zeta\boldsymbol{A}$ with $h = (\bar\rho\bar\gamma)^{-1}$, $\zeta = h\bar\varrho^{-1}\bar\varrho_0$ where $\bar\gamma = \gamma(\bar{x})$ etc. Without any loss of the generality one can assume that $\bar\gamma = 1$ (otherwise we rescale).

Consider $f \in \mathscr{L}^2(X, \mathbb{H})$ with $\mathsf{supp}\, f \cap \mathcal{B}(\bar{x}, \epsilon_0\bar\gamma, X)$. Let us set

$$u(x,t) = \int U(x,y,t)(y)f(y)dy. \tag{9.1.35}$$

Then

$$Pu^{\pm} = 0 \quad \text{in} \quad \mathcal{B}(\bar{x}, \epsilon_0\bar\gamma, X), \tag{9.1.36}$$

$$B_j u^{\pm} = 0 \quad \text{at} \quad \partial X \cap \mathcal{B}(\bar{x}, \bar\gamma, X) \qquad (j = 1, \dots, \mu). \tag{9.1.37}$$

(b) Let us first assume that $\zeta \le c$. Then one can apply the theorem about the finite speed of propagation of singularities[7)] and therefore the following estimate holds:

$$\|F_{t\to h^{-1}\tau}\chi_T(t)u^{\pm}\| \le C_1' h^s \|f\| \qquad \forall x \in \mathcal{B}(\bar{x}, \frac{1}{2}\epsilon_0\bar{\gamma}, X) \qquad \forall \tau : |\tau| \le c \tag{9.1.38}$$

where we have also used the estimate

$$\|u^{\pm}\| \le \|f\| \tag{9.1.39}$$

which follows from the fact that the operator $\exp(itA)$ is unitary in $\mathcal{H}_0$; here $\mathscr{L}^2(X, \mathbb{H})$-norms (with respect to x) are calculated.

(c) Now let us apply the following *additional arguments:* the classical ellipticity of the rescaled problem as stated in (9.1.27), the inequality $\zeta \ge h^{\rho}$ due to (9.1.25) and the standard embedding inequality; then we obtain estimate

$$|D^{\alpha}F_{t\to h^{-1}\tau}\chi_T(t)u^{\pm}| \le C_1' h^s \|f\| \qquad \forall x \in \mathcal{B}(\bar{x}, \frac{1}{2}\epsilon_0\bar{\gamma}, X) \quad \forall \alpha : |\alpha| \le s \quad \forall \tau : |\tau| \le c. \tag{9.1.40}$$

Furthermore, the same *additional arguments* and the classical ellipticity and positivity of the rescaled problem as stated in (9.1.27) easily yield estimate

$$|D^{\alpha}F_{t\to h^{-1}\tau}\chi_T(t)u^{\pm}| \le C_1' h^s (|\tau|+1)^{-s}\|f\| \qquad \forall x \in \mathcal{B}(\bar{x}, \frac{1}{2}\epsilon_0\bar{\gamma}, X) \quad \forall \alpha : |\alpha| \le s \quad \forall \tau \le -C_0. \tag{9.1.41}$$

(d) On the other hand, let $\zeta \ge c$. Then as $\rho\gamma \ge c$ the problem is elliptic in the semiclassical sense in $\mathcal{B}(\bar{x}, \epsilon_0\bar{\gamma})$ due to assumption (9.1.24) and the semiclassical elliptic arguments and the above *additional arguments* yield estimate (9.1.41) for all $\tau \le \epsilon_2\zeta$. Therefore estimate (9.1.41) remains true in this case as well.

[7)] Let us recall that we assumed not just estimate in (9.1.18) but also being in the frameworks of one of the theorems of Part sII and III in X_{reg} and it implies the finite speed of propagation and all other results involved.

(e) Finally, the *additional arguments* alone easily yield the estimate

$$|D^{\alpha}F_{t\to h^{-1}\tau}\chi_T(t)u^{\pm}| \le C_1' h^{-M}(|\tau|+1)^M \|f\| \quad \forall x \in \mathcal{B}(\bar{x}, \frac{1}{2}\epsilon_0\bar{\gamma}, X) \quad \forall \alpha : |\alpha| \le s \quad \forall \tau. \tag{9.1.42}$$

Since $f \in \mathscr{L}^2(X, \mathbb{H})$ is an arbitrary function such that $\mathsf{supp}(f) \cap \mathcal{B}(\bar{x}, \epsilon_0\bar{\gamma}, X) = \emptyset$, estimates (9.1.40)–(9.1.42) yield estimates (9.1.32) and (9.1.33). □

Remark 9.1.9. The same proof implies that estimates (9.1.32) and (9.1.33) remain true for $U'^{\pm}(x, y, t)$ where $U'(x, y, t)$ is the Schwartz kernel of the operator $\exp(it\boldsymbol{A}')$ as long as $x \in X \setminus X_{\mathsf{reg}, \frac{1}{4}}$.

Indeed, in $X \setminus X_{\mathsf{reg}, \frac{1}{4}}$ operator $\boldsymbol{A}'$ coincides with $\boldsymbol{A}$.

Proposition 9.1.10. *In the framework of Proposition 9.1.8 estimates*

$$\|F_{t\to\tau}\chi_T(t) \int \psi_{\mathsf{sing}}(x)\big(U^{\pm} - U'^{\pm}\big)(x, x, t)\, dx\| \le C_1'(|\tau|+1)^{-s} R_2 \tag{9.1.43}$$

and

$$\|F_{t\to\tau}\psi_{\mathsf{sing}}\chi_T(t)\big(\exp(it\boldsymbol{A}) - \exp(it\boldsymbol{A}')\big)\psi_{\mathsf{sing}}\|_1 \le C'(|\tau|+1)^{-s} R_2 \tag{9.1.44}$$

hold for all $\tau \le \frac{1}{2}c$. Recall that $\|.\|_1$ means the trace norm and C_1' does not depend on D but C' does.

Proof. Let ψ satisfy (9.1.19), $\psi = 1$ in $\bar{X} \setminus \bar{X}_{\mathsf{reg}, \frac{6}{16}}$ and $\mathsf{supp}(\psi) \subset \bar{X} \setminus \bar{X}_{\mathsf{reg}, \frac{5}{16}}$. Let us recall that $\mathsf{supp}(\psi_{\mathsf{sing}}) \subset \bar{X} \setminus \bar{X}_{\mathsf{reg}, \frac{7}{16}}$.

Let us consider

$$V^{\pm}(x, y, t) = \psi(x)(U^{\pm} - U'^{\pm})(x, y, t)\psi'(y) \tag{9.1.45}$$

where here and below ψ' is a cut-off function similar to ψ_{sing}. Then

$$PV^{\pm} = F^{\pm}, \quad B_k V^{\pm} = G_k^{\pm}$$

with $P = D_t - \boldsymbol{A}$, $F^{\pm}$ and $G_k^{\pm}$ supported in $\mathcal{X} := \bar{X}_{\mathsf{reg},\frac{6}{16}} \setminus \bar{X}_{\mathsf{reg},\frac{5}{16}}$ and therefore by the Duhamel principle

$$V^+(x,y,t) = \int_0^t dt' \Big(\int U^+(x,z,t') F^+(z,y,t-t')\, dz + \sum_{1\le k\le \mu} \int_{\partial X} \big(\beta_k(z,D_z) U^+(x,z,t')\big)\, G_k^+(z,y,t-t')\, dz' \Big) \tag{9.1.46}$$

where β_k are differential operators on z.

Therefore $V^+ = \sum_n V_n^+$ where V_n^+ are given by formula (9.1.46) with additional factors $\psi_n^2(z)$ in the integrands and $\{\psi_n^2\}$ is a γ-admissible partition of unity in $\mathcal{X}$.

Then Proposition 9.1.8 and Remark 9.1.9 yield that $\mathscr{L}^2$-norms with respect to y of $F_{t\to h^{-1}\tau}\chi_T(t)F^+$ and $F_{t\to h^{-1}\tau}\chi_T(t)G_k^+$ do not exceed

$$C_0 h^{-M}\big(|\tau|+1\big)^M \bar{\gamma}^{-\frac{d}{2}}$$

and

$$C_0 h^{-M} \bar{\varrho}_k \big(|\tau|+1\big)^M \bar{\gamma}^{-\frac{d}{2}}$$

respectively for all $z \in \mathsf{supp}(\psi_n)$ where $h = \bar{\rho}^{-1}\bar{\gamma}^{-1}$ and $\bar{\rho}, \bar{\gamma}, \bar{\varrho}_k$ are ρ, γ, ϱ_k calculated at some point of $\mathsf{supp}(\psi_n)$; all of these and the following estimates hold for all n.

Further, Proposition 9.1.8 and Remark 9.1.9 yield that the $\mathscr{L}^2$-norms with respect to x of

$$\psi'(x) F_{t\to h^{-1}\tau}\chi_T(t) U^+(x,z,t)$$

and

$$\psi'(x) F_{t\to h^{-1}\tau}\chi_T(t) B_k'(z,D_z) U^+(x,z,t)\psi_n(z)$$

do not exceed

$$C_0 h^{-M}\big(|\tau|+1\big)^M \bar{\gamma}^{-\frac{d}{2}}$$

and

$$C_0 h^{-M} \bar{\varrho}_k^{-1} \big(|\tau|+1\big)^M \bar{\gamma}^{1-\frac{d}{2}}$$

respectively. Recall that $z \in \mathsf{supp}(\psi_n)$ and

$$U(x,y,t) = U^{\dagger}(y,x,-t) \tag{9.1.47}$$

because $\boldsymbol{A}$ is a self-adjoint operator in $\mathcal{H}_0$.

Furthermore, Proposition 9.1.8 and Remark 9.1.9 also imply that for $\tau \le \frac{1}{2}c$ these four estimates hold with the exponent M replaced by $-s$ in the right-hand expressions and with constant C_0 replaced by C_1'. Since $\dim \mathbb{H} = \mathsf{D} < \infty$ all these inequalities hold (with constants depending also on D) not only $\mathscr{L}^2$-norms with respect to x or y but also for Hilbert-Schmidt norms of the corresponding operators.

Therefore we conclude that

(9.1.48) Let $W_n^+(x,y,t)$ be given by formula (9.1.46) with the additional factor $\psi_n^2(z)\chi_T(t')\chi_T(t-t')$ in the integrand. Then the trace norm of operator with the Schwartz kernel $F_{t\to h^{-1}\tau} W_n^+ \psi'(x)\psi'(y)$ does not exceed

$$\begin{cases} C'\bar\gamma^{-d} h^s (|\tau|+1)^{-s} & \text{as } \tau \le c, \\ C\bar\gamma^{-d} h^{-M} (|\tau|+1)^{M} & \text{as } \tau \ge c. \end{cases} \tag{9.1.49}$$

Therefore

(9.1.50) The trace norm of the operator with the Schwartz kernel

$$F_{t\to h^{-1}\tau}\chi_{\frac{1}{2}T}(t) W_n^+ \psi'(x)\psi'(y)$$

does not exceed $C'\bar\gamma^{-d} h^s (|\tau|+1)^{-s}$ as $\tau \le \frac{1}{2}c$.

Let us replace in the previous analysis and definitions function χ by $\bar\chi$ equal to 1 on $\operatorname{supp}(\chi)$. Then $\chi_T(t)\big(V_n^+ - W_n^+\big) = 0$ and therefore one can replace W_n^+ by V_n^+ in the statement (9.1.50).

Surely one can replace $h^{-1}\tau$ by τ in this statement as well. Replacing h by $\rho^{-1}\gamma^{-1}$ and summing with respect to n we obtain estimate (9.1.44); (9.1.43) follows from it. Finally, these estimates for sign "+" imply themselves for sign "−". □

Proof of Theorem 9.1.7. Let Π be an arbitrary r-dimensional self-adjoint projector in $\mathcal{H}_0$. Then (9.1.44) yields that

$$|F_{t\to\tau}\chi_T(t)\operatorname{Tr}\Big(\Pi\psi_{\mathsf{sing}}\big(\exp(it\boldsymbol{A}) - \exp(it\boldsymbol{A}')\big)\psi_{\mathsf{sing}}\Big)| \le C'R_2\big(|\tau|+1\big)^{-s} \tag{9.1.51}$$

where here and below we assume that $\tau \le \frac{1}{2}c$.

Observe that the left-hand expression of (9.1.51) equals the left-hand expression of

$$|\int \widehat{\chi_T}(\tau-\lambda)\, d_\lambda\, \mathsf{Tr}\Big(\Pi\psi_{\mathsf{sing}}\big(E(\lambda)-E'(\lambda)\big)\psi_{\mathsf{sing}}\Big)| \le C'R_2\big(|\tau|+1\big)^{-s}$$

where $E'(\lambda)$ is the spectral projector of $\boldsymbol{A}'$. The latter estimate holds ince the trace norm of Π does not exceed Cr. Therefore

$$|\int \widehat{\chi_T}(\tau-\lambda)\, d_\lambda\, \mathsf{Tr}\Big(\Pi\psi_{\mathsf{sing}}E(\lambda)\psi_{\mathsf{sing}}\Big)| \le$$
$$|\int \widehat{\chi_T}(\tau-\lambda)\, d_\lambda\, \mathsf{Tr}\Big(\Pi\psi_{\mathsf{sing}}E'(\lambda)\psi_{\mathsf{sing}}\Big)| + C'R_2\big(|\tau|+1\big)^{-s}.$$

Let us take a Hörmander function χ (and so χ_T is of the same type). Then the standard Tauberian arguments yield that

$$\mathsf{Tr}\Big(\Pi\psi_{\mathsf{sing}}\big(E(\tau)-E(\tau-1)\big)\psi_{\mathsf{sing}}\Big) \le$$
$$C\int \widehat{\chi_T}(\tau-\lambda)\, \mathsf{Tr}\Big(\Pi\psi_{\mathsf{sing}}\, d_\lambda E'(\lambda)\psi_{\mathsf{sing}}\Big) + C'R_2\big(|\tau|+1\big)^{-s}.$$

The right-hand expression of this estimate does not decrease if we replace Π by I and ψ_{sing} by 1. Since $\mathsf{Tr}(E'(\lambda)) = \mathsf{N}^-(\boldsymbol{A}'-\lambda)$ we conclude that

$$\mathsf{Tr}\Big(\Pi\psi_{\mathsf{sing}}\big(E(\tau)-E(\tau-1)\big)\psi_{\mathsf{sing}}\Big) \le$$
$$C\int \widehat{\chi_T}(\tau-\lambda)\, d_\lambda \mathsf{N}^-(\boldsymbol{A}'-\lambda) + C'R_2\big(|\tau|+1\big)^{-s}.$$

Since this estimate is true for all the self-adjoint finite-dimensional projectors in $\mathcal{H}_0$ it is true for Π replaced by I:

$$\mathsf{Tr}\Big(\psi_{\mathsf{sing}}\big(E(\tau)-E(\tau-1)\big)\psi_{\mathsf{sing}}\Big) \le \tag{9.1.52}$$
$$C\int \widehat{\chi_T}(\tau-\lambda)\, d_\lambda \mathsf{N}^-(\boldsymbol{A}'-\lambda) + C'R_2\big(|\tau|+1\big)^{-s}.$$

Thus we got rid of projector Π which we used only to justify the Fourier transform and the replacement of the left-hand expression in (9.1.51).

Picking up $\tau = -n$ in (9.1.52), after summation with respect to $n = 0, 1, 2, \dots$ we obtain estimate

$$\mathsf{Tr}\Big(\psi_{\mathsf{sing}} E(0) \psi_{\mathsf{sing}}\Big)) \le C \sum_{n\ge 0} \int \widehat{\chi_T}(-n-\lambda)\, d_\lambda \mathsf{N}^-(\boldsymbol{A}' - \lambda) + C' R_2. \tag{9.1.53}$$

Let us note that $\sum_{n\ge 0} \widehat{\chi_T}(-n-\lambda)$ does not exceed C_1 and $C_1'(|\lambda|+1)^{-s}$ for $\lambda \ge 0$ and $\lambda \le 0$ respectively.

Therefore the first term in the right-hand expression of (9.1.53) does not exceed

$$C_1' \int_0^\infty (|\lambda|+1)^{-s}\, d_\lambda \mathsf{N}^-(\boldsymbol{A}' - \lambda) + C_1 \int_{-\infty}^0 d_\lambda \mathsf{N}^-(\boldsymbol{A}' - \lambda) \le$$
$$C_1' \int_0^\infty (|\lambda|+1)^{-s}\, d_\lambda \mathsf{N}^-(\boldsymbol{A}' - \lambda) + C_1 \mathsf{N}^-(\boldsymbol{A}')$$

where we have integrated by parts. Then condition (9.1.23) yields that the first term in the right-hand expression in (9.1.53) does not exceed $C_1 R_0$ and therefore

$$\mathsf{Tr}\big(\psi_{\mathsf{sing}} E(0) \psi_{\mathsf{sing}}\big) \le C_1 R_0 + C' R_2. \tag{9.1.54}$$

Namely this estimate was lacking for the proof of Theorem 9.1.7. □

The following problem does not seem to be too hard but still worth publishing:

Problem 9.1.11. As $\mathbb{H}$ is infinite-dimensional space prove the similar results involving auxiliary operators B and B_1 (not to be confused with boundary operators) and Hilbert scale $\mathbb{H}_j$ (see Chapters 2 and 4 for details.

9.1.4 Riesz Means Estimates

Let us instead of $\mathsf{N}^-(A)$ consider Riesz means, namely

$$\mathsf{N}^-_\omega(A) = \int \tau_+^{\omega-1} \mathsf{N}^-(A-\tau)\, d\tau, \qquad \omega > 0. \tag{9.1.55}$$

Then one can apply local asymptotics for Riesz means. However, it would be more convenient for us to write $\mathsf{N}^-_\omega(A)$ in the form

$$\mathsf{N}^-_\omega(A) = \int \tau_+^{\omega-1} \mathsf{N}^-(J^{-\frac12}(A-\tau)J^{-\frac12})\, d\tau \tag{9.1.56}$$

with the same function J as before in order to get better estimates in the singular zone. In this case one can apply the results of Section 6.5.

As before contribution of the partition element ψ of the the regular zone to the main part of the asymptotics is given by $\mathcal{N}_{\psi,\omega}$ prescribed by Section 6.5 while its contribution to the remainder does not exceed $h^{-d}\varrho^{\omega}\nu_{l,\omega}(h)$ as this element is supported in $X_{\mathsf{reg},(l)}$. So the total main part of asymptotics is $\mathcal{N}_{\psi_{\mathsf{reg}},\omega}$ while contribution of the regular zone to the remainder is

$$\mathsf{R}_{\psi,\omega} \le R_{1,\omega} \coloneqq C\sum_{l}\int_{X_{\mathsf{reg}(l)}} \rho^{d}\varrho^{\omega}\vartheta_{l,\omega}\bigl((\rho\gamma)^{-1}\bigr)\,dx. \tag{9.1.57}$$

As in (9.1.21) the standard case was h^{1-d} but now it is $h^{1-d+\omega}\varrho^{\omega}$; so the standard remainder estimate is

$$C\int_{X_{\mathsf{reg}}} \rho^{d-1-\omega}\gamma^{-1-\omega}\varrho^{\omega}\,dx. \tag{9.1.58}$$

Remark 9.1.12. One can easily prove that application of (9.1.56) instead of (9.1.55) does not lead to a different local expression.

Further, the same arguments as in the proof of Theorem 9.1.7 imply

Theorem 9.1.13. *Let conditions* (9.1.5)–(9.1.13) *and* (9.1.15)–(9.1.18) *be fulfilled.*

(i) Then estimate

$$\mathsf{N}^{-}_{\omega}(\boldsymbol{A}) \ge \mathcal{N}_{\omega} - C_0 R_{1,\omega} \tag{9.1.59}$$

holds.

(ii) Moreover, if conditions

$$\mathsf{N}^{-}_{\omega}(\boldsymbol{A}'-t) \le R_{0,\omega}(t+1)^{p} \qquad \forall t\ge 0, \tag{9.1.60}$$

(9.1.24), (9.1.25) *and* (9.1.27) *are fulfilled then the estimate*

$$\mathsf{N}^{-}_{\omega}(\boldsymbol{A}) \le \mathcal{N}_{\omega} + C_0 R_{1,\omega} + C_1 R_{0,\omega} + C' R_{2,\omega} \tag{9.1.61}$$

holds with

$$R_{2,\omega} = \int_{X_{\mathsf{reg},\frac{1}{4}}\cap X_{\mathsf{sing}}} \rho^{d-s}\gamma^{-s}\varrho^{\omega}\,dx, \tag{9.1.62}$$

$C_0 = C_0(d,c,\omega)$, $C_1 = C_1(d,c,p,\omega)$, $C' = C'(d,\mathsf{D},c,M,\omega)$.

Proof. Easy details are left to the reader. □

Problem 9.1.14. Derive similar results for averaging functions φ satisfying (4.2.73)–(4.2.75) rather than $\varphi(\tau) = \tau_+^\omega$.

9.2 Examples to Main Theorems

In this section we consider some rather general examples of contributions of X_{reg} and X_{sing} to the remainder.

9.2.1 Regular Zone: Standard Remainder Estimate

Let us consider the first regular zone. In virtue of Parts II and III one can set

$$\mathcal{N} = \mathcal{N}^{\mathsf{W}} := (2\pi)^{-d} \iint \boldsymbol{n}^-(x,\xi)\, dxd\xi \tag{9.2.1}$$

where $\boldsymbol{n}^-(x,\xi)$ is a number of negative eigenvalues of $a(x,\xi)$ and

$$R_1 = C \int_{X_{\mathsf{reg}} \setminus X_{\mathsf{reg,ell}}} \rho^{d-1}\gamma^{-1}\, dx + C' \int_{X_{\mathsf{reg,ell}}} \rho^{d-s}\gamma^{-s}\, dx \tag{9.2.2}$$

with C' depending on s in the cases described in Examples 9.2.1–9.2.5 below. We assume that $\mathsf{D} < \infty$ is fixed.

Example 9.2.1. Assume that the regularity conditions

$$|D^\beta a_\alpha| \le c\varrho\rho^{-|\alpha|}\gamma^{-|\beta|} \qquad \forall \alpha : |\alpha| \le m, \tag{9.2.3}_1$$

$$|D^\beta b_{j\alpha}| \le c\varrho_j\rho^{-|\alpha|}\gamma^{-|\beta|} \qquad \forall j = 1,\dots,\mu \qquad \forall \alpha : |\alpha| \le m_j, \tag{9.2.3}_2$$

$$|D^\beta J| \le c\varrho_0\gamma^{-|\beta|}, \qquad \forall \beta : |\beta| \le K \tag{9.2.3}_3$$

and

$$|D^\beta \phi| \le c\gamma^{1-|\beta|} \qquad \forall \beta : 1 \le |\beta| \le K \tag{9.2.3}_4$$

are fulfilled; recall that a_α and $b_{j\alpha}$ are coefficients of A and B_j and ϕ appears in (9.1.15)–(9.1.16). Moreover assume that

$$|J^{-1}| \le c\varrho_0^{-1} \tag{9.2.4}$$

and operator $\varrho^{-1}A$ is microhyperbolic at energy level 0 (uniformly, after rescaling $x \mapsto x\gamma^{-1}$, $\xi \mapsto \xi\rho^{-1}$).

Further let us assume that operator $\varrho^{-1}A$ and boundary value problem $\{\varrho^{-1}A, \varrho_1^{-1}B_1, \dots, \varrho_\mu^{-1}B_\mu\}$ are uniformly elliptic in the classical sense. Recall that in virtue of Theorem 7.5.15 we do not need to assume microhyperbolicity of the boundary value problem.

Example 9.2.2. Consider in dimension $d \ge 2$ Schrödinger operator (possibly with magnetic field)

$$A = \sum_{j,k} (D_j - V_j) g^{jk} (D_k - V_k) + V \tag{9.2.5}$$

with real-valued $g^{jk} = g^{kj}$, V_j, V such that

$$c^{-1}|\eta|^2 \le \sum_{j,k} g^{jk}\eta_j\eta_k \le c|\eta|^2 \qquad \forall \eta \in \mathbb{R}^d \tag{9.2.6}$$

and

$$|D^\beta g^{jk}| \le c\gamma^{-|\beta|} \qquad \forall j,k, \tag{9.2.7}_1$$

$$|D^\beta F_{jk}| \le c\rho\gamma^{-|\beta|-1} \qquad \forall j,k, \tag{9.2.7}_2$$

$$|D^\beta V| \le c\rho^2\gamma^{-|\beta|}, \qquad \forall \beta : |\beta| \le K \tag{9.2.7}_3$$

where $F_{jk} = \partial_k V_j - \partial_j V_k$ is the tensor of magnetic field intensity.

Recall that due to results of Section 8.2 we do not need microhyperbolicity.

Furthermore, due to Sections 4.6, 7.5 and 8.2 we can relax smoothness conditions in Examples 9.2.1 and 9.2.2:

Example 9.2.3. Assume ξ-microhyperbolicity in example 9.2.1. Then one can replace condition $(9.2.3)_1$ by

$$|D^\beta a_\alpha| \le c\varrho\beta!\rho^{-|\alpha|}\gamma^{-|\beta|}(1+\varepsilon^{1-|\beta|}) \qquad \text{with } \varepsilon = C(\rho\gamma|\log\rho\gamma|)^{-1} \quad \forall \alpha : |\alpha| \le m, \forall \beta : |\beta| \le \epsilon\rho\gamma; \tag{9.2.8}_1$$

we leave to the reader to rewrite the smoothness conditions to ϕ, $b_{j,\alpha}$ (starting from (7.5.42)) according to Section 7.5 [8)].

[8)] One can relax smoothness conditions to J as well but this is not important since it is an auxiliary weight we introduce as needed and can smoothen it.

Example 9.2.4. Assume that either $d \ge 3$ or that $d = 2$ and

$$|V| + |\nabla V|^{\gamma} \ge \epsilon\rho^2 \tag{9.2.9}$$

in Example 9.2.2. Then we can relax conditions $(9.2.7)_1$–$(9.2.7)_3$ in the similar way.

Example 9.2.5. Furthermore, again due to Sections refsect-4-6, 7.5 and 8.2 we can relax smoothness conditions even further: instead of $(9.2.8)_1$ we can assume $(9.2.3)_1$ with $K = 1$ and

$$|\nabla a_\alpha(x) - \nabla a_\alpha(y)| \le C|\log \mathsf{dist}(x,y)|^{-1}. \tag{9.2.10}$$

Then we can find operators A^+ and A^- bracketing A [9], both satisfying $(9.2.8)_1$ and such that

$$\mathcal{N}^{\mathsf{W}}(A^-) - CR_1 \le \mathcal{N}^{\mathsf{W}}(A) \le \mathcal{N}^{\mathsf{W}}(A^+) \le CR_1. \tag{9.2.11}$$

Other such examples following immediately from Parts II and III we leave to the reader.

9.2.2 Regular Zone: Sharper Asymptotics

In Parts II and III we also derived sharper asymptotics. Their main parts contained next terms some of which were expressed as integrals of functions supported in

$$\Sigma_\tau = \mathsf{supp}\big(\partial_\tau \boldsymbol{n}(x,\xi,\tau)\big) \tag{9.2.12}$$

and others as integrals over

$$\iota\big(\mathsf{supp}\big(\boldsymbol{n}(x,\xi,\tau)\big) \cap T^*X|_Y\big); \tag{9.2.13}$$

recall that $\iota : T^*X|_Y \to T^*Y$. We recall the exact expressions of them in Part V as needed.

Meanwhile the remainder estimate contains two terms:

$$\frac{C}{T_j}\partial_\tau \iint_{\Lambda_j} \boldsymbol{n}(x,\xi,\tau)\, dxd\xi \tag{9.2.14}$$

[9] I.e. such that $A^+ \ge A \ge A^-$ and therefore $\boldsymbol{A}^+ \ge \boldsymbol{A} \ge \boldsymbol{A}^-$ as we use the same matrix J satisfying $(9.2.3)_3$ for all three operators.

and

$$C'' \int_{X_{\mathrm{reg}}} \rho^{d-1-\delta}\gamma^{-1-\delta}\,dx \tag{9.2.15}$$

where (generalized) Hamiltonian trajectories (or billiards) issued from Λ_j in an appropriate time direction should satisfy certain assumptions of Parts II and III. Again we leave exact form of those assumptions to more concrete Part V but we need to mention that while the geometric shape of these trajectories on the energy level 0 does not depend on J, the temp of movement depends on ϱ_0. Really, consider just the case of scalar symbols $a(x,\xi)$ and $J(x)$. Then trajectory is given by equation

$$\begin{cases} \dfrac{dx}{dt} = \quad \varrho_0(x)^{-1}\kappa\partial_\xi a(x,\xi) \\ \dfrac{d\xi}{dt} = -\varrho_0(x)^{-1}\kappa\partial_x a(x,\xi) \end{cases} \tag{9.2.16}$$

where

$$\kappa = \varrho^{-1}\varrho_0\rho\gamma\big|_{x=x(0)} \tag{9.2.17}$$

is a normalization factor, so that initial ball $B(x,\gamma)$ is passed with time $T_0 \asymp 1$. Here we renormalized time with our usual scaling. So, these conditions and estimates are affected by the choice of ϱ_0.

9.2.3 Singular Zone

All results in the singular zone are due to Appendix 9.A.1. We mention only few of them.

Example 9.2.6. Let us assume that

$$(Au,u) \ge \epsilon\|u\|_r^2 - \|\rho^r u\|^2 \qquad \forall u \in \mathfrak{D}(A_B) : \mathsf{supp}(u) \subset \bar{X}_{\mathsf{sing}} \tag{9.2.18}$$

where $m = 2r$ is an order of operator. Therefore due to the Rozenblioum estimate (9.A.11) if on $\partial X_{\mathsf{sing}}$ $\mathfrak{D}(A_B)$ is defined by Dirichlet boundary conditions and if $d > m$ then

$$R_0 = C\int \left(\rho^d + \varrho_0^{d/m}\right) dx. \tag{9.2.19}$$

Meanwhile R_2 is less than R_0, so R_0 estimates the total contribution of X_{sing} to the remainder. Let us recall that the standard remainder estimate in the

regular zone is given by (9.2.2) but the first term may be sharpened; then the total remainder estimate is given by the sum of (9.2.2) and (9.2.19):

$$C\int_{X_{\mathsf{reg}}\setminus X_{\mathsf{reg,ell}}} \rho^{d-1}\gamma^{-1}\,dx + C'\int_{X_{\mathsf{reg,ell}}} \rho^{d-s}\gamma^{-s}\,dx + C\int_{X_{\mathsf{sing}}} \bigl(\rho^d + \varrho_0^{d/m}\bigr)\,dx. \tag{9.2.20}$$

Since the answer should not depend on J we can select $\varrho_0 \le \rho^m$ thus arriving to

$$C\int_{X_{\mathsf{reg}}\setminus X_{\mathsf{reg,ell}}} \rho^{d-1}\gamma^{-1}\,dx + C'\int_{X_{\mathsf{reg,ell}}} \rho^{d-s}\gamma^{-s}\,dx + C\int_{X_{\mathsf{sing}}} \rho^d\,dx. \tag{9.2.21}$$

Note however that selection of J affects sharper asymptotics.

Example 9.2.7. Let all assumptions of the previous example are fulfilled with the exception that on part Y_{N} of $(\partial X)\cap \bar{X}_{\mathsf{sing}}$ $\mathfrak{D}(A_B)$ is defined by some other boundary condition, which does not violate (9.2.18).

Assume that the cone condition (9.A.23)–(9.A.26) is satisfied with the cone of some fixed angle ϕ and height h_j (depending on $X_{\mathsf{sing},j}$). Then in virtue of (9.A.32) remainder estimate is still valid with an extra term

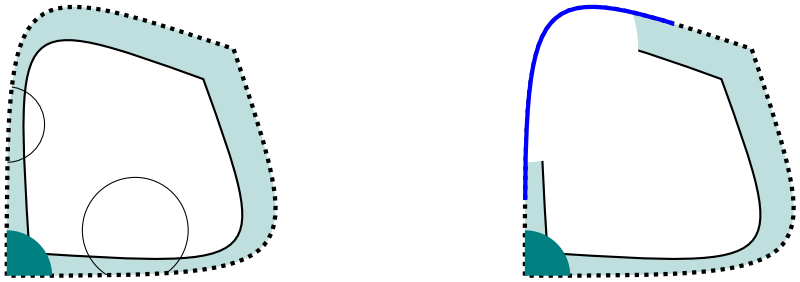
$$C\sum_j h_j^{-d}\,\mathsf{mes}(X_j). \tag{9.2.22}$$

Example 9.2.8. Consider *quasiregular zone* X_{qreg} in which all assumptions of the regular zone are fulfilled with the exception of those to the boundary which may be rather irregular there (see Figure 9.3).

Let us include the *inner zone* $\mathcal{X} = \{x \in X_{\mathsf{qreg}},\ \mathsf{dist}(x,\partial X) \ge \rho^{-1}(x)\}$ into X_{reg} and the *boundary strip* $\mathcal{Y} = \{x \in X_{\mathsf{qreg}},\ \mathsf{dist}(x,\partial X) \le 2\rho^{-1}(x)\}$ into X_{sing}.

Assume that

(9.2.23) After rescaling $x \mapsto (x-y)\gamma(y)^{-1}$ in (rescaled) $\mathcal{B}(y,\gamma(y),X)$ condition (7.5.28) is fulfilled uniformly with respect to y.



(a) (b)

Figure 9.3: On both pictures dotted line shows the part of $\partial X \cap \bar{X}_{\mathsf{qreg}}$ while solid blue line shows the smooth part of the boundary; dark color indicates $X_{\mathsf{sing}} \setminus X_{\mathsf{qreg}}$ and light color indicates $\mathcal{Y}$. Finally, on (a) two balls $\mathcal{B}(y, \gamma(y), X)$ are drawn.

Then due to Theorem 7.5.10 and Proposition 7.5.6(iii) contribution of $\mathcal{B}(y, \gamma(y), X) \cap \mathcal{X}$ to the remainder does not exceed $C\rho^{d-1}\gamma^{d-1}|\log\log(\rho\gamma)|$ [10]. Therefore the total contribution of $\mathcal{X}$ to the remainder does not exceed

$$C \int_{X_{\mathsf{qreg}}} \rho^{d-1}\gamma^{-1} |\log\log(\rho\gamma)|\, dx. \tag{9.2.24}$$

Assume that the Dirichlet boundary conditions are fulfilled there. Then as $d > m$ we can apply Rozenblioum estimate (9.2.12) and conclude that the contribution of X_{sing} to the remainder does not exceed contribution of $X_{\mathsf{sing}} \setminus X_{\mathsf{qreg}}$ plus

$$C \int_{\{x \in X_{\mathsf{qreg}},\ \mathsf{dist}(x, \partial X) \le 2\rho^{-1}(x)\}} \rho^{d}\, dx \tag{9.2.25}$$

(provided $\varrho_0 \le \rho^d$ there). Due to (9.2.23) contribution of $\mathcal{B}(y, \gamma(y), X)$ to this expression does not exceed $C\rho^{d-1}\gamma^{d-1}$ and therefore (9.2.25) does not exceed

$$C \int_{X_{\mathsf{qreg}}} \rho^{d-1}\gamma^{-1}\, dx. \tag{9.2.26}$$

[10]. Really, as $\gamma(y) = 1$, $\rho(y) = h^{-1}$ we are exactly in the framework of this section. General case is reduced to this by rescaling. Obviously γ at this moment is the radius of the ball rather then scaling function inside it.

On the other hand, if $d \le m$ we can apply the boxing described in Subsubsection *9.A.1.2 Bracketing* and since ρ has a constant magnitude in $\mathcal{B}(y,\gamma(y),X)$ we again recover the same remainder estimate.

So, instead of the assumption $X = X_{\mathsf{reg}} \cap X_{\mathsf{sing}}$ we can assume that $X = X_{\mathsf{reg}} \cap X_{\mathsf{qreg}} \cap X_{\mathsf{sing}}$ and calculate contribution of X_{reg} and X_{sing} to the remainder in the usual way and of X_{qreg} by (9.2.24).

Example 9.2.9. (i) If after rescaling $x \mapsto x\gamma^{-1}$ we find ourselves in the framework of condition (7.5.39) uniform with respect to y, we can skip factor $|\log\log(\rho\gamma)|$ in (9.2.24).

(ii) Further, assume that in X_{qreg} cone condition (9.A.23)–(9.A.26) is satisfied with the cone of fixed angle ϕ and height $h = \epsilon\gamma(x)$. Then condition (9.2.23) is fulfilled automatically. In this case we do not need to assume that we have Dirichlet boundary condition but only that (9.2.18) holds.

(iii) Therefore under certain assumptions contribution of X_{qreg} to the remainder does not exceed (9.2.24).

Example 9.2.10. On the other hand, assume that the boundary is less regular; namely we assume only that

$$\mathsf{mes}\big(\{x \in X_{\mathsf{qreg}} \cap \mathcal{B}(y,\gamma(y),X), \mathsf{dist}(x,\partial X) \le t\}\big) \le ct^{\sigma}\gamma(y)^{d-\sigma} \qquad \forall t : \rho(y)^{-1} \le t \le \gamma(y) \tag{9.2.27}$$

is fulfilled uniformly with respect to $y \in X_{\mathsf{qreg}}$ with $\sigma \in (0,1)$ [11].

In this case we can relax smoothness conditions to coefficients as well and due to Theorem 7.5.22 the contribution of $\mathcal{X}$ to the remainder does not exceed

$$C\int_{X_{\mathsf{qreg}}} \rho^{d-\sigma}\gamma^{-\sigma}\,dx. \tag{9.2.28}$$

On the other hand, expression (9.2.25) in this case does not exceed (9.2.28) as well and so the contribution of X_{qreg} to the remainder is estimated by (9.2.28).

This allows us to treat fractal boundaries in Chapter 11.

[11] As $\sigma = 1$ this condition coincides with (9.2.23).

Remark 9.2.11. (i) So far we considered operators which in the regular zone X_{reg} are either regular or "rough". However bracketing operator with the irregular coefficients A between two rough operators $A^- \le A \le A^+$ we conclude that $\mathsf{N}^-(A^+) \le \mathsf{N}(A)\mathsf{N}(A^-)$ and estimating difference $\mathsf{N}^-(A^-) - \mathsf{N}^-(A^+)$ we get remainder estimate for $\mathsf{N}^-(A)$ as well. Actually since $\mathsf{N}^-(A^-)$ and $\mathsf{N}^-(A^+)$ have the same remainder estimates we need to estimate $\mathcal{N}(A^-) - \mathcal{N}(A^+)$.

(ii) The similar approach works for Riesz means as well; we can also apply results of Section 4.7.

9.2.4 Riesz Means

Example 9.2.12. In the framework of Example 9.2.6 remainder estimate (9.2.21) is replaced by

$$C\int_{X_{\mathsf{reg}}\setminus X_{\mathsf{reg,ell}}} \varrho^{\omega}\rho^{d-1-\omega}\gamma^{-1-\omega}\,dx + C'\int_{X_{\mathsf{reg,ell}}} \varrho^{\omega}\rho^{d-s}\gamma^{-s}\,dx + C\int_{X_{\mathsf{sing}}} \varrho^{\omega}\rho^{d}\,dx. \tag{9.2.29}$$

Note however that selection of J affects sharper asymptotics.

Example 9.2.13. In the framework of Example 9.2.7 an extra term (9.2.22) is replaced by

$$C\sum_j h_j^{-d}\int_{X_j} \varrho^{\omega}\,dx. \tag{9.2.30}$$

Example 9.2.14. In the framework of Example 9.2.8 the contributions of X_{qreg} to both the remainder and the principal part are estimated by

$$C\int_{X_{\mathsf{qreg}}} \varrho^{\omega}\rho^{d-1}\gamma^{-1}\,dx. \tag{9.2.31}$$

Remark 9.2.15. In Section 12.6 we discuss Riesz means in domains with corners, angles and critical points and discover that those singularities can produce non-Weyl correction terms in the main part of asymptotics.

9.3 Estimates for Quadratic Forms

Let us prove another version of the basic theorems of this chapter. These new theorems deal with the quadratic forms instead of operators and N^- is now the maximal dimension of the negative subspace lying in $\mathscr{C}_0^K(\mathbb{R}^d, \mathbb{H}) \cap \mathfrak{D}(Q)$. The advantage is that conditions to J become more transparent and we are able to select it properly in X_{sing}.

So, we have two quadratic forms, namely $Q(u) = (Au, u)$ and $Q_0(u) = (Ju, u)$, but we are interested in the negative subspaces of $Q(u)$, and the answer cannot depend on $Q_0(u)$ [12)].

Let us introduce space $\mathcal{H} = \mathcal{H}_{Q_0}$ which is the closure of $\mathscr{C}_0^K(\mathbb{R}^d, \mathbb{H}) \cap \mathfrak{D}(Q)$ in the norm $\|u\|_0 = Q_0(u)^{\frac{1}{2}}$. If the form $Q(u)$ generates a self-adjoint operator $\boldsymbol{A}$ in this space then

$$\mathsf{N}^-(\boldsymbol{A}) = \mathsf{N}^-(Q) \tag{9.3.1}$$

and under appropriate conditions we can apply Theorems 9.1.7 and 9.1.13.

Let us discuss how the answer depends on J. Let us observe first that if we do not consider trajectories then neither conditions nor estimates depend on the choice of J outside of X_{sing}. Surely, if we were to assume that $\boldsymbol{A}$ has a scalar principal symbol (or something like this) then we should take J satisfying this condition because then we have special theorems; so, let us assume that if A has a scalar principal symbol then J is a scalar matrix.

Note that existence of self-adjoint operator requires that

(9.3.2) $\mathfrak{D}(Q)$ is a dense subset of $\mathcal{H}$.

We try to take J in X_{sing} as small as possible (in order to decrease R_0 in estimate (9.1.23)) but we need to fulfill $(9.2.3)_3$ and (9.1.24)–(9.1.25). Note that if we assume that

$$\varrho = \varrho_0 \rho\gamma \qquad \text{in } \ X_{\mathsf{reg}} \setminus X_{\mathsf{reg,ell}} \tag{9.3.3}$$

then in the equations of trajectories (9.2.16) $\kappa = 1$ and therefore these equations and conditions will be more natural. Despite these simplifications the constant R_0 has not yet been calculated and whether the quadratic form $Q(u)$ generates a self-adjoint operator in $\mathcal{H}$ is not yet clear.

[12)] Nevertheless the estimates we will derive can depend on J.

However, in the lower bounds of $\mathsf{N}^-(Q)$ the latter condition is not necessary. Really, these estimates are non-trivial only if $\mathsf{N}^-(Q)<\infty$ but in this case the quadratic form Q is semibounded from below on $\mathcal{H}$ and due (9.3.3) this form really generates an operator on $\mathcal{H}$ which is semibounded from below. Therefore the lower and the upper estimates are provided and in the upper estimates the constant R_0 is deduced from the quadratic form $Q_t(u)=Q(u)-tQ_0(u)$, i.e., conditions (9.1.23) and (9.1.60) should be replaced by

$$\mathsf{N}^-(Q-tQ_0)\le R_0(t+1)^p \qquad \forall t\ge 0 \tag{9.3.4}$$

with Q here restricted to the domain X_{sing}.

Let us consider upper estimates. Namely, let us assume that Q is given by the formula

$$Q(u)= \int_X \sum_{|\alpha|,|\beta|\le r} \langle a_{\alpha\beta}D^\beta u, D^\alpha u\rangle\,dx + \int_{\partial X}\sum_{|\alpha|,|\beta|\le r-1}\langle b_{\alpha\beta}D^\beta u, D^\alpha u\rangle\,dx' \tag{9.3.5}$$

with coefficients satisfying

$$a_{\alpha\beta}=a^\dagger_{\beta\alpha},\quad b_{\alpha\beta}=b^\dagger_{\beta\alpha}\qquad \forall \alpha,\beta, \tag{9.3.6}$$

and

$$\sum_{\alpha,\beta:|\alpha|=|\beta|=r}\langle a_{\alpha\beta}\xi^{\alpha+\beta}v,v\rangle \ge \epsilon_0\varrho\rho^{-2r}|\xi|^{2r}\|v\|^2 \qquad \forall \xi\in\mathbb{R}^d\quad \forall v\in\mathbb{H}. \tag{9.3.7}$$

Further, let us also assume that

(9.3.8) Function $u\in \mathscr{C}_0^K(\mathcal{X},\mathbb{H})$ belongs to $\mathfrak{D}(Q)$ if and only if $B_ju|_{\partial X}=0$ $\forall j=1,\dots,\mu'$ where B_j are differential operators of orders $m_j\le r-1$ (where $m=2r$) and that Q is originally defined on these functions and is expanded to $\mathcal{H}$ as a closure.

Thus the boundary conditions for operator $\boldsymbol{A}$ are $B_ju|_{\partial X}=0$ with $j=1,\dots,\mu$ of two types: original conditions (imposed to the domain of the quadratic form) with $j=1,\dots,\mu'$ and $m_j\le r-1$ and additional conditions with $j=\mu'+1,\dots,\mu$ and $m_j\ge r$ which follow from the variational problem.

Furthermore, let us assume that

(9.3.9) $J, J^{-1} \in \mathscr{L}^{\infty}_{\mathsf{loc}}(\mathcal{X})$ where $\mathcal{X} \supset \bar{X}_{\mathsf{reg}}$ and $\mathcal{X}$ is dense in X.

Then

(9.3.10) The subspaces in which we test whether Q is negative definite are the same for the chosen J and for $J = I$.

All these conditions should be assumed from the beginning which we will do in the statements of theorems. To proceed further, let us first note that under the assumptions of our theorems the contribution of every ball $\mathcal{B}(x, \gamma(x), X)$ with $x \in X_{\mathsf{reg}}$ to $\mathcal{N} + \mathsf{R}$ is either larger than a constant ϵ_1 or is $O(h^s)$ where h means $h_{\mathsf{eff}} = (\rho\gamma)^{-1}$.

We will call these balls *balls of the first type* and *balls of the second type* respectively. Therefore

(9.3.11) If the right-hand expression is finite then the number of balls of the first type is finite and $\rho, \gamma, \varrho, \varrho_0, J$ on the union of these balls are disjoint from 0 and ∞.

Thus, in order to obtain a theorem similar to Theorems 9.1.7 and 9.1.13 but with $\mathsf{N}^-(Q)$ instead of $\mathsf{N}^-(A)$ we need only to prove that Q generates a self-adjoint operator in $\mathcal{H}$. Surely it is sufficient to prove that $\mathsf{N}^-(Q_T) < \infty$ for some $T > 0$ with $Q_T(u) = Q(u) + T(Ju, u)$.

Moreover, note that estimate $N^-(Q) \le M$ is due to the uniform estimate $N^-(Q_t) \le M$ for $Q_t(u) = Q(u) + t(J_1 u, u)$ and for all $t > 0$ and for some function J_1 satisfying the same assumptions as J [13)].

Further, if Q_t generates a self-adjoint operator A_t then one can apply the Theorems 9.1.7 and 9.1.13 to this operator.

Let us note that in balls of the first type for small enough $t > 0$ the same conditions are fulfilled[14)] which were assumed to be fulfilled for $t = 0$. Estimates in balls of the second type and in the singular zone are obvious. Therefore we get the estimate $N^-(Q_t) \le M + \varepsilon(t)$ for $t > 0$, $\varepsilon = 0(1)$ as $t \to +0$, which yields the estimates in question.

Thus, it is sufficient to assume that

[13)] With another weight function ϱ_0' instead of ϱ_0 and with a non-essential constant c' instead of c.

[14)] With constants $2c$ and $\frac{1}{2}\epsilon_0$ instead of c and ϵ.

(9.3.12) Form $Q_{t,T}(u) = Q(u) + t(J_1 u, u) + T(Ju, u)$ is non-negative-definite for every $t > 0$ and sufficiently large $T = T_t$.

But this assumption holds provided

(9.3.13) The quadratic form $Q_T(u) = Q(u) + (J_0 u, u)$ is non-negative definite for some function J_0 satisfying the same assumptions as J [13)].

Really, under this condition one can pick $J_1 = J_0^2 J^{-1}$ and $T = t^{-1}$. Therefore,

(9.3.14) Under condition (9.3.13) one can pass from operators to quadratic forms.

There are many situations in which condition (9.3.13) is obviously fulfilled. Let us list some of them. Assume that

(9.3.15) $\{\psi_n^2\}_{n\ge 0}$ is a partition of unity such that ψ_n satisfy (9.1.19) in X_{reg} uniformly on n, $\mathsf{supp}(\psi_0) \subset X_{\mathsf{sing}} \setminus Y_{\mathsf{reg},\epsilon}$, $\mathsf{supp}(\psi_n) \subset X_{\mathsf{reg}} \setminus Y_{\mathsf{sing},\epsilon}$ where $Y_{\mathsf{reg},\epsilon}$ is $\epsilon\gamma$-neighborhood of $X \setminus X_{\mathsf{sing}}$ and $Y_{\mathsf{sing},\epsilon}$ is $\epsilon\gamma$-neighborhood of $X \setminus X_{\mathsf{reg}}$, $\mathsf{diam}\,\mathsf{supp}(\psi_n) \le \epsilon_2\gamma$ for $n \ge 1$ and the multiplicity of this partition does not exceed $C_0 = C_0(d)$.

Then

$$(9.3.16)\quad Q(u) = \sum_{n\ge 0}\Big(\int_X \psi_n^2 \sum \langle a_{\alpha\beta} D^\beta u, D^\alpha u\rangle \, dx + \int_{\partial X} \psi_n^2 \sum \langle b_{\alpha\beta} D^\beta u, D^\alpha u\rangle \, dx\Big)$$

which in turn equals

$$\begin{aligned}\sum_{n\ge 0}\Big(Q(\psi_n u) &- 2\,\mathsf{Re}\int_X \sum \langle a_{\alpha\beta}[D^\beta, \psi_n]u, D^\alpha \psi_n u\rangle \, dx + \\ &\mathsf{Re}\int_X \sum \langle a_{\alpha\beta}[D^\beta, \psi_n]u, [D^\alpha, \psi_n]u\rangle dx - \\ &2\,\mathsf{Re}\int_{\partial X} \sum \langle b_{\alpha\beta}[D^\beta, \psi_n]u, D^\alpha \psi_n u\rangle \, dx' + \\ &\mathsf{Re}\int_{\partial X} \sum \langle b_{\alpha\beta}[D^\beta, \psi_n],\ [D^\alpha \psi_n]u\rangle \, dx'\Big)\end{aligned}$$

where we sum on α, β inside. Therefore (9.3.16) yields that

$$(9.3.17)\quad Q(u) \geq \sum_{n\geq 0} Q(\psi_n u) - \sum_{|\alpha|\leq r-1} C \int_X \chi\varrho(\rho\gamma)^{-2}\rho^{-2|\alpha|}|D^\alpha u|^2\,dx - \sum_{|\alpha|\leq r-1} C\int_{\partial X} \chi\varrho\rho^{-2|\alpha|-1}|D^\alpha u|^2\,dx' \geq \sum_{n\geq 0}\tilde{Q}(\psi_n u)$$

where

$$(9.3.18)\quad \tilde{Q}(v) = Q(v) - \sum_{|\alpha|\leq r-1} C \int_X \chi\varrho(\rho\gamma)^{-2}\rho^{-2|\alpha|}|D^\alpha v|^2\,dx - \sum_{|\alpha|\leq r-1} C\int_{\partial X} \chi\varrho\rho^{-2|\alpha|-1}|D^\alpha v|^2\,dx',$$

χ is the characteristic function of $X \setminus (2\epsilon\gamma$-neighborhood of $(X_{\mathsf{reg}} \setminus X_{\mathsf{sing}}))$ and we assume from the beginning that

$$(9.3.19)\qquad |D^\sigma a_{\alpha\beta}| \leq c\varrho\rho^{-|\alpha|-|\beta|}\gamma^{1-|\sigma|},$$

$$(9.3.20)\qquad |D^\sigma b_{\alpha\beta}| \leq c\varrho\rho^{-|\alpha|-|\beta|}\gamma^{1-|\sigma|} \qquad \forall\sigma : |\sigma|\leq 1 \quad \forall\alpha,\beta.$$

One can easily see that under conditions (9.3.8)

$$(9.3.21)\qquad \tilde{Q}(u) + 2(J_0 u, u) \geq \frac{1}{2}Q(u) + (J_0 u, u)$$

for appropriate function J_0.

Let us consider $\tilde{Q}(\psi_n u)$ with $n \geq 1$ first. In the case of balls of the first type we consider quadratic forms without boundary conditions and one can see easily that for a large enough T the corresponding quadratic form is positive definite.

Let us consider balls of the second type. The following new difficulty arises: $\psi_n u$ does not necessarily satisfy the boundary conditions defining the domain of the quadratic form Q.

However, this difficulty does not arise provided one of the following conditions is fulfilled:

$$(9.3.22)_1\qquad B(\bar{x}, \gamma(\bar{x})) \subset X \qquad \forall \bar{x} \in X_{\mathsf{reg}};$$

$$(9.3.22)_2\qquad m_j \leq \frac{m}{2} - 1 \implies m_j = 0;$$

in particular, this condition is fulfilled if $m = 2$;

$(9.3.22)_3$ $$B_j = D_1^{j-1} \qquad \forall j = 1, \dots, \mu'.$$

Moreover, this difficulty does not arise in some other cases. Finally, one can avoid this difficulty provided

$(9.3.22)_4$ $\forall j$ with $m_j \le \frac{m}{2} - 1$ operator B_j is a $\mathsf{D}_j \times \mathsf{D}$ matrix operator with $\mathsf{D}_j \le \mathsf{D}$ and the coefficient of $D_1^{m_j}$ has rank D_j at any point of ∂X (where in the local coordinates $X = \{x_1 > 0\}$). Further, (9.3.20) holds for all $\sigma : |\sigma| \le K$ (in contrast to original (9.3.20) with $K = 1$).

Indeed, one can prove easily that under this condition there exists an operator $\mathcal{L}$ such that

(9.3.23) $$\mathsf{supp}(v) \subset \mathcal{B}\big(\bar{x}, (1-2\epsilon)\gamma(\bar{x}), X\big) \implies \mathsf{supp}(\mathcal{L}v) \subset \mathcal{B}\big(\bar{x}, (1-\epsilon)\gamma(\bar{x}), X\big),$$

(9.3.24) $$\sum_{|\alpha|\le q} \rho^{-|\alpha|} \|D^\alpha \mathcal{L} v\| \le C_q \sum_{|\alpha|\le q-1} \rho^{-|\alpha|-1}\gamma^{-1} \|D^\alpha v\|$$

where K in (9.3.20) also depends on q and $q \ge p$ is arbitrary,

(9.3.25) $$B_j(I + \mathcal{L})v|_{\partial X} = 0 \qquad \forall j : m_j \le \frac{m}{2} - 1$$

and

(9.3.26) $$(I + \mathcal{L})v = 0,\ \mathsf{supp}(v) \subset \mathcal{B}\big(\bar{x}, (1-\epsilon)\gamma(\bar{x}), X\big) \implies v = 0.$$

An easy proof is left to the reader.

Further, a similar operator $\mathcal{L}$ is also defined for $n = 0$ with

(9.3.27) $$\mathsf{supp}(v) \subset \mathcal{Z}_2 \implies \mathsf{supp}(\mathcal{L}v) \subset \mathcal{Z}_1, \qquad v = \mathcal{L}v \quad \text{in} \quad \mathcal{Z}_4,$$

(9.3.28) $$\sum_{|\alpha|\le q} \|\varrho^{\frac{1}{2}} \rho^{-|\alpha|} D^\alpha \mathcal{L} v\| \le C_q \sum_{|\alpha|\le q-1} \|\varrho^{\frac{1}{2}} \rho^{-|\alpha|-1}\gamma^{-1} D^\alpha v\|$$

(9.3.29) $$B_j(I + \mathcal{L})v|_{\partial X} = 0 \quad \forall j : m_j \le \frac{m}{2} - 1 \quad \text{provided} \quad B_j v|_{\partial X \cap \mathcal{Z}_3} = 0 \quad \forall j : m_j \le \frac{m}{2} - 1,$$

$$(I+\mathcal{L})v=0,\ \operatorname{supp}(v)\subset \mathcal{Z}_1 \implies v=0 \tag{9.3.30}$$

where $\mathcal{Z}_k$ is $\bar{X}\setminus(k\epsilon\gamma$-neighborhood of $X\setminus X'_{\mathrm{reg},0})$ and $\epsilon>0$ is a small enough constant. This construction is also left to the reader. Recall that ψ_0 differs from 1 only in the regular zone.

Finally, we obtain estimate

$$Q(u)\geq \sum_{n\geq 0}\tilde{Q}\big((I+\mathcal{L})\psi_n u\big). \tag{9.3.31}$$

Therefore in all the cases $(9.3.22)_{1-4}$ the following estimate

$$\mathsf{N}^-(Q)\leq \sum_{n\geq 1}\mathsf{N}^-\big(\tilde{Q};\mathcal{B}(x^n(1-\epsilon)\gamma^n,X\big)+\mathsf{N}^-\big(\tilde{Q},\bar{X}_{\mathsf{sing}}\big) \tag{9.3.32}$$

holds where Z denotes the geometrical domain on which the quadratic form Q is defined (i.e., all the test functions are supported) in the definition of $\mathsf{N}^-(\tilde{Q},Z)$, $x^n\in\operatorname{supp}(\psi_n)$ and $\gamma^n=\gamma(x^n)$ as $n\geq 1$.

Let us recall that under the above conditions estimate

$$\mathsf{N}^-(\tilde{Q},\mathcal{B}(x^n,(1-\epsilon)\gamma^n,X)\leq M(\mathcal{N}_n^-+R_n)$$

holds where $\mathcal{N}_n^-$ and R_n are the contributions of the ball $\mathcal{B}(x^n,\gamma^n,X)$ to the principal part and the remainder estimate of the corresponding asymptotics. To prove this one should consider balls of the first and second type separately. Recall that there is only a finite number of balls of the first type.

Therefore the total contributions of these elements is finite provided the right-hand expression in the upper estimate is finite. Furthermore, condition (9.3.4) yields that the contribution of $n=0$ is finite as well. Therefore,

(9.3.33) In the framework of our assumptions for appropriate function J_0 the number $N^-(Q_1)$ is finite and therefore $Q_T(u)=Q(u)+T(J_0u,u)$ is positive definite for some T.

Thus we arrive to

Theorem 9.3.1. *In the framework of our assumptions Theorems 9.1.7 and 9.1.13 hold for quadratic forms as well.*

9.A Appendices

9.A.1 Variational Estimates

Rozenblioum Estimates

Rozenblioum estimates are far the most important variational estimates we use. Let us consider the following two quadratic forms defined on scalar functions:

$$Q_1(u) = \sum_{|\alpha|=m} \bigl(a(x)D^\alpha u, D^\alpha u\bigr), \tag{9.A.1}$$

and

$$Q_0(u) = \sum_{|\alpha_1|\le m, |\alpha_2|\le m, |\alpha_1|+|\alpha_2|=2r} \bigl(V_{\alpha_1,\alpha_2}(x)D^{\alpha_1}u, D^{\alpha_2}u\bigr) \tag{9.A.2}$$

where

$$m > r \ge 0, \quad m \in \mathbb{Z}, \quad r \in \frac{1}{2}\mathbb{Z}, \tag{9.A.3}$$

$$a \ge 0, \quad V_{\alpha_1,\alpha_2} = V^\dagger_{\alpha_2,\alpha_1}. \tag{9.A.4}$$

Let us assume that

$$a^{-1} \in \mathscr{L}^p, \quad V_{\alpha_1,\alpha_2} \in \mathscr{L}^q, \tag{9.A.5}$$

$$1 \le p \le \infty, \quad 1 \le q \le \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{\theta} = \frac{2}{d}(m-r), \tag{9.A.6}$$

$$m - r \le d,\ p \ge 1; \quad \theta > 1 \implies q > 1. \tag{9.A.7}$$

Moreover, let us assume that

(9.A.8) Q_0 contains only terms with

$$\begin{aligned} &2(m-|\alpha_i|) < d(1+\frac{1}{p}) && \text{for} \quad p > 1,\\ &m - |\alpha_i| \le d && \text{for} \quad p = 1. \end{aligned}$$

Let us introduce main objects of the study:

(9.A.9) Let $\mathsf{N}_0^-(Q,X)$ be a maximal dimension of Q-negative subspace of $\mathscr{C}_0^\infty(X)$

and

(9.A.10) Let $\mathsf{N}^-(Q,X)$ be a maximal dimension of Q-negative subspace of $\mathscr{C}^\infty(\bar{X})$ having compact support where integration is taken over X in definition of Q.

Then the following is estimate due to G. Rozenblioum [5]:

Theorem 9.A.1. *In assumptions* (9.A.1)–(9.A.8) *estimate*

$$\mathsf{N}^-(Q_1-Q_0,\mathbb{R}^d)\le C\|a^{-1}\|^\theta_{\mathscr{L}^p}\cdot\|V\|^\theta_{\mathscr{L}^q} \tag{9.A.11}$$

holds where

$$\|V\|_*:=\sum_{\alpha_1,\alpha_2}\|V_{\alpha_1,\alpha_2}\|_*$$

and $C=C(d,m,r,p,q)$.

Example 9.A.2. The most well known case is $m=1, r=0$ when for $d\ge 3$ one can take $p=\infty, q=\frac12 d$. This is the famous Cwikel-Lieb-Rozenblioum estimate (CLR) (see M. Reed & B. Simon [1] and for $d=2$ one can take arbitrary $q>1$ and $p=q/(q-1)$.

Corollary 9.A.3. *Estimate* (9.A.11) *holds for* $\mathsf{N}^-_0(Q_1-Q_0,X)$ *as* $X\subset\mathbb{R}^d$.

Other estimates of this type are discussed in survey M. S. Birman & M. Z. Solomyak [5].

Bracketing

Let γ be a scaling function on X, and X be covered by balls $\mathcal{B}(x_n,\gamma(x_n),X)$ of multiplicity not exceeding c. Let $\sum_n\psi_n^2=1$ where ψ_n be real-valued functions supported in $\mathcal{B}(x_n,\gamma(x_n),X)$ and satisfy

$$|D^\beta\psi_n|\le c\gamma_n^{-|\beta|} \tag{9.A.12}$$

Note that

$$Q_1(u)\ge\sum_n\Big(\frac12 Q_1(\psi_n u)-Q_1'(\psi_n u)\Big)$$

with

$$Q_1'(u) = C(a\gamma^{-2m}u, u)$$

and as $r = 0$

$$Q_0(u) = \sum_n Q_0(\psi_n u).$$

Then

$$\mathsf{N}_0^-(Q_1 - Q_0, X) \le \sum_n \mathsf{N}_0^-\big(Q_1 - 2Q' - 2Q_0, \mathcal{B}(x_n, \gamma(x_n), X)\big). \tag{9.A.13}$$

Similar inequalities hold for $\mathsf{N}^(Q_1 - Q_0, X)$

$$\mathsf{N}^-(Q_1 - Q_0, X) \le \sum_n \mathsf{N}^-\big(Q_1 - 2Q' - 2Q_0, X; \mathcal{B}(x_n, \gamma(x_n), X)\big) \tag{9.A.14}$$

where $\mathsf{N}^-(Q, X; Y_n)$ denotes Q-negative subspace of $\mathscr{C}^\infty(Y_n)$ supported in Y_n where integration is taken over X in definition of Q.

Neumann Type Problems

Let us now consider these quadratic forms on the space $\mathcal{V} \subset \mathscr{C}^m(\bar{X})$. Let us assume that $X = \bigcup_{j\in\mathcal{J}} X_j$ and $X_j \subset \mathcal{X}_j$ where X_j and $\mathcal{X}_j$ are open domains such that for every j there exists a linear operator $\mathcal{L}_j : \mathcal{V} \to \mathscr{C}_0^m(\mathcal{X}_j)$ satisfying

$$\mathcal{L}_j u = u \qquad \text{in} \quad X_j \quad \forall u \in \mathcal{V}, \tag{9.A.15}$$

with

$$Q_{1j}(\mathcal{L}_j u) \le Q_{1j}'(u) + Q_{2j}'(u) \qquad \forall u \in \mathcal{V} \qquad \forall j \tag{9.A.16}$$

where Q_{1j} and Q_{1j}' are forms of (9.A.1)-type with a replaced by $a_j \ge 0$, $a_j' \ge 0$ respectively and Q_{2j}' are forms of (9.A.2)-type.

Then summing (9.A.16) over j we obtain from (9.A.15) that if the quadratic form $Q - Q_0$ is negative definite on the subspace $\mathcal{W} \subset \mathcal{V}$ then on this subspace

$$\sum_j Q_{1j}(\mathcal{L}_j u) \le Q_1(u) + \sum_j Q_{2j}'(u) \le \\ Q_0(u) + \sum_j Q_{2j}'(u) \le \sum_j Q_0'(\mathcal{L}_j u) \tag{9.A.17}$$

provided

$$\sum_j a'_j = a, \tag{9.A.18}$$

and

(9.A.19) Forms Q_0, Q'_{2j} contain only terms with $\alpha_1 = \alpha_2$, $|\alpha_1| = r$ and $V_{\alpha_1,\alpha_1} \ge 0$ (and the same is true for the coefficients of Q'_{2j}) and

$$Q'_0 = Q_0 + \sum_k Q'_{2k}. \tag{9.A.20}$$

Therefore $\mathsf{N}^-(Q_1 - Q_0, \mathcal{V})$ does not exceed the maximal dimension of the subspace $\mathcal{W} \subset \bigoplus_j \mathscr{C}_0^m(\mathcal{X}_j)$ with the following property:

(9.A.21) For every $w = (w_j)_{j\in\mathcal{J}} \in \mathcal{W}$ there exists $u \in V$ with $w_j = \mathcal{L}_j u$ for all j on which the quadratic form $\sum_j (Q_{1j} - Q'_{0j})(w_j)$ is negative definite.

Dropping the condition $\mathcal{L}_j u = w_j$ on w_j we obtain that

$$\mathsf{N}^-(Q_1 - Q_0, \mathcal{V}) \le \sum_j \mathsf{N}^-(Q_{1j} - Q'_{0j}, \mathcal{W}_j) \tag{9.A.22}$$

with $\mathcal{W}_j = \mathscr{C}_0^m(\mathcal{X}_j)$.

Now one can apply Rozenblioum estimate (9.A.11) in order to estimate every term in the right-hand expression. One should take $\mathscr{L}^p$- and $\mathscr{L}^q$-norms on $\mathcal{X}_j$ in these estimates.

In particular, let $\mathcal{V}$ be the space of all compactly supported $\mathscr{C}^{m+1}$ functions satisfying the Dirichlet condition of order $(m+1)$ on $Y_{\mathsf{D}} \subset \partial X$. Let us assume that

$$Y_{\mathsf{N}} = \partial X \setminus Y_{\mathsf{D}}, \tag{9.A.23}$$

$$X = \bigcup_{j\ge 0} X_j, \tag{9.A.24}$$

$$Z_j = Y_{\mathsf{N}} \cap \bar{X}_j, \qquad Y_{\mathsf{N}} \cap \bar{X}_0 = \emptyset. \tag{9.A.25}$$

Let Γ_j be open cones with vertex 0 in $\mathbb{R}^d$ with angle $\phi > 0$ and altitudes h_j such that the following *cone condition* is fulfilled:

(9.A.26) For every point $\bar{x} \in X : \mathsf{dist}(x, X_j) \le \epsilon_0 h_j$ the connected component of $(\bar{x} + \Gamma_j) \setminus Y_{\mathsf{D}}$ containing $\bar{x}$ does not intersect Y_N.

Recall that $\mathsf{dist}(x, y)$ is a path distance in X.

Then taking the Sobolev extension operators $\mathcal{L}_j$ (see f.e. S. L. Sobolev [1]) associated with cones $\epsilon\Gamma_j$ with a small enough constant $\epsilon > 0$ and cutting of on the distances exceeding $\epsilon_1 h_j$ from X we see that for $j \ge 1$

$$\sum_{|\alpha|=m} \|D^\alpha \mathcal{L}_j u\| \le C \sum_{|\alpha|=m} \|D^\alpha u\|_{X'_j} + C h_j^{-m} \|u\|_{X'_j} \tag{9.A.27}$$

where X'_j is a connected component of $(X_j + 2\epsilon\Gamma_j) \setminus Y_{\mathsf{D}}$, containing $\bar{x}$.

Assuming that

(9.A.28) X'_j form no more than a c-multiple covering of $X \setminus X_0$

one can take $Q_1 \asymp \|.\|^2_m$, $Q'_{1j} \asymp \|.\|^2_m$, $Q_{1j} \asymp \|.\|^2_m$ in the domains X, X'_j and $\mathbb{R}^d$ respectively and $Q'_{2j} = C h_j^{-2m} \|.\|^2_{X'_j}$.

Moreover, assume that

(9.A.29) $X''_j = \{x \in X, \mathsf{dist}(x, Z_j) < h_j\}$ form no more than a c-multiple covering of $X \setminus \mathcal{X}_0$.

Then as $\mathcal{L}_0$ one can take the operator of multiplication by a function ψ such that $\psi = 1$ in X_0, $\psi = 0$ in the ϵh_j-neighborhood of Z_j (for every $j \ge 1$) satisfying the inequalities $|D^\alpha \psi| \le c\gamma^{-|\alpha|} \quad \forall \alpha : |\alpha| \le m$ with $\gamma = \mathsf{dist}(x, Z)$.

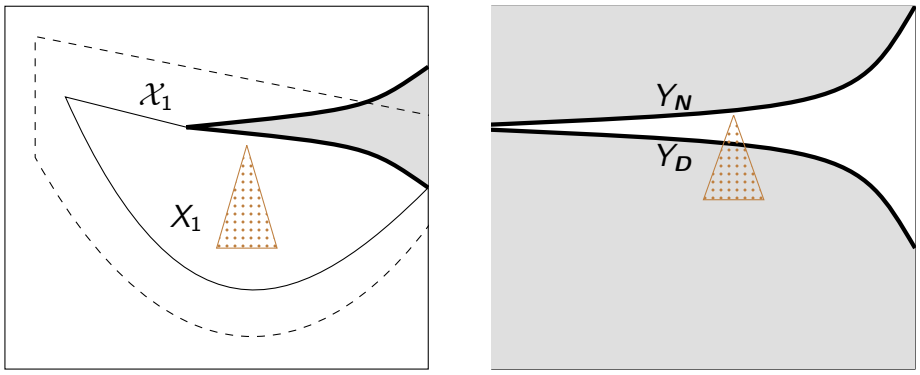
In this case one can take $Q_1(u) \asymp \sum_{|\alpha|=m} (a D^\alpha u, D^\alpha u)$, Q'_{10} and Q_{10} given by the same formula in X, X_0 and $\mathbb{R}^d$ respectively and $Q'_{20} = C \sum_j h_j^{-2m} \|.\|^2_{X''_j}$ provided

$$\epsilon_0 \le a \le c \qquad \text{in} \quad Y'_j \quad \forall j \ge 1. \tag{9.A.30}$$

So, we obtain that $Q'_{0j}(v) = Q_0(v) + (W_j v, v)$ with $W_j = h_j^{-2m} \chi_{X''_j}$ for $j \ge 1$ and $W_0 = \sum_{j \ge 1} h_j^{-2m} \chi_{X''_j}$ where χ_M is the characteristic function of the set M. In fact, without any loss of the generality one can assume that X_j are contained in the $\frac{2}{3} h_j$-neighborhoods of Z_j.

Let us apply the following inequality

$$\mathsf{N}^-(Q_{(1)} + Q_{(2)}) \le \mathsf{N}^-(Q_{(1)}) + \mathsf{N}^-(Q_{(2)}) \tag{9.A.31}$$



(a) (b)

Figure 9.4: Domain X is white, cone is covered by dots. On (a) this cone serves X_1; flipping vertically we get X_2 and corresponding cone. On (b) pretty large cone serves thin domain because of Y_D making the lower part of the cone irrelevant.

(on the same functional spaces) with $Q_{(1)} = \frac{1}{2}Q_{1j} - Q_0$ and $Q_{(2)} = \frac{1}{2}Q_{1j} - Q'_{2j}$.

First of all, let us apply Rozenblioum estimate (9.A.11) to $\mathsf{N}^-(Q_{(1)})$.

(i) Assume that $d > 2m$. Then we can apply the same estimate (9.A.11) to $\mathsf{N}^-(Q_{(2)})$ with $p = \infty$, $\theta = q = d/2m$. After summation over j we end up with estimate

$$\mathsf{N}^-(Q_1 - Q_0) \le C\|a^{-1}\|^\theta_{\mathscr{L}^p}\|V\|^\theta_{\mathscr{L}^q} + C\sum_j h_j^{-d}\,\mathsf{mes}(X''_j). \tag{9.A.32}$$

(ii) For $d \le 2m$ let us assume that

$$j = 0, \dots, J < \infty \quad \text{in the covering in question.} \tag{9.A.33}$$

In this case let us again apply (9.A.31) and take a covering of X''_j by h_j-cubes of multiplicity less than C_0. Furthermore, let us apply estimate $\mathsf{N}^-_(Q_1 - 2Q_0) \le C_1$ in every cube. Then we again obtain estimate (9.A.32) with a constant C depending on J.

So the following statement has been proven:

Proposition 9.A.4. *(i) Under the above conditions estimate* (9.A.32) *holds for* $d > 2m$.

(ii) For $d \le 2m$ *this estimate holds under additional assumption* (9.A.33) *with* C *depending on* J.

Some Useful Estimate

In the end of this subsection let us assume that $a \ge \epsilon_0$ in X_{sing} and let us estimate $\mathsf{N}^-(Q_1 - Jt + J_3)$ in the domain X_{sing} with $J = J_1 + J_2$ where $J_1 = \chi_{X_{\mathsf{reg}} \cap X_{\mathsf{sing}}} \rho^{-2s} \gamma^{-2s-2m}$ $(s \in \mathbb{R})$, $J_2 \ge 0$, $J_3 \ge 0$, with thea Dirichlet condition on $\partial X_{\mathsf{sing}} \cap X$. Here ρ and γ satisfy all assumptions of Section 9.1.

Let us apply estimate (9.A.31). Then in order to estimate $\mathsf{N}^-(Q_1 - 2J_1 t)$ one can use a covering of $X_{\mathsf{reg}} \cap X_{\mathsf{sing}}$ by balls $\mathcal{B}(x, \gamma(x), X)$ with multiplicity less than C_0.

Then in every ball without boundary conditions (because ∂X is regular in the intersection with this ball)

$$\mathsf{N}^-(Q_1 - 2J_1 t + J_3) \le C(t\rho^{-2s}\gamma^{-2s})^{\frac{d}{2m}} + C_0 \tag{9.A.34}$$

in the general case and the last term is absent provided

$$x \in X_{\mathsf{reg},+} = \{x \in X_{\mathsf{reg}}, J_3 \ge \epsilon \rho^{2m} \text{ in } \quad B(x, \bar{\gamma}(x))\}. \tag{9.A.35}$$

Therefore their sum does not exceed

$$Ct^{\frac{d}{2m}} \int_{X_{\mathsf{reg}} \cap X_{\mathsf{sing}}} \rho^{-\frac{ds}{m}} \gamma^{-\frac{ds}{m} - d} \, dx + C \int_{X_{\mathsf{reg},-} \cap X_{\mathsf{sing}}} \gamma^{-d} \, dx \tag{9.A.36}$$

where $X_{\mathsf{reg},-} = X_{\mathsf{reg}} \setminus X_{\mathsf{reg},+}$.

Moreover, the contribution of $X_{\mathsf{sing}} \setminus X_{\mathsf{reg}}$ is equal to 0 (because the form $Q_1 - 2J_1 t + J_3$ is non-negative definite on functions supported there). So in this case we obtain the estimate

$$\mathsf{N}^-(Q_1 - Jt + J_3) \le \tag{9.A.37}$$

$$\mathsf{N}^-(Q_1 - 2J_2 t + J_3) + Ct^{\frac{d}{2m}} \int_{X_{\mathsf{reg}} \cap X_{\mathsf{sing}}} \rho^{\frac{ds}{m}} \gamma^{\frac{ds}{m} - d} \, dx + C \int_{X_{\mathsf{reg},-} - \cap X_{\mathsf{sing}}} \gamma^{-d} \, dx.$$

9.A.2 Riesz Means Estimates

Assume that $r = 0$, $Q_0 = (Wu, u)$ with $W \ge 0$ and $\bar{Q} = \|u\|^2$. Let us consider

$$\mathsf{N}^-_\omega := \int_{-\infty}^0 t_-^\omega \, d_t \mathsf{N}^-(Q_1 - Q_0 - t\bar{Q}) = \omega \int_{-\infty}^0 t_-^{\omega-1} \mathsf{N}^-(Q_1 - Q_0 - t\bar{Q})\, dt. \tag{9.A.38}$$

Then, according to Rozenblioum estimate (9.A.11)

$$\mathsf{N}^-_\omega \le C\|a^{-1}\|^\theta_{\mathscr{L}^p} \int_0^\infty \|(V-t)_+\|^\theta_{\mathscr{L}^q} \cdot t^{\omega-1}\, dt. \tag{9.A.39}$$

In the case $p = \infty$, $q = \theta$ [15)] we immediately obtain estimate

$$\mathsf{N}^-_\omega \le C\|a^{-1}\|^{\frac{d}{2m}}_{\mathscr{L}^\infty} \int W^{\frac{d}{2m}+\omega}\, dx. \tag{9.A.40}$$

On the other hand, let $p < \infty$; then $q > \theta$. Applying the Hölder inequality to the right-hand expression of estimate (9.A.11) we get estimate

$$\mathsf{N}^-_\omega \le C\|a^{-1}\|^\theta_{\mathscr{L}^p} \cdot \Big(\int \phi(t)\, dt\Big)^{1-\frac{\theta}{q}} \Big(\iint t^{\frac{q}{\theta}(\omega-1)} (W-t)_+^q \phi(t)^{\frac{1-q}{\theta}}\, dxdt\Big)^{\frac{\theta}{q}}$$

with arbitrary function $\phi(t) \ge 0$; here we integrate with respect to $t \in (0, \infty]$. Picking up $\phi(t) = t^{-1+\delta}(t+\tau)^{-\delta-\delta'}$ with $\delta > 0, \delta' > 0$ and parameter $\tau \ge 0$ we then obtain that

$$\mathsf{N}^-_\omega \le C\|a^{-1}\|^\theta_{\mathscr{L}^p} \Big[\Big(\int W^{l_1} dx\Big)^{\frac{\theta}{q}} \tau^{-\delta'(1-\frac{\theta}{q})} + \Big(\int W^{l_2} dx\Big)^{\frac{\theta}{q}} \tau^{\delta(1-\frac{\theta}{q})}\Big]$$

with

$$l_1 = q(1+\frac{\omega}{\theta}) + \delta'(\frac{q}{\theta}-1),\ l_2 = q(1+\frac{\omega}{\theta}) - \delta(\frac{q}{\theta}-1).$$

Minimizing with respect to τ and denoting $\rho = (\delta+\delta')(\frac{q}{\theta}-1)$, $\nu = \delta(\delta+\delta')^{-1}$ we obtain the following

Proposition 9.A.5. *(i) As $d > 2m$ estimate* (9.A.40) *holds.*

[15)] This is possible if and only if $d > 2m$.

(ii) As $d \le 2m$ estimate

$$\mathsf{N}_\omega^- \le C\|a^{-1}\|_{\mathscr{L}^p}^\theta \Big(\int W^{l_1}\,dx\Big)^{\frac{\nu\theta}{q}} \Big(\int W^{l_2}\,dx\Big)^{\frac{(1-\nu)\theta}{q}} \tag{9.A.41}$$

holds with arbitrary $\nu \in (0,1)$, $\rho > 0$

$$l_1 = q(1+\frac{\omega}{\theta}) + (1-\nu)\rho, \quad l_2 = q(1+\frac{\omega}{\theta}) - \nu\rho.$$

(iii) In particular, as $d = 2m$ estimate

$$\mathsf{N}_\omega^- \le C_q\|a^{-1}\|_{\mathscr{L}^\infty} \big(\mathsf{mes}(X)\big)^{1-\frac{1}{q}} \Big(\int W^{(\omega+1)q}\,dx\Big)^{\frac{1}{q}} \tag{9.A.42}$$

holds with arbitrarily chosen $q > 1$.

We leave to the reader the following

Problem 9.A.6. Investigate averaging with function φ satisfying (4.2.73)–(4.2.75).

The following problem is really hard (see Comments).

Problem 9.A.7. Investigate if estimate (9.A.40) holds for some $q > 0$ even if $d \le 2m$.

Comments

The principal idea of this chapter was first used in the papers V. Ivrii [13] and V. Ivrii & S. Fedorova [1] where a second order operator was treated via the wave equation, and the finite spatial velocity of the propagation of supports was used instead of the finite spatial velocity of propagation of singularities. The next paper V. Ivrii [14] and all consecutive papers were using semiclassical asymptotics in the regular zone and variational estimates in the singular zone and were based on the slightly more complicated (and essentially more universal and flexible) proof presented here.

In all my papers and in Part V just the Rozenblioum spectral estimate (see G. Rozenblioum [2]) was used. Sometimes even the earlier Birman–Solomyak estimate (see M. S. Birman & M. Z. Solomyak [3–6]) is sufficient.

But we do not want to "close the door" to using other estimates and hence a more abstract form of the basic theorems is presented.

Note that in the special case of the Schrödinger operator the Rozenblioum estimate is the famous Lieb–Cwikel–Rozenblioum estimate, at least five different proofs of which are known (J. Conlon [1], M. Cwikel [1], E. Lieb [1], P. Li & S. T. Yau [1], G. Rozenblioum [2].

Essential in these estimates are Dirichlet boundary conditions. It is well known that Neumann Laplacian could differ drastically from Dirichlet Laplacian in the domains which are not "bounded domains with smooth boundaries". This difference will be exploited in Section 12.7. But in more general setting not just asymptotics and estimates but the whole character of the spectrum could be different. What we have in Appendix 9.A.1 is the only patch. We can think about more systematic use of the methods and results of Yu. Netrusov & Yu. Safarov [1, 2].

Estimates to Riesz means were investigated in many papers, see f.e. E. H. Lieb & W. Thirring [1], L. Erdös [1], A. Laptev & T. Weidl [1], Y. Netrusov & T. Weidl [1].

Surely, in applications in which an operator depends on parameters, the auxiliary functions $\rho, \varrho, \dots$ (and maybe γ) also depend on the parameters, and thus regular and singular zones, etc., also depend on the parameters but constants do not.

Chapter 10

Estimates of the Spectrum in the Interval

In this chapter we obtain upper and lower estimates for

$$\mathsf{N}(A;\tau_1,\tau_2) \coloneqq \mathsf{Tr}\, E(A;\tau_1,\tau_2) \tag{10.0.1}$$

with $-\infty < \tau_1 < \tau_2 < \infty$.

Here $E(A;\tau_1,\tau_2)$ is the spectral projector corresponding to the *open interval* (τ_1,τ_2) rather than to more standard semi-open interval $[\tau_1,\tau_2)$ as before[1)] and $\mathsf{N}(A;\tau_1,\tau_2)$ is number of eigenvalues in (τ_1,τ_2) counting their multiplicities, unless $(\tau_1,\tau_2)\cap \mathsf{Spec}_{\mathsf{ess}}(A)\neq\emptyset$ in which case $\mathsf{N}(A;\tau_1,\tau_2)=\infty$.

These results are useful in the case when $\mathsf{N}^-(A-\tau_1)=\infty$ and also $\mathsf{N}^-(-A+\tau_2)=\infty$. Our analysis is very similar to the analysis of Chapter 9 but here an important role is played by Birman-Schwinger principle. It is rather strange that this extremely powerful and simple tool has been used much less than it should be in the papers devoted to spectral asymptotics.

Further, in this chapter we do not consider Riesz means-like mollifications with respect to spectral parameters[2)].

[1)] This adjustment allows us to cover the case $\tau_1\in \mathsf{Spec}_{\mathsf{ess}}(A)$.

[2)] Riesz means themselves are usually infinite.

10.1 Statement of the Main Theorems

10.1.1 Preliminary Remarks

First of all let us just mention a series of results which follows trivially from the results of the previous chapter applied to operators $A-(\tau_1+0)$ and $A-\tau_2$ with corresponding functions J_1 and J_2, and from the formula

$$\mathsf{N}(A;\tau_1,\tau_2)=\mathsf{N}^-(A-\tau_2)-\mathsf{N}^-(A-(\tau_1+0)). \tag{10.1.1}$$

These results do not work, however, in the numerous cases when there is an indefiniteness of the form "$\infty-\infty$" in the right-hand expression.

Similarly, replacing A, τ_1 and τ_2 by $-A$, $-\tau_2$ and $-\tau_1$ respectively we could apply formula

$$\mathsf{N}(A;\tau_1,\tau_2)=\mathsf{N}^-(-A+\tau_1)-\mathsf{N}^-(-A+(\tau_2-0)) \tag{10.1.2}$$

but again these results do not work, however, in the numerous cases when there is an indefiniteness of the form "$\infty-\infty$" in the right-hand expression.

Formula

$$\mathsf{N}\big(A;\tau_1,\tau_2\big)=\mathsf{N}^-\big((2A-(\tau_2+\tau_1))^2-(\tau_2-\tau_1)^2\big) \tag{10.1.3}$$

may be more useful than both (10.1.1) and (10.1.2) but it may ignore that levels τ_1 and τ_2 are not "of equal rights".

Further, let us mention some results which follow trivially from Parts II and III: if operators $A-\tau_1$ and $A-\tau_2$ satisfy assumptions of the previous chapter with $J_1=J_2=I$ (this is essential only in the restrictions to trajectories) and with the auxiliary functions $\gamma_{(1)}$, $\rho_{(1)}$, $\varrho_{(1)1}$, ... and $\gamma_{(2)}$, $\rho_{(2)}$, $\varrho_{(2)1}$, ... respectively and if in both cases $X_{\mathsf{sing}(1)}=X_{\mathsf{sing}(2)}=\emptyset$ then under additional restriction $\gamma_{(1)}=\gamma_{(2)}=\gamma$, $\rho_{(1)}=\rho_{(2)}=\rho$, ... we can apply a γ-admissible partition of unity for

$$\mathsf{N}(A;\tau_1,\tau_2)=\int \mathsf{tr}\big(e(x,x,\tau_1,\tau_2)\big)\,dx \tag{10.1.4}$$

and obtain estimate

$$|\mathsf{N}(A;\tau_1,\tau_2)-\mathcal{N}(\tau_1,\tau_2)|\le C\big(R_{(1)}+R_{(2)}\big) \tag{10.1.5}$$

where $R_{(i)}$ are remainder estimates for operators $(A-\tau_i)$ while $\mathcal{N}$ is a main part of asymptotics due to Part II; writing $\mathcal{N}(\tau_1,\tau_2)=\mathcal{N}_{(2)}-\mathcal{N}_{(1)}$ we may

get indefiniteness of the form "$\infty - \infty$". Recall that remainder estimates were corresponding to the energy levels rather than to the error $|\mathsf{N} - \mathcal{N}|$ itself.

Note that again $e(.,.,\tau_1,\tau_2)$ is a Schwartz kernel of the modified spectral projector $E(A;\tau_1,\tau_2)$ but as one can notice easily the results of Parts II and III are equally valid for intervals (τ_1,τ_2), $(\tau_1,\tau_2]$, $[\tau_1,\tau_2)$ and $[\tau_1,\tau_2]$.

Furthermore, restriction $\gamma_{(1)} = \gamma_{(2)} = \gamma$, $\rho_{(1)} = \rho_{(2)} = \rho$, ... in this analysis is not necessary.

Indeed, let us take a γ-admissible partition of unity with $\gamma = \max(\gamma_{(1)}, \gamma_{(2)})$. Then for every element of this partition we have an estimate like

$$|\int \frac{1}{L}\phi\big(\frac{\tau-\tau_i}{L}\big)\Big(\int_X \mathsf{tr}\big(e(x,x,\tau,\tau_i)\big)\psi(x)\,dx - \tilde{\mathcal{N}}_\psi(\tau,\tau_i)\Big)\,d\tau| \le CR_{(i),\psi} \tag{10.1.6}$$

where $\tilde{\mathcal{N}}_\psi$ may also contain extra terms in comparison with $\mathcal{N}_\psi$ as L is large enough.

Estimate (10.1.6) is definitely correct as $\gamma_{(i)} \ge \epsilon_0\gamma$ on the given element; if this condition is violated there one can make an additional $\gamma_{(i)}$-admissible subpartition on this element, apply on every element of this subpartition results of Parts II and III and then sum.

Note that as $L \ge |\tau_2 - \tau_1|$ we can replace there τ_i by $\bar\tau = \frac{1}{2}(\tau_1+\tau_2)$. Then adding these estimates with the subtraction of the estimated quantities we arrive to

$$\begin{aligned}|\int_X \mathsf{tr}\big(e(x,x,\tau_1,\tau_2)\big)\psi(x)\,dx - \\ \int \frac{1}{L}\phi\big(\frac{\tau-\bar\tau}{L}\big)\Big(\tilde{\mathcal{N}}_\psi(\tau_1,\tau) - \tilde{\mathcal{N}}_\psi(\tau_2,\tau)\Big)\,d\tau| \le CR_{(1),\psi} + CR_{(2),\psi}\end{aligned} \tag{10.1.7}$$

where we assumed that $\int \phi(\tau)\,d\tau = 1$ and used equality $E(\tau_1,\tau) - E(\tau_2,\tau) = E(\tau_1,\tau_2)$.

Note that

$$\tilde{\mathcal{N}}_\psi(\tau_1,\tau) - \tilde{\mathcal{N}}_\psi(\tau_2,\tau) = \tilde{\mathcal{N}}_\psi(\tau_1,\tau_2) = \mathcal{N}_\psi(\tau_1,\tau_2). \tag{10.1.8}$$

Really, the first equality is due to construction of $\tilde{\mathcal{N}}$. The second equality is true since if $|\tau_2 - \tau_1| \ge C_0 \max(\varrho_{(1)} + \varrho_{(2)})$ then $\tilde{\mathcal{N}}_\psi(\tau_1',\tau_2') = 0$ as

$$\tau_1 + C_0\varrho_{(1)} \le \tau_1' \le \tau_2' \le \tau_2 - C_0\varrho_{(2)}$$

for sufficiently large C_0.

Now (10.1.5) follows from (10.1.7), (10.1.8) after summation over partition.

The results of this type are not very exciting because in the real situations condition $X_{\mathsf{sing}(1)} = X_{\mathsf{sing}(2)} = \emptyset$ is not necessarily fulfilled and that in the conditions on Hamiltonian trajectories restriction $J_1 = J_2 = I$ is not necessarily natural.

In order to circumvent the first restriction we can apply the arguments of Section 9.1 (assuming that $X_{\mathsf{reg}(1)} = X_{\mathsf{reg}(2)} = X_{\mathsf{reg}}$ and $X_{\mathsf{sing}(1)} = X_{\mathsf{sing}(2)} = X_{\mathsf{sing}}$ in the upper estimates; one picks up $J_1 = J_2 = I$). Therefore the following statement holds:

Theorem 10.1.1. *Let the operators $(A - \tau_i)$ satisfy assumptions of Theorem 9.1.7 in $X_{\mathsf{reg}(i)}$ with $J_i = \lambda I$ (where λ is a constant parameter) and with auxiliary functions $\gamma_{(i)}$, $\rho_{(i)}$, ... instead of γ, ρ, ...; conditions depending on j should be fulfilled for both $i = 1$ and $i = 2$.*

Let $\tau_1 < \tau_2$ and $X_{\mathsf{reg}(1)} = X_{\mathsf{reg}(2)} = X_{\mathsf{reg}}$. Then

(i) Estimate

$$\mathsf{N}(A; \tau_1, \tau_2) \ge \mathcal{N}(A; \tau_1, \tau_2) - C\big(R_{*(1)} + R_{*(2)}\big) \tag{10.1.9}$$

holds where $\mathcal{N}(A; \tau_1, \tau_2)$ is prescribed by Parts II and III and $R_{*(1)}$ and $R_{*(2)}$ are taken from the lower estimates for $\mathsf{N}^-(A-\tau_1)$ and $\mathsf{N}^-(A-\tau_2)$ respectively, obtained in the Theorem 9.1.7 in which the conditions were fulfilled.

(ii) Moreover, let $X_{\mathsf{sing}(1)} = X_{\mathsf{sing}(2)} = X_{\mathsf{sing}}$ and let assumptions (9.1.24)–(9.1.25) *be fulfilled in $X_{\mathsf{reg}} \cap X_{\mathsf{sing}}$ with $\varrho_{(i)0} = \lambda$; further, let condition* (9.1.22) *be fulfilled with inequality* (9.1.23) *replaced by inequality*

$$\mathsf{N}(A'; \tau_1 - t\lambda, \tau_2 + t\lambda) \le R_0(1+t)^l \qquad \forall t \ge 0 \tag{10.1.10}$$

with some (arbitrarily large) exponent l [3]*. Then estimate*

$$\mathsf{N}(A; \tau_1, \tau_2) \le \mathcal{N}(A; \tau_1, \tau_2) + C\big(R^*_{(1)} + R^*_{(2)}\big) \tag{10.1.11}$$

holds with $R^*_{(1)}$ and $R^*_{(2)}$ taken from the upper estimates for $\mathsf{N}^-(A - \tau_1)$ and $\mathsf{N}^-(A - \tau_2)$ respectively, and these estimates are not less than CR_0.

[3] We denote the auxiliary operator coinciding with A in the $\epsilon\gamma$-vicinity of $X_{\mathsf{reg}} \setminus X_{\mathsf{sing}}$ by A' instead of $\boldsymbol{A}'$ because $J = I$.

Remark 10.1.2. This theorem does not follow from Theorem 9.1.7 because $\mathcal{N}(A;\tau_1,\tau_2)$, $R_{(1)}$ and $R_{(2)}$ can be finite and $\mathcal{N}^-(A-\tau_i)$ can be infinite at the same time. Usually this occurs in the case when $\tau_1<\tau_2$ belong to the same gap in the semiclassical approximation to the spectrum for $|x|\ge C_0$; examples provided by the Dirac operator be given in Subsection 11.1.3.

Similar examples for the Schrödinger and Dirac operators with constant magnetic field in dimension 2 and higher even dimensions will be provided in Parts VI–X.

10.1.2 Birman-Schwinger Principle

Main Formula

In order to obtain less trivial results let us apply the following identity which is trivial due to *Birman-Schwinger principle:*

$$\mathsf{N}(A;\tau_1,\tau_2)=\mathsf{N}(A-tJ;\tau_1,\tau_2)+ \mathsf{N}\bigl(J^{-1}(A-\tau_1);0,t\bigr)-\mathsf{N}\bigl(J^{-1}(A-\tau_2);0,t\bigr) \tag{10.1.12}$$

provided $\tau_1<\tau_2$, $t>0$ and J is a positive definite operator such that

$$J^{-1}(A-\tau_i) \text{ are self-adjoint operators in the space } \mathcal{H}=\mathcal{H}_J. \tag{10.1.13}$$

As we will prove in Section 10.2 it is sufficient to assume that J is a symmetric, positive[4], bounded and A-compact[5] operator. We will justify formula (10.1.12) in Section 10.2. Then this formula (10.1.12) is also valid for $t<0$ where we define

$$\mathsf{N}(B;0,t):=-\mathsf{N}(B;0,-t) \qquad \text{as} \quad t<0. \tag{10.1.15}$$

Then formula (10.1.12) yields more general formula

$$\mathsf{N}(A;\tau_1,\tau_2)=\mathsf{N}(A-t_1J_1-\ldots-t_nJ_n;\tau_1,\tau_2)- \sum_{1\le k\le n}\mathsf{N}\bigl(J_k^{-1}(A-t_1J_1-\ldots-t_{k-1}J_{k-1}-\tau);0,t_k\bigr)\Big|_{\tau=\tau_1}^{\tau=\tau_2} \tag{10.1.16}$$

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[4] I.e. $(Jv,v)>0 \quad \forall u\ne 0$.

[5] I.e. $J(A+i)^{-1}$ is a compact operator. This condition is stronger than the standard definition

$$\|Jv\|\le\epsilon\|Av\|+C_\epsilon\|v\| \qquad \forall v\in\mathfrak{D}(A) \qquad \forall\epsilon>0 \tag{10.1.14}$$

which will be also useful.

where we assume that

(10.1.17) All operators J_k ($k = 1, \dots, n$) are symmetric, positive definite, bounded and A-compact.

Artificial energy levels in (10.1.16) may spoil our estimate; to overcome this we will mollify with respect to them arriving to the following final formula which will be used:

$$\begin{aligned}(10.1.18)\quad \mathsf{N}(A;\tau_1,\tau_2) = {}& \\ & \int_{\mathbb{R}^n} \mathsf{N}(A - t_1J_1 - \dots - t_nJ_n;\tau_1,\tau_2)\phi_1(t_1)\cdots\phi_n(t_n)\,dt_1\cdots dt_n - \\ & \sum_{1\le k\le n}\int_{\mathbb{R}^k} \mathsf{N}\big(J_k^{-1}(A - t_1J_1 - \dots - t_{k-1}J_{k-1} - \tau);0,t_k\big)\Big|_{\tau=\tau_1}^{\tau=\tau_2}\times \\ & \qquad\qquad \phi_1(t_1)\cdots\phi_k(t_k)\,dt_1\cdots dt_k\end{aligned}$$

provided

$$(10.1.19)\qquad \phi_k \in \mathscr{C}_0^K(\mathbb{R}),\quad \int_{\mathbb{R}}\phi_k(t)dt = 1 \qquad \forall k.$$

Partition and Scaling

Let $\{\psi_j^2\}_{j\ge 0}$ be a partition of unity of the same type as in Chapter 9; then the right-hand expression of (10.1.18) is a sum of terms of the type

$$(10.1.20)_{n+1}\qquad \int_{\mathbb{R}^n} \mathsf{tr}\big(\Gamma\psi^2 e_{A-t_1J_1-\dots-t_nJ_n}\big)(\tau_1,\tau_2)\phi_1(t_1)\cdots\phi_n(t_n)\,dt_1\cdots dt_n$$

and

$$\begin{aligned}(10.1.21)_k\quad \mp\int_{\mathbb{R}^k} \mathsf{tr}\big(\Gamma\psi^2 e_{J_k^{-1}(A-t_1J_1-\dots-t_{k-1}J_{k-1}-\tau)}\big)(0,t_k)\times & \\ \phi_1(t_1)\cdots\phi_k(t_k)\,dt_1\dots dt_k & \end{aligned}$$

with $k = 1, \dots, n$.

Remark 10.1.3. (i) In these expressions for the sake of simplicity we skip the subscript j of ψ and also the subscript $i = 1, 2$ of τ in terms with $k \le n$; in these terms $\tau = \tau_{1,2}$ correspond to signs $\mp$.

(ii) Obviously we can take different partitions of unity for different $k = 1, \dots, n+1$ and for different $i = 1, 2$ if $k \le n$.

Let us assume that J_k are Hermitian positive definite matrices (bounded and A-compact as operators) and that in $\mathcal{B}(y, \gamma(y), X)$

$$|D^\alpha J_k| \le \varrho_k' \gamma^{-|\alpha|}, \qquad |J_k^{-1}| \le {\varrho_k'}^{-1} \qquad \forall \alpha : |\alpha| \le K \tag{10.1.22}$$

with appropriate ϱ_k' and $\gamma = \gamma_{(i)}$, $i = 1, 2$. Later ϱ_k' will be scaling functions. These conditions must be fulfilled in addition to all "standard" conditions which will be formulated later and invoking $\gamma = \gamma_{(i)}$, $\rho = \rho_{(i)}$, $\varrho = \varrho_{(i)}$ and possibly $\varrho_l = \varrho_{l(i)}$, associated with the boundary conditions.

General Operators. Consider first term $(10.1.20)_{n+1}$. Then $h_{\mathsf{eff}} = (\rho\gamma)^{-1}$ is an effective semiclassical parameter. Let us note first of all that after rescaling we get the mollification parameter $L_{\mathsf{eff}} = \varrho_k' \varrho^{-1}$ for t_k and our results of Section 6.5 on spectral asymptotics with the mollification[6] yield that we have a complete[7] asymptotics of term in question provided $L_{\mathsf{eff}} \ge h_{\mathsf{eff}}^{\kappa-\delta}$ i.e.

$$\max_{r \le k} \varrho_r' \ge \varrho h^{\kappa-\delta}, \qquad h = (\rho\gamma)^{-1}, \qquad \text{with } \kappa = \frac{1}{2} \tag{10.1.23}_k$$

where *here* $k = n$ and arbitrarily small exponent $\delta > 0$. Further, in the scalar case we can replace exponent $(\frac{1}{2} - \delta)$ by $(1 - \delta)$ i.e. take $\kappa = 1$ in condition $(10.1.23)_k$.

Moreover, for $k \ge 2$ we have a complete asymptotics of $(10.1.21)_k$ provided condition $(10.1.23)_{k-1}$ is fulfilled; here do not need microhyperbolicity assumption with respect to (x, ξ) because these problems are automatically microhyperbolic with respect to $t_1, \dots, t_k$ and condition $(10.1.23)_k$ means that there is sufficient mollification with respect to t_r with appropriate $r \le k-1$ is necessary.

Finally, consider term $(10.1.21)_1$. Then under condition $(10.1.23)_1$ we need microhyperbolicity condition[8] as $t_1 = 0$ and we do not have complete asymptotics. So let us now assume that

(10.1.24) For $\tau = \tau_i$ $(i = 1, 2)$ microhyperbolicity condition[8] is fulfilled for the original problem $(A - \tau_i, B)$.

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[6] Which obviously hold near boundary as well.

[7] I.e., modulo $O(h^s)$ with arbitrary s.

[8] Or ξ-microhyperbolicity for "rough" operators in the sense of Sections 4.6 and 7.5.

Then modulo $O(h^{1-d})$ we have asymptotics of the term in question.

On the other hand, if condition $(10.1.23)_1$ is violated then this microhyperbolicity condition is fulfilled for the problem $(A - t_1 J_1 - \tau_i, B)$ $(i = 1, 2)$ and we have asymptotics of the term $(10.1.21)_2$ modulo $O(h^{1-d})$ also. Repeating these arguments we conclude that under the microhyperbolicity condition we have asymptotics modulo $O(h^{1-d})$ of all the terms $(10.1.21)_k$ with $k = 1, \dots, n$ (and we have no need to check conditions $(10.1.23)_k$.

Furthermore, the same is true for the modified term $(10.1.20)_{n+1}$ now defined by

$$(10.1.25)_{n+1} \quad \frac{1}{L}\int_{\mathbb{R}^{n+1}} \mathsf{tr}\big(\Gamma \psi^2 e_{A-t_1J_1-\dots-t_nJ_n}\big)(\tau', \tau_i)\times \phi_0\big(\frac{1}{L}(\tau' - \tau_i)\big)\phi_1(t_1)\cdots\phi_n(t_n)\, d\tau' dt_1 \cdots dt_n$$

with $L \ge h^{\frac{1}{2}-\delta}$ (or $L \ge h^{1-\delta}$ in the scalar case).

Moreover, the same assertion is true if the microhyperbolicity condition for $(A-\tau_i, B)$ is replaced by the ellipticity condition for the operator $(A-\tau_i)$; this condition does not include the assumption that the boundary value problem is elliptic (or microhyperbolic).

Finally, if we assume that the boundary value problem $(A - \tau_i, B)$ is also elliptic then we can obtain all the terms $(10.1.21)_k$ $(k = 1, \dots, n)$ and $(10.1.25)_{n+1}$ modulo $O(h^s)$.

Dirac Operator. Let us now treat the Dirac operator. Let us recall that according to Subsections 5.3.2 (smooth case) and 5.4.3 ("rough case") Dirac operator

$$(10.1.26) \qquad A = \sum_{j,l} \frac{1}{2}\sigma_l(\omega^{jl}P_j + P_j\omega^{jl}) + \sigma_0 m + V\cdot I, \qquad P_j = D_j - V_j$$

the usual smoothness (or "roughness") conditions are fulfilled but with

$$(10.1.27) \qquad \varrho = \mathsf{min}\big(\frac{\rho^2}{m}, \rho\big)$$

$V \pm m$ instead of V and weakened condition to V_j:

$$(10.1.28) \qquad |F_{jl}| \le c\rho\gamma^{-1}, \qquad F_{jl} = F_{jk} = \partial_l V_j - \partial_j V_l$$

instead of $|V_j| \le c\rho$. Corresponding conditions should be fulfilled for their derivatives.

Also Dirac operator is accessible to the scaling technique and one does not need any microhyperbolicity condition as $d \ge 3$, or even $d = 2$ but under some smoothness or weakened nondegeneracy conditions, and similarly for $d = 1$.

Remark 10.1.4. We need matrix functions J_k such that operators

$$J_k^{-\frac{1}{2}}\bigl(A - t_1J_1 - ... - t_{k-1}J_{k-1} - \tau_i\bigr)J_k^{-\frac{1}{2}} \tag{10.1.29}$$

are generalized Dirac operators (5.3.71) i.e. that matrices

$$W_k = -t_1J_1 - ... - t_{k-1}J_{k-1} \tag{10.1.30}$$

satisfy after rescaling $(5.3.77)_5$ in the smooth case or similar condition in the "rough" case.

Scalar-Like Operators. Finally, let us treat the arbitrary matrix operator and let us assume that (after normalization, i.e., multiplication by an appropriate number if necessary) this operator satisfies the following condition:

(10.1.31) Interval $[\tau_1 - \epsilon_0, \tau_2 + \epsilon_0]$ contains no eigenvalues of the principal symbol of A different from the eigenvalue $\lambda(x,\xi)$ of constant multiplicity.

Then this condition is also fulfilled for operator $J_k^{-1}A$ provided one of the following three conditions is fulfilled:

$(10.1.32)_1$ λ is a simple eigenvalue of the principal symbol of A;

$(10.1.32)_2$ J_k is scalar;

$(10.1.32)_3$ The restriction of J_k to $\mathsf{Ker}(a^0 - \lambda)$ is scalar (a^0 is the principal symbol of A).

Remark 10.1.5. (i) Condition $(10.1.32)_3$ is more general than $(10.1.32)_{1,2}$ but it is difficult to fulfill $(10.1.32)_3$ outside of the framework of $(10.1.32)_{1,2}$ because J_k does not depend on ξ and $(a^0 - \lambda)$ depends on ξ.

(ii) Application of the matrix J_k which does not satisfy one of the conditions $(10.1.32)_{1-3}$ is possible only if before there was mollification with parameter $L_{\text{eff}} \ge h_{\text{eff}}^{\frac12-\delta}$.

Thus, let us assume that one of conditions $(10.1.32)_{1-3}$ is fulfilled. Assume that second derivatives of coefficients are bounded (after rescaling); then it is true for λ as well. Let us make a γ-admissible partition of unity with

$$\gamma = \epsilon_0\bigl(|\lambda-\tau_i| + |\nabla\lambda|^2\bigr)^{\frac12} + h^{\frac12}. \tag{10.1.33}$$

Then for $\gamma \ge 2h^{\frac12}$ microhyperbolicity condition is fulfilled for the original operator after the rescaling-multiplication procedure; therefore in this case our remainder estimate is justified. For $\gamma \le h^{\frac12-\delta'}$ the condition $L' \ge h'^{\frac12-\delta'}$ is fulfilled where $\delta' = \delta'(\delta) > 0$ is small enough, h' and L' are effective values of h and L after the dilatation-multiplication procedure and we assume that $L \ge h^{1-\delta}$. So in this we can replace condition $L \ge h^{\frac12-\delta}$ by $L \ge h^{1-\delta}$ and thus we can take $\kappa = 1$ in $(10.1.23)_k$.

On the other hand, for rough operators one needs ξ-microhyperbolicity and thus

$$\gamma = \epsilon_0\bigl(|\lambda-\tau_i| + |\nabla_\xi\lambda|^2\bigr)^{\frac12} + h^{\frac12}. \tag{10.1.34}$$

which leads us to $\kappa = \frac23$ in $(10.1.23)_k$. In a more regular case we need some $\kappa \in \bigl(\frac23, 1\bigr)$ depending on regularity.

Finally, we obtain that in the general case the microhyperbolicity condition can be replaced by some non-degeneracy condition of Sections 5.2, 5.4 and

$(10.1.35)_p$ For every $k = 1, \dots, p$ the matrix J_k satisfies one of conditions $(10.1.32)_{1-3}$ and there exists $k \le p$ such that $(10.1.23)_k$ holds with $\kappa \in [\frac23, 1]$ defined by regularity

with some $p \le n+1$.

Moreover, let conditions (9.1.31) and $(9.1.32)_1$ be fulfilled. Then $(10.1.35)_p$ is reduced to the condition

$(10.1.36)_p$ There exists $k \le p$ such that $\varrho'_k \ge \varrho h^{\kappa-\delta} = \varrho(\rho\gamma)^{\delta-\kappa}$.

So far contribution of the ball $\mathcal{B}(y,\gamma(y),X)$ to the remainder is $O(h_{\mathsf{eff}}^{1-d})$. Furthermore, under appropriate conditions to the trajectories contribution of the ball $\mathcal{B}(y,\gamma(y),X)$ to the remainder estimate is $O(h_{\mathsf{eff}}^{1-d+\delta})$ with some small exponent $\delta>0$.

Recall that while geometry of trajectories do not depend on the choice of J_k but the temp of movement depends on it. Not to invoke multiple matrices we should assume that condition $(10.1.23)_1$ is fulfilled.

Remark 10.1.6. In the lacunary zone, which appears in the case of periodic trajectories, restrictions of the type $\varrho_k'\le\eta$ and $\varrho_k'\ge\eta h^{1-\delta}$ are also possible; here $\eta\in(h',h)$ is a width of the spectral gap.

Calculations in the Regular Zone

Let us now treat the terms of the derived asymptotics. For the sake of the notational simplicity we do not write the mollification on $t_1,\dots,t_k$. Then expression $(10.1.21)_k$ with $k=1,\dots,n$ generates

$$(10.1.37)_k\qquad \int_0^{t_k}\mathsf{Tr}\bigl(\mathsf{Res}_{\mathbb{R}}\bigl(z-J_k^{-1}(A-t_1J_1-\dots-t_{k-1}J_{k-1}-\tau)\bigr)^{-1}\psi\bigr)\,dz$$

with $\tau=\tau_1,\tau_2$ and expression $(10.1.20)_{n+1}$ generates

$$(10.1.38)_{n+1}\qquad \int_{\tau_1}^{\tau_2}\mathsf{Tr}\bigl(\mathsf{Res}_{\mathbb{R}}(z-A+t_1J_1+\dots+t_nJ_n)^{-1}\psi\bigr)\,dz$$

where $\mathsf{Res}_{\mathbb{R}}$ is the difference of the limits from the upper and lower complex half-planes with respect to z, we skip numerical factors depending on d only and G^{-1} means the formal resolvent of G (in the context of the formal calculus of pseudodifferential operators) derived according to the rules of Parts II and III.

We need to treat the case $n=1$ only: the general case can be reduced to this case if we replace A by $A-t_1J_1-\dots-t_{k-1}J_{k-1}$ and sum with respect to k. Therefore we can skip the subscript "1" of t and J.

Then term $(10.1.37)_1$ is replaced by

$$(10.1.39)_1\qquad \int_0^{t}\mathsf{Tr}\bigl(\mathsf{Res}_{\mathbb{R}}(Jz-A+\tau)^{-1}J\psi\bigr)\,dz$$

(it is easy to see that the equality $(z - J^{-1}A)^{-1} = (Jz - A)^{-1}J$ holds for formal resolvents also) and term $(10.1.38)_2$ is replaced by

$$(10.1.40)_2 \qquad \int_{\tau_1}^{\tau_2} \mathsf{Tr}\big(\mathsf{Res}_{\mathbb{R}}(z - A + tJ)^{-1}\psi\big)\, dz.$$

Denote by $\Theta(t)$ and $\Theta'_i(t)$ be expressions $(10.1.40)_2$ and $(10.1.39)_1$ at $\tau = \tau_i$ respectively. We claim that

$$\begin{aligned}(10.1.41)\quad \Theta(0) - \Theta(t) - \Theta'_2(t) + \Theta'_1(t) = \\ - \int_0^t \int_{\tau_1}^{\tau_2} \mathsf{Tr}\big(\mathsf{Res}_{\mathbb{R}}(z - A + tJ)^{-2} J (z - A + tJ)^{-1}[A, \psi]\big)\, dz\end{aligned}$$

Indeed, the derivative with respect to t of the left-hand expression is equal to

$$- \int_{\tau_1}^{\tau_2} \mathsf{Tr}\big(\mathsf{Res}_{\mathbb{R}}(z - A + tJ)^{-2} J (z - A + tJ)^{-1}[A, \psi]\big)\, dz$$

because of the chain of equalities

$$\begin{aligned}\partial_t \,\mathsf{Tr}\big(\mathsf{Res}_{\mathbb{R}}(z - A + tJ)^{-1}\psi\big) = \\ - \mathsf{Tr}\big(\mathsf{Res}_{\mathbb{R}}(z - A + tJ)^{-1} J (z - A + tJ)^{-1}\psi\big) = \\ - \mathsf{Tr}\big(\mathsf{Res}_{\mathbb{R}}(z - A + tJ)^{-2} J\psi\big) - \mathsf{Tr}\big(\mathsf{Res}_{\mathbb{R}}(z - A + tJ)^{-2} J (z - A + tJ)^{-1}[A, \psi]\big)\end{aligned}$$

which follow from the properties of the trace and from the fact that J and ψ commute. The first term in the right-hand expression is equal to

$$\partial_z \,\mathsf{Tr}\big(\mathsf{Res}_{\mathbb{R}}(z - A + tJ)^{-1} J\psi\big)$$

and after integration with respect to z it reduces with $\partial_t(-\Theta'_2(t) + \Theta'_1(t))$. So, derivatives with respect to t of both sides of (10.1.41) and for $t = 0$ both expressions vanish.

In the framework of condition $(10.1.35)_1$ these calculations are justified by microhyperbolicity for small ϱ'_1 and by microhyperbolicity and mollifications otherwise; justifications in the frameworks of other conditions are similarly.

Finally, after adding all terms due to $(10.1.21)_k$ $(k = 1, \dots, n)$ and $(10.1.20)_{n+1}$ we obtain that the right-hand expression of (10.1.18) contains main parts of asymptotics of $(\Gamma\psi e_A)(\tau_1, \tau_2)$ described in Parts II and III plus a remainder estimate also described there plus extra terms of the form

$$\sum_{1 \le r \le s} h^{r-d} \int (\mathcal{L}_r \psi)\, dx.$$

Here $\mathcal{L}_r$ are differential operators such that $\mathcal{L}_r 1 = 0$. Let us note that these operators $\mathcal{L}_r$ depend uniquely on A, $J_1, \dots, J_n$, $\phi_1, \dots, \phi_n$ and hence after summation with respect to partition of unity all these extra terms disappear.

All these assertions have been proven provided $\varrho'_k \le \varrho$ for all $k = 1, \dots, n$. On the other hand, as it is true for $k < k_0$ only and $\varrho'_k \ge \varrho$, one can replace in all the terms with $k \ge k_0 + 1$ h by $h' = h\bigl(\varrho/\varrho'_k\bigr)^{\frac{1}{m}}$, etc.

Therefore after summation with respect to partition of unity we obtain that only terms supported in $X_{\mathsf{reg}} \cap X_{\mathsf{sing}}$ are not completely reduced, and the remaining terms do not exceed

$$Ch^{1-d}\Bigl(1 + \frac{(\varrho'_1 + \dots + \varrho'_n)}{\varrho}\Bigr)^{\frac{(d-1)}{m}}. \tag{10.1.42}$$

Finally, let us note that if in some ball $\mathcal{B}(y, \gamma(y), X)$ condition

(10.1.43) τ_i lies in the spectral gap of the problem $(A - t'_1 J_1 - \dots - t'_k J_k, B)$ for all $t'_k \in [0, t_k]$ (or $t'_k \in [t_k, 0]$ for $t_k < 0$) for all $k = 1, \dots, n$ and for all $t_k \in \mathsf{supp}(\phi_k)$

is fulfilled then all the operators $\mathcal{L}_r$ vanish in this ball.

Analysis in the Singular Zone

Let us now treat the zone X_{sing}. First of all we must assume that in every ball in $X_{\mathsf{reg}} \cap X_{\mathsf{sing}} \cap \{\rho\gamma \ge c\}$ [9] either an ellipticity condition or condition

$$\varrho'_1 \ge \varrho h \tag{10.1.44}$$

are fulfilled.

Furthermore, we need to introduce condition similar to (9.1.22). Namely, let us assume that

(10.1.45) There exist a self-adjoint operator A' and symmetric operators J'_k coinciding with A and J_k respectively in X_{sing} and such that for all k the operators J'_k are A'-compact and

$$\mathsf{N}\bigl({J'_k}^{-1}(A' - \tau_i - t_1 J'_1 - \dots - t_{k-1} J'_{k-1}); -t, t\bigr) \le R_0(|t|+1)^{l} \qquad \forall k \quad \forall t_j \in \mathsf{supp}(\phi_j) \quad (j = 1, \dots, k-1) \quad \forall t \in \mathbb{R}. \tag{10.1.46}$$

[9] In the general case; modifications will be given below for the Schrödinger and Dirac operators.

Moreover, we assume that

(10.1.46)* condition (10.1.46) is fulfilled for some sequences $\tau_{i,n} \to \tau_i$ with $\tau_{i,n} \in [\tau_1, \tau_2] \setminus \mathsf{Spec}_{\mathsf{ess}}(A)$.

Finally, let us assume that

$$\int_{X_{\mathsf{sing}}} \mathsf{tr}\big(e_{A-t_1J_1-\dots-t_nJ_n}(x,x,\tau_1,\tau_2)\big)\,dx \le R_0 \qquad \forall t_j \in \mathsf{supp}(\phi_j) \quad (j=1,\dots,n). \tag{10.1.47}$$

Instead of this condition we will usually use the condition

$$\mathsf{N}(A - t_1J_1 - \dots - t_nJ_n; \tau_1, \tau_2) \le R_0 \qquad \forall t_j \in \mathsf{supp}(\phi_j) \quad (j=1,\dots,n). \tag{10.1.48}$$

This condition may seem to be too restrictive because it is associated with X instead of X_{sing}, but in fact this is not so.

10.1.3 Main Theorems

Now we can formulate and prove our main theorems.

General Operators

We start from the general operators.

Theorem 10.1.7. *Let X, X_{reg}, X_{sing}, $\gamma_{(i)}$, $\rho_{(i)}$, $\varrho_{(i)j}$ and ϱ'_k satisfy conditions* (9.1.5)–(9.1.8) *for both $i = 1, 2$* [10]*.*

Let operators A and B be of the forms (9.1.9) *and* (9.1.10) *respectively. Let us assume that in X_{reg}*

(10.1.49)$_1$ *Operators $\rho^m_{(i)}\varrho^{-1}_{(i)1}A$ are classically uniformly elliptic for $i = 1, 2$* [11]*;*

(10.1.49)$_2$ *Boundary value problems*

$$\big(\rho^m_{(i)}\varrho^{-1}_{(i)1}A,\ \rho^{m_j}_{(i)}\varrho^{-1}_{(i)j}B_j,\ j=1,\dots,\mu\big)$$

are classically uniformly elliptic for $i = 1, 2$.

[10] Here X_{reg}, X_{sing} and ϱ'_k do not depend on i.

[11] In which case $\rho^m_{(1)}\varrho^{-1}_{(1)1} \asymp \rho^m_{(2)}\varrho^{-1}_{(2)1}$ and a similar conclusion holds for the next assumption (10.1.49)$_2$.

Further, let us assume that $\tau_1 < \tau_2$ *and*

(10.1.50) *For both* $i = 1, 2$, *for every ball* $\mathcal{B}(x, \gamma^{(i)}(x), X)$ *in*

$$X_{\mathsf{reg}} \cap X_{\mathsf{sing}} \cap \{\rho_{(i)}\gamma_{(i)} \ge c\}$$

either the boundary value problem $\bigl(\varrho_{(i)1}^{-1}(A - \tau_i),\ \varrho_{(i)j}^{-1}B_j,\ j = 1, \dots, \mu\bigr)$ *is elliptic (as an* $h_{(i)}$*-boundary value problem with* $h_{(i)} = \bigl(\rho_{(i)}\gamma_{(i)}\bigr)^{-1}$*) or condition* $(10.1.36)_1$ *is fulfilled*[12]*.*

Furthermore, let us assume that J_k *are bounded and* A*-compact operators. Finally, let us assume that there exist operator* $A'_{(i)}$ *coinciding with* A *in* $X_{(i)\mathsf{sing}}$*, self-adjoint and such that for all* k *the operators* J_k *are* A'*-compact and conditions* $(10.1.46)^*$*,* $(10.1.48)$ *are fulfilled.*

Then

(i) The following estimate holds:

$$\mathsf{N}(A; \tau_1, \tau_2) \ge \mathcal{N}(A; \tau_1, \tau_2) - C \sum_{i=1,2} \bigl(R_{(i)1} + R_{(i)2} + R_{(i)3}\bigr) - CR_0 \tag{10.1.51}$$

where $\mathcal{N}$ *is derived as a semiclassical approximation in construction in* X_{reg} *for the sum of* $\mathsf{N}\bigl(J_k^{-1}(A - t_1J_1 - \dots - t_{k-1}J_{k-1}, 0, t_k\bigr)$ *with mollification, according to rules of the Part II,* $R_{(i)1}$ *is the corresponding remainder estimate,* $R_{(i)2}$ *is given by* (9.1.30) *i.e. one can define*

$$R_{(i)2} = \int_{X_{\mathsf{reg}} \cap X_{\mathsf{sing}}} \rho_{(i)}^{d-s}\gamma_{(i)}^{-s}\, dx, \tag{10.1.52}$$

$$R_{(i)3} = \int_{X_{\mathsf{reg}} \cap X_{\mathsf{sing}}} \bigl(\varrho'_1 + \dots + \varrho'_n\bigr)^{\frac{(d-1)}{m}} \varrho^{-\frac{(d-1)}{m}} \rho_{(i)}^{d-1}\gamma_{(i)}^{-1}\, dx, \tag{10.1.53}$$

(ii) On the other hand, if also condition (10.1.47) *is fulfilled then*

$$\mathsf{N}(A; \tau_1, \tau_2) \le \mathcal{N}(A; \tau_1, \tau_2) + C \sum_{i=1,2} \bigl(R_{(i)1} + R_{(i)2} + R_{(i)3}\bigr) + CR_0. \tag{10.1.54}$$

The proof which, after (10.1.12) is established, repeats almost exactly the proofs of Theorems 9.1.7 and 9.1.13 will be presented in Subsection 10.2.3.

[12] I.e., $\varrho'_1 \ge \varrho_{(i)1}h_{(i)}^{1-\delta}$ with $\delta > 0$.

Dirac Operator

In the case of the Dirac operator one can get effective expressions for $R_{j(i)}$ and deduce the following statements:

Theorem 10.1.8. *Let X, X_{reg}, X_{sing}, γ and ρ satisfy conditions* (9.1.5)–(9.1.8). *Let operator A be of the form* (10.1.26), *$d \geq 2$, $\tau_1 < \tau_2$. Let in X_{reg} let the following conditions be fulfilled:*

$$|D^\alpha \omega^{jk}| \leq c\gamma^{-|\alpha|}, \tag{10.1.55}$$

$$|D^\alpha F_{jk}| \leq c\rho\gamma^{-|\alpha|-1}, \tag{10.1.56}$$

$$|D^\alpha V| \leq c\min(\rho, \frac{1}{m}\rho^2)\gamma^{-|\alpha|} \qquad (\alpha \neq 0) \qquad \forall \alpha : |\alpha| \leq K, \tag{10.1.57}$$

and

$$(V - \tau_2 - m)_+ \leq c\min(\rho, \frac{1}{m}\rho^2), \tag{10.1.58$)_2$}$$

$$(V - \tau_1 + m)_- \leq c\min(\rho, \frac{1}{m}\rho^2), \tag{10.1.58$)_1$}$$

$$|\det(\omega^{jk})| \geq \epsilon_0, \tag{10.1.59}$$

$$B\big(x, \gamma(x)\big) \subset X. \tag{10.1.60}$$

Then

(i) Estimate

$$\mathsf{N}(A, \tau_1, \tau_2) \geq \mathsf{N}^{\mathsf{W}}(A; \tau_1, \tau_2) - C\big(R_1 + R_2\big) \tag{10.1.61}$$

holds with

$$R_1 = \int_{X_{\mathsf{reg}} \setminus X_{\mathsf{reg,ell}}} \rho^{d-1}\gamma^{-1}\, dx, \tag{10.1.62}$$

$$R_2 = \int_{X_{\mathsf{reg,ell}}} \rho^{d-s}\gamma^{-s}\, dx, \tag{10.1.63}$$

with arbitrarily large exponent s where $X_{\mathsf{reg,ell}}$ consists of points X_{reg} satisfying both of the following conditions

$$V - \tau_2 - m \leq -\epsilon_0 \min\big(\rho, \frac{1}{m}\rho^2\big), \tag{10.1.64$)_2$}$$

$$-V + \tau_1 - m \leq -\epsilon_0 \min\big(\rho, \frac{1}{m}\rho^2\big). \tag{10.1.64$)_1$}$$

(ii) Estimate

$$\mathsf{N}(A;\tau_1,\tau_2) \le \mathsf{N}^{\mathsf{W}}(A;\tau_1,\tau_2) + C\big(R_1+R_2+R_3+R_4+R_5\big) \tag{10.1.65}$$

holds with R_1, R_2 *defined above,*

$$R_3 = \int_{X_{\mathsf{reg}}\cap X_{\mathsf{sing}}} \rho^d\,dx + \begin{cases} C\displaystyle\int_{X_{\mathsf{sing}}} \tilde\rho^d\,dx & \text{for } d\ge 3,\\ C_p\big(\displaystyle\int_{X_{\mathsf{sing}}} \tilde\rho^{2p}\,dx\big)^{\frac{1}{p}}(\mathsf{mes}(X_{\mathsf{sing}}))^{1-\frac{1}{p}} & \\ \qquad\qquad \text{for } d=2,\ p>1, & \end{cases} \tag{10.1.66}$$

$$R_4 = \Big(\int (\rho+m)^{\delta}\chi_Y\gamma^{-d}\,dx\Big)^{\frac{1}{2}}\Big(\int (\rho+m)^{-\delta}\chi_Y\gamma^{-d}\,dx\Big)^{\frac{1}{2}} \tag{10.1.67}$$

with arbitrarily small $\delta>0$ *and we assume that*

$$\bar X_{\mathsf{sing}}\cap \partial X = \emptyset, \tag{10.1.68}$$

and in X_{sing} *conditions* $(10.1.58)_{1,2}$ *with* ρ *replaced by* $\tilde\rho$ *and* (10.1.59) *are fulfilled and also there*

$$|V_j| \le c\tilde\rho, \tag{10.1.69}$$

$$|D_j\omega^{kl}| \le c\tilde\rho. \tag{10.1.70}$$

Finally,

$$R_5 = \min\Big(\frac{Mm^{\frac{d}{2}}}{|\tau_2-\tau_1|}, M^{\frac12}m^{\frac{d}{2}}(\mathsf{mes}(X_{\mathsf{reg}}\cap X_{\mathsf{sing}}))^{\frac12}\Big), \tag{10.1.71}$$

$$M = \int_{X_{\mathsf{reg}}\cap X_{\mathsf{sing}}} \gamma^{-d}\,dx \tag{10.1.72}$$

and only those connected components of X_{sing} *where condition* $\rho\le cm$ *is violated are taken into account.*

The proof of this theorem will be given in Subsection 10.2.4.

Remark 10.1.9. (i) For $d \ge 3$ estimates (10.1.61), (10.1.65) hold with

$$R_1 = \sum_{\pm} \int_{\Sigma_\pm \cap T^* X_{\mathsf{reg}}} dx d\xi : dH_\pm + R_1', \tag{10.1.73}$$

$$R_1' = \int_{X_{\mathsf{reg}} \setminus X_{\mathsf{reg,ell}}} \rho^{d-1-\delta} \gamma^{-1-\delta}\, dx, \tag{10.1.74}$$

$$H_\pm = V \pm \bigl(m^2 + g^{jk}(\xi_j - V_j)(\xi_k - V_k)\bigr)^{\frac{1}{2}}, \tag{10.1.75}$$

$\Sigma_\pm = \{H_\pm = 0\}$. Moreover, thus modified estimate remains true for $d = 2$ provided either $V_j = 0$ or microhyperbolicity condition is fulfilled. Finally, under appropriate modifications (see Section 5.3.2) this estimate remains true for $d = 2$ in the general case.

(ii) One can improve R_1 in the standard way under assumptions for trajectories. While ϱ_1' defining temp along trajectories could be selected in different ways, it affects contribution of $X_{\mathsf{reg}} \cap X_{\mathsf{sing}}$.

(iii) All these results hold for "rough" Dirac operator (see Subsection 5.4.3) as long as contribution of X_{reg} to the remainder estimate does not change.

(iv) Moreover, in the case when $X_{\mathsf{reg}} \cap X_{\mathsf{sing}}$ contains few connected components one can calculate R_4 and R_5 for every component and then sum the results. As $d = 2$ the same statement holds for R_3.

(v) There will be some improvement for R_4 (see the second term in (10.2.73)).

(vi) Dirac and Schrödinger operators with the strong magnetic field will be covered later.

(vii) The method of this chapter and the previous chapters provide estimates for $\mathsf{N}(A; \tau_1, \tau_2)$ averaged with respect to τ_1 or (and) τ_2 with improved contribution of X_{reg} to the remainder estimate.

(viii) For principally massless Dirac operator one can just take $J = I$.

Consider principally massless operator

$$A = \sum_{j,l} \frac{1}{2} \sigma_l (\omega^{jl} P_j + P_j \omega^{jl}) + V \cdot I, \qquad P_j = D_j - V_j \tag{10.1.76}$$

which is not necessarily of the form (10.1.26) with $m = 0$ since matrix σ_0 may not exist.

It is not necessarily of the type (10.1.26) since σ_0 does not necessarily exist. However we can always extend σ_j by $\begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}$ and $\sigma_0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ fits the bill. Therefore we arrive to

Corollary 10.1.10. *Let all assumptions of Theorem 10.1.8 be fulfilled with* $m = 0$. *Then estimates* (10.1.61) *and* (10.1.65) *hold with* $R_5 = 0$.

10.2 Functional-Analytic Arguments and Proofs of the Main Theorems

10.2.1 Self-Adjointness of the Dirac Operator

Let us begin with the self-adjointness of the (generalized) Dirac operator. First of all let us prove

Theorem 10.2.1. *Let us assume that* A *is an operator of the form* (10.1.26) *or* (10.1.76) *with real-valued coefficients* ω^{jl}, V_j *and* V. *Let us assume that* (10.1.59) *holds and*

(10.2.1) $\omega^{jl} \in \mathscr{C}(X)$, $D_j\omega^{jl}, V_j, V \in \mathscr{L}^p_{\mathsf{loc}}(X)$ *with* $p = d$ *for* $d \geq 3$ *and with some* $p > 2$ *for* $d = 2$.

Then the operator A *with the domain*

$$\mathfrak{D}(A) = \bigl\{u \in \mathscr{L}^2(X, \mathbb{H}) \cap \mathscr{H}^1_{\mathsf{loc}}(X, \mathbb{H}) : Au \in \mathscr{L}^2(X, \mathbb{H})\bigr\} \tag{10.2.2}$$

is self-adjoint in $\mathscr{L}^2(X, \mathbb{H})$.

Proof. (a) Let us first observe that if $u \in \mathscr{H}^1_{\mathsf{loc}}$ then Sobolev embedding theorem yields that $u \in \mathscr{L}^q_{\mathsf{loc}}$ and

$$\|u\|_{\mathscr{L}^q(Y)} \leq C_Y \|u\|_{\mathscr{H}^1(Y)} \tag{10.2.3}$$

on every compact Y where here and below $q = d/(d-2)$ for $d \geq 3$ and $q = p/(p-2)$ with arbitrary $p > 2$ for $d = 2$. Therefore (10.2.1) yields that $V_j u, Vu, \omega^l u \in \mathscr{L}^2_{\mathsf{loc}}(x, \mathbb{H})$ where $\omega^l = i\sum_j D_j\omega^{jl}$.

(b) Next let us prove that the operator A with the indicated domain is closed. Let us first assume that $\max_j \bar\gamma_j < \infty$ [13)].

Let us rewrite V, V_j in the form $V = V' + V''$ with $\|V''\|_{\mathscr{L}^p} \le \varepsilon$, $\|V'\|_{\mathscr{L}^\infty} \le C_\varepsilon$, etc., and let us rewrite ω^{jl} in the form $\omega^{jl} = \omega^{jl\prime} + \omega^{jl\prime\prime}$ with $\|\omega^{jl\prime\prime}\|_{\mathscr{L}^\infty} \le \varepsilon$, $\|\omega^{jl\prime}\|_{\mathscr{C}^2} \le C_\varepsilon$ with arbitrarily small $\varepsilon > 0$. Then

$$\|Au - A_1 u\| \le C_0\varepsilon\|u\|_{\mathscr{H}^1} + C'_\varepsilon\|u\| \qquad \forall u \in \mathfrak{D}(A) \tag{10.2.4}$$

where $A_1 = \sum_{j,l} \omega^{jl\prime}\sigma_l h D_j$ and C_0 does not depend on ε. Moreover, properties of $\omega^{jl\prime}$ and σ_l yield that

$$\left|\|A_1 u\|^2 - \int \sum_{j,k} g^{jk\prime}(D_k u)^\dagger (D_j u)\, dx\right| \le C'_\varepsilon\varepsilon'\|u\|^2_{\mathscr{H}^1} + C''_{\varepsilon\varepsilon'}\|u\|^2 \qquad \forall u \in \mathfrak{D}(A)$$

where $g^{jk\prime} = \sum_l \omega^{jl\prime}\omega^{kl\prime}$ and therefore $|g^{jk} - g^{jk\prime}| \le C_0\epsilon$; here $\varepsilon' > 0$ is arbitrarily small and C'_ε does not depend on ε'. This inequality and (10.2.4) yield inequality

$$\left|\|Au\|^2 - \int \sum_{j,k} g^{jk}(D_k u)^\dagger (D_j u)\, dx\right| \le C_0\varepsilon\|u\|^2_{\mathscr{H}^1} + C'_\varepsilon\|u\|^2 \qquad \forall u \in \mathfrak{D}(A)$$

and then (10.1.59) yields estimate

$$\|u\|_{\mathscr{H}^1} \le C\big(\|Au\| + \|u\|\big) \qquad \forall u \in \mathfrak{D}(A). \tag{10.2.5}$$

This estimate immediately yields that the operator A is closed provided $\max_j \bar\gamma_j < \infty$.

Now let us consider the case $\max_j \bar\gamma_j = \infty$. Let $\psi \in \mathscr{C}_0^1(X)$ be a scalar function. Then $A\psi u = \psi Au + \psi' u$ with a matrix-valued function $\psi' \in \mathscr{C}_0(X)$. Then the above arguments yield the estimate

$$\|\psi u\|_{\mathscr{H}^1} \le C\big(\|Au\| + \|u\|\big) \qquad \forall u \in \mathfrak{D}(A) \tag{10.2.6}$$

where C depends on ψ. But then A is again obviously closed.

(c) Let us prove that the operator A is symmetric. This is obvious for $\max_j \bar\gamma_j < \infty$. For $\max_j \bar\gamma_j = \infty$ equality

$$(Au, v) = (u, Av) \qquad u, v \in fD(A) \tag{10.2.7}$$

[13)] Recall that $X = \mathbb{R}^d/(\bar\gamma_1\mathbb{Z} \times ... \times \bar\gamma_d\mathbb{Z})$.

holds for compactly supported v. Let us replace v by $\psi(x't^{-1})v$ where $\psi \in \mathscr{C}_0^1(X')$ equals 1 in a neighborhood of 0 and $X' \ni x' = (x_j)_{j\in J}$ with $J = \{j, \bar\gamma_j = \infty\}$. For t tending to $+\infty$ we obtain the same equality (10.2.7) for arbitrary $v \in \mathfrak{D}(A)$.

(d) To prove that A is self-adjoint let us consider the problem

$$D_t u = Au, \qquad u|_{t=0} = f. \tag{10.2.8}$$

We are going to prove that for every $f \in \mathfrak{D}(A)$ there exists a solution $u \in \mathscr{C}(\mathbb{R}, \mathscr{L}^2(X, \mathbb{H})) \cap \mathscr{L}^\infty(\mathbb{R}, \mathfrak{D}(A))$ of this problem.

Let us assume first that $\max_j \bar\gamma_j < \infty$. Then problem (10.2.8) is obviously well-posed provided $\omega^{jl}, V_j, V \in \mathscr{C}^2$. In this case $\mathfrak{D}(A) = \mathscr{H}^1(X, \mathbb{H})$.

Let $\omega^{jl}_\delta, V_{j\delta}, V_\delta$ be sequences of mollifications of ω^{jl}, V_j, V respectively with mollifying function $\psi_\delta(x) = \delta^{-d}\psi(x\delta^{-1})$ where $\psi \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ is a real-valued non-negative function with $\int \psi\, dx = 1$. Then for $\delta \to +0$, ω^{jl}_δ tends to ω^{jl} in $\mathscr{C}$ and ω^{l}_δ, $V_{j\delta}$, V_δ tend to ω^{l}, V_j, V respectively in $\mathscr{L}^p$. Moreover, $\|V''_\delta\|_{\mathscr{L}^p} \le C_0\varepsilon$ and $\|V'_\delta\|_{\mathscr{L}^\infty} \le C_\varepsilon$, etc., where V' and V'' were introduced in part (b).

Let u_δ be a sequence of solutions of problem (10.2.8) with the operator A_δ instead of A. Then the arguments of part (b) and the equalities $\|u_\delta\| = \|f\|$, $\|A_\delta u_\delta\| = \|A_\delta f\|$ $\forall t$ yield that for all $\delta > 0$ small enough,

$$\|D_t u_\delta\| + \|u_\delta\|_{\mathscr{H}^1} \le C(\|Af\| + \|f\|) \qquad \forall t \in \mathbb{R}$$

where the constant C does not depend on δ. Then the standard compactness arguments yield that one can choose a subsequence u_δ tending to a function u in $\mathscr{C}(\mathbb{R}, \mathscr{L}^2(X, \mathbb{H}))$ which lies in the required solution space and, moreover, $A_\delta u_\delta \to Au$ in $\mathscr{D}'(\mathbb{R}\times X, \mathbb{H})$ and therefore u is a solution of (10.2.8). The symmetry of A yields that this solution is unique and

$$\|u\| = \|f\|, \ \|Au\| = \|Af\| \qquad \forall t \in \mathbb{R}. \tag{10.2.9}$$

Now let $\max_j \bar\gamma_j = \infty$. Let us first take f supported in the domain $\{x\colon |x'| \le R\}$. Let us replace $\bar\gamma_j = \infty$ by $\bar\gamma_j = CR^2$ and let us solve problem (10.2.8). Then $|\omega^{jl}| \le c$ yields that the propagation speed for operators A_δ does not exceed C_0 and therefore the same is true for the operator A. Then u is supported in $\{|x'| \le \frac12 CR^2\}$ for $|t| \le R$ and sufficiently large R;

moreover, $u(.,t)$ does not depend on $R \geq |t|$. Letting R tend to $+\infty$ we conclude that problem (10.2.8) has a solution in the required space for every compactly supported $f \in \mathfrak{D}(A)$ and the propagation speed does not exceed C_0 and estimate (10.2.9) holds. But then the compactness condition is not necessary.

Now Theorem 10.2.1 is due to Proposition 10.2.2 below. □

Proposition 10.2.2. *Let A be a symmetric closed operator with the dense domain in the Hilbert space $\mathcal{H}$ and let $U(t)$ be a group of unitary operators in $\mathcal{H}$ such that for $f \in \mathfrak{D}(A)$*

(a) $U(t)f \in \mathfrak{D}(A) \quad \forall t \in \mathbb{R}$ and $\|AU(t)f\| \leq M_f e^{ct}$ with some constant $M = M_f$;

(b) $U(t)f \to f$ as $t \to 0$;

(c) $D_t U(t)f = AU(t)f$ in the weak sense.

Then A is a self-adjoint generator of $U(t)$.

Proof. Let us set

$$R_\pm = \mp i \int_0^\infty U(\pm t) e^{-\lambda t} dt$$

with $\lambda > c$. Obviously, the operator norm of $R_\pm$ in $\mathcal{H}$ does not exceed λ^{-1} and for $f \in \mathfrak{D}(A) \quad R_\pm f \in \mathfrak{D}(A)$ and $(A \pm i\lambda)R_\pm f = f$. Then $R_\pm f \in \mathfrak{D}(A)$ $\forall f \in \mathcal{H}$ and both defect indices vanish; here we used that A is closed and $\mathfrak{D}(A)$ is dense. Therefore A is self-adjoint. But then a generator of $U(t)$ coincides with A. □

Remark 10.2.3. (i) Obviously, symmetric operator A is defined under conditions weaker than in Theorem 10.2.1. For example, one can take its domain $\mathfrak{D}(A) = \mathscr{C}_0^1(X)$ provided $\omega^{jl}, \omega^l, V_j, V \in \mathscr{L}^2_{\mathsf{loc}}(X)$.

Moreover, let Y be a measure zero closed subset of X. Assume that $\omega^{jl}, \omega^l, V_j, V \in \mathscr{D}'(X) \cap \mathscr{L}^2_{\mathsf{loc}}(X \setminus Y)$. Then one can take $\mathfrak{D}(A) = \mathscr{C}_0^1(X \setminus Y)$. Obviously, A is a symmetric operator with domain dense in $\mathscr{L}^2(X, \mathbb{H})$.

(ii) Further, if $V_j = 0$ and <u>either</u> d is odd <u>or</u> d is even and $m = 0$ then operator (10.1.66) has a self-adjoint extension. In fact, there exists a unitary matrix β such that $\beta^\dagger \bar{\sigma}_j \beta = \sigma_j$ with $j = 0, \dots, d$ as d is odd and with

$j = 1, \dots, d$ as d is even where the bar means a complex conjugation[14]. But then $\beta^\dagger \bar{A} \beta = A$ and therefore both defect indices of A coincide.

(iii) Similarly, if $V = 0$, $m = 0$, then $\beta^\dagger A \beta = -A$ with $\beta = \sigma_0$ and therefore both defect indices of A again coincide[14].

On the other hand, if $V = 0$ and $m > 0$ then

$$\|Au\|^2 = \|(A - m\sigma_0)u\|^2 + m^2\|u\|^2$$

and inequality

$$\|Au\| \geq m\|u\| \qquad \forall u \in \mathfrak{D}(A) \tag{10.2.10}$$

holds. Therefore both defect indices of A coincide in this case also; moreover, there exists a self-adjoint extension of A satisfying (10.2.10).

Thus, if $V = 0$ then there exists a self-adjoint extension of A satisfying (10.2.10) [13].

(iv) For $d = 2$ let us consider an operator A of the form

$$A_1 = \sigma_1(D_r + \frac{i}{2}r^{-1}) + \sigma_2 r^{-1}(D_\phi + \alpha) + \sigma_0 m \tag{10.2.11}$$

in the polar coordinates with $\mathfrak{D}(A) = \{u \in \mathscr{H}^1(\mathbb{R}^2, \mathbb{H}), r^{-1}u \in \mathscr{L}^2(\mathbb{R}^2, \mathbb{H})\}$. One can see easily that A is self-adjoint for $\alpha \notin \mathbb{Z}$ and that

$$q^\dagger(\sigma_1 \cos\phi - \sigma_2 \sin\phi)q = \sigma_1, \quad q^\dagger(\sigma_1 \sin\phi + \sigma_2 \cos\phi)q = \sigma_2, \quad q^\dagger \sigma_0 q = \sigma_0$$

for the unitary matrix-valued function $q = \operatorname{diag}(e^{i\zeta\phi}, 1)$ and σ_0, σ_1, σ_2 defined by (5.A.28). Moreover,

$$A_2 = q^\dagger A_1 q = \sigma_1(D_1 - \alpha x_2|x|^{-2}) + \sigma_2(D_2 + \alpha x_1|x|^{-2}) + \sigma_0 m. \tag{10.2.12}$$

Then for $\alpha \notin \mathbb{Z}$ the operator A_2 (with the same domain as A_1) is self-adjoint. Moreover, magnetic intensity $F = 0$ for $x \neq 0$ but for $\alpha \notin \mathbb{Z}$ one can remove the vector potential by means the of a gradient transform only in a simply connected subdomain in $\mathbb{R}^2 \setminus 0$.

Furthermore, one can easily see that $A_2(\alpha')$ and $A_2(\alpha)$ are unitarily equivalent if and only if $\alpha' - \alpha \in \mathbb{Z}$.

(v) Other examples of self-adjoint Dirac operators with the magnetic field will be given in Parts VI–X.

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[14] These arguments however do not work as d is odd but operator A is principally massless operator (10.1.76).

10.2.2 Abstract Theory

Let us now consider the abstract theory. We will use the following definition:

Definition 10.2.4. Let A be a self-adjoint operator and J a symmetric operator in the separable Hilbert space $\mathcal{H}$ such that $\mathfrak{D}(J) \supset \mathfrak{D}(A)$. Then

(i) Operator J is A-*bounded* if

$$\|Ju\| \le C\|Au\| + C'\|u\| \quad \forall u \in \mathfrak{D}(A). \tag{10.2.13}$$

The smallest C_0 such that (10.2.13) holds with $C = C + \epsilon$ and $C' = C'_\epsilon$ for arbitrarily small $\epsilon > 0$ is called A-*bound* of J.

(ii) Operator J is A-*compact* if $J(A+i)^{-1}$ is a compact operator.

Proposition 10.2.5. *Let A be a self-adjoint operator and J_k be symmetric operators in the separable Hilbert space $\mathcal{H}$. Assume that operator J_1 is A-compact and A-bound of J_2 is less than 1. Then*

(i) A-bound of J_1 equals 0 and therefore operators $A + J_k$ are self-adjoint.

(ii) Operator $J_1(A + J_2 + i)^{-1}$ is compact.

(iii) $\mathsf{Spec}_{\mathsf{ess}}(A + J_1) = \mathsf{Spec}_{\mathsf{ess}}(A)$.

Proof. Recall that operator J_1 is A-compact if $J_1(A+i)^{-1}$ is compact; then for every $\epsilon > 0$ there exists a finite-dimensional operator $K = K_\epsilon$ such that $\|J_1(A+i)^{-1} - K\| \le \epsilon$. This operator is of the form

$$Ku = \sum_{1 \le j \le N} f_j \langle u, g_j \rangle$$

and without any loss of the generality one can assume that $g_j \in \mathfrak{D}(A)$ (otherwise they can be replaced by nearby elements of $\mathfrak{D}(A)$). But then $K(A+i)$ is a bounded operator and therefore

$$\|J_1 u\| \le \|K(A+i)u\| + \epsilon\|(A+i)u\| \le \epsilon\|Au\| + C'_\epsilon\|u\| \qquad \forall u \in \mathfrak{D}(A)$$

and therefore A-bound of J_1 equals 0. Statement (i) is proven.

Statement (ii) is due to equality

$$J_1(A+i)^{-1} - J_1(A+J_2+i)^{-1} = J_1(A+i)^{-1} \cdot J_2(A+J_2+i)^{-1}$$

with compact operator $J_1(A+i)^{-1}$ and bounded operator $J_2(A+J_2+i)^{-1}$ (because the A-bound of J_2 is less than 1).

Let us prove that if $\tau \in \mathsf{Spec}_{\mathsf{ess}}(A)$ then $\tau \in \mathsf{Spec}_{\mathsf{ess}}(A+J_1)$; this assertion combined with Statements (i) and (ii) with $J_2 = J_1$ yield the inverse implication and thus Statement (iii). If $\tau \in \mathsf{Spec}_{\mathsf{ess}}(A)$ then there exists a sequence $u_n \in \mathfrak{D}(A)$ such that $\|u_n\| = 1$, $u_n \overset{w}{\to} 0$ and $\|(A-\tau)u_n\| \to 0$. Let $v_n = (A+i)u_n$. Then $1 \le \|v_n\| \le M$ obviously and

$$\langle v_n, f\rangle = \langle u_n, (A-i)f\rangle \to 0 \qquad \text{as} \quad f \in \mathfrak{D}(A).$$

Then $v_n \overset{w}{\to} 0$ and then $J_1 u_n = J_1(A+i)^{-1} v_n \overset{s}{\to} 0$ because operator $J_1(A+i)^{-1}$ is compact. Hence $\|(A+J_1-\tau)u_n\| \to 0$ and therefore $\tau \in \mathsf{Spec}_{\mathsf{ess}}(A+J_1)$.□

Proposition 10.2.6. *Let A be a self-adjoint operator and let J be a symmetric operator in the separable Hilbert space $\mathcal{H}$. Assume that J is A-bounded. Then*

(i) Let $\mathsf{dist}\big(\tau_0, \mathsf{Spec}(A)\big) \ge \epsilon > 0$. *Then* $\mathsf{dist}\big(\tau_0, \mathsf{Spec}(A+tJ)\big) \ge \frac{1}{2}\epsilon$ *for* $|t| \le \delta = \delta(\epsilon) > 0$.

(ii) If $\mathsf{dist}\big(\tau_0, \mathsf{Spec}(A) \setminus \tau_0\big) \ge \epsilon > 0$ *then*

$$\mathsf{Spec}(A+tJ) \cap [\tau_0 - \frac{1}{2}\epsilon, \tau_0 + \frac{1}{2}\epsilon] \subset [\tau_0 - C|t|, \tau_0 + C|t|]$$

for $|t| \le \delta = \delta(\epsilon) > 0$.

(iii) Moreover, in the framework of (ii) the spectral projector of the operator $A+tJ$ corresponding to the interval $[\tau_0 - \frac{1}{2}\epsilon, \tau_0 + \frac{1}{2}\epsilon]$ depends analytically on t.

(iv) In particular, if τ_0 is an eigenvalue of A with multiplicity $n < \infty$ then for $|t| \le \delta$

$$\mathsf{Spec}(A+tJ) \cap [\tau_0 - \frac{1}{2}\epsilon, \tau_0 + \frac{1}{2}\epsilon] = \big\{\tau_1(t), \dots, \tau_n(t)\big\}$$

where $\tau_k(t)$ are Lipschitz functions and the number of eigenvalues of $A+tJ$ in the interval $[\tau_0 - \frac{1}{2}\epsilon, \tau_0 + \frac{1}{2}\epsilon]$ (counting multiplicities) is equal to n.

Proof. Both Statements (i) and (ii) immediately follow from the inequality

$$\|(A-\tau)u\| \ge \mathsf{dist}\big(\tau, \mathsf{Spec}(A)\big) \cdot \|u\| \qquad \forall u \in \mathfrak{D}(A)$$

for self-adjoint operators.

Rewriting the spectral projector in question in the form

$$\Pi(t) = \frac{1}{2\pi i}\int_{\Gamma}(z - A - tJ)^{-1}dz$$

where $\Gamma = \{z \in \mathbb{C}, |z - \tau_0| = \frac{3}{4}\epsilon\}$ is a counter-clockwise oriented contour we obtain Statement (iii) and the assertion concerning the number of eigenvalues in Statement (iv). Moreover, we conclude that

$$\mathsf{Spec}_{\mathsf{ess}}(A + tJ) \cap [\tau_0 - \frac{1}{2}\epsilon, \tau_0 + \frac{1}{2}\epsilon] = \emptyset.$$

Finally, the Lipschitz property of τ_k easily follows from the arguments of the proof of Statements (i) and (ii). □

Proposition 10.2.7. *In the framework of Proposition 10.2.6(iv) assume that*

$$\langle Jv, v\rangle \ge \epsilon\|v\|^2 \qquad \forall u \in \mathsf{Ker}(A - \tau_0). \tag{10.2.14}$$

Then

$$\tau_k(t') \ge \tau_k(t) + \delta_1(t' - t) \qquad \forall k = 1, \dots, n \quad \forall t, t' \in [-\delta, \delta] : t' > t \tag{10.2.15}$$

with $\delta_1 = \delta_1(\epsilon) > 0$.

Proof. Obviously it is sufficient to prove (10.2.15) under additional condition $t' < t + \delta_2(t)$ with $\delta_2(t) > 0$.

Consider $t = \bar{t}$. Let $\bar{\tau}$ be an eigenvalue of $A + \bar{t}J$ of multiplicity r and let $\bar{\Pi}$ be the corresponding spectral projector.

Let τ' be an eigenvalue of $A + t'J$ close to $\bar{\tau}$ and let $|t' - \bar{t}|$ be small enough; then Proposition 10.2.6(iii) yields that

$$\|(I - \bar{\Pi})u\| \le M|t' - \bar{t}| \cdot \|u\| \qquad \text{with} \quad M = M(\bar{t}). \tag{10.2.16}$$

We know that $|\tau' - \bar{\tau}| \le C|t' - \bar{t}|$. This inequality combined with (10.2.16), equalities

$$(A + t'J - \tau')v = 0, \quad (A + \bar{t}J - \bar{\tau})\bar{\Pi}v = 0 \quad \text{for} \quad v \in \mathsf{Ker}(A + t'J - \tau')$$

and the A-boundedness of the operator J yield that

$$\|A(I-\bar\Pi)v\| \le M|t'-\bar t|\cdot\|v\| \quad \text{for} \quad v\in \mathsf{Ker}(A+t'J-\tau'). \tag{10.2.17}$$

Then the same inequality (10.2.17) holds for J instead of A (we increase M if necessary). These inequalities and the equality $(A+\bar tJ-\bar\tau)\bar\Pi=0$ yield that

$$\begin{aligned}\big|(t'-\bar t)\langle Jv,v\rangle+(\tau'-\bar\tau)\|v\|^2\big| = \\ |\langle (A+\bar tJ-\bar\tau)v,v\rangle| \le M|t'-t|^2\|v\|^2.\end{aligned} \tag{10.2.18}$$

On the other hand, inequality (10.2.14) remains true (with $\frac12\epsilon$ instead of ϵ) for $u\in\mathsf{Ker}(A+t'J-\tau')$ also with $|t'|\le\delta$, $|\tau'-\bar\tau|\le\delta$. This inequality, (10.2.18) and Proposition 10.2.6(iii) yield that

$$\mathsf{Spec}(A+t'J)\cap[\tau_0-\delta_3,\tau_0+\delta_1(t'-\bar t)]=\emptyset \qquad \text{for}\quad t'\in[\bar t,\bar t+\delta_2(\bar t)]$$

where $\delta_3=\delta_3(\bar t)>0$ while $\delta_1>0$ does not depend on $\bar t$. So (10.2.15) is proven for $t'<\bar t+\delta_2(\bar t)$. □

Now we are ready to prove the main statement:

Theorem 10.2.8 (Birman-Schwinger principle). *Let A be self-adjoint operator and let J be a symmetric operator in the separable Hilbert space $\mathcal{H}$. Assume that J is A-compact operator.*

Let us assume that $\bar\tau_1,\bar\tau_2\in\mathbb{R}$, $\bar\tau_1<\bar\tau_2$, $\bar t>0$ and

$$(A+tJ-\bar\tau_i)u=0 \implies \varsigma_i\langle Ju,u\rangle\ge\epsilon\|u\|^2 \qquad \forall i=1,2\quad \forall t\in[0,\bar t] \tag{10.2.19}$$

with $\varsigma_i=\pm1$, $\epsilon>0$. Then

(i) The following equality holds:

$$\begin{aligned}\mathsf{N}\big(A;[\bar\tau_1,\bar\tau_2]\big)=\mathsf{N}\big(A+\bar tJ;[\bar\tau_1,\bar\tau_2]\big)+ \\ \varsigma_2\tilde{\mathsf{N}}\big((A-\bar\tau_2),-J;[0,\bar t]\big)-\varsigma_1\tilde{\mathsf{N}}\big((A-\bar\tau_1),-J;[0,\bar t]\big)\end{aligned} \tag{10.2.20}$$

where $\tilde{\mathsf{N}}(B,-J;\mathcal{J})$ is the number of eigenvalues $t\in\mathcal{J}$ of the problem

$$(B+tJ)u=0 \qquad u\in\mathfrak{D}(B)\setminus 0 \tag{10.2.21}$$

counting their multiplicities[15)].

On the other hand, $\mathsf{N}(B,\mathcal{J})$ *is the cardinality of the spectrum of* B *in* $\mathcal{J}$ *and* $\mathsf{N}(B,\mathcal{J})=\infty$ *if* $\mathsf{Spec}_{\mathsf{ess}}(B)\cap\mathcal{J}\neq\emptyset$.

(ii) Further, equality (10.2.20) *remains true if one replaces the interval* $\mathcal{J}$ *in the left-hand and right-hand expressions: one removes* $\bar\tau_1$ *and (or)* $\bar\tau_2$ *from* $\mathcal{J}$ *and replaces* $[0,\bar t]$ *by* $(0,\bar t)$ *in the third and (or) second terms in the right-hand expression; however, in order to avoid complications with the domain of* u*, one should assume that* $\bar\tau_i\notin\mathsf{Spec}_{\mathsf{ess}}(A)$.

(iii) Furthermore, $\tilde{\mathsf{N}}\bigl(A-\bar\tau_i,-J;[0,\bar t]\bigr)<\infty$ *provided* $\bar\tau_i\notin\mathsf{Spec}_{\mathsf{ess}}(A)$.

Remark 10.2.9. If $\mathsf{Spec}_{\mathsf{ess}}(A)\cap[\bar\tau_1,\bar\tau_2]\neq\emptyset$ then $\mathsf{Spec}_{\mathsf{ess}}(A+tJ)\cap[\bar\tau_1,\bar\tau_2]\neq\emptyset$ due to Proposition 10.2.5(iii) and both sides of (10.2.20) are infinite: indeed, it will follow from Statement (iii) of the Theorem that $\tilde{\mathsf{N}}\bigl((A-\bar\tau_i),-J;[0,\bar t]\bigr)<\infty$ for $i=1,2$ and therefore there is no indefiniteness in the right-hand expression).

Proof of Theorem 10.2.8. In virtue of Remark 10.2.9 in the proof of Statements (i) and (i) we can assume without any loss of the generality that

$$\mathsf{Spec}_{\mathsf{ess}}(A)\cap[\bar\tau_1,\bar\tau_2]=\emptyset. \tag{10.2.22}$$

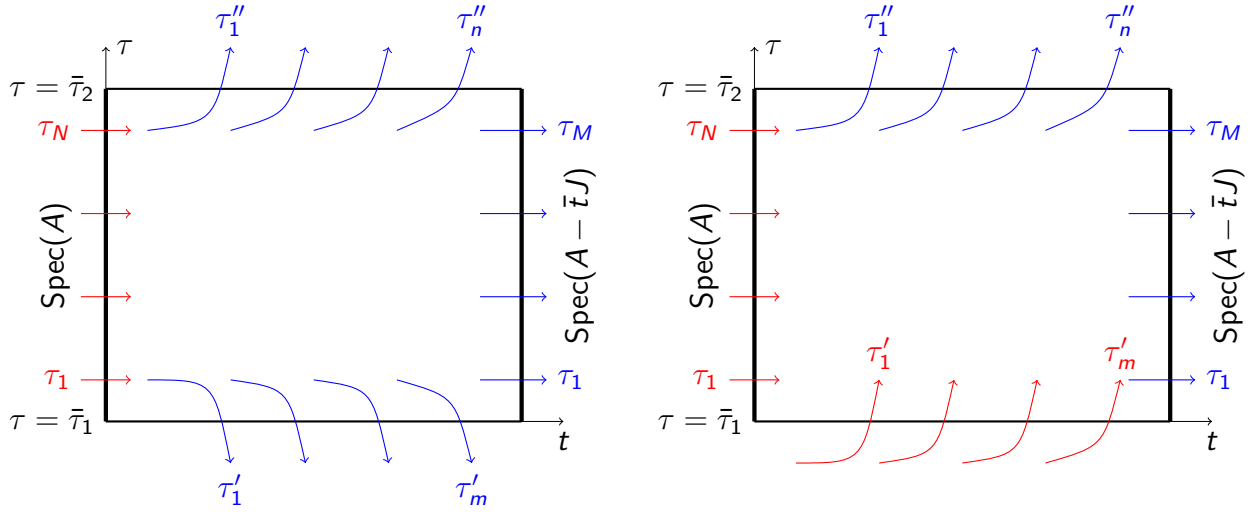
Then

$$\mathsf{Spec}(A+tJ)\cap[\bar\tau_1,\bar\tau_2]=\bigl\{\tau_k(t),\,t\in[t'_k,t''_k],\,k=0,\dots,N\bigr\}$$

where $N\le\infty$ and for each $\tau\in[\bar\tau_1,\bar\tau_2]$ $\#\{k\colon\tau_k(t)=\tau\}$ is equal to the multiplicity of τ as an eigenvalue of $(A+tJ)$.

Furthermore, due to Propositions 10.2.5 and 10.2.5 $0\le t'_k<t''_k\le\bar t$ and $\tau_k(t)$ are Lipschitz functions on the intervals $(\tilde t'_k,\tilde t''_k)\supset[t'_k,t''_k]$ such that for $t\in(\tilde t'_k,\tilde t''_k)$ $\tau_k(t)\in[\bar\tau_1,\bar\tau_2]$ if and only if $t\in[t'_k,t''_k]$. Finally, $\varkappa_i\tau_k(t)$ are monotone increasing function of t in neighborhoods of $t_{ik}\in[t'_k,t''_k]$ with $\tau_k(t_{ik})=\bar\tau_i$.

Then the various terms in (10.2.20) are the numbers of graphs $\{\tau=\tau_k(t)\}$ intersecting respectively the upper, lower, right and left sides of the rectangle $\{0\le t\le\bar t,\ \bar\tau_1\le\tau\le\bar\tau_1\}$ on Figure 10.1, taken with the correct signs and



(a) $\varsigma_2 > 0, \varsigma_1 < 0$; $N = M + m + n$ (b) $\varsigma_2 > 0, \varsigma_1 > 0$; $N + m = M + n$

Figure 10.1: : As $\varsigma_2 \gtrless 0$ eigenvalues exit/enter $[\bar\tau_1, \bar\tau_2]$ through $\tau = \bar\tau_2$ and as $\varsigma_1 \lessgtr 0$ eigenvalues exit/enter $[\bar\tau_1, \bar\tau_2]$ through $\tau = \bar\tau_1$ as t grows. As $\varsigma_2 < 0$ pictures should be modified in the obvious way.

equality (10.2.20) and its modifications in Statement (ii) are due to obvious geometric arguments.

Statements (i) and (ii) are proven; let us prove Statements (iii). If $\tilde{\mathsf{N}}(A - \bar\tau, -J; [0, \bar t]) = \infty$ then there would exist sequences $t_n \in [0, \bar t]$ and $u_n \in \mathfrak{D}(A)$ such that $(A - \bar\tau + t_n J)u_n = 0$ and $\langle Ju_n, u_m\rangle = \delta_{nm}$. Then $\|u_n\| \le C$ by (10.2.19) and then the equality $(A + t_n J - \bar\tau)u_n = 0$ and the fact that the A-bound of J is 0 imply that $\|Au_n\| \le C_1$. Then the compactness of the operator $J(A + i)^{-1}$ yields that the sequence Ju_n is strongly convergent (we take a subsequence if necessary) but this contradicts the equality $\langle Ju_n, u_m\rangle = \delta_{nm}$. □

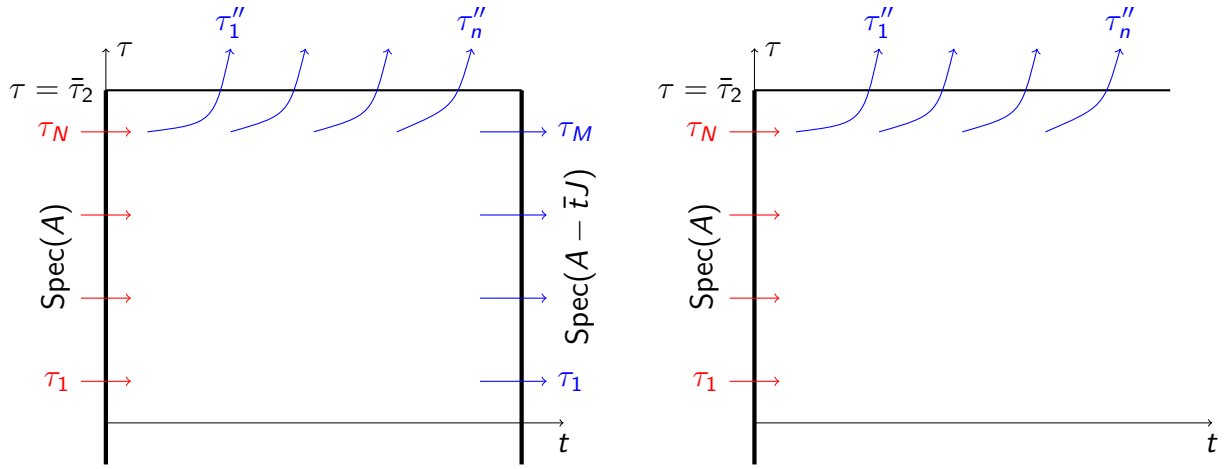
Remark 10.2.10. (i) As A is semi-bounded from below we are in the framework of Chapter 9 and Birman-Schwinger principle is illustrated on Figure 10.2(a); we can take $\bar\tau_1 = -\infty$ resulting in

$$(10.2.20)' \quad \mathsf{N}\big(A; [\bar\tau_1, \bar\tau_2]\big) = \mathsf{N}\big(A + \bar t J; [\bar\tau_1, \bar\tau_2]\big) + \varsigma_2 \tilde{\mathsf{N}}\big((A - \bar\tau_2), -J; [0, \bar t]\big).$$

[15] I.e., the dimensions of the subspaces of $\mathcal{H}$ on which (10.2.21) holds with given t; in this definition neither continuous nor fancier types of the spectrum are discussed, just the bare dimension of the null space.

(ii) And then we can take $\bar{t} = \infty$ important subcase is of $\bar{t} = \infty$ as illustrated on Figure 10.2(b):

$$\mathsf{N}\big(A; [\bar{\tau}_1, \bar{\tau}_2]\big) = \varsigma_2 \tilde{\mathsf{N}}\big((A - \bar{\tau}_2), -J; [0, \bar{t}]\big). \tag{10.2.20}''$$



(a) $\varsigma_2 > 0, N = M + n$ (b) $\varsigma_2 > 0, N = n, \bar{t} = \infty$

Figure 10.2: : Case of A bounded from below.

Remark 10.2.11. Let A be a self-adjoint operator and J be a symmetric operator in the separable Hilbert space $\mathcal{H}$. Let $\mathfrak{D}(J) \supset \mathfrak{D}(A)$ and $J(A+i)^{-1}$ be a compact operator. Moreover, let us assume that $J \geq 0$, $\mathsf{Ker}\, J = 0$ and

$$\|(A - \bar{\tau}_i)u\| + \|Ju\| \geq \epsilon_1 \|u\| \qquad \forall u \in \mathfrak{D}(A) \tag{10.2.23}$$

with $\epsilon_1 > 0$. Then (10.2.19) holds.

In fact, otherwise there exist sequences $t_n \in [0, \bar{t}]$ and $u_n \in \mathfrak{D}(A)$ such that $Au_n = -t_n Ju_n$ (we assume for the sake of simplicity that $\bar{\tau}_i = 0$), $\|u_n\| = 1$, $\langle Ju_n, u_n\rangle \to 0$. Then estimate $\|Ju\| \leq \delta\|Au\| + C_\delta\|u\|$ (with arbitrarily chosen $\delta > 0$) yields that $\|Au_n\| \leq C$ and since $\|u\|_n = 1$ we obtain that the sequence Ju_n is strongly convergent (we take a subsequence if necessary). Then the sequence Au_n is also strongly convergent. Then (10.2.23) yields that the sequence u_n is also strongly convergent (because a Cauchy sequence is a converging sequence and conversely).

Let $u = \lim u_n$. Then $u \in \mathfrak{D}(A)$, $\|u\| = 1$ and $\langle Ju, u\rangle = \lim\langle Ju_n, u_n\rangle = 0$; this contradicts the conditions $J \geq 0$ and $\operatorname{Ker} J = 0$.

Proposition 10.2.12. *Let assumptions of Remark 10.2.11 be fulfilled. Let us replace assumption* (10.2.23) *by assumption*

$$\|(A - \bar\tau_i)u\| + \|J^{\frac12}u\| \geq \epsilon_1\|u\| \qquad \forall u \in \mathfrak{D}(A); \tag{10.2.24}$$

one can easily see that this condition is stronger. Then for small enough $\delta > 0$ operators $((A-\bar\tau_i)\pm i\delta J)^{-1}$, $(A-\bar\tau_i)((A-\bar\tau_i)\pm i\delta J)^{-1}$ and $J((A-\bar\tau_i)\pm i\delta J)^{-1}$ are bounded (their norms depend on δ) and therefore operator $J((A-\bar\tau_i)\pm i\delta J)^{-1}$ is compact.

Moreover, one can take arbitrarily large $\delta > 0$ provided J is a bounded operator.

Proof. Without any loss of the generality one can assume that $\bar\tau_i = 0$. If $(A + i\delta J)u = f$ then

$$\|J^{\frac12}u\|^2 \leq \delta^{-1}\|u\| \cdot \|f\|, \quad \|Au\| \leq \delta\|Ju\| + \|f\| \leq C\delta(\|Au\| + \|u\|) + \|f\|$$

and hence $\|u\| + \|Au\| \leq C_\delta\|f\|$. This inequality yields the boundedness of the indicated operators. Moreover, if J is a bounded operator then $\|Ju\| \leq C\|J^{\frac12}u\|$ and therefore $\|Au\| \leq C\delta\|J^{\frac12}u\| + \|f\|$; this inequality combined with the inequalities $\|J^{\frac12}u\|^2 \leq \delta^{-1}\|u\| \cdot \|f\|$ and (10.2.24) yield that $\|u\| + \|Au\| \leq C_\delta\|f\|$ for arbitrary $\delta > 0$. □

Proposition 10.2.13. *Let assumptions of Remark 10.2.11 be fulfilled and let J be a bounded operator (one can easily check that conditions* (10.2.23) *and* (10.2.24) *are equivalent in this case). Then*

(i) Operator $B = J^{-\frac12}(A - \bar\tau_i)J^{-\frac12}$ with domain

$$\mathfrak{D}(B) = \big\{u \in \mathfrak{D}(J^{-\frac12}) : \ J^{-\frac12}u \in \mathfrak{D}(A) \ \text{ and } \ (A - \bar\tau_i)J^{-\frac12}u \in \mathfrak{D}(J^{-\frac12})\big\}$$

is self-adjoint.

(ii) Operators $(B \pm i)^{-1}$ are compact.

(iii) Eigenvalues of the operator $-B$ and those of the problem

$$(A - \bar\tau_i)u + tJu = 0 \qquad u \in \mathfrak{D}(A) \tag{10.2.25}$$

coincide (counting multiplicities) and

$$\mathsf{Ker}(B+tJ) = J^{\frac12}\,\mathsf{Ker}((A-\bar\tau_i)+tJ). \tag{10.2.26}$$

Proof. The obvious proof is left to the reader. □

10.2.3 Proof of Theorem 10.1.7

Thus *Theorem 10.1.7 has been proven under additional condition* (10.2.23) *which should be checked for operator* A *replaced by*

$$A'_k := A - t_1J_1 - ... - t_{k-1}J_{k-1},$$

$J = J_k$ *and* $\bar\tau_i = \tau_i$ *for every* $k = 1, ..., n$*:*

$$\|(A-\tau_i - t_1J_1 - ... - t_{k-1}J_{k-1})u\| + \|J_ku\| \ge \epsilon_1\|u\| \qquad u\in\mathfrak{D}(A)\quad \forall k = 1, ..., n. \tag{10.2.27}$$

Really, by the construction operators $J_k \ge 0$ are bounded[16] operators with $\mathsf{Ker}\, J_k = 0$ and therefore in virtue of Proposition 10.2.11

$$\tilde{\mathsf{N}}\big((A-\tau_2), -J; [0,\bar t]\big) = \mathsf{N}\big(-J^{-\frac12}(A-\tau_2)J^{-\frac12}; [0,\bar t]\big) = \mathsf{N}\big(-J^{-1}(A-\tau_2); [0,\bar t]\big) \tag{10.2.28}$$

and formula (10.2.20) becomes

$$\mathsf{N}\big(A; [\tau_1,\tau_2]\big) = \mathsf{N}\big(A+\bar tJ; [\tau_1,\tau_2]\big) + \varsigma_2\mathsf{N}\big(-J^{-\frac12}(A-\tau_2)J^{-\frac12}; [0,\bar t]\big) - \varsigma_1\mathsf{N}\big(J^{-\frac12}(A-\tau_1)J^{-\frac12}; [0,\bar t]\big). \tag{10.2.29}$$

Due to (10.2.27) we can replace here A by A'_k and J_1 by J_k and $\bar t$ by $-t_k$.

Therefore we justified (10.1.12) and thus (10.1.18) and then the arguments of Subsection 10.1.2 prove the statement of Theorem 10.1.7.

One can easily check that

(10.2.30) Condition (10.2.23) is fulfilled provided

$$\tau_1, \tau_2 \notin \mathsf{Spec}_{\mathsf{ess}}(A). \tag{10.2.31}$$

[16] One can see easily that otherwise $R_{(i)1} = \infty$ and then the statement of the theorem is empty.

Moreover, Proposition 10.2.5(iii) yields that in this case condition (10.2.27) is fulfilled as well. Therefore *Theorem 10.1.7 has been proven under additional condition* (10.2.31).

Finally, let us consider the general case. Let us replace condition (10.2.31) by condition

(10.2.32) There exist sequences $\tau_{1,n} \to \tau_1 + 0$ and $\tau_{2,n} \to \tau_2 - 0$ with $\tau_{1,n}, \tau_{2,n} \notin \mathsf{Spec}_{\mathsf{ess}}(A)$;

this condition is fulfilled in virtue of (10.2.24). Under this condition we can apply Theorem 10.1.7 to intervals $(\tau_{1,n}, \tau_{2,n})$.

Let us note that $\mathsf{Spec}\, a^0(x,\xi) = \{\lambda_1(x,\xi), \dots, \lambda_D(x,\xi)\}$ with Lipschitz functions λ_j such that $|\lambda_j(x,\xi)| \to \infty$ as $|\xi| \to \infty$ and therefore

$$x \in X_{(1)\mathsf{reg},\mathsf{ell}} \cap X_{(2)\mathsf{reg},\mathsf{ell}} \implies \mathsf{Spec}\, a^0(x,\xi) \cap [\tau_1, \tau_2] = \emptyset. \tag{10.2.33}$$

On the other hand, it is easy to check (e.g., see the next Subsection 10.2.4) that if $R_{(1)1}+R_{(2)1} < \infty$ (otherwise the statement of Theorem 10.1.7 is empty) then $X_{(1)\mathsf{reg}} \setminus X_{(1)\mathsf{reg},\mathsf{ell}}$ and $X_{(2)\mathsf{reg}} \setminus X_{(2)\mathsf{reg},\mathsf{ell}}$ are compact sets. Therefore all conditions in $X_{(i)\mathsf{reg}} \setminus X_{(i)\mathsf{reg},\mathsf{ell}}$ (i.e., the microhyperbolicity condition, etc.) remain true (with $\gamma' = \frac{1}{2}\gamma$ and the same $\rho, \dots$) for $\tau_{i,n}$ instead of τ_i provided n is large enough.

Moreover, the ellipticity condition which defines $X_{(i)\mathsf{reg},\mathsf{ell}}$ remains true (with $\gamma' = \frac{1}{2}\gamma$ and $\rho' \ge \rho$ and appropriately changed other parameters) for τ_i' instead of τ_i provided $|\tau_i' - \tau_i|$ is small enough.

Therefore every term of the remainder estimate in the zone $X_{(i)\mathsf{reg}}$ for the interval $(\tau_{1,n}, \tau_{2,n})$ tends to the corresponding term of the remainder estimate for the interval (τ_1, τ_2) and $\mathcal{N}(\tau_{1,n}, \tau_{2,n})$ tends to $\mathcal{N}(\tau_1, \tau_2)$ (even if it is infinite).

Moreover, we can take the same R_0 for the interval $(\tau_{1,n}, \tau_{2,n})$ as for the original interval (τ_1, τ_2) (see conditions (10.1.44)–(10.1.46)). Therefore *Theorem 10.1.7 has been proven in the full generality.*

10.2.4 Proof of Theorem 10.1.8

Functional-Analytic Arguments

Let us return to the Dirac operator. In order to prove Theorem 10.1.8 one should derive some estimates of (10.1.46)- and (10.1.48)-type and construct the admissible matrix functions J_k.

Let us first consider the Dirac operator in the context of Theorem 10.1.7. Let us note first of all that if $R_1 < \infty$ (otherwise all the statements of this theorem are void) then $X_{\mathsf{reg}} \setminus X_{\mathsf{reg,ell}}$ are compact sets and on these sets ρ and γ are bounded and disjoint from 0. Really, otherwise there exists a family of non-intersecting balls $B(x^n, \gamma(x^n))$ with $n = 1, 2, \dots$, lying in this set, the contribution of each of these balls to R_1 is larger than $\epsilon > 0$.

On the other hand, the definition of $X_{\mathsf{reg,ell}}$ immediately yields that $\rho \le c'm$ on this set. So we can assume that

$$\rho \le M,\ \gamma \ge \epsilon \qquad \text{at} \quad X_{\mathsf{reg,ell}} \tag{10.2.34}$$

and

$$\varepsilon \le \rho \le M,\ \epsilon \le \gamma \le M \qquad \text{at} \quad X_{\mathsf{reg}} \tag{10.2.35}$$

where ε, M should not enter in the estimates.

Now we need to prove the A-compactness property.

Proposition 10.2.14. *Let all the conditions of Theorem 10.1.8 associated with the zone X_{reg} be fulfilled. Let us assume that the operator A is self-adjoint. Moreover, let $\rho \le M$ at X_{reg} and $J \in \mathscr{L}^\infty(X)$ be supported in X_{reg}. Let*

$$\forall \delta > 0 \qquad \mathsf{mes}(\{x, |J(x)| > \delta\}) < \infty. \tag{10.2.36}$$

Then operator $J(A+i)^{-1}$ is compact in $\mathscr{L}^2(X, \mathbb{H})$.

Proof. The results of Subsection 5.3.2 and the boundedness of ρ yield the estimate

$$|e(x, x, 0, \tau)| \le M(|\tau|^d + 1) \qquad \forall x \in X_{\mathsf{reg}} \quad \forall \tau \in \mathbb{R} \tag{10.2.37}$$

where the constant M is increased if necessary and e assumed without losing generality that $\tau_1 = -\tau_2$.

Therefore $Q := J^{\frac12}\big(E(\tau) - E(0)\big)J^{\frac12}$ is a trace operator with the trace norm not exceeding $M(|\tau|^d + 1)$ provided $J \in \mathscr{L}^1 \cap \mathscr{L}^\infty$ where $E(\tau)$ is the spectral projector of A; recall that J is supposed to be supported in X_{reg} here.

This operator is of the form $Q = SS^*$ and therefore $S = J^{\frac12}\big(E(\tau) - E(0)\big)$ is a Hilbert-Schmidt operator with the Hilbert-Schmidt norm not exceeding $M(|\tau|^d + 1)$. Then equalities

$$(A+i)^{-p} = \int_{-\infty}^{\infty} (\tau+i)^{-p} d_\tau E(\tau) = \int_{-\infty}^{\infty} p(\tau+i)^{-p-1}\big(E(\tau) - E(0)\big)\, d\tau$$

yield that

(10.2.38) For $p > d$ operator $J(A+i)^{-p}$ is also a Hilbert-Schmidt operator.

Let us note that for $\varepsilon > 0$

$$J(A+i)^{-1} = J(A+i)^{-1}(I+i\varepsilon A)^{-d} + JB_\varepsilon \tag{10.2.39}$$

with

$$B_\varepsilon = (A+i)^{-1}(I+i\varepsilon A)^{-d}\big((I+i\varepsilon A)^d - I\big); \tag{10.2.40}$$

then due to (10.2.38) the first term in the right-hand expression of (10.2.39) is a Hilbert-Schmidt operator and B_ε is a product of $\varepsilon A(A+i)^{-1}$ and an operator with the uniformly bounded operator norm. Hence the operator norm of B_ε does not exceed $C\varepsilon$. Therefore $J(A+i)^{-1}$ is approximated in the operator norm by compact operators and therefore is also compact. *Thus, for $J \in \mathscr{L}^1 \cap \mathscr{L}^\infty$ the proposition is proven.*

In the general case, for every $\varepsilon > 0$ there exists a representation $J = J' + J''$ with $J' \in \mathscr{L}^1 \cap \mathscr{L}^\infty$ and $\|J''\|_{\mathscr{L}^\infty} \le \varepsilon$. Then $J'(A+i)^{-1}$ is a compact operator and $\|J''(A+i)^{-1}\| \le \varepsilon$ and so $J(A+i)^{-1}$ is approximated by compact operators and so is also compact. □

Proposition 10.2.15. *Let all the conditions of Theorem 10.1.8 associated with the zone X_{sing} be fulfilled. Let $R_3 < \infty$ and $\gamma \ge \epsilon$ on $X_{\mathsf{reg}} \cap X_{\mathsf{sing}}$. Let $J \in \mathscr{L}^p(X)$ with $p = d$ for $d \ge 3$ and with $p > 2$ for $d = 2$* [17]*.*

Suppose $\mathsf{supp}(J) \subset X_{\mathsf{sing}}$ *and* $\mathsf{dist}(\mathsf{supp}(J), \partial X_{\mathsf{sing}} \setminus X_{\mathsf{reg}}) \ge \epsilon$. *Then operator $J(A+i)^{-1}$ is compact.*

Proof. Conditions of this proposition yield that there exists a function ϕ supported in the $\frac{2}{3}\epsilon$-neighborhood of $\mathsf{supp}(J)$ and equal 1 in the $\frac{1}{3}\epsilon$-neighborhood of $\mathsf{supp}(J)$ such that $0 \le \psi \le 1$ and $|D^\alpha \phi| \le C \quad \forall \alpha : |\alpha| \le K$.

Let us consider equation $(A+i)v = f \in \mathscr{L}^2(X, \mathbb{H})$ with v supported in X_{sing}. Let us apply the arguments used in the proof of Theorem 10.2.1, and also condition $R_3 < \infty$. One can prove easily that if $v \in \mathscr{H}^1$ then

$$\|Dv\| \le M(\|f\| + \|v\|). \tag{10.2.41}$$

Let us consider $v \in \mathscr{H}^1_{\mathsf{loc}} \cap \mathscr{L}^2(X, \mathbb{H})$ and take a function ψ with $|\psi| + |\nabla\psi| \le C$. Then $(A+i)\psi v = \psi f + [A, \psi]v$ and (10.2.41) yields

[17] Let us recall that for $d = 2$ the condition $R_3 < \infty$ also means that $\mathsf{mes}(X_{\mathsf{sing}}) < \infty$.

estimate $\|D\psi v\| \le M(\|f\| + \|v\|)$. Since this estimate holds for every ψ described above we obtain that $v \in \mathscr{H}^1$ and estimate (10.2.41) holds.

Similarly, if $u \in \mathfrak{D}(A)$ and $(A+i)u = f$ then

$$\|\phi u\|_{\mathscr{H}^1} \le M(\|f\| + \|u\|) \le (M+1)\|f\|$$

and therefore the operator $\phi(A+i)^{-1} : \mathscr{L}^2(X, \mathbb{H}) \to \mathscr{H}^1(X, \mathbb{H})$ is bounded. On the other hand, it is well known that if $J \in \mathscr{L}^p$ with p indicated above (and $\mathsf{mes}(X_{\mathsf{sing}}) < \infty$ for $d = 2$) then the operator $J : \mathscr{H}^1(X_{\mathsf{sing}}, \mathbb{H}) \to \mathscr{L}^2(X, \mathbb{H})$ is compact. □

The final arguments of the proof of Proposition 10.2.14 obviously yield

Corollary 10.2.16. *The assertion of Proposition 10.2.15 remains true if condition $J \in \mathscr{L}^p$ is replaced by condition*

(10.2.42) $V = V' + V''$, $V \in \mathscr{L}^p$, $V'' \in \mathscr{L}^\infty$ *and satisfies*(10.2.36).

Estimates for a Number of Crossing Eigenvalues

Let us start from the construction of matrices J_k. One can see easily that there exists a function ζ' on X_{reg} such that $\zeta'(x) > 0$, $c^{-1} \le \zeta'(x)/\zeta'(y) \le c$ for $y \in B\big(x, \epsilon_1\gamma(x)\big)$, $x \in X'_{\mathsf{reg}}$ where X'_{reg} is a domain intermediate between X_{reg} and $X_{\mathsf{reg}} \setminus X_{\mathsf{sing}}$, $\zeta' \asymp \min(\rho, \frac{1}{m}\rho^2)$. Let us set $\zeta = \zeta'$ in X_{reg} and $\zeta = \eta\zeta''$ in $X_{\mathsf{sing}} \setminus X_{\mathsf{reg}}$ where $\zeta'' \in \mathscr{S}(X)$ is a fixed function, $\zeta''(x) > 0$ and $\mathbb{R} \ni \eta \to +0$ in what follows. Let us note that $\zeta \in \mathscr{L}^\infty$ provided $R_1 < \infty$.

Let us choose

$$J_1 = \begin{pmatrix} \zeta I & 0 \\ 0 & \epsilon_2\zeta I \end{pmatrix}, \qquad J_2 = \begin{pmatrix} \epsilon_2\zeta I & 0 \\ 0 & \zeta I \end{pmatrix}$$

and let us take ϕ_1 supported in $[c_0, 2c_0]$ and ϕ_2 supported in $[-2c_0, -c_0]$, where $\epsilon_2 > 0$ is small enough and c_0 is large enough constants, such that

$$(10.2.43)_2 \qquad m - V - \tau_2 + t_1 j + \epsilon_2 t_2 \zeta \ge \frac{\epsilon}{2}\min(\rho, \frac{1}{m}\rho^2),$$

$$(10.2.43)_1 \qquad -m + V - \tau_1 + t_1\epsilon_2 j + t_2\zeta \le -\frac{\epsilon}{2}\min(\rho, \frac{1}{m}\rho^2)$$

$$\text{in } X_{\mathsf{reg}} \qquad \forall t_k \in \mathsf{supp}(\phi_k).$$

Let us apply the arguments of Section 10.1. Then in order to estimate $\mathsf{N}(A, \tau_1, \tau_2)$ from below we need to estimate R_0 in condition (10.1.46); the compactness of the operator $J(A+i)^{-1}$ obviously follows from $R_1 < \infty$ and Proposition 10.2.14.

Further, in order to estimate R_0 we need to estimate $\mathsf{N}(\tilde{A}-\tau_i, -\tilde{J}_1, (-t, t))$ and $\mathsf{N}(\tilde{A}+t\tilde{J}_1-\tau_i, -J_2, (-t, t))$ where Proposition 10.2.13(iii) yields that we can replace $\tilde{\mathsf{N}}$ by N. Here $\tilde{A}$ and $\tilde{J}_k$ mean corresponding operators extended from X_{sing} [18]. Let us consider the first number; one can analyze the second number in the same way. Let us assume for the sake of simplicity that $\tau_i = 0$ (otherwise we replace V by $V - \tau_i$).

We know that $\mathsf{N}(\tilde{A}, -\tilde{J}_1; (-t, t))$ for $t \ge 0$ is the maximal dimension of the subspace $\mathcal{L}' \subset \mathscr{C}_0^1(X, \mathbb{H})$ on which inequality

$$\|\tilde{J}_1^{-\frac{1}{2}}\tilde{A}u\|^2 \le t^2\|\tilde{J}_1^{\frac{1}{2}}u\|^2 \tag{10.2.44}$$

holds. Therefore we should estimate the maximal dimension of the subspace $\mathcal{L} \subset \mathscr{C}_0^1(X, \mathbb{H})$ on which inequality

$$\|\tilde{\zeta}^{-\frac{1}{2}}\tilde{A}u\|^2 \le C_0 t^2\|\tilde{\zeta}^{\frac{1}{2}}u\|^2 \tag{10.2.45}$$

holds where $\tilde{\zeta}$ is an admissible continuation of ζ from X''_{sing} (which is a domain intermediate between X_{sing} and $X_{\mathsf{sing}} \setminus X'_{\mathsf{reg}}$) to $\mathbb{R}^d$ and none of the constants below depend on η.

Let $\tilde{A}$ be an operator of (5.3.71) type:

$$\tilde{A} = \tilde{A}_0 + \tilde{V}, \qquad \tilde{A}_0 = \sum \frac{1}{2}\sigma_l(\omega^{kl}\tilde{P}_j + \tilde{P}_k\omega^{kl}) + \sigma_0(m+W) \tag{10.2.46}$$

where the coefficients ω^{kl} are the same as in A,

$$\tilde{P}_k = D_k - \tilde{V}_k, \quad |\tilde{V}_k| \le c\tilde{\rho}, \quad |\tilde{V}| \le m + c\min(\rho, \frac{1}{m}\rho^2) \tag{10.2.47}$$

and $\tilde{V}_k$, $\tilde{V}$ are supported in X_{sing},

$$W \ge 0, \qquad |D^\alpha W| \le c\rho\gamma^{-|\alpha|} \quad \forall \alpha : |\alpha| \le K, \tag{10.2.48}$$

$W \asymp \rho$ for $\rho \ge cm$ outside of X_{sing} and W vanishes in X'_{sing} and also W vanishes for $\rho \le \epsilon_1 m$.

[18] Or rather from its connected component, and later we sum the results obtained for these components.

(10.2.49) *For notational simplicity in what follows we skip tilde writing* A, P, W *instead of* $\tilde{A}$, $\tilde{P}$, $\tilde{V}$.

Let us consider a partition of unity $1 = \sum_{\nu\ge 0} \psi_\nu^2$ in X_{sing} where ψ_ν are supported in the $\epsilon\gamma$-vicinity of X_{reg} and γ-admissible for $\nu \ge 1$, and where ψ_0 is supported in X_{sing}. Then the left-hand expression in (10.2.45) is equal to

$$2\|\zeta^{-\frac{1}{2}} Au\|^2 \ge 2\sum_{\nu\ge 0} \lambda_\nu \|\psi_\nu Au\|^2 \ge \sum_{\nu\ge 0} \lambda_\nu \|A\psi_\nu u\|^2 - 2\sum_{\nu\ge 0} \lambda_\nu^2 \|[A,\psi_\nu]u\|^2 \tag{10.2.50}$$

provided

$$\zeta\lambda_\nu^2 \le 1 \qquad \text{on} \quad \mathsf{supp}(\psi_\nu). \tag{10.2.51}$$

Note that

$$[A,\psi_\nu] = \sum_{k,l} \sigma_l \omega^{kl} [P_j,\psi_\nu]$$

and therefore

$$[A,\psi_\nu]^*[A,\psi_\nu] \asymp \sum_l |[P_l,\psi_\nu]|^2 \le C|\nabla\psi_\nu|^2$$

and

$$\sum_{\nu\ge 0} \lambda_\nu^2 \|[A,\psi_\nu]u\|^2 \le C\sum_{\nu\ge 0} \lambda_\nu^2 \|(\nabla\psi_\nu)u\|^2 = C\sum_{\mu,\nu\ge 0} \lambda_\mu^2 \|(\nabla\psi_\mu)\psi_\nu u\|^2. \tag{10.2.52}$$

Using the finite multiplicity of the covering we can estimate the sum of terms with $\mu \ge 1, \nu \ge 1$ and $\nu = \mu = 0$ by

$$C\sum_{\nu\ge 0} \lambda_\nu^2 \|\chi_Y \gamma^{-1}\psi_\nu u\|^2 \tag{10.2.53}$$

provided

$$\mathsf{supp}(\psi_\nu) \cap \mathsf{supp}(\psi_\mu) \ne \emptyset \implies \lambda_\nu \le c\lambda_\mu \qquad \forall \mu \ge 1, \nu \ge 1 \tag{10.2.54}$$

where Y is $\epsilon\gamma$-vicinity of X_{reg} intersected with X_{sing} and χ_Y is its characteristic function. However the sum in the right-hand expression of (10.2.52) contains also terms with $\nu = 0$ and $\mu \ge 1$ and inversely and we want to "untangle" these indices. We assume that

$$|\nabla\psi_\nu| \le \bigl(\alpha + \Phi(\alpha)\psi_\nu\bigr)\gamma^{-1} \qquad \text{for } \alpha \in (0,1), \tag{10.2.55}$$

with slowly growing to $+\infty$ function $\Phi(\alpha) \ge 1$ as $\alpha \to +0$. We discuss the possible choice of Φ and its implication later.

Then the sums of the terms with $\nu = 0, \mu \ge 1$ and with $\nu \ge 1, \mu = 0$ in the right-hand expression of (10.2.52) do not exceed

$$C\sum_{\nu\ge 1}\bigl(\lambda_\nu\Phi(\alpha_\nu) + \lambda_0\beta_\nu\bigr)^2\|\chi_Y\gamma^{-1}\psi_\nu u\|^2 + C\sum_{\nu\ge 1}\bigl(\lambda_0\Phi(\beta_\nu) + \lambda_\nu\alpha_\nu\bigr)^2\|\chi_Y\gamma^{-1}\bar\psi_\nu\psi_0 u\|^2 \tag{10.2.56}$$

where $\bar\psi_\nu$ is a characteristic function of $\mathsf{supp}(\psi_\nu)$.

This expression is greater than (10.2.53). Combining with (10.2.50) we conclude that

$$\sum_{\nu\ge 0}\lambda_\nu^2\|A\psi_\nu u\|^2 \le 2\|\zeta^{-\frac12}Au\|^2 + C\sum_{\nu\ge 1}\bigl(\lambda_\nu\Phi(\alpha_\nu) + \lambda_0\beta_\nu\bigr)^2\|\chi_Y\gamma^{-1}\psi_\nu u\|^2 + C\sum_{\nu\ge 1}\bigl(\lambda_0\Phi(\beta_\nu) + \lambda_\nu\alpha_\nu\bigr)^2\|\chi_Y\gamma^{-1}\bar\psi_\nu\psi_0 u\|^2. \tag{10.2.57}$$

This conclusion allows us to "localize" variational problem. Note that

$$\|A_0\psi_\nu u\|^2 = \|A\psi_\nu u\|^2 + \|V\psi_\nu u\|^2 - 2\,\mathsf{Re}(\psi_\nu Au, V\psi_\nu u), \tag{10.2.58}$$

because $[A, \psi_\nu]$ is a skew-Hermitian matrix valued function and V, ψ_ν are real-valued and scalar. Then

$$\|A_0\psi_\nu u\|^2 \le 2\|A\psi_\nu u\|^2 + \|V\psi_\nu u\|^2 + \lambda_\nu^{-2}\|\zeta^{-\frac12}\psi_\nu Au\|^2 + \lambda_\nu^2\|\zeta^{\frac12}V\psi_\nu u\|^2. \tag{10.2.59}$$

Multiplying by λ_ν^2, summing on ν and applying (10.2.57) we conclude that

$$\sum_{\nu\ge 0}\lambda_\nu^2\|A_0\psi_\nu u\|^2 \le 2\|\zeta^{-\frac12}Au\|^2+\sum_{\nu\ge0}\lambda_\nu^2\|V\psi_\nu u\|^2+ C\sum_{\nu\ge0}\lambda_\nu^4\|\zeta^{\frac12}V\psi_\nu u\|^2+C\sum_{\nu\ge1}\bigl(\lambda_\nu\Phi(\alpha_\nu)+\lambda_0\beta_\nu\bigr)^2\|\chi_Y\gamma^{-1}\psi_\nu u\|^2+ C\sum_{\nu\ge1}\bigl(\lambda_0\Phi(\beta_\nu)+\lambda_\nu\alpha_\nu\bigr)^2\|\chi_Y\gamma^{-1}\bar\psi_\nu\psi_0 u\|^2. \tag{10.2.60}$$

and we can replace in the right-hand expression $\|\zeta^{-\frac12}Au\|^2$ by $t^2\sum_{\nu\ge0}\|\zeta^{\frac12}\psi_\nu u\|^2$ provided $\|\zeta^{-\frac12}Au\|\le t\|\zeta^{\frac12}u\|$.

Therefore

(10.2.61) Maximal dimension $\mathsf{N}(t)$ of subspace $\mathcal{L}\subset \mathscr{C}_0^1(X_{\mathsf{sing}})$ on which $\|\zeta^{-\frac12}Au\|\le t\|u\|$ does not exceed the sum of maximal dimensions $\mathsf{N}_\nu(t)$ of subspaces $\mathcal{L}_\nu\subset\mathscr{C}_0^1(\mathsf{supp}(\psi_\nu))$ on which

$$\|A_0u_\nu\|^2\le 2t^2\lambda_\nu^{-2}\|\zeta^{\frac12}u_\nu\|^2+\|V\psi_\nu u\|^2+C\lambda_\nu^2\|\zeta^{\frac12}Vu_\nu\|^2+ C\bigl(\Phi(\alpha_\nu)+\lambda_\nu^{-1}\lambda_0\beta_\nu\bigr)^2\|\chi_Y\gamma^{-1}u_\nu\|^2 \qquad \text{as}\ \ \nu\ge1 \tag{10.2.62}$$

and

$$\|A_0u_0\|^2\le 2t^2\lambda_0^{-2}\|\zeta^{\frac12}u_0\|^2+\|Vu_0\|^2+ C\lambda_0^2\|\zeta^{\frac12}Vu_0\|^2+C\sum_{\nu\ge1}\bigl(\Phi(\beta_\nu)+\lambda_\nu\lambda_0^{-1}\alpha_\nu\bigr)^2\|\chi_Y\gamma^{-1}\bar\psi_\nu u_0\|^2. \tag{10.2.63}$$

One can see easily that

$$A_0^2=\sum_{l,k}P_lg^{lk}P_k+(m+W)^2+\sum_l(\theta^lP_l+P_l\theta^l)+\theta \tag{10.2.64}$$

where θ^l,θ are Hermitian matrices such that $|\theta^l|\le C\rho$, $|\theta|\le C\tilde\rho^2+C\rho\gamma^{-1}\chi_Y$. Therefore

$$\|A_0v\|^2\ge \epsilon_1\sum\|P_jv\|^2+\|(m+W)v\|^2-C\|\tilde\rho v\|^2-C\|\rho^{\frac12}\gamma^{-\frac12}\chi_Yv\|^2 \tag{10.2.65}$$

with $\tilde\rho$ supported in X_{sing} and due to (10.2.47)

(10.2.66) Estimate (10.2.65) remains true with P_j replaced by D_j in the right-hand expression.

Then on $\mathcal{L}_\nu$ with $\nu \ge 1$

$$\epsilon_1\|\nabla v\|^2 \le 2t^2\lambda_\nu^{-2}\|\zeta^{\frac12}v\|^2 + C\big(\Phi(\alpha_\nu)+\lambda_\nu^{-1}\lambda_0\beta_\nu\big)^2\|\chi_Y\gamma^{-1}v\|^2 + C\lambda_\nu^2\|\zeta^{\frac12}Vv\|^2 + C\|\tilde\rho v\|^2 + C\|\chi_Y\rho^{\frac12}\gamma^{-\frac12}v\|^2 \tag{10.2.67}$$

where originally we had $\|(m+W)u_\nu\|^2$ and $\|Vu\|^2$ in the left- and right-hand expressions respectively and used that $V^2 \le m^2 + c\tilde\rho^2$, $W \ge 0$.

Let us apply Rozenblioum variational estimate (9.A.11) . As $d \ge 3$ we conclude that for $\nu \ge 1$

$$\mathsf{N}_\nu(t) \le C\int \Big(t^d\lambda_\nu^{-d}\zeta^{\frac d2} + \lambda_\nu^d\zeta^{\frac d2}|V|^d + \tilde\rho^d + \rho^{\frac d2}\gamma^{-\frac d2} + \big(\Phi(\alpha_\nu)+\lambda_\nu^{-1}\lambda_0\beta_\nu\big)^d\gamma^{-d}\Big)\bar\psi_\nu\,dx. \tag{10.2.68}$$

Similarly starting from (10.2.63) instead of (10.2.62) we conclude that

$$\mathsf{N}_0(t) \le C\int \Big(t^d\lambda_0^{-d}\zeta^{\frac d2} + \lambda_0^d\zeta^{\frac d2}|V|^d + \tilde\rho^d + \rho^{\frac d2}\gamma^{-\frac d2}\chi_Y\Big)\bar\psi_0\,dx + C\sum_{\nu\ge1}\int \big(\Phi(\beta_\nu)+\lambda_0^{-1}\lambda_\nu\alpha_\nu\big)^d\gamma^{-d}\bar\psi_\nu\bar\psi_0\,dx. \tag{10.2.69}$$

Then

$$\mathsf{N}(t) \le \sum_{\nu\ge0}\mathsf{N}_\nu \le C(t^d+1)R \tag{10.2.70}$$

with

$$R = \int \Big(\lambda_0^{-d}\zeta^{\frac d2} + \lambda_0^d\zeta^{\frac d2}|V|^d + \tilde\rho^d + \rho^{\frac d2}\gamma^{-\frac d2}\chi_Y\Big)\bar\psi_0\,dx + \sum_{\nu\ge1}\int \Big(\lambda_\nu^{-d}\zeta^{\frac d2} + \lambda_\nu^d\zeta^{\frac d2}|V|^d + \tilde\rho^d + \rho^{\frac d2}\gamma^{-\frac d2}\chi_Y + \big(\Phi(\alpha_\nu)+\lambda_\nu^{-1}\lambda_0\beta_\nu + \Phi(\beta_\nu)+\lambda_0^{-1}\lambda_\nu\alpha_\nu\big)^d\gamma^{-d}\Big)\bar\psi_\nu\,dx. \tag{10.2.71}$$

Our purpose now is to make (10.2.71) as small as possible in the framework of our construction. First of all recall that we can make ζ on $\bar\psi_0$ arbitrarily small function without affecting estimates; so we can skip first two terms in the first line. Since Φ is slowly growing at 0 almost optimal choice is $\alpha_\nu = \min(\lambda_0\lambda_\nu^{-1}, 1)$ and $\beta_\nu = \min(\lambda_0^{-1}\lambda_\nu, 1)$ and (10.2.71) becomes

$$R = \int \Big(\tilde\rho^d + \rho^{\frac{d}{2}}\gamma^{-\frac{d}{2}}\chi_Y\Big)\,dx + \sum_{\nu\ge 1}\int \Big(\lambda_\nu^{-d}\zeta^{\frac{d}{2}} + \lambda_\nu^d\zeta^{\frac{d}{2}}|V|^d + \bar\Phi(\lambda_0^{-1}\lambda_\nu)^d\gamma^{-d}\Big)\bar\psi_\nu\,dx$$

with

$$\bar\Phi(t) = \Phi\big(\min(t, t^{-1})\big). \tag{10.2.72}$$

Estimating $|V| \le c(m+\rho)$, $\zeta \le c\rho^2(m+\rho)^{-1}$ and selecting $\lambda_\nu \asymp (m+\rho)^{-\frac12}$ on $\operatorname{supp}(\psi_\nu)$ to optimize two first terms in the second line here we arrive to

$$R = \int \big(\tilde\rho^d + \rho^d\chi_Y\big)\,dx + \int \bar\Phi\big(\lambda_0(\rho+m)^{\frac12}\big)^d\gamma^{-d}\chi_Y\,dx. \tag{10.2.73}$$

Estimates for a Number of Crossing Eigenvalues (End)

Let us discuss the possible choice of Φ. Note that if ϕ_ν^2 is some partition then partition ψ_ν^2 with $\psi_\nu = \varphi(\phi_\nu)\big(\sum\varphi(\phi_\nu)^2\big)^{-\frac12}$ satisfies (10.2.55) provided

$$\varphi(t) = 0 \text{ for } t<0, \quad \varphi>0 \text{ for } t>0, \quad |\frac{d}{dt}\varphi(t)| \le c \tag{10.2.74}$$

and

$$|\frac{d}{dt}\varphi(t)| \le \alpha + \Phi(\alpha)\varphi(t) \qquad \forall \alpha, t\in(0,1). \tag{10.2.75}$$

Remark 10.2.17. One can prove easily that function φ satisfying (10.2.74)–(10.2.75) exists provided

$$\int_0^1 \frac{d\beta}{\min_{\alpha>0}(\alpha+\Phi(\alpha)\beta)} < \infty. \tag{10.2.76}$$

Therefore one can take $\Phi(\alpha) = C_\delta\alpha^{-2\delta}$ with arbitrarily small exponent $\delta>0$ in which case the second term in (10.2.73) becomes

$$\lambda_0^{2\delta}\int(\rho+m)^\delta\chi_Y\gamma^{-d}\,dx + \lambda_0^{-2\delta}\int(\rho+m)^{-\delta}\chi_Y\gamma^{-d}\,dx.$$

Optimizing by λ_0 we arrive to term R_4 defined by (10.1.67).

However, one can pick $\Phi(\alpha)$ $\Phi(\alpha) = \log^{1+\delta}\alpha^{-1}$ or even

$$\Phi(\alpha) = \log\alpha^{-1}\cdot \log_{(2)}\alpha^{-1}\cdots \log_{(n)}\alpha^{-1}\cdot \log^{1+\delta}_{(n+1)}\alpha^{-1}$$

with arbitrarily small exponent $\delta > 0$, arbitrary n and $\log_{(n)}$ denoting the n-tuple logarithm (as $\alpha \ll 1$) which implies

Remark 10.2.18. If

$$c^{-1}\lambda \le \rho + m \le c\lambda \qquad \text{in } \ X_{\mathsf{reg}}\cap X_{\mathsf{sing}} \tag{10.2.77}$$

with some parameter λ then the second term optimizes to $C\int \chi_Y \gamma^{-d}\,dx$ which is less than the first term.

Further, if X_{sing} consists of several connected components $X_{\mathsf{sing},l}$ then one can use different parameters $\lambda = \lambda_l$ in $X_{\mathsf{reg}}\cap X_{\mathsf{sing},l}$.

Remark 10.2.19. For $d = 2$ we should slightly modify Rozenblioum estimate; however this modification affects only term with $\nu = 0$ and only contribution of $X_{\mathsf{sing}}\setminus X_{\mathsf{reg}}$. However, applying the same arguments as in Appendix *9.A.1.2 Bracketing* we see that one needs only to replace the first term in (10.2.73) by (10.1.66).

Further, if $d = 2$ and X_{sing} consists of several connected components one can instead calculate R_3 for each of them and then add results.

Remark 10.2.20. Let us recall that deriving (10.2.67) we used estimate $(m+W)^2 - V^2 \le c\rho^2$ in X_{reg}. However in $X_{\mathsf{reg,ell}}$ one can replace estimate by $(m+W)^2 - V^2 \le c - \epsilon\rho^2$ and, moreover, from the very beginning one can take $\zeta = \rho\gamma^{-1}m^{-1}$ here and therefore one can get slightly better remainder estimate than (10.2.73): from integral of ρ^d is excluded $X_{\mathsf{reg,ell}}\cap\{\rho\gamma \ge C_\epsilon\}$ and from integral of $\bar{\Phi}\big(\lambda_0(\rho+m)^{\frac12}\big)^d\gamma^{-d}$ is excluded $X_{\mathsf{reg,ell}}\cap\{\rho\gamma \ge C_\epsilon\}$ where $\bar{\Phi}\big(\lambda_0(\rho+m)^{\frac12}\big) \le \epsilon\rho\gamma$.

Estimates for a Number of Remaining Eigenvalues

Thus, we derived asymptotics for terms counting eigenvalues crossing τ_1 and τ_2 i.e. the last line in the right-hand expression of (10.1.18). The main part of the result is given by Weyl formula (cut outside of X_{reg}) and the remainder is estimated.

In order to prove the upper estimate (10.1.65) we need also to derive an upper estimate for $\mathsf{N}(B;\tau_1,\tau_2)$ which is the second line in (10.1.18) where here and in what follows

$$B = A - t_1 J_1 - \dots - t_n J_n. \tag{10.2.78}$$

So B is a generalized Dirac operator (5.3.71) with $J = I$ satisfying all the conditions of Subsection 5.3.2 with the following modifications: in conditions on $D^\alpha V$, $D^\alpha W$ with $|\alpha| \ge 1$ there is an additional factor $\bar{t}$ in the right-hand expression, conditions on V (without derivatives) remain true and

$$|V| \le \bigl(m - \bar{t}\,\mathsf{min}(\rho, \frac{1}{m}\rho^2)\bigr)_+ + \mathsf{min}(\rho, \frac{1}{m}\rho^2) \qquad \text{in } X'_{\mathsf{reg}} \tag{10.2.79}$$

$$\bar{t}\,\mathsf{min}(\rho, \frac{1}{m}\rho^2) > m \implies W \ge \bar{t}\,\mathsf{min}(\rho, \frac{1}{m}\rho^2) \qquad \text{in } X'_{\mathsf{reg}}, \tag{10.2.80}$$

$$c\bar{t}\,\mathsf{min}(\rho, \frac{1}{m}\rho^2) \ge W \ge 0 \qquad \text{in } X \tag{10.2.81}$$

where $\bar{t}$ is arbitrarily large and c does not depend on $\bar{t}$, X'_{reg} is the $\epsilon\gamma$-neighborhood of $X_{\mathsf{reg}} \setminus X_{\mathsf{sing}}$; however we can still denote it by X_{reg} without breaking estimates. All these conditions are provided by an appropriate choice of $J_1, \dots, J_n$ and $\phi_1, \dots, \phi_n$. Also in X_{sing} one should replace ρ by $\tilde{\rho}$.

Note that conditions (10.2.79)–(10.2.79) mean that ellipticity condition in X_{reg} is fulfilled. More precisely:

$$\tau_1 - \frac{k}{m}\rho^2 + m \ge V \ge \tau_2 - m + \frac{k}{m}\rho^2 \qquad \text{as } \ m \ge k_1\rho, \tag{10.2.82}$$

$$W \ge k\rho \qquad \text{as } \ m \le k_1\rho, \tag{10.2.83}$$

and thus in both cases

$$\tau_1 - k\,\mathsf{min}(\rho, \frac{1}{m}\rho^2) + m + W \ge V \ge \tau_2 - m - W + k\,\mathsf{min}(\rho, \frac{1}{m}\rho^2). \tag{10.2.84}$$

Let us divide connected components of X_{sing} into two types. Components with $\tilde{\rho} \le cm$ at every point are components of the first type.

Remark 10.2.21. Note that we could choose $J_1, \dots, J_n$ such that conditions (10.2.79)–(10.2.81) are also fulfilled on components of the first type but with ρ replaced by $\tilde{\rho}$ and all estimates of the previous subsubsection remain true.

One can see easily that $\mathsf{N}(B;(\tau_1,\tau_2))$ equals the maximal dimension of the subspace $\mathcal{L}\subset \mathscr{C}_0^1(X,\mathbb{H})$ on which

$$\|(B-\bar\tau)u\|^2 < \tau^2\|u\|^2 \qquad (u\neq 0) \tag{10.2.85}$$

where $\bar\tau = \frac{1}{2}(\tau_2+\tau_1)$, $\tau = \frac{1}{2}(\tau_2-\tau_1)$.

Let $\{\psi_\nu^2\}_{k=1,2}$ be a partition of unity such that $|\nabla\psi_\nu|\le c\gamma^{-1}$ and ψ_ν be supported in X_ν where X_1 is the union of X_{reg} and all components of X_{sing} the first type and X_2 is the union of all components of X_{sing} of the second type.

Then one can see easily that

$$\sum_\nu \|(B-\bar\tau)\psi_\nu u\|^2 = \|(B-\bar\tau)u\|^2 + \sum_\nu \|[B,\psi_\nu]u\|^2, \tag{10.2.86}$$

$$\|[B,\psi_\nu]u\| \le C_0\|\gamma^{-1}\chi_Y u\| \tag{10.2.87}$$

where χ_Y is the characteristic function of $Y = X_{\mathsf{reg}}\cap X_{\mathsf{sing}}$.

Therefore

(10.2.88) $\dim\mathcal{L}$ does not exceed $\max\dim\mathcal{L}_1 + \max\dim\mathcal{L}_2$ where $\mathcal{L}_\nu$ is a subspace of $\mathscr{C}_0^1(X_\nu,\mathbb{H})$ on which inequality

$$\|(B-\bar\tau)u\|^2 < \tau^2\|u\|^2 + C_0\|\gamma^{-1}\chi_Y u\|^2 \quad (u\neq 0) \tag{10.2.89}$$

is fulfilled.

Obviously there exists an extension B' of B from X_1 to X such that conditions (10.2.79)–(10.2.81) are fulfilled on X. Then $\dim\mathcal{L}_1$ does not exceed the cardinality of the spectrum of the problem

$$(B'-\bar\tau)v + \lambda Jv = 0 \qquad v\in\mathfrak{D}(B') \tag{10.2.90}$$

in the interval $(-1,1)$ where $J = (\tau^2 + c\gamma^{-2}\chi_Y)^{\frac{1}{2}}$ and $J\in\mathscr{L}^\infty$ provided the right-hand expression of the estimate in question is finite.

Dividing v into $\frac{1}{2}\mathsf{D}$-dimensional components v_1 and v_2 we can rewrite operator $B'-\bar\tau+\tau J$ in the form

$$B'-\bar\tau+\tau J = \begin{pmatrix} \zeta_1 & T\\ T^* & -\zeta_2\end{pmatrix}$$

where conditions (10.2.79)–(10.2.81) fulfilled in X yield that

$$\zeta_j > \epsilon_0(1-|\tau|),\ |\tau|\le 1 \qquad \forall j = 1,2$$

provided $\bar{t}$ is large enough. Then $(-1,1)$ does not intersect the spectrum of the problem (10.2.90) and therefore $\mathsf{dim}\,\mathcal{L}_1 = 0$.

Let us estimate $\mathsf{dim}\,\mathcal{L}_2$. Let us note that the conditions on V can be rewritten in the form

$$\tau - m - c\,\mathsf{min}(\tilde{\rho}, \frac{1}{m}\tilde{\rho}^2) \le V - \bar{\tau} \le m - \tau + c\,\mathsf{min}(\tilde{\rho}, \frac{1}{m}\tilde{\rho}^2). \tag{10.2.91}$$

In particular

$$\tau \ge m \implies |V-\tau| + |\tau - m| \le c\,\mathsf{min}(\tilde{\rho}, \frac{1}{m}\tilde{\rho}^2). \tag{10.2.92}$$

Let us consider operator $B_0 = B - V\cdot I - \sigma_0 W$. Standard arguments and assumptions (10.1.69), (10.1.70) easily yield that

$$\|B_0 u\|^2 \ge \epsilon_1 \|Du\|^2 + m^2\|u\|^2 - C\|\tilde{\rho}u\|^2 \tag{10.2.93}$$

On the other hand, on $\mathcal{L}_2$ estimates

$$\begin{aligned}\|B_0u\|^2 \le (1+\varepsilon^{-2})\|(B-\bar{\tau})u\|^2 + (1+\varepsilon^2)\|(V+\sigma_0 W - \bar{\tau})u\|^2 \le \\ \big((1+\varepsilon^{-2})\tau^2 + (1+\varepsilon^2)(m-\tau)^2\big)\|u\|^2 + \\ C(1+\varepsilon^{-2})\|\chi\gamma^{-1}u\|^2 + C(1+\varepsilon^2)\|\tilde{\rho}u\|^2\end{aligned}$$

are fulfilled where we have applied (10.2.91) and the restrictions on W. These inequalities, (10.2.93) and (10.1.69) yield that

$$\|Du\|^2 \le C(\mathcal{K}u, u) \qquad \text{on} \quad \mathcal{L}_2 \tag{10.2.94}$$

where

$$\mathcal{K} = 9\big((m-\tau)\varepsilon - \varepsilon^{-1}\tau\big)^2 + \tilde{\rho}^2(1+\varepsilon^2) + \chi\gamma^{-2}(1+\varepsilon^{-2}). \tag{10.2.95}$$

If $\tau \ge 2m$ then (10.2.91) yield that $m+\tau \le c\tilde{\rho}$ in X_2 and therefore for $\varepsilon = 1$ one can skip the first and the third terms in $\mathcal{K}$; then Rozenblioum estimate yields that $\mathsf{dim}\,\mathcal{L}_2$ does not exceed CR_3 defined by (10.1.66); as $d=2$ a standard trick with an extra partition is needed to get $\int \rho^d\,dx$ rather than the same expression as with $\tilde{\rho}$.

Let us consider the case $\tau \le 2m$. One can assume without any loss of the generality that $\tau_2 - \tau_1 \le \frac{1}{2}m$; otherwise one can divide the interval (τ_1, τ_2) into fewer than 10 intervals (τ_1', τ_2') with $\frac{1}{4}m \le |\tau_1' - \tau_2'| \le \frac{1}{2}m$. Moreover, if $\tau_2 - \tau_1 \ge \frac{1}{4}m$ then the value of $\varepsilon > 0$ for which the first term in $\mathcal{K}$ vanishes is disjoint from 0 and ∞ and in this case estimate $\mathsf{dim}\, \mathcal{L}_2 \le CR_3$ remains true.

Let us now consider the case $\tau \le \frac{1}{4}m$. Then Rozenblioum estimate yields

$$\mathsf{dim}\, \mathcal{L}_2 \le C\Big(\big((m-\tau)\varepsilon - \varepsilon^{-1}\tau\big)^2 + a^2\varepsilon^2 + b^2\varepsilon^{-2}\Big)^{\frac{d}{2}}(\mathsf{mes}(X_2)) + CR_3 \tag{10.2.96}$$

with

$$a = \begin{cases} \Big(\dfrac{1}{\mathsf{mes}(X_2)}\displaystyle\int_{X_2} \rho^d\, dx\Big)^{\frac{1}{d}} & (d \ge 3), \\ \Big(\dfrac{1}{\mathsf{mes}(X_2)}\displaystyle\int_{X_2} \rho^{2p}\, dx\Big)^{\frac{1}{2p}} & (d = 2, p > 1), \end{cases} \tag{10.2.97}$$

$$b = \Big(\frac{1}{\mathsf{mes}(X_2)}\int_{X_2} \chi\gamma^{-d}\, dx\Big)^{\frac{1}{d}} \qquad (d \ge 2) \tag{10.2.98}$$

(for $d = 2$ the procedure of covering of $X_1 \cap X_2$ by balls $B(x_n, \epsilon\gamma(x_n))$ is applied; see above).

Optimizing with respect to ε we obtain from (10.2.96) the final estimate

$$\mathsf{dim}\, \mathcal{L}_2 \le CR_3 + CR_5 \tag{10.2.99}$$

with R_5 defined by (10.1.68).

For $\tau \ge \epsilon_0 m$ this estimate coincides with (10.2.96).

Theorem 10.1.8 has been proven.

Remark 10.2.22. The contribution of X_2 to the remainder in Theorem 10.1.8 is estimated reasonably as $\tilde\rho \ge cm$ everywhere on X_2. One can get better estimates using Birman-Schwinger principle while reducing $\tilde\rho$ and estimating number of "escaping" eigenvalues. Further development in this direction is left to the reader.

10.2.5 From Rough to Irregular Operators

Remark 10.2.23. (i) While we formulated our results for smooth operators, the same results hold for rough operators (in the assumptions of the corresponding theorems).

(ii) Approximation of operators with irregular coefficients by rough operators is more subtle in the case when we calculate $\mathsf{N}(A, \tau_1, \tau_2)$ because in contrast to the case $\mathsf{N}^-(A-\tau)$ simple bracketing does not work here.

However if we know that $A' - \eta < A < A' + \eta$ [19] we conclude that

$$\mathsf{N}(A', \tau_1 + \eta, \tau_2 - \eta) \le \mathsf{N}(A, \tau_1, \tau_2) \le \mathsf{N}(A', \tau_1 - \eta, \tau_2 + \eta) \tag{10.2.100}$$

and therefore

$$\begin{aligned} -\mathsf{N}(A', \tau_1, \tau_1 + \eta) - \mathsf{N}(A', \tau_2 - \eta, \tau_2) \le \\ \mathsf{N}(A, \tau_1, \tau_2) - \mathsf{N}(A', \tau_1, \tau_2) \le \\ \mathsf{N}(A', \tau_1 - \eta, \tau_1) + \mathsf{N}(A', \tau_2, \tau_2 + \eta) \end{aligned} \tag{10.2.101}$$

and this conclusion could be applied either directly or through Birman-Schwinger principle.

Comments

Due to its nature and to the fact that M. Sh. Birman for many years has the leader of the school using variational approach to eigenvalue asymptotics Birman-Schwinger principle has been used mainly in such papers. One should mention many papers of M. Sh. Birman and M. Z. Solomyak and their students: G. Rozenblioum, Yu. Safarov, A. Sobolev and many others.

Extremely surprising is that this powerful method had not find its way to the papers studying sharp asymptotics much sooner; at least the author is not aware of any such papers prior to his own. Furthermore, it took some time and advice of M. Z. Solomyak to realize the tremendous power of Birman-Schwinger principle which allows to analyze often very easily many asymptotics which either could not be analyzed directly or at least require some difficult and ingenious arguments. With Birman-Schwinger principle

[19] Here A is an original operator with the irregular coefficients, A' is its rough approximation and $\eta > 0$ is a small parameter.

we just reduce asymptotics with respect to some parameters (f.e. spectral parameter) to asymptotics with respect to others, most notably semiclassical parameter.

In the framework of this chapter this method was first used by V.Ivrii [17]. We will use Birman-Schwinger principle systematically either directly or through Chapters 9 and 10.

Part V

Asymptotics of Spectra

Chapter 11

Weyl Asymptotics of Spectra

This chapter is devoted to applications of the results of Parts II, III and IV to the eigenvalue asymptotics. Now we consider operator A depending on parameters; then all the functions $\rho, \tilde{\rho}, \varrho,$ and γ with different indices and zones X_{reg} and X_{sing} depend on these parameters as well. Further, the main part of the asymptotics N^{W} (given by Weyl formula but may be with corrections) and the remainder R depend on this parameter as well; both of them tend to infinity but under appropriate assumptions $R \ll \mathsf{N}^{\mathsf{W}}$. Further, as parameters tend to their limits X_{sing} shrinks.

In Section 11.1 we treat semiclassical asymptotics. In this case we consider operator $A_h = A(x, hD, h)$ and select $\rho_h = h^{-1}\rho$, $\gamma_h = \gamma$ and $h_{\mathsf{eff}} = (\rho_h\gamma_h)^{-1} = h\rho^{-1}\gamma^{-1}$; then $X_{\mathsf{reg},h} = \{\rho_h\gamma_h \ge 1\} = \{x : \rho\gamma \ge h\}$ and $X_{\mathsf{sing}} = \{\rho\gamma \le 2h\}$.

In Sections 11.2–11.6 we consider asymptotics with respect to the spectral parameter τ; in this case $A_\tau = A - \tau$ and usually $\rho_\tau = |\tau|^{1/m} + \rho$, $\gamma_\tau = \gamma$ as we assume that A satisfies some conditions with functions ρ, γ and $\varrho = \rho^m$. Then $h_{\mathsf{eff}} = \big(|\tau|^{1/m} + \rho\big)^{-1}\gamma^{-1}$ and $X_{\mathsf{reg},\tau} = \{\big(|\tau|^{1/m} + \rho\big)\gamma \ge 1\}$ and $X_{\mathsf{sing},\tau} = \{\big(|\tau|^{1/m} + \rho\big)\gamma \le 2\}$.

More precisely, in Sections 11.2–11.5 we consider asymptotics of the eigenvalues tending to infinity, then $\tau \to +\infty$ but there are different scenarios:

- in Section 11.2 domain X is "essentially" bounded (i.e., exits to infinity are thin enough) and the coefficients are weakly singular;
- in Section 11.4 the coefficients are strongly singular and the coercivity condition is fulfilled but the domain is still "essentially" bounded;

- in Section 11.5 the domain is "very unbounded" and the spectrum is discrete only because the lower order terms are coercive. In particular, we treat the Schrödinger operator with potential V tending to $+\infty$ at infinity and with the spectral parameter τ tending to $+\infty$, usually like $|x|^{2\mu}$ with $\mu > 0$ and the growths of both $\mathsf{N}^{\mathsf{W}}_{\tau}$ and R_{τ} depend on μ.

Further, in Section 11.6 we treat asymptotics of the eigenvalues tending to the boundary of the essential spectrum. In particular, we treat the Schrödinger and Dirac operators with potentials tending to 0 at infinity, usually like $|x|^{2\mu}$ with $\mu \in (-1, 0)$ and the spectral parameter τ tending to -0 and $M - 0$ or $-M + 0$ respectively (where M is the mass in the case of the Dirac operator). Then the rates of the growth of both $\mathsf{N}^{\mathsf{W}}_{\tau}$ and R_{τ} depend on μ.

Section 11.7 is devoted to the multiparameter asymptotics; for example, there is the semiclassical parameter $h \to +0$ and the spectral parameter τ tending either to $+\infty$ or $-\infty$ (for the Schrödinger and Dirac operators in bounded domains or for the Schrödinger operator in unbounded domains), or to -0, $M - 0$ or $-M + 0$ (for the Schrödinger and Dirac operators in unbounded domains), or to $\inf V + 0$, $\inf V + M + 0$, $\sup V - M - 0$ (for the Schrödinger and Dirac operators) where for the Schrödinger operator $\inf V$ is not necessarily finite. Moreover, the case of a vector semiclassical parameter $\boldsymbol{h} = (h_1, \dots, h_d)$ is treated as well.

For semi-bounded operators, Riesz means are also treated.

Finally, in Section 11.8 we consider asymptotics of the spectrum of the negative order pseudodifferential operators tending to 0. While these operators are not exactly covered by the general theory and require special analysis, some special cases (like $\Delta^{-m} V$) are and the general results are the same.

For simplicity we assume the sufficient smoothness but the reader can generalize easily these results applying results for rough and irregular operators.

11.1 Semiclassical Eigenvalue Asymptotics

In this section the operator is fixed and only the semiclassical parameter $h \to +0$ varies.

11.1.1 General Analysis

Standard Remainder Estimates

Let us consider first operators which are semi-bounded from below and apply the results of Chapter 9 derived on the basis of the results of Parts II and III. This means that $\tau_1 = -\infty$ and without any loss of the generality one can assume that $\tau_2 = 0$.

Let us consider operator

$$A_h = \sum_{|\alpha|,|\beta|\le m} (hD)^\alpha a_{\alpha,\beta}(hD)^\beta \tag{11.1.1}$$

with coefficients satisfying

$$a_{\alpha,\beta} = a^\dagger_{\beta,\alpha}, \tag{11.1.2}$$

$$\|D^\sigma a_{\alpha,\beta}\| \le c\varrho\rho^{-|\alpha|-|\beta|}\gamma^{-|\sigma|} \qquad \forall \alpha,\beta : |\alpha| \le m, |\beta| \le m\ \forall \sigma : |\sigma| \le K \tag{11.1.3}$$

and

$$\sum_{|\alpha|=|\beta|=m} \langle a_{\alpha,\beta} v_\alpha, v_\beta\rangle \ge \epsilon_0 \varrho\rho^{-2m} \sum_{|\alpha|=m} \|v_\alpha\|^2 \qquad \forall v = (v_\alpha)_{|\alpha|=m} \quad \forall x : \rho\gamma \le h_0 \tag{11.1.4}$$

where scaling function γ satisfies $(9.1.6)_1$ on X and

$$\mathsf{mes}(\mathcal{X}_{\mathsf{sing}}) = 0 \qquad \text{where} \quad \mathcal{X}_{\mathsf{sing}} = \{x, \gamma(x) = 0\}, \tag{11.1.5}$$

functions ρ, ϱ satisfying $(9.1.6)_2$ and $(9.1.6)_3$ are defined in $\mathcal{X}_{\mathsf{reg}} = X \setminus \mathcal{X}_{\mathsf{sing}}$, and $h_0 = \mathsf{const} > 0$.

The sets $\mathcal{X}_{\mathsf{reg}}$ and $\mathcal{X}_{\mathsf{sing}}$ are called *regular* and *singular sets* respectively. These sets do not coincide with but are obviously related to the *regular* $X_{\mathsf{reg},h} = \{x, \rho\gamma \ge h\}$ and *singular zones* $X_{\mathsf{sing},h} = \{x, \rho\gamma \le 2h\}$ and respectively: $X_{\mathsf{reg},h} \subset \mathcal{X}_{\mathsf{reg}}$ and expands to it, $X_{\mathsf{sing},h} \supset \mathcal{X}_{\mathsf{sing}}$ and shrinks to it as $h \to +0$.

In particular, for the Schrödinger operator

$$A = \sum_{1\le j,k\le d} (hD_j - V_j) g^{jk} (hD_k - V_k) + V \tag{11.1.6}$$

with real-valued functions $g^{jk} = g^{kj}$, V_j and V we always take $\varrho = \rho^2$ and conditions (11.1.3)–(11.1.4) become

$$|D^\alpha g^{jk}| \le c\gamma^{-|\alpha|}, \quad |D^\alpha V_j| \le c\rho\gamma^{-|\alpha|}, \quad |D^\alpha V| \le c\rho^2\gamma^{-|\alpha|} \tag{11.1.7}$$

and

$$\epsilon_0|\xi|^2 \le \sum_{j,k} g^{jk}\xi_j\xi_k \le c|\xi|^2 \qquad \forall \xi \in \mathbb{R}^d. \tag{11.1.8}$$

Moreover, we assume that conditions (9.1.15) (the boundary regularity condition) are fulfilled in $\mathcal{X}_{\mathsf{reg}}$. For the sake of simplicity we assume that the Dirichlet boundary conditions are given in which case the ellipticity condition for the boundary value problem is fulfilled automatically. However later we discuss Neumann-like problems as well.

Then one can apply Theorem 9.1.7 with $X_{\mathsf{reg}} = X_{\mathsf{reg},h} = \{x \in X, \rho\gamma > h\}$ and $X_{\mathsf{sing}} = X_{\mathsf{sing},h} = \{x \in X, \rho\gamma \le 2h\}$ provided $h \le h_0$. Here conditions (11.1.2)–(11.1.4) yield the following estimate on X_{sing}:

$$(A_h u, u) \ge \epsilon_0 \int \varrho\Big(h^{2m}\rho^{-2m} \sum_{|\alpha|=m} |D^\alpha u|^2 - C_0 \sum_{|\alpha|\le m-1} h^{2|\alpha|}\rho^{-2|\alpha|}|D^\alpha u|^2\Big)\, dx \tag{11.1.9}$$

for all compactly supported functions $u \in \mathscr{C}^{2m}(\bar{X}_{\mathsf{sing}}, \mathbb{H})$ satisfying the Dirichlet boundary condition. Therefore Rozenblioum estimate (9.A.11) yields that the contribution of the zone X_{sing} to the upper estimate of N_h^- in Theorem 9.1.7 does not exceed CR_0h^{-d} with

$$R_0 = C\Big(\int_{X_{\mathsf{sing}}} \varrho^{-p}\rho^{2mp}\, dx\Big)^{\frac{d}{2mp}} \Big(\int_{X_{\mathsf{sing}}} \varrho^{q}\, dx\Big)^{\frac{d}{2mq}} \Big(\mathsf{mes}(X_{\mathsf{sing}})\Big)^{1-\frac{d}{2m}(\frac{1}{p}+\frac{1}{q})} \tag{11.1.10}$$

where p, q are arbitrarily chosen numbers such that

$$\frac{1}{p} + \frac{1}{q} \le \frac{2m}{d}, \qquad p \ge 1, q \ge 1 \tag{11.1.11}$$

and

$$\text{either} \quad \frac{1}{p}+\frac{1}{q}<\frac{2m}{d}, \tag{11.1.12}$$

$$\text{or} \quad \frac{1}{p}+\frac{1}{q}=\frac{2m}{d}, d\geq m \text{ and } d>m \implies q>1$$

$$\text{and } p>1 \implies 2m<\frac{d}{p}(p+1).$$

Recall that X_{sing} does not contribute to the lower estimate of N_h^-.

At the same time the contribution of the zone X_{reg} to both lower and upper estimates does not exceed $Ch^{1-d}R_1 + Ch^{s-d}R_2$ where

$$R_1 = \int_{X_{\mathsf{reg},h}\setminus X_{\mathsf{reg,ell},h}} \rho^{d-1}\gamma^{-1}\,dx, \tag{11.1.13}$$

$$R_2 = \int_{X_{\mathsf{reg},h}} \rho^{d-s}\gamma^{-s}\,dx \tag{11.1.14}$$

with arbitrarily chosen s and we assume that in every ball $\mathcal{B}\big(y,\gamma(y),X\big)$ with $y\in X_{\mathsf{reg}}$ one of the conditions of Parts II and III providing remainder estimate $O(h_{\mathsf{eff}}^{1-d})$ [1)] is fulfilled; here $h_{\mathsf{eff}}=h(\rho\gamma)^{-1}$.

Further, the principal term of the upper and lower estimates is $\mathsf{N}_h^{\mathsf{W}} = h^{-d}\mathsf{N}^{\mathsf{W}}$ with the *cut Weyl expression*

$$\mathsf{N}^{\mathsf{W}} = (2\pi)^{-d}\int_{T^*X_{\mathsf{reg},h}} \boldsymbol{n}(x,\xi,-\infty,0)\,dxd\xi \tag{11.1.15}$$

where $\boldsymbol{n}(x,\xi,.,.)$ is the eigenvalue counting function for the symbol

$$a(x,\xi)=\sum_{\alpha,\beta} a_{\alpha,\beta}\xi^{\alpha+\beta}; \tag{11.1.16}$$

recall that both X_{reg} and X_{sing} and thus N^{W} depend on h.

So the following statement is proven:

Theorem 11.1.1. *Let A be a self-adjoint operator (with the Dirichlet boundary conditions) of the form* (11.1.1) *in $\mathscr{L}^2(X,\mathbb{H})$ and let conditions* (11.1.2)–(11.1.5) *be fulfilled. Moreover, let us assume that, in every ball $\mathscr{B}\big(y,\gamma(y),X\big)$*

[1)] To be discussed in Remarks 11.1.3 and 11.1.4.

with $y \in \mathcal{X}_{\mathsf{reg}}$, one of the conditions of Parts II and III providing remainder estimate $O(h_{\mathsf{eff}}^{1-d})$ [1)] is fulfilled. Then the estimates

$$h^{-d}\mathsf{N}^{\mathsf{W}} - Ch^{1-d}R_1 \le \mathsf{N}^{-}(A_h) \le h^{-d}\mathsf{N}^{\mathsf{W}} + Ch^{1-d}R_1 + Ch^{s-d}R_2 + CR_0h^{-d} \tag{11.1.17}$$

hold with N^{W}, R_1, R_2 and R'' given by (11.1.15), (11.1.13), (11.1.14) *and* (11.1.10) *respectively.*

Corollary 11.1.2. *(i) For N^{W} in formula* (11.1.15) *let us replace the domain $X_{\mathsf{reg},h}$ by the domain $\mathcal{X}_{\mathsf{reg}}$ (or X); then the error does not exceed $C\int_{X_{\mathsf{sing},h}} \rho^d\,dx \le CR_0$ due to Hölder inequality. Therefore*

$$|\mathsf{N}^{-}(A_h) - \mathsf{N}^{\mathsf{W}}h^{-d}| \le Ch^{1-d}R_1 + Ch^{s-d}R_2 + Ch^{-d}R_0 \tag{11.1.18}$$

where N^{W} is now given by (uncut) Weyl expression and no longer depends on h.

(ii) This estimate yields that if

$$\rho^{d-s}\gamma^{-s} \in \mathscr{L}^1(\{X,\ \rho\gamma \ge h_0\}), \tag{11.1.19}$$

$$\rho^{d-1}\gamma^{-1} \in \mathscr{L}^1(X \setminus X_{\mathsf{reg},\mathsf{ell},h_0}) \tag{11.1.20}$$

and

$$R_0 = O(h) \qquad \text{as } h \to +0, \tag{11.1.21}$$

then

$$\mathsf{N}^{-}(A_h) = h^{-d}\mathsf{N}^{\mathsf{W}} + O(h^{1-d}). \tag{11.1.22}$$

These asymptotics will be improved and generalized below, where other asymptotics due to the results of Chapter 9 will be derived.

(iii) Let us assume that

$$\varrho \asymp \rho^k \quad \text{in } \mathcal{X}_{\mathsf{reg}} \tag{11.1.23}$$

and that one of the following conditions are fulfilled:

$$(11.1.24)_1 \qquad d > 2m, \qquad k \in [0, 2m];$$

$$(11.1.24)_2 \qquad m < d \le 2m, \qquad k \in [2m-d, d);$$

$$(11.1.24)_3 \qquad d = m, \qquad k = d.$$

Then one can choose p, q *satisfying condition* (11.1.11) *with equality and condition* (11.1.12) *and such that*

$$R_0 \asymp \int_{X_{\mathsf{sing},h}} \rho^d \, dx \le 2h \int_{X_{\mathsf{sing},h}} \rho^{d-1}\gamma^{-1} \, dx.$$

This inequality and (11.1.20) *yield that* $R_0 = o(h)$ *and therefore condition* (11.1.21) *is fulfilled. In particular, for* $d > 2m$ *one can take* $k = 2m$.

Remark 11.1.3. Let us list some cases when according to Parts II and III contribution of $\mathcal{B}(y, \gamma(y), X)$ to the remainder is $O(h^{1-d}\rho^{d-1}\gamma^{d-1})$; these conditions are uniform with respect to y (after rescaling):

(11.1.25) Assumptions (11.1.3), (11.1.4) and (9.1.15) are fulfilled with $K = K(d)$ and operator A is microhyperbolic on the energy level 0;

(11.1.26) A is Schrödinger operator, $d \ge 2$, assumptions (11.1.7), (11.1.8) and (9.1.15) are fulfilled with $K = K(d)$;

(11.1.27) A is "rough" operator described in Section 4.6 as $B(y, \gamma(y)) \subset X_{\mathsf{reg}}$ and in Section 7.5 otherwise.

Through bracketing $A^- \le A \le A^+$ between two rough operators such that contributions to N^{W} of $\mathcal{B}(y, \gamma(y), X)$ for A^- and A^+ differ by $O(h_{\mathsf{eff}}^{1-d})$ we cover also

(11.1.28) Assumptions (11.1.3), (11.1.4) are fulfilled with $K = 1$, $\nabla a_{\alpha,\beta}$ are continuous with continuity modulus $C\varrho\rho^{-|\alpha|-|\beta|}\gamma^{-1}|\log(|x-y|\gamma^{-1})|^{-1}$ and either $B(y, \gamma(y)) \subset X_{\mathsf{reg}}$ or extra conditions of Theorem 7.5.15 to the boundary are fulfilled, operator A is ξ-microhyperbolic on the energy level 0;

(11.1.29) A is Schrödinger operator, assumptions (11.1.7), (11.1.8) are fulfilled with $K = 1$, ∇g^{kl}, ∇V_k and ∇V are continuous with continuity moduli $C\rho^p\gamma^{-1}|\log(|x-y|\gamma^{-1})|^{-1}$, $p = 0, 1, 2$ respectively and either $B(y, \gamma(y)) \subset X_{\mathsf{reg}}$ or extra conditions of Theorem 7.5.15 to the boundary are fulfilled, $d \ge 3$ or $|V| + \gamma|\nabla V| \ge \epsilon\rho^2$;

(11.1.30) A is Schrödinger operator, $d = 2$, assumptions (11.1.7), (11.1.8) are fulfilled with $K = 2$, $\nabla^2 g^{kl}$, $\nabla^2 V_k$ and $\nabla^2 V$ are continuous with continuity moduli $C\rho^p\gamma^{-2}|\log(|x-y|\gamma^{-1})|^{-2}$, $p = 0, 1, 2$ respectively and either $B(y, \gamma(y)) \subset X_{\mathsf{reg}}$ or extra conditions of Theorem 7.5.15 to the boundary are fulfilled.

Remark 11.1.4. Actually we can weaken assumptions (11.1.28)–(11.1.30) because we can bracket symbol $a(x, \xi)$ between $a(x, \xi) \pm b_\varepsilon(x)$ where

$$b_\varepsilon(x) = C_0 \max_{\alpha,\beta} \operatorname{osc}(\nabla a_{\alpha,\beta}, x, \varepsilon) * \phi_\varepsilon \tag{11.1.31}$$

where

$$\operatorname{osc}(a, x, \varepsilon) = \max_{y:|x-y|\le\varepsilon} a - \min_{y:|x-y|\le\varepsilon} a \tag{11.1.32}$$

ϕ_ε is a standard mollification function. An easy proof is left to the reader.

Then for ξ-microhyperbolic operator contributions of $\mathcal{B}(y, \gamma(y), X)$ to N^{W} for A^- and A^+ differ by no more than

$$Ch_{\mathsf{eff}}^{-d} \max_{\alpha,\beta} \int \operatorname{osc}(\nabla a_{\alpha,\beta}, x, \varepsilon)\, dx \tag{11.1.33}$$

with an integral taken over $\mathcal{B}(y, \gamma(y), X)$ rescaled and in the end of the day results in an additional error

$$R_3 = Ch_{\mathsf{eff}}^{1-d} \max_{\alpha,\beta} \int_{X_{\mathsf{reg}} \setminus X_{\mathsf{reg,ell}}} \rho^d \operatorname{osc}\bigl(\nabla a_{\alpha,\beta}, x, \rho^{-1}\log(\rho\gamma)\bigr)\, dx. \tag{11.1.34}$$

So, to estimate (11.1.33) by $O(h_{\mathsf{eff}}^{1-d})$ we actually need to impose estimate $O(|\log\varepsilon|^{-1})$ to $\mathscr{L}^1$-norm rather than $\mathscr{L}^\infty$-norm of $\operatorname{osc}(\nabla a_{\alpha,\beta}, x, \varepsilon)$.

We leave to the reader the following easy problem:

Problem 11.1.5. Apply rescaling to modified expressions (11.1.33) and (11.1.34) for Schrödinger operator ($d \ge 3$) without condition $|V| \ge \epsilon\rho^2$.

Remark 11.1.6. Some of the conditions of Theorem 11.1.1 and Corollary 11.1.2 are too restrictive but we can weaken them.

(i) First of all, one can assume that $X = \mathcal{X}_{\mathsf{reg}} \cup \mathcal{X}_{\mathsf{sing}} \cup \mathcal{Y}$ where the domains $(\mathcal{X}_{\mathsf{reg}} \cup \mathcal{X}_{\mathsf{sing}})$ and $\mathcal{Y}$ are properly overlapping (with respect to the scaling

function γ), and that in $(\mathcal{X}_{\mathsf{reg}} \cup \mathcal{X}_{\mathsf{sing}})$ all the previous conditions are fulfilled and we assume that in $\mathcal{Y}$ the inequality

$$\sum_{\alpha,\beta} \langle a_{\alpha,\beta} v_\alpha, v_\beta \rangle \ge \epsilon_0 \sum_{|\alpha|=m} \varrho_1 \|v_\alpha\|^2 - C \sum_{|\alpha|\le m-1} \varrho_1 \rho_1^{2|\alpha|} \|v_\alpha\|^2 \tag{11.1.35}$$

holds with appropriate measurable functions ϱ_1 and ρ_1. Then one can take $X_{\mathsf{reg}} = \{x \in \mathcal{X}_{\mathsf{reg}}, \rho\gamma \ge h\}$ and $X_{\mathsf{sing}} = \{x \in \mathcal{X}_{\mathsf{reg}}, \rho\gamma \le 2h\} \cup X_{\mathsf{sing}} \cup \mathcal{Y}$ in which case R_0 is given by the formula

$$\begin{aligned} R_0 = C\Big(\int_{\mathcal{X}_{\mathsf{reg}} \cap \{\rho\gamma \le 2h\}} \rho^{-p} \rho^{2mp}\, dx + \int_{\mathcal{Y}} \varrho_1^{-p} dx\Big)^{\frac{d}{2mp}} \times \\ \Big(\int_{\mathcal{X}_{\mathsf{reg}} \cap \{\rho\gamma \le 2h\}} \varrho^q\, dx + \int_{\mathcal{Y}} \varrho_1^q \rho_1^{2mq}\, dx\Big)^{\frac{d}{2mq}} \times \\ \big(\mathsf{mes}(\mathcal{X}_{\mathsf{reg}} \cap \{\rho\gamma \le 2h\} \cup \mathcal{Y})\big)^{1-\frac{d}{2m}(\frac{1}{p}+\frac{1}{q})}. \end{aligned} \tag{11.1.36}$$

(ii) Further, one can assume that the Dirichlet boundary condition is given only on $\partial X \cap (\bar{\mathcal{X}}_{\mathsf{reg}} \cap \{\rho\gamma \le h_0\} \cup \bar{\mathcal{Y}})$; on other part of ∂X it could be different (but still appropriate boundary condition).

Furthermore, assume that even in $\partial X_{\mathsf{sing},h}$ Dirichlet boundary condition is assumed only on Y_{D} while on $Y_{\mathsf{N}} = \partial X_{\mathsf{sing},h} \setminus Y_{\mathsf{D}}$ another boundary condition is given, not necessary Neumann one but satisfying

$$\begin{aligned} (Au,u)_{X_{\mathsf{sing},h}} \ge \\ \epsilon \sum_{\alpha:|\alpha|=m} h^{2m} \|D^\alpha u\|^2 - C \sum_{\alpha:|\alpha|\le m-1} h^{2|\alpha|} \|\rho_1^{m-|\alpha|} D^\alpha u\|^2 \\ \forall u \in \mathfrak{D}(A):\ u = 0 \qquad \text{on}\ \ X \setminus X_{\mathsf{sing},h} \end{aligned} \tag{11.1.37}$$

Finally, let us assume that conditions (9.A.23)–(9.A.26) are fulfilled for X replaced by $X_{\mathsf{sing},h}$; recall that the inner boundary $\partial X_{\mathsf{sing},h} \cap \overset{\circ}{X}$ is considered as part of Y_{D}.

Then the previous estimate remain true with $\varrho_1 = 1$ and with ρ_1 replaced by

$$\rho_2 = \rho_1 + \sum_j h_j^{-1} \chi_j \tag{11.1.38}$$

in the estimates from above where χ_j is a characteristic function of X_j. In some examples specific arguments will improve even this estimate.

(iii) All these results remain true if we replace hD_j by $hD_j - V_j$ in the operator A where V_j are real-valued functions supported in $X_{\mathrm{reg},h}$ such that

$$|D^\alpha (D_j V_k - D_k V_j)| \le c\rho\gamma^{-|\alpha|-1} \qquad \forall \alpha : |\alpha| \le K. \tag{11.1.39}$$

In fact, after an appropriate gradient transform in every ball $\mathcal{B}(x, \epsilon\gamma(x), X)$ lying in $X_{\mathrm{reg},h}$ condition

$$|D^\alpha V_j| \le c\rho\gamma^{-|\alpha|} \qquad \forall \alpha : |\alpha| \le K \tag{11.1.40}$$

is fulfilled.

(iv) If the operator $(A - \tau)$ satisfies all the conditions of Theorem 11.1.1 uniformly with respect to $\tau \in [\tau_1, 0]$ with $\tau_1 < 0$ then all the estimates and asymptotics of Theorem 11.1.1 and Corollary 11.1.2 remain true for this operator uniformly with respect to $\tau \in [\tau_1, 0]$. In particular, for the Schrödinger operator with $d \ge 2$ the condition of this statement is fulfilled automatically for $\tau_1 = -\infty$.

Remark 11.1.7. Let us consider the Riesz means

$$\mathsf{N}^-_{(\vartheta)}(A) = \int \tau_-^{\vartheta-1} \mathsf{N}^-(A - \tau)\, d\tau \tag{11.1.41}$$

of order $\vartheta > 0$ (i.e., we calculate them for the spectral parameter $\tau = 0$). Let us include in $\mathsf{N}^{\mathsf{W}}_{(\vartheta)}$ (depending on h for $A = A_h$) a sufficient number of terms. So, in principle $\mathsf{N}^{\mathsf{W}}_{(0)}$ does not necessarily coincide with N^{W} (in more sharp asymptotics it may contain one extra term).

(i) Then the remainder estimates in the local semiclassical spectral asymptotics zone $X_{\mathrm{reg},h}$ yield that terms $h^{1-d}R_1$ and $h^s R_2$ should be replaced by $h^{1+\vartheta-d}R_{1(\vartheta)}$ and $h^s R_{2(\vartheta)}$ with

$$R_{1(\vartheta)} = \int_{X_{\mathrm{reg},h} \setminus X_{\mathrm{reg,ell},h}} \varrho^\vartheta \rho^{d-\vartheta-1} \gamma^{-\vartheta-1}\, dx, \tag{11.1.42}$$

$$R_{2(\vartheta)} = \int_{X_{\mathrm{reg}}} \varrho^\vartheta \rho^{d-s} \gamma^{-s}\, dx. \tag{11.1.43}$$

To do this we need $O(h^{1+\vartheta-d}_{\mathsf{eff}})$ remainder estimate which requires a stronger regularity: $K = 1 + \lfloor \vartheta \rfloor$ and the K-th derivatives should be continuous with the continuity modulus $O(|x-y|^{\vartheta - \lfloor\vartheta\rfloor} |\log |x-y||^{-1})$ and the boundary should be a bit smoother.

(ii) Further, in virtue of SubSection 9.2.4 and Appendix 9.A.2 term $h^{-d}R_0$ with R_0 defined by (11.1.10) should be replaced by $h^{-d}R_{0(\vartheta)}$ with $R_{0(\vartheta)}$ defined by the same formula but with the second factor in the right-hand expression should be given in the form

$$\int_0^\infty \tau^{\vartheta-1}\Big(\int_{X_{\mathsf{sing},h}} (\varrho_1-\tau)_+^q \, dx\Big)^{\frac{d}{2mq}} \, d\tau. \tag{11.1.44}$$

In particular, if one can take $q=\frac{d}{2m}$ and $p=\infty$ then

$$R_{0(\vartheta)} = C\int_{X_{\mathsf{sing},h}} \varrho^{\frac{d}{2m}+\vartheta}\, dx \times \sup_{X_{\mathsf{sing},h}} \big(\varrho^{-\frac{d}{2m}}\rho^d\big). \tag{11.1.45}$$

Furthermore, in virtue of Appendix 9.A.2 we can use this formula even if $d=2m$ but $\vartheta>0$ which is the case for the Schrödinger operator with $d=2$.

(iii) In particular, replacing conditions (11.1.19)–(11.1.21) by conditions

$$\varrho_1^\vartheta \rho^{d-s}\gamma^{-s} \in \mathscr{L}^1(\{x,\ \rho\gamma\ge h_0\}), \tag{11.1.46}$$

$$\varrho_1^\vartheta \rho^{d-1-\vartheta}\gamma^{-1-\vartheta} \in \mathscr{L}^1(X\setminus X_{\mathsf{reg},\mathsf{ell},h_0}) \tag{11.1.47}$$

and

$$R_{0(\vartheta)} = O(h^{1+\vartheta}) \qquad \text{as}\quad h\to +0, \tag{11.1.48}$$

we obtain t asymptotics

$$\mathsf{N}^-_{(\vartheta)}(A_h) = h^{-d}\mathsf{N}^{\mathsf{W}}_{(\vartheta)} + O(h^{1+\vartheta-d}). \tag{11.1.49}$$

In particular, in the framework of (11.1.23)–(11.1.24) (upgraded to $d\ge 2m, k\in[0,2m]$ as $\vartheta>0$) conditions (11.1.48) and (11.1.50) below follow from (11.1.47).

Non-Standard Remainder Estimates

So far we assumed that the remainder estimate in the local spectral asymptotics is $O(h_{\mathsf{eff}}^{1-d+\vartheta})$ as $\vartheta\ge 0$; $\vartheta=0$ refers to the most interesting case of the asymptotics without averaging.

Sharp and Sharper Asymptotics However the remainder estimate in the local spectral asymptotics could be better than this: it could be $o(h_{\mathsf{eff}}^{1+\vartheta-d})$ or even $O(h_{\mathsf{eff}}^{\kappa+\vartheta-d})$ with $\kappa > 1$. In the former case conditions (11.1.46)–(11.1.47) imply that the contribution of $X_{\mathsf{reg},h}$ to the remainder is $o(h^{1-d})$. Adding assumption

$$R_{0(\vartheta)} = o(h^{1+\vartheta}) \qquad \text{as} \quad h \to +0, \tag{11.1.50}$$

we arrive to asymptotics

$$\mathsf{N}^{-}_{(\vartheta)}(A) = h^{-d}\mathsf{N}^{\mathsf{W}}_{(\vartheta)} + o(h^{1+\vartheta-d}). \tag{11.1.51}$$

In the latter case conditions (11.1.47)–(11.1.48) should be upgraded to

$$\varrho_1^{\vartheta}\rho^{d-\kappa-\vartheta}\gamma^{-\kappa-\vartheta} \in \mathscr{L}^1(X \setminus X_{\mathsf{reg},\mathsf{ell},h_0}) \tag{11.1.52}$$

and

$$R_{0(\vartheta)} = O(h^{\kappa+\vartheta}) \qquad \text{as} \quad h \to +0 \tag{11.1.53}$$

respectively. Note that in the framework of (11.1.23)–(11.1.24) (upgraded to $d \ge 2m, k \in [0, 2m]$ as $\vartheta > 0$) condition (11.1.53) follows from (11.1.52).

Then we arrive to

$$\mathsf{N}^{-}_{(\vartheta)}(A) = h^{-d}\mathsf{N}^{\mathsf{W}}_{(\vartheta)} + O(h^{\kappa+\vartheta-d}). \tag{11.1.54}$$

Remark 11.1.8. (i) Let us recall that sharp local asymptotics are due to corresponding non-periodicity condition:

$$\mu_0\big((\Pi_+ \cup \Lambda_+) \cap (\Pi_- \cup \Lambda_-)\big) = 0 \tag{11.1.55}$$

where μ_0 is a measure introduced on Σ_0 and

$$\Pi_{\pm} = \{z : \exists t : \pm t > 0 : \Psi_t(z) \ni z\} \tag{11.1.56}$$

is the set of points "periodic" with respect to Hamiltonian flow Ψ_t with reflections, possibly branching and $\Lambda_{\pm}$ is the set of dead-end trajectories.

(ii) Recall, that we need also a bit more smoothness: for example, as $\vartheta = 0$ the first derivatives of the coefficients should be continuous with the continuity modulus $o(|\log|x-y||^{-1})$ rather than $O(|\log|x-y||^{-1})$ (after rescaling) and condition to the smoothness of the boundary should be also a bit stronger.

As $\vartheta > 0$ smoothness conditions grow stronger: $K = 1 + \lfloor \vartheta \rfloor$ and the K-th derivatives should be continuous with the continuity modulus $o(|x-y|^{\vartheta - \lfloor \vartheta \rfloor} |\log |x-y||^{-1})$.

We refer to these assumptions as *improved regularity conditions*.

(iii) Also a bit less degeneration is allowed: as we consider Schrödinger operator with $d = 2$ and $\vartheta = 0$ we need to assume that

$$\mathsf{mes}(\{x, V(x) = 0, F = |D_1 V_2 - D_2 V_1| \neq 0\}) = 0. \tag{11.1.57}$$

We refer to these assumptions as *improved non-degeneracy condition*

Remark 11.1.9. Recall that for sharper remainder estimates these improved conditions are required in the stronger form but since non-periodicity condition is so difficult to check we will consider only some special cases.

Less Sharp Asymptotics However, either due to irregularity (as in Theorem 4.6.21 and similar) or to degeneration or both the local asymptotics may have remainder estimate not as good as $O(h_{\mathsf{eff}}^{1-d+\vartheta})$; for example it could be $O(h_{\mathsf{eff}}^{\kappa-d+\vartheta})$ with $\kappa < 1$. Then again under assumptions (11.1.46), (11.1.52), (11.1.53) we arrive to asymptotics (11.1.54), this time with $\kappa < 1$.

While estimates from above relied upon taking $T \asymp 1$ in $X_{\mathsf{reg}} \cap X_{\mathsf{sing}}$ which is not necessary the case now, it does not matter as $\rho\gamma \asymp h$ in $X_{\mathsf{reg}} \cap X_{\mathsf{sing}}$.

11.1.2 Examples

Let us consider a series of examples.

Example 11.1.10. Let $d \geq 2$ and let X be a compact domain (or manifold) with the boundary. Let assumptions of Theorem 11.1.1 be fulfilled with $\rho \asymp 1$ and $\gamma \asymp 1$. Then asymptotics (11.1.22) holds.

Further, under appropriate smoothness condition (see Remark 11.1.8(i)) asymptotics (11.1.49) holds.

Furthermore, under non-periodicity condition (11.1.55)–(11.1.56), improved smoothness and non-degeneracy conditions of Remark 11.1.8 asymptotics (11.1.54) holds. Here $\Lambda_{\pm} = \emptyset$.

Finally, under corresponding stronger non-periodicity conditions sharper asymptotics hold.

The conditions of our theorems are local and all our analysis is local (until the end of the subsection). So we assume that outside of the singularities indicated below everything is well (in the sense of Example 11.1.10).

Example 11.1.11 (Conical vertex). Let $d \ge 2$ and in a neighborhood of 0 let the domain X coincide with the cone $X^0 = \{(r,\theta) \in \mathbb{R}^+ \times U\}$ where (r,θ) are spherical coordinates, U is a domain on the sphere $\mathbb{S}^{d-1}$ with the boundary ∂U.

Moreover, let the conditions of Theorem 11.1.1 be fulfilled with $\varrho = \rho^{2m}$, $\rho = r^{\mu}$, $\gamma = \epsilon r$ where here and below $\epsilon > 0$ is a small constant.

(i) Then ρ^d and (or) $\rho^{d-1}\gamma^{-1}$ belong to $\mathscr{L}^1(X)$ if and only if $\mu > -1$. Let us assume that

$$\mu > \max(-1, -\frac{d}{2m}). \tag{11.1.58}$$

Then asymptotics (11.1.22) holds.

(ii) Further, under non-periodicity condition and improved regularity and non-degeneracy condition asymptotics (11.1.54) holds. Here $\Lambda_{\pm}$ are sets of points such that $\Psi_t(z)$ "falls" to 0 in the finite time. For example, if problem is spherically symmetric then $\Lambda_{\pm}$ consists of points with $\xi \parallel x$ (i.e. with angular momentum equal 0).

(iii) On the other hand, $\varrho^{\vartheta}\rho^{d-1-\vartheta}\gamma^{-1-\vartheta} \in \mathscr{L}^1(X)$ if and only if

$$(\mu + 1 - 2\mu m)\vartheta < (d-1)(\mu+1) \tag{11.1.59}$$

and if this condition is fulfilled and if also $\mu > -\frac{d}{2m}$ then asymptotics (11.1.49) holds for given ϑ.

Moreover, under non-periodicity condition (11.1.55)–(11.1.56), improved smoothness and non-degeneracy conditions of Remark 11.1.8 asymptotics (11.1.54) holds.

(iv) Furthermore, in the case

$$(\mu + 1 - 2\mu m)\vartheta = (d-1)(\mu+1) \tag{11.1.60}$$

we obtain asymptotics

$$\mathsf{N}^-_{(\vartheta)}(A) = h^{-d}\mathsf{N}^{\mathsf{W}}_{(\vartheta)} + O(h^{1-d+\vartheta}\log h); \tag{11.1.61}$$

here we use Theorem 11.1.1 instead of Corollary 11.1.2 [2)].

(v) Finally, in the case

$$(\mu+1-2\mu m)\vartheta > (d-1)(\mu+1), \quad 2\mu m\vartheta + d(\mu+1) > 0 \tag{11.1.62}$$

we obtain asymptotics

$$\mathsf{N}^{-}_{(\vartheta)}(A) = h^{-d}\mathsf{N}^{\mathsf{W}}_{(\vartheta)} + O(h^{p}) \tag{11.1.63}$$

with

$$p = \frac{2\mu}{\mu+1} m\vartheta \in (-d,\ 1+\vartheta-d) \tag{11.1.64}$$

where only terms with the degree of h less than $p+d$ are included in $\mathsf{N}^{\mathsf{W}}_{(\vartheta)}$.

Remark 11.1.12. (i) We assumed that $\mu > -d/2m$ only to guarantee that

$$a_{\alpha\beta} \in \mathscr{L}^{1}(X \cap \{|x| \le \epsilon\}) \tag{11.1.65}$$

and one can replace it by the latter, while also taking care that Rozenblioum estimate works; we leave details to the reader.

(ii) On the other hand, as $\mu \le -1$ it may happen that the corresponding quadratic form is not semi-bounded from below and thus operator A is not defined; even if we define it, its essential spectrum may occupy the whole $\mathbb{R}$.

Remark 11.1.13. (i) One can easily see that in the domain $X_{\mathsf{sing},h} = \{x \in X, r \le \epsilon h^{\frac{1}{\mu+1}}\}$ inequality

$$(A_h u, u) \ge \epsilon_0 \sum_{|\alpha|=m} h^{2m}\|D^{\alpha}u\|^2 \tag{11.1.66}$$

$$\forall u \in \mathscr{C}^{m}(\bar{X}) : \operatorname{supp} u \subset B(0, \epsilon h^{\frac{1}{\mu+1}})$$

[2)] Here and below $\mathsf{N}^{\mathsf{W}}_{(\vartheta)}$ is given by the Weyl formula but in general we only integrate over the domain $X_{\mathsf{reg},h} = \{x \in X, |x| \ge \epsilon h^{1/(\mu+1)}\}$ so we have cut Weyl expression. Moreover, one can skip all the terms with $n-d > p$ in the local semiclassical spectral asymptotics and in the terms with $n-d < p$ integrate over X; $p = 1-d+\vartheta$ under condition (11.1.60).

holds. This inequality follows from the inequalities

$$h^k \sum_{|\alpha|=k} \|D^\alpha u\| \ge \epsilon_0 \varepsilon^{-\delta} \|r^{\mu k} u\| \qquad \forall k = 1, \dots, m \tag{11.1.67}$$

for the same u as in (11.1.66). Here $\delta > 0$ is a small enough exponent and ϵ_0 does not depend on ε.

In fact, for $h = \epsilon = 1$ estimate (11.1.67) follows easily from condition (11.1.58), and the estimate in the general case follows from the estimate in this special case by means of rescaling.

Therefore one can take $\rho = 0$, $\varrho = r^{2\mu m}$ in $X_{\mathsf{sing},h}$ and this simplifies the estimate of R_0. Further, we obtain that for

$$m\vartheta + d(\mu + 1) = 0 \tag{11.1.68}$$

asymptotics (11.1.63) holds with $p = -d$ and with $\mathsf{N}^{\mathsf{W}}_{(\vartheta)}$ given only by the integration of the $n = 0$ term over the domain $X_{\mathsf{reg},h}$. Here $\mathsf{N}^{\mathsf{W}}_{(\vartheta)} = O(\log h)$; moreover, if

$$\inf_{v:\|v\|=1,\xi\in\mathbb{R}^d} \langle a(x,\xi)v, v\rangle \le -\epsilon_0 \rho^{2m} \qquad \forall x \in \Gamma \cap \{|x| \le \epsilon\} \tag{11.1.69}$$

for some non-empty cone $\Gamma \subset \mathbb{R}^+ \times U$ with vertex in 0 then $\mathsf{N}^{\mathsf{W}}_{(\vartheta)} \asymp \log h$.

(ii) Further, let us note that under condition (11.1.60) the main contribution to the remainder estimate in (11.1.63) is delivered by the zone

$$X_{h,\varepsilon} \coloneqq \{X, ch^{\frac{1}{\mu+1}} \le |x| \le \varepsilon_h\} \tag{11.1.70}$$

with $\varepsilon_h = o(1)$ for $h \to +0$ (the contribution of the other zones is "o" with respect to the contribution of this zone). In order to improve the remainder estimate let us assume that the operator A stabilizes at 0 to the quasihomogeneous operator A^0, i.e., that

$$D^\alpha(a_\beta - a^0_\beta) = o(|x|^{\mu|\beta|-|\alpha|}) \qquad \forall \beta : |\beta| \le 2m \quad \forall \alpha : |\alpha| \le K \tag{11.1.71}$$

for $|x| \to 0$ where a^0_β are positively homogeneous of degrees $\mu|\beta|$. Further, let us assume that the symbol

$$a^0(x,\xi) = \sum_\beta a^0_\beta(x)\xi^\beta \tag{11.1.72}$$

satisfies non-periodicity condition (11.1.55)–(11.1.56) to Hamiltonian flow. Here and below Σ^0_τ, μ^0_τ, $\Pi^0_\pm$, $\Lambda^0_\pm$ are constructed for a^0.

Then asymptotics

$$\mathsf{N}^-_{(\vartheta)}(A) = h^{-d}\mathsf{N}^{\mathsf{W}}_{(\vartheta)} + o(h^{1-d+\vartheta}\log h) \tag{11.1.73}$$

holds. Here in the term with $n = -d+1+\vartheta$ we integrate over the domain $X_{\mathsf{reg},h}$ and this term is $O(h^{1+\vartheta}\log h)$; in all the other terms we can integrate over X.

Example 11.1.14. We can treat the case when the singularity is more complicated. Namely, for the Schrödinger operator we can consider positively homogeneous g^{jk}, V_j, V and also ∂U which are not necessarily as regular as required by Theorem 11.1.1 on $\mathbb{S}^{d-1}$.

Let us assume that on $\mathbb{S}^{d-1}$ all the conditions are fulfilled with $\hat\rho$ and $\hat\gamma$ instead of ρ and γ. Then we can take $\rho = \hat\rho(\theta)r^\mu$ and $\gamma = \epsilon\hat\gamma(\theta)r$ with a small enough constant $\epsilon > 0$.

Let us note that $\rho^{2\vartheta+d} \in \mathscr{L}^1$ if and only if $\hat\rho^{2\vartheta+d} \in \mathscr{L}^1(U)$ and $(2\vartheta+d)\mu + d > 0$; moreover, let us note that $\rho^{\vartheta+d-1}\gamma^{-\vartheta-1} \in \mathscr{L}^1$ if and only if $\hat\rho^{\vartheta d-1}\hat\gamma^{-\vartheta-1} \in \mathscr{L}^1(U)$ and $(\mu-1)\vartheta + (\mu+1)(d-1) > 0$.

Therefore a more complicated structure is possible: one can treat a hierarchy of singularities: "vertices of singularities," "edges of singularities," "faces of singularities," etc.

We leave to the reader

Problem 11.1.15. Analyze the case in which

$$\gamma = \epsilon\min(|x_{(1)}|, \dots, |x_{(l)}|), \quad \rho = \prod_{S\subset\{1,\dots,l\}} \zeta_S^{\mu_S}, \quad \zeta_S = \sum_{j\in S}|x_{(j)}| \tag{11.1.74}$$

where $x = (x_{(1)}, \dots, x_{(l)}) \in \mathbb{R}^{d_1}\times\dots\times\mathbb{R}^{d_l}$ is an appropriate partition.

Example 11.1.16 (Conical edge). Let us consider a more general case. Let X be a bounded domain locally coinciding with $\mathbb{R}^{d'}\times\mathbb{R}^+\times U$ where spherical coordinates (r,θ) are introduced in $\mathbb{R}^{d''}$ ($d'' = d-d'$), and where $U \subset \mathbb{S}^{d''-1}$, $0 \le d' \le d-1$. Let conditions of Theorem 11.1.1 be fulfilled with $\varrho = \rho^{2m}$, $\rho = r^\mu$, $\gamma = \epsilon r$ and let

$$\mu > \max(-1, -\frac{d''}{2m}). \tag{11.1.75}$$

Then

(i) For

$$(\mu+1-2\mu m)\vartheta < (\mu+1)(d-1)-d' \tag{11.1.76}$$

asymptotics (11.1.49) holds. Further, under non-periodicity condition and improved regularity and non-degeneracy condition asymptotics (11.1.54) holds. Here $\Lambda_\pm$ are sets of points such that $\Psi_t(z)$ "falls" to $\mathbb{R}^{d'}\times 0$ in the finite time.

(ii) On the other hand, for

$$(\mu+1-2\mu m)\vartheta = (\mu+1)(d-1)-d' \tag{11.1.77}$$

asymptotics (11.1.61) holds and for

$$(\mu+1-2\mu m)\vartheta > (\mu+1)(d-1)-d',\quad 2\mu m\vartheta + d\mu + d'' \ge 0 \tag{11.1.78}$$

asymptotics (11.1.63) holds with

$$p=\frac{2\mu}{\mu+1}(m\vartheta - d') \tag{11.1.79}$$

where for

$$2\mu m\vartheta + d\mu + d'' = 0 \tag{11.1.80}$$

only one term in $\mathsf{N}^{\mathsf{W}}_{(\vartheta)}$ remains and integration is taken over $X_{\mathsf{reg},h}$. In this case $\mathsf{N}^{\mathsf{W}}_{(\vartheta)} = O(\log h)$ and $\mathsf{N}^{\mathsf{W}}_{(\vartheta)} \asymp \log h$ under condition (11.1.69) fulfilled in $V\times\Gamma\cap\{x, |x''|\le\epsilon\}$ where V is a neighborhood of 0 in $\mathbb{R}^{d'}$ and Γ is a non-empty cone in $\mathbb{R}^+\times U$ with vertex at 0.

Let us note that in this example (in contrast to Example 11.1.11) the variants are possible even for $\vartheta = 0$.

(iii) Let $\vartheta = 0$. Then asymptotics (11.1.22) and (11.1.61) are replaced by asymptotics (11.1.63) with $p = -d+1-\delta$ with arbitrarily small exponent $\delta > 0$ while asymptotics (11.1.63) with $p < -d+1$ remains true.

Consider conical singularity with the critical exponents:

Example 11.1.17. Let $0 \in \bar{X}$ and all the conditions be fulfilled with $\gamma = \epsilon r$, $\rho = r^{\mu} |\log r|^{-\nu}$, $r = |x|$ where we assume that

$$\mu > -\frac{2m}{d},\ \mu \ge -1,\ \vartheta \ge 0,\ \nu > 0, \qquad 2m\mu\vartheta + (d - \vartheta - 1)(\mu + 1) = 0. \tag{11.1.81}$$

Then repeating the arguments of the previous examples with

$$X_{\mathsf{sing},h} = \{x \in X, |x| \le \epsilon h^{\frac{1}{\mu+1}} |\log h|^{\frac{\nu}{\mu+1}}\} \qquad \text{for } \mu > -1 \tag{11.1.82}$$

and

$$X_{\mathsf{sing},h} = \{x \in X, \log|x| \le -Ch^{-\frac{1}{\nu}}\} \qquad \text{for } \mu = -1 \tag{11.1.83}$$

and with corresponding $X_{\mathsf{reg},h}$ we conclude that

(i) Under condition

$$(2m\vartheta + d - \vartheta - 1)\nu > 1 \tag{11.1.84}$$

asymptotics (11.1.49) holds; moreover, under standard non-periodicity condition and improved regularity and non-degeneracy conditions asymptotics (11.1.51) holds.

(ii) On the other hand, under condition

$$(2m\vartheta + d - \vartheta - 1)\nu = 1 \tag{11.1.85}$$

asymptotics (11.1.61) holds.

(iii) Finally, if

$$(2m\vartheta + d - \vartheta - 1)\nu < 1 \tag{11.1.86}$$

and if either

$$\mu > -1 \tag{11.1.87}_1$$

or

$$\mu = -1 (\implies \vartheta = 0),\ \nu \ge \frac{1}{d} \tag{11.1.87}_2$$

then asymptotics (11.1.63) holds with

$$p = 2m\vartheta - \vartheta - \frac{1}{\nu}. \tag{11.1.88}$$

For $\mu = -1$, $\vartheta = 0$, $\nu = \frac{1}{d}$ we see that $\mathsf{N}^{\mathsf{W}} = O(\log h)$ and $\mathsf{N}^{\mathsf{W}} \asymp \log h$ provided condition (11.1.69) is fulfilled in some non-empty cone with vertex at 0 (for $|x| \le \epsilon$).

(iv) Moreover, in the first critical case (11.1.85) one can improve asymptotics (11.1.61). We do not do a careful analysis of the Hamiltonian trajectories here but refer to the arguments of the proofs of Example 11.5.21 (see below). Let us assume that the coefficients a_β of A stabilize to functions $a^0_\beta \big|\log|x|\big|^{-\nu|\beta|}$ in the sense that

$$D^\sigma\big(a_\beta \big|\log|x|\big|^{\nu|\beta|} - a^0_\beta\big) = o\big(|x|^{\mu|\beta|-|\sigma|}\big) \qquad \forall \beta : |\beta| \le 2m\ \forall \sigma : |\sigma| \le 1 \tag{11.1.89}$$

where a^0_β are positively homogeneous of degree $\mu|\beta|$. Moreover, let us assume that the quasihomogeneous symbol (11.1.72) satisfies the standard non-periodicity condition.

Then one can easily prove that asymptotics (11.1.73) holds with all the explanations made in Remark 11.1.13.

(v) For $\nu > 0$ satisfying the critical condition (11.1.85) one can treat the case in which $\rho = r^\mu |\log r|^{-\nu} \big|\log|\log r|\big|^{-\nu'}$ and analyze all the possible cases. In this case the conditions on ν are replaced by similar conditions on ν'. One can continue this process of complication.

(vi) On the other hand, one can treat the case in which $\rho = r^\mu |\log r|^{-\nu}$ even if equality in (11.1.81) is violated, i.e., μ is not the critical value. In this case for $2m\mu\vartheta + (d - \vartheta - 1)(\mu + 1) > 0$ asymptotics (11.1.49) (and even (11.1.51) under the standard non-periodicity condition to the Hamiltonian flow) holds and for $2m\mu\vartheta + (d - \vartheta - 1)(\mu + 1) < 0$ asymptotics (11.1.63) holds with the remainder estimate $O(h^p |\log h|^q)$ instead of $O(h^p)$. The calculation of q is left to the reader.

(vii) In all the above statements the condition $\nu > 0$ is not necessary if $\mu > -1$.

(viii) Statement (ii) of Example 11.1.16 remains true (with obvious modifications).

Different combinations of Statements (v) and (vi) are possible.

Example 11.1.18 (Quasiconical singularity). Let $0 \in \bar{X}$ and let us consider operator with coefficients $a_{\alpha\beta}$ which are L-*quasihomogeneous of degrees* $(2m - |\alpha| - |\beta|)\mu$ respectively, smooth away from 0. In a neighborhood of zero let X be an L-*quasiconical* domain and let ∂X be smooth away from 0. Here $L = (L_1, \dots, L_d)$ is an exponent vector with $L_j > 0$ $\forall j$ and $\max_j L_j = 1$.

Let us assume that either a Dirichlet or some other appropriate boundary condition is given and in the second case let us assume that the domain X satisfies the standard cone condition.

In this case $\gamma(x) = \epsilon[x]_L$, $\rho(x) = \gamma^{\mu}$, $\varrho(x) = \gamma^{2m\mu}$ with L-*quasihomogeneous norm* $[x]_L = \sum_j |x_j|^{\frac{1}{L_j}}$. Then repeating the arguments of Example 11.1.11 and assuming that

$$\mu > \max\bigl(-1, -\frac{L^*}{2m}\bigr), \qquad L^* = L_1 + \cdots + L_d \tag{11.1.90}$$

we conclude that

(i) For

$$(\mu + 1 - 2\mu m)\vartheta < (d-1)\mu + L^* \tag{11.1.91}$$

asymptotics (11.1.49) holds.

Moreover, under non-periodicity condition and improved regularity and non-degeneracy conditions asymptotics (11.1.51) holds.

On the other hand, for

$$(\mu + 1 - 2\mu m)\vartheta = (d-1)\mu + L^* \tag{11.1.92}$$

asymptotics (11.1.61) holds and for

$$(\mu + 1 - 2\mu m)\vartheta > (d-1)\mu + L^*, \quad 2m\mu\vartheta + d\mu + L^* > 0 \tag{11.1.93}$$

asymptotics (11.1.63) holds with $p = (2m\mu\vartheta + L^* - d)(\mu + 1)^{-1}$.

However the author have failed to solve

Problem 11.1.19. (i) Apply more advanced arguments of Remark 11.1.13.

(ii) Investigate the second critical case $2m\mu\vartheta + d\mu + L^* = 0$.

We leave to the reader to

Problem 11.1.20. Investigate the case $\rho = r^{\mu}$, $\gamma = \epsilon r^{1+\nu}$ with $\nu > 0$ arising in the analysis of the Schrödinger operator with potential $V = \rho^{2\mu} \sin r^{-\nu}$ and other examples of this type.

Let us treat exits to infinity. We assume that

(11.1.94) Either $X_{\mathsf{sing},h} = \emptyset$ or a Dirichlet boundary condition is given on $\partial X \cap \{|x| \ge c\}$.[3)]

Example 11.1.21 (Exit to infinity). (a) In a neighborhood of infinity let X coincide with $\{x, x_1 > 0, x' = F(x_1)\Omega\}$ where $\Omega \subset \mathbb{R}^{d-1}$ is a bounded domain satisfying assumptions of Theorem 11.1.1 with $\gamma(x') \asymp 1$, $x' = (x_2, \dots, x_d)$, $t\Omega = \{tx', x' \in \Omega\}$ and

$$F(x_1) \asymp x_1^{\beta}, \qquad \beta \le 1, \tag{11.1.95}$$

$$|D^{\alpha} F| \le c F^{1-|\alpha|} \qquad \forall \alpha : |\alpha| \le 2. \tag{11.1.96}$$

For $\beta = 1$ we have a conical exit, for $\beta \in (0,1)$ we have a slowly expanding quasiconical exit, for $\beta = 0$ we have a cylindrical exit and for $\beta < 0$ we have a constricting exit and these exits could be corrugated along the x_1 axis.

(b) Similarly, let us consider the case in which X coincides with $\{x, x' = F(|x''|)\Omega\}$ in a neighborhood of infinity where $x = (x'', x') \in \mathbb{R}^{d''} \times \mathbb{R}^{d'}$, $0 < d' < d = d' + d''$ and the function F satisfies conditions (11.1.95)–(11.1.96).

Then geometry yields that (at least for $|x''| \ge c$)

(11.1.97) X satisfies conditions of Theorem 11.1.1 with $\gamma \asymp F(|x''|)$.

Actually all we need is (11.1.96) with $K = 1$ and (11.1.97). Moreover, let us assume that the other conditions are fulfilled with $\gamma = \epsilon F(|x''|)$ and $\rho = |x''|^{\mu}$. Then

(i) If $\mu + \beta \ge 0$ one can take $X_{\mathsf{reg},h} = X$ and $X_{\mathsf{sing},h} = \emptyset$. Then under the condition

$$2m\mu\vartheta + d\mu + d'\beta + d'' < 0 \tag{11.1.98}$$

[3)] In the next sections we discuss how to weaken this assumption (albeit it is impossible to get rid of it).

(which provides that $\mathsf{N}^{\mathsf{W}}_{(\vartheta)}$ converges) asymptotics (11.1.49) holds. Note that $\mu+\beta\ge 0$ and (11.1.98) imply that $\mu<0$, $\vartheta>0$.

Moreover, under the standard non-periodicity condition and improved regularity and non-degeneracy conditions asymptotics (11.1.51) holds.

(ii) Let $\mu+\beta<0$. Then one can take $X_{\mathsf{reg},h}=X\cap\{r\le Ch^{1/(\mu+\beta)}\}$ and $X_{\mathsf{sing},h}=X\cap\{r\ge \frac{1}{2}Ch^{1/(\mu+\beta)}\}$ where $r=|x''|$. Let us assume that a Dirichlet condition is given on $\partial X\cap\{|x''|\ge C\}$ [3]. Then inequality

$$h^m\sum_{\alpha:|\alpha|=m}\|D'^{\alpha}u\|_{(x'')}\ge \epsilon_0 F(r)^{-m}\|u\|_{(x'')} \tag{11.1.99}$$

(where $\|.\|_{(x'')}$ means the $\mathscr{L}^2$-norm over sections of X at points x'') yields that under condition

$$(2m\mu-\beta-\mu)\vartheta+(d-1)\mu-\beta+d'\beta+d''<0 \tag{11.1.100}$$

asymptotics (11.1.49) holds.

Moreover, under standard non-periodicity condition and improved regularity and non-degeneracy conditions asymptotics (11.1.51) holds.

(iii) On the other hand, for

$$(2m\mu-\beta-\mu)\vartheta+(d-1)\mu-\beta+d'\beta+d''=0 \tag{11.1.101}$$

asymptotics (11.1.61) holds and for

$$\begin{aligned}&(2m\mu-\beta-\mu)\vartheta+(d-1)\mu-\beta+d'\beta+d''>0,\\ &\qquad\qquad 2m\mu\vartheta+d\mu+d'\beta+d''\le 0\end{aligned} \tag{11.1.102}$$

asymptotics (11.1.63) holds with $p=(2m\mu\vartheta+d''-d''\beta)(\mu+\beta)^{-1}$; here in the case of equality in the last condition the integral in $\mathsf{N}^{\mathsf{W}}_{(\vartheta)}$ is calculated over the domain $X_{\mathsf{reg},h}$ and $\mathsf{N}^{\mathsf{W}}_{(\vartheta)}=O(\log h)$.

Moreover, let condition (11.1.69) be fulfilled for $|x''|\ge c, x''\in\Gamma$ and $x'\in F(|x''|)\Omega'$ where $\Gamma\subset\mathbb{R}^{d''}$ is a non-empty cone and $\Omega'\subset\Omega$ is a non-empty domain. Then $\mathsf{N}^{\mathsf{W}}_{(\vartheta)}\asymp\log h$.

We leave to the reader to analyze the case of more general dependence of F on x''.

Example 11.1.22 (Conical exit to infinity). (i) In a neighborhood of infinity let the domain X coincide with the cone $X^0 = \{(r,\theta) \in \mathbb{R}^+ \times U\}$ and let the conditions be fulfilled with $\gamma = \epsilon r$, $\rho = r^\mu$. Then if

$$\mu \geq -1, \qquad 2m\mu\vartheta + (\mu+1)d < 0 \tag{11.1.103}$$

then asymptotics (11.1.49) holds.

Moreover, under standard non-periodicity condition and improved regularity and non-degeneracy conditions asymptotics (11.1.51) holds.

(ii) Let us treat the case

$$\mu < \min(-1, -\frac{2m}{d}); \tag{11.1.104}$$

then estimates (11.1.66) and (11.1.67) hold for functions u supported in $\mathbb{R}^d \setminus B(0, \epsilon^{-1}h^{1/(\mu+1)})$ and therefore under condition $(11.1.59)^\#$ asymptotics (11.1.49) holds.

Moreover, under standard non-periodicity condition and improved regularity and non-degeneracy conditions asymptotics (11.1.51) holds.

Here the superscript "#" means that the condition is replaced by its opposite (i.e., ">" by "<" and v.v., and "=" remains), $|x| \to 0$ is replaced by $|x| \to \infty$ and $|x| \leq \epsilon$ is replaced by $|x| \geq C$.

(iii) Further, under condition $(11.1.60)^\#$ we obtain asymptotics (11.1.61) and under condition $(11.1.62)^\#$ we obtain asymptotics (11.1.63) with p given by the same formula as before.

Finally, under condition $(11.1.68)^\#$ asymptotics (11.1.63) remains true with the same clarifications as before.

Moreover, under conditions $(11.1.60)^\#$, $(11.1.71)^\#$ and the standard non-periodicity condition and improved regularity and non-degeneracy conditions we obtain asymptotics (11.1.73).

Example 11.1.23 (Conical exit to infinity. II). Let $\gamma = \epsilon r$, $\rho = r^\mu \log^{-\nu} r$ where

$$\mu \leq -1. \qquad \mu < -\frac{2m}{d} \tag{11.1.105}$$

satisfies (11.1.59).

Then all the results of Example 11.1.17 remain true and the inequalities on ν do not change and in the stabilization conditions and in (11.1.69) $|x| \to 0$ is replaced by $|x| \to \infty$ and $|x| \le \epsilon$ is replaced by $|x| \ge C$. Again we need to change all inequalities to their opposites.

Example 11.1.24 (Quasiconical exit to infinity). In a neighborhood of infinity let X be an L-quasiconical domain and $\partial X \in \mathscr{C}^2$. Then in a neighborhood of infinity one can take $\gamma = \epsilon[x]^{\delta}_{\delta L}$ where we assume that $L = (L_1, \dots, L_d)$, $L_j \ge 1 \ \ \forall j$ and $\delta \le \min_j L_j^{-1}$. Let us assume that $\rho = \gamma^{\mu}$ and $\varrho = \gamma^{2\mu m}$.

In this case all the statements of Example 11.1.18 remain true with the inequalities on μ replaced by their opposites.

We leave to the reader the following rather easy problem:

Problem 11.1.25. Investigate the case in which $X = \mathbb{R}^{d'} \times \mathbb{T}^{d''}$ in a neighborhood of infinity and $\gamma = 1$.

11.1.3 Non-Semi-Bounded oOperators

Let us now consider operators which are non-semi-bounded. In this case we assume that $\vartheta = 0$ unless otherwise indicated. Since the order of operator is not necessarily even anymore, denote it by m rather than by $2m$.

Theorem 11.1.1 and Corollary 11.1.2, as well as Example 11.1.10, remain true in this case with obvious modifications. Moreover, one can treat the Dirac operator in $X = \mathbb{R}^d$ assuming that for $|x| \ge c$

$$V + M - \tau_2 \ge \epsilon_0 \rho^2, \qquad M - V + \tau_1 \ge \epsilon_0 \rho^2 \tag{11.1.106}$$

and

$$\rho\gamma \ge \epsilon_0, \tag{11.1.107}$$

$$\int \rho^{d-s}\gamma^{-s}\, dx < \infty \tag{11.1.108}$$

because these conditions ensure that $\{|x| \ge c\} \subset X_{\mathsf{reg,ell}}$. Recall that M is the mass.

We leave to the reader

Problem 11.1.26. (i) Include a neighborhood of infinity in $X_{\mathsf{sing},h}$ and skip condition (11.1.108);

(ii) Treat the case $X = \mathbb{R}^{d'} \times \mathbb{T}^{d''}$ with $d' > 0, d'' > 0$.

Furthermore, one can treat the *generalized Dirac* operator

$$A = \begin{pmatrix} V_1 & \mathcal{L} \\ \mathcal{L}^* & V_2 \end{pmatrix} \tag{11.1.109}$$

where V_k are matrix-valued functions, $\mathcal{L}$ is a uniformly elliptic operator of order m and for $|x| \ge c$

$$V_2 - \tau_2 \ge \epsilon_0 \rho^{2m}, \qquad V_1 - \tau_1 \le -\epsilon_0 (\rho+1)^{2m}, \qquad \rho\gamma \ge \epsilon_0. \tag{11.1.110}$$

Moreover, we assume that the coefficients of $\mathcal{L}$ satisfy conditions

$$\|D^\alpha b_\beta\| \le c\rho^{m-|\beta|}\gamma^{-|\alpha|} \qquad \forall \beta : |\beta| \le m \tag{11.1.111}$$

and

$$\|D^\alpha V_2\| \le c\rho^{2m}\gamma^{-|\alpha|}, \qquad \|D^\alpha V_1\| \le c(\rho+1)^{2m}\gamma^{-|\alpha|} \qquad \forall \alpha : 1 \le |\alpha| \le K \tag{11.1.112}$$

We can assume instead that the corresponding roughness conditions are fulfilled.

Then

(11.1.113) All examples with singularities at 0 [4] remain true for $X = \mathbb{R}^d$ if in the inequalities for exponents we replace $2m$ by m; however, we do not modify the inequalities (11.1.58) and (11.1.75) to reflect that m rather than $2m$ is an order of the operator.

Further,

(11.1.114) All examples with singularities at infinity [5] remain true without modification in the inequalities on exponents provided in a neighborhood of infinity

$$V_1 - \tau_1 \le -\epsilon_0, \qquad V_2 - \tau_2 \le -\epsilon_0 \rho^{2m}. \tag{11.1.115}$$

Recall that in these examples $\rho \le c$.

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[4] I.e. Examples 11.1.10, 11.1.11 and 11.1.14–11.1.18.

[5] I.e. Examples 11.1.21–11.1.24.

Finally, let us note that under conditions (in X)

$$V_1 - \tau_1 \le -\epsilon_0, \qquad V_2 - \tau_1 \ge \epsilon_0, \qquad \rho \le c \tag{11.1.116}$$

one can treat

$$\mathsf{N}^-_{(\vartheta)}(A) = \int_{\tau_1}^{\tau_2} (\tau_2 - \tau)_+^{\vartheta-1} d\tau \tag{11.1.117}$$

with $\vartheta > 0$. Then Example 11.1.10 and all the examples with singularities at infinity[5)] remain true.

We leave to the reader

Problem 11.1.27. (i) Justify these claims (11.1.113) and (11.1.114) basing on the results of Chapter 10.

(ii) Based on Remark 10.2.23 pass from operators with rough coefficients to operators with irregular coefficients. In particular, using the fact that eigenvalues are monotone with respect to V_1, V_2 consider the case when only V_1 and V_2 are irregular and either τ_1 or τ_2 belong to spectral gap.

11.2 Large Eigenvalues for Operators with Weakly Singular Potentials

[Large Eigenvalues for Weakly Singular Potentials]Large Eigenvalues for Operators with Weakly Singular Potentials

This section is devoted to the General scheme and main theorems.

11.2.1 General Scheme

In this and the following two sections we consider the asymptotics of the eigenvalues of operator A, tending to $+\infty$. In fact, one can treat more general problem

$$(A - \lambda J)u = 0 \qquad u \in \mathfrak{D}(A) \tag{11.2.1}$$

where A and J are operators of orders $2m$ and $n < 2m$ respectively ($m \in \frac{1}{2}\mathbb{Z}$). Recall that in this problem one should assume that

(11.2.2) A is a self-adjoint operator, J is a symmetric operator and J is A-compact[6)] and at least one of operators A and J is positive definite.

Surely, the case $J = I$ is the most natural and interesting from the point of view of miscellaneous, in particular physical applications; in the next sections that case will be treated but in this section we consider the more general case (11.2.1) as well.

In this section we assume that A is semi-bounded from below; at the end of the section $J = I$ and A is not necessarily semi-bounded. Note that Birman-Schwinger principle yields that

(11.2.3) If the operator J is positive definite then the the number of eigenvalues of the problem (11.2.1) less than τ is equal to $\mathsf{N}^-(\tau)$ which is the number of negative eigenvalues of the operator $(A - \tau J)$

and

(11.2.4) If the operator A is positive definite then the number of eigenvalues $\lambda \in (0, \tau)$ of the problem (11.2.1) (with $\tau > 0$) is equal to $\mathsf{N}^-(\tau) - \mathsf{N}^-(+0)$ where $\mathsf{N}^-(\tau)$ is again the number of negative eigenvalues of the operator $(A - \tau J)$ and $\mathsf{N}^-(+0) = \mathsf{lim}_{\tau \to +0}\, \mathsf{N}^-(\tau)$.

There is no contradiction since as both A and J are positive definite problem (11.2.1) has only positive eigenvalues.

Therefore in the both cases we should consider the asymptotics of $\mathsf{N}^-(\tau)$ as $\tau \to +\infty$.

Let us assume that A is given by (11.1.1) with $h = 1$, $m \in \mathbb{Z}$, that conditions (11.1.2) and (11.1.3) are fulfilled and

$$J = \sum_{|\alpha|,|\beta| \le m, |\alpha|+|\beta| \le n} D^\alpha j_{\alpha,\beta} D^\beta, \tag{11.2.5}$$

with the coefficients satisfying

$$|D^\sigma a_{\alpha,\beta}| \le c\varrho\rho^{-|\alpha|-|\beta|}\gamma^{-|\sigma|}, \tag{11.2.6}$$

$$|D^\sigma j_{\alpha,\beta}| \le c\varrho_0\rho^{-|\alpha|-|\beta|}\gamma^{-|\sigma|} \qquad \forall \sigma : |\sigma| \le K \tag{11.2.7}$$

[6)] This condition will be fulfilled automatically in the frameworks of the theorems below.

or corresponding "rough" conditions.

Further, let us assume that the boundary ∂X is smooth enough after rescaling of $\mathcal{B}(y,\gamma(y),X)$ and that the coefficients of the boundary operator $B=(B_1,\dots,B_l)$ satisfy condition $(9.2.3)_2$. Furthermore, let us assume that the operator $\varrho^{-1}\rho^{2m}A$ is uniformly elliptic and that the boundary value problem $(\varrho^{-1}\rho^{2m}A, \varrho_{(1)}^{-1}B_1,\dots,\varrho_{(l)}^{-1}B_l)$ is uniformly elliptic as well.

One can easily see that in the zone $\{x\in X, \rho_\tau\gamma_\tau\ge 1\}$ operator $(A-\tau J)$ satisfies all the conditions of Section 9.1[7] with functions

$$\gamma_\tau=\gamma,\quad \rho_\tau=\rho\bigl(1+\frac{\tau\varrho_0}{\varrho}\bigr)^{\frac{1}{2m-n}},\quad \varrho_\tau=\varrho\bigl(1+\frac{\tau\varrho_0}{\varrho}\bigr)^{\frac{2m}{2m-n}},\quad \varrho_{\tau(j)}=\varrho_{(j)}\bigl(\frac{\tilde\rho}{\rho}\bigr)^{-m_j} \tag{11.2.8}$$

instead of γ, ρ, ϱ and $\varrho_{(j)}$ respectively[8].

Let us treat the microhyperbolicity or non-degeneracy condition in the zone

$$X'_{\mathsf{reg}}=\bigl\{x\in X, \tau\varrho_0\ge C\varrho\bigl(1+(\rho\gamma)^{n-2m}\bigr)\bigr\} \tag{11.2.9}$$

in which

$$\rho_\tau\gamma\ge C' \quad\text{and}\quad \rho_\tau\ge C'\rho \tag{11.2.10}$$

provided $C=C(C')$. We easily see that for large enough C' in this zone one can replace the complete symbols of the operators A, B and J by their senior symbols[9] in the verification of such condition and after rescaling with respect to x and ξ [10] we easily see that

(11.2.11) One needs to check this non-degeneracy condition with $\gamma\partial_x$ instead of ∂_x and with ∂_ξ preserved for the symbol $\bigl(\tilde a(x,\xi)-\tilde\jmath(x,\xi),\tilde b(x,\xi)\bigr)$

where tilde means that one needs to multiply a, j and b_k by ϱ^{-1}, ϱ_0^{-1} and $\varrho_{(k)}^{-1}$ respectively at point x; these factors are "well-differentiable" (in the corresponding scale) with respect to x.

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[7] With the possible exception of the microhyperbolicity, or ξ-microhyperbolicity, or some other non-degeneracy condition.

[8] We use the notation $\varrho_{(k)}$ instead of the ϱ_k of Section 9.1 in order to avoid misunderstanding.

[9] I.e., the higher degree homogeneous symbols.

[10] I.e., rescaling with respect to x and multiplication of the operators by appropriate factors.

Let us first check the conditions which are not associated with the symbol $\tilde{b}$ of the boundary operator. Note that for $|\xi| \ge \epsilon$ with an arbitrarily small constant $\epsilon > 0$ the ellipticity of $\tilde{a}(x,\xi)$, the homogeneity of $\tilde{a}(x,\xi)$ and $\tilde{\jmath}(x,\xi)$ of degrees $2m$ and n with respect to ξ and the Euler identity yield that ξ-microhyperbolicity condition is fulfilled. Therefore *the remaining check should be done for $\xi = 0$ only*.

For general matrix operators one needs to check ξ-microhyperbolicity or microhyperbolicity condition and since $\tilde{a}$ with its first order derivatives vanishes at $\xi = 0$ this condition should be checked for $\tilde{\jmath}(x,\xi)$.

The three following cases are possible:

(i) $n = 0$; then microhyperbolicity condition means exactly that

(11.2.12) $n = 0$ and for every $x \in X_{\mathsf{reg}}$ there exists $\ell \in T_x X$ such that $|\ell| = 1$ and

$$\gamma\langle \ell j(x)v, v\rangle \ge \epsilon_0 \varrho_0 \|v\|^2 - c\varrho_0^{-1}\|j(x)v\|^2 \qquad \forall v \in \mathbb{H} \tag{11.2.13}$$

where $j(x) = J(x)$ and we returned to j from $\tilde{\jmath}$. In particular, this condition is fulfilled automatically if

(11.2.14) $n = 0$ and $\quad \|j(x)v\| \ge \epsilon\varrho_0\|v\| \qquad \forall x \in X' \quad \forall v \in \mathbb{H}$;

(ii) $n = 1$; in this case $\tilde{\jmath}(x,0) = 0$ and one should differentiate with respect to ξ only; then microhyperbolicity condition means exactly that

(11.2.15) $n = 1$ and for every $x \in X_{\mathsf{reg}}$ there exists $\xi \in T_x^* X$ such that $|\xi| = 1$ and

$$\langle j(x,\xi)v, v\rangle \ge \epsilon_0\varrho_0\rho^{-1}\|v\|^2 \qquad \forall v \in \mathbb{H}$$

where $j(x,\xi)$ is the senior symbol of $J(x,D_x)$.

(iii) $n \ge 2$; in this case $\tilde{\jmath}$ and its first order derivatives vanish at $\xi = 0$ and the microhyperbolicity condition is surely violated.

Note that ξ-microhyperbolicity condition is fulfilled in the cases (11.2.14) and (11.2.15) only; in this case remainder estimate $O(h_{\mathsf{eff}}^{1-d})$ is ensured under regularity conditions marginally stronger than $K = 1$ (see Sections 4.6 and 7.5); we refer to them as *minimal regularity assumptions*.

For scalar symbols a and j [11] the following additional possibilities arise:

(iv) $m = 1$, $d \ge 2$; then $(A - \lambda J)$ after all rescaling is the Schrödinger operator and no microhyperbolicity is needed; so we assume that

(11.2.16) $m = 1$, $n = 0, 1$, a and j are scalar symbols and $d \ge 2$; further, as either $d \ge 3$ or $d = 2$ or (11.2.12) holds we recover local remainder estimate $O(h_{\mathrm{eff}}^{1-d})$ under minimal regularity assumptions;

(v) $m = 1$, $d = 1$, $n = 0$; we recover local remainder estimate $O(h_{\mathrm{eff}}^{1-d})$ as $K = K(p)$ provided

(11.2.17) $m = 1$, $d = 1$, $n = 0$ and for every $x \in X_{\mathrm{reg}}$

$$\sum_{\alpha:|\alpha|\le p} \gamma^{|\alpha|} |D^\alpha j| \ge \epsilon_0 \varrho_0; \tag{11.2.18}$$

as $p = 1$ we recover local remainder estimate $O(\log(h_{\mathrm{eff}}))$ under minimal regularity assumptions[12];

(vi) $m = 1$, $d = 1$, $n = 1$; then

$$J = \frac{1}{2}(\langle D_x, j(x)\rangle + \langle j(x), D_x\rangle) + j'(x) \tag{11.2.19}$$

and different variants of non-degeneracy condition are possible:

(11.2.20) $m = 1$, $d = 1$, $n = 1$ and $|j(x)| \ge \epsilon_0 \varrho_0 \rho^{-1}$ for every $x \in X_{\mathrm{reg}}$ ensures local remainder estimate $O(\log(h_{\mathrm{eff}}))$ under under minimal regularity assumptions;

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[11] It is sufficient to assume that either the eigenvalues of $a(x,\xi) - j(x,\xi)$ close to 0 are simple in a neighborhood of $\{\xi = 0\}$ or these eigenvalues are of constant multiplicity there and $\rho\gamma \le C$ in the latter case. For $n \ge 1$ that automatically yields that $(a - j)$ is a scalar symbol and the homogeneity of a and j yield that a and j are scalar symbols.

For $n = 0$ this conjecture is not necessary. In this case in the appropriate local coordinates $j = \begin{pmatrix} j'' & 0 \\ 0 & j'I \end{pmatrix}$, $a = \begin{pmatrix} * & * \\ * & a'I \end{pmatrix}$ where j'' is a non-degenerate matrix and j' and a' are scalar symbols and under the above condition all the following conditions should be checked for symbols a' and j'.

[12] As $p \ge 2$ we need to assume that A and J are scalar operators.

while

(11.2.21) $m = 1$, $d = 1$, $n = 1$ and (11.2.18) ensures local remainder estimate $O(1)$ as $K = K(p)$ [12].

(vii) If $m \ge 2$ then the first and the second order derivatives of a vanish at $\xi = 0$ and we obtain the following cases ensuring local remainder estimate $O(h_{\mathsf{eff}}^{1-d})$ under smoothness assumptions:

(11.2.22) $m \ge 2$, $d \ge 2$, $n = 0$ and for every $x \in X_{\mathsf{reg}}$ inequality

$$|j(x)| + \gamma |\nabla j(x)| \le \epsilon_0 \varrho_0 \tag{11.2.23}$$

yields that $\mathsf{Hess}\, j(x)$ has two eigenvalues with absolute values greater than $\epsilon_0 \varrho_0 \gamma^{-2}$;

(11.2.24) $m \ge 2$, $d \ge 2$, $n = 0$ and for every $x \in X_{\mathsf{reg}}$ (11.2.18) holds with $p = 1$ or $p = 2$ [12];

(11.2.25) $m \ge 2$, $n = 1$ and for every $x \in X_{\mathsf{reg}}$ inequality

$$|j(x)| + \gamma |(Dj)(x)| \ge \epsilon_0 \varrho_0 \rho^{-1} \tag{11.2.26}$$

holds where here J is defined by (11.2.19), $j = j(x)$ is a vector field and (Dj) is the matrix of its derivatives, $j' = j'(x)$.

(viii) Finally, as $m \ge 2$ we can consider $n = 2$ and arrive to

(11.2.27) $m \ge 2$, $d \ge 2$, $n = 2$ and

$$\sum_{\alpha : |\alpha| = 2} |j_\alpha| \ge \epsilon_0 \varrho_0 \rho^{-2} \tag{11.2.28}$$

for every $x \in X_{\mathsf{reg}}$ where j_α are the coefficients of the senior symbol of J.

Remark 11.2.1. (i) For $\vartheta > 0$ we recover a local remainder estimate $O(h_{\mathsf{eff}}^{1-d+\vartheta} \tau^{\vartheta})$ under corresponding regularity assumptions.

(ii) Let both A and J be scalar operators. Then in the smooth case we do not need any non-degeneracy assumptions to recover a local remainder estimate $O(h_{\mathsf{eff}}^{1-d+\vartheta}\tau^{\vartheta})$ as $\vartheta > 0$ and $O(h_{\mathsf{eff}}^{1-d-\delta})$ with arbitrarily small $\delta > 0$ as $\vartheta = 0$; "smooth" means "with $K = K(d, \vartheta)$ or $K = K(d, \delta)$ respectively".

Remark 11.2.2. To avoid $\mathsf{N}^{\mathsf{W}} = 0$ we need to assume that

(11.2.29) $j(x, \xi)$ (the principal symbol of J) is not a non-positive definite matrix (as long as we assume that $a(x, \xi)$ is a positive definite matrix).

We leave to the the following

Problem 11.2.3. Investigate the opposite case when according to Chapter 7 the local remainder estimate may be $O(h_{\mathsf{eff}}^{2-d})$ and $\mathsf{N}^{\mathsf{W}} \asymp h_{\mathsf{eff}}^{1-d}$.

Let us discuss conditions associated with the boundary value problem. Recall that this condition for 1-dimensional operator $f = \tilde{a} - \tilde{\jmath}$ is of the form

$$\mathsf{Re}\,\langle \ell f v, v\rangle_{+} + \mathsf{Re}\,\langle \ell \tilde{b} v, \beta v\rangle_{0} \ge \epsilon_0 \|v\|_{+}^2 \qquad \forall v \in \mathscr{S}'(\mathbb{R}^+, \mathbb{H}) : fv = 0 \text{ and } bv|_{x_1=0} = 0 \tag{11.2.30}$$

where we use the notation of Chapter 7 and make a rescaling; here $f = f(z, D_1)$, $b = b(z, D_1)$ and $\beta = \beta(z, D_1)$ is defined by equality

$$\langle fu, v\rangle_{+} + \langle \tilde{b} u, \beta v\rangle_{0} = \langle u, fv\rangle_{+} + \langle \beta u, \tilde{b} v\rangle_{0} \qquad \forall u, v \in \mathscr{S}(\mathbb{R}^+, \mathbb{H}) \tag{11.2.31}$$

where $z = (x', \xi')$.

Let us note that

(11.2.32) Condition (11.2.30) survives substitution $D_1 \mapsto \theta(z) D_1$ with $\theta \asymp 1$.

In fact, it is obvious for $\theta = \mathsf{const}$ and therefore we need check this statement only for θ equal to 1 at the point in question. In this case additional term

$$(\ell\theta)(z) \times \mathsf{Re}\big(\langle f' v, v\rangle_{+} + \langle \tilde{b}' v, \beta v\rangle_{0}\big) \tag{11.2.33}$$

in the left-hand expression of (11.2.30) appears where the prime means derivative with respect to θ. Let us introduce $v_\theta(x_1) = v(\theta^{-1} x_1)$; then

$f_\theta v_\theta = 0$ and $\tilde{b}_\theta v_\theta|_{x_1=0} = 0$. Therefore treating θ as an independent variable, differentiating and taking $\theta = 1$ we obtain that

$$f'v + fv' = 0, \qquad \tilde{b}'v + \tilde{b}v' = 0 \tag{11.2.34}$$

where the prime on v means derivative on θ also and therefore $v' = -x_1\partial_1 v$. Then (11.2.34), (11.2.31) and the equalities $fv = 0$, $\tilde{b}v|_{x_1=0} = 0$ obviously yield that the second factor in (11.2.30) vanishes; (11.2.32) has been proven.

Let us consider first $|\xi'| \ge \epsilon_0$ and let us introduce $\ell = \langle \xi', \partial_{x'}\rangle$, $\theta = |\xi'|$. Then 1-dimensional operators $\tilde{a}(x', \xi', \theta D_1)$ and $\tilde{\jmath}(x', \xi', \theta D_1)$ are positive homogeneous of degrees $2m$ and n on ξ' respectively. Moreover, operators $\tilde{b}_k(x', \xi', \theta D_1)$ are positive homogeneous of degrees m_k on ξ' and $b_k v = 0$. Therefore the Euler identity yields that the left-hand expression of (11.2.30) is equal to

$$2m\langle \tilde{a}v, v\rangle_+ - n\langle \tilde{\jmath}v, v\rangle_+.$$

Since $(\tilde{a} - \tilde{\jmath})v = 0$ we obtain that this expression is equal to

$$(2m - n)\langle \tilde{a}v, v\rangle_+ = (2m - n)\langle \tilde{\jmath}v, v\rangle_+.$$

Therefore we arrive to

Proposition 11.2.4. *As $|x'| \ge \epsilon_0$ condition* (11.2.30) *is fulfilled provided either*

$$\langle \tilde{a}v, v\rangle_+ \ge \epsilon_0 \|v\|_+^2 \qquad \forall v \in \mathscr{S}(\mathbb{R}^+, \mathbb{H}) : \ \tilde{b}v = 0 \ \textit{at} \ \ x_1 = 0 \tag{11.2.35}$$

or the same condition is fulfilled for $\tilde{\jmath}$:

$$\langle \tilde{\jmath}v, v\rangle_+ \ge \epsilon_0 \|v\|_+^2 \qquad \forall v \in \mathscr{S}(\mathbb{R}^+, \mathbb{H}) : \ \tilde{b}v = 0 \ \textit{at} \ \ x_1 = 0 \tag{11.2.36}$$

If A is a semi-bounded operator (under the boundary condition $Bv = 0$) then the ellipticity of the boundary value problem (A, B) yields that condition (11.2.35) is fulfilled. *In what follows when we consider non-semi-bounded operators A we assume that $J = I$* and then condition (11.2.36) is fulfilled automatically.

Let us consider the case $\xi' = 0$. Let us prove that

(11.2.37) As $|\xi'| = 0$ the kernel of the problem

$$fv = 0, \qquad \tilde{b}v = 0 \quad \text{at} \ \ x_1 = 0, \qquad v \in \mathscr{S}(\mathbb{R}^+, \mathbb{H}) \tag{11.2.38}$$

is trivial.

Then condition (11.2.30) will be automatically fulfilled.

Let us note that if $\xi' = 0$ then $\tilde{a} = a_0 D_1^{2m}$, $\tilde{j} = j_0 D_1^n$, $\tilde{b}_k = b_{0k} D_1^{m_k}$ where a_0, j_0, b_{0k} are matrices (depending on z). Let us replace D_1 by θD_1 and let us introduce $v_\theta(x_1) = v(\theta^{-1}x_1)$ with an arbitrary parameter θ. The above arguments yield the following equality

$$\langle f' v, v\rangle_+ + \langle \tilde{b}' v, \beta v\rangle_0 = 0$$

for any solution of the problem (11.2.38). Let us note that in our case $\tilde{b}' v = 0$ at $x_1 = 0$. Then equalities $fv = 0$, $f = \tilde{a} - \tilde{j}$ and $f' = 2m\tilde{a} - n\tilde{j}$ obviously yield that

$$\langle \tilde{a} v, v\rangle_+ = 0, \qquad \langle \tilde{j} v, v\rangle_+ = 0.$$

The first equality is impossible for $v \neq 0$ if A is a semi-bounded elliptic operator and the second equality is impossible for $v \neq 0$ if $J = I$. So, (11.2.37) and thus (11.2.30) are established.

Thus we arrive to the following

Proposition 11.2.5. *Let one of the following assumptions be fulfilled:*

(a) Either A and J are matrix matrix operators satisfying one of the conditions (11.2.12)–(11.2.15) *or A and J are operators with scalar senior symbols satisfying condition* (11.2.20)*;*

(b) $B\big(x,\gamma(x)\big)\cap\partial X = \emptyset$ and A and J are operators with scalar senior symbols satisfying one of the conditions (11.2.16), (11.2.17), (11.2.21)–(11.2.15)*;*

(c) $d \geq 2$ and A is the Schrödinger operator with a Dirichlet or Neumann boundary condition[13] *on $\partial X \cap B\big(x,\gamma(x)\big)$.*

Then the contribution of $\mathcal{B}\big(x,\gamma(x),X\big) \subset X'_{\mathrm{reg}}$ to the principal part of the asymptotics for $\mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau)$ is $O(h_{\mathrm{eff}}^{-d}\tau^\vartheta)$ [14] *and its contribution to the remainder is $O(h_{\mathrm{eff}}^{1-d+\vartheta}\tau^\vartheta)$ with*

$$h_{\mathrm{eff}} = \Big(\frac{\tau\varrho_0}{\varrho}\Big)^{-\frac{1}{2m-n}}\rho^{-1}\gamma^{-1} \tag{11.2.39}$$

[13] Or a Robin boundary condition with appropriate restrictions on lower order coefficients.

[14] Recall that under assumption (11.2.29) it is $\asymp h_{\mathrm{eff}}^{-d}\tau^\vartheta$.

and therefore the total contribution of X'_{reg} *defined by* (11.2.9) *to the remainder does not exceed*

$$R_{1(\vartheta)}(\tau) = \tau^{\frac{d-1-\vartheta}{2m-n}+\vartheta} \int_{X'_{\mathsf{reg}}} \Big(\frac{\varrho_0}{\varrho}\Big)^{\frac{d-1-\vartheta}{2m-n}} \rho^{d-1-\vartheta}\gamma^{-1-\vartheta}\, dx. \tag{11.2.40}$$

11.2.2 Weak Singularity

Now consider the contribution of

$$X''_{\mathsf{reg}} = X_{\mathsf{reg}} \setminus X'_{\mathsf{reg}} = \{\rho\gamma \ge 1,\ \tau\varrho_0 \le C_0\varrho\}; \tag{11.2.41}$$

we estimate the contribution of $\mathcal{B}\big(x, \gamma(x), X\big) \subset X''_{\mathsf{reg}}$ to both the main part and remainder estimate by $O(h_{\mathsf{eff}}^{-d}\varrho^{\vartheta}\varrho_0^{-\vartheta})$ with $h_{\mathsf{eff}} = \rho^{-1}\gamma^{-1}$ and the total contribution of X''_{reg} does not exceed

$$R_{2(\vartheta)}(\tau) = \int_{X''_{\mathsf{reg}}} \rho^d \varrho^{\vartheta}\varrho_0^{-\vartheta}\, dx \tag{11.2.42}$$

provided

$$\vartheta > 0 \implies J = I \quad \text{i.e.}\ \ n = 0,\ \varrho_0 = 1. \tag{11.2.43}$$

As A and J are scalar operators we can use Remark 11.2.1 and vastly improve the estimate of the contribution of X''_{reg} to the remainder.

As we use such crude estimate in X''_{reg} and hope to recover a reasonable result we will need to assume that singularity is weak.

Further, we need to estimate the contribution of X_{sing}. If we can apply Rozenblioum estimate we arrive to the following

Proposition 11.2.6. *(i) As* $\vartheta = 0$ *the contribution of* X_{sing} *to asymptotics does not exceed*

$$CR_0 + CR'_0 \tag{11.2.44}$$

where R_0 *is given by* (11.1.10) *with exponents satisfying* (11.1.11) *and* (11.1.12) *and*

$$\begin{aligned} R'_0 = \tau^{\frac{d}{2m-n}} &\Big(\int_{X_{\mathsf{sing}}} \varrho^{-p'}\rho^{2mp'}\, dx\Big)^{\frac{d}{(2m-n)p'}} \times \\ &\Big(\int_{X_{\mathsf{sing}}} \varrho_0^{q'}\rho^{-nq'}\, dx\Big)^{\frac{d}{(2m-n)q'}} \times \big(\mathsf{mes}(X_{\mathsf{sing}})\big)^{1-\frac{d}{2mq'}(\frac{1}{p'}+\frac{1}{q'})} \end{aligned} \tag{11.2.45}$$

with exponents p' *and* q' *satisfying conditions* (11.1.11)$^{\#}$ *and* (11.1.12)$^{\#}$ *where the* $^{\#}$ *means that we replace* $2m$, p *and* q *by* $2m-n$, p' *and* q' *respectively.*

We assume that a Dirichlet boundary condition is given on $\partial X \cap \bar{X}_{\mathsf{sing}}$. *Otherwise we should assume that a cone condition is fulfilled (with constant angle and height of the cone),* $\varrho \asymp \rho^{2m}$ *and* $\mathsf{mes}(X_{\mathsf{sing}}) < \infty$.

(ii) As $\vartheta > 0$ *and* $J = I$ *the contribution of the zone* X_{sing} *to the asymptotics does not exceed*

$$CR_{0(\vartheta)} + CR'_{0(\vartheta)} \tag{11.2.46}$$

where

$$R_{0(\vartheta)} = \int_0^\infty t^{\vartheta-1}\Big(\int_{X_{\mathsf{sing}}} (\varrho - t)_+^q \, dx\Big)^{\frac{d}{2mq}} dt \times \Big(\int_{X_sing} \varrho^{-p}\rho^{2mp}\, dx\Big)^{\frac{d}{2mp}} \times \big(\mathsf{mes}(X_{\mathsf{sing}})\big)^{1-\frac{d}{2m}(\frac{1}{p}+\frac{1}{q})}, \tag{11.2.47}$$

$$R'_{0(\vartheta)} = \tau^{\vartheta+\frac{d}{2m}}\Big(\int_{X_{\mathsf{sing}}} \varrho^{-p}\rho^{2mp}\, dx\Big)^{\frac{d}{2mp}} \times \big(\mathsf{mes}(X_{\mathsf{sing}})\big)^{1-\frac{d}{2mp}} \tag{11.2.48}$$

and exponents p *and* q *satisfy conditions* (11.1.11)–(11.1.12).

11.2.3 Main Theorems

Let us start from the standard asymptotics. Then we improve the remainder estimate under conditions on the Hamiltonian trajectories in X'_{reg}. Resulting from this analysis we obtain the following statements:

Theorem 11.2.7. *Let* $X = \mathcal{X}_{\mathsf{reg}} \cup \mathcal{X}_{\mathsf{sing}}$, $\mathsf{mes}(\mathcal{X}_{\mathsf{sing}}) = 0$. *Let us assume that in* $\mathcal{X}_{\mathsf{reg}}$ *all the conditions* (9.1.5)–(9.1.13), (9.1.15) *(excluding restriction* $\rho\gamma \ge 1$*) are fulfilled.*

Further, let us assume that $\varrho^{-1}\rho^{2m}a^0(x,\xi)$ *is uniformly classically elliptic and the corresponding boundary value problem for it is also uniformly classically elliptic at the boundary points.*

Furthermore, let us assume that in each ball $\mathcal{B}(x,\gamma(x),X) \subset \mathcal{X}_{\mathsf{reg}}$ *one of the assumptions (a), (b) or (c) of Proposition 11.2.5 is fulfilled. Then*

(i) The following lower estimate holds:

$$\mathsf{N}(0,\tau) \ge \mathsf{N}^{\mathsf{W}}(0,\tau) - CR_{1(0)}(\tau) \tag{11.2.49}$$

where $\mathsf{N}^{\mathsf{W}}(0,\tau)$ *is defined by cut Weyl formula (with integration over* X'_{reg}*):*

$$\mathsf{N}^{\mathsf{W}}(0,\tau) = \varkappa_0(\tau)\tau^{\frac{d}{2m-n}}, \tag{11.2.50}$$

$$\varkappa_0(\tau) = (2\pi)^{-d}\int_{X'_{\mathsf{reg}}(\tau)} \boldsymbol{n}(x,\xi;0,1)\,dxd\xi, \tag{11.2.51}$$

$\boldsymbol{n}(x,\xi;0,1)$ *is the eigenvalue counting function for the symbol* $j^{-1}(x,\xi)a(x,\xi)$ *on the interval* $(0,1)$*,* $R_{1(0)}(\tau)$ *is defined by* (11.2.40) *with* $\vartheta = 0$*;* $X'_{\mathsf{reg}}(\tau)$ *is defined by* (11.2.10)*.*

(ii) Further, let us assume that in

$$X_{\mathsf{sing}}(\tau) = X \setminus \bigl(X'_{\mathsf{reg}}(\tau) \cup X''_{\mathsf{reg}}(\tau)\bigr) \tag{11.2.52}$$

we can apply Rozenblioum estimate. Then the upper estimate holds

$$\mathsf{N}(0,\tau) \le \mathsf{N}^{\mathsf{W}}(0,\tau) + CR_{1(0)}(\tau) + CR_{2(0)}(\tau) + CR_0(\tau) + CR'_0(\tau) \tag{11.2.53}$$

where $R_{2(0)}(\tau)$*,* $R_0(\tau)$*,* $R'_0(\tau)$ *are defined by* (11.2.42)*,* (11.1.10)*,* (11.2.45) *respectively with* $X''_{\mathsf{reg}}(\tau)$ *and* $X_{\mathsf{sing}}(\tau)$ *defined by* (11.2.42) *and* (11.2.52)*.*

Similarly, as $\vartheta > 0$ we arrive to

Theorem 11.2.8. *Let assumptions of Theorem 11.2.7 be fulfilled. Moreover, let us assume that* $J = I$ *and* A *is semi-bounded from below. Then*

(i) The following lower estimate holds:

$$\mathsf{N}^{-}_{(\vartheta)}(\tau) \ge \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) - CR_{1(\vartheta)}(\tau) \tag{11.2.54}$$

where $\mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau)$ *is defined by cut Weyl formula (with integration over* X'_{reg}*),* $R_{1(\vartheta)}(\tau)$ *is defined by* (11.2.40)*.*

item Further, let us assume that in $X_{\mathsf{sing}}(\tau)$ *defined by* (11.2.52) *we can apply Rozenblioum estimate. Then the upper estimate holds*

$$\begin{aligned}\mathsf{N}^{-}_{(\vartheta)}(\tau) \ge \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau)+\\ CR_{1(\vartheta)}(\tau) + CR_{2(\vartheta)}(\tau) + CR_{0(\vartheta)}(\tau) + CR'_{0(\vartheta)}(\tau)\end{aligned} \tag{11.2.55}$$

where $R_{2(\vartheta)}(\tau)$*,* $R_{0(\vartheta)}(\tau)$*,* $R'_{0(\vartheta)}(\tau)$ *are defined by* (11.2.42)*,* (11.2.46)*,* (11.2.45) *respectively with* $X''_{\mathsf{reg}}(\tau)$ *defined by* (11.2.42)*.*

Obviously, if X is a compact domain with the smooth boundary and $J(x) = I$ we conclude that

$$\mathsf{N}^{-}_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O\bigl(\tau^{\frac{d-1-\vartheta}{2m}+\vartheta}\bigr) \qquad \text{as } \tau \to +\infty \tag{11.2.56}$$

with

$$\mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) \asymp \tau^{\frac{d}{2m}+\vartheta}. \tag{11.2.57}$$

Our goal now is to consider the *weakly singular* case when this conclusion holds. Since X'_{reg} extends to $\mathcal{X}_{\mathsf{reg}}$ as $\tau \to +\infty$, we need to assume that

$$\bigl(\frac{\varrho_0}{\varrho}\rho^{2m-n}\bigr)^{\frac{d}{2m-n}} \in \mathscr{L}^1(X) \tag{11.2.58}$$

and

$$\bigl(\frac{\varrho_0}{\varrho}\rho^{2m-n}\bigr)^{\frac{d-1-\vartheta}{2m-n}} \gamma^{-1-\vartheta} \in \mathscr{L}^1(X). \tag{11.2.59}$$

Corollary 11.2.9. *Assume that conditions of Theorem 11.2.7 as $\vartheta = 0$ or Theorem 11.2.8 as $\vartheta > 0$ are fulfilled. Further, let conditions* (11.2.58)–(11.2.59) *be fulfilled. Then*

$$R_{2(\vartheta)}(\tau) = o\bigl(\tau^{\frac{d-1-\vartheta}{2m-n}+\vartheta}\bigr) \qquad \textit{as } \tau \to +\infty \tag{11.2.60}$$

and assuming that

$$R_{0(\vartheta)}(\tau) + R'_{0(\vartheta)}(\tau) = O\bigl(\tau^{\frac{d-1-\vartheta}{2m-n}+\vartheta}\bigr) \qquad \textit{as } \tau \to +\infty \tag{11.2.61}$$

we conclude that (11.2.56)–(11.2.57) *hold.*

Remark 11.2.10. (i) One can easily see that extending integration in $\mathsf{N}^{\mathsf{W}}(\tau)$ from X_{reg} to X causes an error not exceeding (11.2.44). So we can replace cut Weyl expression by the full Weyl expression.

(ii) Also skipping lower degree terms in N^{W} leads to an error not exceeding $CR_{1(0)}(\tau)$; this leads to (11.2.50)–(11.2.51).

(iii) Combining Statements (i) and (ii) we arrive to (11.2.51) with integration over X (then $\varkappa_0$ does not depend on τ).

(iv) Some of these conclusions remain true for $\vartheta > 0$, at least with some modifications: for example, (11.2.50)–(11.2.51) becomes

$$\mathsf{N}^{\mathsf{W}}_{(\vartheta)}(0,\tau) = \sum_{0\le n<1+\vartheta} \varkappa_n(\tau)\tau^{\frac{d-n}{2m-n}+\vartheta} \tag{11.2.62}$$

but extension of integration in $\varkappa_n(\tau)$ terms is justified only under assumption (11.2.59) with ϑ replaced by n (and some other assumptions may be required as well).

(v) Theorems 11.2.7 and 11.2.8 and Corollary 11.2.9 remain true for compact $\mathscr{C}^K$ manifolds.

Remark 11.2.11. In the case $\varrho = \rho^{2m}$ and $2m-n<d$ one can set $q' = \frac{d}{2m-n}$ and then (11.2.59) yields (11.2.61).

We can slightly improve the results of above theorems under assumptions to Hamiltonian trajectories:

Theorem 11.2.12. *Assume that conditions of Theorem 11.2.7 as $\vartheta = 0$ or Theorem 11.2.8 as $\vartheta > 0$ are fulfilled.*

Further, let us assume that non-periodicity condition (11.1.55)–(11.1.56) *and improved smoothness and non-degeneracy conditions of Remark 11.1.8 hold for symbol $a^0(x,\xi) - j^0(x,\xi)$ on the energy level 0. Then sharp versions of asymptotics* (11.2.54), (11.2.55) *hold where "sharp version" means that $R_{1(\vartheta)}(\tau)$ is replaced by*

$$R'_{1(\vartheta)}(\tau) = o(1)R_{1(\vartheta)}(\tau) + R_{1(\vartheta-\delta)}(\tau) \tag{11.2.63}$$

where $\delta > 0$ is sufficiently small exponent and formula (11.1.39) *defining $R_{1(\vartheta)}$ is extended to $\vartheta < 0$ if needed; also in $\mathsf{N}^{\mathsf{W}}_{(\vartheta)}$ one extra term should be included if needed.*

Sharper asymptotics under more restrictive conditions on the Hamiltonian flow will be treated later.

11.2.4 Examples

Consider some examples. Let us start with examples in which $J = I$, $\varrho = \rho = 1$. Then conditions (11.2.58), (11.2.59) mean respectively that

$$1 \in \mathscr{L}^1(X) \qquad \text{i.e.} \quad \mathsf{mes}(X) < \infty, \tag{11.2.64}$$

$$\gamma^{-1-\vartheta} \in \mathscr{L}^1(X). \tag{11.2.65}$$

In the first examples the coefficients of A are regular and γ is completely defined by the geometry of the domain X.

Example 11.2.13 (Quasipolygonal and quasipolyhedral domains). Let X be a bounded domain with ∂X contained in the union of a finite number of $\mathscr{C}^K$ hypersurfaces[15] $\tilde{S}_l$ (with $l = 1, \dots, L$).

Let $S_l = \tilde{S}_l \cap \partial X$ and δS_l be the relative boundary of S_l in $\tilde{S}_l$. Introduce the *singular set* of ∂X

$$\mathcal{K} = \bigcup_{l \neq k} (S_l \cap S_k) \cup \bigcup_{l} \delta S_k \tag{11.2.66}$$

Obviously, we can take

$$\gamma(x) \asymp \min\Big(\min_{l \neq k} \gamma_{lk}(x), \mathsf{dist}(x, \mathcal{K})\Big), \tag{11.2.67}$$

with

$$\gamma_{lk}(x) = \max\big(\gamma_l(x), \gamma_k(x)\big) \tag{11.2.68}$$

where $\gamma_l(x) = \mathsf{dist}(x, S_l)$.

Then condition (11.2.65) needs to be verified for γ_{lk} for all $l \neq k$ and for $\gamma'(x) = \mathsf{dist}(x, \mathcal{K})$ separately.

First of all,

(11.2.69) If the hypersurfaces $\tilde{S}_l$ and $\tilde{S}_k$ are transversal at $S_k \cap S_l$ then condition (11.2.65) is fulfilled for $\gamma = \gamma_{kl}$ and $\vartheta < \vartheta^*(= 1)$.

Moreover,

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[15] Actually we need only regularity required to produce local remainder estimate $O(h_{\mathsf{eff}}^{1-d+\vartheta})$ or $o(h_{\mathsf{eff}}^{1-d+\vartheta})$ in case we are interested in the sharp asymptotics.

(11.2.70) If $\mathcal{K}$ is contained in the union of a finite number of $\mathscr{C}^K$ manifolds of codimension 2 then condition (11.2.65) is fulfilled for $\gamma = \gamma'$ for $\vartheta < \vartheta^* = 1$ as well.

One can see easily that the above assumptions yield the cone condition. Therefore

(i) For appropriate boundary conditions[16)] asymptotics (11.2.56) holds for $\vartheta < \vartheta^*$.

Moreover, under standard non-periodicity condition (11.1.55)–(11.1.56) to the Hamiltonian flow of the symbol $\mathsf{a}^0(x,\xi)$ on the energy level 1 asymptotics

$$\mathsf{N}^-_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + o\bigl(\tau^{\frac{d-1-\vartheta}{2m}+\vartheta}\bigr) \qquad \text{as } \tau \to +\infty \tag{11.2.56)*}$$

holds.

(ii) Further, our analysis yields that for $\vartheta = \vartheta^*$ asymptotics

$$\mathsf{N}^-_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O\bigl(\tau^{\frac{1}{2m}(d-\vartheta-1)+\vartheta}\log\tau\bigr) \tag{11.2.71}$$

holds as $\tau \to +\infty$ and for $\vartheta > \vartheta^*$ asymptotics

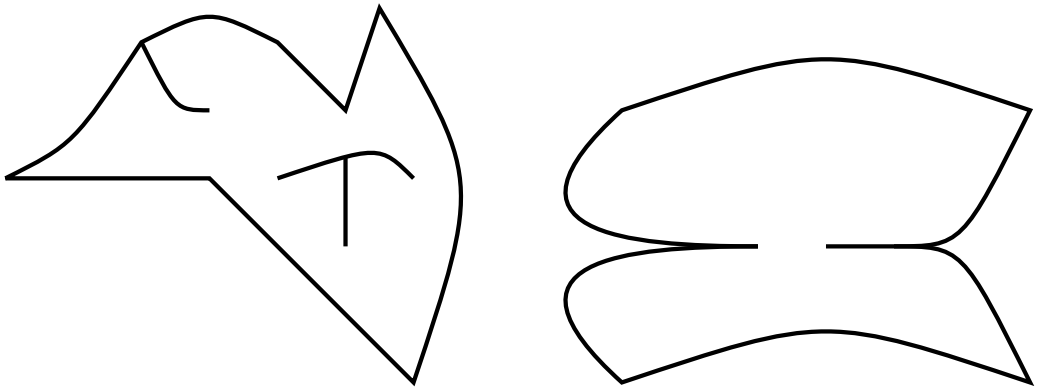
$$\mathsf{N}^-_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O\bigl(\tau^{\frac{1}{2m}(d-\vartheta^*-1)+\vartheta}\bigr) \tag{11.2.72}$$

holds as $\tau \to +\infty$.

Later we will show that under appropriate assumptions sharper asymptotics than (11.2.71)–(11.2.72) holds and in the latter case some extra non-Weyl term(s) associated with the singularities at $\mathcal{K}$ come into play.

Example 11.2.14 (Spikes and conjugations). Let us drop transversally condition (11.2.69) thus allowing *spikes*. Consider first spikes directed inward as on Figure 11.1(b).

[16)] Such as Dirichlet, Neumann and Robin boundary conditions, or boundary conditions which are smooth, elliptic, regular (see condition (7.1.14)) on $S_l \setminus \mathcal{K}$ and such that A is self-adjoint and $(Au,u) \ge \epsilon_0\|u\|^2_m - c\|u\|^2_0$.



(a) (b)

Figure 11.1: Conclusions of Example 11.2.13 holds despite on (b) transversally condition violated

(i) Assume first that locally only two surfaces S_k and S_l are involved and in appropriate coordinates

$$S_j = \{x_d = f_j(x'),\ x' \in X'\}, \qquad (f_j - f_i)|_{\partial X'} = 0 \tag{11.2.73}$$

where $X' \subset \mathbb{R}^{d-1}$, $x' = (x_1, \dots, x_{d-1})$, $|\nabla f_j| \le \epsilon$. Then assuming that locally $\mathsf{C}X$ is between surfaces

$$\mathsf{C}X = \{f_k(x') \le x_d \le f_l(x'),\ x' \in X'\} \cup \{f_l(x') \le x_d \le f_k(x'),\ x' \in X'\} \tag{11.2.74}$$

we conclude that $\gamma(x) \asymp \gamma(x') = \mathsf{dist}(x, \mathcal{K})$ with

$$\mathcal{K} = \{x : x' = \mathcal{K}', x_d = f_j(x')\}, \quad \mathcal{K}' = \partial X' \cup \{x', f_j(x') = f_k(x')\}. \tag{11.2.75}$$

Then condition (11.2.65) is fulfilled for $\vartheta < \vartheta^* = 1$ provided condition (11.2.70) is fulfilled. Moreover, cone condition is fulfilled as well and then all conclusions of Example 11.2.13 remain valid.

(ii) Further, the same conclusions obviously remain true as locally there is *conjugation* instead of spike

$$S_j = \{x_d = f_j(x'),\ x' \in X'_j\}, \qquad (f_j - f_k)|_{\partial X'_j \cap \partial X'_k} = 0 \quad \text{for} \quad k \ne j \tag{11.2.76}$$

as X'_j form locally a partition of the vicinity of 0 in $\mathbb{R}^{d-1}$.

(iii) Furthermore, we can consider inward spike or conjugation and add extra surfaces to ∂X assuming that each new surface is either transversal to existing surfaces or further limits $\complement X$. For example, we can consider multidimensional inward spike:

$$\complement X = \{x_1 > 0, x' \in f(x_1)X'\} \tag{11.2.77}$$

with $f > 0, f'_j > 0$ as $x_1 > 0$, $f(0) = f'(0) = 0$ and X' bounded convex quasipolygon of Example 11.2.14.

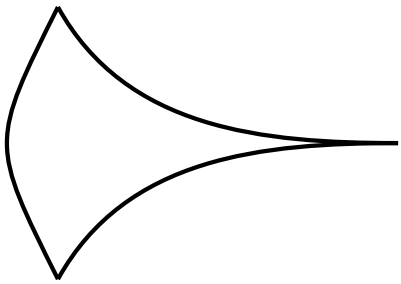


Figure 11.2: Outward spike

Example 11.2.15 (Spikes. II). Outward spike are completely different matter. Assume first that locally only two surfaces S_k and S_l are involved and in appropriate coordinates (11.2.73) and (11.2.74) are fulfilled with $\complement X$ replaced by X

$$X = \{f_k(x') < x_d < f_l(x'),\ x' \in X'\} \cup \{f_l(x') < x_d < f_k(x'),\ x' \in X'\} \tag{11.2.78}$$

as on Figure 11.2. Obviously, without any loss of the generality one can assume that $f_k = f$, $f_l = -f$ and then $\gamma_{kl} \asymp |f|$. Therefore condition (11.2.65) becomes

$$|f|^{-\vartheta} \in \mathscr{L}^1(X') \tag{11.2.79}$$

which obviously is fulfilled for $\vartheta = 0$, violated for $\vartheta \geq 1$ and may be fulfilled or not for any particular $\vartheta \in (0,1)$; f.e. as $f = f(x_1) \asymp x_1^r$ for $x_1 > 0$ ($r > 1$) this condition holds for $\vartheta < \vartheta^* = r^{-1}$ while for $f = O(x_1^\infty)$ at 0 this condition holds only for $\vartheta = 0$.

So, if (11.2.79) is fulfilled then asymptotics (11.2.56) holds for appropriate boundary conditions. Moreover, under the standard non-periodicity condition (11.1.55)–(11.1.56) to the Hamiltonian flow of the symbol $a^0(x,\xi)$ on the energy level 1 asymptotics (11.2.56)* holds.

Problem 11.2.16. (i) Derive asymptotics similar to (11.1.69) or (11.1.70) for particular functions f as condition (11.2.79) is violated.

(ii) Consider multidimensional outward spikes like one similar to (11.2.77):

$$X = \{x_1 > 0, x' \in f(x_1)X'\} \tag{11.2.80}$$

with $f > 0, f_j' > 0$ as $x_1 > 0$, $f(0) = f'(0) = 0$ and X' bounded convex quasipolygon of Example 11.2.14.

(iii) Consider the mixture of inward-outward spikes like (11.2.80) but without any convexity assumptions.

Remark 11.2.17. (i) n the case of "outward spike" the cone condition now is not ensured, so "appropriate boundary conditions" in 11.2.16(i), (ii) includes cone condition. In particular cone condition is fulfilled if at least on one side of the spike we have a Dirichlet boundary condition. There is not much known about more general conditions. Still for 2-dimensional Laplace operator with Neumann boundary condition Yu. Netrusov and Yu. Safarov [1, 2] in pretty special case derived variational estimate which allows to allow Neumann boundary condition on the whole boundary.

(ii) Later in Section 12.7 we will derive spectral asymptotics for multidimensional Neumann Laplacian and those methods could be applied in some of the examples discussed here to estimate the contribution of zone $\{x : \gamma(x) \le \tau^{-1/2}\}$ to the asymptotics thus extending corresponding results (see Subsection 12.7.8). However there will be also plenty of examples when conclusions valid for Dirichlet Laplacian are wrong for Neumann Laplacian.

In the series of examples below $\gamma(x) \asymp \mathsf{dist}(x, \mathcal{K})$ where $\mathcal{K}$ is the union of a finite number of $\mathscr{C}^K$ manifolds of codimension $\delta \ge 2$. Namely, we consider domains with conical singularities of the boundary, domains the boundaries of which contain components of codimension $\delta \in [2, 4m-1]$ [17]. Then under appropriate assumptions critical exponent is $\vartheta^* = \delta - 1$.

[17] For operators of order $2m$ a boundary operator of degree less than $2m - \frac{1}{2}\delta$ can be considered on a δ-codimensional submanifold.

Example 11.2.18 (Scheme). Let X' be a domain in $\mathbb{R}^{d'}$. Assume that

(11.2.81) X' satisfies condition (9.1.15) with γ replaced by $\gamma' = \gamma'(x')$ such that $\gamma' \le c$ and $\gamma'^{-1-\vartheta} \in \mathscr{L}^1(X')$

Let $X = \{x = (x', x''), x'' \in \mathbb{R}^{d''}, x' \in f(x'')X'\}$ [18] where the function f satisfies condition

$$|D^\alpha f| \le c f^{1-|\alpha|} \qquad \forall \alpha : |\alpha| \le K. \tag{11.2.82}$$

Then for the domain X the function

$$\gamma(x) = \epsilon \gamma'(x' f^{-1}(x'')) f(x'') \tag{11.2.83}$$

is admissible and in this case condition (11.2.65) is equivalent to

$$f^{d'-\vartheta-1} \in \mathscr{L}^1; \tag{11.2.84}$$

in particular, this condition as $\vartheta = 0$ is equivalent to the condition

(11.2.85) the $(d-1)$-dimensional measure of ∂X is finite.

Example 11.2.19 (Cusps). This scheme described in Example 11.2.18 includes cusps (exits to infinity): if $f \asymp |x''|^\nu$ with $\nu < 0$ for $|x''| \ge 1$ then the following cases are possible:

(a) Let $\vartheta < \vartheta^*$ with

$$\vartheta^* = d' - 1 - d''\nu^{-1} \iff (d' - 1 - \vartheta)\nu + d'' = 0. \tag{11.2.86}$$

Then asymptotics (11.2.56) holds.

Moreover, under standard non-periodicity condition (11.1.55)–(11.1.56) to the Hamiltonian flow of the symbol $a^0(x, \xi)$ on the energy level 1 asymptotics (11.2.56)* holds.

(b) For $\vartheta = \vartheta^*$ asymptotics with the remainder estimate $O\big(\tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta} \log \tau\big)$ holds.

[18] For $d'' = 1$ one can take only one semiaxis, i.e., replace $\mathbb{R}$ by $\mathbb{R}^\pm$.

(c) For $\vartheta > \vartheta^*$, $d'\nu + d'' < 0$, asymptotics with the remainder estimate $O\bigl(\tau^{\frac{1}{2m}(d-1-\vartheta^*)+\vartheta}\bigr)$ holds.

Moreover, if $\vartheta^* \in \mathbb{Z}^+$ then the last term of these asymptotics is of the form $\varkappa_{\vartheta,\vartheta^*+1}(\tau)\tau^{\frac{1}{2m}(d-1-\vartheta^*)+\vartheta}$ with $\varkappa_{\vartheta,\vartheta^*+1}(\tau) = O(\log\tau)$ while all other terms in this case and all terms in the other cases are of the form $\varkappa_{\vartheta,n}\tau^{\frac{1}{2m}(d-n)+\vartheta}$.

(d) If $d'\nu + d'' = 0$ then the previous statement remains true with only the term $\varkappa_{\vartheta,0}(\tau)\tau^{\frac{d}{2m}+\vartheta}$ in the principal part with $\varkappa_{\vartheta,0}(\tau) \asymp \log\tau$.

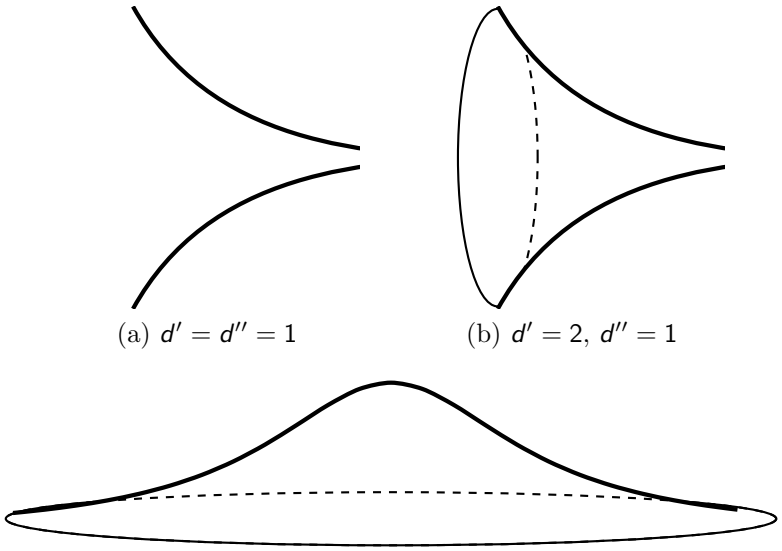


Figure 11.3: Cusps; in (c) X bounded by a drawn "infinitely wide hat" and horizontal plane

Remark 11.2.20. (i) As $d'' \geq 2$ we can consider more general cusp bases than $X'' = \mathbb{R}^{d''}$, most notably we can consider smooth consider a cone $X'' \subset \mathbb{R}^{d''}$ with a smooth boundary; rounding near $\partial X''$ here is obvious.

(ii) We treat Example 11.2.19 in Section 12.2 again and obtain non-Weyl asymptotics with a more precise remainder estimate in cases (b)–(d) and and also in the case $d'\nu + d'' > 0$. We will consider also a generalization mention above.

(iii) In Example 11.2.19 we need to assume that a Dirichlet boundary condition is given on a significant part of ∂X so the cone condition if fulfilled. For example, we can assume that the Dirichlet boundary condition is given as $x' \in Y_{\mathsf{D}} \subset \partial X'$ and the following infinite cone condition is fulfilled:

(11.2.87) there exist an infinite cone $\Gamma \subset \mathbb{R}^{d'}$ with a vertex at 0 such that for each $x' \in X'$ connected component of $(x' + \Gamma) \cap X'$ is disjoint from $\partial X' \setminus Y_{\mathsf{D}}$.

(iv) Without the assumption of (iii) (f.e. for Neumann Laplacian in domain with cusps we need to assume that the cusp is very thin: $|\log f| \asymp |x''|^{\kappa}$ with $\kappa > 0$. Otherwise massive essential spectrum appears. We treat the Neumann Laplacian in domain with "piggy-tail" cusps in Section 12.7 and derive possibly non-Weyl asymptotics.

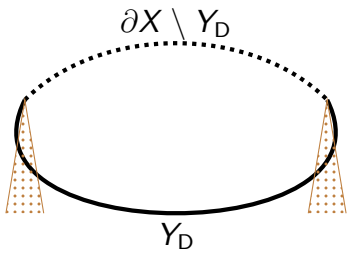


Figure 11.4: Infinite cone condition

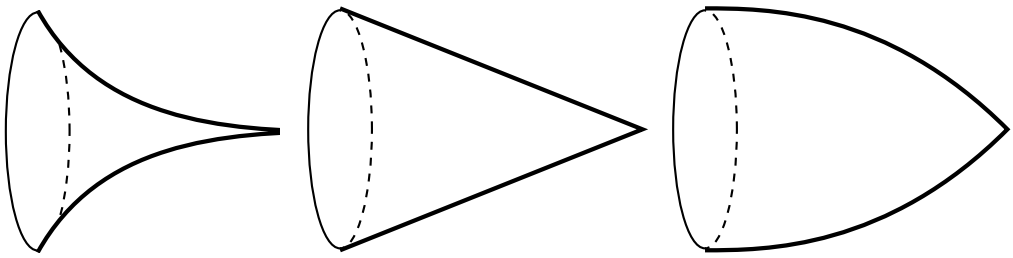
Example 11.2.21 (Conical and sharp quasiconical singularities). (i) The scheme described in Example 11.2.18 covers conical singularities as well as sharp quasiconical singularities directed outward. Namely let

$$X = \{x = (x', x''),\ x' \in |x''|^{\nu} \cdot X',\ |x''| \le 1\} \tag{11.2.88}$$

with $\nu \ge 1$; for $d'' = 1$ one can take only half of this singularity (lying in $\{x, x_1 > 0\}$ where here $x'' = (x_1)$ for $d'' = 1$)). Then we have conical and quasiconical singularities as $\nu = 1$ and $\nu > 1$ respectively. Here we use X', γ' and γ described in Example 11.2.19.

The classification coincides with that in Example 11.2.19 but in all the inequalities on the exponents "<" should be replaced by ">" and v.v., and case (d) is impossible.

(ii) If the domain X described in (i) is contained in the ball $B(0,R)$ then $\gamma(x) = \max\bigl(\gamma_{\mathsf{old}}(x)), \operatorname{dist}(x,\partial X)\bigr)$ works for the domain $\complement X$ where γ_{old} is defined as γ in (i). One can easily see that the classification given in (i) remains true. It allows us to cover sharp quasiconical singularities directed inward.



(a) acute (b) cone (c) blunt

Figure 11.5: Conical and quasiconical singularities, could be directed inward and outward

Example 11.2.22 (Blunt quasiconical singularity). (i) Let X' and γ' be as in Example 11.2.19 again, let $0 \in X'$ and let $X = \{x = (x', x''), x'' \in B(0,R), x' \in f(|x''|)X'\}$ where $f(0) = 0$ and f is a monotone decreasing function such that the inverse function f satisfies condition (11.2.82) in dimension 1. Then for $x'' \in B(0, \frac{1}{2}R)$ one can take $\gamma(x) = \epsilon|x|\gamma'(x'f^{-1}(|x''|))$. Then condition (11.2.65) fulfilled if and only if

$$f^{d'}(r)r^{-\vartheta-1+d''-1} \in \mathscr{L}^1([0,1]). \tag{11.2.89}$$

This example covers various blunt quasiconical singularities directed outward of the type $X = \{x = (x', x''), |x'| \le |x''|^\nu\}$ with $\nu \in (0,1]$.

(ii) One can replace X by $B(0,R) \setminus X$ with the γ defined above. Again condition (11.2.89) is crucial. This example covers various blunt quasiconical singularities directed inward.

Remark 11.2.23. In Examples 11.2.21 and 11.2.22 cone condition poses restrictions, especially as $\nu > 1$ in their parts (i). And again under certain assumptions we will be able to allow Neumann Laplacian (see Section 12.7).

Example 11.2.24 (Clews). We can consider domain containing "disk with an infinite cut" $r = r(\theta)$ with $\theta_0 \le \theta < \infty$ with $r(\theta) = \theta^{-\alpha}$ as $\theta \to \infty$ (we use polar coordinates here). As $\alpha \le 1$ the length of the cut is infinite.

On this cut Dirichlet boundary condition should be imposed. We leave details to the reader.

Example 11.2.25 (Combs). (i) Let

$$X = \{x = (x_1, x_2, x''), 0 < x_1 < 1,\ -1 < x_2 < f(x_1), |x''| < 1\} \tag{11.2.90}$$

where $f(x_1)$ is the jump function given by $f(x_1) = h_i = \mathsf{const} \ge 0$ for $x_1 \in I_i$ and $f(x_1) = 0$ for $x_1 \in I_\infty$ where $I_i \cap I_j = \emptyset$ for $i \ne j$, I_i with $i < \infty$ are open intervals of lengths l_i, $I_\infty = [0,1] \setminus \cup_{i<\infty} I_i$ and $\mathsf{mes}(I_\infty) = 0$.

Then the subdomain $X \cap \{x_2 > 0\}$ consists of separate "boxes" in each of which one should take γ according to scheme described in Example 11.2.18. Then for $\vartheta \in [0,1)$ the contribution of the i-th box to $\int \gamma^{-1-\vartheta} dx$ is of the order $h_i l_i^{-\vartheta} + l_i^{1-\vartheta}$.

On the other hand, on $X \cap \{x_2 \le 0\}$ one can take $\gamma(x) = \frac{1}{2}(-x_2 + \delta(x_1))$ where $\delta(x_1) = \mathsf{dist}(x_1, I_\infty)$. Then the contribution of this subdomain is of the order

$$\int |\log \delta(x_1)| dx_1 \asymp \sum_i -l_i \log l_i \qquad \text{for } \vartheta = 0,$$

$$\int \delta(x_1)^{-\vartheta} dx_1 \asymp \sum_i l_i^{1-\vartheta} \qquad \text{for } \vartheta > 0.$$

Therefore condition (11.2.65) with $\vartheta = 0$ means that

$$\sum_i h_i < \infty, \quad \sum_i -l_i \log l_i < \infty \tag{11.2.91}$$

and condition (11.2.65) with $\vartheta > 0$ means that

$$\sum_i h_i l_i^{-\vartheta} < \infty, \qquad \sum_i l_i^{1-\vartheta} < \infty. \tag{11.2.92}$$

(ii) Recall that under the cone condition, a Dirichlet boundary value problem can be replaced by an arbitrary regular boundary value problem. Recall that this condition does not mean that the cone Γ lies completely in X; we assume that the intersection of Γ with X is connected and a Dirichlet condition is given on $\Gamma \cap \partial X$.

Example 11.2.26 (Trees). One can consider example when X is a *tree*: for example it consists of cylindrical pipes $P_{i,j}$ of the lengths $l_{i,j}$ and radii r_i ($d''=1$, $d'=d-1$) where $i=1,\dots,\infty$, $j=1,\dots,J(i)$ and each pipe $P_{i,*}$ has no more than c pipes $P_{i+1,j}$ on its end. Let us assume that $l_i \ge r_i$ and $r_{i+1}\le r_i \le c r_{i+1}$ for all i.

Assume that the cross-section of pipes and conjugations are regular enough so we can select $\epsilon_0 r_i$ as γ around $P_{i,*}$. Then the total volume is finite if and only if

$$\sum_i r_i^{d-1} L_i < \infty, \tag{11.2.93}$$

and condition (11.2.65) holds if and only if

$$\sum_i r_i^{d-2-\vartheta} L_i < \infty, \tag{11.2.94}$$

where $L_i = \sum_{1\le j\le J(i)} l_{i,j}$.

These pipes are not necessarily cylindrical: we can take any bounded domain with $\mathscr{C}^K$ boundary and make $P_{i,j}$ by scaling it l_i times along $x''=(x_1)$ and r_i times along x'. These trees could be bounded as $\sum_i l_i^* < \infty$ or extended to infinity as $\sum_i l_i^* = \infty$, $l_i^* = \max_{1\le j\le J(i)} l_{i,j}$.

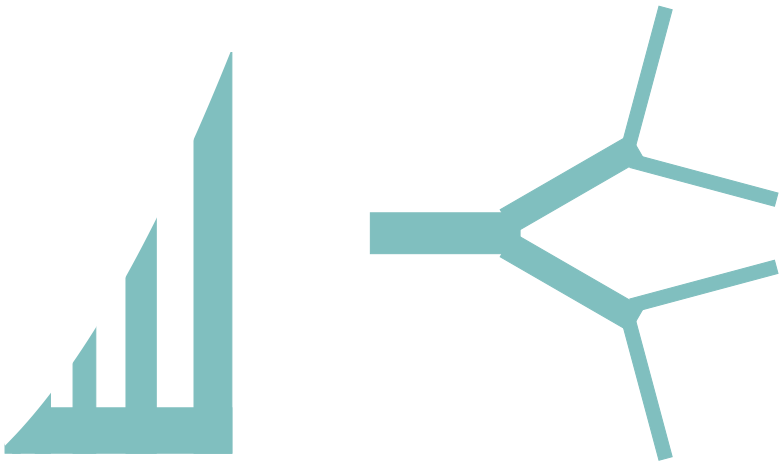
Let us discuss mixed boundary value problems assuming the cone condition is fulfilled in the indicated form.

Let the geometry of the domain (and the singularities of the coefficients of the operator A) are compatible with the scaling function $\gamma(x)=\gamma_0(x)$. Then in order to take into account the singularities of the boundary value problem we should replace $\gamma_0(x)$ by $\min(\gamma_0(x),\gamma_1(x))$ where scaling function $\gamma_1(x)$ corresponds to the singularities of the boundary value problem.

If $\gamma_0(y)\le 2\operatorname{dist}(y,\partial X)$ then one can take $\gamma(x)\asymp\gamma_0(x)$ in $\big(y,\frac{1}{3}\gamma(y)\big)$. Otherwise one can straighten ∂X in $\mathcal{B}\big(y,\frac{2}{3}\gamma(y),X\big)$ by means of a map which is uniformly smooth in the γ_0-scale with Jacobian disjoint from 0. Therefore without any loss of generality one can assume that

$$X\cap\mathcal{B}\big(y,\gamma(y),X\big) = \{x_1>0\}\cap B\big(y,\gamma(y)\big). \tag{11.2.95}$$

Then $\gamma_1(x)\asymp x_1+\gamma_1(x')$ in $X\cap\mathcal{B}\big(y,\frac{1}{2}\gamma(y),X\big)$ where here $\gamma_1(x)$ is defined only on ∂X in order to take into account the singularities of the boundary



(a) Comb (b) Tree

Figure 11.6: Comb (Example 11.2.25) and tree (Example 11.2.26) with infinite numbers of teeth and branches

value problem. Assuming that $\gamma_0^{-1-\vartheta} \in \mathscr{L}^1(X)$ we conclude that

$$\gamma_1^{-1-\vartheta} \in \mathscr{L}^1(X) \iff \begin{cases} \log \gamma_1 \in \mathscr{L}^1(\partial \mathcal{X}_{\mathsf{reg}}) & \text{for } \vartheta = 0, \\ \gamma_1^{-\vartheta} \in \mathscr{L}^1(\partial \mathcal{X}_{\mathsf{reg}}) & \text{for } \vartheta > 0 \end{cases} \tag{11.2.96}$$

where the standard surface measure is used on $\partial \mathcal{X}_{\mathsf{reg}}$. In particular, this condition is fulfilled for $\vartheta \in [0, \delta)$ as $\gamma_1(x) \geq \epsilon_0 \mathsf{dist}(x, \mathcal{K})$ where $\mathcal{K}$ is a subset of ∂X with the following property:

(11.2.97) For every $y \in \mathcal{X}_{\mathsf{reg}}$ $\mathcal{K} \cap B\big(y, \gamma_0(y), X\big)$ is contained in $\bigcup_{1 \leq l \leq L} \mathcal{K}_l$ where $\mathcal{K}_l$ (depending on y also) are uniformly $\mathscr{C}^K$ submanifolds (in the γ_0-scale) of $\partial \mathcal{X}_{\mathsf{reg}}$ of codimension δ.

This analysis provides us immediately with a series of examples for the Laplacian when on part of the boundary the Dirichlet condition is given and on the remaining part the Neumann boundary condition is given and $\vartheta < 1$.

Example 11.2.27. Let us consider a boundary value problem of variational origin with the functional

$$\mathbf{\Phi}(u) = \mathbf{\Phi}_{\mathsf{inn}}(u) + \mathbf{\Phi}_{\mathsf{bound}}(u) = \int_X \sum_{\alpha,\beta:|\alpha|\le m,|\beta|\le m} a_{\alpha,\beta}\langle a_{\alpha,\beta} D^\alpha u, D^\beta u\rangle \, dx + \int_{\partial X} \sum_{\alpha,\beta:|\alpha|\le m-1,|\beta|\le m-1} a_{\alpha,\beta}\langle b_{\alpha,\beta} D^\alpha u, D^\beta u\rangle \, dS \tag{11.2.98}$$

where dS is a standard measure at ∂X. Let us assume that the boundary terms of this functional are regular in the sense that after the rescaling $x \mapsto (x-\bar{x})\gamma(\bar{x})^{-1}$ and multiplication by $\gamma(\bar{x})^{2m-d}$ we obtain a functional the coefficients of which (and all their derivatives of order $\le K$) are bounded in $B(0,1)$. The same should be true for the boundary conditions which define $\mathfrak{D}(\mathbf{\Phi})$ but the l-th condition has its own weight $\varrho_{(l)}$ for multiplication and is given on the set $\Xi_l \subset \partial X$ $(l = 1, \dots, L)$.

Let us assume that the cone condition is fulfilled. Then $\mathcal{K}$ is the union of the relative (in ∂X) boundaries of Ξ_l. However, the possibly irregular structure of $\mathcal{K}$ prevents us from immediately applying the above arguments. Nevertheless, for $\vartheta = 0$ we avoid this difficulty by means of variational arguments. Namely, let us consider a covering of $\partial\Xi_l$ of finite multiplicity by balls $B\bigl(x^k, \epsilon\gamma(x^k)\bigr)$ lying in the $2\epsilon\gamma$-neighborhood of $\partial\Xi_l$; let $\omega_{l,\epsilon}$ be the union of these balls. Let us replace Ξ_l by $\Xi_l' = \Xi_l \cap \omega_{l,\epsilon}$ or by $\Xi_l'' = \Xi_l \setminus \omega_{l,\epsilon}$. We do the same with the set Ξ_0 where the Dirichlet boundary condition is given.

In the first case (when all the sets Ξ_l expand) the domain of $\mathbf{\Phi}$ shrinks and $\mathsf{N}^-(\tau)$ decreases, and in the second case (when all the sets Ξ_l shrink) the domain of $\mathbf{\Phi}$ expands and $\mathsf{N}^-(\tau)$ increases. Further, the cone condition remains true in both cases for small enough $\epsilon > 0$. Furthermore, in both cases the required regularity conditions are now fulfilled (because $\partial\Xi_l'$ and $\partial\Xi_l''$ intersected with ∂X_r lie in the union of spheres $S\bigl(x^k, \epsilon\gamma(x^k)\bigr)$ and every ball $B\bigl(\bar{x}, \gamma(\bar{x})\bigr)$ intersects no more than $C = C_\epsilon$ of these spheres. Therefore

$$\mathsf{N}'^-(\tau) \le \mathsf{N}^-(\tau) \le \mathsf{N}''^-(\tau) \tag{11.2.99}$$

and $\mathsf{N}'^-(\tau) = \varkappa_0 \tau^{\frac{d}{2m}} + O\bigl(\tau^{\frac{1}{2m}(d-1)}\bigr)$ and the same is true for $\mathsf{N}''^-(\tau)$ with the same coefficient $\varkappa_0$ (because it does not depend on the boundary conditions).

Therefore *the same is true for the problem in question:*

$$\mathsf{N}^-(\tau) = \varkappa_0 \tau^{\frac{d}{2m}} + O\bigl(\tau^{\frac{1}{2m}(d-1)}\bigr) \qquad \text{as } \tau \to +\infty. \tag{11.2.100}$$

Moreover, under the standard non-periodicity condition to Hamiltonian trajectories[19] asymptotics

$$\mathsf{N}'(\tau) = \varkappa_0 \tau^{\frac{d}{2m}} + \varkappa'_1 \tau^{\frac{1}{2m}(d-1)} + o\bigl(\tau^{\frac{1}{2m}(d-1)}\bigr)$$

holds and the same is true for $\mathsf{N}''(\tau)$. Here $\varkappa'_1$ is a monotone non-increasing and $\varkappa''_1$ is a monotone non-decreasing function of $\epsilon \in (0, \epsilon_0)$ and $\varkappa'_1 \le \varkappa''_1$. Let us recall that the coefficient $\varkappa_1$ consists of two terms; one of them does not depend on boundary conditions and the other is an integral over ∂X_r of some function of the principal coefficients of the operator and the boundary operators. Therefore, assuming that

$$\mathsf{mes}_{\partial X}(\partial \Xi_l) = 0 \qquad \forall l = 0, \dots, L \tag{11.2.101}$$

we obtain that the limits of $\varkappa'_1$ and $\varkappa''_1$ as $\epsilon \to +\infty$ coincide and therefore the asymptotics

$$\mathsf{N}^-(\tau) = \varkappa_0 \tau^{\frac{d}{2m}} + \varkappa_1 \tau^{\frac{1}{2m}(d-1)} + o\bigl(\tau^{\frac{1}{2m}(d-1)}\bigr) \qquad \text{as } \tau \to +\infty \tag{11.2.102}$$

holds.

Let us treat the case in which the boundary is very irregular.

Example 11.2.28. Let us assume that the operator A satisfies our conditions with $\gamma = \mathsf{const} > 0$. Then one can take $\gamma(x) = \mathsf{dist}(x, \partial X)$ in which case the remainder estimate in the asymptotics of $\mathsf{N}^-_{(\vartheta)}(\tau)$ is "O" of

$$\begin{aligned} \tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta} \int_{r \ge \tau^{\frac{1}{2m}}} r^{-1-\vartheta}\, d\mu(r) + \tau^{\frac{d}{2m}+\vartheta} \mu\bigl(\tau^{-\frac{1}{2m}}\bigr) \asymp \\ \tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta} \int_{r \ge \tau^{\frac{1}{2m}}} r^{-2-\vartheta} \mu(r)\, dr + \tau^{\frac{d}{2m}+\vartheta} \mu\bigl(\tau^{-\frac{1}{2m}}\bigr) \end{aligned} \tag{11.2.103}$$

with $\mu(r) = \mathsf{mes}(\{x \in X, \mathsf{dist}(x, \partial X) \le r\})$; we assume that $\mathsf{mes}(X) < \infty$. In particular, if $\mu(r) = O(r^\delta)$ as $r \to +0$ then for $\vartheta < \delta - 1$ [17] we obtain

[19] This condition is not associated with $\partial \Xi'_l$ or $\partial \Xi''_l$ because the $2d$-dimensional measures of points lying on trajectories issuing from these sets obviously vanish.

the remainder estimate $O\bigl(\tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta}\bigr)$; moreover, under standard non-periodicity condition to the Hamiltonian trajectories we obtain the remainder estimate $o\bigl(\tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta}\bigr)$.

On the other hand, we obtain the remainder estimate $O\bigl(\tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta}\log\tau\bigr)$ for $\vartheta = \delta - 1$ and we obtain the remainder estimate $O\bigl(\tau^{\frac{1}{2m}(d-\delta)+\vartheta}\bigr)$ for $\vartheta > \delta - 1$.

For $\vartheta = 0, \delta = 1$ and for $\vartheta = 0, \delta \in (0,1)$ we obtain the results of the papers J. Brüning [1] and J. Fleckinger, M. Lapidus [1] respectively. Finally, in these two cases under the condition $\mu(r) = o(r^\delta)$ as $r \to +0$ we obtain "o" in the remainder estimates.

Remark 11.2.29. Condition $\mu(r) = O(r^\delta)$ as $r \to +0$ means exactly that *inner Minkowski dimension of ∂X does not exceed $d - \delta$*. So called *Weyl-Berry conjecture* (in sanitized form: originally M. Berry thought about Hausdorff dimension) claims the second term $\kappa_\delta \tau^{(d-\delta)/2}$ and the remainder estimate $o(\tau^{(d-\delta)/2})$ for Dirichlet (Euclidean) Laplacian.

However, as J. Brossard and R. Carmona [1] had shown it is almost never the case as capacity enters the game (recall that Weyl conjecture is almost always true for sure for Euclidean domains and may be even always true).

11.3 Large Eigenvalues for Operators Wwith Weakly Singular Potentials (End)

11.3.1 Operators with Non-Periodic Flows

We start this subsection by discussing existence and some properties of the Hamiltonian billiard flow for second order operators.

Then we consider examples in which restrictive conditions on the Hamiltonian flow imply asymptotics with highly accurate remainder estimates (those we used to call *sharper*). $\vartheta = 0$. Here we do not treat smooth closed manifolds without boundary; the reader can find all those results in A. Volovoy [1, 2] and in the earlier papers P. Bérard [1], B. Randoll [1]. For the Laplace-Beltrami operator these statements yield the following

Example 11.3.1. (i) For the Laplace-Beltrami operator on a compact closed Riemannian $\mathscr{C}^K$ manifold of non-positive sectional curvature (for $d \ge 3$) or

without conjugate points (for $d = 2$) asymptotics of $\mathsf{N}^-(\tau)$ with remainder estimate $O\bigl(\tau^{\frac{1}{2}(d-1)}\log^{-1}\tau\bigr)$ holds.

(ii) For a certain class of surfaces of rotation and for a certain class of graded Lie algebras asymptotics of $\mathsf{N}^-(\tau)$ with the remainder estimate $O\bigl(\tau^{\frac{1}{2}(d-1-\delta)}\bigr)$ holds with $\delta > 0$.

Remark 11.3.2. In the context of our book the results on Hamiltonian flows obtained in these papers yield asymptotics for $\mathsf{N}^-_{(\vartheta)}(\tau)$ with remainder estimates $O\bigl(\tau^{\frac{1}{2}(d+\vartheta-1)}\log^{-\vartheta-1}\tau\bigr)$ and $O\bigl(\tau^{\frac{1}{2}(d+\vartheta-1-\delta)}\bigr)$ respectively.

Existence of the Billiard Flow

Let us consider the Laplace(-Beltrami) operator in compact domains or on compact manifolds with singularities. The following statement is necessary:

Theorem 11.3.3. *Let X be a bounded domain in $\mathbb{R}^d$ which is a Riemannian manifold with boundary with metric $g(x,\xi)$ satisfying conditions*

$$g^{jk} = g^{kj}, \qquad \epsilon_0|\xi|^2 \le g(x,\xi) = \sum_{jk} g^{jk}\xi_j\xi_k \le c|\xi|^2, \tag{11.3.1}$$

$$|D^\alpha g^{jk}| \le c\gamma^{-|\alpha|} \qquad \forall \alpha : |\alpha| \le K \quad \forall j,k \tag{11.3.2}$$

and with scaling function $\gamma(x)$ such that $\gamma^{-1} \in \mathscr{L}^1(X)$.

Then there exist a set of measure 0 and also of the first Baire category $\Lambda \subset S^*\bar{X}$ and a continuous Hamiltonian flow Ψ_t on $S^*\bar{X}\setminus\Lambda$ with reflection at the boundary such that all its trajectories lie in $S^*\bar{X}_r$, $\bar{X}_{\mathsf{reg}} = \{x \in \bar{X}, \gamma(x) > 0\}$ and are transversal to ∂X; as usual $\bar{X} = X \cup \partial X$.

This flow conserves the natural measure $d\mu = dxd\xi : d\sqrt{g}$ on S^*X.

Proof. The proof coincides with the arguments of Chapter 6 in the book I. P. Cornfeld, S. V. Fomin and Y. G. Sinai, [1]. This proof follows the following scheme: first we establish that the flow where it is defined preserves measure μ.

Then we consider the set Λ_{t_1,t_2} of all points such that Ψ_t exists on time interval $[0,t_1]$ but breaks on $(t_1,t_2]$ (as $0 < t_1 < t_2$). Invariance of the measure μ implies that $\mu(\Lambda_{T,T+\varepsilon}) = \mu(\Lambda_{0,\varepsilon})$ and therefore

$$\mu(\Lambda_{0,T}) \le \mu(\Lambda_{0,\varepsilon})\lceil\frac{T}{\varepsilon}\rceil. \tag{11.3.3}$$

On the other hand, due to the bounded speed of propagation $\Lambda_{0,\varepsilon}$ is covered by $\{(x,\xi) : \gamma(x) \le C_1\varepsilon\}$ and by union of subsets

$$\big\{(x,\xi) :\ \gamma(x) \asymp \gamma \mathsf{dist}(x,\partial X) \le C_0\varepsilon\big\}$$

which trajectories have an angle less than δ with the boundary; here $\delta = \delta(\varepsilon\gamma^{-1}) = o(1)$ as $\varepsilon \to +0$, with $\gamma \ge C_1\varepsilon$.

Actually those points belong to $C_0\delta\varepsilon$-vicinity of ∂X. Obviously μ-measure of the first set does not exceed $C\,\mathsf{mes}(\{x, \gamma(x) \le C_0\varepsilon\})$ and μ-measure of the second set does not exceed $C\delta^2\varepsilon\gamma^{-1}\,\mathsf{mes}(\{x, \gamma(x) \asymp \gamma\})$.

Note that in virtue of $\gamma^{-1} \in \mathscr{L}^1$ we estimate the measure of the first set

$$\mathsf{mes}(\{x, \gamma(x) \le C_1\varepsilon\}) = o(\varepsilon) \tag{11.3.4}$$

and the measure of the second set by

$$C\int_{x:\gamma(x)\ge C_1\varepsilon} \delta^2\big(\frac{\varepsilon}{\gamma(x)}\big)\gamma^{-1}(x)\,dx = o(\varepsilon);$$

so $\mu(\Lambda_{0,\varepsilon}) = o(\varepsilon)$ as $\varepsilon \to +0$.

Combining with (11.3.3) we conclude that $\mu(\Lambda_{0,T}) = o(1)$ and thus $\mu(\Lambda_{0,T}) = 0$. Negative time direction is covered by the same way. □

Remark 11.3.4. Let us note that if we are interested only in trajectories of length not exceeding T then in the statement of this lemma one can replace Λ by a closed nowhere dense subset Λ_T. However, $S^*\bar{X} \setminus \Lambda_T$ is not invariant with respect to the flow Ψ_t.

Properties of Hamiltonian Billiard Flows and Asymptotics

Recall that to recover sharper remainder estimate we need to estimate the "divergence" of the billiard flow and its non-periodicity. The first notion is covered by the following

Definition 11.3.5. Let X be a compact γ-admissible Riemannian manifold with boundary where γ is an admissible scaling function such that $\gamma^{-1} \in \mathscr{L}^1(X)$.

We say that

(i) Ψ_t is *a power geodesic flow with reflections*is given on X if for every $T \geq 0$ there exists a closed set $\Xi_T \subset \Sigma$ with $\mu(\Xi_T) \leq CT^{-1}$ such that through every point $z \in \Sigma \backslash \Xi_T$ there passes a geodesic trajectory $\Psi_t(z) = \big(x(t), \xi(t)\big)$ with reflections from ∂X with $t \in [0, T]$ or $t \in [-T, 0]$, along which $\gamma(x(t)) \geq \varepsilon(T)$, the angle with ∂X is not less than $\varepsilon(T)$ at every point where the distance from the boundary is less than $\varepsilon(T)$, and also $|D\Psi_t| \leq \varepsilon(T)^{-1}$ where $\varepsilon(T) = T^{-L}$ with large enough C, L.

(ii) Similarly, we say that Ψ_t is *an exponential geodesic flow with reflections* if (i) holds with $\varepsilon(T) = e^{-CT}$ respectively.

We have a non-trivial *non-periodicity* condition. In order to take it into account we introduce

Definition 11.3.6. In the framework of Definition 11.3.5 we say that Ψ_t is a *non-periodic power (or exponential) geodesic flow* if in addition to the conditions of definition 11.3.5

$$\mu\big(\big\{z \in \Sigma \setminus \Xi_T, \exists t \in (0, T) : |t|^{-1}\mathsf{dist}(z, \Psi_t(z)) \leq \varepsilon(T)\big\}\big) = O(T^{-1}) \tag{11.3.5}$$

where we increase C and L in the definition of $\varepsilon(T)$ if necessary.

We conclude immediately that

Theorem 11.3.7. *(i) Let X be a domain with non-periodic power geodesic flow. Let γ_0 be scaling function describing singularities of domain and $\gamma \leq \gamma_0$ be scaling function describing singularities of operator and boundary conditions. Then as*

$$\gamma^{-1-\vartheta-\delta_1} \in \mathscr{L}^1(X) \tag{11.3.6}$$

with arbitrarily small exponent $\delta_1 > 0$, asymptotics

$$\mathsf{N}^-_{(\vartheta)} = \mathsf{N}^{\mathsf{W}}_{(\vartheta)} + O\big(\tau^{\frac{d-1-\vartheta}{2m}+\vartheta-\delta}\big) \tag{11.3.7}$$

holds with $\delta > 0$ as $\tau \to +\infty$.

(ii) Let X be a domain with non-periodic exponential geodesic flow. Let γ_0 be scaling function describing singularities of domain and $\gamma \leq \gamma_0$ be scaling

function describing singularities of operator and boundary conditions. Then as

$$\gamma^{-1-\vartheta}|\log\gamma|^{1+\vartheta}\in \mathscr{L}^1(X) \tag{11.3.8}$$

with arbitrarily small exponent $\delta_1>0$, asymptotics

$$\mathsf{N}^-_{(\vartheta)}=\mathsf{N}^{\mathsf{W}}_{(\vartheta)}+O\big(\tau^{\frac{d-1-\vartheta}{2m}+\vartheta}|\log\tau|^{-1-\vartheta}\big) \tag{11.3.9}$$

holds as $\tau\to+\infty$.

Examples

Polygons and Polyhedra Recall that according to Theorem 7.4.11 in a compact Euclidean polyhedral domain in $\mathbb{R}^d$ $|D\Psi_t(x,\xi)|\le C(|t|+1)$.

Theorem 11.3.8. *Let X be a polyhedral domain with cuts; assume that X is γ-admissible with $\gamma^{-1-\sigma}\in\mathscr{L}^1(X)$ with $\sigma>0$. Then there is a non-periodic power geodesic flow.*

Proof. Note first that

$$\mathsf{mes}\big(\big\{z=(x,\xi)\in\Sigma, \mathsf{dist}(x,\partial X)\le\varepsilon\big\}\big)\le C\varepsilon. \tag{11.3.10}$$

Let us cover set $\big\{z=(x,\xi)\in\Sigma, \mathsf{dist}(x,\partial X)\ge\varepsilon\big\}$ with no more than $C\varepsilon^{1-4d}$ balls of radius $O(\varepsilon^2)$ and let us divide the time interval $[0,T]$ into no more than $O(T\varepsilon^{-2})$ intervals of length $O(\varepsilon^2)$. Let $\varepsilon=O(T^{-L})$ with large enough exponent L.

Then if the distance between the Ψ_t-image of some ball and this ball is less than ε^2 for some t from the above subinterval then for all points (x,ξ) of this ball and for all t of this subinterval the geodesic flow is given by the formula

$$\Psi_t(x,\xi)=(y+\Pi x+t\Pi\xi,\Pi\xi)$$

with constant y and Π such that $\mathsf{dist}(\xi,\Pi\xi)\le\varepsilon$ for all ξ in question. Here $\Pi\in O(d)$ as in the proof of Theorem 7.4.11.

Taking into account that for every x the $\mathbb{S}^{d-1}$-measure of the set

$$\big\{\xi\in\mathbb{S}^{d-1}, |y+\Pi x-x+t\Pi\xi|\le\varepsilon^N \quad \text{for some } t \text{ with } |t|\ge\varepsilon\big\}$$

does not exceed $O(\varepsilon^N)$ we obtain after summation over all the balls and subintervals that (11.3.5) holds. □

Example 11.3.9. Obviously in Euclidean polyhedral domains (without cuts) $\gamma_0^{-1-\vartheta} \in \mathscr{L}^1(X)$ as $\vartheta < 1$. So the only condition to be checked is $\gamma^{-1-\vartheta} \in \mathscr{L}^1(X)$ where $\gamma = \mathsf{dist}(x, \mathcal{K})$ and $\mathcal{K}$ is the set of the cut ends and transition points. This condition is equivalent to $\gamma^{-\vartheta} \in \mathscr{L}^1(\partial X)$ as $\vartheta > 0$ and $|\log \gamma| \in \mathscr{L}^1(\partial X)$ as $\vartheta = 0$.

Elliptical Domains According to Theorem 7.4.12 in Euclidean elliptical domains (or domains between two confocal ellipses) Ψ_t is power growth geodesic flow with singularities (see (7.4.32)– (7.4.34) for more details).

Theorem 11.3.10. *Let X be a domain in $\mathbb{R}^2$ bounded by one or two confocal ellipses. Then there is a non-periodic power geodesic flow.*

Proof. Obviously (11.3.5) is equivalent to

$$\mathsf{mes}_{\Sigma'_2}\big(\big\{z' \in \Sigma'_2 : \exists k = 1, \dots, n : \mathsf{dist}(z', \Phi^k z') \le \varepsilon = n^{-M}\big\}\big) = O(n^{-\sigma}) \quad \text{as } n \to +\infty \tag{11.3.11}$$

for appropriate M and $\sigma > 0$. Recall that $\Sigma'_2 = \Sigma|_{Y_2} = B^* Y_2$ where Y_2 is an outer ellipse and we use other notations of the proof of Theorem 7.4.12.

(a) Let us first consider the zone $\{(x, \xi), \beta > 0\}$. Let us consider the lines $\{\eta = \mathsf{const}\}$. These lines are analytically parameterized by a cyclic parameter $u \in \mathbb{R}/2\pi\mathbb{Z}$ (see the proof of theorem-7.4.12)[20].

Let $w \in \mathbb{R}/\mathbb{Z}$ be a new cyclic coordinate: $w = w_\eta(u) = \nu_\eta((0, u))m_\eta^{-1}$ for $u \in [0, 2\pi]$ and then by periodicity. Here $m_\eta = \nu_\eta([0, 2\pi])$. One can easily see that $m_\eta \asymp \log|\beta|$ as $\beta \to +0$ for $a_2 > b_2$ and $m_\eta \asymp \beta^{-1}$ as $\beta \to +0$ for $a_2 = b_2$.

For every fixed η the map Φ_η of this line is analytic and preserves the measure $dw_\eta = m_\eta^{-1} d\nu_\eta$ and the orientation[21] and therefore Φ_η is a shift $\Phi_\eta : w_\eta \to w_\eta + f(\eta)$.

[20] More precisely, this line consists of two connected components. Billiard map Φ transforms every component into itself and the x-projection is an analytic diffeomorphism of this component to the ellipse Y_2. Every component is parameterized by u. Permutation of these components is equivalent to replacing Φ by Φ^{-1}.

[21] Moreover, it is well known (see I.P. Cornfeld, S. V. Fomin and Y. G. Sinai, [1]) that for generic η, Φ_η is an ergodic map.

Then in the proof of (11.3.11) for $\Sigma_2' \cap \{\beta > 0\}$ one needs to prove that

$$Cn\,\mathsf{mes}\big(\{\eta : \exists k, l = 1, \dots, n : |f(\eta) - \frac{k}{l}| \le \varepsilon = n^{-L}\}\big) = O(n^{-\sigma}) \quad \text{as } n \to +\infty \tag{11.3.12}$$

for admissible L and $\sigma > 0$; here mes means Lebesgue measure.

Obviously, $f(\eta)$ is a continuous function. One can easily prove that

$$\epsilon\beta^p \le |\frac{df}{d\eta}| \le c\beta^{-p} \qquad \text{for } |\eta| \le a_2 - \epsilon_0, \tag{11.3.13}$$

i.e., for $\beta \in (0, \beta_0)$ with small enough $\epsilon_0 > 0, \beta_0 > 0$. Recall that $\beta = b_2^2 - a_2^2 b_2^2 \eta^2$. Therefore (11.3.12) holds for the η indicated here.

On the other hand, Φ is an analytic map for $\beta \ne b_1^2$ (the case when rays are tangent to the inner ellipse Y_1). Therefore function $f(\eta)$ is analytic for $|\eta| < \eta_1 = a_2^{-1} b_2^{-1} (b_2^2 - b_1^2)^{\frac{1}{2}}$ and for $\eta_1 < |\eta| \le a_2^{-1}$. Obviously, $f(\eta)$ is not constant in each of these three intervals. One can easily see that $f_\pm(\zeta) = f(\bar\eta \pm \zeta^2)$ is an analytic function for $|\zeta| \le \epsilon_0$ and $\bar\eta = +\eta_1, -\eta_1$.

However one can easily see that

$$\mathsf{mes}(\{\zeta : |f(\zeta)| \le \varepsilon\}) = O(\varepsilon^\delta) \tag{11.3.14}$$

holds for every $\mathscr{C}^K$ function on the closed segment provided

$$\sum_{1 \le j \le K-1} |f^{(j)}| \ge \epsilon_0. \tag{11.3.15}$$

Therefore (11.3.12) holds.

(b) Let us treat the zone $\{(x, \xi) : 0 > \beta > b_2^2 - a_2^2\}$. In this case every line $\{\eta = \mathsf{const}\}$ consists of two connected components and every component is equivalent to $\mathbb{S}^1$ and the x-projection maps every component to one of two segments on the ellipse Y_2 describing by the inequality $x_2^2 \ge -\beta b_2^2 (a_2^2 - b_2^2)^{-1}$. The inverse map is two-valued.

Let us introduce on this connected component the cyclic coordinate u' which can be done in the following way: let us consider the confocal

hyperbola described by the equation $\frac{x_1^2}{a'^2} - \frac{x_2^2}{b'^2}$ with $a'^2 + b'^2 = a_2^2 - b_2^2$. This hyperbola intersects Y_2 in four points $(\bar{x}_1, \bar{x}_2)$ with

$$\bar{x}_1^2 = a_2^2(a_2^2 - b_2^2 + \beta)(a_2^2 - b_2^2)^{-1}, \quad \bar{x}_2^2 = -b_2^2\beta(a_2^2 - b_2^2)^{-1}.$$

Let us introduce parameter $u' \in \mathbb{S}^1$ on this hyperbola: $x_1 = a' \sec u'$, $x_2 = b' \tan u'$. Through every point of the hyperbola corresponding to this parameter there passes a tangent straight line hitting the corresponding segment of the ellipse. Then (x_1, x_2) is an intersection point and (ξ_1, ξ_2) is a unit vector along this tangent line, directed inside and outside for $\eta < 0$ and $\eta > 0$ respectively.

Let us introduce the parameter $w' \in \mathbb{R}/\mathbb{Z}$ in the same way as before. Again we take $dw' = m_\eta^{-1} d\nu_\eta$ with $m_\eta \asymp \log|\beta|$ as $\beta \to -0$ and $m_\eta \asymp 1$ as $\beta \to b_2^2 - a_2^2$. Moreover, in the latter case m_η is an analytic function of $\zeta' = (\beta + a_2^2 - b_2^2)^{\frac{1}{2}}$. One can easily see that if there is no inner ellipse Y_1 then Φ transforms one connected component of every line $\{\eta = \mathsf{const}\}$ to another and therefore for $\beta < -n^{-\sigma}$ the inequality $\mathsf{dist}(z', \Phi^k(z')) < \epsilon$ is impossible for odd k and the measure of the set $\{|\beta| \le n^{-\sigma}\}$ is $O(n^{-\delta})$.

On the other hand, if there is an inner ellipse Y_1 then the map Φ preserves both connected components of $\eta = \mathsf{const}$ but changes their orientations, i.e., the map is of the form $w_\eta \to \frac{1}{2}f(\eta) - w_\eta$ and therefore for odd k the $d\nu_\eta$-measure of points with $\mathsf{dist}(z', \Phi^k z') \le \varepsilon$ is $O(n^p\varepsilon)$. Therefore *in both cases only even k should be treated.*

Furthermore, on every connected component map Φ^2 is of the form $w_\eta \to w_\eta + f(\eta)$ where function f is again continuous and satisfies (11.3.13), this time for $|\eta| \ge a_2^{-1} + \epsilon_0$, i.e. $\beta \in (-\beta_0, 0)$ with small enough ϵ_0 and β_0.

This function is analytic for $a_2^{-1} < |\eta| < b_2^{-1}$, i.e., for $\beta \in (0, b_2^2 - a_2^2)$, and the functions $f(\pm(b_2^{-1} - \zeta^2))$ are analytic with respect to ζ with $|\zeta| \le \epsilon_0$. Therefore we obtain that (11.3.15) and thus (11.3.14) and (11.3.12) remain true in this case. □

Example 11.3.11. Let X be an elliptical domain or domain between two confocal ellipses. Then

(i) If the boundary problem is regular then asymptotics (11.3.7) holds for any $\vartheta \ge 0$.

(ii) We can allow singularities: transitions between different boundary value problems and cuts along some segments on semi-axis of the ellipses[22)]

Then the only condition to be checked is $\gamma^{-1-\vartheta} \in \mathscr{L}^1(X)$ where $\gamma = \mathsf{dist}(x, \mathcal{K})$ and $\mathcal{K}$ is the set of the cut ends and transition points. This condition again is equivalent to $\gamma^{-\vartheta} \in \mathscr{L}^1(\partial X)$ and $|\log \gamma| \in \mathscr{L}^1(\partial X)$ as $\vartheta = 0$.

Example 11.3.12. Let X be a ball or domain between two concentric spheres in $\mathbb{R}^d$.

Then

(i) If the boundary problem is regular then asymptotics (11.3.7) holds for any $\vartheta \geq 0$.

(ii) We can allow singularities: transitions between different boundary value problems. Then the only condition to be checked is $\gamma^{-1-\vartheta} \in \mathscr{L}^1(X)$ where $\gamma = \mathsf{dist}(x, \mathcal{K})$ and $\mathcal{K}$ is the set of the cut ends and transition points. Again, this condition is equivalent to $\gamma^{-\vartheta} \in \mathscr{L}^1(\partial X)$ for $\vartheta > 0$ and $|\log \gamma| \in \mathscr{L}^1(\partial X)$ as $\vartheta = 0$.

There are many open problems:

Problem 11.3.13. (i) Investigate domains bounded by one or two ellipses and one or two hyperbolas (all with the same foci).

(ii) Investigate domains whose boundary consists of segments of ellipses and hyperbolas (all with the same foci); investigate if one can add cuts along some segments on semi-axis.

(iii) Investigate ellipsoids of revolutions (i.e. with semi-axis $a_1 = a_2 = \dots = a_p = a > a_{p+1} = \dots = a_d = b$) or domains between two confocal ellipsoids of revolution (with the obvious meaning).

(iv) Investigate general ellipsoids.

Miscellaneous Remarks

Remark 11.3.14. Since the billiard trajectories of the principal symbol a^m coincide with those of a, all our discussion remains true for operators with the senior part Δ^m.

[22)] One can show easily that the reflection from cuts does not violate power non-periodicity.

Remark 11.3.15. (i) Billiard geodesic flow has power growth for certain Lie groups. One can easily see that in this case

$$\mu\big(\{z, \exists t \in [0, T] : \gamma(\Psi_t(z)) \le \varepsilon\}\big) \le C\varepsilon^\sigma T$$

and $\mu(\mathcal{Q}_{T,\varepsilon}) \le C\varepsilon T$ where $\mathcal{Q}_{T,\varepsilon} = \{z, \exists t \in [0, T]$ such that the distance from x to ∂X and the angle between $\frac{dx}{dt}$ and $T\partial X$ do not exceed $\epsilon\varepsilon$, and $\gamma(\Psi_t(z)) \ge \varepsilon\}$.

(ii) Billiard geodesic flow has power growth for manifolds with periodic geodesic flow (such manifolds can be either smooth or γ-admissible with $\gamma^{-1-\sigma} \in \mathscr{L}^1(X)$ for arbitrarily small $\sigma > 0$). We will analyze them in the next subsection.

(iii)

(iv) Recall that according to Theorem 7.4.10 in Euclidean strongly convex domains with $\mathscr{C}^3$-boundary geodesic billiard flow has (no more than) exponential growth. The same is true for compact Riemannian manifolds with geodesically strongly concave boundary and for compact closed Riemannian manifolds.

Remark 11.3.16. One can construct new manifolds of the corresponding type by the following methods:

(a) Taking a Cartesian product.

(b) Taking a factor manifold of X with respect to a finite group of its isometries.

(c) Considering reflections, i.e., metric preserving diffeomorphisms $S : \bar{X} \to \bar{X}$ such that $S^2 = Id$ and $S(x) = x \implies dS(x) \ne dx$ (X can have cuts so S does not necessarily preserve X) and removing from X some appropriate subset K of the set of fixed points of S. The condition on K is $\delta^{-1-\sigma} \in \mathscr{L}^1(X)$ for $\delta(x) = \mathsf{dist}(x, K)$.

Remark 11.3.17. (i) Let us note that $\mathsf{dist}(z, \Psi_t(z)) \asymp |t|$ for $|t| \le \epsilon\gamma(x)$, $z = (x, \xi)$. Therefore the non-periodicity condition should be checked only for $|t| \ge \epsilon\varepsilon(T)$.

(ii) Let us consider the Cartesian product X of two Riemannian manifolds X' and X'' with dimensions ≥ 1. Let us note that $\Psi_t(x', x'', \xi', \xi'') = (\Psi'_{t|\xi'|}(x', \xi'), \Psi''_{t|\xi''|}(x', \xi''))$ for $(x', x'', \xi', \xi'') \in S^*X$.

Then, if inequality $\mathsf{dist}(z', \Psi_t(z')) \geq \varepsilon|t|$ is fulfilled for all $t \in (0, T)$ for a point $z' = (x', \xi'_0) \in S^*X'$ then the same inequality with ε^2 instead of ε is fulfilled for a point $(x', x'', \xi', \xi'') \in S^*X$ with $\xi' = |\xi'|\xi'_0$ and $|\xi'| \geq \varepsilon$.

Note that $\mu\{(x, \xi) : |\xi'| \leq \varepsilon = O(\varepsilon)$. Therefore one should treat only points $(x', x'', \xi', \xi'') \in S^*X$ such that the non-periodicity condition is violated at both points $z' = (x', \xi'|\xi'|^{-1}) \in S^*X'$ and $z'' = (x'', \xi''|\xi''|^{-1}) \in S^*X''$. Let

$$T' = \mathsf{min}\big(t \geq \epsilon\varepsilon : \mathsf{dist}(z', \Psi'_t(z')) \leq \varepsilon\big)$$

and let define T'' in a similar way. Recall that we can assume that $|\xi'| \geq \varepsilon$ and $|\xi''| \geq \varepsilon$. Then taking into account condition on the Jacobi matrix we obtain that $\mathsf{dist}(\Psi'_t(z'), z') \geq \varepsilon^2$ provided $|\frac{t}{T'} - m| \geq \varepsilon\ \forall m \in \mathbb{Z}$ and $|t| \leq T$ (where $\varepsilon = \varepsilon(T)$. Therefore one needs to estimate the measure of the set

$$\mathcal{Q} = \big\{\phi \in \mathbb{S}^1 : \exists m', m'' \in \mathbb{Z} : \\ |m'| \leq \varepsilon^{-\sigma}, |m''| \leq \varepsilon^{-\sigma}, |m'T' \cos\phi + m''T'' \sin\phi| \leq \varepsilon\big\}$$

where ϕ is associated with ξ by the relations $|\xi'| = \cos\phi$, $|\xi''| = \sin\phi$ and one can take $\sigma > 0$ arbitrarily small. However, $\mathsf{mes}(\mathcal{Q}) = O(\varepsilon^\sigma)$ obviously.

Therefore for Cartesian products the non-periodicity condition always is fulfilled.

(iii) The above arguments also yield that $\mathsf{dist}(G^*z, \Psi_t(z)) \leq \varepsilon^2$ implies that $\mathsf{dist}(G^{*k}z, \Psi_{kt}(z)) \leq \varepsilon$ for constant k and the isometry $G^* : S^*\bar{X} \to S^*\bar{X}$ induced by the isometry $G : \bar{X} \to \bar{X}$. Therefore Definition 11.3.6 is stable under the above constructions of new manifolds in Remark 11.3.16(b) and (c) .

11.3.2 Operators with Periodic Flows

Regular Operators on Closed Manifolds

Let us now consider a closed manifold X and a scalar operator A on X with periodic Hamiltonian trajectories of the principal symbol. As before we assume that the manifold and operator are smooth enough; furthermore,

as before we divide the operator A by τ, introduce $h = \tau^{-\frac{1}{2m}}$ and obtain an h-pseudodifferential operator. Then if all the Hamiltonian trajectories on the energy level $\Sigma = \{a(x,\xi) = 1\}$ are periodic with period T_0 then every trajectory is T-periodic with $T(x,\xi) = T_0 a^{\frac{1}{2m}-1}(x,\xi)$ and then the Hamiltonian flow of the symbol $a^{\frac{1}{2m}}$ is $2mT_0$-periodic.

Without any loss of the generality we assume that $2mT_0 = 1$. According to Section 6.2 we must replace operator A by operator $\bar{A} = (\tau^{-1}A)^{\frac{1}{2m}}$ (one should use this formula only in a neighborhood of Σ).

Then conditions (6.2.2), (6.2.4) and (6.2.7) with $T_0 = 1$ are fulfilled with $\Omega_0 \supset \Sigma$. Calculating the action by formula (6.2.6) we obtain that $\alpha_1 = 0$ because symbol $\bar{a}$ is positively homogeneous of degree 1 on ξ. In what follows all the objects related to the operator $\bar{A}$ are denoted by the same letters as the corresponding objects related to A but equipped with a bar. Then according to (6.2.12)

$$b_0(x,\xi) = \frac{1}{2mT_0}\int_0^{T_0} (a^s \circ \Psi_t)(x,\xi)\,dt - \frac{\pi\alpha_0}{2} \qquad \text{on } \Sigma \tag{11.3.16}$$

and is positively homogeneous of degree 0 on ξ in a neighborhood of Σ. Here α_0 is the Maslov index.

Then results of Subsubsection *6.2.3.3 Spectral Gaps (the First Attempt)* and Subsection 6.4 yield

Theorem 11.3.18. *Let X be a closed compact $\mathscr{C}^K$ manifold and A an elliptic self-adjoint differential operator*[23)] *of order $2m$. Let the Hamiltonian flow of $a(x,\xi)$ be T_0-periodic on Σ.*

Let $\mathcal{L}$ be the image of the map $\Sigma \ni (x,\xi) \to b(x,\xi) \overset{\iota}{\to} \mathbb{R}/\mu\mathbb{Z}$ where $\mu = \pi(mT_0)^{-1}$ and $\iota : \mathbb{R} \to \mathbb{R}/\mu\mathbb{Z}$ is a natural map. Assume $\mathcal{L}$ does not coincide with $\mathbb{R}/\mu\mathbb{Z}$; then $\iota^{-1}\mathcal{L}$ is a union of intervals $\{I_n\}$ of length $O(1)$ such that $\omega \in I_n \implies |\omega| \asymp n$ for $|n| \ge 1$.

It follows that

$$\mathsf{Spec}(A^{\frac{1}{2m}}) \subset \bigcup_{n\ge 1} J_n \cup [-C, C] \tag{11.3.17}$$

where J_n are the $Cn^{\delta-1}$-neighborhoods of I_n, $\delta > 0$ is arbitrarily small and $C = C_\delta$; without loss of generality we assume that $A \ge I$.

Moreover, for $b_0 = \mathsf{const}$ one can take $\delta = 0$.

[23)] Or a classical pseudodifferential operator. In all cases we assume that $a \ge 0$.

Theorems 6.3.6 and 6.3.12 yield then

Theorem 11.3.19. *Let the conditions of Theorem 11.3.18 be fulfilled and let* $\omega_0 \in \mathcal{L}$,

$$(x,\xi) \in \Sigma,\ b_0(x,\xi) = \omega_0 \implies \text{either } \nabla_\Sigma b_0(x,\xi) \neq 0 \text{ or } \operatorname{rank} \operatorname{Hess}_\Sigma b_0(x,\xi) \geq r. \tag{11.3.18}$$

Further, let us assume that <u>*either*</u> *the Hamiltonian flow* Ψ_t *has no subperiodic trajectory on* Σ <u>*or*</u>

$$\operatorname{dist}\big(\Psi_t(x,\xi),(x,\xi)\big) \geq t(T_0 - t)\rho(x,\xi) \qquad \forall t \in (0, T_0)\ (x,\xi) \in \Sigma \tag{11.3.19}$$

where

$$|\nabla \rho| \leq c \quad \text{and} \quad \mu\big\{(x,\xi), \rho(x,\xi) \leq h\big\} = O(h^q) \quad \text{as } h \to +0. \tag{11.3.20}$$

Then for $\tau \to +\infty$ *such that* $|\iota \tau^{\frac{1}{2m}} - \omega_0| \leq \epsilon_0$ *(where* $\epsilon_0 > 0$ *is small enough) asymptotics*

$$\mathsf{N}^-(\tau) = \varkappa_0 \tau^{\frac{d}{2m}} + \big(\varkappa_1 + F(\tau^{\frac{1}{2m}})\big)\tau^{\frac{d-1}{2m}} + O\big(\tau^{\frac{d-1-p}{2m}}\big) \tag{11.3.21}$$

holds with

$$F(z) = (2mT_0)^{-1}(2\pi)^{-d} \int_\Sigma \Upsilon\Big(2mT_0\big(z - b(x,\xi)\big)\Big)\, dx d\xi : da, \tag{11.3.22}$$

$p = \min(1, \frac{r}{2} - \delta, \frac{q}{2} - \delta)$ *with an arbitrary small exponent* $\delta > 0$. *Here and below* $\Upsilon(z)$ *is a* 2π*-periodic function on* $\mathbb{R}$ *equal to* $\pi - z$ *on* $[0, 2\pi)$.

The following should be rather easy:

Problem 11.3.20. (i) Show that for generic differential and pseudodifferential symbols a^s condition (11.3.18) is fulfilled with $r = 2d - 2$ for all ω_0. In fact the function $b_0(x,\xi)$ is given *on* $\Sigma' = \Sigma/\Psi_t$ (the factor of Σ with respect to the Hamiltonian flow) which is a $2d-2$-dimensional manifold *rather than on* Σ.

(ii) Derive more precise asymptotics are possible as well and improved asymptotics of $\mathsf{N}^-_{(\vartheta)}(\tau)$ with remainder estimates which could be as good as $O(\tau^{(d-2-2\vartheta)/2m})$ (under an appropriate non-degeneracy condition).

Surely, Theorem 11.3.19 is applicable to a Laplace-Beltrami operator on a Riemannian manifold with periodic geodesic flow which is perturbed by a first-order operator. For a generic vector $(V_1, \dots, V_d)$ the field corresponding to the perturbation condition (11.3.18) is fulfilled with $r = 2d - 2$.

In particular, for the Laplace-Beltrami operator on the standard sphere $\mathbb{S}^d$ one can see easily that $\alpha_0 \equiv 2(d-1) \mod 4\mathbb{Z}$ and $T_0 = \pi$. Let us consider this operator with perturbation more carefully. Let $j : \mathbb{S}^d \to \mathbb{S}^d$ be the antipodal map $x \to -x$ and let $j^* : S^*\mathbb{S}^d \to S^*\mathbb{S}^d$ be the adjoint map $(x,\xi) \to (-x,-\xi)$. Then (11.3.16) yields that b_0 depends only on the even part of a^s: $a^s_{\mathsf{even}} = \frac{1}{2}(a^s + a^s \circ j^*)$. Let us now assume that a^s is skew-symmetric with respect to j^*:

$$a^s = -a^s \circ j^*. \tag{11.3.23}$$

Then $b_0 = 0$ and B is a (-1)-order operator (i.e., $\eta = h = \tau^{-\frac{1}{2}}$: see Section 6.2). Let us calculate then the principal symbol b_{-1} of the operator B. Given that the eigenvalues of unperturbed Laplace-Beltrami A_0 on $\mathbb{S}^d$ are $n(n+d-1)$ with $n \in \mathbb{Z}^+$ it will be more convenient for us to set

$$\bar{A} = h\Big(\big(A + \frac{1}{4}(d-1)^2\big)^{\frac{1}{2}} - \frac{1}{2}(d-1)\Big) \tag{11.3.24}$$

here instead of $\bar{A} = A^{\frac{1}{2}}$. Let

$$\bar{A}_0 = h\Big(\big(A_0 + \frac{1}{4}(d-1)^2\big)^{\frac{1}{2}} - \frac{1}{2}(d-1)\Big) \tag{11.3.25}$$

be the corresponding unperturbed operator.

Let $U(t) = \exp(ih^{-1}t\bar{A})$ and $U_0(t) = \exp(ih^{-1}t\bar{A}_0)$. Then

$$U_0(2\pi) = I, \qquad U_0(\pi) = J \quad \text{where} \quad Jv = v \circ j. \tag{11.3.26}$$

So *we want to calculate* $U(2\pi)$ *modulo operators of order* (-2) because near Σ the orders in the sense of standard pseudodifferential operators and in the sense of h-pseudodifferential operators coincide. Let us calculate $U(\pi)J$ modulo operators of order (-1) first. To do it we consider $V(t) = U(t)U_0(-t)$ which is a standard (h-)pseudodifferential operator. Then

$$\frac{d}{dt}V(t) = ih^{-1}U(t)(\bar{A} - \bar{A}_0)U_0(-t) = iV(t)U_0(t)QU_0(-t)$$

where $Q = h^{-1}(\bar{A} - \bar{A}_0)$ is an operator of order 0. Considering principal symbols we obtain that $\partial_t v = iv(b \circ \Psi_t)$. Since $v|_{t=0} = 1$ we obtain that

$$v(t) = \exp(-i \int_0^t I \circ \Psi_{t'}\, dt')$$

where $q = \frac{1}{2} a^s a^{-\frac{1}{2}}$ is the principal symbol of Q. Therefore (11.3.26) yields that

(11.3.27) $U(\pi)J$ is an h-pseudodifferential operator with the principal symbol $\exp(i\zeta \cdot \int_0^\pi q \circ \Psi_t\, dt)$

with $\zeta = 1$.

Let us consider operator $\bar{A}_\zeta = \bar{A}_0 + \zeta h Q$ and let us consider the corresponding unitary group $U_\zeta(t)$. Differentiating equalities

$$(hD_t - \bar{A}_\zeta)U_\zeta(t) = 0, \qquad U_\zeta(0) = I$$

with respect to ζ we obtain that

$$(hD_t - \bar{A}_\zeta)U'_\zeta(t) = hQU_\zeta, \qquad U'_\zeta(0) = 0$$

where the prime means the derivative with respect to ζ. Then

$$U'_\zeta(t) = i \int_0^t U_\zeta(t - t')QU_\zeta(t')\, dt'.$$

Let us set $t = 2\pi$ and divide interval $[0, 2\pi]$ of integration into subintervals $[0, \pi]$ and $[\pi, 2\pi]$ and replace t' by t in the first subinterval and by $t + \pi$ in the second one. We obtain that

$$U'_\zeta(2\pi) = iU_\zeta(2\pi) \int_0^\pi U_\zeta(-t)\tilde{Q}U_\zeta(t)\, dt \tag{11.3.28}$$

with $\tilde{Q} = Q + U_\zeta(-\pi)QU_\zeta(\pi)$. We know that $U_\zeta(\pi) = V_\zeta J$ and $U_\zeta(-\pi) = JV_\zeta^*$ where V_ζ is a unitary operator. Therefore

$$\tilde{Q} = Q + JQJ + JV_\zeta^*[Q, V_\zeta]J.$$

Then (11.3.23) yields that $\tilde{Q}$ is an operator of order (-1) with the principal symbol

$$\tilde{q} = 2q_{\mathrm{even}} + \zeta\{q, \int_0^\pi q \circ \Psi_{t'}\, dt'\}$$

where q_{even} is the even (with respect to j^*) part of q^s and the second term is also even. Therefore (11.3.28) yields that $-iU_\zeta(-2\pi)U'_\zeta(2\pi)$ is an operator of order (-1) with principal symbol

$$\int_0^\pi \tilde{q}\circ\Psi_{-t}\,dt = \int_0^{2\pi} q^s\circ\Psi_t\,dt + \zeta\int_0^\pi\int_0^\pi \{q\circ\Psi_t, q\circ\Psi_{t+t'}\}\,dtdt'$$

where we changed variables in the integration in the obvious way. Integrating over $\zeta\in[0,1]$ we obtain that

(11.3.29) $U(2\pi)-I$ is an operator of order (-1) with principal symbol $2\pi i b_{-1}$ where

$$b_{-1} = \frac{1}{2\pi}\Big(\int_0^{2\pi} q^s\Psi_t dt + \int_0^\pi\int_0^\pi \{q\circ\Psi_t, q\circ\Psi_{t+t'}\}\,dtdt'\Big). \tag{11.3.30}$$

Now theorems 6.3.6 and 6.3.12 yield

Theorem 11.3.21. *Let $A=\Delta+A'+A''$ where Δ is the (positive) Laplace-Beltrami operator on the standard sphere $\mathbb{S}^d$, $A'=A'(x,D)$ and $A''=A''(x,D)$ are symmetric operators of first and zero order respectively and A' anti-commutes with J (in which case a^s satisfies* (11.3.23)*). Let $b_{-1}(x,\xi)$ be defined by* (11.3.30) *with $q=\frac12 a' a^{-\frac12}$, $q^s=\frac12 a'' a^{-\frac12}-\frac18 a'^2 a^{-\frac32}$ where a' and a'' are the principal symbols of A' and A'' respectively.*

Let $\mathcal{L}'=[\lambda_1,\lambda_2]$ be the image of the map $\Sigma\ni(x,\xi)\to b_{-1}(x,\xi)\in\mathbb{R}$ and let $f(\tau)=\big(\tau+\frac14(d-1)^2\big)^{\frac12}-\frac12(d-1)$. Then

(i) $f(\operatorname{Spec} A)\subset\bigcup_{n\ge1} J'_n\cup[-C,C]$ where

$$J'_n=\big[n+\lambda_1 n^{-1}-Cn^{\delta-2}, n+\lambda_2 n^{-1}+Cn^{\delta-2}\big], \tag{11.3.31}$$

$C=C(\delta)$ is large enough, $\delta>0$ is arbitrarily small and for $\lambda_1=\lambda_2$ one can take $\delta=0$.

(ii) Let $\omega_1\in\mathcal{L}'$ and condition (11.3.18) *be fulfilled with ω_1 and b_{-1} instead of ω_0 and b_0. Then for $\tau\to+\infty$ such that*

$$f(\tau)\subset\bigcup_{n\ge1}[n+(\omega_1-\epsilon)\frac{1}{n}, n+(\omega_1+\epsilon)\frac{1}{n}]$$

with a sufficiently small constant $\epsilon > 0$ *asymptotics*

$$\mathsf{N}^-(\tau) = \varkappa_0 \tau^{\frac{d}{2}} + \varkappa_1 \tau^{\frac{d-1}{2}} + F\bigl(f(\tau), \tau\bigr)\tau^{\frac{d-1}{2}} + O\bigl(\tau^{\frac{d-1-p}{2}}\bigr) \tag{11.3.32}$$

holds with $p = \mathsf{min}(1, \frac{r}{2} - \delta)$ *with an arbitrarily small exponent* $\delta > 0$ *and with*

$$F(z,\tau) = 2(2\pi)^{-d-1} \int_{\Sigma} \Upsilon\Bigl(2\pi\bigl(z - \tau^{-\frac{1}{2}} b_{-1}(x,\xi)\bigr)\Bigr)\, dxd\xi : da. \tag{11.3.33}$$

The following seems to be easy:

Problem 11.3.22. (i) Prove that for an arbitrary fixed symbol a'' and for a generic differential symbol (i.e., vector field) as well as for a generic pseudodifferential symbol a' satisfying (11.3.23) the modified[24)] condition (11.3.18) is fulfilled with $r = 2d - 2$ for all $\omega_1 \in \mathbb{R}$.

(ii) Prove that the same is true for an arbitrary fixed a'' satisfying (11.3.23) and for a generic differential symbol (i.e., potential) as well as for a generic pseudodifferential symbol a''.

Let us now assume that

$$a' = 0, \quad a'' \circ j^* = -a''. \tag{11.3.34}$$

Then $b_{-1} = 0$ and the operator $U(2\pi) - I$ obviously is an operator of order ≤ -2 (we will prove that its order is actually (-3)) and our goal is to calculate its principal symbol. Now the calculations are simpler because the perturbation is weaker. Let $\bar{A}$ and $\bar{A}_0$ be defined as above. Then $\bar{A} = \bar{A}_0 + Q' + Q''$ [25)] where Q' is an operator of order (-1) with principal symbol $q' = -\frac{1}{2} a^{-\frac{1}{2}} a''$ anti-commuting with J and Q'' is an operator of order (-3) with principal symbol $q'' = -\frac{1}{8} a^{-\frac{3}{2}} a''^2$ [26)]. Then the equalities

$$(D_t - \bar{A}_0)U(t) = QU(t), \qquad U(0) = I$$

[24)] I.e., with ω_1, b_{-1} instead of ω_0, b_0.

[25)] For the sake of simplicity we do not write the factor $\tau^{-\frac{1}{2}}$ which in any case is reduced in the final formula.

[26)] In contrast with the above case we did not include a lower order operator commuting with J in A because we are mainly interested in differential operators. This generalization is left to the reader.

with $Q = Q' + Q''$ yield that

$$U(t) = U_0(t) + i \int_0^t U(t-t')QU(t')\,dt'$$

(this is similar to the calculation of V). After iteration we obtain $U(t)$ as an asymptotic series of operators with order decreasing by step 1. Let us set $t = 2\pi$. We then obtain that modulo operators of order (-4)

$$U(t) \equiv I + i\int_0^{2\pi} Q(t_1)\,dt_1 - \iint_{\Pi_2} Q(t_1)Q(t_2)\,dt_1dt_2 - i\iiint_{\Pi_3} Q(t_1)Q(t_2)Q(t_3)\,dt_1dt_2dt_3 \tag{11.3.35}$$

with $\Pi_k = \{t = (t_1, \dots, t_k) \in \mathbb{R}^k, 0 \le t_1 \le \dots \le t_k \le 2\pi\}$ and $Q(t) = U_0(-t)QU_0(t)$. Moreover, in the third and fourth terms we can replace Q by Q'.

Dividing the interval of integration in the second term into subintervals $[0,\pi]$ and $[\pi, 2\pi]$ and replacing t_1 by t and $t+\pi$ on these subintervals respectively we obtain that this term is a pseudodifferential operator with principal symbol $i\int_0^{2\pi} q'' \circ \Psi_t\,dt$.

The third term in the right-hand expression is equal to

$$-\frac{1}{2}\iint_{\Pi_2} \big(Q(t_1)Q(t_2) + Q(t_2)Q(t_1)\big)\,dt_1dt_2 - \frac{1}{2}\iint_{\Pi_2} [Q(t_1), Q(t_2)]\,dt_1dt_2.$$

The first term here equals half of $-Q(t_1)Q(t_2)$ integrated over the square $(t_1, t_2) \in [0, 2\pi]^2$; therefore it is equal to $-\frac{1}{2}(\int_0^{2\pi} Q(t)dt)^2$ and therefore is negligible as its principal symbol is 0.

The second term is an operator of order (-3) and after simple calculations we obtain that its principal symbol is

$$i\int_0^{\pi}\int_0^{\pi} \{q\circ\Psi_t, q\circ\Phi_{t+t'}\}\,dtdt'.$$

Finally, the third term in (11.3.35) is obviously an operator of order (-3). Making permutations of (t_1, t_2, t_3) and summing we obtain that modulo an operator of order (-4) this term is one-sixth of $Q(t_1)Q(t_2)Q(t_3)$ integrated over the cube $[0, 2\pi]^3$ and is therefore negligible.

Thus,

(11.3.36) $(U(2\pi)-I)$ is an operator of order (-3) with the principal symbol $2\pi i b_{-3}(x,\xi)$ where b_{-3} is given by (11.3.30) with $q' = \frac{1}{2}a''a^{-\frac{1}{2}}$ and $q'' = -\frac{1}{8}a''^2a^{-\frac{3}{2}}$ instead of q and q^s respectively.

Therefore Theorem 6.2.3 yields

Theorem 11.3.23. *Suppose that conditions of Theorem 11.3.21 and condition* (11.3.34) *are fulfilled. Let $\mathcal{L}'' = [\lambda_1, \lambda_2]$ be the image of the map $\Sigma \ni (x,\xi) \to b_{-3}(x,\xi) \in \mathbb{R}$. Then*

(i) $f(\operatorname{Spec} A) \subset \bigcup_{n\ge 1} J''_n \cup [-C, C]$ *where*

$$J''_n = \big[n + \lambda_1 n^{-3} - Cn^{\delta-4}, n + \lambda_2 n^{-3} + Cn^{\delta-4}\big], \tag{11.3.37}$$

$C = C(\delta)$ *is small enough,* $\delta > 0$ *is arbitrarily small and for* $\lambda_1 = \lambda_2$ *one can take* $\delta = 0$.

(ii) Let $\omega_2 \in \mathcal{L}''$ *and condition* (11.3.18) *be fulfilled with* ω_2 *and* b_{-3} *instead of* ω_0 *and* b_0. *Then for* $\tau \to +\infty$ *such that*

$$f(\tau) \subset \bigcup_{n\ge 1} [n + (\omega_2 - \epsilon)n^{-3}, n + (\omega_2 + \epsilon)n^{-3}]$$

with a sufficiently small constant $\epsilon > 0$ *asymptotics* (11.3.32) *holds with the same* p *as above and with*

$$F(z,\tau) = 2(2\pi)^{-d-1} \int_\Sigma \Upsilon\Big(2\pi\big(z - \tau^{-\frac{3}{4}} b_{-3}(x,\xi)\big)\Big)\, dxd\xi : da. \tag{11.3.38}$$

We leave to the reader

Problem 11.3.24. (i) Prove that for a generic differential symbol (i.e., potential) as well as for a generic pseudodifferential symbol a'' satisfying (11.3.34) the modified[27] condition (11.3.18) is fulfilled with $r = 2d-2$ for all $\omega_2 \in \mathbb{R}$.

(ii) Thus, in all the above cases for generic perturbations the remainder estimate is $O\big(\tau^{\frac{d-2}{2}}\big)$ for $d \ge 3$ and $O\big(\tau^{\frac{1}{2}(d-2+\delta)}\big)$ with an arbitrarily small exponent $\delta > 0$ for $d = 2$. Applying the more precise results of Section 6.2 we obtain that for $d = 2$ the remainder estimate is $O(1)$ if the principal symbol b has no saddle points at the corresponding level and $O(\log \tau)$ otherwise.

[27] I.e., with ω_2, b_{-3} instead of ω_0, b_0.

(iii) Derive the remainder estimate $O(1)$ in the latter case as well by including in the principal symbol the next symbol too (as we have seen it does not necessarily vanish even if the principal symbol vanishes identically).

(iv) Consider Riesz means asymptotics to the reader. Recall that the best possible remainder estimate is $O\bigl(\tau^{\frac{1}{2}(d-1-n\vartheta)}\bigr)$ where $n = 0, 1, 3$ is the order of B taken with the opposite sign.

11.3.3 Singular operators on Closed Manifolds

Let us now consider the singular case when not only an operator but also a manifold itself has singularities. Singularities are described by scaling function $\gamma(x)$:

Definition 11.3.25. (i) Let X be a Riemannian manifold (with or without boundary) such that $\bar{X}$ a closure of X in the Riemannian metrics is compact. Then we call X *singular compact Riemannian manifold*.

(ii) If the regular part of the boundary is empty, we call X *singular compact closed Riemannian manifold*.

(iii) Let γ be a scaling function on $\bar{X}$, such that $|\gamma(x,y) \le \frac{1}{2}\mathsf{dist}(x,y)$, $\partial X = \{x, \gamma(x) = 0\}$, $X = \{x, \gamma(x) > 0\}$ and let metrics on X be γ-admissible. Then we call X (singular) *γ-admissible compact Riemannian manifold*.

However we will need also a γ-admissible function $\zeta(x,\xi)$ describing how close trajectory $\Psi_t(x,\xi)$ can approach $\{\gamma = 0\}$ (see condition (11.3.39) below):

Theorem 11.3.26. *Let X be a γ-admissible compact closed Riemannian manifold, A a scalar elliptic γ-admissible operator of order $2m$ with the lower order terms satisfying the standard conditions of this section. Let us assume that $\gamma^{-1-\delta} \in \mathscr{L}^1(X)$ with $\delta > 0$.*

Further, let us assume that the Hamiltonian flow of $\mathsf{a}(x,\xi)$ is T_0-periodic on Σ. Furthermore, let us assume that a γ-admissible bounded function ζ is given on Σ such that

$$|D\Psi_t(z)| \le C\zeta^{-n}(z), \quad \gamma(\Psi_t(z)) \ge \epsilon\zeta^n(z) \qquad \forall z \in \Sigma \quad \forall t. \tag{11.3.39}$$

Finally, let us assume that conditions (11.3.18) *with* $r \ge 1$ *(for some fixed* ω_0*) and* (11.3.19)–(11.3.20) *(which is condition to subperiodic trajectories) with* $q > 0$ *are fulfilled.*

Then for $\tau \to +\infty$ *such that* $|\iota\tau^{\frac{1}{2m}} - \omega_0| \le \epsilon$ *asymptotics* (11.3.21) *holds with sufficiently small exponent* $p > 0$.

Proof. Dividing Σ into the zones $\{\zeta \ge \frac{1}{2}\tau^{-\delta}\}$ and $\{\zeta \le \tau^{-\delta}\}$ we can apply results of Section 6.2 in the former zone and the standard asymptotics in the latter one. □

Problem 11.3.27. We derived asymptotics which is marginally better than the standard one only as $\vartheta \le p$. The problem which is definitely rather difficult and very interesting is to derive asymptotics with the better remainder estimates; ideally one should derive asymptotics with the remainder estimates which are as good as in the regular case (surely one needs to impose more restricting conditions).

Example 11.3.28. (i) Consider *Zoll-Tannery manifold* with the rotational symmetry (see Example 8.3.9(iii). Recall that these manifolds are isometric (up to a factor) to a sphere $\mathbb{S}^d$ with the metrics $(r+h(\cos(s)))^2\, ds^2+\sin^2(s)\, d\theta^2$ with $s \in [0,\pi]$, $r = p/q$, $p, q \in \mathbb{N}$, $\gcd(p,q) = 1$ and odd function $h : (-1,1) \to (-r,r)$. Then the length of equator is 2π, the length of meridian $2\pi r$ and the length of general geodesics $2\pi p$. See A. Besse [1], Theorem 4.13.

Then $T_0 = 2\pi q$. Therefore one can take $A = \Delta^m$ (plus lower order terms) where Δ is the Laplace-Beltrami operator; these lower order terms must satisfy the standard assumptions of this section and also provide condition (11.3.12) with $p \ge 1$. Singularities (if there are any) are located at $s = 0, \pi$.

However these singularities do not prevent us from recovering remainder estimate $O(\tau^{(d-2)/2m})$ as long as non-degeneracy condition to the lower order terms is fulfilled. This asymptotics contains non-Weyl correction term.

(ii) Consider the same example replace rational r by irrational r; then we are basically in the framework of the Subsection 11.3.1 and can recover remainder estimate $O(\tau^{(d-1)/2m-\delta})$ without any condition to the lower order terms.

Repeating the arguments of Theorems 11.3.21 and 11.3.26 we can generalize them to operators with the singular perturbations on $\mathbb{S}^d$.

Remark 11.3.29. Let us consider the Laplace-Beltrami operator of Example 11.3.28 perturbed by a potential $V \in \mathscr{L}^s$ such that $\|V\|_{\mathscr{L}_s} < \epsilon$ with $s > d = 2$ or $s = d \geq 3$. Then the standard functional-analytic arguments yield that the cluster character of the spectrum remains true. The size of a cluster is $O(\epsilon\tau^{\frac{d}{2s}})$ and the distance between clusters is $\asymp \tau^{\frac{1}{2}}$; here $\epsilon = \|V\|_{\mathscr{L}^s}$ is assumed to be small only for $s = d$.

11.3.4 Manifolds with the Boundary

The possibility of singularities increases the possibilities even if the boundary is smooth.

Example 11.3.30. Consider manifolds of Example 8.3.9 which have boundaries (and these boundaries may also have singularities). Therefore one can take $A = \Delta^m$ (plus lower order terms) where Δ is the Laplace-Beltrami operator with the appropriate boundary conditions.

Then

(i) If r is rational the lower order terms and boundary conditions must satisfy the standard assumptions of this section and also provide condition (11.3.12) with $p \geq 1$. Then asymptotics with remainder estimate $O(\tau^{(d-1)/2m-\delta})$ holds; this asymptotics contains non-Weyl correction term.

(ii) If r is irrational we can recover remainder estimate $O(\tau^{(d-1)/2m-\delta})$ without any condition to the lower order terms and boundary conditions.

11.3.5 The Indefinite Case

In this subsection we discuss the case when the principal symbol is neither positive nor negative definite. We assume that the lower order terms are appropriate: conditions on these terms will be discussed in the next Subsection 11.3.6.

All the results of Sections 11.2 and 11.3 remain true for operators with elliptic principal symbols which are neither positive nor negative definite. In this case $\mathsf{N}^-(\tau)$ is the number of eigenvalues in the interval $(0, \tau)$ for $\tau > 0$ and $\mathsf{N}^+(\tau)$ is the number of eigenvalues in the interval $(\tau, 0)$ for $\tau < 0$ and we consider asymptotics of $\mathsf{N}^\mp(\tau)$ for $\tau \to \pm\infty$.

Surely, for $\tau \to +\infty$ and for $\tau \to -\infty$ we may obtain different asymptotics and the conditions on Hamiltonian trajectories may be different;

however in two most interesting examples – Dirac and Maxwell operators (see Example 8.4.4) results should be the same modulo remainder estimate and equal to $\frac{1}{2}\mathsf{N}^-(A^2-\tau^2)$; this would make those results look trivial.

However it is not the case, at least not for the Maxwell operator which should be considered on the subspace defined by under-defined system and as domain is irregular (for example, if it has edges directed inside, like in $(\mathbb{R}^d \setminus \mathbb{R}^{+2}\times\mathbb{R}^{d-2})$) one should be really careful ; in particular formally Maxwell system and system of hydrodynamics[28)] look the same but in such domains results are different. So, we suggest very difficult and interesting problem

Problem 11.3.31. Accurate analysis with the results of the same sharpness as in this Sections 11.2 and 11.3 for

(i) Maxwell system:

(a) isotropic (already difficult);

(b) anisotropic (much more difficult);

(ii) System of hydrodynamics[28)].

Further, all the results for $\vartheta>0$ remain true if we set

$$\mathsf{N}^{\mp}_{(\vartheta)}(\tau)=\pm\int_0^{\tau}|\tau'|^{\vartheta-1}N(\tau')d\tau' \tag{11.3.40}$$

for $\pm\tau>0$.

Furthermore, the results of Subsection 11.2.3 should remain true. In this case (instead of the assumption that $A(x,D)$ is a scalar operator) we assume that $a(x,\xi)$ has only one positive (or negative) eigenvalue $\lambda(x,\xi)$ which is simple provided we treat asymptotics as $\tau\to+\infty$ ($\tau\to-\infty$ respectively). In this case all the conditions of these subsections are assumed to be fulfilled for the Hamiltonian flow generated by $\lambda(x,\xi)$.

Moreover, to obtain conditions on lower order terms one should find pseudodifferential operator Q such that

$$Q^*Q\equiv QQ^*\equiv I \qquad \text{and} \qquad Q^*AQ\equiv\begin{pmatrix}A_1 & 0\\ 0 & A_2\end{pmatrix}$$

[28)] Without viscosity.

where A_1 is a scalar operator with principal symbol $\lambda(x,\xi)$ and A_2 an operator with principal symbol of the opposite sign. Then all the conditions should be imposed on the lower order terms of A_1.

Moreover, we can treat the Dirac operator; nevertheless the eigenvalues of its principal symbol are simple only for $D=2 \implies d=1,2,3$ (for $d=3$ one has a principally massless particle). However, one needs to check that all the symbols associated with A_1 and taken into account are scalar. Similar notes remain true for isotropic Maxwell and hydrodynamics systems and I suggest another very difficult and interesting problem

Problem 11.3.32. Accurate analysis with the results of the same sharpness as in this subsection 11.3.2 for

(i) Dirac operator;

(ii) Isotropic Maxwell system;

(iii) System of hydrodynamics[28)];

In order to prove the above statements one should apply the results of Parts II and III to the interval $(\frac12\tau,\tau)$ (we consider the case $\tau>0$ for the sake of simplicity). Then the routine remainder estimate $O(\tau^\nu)$ or $O(\tau^\nu \log^\sigma \tau)$ is obtained by standard methods. Let us replace τ by $2^{-j}\tau$ (with $j=0,\dots,n$ such that $\tau \asymp 2^n$) and sum all these asymptotics; then we obtain asymptotics for the number of eigenvalues lying in (τ_1,τ) with the same remainder estimate for $\nu>0$ and with the remainder estimate $O(\log^{\sigma+1}\tau)$ for $\nu=0, \sigma\ge 0$. On the other hand it is relatively easy to prove that $\mathsf{N}(0,\tau_1)<\infty$.

Moreover, in order to improve the final remainder estimate in the case when it grows slowly, one can apply the following arguments to obtain the remainder estimate $O(\tau^{-s})$ with arbitrarily chosen s for mollifications of $\mathsf{N}(\tau',\tau'')$ (the number of eigenvalues in the interval (τ',τ'')) with respect to τ' and τ'' with mollifying functions $\tau^{-1}\phi_1(\tau'\tau^{-1})$ and $\tau^{-1}\phi_2(\tau''\tau^{-1})$ where ϕ_1 and ϕ_2 are supported near $\frac12$ and 1 respectively. Replacing τ by $2^{-j}\tau$ and making a partition of unity one obtains the remainder estimate $O(1)$ for $\mathsf{N}(0,\tau')$ mollified near $\frac12\tau$. On the other hand, for $N(\tau',\tau)$ (even without mollification) we have the appropriate remainder estimate provided $\tau'\asymp\tau$.

So, semiclassical arguments look rather straightforward but there is also analysis in the singular zone and for indefinite operators it may become rather non-trivial. For $\vartheta>0$ one can apply analysis of the same type.

11.3.6 Conditions to Lower Order Terms

Let us discuss conditions to the lower order terms starting from a series of remarks:

Remark 11.3.33. All our results were proven for (ρ,γ)-admissible lower order terms provided

$$\rho^{d+2m\vartheta}\in \mathscr{L}^1(X), \tag{11.3.41}$$

$$\rho^{2m}\in \mathscr{L}^1(X), \tag{11.3.42}$$

$$\rho^{2m}\in \mathscr{L}^2(X), \tag{11.3.42*}$$

where (11.3.42) and (11.3.42)* are conditions in the semi-bounded and non-semibounded cases respectively. Here $2m$ is the order of the operator and $m\in \frac{1}{2}\mathbb{Z}$.

Recall that so far ρ was not mentioned at this section.

We can give a series of admissible lower order terms for all the types of operators treated above. However, it is very simple in the context of (ρ,γ)-admissibility. We leave this analysis to the reader and discuss how one can weaken these conditions which may be excessive (in contrast to Section 11.1 because now ρ does not enter in the assumptions on the principal part.

Example 11.3.34 (singularity at origin). (i) Let $\gamma=\epsilon|x|$ and $\rho=|x|^{\mu}$. Then conditions (11.3.42), (11.3.42) and (11.3.42)* mean respectively that

$$\mu>-d(d+2m\vartheta)^{-1},\qquad \mu>-\frac{d}{2m},\qquad \mu>-\frac{d}{4m}. \tag{11.3.43}$$

Let μ^* be the maximal forbidden value. Then for $\mu=\mu^*$ we can treat $\rho=|x|^{\mu}\bigl|\log|x|\bigr|^{\nu}$ with appropriate ν, etc.

However, arguments associated with estimates (11.1.66)–(11.1.67) imply that it is sufficient to assume that

$$\mu>\max(-1,-\frac{d}{2m}),\qquad \mu>\max(-1,-\frac{d}{4m}) \tag{11.3.44}$$

in the semi-bounded and non-semi-bounded cases respectively, i.e. that one should check (11.3.43) only with $\vartheta=0$.

Really, in such vicinity operator A_0 or A_0^2 (where A_0 contains no lower order terms) is larger in operator sense than $r^{-2m+\delta}$, $r^{-4m+\delta}$ with an arbitrarily small exponent $\delta > 0$ respectively.

Moreover, for $d > 2m$, $d > 4m$ respectively in these cases one can even take $\mu = -1$ assuming that in a neighborhood of 0

$$|a_{\alpha',\alpha''}| \le \epsilon \rho^{2m-|\alpha'|-|\alpha''|} \qquad \forall \alpha', \alpha'' : |\alpha'| \le m, |\alpha''| \le m, |\alpha'| + |\alpha''| \ge 1, \tag{11.3.45}$$

$$|a_\alpha| \le \epsilon \rho^{2m-|\alpha|} \qquad \forall \alpha : 1 \le |\alpha| \le 2m \tag{11.3.45}^*$$

respectively with small enough $\epsilon = \mathsf{const} > 0$.

(ii) Let us assume that in a neighborhood of 0 domain X coincides with the cone $\{(r,\theta) \in \mathbb{R}^+ \times U\}$ where $\partial U \in \mathscr{C}^K$. Further, let us assume that the Dirichlet boundary condition is given on a part of the boundary $\{(r,\theta) : r \le r_0, \theta \in U_1 \subset \partial U\}$ and that $r_0 > 0$ and $U_1 \ne \emptyset$ contains the ∂U-neighborhood of some point. Then one even does not need condition $\mu > -\frac{d}{2m}$ in the semi-bounded case.

Really, in such vicinity operator A_0 is larger in operator sense than ϵr^{-2m} with sufficiently small constant $\epsilon > 0$.

(iii) Moreover, for the Schrödinger operator with either $d \ge 3$ and with the Dirichlet boundary condition or with $d \ge 2$ and a domain and boundary conditions of the type described in (ii), variational arguments yield that one can assume that in a neighborhood of 0

$$V \ge -\epsilon\rho^2, \qquad |V_j| \le \epsilon\rho \tag{11.3.46}$$

with $\rho = |x|^{-1}$.

(iv) Let us treat the Schrödinger operator with $\mu = -1$, $d \ge 3$ assuming that either the Dirichlet boundary condition is given on ∂X or the domain and the boundary condition are of the type described in (ii). Then

$$(Au,u) \ge \sum_j \epsilon_0 \|(D_j - V_j)u\|^2 + (Vu,u). \tag{11.3.47}$$

Let us introduce spherical coordinates (r,θ) and remove the radial component of the magnetic potential by means of a gradient transform. We obtain that for functions supported in the vicinity of 0

$$(Au,u) \ge \epsilon_0 \|D_r u\|^2 + \epsilon_0 \sum_j \|r^{-1}\mathcal{L}_j u\|^2 + (Vu,u) - \varepsilon \|r^{-1}u\|^2 \tag{11.3.48}$$

where $\mathcal{L}_j$ are first-order operators with bounded coefficients on $\mathbb{S}^{d-1}$ and $\varepsilon > 0$ is arbitrarily small (the vicinity depends on ε). Let us consider

$$\|D_r u\|^2 = \int\int_0^{r_0} r^{d-1} |D_r u|^2 \, dr d\theta.$$

In order to do this we treat $\|r^\nu D_r u\|_1^2$ where $\|.\|_1$ and $(.,.)_1$ mean respectively $\mathscr{L}^2$-norm and inner product on $[0,r_0]$, $\nu = \frac{1}{2}(d-1)$. Substituting $u = r^{-\nu}v$ we obtain that

$$\begin{aligned}\|r^\nu \partial_r u\|_1^2 = \|(\partial_r - \nu r^{-1})v\|_1^2 = \|\partial_r v\|_1^2 + \nu^2 \|r^{-1}v\|_1^2 - \\ 2\nu \operatorname{Re}(\partial_r v, r^{-1}v)_1 = \|\partial_r v\|_1^2 + (\nu^2 - \nu)\|r^{-1}v\|_1^2.\end{aligned} \tag{11.3.49}$$

Let $d > 3$; then $\nu^2 > \nu$ and we conclude that in this case

$$\|r^\nu \partial_r u\|_1^2 \ge \epsilon_1 \|r^{\nu-1}u\|_1^2. \tag{11.3.50}$$

On the other hand, let $d = 3$. Then $\nu = 1$ in which case $\|r^\nu \partial_r u\|^2 = \|\partial_r v\|^2$ but then obviously (11.3.50) holds again. Therefore

(11.3.51) For $d \ge 3$ the second of conditions (11.3.46) is not necessary.

On the other hand, for $d = 2$, $\nu = \frac{1}{2}$ and $\nu^2 - \nu = -\frac{1}{4}$. In this case only one operator $\mathcal{L}_1 = D_\theta - w(r,\theta)$ is present and if for $\theta = \theta_1$ a Dirichlet boundary condition is given then $\|\mathcal{L}_1 u\|_2 \ge \epsilon_1 \|u\|_2$ where $\|.\|_2$ is $\mathscr{L}^2$-norm with respect to θ.

On the other hand, if $U = \{\theta, \theta_1 < \theta < \theta_2\}$ and if for both $\theta = \theta_1$ and $\theta = \theta_2$ the Neumann boundary condition is given then this estimate fails. However, if X is a complete neighborhood of 0 then this estimate remains true provided

$$\Big|\int_0^{2\pi} w(\theta,r)d\theta - 2\pi m\Big| \ge \epsilon_0 \qquad \forall m \in \mathbb{Z} \quad \forall r \in (0,r_0). \tag{11.3.52}$$

Conversely, if (11.3.52) fails for every $\epsilon_0 > 0$ then the previous inequality also fails. We conclude that

(11.3.53) Let $d = 2$ and either $U = \{\theta_1 < \theta < \theta_2\}$ and for θ_1 or for θ_2 a Dirichlet boundary condition is given, or X contains a complete neighborhood of 0 and (11.3.52) holds. Then the second of conditions (11.3.46) is not necessary.

(v) All these arguments remain true in the case $\gamma = \epsilon|x''|$, $\rho = |x''|^\mu$ with d replaced by d'' in the conditions.

Example 11.3.35. Using the arguments of Remark 11.1.13 we see that in the context of Example 11.2.19 one can assume that $\rho f(|x''|) \le \epsilon$ (with small enough constant $\epsilon > 0$) instead of assumption (11.3.42).

Remark 11.3.36. Certain generalizations can be obtained from functional-analytic arguments. For example, consider Schrödinger operator and asymptotics of $\mathsf{N}^-(\tau)$. Then

(i) Asymptotics with the remainder estimate $O\bigl(\tau^{\frac{1}{2}(d-1)}\bigr)$ remain true if one replaces potentials V and V_j by $V + V'$ and $V_j + V_j'$ with $V' \in \mathscr{L}^d(X)$, $V_j' \in \mathscr{L}^\infty(X)$.

(ii) Moreover, asymptotics with the remainder estimate $O\bigl(\tau^{\frac{1}{2}(d-1-\sigma)}\bigr)$ remain true provided $V_j' = 0$, $V' \in \mathscr{L}^{d(1-\sigma)^{-1}}(X)$.

(iii) Therefore approximating $V' \in \mathscr{L}^d(X)$ by the potentials $V_\varepsilon' \in \mathscr{L}^\infty(X)$ we obtain that asymptotics with the remainder estimate $o\bigl(\tau^{\frac{1}{2}(d-1)}\bigr)$ remain true for such perturbations.

(iv) Moreover, similar approximation arguments yield that asymptotics with the the remainder estimate $O\bigl(\tau^{\frac{1}{2}(d-1)} \log^{-\nu} \tau\bigr)$ remain true provided $V'|\log(|V'|+1)|^\nu \in \mathscr{L}^d(X)$.

(v) On the other hand, remainder estimates weaker than $O\bigl(\tau^{\frac{1}{2}(d-1)}\bigr)$ remain true under weaker conditions. However, our weakly singular potentials (i.e., the potentials which we considered in this section) do not necessarily belong to the indicated classes. Thus, a combination of the two methods is the most effective.

11.4 Large Eigenvalues for Operators in Essentially Unbounded Domains

11.4.1 Coercivity Condition and Standard Asymptotics

In this section, under the assumptions of the previous section concerning the domain X and the principal symbol, we consider operators with the lower order terms essential from the point of view of the domain of the operator and asymptotics[29]. These terms could be more singular than those studied before but they must satisfy *coercivity condition*.

We start with operators with positive principal symbol. In this case the coercivity condition is

$$\langle A(x,\xi)v, v\rangle \ge \epsilon_0(|\xi|+\rho)^{2m}\|v\|^2 \qquad \forall \xi \quad \forall v, \tag{11.4.1}$$

$$\sum_{\alpha,\beta:|\alpha|\le m,|\beta|\le m} \langle a_{\alpha,\beta}w_\alpha, w_\beta\rangle \ge \epsilon_0 \sum_{\alpha:|\alpha|\le m} \|w_\alpha\|^2\rho^{2m-2|\alpha|} \qquad \forall w=(w_\alpha)_{\{|\alpha|\le m\}} \tag{11.4.2}$$

where $\epsilon_0>0$,

$$A(x,\xi)=\sum_{\alpha,\beta:|\alpha|\le m,|\beta|\le m} a_{\alpha,\beta}\xi^{\alpha+\beta} \tag{11.4.3}$$

is the complete (symmetric) symbol of an operator A of the form (11.1.1) with $h=1$ and the first condition (11.4.1) is assumed to be fulfilled for all $x\in X$ while the second condition (11.4.2) (which is surely stronger) is assumed to be fulfilled only in $\{x\in X, \rho\gamma\ge\epsilon_0, \gamma\le\epsilon_0\}$. However, in the certain cases this condition is assumed to be fulfilled in a part of X which larger than this. Again we have assumed that

$$\varrho=\rho^{2m} \tag{11.4.4}$$

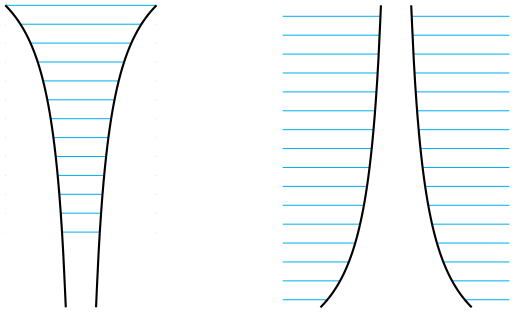
for the sake of simplicity. We leave to the reader to consider the more general case. In what follows we treat lower order terms which are sums of coercive

[29] Thus, it would be better to speak about senior symbols and junior terms.

and admissible indefinite (lower order) terms. For the Schrödinger operator (11.1.6) conditions (11.4.1) and (11.4.2) coincide with the condition

$$V \ge \epsilon_0 \rho^2 \qquad \forall x \tag{11.4.5}$$

provided condition (11.1.6) is fulfilled.



(a) (b)

Figure 11.7: Non-coercive weakly singular potential (a) and coercive singular potential (b); blue lines show symbolically energy levels of the corresponding Schrödinger operator. For weakly singular potential the lowest energy level on (a) exists.

As before we set $\gamma_\tau = \gamma$ and $\rho_\tau = \rho + \tau^{\frac12}$ and consider domain

$$X_{\mathsf{reg}} = \{x \in X,\ \tau^{\frac{1}{2m}}\gamma(x) \ge 1,\ \exists \xi\, \exists v \ne 0 :\ \ \langle A(x,\xi)v, v\rangle \le (1+\epsilon_1)\tau \|v\|^2\}$$

with an arbitrarily small constant $\epsilon_1 > 0$. Then condition (11.4.1) yields that $\rho \le C\tau^{\frac{1}{2m}}$.

As before we apply the results of Parts II and III in X_{reg} to obtain local semiclassical spectral asymptotics for $\mathsf{N}^-(\tau)$ (case $\vartheta > 0$ will be considered later) in the ball $\mathcal{B}\big(x, \gamma(x), X\big)$ with the remainder estimate $O\big(\tau^{\frac{1}{2m}(d-1)}\gamma^{d-1}\big)$ provided $\rho \le \epsilon_2 \tau^{\frac{1}{2m}}$, in which case operator is ξ-microhyperbolic.

Moreover, under condition to the Hamiltonian flow generated by the senior symbol this remainder estimate can be improved to $o\big(\tau^{\frac{1}{2m}(d-1)}\gamma^{d-1}\big)$ or sharper.

On the other hand, for balls with $\rho \ge \epsilon_2 \tau^{\frac{1}{2m}}$ the remainder estimate (as well as the principal part of the asymptotics) is $O\big(\tau^{\frac{d}{2m}}\gamma^d\big)$ in the general case; however it could be improved to $O\big(\tau^{\frac{1}{2m}(d-1+\delta)}\gamma^{d-1+\delta}\big)$ with an arbitrarily small exponent $\delta > 0$ for the scalar operators.

In order to obtain the remainder estimate $O\bigl(\tau^{\frac{1}{2m}(d-1)}\gamma^{d-1}\bigr)$ in the local spectral asymptotics in this zone one should check microhyperbolicity or another non-degeneracy condition there. In particular, for the Schrödinger operator with $d \ge 2$ we obtain this remainder estimate without any additional conditions. This improvement and condition will be important in the next sections but not now.

On the other hand, in the zone

$$X_+ = \bigl\{x \in X,\ \rho\gamma \ge \epsilon_1, \forall \xi\ \forall v\ \langle A(x,\xi)v, v\rangle \ge (1+\epsilon_1)\tau\|v\|^2\bigr\}$$

we obtain the remainder estimate[30] $O(\rho^{-s}\gamma^{-s})$ with arbitrarily chosen exponent s provided either $B\bigl(x,\gamma(x)\bigr)$ does not intersect ∂X, or m-th order Dirichlet boundary condition is given on ∂X.

Moreover, one can obtain the same answer provided[31]

$$(Au,u) \ge \sum_{\alpha,\beta:|\alpha|,|\beta|\le m} (a_{\alpha,\beta}D^\alpha u, D^\beta u) \quad \forall u \in \mathfrak{D}(A). \tag{11.4.6}$$

Let us note that $X_{\mathsf{reg}} \cup X_+ \supset \bigl\{x \in X, \tau^{\frac{1}{2m}}\gamma \ge C_0\bigr\}$. Thus we include X_+ either into semiclassical zone X_{reg} or into variational zone X_{sing}.

Let us assume that condition (11.4.2) is fulfilled in the domain

$$\mathcal{X} \supset \{x \in X,\ \rho\gamma \le \epsilon_1,\ \gamma \le \epsilon_1\}.$$

Then for u supported in $\mathcal{X}$ the following estimate

$$\bigl((A-\tau I)u,u\bigr) \ge -C\tau \int_{\mathcal{X}_-} \|u\|^2 dx$$

holds with $\mathcal{X}_- = \mathcal{X} \setminus \mathcal{X}_+$,

$$\mathcal{X}_+ = \bigl\{x \in \mathcal{X}, \sum_{\alpha,\beta:|\alpha|\le m,|\beta|\le m} \langle a_{\alpha,\beta}(x)w_\alpha, w_\beta\rangle \ge \tau \sum_{\alpha:|\alpha|\le m} \|w_\alpha\|^2 \quad \forall w_\alpha\bigr\}.$$

Surely $\mathcal{X}_+ \supset \bigl\{x \in X, \rho \ge C\tau^{\frac{1}{2m}}\bigr\}$. Therefore we conclude that

(11.4.7) Contribution of $\mathcal{X}$ to the remainder does not exceed $C\tau^{\frac{d}{2m}}\mathsf{mes}(\mathcal{X}_-)$ provided either the Dirichlet boundary condition or any other condition providing (11.4.6) on $\partial X \cap \bar{\mathcal{X}}$ is given; in the latter case we assume that the cone condition is fulfilled.

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[30] The principal part vanishes here.

[31] This inequality is a restriction on the boundary conditions.

Let us note that

Remark 11.4.1. $\mathcal{X}$ and X_+ overlap and therefore we can subtract from X_+ and add to $\mathcal{X}$ zone

$$\tilde{\mathcal{X}} = \{x \in X, \mathsf{dist}(x, X \setminus \mathcal{X}) \ge \epsilon_1 \gamma\}.$$

So we arrive to the estimate

$$\bigl|\mathsf{N}^-(\tau) - \mathsf{N}^{\mathsf{W}}(\tau)\bigr| \le C\bigl(R_1(\tau) + R_2(\tau) + R_3(\tau)\bigr) \tag{11.4.8}$$

with

$$\mathsf{N}^{\mathsf{W}}(\tau) = (2\pi)^{-d} \iint_{X_{\mathsf{reg}} \times \mathbb{R}^d} \boldsymbol{n}(x,\xi;\tau)\, dx d\xi \tag{11.4.9}$$

where $\boldsymbol{n}(x,\xi;\tau)$ is now the eigenvalue counting function for the complete symbol $A(x,\xi)$,

$$R_1(\tau) = \int_{X_{\mathsf{reg}}} \tau^{\frac{d-1}{2m}} \gamma^{-1}\, dx, \tag{11.4.10}$$

$$R_2(\tau) = \tau^{\frac{d}{2m}}\, \mathsf{mes}(\mathcal{X}_- \cup \tilde{\mathcal{X}}), \tag{11.4.11}$$

$$R_3(\tau) = \int_{X_+ \setminus \tilde{\mathcal{X}}} \rho^{d-s} \gamma^{-s}\, dx, \tag{11.4.12}$$

$\tilde{\mathcal{X}} = \bigl\{x \in X_{\mathsf{reg}},\ \rho \ge \epsilon_2 \tau^{\frac{1}{2m}}\bigr\}$ in the general case. This zone can be reduced (or even voided) under microhyperbolicity or other non-degeneracy condition; in particular $\tilde{\mathcal{X}} = \emptyset$ for the Schrödinger operator with $d \ge 2$.

Similarly, for $\vartheta > 0$ estimate

$$\bigl|\mathsf{N}^-_{(\vartheta)}(\tau) - \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau)\bigr| \le C\bigl(R_{1(\vartheta)}(\tau) + R_{2(\vartheta)}(\tau) + R_{3(\vartheta)}(\tau)\bigr) \tag{11.4.13}$$

holds with $\mathsf{N}^-_{(\vartheta)}(\tau) = \tau_+^{\vartheta-1} * \mathsf{N}^-(\tau)$, $\mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) = \tau_+^{\vartheta-1} * \mathsf{N}^{\mathsf{W}}(\tau)$, $\mathsf{N}^{\mathsf{W}}(\tau)$ containing now lower order terms as well,

$$R_{1(\vartheta)}(\tau) = \int_{X_{\mathsf{reg}}} \tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta} \gamma^{-1-\vartheta}\, dx, \tag{11.4.14}$$

$$R_{2(\vartheta)}(\tau) = \tau^{\frac{d}{2m}+\vartheta}\, \mathsf{mes}(\mathcal{X}_- \cup \tilde{\mathcal{X}}), \tag{11.4.15}$$

$$R_{3(\vartheta)}(\tau) = \int_{X_+ \setminus \tilde{\mathcal{X}}} \rho^{2m\vartheta+d-s} \gamma^{-s}\, dx. \tag{11.4.16}$$

In particular, we have proven

Proposition 11.4.2. *Let*

$$\gamma^{-1-\vartheta} \in \mathscr{L}^1(X), \tag{11.4.17}$$

$$\rho^{2m\vartheta+d-s}\gamma^{-s} \in \mathscr{L}^1(X \setminus \tilde{\mathcal{X}}) \tag{11.4.18}$$

and

$$\mathsf{mes}(\mathcal{X}_- \cup \tilde{\mathcal{X}}) = O\bigl(\tau^{-\frac{1}{2m}(\vartheta+1)}\bigr); \tag{11.4.19}$$

then

$$\mathsf{N}^-_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O\bigl(\tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta}\bigr) \qquad \text{as } \tau \to +\infty. \tag{11.4.20}$$

For the Schrödinger operator with $d \geq 2$ and other scalar operators with $\vartheta > 0$ condition (11.4.19) is fulfilled automatically, but in the general case this condition is very unpleasant and it is best to check the microhyperbolicity condition for $(A - \tau)$ in $\tilde{\mathcal{X}}_-$ and then reduce this zone.

Remark 11.4.3. Let us discuss the principal differences between weakly and strongly singular potentials:

(i) For strongly singular potentials coercivity condition is required; without it operator may be even not essentially self-adjoint;

(ii) For weakly singular potentials small vicinity of the singular point is a singular zone; for strongly singular potential it belongs to a classically forbidden (due to coercivity) part of regular zone;

(iii) Finally, in the general case the principal parts of the asymptotics $\mathsf{N}^{\mathsf{W}}(\tau)$ and $\mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau)$ cannot be decomposed into a power series of $\tau^{\frac{1}{2m}}$ as in the previous section. Usually this decomposition leads to much worse remainder estimate than the original one.

11.4.2 Sharp Remainder Estimates

Let us discuss an improvement of the remainder estimate in the zone $X_{\mathsf{reg}} \cap \{\rho \leq C\tau^{\frac{1}{2m}}\}$; thus we consider the term $R_1(\tau)$ or $R_{1(\vartheta)}(\tau)$. Let us write our results as a series of remarks:

Remark 11.4.4. (i) Let the standard non-periodicity condition to the (billiard) Hamiltonian trajectories is fulfilled in X_{reg} for the symbol $a(x,\xi)$ on the energy level 1.

Then for any fixed $\bar\gamma > 0$ one can replace the contribution of zone $\{X, \gamma(x) \ge \bar\gamma\}$ to the remainder by $R^*_{1(\vartheta)}(\tau) = o\bigl(R_{1(\vartheta)}(\tau)\bigr)$.

(ii) Further, let condition (11.4.17) be fulfilled; then $R_{1(\vartheta)}(\tau) \asymp \tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta})$ and we can replace it by $R^*_{1(\vartheta)}(\tau) = o\bigl(\tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta}\bigr)$.

So, under condition (11.4.17) and improved assumption (11.4.19)

$$\mathsf{mes}(\mathcal{X}_- \cup \tilde{\mathcal{X}}) = o\bigl(\tau^{-\frac{1}{2m}(\vartheta+1)}\bigr) \tag{11.4.19*}$$

we can improve asymptotics (11.4.20) to

$$\mathsf{N}^-_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + o\bigl(\tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta}\bigr) \qquad \text{as } \tau \to +\infty. \tag{11.4.20*}$$

In particular, for $\vartheta = 0$ the two-term asymptotics holds with the remainder estimate $o(\tau^{(d-1)/2m})$.

(iii) All the examples of Subsection 11.3.1 remain true. However, the principal part of these asymptotics should be given by the formulas of the present section.

Remark 11.4.5. (i) Let the enhanced non-periodicity condition to the (billiard) Hamiltonian trajectories is fulfilled in X_{reg} for the symbol $a(x,\xi)$ on the energy level 1. Namely, let us assume that this flow is of either exponential or power type and that the corresponding non-periodicity condition is fulfilled. Then one can replace $R_{1(\vartheta)}(\tau)$ by

$$R^{**}_{1(\vartheta)}(\tau) = \int_{X_{\mathsf{reg}}} \tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta} \gamma^{-1-\vartheta} \bigl(\log(\tau^{1/2m}\gamma)\bigr)^{-1-\vartheta}\, dx, \tag{11.4.14**}$$

or

$$R^{***}_{1(\vartheta)}(\tau) = \int_{X_{\mathsf{reg}}} \tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta} \gamma^{-1-\vartheta} (\tau^{1/2m}\gamma)^{-\delta}\, dx \tag{11.4.14***}$$

respectively.

(ii) So, under condition (11.4.17) and further improved assumption (11.4.19)

$$\mathsf{mes}(\mathcal{X}_- \cup \tilde{\mathcal{X}}) = O\bigl(\tau^{-\frac{1}{2m}(\vartheta+1)} |\log\tau|^{-1-\vartheta}\bigr) \tag{11.4.19**}$$

or

$$\mathsf{mes}(\mathcal{X}_- \cup \tilde{\mathcal{X}}) = O\bigl(\tau^{-\frac{1}{2m}(\vartheta+1)-\delta}\bigr) \tag{11.4.19***}$$

we can improve asymptotics (11.4.20) to

$$\mathsf{N}^{-}_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O\bigl(\tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta}|\log\tau|^{-1-\vartheta}\bigr) \tag{11.4.20)**}$$

or

$$\mathsf{N}^{-}_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O\bigl(\tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta-\delta}\bigr) \tag{11.4.20)***}$$

respectively.

In particular, for $\vartheta = 0$ the two-term asymptotics holds with the remainder estimate $O(\tau^{(d-1)/2m}(\log\tau)^{-1})$ or $O(\tau^{(d-1)/2m-\delta})$ respectively.

Remark 11.4.6. In the context of Subsection 11.3.2 (periodic billiard Hamiltonian flow) under condition (11.4.19)*** asymptotics

$$\mathsf{N}^{-}(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + \tau^{\frac{d-1}{2m}}\bigl(\varkappa_1 + F(\tau^{\frac{1}{2m}})\bigr) + O\bigl(\tau^{\frac{d-1}{2m}-\delta}\bigr) \tag{11.4.21}$$

holds as $\tau \to +\infty$.

Remark 11.4.7. Non-periodicity condition may be useful even with weakened assumptions (11.4.17)–(11.4.19).

11.4.3 Examples

Let us consider some examples. We start from the Schrödinger operator $\Delta + V(x)$ with potential $V(x) = \kappa|x|^{2\mu}$, $\kappa = \mathsf{const} \neq 0$, $\mu \leq -1$, singular at origin, gradually adding the complexity. Note that as $\mu < -1$ operator V is not Δ-bounded (see Definition 10.2.4) and as $\mu = -1$ it may be Δ-bounded[32)] but it's Δ-bound is not 0.

So, as $\mu < -1$, we *must* assume that $0 \notin X$ and $\kappa > 0$, defining operator $\Delta + V$ through quadratic form; as $\mu = -1$ we may[32)] allow $\kappa \geq \kappa_0$ with $\kappa_0 < 0$.

Then we can consider the case of singularity at origin when $\gamma(x) \asymp |x|$ and $\rho \asymp |x|^{\mu}$ with $\mu \leq -1$ and to generalize it to singularity at $\{x, x'' = 0\}$ of codimension d''.

Example 11.4.8 (Power singularities). (i) Let all the conditions be fulfilled with $\gamma \asymp |x''|$, $\rho \asymp |x''|^{\mu}$, $\mu \leq -1$. There one can take $\mathcal{X} = \emptyset$ for $\mu < -1$ and $\mathcal{X} = \{x \in X, |x''| \leq \epsilon_0\}$ for $\mu = -1$.

[32)] It is the case unless $d = 2$ and 0 is surrounded by X.

Then in the general case we should take the transitional zone $\tilde{\mathcal{X}} = \{\epsilon\tau \le |x''|^{2m\mu} \le C\tau\}$ arriving to

(a) For $d'' > \vartheta + 1$, $\mu \ge -d''(\vartheta+1)^{-1}$ asymptotics (11.4.20) holds.

Moreover, for $\mu > -d''(\vartheta+1)^{-1}$ under the standard non-periodicity condition to the Hamiltonian trajectories asymptotics (11.4.20)* holds.

Note that for $\vartheta = 0$ and $\mu > -d''$ one can replace full Weyl expression (including all terms of A) $\mathsf{N}^{\mathsf{W}}(\tau)$ by the standard two-term expression (one needs the second term only for sharp asymptotics) $\varkappa_0\tau^{\frac{d}{2m}} + \varkappa_1\tau^{\frac{d-1}{2m}}$ with the standard coefficients $\varkappa_0, \varkappa_1$ and with the same remainder estimate as above.

On the other hand, for $\mu = -d''$, $\mathsf{N}^{\mathsf{W}}(\tau) = \varkappa_0\tau^{\frac{d}{2m}} + O\bigl(\tau^{\frac{d-1}{2m}}\log\tau\bigr)$ in the general case and $\mathsf{N}^{\mathsf{W}}(\tau) = \varkappa_0\tau^{\frac{d}{2m}} + O\bigl(\tau^{\frac{d-1}{2m}}\bigr)$ in the case of a scalar principal symbol.

(b) For $d'' = \vartheta + 1$, $\mu = -1$ asymptotics

$$\mathsf{N}^-_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O\bigl(\tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta}\log\tau\bigr) \tag{11.4.22}$$

holds and for $\vartheta = 0$, $d'' = 1$, $\mu = -1$ one can replace $\mathsf{N}^{\mathsf{W}}(\tau)$ by $\varkappa_0\tau^{\frac{d}{2m}}$.

(c) If $\mu < -d''(\vartheta+1)^{-1}$ then asymptotics

$$\mathsf{N}^-_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O\bigl(\tau^{\frac{d}{2m}+\frac{d''}{2m\mu}+\vartheta}\bigr) \tag{11.4.23}$$

holds.

Let us note that $\mu \ge -d''(\vartheta+1)^{-1}$, $\mu \le -1$ yield $d'' \le \vartheta + 1$ and therefore only the listed cases are possible.

(ii) One can slightly improve the above results in the number of cases. For the Schrödinger operator with $d \ge 2$ and general scalar operators with $\vartheta > 0$ one can extend X_{reg} and take $\tilde{\mathcal{X}} = \emptyset$. Moreover, one can easily see that if X is a conic with respect to the x'' domain in a neighborhood of $\{x'' = 0\}$ and

$$D^\alpha(a_{\beta',\beta''} - a^0_{\beta',\beta''}) = o(\rho^{2m-|\beta'|-|\beta''|}\gamma^{-|\alpha|}) \qquad \forall\beta', \beta'' \quad \forall\alpha : |\alpha| \le 1 \tag{11.4.24}$$

as $|x''| \to 0$ with functions $a^0_{\beta',\beta''}$ which are positively homogeneous of degrees $(2m - |\beta'| - |\beta''|)\mu$ then[33] checking the microhyperbolicity condition with

$$\ell = \langle\xi, \partial_\xi\rangle + \mu^{-1}\langle x'', \partial_{x''}\rangle$$

[33] We call conditions similar to (11.4.24) *stabilization conditions*.

we see that we can take $\tilde{X} = \emptyset$ as well.

In all these we obtain that

(a) For $d'' > \vartheta + 1$ asymptotics (11.4.20) holds. Moreover, under the standard non-periodicity condition to Hamiltonian trajectories asymptotics (11.4.20)* holds.

(b) For $d'' = \vartheta+1$ asymptotics (11.4.22) holds and for $d'' < \vartheta+1$ asymptotics (11.4.23) holds.

Let us note that in both cases for $\mu \le -d''$, even for the Schrödinger operator, it is possible that

$$\mathsf{N}^{\mathsf{W}}(\tau) - \varkappa_0 \tau^{\frac{d}{2m}} - \varkappa_1 \tau^{\frac{d-1}{2m}} \asymp \tau^{\frac{d}{2m}+\frac{d''}{2m\mu}}$$

and therefore for $\mu < -d''$ we cannot replace $\mathsf{N}^{\mathsf{W}}(\tau)$ by $\varkappa_0 \tau^{\frac{d}{2m}}$ and preserve the remainder estimate. Moreover, for $\mu = -d''$ we cannot replace $\mathsf{N}^{\mathsf{W}}(\tau)$ by $\varkappa_0 \tau^{\frac{d}{2m}} + \varkappa_1 \tau^{\frac{d-1}{2m}}$ and preserve the remainder estimate $o\bigl(\tau^{\frac{d-1}{2m}}\bigr)$.

Example 11.4.9 (Power singularities again). Let us consider Schrödinger operator with $d \ge 2$ [34] and let us assume that in a neighborhood of $\{x'' = 0\}$ domain X coincides with $X^0 = \{x = (x', r, \theta) \in \mathbb{R}^{d''} \times \mathbb{R}^+ \times U\}$ where $U \subset \mathbb{S}^{d''-1}$ is a $\tilde{\gamma}$-admissible domain with $\tilde{\gamma} = \tilde{\gamma}(x', \theta)$.

Let us assume that $g^{jk} = g^{jk}(x', \theta)$, $V_j = v_j(x', \theta)|x''|^\mu$, $V = v(x', \theta)|x''|^{2\mu}$ and $\mu \le -1$. Further, let us assume that $v \ge 0$ and the functions g^{jk}, v_j and v on $\mathbb{R}^{d'} \times \mathbb{S}^{d''-1}$ satisfy standard conditions (11.1.6)–(11.1.8) and (11.4.5) with $\tilde{\rho} = \sqrt{v}$ and with $\tilde{\gamma}$ instead of ρ and γ. Then

(i) Under additional assumption $|v_j| \le c\tilde{\rho}$ one can take $\rho = \tilde{\rho}|x''|^\mu$ and $\gamma = \epsilon_1 \tilde{\gamma}|x''|$ with small enough constant $\epsilon_1 > 0$.

(a) Then we obtain that under assumption

$$\tilde{\gamma}^{-1-\vartheta} \in \mathscr{L}^1(\mathbb{R}^{d'} \times U) \tag{11.4.25}$$

asymptotics (11.4.20) holds for $d'' > \vartheta + 1$.

(b) Moreover, under the standard non-periodicity condition to the Hamiltonian trajectories asymptotics (11.4.20)* holds.

[34] One can consider general scalar operators for $\vartheta > 0$.

(c) On the other hand, under condition (11.4.25) for $d'' = \vartheta + 1$ and $d'' < \vartheta + 1$ we obtain asymptotics (11.4.22) and (11.4.23) respectively.

(ii) Without condition $|v_j| \le c\tilde{\rho}$ we cannot use variational estimates. In this case we should assume that $\mu < -1$ and

$$\tilde{\rho}\tilde{\gamma} \ge \epsilon_0 > 0. \tag{11.4.26}$$

These conditions forbid the degeneration of v but permit certain singularities. Then all the statements of the previous assertion remain true.

(iii) Let $\mu < -1$, condition (11.4.26) be fulfilled and $V_j = v_j(x', \theta)|x''|^{\mu'}$ with $\mu' < \mu$ (the case $\mu' > \mu$ is trivial).

Then one can take $\rho = \tilde{\rho}|x'|^{\mu}$, $\gamma = \epsilon_1\tilde{\gamma}|x''|^{-\mu'+\mu+1}$. Then $\rho\gamma \ge \epsilon_2|x''|^{-\sigma}$ with $\sigma > 0$ provided $2\mu + 1 < \mu'$.

(a) Then, under assumption (11.4.25) we obtain asymptotics (11.4.20) for $d'' > (1 + \vartheta)(1 + \mu - \mu')^{-1}$.

(b) Moreover, under the standard non-periodicity condition to Hamiltonian trajectories we obtain asymptotics (11.4.20)*.

(c) On the other hand, for $d'' = (1+\vartheta)(1+\mu-\mu')^{-1}$ we obtain asymptotics (11.4.22) and for $d'' < (1 + \vartheta)(1 + \mu - \mu')^{-1}$ we obtain asymptotics

$$\mathsf{N}^{-}_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O\bigl(\tau^{\frac{d-\eta}{2}+\vartheta}\bigr) \tag{11.4.27}$$

with $\eta = \mu^{-1}\bigl((1 + \vartheta)(1 + 2\mu - \mu') - d''\bigr) \in (0, 1 + \vartheta)$.

Here we cannot consider the case $\mu' \le 2\mu + 1$ because the magnetic field is too strong near the origin. This case is left to the next part.

Let us note that every time assertions with large enough s are used, K (in the conditions on coefficients) depends on μ, μ', etc. (via s).

Example 11.4.10 (Exponential singularities). Let X, $g^{jk} = g^{jk}(x', \theta)$, $v_j = v_j(x', \theta)$ and $v = v(x', \theta)$ be as in statements (i) or (ii) of the previous Example 11.4.9. Let $V_j = v_j \exp(|x''|^{-\nu})$, $V = v \exp(2|x''|^{-\nu})$ with $\nu > 0$.

Then all the conditions will be fulfilled with $\rho = \tilde{\rho} \exp(|x''|^{-\nu})$ and $\gamma = \epsilon_1\tilde{\gamma}|x''|^{\nu+1}$. We obtain that

(a) For $(1+\nu)(1+\vartheta) < d''$ asymptotics (11.4.20) holds and under the standard non-periodicity condition to Hamiltonian trajectories asymptotics (11.4.20)* holds.

(b) Further, for $(1+\nu)(1+\vartheta) = d''$ asymptotics

$$\mathsf{N}^-_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O\bigl(\tau^{\frac{d-1+\vartheta}{2}} \log\log\tau\bigr) \tag{11.4.28}$$

holds and

(c) For $(1+\nu)(1+\vartheta) > d''$ asymptotics

$$\mathsf{N}^-_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O\bigl(\tau^{\frac{d-1+\vartheta}{2}} \log^{\sigma}\tau\bigr) \tag{11.4.29}$$

holds with $\sigma = \nu^{-1}\bigl((1+\nu)(1+\vartheta) - d''\bigr)$.

We leave it to the reader the following series of rather easy exercises:

Problem 11.4.11. (i) Consider examples in the spirit of Example 11.4.9(iii) with exponential singularities, i.e., with $V_j = v_j|x''|^{-l}\exp(|x''|^{-\nu})$ or with $V_j = v_j \exp(k|x''|^{-\nu})$, $k \in (1,2)$.

(ii) For different examples estimate $\bigl(\mathsf{N}^{\mathsf{W}}(\tau) - \varkappa_0\tau^{\frac{d}{2}}\bigr)$ from above and below.

(iii) Consider quasiconical domains and domains with cusps for potentials which grow at 0 or infinity.

(iv) Consider general scalar operators with $\vartheta > 0$.

11.4.4 Indefinite Perturbations

Now we consider the case when a strongly singular coercive potential is perturbed by a less singular indefinite potential. Let us assume that conditions (11.1.3)–(11.1.4) are fulfilled with $\rho+\rho'$ instead of ρ and with $\varrho = (\rho+\rho')^{2m}$ where ρ and ρ' are γ-admissible weight functions. We denote such conditions by (11.1.3)$'$–(11.1.4)$'$.

Meanwhile we replace conditions (11.4.1) and (11.4.2) by conditions

$$\langle A(x,\xi)v, v\rangle \ge \epsilon_0\bigl((|\xi|+\rho)^{2m} - C\rho'^{2m}\bigr)\|v\|^2 \qquad \forall\xi \quad \forall v \tag{11.4.1}'$$

and

$$(11.4.2)' \qquad \sum_{\alpha,\beta:|\alpha|\le m,|\beta|\le m} \langle a_{\alpha,\beta}(x)v_\alpha, v_\beta\rangle \ge \epsilon_0 \sum_{\alpha:|\alpha|\le m} \rho^{2m-2|\alpha|}\|v_\alpha\|^2 - C\sum_{\alpha:|\alpha|\le m-1} \rho'^{\,2m-2|\alpha|}\|v_\alpha\|^2 \qquad \forall v = (v_\alpha)_{\{|\alpha|\le m\}}$$

respectively.

Remark 11.4.12. Then the arguments of this and previous sections show that all the estimates of this section remain true with all domains defined by ρ if to their right-hand expressions we add

$$(11.4.30) \qquad R'(\tau) = C\Big(\int_{\mathcal{Z}} (\rho'^{\,2m})_+^q \, dx\Big)^{\frac{d}{2mq}} \times \big(\mathsf{mes}(\mathcal{Z})\big)^{1-\frac{d}{2mq}}$$

for $\vartheta = 0$, and

$$(11.4.31) \quad R'_{(\vartheta)}(\tau) = C\int_0^\infty t^{\vartheta-1}\Big(\int_{\mathcal{Z}} (\rho'^{\,2m} - t)_+^q \, dx\Big)^{\frac{d}{2mq}} dt \times \big(\mathsf{mes}(\mathcal{Z})\big)^{1-\frac{d}{2mq}}$$

for $\vartheta > 0$ where $q \ge \frac{d}{2m}$, $q \ge 1$ and one of these inequalities is strict, and

$$\mathcal{Z} = \{x \in X, \rho \le C\rho'\}.$$

Example 11.4.13. Consider a power singularity under the assumptions of Example 11.4.9(i) with potentials V, V_j replaced by $V + V'$, $V_j + V'_j$ respectively where $V' = w(x', \theta)|x''|^{2\nu}$, $V'_j = w_j(x', \theta)|x''|^\nu$, w and w_j are also functions on $\mathbb{S}^{d''-1} \times \mathbb{R}^{d'}$ satisfying conditions (11.1.6)–(11.1.8) with $\tilde\rho'$, $\tilde\gamma$.

Here $\tilde\rho'$ is a $\tilde\gamma$-admissible weight function on $\mathbb{S}^{d''-1} \times \mathbb{R}^{d'}$ and $\nu > \mu$, $\mu \le -1$, $\nu < 0$. Let us set $\gamma = \epsilon_1\tilde\gamma|x''|$, $\rho = \tilde\rho|x''|^\mu$, $\rho' = \tilde\rho'|x''|^\nu$ with $\tilde\rho = \sqrt{v}$. Then term (11.4.30) or (11.4.31) should be added to the right-hand expressions of our estimates, and one can see easily that this term does not exceed existing remainder estimate provided one of the following conditions is fulfilled:

$$(11.4.32)_1 \quad \iint_{\{\tilde\rho'^{\,2}\ge\epsilon v\}} \tilde\rho'^{\,-a} v^{\frac{b}{2}} \, d\theta dx' < \infty \qquad \text{if } \ b < 0,\ d \ge 3$$

with $a = (d'' + d\mu + 2\vartheta\mu)(\nu-\mu)^{-1}$, $b = (d'' + d\nu + 2\vartheta\nu)(\nu-\mu)^{-1}$,

$$(11.4.32)_2 \quad \iint_{\{\tilde\rho'^2 \ge \epsilon v\}} \tilde\rho'^{\,d+2\vartheta}\bigl(\log\frac{\tilde\rho'^2}{v}\bigr)_+ d\theta dx' < \infty \qquad \text{if } \ b=0,\ d\ge 3$$

$$(11.4.32)_3 \quad \iint_{\{\tilde\rho'^2 \ge \epsilon v\}} \tilde\rho'^{\,d+2\vartheta}\, d\theta dx' < \infty \qquad \text{if } \ b>0,\ d\ge 3,$$

$$(11.4.32)_4 \quad \iint_{\{\tilde\rho'^2 \ge \epsilon v\}} \tilde\rho'^{\,-a-\delta\bar a} v^{\frac{b}{2}+\delta}\, d\theta dx' < \infty \qquad \text{if } \ b\le 0,\ d=2,$$

$$(11.4.32)_5 \quad \iint_{\{\tilde\rho'^2 \ge \epsilon v\}} \tilde\rho'^{\,d+2\vartheta+\delta}\, d\theta dx' < \infty \qquad \text{if } \ b>0,\ d=2,$$

where $\delta > 0$ is arbitrarily small, $\bar a = \mu(\nu-\mu)^{-1}$, $\bar b = \nu(\nu-\mu)^{-1}$.

We leave to the reader

Problem 11.4.14. Modify and analyze the same way the other examples of this section.

11.4.5 Operators with Indefinite Principal Symbols

We leave to the reader

Problem 11.4.15. Analyze operators with indefinite principal symbols.

In this case one should replace conditions (11.4.1), (11.4.2) by

$$(11.4.1)^* \qquad \|A(x,\xi)v\| \ge \epsilon_0\bigl(|\xi|+\rho\bigr)^{2m}\|v\| \qquad \forall \xi \quad \forall v$$

and

$$(11.4.2)^* \qquad \Bigl\|\sum_{\alpha:|\alpha|\le 2m} a_\alpha(x) v_\alpha\Bigr\| \ge \epsilon_0 \sum_{\alpha:|\alpha|\le 2m} \rho^{2m-|\alpha|}\|v_\alpha\| \qquad \forall v = (v_\alpha)_{\{\alpha:|\alpha|\le 2m\}}$$

respectively. Moreover, one should replace conditions $(11.4.1)'$, $(11.4.2)'$ by

$$(11.4.1)^{*\prime} \qquad \|A(x,\xi)v\| \ge \epsilon_0\bigl(\bigl(|\xi|+\rho\bigr)^{2m} - C\rho'^{\,2m}\bigr)\|v\| \qquad \forall \xi \quad \forall v$$

and

$$(11.4.2)^{*\prime} \quad \| \sum_{\alpha:|\alpha|\le 2m} a_\alpha(x) v_\alpha \| \ge \epsilon_0 \sum_{\alpha:|\alpha|\le 2m} \rho^{2m-|\alpha|} \|v_\alpha\| - C \sum_{\alpha:|\alpha|\le 2m-1} \rho'^{2m} \|v_\alpha\| \quad \forall v = (v_\alpha)_{\{\alpha:|\alpha|\le 2m\}}$$

respectively.

Unfortunately we are not aware of any examples of this type which are interesting from the physical point of view. The reader can easily construct artificial examples (like generalized Dirac operator) and analyze them.

11.5 Large Eigenvalues for Operators in Essentially Unbounded Domains (End)

Now we consider *essentially unbounded domains* which are domains containing massive[35)] exits to infinity. In domains of this type the spectrum of the operator in question is discrete only because the operator contains lower degree terms which are coercive with the function ρ tending to ∞ at infinity (in almost all directions).

Therefore even the order of the asymptotics of $\mathsf{N}^{\mathsf{W}}(\tau)$ depends on the lower degree terms (which are not anymore "lower order terms" from our point of view).

We treat here only operators with positive definite senior symbols and leave the indefinite case to the reader.

11.5.1 Standard Results

We begin with potentials of the power growth at infinity. As the first example we use only the results of Parts II and III to obtain the following well known assertion:

Theorem 11.5.1. *Let X be an unbounded connected domain in $\mathbb{R}^d$ and let conditions* (11.1.1)–(11.1.3), (11.4.1) *and* (11.4.4) *be fulfilled with $\gamma = \epsilon_1\langle x\rangle$, $\rho = \langle x\rangle^\mu$, $\mu > 0$ where here and below $\langle x\rangle = (|x|^2+1)^{\frac{1}{2}}$.*

[35)] E.g., conical. Domains with cusps even having infinite volume do not qualify as essentially unbounded and will be treated in the next chapter.

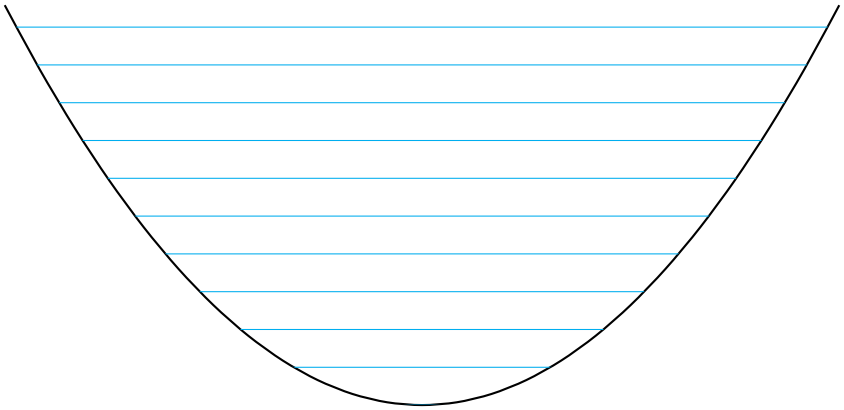

Figure 11.8: Coercive at infinity potential; blue lines show symbolically energy levels of the corresponding Schrödinger operator.

(i) Let us assume that for every $\tau > c$ one of the non-degeneracy conditions is fulfilled after rescaling for the symbol $A(x,\xi)\tau^{-1}$ on the energy level 1 uniformly with respect to τ, x, ξ (this assumption is only restrictive for $|x| \ge C_0$). Then asymptotics

$$N^-(\tau) = N^W(\tau) + O\big(\tau^{l(d-1)}\big) \tag{11.5.1}$$

holds as $\tau \to +\infty$ where $N^W(\tau)$ is given by (11.4.9) *and $N^W \asymp \tau^{ld}$, $l = \frac{1}{2m\mu}(\mu+1)$.*

(ii) Let us assume that for every $\tau > c$ one of the corresponding non-degeneracy conditions[36] *is fulfilled for the symbol $A(x,\xi)\tau^{-1}$ on the energy level 1 uniformly with respect to τ, x, ξ. Then for $\vartheta \ge 0$ asymptotics*

$$N^-_{(\vartheta)}(\tau) = N^W_{(\vartheta)}(\tau) + O\big(R_{(\vartheta)}(\tau)\big) \tag{11.5.2}$$

holds as $\tau \to +\infty$ with

$$R_{(\vartheta)}(\tau) = \begin{cases} \tau^{l(d-1)+\vartheta(1-l)} & \text{for } \vartheta < d-1, \\ \tau^{d-1}\log\tau & \text{for } \vartheta = d-1, \\ \tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta} & \text{for } \vartheta > d-1 \end{cases} \tag{11.5.3}$$

where $N^W_{(\vartheta)}(\tau) \asymp \tau^{ld+\vartheta}$.

[36] Recall that for operators with the scalar symbols some very weak non-degeneracy condition is needed only as $\vartheta = 0$.

Proof. Proof trivially follows from the corresponding results of Parts II and III (also see Introduction). □

Remark 11.5.2. (i) Obviously domain $\{|x| \asymp \tau^{1/(2m\mu)}\}$ provides the main contribution to the asymptotics. Further, as $\vartheta < \vartheta^* = (d-1)$ this domain also provides the main contribution to the remainder estimate.

(ii) However as $\vartheta > \vartheta^*$ inner domain $\{|x| \asymp 1\}$ provides the the main contribution to the remainder estimate while for $\vartheta = \vartheta^*$ intermediate domain $\{\epsilon^{-1} \le |x| \le \epsilon\tau^{1/(2m\mu)}\}$ provides the main contribution to the remainder estimate.

Therefore to improve remainder estimate one needs non-periodicity assumption for the corresponding Hamiltonian flow.

Theorem 11.5.3. *Let assumptions of Theorem 11.5.1 be fulfilled. Moreover, let $X \cap \{x \ge c_1\} = X^0 \cap \{x \ge c_1\}$ where X^0 is a conical (with respect to x) domain with $\partial X^0 \setminus 0 \in \mathscr{C}^K$ and let the stabilization conditions*

$$D^\alpha(a_\beta - a^0_\beta) = o\big(|x|^{(2m-|\beta|)\mu-|\alpha|}\big) \qquad \forall \beta : |\beta| \le 2m \quad \forall \alpha : |\alpha| \le 1 \tag{11.5.4}$$

be fulfilled as $|x| \to \infty$ with functions $a^0_\beta \in \mathscr{C}^K(\bar{X}^0 \setminus 0)$ which are positive homogeneous of degrees $(2m-|\beta|)\mu$ with respect to x.

(i) Further, let us assume that on the energy level $\tau = 1$ for the quasihomogeneous symbol

$$A^0(x,\xi) = \sum_{|\beta|\le 2m} a^0_\beta(x)\xi^\beta \tag{11.5.5}$$

the standard non-periodicity condition to the Hamiltonian trajectories in T^*X^0 is fulfilled and that these permitted trajectories[37] *never hit $0 \times \mathbb{R}^d$.*

Then for $\vartheta < \vartheta^* = (d-1)$ asymptotics

$$\mathsf{N}^-_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + o\big(R_{(\vartheta)}(\tau)\big) \tag{11.5.6}$$

holds $\tau \to +\infty$ with $R_{(\vartheta)}(\tau) = \tau^{l(d-1)+\vartheta(1-l)}$ according to (11.5.3).

[37] I.e. trajectories originated at $\{A^0(x,\xi) = 1\} \setminus \Lambda$ where Λ is an exceptional set.

(ii) On the other hand, let $\vartheta > \vartheta^*$, and the standard non-periodicity condition to Hamiltonian trajectories be fulfilled on the energy level $\tau = 1$ for the symbol

$$a(x,\xi) = \sum_{|\beta|=2m} a_\beta(x)\xi^\beta \tag{11.5.7}$$

in the domain X. Then asymptotics (11.5.6) *also holds as $\tau \to +\infty$ but this time with $R_{(\vartheta)}(\tau) = \tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta}$ according to* (11.5.3).

(iii) Finally, let $\vartheta = \vartheta^*$ and stabilization condition (11.5.4) *and the standard non-periodicity condition to Hamiltonian trajectories is fulfilled on the energy level $\tau = 1$ for the symbol*

$$a^0(x,\xi) = \sum_{|\beta|=2m} a^0_\beta(x)\xi^\beta \tag{11.5.8}$$

in the domain X^0 and these permitted[37] trajectories never hit $0 \times \mathbb{R}^d$. Then asymptotics (11.5.6) *holds as $\tau \to +\infty$ but this time with $R_{(\vartheta)}(\tau) = \tau^{d-1}\log\tau$ according to* (11.5.3).

Remark 11.5.4. (i) If only stabilization condition (11.5.4) is fulfilled then the microhyperbolicity condition is automatically fulfilled for the symbol $A(x,\xi)\tau^{-1}$ on the energy level 1 and

$$\mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) = \varkappa_{0(\vartheta)}\tau^{ld}\big(1+o(1)\big) \tag{11.5.9}$$

as $\tau \to +\infty$.

(ii) As $\vartheta = 0$ asymptotics must include the second term $\varkappa_1\tau^{l(d-1)}$ where for the Schrödinger operator,

$$\varkappa_0 = (2\pi)^{-d}\omega_d \int_{X^0} \big(1-V^0\big)_+^{\frac{d}{2}} \sqrt{g^0}\,dx \tag{11.5.10}$$

and

$$\varkappa_1 = \mp\frac{1}{4}(2\pi)^{1-d}\omega_{d-1}\int_{\partial X^0} \big(1-V^0\big)_+^{\frac{d-1}{2}}\,dS^0 \tag{11.5.11}$$

for Dirichlet and Neumann boundary conditions respectively, dS^0 is the Riemannian density on ∂X^0 generated by the induced Riemannian metrics, $g^0 = \det(g^{jk0})^{-1}$.

Proof of Theorem 11.5.3. (i) One can see easily that as $\vartheta < \vartheta^*$ the contribution of zone $\{|x| \le \varepsilon\tau^{\frac{1}{2m\mu}}\}$ to the remainder does not exceed $\varepsilon^\delta R_{(\vartheta)}(\tau)$ and therefore only zone $\{\varepsilon\tau^{\frac{1}{2m\mu}} \le |x| \le C_0\tau^{\frac{1}{2m\mu}}\}$ should be considered: in all cases zone $\{|x| \ge C_0\tau^{\frac{1}{2m\mu}}\}$ is elliptic.

Let us apply the theorems of Parts II and III and we examine the behavior of the billiards generated by the symbol $A(x,\xi)$ on $\Sigma_\tau = \{(x,\xi), \tau \in \mathsf{Spec}\, A(x,\xi)\}$. We take

$$\Lambda_1 = \Lambda_{1,\tau,\epsilon,T} = \big\{(x,\xi) \in \Sigma_\tau, |x| \ge \varepsilon\tau^{\frac{1}{2m\mu}}, \quad |x\tau^{-\frac{1}{2m\mu}} - y| + |\xi\tau^{-\frac{1}{2m}} - \eta| \ge \epsilon \quad \forall (y,\eta) \in \Lambda^0_T\big\} \tag{11.5.12}$$

where $\Lambda^0_T = \Sigma^0 \setminus \Lambda^1_T$, $\Sigma^0 = \{(y,\eta), 1 \in \mathsf{Spec}\, A^0(y,\eta)\}$ and for every point of Λ^1_T the Hamiltonian frow of A^0 (with reflection from ∂X^0) for time T in an appropriate time direction does not enter ε-vicinity of $0 \times \mathbb{R}^d$ and periodicity is violated with at least divergence ε; we fix arbitrary small ε in advance.

By assumption $\mathsf{mes}(\Lambda^0_T) = 0$, on the other hand, Λ^0_T is a relatively closed subset of $\Sigma^0 \setminus \{x = 0\}$ for every T. The remaining arguments repeat those in the proof of Theorem 11.1.1: we check that the conditions on billiards generated by $A^0(x,\xi)$ and $A(x,\xi)$ coincide provided $\tau \ge C_{\varepsilon,T}$.

(ii) On the other hand, for $\vartheta > \vartheta^*$ the principal contribution to the remainder estimate is given by the zone $\{x \in X, |x| \le r\}$ (with arbitrarily large fixed r) in the sense that the contribution of the remaining part does not exceed $r^{-\delta}R_{(\vartheta)}(\tau)$.

Therefore, to prove Statement (ii) one should treat Hamiltonian billiards in this zone, in which one can replace $A(x,\xi)$ by $a(x,\xi)$ for large τ.

(iii) Finally, for $\vartheta = \vartheta^*$ the principal contribution to the remainder estimate is given by the zone $\{r \le |x| \le \varepsilon\tau^{\frac{1}{2m\mu}}\}$ for every fixed ε and r in the sense that the contribution of the remaining part does not exceed $(\varepsilon^\delta + r^{-\delta})R_{(\vartheta)}(\tau)$.

Therefore, to prove Statement (iii) one should treat Hamiltonian billiards in this zone, in which one can replace $A(x,\xi)$ by $a^0(x,\xi)$ for large τ, $|x|$. □

Let us consider scalar operators. Observing that the microhyperbolicity condition is automatically fulfilled for $|x| \le \epsilon\tau^{\frac{1}{2m\mu}}$ and recalling the results of Chapter 5 we arrive immediately to

Theorem 11.5.5. *Let X be an unbounded connected domain in $\mathbb{R}^d$ and let conditions* (11.1.1)–(11.1.3), (11.4.1), (11.4.4) *be fulfilled with $\gamma = \epsilon_1\langle x\rangle$, $\rho = \langle x\rangle^\mu$, $\mu > 0$.*

Moreover, let A be a scalar operator. Then

(i) Asymptotics

$$\mathsf{N}^-(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + O\bigl(\tau^{l(d-1)+\delta}\bigr) \tag{11.5.13}$$

holds as $\tau \to +\infty$ with an arbitrarily small exponent $\delta > 0$ and asymptotics (11.5.2) *holds for $\vartheta > 0$.*

(ii) Moreover, for $0 < \vartheta < \vartheta^* = (d-1)$, $\vartheta > \vartheta^*$, $\vartheta = \vartheta^*$ respectively Statements (i)–(iii) of Theorem 11.5.3 remain true.

11.5.2 Strongly Non-Periodic Hamiltonian Flow

In what follows we consider homogeneous coefficients with singularities and (what is more essential here) with degenerations. However, we now improve the remainder estimates under stronger conditions to Hamiltonian trajectories.

Theorem 11.5.6. *Let the conditions of Theorem 11.5.1 be fulfilled. Let A be a scalar operator. Assume that*

(11.5.14) *Either $X^0 = \mathbb{R}^d$ or A is the Schrödinger operator or its power.*

Further, let the strong stabilization condition

$$D^\alpha(a_\beta - a^0_\beta) = O\bigl(|x|^{2m-|\beta|)\mu-|\alpha|-\delta}\bigr) \qquad \forall \beta : |\beta| \le 2m \quad \forall \alpha : |\alpha| \le 1 \tag{11.5.4)*}$$

be fulfilled as $|x| \to \infty$ with some $\delta > 0$. Furthermore, let us assume that the Hamiltonian flow (with reflections at ∂X^0) is of non-periodic power type[38] *and that along trajectories $(x(t), \xi(t))$ with $(x(0), \xi(0)) \in \Lambda^0_{\varepsilon,T} \subset \Sigma_1$, $|x(0)| = 1$, $\varepsilon = T^{-M}$, condition $|x(t)| \ge T^{-M_1}$ is fulfilled.*

[38] We are aware of no example when the flow is of non-periodic exponential type but is not of non-periodic power type. We would appreciate construction of such example. We leave to the reader to formulate the statement in the case of non-periodic exponential type.

Then asymptotics

$$\mathsf{N}^{-}(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + \bigl(\varkappa_1 + O(\tau^{-\delta'})\bigr)\tau^{l(d-1)} \tag{11.5.15}$$

holds as $\tau \to +\infty$. *Moreover, for* $0 < \vartheta \le d-1-\delta$ *asymptotics*

$$\mathsf{N}^{-}_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O\bigl(\tau^{l(d-1)+\vartheta(1-l)-\delta'}\bigr) \tag{11.5.16}$$

holds as $\tau \to +\infty$ *with* $\delta' = \delta'(d, m, \delta, l) > 0$.

Example 11.5.7. (i) Growth and non-periodicity conditions of Theorem 11.5.6 to the Hamiltonian flow are fulfilled for the Schrödinger operator with $X^0 = \mathbb{R}^d$ and $A^0(x,\xi) = \bigl(|\xi|^2 + |x|^{2\mu}\bigr)$, $0 < \mu \ne 1$.

(ii) Therefore the statement of Theorem 11.5.6 is true for this operator (provided all the other conditions of this theorem, including (11.5.4)*, are fulfilled) and for the $2m$-order operator with $A^0(x,\xi) = (|\xi|^2 + |x|^{2\mu})^m$, and one can replace $X^0 = \mathbb{R}^d$ by the X^0 obtained after (a sequence of) the following transformations:

(a) We take the factor manifold of X^0 with respect to a finite rotation group (with center at the origin),

(b) We consider the reflection $S : \bar{X}^0 \to \bar{X}^0$ with respect to a hyperplane and remove from X^0 some subset $\mathcal{K}$ of this plane provided $\tilde{\gamma}^{-1-\vartheta} \in \mathscr{L}^1(\mathbb{S}^{d-1})$ with $\tilde{\gamma} = \mathsf{dist}(x, \mathcal{K})$.

Proof. Obviously it is sufficient to prove Statement (i). Moreover, the potential $V^0 = |x|^{2\mu}$ is spherically symmetric and therefore an angular momentum is constant along every Hamiltonian trajectory and therefore the x-projection of this trajectory lies in some 2-dimensional plane. Therefore it is sufficient to treat only the case $d = 2$. We introduce polar coordinates (r, θ) and use the fact that the *angular momentum* $M = \dot{\theta} r^2$ is constant where $\dot{\theta} = \frac{d\theta}{dt}$, etc.

Therefore the energy is equal to

$$E = \frac{1}{2}\dot{r}^2 + \frac{1}{2}M^2 r^{-2} + r^{2\mu} = \frac{1}{2}\dot{r}^2 + W(r, M) \tag{11.5.17}$$

and is also constant and therefore r obeys equation of oscillations

$$\ddot{r} = -W'(r, M) \tag{11.5.18}$$

with the *effective potential* $W(r) = \frac{1}{2}M^2r^{-2} + r^{2\mu}$.

Then a trajectory with $|M| \ge \varepsilon$ lies permanently in the zone $\{(\theta, r), r \ge M\sqrt{\frac{2}{E}}\}$. One can easily see that

$$|M| \le M_0(E) = (2\mu)^{\frac{1}{2}}\big(\frac{E}{\mu+1}\big)^{\frac{1}{2\mu}(\mu+1)} \tag{11.5.19}$$

and every trajectory oscillates between the largest and the smallest values $r = r_2(E, M)$ and $r = r_1(E, M)$ respectively where $r_1 < r_2$ [39)] are roots of the equation $W(r, M) = E$ (i.e. $\frac{1}{2}M^2r^{-2} + r^{2\mu} = E$) and the period of the oscillation is

$$T_\Pi = T_\Pi(E, M) = 2\int_{r_1(E,M)}^{r_2(E,M)} \big(2E - 2W(r, M)\big)^{-\frac{1}{2}}\, dr \tag{11.5.20}$$

while the increment of θ during a period is

$$\theta_\Pi = 2\int_{r_1(E,M)}^{r_2(E,M)} Mr^{-2}\big(2E - 2W(r, M)\big)^{-\frac{1}{2}}\, dr. \tag{11.5.21}$$

One can easily see that

(11.5.22) If $\varepsilon \le |M| \le M_0(E) - \varepsilon$ and $E \asymp 1$ then

$$|r_2(E, M) - r_1(E, M)| \ge \epsilon\varepsilon^p, \quad r_1(E, M) \ge \epsilon\varepsilon^p, \quad T_\Pi(E, M) \ge \epsilon\varepsilon^p, \quad \theta_\Pi \ge \epsilon\varepsilon^p$$

and the functions $f(E, M) = r_1(E, M), r_2(E, M), T_\Pi(E, M), \theta_\Pi(E, M)$ satisfy the inequalities

$$|D^\beta_{E,M} f| \le C_{|\beta|}\varepsilon^{-p|\beta|} \qquad \forall \beta$$

with appropriate constants $p, \epsilon > 0$ and $C_{|\beta|}$; $\varepsilon > 0$ is a small parameter here.

On the other hand,

(11.5.23) The $dxd\xi : dA^0$-measure of the set of points

$$\big\{(r, \theta) : E = 1, |M| \le \varepsilon \text{ or } E = 1, |M| \ge M_0(E) - \varepsilon\big\}$$

is less than $C\varepsilon^\delta$ with $\delta > 0$.

[39)] And $r_1(E, M) = r_2(E, M)$ if and only if $|M| = M_0(E)$.

These assertions (11.5.22) and (11.5.2) yield that the Hamiltonian flow is of power type. Let us prove that as $\mu \neq 1$ this flow is of non-periodic power type.

The arguments above yield that it is sufficient to prove that

(11.5.24) For $k, n \in \mathbb{Z}^+, m \le \varepsilon^{-1}, n \le \varepsilon^{-1}$ the measure of the set of points (x, ξ) with $E(x, \xi) = 1$ such that

$$|\theta_\Pi(1, M(x, \xi)) - 2\pi \frac{k}{n}| \le \varepsilon^L \tag{11.5.25}$$

does not exceed $O(\varepsilon^{\frac{L}{q}})$ where L is arbitrarily large.

Obviously, $\theta_\Pi(1, M)$ is an analytic function of $M \in (0, M_0(1))$; further, $\theta_\Pi(1, M_0(1) - \eta^2)$ is an analytic function of $\eta \in (-\epsilon_0, \epsilon_0)$.

Furthermore, it is well known (see, e.g., V. I. Arnold [2]) that

(11.5.26) For $\mu > 0, \mu \neq 1$ function $\theta_\Pi(1, M)$ is not constant.

Therefore the measure of points (x, ξ) with $E(x, \xi) = 1$ and $|M(x, \xi)| \ge \epsilon_1$, satisfying (11.5.25) is $O(\varepsilon^{\frac{L}{q}})$ as $\varepsilon \to +0$. Here $\epsilon_1 > 0$ is an arbitrary constant.

On the other hand, one can easily see that θ_Π is continuous at $[0, M_0(1)]$ and $\theta_\Pi(+0) = \pi$. Therefore in the zone $\{(x, \xi), 0 < M(x, \xi) \le \epsilon\epsilon_0\}$ inequality (11.5.25) can be fulfilled only with $k \asymp n$.

Further, one can easily see that for $M \in (0, \epsilon_0)$, $\theta_\Pi(1, M) = \mathcal{F}(M, M^{2\mu})$ where $\mathcal{F}$ is an analytic function of two variables in a neighborhood of $(0, 0)$. Furthermore, $\theta_\Pi(1, M) \neq \mathsf{const}$ and therefore $|\nabla_M \theta_\Pi(1, M)| \ge \epsilon_1 M^p$ for $M \in (0, \epsilon_1)$ with appropriate $\epsilon_1 > 0$ and p and therefore the measure of the set of points with $E(x, \xi) = 1$ and $M(x, \xi) \in (0, \epsilon_1)$ satisfying (11.5.25) is $O(\varepsilon^{\frac{L}{p}})$. □

Remark 11.5.8. (i) For $\vartheta > d - 1 + \delta$ one can easily improve the remainder estimate $O(\tau^{\frac{1}{2m}(d-1\vartheta)+\vartheta})$ to $O(\tau^{\frac{1}{2m}(d-1\vartheta)+\vartheta-\delta'})$ under a corresponding condition on the Hamiltonian flow of $a(x, \xi)$.

In order to reach uniformity one can assume that stabilization condition $(11.5.4)^*$ is fulfilled only for terms with $|\alpha| = 2m$. In this case only these terms should be scalar but if some junior terms are not scalar one should check that some non-degeneracy condition is fulfilled.

(ii) Moreover, one can consider the case $\vartheta = d-1$. It is easy to replace the factor $\log\tau$ by $\log^{1-\delta'}\tau$ but better improvements are possible only under much more complicated and restrictive conditions.

11.5.3 Periodic Hamiltonian Flow

Let A be a scalar operator and condition (11.5.14) be fulfilled. Further, let us assume that

(11.5.27) The Hamiltonian trajectories of $A^0(x,\xi)$ are periodic with period $T(A^0(x,\xi))$ and

$$A = A^{0\mathrm{w}}(x,D) + A'^{\mathrm{w}}(x,D), \tag{11.5.28}$$

where

$$D^\beta(a'_\alpha - a^1_\alpha) = O\big(|x|^{(2m-|\alpha|)\mu-\delta_1-\delta'-|\beta|}\big) \qquad \forall\alpha : |\alpha|\le 2m \quad \forall\beta : |\beta|\le K \qquad \text{as} \quad |x|\to\infty \tag{11.5.29}$$

where $a^1_\alpha(x)$ are positively homogeneous of degrees $(2m-|\alpha|)\mu-\delta_1$ with respect to x and are smooth away from 0, $0<\delta_1<2(\mu+1)$, $\delta'>0$.

Let us introduce function

$$b(x,\xi) = \frac{I}{T_0}\int_0^{T_0} A^1\circ\Psi_t dt \qquad \text{with} \quad A^1(x,\xi) = \sum_{|\alpha|\le 2m} a^1_\alpha(x)\xi^\alpha \tag{11.5.30}$$

on the surface $\Sigma^0 = \big\{(x,\xi), A^0(x,\xi)=1\big\}$; recall that $I = \frac{1}{2m\mu}(\mu+1)$ and $T_0 = T(1)$.

It follows from the quasihomogeneity of A^0 with respect to (x,ξ) that the action vanishes[40)].

Then the results of Sections 6.2, 8.3 immediately yield the following theorem:

[40)] In fact, the action is equal to

$$\frac{1}{T_0}\int_0^{T_0}\big(k\langle\xi, A^0_\xi\rangle + (1-k)\langle x, A^0_x\rangle - A\big)\circ\Psi_t dt$$

with an arbitrary constant k. Let us take $k=\mu(\mu+1)^{-1}$. Then the Euler identity yields that the integrand vanishes for $I=1$. On the other hand, we must assume that $I=1$: otherwise we should replace A by A^I in order to obtain $T=T_0$ for all (x,ξ).

Theorem 11.5.9. *Let conditions* (11.5.14) *and* (11.5.27)–(11.5.30) *be fulfilled. Further, let us assume that*

$$\mathsf{mes}_{\Sigma}\big(\{(x,\xi)\in\Sigma, |b(x,\xi)-\omega|\le\varepsilon\}\big) = O(\varepsilon^{\delta}) \qquad \text{as } \varepsilon\to+0 \tag{11.5.31}$$

with $\delta>0$ *and*

$$\mathsf{dist}(\Psi_t(x,\xi),(x,\xi)) \ge t(T_0-t)\gamma(x,\xi) \qquad \forall (x,\xi)\in\Sigma \quad \forall t\in(0,T_0) \tag{11.5.32}$$

with $\gamma\ge 0$ *such that* $\mathsf{mes}\big(\{(x,\xi),\gamma(x,\xi)\le\varepsilon\}\big)=O(\varepsilon^{\delta})$ *as* $\varepsilon\to+0$.

Then

(i) If $\delta_1<\mu+1$ *and condition* (11.5.31) *is fulfilled for every* $\omega\in\mathbb{R}$ *then asymptotics*

$$\mathsf{N}^{-}(\tau)=\mathsf{N}^{\mathsf{W}}(\tau)+\kappa_1\tau^{l(d-1)}+O\big(\tau^{l(d-1)-\delta''}\big) \tag{11.5.33}$$

holds as $\tau\to+\infty$ *with* $\delta''>0$.

(ii) If $\delta_1\in[\mu+1,2\mu+2)$ *then for* $\tau\to+\infty$ *such that* $\tau^l\in\bigcup_{n\in\mathbb{Z}^+}J_n$ *asymptotics*

$$\mathsf{N}^{-}(\tau)=\mathsf{N}^{\mathsf{W}}(\tau)+\big(\kappa_1+F(\tau)\big)\tau^{l(d-1)}+O\big(\tau^{l(d-1)-\delta''}\big) \tag{11.5.34}$$

holds with $\delta''>0$ *where*

$$J_n=\big[\nu(n+\frac{1}{4}\omega_0)+(\omega-\epsilon_1)(\nu n)^{l-\frac{\delta_1}{2m\mu}}, \nu(n+\frac{1}{4}\omega_0)+(\omega+\epsilon_1)(\nu n)^{l-\frac{\delta_1}{2m\mu}}\big],$$

ω_0 *is the Maslov index of the trajectory* $\Psi_t(x,\xi)$ *with* $t\in(0,T_0)$, $\nu=\frac{2\pi l}{T_0}$ *and* $\epsilon_1>0$ *is small enough,*

$$F(\tau)=(2\pi)^{-d}lT_0^{-1}\int_{\Sigma}\Upsilon\Big(T_0l^{-1}\big((\tau^l-\frac{1}{4}\pi\omega_0)-\tau^{l-\frac{\delta_1}{2m\mu}}b\big)\Big)\times dxd\xi : dA^0. \tag{11.5.35}$$

Let us consider the perturbed harmonic oscillator $A_0=\frac{1}{2}(|D|^2+|x|^2)$ in $\mathbb{R}^d$ more carefully. In this case $\mu=l=m=1$, $T_0=2\pi$, $\omega_0\equiv 2d \mod 4\mathbb{Z}$ and the x-projections of the Hamiltonian trajectories are ellipses centered at the origin.

Moreover,

(11.5.36) $\operatorname{Spec} A_0 = \{\tau_n = \frac{d}{2} + n, n \in \mathbb{Z}^+\}$. The multiplicity of τ_n is $P(n)$ where $P(.)$ is a polynomial of degree $d-1$.

Functional-analytic arguments yield that

(11.5.37) If the perturbing operator is a potential V with $\|V\|_{\mathscr{L}^\infty} \le \epsilon$ then the spectrum consists of clusters of length $\le \epsilon$.

We leave to the reader

Problem 11.5.10. Prove that for generic functions a^1_α ($|\alpha| \le k$) which are positively homogeneous of degree $(2-|\alpha|-\delta_1)$ condition (11.5.31) is fulfilled for every ω.

For such perturbations the theory of Section 6.2 yields that

(11.5.38) Asymptotics (11.5.33) holds for $0 < \delta_1 < 2$ and asymptotics (11.5.34)–(11.5.35) holds for $\delta_1 \ge 2$ where $\delta_k > 0$ ($k = 0, 1, 2$) are now arbitrary.

In particular, one can take $k = 0$ and a perturbing potential $V' = v_0 r^{2\mu-\delta_1}$ with arbitrary $v_0 \ne 0$ and $0 < \delta_1 \ne 2$.

Let us consider the case $b \equiv 0$. Then we have two possibilities.

The first (rather trivial) possibility is to assume that $A = A^{0\mathrm{w}} + A^{1\mathrm{w}} + A''^{\mathrm{w}}$ where A'' (instead of A') satisfies condition (11.5.29) with a^2_α, δ_2 instead of a^1_α, δ_1 respectively, $\delta_2 \in (\delta_1, 2\delta_1)$. Then one can construct the symbol $b(x,\xi)$ by formula (11.5.30) with A^2 instead of A^1, assume that condition (11.5.31) is fulfilled and obtain asymptotics (11.5.33) or (11.5.34)–(11.5.35) with δ_2 instead of δ_1.

However, for $\delta_2 = 2\delta_1$ the situation is more complicated because in this case b contains a term which is quadratic with respect to A^1.

So, let us assume that

$$A^1 \circ j^* = -A^1 \tag{11.5.39}$$

where j^* is the antipodal map $(x,\xi) \to (-x,-\xi)$. Repeating the arguments of the proof of Theorem 11.3.21 we again obtain formula (11.3.27):

$$b = \frac{1}{2\pi}\Big(\int_0^{2\pi} A^2 \circ \Psi_t \, dt + \int_0^\pi \int_0^\pi \{A^1 \circ \Psi_t, A^1 \circ \Psi_{t+t'}\} \, dt dt'\Big) \tag{11.5.40}$$

is a positive homogeneous symbol of degree $(2-2\delta_1)$ with respect to (x,ξ).

Therefore we have proven

Theorem 11.5.11. *Let* $X^0 = \mathbb{R}^d$, $A = A^{0\mathsf{w}} + A^{1\mathsf{w}} + A''^{\mathsf{w}}$ *where* A'' *satisfies* (11.5.29) *with* a^2_α *and* $\delta_2 > 0$ *instead of* a^1_α *and* δ_1. *Here* a^1_α *and* a^2_α *are functions smooth for* $x \neq 0$ *and positively homogeneous of degrees* $(2-|\alpha|-\delta_1)$, $(2-|\alpha|-\delta_2)$, *and* $\delta_2 = 2\delta_1$.

Further, let condition (11.5.39) *be fulfilled and let the symbol* b *be defined by* (11.5.40). *Then*

(i) If $\delta_2 < 2$ *and condition* (11.5.31) *is fulfilled for every* $\omega \in \mathbb{R}$ *then asymptotics* (11.5.33) *holds for* $\tau \to +\infty$.

(ii) If $\delta_2 \geq 2$ *and condition* (11.5.31) *is fulfilled for some* $\omega \in \mathbb{R}$ *then asymptotics* (11.5.34)–(11.5.35) *holds for* $\tau \to +\infty$ *and belonging to the intervals* J_n *indicated in Theorem 11.5.9(ii) with* $\omega_0 = 2(d-1)$, $\nu = 1$. *In these asymptotics as well as in the definition of the intervals* J_n *one should replace* δ_1 *with* δ_2.

We leave to the reader

Problem 11.5.12. (i) Prove that for generic positively homogeneous functions a^2_α ($|\alpha| \leq k$) with $k = 0, 1, 2$ condition (11.5.33) is fulfilled for all $\omega \in \mathbb{R}$.

(ii) Consider Riesz means with $\vartheta < \vartheta^* = (d-1)$.

Remark 11.5.13. It does not look right that in our results here $\delta'' > 0$ is unspecified small exponent. Later under certain conditions we will bring our results to be more sharp (up to being consistent with results of Section 6.2.

11.5.4 Singularities and Degenerations

Let us consider operators with homogeneous terms with singularities and (what is now more interesting) degenerations. In order to avoid non-degeneracy conditions we consider only the Schrödinger operator.

Theorem 11.5.14. *(i) Let* $d \geq 3$, $X = \{x = (r, \theta) \in \mathbb{R}^+ \times U\}$ *where* $U \subset \mathbb{S}^{d-1}$. *Let* $g^{jk} = g^{jk}(\theta)$, $V_j = v_j(\theta) r^\mu$, $V = v(\theta) r^{2\mu}$ *with* $\mu > 0$.

Further, let $v \geq 0$ *and let* U, g^{jk}, v_j, v *and* $\tilde{\rho} = \sqrt{v}$, $\tilde{\gamma} = \tilde{\gamma}(\theta)$ *satisfy conditions* (11.1.6)–(11.1.6) *and* (11.4.5).

Finally, let

$$v^{-\frac{d}{2\mu}}, v^{-\frac{d-1}{2\mu}}\tilde{\gamma}^{-1} \in \mathscr{L}^1(U). \tag{11.5.41}$$

Then asymptotics (11.5.1) *holds and one can substitute* $\mathsf{N}^{\mathsf{W}}(\tau)$ *by* $\varkappa_0\tau^{Id}$ *with* $I = \frac{1}{2\mu}(\mu+1)$ *and with the coefficient* $\varkappa_0$ *given by* (11.5.10).

Moreover, if $0 < \vartheta < \vartheta^* = (d-1)$ *and*

$$v^{-\frac{1}{2\mu}(d-1-\vartheta)}\tilde{\gamma}^{-1-\vartheta} \in \mathscr{L}^1(U) \tag{11.5.42}$$

then asymptotics (11.5.2)–(11.5.3) *holds.*

(ii) Let us assume that there exists a set $\Xi \subset \Sigma = \{(x,\xi), a(x,\xi) = 1\}$ *such that* $\mathsf{mes}_\Sigma(\Xi) = 0$ *and through every point of* $\Sigma \setminus \Xi$ *there passes an infinitely long non-periodic Hamiltonian billiard generated by the Hamiltonian*

$$H^0 = \tilde{\gamma}|x|\big(g^{jk}(\xi_j - V_j)(\xi_k - V_k) + V - 1\big). \tag{11.5.43}$$

Then asymptotics (11.5.6) *holds and one can substitute* $\mathsf{N}^{\mathsf{W}}(\tau)$ *by* $\varkappa_0\tau^{Id}$.

Moreover, if $0 < \vartheta < \vartheta^*$ *and*

$$v^{-\frac{1}{2\mu}(d-1-\vartheta)}\tilde{\gamma}^{-1-\vartheta} \in \mathscr{L}^1(U) \tag{11.5.44}$$

then asymptotics (11.5.6), (11.5.3) *holds.*

Proof. One can easily prove this theorem by the same method as Theorem 11.5.1 but also with analysis in the zone $X_{\mathsf{sing}} = \{\gamma \cdot (\tau^{\frac{1}{2}} + \rho) \le C_0\}$ with $\gamma = \tilde{\gamma}r$, $\rho = \tilde{\rho}r^\mu$.

In the calculation of $\mathsf{N}^{\mathsf{W}}(\tau)$ we use spherical coordinates and the change of variable $r = \tau^{\frac{1}{2\mu}}r'$. □

Remark 11.5.15. It should be noted that if in (11.5.43) we skip the regularizing factor $\tilde{\gamma}r$ in H^0 (i.e., replace H^0 by H^0_{st}) then the conditions of the Statement (ii) do not change because it follows from the assumptions of the Statement (i) that the set of the points through which pass Hamiltonian billiards generated by H^0_{st} and reaching $\{\tilde{\gamma}r = 0\}$ in a finite time is a set of measure 0.

We leave to the reader the following rather easy

Problem 11.5.16. Generalize Theorem 11.5.14 to operators different from the Schrödinger operator.

Example 11.5.17. Let $X = \mathbb{R}^d$, $d \ge 3$ and as $|x| \ge 1$ $V \asymp |x''|^{2\nu}|x|^{2\mu-2\nu}$ where $x = (x'; x'') = (x_1, \dots, x_{d'}; x_{d'+1} \dots, x_d)$, $d'' = d - d'$, $0 < d'' < d$ (in which case one should try $\rho(x) = |x''|^{\nu} \cdot |x|^{\mu-\nu}$, $\gamma(x) = \frac{1}{2}|x''|$).

Let us assume that the conditions to g^{jk}, V_j, V are fulfilled with the indicated γ and ρ. Then

(i) $v^{-\frac{d}{2\mu}} \in \mathscr{L}^1(\mathbb{S}^{d-1})$ and $v^{-\frac{1}{2\mu}(d-1-\vartheta)}\tilde{\gamma}^{-1-\vartheta} \in \mathscr{L}^1(\mathbb{S}^{d-1})$ if and only if

$$\nu < \nu_0^* = \mu\frac{d''}{d} \tag{11.5.45}$$

and

$$\nu < \nu_{1+\vartheta}^* = \mu(d'' - 1 - \vartheta)(d - 1 - \vartheta)^{-1} \tag{11.5.46}$$

respectively.

Therefore for $\nu < \mu(d''-1)(d-1)^{-1}$ asymptotics (11.5.1)[41] holds, and for $0 < \vartheta < d'' - 1$ under assumption (11.5.46) asymptotics (11.5.2)–(11.5.3) [42] holds.

(ii) Let $\vartheta = 0$. In order to treat the case $\nu \in [\nu_1^*, \nu_0^*]$ when condition (11.5.45) holds (or reaches the borderline value) and (11.5.46) fails, recall that the original remainder estimate was

$$R(\tau) = \int_{\{V \le C\tau, \sqrt{\tau}\gamma \ge 1\}} \tau^{\frac{d-1}{2}}\gamma^{-1}\,dx + \int_{\{V \le C\tau, \sqrt{\tau}\gamma \le 1\}} \tau^{\frac{d}{2}}\,dx.$$

Introducing spherical coordinates and integrating over r we obtain the remainder estimate

$$R(\tau) = \tau^{l(d-1)} \int_{\{v \le C\tau^{\mu+1}\tilde{\gamma}^{2\mu}\}} v^{-\frac{d-1}{2\mu}}\tilde{\gamma}^{-1}\,d\theta + \tau^{ld} \int_{\{v \ge C\tau^{\mu+1}\tilde{\gamma}^{2\mu}\}} v^{-\frac{d}{2\mu}}\,d\theta.$$

We can replace v by $|x''|^{2\nu} \cdot |x|^{2\mu-2\nu}$ here in which case we arrive to asymptotics

$$\mathsf{N}^-(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + O\bigl(\tau^{l(d-1)}\log\tau\bigr) \tag{11.5.47}$$

[41] As well as asymptotics (11.5.6) under the standard condition on Hamiltonian trajectories.

[42] As well as asymptotics (11.5.6), (11.5.3) under the standard condition on Hamiltonian trajectories. Both these footnotes remain true below.

as $\tau \to +\infty$ provided $\nu = \nu_1^*$ and the asymptotics

$$\mathsf{N}^-(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + O\bigl(\tau^{l(d-\eta)}\bigr) \tag{11.5.48}$$

as $\tau \to +\infty$ provided $\nu \in (\nu_1^*, \nu_0^*)$ and $\eta = (-\nu d + \mu d'')(\mu - \nu)^{-1} \in (0,1)$.

Here again one can replace $\mathsf{N}^{\mathsf{W}}(\tau)$ by $\varkappa_0 \tau^{ld}$ with $\varkappa_0$ given by (11.5.10).

(iii) Similarly, for $\nu = \nu_{\vartheta+1}^* = \mu(d'' - \vartheta - 1)(d - \vartheta - 1)^{-1}$ asymptotics

$$\mathsf{N}^-_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O\bigl(\tau^{l(d-1-\vartheta)+\vartheta} \log \tau\bigr) \tag{11.5.49}$$

holds as $\tau \to +\infty$ and for $\nu \in (\nu_{\vartheta+1}^*, \nu_0^*)$ asymptotics

$$\mathsf{N}^-_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O\bigl(\tau^{l(d-\eta)+\vartheta}\bigr) \tag{11.5.50}$$

holds as $\tau \to +\infty$ with $\eta = \bigl((d'' - \vartheta)\mu - (d - \vartheta)\nu\bigr)(\mu - \nu)^{-1} \in (0, \vartheta + 1)$.

(iv) Finally, for $\nu = \nu_0^*$ asymptotics

$$\mathsf{N}^-(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + O(\tau^{ld}) \tag{11.5.51}$$

holds as $\tau \to +\infty$ where

$$\mathsf{N}^{\mathsf{W}}(\tau) = (2\pi)^{-d}\omega_d \int_{\gamma \ge \epsilon\tau^{-\frac{1}{2}}} (\tau - V)_+^{\frac{d}{2}} \sqrt{g}\, dx \tag{11.5.52}$$

and $\mathsf{N}^{\mathsf{W}}(\tau) \asymp \tau^{ld} \log \tau$ [43]. The results for $\vartheta > 0$ obviously follow from this statement.

In order to prove this statement one needs only note that in the domain $X_{\mathsf{sing}} = \{x, |x''| \le \epsilon\tau^{-\frac{1}{2}}\}$ one can take $\tilde\rho = 0$ in the inequality

$$\bigl((Au - \tau), u\bigr) \ge \epsilon \|u\|_m^2 - C\|\tilde\rho^m u\|^2 \tag{11.5.54}$$

for all $u \in \mathfrak{D}(A)$ supported in $\bar X_{\mathsf{sing}}$. Moreover, this estimate yields that

(v) All the results of Statements (i)–(iv) remain true for $d = 2$.

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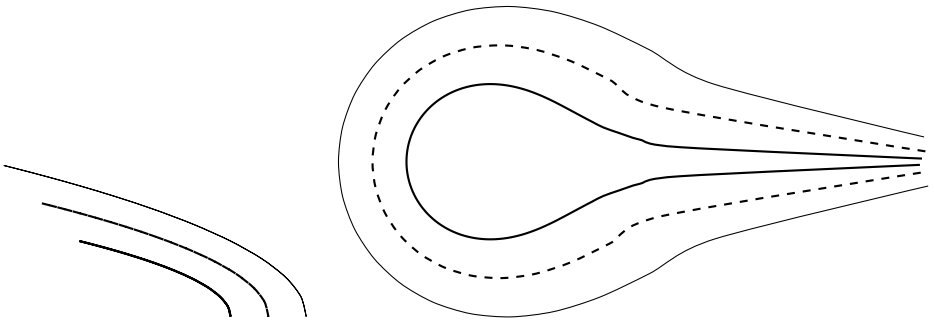
[43] Moreover, under stabilization condition (11.5.4)

$$\mathsf{N}^{\mathsf{W}}(\tau) = \bigl(\bar\varkappa + o(1)\bigr)\tau^{ld} \log \tau. \tag{11.5.53}$$

This statement remains true below.



(a) (b)

Figure 11.9: Level lines for non-degenerate (a) and degenerate in one direction (b) potential. In the latter case $V \le \tau$ forms a cusp but in contrast to cusps studied in Section 11.2 and 11.3 this cusp grows and fattens as $\tau \to +\infty$.

It is interesting to perturb the potentials treated above by potentials which have a lesser "average" growth at infinity bet does not have this degeneration. Namely, we can take perturbing potentials satisfying condition $(11.1.7)_3$ with the same γ as above and with $\rho' = (|x|+1)^{\mu'}$, $0 < \mu' < \mu$ instead of ρ:

$$|D^\alpha V'| \le c\rho'^2 \gamma^{-|\alpha|} \tag{11.5.55}$$

Then two different situations may arise:

(a) The perturbing potential is also coercive i.e., satisfies (11.4.5) with ρ replaced by ρ'

$$V' \ge \epsilon_0 \rho'^2 \qquad \forall x \tag{11.5.56}$$

in which case the perturbing potential "helps" to the unperturbed one. In this case the above analysis can be applied with no change.

(b) The perturbing potential is indefinite.

In the former case (a) we leave the detailed analysis to the reader and formulate the following statements:

Theorem 11.5.18 (Coercive perturbations). *(i) Let* $d \ge 3$ *and* X, g^{jk}, V_j^0 *and* V^0 *be as in Theorem 11.5.14. Let* $V_j = V_j^0 + V_j'$, $V = V^0 + V'$ *where* V_j' *and* V' *satisfy* (11.5.55) *and* (11.5.56) *with* $\rho' = \langle x\rangle^{\mu'}$, $0 < \mu' < \mu$. *Then asymptotics* (11.5.1) *holds with* $\mathsf{N}^{\mathsf{W}}(\tau) = \varkappa_0 \tau^{ld}\big(1 + o(1)\big)$ *as* $\tau \to +\infty$.

(ii) Moreover, if the Hamiltonian H^0 *given by* (11.5.43) *satisfies the conditions of the Statement (ii) of Theorem 11.5.14 then asymptotics* (11.5.6), (11.5.3) *holds as* $\vartheta = 0$.

(iii) Finally, if $0 < \vartheta < (d-1)$ *and* (11.5.42) *is fulfilled then in the frameworks of (i) and (ii) asymptotics* (11.5.2)–(11.5.3) *and* (11.5.6), (11.5.3) *hold respectively.*

Example 11.5.19. Let X, g^{jk}, V_j^0 and V^0 be as in Example 11.5.17. Let $V_j = V_j^0 + V_j'$, $V = V^0 + V'$ where V_j' and V' satisfy (11.5.55) and (11.5.56) with $\rho' = \langle x\rangle^{\mu'}$, $0 < \mu' < \mu$, and $\gamma = \epsilon|x''|$. Then

(i) If $\nu < \nu_{\vartheta+1}$ defined by (11.5.46), $0 \le \vartheta < (d''-1)$ then asymptotics (11.5.2)–(11.5.3) holds.

(ii) In the case of threshold value $\nu = \nu_{\vartheta+1}$, $0 \le \vartheta < (d''-1)$ asymptotics (11.5.47) holds.

(iii) In the case of above threshold value $\nu_{\vartheta+1} < \nu < \nu_0$, with ν_0 defined by (11.5.45), $0 \le \vartheta < (d''-1)$ asymptotics (11.5.50) holds with η given in Example 11.5.17(ii)$'$ provided $\mu' \le \mu'_* = (\mu-\nu)(\nu+1)^{-1}$.

(iv) On the other hand, for $\mu' > \mu'_*$ the remainder estimate is $O(\tau^q)$ with

$$q = \frac{1}{2}(d-\vartheta-1) + \frac{1}{2\mu'\nu}\big(d'\nu - (\mu-\nu+\mu')(d''-\vartheta-1)\big)$$

for $\vartheta < (d''-1)$ and with $q = \frac{d'}{2} + \frac{d'}{2\mu'} + \vartheta$ for $\vartheta > d''-1$.

For $\vartheta = (d''-1)$ the remainder estimate i $O(\tau^q \log\tau)$ with the same q.

(v) All these results remain true for $\nu = \nu_0$ and $\vartheta = 0$ with $\mathsf{N}^{\mathsf{W}}(\tau) \asymp \tau^{lq}\log\tau$.

(vi) Moreover, for $\nu > \nu_0$, $\vartheta = 0$ and $\mu' > \mu'_*$ these results remain true with $\mathsf{N}^{\mathsf{W}}(\tau) \asymp \tau^p$ with $p = \frac{1}{2}(d + \frac{d}{\mu'} - \frac{d''\mu}{\mu'\nu} + \frac{d''}{\nu})$. Then $p-q = \frac{1}{2\nu}\big(\nu+1-\frac{1}{\mu'}(\mu-\nu)\big) > 0$.

Let us now consider indefinite perturbations. In this case the perturbation cannot be too strong.

Theorem 11.5.20 (Indefinite perturbations). *(i) Let $d \ge 3$ and let X, g^{jk}, V_j^0 and V^0 be as in Theorem 11.5.14. Let $V_j = V_j^0 + V_j'$, $V = V^0 + V'$ where V_j' and V' satisfy* (11.5.55) *with $\rho' = \langle x\rangle^{\mu'}$, $0 < \mu' < \mu$ and $\gamma = \tilde\gamma|x|$. Then, if*

$$v^{-nd-\mu'\vartheta(\mu-\mu')^{-1}} \in \mathscr{L}^1(U) \tag{11.5.57}$$

and

$$v^{-n(d-\vartheta-1)-\mu'\vartheta(\mu-\mu')^{-1}}\tilde\gamma^{-\vartheta-1} \in \mathscr{L}^1(U) \tag{11.5.58}$$

with $n = \frac12(\mu'+1)(\mu-\mu')^{-1}$, $0 \le \vartheta < (d-1)$ then asymptotics (11.5.2)–(11.5.3) *holds and for $\vartheta = 0$ $\mathsf{N}^{\mathsf{W}}(\tau) = \varkappa_0\tau^{ld} + o(\tau^{l(d-1)})$.*

(ii) Moreover, if the Hamiltonian H^0 given by (11.5.43) *satisfies the conditions of Theorem 11.5.14(ii), then asymptotics* (11.5.6), (11.5.3) *holds.*

Proof. The proof is similar to the proof of Theorem 11.5.14 but the domain $\{V \le \tau\}$ consists of two overlapping zones $\{V^0 \le C\tau, \rho' \le C\tau^{\frac12}\}$ and $\{V^0 \le C\rho'^2, \tau \le C\rho'^2\}$.

Then $\tilde\rho(\tau) \asymp \tau^{\frac12}$ and $\tilde\rho(\tau) \asymp \rho'$ in the first and second zones respectively.□

Example 11.5.21. Let X, g^{jk}, V_j^0 and V^0 be as in Example 11.5.17. Let $V_j = V_j^0 + V_j'$, $V = V^0 + V'$ where V_j' and V' satisfy (11.5.55) with $\rho' = \langle x\rangle^{\mu'}$, $0 < \mu' < \mu$ and $\gamma = \epsilon|x''|$.

Let $\mu > \mu' + \nu$ (otherwise the potential V can be semi-bounded from above or even tend to $-\infty$ in the pipe $\{|x''| \le \epsilon\}$). Moreover, let us assume that

$$\nu < \frac{\mu-\mu'}{\mu'+1}. \tag{11.5.59}$$

Then considering the zone

$$X'' = \bigl\{x : |x''| \le z(x,\tau) = \epsilon\min\bigl(\tau^{-\frac12}, |x|^{-(\mu-\nu)(\nu+1)^{-1}}\bigr)\bigr\}$$

we see that we can take $\tilde\rho(\tau) = (\rho' - \epsilon_1 z^{-1})_+$ in condition (11.5.54) where $\epsilon_1 > 0$ does not depend on ϵ. Then (11.5.54) yields that we can take $\tilde\rho(\tau) = 0$ in X'' for $\tau \ge \epsilon^{-1}$ and small enough ϵ.

On the other hand, (11.5.59) yields that

$$(\tau^{\frac12}+\rho)\gamma \ge \epsilon_2 > 0, \quad \rho' \le C(\tau^{\frac12}+\rho)^{1-\delta}$$

outside of X'' where $\delta > 0$, $\rho = |x''|^{\nu}\cdot |x|^{\mu\mu-\nu}$, $\gamma = \epsilon_3|x''|$. Using these assertions one can easily prove that under condition (11.5.59) all the statements of Example 11.5.17 remain true.

We leave to the reader

Problem 11.5.22. (i) Consider the Schrödinger operator with coefficients satisfying conditions (11.1.6)–(11.1.8), (11.4.5) with $\rho = \langle x\rangle^{\mu}$, $\gamma = \epsilon\langle x\rangle^{1-\delta}$, $\mu > 0$, $\delta > 0$, $\mu + 1 > \delta$.

In this case the asymptotics $\mathsf{N}^{-}_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O(\tau^{q})$ appear with $\mathsf{N}^{\mathsf{W}}(\tau) \asymp \tau^{p}$ in domains conical at infinity.

(ii) Consider domains or Riemannian manifolds of type (11.5.69) (see below). Then asymptotics appear with other exponents p and q. In this example the magnetic field can grow strongly enough at infinity.

11.5.5 Quasihomogeneous Potentials

Let us now consider the quasihomogeneous case.

Example 11.5.23. (i) Let X be an L-quasiconical domain in $\mathbb{R}^d$ and let g^{jk}, V_j, V be L-quasihomogeneous on x of degrees 0, μ and 2μ respectively. Here $\mu > 0$ and $L = (L_1, \dots, L_d)$ with $L_j = 1$ for $1 \le j \le d'$ and $L_j > 1$ for $d'+1 \le j \le d$, $0 < d' < d$.

Let us assume that $\partial X \setminus 0 \in \mathscr{C}^{K}$ and $g^{jk}, V_j, V \in \mathscr{C}^{K}(\bar X \setminus 0)$ satisfy (11.1.8) and $V(x) > 0$ for $x \ne 0$.

Then all of the conditions (11.1.6)–(11.1.8) and (11.4.5) are fulfilled with

$$\rho = [x]_L^{\mu}, \qquad \gamma = \epsilon [x]_L \tag{11.5.60}$$

for

$$[x]_L \coloneqq \Big(\sum_j |x|^{\frac{\bar L}{L_j}}\Big)^{\frac{1}{\bar L}} \ge 1; \tag{11.5.61}$$

for $[x]_L < 1$ one should take $\gamma = [x]_L^{\bar L}$ with $\bar L = \max_j L_j$ here.

Then asymptotics

$$\mathsf{N}^-(\tau) = \varkappa_0 \tau^p + O(\tau^q) \tag{11.5.62}$$

holds with $p = \frac{d}{2} + \frac{L^*}{2\mu}$, $q = \frac{1}{2}(d-1) + \frac{1}{2\mu}(L^*-1)$, $L^* = \sum_j L_j$ and with the coefficient

$$\varkappa_0 = (2\pi)^{-d}\omega_d \int (1-V)_+^{\frac{d}{2}} \sqrt{g}\, dx. \tag{11.5.63}$$

We leave the analysis for $\vartheta > 0$ to the reader.

(ii) In order to improve these asymptotics let us consider the Hamiltonian billiards generated by the Hamiltonian

$$H = [x]_L^{1-\mu}\Big(\sum_{j,k} g^{jk}(\xi_j - V_j)(\xi_k - V_k) + V - \tau\Big). \tag{11.5.64}$$

Obviously, the contribution of the zones $\{x \in X, |x_i| \le \epsilon \tau^{\frac{L_j}{2\mu}}\}$ to the remainder estimate does not exceed $\epsilon_1 \tau^q$ with $\epsilon_1 \to 0$ as $\epsilon \to +0$ $(i = 1, \dots, d)$. So we consider billiards starting from complements to these zones.

One can easily see that along Hamiltonian trajectories

$$\frac{d\xi_i}{dt} = o\big(\tau^{\frac{1}{2}}\big), \qquad \frac{dx_i}{dt} = o\big(\tau^{\frac{L_i}{2\mu}}\big) \tag{11.5.65}$$

as $\tau \to +\infty$ for $i \ge d'+1$ provided $[x]_L \ge \epsilon \tau^{\frac{1}{2\mu}}$. Let us fix arbitrarily large T and arbitrarily small $\epsilon > 0$ and treat billiards of the length T.

Let us assume that $x' \ne 0$, $\nu' \ne 0$ on $\partial X \setminus 0$ where $x' = (x_1, \dots, x_{d'})$ and $\nu' = (\nu_1, \dots, \nu_{d'})$, $\nu = (\nu_1, \dots, \nu_d) = (\nu', \nu'')$ is a unit normal to ∂X. Then in the indicated zone $|\nu''| = o(1)$ and therefore the increment of ξ_i when a billiard reflects from ∂X is $o(\tau^{\frac{1}{2}})$ for $i \ge d'+1$.

Therefore $\tau^{-\frac{1}{2}}\xi_i$ and $\tau^{-\frac{L_i}{2\mu}} x_i$ with $i \ge (d'+1)$ are almost constant along billiards (if the length and the number of reflections are bounded).

Therefore if $(\bar{x}', \bar{\xi}')$ is a good starting point for a billiard generated by the Hamiltonian $H_{(\bar{x}'', \bar{\xi}'')} = H(x', \bar{x}'', \xi', \bar{\xi}'')$ in the cross-section domain $X_{(\bar{x}'')} = \{x' : (x', \bar{x}'') \in X\}$ then $(\bar{x}, \bar{\xi})$ is a good starting point for the original Hamiltonian.

Therefore the phase space measure of points which cannot be starting points does not exceed $\epsilon_2\tau^q$ with arbitrarily small $\epsilon_2 > 0$. Here $\tau \ge \tau_0(\epsilon_2)$.

Let us treat the non-periodicity. Let us assume that

$$g^{jk} = 0, \quad V_k = 0 \quad \text{for } j \le d',\ k \ge d'+1. \tag{11.5.66}$$

Then one can easily show that $|x''(0)-x''(t)| \ge \varepsilon|t|\tau^{\frac{1}{2\mu}}$ provided $t \in [-T, T]$, $|\xi''| \ge C_{\epsilon,T}\varepsilon\tau^{\frac{1}{2}}$, $\tau \ge \tau_0(\epsilon, \varepsilon, T)$ where ϵ is defined by the above conditions including the transversality condition. Therefore the non-periodicity condition is also fulfilled if we exclude a subset of measure $\epsilon_2\tau^q$ from the starting point set.

Therefore asymptotics

$$\mathsf{N}^-(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + \bigl(\varkappa_1 + o(1)\bigr)\tau^q \tag{11.5.67}$$

holds with the coefficient

$$\varkappa_1 = \mp\frac{1}{4}(2\pi)^{1-d}\omega_{d-1}\int_{\partial X}(1-V)_+^{\frac{d-1}{2}}\, dS'\sqrt{g''}\, dx'' \tag{11.5.68}$$

where dS' and $\sqrt{g''}dx''$ are Riemannian densities on the projections $(\partial X)_{x''} = \{x' : (x', x'') \in \partial X\}$ and $X_{x'} = \{x'' : (x', x'') \in X\}$ generated by the metrics $(g^{jk})_{j,k\le d'}$ and $(g^{jk})_{j,k\ge d'+1}$ respectively (these metrics and densities depend on x). Here the sign "$\mp$" corresponds to Dirichlet and Neumann boundary condition respectively.

Remark 11.5.24. Surely, these statements remain true in the case when the operator tends to an L-quasihomogeneous operator or if assumptions are fulfilled with ρ and γ defined by (11.5.60)–(11.5.61); in the latter case only Statement (i) remains true. In the former case $\mathsf{N}^{\mathsf{W}}(\tau) = \bigl(\varkappa_0 + o(1)\bigr)\tau^p$ without exact equality.

11.5.6 Quickly Growing Potentials

In this subsection we consider potentials with more than power growth at infinity. Then the volume of the domain $\{x : V(x) \le \tau\}$ grows slowly as $\tau \to +\infty$ and one should expect that

$$N^{\mathsf{W}}_{(\vartheta)} \asymp \mathsf{mes}(\{x : V(x) \le \tau\}) \cdot \tau^{\frac{d}{2}} \tag{11.5.69}$$

have slightly larger magnitude than in the case of the essentially bounded domain.

(a) (b)

Figure 11.10: Domains of the type (11.5.69)

Theorem 11.5.25. *(i) Let $d \ge 2$ and let X be a domain in $\mathbb{R}^d$ such that*

$$\mathsf{vol}(\{x \in X, |x| \asymp r\}) \asymp r^n \quad \text{as} \quad r \to +\infty, \tag{11.5.70}$$

$n \le d$.

Let conditions (11.1.6)–(11.1.8) *and* (11.4.5) *be fulfilled with $\gamma = \epsilon_1 \langle x\rangle^{1-\mu}$, $\rho = \exp\langle x\rangle^{\mu}$, $\mu > 0$.*

Then asymptotics (11.5.2) *holds with the principal part*

$$\mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) \asymp \tau^{\frac{d}{2}+\vartheta}\sigma(\tau), \tag{11.5.71}$$

$$\sigma(\tau) = \begin{cases} \log^n \tau & \text{for } n > 0, \\ \log\log\tau & \text{for } n = 0, \\ 1 & \text{for } n < 0. \end{cases} \tag{11.5.72}$$

and remainder estimate

$$R_{(\vartheta)}(\tau) = \tau^{\frac{1}{2}(d+\vartheta-1)}\omega_{(\vartheta)}(\tau), \tag{11.5.73}$$

$$\omega_{(\vartheta)}(\tau) = \begin{cases} \log^q \tau & \text{for } q := (n-(1-\mu)(1+\vartheta))/\mu > 0, \\ \log\log\tau & \text{for } q = 0, \\ 1 & \text{for } q < 0. \end{cases} \tag{11.5.74}$$

(ii) Moreover, let $q \ge 0$ and for every fixed $k > 0$ let condition

$$\mathsf{vol}\big\{x \in X, |x| \le r,\ \mathsf{dist}(x, \partial X) \le k\gamma(x)\big\} = o(r^n) \tag{11.5.75}$$

as $r \to +\infty$

be fulfilled. Furthermore, let

$$|D^\alpha g^{jk}| = o(\gamma^{-1}) \qquad \text{as } |x| \to +\infty \quad \forall j,k,\alpha : |\alpha| = 1, \tag{11.5.76}$$

and

$$|D^\alpha V_j| \le C\rho\gamma^{-|\alpha|} \qquad \forall j,\alpha : |\alpha| \le 1. \tag{11.5.77}$$

Then asymptotics (11.5.6) *holds with* $R_{(\vartheta)}(\tau)$ *defined above and with* $\varkappa_1 = 0$ *for* $\vartheta = 0$.

Proof. One can easily prove Statement (i) in the standard way.

In order to prove Statement (ii) let us note that if $q > 0$ then for $\tau \ge \tau_0(\varepsilon)$ the contributions of the zones $\{x \in X, |x| \le \varepsilon \log^{\frac{1}{\mu}} \tau^{\frac{1}{2}}\}$ and $\{x \in X, |x| \ge (1-\varepsilon) \log^{\frac{1}{\mu}} \tau^{\frac{1}{2}}\}$ to the remainder do not exceed $\varepsilon_1 R_{(\vartheta)}(\tau)$ with $\varepsilon_1 \to 0$ as $\varepsilon \to 0$.

Moreover, condition (11.5.75) yields that the contribution of the zone $\{X, \operatorname{dist}(x, \partial X) \le k\gamma(x)\}$ to the remainder is $o(R_{(\vartheta)}(\tau))$ and therefore one need consider only billiards generated by the improved Hamiltonian and starting in the zones

$$X_{k,\varepsilon,\tau} = \big\{X \in X, \varepsilon \le |x| \log^{-\frac{1}{\mu}} \tau^{\frac{1}{2}} \le 1-\varepsilon, \operatorname{dist}(x, \partial X) \ge k(\log \tau^{\frac{1}{2}})^{\frac{1}{\mu}-1}\big\}.$$

Then for $t \in [-T, T]$ and $\tau \ge \tau_0(T, k, \varepsilon)$, $k \ge k_0(T, \varepsilon)$ these billiards lie in $X_{\frac{k}{2},\frac{\varepsilon}{2},\tau}$, they do not hit ∂X, and along them

$$\tau^{-\frac{1}{2}} V_j \equiv \tau^{-1} V \equiv 0, \quad g^{jk} \equiv \text{const}, \quad \gamma(\log \tau)^{1-\frac{1}{\mu}} \equiv \text{const},$$
$$\tau^{-\frac{1}{2}} \xi_j \equiv \text{const}, \quad \gamma^{-1} \frac{dx_j}{dt} \equiv 2\tau^{-\frac{1}{2}} \sum_k g^{jk}\xi_k, \quad \tau^{-1} \sum_{j,k} g^{jk}\xi_j\xi_k \equiv 1$$

modulo $o(1)$ as $\tau \to +\infty$.

The rest of the proof is similar to the proof of Theorems 11.2.7 and 11.2.8. The case $q = 0$ is treated in the same way with the removal of the zones $\{x \in X, |x| \le R\}$, $\{x \in X, |x| \ge \epsilon \log^{\frac{1}{\mu}} \tau\}$ and $\{x \in X, \operatorname{dist}(x, \partial X) \le k\gamma(x)\}$ with arbitrarily small $\epsilon > 0$ and arbitrarily large R. □

This theorem could be improved.

Theorem 11.5.26. *In the framework of the previous Theorem 11.5.25 assume that g^{jk} and the boundary satisfy smoothness assumptions with $\gamma(x) = \epsilon\langle x\rangle$ (then $n = d$). Then one can redefine* (11.5.74)

$$\omega_{(\vartheta)}(\tau) = \log^q \tau \qquad \text{for } q := (d - 1 - (1-\mu)\vartheta)/\mu. \tag{11.5.78}$$

Proof. One can see easily that we can then replace $\gamma(x) = \langle x\rangle^{1-\mu}$ by

$$\gamma_1(x) = \epsilon \min\big((\tau/\rho^2)^{\delta}\gamma(x),\ \langle x\rangle\big). \tag{11.5.79}$$

Indeed, one can check easily that it is scaling function ($|\nabla\gamma| \le \epsilon_1$) and that $|\nabla^\alpha V| \le C\tau\gamma_1^{-|\alpha|}$, $|\nabla^\alpha V_j| \le C\tau^{1/2}\gamma_1^{-|\alpha|}$ for $|\alpha| \le K$.

Then contribution of the zone $\{x : V(x) \le 2\tau\}$ to the remainder does not exceed $C\int \tau^{(d-1-\theta)/2}\gamma_1^{-1-\theta}\,dx$. Plugging $\gamma_1(x) = (\tau/\rho^2)^{\delta}\gamma(x)$ and $dx = r^{d-1}\,dr$ we get (11.5.74) with $q = d - 1 - (1-\mu)\vartheta$ and plugging $\gamma_1 = \langle x\rangle$ we get even better remainder estimate. □

In the same way the following theorem is proven:

Theorem 11.5.27. *Let $d \ge 2$, $X = \mathbb{R}^+ \times X'$ where X' is a bounded domain in $\mathbb{R}^{d-1} \ni x'$ with $\mathscr{C}^K$ boundary. Let all the conditions* (11.1.6)–(11.1.8) *and* (11.4.5) *be fulfilled with $\rho = \exp x_1$ and $\gamma = \epsilon_0 > 0$ and let*

$$g^{jk} - g^{jk0} = o(1), \qquad D_1 g^{jk} = o(1) \qquad \text{as } x_1 \to +\infty \tag{11.5.80}$$

where $g^{jk0} = g^{jk0}(x')$ and $g^{1k0} = 0$ for $k \ge 2$.

Further, let conditions (11.5.76)–(11.5.77) *be fulfilled. Then for every $\vartheta > 0$ asymptotics* (11.5.6) *holds with*

$$R_{(\vartheta)}(\tau) = \tau^{\frac{1}{2}(d+\vartheta-1)}\log\tau. \tag{11.5.81}$$

Further, for $\vartheta = 0$ the principal part is

$$\mathsf{N}^{\mathsf{W}}(\tau) + \varkappa_1 \tau^{\frac{1}{2}(d-1)}\log\tau \tag{11.5.82}$$

with coefficient

$$\varkappa_1 = \mp\frac{1}{8}(2\pi)^{1-d}\omega_{d-1}\int_{\partial X'} \frac{dS'}{\sqrt{g^{110}}} \tag{11.5.83}$$

where dS' is the Riemannian density on $\partial X'$ generated by the metric $(g^{jk0})_{j,k\ge 2}$.

Furthermore,

$$\mathsf{N}^{\mathsf{W}}(\tau) = \bigl(\bar{\varkappa}_0 + o(1)\bigr)\tau^{\frac{d}{2}}\log\tau \tag{11.5.84}$$

with the coefficient

$$\bar{\varkappa}_0 = \frac{1}{2}(2\pi)^{-d}\omega_d \int_{X'} \sqrt{g^{0\prime}}\frac{dx'}{\sqrt{g^{110}}} \tag{11.5.85}$$

where $\sqrt{g^{0\prime}}dx'$ *is a Riemannian density on* X' *generated by the same metric.*

The following problems look rather interesting but not very difficult:

Problem 11.5.28. (i) Consider more general operators than the Schrödinger operator.

(ii) Consider $X = \mathbb{R}^+ \times X'$, where X' is a bounded domain in $\mathbb{R}^{d-1}$ with $^{\prime}\mathscr{C}^{K}$-boundary, $\rho = \exp(x_1^{\mu})$, $\gamma \asymp x_1^{\sigma}$, $\sigma = \max(0, 1-\mu)$, $\mu > 0$.

(iii) Consider potentials with the degenerations and singularities (in the spirit of Section 11.5.4.

(iv) Consider other types of super-power growth.

11.5.7 Slowly Growing Potentials

Finally, let us consider potentials with slow growth at infinity. Here we should be very careful and specific since even formula (11.5.69) may be not valid anymore.

Namely, let $V = v(\theta)\log r$ in spherical coordinates (r,θ). We use $\rho_\tau = (\tau - V)_+^{\frac{1}{2}}$ and $\gamma_\tau = \epsilon_0 r \min(1, (\tau - V)_+ \log^{-1} r)$. Then we obtain

Theorem 11.5.29. *Let* $d \ge 3$ *and let* $g^{jk} = g^{jk}(\theta)$, $V_j = 0$, $V = v(\theta)\log r$, $X = \{\theta \in U\}$ *for* $r \ge c$ *where* (r,θ) *are spherical coordinates,* $g^{jk}, v, \partial U \in {}^{\prime}\mathscr{C}^{K}$ *and* (g^{jk}) *is a positive definite matrix. Let* $v_0 = \min_U v > 0$. *Then*

(i) Asymptotics (11.5.2) *holds with*

$$\mathsf{N}^{\mathsf{W}}_{(\vartheta)} \asymp \int_U \exp\Bigl(\frac{\tau d}{v}\Bigr)\,d\theta, \tag{11.5.86}$$

$$R_{(\vartheta)}(\tau) = \tau^{1+\vartheta}\int_{U'} \exp\Bigl(\frac{\tau(d-1-\vartheta)}{v}\Bigr)\,d\theta \qquad \text{for } \vartheta < d-1. \tag{11.5.87}$$

Moreover,

$$\mathsf{N}^{\mathsf{W}}(\tau) = a \int_{U'} v^{\frac{d}{2}} \exp\Big(\frac{\tau\ d}{v}\Big)\sqrt{g}\, d\theta \tag{11.5.88}$$

with $a = (2\pi)^{-d}\omega_d d^{-(1+\frac{d}{2})}\Gamma(1+\frac{d}{2}) = 2^{1-d}\pi^{-\frac{d}{2}} d^{-(1+\frac{d}{2})}$ *and* $U' = \{\theta \in U, v \le v_0 + \varepsilon\}$ *with arbitrarily small* $\varepsilon > 0$.

(ii) In particular, if $v = \mathsf{const} = v_0$ *then the principal part is* $\asymp \exp\big(\frac{\tau d}{v_0}\big)$ *and the remainder estimate is* $O\big(\tau^{1+\vartheta}\exp(\tau(d-\vartheta-1)v_0^{-1})\big)$ [44].

On the other hand, if $v(\theta) = v_0 \implies \theta \in \partial U, \nabla v(\theta) \ne 0$ *and* $\mathsf{rank}\,\mathsf{Hess}\, v|_{\partial U}(\theta) = d-2$ *then the principal part is* $\asymp \tau^{-\frac{d}{2}}\exp(\tau d v_0^{-1})$ *and the remainder estimate is* $O\big(\tau^{1+\vartheta-\frac{d}{2}}\exp(\tau(d-\vartheta-1)v_0^{-1}))\big)$.

In other cases the principal part of the asymptotics and the remainder estimate have intermediate orders.

We leave to the reader

Problem 11.5.30. (i) Investigate the case $\vartheta \ge (d-1)$. Note that one can take $\tilde\rho(\tau) = \tau^{\frac12} + (\log r)^{\frac12}$, $\tilde\gamma(\tau) = \epsilon_0 r$ and for $d \ge 2$ and smooth ∂X, g^{jk}, V_j and V we obtain the remainder estimate $R_{(\vartheta)}(\tau) = C\tau^d$ for $\vartheta = d-1$ and $R_{(\vartheta)}(\tau) = C\tau^{\frac12(d+\vartheta-1)}$ for $\vartheta > (d-1)$.

(ii) Consider potential $V = v(\theta)\log^\mu r$.

(iii) Consider potentials with degenerations and singularities. Obviously in this case the degeneration should be very weak, something like

$$(\log(|x|^2+2))^\mu \cdot (\log(|x''|^2+2))^{-\nu}.$$

11.5.8 Non-Dirichlet Boundary Conditions

All our results have been obtained for Dirichlet boundary conditions. Surely, they remain true for other boundary conditions provided these conditions are regular in the obvious sense and the domain satisfies the cone condition. In particular, for the Schrödinger operator Neumann conditions (and Robin

[44] One can take $\gamma_\tau = \epsilon_0 r\,\mathsf{min}(1, (\tau - V)_+)$ in this case and drop factor $\tau^{1+\vartheta}$ in the remainder estimate.

conditions provided the free term coefficient is regular in the obvious sense) are regular.

Then we obtain that if the domain satisfies the cone condition and if the asymptotics do not distinguish Dirichlet and Neumann boundary conditions then the asymptotics remain true for mixed boundary condition when Dirichlet and Neumann boundary conditions are given on $Y_0 \subset \partial X$ and $Y = \partial X \setminus Y_0$ respectively.

Let us consider more accurate asymptotics distinguishing Dirichlet and Neumann the boundary conditions. Then, repeating arguments of Subsection 11.2.3 we obtain that these asymptotics remain true for mixed boundary conditions with properly defined second coefficient provided either

(a) We are in the frameworks of Theorems 11.5.3, 11.5.14(ii)), 11.5.18(ii) and

$$\mathsf{mes}_{\partial X}\big((\bar{Y}_0 \cap \bar{Y}_1 \cap \{|x| \ge c\})_{\mathsf{cone}}\big) = 0 \tag{11.5.89}$$

or

(b) we are in the framework of Example 11.5.23(ii) and

$$\mathsf{mes}_{\Pi'\partial X}\big(\Pi'(\bar{Y}_0 \cap \bar{Y}_1 \cap \{|x| \ge c\})_{\mathsf{cone}}\big) = 0 \tag{11.5.90}$$

where Π' is the projection $\mathbb{R}^d \ni x \to x' \in \mathbb{R}^{d'}$ or

(c) We are in the framework of Theorem 11.5.27 and

$$\mathsf{mes}_{\partial X'}\big(\Pi'(\bar{Y}_0 \cap \bar{Y}_1 \cap \{|x| \ge c\})\big) = 0 \tag{11.5.91}$$

with the same Π'. Here M_{cone} is the conical hull of the set M.

Moreover, applying the arguments of Example 11.2.27 we obtain that asymptotics (11.5.16) remains true in the context of Example 11.5.7 provided

$$\mathsf{mes}_{\partial X}\{x \in \partial X, \mathsf{dist}(x, Y) \le \varepsilon\} = O(\varepsilon^{\delta}) \qquad \text{as } \varepsilon \to +0 \tag{11.5.92}$$

with $\delta > 0$ and $Y = (\bar{Y}_0 \cap \bar{Y}_1 \cap \{|x| \ge c\})_{\mathsf{cone}}$.

Finally, we would like to note that in almost all the above examples the remainder estimate is $O(\tau^{\frac{1}{2}(d+\vartheta-1)})$ for large enough ϑ (for the Schrödinger operator). In this case the principal contribution to the remainder estimate is provided by the zone $\{x \in X, |x| \le R\}$ with arbitrarily large R. In this

case the standard non-periodicity condition to the Hamiltonian $\sum_{j,k} g^{jk}\xi_j\xi_k$ on the energy level $\tau = 0$ provides the remainder estimate $o\bigl(\tau^{\frac{1}{2}(d+\vartheta-1)}\bigr)$ (see, e.g., Theorem 11.5.3).

The situation in which the principal contributions to the principal part of the asymptotics and to the remainder estimate are provided by different zones is not very exotic.

11.6 Asymptotics of Eigenvalues near -0

In this section we consider Schrödinger operators with potentials decaying at infinity. Under certain rather general assumptions its continuous spectrum is $[0,\infty)$ and the point -0 is the only possible limit point of the discrete spectrum. Moreover, there are no positive eigenvalues.

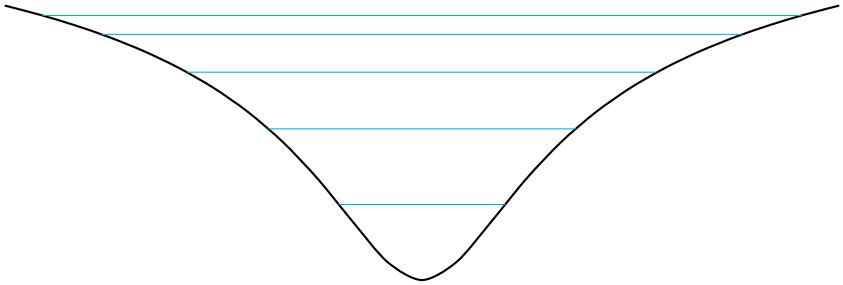


Figure 11.11: Slowly decaying at infinity potential; blue lines show symbolically energy levels of the corresponding Schrödinger operator.

Further, as potential is negative (at least in massive zones so that $\mathsf{mes}(\{x, V(x) < 0\}) = \infty$ and potential decays slowly ($|x|^{-2}$ is a threshold) -0 is a limit point of the discrete spectrum. In this case we obtain asymptotics of $\mathsf{N}^-(\tau)$ as $\tau \to -0$.

We treat more general higher-order operators and non-semi-bounded operators as well. In particular, we treat the Dirac operator with positive mass M where a spectral gap $(-M, M)$ appears in the case of $V = 0$ (while continuous spectrum occupies $(-\infty, -M] \cup [M, \infty)$ and in the general case of decaying potential there could be a discrete spectrum inside the spectral gap and if V decays slowly enough these eigenvalues accumulate to $-M + 0$ and/or $M - 0$. So we will treat *asymptotics of eigenvalues accumulating to the border of the essential spectrum*.

Later we consider much more subtle mechanism when the similar situation appears: Schrödinger and Dirac operators with magnetic field or operator with periodic coefficients. Essential spectrum may occupy spectral bands or separate points (separating spectral gaps). In both cases adding decaying potential one may create discrete spectrum inside the spectral gap and these eigenvalues may accumulate to its borders.

11.6.1 Standard Results

We start with semi-bounded operators. First of all the following statement holds:

Theorem 11.6.1 [45]**.** *Let X be an unbounded connected domain in $\mathbb{R}^d$ and let conditions* (11.1.1)–(11.1.3) *be fulfilled with $\gamma = \epsilon_1\langle x\rangle$, $\rho = \langle x\rangle^{\mu}$, $\mu \in (-1, 0)$.*

Let us assume that

$$\langle a(x,\xi)v, v\rangle \geq \epsilon_0|\xi|^{2m}\|v\|^2 \qquad \forall\xi\ \forall v \tag{11.6.1}$$

where $a(x,\xi)$ is the senior symbol of A.

Further, let us assume that for every $\tau \in (-\epsilon, 0]$ one of non-degeneracy conditions is fulfilled for the symbol $(A(x,\xi)\tau^{-1} - 1)$ uniformly with respect to (τ, x, ξ) (this condition is restrictive only for $|x| \geq C_0$). Then

(i) Asymptotics

$$\mathsf{N}^-(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + O\bigl(\tau^{l(d-1)}\bigr) \tag{11.6.2}$$

holds as $\tau \to -0$, where $\mathsf{N}^{\mathsf{W}}(\tau)$ is given by (11.4.9) *and $\mathsf{N}^{\mathsf{W}} = O(\tau^{ld})$ with $l = \frac{1}{2m\mu}(\mu + 1) < 0$.*

(ii) Moreover, for $0 < \vartheta \leq \vartheta_0^* = -ld$ asymptotics

$$\mathsf{N}^-_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O\bigl(R_{(\vartheta)}(\tau)\bigr) \tag{11.6.3}$$

holds as $\tau \to -0$ with $\mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) = \tau_+^{\vartheta-1} * \mathsf{N}^{\mathsf{W}}$ where in this case N^{W} should contain also lower order terms and

$$R_{(\vartheta)}(\tau) = \begin{cases} |\tau|^{l(d-1)+\vartheta(1-l)} & \text{for } \vartheta < \vartheta_1^*, \\ \log|\tau| & \text{for } \vartheta = \vartheta_1^*, \\ 1 & \text{for } \vartheta > \vartheta_1^* \end{cases} \tag{11.6.4}$$

[45] Cf. Theorem 11.5.1.

where $\vartheta_1^* = -l(d-1)(1-l)^{-1}$.

One can see easily that $0 < \vartheta_1^* < \vartheta_0^*$ *and that*

$$\mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) = \begin{cases} O\big(|\tau|^{ld+\vartheta}\big) & \text{for } \vartheta < \vartheta_0^* \\ O\big(\log|\tau|\big) & \text{for } \vartheta = \vartheta_0^* \end{cases} \tag{11.6.5}$$

as $\tau \to -0$ *while for* $\vartheta > \vartheta_0^*$ *we have estimate* $\mathsf{N}^{\mathsf{W}}_{(\vartheta)} = O(1)$.

(iii) Furthermore, if

$$\begin{aligned} &\mathsf{mes}\big(\{x \in X : |x| \le r \quad \& \quad \exists \xi \in \mathbb{R}^d : \\ &\qquad \mathsf{Spec}\, A(x,\xi) \cap (-\infty, -\epsilon_1 \rho^{2m}] \ne \emptyset\}\big) \asymp r^d \qquad \text{as } r \to +\infty \end{aligned} \tag{11.6.6}$$

for some $\epsilon_1 > 0$ *then*

$$\mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) \asymp \begin{cases} |\tau|^{ld+\vartheta} & \text{for } \vartheta < \vartheta_0^* \\ \log|\tau| & \text{for } \vartheta = \vartheta_0^* \end{cases} \tag{11.6.7}$$

as $\tau \to -0$.

For the Schrödinger operator condition (11.6.7) *is equivalent to*

$$\mathsf{mes}\big(\{x \in X : |x| \le r, V \le -\epsilon_1 \rho^2\}\big) \asymp r^d \qquad \text{as } r \to +\infty. \tag{11.6.8}$$

Theorem 11.6.2 [46]**.** *Let assumptions of Theorem 11.6.1 be fulfilled.*

Further, let us assume that $X \cap \{|x| \ge r_0\} = X^0 \cap \{|x| \ge r_0\}$ *where* X^0 *is a non-empty conical domain with* $\partial X^0 \backslash 0 \in \mathscr{C}^K$ *and that stabilization conditions* (11.5.4) *are fulfilled. Let us assume that as* $\tau = -1$ *the standard non-periodicity condition to Hamiltonian trajectories for the quasihomogeneous symbol*

$$A^0(x,\xi) = \sum_{|\beta| \le 2m} a^0_\beta(x)\xi^\beta \tag{11.6.9}$$

satisfies, and the permitted[37] *trajectories never hit* $0 \times \mathbb{R}^d$. *Then*

[46] Cf. Theorem 11.5.3.

(i) Asymptotics

$$\mathsf{N}^-(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + \bigl(\varkappa_1 + o(1)\bigr)|\tau|^{l(d-1)} \tag{11.6.10}$$

holds as $\tau \to -0$. *In particular, for the Schrödinger operator*

$$\varkappa_1 = \mp\frac{1}{4}(2\pi)^{1-d}\omega_{d-1}\int_{\partial X^0}\bigl(-1-V^0\bigr)_+^{\frac{d-1}{2}}\,dS^0 \tag{11.6.11}$$

for Dirichlet or Neumann boundary conditions respectively where the notation of (11.5.10)–(11.5.11) *is used.*

(ii) Moreover, for $0 < \vartheta < \vartheta_1^*$ *asymptotics*

$$\mathsf{N}^-_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + o\bigl(R_{(\vartheta)}(\tau)\bigr) \qquad \text{as } \tau \to -0. \tag{11.6.12}$$

holds.

(iii) Finally, let $\vartheta = \vartheta_1^*$*; the condition on the domain and the stabilization condition remain true. Moreover, let the non-periodicity condition to Hamiltonian trajectories be fulfilled for the symbol* $A^0(x,\xi)$ *on the energy level* 0 *and let the permitted*[37] *trajectories never hit* $0 \times \mathbb{R}^d$.

Then asymptotics (11.5.6) *also holds (so the remainder estimate is* $o(\log|\tau|)$ *rather than* $O(\log|\tau|)$*).*

Proof of Theorems 11.6.1 and 11.6.2. The proof is similar to the proof of theorems 11.5.1 and 11.5.3 but now in the zone X'_{reg}, $\rho_\tau \asymp \rho$ (instead of $\rho_\tau \asymp |\tau|^{\frac{1}{2m}}$) and $X'_{\mathsf{reg}} \subset \{\rho \ge \epsilon_2|\tau|^{\frac{1}{2m}}\}$ with appropriate $\epsilon_2 > 0$.

Further, every bounded domain gives a finite contribution to the remainder estimate (and therefore in the contrast to 11.5.3 there is no assertion about improved asymptotics as $\vartheta > \vartheta_1^*$), Furthermore, the reduced Hamiltonian is now $|x|^{1-2m\mu+\mu}A^0(x,\xi)$.

We would like to emphasize that for $\vartheta > \vartheta_0^*$ we obtain the estimate $\mathsf{N}^-_{(\vartheta)}(-0) < \infty$ instead of asymptotics. This remark remains true in many examples below.

In the framework of Theorem 11.6.2 the remainder estimate is $o\bigl(|\tau|^{l(d-1)+\vartheta(1-l)}\bigr)$ and $o(\log|\tau|)$ in asymptotics (11.5.6) for $\vartheta < \vartheta_1^*$ and $\vartheta = \vartheta_1^*$ respectively.

Finally, the main contributions to the remainder estimate are made by the zones $\{x, |x| \ge \varepsilon|\tau|^{\frac{1}{2m}}\}$ and $\{x, R \le |x| \le \varepsilon|\tau|^{\frac{1}{2m}}\}$ for $\vartheta < \vartheta_1^*$ and $\vartheta = \vartheta_1^*$ respectively with arbitrarily small $\varepsilon > 0$ and arbitrarily large R. This difference leads to different non-periodicity conditions. □

Remark 11.6.3. If only stabilization conditions (11.5.4) (including the condition $X \cap \{|x| \ge r_0\} = X^0 \cap \{|x| \ge r_0\}$) are fulfilled then the non-degeneracy condition is automatically fulfilled and $\mathsf{N}^{\mathsf{W}}(\tau) = \varkappa_0 |\tau|^{ld}\big(1+o(1)\big)$ as $\tau \to -0$ where for the Schrödinger operator

$$\varkappa_0 = (2\pi)^{-d}\omega_d \int_{X^0} \big(-1 - V^0\big)_+^{\frac{d}{2}} \sqrt{g^0}\,dx \tag{11.6.13}$$

in the notation of (11.5.10)–(11.5.11).

For scalar operators we get the following

Theorem 11.6.4 [47]**.** *Let X be an unbounded connected domain in $\mathbb{R}^d$ and let conditions* (11.1.1)–(11.1.3) *be fulfilled with $\gamma = \epsilon_1\langle x\rangle$, $\rho = \langle x\rangle^{\mu}$, $\mu \in (-1, 0)$.*

Let condition (11.6.1) *be fulfilled and let A be a scalar operator. Then asymptotics*

$$\mathsf{N}(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + O\big(\tau^{l(d-1)-\delta}\big) \tag{11.6.14}$$

hold as $\tau \to -0$, and for $\vartheta > 0$ asymptotics (11.6.3) *holds.*

11.6.2 Strongly Non-Periodic Hamiltonian fFlow

In what follows we consider homogeneous potentials with singularities. Now we obtain a more precise remainder estimate under more restrictive conditions on the Hamiltonian flow. One can prove easily the following

Theorem 11.6.5 [48]**.** *Let the conditions of Theorem 11.6.1 be fulfilled and let A be a scalar operator. Moreover, let us assume* (11.5.14) *is fulfilled i.e. that* <u>*either*</u> *$X^0 = \mathbb{R}^d$* <u>*or*</u> *A is the Schrödinger operator or its power.*

Further, let stabilization condition $(11.5.4)^*$ *be fulfilled. Furthermore, let us assume that the Hamiltonian flow (with reflections at ∂X^0) is of non-periodic power type*[49] *at negative energy levels and that along the trajectories $(x(t), \xi(t))$ with $(x(0), \xi(0)) \in \Lambda^0_{\varepsilon, T} \subset \Sigma_{-1}$, $|x(0)| = 1$, $\varepsilon = T^{-M}$ condition $|x(t)| \ge T^{-M_1}$ is fulfilled.*

Then asymptotics

$$\mathsf{N}^-(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + \big(\varkappa_1 + O(|\tau|^{\delta'})\big)\tau^{l(d-1)} \tag{11.6.15}$$

[47] Cf. Theorem 11.5.5.

[48] Cf. Theorem 11.5.6.

[49] Footnote [38] of the previous section remains true.

holds as $\tau \to -0$ *with* $\delta' > 0$. *Moreover, for* $0 < \vartheta \le \vartheta_1^* - \delta$ *with* $\delta > 0$ *asymptotics*

$$\mathsf{N}^-_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O\big(|\tau|^{l(d-1)+\vartheta(1-l)+\delta'}\big) \tag{11.6.16}$$

holds as $\tau \to -0$ *with* $\delta' = \delta'(d, m, \delta, L) > 0$.

Let us consider the following

Example 11.6.6[50]. (i) The condition of Theorem 11.6.5 to the Hamiltonian flow is fulfilled for the Schrödinger operator with $X^0 = \mathbb{R}^d$ and $A^0(x,\xi) = |\xi|^2 - |x|^{2\mu}$ with $\mu \ne -\frac{1}{2}$. So we consider $\mu \in (-1, -\frac{1}{2}) \cup (-\frac{1}{2}, 0)$.

(ii) Therefore the statement of Theorem 11.6.5 is true for this operator (provided all the other conditions of this theorem, including (11.5.4)*, are fulfilled) and for any $2m$-order operator with $A^0(x,\xi) = (|\xi|^2 - |x|^{2\mu})^m$, and one can replace $X^0 = \mathbb{R}^d$ by the X^0 obtained after (a sequence of) the transformations (a),(b) of Example 11.5.7(ii).

11.6.3 Periodic Hamiltonian Flow

As in the previous section let us consider the case of periodic Hamiltonian flow:

Theorem 11.6.7[51]. *Let* $X^0 = \mathbb{R}^d$ *and conditions* (11.5.27)–(11.5.27), (11.5.32) *with* $\mu \in (-1, 0)$ *be fulfilled in the domain of negative energies. Let us introduce the symbol*

$$b(x,\xi) = -\frac{l}{T_0}\int_0^{T_0} A^1 \circ \Psi_t\, dt \tag{11.6.17}$$

where T_0 *is the period on the energy level* -1 *(compare with* (11.5.30)*; we have changed sign because* $A^0(x,\xi) = -1$ *now). Then*

(i) If $\delta_1 < (\mu + 1)$ *and condition* (11.5.31) *is fulfilled for all* ω *then asymptotics*

$$\mathsf{N}^-(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + O\big(|\tau|^{l(d-1)+\delta''}\big) \tag{11.6.18}$$

holds as $\tau \to -0$ *with* $\delta'' > 0$.

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[50] Cf. Example 11.5.7.

[51] Cf. Theorem 11.5.9.

(ii) If $\delta_1 \in [\mu+1, 2\mu+2]$ *then for* $\tau \to -0$ *such that* $|\tau|^l \in \bigcup_{n\in\mathbb{Z}^+} J_n$ *asymptotics*

$$\mathsf{N}^-(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + F(\tau)|\tau|^{l(d-1)} + O\big(|\tau|^{l(d-1)+\delta''}\big) \tag{11.6.19}$$

holds with $\delta'' > 0$ *and*

$$F(\tau) = (2\pi)^{-d} l T_0^{-1} \int_{\Sigma_{-1}} \Upsilon\Big(T_0 l^{-1}\big((|\tau|^l - \frac{\pi}{4}\omega_0) - |\tau|^{l-\frac{\delta_1}{2m\mu}} b\big)\Big)\, dx d\xi : dA^0. \tag{11.6.20}$$

The intervals J_n *are defined in Theorem 11.5.9(ii) with* $\nu = -\frac{2\pi l}{T_0}$, ω_0 *replaced by* $-\omega_0$ *where* ω_0 *is the Maslov index of the trajectory* Ψ_t *with* $t \in (0, T_0)$.

Let us treat the perturbed Schrödinger operator with the Coulomb potential more carefully. In this case $A_0 = |D|^2 - |x|^{-1}$ and $\mu = -\frac{1}{2}$, $l = -\frac{1}{2}$, $T_0 = \frac{\pi}{2}$, $\nu = 2$ and the Hamiltonian trajectories are Keplerian ellipses (with one focus located at the origin). Moreover, $\omega_0 \equiv 2(d-1) \mod 4\mathbb{Z}$ and

$$\mathsf{Spec}(A_0) = \{\tau_n = -1(d+1+2n)^2,\ n \in \mathbb{Z}^+\} \tag{11.6.21}$$

and the multiplicity of τ_n is $P(n)$ where $P(n)$ is a polynomial of degree $(d-1)$ in n. Let us introduce the function $f(\tau) = \frac{1}{2}(-\tau)^{-\frac{1}{2}} - \frac{1}{2}$.

We leave to the reader

Problem 11.6.8. Prove that for generic positively homogeneous functions a^1_β (with $|\beta| \le k$) of degree $(-1+2|\beta|-\delta_1)$ condition (11.5.31) is fulfilled for all $\omega \in \mathbb{R}$ and therefore asymptotics (11.6.18) holds for $0 < \delta_1 < \frac{1}{2}$ while asymptotics (11.6.19)–(11.6.20) holds for $\delta_1 \ge \frac{1}{2}$, and now there is no restriction of the type "$\delta_1 \le 2\mu+2$" on δ_1 and $k = 0, 1, 2$ is arbitrary.

Moreover, prove that one can take $V' = v_0|x|^{-1-\delta}$ with an arbitrary exponent $\delta > 0$ and an arbitrary constant $v_0 \ne 0$.

Let us assume that $b = 0$. There are two possibilities in this case: the trivial possibility is to take $\delta_2 \in (\delta_1, 2\delta_1)$ and the non-trivial one is to take $\delta_2 = 2\delta_1$. In the first case all the previous statements remain true with δ_1 replaced by δ_2 etc.

In the second case we assume that condition (11.5.39) is fulfilled in which case the symbol $b(x,\xi)$ is given by formula (11.3.27) with b and b^s replaced

by $-\frac{1}{2}A^1(-A^0)^{-\frac{1}{2}}$ and $-\frac{1}{2}A^2(-A^0)^{-\frac{1}{2}}+\frac{3}{4}(A^1)^2(-A^0)^{-\frac{3}{2}}$ respectively and with integration from 0 to π, $\frac{\pi}{2}$ instead of 2π, π respectively.

Further, the numerical factor on every integral is $\frac{1}{\pi}$ instead of $\frac{1}{2\pi}$. Furthermore, one can move A^0 out of the Poisson brackets and then take $A^0=-1$. Therefore

$$b(x,\xi)=\frac{1}{2\pi}\int_0^{\pi}\bigl(A^2-\frac{3}{2}(A^1)^2\bigr)\circ\Psi_t\,dt+ \frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\{A^1\circ\Psi_t, A^1\circ\Psi_{t+t'}\}\,dtdt'. \tag{11.6.22}$$

Therefore the following assertion has been proven:

Theorem 11.6.9. *Let $x^0=\mathbb{R}^d$, $A=A^{0\mathsf{w}}+A^{1\mathsf{w}}+A''^{\mathsf{w}}$ where A'' satisfies condition* (11.5.29) *with a^2_β, δ_2 instead of a^1_β, δ_1, $\delta_2=2\delta_1$ and a^j_β are functions which are positively homogeneous of degrees $(-1+2|\beta|-\delta_j)$ with respect to x, smooth for $x\ne 0$. Let condition* (11.5.39) *be fulfilled and let us define the symbol b by* (11.6.22). *Then*

(i) If $\delta_2<\frac{1}{2}$ and condition (11.5.31) *is fulfilled for every $\omega\in\mathbb{R}$ then asymptotics* (11.6.18) *hold as $\tau\to -0$.*

(ii) If $\delta_2\ge\frac{1}{2}$ and condition (11.5.31) *is fulfilled for some $\omega\in\mathbb{R}$ then asymptotics* (11.6.19)–(11.6.20) *holds as $\tau\to -0$ and $(-\tau)^{-\frac{1}{2}}\in\bigcup_{n\in\mathbb{Z}^+}$ where J_n are given in Theorem 11.5.9(ii) with $\omega_0=2(d-1), \mu=2$ and with δ_1 replaced by δ_2 everywhere.*

One can easily prove that for fixed a^1_β with $|\beta|\le 2$ satisfying (11.5.39) and for generic a^2_β with $|\beta|\le k$ condition (11.5.33) is fulfilled for all $\omega\in\mathbb{R}$; here $k=0,1,2$ is arbitrary.

11.6.4 Singularities

Let us consider homogeneous lower order terms with singularities (for the Schrödinger operator only):

Theorem 11.6.10 [52]**.** *Let $d\ge 2$, $X=\{(r,\theta)\in\mathbb{R}^+\times U\}$ where $U\subset\mathbb{S}^{d-1}$ and let $g^{jk}=g^{jk}(\theta)$, $V_j=v_j(\theta)r^\mu$, $V=v(\theta)r^{2\mu}$, $\mu\in(-1,0)$. Further, let*

[52] Cf. Theorem 11.5.14.

$\tilde\rho = \tilde\rho(\theta)$ and $\tilde\gamma = \tilde\gamma(\theta)$ be functions on U and let conditions (11.1.6)–(11.1.8) *be fulfilled.*

Furthermore, assume that <u>*either*</u> *$d \ge 3$* <u>*or*</u> *$\tilde\rho\tilde\gamma \ge \epsilon_0 > 0$. Finally, let*

$$\tilde\rho^{-\frac{d}{\mu}} \in \mathscr{L}^1(U), \tag{11.6.23}$$

$$\tilde\rho^{-\frac{1}{\mu}(d-1)}\tilde\gamma^{-1} \in \mathscr{L}^1(U). \tag{11.6.24}$$

Moreover, let $v(\bar\theta) < 0$ for some $\bar\theta \in U$.

Then asymptotics (11.6.2) *holds and $\mathsf{N}^{\mathsf{W}}(\tau) = \varkappa_0|\tau|^{Id}$ with the coefficient $\varkappa_0$ given by* (11.6.13).

Further, if $\vartheta \in (0, \vartheta_1^*)$, $\vartheta_1^* = -I(d-1)(1-I)^{-1} = (1+\mu)(d-1)(1-\mu)^{-1}$ and if

$$\tilde\rho^{-\frac{1}{\mu}(d-1-\vartheta)}\tilde\gamma^{-1-\vartheta} \in \mathscr{L}^1(U) \tag{11.6.25}$$

then asymptotics (11.6.3)–(11.6.4) *holds.*

Furthermore, let us assume that there exists a set $\Lambda \subset \Sigma = \Sigma_{-1}$ with $\mathsf{mes}_\Sigma(\Lambda) = 0$ such that through every point of $(\Sigma \setminus \Lambda)$ there passes an infinitely long non-periodic billiard generated by the Hamiltonian

$$H^0 = \tilde\gamma r^{1-\mu}\bigl(\tilde\rho + r^{-\mu}\bigr)^{-1}\Bigl(\sum_{j,k} g^{jk}(\xi_j - V_j)(\xi_k - V_k) + V + 1\Bigr) \tag{11.6.26}$$

which never hits $0 \times \mathbb{R}^d \cup \Lambda$. Then asymptotics (11.6.10) *holds with the coefficient $\varkappa_0$ given by* (11.6.11).

Moreover, let $\vartheta \in (0, \vartheta_1^*)$ and (11.6.25) *be fulfilled. Then asymptotics* (11.6.12) *holds.*

Example 11.6.11[53]. Let $x = \mathbb{R}^d$, $d \ge 2$ and $\rho = |x''|^\nu \cdot |x|^{\mu-\nu}$, $\gamma = \frac12|x''|$ with $\mu \in (-1, 0)$, $\nu \in (\mu, 0)$. Then

(i) Conditions (11.6.23) and (11.6.25) are fulfilled if and only if $\nu > \nu_0 = \mu d'' d^{-1}$ and $\nu > \nu_{\vartheta+1} = \mu(d'' - 1 - \vartheta)(d - 1 - \vartheta)^{-1}$ respectively.

Therefore for $\nu > \nu_1$ asymptotics (11.6.2) holds while for $\nu > \nu_{\vartheta+1}$, $\vartheta < \vartheta_1^*$ asymptotics (11.6.3) holds[54].

[53] Cf. Example 11.5.17.

[54] And under non-periodicity condition on the Hamiltonian flow asymptotics (11.6.10) holds. Further, asymptotics (11.6.12) (for $\nu > \nu_{\vartheta+1}$) holds.

(ii) The case $\nu \in [\nu_0, \nu_1)$ is treated in the same way as case (ii) of Example 11.5.17. We obtain that for $\nu = \nu_1$ asymptotics

$$\mathsf{N}^-(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + O\bigl(|\tau|^{l(d-1)} \log|\tau|\bigr) \tag{11.6.27}$$

holds as $\tau \to -0$ and for $\nu \in (\nu_0, \nu_1)$ asymptotics

$$\mathsf{N}(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + O\bigl(|\tau|^{q}\bigr) \tag{11.6.28}$$

holds as $\tau \to -0$ with $q = -\frac{1}{2}(\mu+1)d'(\nu-\mu)^{-1} = l(d-\eta)$, $\eta = (\mu d'' - \nu d)(\mu-\nu)^{-1} \in (0,1)$.

In these asymptotics $\mathsf{N}^{\mathsf{W}}(\tau) = \varkappa_0 |\tau|^{ld}$.

(iii) Similarly, for $\nu = \nu_{\vartheta+1}$, $\vartheta \in (0, \vartheta_1^*)$ asymptotics

$$\mathsf{N}^-_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O\bigl(|\tau|^{l(d-1-\vartheta)+\vartheta} \log|\tau|\bigr) \tag{11.6.29}$$

holds as $\tau \to -0$ and for $\nu \in (\nu_0, \nu_{\vartheta+1})$ asymptotics

$$\mathsf{N}^-_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O\bigl(|\tau|^{q+\vartheta}\bigr) \tag{11.6.30}$$

holds as $\tau \to -0$ with the same q as before.

(iv) Finally, for $\nu = \mu d'' d^{-1}$ asymptotics

$$\mathsf{N}(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + O\bigl(|\tau|^{ld}\bigr) \tag{11.6.31}$$

holds as $\tau \to -0$ with $\mathsf{N}^{\mathsf{W}}(\tau)$ given by (11.5.52) and moreover $\mathsf{N}^{\mathsf{W}}(\tau) \asymp |\tau|^{ld} \log|\tau|$. Moreover, footnotes [41], [42] section hold.

We now have no need of the coercivity condition and therefore Theorems 11.5.18 and 11.5.20 and Examples 11.5.19 and 11.5.21 are almost senseless. However, the following problems make sense:

Problem 11.6.12. (i) Examine potentials of the type $V^0 + V'$ where V^0 is a slowly decreasing and not very singular potential and V' decreases faster and is more singular, etc.

(ii) Let us note that for $d \ge 3$ in the zone $\{V \ge 0\}$ no condition excluding $|V_j| \le c\sqrt{V}$ is necessary. Consider examples with $\rho = \langle x\rangle^{\mu}$, $\gamma = \langle x\rangle^{1-\sigma}$ with $\mu < 0$, $\sigma \in (0, 1+\mu)$, where asymptotics with a worse remainder estimate may be obtained. In this case the magnetic field can decrease more slowly at infinity.

11.6.5 Quasihomogeneous Potentials

Let us consider non-homogeneous operators. Repeating the arguments of Example 11.5.23 with obvious modifications we obtain

Example 11.6.13. (i) Let X be an L-quasiconical domain in $\mathbb{R}^d$ and let g^{jk}, V_j, V be positively L-quasihomogeneous with respect to x of degrees 0, μ, 2μ respectively where $\mu \in (-1,0)$, $L = (L_1, \dots, L_d)$ with $L_j = 1$ for $1 \le j \le d'$ and $L_j > 1$ for $d'+1 \le j \le d$, $1 \le d' < d$.

Let us assume that $g^{jk}, V_j, V \in \mathscr{C}^K(\bar{X} \setminus 0)$ satisfy condition (11.1.8) and that $V(\bar{\theta}) < 0$ for some $\bar{\theta} \in X$. Then conditions (11.1.7) are fulfilled with $\rho = [x]_L^{\mu}$, $\gamma = \epsilon [x]_L$ for $[x]_L \ge 1$; for $[x]_L \le 1$ one should take $\gamma = \epsilon [x]_L^{\bar{L}}$ with $\bar{L} = \max_j L_j$. Then the asymptotics

$$\mathsf{N}^-(\tau) = \varkappa_0 |\tau|^p + O\big(|\tau|^q\big) \tag{11.6.32}$$

holds as $\tau \to -0$ with $p = \frac{d}{2} + \frac{L^*}{2\mu}$, $q = \frac{1}{2}(d-1) + \frac{1}{2\mu}(L^*-1)$, $L^* = \sum_j L_j$ and

$$\varkappa_0 = (2\pi)^{-d} \omega_d \int \big(-1 - V\big)_+^{\frac{d}{2}} \sqrt{g}\, dx. \tag{11.6.33}$$

(ii) Moreover, let us assume that $x' \ne 0$, $\nu' \ne 0$ on $\partial X \setminus 0$ where ν is a unit normal to ∂X and $x' = (x_1, \dots, x_{d'})$, $\nu' = (\nu_1, \dots, \nu_{d'})$. Let us assume that $g^{jk} = 0$, $V_k = 0$ for $j \le d'$, $k \ge d'+1$. Then asymptotics

$$\mathsf{N}^-(\tau) = \varkappa_0 |\tau|^p + \big(\varkappa_1 + o(1)\big)|\tau|^q \tag{11.6.34}$$

holds with

$$\varkappa_1 = \mp \frac{1}{4} (2\pi)^{1-d} \omega_{1-d} \int \big(-1 - V\big)_+^{\frac{d-1}{2}}\, dS' \sqrt{g''} dx'' \tag{11.6.35}$$

where dS' and $\sqrt{g''}dx''$ are the Riemannian densities introduced in Example 11.5.23.

11.6.6 Quickly Decreasing Potentials

Let us consider potentials which decrease rather quickly. We considered homogeneous potentials decreasing as $\langle x\rangle^{2\mu}$ with $\mu \in (-1,0)$ and $N^-(0) < \infty$ as either $\mu < -1$ or $\mu = -1$, $V \ge -\epsilon \langle x\rangle^{2\mu}$ with sufficiently small $\epsilon > 0$. So "rather quickly" should mean "a bit slower than $\langle x\rangle^{-2}$" .

Theorem 11.6.14. *Let X be an unbounded domain in $\mathbb{R}^d$ and let conditions* (11.1.1)–(11.1.3) *and* (11.6.1) *be fulfilled with*

$$\gamma = \epsilon\langle x\rangle, \qquad \rho = \langle x\rangle^{-1}\log^{\nu}\langle x\rangle \tag{11.6.36}$$

where we now take $\langle x\rangle = (|x|^2+4)^{\frac{1}{2}}$, $\nu > 0$.

Further, let us assume that for every $\tau \in (-\epsilon, 0]$ one of non-degeneracy conditions is fulfilled for the symbol $(A(x,\xi)\tau^{-1}-1)$ uniformly with respect to (τ, x, ξ) (this condition is restrictive only for $|x| \ge C_0$). Then

(i) Asymptotics

$$\mathsf{N}^-(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + O\big(\big|\log|\tau|\big|^{\nu(d-1)+1}\big) \tag{11.6.37}$$

holds as $\tau \to -0$ where $\mathsf{N}^{\mathsf{W}}(\tau) = O\big(\big|\log|\tau|\big|^{\nu d+1}\big)$ and under condition (11.6.8) $\mathsf{N}^{\mathsf{W}}(\tau) \asymp \big|\log|\tau|\big|^{\nu d+1}$.

(ii) Moreover, let $X \cap \{|x| \ge r_0\} = X^0 \cap \{|x| \ge r_0\}$ where X^0 is a conical domain with $\partial X^0 \setminus 0 \in \mathscr{C}^K$ and let stabilization conditions (11.5.4) *be fulfilled for functions $a_\beta \log^{-\nu(2m-|\beta|)}$ where the functions $a^0_\beta \in \mathscr{C}^K(\bar X^0 \setminus 0)$ are positively homogeneous of degrees $-(2m-|\beta|)$ on x.*

Then the microhyperbolicity condition is fulfilled automatically. Further, if the non-periodicity condition is fulfilled for the quasihomogeneous symbol

$$A^0(x,\xi) = |x|^{2m}\sum_{|\beta|\le 2m} a^0_\beta(x)\xi^\beta \tag{11.6.38}$$

on the energy level 0 then the asymptotics

$$\mathsf{N}^-(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + \big(\varkappa_1 + o(1)\big)\big|\log|\tau|\big|^{\nu(d-1)+1} \tag{11.6.39}$$

holds where, for the Schrödinger operator,

$$\varkappa_1 = \mp 2^{-\nu(d-1)-3}(2\pi)^{1-d}\omega_{d-1}\int_{\partial U}\big(V^0\big)_-^{\frac{d-1}{2}}\,dS, \tag{11.6.40}$$

$U = X^0 \cap \mathbb{S}^{d-1}$ is a domain on $\mathbb{S}^{d-1}$, $dS = dS^0 : dr$ is the natural density on ∂U, dS^0 is the Riemannian density on ∂X^0 generated by the metric g^{jk0} (compare with (11.6.12)*).*

(iii) On the other hand, if only the stabilization condition is fulfilled then $\mathsf{N}^{\mathsf{W}}(\tau) = \big(\varkappa_0 + o(1)\big)\big|\log|\tau|\big|^{\nu d+1}$ *where, for the Schrödinger operator,*

$$\varkappa_0 = 2^{-\nu d-1}(2\pi)^{-d}\omega_d \int_U \big(V^0\big)_-^{\frac{d}{2}} \sqrt{g^0}\, d\theta, \tag{11.6.41}$$

$d\theta = dx : dr$, $g^0 = \det(g^{jk0})^{-1}$ *(compare with (*11.6.11*)).*

Proof. Statement (i) follows immediately from our previous arguments; moreover, now $\rho\gamma = \log^{\nu}\langle x\rangle \to \infty$ as $|x| \to \infty$ and $X'' = \emptyset$.

In order to prove Statement (ii) let us note that the principal contribution to the remainder estimate is given by the zone $\{R \le |x| \le \epsilon\tau^{-\frac{1}{2m}}\}$ with arbitrary fixed R and $\epsilon > 0$. We need to examine the Hamiltonian trajectories starting in this zone and associated with the symbol

$$A_\tau(x,\xi) = \langle x\rangle^{2m}\log^{-\nu}\langle x\rangle\big(A(x,\xi)+\tau\big) \tag{11.6.42}$$

on $\Sigma_\tau = \{\det A_\tau(x,\xi) = 0\}$. Let $\Sigma^0 \supset \Lambda_T$ be a closed set of measure 0 such that through every point of $(\Sigma^0 \setminus \Lambda_T)$ there passes a non-periodic Hamiltonian trajectory (with reflections) of length $2T$ associated with the symbol A^0. Then one can take

$$\Lambda_{1,T,\epsilon,\varepsilon,\tau,R} = \big\{(x,\xi)\in\Sigma_\tau, R\le|x|\le\epsilon\tau^{-\frac{1}{2m}}, \\ \operatorname{dist}\big(\frac{x}{|x|},\partial X^0\big) > 0, \operatorname{dist}\big((\frac{x}{|x|},\xi|x|\log^{-\nu}|x|),\Lambda_T\big) > \varepsilon\big\}$$

with $R = R(\epsilon,\varepsilon,T)$. Moreover, one can see that $\log|x(t)|\log^{-1}|x(0)| = 1+o(1)$ as $|x(0)|\to\infty$ uniformly with respect to τ and $t\in[-2T,2T]$ with arbitrary fixed T. We leave the details to the reader. □

We leave to the reader the following set of rather easy problems:

Problem 11.6.15. (i) Prove a statement similar to Theorem 11.6.10 with

$$g^{jk} = g^{jk}(\theta), \qquad V_j = v_j(\theta)r^{-1}\log^{\nu} r, \qquad V = v(\theta)r^{-2}\log^{2\nu} r,$$

$X = (1,\infty)\times U$.

(ii) Consider the case when

$$\rho = \langle x\rangle^{\mu}\log^{\nu}\langle x\rangle, \tag{11.6.43}$$

$\vartheta \ge 0$ with either $\mu > -d(d+2m\vartheta)^{-1}$ or $\mu = -d(d+2m\vartheta)^{-1}$, $\nu \ge -1(d+2m\vartheta)^{-1}$.

Prove that under these conditions the principal part of the asymptotics is $\asymp I$ with either $I = |\tau|^p$ (with $p<0$) or $I = \bigl|\log|\tau|\bigr|^q$ (with $q>0$) or $I = \log\bigl|\log|\tau|\bigr|$ and the remainder estimate is either from the same list or $O(1)$.

(iii) Moreover, consider the case of potentials decreasing even more quickly when

$$\rho = \langle x\rangle^{-1}\bigl(\underbrace{\log\dots\log}_{r \text{ times}}\langle x\rangle\bigr)^{\nu} \tag{11.6.44}$$

when the principal part of the asymptotics and the remainder estimate are of the orders $\log|\tau| \times \zeta(\tau)^{\nu d}$ and $\log|\tau| \times \zeta(\tau)^{\nu(d-1)}$ respectively with $\zeta(\tau) = \underbrace{\log\dots\log}_{r \text{ times}}|\tau|$.

(iv) Prove that in the framework of Theorem 11.6.14 $\mathsf{N}^{-}_{(\vartheta)}(-0) < \infty$ for $\vartheta > 0$.

11.6.7 Slowly Decreasing Potentials

Let us consider the slowly decreasing potential $V = v(\theta)\log^{-1} r$ in the spherical coordinate system (r,θ). Using

$$\tilde\rho_\tau = (\tau - V)_+^{\frac12}, \qquad \tilde\gamma_\tau = \epsilon_0 r \min(1, (\tau-V)_+ \log r) \tag{11.6.45}$$

for $r \ge 2$ we easily prove

Theorem 11.6.16. *Let $d \ge 3$ and $X \cap \{r \ge r_0\} = \{r \ge r_0, \theta \in U\}$. For $|x| \ge r_0$ let*

$$g^{jk} = g^{jk}(\theta), \qquad V_j = 0, \qquad V = v(\theta)\log^{-1} r \tag{11.6.46}$$

where (r,θ) are spherical coordinates, $g^{jk}, v, \partial X \in \mathscr{C}^K$ and the matrix (g^{jk}) is positive definite. Let $v_0 = \min_U v < 0$. Then

(i) Asymptotics (11.6.3) *holds with*

$$\mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) \asymp |\tau|^{d+2\vartheta}\int_U \exp\bigl(\frac{vd}{\tau}\bigr)\,d\theta, \tag{11.6.47}$$

$$R_{(\vartheta)}(\tau) = |\tau|^{d-2}\int_U \exp\bigl(\frac{v}{\tau}(d-\vartheta-1)\bigr)\,d\theta \tag{11.6.48}$$

for $\vartheta < d-1$. Moreover, one can take

$$\mathsf{N}^{\mathsf{W}}(\tau) = a\int_{U'} |\tau|^d v_-^{-\frac{d}{2}} F_d\bigl(-\frac{\tau}{v_-}\bigr)\exp\bigl(-\frac{vd}{\tau}\bigr)\sqrt{g}\,d\theta \tag{11.6.49}$$

where $U' = U\cap\{v < v_0+\varepsilon\}$ with arbitrarily small fixed $\varepsilon > 0$, $a = 2^{1-d}\pi^{-\frac{d}{2}}d^{-\frac{d}{2}+1}$,

$$F_d(s) = \frac{1}{\Gamma(\frac{d}{2}+1)}\int_0^{\frac{d}{s}-\varepsilon} t^{\frac{d}{2}}\bigl(1-\frac{st}{d}\bigr)^{-\frac{d}{2}}e^{-t}dt = 1+O(s) \tag{11.6.50}$$

as $s\to+0$.

(ii) In particular, for $v = v_0 = \mathsf{const}$ the principal part of the asymptotics is $\asymp |\tau|^{d+2\vartheta}\exp(\frac{v_0 d}{\tau})$ and the remainder estimate is $O(|\tau|^{d-2}\exp(\frac{v_0}{\tau}(d-1-\vartheta)))$.

Moreover, in this case one can take

$$\gamma_\tau = \epsilon_0 r\min\bigl(1,(\tau-V)_+\log^2 r\bigr)$$

and obtain a remainder estimate with the factor $|\tau|^{d+\vartheta-1}$ instead of $|\tau|^{d-2}$.

On the other hand, if $v = v(\theta) = v_0 \implies \theta\in\partial U, \nabla v(\theta)\ne 0$ and $\mathsf{rank}\,\mathsf{Hess}\, v|_{\partial U} = d-1$ then the principal part is $\asymp |\tau|^{\frac{3}{2}+2\vartheta}\exp(\frac{v_0 d}{\tau})$ and the remainder estimate is $O(|\tau|^{\frac{3}{2}d-3}\exp\bigl(\frac{v_0}{\tau}(d-1-\vartheta)\bigr))$. In other cases the principal part and the remainder estimate are of intermediate orders.

Problem 11.6.17. (i) Consider the case $\vartheta\ge(d-1)$. Then one can take $\rho_\tau = \log^{-\frac{1}{2}} r + |\tau|$ and $\gamma_\tau = \epsilon_0 r$ and for $d\ge 2$ one can derive the remainder estimates $O(|\tau|^d)$ for $\vartheta = (d-1)$ and $O(1)$ for $\vartheta > (d-1)$.

(ii) Consider potential $V = v(\theta)\log^{-\nu} r$ with arbitrary $\nu > 0$.

11.6.8 General Boundary cConditions

Remark 11.6.18. All the above results remain true for the Neumann boundary condition provided the domain satisfies the cone condition. Let us assume that this (cone) condition is fulfilled. Moreover, all the above results remain true for mixed boundary conditions (provided the asymptotics do not distinguish Dirichlet and Neumann boundary conditions) and also in the following cases:

(i) We are in the frameworks of Theorem 11.6.2 or 11.6.1011.6.10, and condition (11.5.89) is fulfilled.

(ii) We are in the frameworks of Example 11.6.13(ii), and condition (11.5.90) is fulfilled.

Moreover, in the context of Theorem 11.6.5 asymptotics (11.6.16) remain true if condition (11.5.91) is fulfilled.

11.6.9 Non-sSemi-Bounded Operators

Finally, let us consider non-semi-bounded operators. We consider the Dirac operator with mass $M > 0$ (otherwise in the context of this section the spectrum coincides with $\mathbb{R}$). Moreover, let $V \to 0$, $V_j \to 0$ as $|x| \to \infty$.

Then $\mathsf{Spec}_{\mathsf{ess}}(A) = (-\infty, M] \cup [M, \infty)$ and the spectral gap $(-M, M)$ contains either a finite or infinite number of eigenvalues. In the second case these eigenvalues tend to $-M + 0$ and (or) to $M - 0$ and we can consider the corresponding eigenvalue asymptotics. Let us consider $M - 0$. In order to replace it by -0 we replace the operator A by $A - M \cdot I$. Let us fix $\tau_1 \in (-2M, 0)$ and let us consider

$$\mathsf{N}^-(\tau) \coloneqq \mathsf{N}(\tau_1, \tau) \tag{11.6.51}$$

with $\tau \to -0$. Then

Remark 11.6.19. Under described modifications

(i) Theorems 11.6.1, 11.6.2, 11.6.14 and 11.6.5 and Examples 11.6.13, 11.6.6 remain true and one can check conditions on the Hamiltonian flow for the corresponding Schrödinger operator with the symbol

$$H = \sum_{j,k} g^{jk}(\xi_j - V_j)(\xi_k - V_k) + (V + M)^2 - M^2. \tag{11.6.52}$$

In this Hamiltonian one can skip the term V^2 which is negligible at infinity (as far as non-periodicity condition is concerned) in which case one can consider the Hamiltonian

$$H = \sum_{j,k} g^{jk}(\xi_j - V_j)(\xi_k - V_k) + 2MV. \tag{11.6.53}$$

Now, however, due to term V^2

$$\mathsf{N}^{\mathsf{W}}(\tau) = \varkappa_0 |\tau|^p \big(1 + O(|\tau|^{\delta})\big) \tag{11.6.54}$$

with $\delta > 0$ instead of $\mathsf{N}^{\mathsf{W}}(\tau) = \varkappa_0 |\tau|^p$.

(ii) Theorem 11.6.10 (and Example 11.6.11(i) for $d \ge 3$) remain true. In order to prove this let us note that only the analysis in the zone $\{\tilde\rho r^\mu \ge 1\}$ should be changed. In this zone instead of $\rho = \tilde\rho r^\mu$ one should set $\rho = \tilde\rho^2 r^{2\mu}$.

We include this zone in X_{sing}. In this zone $\rho^d \in \mathscr{L}^1$ due to assumption $\tilde\rho^{-\frac{d}{2\mu}} \in \mathscr{L}^1(U)$. Therefore the contribution of this zone to the asymptotics and to the remainder estimate is $O(1)$.

(iii) Theorem 11.6.16 also remains true. However, $\mathsf{N}^{\mathsf{W}}(\tau)$ should be calculated for the Dirac operator, i.e., for the Schrödinger operator with the symbol (11.6.52) and with the spectral parameter equal to 0.

In the proof one should replace $\rho_\tau = (\tau - V)_+^{\frac12}$ by

$$\rho_\tau = \big(\tau - V + C_1 r^{-1} \log^{-2} r\big)_+^{\frac12} \tag{11.6.55}$$

with large enough C_1 in the zone $X_{\mathsf{sing}} \cap \{\tau - V \le C|\tau|^{\frac23} r^{-\frac23}\}$. This does not increase the remainder estimate.

Let us consider Theorems 11.5.9 and 11.6.9. For simplicity we assume that

$$|v| \le 2M - \epsilon_0 \qquad \forall x. \tag{11.6.56}$$

We leave to the reader the following easy

Problem 11.6.20. Conduct analysis below without assumption (11.6.56).

Under assumption (11.6.56) for appropriate τ_1,

$$\mathsf{N}^-(\tau) = \tilde{\mathsf{N}}^-(\tau) - \tilde{\mathsf{N}}^-(\tau_1) \tag{11.6.57}$$

where $\tilde{\mathsf{N}}^-(\tau)$ is the number of the negative eigenvalues for the operator

$$A(\tau) = \mathcal{L}^*(2M - V + \tau)^{-1}\mathcal{L} + V - \tau \tag{11.6.58}$$

with

$$\mathcal{L} = \sum_j \Big(\sum_k \omega^{jk}(D_k - V_k) - \frac{i}{2}\omega^j\Big)\sigma'_j \tag{11.6.59}$$

where we assume that

$$\sigma_0 = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_j = \begin{pmatrix} 0 & \sigma_j'^{*} \\ \sigma'_j & 0 \end{pmatrix} \quad (j \ge 1)$$

and use the notation $\omega^j = \sum_k \partial_k \omega^{jk}$.

If we could take $\rho = \langle x\rangle^{-\frac{1}{2}}$, $\gamma = \epsilon_0\langle x\rangle$ for the original Dirac operator and if

$$D^\alpha(\omega^{jk} - \delta^{jk}) = O\big(|x|^{-\delta_0-|\alpha|}\big), \tag{11.6.60}$$

$$D^\alpha F_{jk} = O\big(|x|^{-\frac{3}{2}-\delta_3-|\alpha|}\big) \qquad \text{as } |x| \to \infty \quad \forall \alpha : |\alpha| \le K \tag{11.6.61}$$

(where $F_{jk} = i(\partial_j V_k - \partial_k V_j)$ is the tensor magnetic intensity) then the operator $A(\tau)$ is a scalar operator plus perturbation $\sum_j A'_j(D_j - V_j) + A'_0$ where

$$D^\alpha A'_j = O\big(|x|^{\max(-1-\delta_0,-2)-|\alpha|}\big), \tag{11.6.62}$$

$$D^\alpha A'_0 = O\big(|x|^{\max(-1-\delta_0,-2,-\frac{1}{2}-\delta_3)-|\alpha|-1}\big) \quad \text{as } |x| \to \infty \tag{11.6.63}$$

and in the context of the previous subsections this error does not matter provided that under the assumptions of Theorems 11.6.7 and 11.6.9

$$\delta_\iota < \frac{3}{2}, \quad \delta_\iota < \frac{1}{2} + \delta_0, \quad \delta_\iota < \frac{1}{2} + \delta_3 \tag{11.6.64}_{1,2,3}$$

respectively with $\iota = 1, 2$ (because the "order" of the operator $D_j - V_j$ is $-\frac{1}{2} + \delta'$ with arbitrarily small $\delta' > 0$ if we take the "order" of $|x|$ equal to 1). Then the operator $A(\tau)$ differs by negligible terms from the Schrödinger operator

$$B(\tau) = \sum_{j,k}(D_j - V_j)g^{jk}(2M - V + \tau)^{-1}(D_k - V_k) + V - \tau \tag{11.6.65}$$

and moreover, for

$$\delta_\iota < 1 \qquad (\iota = 1, 2) \tag{11.6.66}$$

one can replace this operator by the operator

$$B'(\tau) = (2M)^{-1} \sum_{j,k} (D_j - V_j) g^{jk} (D_k - V_k) + V - \tau. \tag{11.6.67}$$

So we arrive to

Theorem 11.6.21. *Let conditions* (11.6.60)–(11.6.63), $(11.6.64)_{1,2,3}$ *and* (11.6.66) *be fulfilled.*

Furthermore, let assumptions of some statement of Theorem 11.6.7 or Theorem 11.6.9 be fulfilled for the Schrödinger operator $B'(\tau)$ *defined by* (11.6.65) *(or equivalently by* (11.6.67)*); then* $\iota = 1, 2$ *respectively.*

Then the corresponding asymptotics holds for the Dirac operator. Namely, asymptotics (11.6.18)*) holds for* $\delta_\iota < \frac{1}{2}$ *(then condition* (11.6.64)*) is fulfilled automatically) and asymptotics* (11.6.19)*) holds for* $\delta_\iota \in (\frac{1}{2}, 1)$.

In these asymptotics $\mathsf{N}^{\mathsf{W}}(\tau)$ *is calculated for the Dirac operator and* $F(\tau)$ *is calculated for the Schrödinger operator and then it is multiplied by* $\frac{1}{2}\mathsf{D}$.

On the other hand, let condition (11.6.66) be violated but condition $(11.6.63)_{1,2,3}$ be (almost) fulfilled. Then one should replace $B(\tau)$ by

$$B''(\tau) = (2M)^{-1} \sum_{j,k} (D_j - V_j) g^{jk} (D_k - V_k) + V - \tau - (2M)^{-1} (V - \tau)^2 \tag{11.6.68}$$

and in the last term one can replace V by V^0 where V^0 is a smooth function equal to $-1(2M|x|)^{-1}$ for $|x| \ge c$. Changing variables $x \to \zeta x$ and multiplying by $2M\zeta^2$ with $\zeta = 1 - \tau M^{-1}$ we obtain, modulo negligible terms, operator

$$B'''(\tau') = \sum_{j,k} (D_j - V_j) g^{jk} (D_k - V_k) + W - \tau' \tag{11.6.69}$$

with the new potential

$$W = 2MV - (V^0)^2 \tag{11.6.70}$$

and with the modified spectral parameter

$$\tau' = \tau\big(1 - \frac{3\tau}{2M}\big) \tag{11.6.71}$$

This yields that

Theorem 11.6.22. *Let assumptions of some statement of either Theorem 11.6.7 or Theorem 11.6.9 be fulfilled for the Schrödinger operator* $B'''(0)$ *defined by* (11.6.69).

Let conditions (11.6.60)–(11.6.63), $(11.6.63)_{1,2,3}$ *be fulfilled but* $\delta_\iota \in [1, \frac{3}{2})$.

Then asymptotics (11.6.19) *remains true where* $\mathsf{N}^{\mathsf{W}}(\tau)$ *is calculated for the Dirac operator and* $F(\tau')$ *is calculated for the Schrödinger operator* $B'''(\tau')$ *and is then multiplied by* $\frac{1}{2}\mathsf{D}$, *and* τ' *is defined by* (11.6.71).

Let us weaken condition $(11.6.64)_{1,2,3}$.

Remark 11.6.23. For $d = 2$ one can assume without any loss of the generality that $\mathsf{D} = 2$. Then $A(\tau)$ is surely a scalar operator but we have not taken into account some terms. Taking them into account and applying the above analysis we can discard condition $(11.6.64)_{1,2,3}$ completely (in which case we have additional corrections for W and τ').

In the general case we assume that conditions (11.6.60)–(11.6.62) are fulfilled. If $\delta_1 \ge 1$ then the non-scalar terms in $A(\tau)$ are

$$\frac{1}{2}(2M)^{-3} r^{-3} \sum_{l \ne k} \sigma'^{*}_l \sigma'_k F_{lk} \tag{11.6.72}$$

modulo operators of the form $\sum_j b'_j D_j + b'_0$ where b'_j, b'_0 satisfy condition (11.6.60), (11.6.61) with "-2" replaced by "-3." Here we used the skew-symmetry of the matrices $\sigma'^{*}_l \sigma'_k$ with $l \ne k$; recall that $F_{lk} = (\partial_k A_l - = \partial_l A_k)$.

This statement remains true for $\delta_2 \ge 1$ (i.e., for $\delta_1 \ge \frac{1}{2}$) in the framework of Theorem 11.6.7. However, in this case the operator $A(\tau)$ also contains terms of type $b''_j D_j + b''_0$ with b''_j, b''_0 satisfying conditions (11.6.62)–(11.6.63) with "-2" replaced by "$-2-\delta_1$" and this operator skew-commutes with the antipodal map. In both cases these terms are negligible provided

$$\delta_\iota < 2, \quad \delta_\iota < \frac{1}{2} + \delta_3 \quad (\iota = 1, 2). \tag{11.6.73}_{1,2}$$

However, the second inequality is not necessary: for $\iota = 1$ it is fulfilled automatically and for $\iota = 2$ it is associated with terms of the type

$$-i\,\mathsf{const} \sum_{l \ne k} \sigma'^{*}_l \sigma'_l F_{lk}.$$

However, in this case $V_l \circ j^* = V_l$ and therefore $F_{lk} \circ j^* = -F_{lk}$ and the same is true for this extra term. Then one should check the condition $-2-\delta_1 < -1 < -2\delta_1$ which follows from the first inequality.

Let us consider Schrödinger operator with a (perturbed) Coulomb potential. Let us consider

$$\int_0^{T_0} r^{-3}(x_j\xi_k - x_k\xi_j)\circ \Psi_t \, dt$$

where Ψ_t is the Hamiltonian flow generated by $|\xi|^2 - |x|^{-1}$. One can easily see that this integral is equal to $\pi\mu_{jk}\mu^{-3}$ where μ_{jk} are components of the angular momentum tensor

$$\mu_{jk} = (x_j\xi_k - x_k\xi_j) \qquad \text{and} \qquad \mu = \Big(\sum_{j<k}\mu_{jk}^2\Big)^{\frac12}. \tag{11.6.74}$$

This calculation is easy for trajectories lying in the (x_1, x_2)-plane and then one can change coordinates.

Then we obtain that the contribution of the non-scalar terms to the operator $U(\pi,\tau)$ [55] is the multiplication by

$$I - i\pi\sum_{j,k}\sigma_j'^{*}\sigma_k'\mu_{jk}\mu^{-3} + O\big(|\tau|^{\frac32-\delta'}\big)$$

with arbitrarily small $\delta' > 0$. Applying corollary 5.A.7 and calculating the eigenvalues of μ_{jk} we obtain that in order to take into account non-scalar terms for $d \ge 3$ we can treat the family of operators

$$B'''(\tau') + k\nu|\tau|^{\frac32}M^{-2} \tag{11.6.75}$$

with an appropriate constant k and with $\nu \in \{-1, 0, 1\}$ of the corresponding multiplicities where M^s is an operator with the symbol μ^s with $\mu = \mu(x,\xi)$ defined by (11.6.74). One should check conditions for every such operator, calculate and sum $\mathsf{N}_\nu^-(\tau)$ with the corresponding multiplicities.

For $d = 2$ and $\mathsf{D} = 2$ we have $\nu = 0, 1$ and the operators

$$B'''(\tau') + 3\varsigma\nu|\tau'|^{\frac32}\mathcal{M}M^{-3} \tag{11.6.76}$$

with $\mathcal{M} = x_1D_2 - x_2D_1$, $\varsigma = i\sigma_0\sigma_1\sigma_2 = \pm 1$.

We leave the following rather easy problem to the reader:

Problem 11.6.24. Analyze the Schrödinger operator with a (perturbed) Coulomb potential in details.

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[55] Here $U(\pi,\tau) = \exp\big(i\pi(-A(\tau)\big)^{-\frac12}\big)$ and π is the period of the corresponding Hamiltonian flow for $\tau = 0$.

11.7 Multiparameter Asymptotics

11.7.1 Problem Set-up

In this section we consider operators depending on more than one parameter. First of all, we consider semiclassical asymptotics in the case when the scalar semiclassical parameter h is replaced by a vector $\boldsymbol{h} = (h_1, \dots, h_d)$ with $h_j \to +0$.

Next we assume that there are two parameters: the semiclassical parameter $h \to +0$ and the spectral parameter τ. Let us recall that each of the semiclassical asymptotics of Section 11.1 was uniform with respect to the energy level τ lying in the interval in which conditions of the corresponding theorem were fulfilled[56] uniformly. Therefore there are two principally different situations in which non-trivial results are possible. We will explain them for the Schrödinger operator.

In the first situation τ increases and tends to a value τ^* such that the conditions of the corresponding theorem of Section 11.1 are violated for $\tau = \tau^*$ and therefore the asymptotics are no longer uniform. There are two cases:

(a) $\tau \to +\infty$ as in Sections 11.2–11.5 or

(b) $\tau \to \tau^* - 0$ with $\tau^* < \infty$ as in Section 11.6 where in the this case $\tau^* = \liminf_{|x|\to\infty} V$. In this case τ^* is the bottom of the essential spectrum and $\tau^* - 0$ is either a limit point of the discrete spectrum for every fixed $h > 0$ or there is only a finite discrete spectrum for every fixed $h > 0$ and in the second case the number of eigenvalues and (or) the remainder estimate near $\tau^* - 0$ are unusually large ($\gg \mathsf{const} \cdot h^{-d}$ and $\gg \mathsf{const} \cdot h^{1-d}$ respectively).

In all these cases we obtain estimates for the number of eigenvalues and for the remainder which are *worse* than for "normal" τ.

In the second situation τ decreases to the special value $\tau^* = \inf V$ and we obtain estimates for the number of eigenvalues and for the remainder which are *better* than for "normal" τ. Again there are two different cases:

(c) either $\inf V$ is finite in which case $\tau \to \tau^* + 0$ or

[56] In this subsection for the sake of simplicity we consider only the Schrödinger and Dirac operators and we no not need to check non-degeneracy conditions.

(d) $\inf V = -\infty$ but $-\infty$ is not a limit point of the spectrum for every fixed $h > 0$. Then for $h \to +0$ the bottom eigenvalues of the operator "fall" to $-\infty$ and we consider this process. Only the last case is impossible for the Dirac operator.

11.7.2 Semiclassical Asymptotics with a Vector Semiclassical Parameter

Let us consider the Schrödinger operator

$$A = A_{\boldsymbol{h}} = \sum_{j,k} P_j g^{jk} P_k + V, \qquad P_j = h_j - V_j \tag{11.7.1}$$

or the Dirac operator of the same type as before. Changing coordinates $x_{j\,\mathrm{new}} = x_{j\,\mathrm{old}} h_j^{-1}$ we obtain operator

$$A' = A'_{\boldsymbol{h}} = \sum_{j,k} (D_j - V_{j\boldsymbol{h}}) g^{jk}_{\boldsymbol{h}} (D_k - V_{k\boldsymbol{h}}) + V_{\boldsymbol{h}} \tag{11.7.2}$$

in the domain $X_{\boldsymbol{h}} = \{x, f_{\boldsymbol{h}}(x) \in X\}$ where X is the original domain and $v_{\boldsymbol{h}} = v \circ f_{\boldsymbol{h}}$ for every function v where $f_{\boldsymbol{h}}(x) = h \bullet x := (h_1 x_1, \dots, h_d x_d)$, $\boldsymbol{h} = (h_1, \dots, h_d)$. One can easily see that if (some of) the conditions (11.1.6)–(11.1.8), (11.4.5) were fulfilled for $X, \rho, \gamma, g^{jk}, V_j, V$ then the same conditions remain true for $X_{\boldsymbol{h}}, \rho_{\boldsymbol{h}}, h^{-1}\gamma_{\boldsymbol{h}}, V_{j\boldsymbol{h}}, V_{\boldsymbol{h}}$ where $h = \max_j h_j$. Moreover, if the cone condition was fulfilled for the domain X then it remains true for $X_{\boldsymbol{h}}$ provided $h \le 1$. This immediately yields

Theorem 11.7.1. *(i) Let $d \ge 3$ and let conditions* (11.1.6)–(11.1.8) *be fulfilled for $g^{jk}, V_j, V, \rho, \gamma$. Let $\gamma > 0$ a.e. and $V_-^{\frac{d}{2}}$, $\rho^{d-1}\gamma^{-1} \in \mathscr{L}^1(X)$. Then asymptotics*

$$\mathsf{N}^-(A_{\boldsymbol{h}}) = (h_1 \dots h_d)^{-1} \bigl(\mathsf{N}^{\mathsf{W}} + O(h)\bigr) \tag{11.7.3}$$

holds as $h = \max_j h_j \to +0$ where as usual

$$\mathsf{N}^{\mathsf{W}} = (2\pi)^{-d} \omega_d \int V_-^{\frac{d}{2}} \sqrt{g}\, dx. \tag{11.7.4}$$

(ii) Moreover, let us assume that

$$g^{jk} = V_j = 0 \qquad \text{for } \ j \ge d' + 1, k \le d', \tag{11.7.5}$$

$0 < d' < d$ *and that*

$$|\nu'| \ge \epsilon_0 > 0 \qquad \text{on } \partial X \cap \{V \le 0\} \tag{11.7.6}$$

where $\nu = (\nu_1, \dots, \nu_{d'}; \nu_{d'+1}, \dots, \nu_d) = (\nu'; \nu'')$ *is a unit normal to* ∂X. *Then asymptotics*

$$\mathsf{N}^-(A_{\boldsymbol{h}}) = (h_1 \dots h_d)^{-1}\big(\mathsf{N}^{\mathsf{W}} + \varkappa_1 h + o(h)\big) \tag{11.7.7}$$

holds as $h = \max_j h_j \to +0$, $h_j = o(h)$ $\forall j \ge d'+1$ *where*

$$\varkappa_1 = \mp\frac{1}{4}(2\pi)^{1-d}\omega_{d-1}\int_{\partial X} V_-^{\frac{d-1}{2}}\, dS'\sqrt{g''}dx'' \tag{11.7.8}$$

for Dirichlet and Neumann boundary conditions respectively and the Riemannian densities dS', $\sqrt{g''}dx''$ *were introduced in Example 11.5.23(ii) (see* (11.5.68)*).*

(iii) Furthermore, if $\vartheta > 0$, $V_-^{\frac{d}{2}+\vartheta}, \rho^{d-1+\vartheta} \in \mathscr{L}^1(X)$ *we have, in the frameworks of Statements (i) and (ii), asymptotics*

$$\mathsf{N}^-_{(\vartheta)}(A_{\boldsymbol{h}}) = (h_1 \dots h_d)^{-1}\big(\mathsf{N}^{\mathsf{W}}_{\vartheta,\boldsymbol{h}} + O(h^{1+\vartheta})\big) \tag{11.7.9}$$

and

$$\mathsf{N}^-_{(\vartheta)}(A_{\boldsymbol{h}}) = (h_1 \dots h_d)^{-1}\big(\mathsf{N}^{\mathsf{W}}_{\vartheta,\boldsymbol{h}} + o(h^{1+\vartheta})\big) \tag{11.7.10}$$

respectively.

(iv) The same results remain true for $d = 2$, $d' = d'' = 1$ *under additional restrictions* $\mathsf{mes}(Y) < \infty$ *where* $Y = \{X, \rho\gamma \le h_0\} < \infty$ *for some* $h_0 > 0$ *and* $V_-^{p+\vartheta} \in \mathscr{L}^1(X'')$ *for some* $p > 1$.

(v) Under the cone condition all these results remain true for Neumann boundary conditions and also for the Dirac operator (with the reasonable boundary conditions). In the case of Dirac operator we consider eigenvalues lying in the interval $[\tau_1, \tau_2]$ *with arbitrary fixed* τ_1, τ_2.

Proof. Statement (i) follows immediately from Theorem 9.1.7.

Statement (ii) follows from Theorem 9.1.7 with an improved remainder estimates under non-periodicity assumption. Analysis of the Hamiltonian billiards is based on the same arguments as in Example 11.5.23(ii); namely ξ_j

and $h_j x_j$ with $j \ge d'+1$ are "almost constant" (constant modulo $o(1)$) along these billiards (which is not true for $j \le d'$). But then $\frac{d}{dt} x_j \equiv \sum_{k \ge d'+1} g^{jk}\xi_k \mod o(1)$ as $j \ge d'+1$ and for generic ξ non-periodicity condition is fulfilled automatically.

The other Statements (iii)–(v) are proven by a similar method. □

11.7.3 Asymptotics of Eigenvalues near $+\infty$

Here and below we consider the Schrödinger operator A_h and consider the number of its eigenvalues lesser than τ. For the Dirac operator A_h (we use the same notation) we consider the number of eigenvalues lying in the interval $[\tau_1, \tau)$ with fixed τ_1.

Applying the arguments of Sections 11.2–11.5 with $\rho_{\tau,h} = h^{-1}(\rho + \sqrt{\tau})$ and $\gamma_{\tau,h} = \gamma$ we obtain modifications of the statements of these sections. However, we cannot anymore replace $\mathsf{N}^{\mathsf{W}}(\tau)$ by $\varkappa_0 \tau^{\frac{d}{2m}}$ in these assertions. In particular, we obtain the following statements:

Theorem 11.7.2. *(i) For the Schrödinger operator, let conditions* (11.1.6)–(11.1.8) *be fulfilled with auxiliary functions ρ, γ. Let $\vartheta \ge 0$ and*

$$\rho^{d+2\vartheta}, 1, \gamma^{-\vartheta-1} \in \mathscr{L}^1(X). \tag{11.7.11}$$

For $d = 2$ let us assume also that

$$\rho^{2p+2\vartheta} \in \mathscr{L}^1(X) \qquad \text{with some } \ p > 1. \tag{11.7.12}$$

Moreover, in the case of the Neumann condition let us in addition assume that the cone condition is also fulfilled[57]*. Then asymptotics*

$$\mathsf{N}^{-}_{(\vartheta)}(\tau, h) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau, h) + O\bigl(\tau^{\frac{1}{2}(d+\vartheta-1)} h^{1-d+\vartheta}\bigr) \tag{11.7.13}$$

holds as $\tau \to +\infty, h \to +0$. In particular, for $\vartheta = 0$

$$\mathsf{N}^{\mathsf{W}}(\tau, h) = \mathsf{N}^{\mathsf{W}}(\tau) h^{-d}. \tag{11.7.14}$$

(ii) Further, if the set of all points which are periodic with respect to the geodesic flow with reflections is of measure 0 then asymptotics

$$\mathsf{N}^{-}_{(\vartheta)}(\tau, h) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau, h) + o_\tau\bigl(\tau^{\frac{1}{2}(d+\vartheta-1)} h^{1-d+\vartheta}\bigr) \tag{11.7.15}$$

[57] We leave the analysis of mixed boundary conditions to the reader.

holds as $\tau \to +\infty, h \to +0$ *where* $M' = o_\zeta(M)$ *means that* $M'M^{-1} \le \varepsilon(\zeta)$ *with* $\varepsilon(\zeta) = o(1)$ *as* ζ *tends to the described limit.*

In particular, for $\vartheta = 0$ *asymptotics*

$$\mathsf{N}^-(\tau, h) = \mathsf{N}^{\mathsf{W}}(\tau)h^{-d} + \big(\varkappa_1 + o_\tau(1)\big)\tau^{\frac{1}{2}(d-1)}h^{1-d} \tag{11.7.16}$$

holds as $\tau \to +\infty, h \to +0$ *with the coefficient*

$$\varkappa_1 = \mp\frac{1}{4}(2\pi)^{1-d}\omega_{d-1}\,\mathsf{vol}'\big(\partial X \cap \{\gamma > 0\}\big) \tag{11.7.17}$$

for Dirichlet and Neumann boundary conditions respectively.

(iii) Furthermore, let us assume that $\gamma^{-1-\theta} \in \mathscr{L}^1(X)$ *with* $\theta > \vartheta$ *and that the geodesic flow is* <u>*either*</u> *of non-periodic power type* <u>*or*</u> *of non-periodic exponential type. Then these asymptotics hold the remainder estimates*

$$O\big(\tau^{\frac{1}{2}(d+\vartheta-1-\delta')}h^{1-d+\vartheta}\big) \tag{11.7.18}$$

or

$$O\big(\tau^{\frac{1}{2}(d+\vartheta-1)}h^{1-d+\vartheta}\,\mathsf{log}^{-1-\vartheta}\,\tau\big) \tag{11.7.19}$$

respectively.

(iv) All these asymptotics remain true if we drop condition $\rho^{d+2\vartheta} \in \mathscr{L}^1(X)$ *replacing it by the coercivity condition* (11.4.5).

(v) Statements (i)–(iii) remain true for $\vartheta = 0$ *for the Dirac operator*[58]*. In these assertions one should replace* $\mathsf{N}^-(\tau, h)$ *and* $\mathsf{N}^{\mathsf{W}}(\tau_2)$ *by* $\mathsf{N}(\tau_1, \tau_2, h)$ *and* $\mathsf{N}^{\mathsf{W}}(\tau_1, \tau_2)$ *respectively and take* $\tau = |\tau_1| + |\tau_2| \to \infty$*. One should assume that the boundary conditions are elliptic and the domain satisfies the cone condition.*

We leave to the reader the following easy set of problems

Problem 11.7.3. (i) Modify in the same way the complete set of Theorems 11.2.7, 11.2.8, 11.2.12 and Corollary 11.2.9.

(ii) Consider Examples 11.2.13, 11.2.14, 11.2.15, 11.2.17, 11.2.18, 11.2.19, 11.2.21, 11.2.22, 11.2.25, 11.2.26, 11.2.27, 11.3.34, 11.3.35, 11.4.8, 11.4.9, 11.4.10 and translate them into the framework of the present two-parameter theory (automatically or with minor modifications). The same for all other results of this type.

[58] For $\vartheta > 0$ one should modify $\mathsf{N}_{(\vartheta)}(\tau_1, \tau_2, h)$ in the obvious manner.

(iii) Modify Theorems 11.3.7, 11.3.18, 11.3.19, 11.3.21, 11.3.23, 11.3.26 etc.

Here one should assume that $h \ge \tau^{-n}$, and the principal part of the asymptotics is of the form $\mathsf{N}^{\mathsf{W}}(\tau)h^{-d}$ and the second term is of the form

$$\big(\varkappa_1 + F(\tau h^{-m})\big)h^{1-d}\tau^{\frac{1}{2m}(d-1)} \tag{11.7.20}$$

and there is additional factor h^{1-d} in the remainder estimate.

Let us consider the statements of Section 11.5. For the sake of simplicity let us consider only the Schrödinger operator. Then the following statements hold:

Theorem 11.7.4. *(i) In the framework of Theorem 11.5.1 (with the additional assumption that A is the Schrödinger operator) asymptotics*

$$\mathsf{N}^{-}_{(\vartheta)}(\tau, h) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau, h) + O\big(R_{(\vartheta)}(\tau)h^{1-d+\vartheta}\big) \tag{11.7.21}$$

holds as $\tau \to +\infty, h \to +0$ where $R_{(\vartheta)}(\tau)$ is defined by (11.5.3).

(ii) Further, under the additional assumptions of Theorem 11.5.3(i) asymptotics

$$\mathsf{N}^{-}_{(\vartheta)}(\tau, h) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau, h) + o_{\tau}\big(R_{(\vartheta)}(\tau)h^{1-d+\vartheta}\big) \tag{11.7.22}$$

holds as $\tau \to +\infty, h \to +0$.

(iii) Furthermore, in the framework of Theorem 11.5.3(ii) asymptotics

$$\mathsf{N}^{-}_{(\vartheta)}(\tau, h) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau, h) + o_{\tau h^{-2}}\big(R_{(\vartheta)}(\tau)h^{1-d+\vartheta}\big) \tag{11.7.23}$$

holds as $\tau \to +\infty, h \to +0$. Moreover, the principal part in asymptotics (11.7.21) *with $\vartheta = 0$ is $\mathsf{N}^{\mathsf{W}}(\tau)h^{-d}$ and in* (11.7.22), (11.7.23) *the principal part is $\mathsf{N}^{\mathsf{W}}(\tau)h^{-d} + \varkappa_1\tau^{l(d-1)}h^{1-d}$ with the coefficient $\varkappa_1$ given by* (11.5.11).

(iv) In the framework of Theorem 11.5.6 for $\vartheta \le d-1-\delta$ with $\delta > 0$ asymptotics

$$\mathsf{N}^{-}_{(\vartheta)}(\tau, h) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau, h) + O\big(\tau^{l(d-1)+\vartheta(1-l)-\delta'}h^{1-d}\big) \tag{11.7.24}$$

holds as $\tau \to +\infty, h \to +0$. Here for $\vartheta = 0$ the principal part is the same as in (11.7.22). *In particular, Example 11.5.7 applies.*

(v) In the frameworks of Theorems 11.5.14, 11.5.18 and 11.5.20 asymptotics (11.7.21) *remains true. Moreover, under the conditions of these theorems on the Hamiltonian trajectories with reflections asymptotics* (11.7.22) *remains true.*

(vi) In the frameworks of Example 11.5.23(i) and (ii) asymptotics

$$\mathsf{N}(\tau, h) = \mathsf{N}^{\mathsf{W}}(\tau) h^{-d} + O(\tau^{q} h^{1-d}), \tag{11.7.25}$$

and

$$\mathsf{N}(\tau, h) = \mathsf{N}^{\mathsf{W}}(\tau) h^{-d} + \bigl(\varkappa_1 + o_\tau(1)\bigr)\tau^{q} h^{1-d}, \tag{11.7.26}$$

hold respectively with the coefficient $\varkappa_1$ *given by* (11.5.68).

(vii) In the frameworks of Theorem 11.5.25(i) and (ii) the asymptotics

$$\mathsf{N}^{-}_{(\vartheta)}(\tau, h) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) h^{-d} + O\bigl(\tau^{\frac{1}{2}(d+\vartheta-1)} h^{1-d+\vartheta}\omega(\tau)\bigr), \tag{11.7.27}$$

and

$$\mathsf{N}^{-}_{(\vartheta)}(\tau, h) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) h^{-d} + o\bigl(\tau^{\frac{1}{2}(d+\vartheta-1)} h^{1-d+\vartheta}\omega(\tau)\bigr) \tag{11.7.28}$$

hold respectively with

$$N^{\mathsf{W}}_{(\vartheta)}(\tau, h) \asymp \tau^{\frac{d}{2}+\vartheta} \log^{\frac{n}{\mu}} \tau h^{-d} \tag{11.7.29}$$

and $\omega(\tau)$ *defined by* (11.5.74). *For* $\vartheta = 0$ *we have* $\mathsf{N}^{\mathsf{W}}(\tau, h) = \mathsf{N}^{\mathsf{W}}(\tau) h^{-d}$.

Moreover, asymptotics (11.7.28) *remains true in the framework of Theorem 11.5.27 and for* $\vartheta = 0$

$$\mathsf{N}^{\mathsf{W}}_{(0)}(\tau, h) = \mathsf{N}^{\mathsf{W}}(\tau) h^{-d} + \varkappa_1 \tau^{\frac{1}{2}(d-1)} h^{1-d} \log \tau \tag{11.7.30}$$

where the coefficient $\varkappa_1$ *is defined by* (11.5.83).

(viii) In the framework of Theorem 11.5.29(i) asymptotics with the principal part

$$\mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau, h) \asymp \mathsf{N}^{\mathsf{W}}(\tau) h^{-d}, \qquad \mathsf{N}^{w}_{(0)}(\tau, h) = \mathsf{N}^{\mathsf{W}}(\tau) h^{-d} \tag{11.7.31}$$

and with remainder estimate $h^{1-d+\vartheta} R_{(\vartheta)}(\tau)$ *holds where* $\mathsf{N}^{\mathsf{W}}(\tau)$ *and* $R_{(\vartheta)}(\tau)$ *are defined in the theorem.*

Moreover, the improvements made in Statement (ii) and in footnote [44] *remain true.*

We leave to the reader the following easy set of problems:

Problem 11.7.5. (i) Analyze Examples 11.5.17, 11.5.19 and 11.5.21.

(ii) Modify Theorems 11.5.5, 11.5.9 and 11.5.11.

Note that remark in Problem 11.7.3(iii) also remains true in this case.

11.7.4 Asymptotics of Eigenvalues near -0 and Related asymptotics

Applying the arguments of Section 11.6 with $\rho_{\tau,h} = h^{-1}(\rho + \sqrt{|\tau|})$ and $\gamma_{\tau,h} = \gamma$ we obtain

Theorem 11.7.6. *Again, we treat only the Schrödinger operator.*

(i) In the framework of Theorem 11.6.1 asymptotics

$$\mathsf{N}^{-}_{(\vartheta)}(\tau, h) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau, h) + O\big(h^{1-d+\vartheta} R_{(\vartheta)}(\tau)\big) \tag{11.7.32}$$

holds as $\tau \to -0$, $h \to +0$ with $R_{(\vartheta)}(\tau)$ defined by (11.6.4).

(ii) Moreover, in the framework of Theorem 11.6.2(i) asymptotics

$$\mathsf{N}^{-}_{(\vartheta)}(\tau, h) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau, h) + o_{\tau}\big(h^{1-d+\vartheta} R_{(\vartheta)}(\tau)\big) \tag{11.7.33}$$

holds as $\tau \to -0$, $h \to +0$.

For $\vartheta = 0$ the principal parts of these asymptotics are $\mathsf{N}^{\mathsf{W}}(\tau) h^{1-d}$ and

$$\mathsf{N}^{\mathsf{W}}(\tau) h^{1-d} + \varkappa_1 |\tau|^{l(d-1)} h^{1-d} \tag{11.7.34}$$

respectively with the coefficient $\varkappa_1$ given by (11.6.11).

(iii) In the framework of Theorem 11.6.5 with $\vartheta \le \vartheta_1 - \delta$, $\delta > 0$ asymptotics

$$\mathsf{N}^{-}_{(\vartheta)}(\tau, h) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau, h) + O(h^{1-d+\vartheta} |\tau|^{l(d-1)+\vartheta(1-l)+\delta'}) \tag{11.7.35}$$

holds with $\delta' > 0$. In these asymptotics the principal part for $\vartheta = 0$ is $\mathsf{N}^{\mathsf{W}}(\tau) h^{1-d} + \varkappa_1 |\tau|^{l(d-1)} h^{1-d}$. In particular, Example 11.6.6 applies.

(iv) In the framework of Theorem 11.6.10 asymptotics (11.7.32) *remains true and under a condition on Hamiltonian trajectories with reflection asymptotics* (11.7.33) *also remains true.*

(v) In the frameworks of Example 11.6.13(i) and (ii) asymptotics

$$\mathsf{N}^{-}(\tau, h) = \mathsf{N}^{\mathsf{W}}(\tau)h^{-d} + O(|\tau|^{q}h^{1-d}), \tag{11.7.36}$$

and

$$\mathsf{N}^{-}(\tau, h) = \mathsf{N}^{\mathsf{W}}(\tau)h^{-d} + \bigl(\varkappa_1 + o_\tau(1)\bigr)|\tau|^{q}h^{1-d} \tag{11.7.37}$$

hold with the coefficient $\varkappa_1$ *given by* (11.6.35).

(vi) In the frameworks of Theorems 11.6.14(i) and (ii) asymptotics

$$\mathsf{N}(\tau, h) = \mathsf{N}^{\mathsf{W}}(\tau)h^{-d} + O\bigl(\bigl|\log|\tau|\bigr|^{\nu(d-1)+1}h^{1-d}\bigr), \tag{11.7.38}$$

and

$$\mathsf{N}(\tau, h) = \mathsf{N}^{\mathsf{W}}(\tau)h^{-d} + \bigl(\varkappa_1 + o_\tau(1)\bigr)\bigl|\log|\tau|\bigr|^{\nu(d-1)+1}h^{1-d} \tag{11.7.39}$$

respectively hold with the coefficient $\varkappa_1$ *given by* (11.6.40).

(vii) Moreover, Statement (vi) remains true for $\nu = 0$ *with* $o_\tau(1)$ *replaced by* $o(1)$ *in asymptotics* (11.7.39).

Remark 11.7.7. Statement Statement (vii) dealing with potentials decaying as $|x|^{-2}$ has no analog in Section 11.6 because for $\nu = 0$ for every small enough fixed $h > 0$ the eigenvalues of A_h tend to -0 but in (11.7.38) both the principal part and the remainder estimate are of the same order $\log|\tau|$ and (11.7.38) is only an estimate (or a couple of estimates: from above and below), not asymptotics.

We leave to the reader

Problem 11.7.8. (i) Modify Example 11.6.11.

(ii) In the framework of Problem 11.6.15 derive asymptotics with the principal parts

$$N^{\mathsf{W}}_{(\vartheta)}(\tau, h) \asymp \mathsf{N}^{W}_{\vartheta,h}(\tau)h^{-d}, \qquad N^{\mathsf{W}}_{(0)}(\tau, h) = \mathsf{N}^{\mathsf{W}}(\tau)h^{-d}$$

and remainder estimate $O\bigl(h^{1-d+\vartheta}R_{(\vartheta)}(\tau)\bigr)$ where $\mathsf{N}^{\mathsf{W}}(\tau)$, $R_{(\vartheta)}(\tau)$ are defined in that problem.

(iii) Modify Theorems 11.6.4, 11.6.7 and 11.6.9.

Note that remark in Problem 11.7.3(iii) also remains true in this case as $h \ge |\tau|^{n}$.

Let us consider the situation in which the conditions of Theorem 11.6.14 are fulfilled with $\nu < 0$ instead of $\nu > 0$.

In this case for every fixed $h > 0$ the discrete spectrum is finite. However, $V_-^{\frac{d}{2}} \notin \mathscr{L}^1(X)$ for $\nu \geq -d^{-1}$ provided condition (11.6.8) is fulfilled. In this case the above arguments lead us to the following

Theorem 11.7.9. *(i) Under the assumptions of Theorem 11.6.14(i) with* $\nu \in (-d^{-1}, 0)$ *asymptotics* (11.7.38) *holds for the Schrödinger operator for* $\tau \to -0$, $h \to +0$ *provided*

$$h = o\big(\big|\log|\tau|\big|^{\nu}\big). \tag{11.7.40}$$

In this asymptotics $\mathsf{N}^{\mathsf{W}}(\tau) = O\big(\big|\log|\tau|\big|^{\nu d+1}\big)$ *and under condition* (11.6.8) $\mathsf{N}^{\mathsf{W}}(\tau) \asymp \big|\log|\tau|\big|^{\nu d+1}$.

(ii) Under the assumptions of Theorem 11.6.14(i) with $\nu = -d^{-1}$ *asymptotics* (11.7.38) *holds for the Schrödinger operator for* $\tau \to -0$, $h \to +0$ *provided*

$$h = O\big(\big|\log|\tau|\big|^{\nu}\big). \tag{11.7.41}$$

In this asymptotics $\mathsf{N}^{\mathsf{W}}(\tau) = O\big(\log\big|\log|\tau|\big|\big)$ *and under condition* (11.6.8) $\mathsf{N}^{\mathsf{W}}(\tau) \asymp \log\big|\log|\tau|\big|$.

(iii) Under the assumptions of Theorem 11.6.14(ii) with $\nu \in [-d^{-1}, 0)$ *asymptotics* (11.7.39) *holds for the Schrödinger operator for* $\tau \to -0$, $h \to +0$ *satisfying* (11.7.40). *In this asymptotics coefficient* $\varkappa_1$ *is given by* (11.6.40).

Let us consider the case

$$h \geq \epsilon_0\big|\log|\tau|\big|^{\nu}. \tag{11.7.42}$$

Then the singular zone $X_{\mathsf{sing}} = \big\{(\rho + \sqrt{|\tau|})\gamma \leq \epsilon h\big\}$ is not necessarily empty. Considering only the lower estimate we obtain that under condition (11.6.8)

$$\mathsf{N}^-(\tau, h) \geq \epsilon_1 h^{\frac{1}{\nu}} \qquad \text{for } \nu \in (-\frac{1}{d}, 0), \tag{11.7.43}$$

and

$$\mathsf{N}^-(\tau, h) \geq \epsilon_1 h^{\frac{1}{\nu}} |\log h| \qquad \text{for } \nu = -\frac{1}{d} \tag{11.7.44}$$

with $\epsilon_1 > 0$.

On the other hand, for $d \ge 3$ one can apply a variational estimate in X_{sing} (with $\rho = 0$ for $\log^\nu r \le \epsilon_2 h$. Then we obtain estimate

$$\mathsf{N}^-(\tau, h) = O(h^{\frac{1}{\nu}}) \qquad \text{for } \ \nu \in (-\frac{1}{d}, 0), \tag{11.7.45}$$

and even asymptotics

$$\mathsf{N}^-(\tau, h) = \big(\mathsf{N}^{\mathsf{W}}_h(\tau) + O(1)\big) h^{-d} \qquad \text{for } \nu = -\frac{1}{d} \tag{11.7.46}$$

for $\tau \to -0$, $h \to +0$ with

$$\mathsf{N}^{\mathsf{W}}_h(\tau) = (2\pi)^{-d}\omega_d \int_{\{|\log|x|| \le Ch^{-\frac{1}{\nu}}\}} \big(-\tau - V\big)^{\frac{d}{2}}_+ \sqrt{g}\, dx \tag{11.7.47}$$

with an arbitrarily chosen constant C. Under condition (11.6.8) this expression is $\asymp \min\big(\log\big|\log|\tau|\big|, |\log h|\big)$ for $\nu = -d^{-1}$.

Let us recall that the asymptotics and estimates (11.7.43)–(11.7.46) were obtained under the restriction (11.7.42).

Thus we arrive to

Remark 11.7.10. (i) If $d \ge 3$, $\nu \in (-d^{-1}, 0)$ then under condition (11.6.8) $\mathsf{N}^-(\tau, h) \asymp h^{\frac{1}{\nu}}$ for $\tau \to -0$, $h \to +0$ satisfying (11.7.42). One can easily obtain this result by purely variational arguments.

(ii) For $d \ge 3$, $\nu = -d^{-1}$ we combine Statement (ii) of Theorem 11.7.9 and (11.7.46) to obtain

$$\mathsf{N}^-(\tau, h) = \mathsf{N}^{\mathsf{W}}_h(\tau) h^{-d} + O\Big(\min\big(h^{-d}, h^{1-d}\big|\log|\tau|\big|^{\frac{1}{d}}\big)\Big) \tag{11.7.48}$$

with $\mathsf{N}^{\mathsf{W}}_h(\tau)$ given by (11.7.47) and $\tau \to -0$, $h \to +0$.

We leave to the reader the following

Problem 11.7.11. (i) Treat the case $\nu \in [-(d-1)^{-1}, -d^{-1})$. In this case the principal part is standard and $\asymp h^{-d}$ but the remainder estimate is worse than usual. In this case we also obtain asymptotics for $\tau = 0$ in the context of Section 11.1.

(ii) rove that all the results of this subsection remain true for the Dirac operator with

$$\tau_1^0 = -M + \limsup_{|x|\to\infty} V < \tau_2^0 = M + \liminf_{|x|\to\infty} V. \tag{11.7.49}$$

One need only to replace $\mathsf{N}^-(\tau; h)$ by $\mathsf{N}(\tau_1, \tau; h)$ with fixed $\tau_1 \in (\tau_1^0, \tau_2^0)$ and respectively modify the formulas for N^{W}, $\mathsf{N}_h^{\mathsf{W}}$, $\varkappa_0$ and $\varkappa_1$; without any loss of the generality one can assume that $\tau_2^0 = 0$ and then $\tau \to -0$.

11.7.5 Asymptotics of Eigenvalues near $+0$

Arguments of the same type as above permit us also to treat the case $\tau \to +0$, $h \to +0$ and to obtain asymptotics of $\mathsf{N}^-(\tau, h)$ assuming that $\inf V = 0$ for the Schrödinger operator.

Similar results may be obtained for the Dirac operator for $\mathsf{N}(\tau_1, \tau; h)$ provided (11.7.49) holds, $\tau_1 \in (\tau_1^0, \tau_2^0)$ and

$$\tau_1^0 = -M + \sup V < \tau_1 \le \tau_2^0 = M + \inf V;$$

without any loss of the generality one can assume that $\tau_2^0 = 0$ and then $\tau \to +0$.

We formulate our statements only for the Schrödinger operator with Dirichlet boundary conditions. For the the Schrödinger operator with Neumann boundary conditions and for the Dirac operator with any self-adjoint boundary conditions these statements remain true as long as the domain X satisfies the cone condition.

It is easy to prove the following

Theorem 11.7.12. *(i) Let X be a bounded domain and let Y a $\mathscr{C}^1$ manifold of codimension d'' in $\mathbb{R}^d$, $2 \le d'' \le d$ such that $\bar{X} \cap Y \ne \emptyset$.*

Let conditions (11.1.6)–(11.1.8), (11.4.5) *be fulfilled with $\gamma = \epsilon r(x)$, $\rho(x) = r^\mu(x)$, $\epsilon > 0$, $\mu > 0$, $r(x) = \operatorname{dist}(x, Y)$. Then asymptotics*

$$\mathsf{N}^-(\tau, h) = \mathsf{N}^{\mathsf{W}}(\tau) h^{-d} + O\bigl(\tau^q h^{1-d}\bigr) \tag{11.7.50}$$

holds for $\tau \to +0$, $h \to +0$ with $\mathsf{N}^{\mathsf{W}}(\tau) = O(\tau^p)$, $p = \frac{1}{2}d + \frac{1}{2\mu}d''$ and $q = \frac{1}{2}(d-1) + \frac{1}{2\mu}(d''-1)$ [59),60)].

[59)] Surely we obtain asymptotics only for $h = o(\tau^l)$ with $l = \frac{1}{2\mu}(\mu+1)$, i.e., only if τ tends to $+0$ slowly enough. Otherwise we obtain only an upper estimate and for $h \ge \epsilon_0 \tau^l$ with a small enough constant ϵ_0 we obtain a lower estimate for $\mathsf{N}^-(\tau, h)$. This remark remains true for other asymptotics of this subsection.

[60)] We did not study the lowest eigenvalue but under rather general assumptions one can prove that it is $\tau_1 > 0 : \tau_1 \asymp h^{1/l}$.

Moreover, $\mathsf{N}^{\mathsf{W}}(\tau) \asymp \tau^{p}$ *provided*

$$\operatorname{vol}\bigl(X \cap \{r(x) < \bar{r}\}\bigr) \asymp r^{n} \quad \text{as } r \to 0 \tag{11.7.51}$$

with $n = d''$.

(ii) Further, let us assume that $Y = \{0\}$, $X \cap \{|x| \le r_0\} = X^0 \cap \{|x| \le r_0\}$ *where* $r_0 > 0$ *and* X^0 *is the non-empty open cone with* $\partial X^0 \setminus 0 \in \mathscr{C}^{K}$.

Let stabilization conditions (11.5.4) *be fulfilled for* $|x| \to 0$ *with functions* g^{jk0}, V_j^0 *and* V^0 *positively homogeneous of degrees* 0, μ *and* 2μ *respectively.*

Finally, let the standard non-periodicity condition be fulfilled for the domain X^0 *and the Hamiltonian*

$$H^0_{\mathsf{st}} = \sum_{j,k} g^{jk0}(\xi_j - V_j^0)(\xi_k - V_k^0) + V^0 - 1 \tag{11.7.52}$$

on the energy level 0. *Then the asymptotics*

$$\mathsf{N}^{-}(\tau, h) = \mathsf{N}^{\mathsf{W}}(\tau) h^{-d} + \bigl(\varkappa_1 + o(1)\bigr)\tau^{l(d-1)} h^{1-d} \tag{11.7.53}$$

holds as $\tau \to +0$, $h = o(\tau^{l})$ *with the coefficient* $\varkappa_1$ *given by* (11.5.11).

Note that under the stabilization conditions alone, $\mathsf{N}^{\mathsf{W}} = \bigl(\varkappa_0 + o(1)\bigr)\tau^{ld}$ *with the coefficient* $\varkappa_0$ *given by* (11.5.10).

(iii) Finally, let the stabilization conditions

$$D^{\alpha}(a_\beta - a^0_\beta) = O\bigl(|x|^{(2m-|\beta|)\mu - |\alpha| + \delta}\bigr) \qquad \forall \beta : |\beta| \le 2m \quad \forall \alpha : |\alpha| \le 1 \tag{11.7.54}$$

be fulfilled as $|x| \to 0$ *with some* $\delta > 0$. *Let the Hamiltonian flow generated by* H^0_{st} *near* 0 *be of non-periodic power type. Then the asymptotics*

$$\mathsf{N}^{-}(\tau, h) = \mathsf{N}^{\mathsf{W}}(\tau) h^{-d} + \bigl(\varkappa_1 + O(h^{\delta'}\tau^{-l\delta'})\bigr) h^{1-d}\tau^{l(d-1)} \tag{11.7.55}$$

with $\delta' > 0$ *holds for* $\tau \to +0$, $h = o(\tau^{l})$.

We leave to the reader the following easy Problem 11.7.13 and the proofs of Theorems 11.7.14–11.7.16 below.

Problem 11.7.13. (i) Generalize Statement (ii) of Theorem 11.7.12 to the case $d'' < d$. The arguments of Example 11.5.23(ii) should be applied here.

(ii) Consider the case of periodic Hamiltonian flow.

(iii) Consider the case $d''=1$ when the logarithmic factor appears in the remainder estimate.

(iv) Consider the general operators and the case $\vartheta>0$.

Theorem 11.7.14. *Let X be a bounded domain, $0\in X$ and*

$$X\cap\{|x|\le r_0\}=X^0\cap\{|x|\le r_0\} \tag{11.7.56}$$

where X^0 is an L-quasiconical domain in $\mathbb{R}^d$. Let g^{jk}, V_j and V be L-quasihomogeneous with respect to x of degrees 0, μ, 2μ respectively with $\mu>0$, $L=(L_1,\dots,L_d)$, $L_j=1$ for $j\le d'$ and $L_j<1$ for $j\ge d'+1$, $0<d'<d$.

Let us assume that $\partial X\setminus 0\in \mathscr{C}^K$ and $g^{jk}, V_j, V\in \mathscr{C}^K(\bar X\setminus 0)$ satisfy condition (11.1.8)[61] *Finally, let us assume that*

$$V(x)>0\qquad \forall x\in \bar X\setminus 0. \tag{11.7.57}$$

Then

(i) Asymptotics

$$\mathsf{N}^-(\tau,h)=\varkappa_0\tau^p h^{-d}+O(\tau^q h^{1-d}) \tag{11.7.58}$$

holds for $\tau\to+0$, $h\to+0$ with $p=\frac12 d+\frac{1}{2\mu}L^*$, $q=\frac12(d-1)+\frac{1}{2\mu}(L^*-1)$, $L^*=L_1+\dots+L_d$ and with the coefficient $\varkappa_0$ given by (11.5.63) *with X replaced by X^0.*

(ii) Under the additional conditions of Example 11.5.23(ii) asymptotics

$$\mathsf{N}^-(\tau,h)=\varkappa_0\tau^p h^{-d}+\bigl(\varkappa_1+o(1)\bigr)\tau^q h^{1-d} \tag{11.7.59}$$

holds for $\tau\to-0$, $h=o(\tau^{\frac12+\frac{1}{2\mu}})$ with the coefficient $\varkappa_1$ given by (11.5.67).

Theorem 11.7.15. *Let X be a bounded domain and let Y be a $\mathscr{C}^1$ manifold of codimension d'' in $\mathbb{R}^d$, $2\le d''\le d$, $\bar X\cap Y\ne\emptyset$.*

Let conditions (11.1.6)–(11.1.8), (11.4.5) *be fulfilled with $\gamma=\epsilon r(x)$, $\rho=\exp\bigl(-r(x)^\nu\bigr)$ with $\nu<0$, $r(x)=\operatorname{dist}(x,Y)$.*

Let us assume that condition (11.7.51) *is fulfilled with an exponent n which is now not necessarily equal to d''. Then*

[61] Then one can take $\gamma=\epsilon[x]_L$, $\rho=\epsilon[x]_L^\mu$.

(i) Asymptotics

$$\mathsf{N}^-(\tau,h)=\mathsf{N}^{\mathsf{W}}(\tau)h^{-d}+O\bigl(\tau^{\frac{1}{2}(d-1)}|\log\tau|^{(\nu+n-1)\nu^{-1}}h^{1-d}\bigr) \tag{11.7.60}$$

holds for $\tau\to+0$, $h\to+0$ *and* $\mathsf{N}^{\mathsf{W}}(\tau)\asymp\tau^{\frac{1}{2}d}|\log\tau|^{n\nu^{-1}}$.

(ii) Moreover, let $Y=\{0\}$, $n=d$. *Let condition* (11.7.59) *be fulfilled for* $r\to+0$ *and every fixed* k *and let conditions* (11.5.76)–(11.5.77) *be fulfilled for* $|x|\to 0$. *Then asymptotics*

$$\mathsf{N}^-(\tau,h)=\mathsf{N}^{\mathsf{W}}(\tau)h^{-d}+o\bigl(\tau^{\frac{1}{2}(d-1)}|\log\tau|^{(\nu+n-1)\nu^{-1}}h^{1-d}\bigr) \tag{11.7.61}$$

holds for $\tau\to+0$, $h=o\bigl(\tau^{\frac{1}{2}}|\log\tau|^{(1-\nu)\nu^{-1}}\bigr)$.

Theorem 11.7.16. *Let* $d\ge 3$, $X=(0,\frac{1}{2})\times U$ *where* U *is a domain on* $\mathbb{S}^{d-1}$ *with* $\mathscr{C}^K$ *boundary,* $g^{jk}=g^{jk}(\theta)\in\mathscr{C}^K(\bar U)$ *satisfy* (11.4.5). *Let* $V_j=0$, $V=v(\theta)|\log r|^{-1}$ *with* $v\in\mathscr{C}^K(\bar U)$ *and* $\min_{\bar U}v=v_0>0$. *Then asymptotics*

$$\mathsf{N}^-(\tau,h)=\mathsf{N}^{\mathsf{W}}(\tau)h^{-d}+O\Bigl(h^{1-d}\tau^{d-2}\int_U\exp\bigl(-v(d-1)\tau^{-1}\bigr)\,d\theta\Bigr) \tag{11.7.62}$$

holds as $\tau\to+0$, $h\to+0$ *with*

$$\mathsf{N}^{\mathsf{W}}(\tau)=a\int_{U'}\tau^d v^{-\frac{d}{2}}F_d\bigl(\tfrac{\tau}{v}\bigr)\exp(-vd\tau^{-1})\sqrt{g}\,d\theta \tag{11.7.63}$$

where $U'=U\cap\{v<v_0+\varepsilon\}$, $\varepsilon>0$ *is arbitrarily small, and* a *and* F_d *are introduced in Theorem 11.6.14 in which the orders of* $\mathsf{N}^{\mathsf{W}}(\tau)$ *and the remainder estimate are also discussed.*

11.7.6 Asymptotics of Eigenvalues near $-\infty$

Let us consider asymptotics of $\mathsf{N}(\tau,h)$ for $\tau\to-\infty$, $h\to+0$. Recall that we consider the Schrödinger operator and now we assume that

$$\inf V=-\infty; \tag{11.7.64}$$

thus it is still the case"$\tau\searrow\inf V$". The results of this subsection surely do not make any sense for the Dirac operator. The same arguments as above give us the following

Theorem 11.7.17. *(i) Let X be a bounded domain in $\mathbb{R}^d$, $0 \in \bar{X}$ and let conditions* (11.1.6)–(11.1.8) *be fulfilled with $\gamma = \epsilon|x|$, $\rho = |x|^\mu$, $\mu \in (-1, 0)$. Then asymptotics*[59),60)]

$$\mathsf{N}^-(\tau, h) = \mathsf{N}^{\mathsf{W}}(\tau)h^{-d} + O\big(|\tau|^{l(d-1)}h^{1-d}\big) \tag{11.7.65}$$

holds for $\tau \to -\infty$, $h \to +0$ and $\mathsf{N}^{\mathsf{W}}(\tau) = O(|\tau|^{ld})$.

Moreover, $\mathsf{N}^{\mathsf{W}}(\tau) \asymp |\tau|^{ld}$ provided

$$\mathsf{mes}\big(\{x \in X, |x| \le r, V \le -\epsilon_0\rho^2(x)\}\big) \asymp r^d \qquad \text{for } r \to +0. \tag{11.7.66}$$

(ii) Further, let $X \cap \{|x| \le r_0\} = X^0 \cap \{|x| \le r_0\}$ with some $r_0 > 0$ and an open non-empty cone X^0 such that $\partial X^0 \setminus 0 \in \mathscr{C}^K$. Let stabilization conditions (11.5.4) *be fulfilled for $|x| \to 0$ with functions $g^{jk0}, V_j^0, V^0 \in \mathscr{C}^K(\bar{X}^0 \setminus 0)$.*

Let the standard non-periodicity condition to the Hamiltonian flow be fulfilled on the energy level 0 for the Hamiltonian

$$H^0(x, \xi) = |x|^{1-\mu}\big(\sum_{j,k} g^{jk0}(\xi_j - V_j^0)(\xi_k - V_k^0) + V^0 + 1\big). \tag{11.7.67}$$

Then asymptotics

$$\mathsf{N}^-(\tau, h) = \mathsf{N}^{\mathsf{W}}(\tau)h^{-d} + \big(\varkappa_1 + o(1)\big)|\tau|^{l(d-1)}h^{1-d} \tag{11.7.68}$$

holds for $\tau \to -\infty$, $h = o\big(|\tau|^l\big)$ with the coefficient $\varkappa_1$ given by (11.6.11).

(iii) Furthermore, let stabilization conditions (11.7.54) *with $\delta > 0$ be fulfilled for $x \to 0$ and let the Hamiltonian flow corresponding to the Hamiltonian*

$$H^0_{\mathsf{st}}(x, \xi) = \sum_{j,k} g^{jk0}\big(\xi_j - V_j^0\big)(\xi_k - V_k^0) + V^0 + 1 \tag{11.7.69}$$

be of power non-periodic type near the energy level 0. Then asymptotics

$$\mathsf{N}^-(\tau, h) = \mathsf{N}^{\mathsf{W}}(\tau)h^{-d} + \big(\varkappa_1 + O(h^{\delta'}|\tau|^{-l\delta'})\big)|\tau|^{l(d-1)}h^{1-d} \tag{11.7.70}$$

with $\delta' > 0$ holds for $\tau \to -\infty$, $h \to +0$.

We leave to the reader the following easy Problem 11.7.18 and Theorems 11.7.19, 11.7.20 and 11.7.22 below.

Problem 11.7.18. Treat the case of a singularity at a manifold of codimension less than d.

Theorem 11.7.19. *(i) Let conditions of Theorem 11.7.14 be fulfilled with $\mu \in (-1,0)$. Then asymptotics*

$$\mathsf{N}^-(\tau,h) = \varkappa_0 |\tau|^p h^{-d} + O\big(|\tau|^q h^{1-d}\big) \tag{11.7.71}$$

holds as $\tau \to -\infty$, $h \to +0$ with the same exponents p, q and coefficient $\varkappa_0$ as in Theorem 11.7.14 (i.e. given by (11.5.63) *with X replaced by X^0).*

(ii) Further, under the additional restrictions of Theorem 11.7.14(ii) asymptotics

$$\mathsf{N}^-(\tau,h) = \varkappa_0 |\tau|^p h^{-d} + \big(\varkappa_1 + o(1)\big)|\tau|^q h^{1-d} \tag{11.7.72}$$

holds for $\tau \to -\infty$, $h = o\big(|\tau|^{\frac{1}{2\mu}(\mu+1)}\big)$ with the coefficient $\varkappa_1$ given by (11.5.68).

Theorem 11.7.20. *Let X be a bounded domain in $\mathbb{R}^d$, $0 \in \bar{X}$ and let conditions* (11.1.6)–(11.1.8) *be fulfilled with $\gamma = \epsilon|x|$, $\rho = |x|^{-1}\big(\big|\log|x|\big|+1\big)^{\nu}$, $\nu \le -d^{-1}$. Then*

(i) For $\nu < -(d-1)^{-1}$ asymptotics

$$\mathsf{N}^-(\tau,h) = \mathsf{N}^{\mathsf{W}}(\tau) h^{-d} + O\big((\log|\tau|)^{\nu(d-1)+1} h^{1-d}\big) \tag{11.7.73}$$

holds for $\tau \to -\infty$, $h \to +0$.

(ii) For $\nu = -(d-1)^{-1}$ the asymptotics

$$\mathsf{N}^-(\tau,h) = \mathsf{N}^{\mathsf{W}}(\tau) h^{-d} + O\big(\log(\log|\tau| \cdot h^{-\nu^{-1}}) h^{1-d}\big) \tag{11.7.74}$$

holds for $\tau \to -\infty$, $h = o(\log^{\nu}|\tau|)$.

(iii) Furthermore, let $X \cap \{|x| \le r_0\} = X^0 \cap \{|x| \le r_0\}$ with $r_0 > 0$ and a non-empty open cone X^0 with $\partial X^0 \setminus 0 \in \mathscr{C}^K$. Let the functions g^{jk}, $V_j\big|\log|x|\big|^{-\nu}$, $V\big|\log|x|\big|^{-2\nu}$ satisfy stabilization conditions (11.5.4) *for $|x| \to 0$ with $\mu = -1$ and with functions g^{jk0}, V_j^0, V^0 positively homogeneous of degrees 0, -1, -2 respectively.*

Moreover, let the standard non-periodicity condition be fulfilled with the Hamiltonian

$$H^0(x,\xi) = |x|^2\Big(\sum_{j,k} g^{jk0}(\xi_j - V^0_j)(\xi_k - V^0_k) + V^0\Big) \tag{11.7.75}$$

on the energy level 0. *Then in the framework of Statement (i) asymptotics*

$$\mathsf{N}^-(\tau,h) = \mathsf{N}^{\mathsf{W}}(\tau)h^{-d} + \big(\varkappa_1 + o(1)\big)(\log|\tau|)^{\nu(d-1)+1}h^{1-d} \tag{11.7.76}$$

holds for $\tau\to-\infty$, $h = o(\log^\nu|\tau|)$ *and in the framework of Statement (ii) asymptotics*

$$\mathsf{N}^-(\tau,h) = \mathsf{N}^{\mathsf{W}}(\tau)h^{-d} + \big(\varkappa_1 + o(1)\big)\log(\log|\tau|\cdot h^{-\nu^{-1}})h^{1-d} \tag{11.7.77}$$

holds for $\tau\to-\infty$, $h = o(\log^\nu|\tau|)$.

(iv) Furthermore, for $\nu\in(-(d-1)^{-1},-d^{-1})$ *asymptotics*

$$\mathsf{N}^-(\tau,h) = \mathsf{N}^{\mathsf{W}}(\tau)h^{-d} + O\big(h^{\nu^{-1}}\big) \tag{11.7.78}$$

holds for $\tau\to-\infty$, $h = o(\log^\nu|\tau|)$.

In all these asymptotics $\mathsf{N}^{\mathsf{W}}(\tau) = O(\log^{\nu d+1}|\tau|)$ *and* $\mathsf{N}^{\mathsf{W}}(\tau)\asymp\log^{\nu d+1}|\tau|$ *under condition* (11.7.66).

(v) Let $\nu = -d^{-1}$. *Furthermore, suppose that* <u>*either*</u> $d\ge 3$ <u>*or*</u> $0\in\partial X$ *and a Dirichlet condition is given on* ∂X. *Then asymptotics*

$$\mathsf{N}^-(\tau,h) = \big(\mathsf{N}^{\mathsf{W}}(\tau,h) + O(1)\big)h^{-d} \tag{11.7.79}$$

holds for $\tau\to-\infty$, $h = o(\log^\nu|\tau|)$ *and* $\mathsf{N}^{\mathsf{W}}(\tau,h)$ *is give by* (11.7.47), $\mathsf{N}^{\mathsf{W}}(\tau,h) = O\big(\log(\log|\tau|\cdot h^{\nu^{-1}})\big)$.

Moreover, under condition (11.7.66) $\mathsf{N}^{\mathsf{W}}(\tau,h)\asymp\log(\log|\tau|\cdot h^{\nu^{-1}})$.

Remark 11.7.21. In the proof of the last statement of this theorem we also use the arguments of Remark 11.1.13(i).

Theorem 11.7.22. *Let* $d\ge 3$, X *be a conical domain with* $\partial X\setminus 0\in\mathscr{C}^K$, *and let* $g^{jk} = g^{jk}(\theta)\in\mathscr{C}^K(\mathbb{S}^{d-1})$ *satisfy condition* (11.4.5). *Let* $V_j = 0$, $V = v(\theta)|\log r|$ *with* $v\in\mathscr{C}^K(\mathbb{S}^{d-1})$ *and* $\min_{\bar X} v = v_0 < 0$. *Then asymptotics*

$$\mathsf{N}^-(\tau,h) = \mathsf{N}^{\mathsf{W}}(\tau)h^{-d} + O\Big(h^{1-d}\tau\int_U \exp\big(-\tau(d-1)v^{-1}\big)\,d\theta\Big) \tag{11.7.80}$$

holds as $\tau \to -\infty$, $h \to +0$ *with*

$$\mathsf{N}^{\mathsf{W}}(\tau) = a \int_U \exp(-\tau dv^{-1}) \sqrt{g}\, d\theta \tag{11.7.81}$$

where $U = X \cap \mathbb{S}^{d-1}$ *and the coefficient* a *is introduced in Theorem 11.5.29. The orders of* $\mathsf{N}^{\mathsf{W}}(\tau)$ *and the remainder estimate and more accurate results are discussed in Statement (ii) of that theorem.*

11.8 Distributions of Spectra for Negative Order Operators

Let $A = A^{\mathsf{w}}(x, D)$ be a classical pseudodifferential operator of order $m < 0$ on a compact closed manifold X. We are interested in the asymptotics of the eigenvalue counting function

$$\mathsf{N}^{\pm}(\tau) = \#\{\lambda > \tau,\ \pm\lambda \text{ is an eigenvalue of } A\} \tag{11.8.1}$$

and

$$\mathsf{N}^{\pm}_{(\vartheta)}(\tau) = \int (\lambda - \tau)^{\vartheta}_{+} d_{\lambda} \mathsf{N}^{\pm}(\lambda) \tag{11.8.2}$$

as $\tau \to +0$. First of all, the following theorem holds:

Theorem 11.8.1. *(i) Let* $a(x, \xi)$ *be a microhyperbolic symbol on the energy level* 0 *(in a direction depending on the point). Then asymptotics*

$$\mathsf{N}^{\pm}(\tau) = \varkappa^{\pm} \tau^{\frac{d}{m}} + O\bigl(\tau^{\frac{d-1}{m}}\bigr) \tag{11.8.3}$$

holds as $\tau \to +0$ *with*

$$\varkappa^{\pm} = (2\pi)^{-d} \int \boldsymbol{n}^{\pm}(x, \xi; 1)\, dx d\xi \tag{11.8.4}$$

where $\boldsymbol{n}^{\pm}(x, \xi; \tau)$ *is an eigenvalue counting function for* $a(x, \xi)$ *i.e. the number of eigenvalues of the symbol* $\pm a(x, \xi)$ *greater than* τ *and* a *is the principal symbol of* A.

(ii) Moreover, asymptotics

$$\mathsf{N}^{\pm}_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W},\pm}_{(\vartheta)}(\tau) + O\bigl(R_{(\vartheta)}(\tau)\bigr) \tag{11.8.5}$$

holds with

$$\mathsf{N}^{\mathsf{W},\pm}_{(\vartheta)}(\tau) = \sum_k \varkappa^{\pm}_{\vartheta k} \tau^{\frac{d-k}{m}} \tag{11.8.6}$$

and

$$R_{(\vartheta)}(\tau) = \begin{cases} \tau^{\frac{1}{m}(d-1-\vartheta)+\vartheta} & \text{for } \vartheta(1-m) < d-1, \\ |\log \tau| & \text{for } \vartheta(1-m) = d-1, \\ 1 & \text{for } \vartheta(1-m) > d-1. \end{cases} \tag{11.8.7}$$

(iii) Further, let $\vartheta(1-m) < d-1$ *and let the standard non-periodicity condition be fulfilled for the Hamiltonian trajectories of the symbol* $\pm a$ *on the energy level* 1*. Then asymptotics*

$$\mathsf{N}^{\pm}(\tau) = \varkappa^{\pm}\tau^{\frac{d}{m}} + \bigl(\varkappa_1 + o(1)\bigr)\tau^{\frac{d-1}{m}} \tag{11.8.8}$$

and asymptotics

$$\mathsf{N}^{\pm}_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W},\pm}_{(\vartheta)}(\tau) + o\bigl(R_{(\vartheta)}(\tau)\bigr) \tag{11.8.9}$$

hold as $\tau \to +0$*.*

(iv) Furthermore, let $\vartheta(1-m) = d-1$ *and the standard non-periodicity condition be fulfilled for the Hamiltonian trajectories of the symbol* $\pm a$ *on the energy level* 0*. Then asymptotics* (11.8.9) *holds.*

(v) On the other hand, if the symbol $\pm a(x,\xi)$ *is negative-definite for every* (x,ξ) *(which under the microhyperbolicity condition means precisely that* $\varkappa^{\pm} = 0$*) then*

$$\mathsf{N}^{\pm}(\tau) = O(1). \tag{11.8.10}$$

Proof. We leave all the details of the proof to the reader and indicate only the rescaling which should be applied in this proof: namely, we apply the rescaling $(\epsilon, \epsilon r)$ with respect to (x,ξ) where $r = |\xi|$ and $h_{\mathsf{eff}} = r + |\tau|^{\frac{1}{m}}$ afterwards. □

Remark 11.8.2. Asymptotics (11.8.3) and (11.8.5) are standard results under ellipticity condition

$$|a(x,\xi)v| \ge \epsilon_0 |v| \cdot |\xi|^m \qquad \forall (x,\xi) \quad \forall v \in \mathbb{H} \tag{11.8.11}$$

and asymptotics (11.8.8) and (11.8.9) for $\vartheta(1-m) < d-1$ under this condition and non-periodicity condition on the energy level 1.

Let us get rid of the microhyperbolicity condition. In the general case we can do nothing better than introduce the rescaling

$$(x_1, x', \xi_1, \xi') \mapsto (\zeta^{-2}x_1, \zeta^{-1}x', r^{-1}\xi_1, \zeta^{-1}r^{-1}\xi') \qquad \text{with} \quad r \asymp |\xi| \tag{11.8.12}$$

where the coordinate system is taken such that $\xi' = 0$ at some point of the element of the partition and

$$\zeta = \min_j(|\alpha_j| + \tau r^{-m} + r^{-\frac{1}{2}}) \qquad \text{for} \quad \tau \le Cr^m \tag{11.8.13}$$

and $\alpha_j = \alpha_j(x, \xi)$ are eigenvalues of $a(x, \xi|\xi|^{-1})$. Further

(11.8.14) Here and below we set $\zeta = 1$ for $\tau \ge r^m$.

Then $h_{\text{eff}} = r^{-1}\zeta^{-2}$ and the contribution of the zone $\{(x, \xi) : \ h_{\text{eff}} \ge 1\}$ to the asymptotics does not exceed $CR'_{(\vartheta)}$ with

$$R'_{(\vartheta)} = \begin{cases} \tau^{l+\vartheta} & \text{for } \vartheta < -l, \\ |\log \tau| & \text{for } \vartheta = -l, \\ 1 & \text{for } \vartheta > -l, \end{cases} \qquad l = -\frac{2d}{1-2m}. \tag{11.8.15}$$

Then one can easily obtain the following

Theorem 11.8.3. *(i) In the general case the following asymptotics holds:*

$$\mathsf{N}^{\pm}_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W},\pm}_{(\vartheta)}(\tau) + O\big(R_{(\vartheta)}(\tau) + R'_{(\vartheta)}(\tau)\big) \tag{11.8.16}$$

where $R_{(\vartheta)}$ and $R'_{(\vartheta)}$ are defined by (11.8.7) *and* (11.8.16) *respectively.*

(ii) Moreover, suppose that $\vartheta < \big(d(1-2m)^{-1} - 1\big)$ [62] *and let the standard non-periodicity condition be fulfilled for the Hamiltonian trajectories of the symbol $\pm a$ on the energy level 1. Then asymptotics* (11.8.8) *and* (11.8.9) *hold.*

(iii) On the other hand, if the symbol $\pm a(x, \xi)$ is non-positive definite for every $(x, \xi)(\Leftrightarrow \varkappa^{\pm} = 0)$ then estimate

$$\mathsf{N}^{\pm}(\tau) = O(\tau^l) \tag{11.8.17}$$

holds.

[62] Which implies that $d > 1 - 2m$ and $\vartheta(1-m) < (d-1)$ i.e. $R_{(\vartheta)} \gg R'_{(\vartheta)}$.

To improve results of Theorem 11.8.3 let us assume that

$$|D^{\alpha}_{(x,\xi)}a(x,\xi)| \le c\gamma^{\mu-|\alpha|} \qquad \forall \alpha : |\alpha| \le \mu \quad \forall (x,\xi) : |\xi| = 1, \tag{11.8.18}$$

$$|a(x,\xi)v| \ge \epsilon_0 \gamma^{\mu}|v| \quad \forall v \in \mathbb{H} \quad \forall (x,\xi) : |\xi| = 1, \tag{11.8.19}$$

where $\gamma(x,\xi) = \mathsf{dist}\big((x,\xi),\Lambda\big)$ and Λ is the union of a finite number of $\mathscr{C}^K$ manifolds of codimension $\ge p$. Then one can make rescalings

$$(x,\xi) \mapsto (\zeta^{-1}x, \zeta^{-1}r^{-1}\xi) \tag{11.8.20}$$

with

$$\zeta = \epsilon\big(\gamma + \tau^{\frac{1}{\mu}} r^{-\frac{m}{\mu}} + r^{-\frac{1}{2}}\big) \tag{11.8.21}$$

and obtain

Theorem 11.8.4. *Under assumptions* (11.8.18)–(11.8.19) *Statements (i)–(ii) of Theorem 11.8.3 hold with* $R'_{(\vartheta)}$ *defined by* (11.8.15) *but this time with* $l = -(2d-p)(\mu-2m)^{-1}$.

Let us now treat the scalar case. First of all, for $\vartheta > 0$ one can apply exactly the construction of the proof of Theorem 11.8.1 (no microhyperbolicity-type condition is necessary at all). On the other hand, for $\vartheta = 0$ one can apply rescaling (11.8.20) with

$$\zeta = \epsilon\Big(\sum_{|\alpha|<M} |D^{\alpha}a|^{\frac{M}{M-|\alpha|}} + |\tau r^{-m}|\Big)^{\frac{1}{M}} + \epsilon r^{-\frac{1}{2}} \tag{11.8.22}$$

with $M = 2$. Then microhyperbolicity is fulfilled for $h_{\mathsf{eff}} \le h_0$. After calculations one gets the remainder estimate $O\big(\tau^{\frac{d-1}{m}} + \tau^{l}\big)$ where this time $l = -d(1-m)^{-1}$ which is good for $m < (1-d)$.

On the other hand, taking arbitrarily large M one can refer to the weakest possible non-degeneracy condition (5.2.126) and after easy calculations get the remainder estimate $O\big(\tau^{\frac{1}{m}(d-1+\delta)}\big)$ with arbitrarily small $\delta > 0$. Therefore the following statement holds:

Theorem 11.8.5. *Let A be a scalar operator. Then*

(i) For $\vartheta = 0$ asymptotics (11.8.3) *hold if $m \le (1-d)$; further, asymptotics*

$$\mathsf{N}^{\pm}(\tau) = \varkappa^{\pm}\tau^{\frac{d}{m}} + O\big(\tau^{\frac{1}{m}(d-1+\delta)}\big) \tag{11.8.23}$$

holds with arbitrarily small exponent $\delta > 0$ if $m > (1-d)$.

(ii) For $\vartheta > 0$ *asymptotics* (11.8.5) *holds.*

(iii) Suppose that either $\vartheta > 0$ *or* $m \le (1-d)$ *and and that the standard non-periodicity condition is fulfilled for the Hamiltonian trajectories of the symbol* $\pm a$ *on the energy level* 1*. Then asymptotics* (11.8.8), (11.8.9) *hold.*

Comments

While some of the results in the simplest situations were established by different authors, all the results of Sections 11.1–11.7 were proven as the corollaries of the general theorems of Chapter 9 by the author during 1985–1986.

Results of Section 11.8 are brand new.

Chapter 12

Miscellaneous Asymptotics of Spectra

12.1 Introduction

In this chapter we apply the results and methods of the previous chapters in order to derive eigenvalue asymptotics for the miscellaneous problems, mostly to extend the results of the previous Chapter 11 to the cases when these results in their original forms are no longer valid.

In Section 12.2 we treat operators in the domains with the "thick" cusps. Let us recall that if the cusp is not thin enough then asymptotics

$$\mathsf{N}^{-}(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + O\bigl(\tau^{(d-1)l}\bigr) \tag{12.1.1}$$

(with $l = 1/(2m)$ where $2m$ is an order of operator) no longer follows from the results of Sections 11.2 and 11.3 and if the cusp is even thicker then even estimate

$$\mathsf{N}^{-}(\tau) = O\bigl(\tau^{dl}\bigr) \tag{12.1.2}$$

no more follows from these results. It occurs that these asymptotics and estimates are no longer valid and should be replaced by asymptotics either containing non-Weyl correction term or even having non-Weyl principal part.

Similarly, in Section 12.3 we consider operators in $\mathbb{R}^d$ with potentials growing at infinity as $|x|^{2m\mu}$ but degenerate in some directions (such as the Schrödinger operator with a positively homogeneous of degree 2μ potential

$V_\infty \ge 0$, which vanishes on some manifold Θ). We proved in Section 11.5 asymptotics (12.1.1) with $I = (\mu+1)/(2m\mu)$ assuming that the degeneration is "mild" i.e. neither it is too strong nor $\dim \Theta$ is too large or, in other words, that the *valley* $\{x : V(x) \le \tau\}$ has only thin cusps for any fixed $\tau > 0$. If the degeneration is stronger then this asymptotics no longer follows from the results of Section 11.5 and if the degeneration is even stronger then even estimate (12.1.2) no more follows from these results.

Again it occurs that these asymptotics are no longer valid and should be replaced by asymptotics either containing non-Weyl correction term or even having non-Weyl principal part.

Further, in Section 12.4 we consider operators in $\mathbb{R}^d$ with potentials decaying at infinity slowly enough (as $|x|^{2m\mu}$ with $\mu \in (-1, 0)$) and consider asymptotics of $\mathsf{N}^-(\tau)$ as $\tau \to -\infty$. However these potentials are falling to $-\infty$ along some manifold Θ. We proved in Section 11.6 asymptotics (12.1.1) with $I = (\mu + 1)/(2m\mu)$ assuming that the singularity is "mild" i.e. neither it is too strong nor $\dim \Theta$ is too large or, in other words, that the *gorge* $\{x : V(x) \le \tau\}$ has only thin cusps for any fixed $\tau < 0$. If the singularity is stronger then this asymptotics no longer follows from the results of Section 11.5 and if the singularity is even stronger then even estimate (12.1.2) no more follows from these results.

Again it occurs that these asymptotics are no longer valid and should be replaced by asymptotics either containing non-Weyl correction term or even having non-Weyl principal part.

Furthermore, Section 12.5 is devoted to maximally hypoelliptic operators with symplectic degeneration manifold Λ. In this case we derived no similar results in Chapter 11 but one can prove easily that if degeneration is mild then asymptotics (12.1.1) holds (with $I = 1/(2m)$) as if operator was elliptic; if degeneration is not that mild then this asymptotics holds with worse remainder estimate and if degeneration is even more severe then even estimate (12.1.2) is no longer valid. So, in Section 12.5 we derive correct asymptotics in the latter cases; again these asymptotics contain non-Weyl correction term or even non-Weyl principal part.

Moreover, Section 12.6 is devoted to asymptotics of Riesz means for operators with singularities at separate points: it may be conical singularities of the metrics and domain and weak singularities of lower order terms. Then while (12.1.1) with $I = 1/(2m)$ holds, asymptotics

$$\mathsf{N}^-_{(\vartheta)}(\tau) = \mathsf{N}^{\mathsf{W}}_{(\vartheta)}(\tau) + O\bigl(\tau^{(d-1-\vartheta)I+\vartheta}\bigr) \tag{12.1.3}$$

fails for $\theta \ge \bar{\vartheta}_1$ and for even larger ϑ even estimate $\mathsf{N}^-_{(\vartheta)}(\tau) = O\bigl(\tau^{d\ell+\vartheta}\bigr)$ is no longer valid and again our goal is to derive correct asymptotics.

Further, in Section 12.7 we consider Neumann Laplacian in domain with cusps (and its certain generalizations). Then we need to assume that the cusps are really very thin: with the width decaying as $\exp(-|x|^\nu)$ with $\nu > 1$; otherwise essential spectrum appears. Still, only as $\nu > \bar{\nu}_1$ we derive asymptotics (12.1.1) and only as $\nu > \bar{\nu}_0$ estimate (12.1.2) is valid; our goal here again is to derive correct asymptotics.

Finally, in Section 12.8 we consider operator $A_t = A - tW$ where A is a periodic operator, W is a potential decaying at infinity. It is known that the spectrum of A consists of *spectral bands* (of essential, usually continuous) spectrum separated by *spectral gaps* (containing no spectrum of A). As $t \ne 0$ the essential spectrum of A_t coincide with one of A but discrete eigenvalues may appear inside spectral gaps. We fix a point E either inside the spectral gap or on its boundary and consider $\mathsf{N}_E(\tau)$ the number of eigenvalues of A_t crossing E as t grows from 0 to τ [1)] and consider asymptotics of $\mathsf{N}_E(\tau)$ as $\tau \to +\infty$.

We consider also the "lite" version of this theory: when A is not a periodic but operator with the constant coefficients.

All these problems have certain common features:

(a) We do not apply the results of Chapters 9 and 10;

(b) The asymptotics either contain non-Weyl terms or are completely non-Weyl (otherwise they would follow from the results of Chapters 9 and 10); however it is not necessarily true for Sections 12.5, 12.7 and 12.8.

(c) Most of the problems in question differ from problems which can be treated by means of results of Chapters 9 and 10 by some parameters (exponents, dimensions, orders of singularity, of degeneration, of Riesz means, etc.).

(d) In some sections we apply the theory of Chapters 2 and 4 for operators with the operator-valued symbols: namely, in Section 12.2 let $x' \in Z$ be the base of the cusp and $x'' \in \sigma(x')Y$ be the shape of the cross-section of the cusp respectively and let $d' = \dim Z$, $d'' = \dim Y$. After a coordinate

[1)] With eigenvalues crossing while increasing counted with the sign "+" and with eigenvalues crossing while decreasing counted with the sign "−".

transform $x'' \mapsto x''\sigma(x')^{-1}$ we obtain a cylindrical cusp $Z \times Y$ and treat operator $\boldsymbol{A}(x', D') = \boldsymbol{A}(x', D'; x'', D'')$ given on the base of the cusp for which the auxiliary Hilbert space is $\mathscr{L}^2(Y)$.

Under appropriate assumptions this is a perturbed operator

$$a(x', D', x'', \sigma^{-1}(x')D'') = \sigma^{-2m}(x')a^0(x', \sigma(x')D', x'', D'') \tag{12.1.4}$$

and it satisfies non-degeneracy condition on the energy level τ; here a is the principal symbol of A. In particular, as $A = \Delta' + \Delta''$ this unperturbed operator is $\Delta' + \sigma^{-2}(x')\Delta''$ and if we deal with unperturbed operator we can decompose it into the direct sum of operators $A_k = \Delta' + \sigma^{-2}(x')\lambda_k$ where $\lambda_1 \le \lambda_2 \le \dots$ are eigenvalues of Δ'' (under inherited boundary conditions; we assume that those are Dirichlet conditions albeit one can consider more general boundary conditions as well).

As $\lambda_1 > 0$ all these operators are d'-dimensional Schrödinger operators and their asymptotics are given by Weyl formula with

$$\mathsf{N}^{\mathsf{W}}_k(\tau) = O\bigl(\tau^{d'l}\lambda_k^{-d'/(2m)}\bigr) \tag{12.1.5}$$

and with the remaining estimate

$$R^{\mathsf{W}}_k(\tau) = O\bigl(\tau^{(d'-1)l}\lambda_k^{-(d'-1)/(2m)}\bigr) \tag{12.1.6}$$

where $l = (\mu+1)/(2m\mu)$ as $\sigma(x') \asymp |x'|^{-\mu}$.

Now we need to sum all the main parts and we arrive to asymptotics with the principal part either $\asymp \tau^{d/(2m)}$ or $\asymp \tau^{d'l}$ whatever is greater but as degrees are equal we get principal part of magnitude $\asymp \tau^{d'l}|\log\tau|$.

Similarly, we need to sum all the remainder estimates and we arrive to $O(\tau^{(d-1)/(2m)})$ or $O(\tau^{d'l})$ whatever is greater but as degrees are equal we get the remainder $O(\tau^{(d'-1)l}|\log\tau|$.

Thus in Section 12.2 we extend this result to the case when operator in question does not decompose into direct sum and to the higher order operators. The main issue here is not only to express the principal part of the asymptotics (possibly containing more than one term) but also simplify it if possible invoking eigenvalue counting function $\boldsymbol{n}_\infty(\tau)$ of one axillary operator rather than the family $\boldsymbol{n}(x', \xi; \tau)$.

The similar approach works in Section 12.3 where in the case of the Schrödinger operator with potential $V(x)$, non-negative and positively homogeneous of degree $2\mu > 0$, x' are coordinates on $Z = \{x : V(x) = 0\}$

and x'' are coordinates in $Z^{\perp}$. Similarly in Section 12.4 in the case of the Schrödinger operator with potential $V(x)$, positively homogeneous of degree $2\mu \in (-2,0)$, x' are coordinates on $Z = \{x : V(x) = -\infty\}$ and x'' are coordinates in $Z^{\perp}$.

In Section 12.5 we consider pseudodifferential operators with the phase space Λ and with auxiliary Hilbert space $\mathscr{L}^2(\mathbb{R}^{d''})$ with $d'' = \frac{1}{2}\operatorname{codim}\Lambda$.

In Sections 12.2–12.5 there are zones where we apply the standard Weyl asymptotics of Part II, zones where we can obtain the same asymptotics with improved remainder estimates (these enhancements are due to results on the propagation of singularities for operators with the operator-valued symbols and zones where results of Chapter 4 are applicable; the second and third zones are overlapping, and one can combine the asymptotics obtained.

In Section 12.6, instead of operators with the operator-valued symbols we treat model operators with positively quasihomogeneous symbols; for these operators

$$\int \Big(e_{(\vartheta)}(x,x,0) - e^{\mathsf{W}}_{(\vartheta)}(x,x,0)\Big)\,dx = \varkappa h^{-\kappa} \tag{12.1.7}$$

converges as $\vartheta > \bar{\vartheta}$ and the equality is due to quasihomogeneity; here $e(.,.,\tau)$ is the Schwartz kernel of the spectral projector, $e^{\mathsf{W}}(.,.,\tau)$ is its Weyl approximation, and $_{(\vartheta)}$ means a Riesz mean of order ϑ with respect to τ; the right-hand expression is *Scott correction term*. We consider also similar expressions with homogeneous factor in the integral and basically result is the same albeit in some cases we need to include cut-off in the left-hand expression and the right-hand expression becomes $(\varkappa|\log h| + \varkappa_1)h^{-\kappa}$. We cannot represent the left-hand expression in (12.1.7) as the difference as both integrals diverge at infinity.

We apply then Weyl method to the difference of the spectral functions of the original and the model operators and derive sharp asymptotics for $\int e_{(\vartheta)}(x,x,0)\psi(x)\,dx$ with a fixed supported near 0 function $\psi \in \mathscr{C}_0^{\infty}$; this asymptotics contains Weyl part and one or more Scott correction terms (which could be actually larger than the Weyl part).

Section 12.7 starts as Section 12.2, but $0 = \lambda_1 < \lambda_2$ (in more general situations $0 = \lambda_1 = ... = \lambda_k < \lambda_{k+1}$ and spectrum seems to become essential; however when we change variables $x'' \mapsto \sigma^{-1}(x')x''$ we get a weighted $\mathscr{L}^2$-space $\mathscr{L}^2(Z, \mathscr{L}^2(Y), \sigma^d(x')dx')$; considering quadratic form and substituting $u = \sigma^{-d/2}v$ we obtain operator in the standard $\mathscr{L}^2(Z, \mathscr{L}^2(Y))$-space and

this operator acquires an extra potential

$$V \sim \frac{d^2}{4} \sum_{j,k} g^{jk} (\partial_j \log \sigma)(\partial_k \log \sigma); \tag{12.1.8}$$

this potential grows at infinity as $|\nabla \log \sigma|^2 \asymp |x|^{2\mu}$ provided σ behaves like $\exp(-|x'|^\nu)$ with $\nu = \mu + 1$. Then the results similar to those of Section 12.2 are derived but since the Schrödinger operator corresponding to $\lambda_j = 0$ is so different from all others (which now have potentials with superexponential growth) logarithmic factor does not appear as two degrees collide.

In Section 12.8 we consider operator $A - tW$ where A is periodic operator and W is a decaying potential, and count eigenvalues crossing E which belongs either to spectral gap or its border. In this case we use Gelfand decomposition so A becomes $A(\xi)$ operator acting on $\mathscr{L}^2(\mathcal{Q})$ where $\mathcal{Q}$ is an elementary cell corresponding to periods lattice and $\xi \in \mathcal{Q}'$ is a quasimomentum where $\mathcal{Q}'$ is a dual cell. Meanwhile W becomes a negative order pseudodifferential operator with respect to ξ and we again apply the theory of operators with the operator-valued symbols. However in contrast to what we had before, dimension does not decrease.

12.2 Operators in Domains with Thick Cusps

12.2.1 Problem Set-up

Let us consider a uniformly elliptic operator of order $2m$

$$A = A^{\mathrm{w}}(x, D), \qquad A(x, \xi) = \sum_{|\alpha| \le 2m} a_\alpha(x) \xi^\alpha \tag{12.2.1}$$

where

(12.2.2) $a_\alpha(x)$ are Hermitian $\mathsf{D} \times \mathsf{D}-$matrix valued functions such that

$$|D^\beta a_\alpha| \le c \langle x \rangle^{-|\beta'| + \mu(2m - |\alpha|)} \tag{12.2.3}$$

$$\forall \alpha : |\alpha| \le 2m \, \forall \beta : |\beta| \le K, \beta'' = 0,$$

$$D^{\beta}a_{\alpha} = o\big(|x'|^{-|\beta'|+\mu(2m-|\alpha|+|\beta''|)}\big) \qquad \text{as} \quad |x'|\to\infty \tag{12.2.4}$$
$$\forall \alpha : |\alpha| \le 2m \ \forall \beta : |\beta| \le K, 2m-|\alpha|+|\beta''|>0,$$

$$\langle a(x,\xi)v, v\rangle \ge \epsilon |v|^2 \qquad \forall v \quad \forall (x,\xi) \tag{12.2.5}$$

where

$$a(x,\xi) = \sum_{|\alpha|=2m} a_{\alpha}(x)\xi^{\alpha} \tag{12.2.6}$$

is the principal symbol of A.

We consider this operator in the domain $X \subset \mathbb{R}^d$ such that

$$X = X_0 \cup X_{\mathsf{cusp}},\ \partial X \in \mathscr{C}^K,\ X_0 \Subset \mathbb{R}^d, \tag{12.2.7}$$
$$X_{\mathsf{cusp}} = \big\{x=(x',x'') : x'\in Z,\ x''\in \sigma(x')Y\big\}, \quad Y \Subset \mathbb{R}^{d''},\ \partial Y \in \mathscr{C}^K$$

with

$$\sigma(x') \asymp \langle x'\rangle^{-\mu}\ \mu>0, \tag{12.2.8}$$

$$|D^{\beta'}\sigma| \le c\langle x'\rangle^{-\mu-|\beta'|} \quad \forall \beta' : |\beta'|\le K \tag{12.2.9}$$

and

(12.2.10) Either $Z=\mathbb{R}^{d'}$ as $d'\ge 1$ or $Z=\mathbb{R}^{\pm}$ as $d'=1$ [2)].

Definition 12.2.1. We say that X_0 is the *compact part* of X, X_{cusp} is the *cusp*, Z and Y are the *cusp base and cusp cross-section* respectively and σ is the *cusp form defining function*.

Let us assume that

(12.2.11) A Dirichlet boundary condition (of order m) is given on ∂X.

One can prove easily that

(12.2.12) Under these conditions (12.2.1)–(12.2.11) A is a self-adjoint operator in $\mathscr{L}^2(X,\mathbb{H})$, $\mathbb{H}=\mathbb{C}^{\mathsf{D}}$ and has a compact resolvent.

[2)] Analysis of more general cusps will be formulated as Problem 12.2.23.

Moreover, the following statement follows from the results of Sections 11.2 and 11.3:

Theorem 12.2.2. *Let conditions* (12.2.1)–(12.2.11) *be fulfilled. Then*

(i) As $\sigma^{d''-1} \in \mathscr{L}^2(Z)$ *which is equivalent to*

$$\mu > \mu_1 := d'(d''-1)^{-1}, \qquad \mu_0 := d'd''^{-1} \tag{12.2.13}$$

then Weyl asymptotics

$$\mathsf{N}^-(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + O(\tau^{\frac{1}{2m}(d-1)}) \tag{12.2.14}$$

holds as $\tau \to \infty$ *with*

$$\mathsf{N}^{\mathsf{W}}(\tau) \asymp \tau^{\frac{d}{2m}}. \tag{12.2.15}$$

Here $\mathsf{N}^{\mathsf{W}}(\tau)$ *is given by the standard Weyl formula, and moreover, one can replace it here by* $\kappa_0\tau^{\frac{d}{2m}}$ *where* κ_0 *and* κ_1 *are standard coefficients which are finite for* $\mu > \mu_0$, $\mu > \mu_1$ *respectively.*

(ii) Moreover, under the standard non-periodicity condition to the generalized Hamiltonian flow of $\mathsf{a}(x,\xi)$ *with reflections at* ∂X *(and perhaps with branching at the points of reflection for* $m \ge 2$*) asymptotics*

$$\mathsf{N}^-(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + \kappa_1\tau^{\frac{1}{2m}(d-1)} + o\bigl(\tau^{\frac{1}{2m}(d-1)}\bigr) \tag{12.2.16}$$

holds as $\tau \to \infty$.

(iii) For $\mu = \mu_1$ *asymptotics*

$$\mathsf{N}^-(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + O\bigl(\tau^{\frac{1}{2m}(d-1)}\log\tau\bigr) \tag{12.2.17}$$

holds as $\tau \to \infty$ *and for* $\mu \in [\mu_0, \mu_1)$ *asymptotics*

$$\mathsf{N}^-(\tau) = \mathsf{N}^{\mathsf{W}}(\tau) + O\bigl(\tau^{\frac{1}{2m}(d-q)}\bigr) \tag{12.2.18}$$

holds as $\tau \to \infty$ *with* $q = d'\mu^{-1} + d'' \in [0,1)$*; here* $q = 0 \iff \mu = \mu_0$, $q = 1 \iff \mu = \mu_1$, *and for* $\mu > \mu_0$ (12.2.15) *holds. Further, for* $\mu = \mu_0$

$$\mathsf{N}^{\mathsf{W}}(\tau) \asymp \tau^{\frac{d}{2m}}\log\tau \tag{12.2.19}$$

is given by cut Weyl formula and one can replace it by $\kappa_0^*\tau^{d/2m}\log\tau$.

Remark 12.2.3. (i) Surely, our conditions here are more restrictive than necessary but the case $\mu > \mu_1$ is not our goal here. Our goal is to treat the case $\mu \le \mu_1$ and to derive asymptotics with better remainder estimates. Namely, we obtain asymptotics with the remainder estimate $O\bigl(\tau^{\frac{1}{2m}(d-1)}\bigr)$ and even $o\bigl(\tau^{\frac{1}{2m}(d-1)}\bigr)$ under the standard non-periodicity condition for $d'=1$ and for $d'\ge 2$, $\mu > \mu^*$ with $\mu^* < \mu_1$; otherwise we obtain asymptotics with a worse remainder estimate.

Surely, one should replace $\mathsf{N}^{\mathsf{W}}(\tau)$ by another expression and some additional conditions appear.

(ii) For Riesz means the crucial values in the Weyl asymptotics are

$$\mu_{\vartheta+1} = d'(d''-\vartheta-1)^{-1} \tag{12.2.13$'$}$$

and μ_0 respectively: namely, for $\mu > \mu_{\vartheta+1}$ we get the remainder estimate $O\bigl(\tau^{\frac{1}{2m}(d-\vartheta-1)+\vartheta}\bigr)$ or even "o" instead of "O" under the standard non-periodicity condition. Further, for $\mu = \mu_{\vartheta+1}$ the remainder estimate is $O\bigl(\tau^{\frac{1}{2m}(d-\vartheta-1)+\vartheta}\log\tau\bigr)$ and for $\mu_0 < \mu < \mu_{\vartheta+1}$ it is $O\bigl(\tau^{\frac{d'(\mu+1)}{2m\mu}+\vartheta}\bigr)$.

One can easily see that this exponent lies in $\bigl(\frac{1}{2m}(d-\vartheta-1)+\vartheta, \frac{d}{2m}+\vartheta\bigr)$. At the same time the principal Weyl part is of order $\tau^{\frac{d}{2m}+\vartheta}$ and $\tau^{\frac{d}{2m}+\vartheta}\log\tau$ for $\mu > \mu_0$ and $\mu = \mu_0$ respectively.

12.2.2 Sharp Asymptotics

Standard Remainder Estimate

First of all let us divide $\mathsf{N}^-(\tau)$ into the sum

$$\mathsf{N}^-(\tau) = \mathsf{N}_0^-(\tau) + \mathsf{N}_{\mathsf{cusp}}^-(\tau) \tag{12.2.20}$$

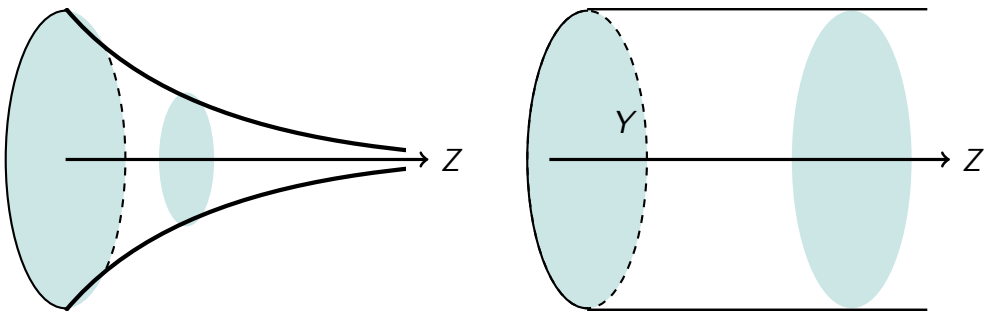
where

$$\mathsf{N}_\bullet^-(\tau) = \int \psi_\bullet(x')\,\mathsf{tr}\bigl(e(x,x,\tau)\bigr)\,dx, \tag{12.2.21}$$

$e(x,y,\tau)$ is the Schwartz kernel of the spectral projector of A and $\psi_0, \psi_{\mathsf{cusp}} \in \mathscr{C}^K$ are supported in $\{|x'|\le 3c\}$, $\{|x'|\ge 2c\}$ respectively, $\psi_0+\psi_{\mathsf{cusp}}=1$. We assume that $X_0 \subset \{|x'|\le \frac{3}{2}c\}$.

Then

(12.2.22) Weyl asymptotics holds for $\mathsf{N}_0^-(\tau)$ with the proper remainder estimate $O\bigl(\tau^{\frac{1}{2m}(d-1)}\bigr)$ (and even $o\bigl(\tau^{\frac{1}{2m}(d-1)}\bigr)$ under the standard non-periodicity condition to the Hamiltonian billiards).



(a) Cusp (b) Cylinder

Figure 12.1: Cusp transforms to a cylinder

So we need to derive asymptotics of $\mathsf{N}^-_{\mathsf{cusp}}(\tau)$. Let us introduce the coordinates (x', x'') with $x''_{\mathsf{new}} = \sigma(x')^{-1}x''$. In these coordinates cusp X_1 transforms to the cylinder $Z \times Y$ and the differential expression A transforms to a differential expression which is symmetric with weight $\sigma(x')^{d''}$ provided our test functions vanish for $|x'| \le \frac{1}{3}c$. In order to get rid of this weight we multiply the differential expression by $\sigma(x')^{\frac{1}{2}d''}$ and $\sigma(x')^{-\frac{1}{2}d''}$ on the left and right respectively, obtaining

$$\tilde{A} = \tilde{A}^{\mathsf{w}}(x, D), \quad \tilde{A}(x,\xi) = \sum_{|\alpha|\le 2m} a_\alpha (\xi' - b'\beta)^{\alpha'} \xi''^{\alpha''} \sigma(x')^{-|\alpha''|}, \tag{12.2.23}$$

$$b_k = \partial_k(\log\sigma) \quad k = 1, \dots, d', \qquad \beta = \langle x'', \xi''\rangle$$

in the coordinates $x = (x', x'')$. We treat this operator as an operator in $\mathscr{L}^2(Z, \mathbb{H})$ with $\mathbb{H} = \mathscr{L}^2(Y, \mathbb{H})$. Surely, we cannot discuss self-adjointness now because we cannot guarantee that our functions vanish for $|x'| \le \frac{5}{3}c$.

Recall that we are interested in the spectral projector $E(\tau)$ of A corresponding to the spectral parameter τ. Let us consider the $\gamma(\bar{x}')$-neighborhood of the point $\bar{x}'$ with $\gamma(x') = \frac{1}{2}|x'|$ where we always assume that $|\bar{x}'| \ge c_1$ with a large enough constant c_1. Let us rescale $x' \mapsto (x' - \bar{x}')\bar{\gamma}^{-1}$ and the make a corresponding multiplication.

Let us first treat case of $\bar{\sigma} \le \epsilon\tau^{-\frac{1}{2m}}$ (i.e., $|x'| \ge \epsilon_1^{-1}\tau^{\frac{1}{2\mu m}}$) where here and below $\epsilon_j > 0$ are sufficiently small and $\bar{f} = f(\bar{x}')$ for any function f. Multiplying operator A by $\bar{\sigma}^{2m}$ after rescaling we obtain an h-differential operator with $h = \bar{\sigma}\bar{\gamma}^{-1} < 1$; we are interested in the operator's spectral projector corresponding to the spectral parameter $\tau' = \tau\bar{\sigma}^{2m} < \epsilon_2$.

One can easily see that this operator $\boldsymbol{A}(x', D'; x'', D'')$ considered as operator with the operator-valued symbol $\boldsymbol{A}(x', \xi'; x'', D'')$ is of the type considered in Chapter 4 provided

(12.2.24) $B = (L_{(m)\mathsf{D}})$, $B_1 = (L_{(M)\mathsf{N}})^{mM^{-1}}$ where $L_M = (\Delta'' + I)^M$ and the subscripts "D" and "N" mean m-th order Dirichlet and Neumann boundary conditions respectively (of the proper order), Δ'' is the positive Laplacian in Y and M is taken large in order to obtain large $I = I(M)$ [3].

Then for $j \in [0, I]$ we see that $\mathbb{H}_j = \mathfrak{D}(B_1^j) = \mathscr{H}^{2mj}(Y)$ are Sobolev spaces and for $j \in [-I, 0]$ $\mathbb{H}_j = B_1^{-j}(\mathbb{H})$ are the spaces dual to $\mathscr{H}^{-2mj}(Y)$.

In particular, one can easily check that conditions (4.2.7), $(4.2.8)_I$, $(4.2.13)_{\frac{1}{2}}$ (i.e. $(4.2.13)_I$ with $I = \frac{1}{2}$), $(4.2.14)_+$, (4.2.15), (4.2.17) and (4.2.21) are fulfilled.

Further, we see that

(12.2.25) Operator $\boldsymbol{A}(x', D'; x'', D'')$ [4] is elliptic on the energy level s $\le \epsilon_2$.

---

[3] See Chapter 4 for B, B_1 and I.

[4] Considered as an operator with the operator-valued symbol.

One can check this easily by dropping lower order terms (with $|\alpha| < 2m$) in $\tilde{A}$ and assuming that D'' and a_α commute and the energy level is non-positive. This procedure is justified by condition (12.2.4) and the appropriate choice of $\epsilon_2 > 0$.

Furthermore, since A is a self-adjoint operator, $\|\exp(itA)\| = 1$ and $\|E(\tau)\| \le 1$. On the other hand, $\exp(itA)$ and $E(\tau)$ generate operator-valued operators in $B\bigl(\bar{x}', \gamma(\bar{x}')\bigr)$ via their Schwartz kernels and one can apply the theory of Chapter 4 (namely, Propositions 4.2.8–4.2.10). This trick permits us to avoid the discussion of the self-adjointness of $\tilde{A}$, etc.

Proposition 12.2.4. *Let conditions* (12.2.1)–(12.2.11) *be fulfilled and let* c_0 *be large enough. Then*

(i) Estimate

$$|e(x,y,\tau)| \le C'_s \langle x'\rangle^{-s}\langle y'\rangle^{-s} \qquad \forall \tau \ge 1 \quad \forall x, y \in X : |x'| + |y'| \ge c_0 \tau^{\frac{1}{2\mu m}} \tag{12.2.26}$$

holds.

(ii) Contribution of the far cusp zone $\{|x'| \ge c_0 \tau^{1/2m\mu}\}$ *to the asymptotics is* $O(\tau^{-s})$.

Proof. Then the results of Chapter 4 immediately yield that

$$\|B_1^j e(x,y,\tau) B_1^k\| \le C'_{jks} \langle x'\rangle^{-s}\langle y'\rangle^{-s} \qquad \forall j,k \le I \ \forall s \tag{12.2.27}$$

where the norm in the left-hand expression is the operator norm in $\mathbb{H}$ and therefore as $I = I(d)$ (and thus $K = K(d)$) we arrive to estimate (12.2.26) thus proving Statement (i).

Statement (ii) follows from Statement (i) immediately. □

Let us now treat the *main cusp zone*

$$\bigl\{c_0 \le |x'| \le \bar{\gamma}(\tau) := c_0 \tau^{\frac{1}{2m\mu}}\bigr\}. \tag{12.2.28}$$

Here we should multiply the operator by τ^{-1} and so we should consider the operator $\tau^{-1}A$ on the energy level 1. One can easily check that we are again in the framework of Chapter 4 with $h = \tau^{-\frac{1}{2m}}\bar{\gamma}^{-1}$ and

$$B = \tau^{-1}\bar{\sigma}^{-2m}\bigl(L_{(M)\mathsf{D}}\bigr)^{mM^{-1}} + I, \quad B_1 = \tau^{-1}\bar{\sigma}^{-2m}\bigl(L_{(M)\mathsf{N}}\bigr)^{mM^{-1}} + I \tag{12.2.29}$$

with the previous definition of $L_{(M)\mathsf{D}}$ and $L_{(M)\mathsf{N}}$.

One can easily check that this operator $\boldsymbol{A}(x', D'; x'', D'')$ [4)] is elliptic on the energy levels ≤ 1 for $|\xi'| \ge C_0$.

Moreover, one can easily check that this operator[4)] is microhyperbolic on the energy level 1 in the direction $\ell \in T_{(x',\xi')} T^*Z$ in each of the following cases[5)]:

(a) $m = 1$, $a_\alpha = o(1)$ for $|\alpha'| = |\alpha''| = 1$, $|\xi'| \ge \epsilon$, $\ell = \langle \xi', \partial_{\xi'} \rangle$;

(b) $m = 1$, $|\xi'| \le \varepsilon$ with a small enough constant $\varepsilon > 0$, $|\ell\sigma| \ge \epsilon\sigma |x'|^{-1}$, $\ell = \langle \ell', \partial_{x'} \rangle$ with an appropriate vector ℓ' such that $|\ell'| \le c$ and

$$\ell a_\alpha = o\bigl(|x'|^{-1+(2m-|\alpha|)}\bigr) \quad \text{as} \quad |x'| \to \infty \quad \forall \alpha : |\alpha| < 2m \tag{12.2.30}$$

(here this condition should be checked for $\alpha' = 0$ only);

(c) $m \ge 1$, $\ell\sigma = \sigma\bigl(1 + o(1)\bigr)|x'|^{-1}$ as $|x'| \to \infty$ with an appropriate vector ℓ, $|\ell| \le c$, (12.2.30) holds and $\ell = \langle \xi', \partial_{\xi'} \rangle + \langle \ell', \partial_{x'} \rangle$.

All these conditions are given in the original coordinates and in Conditions (a) and (c) one can take ℓ' depending on x'.

Then we arrive to

Proposition 12.2.5. *The microhyperbolicity condition is fulfilled for any ξ' provided*[5)] *one of the following conditions holds:*

(12.2.31) $m = 1$, $a_\alpha = o(1)$ *for* $|\alpha'| = |\alpha''| = 1$, $|\ell\sigma| \ge \epsilon\sigma |x|^{-1}$ *and* (12.2.30) *holds for* $\alpha' = 0$

(12.2.32) $a = \bar{a}^m + a'$ *where second-order symbol* $\bar{a}$ *satisfies* (12.2.31) *and all coefficients of* a' *satisfy*

$$a'_\alpha = o\bigl(|x'|^{\mu(2m-|\alpha|)}\bigr), \tag{12.2.33}$$

$$\ell a'_\alpha = o\bigl(|x'|^{-1+\mu(2m-|\alpha|)}\bigr) \tag{12.2.34}$$

(12.2.35) $m \ge 1$, $\ell\sigma = \sigma\bigl(1 + o(1)\bigr)|x'|^{-1}$ *and* (12.2.30) *holds.*

[5)] These conditions are sufficient but not necessary.

In what follows we assume that operator $\boldsymbol{A} = \tau^{-1}\tilde{A}$ is microhyperbolic on the energy level 1 [4) 6)].

Then, applying Theorem 4.4.8 as this operator is ξ'-microhyperbolic and Theorem 4.4.9 as this operator is (x', ξ')-microhyperbolic (i.e. in the general case) we obtain that if $\psi = \psi(x')$ is a γ-admissible function supported in $\{|x' - \bar{x}'| \le \frac{1}{2}\bar{\gamma}\}$ then

$$\|B_1^j \int \psi(x')\Big(e(x,y,\tau)_{x'=y'} - (2\pi)^{-d'}\int e(x',\xi',\tau)\,d\xi'\Big)\,dx' \cdot B_1^k\| \le C\tau^{\frac{1}{2m}(d'-1)}\gamma^{d'-1} \qquad \forall j,k \le l \tag{12.2.36}$$

where the norm in the left-hand expression is the operator norm in $\mathbb{H}$ and

(12.2.37) $\boldsymbol{e}(x',\xi',\tau)$ is the spectral projector of the operator $\boldsymbol{a}(x',\xi') := \boldsymbol{a}(x',\xi';x'',D'')$; the latter operator is an *operator-valued symbol* of $\tilde{A}(x,D)$ and we do not include factor τ^{-1} into operator or symbol.

Recall that $\tilde{A}$ is the operator $\sigma^{\frac{1}{2}d''}A\sigma^{-\frac{1}{2}d''}$ in the cylindrical coordinates (x', x'') (without division by τ)). Thus we obtain $\boldsymbol{a}(x',\xi';x'',D'')$ by replacing D' with ξ' in $\tilde{A}(x,D)$. So operator $\boldsymbol{a}(x',\xi')$ acts in $\mathbb{H} = \mathscr{L}^2(Y,\mathbb{H})$ and $\mathfrak{D}(\boldsymbol{a}) = \mathscr{H}^{2m}(Y,\mathbb{H}) \cap \mathscr{H}_0^m(Y,\mathbb{H})$.

Obviously, this operator is self-adjoint and therefore $\boldsymbol{e}(x',\xi',\tau)$ is well-defined.

Recalling the definition of B_1 one can easily see that

(12.2.38) As $l > \frac{1}{2m}d''$ B_1^{-l} is a trace operator in $\mathbb{H}$ with trace norm not exceeding $C\tau^{\frac{d''}{2m}}\sigma^{d''}$.

Therefore we immediately obtain the following estimate:

$$\Big|\int \psi(x')\,\mathsf{tr}\big(e(x,x,\tau)\big)\,dx - (2\pi)^{-d'}\iint \psi(x')\boldsymbol{n}(x',\xi',\tau)\,dx'd\xi'\Big| \le C\tau^{\frac{1}{2m}(d-1)}\gamma^{d'-1-\mu d''} \tag{12.2.39}$$

where

[6)] Which is the case provided[5)] one of conditions (12.2.31), (12.2.32) or (12.2.35) is fulfilled.

(12.2.40) $\boldsymbol{n}(x',\xi',\tau) = \mathsf{Tr}_{\mathbb{H}}\, \boldsymbol{e}(x',\xi',\tau)$ is an eigenvalue counting function of the operator $\boldsymbol{a}(x',\xi')$ in $\mathbb{H}$.

Really, due to (12.2.38) the trace norm (in $\mathbb{H}$) of the operator in the left-hand expression of (12.2.36) stripped of B_1^j and B_1^k does not exceed $C\tau^{\frac{1}{2m}(d'-1)}\gamma^{d'-1} \times \tau^{\frac{d''}{2m}}\sigma^{d''}$ which is exactly the right-hand expression of (12.2.39).

Summing over a partition of the unity in the main cusp zone (12.2.28) we obtain in the right-hand expression

$$C\tau^{\frac{1}{2m}(d-1)} \int_1^{\bar\gamma(\tau)} \gamma^{d'-1-\mu d''}\, d\gamma$$

which in turn is equal to

$$R^*(\tau) = \begin{cases} \tau^{\frac{1}{2m}(d-1)} & \text{for } \ d' < \mu d''+1, \\ \tau^{\frac{1}{2m}(d-1)}\log\tau & \text{for } \ d' = \mu d''+1, \\ \tau^{\frac{1}{2m\mu}(d'-1)(\mu+1)} & \text{for } \ d' > \mu d''+1. \end{cases} \tag{12.2.41}$$

Then, taking into account results in the zone $\{|x'|\le 2c_0\}$ (Weyl asymptotics) and far cusp zone $\bigl\{|x'|\ge \frac{1}{2}\bar\gamma(\tau)\bigr\}$ (Proposition 12.2.4) we obtain the following basic statement:

Theorem 12.2.6. *Let conditions* (12.2.1)–(12.2.11) *be fulfilled and let microhyperbolicity condition for* $\boldsymbol{a}(x',\xi')$ *be fulfilled*[6]*. Then asymptotics*

$$\mathsf{N}^-(\tau) = \mathcal{N}(\tau) + O\bigl(R^*(\tau)\bigr) \tag{12.2.42}$$

holds as $\tau\to+\infty$ *where* $R^*(\tau)$ *is given by* (12.2.41) *and*

$$\mathcal{N}(\tau) = (2\pi)^{-d}\iint \psi_0(x')n(x,\xi,\tau)\,dxd\xi + (2\pi)^{-d'}\iint \bigl(1-\psi_0(x')\bigr)\boldsymbol{n}(x',\xi',\tau)\,dx'd\xi', \tag{12.2.43}$$

$n(x,\xi,\tau)$ *is the eigenvalue counting function for symbol* $\boldsymbol{a}(x,\xi)$, $\psi_0\in \mathscr{C}^K$ *is supported in* $\{|x'|\le 2c_0\}$ *and* $\psi_0 = 1$ *for* $|x'|\le c_0$.

Moreover,

$$\mathcal{N}(\tau) \asymp \begin{cases} \tau^{\frac{d}{2m}} & \textit{for } \ d' < \mu d'', \\ \tau^{\frac{d}{2m}}\log\tau & \textit{for } \ d' = \mu d'', \\ \tau^{\frac{1}{2m\mu}d(\mu+1)} & \textit{for } \ d' > \mu d''. \end{cases} \tag{12.2.44}$$

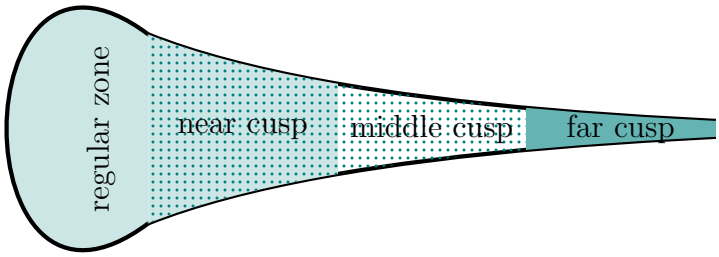


Figure 12.2: In both the *regular zone* and the *near cusp* we apply standard Weyl asymptotics (albeit in the near cusp we use propagation results for operators with the operator-valued symbols), in the *middle cusp* we use Weyl asymptotics for operators with the operator-valued symbols and *far cusp* is an elliptic zone for operator in question considered as operator with the operator-valued symbol; all zones overlap.

Improved Remainder Estimates

Assume first that $d' < 1+\mu d''$. Then the main contribution to the remainder estimate is given by the zone $\{|x'| \le t\}$ with an arbitrarily large constant t (the contribution of the complement does not exceed $\tau^{\frac{1}{2m}(d-1)}t^{-\delta}$ with $\delta > 0$); on the other hand, in this zone under the standard non-periodicity condition to the Hamiltonian billiards we have the remainder estimate $o\bigl(\tau^{\frac{1}{2m}(d-1)}\bigr)$.

Therefore the following statement holds:

Theorem 12.2.7. *Let conditions* (12.2.1)–(12.2.11) *be fulfilled and let microhyperbolicity condition for* $a(x',\xi')$ *be fulfilled*[6)]*.*

Finally, let $d' < 1+\mu d''$ *and let the standard non-periodicity condition to the Hamiltonian billiards be fulfilled. Then asymptotics*

$$\mathsf{N}^-(\tau) = \mathcal{N}(\tau) + o\bigl(\tau^{\frac{1}{2m}(d-1)}\bigr) \tag{12.2.45}$$

holds as $\tau \to +\infty$ *where* <u>now</u>

$$\begin{aligned}\mathcal{N}(\tau) = (2\pi)^{-d}\iint \psi_t(x')\bigl(n(x,\xi,\tau) + \kappa_1\tau^{\frac{d-1}{2m}}\bigr)\,dxd\xi + \\ (2\pi)^{1-d}\int \psi_t(x')\tilde{\kappa}_1\tau^{\frac{d-1}{2m}}\,dS + \\ (2\pi)^{-d'}\iint \bigl(1-\psi_t(x')\bigr)n(x',\xi',\tau)\,dx'd\xi',\end{aligned} \tag{12.2.46}$$

$$(12.2.47)_1 \qquad (2\pi)^{-d}\int \psi_t \kappa_1 \tau^{\frac{1}{2m}(d-1)}\, dx$$

and

$$(12.2.47)_2 \qquad (2\pi)^{1-d}\int_{\partial X} \psi_t \tilde{\kappa}_1 \tau^{\frac{1}{2m}(d-1)}\, dS$$

are the standard second terms in the Weyl asymptotics, $\psi_t(x') := \psi_0(\frac{x'}{t})$ *where* $t = t(\tau)$ *is an appropriate function such that* $t(\tau) \to +\infty$ *as* $\tau \to +\infty$ [7].

On the other hand, let $d' > 1 + d''\mu$. Then the main contribution to the remainder estimate in Theorem 12.2.6 is provided by the zone

$$(12.2.48) \qquad \{\varepsilon \le \tau^{-\frac{1}{2m\mu}}|x'| \le c_0\}$$

with arbitrarily small $\varepsilon > 0$; contribution of zone $\{|x'| \le 2\varepsilon\tau^{\frac{1}{2m\mu}}\}$ does not exceed $\varepsilon^\delta R^*(\tau)$.

However, in zone (12.2.48) $\boldsymbol{n}(x', \xi', \tau)$ does not exceed $\nu(\varepsilon) = \varepsilon^M$ and therefore we have effectively here matrix- rather than operator-valued operators.

Applying Theorems 4.4.11 and 4.4.12 and replacing $\boldsymbol{a}$ if necessary by another operator in $\mathbb{H}$ with eigenvalues asymptotically close to those of $\boldsymbol{a}$ we obtain the following statement:

Theorem 12.2.8. *Let conditions* (12.2.1)–(12.2.11) *be fulfilled and let microhyperbolicity condition for* $\boldsymbol{a}(x', \xi')$ *be fulfilled* [6].

Further, let $d' > 1 + \mu d''$ *and let the stabilization conditions*

$$(12.2.49)_1 \quad D^{\beta'}(a_\alpha - a_{\infty,\alpha}) = o\big(|x'|^{-|\beta'|}\big) \quad \forall \alpha : |\alpha| = 2m, \forall \beta' : |\beta'| \le K,$$

$$(12.2.49)_2 \quad D^{\beta'}(\sigma - \sigma_\infty) = o\big(|x'|^{-\mu-|\beta'|}\big) \qquad \forall \beta' : |\beta'| \le K$$

be fulfilled as $|x'| \to \infty$ *where* $a_{\infty,\alpha} = a_{\infty,\alpha}(x')$, $\sigma_\infty(x')$ *are positively homogeneous functions of* x' *of degrees* 0, $-\mu$ *respectively (surely,* $a_{\infty,\alpha}$ *can be matrix-valued).*

---

[7] If $t(\tau)$ works then every function $t'(\tau) \le t(\tau)$ such that $t'(\tau) \to +\infty$ as $\tau \to +\infty$ also works.

Let us consider operator-valued symbol

$$\boldsymbol{a}_\infty(x',\xi';D'') = \sum_{|\alpha|=2m} a_{\infty,\alpha}\sigma_\infty^{-|\alpha''|}\xi'^{\alpha'}D''^{\,\alpha''} \tag{12.2.50}$$

in $\mathbb{H}$*; let* $\zeta_j(x',\xi')$ *be its (non-decreasing) eigenvalues*[8]*.*

Finally, let us consider $\boldsymbol{a}_{\infty,j}(x',\xi') = \boldsymbol{a}_\infty(x',\xi')\boldsymbol{e}_\infty(x',\xi',\zeta_j(x',\xi'))$ *where* $\boldsymbol{e}_\infty(x',\xi',\tau)$ *is the spectral projector for* $\boldsymbol{a}_\infty(x',\xi')$ *and assume that*

(12.2.51) *For each* j *the standard non-periodicity condition is fulfilled for matrix-valued symbol* $\boldsymbol{a}_{\infty,j}$ *which means that with the exception of measure* 0 *on* $\Sigma_j = \{1 \in \operatorname{Spec} a_{\infty,j}(z)\}$ $\Psi_{j,t}(z) \not\ni z$ *and* $\Psi_{j,t}(z) \cap 0 \times \mathbb{R}^{d'} = \emptyset$ *for all* t *of the proper sign (depending on* $z = (x',\xi')$*) where* $\Psi_{j,t}$ *is a generalized Hamiltonian flow for* $\boldsymbol{a}_{\infty,j}$*.*

Then asymptotics

$$\mathsf{N}^-(\tau) = \mathcal{N}(\tau) + \bigl(\kappa_1^* + o(1)\bigr)\tau^{\frac{1}{2m\mu}(d'-1)(\mu+1)} \tag{12.2.52}$$

hold as $\tau \to +\infty$ *with an appropriate coefficient* κ_1^* *(given in Remark 4.4.13).*

Condition (12.2.51) looks extremely cumbersome and impossible to check. However in the following rather special example this condition becomes much simpler:

Example 12.2.9. Let A satisfy (12.2.31) with $\bar{a} = |\xi|^2$; then eigenvalues of $\boldsymbol{a}_\infty$ are $\zeta_j(x',\xi') = \bigl(|\xi'|^2 + \sigma^{0\,-2}(x')\lambda_j\bigr)$ and they have constant multiplicities, λ_j are eigenvalues of $\Delta_{Y,\mathsf{D}}$.

Then one needs to check the standard non-periodicity condition for symbols $\zeta_j(x',\xi')$ and all of them are equivalent to the same standard non-periodicity condition for a scalar symbol

$$\zeta(x',\xi') = |\xi'|^2 + \sigma_\infty(x')^{-2} \tag{12.2.53}$$

(on the energy level 1).

In this case we need to check only check the standard non-periodicity condition for (12.2.53). Note that

(12.2.54) As $d' \ge 2$ the generic trajectory does not hit $0 \times \mathbb{R}^{d'}$.

[8] Then obviously $\zeta_j(tx', t^\mu\xi') = t^{2m\mu}\zeta_j(x',\xi')$.

Really, $\mu(\Sigma \cap \{|x'| \le 2\varepsilon\}) = O(\varepsilon^{d'})$ where $\mu = dx'd\xi' : d\zeta$ is the standard measure on $\Sigma = \{(x',\xi') : \zeta(x',\xi') = 1\}$ and trajectory intersecting $\{|x'| \le \varepsilon\}$ needs at least $\epsilon_0\varepsilon$-time to leave $\{|x'| \le 2\varepsilon\}$; therefore the μ-measure of points such that Ψ_t touches $\{|x'| \le \varepsilon\}$ for some $t : |t| \le T$ does not exceed $CT\varepsilon^{d'-1}$.

In particular, the standard non-periodicity condition for $\zeta(x',\xi')$ is fulfilled as $\sigma_\infty = |x'|^{-\mu}$, $\mu > 0$ and $\mu \ne 1$; recall that $d' \ge 2$.

Finally, let us consider the case $d' = 1 + d''\mu$. Considering diagonal operator with diagonal elements $\zeta_j^{\mathsf{w}}(x', D')$ one can prove easily the remainder estimate $o(\tau^{(d-1)/2m}\log\tau)$ (not subject to any non-periodicity condition). This leads us to conjecture which we were unable to prove

Conjecture 12.2.10. *As $d' = 1+d''\mu$ and stabilization conditions* $(12.2.49)_{1,2}$ *are fulfilled, remainder estimate is in fact $o(\tau^{(d-1)/2m}\log\tau)$.*

We leave to the reader the following easy

Problem 12.2.11. (i) Generalize Theorem 12.2.8 to the case when a_∞ depends on x''; then one should assume that $a_\infty(x', \sigma_\infty(x')x'')$ is positively homogeneous of degree 0 with respect to x'; then $\mathsf{a}_\infty = \mathsf{a}_\infty(x',\xi';x'',D'')$ and one should take Weyl quantization in the right-hand expression of (12.2.50).

(ii) Treat Riesz means and derive asymptotics with the remainder estimates

$$R^*_{(\vartheta)}(\tau) = \begin{cases} \tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta} & \text{for } d' < \mu d'' + 1 + \vartheta, \\ \tau^{\frac{1}{2m}(d-1-\vartheta)+\vartheta}\log\tau & \text{for } d' = \mu d'' + 1 + \vartheta, \\ \tau^{\frac{1}{2m\mu}(d'-1-\vartheta)(\mu+1)+\vartheta} & \text{for } d' > \mu d'' + 1 + \vartheta. \end{cases} \tag{12.2.55}$$

and in the analogs of Theorems 12.2.7 and 12.2.8 one should assume that $d' < \mu d'' + \vartheta + 1$, $d' > \mu d'' + \vartheta + 1$ respectively.

12.2.3 Effective Asymptotics

Scheme of Effective Asymptotics

The main defect of Theorems 12.2.6–12.2.8 is that the expression for $\mathcal{N}(\tau)$ is not effective and includes the eigenvalue counting function for the *family* of auxiliary operators $\mathsf{a}(x',\xi') = \mathsf{a}(x',\xi';x'',D'')$. We would like to derive a formula including the eigenvalue counting function for *one* auxiliary

operator or at least for a family of auxiliary operators with a smaller number of parameters (under certain conditions). We are aware that for a thick cusp (i.e., with $\sigma^{d''-1-\vartheta} \notin \mathscr{L}^1(Z)$) we cannot completely get rid of the auxiliary operators in the final answer.

Let us start with a more accurate Weyl analysis. Let $U(x,y,t)$ be the Schwartz kernel of the operator $\exp(it\tau^{-1}A)$. Then

$$\tau D_t U = A_x U, \qquad U|_{t=0} = \delta(x-y)I_{\mathbb{K}} \tag{12.2.56}$$

with $\mathbb{K} = \mathbb{C}^{\mathsf{D}}$.

Let us fix a point $\bar{x} \in X$ and consider its $\bar{\sigma}$-neighborhood. As usual, let us rescale $x_{\mathsf{new}} = (x_{\mathsf{old}} - \bar{x})\bar{\sigma}^{-1}$ and divide A by τ; then we transform A into an h_1-differential operator A_1 with $h_1 = \bar{\sigma}^{-1}\tau^{-\frac{1}{2m}}$. Introducing a new time scale $t_1 = th_1$ we obtain from (12.2.56) the scaled problem

$$h_1 D_{t_1} U = A_{1x} U, \qquad U|_{t_1=0} = \bar{\sigma}^{-d}\delta(x-y)I_{\mathbb{K}} \tag{12.2.56}_1$$

where we skip the subscript "new" at x and y. Then one can apply the results of Section 4.3; moreover, on the energy level 1 the microhyperbolicity condition is fulfilled provided h_1 is small enough.

Let $\bar{\chi} \in \mathscr{C}_0^K([-1,1])$, $\bar{\chi} = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and let ψ' be $\bar{\sigma}$-admissible and be supported in the $\bar{\sigma}$-neighborhood of $\bar{x}$. Further, let $h_1 \le \varepsilon$ (i.e., $\bar{\sigma} \ge c_0\tau^{-\frac{1}{2m}}$), and let $T_1 = \varepsilon h_1^{-1}$ where here and below $\varepsilon > 0$ is a small enough constant. Then according to Section 4.3 we obtain the following estimate:

$$\begin{aligned} \Big| \int \Big(F_{t\to\lambda}\bar{\chi}_{T_1}(t)\Gamma_x U - \sum_{0\le j\le J-1} \kappa'_j(x,\lambda\tau)^{\frac{1}{2m}(d-j)-1}\tau \Big)\psi'\,dx - \\ \int_{\partial X} \sum_{1\le j\le J-1} \tilde{\kappa}'_j(x,\lambda\tau)^{\frac{1}{2m}(d-j)-1}\tau\psi'\,dS \Big| \le C\tau^{\frac{1}{2m}(d-J)}\bar{\sigma}^{d-J} \end{aligned} \tag{12.2.57}$$

where $\lambda \in [\epsilon_0, c]$ here and below, dS is the standard Riemannian density on ∂X, κ'_j, $\tilde{\kappa}'_j$ are the standard Weyl coefficients and

$$|\kappa'_j| \le C\bar{\sigma}^{-j}, \qquad |\tilde{\kappa}'_j| \le C\bar{\sigma}^{1-j}. \tag{12.2.58}$$

Recall that one can obtain these coefficients in the following way:

$$\mathsf{Tr}\big(\mathsf{Res}_{\mathbb{R}}\,\psi'(\tau - A)^{-1}\big) \sim \int \sum_j \psi'\kappa'_j\tau^{\frac{1}{2m}(d-j)-1}\,dx \tag{12.2.59}$$

and

$$\mathsf{Tr}\big(\mathsf{Res}_{\mathbb{R}}\, \psi'\big((\tau - A)_D^{-1} - (\tau - A)^{-1}\big)\big) \sim \sum_j \int_{\partial X} \psi' \tilde{\kappa}'_j \tau^{\frac{1}{2m}(d-j)-1}\, dS \tag{12.2.60}$$

where $(\tau - A)^{-1}$ and $(\tau - A)_{\mathsf{D}}^{-1}$ are parametrices of the pseudodifferential operator $(\tau - A)$ in terms of the formal calculus of pseudodifferential operators and the Dirichlet problem for the same operator in terms of the formal calculus of pseudodifferential operators (with the transmission property) and Green operators (see, e.g., Section 1.4), and for the sake of simplicity the ordinary matrix trace tr is included in the operation Γ_x in (12.2.57).

Therefore under the same conditions for the $\bar{\gamma}$-admissible function $\psi = \psi(x')$ supported in the $\bar{\gamma}$-neighborhood of $\bar{x}'$ estimate

$$\begin{aligned} \Big| \int \Big(F_{t\to\lambda} \bar{\chi}_{T_1}(t) \Gamma_x U - \sum_{0\le j\le J-1} \kappa'_j(x, \lambda\tau)^{\frac{1}{2m}(d-j)-1} \tau \Big) \psi\, dx - \\ \int_{\partial X} \sum_{1\le j\le J-1} \tilde{\kappa}'_j(x, \lambda\tau)^{\frac{1}{2m}(d-j)-1} \tau \psi'\, dS \Big| \le C \tau^{\frac{1}{2m}(d-J)} \bar{\sigma}^{d''-J} \bar{\gamma}^{d'} \end{aligned} \tag{12.2.61}$$

holds where the right-hand expression acquired factor $\bar{\gamma}^{d'} \bar{\sigma}^{-d'}$.

So far there is nothing beyond analysis of Sections 11.2 and 11.3. However, let us take another point of view. Namely, let us treat the $\bar{\gamma}$-neighborhood of $\bar{x}'$, rescale $x'_{\mathsf{new}} = (x'_{\mathsf{old}} - \bar{x}')\bar{\gamma}^{-1}$ and divide A by τ; earlier we transformed X_{cusp} to a cylinder in the routine way.

Then A is transformed to a h_2-differential operator (with respect to x') A_2 with the operator-valued symbol; surely $h_2 = \bar{\gamma}^{-1}\tau^{-\frac{1}{2m}} \ll 1$ here. Introducing the new time scale $t_2 = t h_2 \tau^{-1}$ we obtain from (12.2.56) another scaled problem

$$\tau D_{t_2} U = A_{2x} U, \qquad U|_{t_2=0} = \bar{\gamma}^{-d'} \delta(x' - y') I_{\mathbb{H}}. \tag{12.2.56)$_2$}$$

Assume that A_2 is a microhyperbolic operator with the operator-valued symbol[6]. Let us apply the strongest version of Theorem 2.1.19 to estimate

$$\Big| \int F_{t\to\lambda} \chi_T(t) \Gamma_x U \psi'\, dx \Big| \tag{12.2.62}$$

with $\chi \in \mathscr{C}_0^K([-1,1])$ vanishing on $[-\frac{1}{2}, \frac{1}{2}]$.

Namely, formula (4.3.52) yields that in the right-hand expressions in (2.1.58) and (2.1.59) one can add the factor $h = h_2$, i.e., one can take the right-hand expressions to be $h^{1-d-|\alpha|}\bigl(\frac{h}{T}\bigr)^s$.

Therefore scaling we obtain that

Proposition 12.2.12. *Let A_2 be a microhyperbolic operator with the operator-valued symbol, $\chi \in \mathscr{C}_0^K([-1,1])$ vanishing on $[-\frac12,\frac12]$ and ψ $\bar\gamma$-admissible and supported in the $\bar\gamma$-neighborhood of $\bar x'$.*

Then estimate

$$\|F_{t\to\lambda}B_1^j\chi'_T\Gamma'\psi UB_1^k\| \le Ch_2^{-d'}T^{-s} \tag{12.2.63}$$

holds for any $T \in [\epsilon_0, T_2]$, $T_2 = \epsilon h_2^{-1}$; here Γ' means that we take $x' = y'$ and integrate over x', and we take the operator norm in $\mathbb{H}$ in the left-hand expression.

Applying Sobolev embedding theorem we take the trace norm in the left-hand expression with the price of additional factor $h_1^{-d''}$ in the right-hand expression and shedding of some factors B_1^j and B_1^k; we still assume that $\bar\sigma \ge c_0\tau^{-\frac{1}{2m}}$.

Therefore we conclude that

(12.2.64) Expression (12.2.62) does not exceed $Ch_2^{-d'}T^{-s}h_1^{-d''}$.

Taking an appropriate partition of unity in the interval $[T_1', T_2']$ we obtain estimate

$$\bigl|F_{t\to\lambda}\bigl(\bar\chi_{T_2'}(t)-\bar\chi_{T_1'}(t)\bigr)\Gamma\psi U\bigr| \le Ch_2^{-d'}h_1^{-d''}T_1'^{-s} \tag{12.2.65}$$

for $\epsilon \le T_1' < T_2' \le T_2$.

Let us set $T_j' = T_j$ $(= \varepsilon h_j^{-1})$. Then estimates (12.2.65) and (12.2.61) yield that

(12.2.66) If A_2 is a microhyperbolic operator with the operator-valued symbol then estimate (12.2.61) remains true with T_1 replaced by T_2.

Applying the routine Tauberian technique we then obtain then the following estimate

$$\Big|\Big(\int \psi\big(\mathsf{tr}(e(x,x,\tau)) - \sum_{0\le j\le J-1} \kappa_j(x)\tau^{\frac{1}{2m}(d-j)}\big)\,dx - \int_{\partial X}\psi \sum_{1\le j\le J-1} \tilde{\kappa}_j(x)\tau^{\frac{1}{2m}(d-j)}\,dS\Big)\Big|_{\tau=\tau'}^{\tau=\tau}\Big| \le C\bar{\sigma}^{d''}\bar{\gamma}^{d'-1}\tau^{\frac{1}{2m}(d-1)} + C\bar{\sigma}^{d''-J}\bar{\gamma}^{d'}\tau^{\frac{1}{2m}(d-J)} \tag{12.2.67}$$

provided $\tau \ge \epsilon\bar{\sigma}^{-2m}$, $\tau' \asymp \tau$ and $J \le d$.

For $J > d$ one should replace in the left-hand expression the terms with $j = d$ by $\kappa_d \log\tau$ and $\tilde{\kappa}_d \log\tau$. Here and below the Weyl coefficients κ_j and $\tilde{\kappa}_j$ satisfy (12.2.58) [9].

Thus, *using propagation of singularities for operators with the operator-valued symbols we have improved the Weyl asymptotics in the cusp zone* $\{x : \sigma(x') \ge \tau^{-\frac{1}{2m}}\}$.

However now we need to take into account many terms in the decomposition which undermines our intention to get an effective asymptotics. To remedy this we consider A as a perturbation of some other operator A^0. So let A^0 be another operator in the domain X^0 with the same cusp part X_{cusp} also satisfying all the conditions above with the same exponents. Moreover, let us assume that the operators A and A^0 are close at infinity in the sense that

$$\big|D^\beta(a_\alpha - a^0_\alpha)\big| \le C\eta\langle x'\rangle^{-|\beta'|+\mu(2m-|\alpha|+|\beta''|)} \qquad \forall\alpha : |\alpha| \le 2m \quad \forall\beta : |\beta| < K \tag{12.2.68}$$

with

$$\eta = \eta(x') \asymp \langle x'\rangle^{-\nu}, \quad \nu > 0. \tag{12.2.69}$$

Remark 12.2.13. In most of the arguments below one need only assume that η is a γ-admissible weight function and $\eta = o(1)$ as $|x'| \to \infty$.

All the objects associated with A^0 have an additional superscript "0".

Let us introduce the formal differential expression $A^\theta = A^0 + \theta(A - A^0)$ with $\theta \in [0,1]$; then $A^1 = A$. We also assign Dirichlet boundary conditions to

[9] A similar estimate also holds for Riesz means. Namely, the first term in the right-hand expression of (12.2.67) should be multiplied by $\tau^\vartheta\bar{\sigma}^\vartheta\bar{\gamma}^{-\vartheta}$ and the second term should be multiplied by τ^ϑ.

A^θ. Below all the objects associated with A^θ have the additional superscript "θ". Then applying estimate (12.2.67) to both operators we obtain estimate

$$\begin{aligned}(12.2.70)\quad \Big|\Big(\int \psi\Big(e(x,x,\tau)-e^0(x,x,\tau)-\\ \sum_{0\le j\le J-1}\big(\kappa_j(x)-\kappa_j^0(x)\big)\tau^{\frac{1}{2m}(d-j)}\Big)\,dx-\\ \int_{\partial X}\psi\sum_{1\le j\le J-1}\big(\tilde\kappa_j(x)-\tilde\kappa_j^0(x)\big)\tau^{\frac{1}{2m}(d-j)}\,dS\Big)\Big|_{\tau=\tau'}^{\tau=\tau}\Big|\le\\ C\bar\sigma^{d''}\bar\gamma^{d'-1}\tau^{\frac{1}{2m}(d-1)}+C\bar\sigma^{d''-J}\bar\gamma^{d'}\tau^{\frac{1}{2m}(d-J)}.\end{aligned}$$

provided $J\le d$; otherwise terms with $j=d$ contain the additional factor $\log\tau$.

Moreover, differentiating representations (12.2.59), (12.2.60) with respect to θ one can easily see that

$$(12.2.71)\qquad |\partial_\theta\kappa_j^\theta(x)|\le c_1\bar\eta\bar\sigma^{-j},\qquad |\partial_\theta\tilde\kappa_j^\theta(x)|\le c_1\bar\eta\bar\sigma^{1-j}$$

in the same notations as before; we first obtain these estimates for the coefficients $\kappa_j'^\theta$, $\tilde\kappa_j'^\theta$.

Integrating on θ from 0 to 1 and passing from ψ' to ψ we obtain that

$$(12.2.72)\qquad |\kappa_j(x)-\kappa_j^0(x)|\le c_1\bar\eta\bar\sigma^{1-j},\qquad |\tilde\kappa_j(x)-\tilde\kappa_j^0(x)|\le c_1\bar\eta\bar\sigma^{1-j}$$

These estimates and (12.2.70) yield that

$$\begin{aligned}(12.2.73)\quad \Big|\Big(\int \psi\Big(e(x,x,\tau)-e^0(x,x,\tau)-\\ \sum_{0\le j\le J-1}\big(\kappa_j(x)-\kappa_j^0(x)\big)\tau^{\frac{1}{2m}(d-j)}\Big)\,dx-\\ \int_{\partial X}\psi\Big(\sum_{1\le j\le J-1}\big(\tilde\kappa_j(x)-\tilde\kappa_j^0(x)\big)\tau^{\frac{1}{2m}(d-j)}\,dS\Big)\Big|_{\tau=\tau'}^{\tau=\tau}\Big|\le\\ C\bar\sigma^{d''}\bar\gamma^{d'-1}\tau^{\frac{1}{2m}(d-1)}+C\bar\sigma^{d''-s}\bar\gamma^{d'}\tau^{\frac{1}{2m}(d-s)}+C\bar\eta\bar\sigma^{d''-J}\bar\gamma^{d'}\tau^{\frac{1}{2m}(d-J)}.\end{aligned}$$

with arbitrarily large exponent s where K surely depends on s.

This estimate is accurate enough to obtain the final theorems under certain restrictions on ν from below (we leave this easy task to the reader).

However, in order to prove these theorems under weaker conditions one needs a more refined analysis. Namely, *we want to shed of the second term in the right-hand expression of* (12.2.73) which is not nice for h_1 close to 1 (i.e., for $\bar\gamma$ close to $\tau^{\frac{1}{2m\mu}}$).

Remark 12.2.14. Without any loss of the generality we can assume that $X = X^0$. Really, one can arrange this by replacing both X and X^0 with $X^1 := X_{\mathsf{cusp}} \cap \{|x'| \ge c_0\}$ (or possibly $X_{\mathsf{cusp}} \cap \{x_1 \ge c_0\}$ in the case $d' = 1$) and taking Dirichlet boundary conditions on ∂X; surely the operators remain self-adjoint; moreover as $\mathfrak{D}(A^0) = \mathfrak{D}(A^1)$.

Surely, as we do this function $e^\theta(x, y, \tau)$ changes because we changed the geometrical domains. However the routine Tauberian arguments yield that in the frameworks of Theorems 12.2.6–12.2.8 the errors appearing in $\int \psi e^\theta(x, x, \tau)\, dx$ (with ψ supported in $\{|x'| \ge 2c_0\}$) do not exceed the remainder estimates in the asymptotics of these terms in the corresponding theorems and since we are not going to improve these remainder estimates but to provide a better principal part everything is fine.

Then operators $A^\theta = A^0 + \theta(A - A^0)$ with $\mathfrak{D}(A^\theta) = \mathfrak{D}(A^0)$ are self-adjoint and all the functions $e^\theta(x, y, \tau)$ and $U^\theta(x, y, t)$ do not lose their meanings as Schwartz kernels of $E^\theta(\tau)$ and $\exp(i\tau^{-1}tA^\theta)$ respectively as $\theta \in [0, 1]$; $E^\theta(.)$ is the spectral projector of A^θ.

Then

$$\partial_\theta \exp\bigl(it\tau^{-1}A^\theta\bigr) = i\tau^{-1}\int_0^t \exp\bigl(it'\tau^{-1}A^\theta\bigr)\cdot\bigl(A^1 - A\bigr)\cdot\exp\bigl(i\tau^{-1}(t-t')A^\theta\bigr)\,dt' \tag{12.2.74}$$

and one can easily prove that for $\lambda \in [\epsilon, c]$

$$\bigl|F_{t\to\lambda}\bar\chi_T(t)\partial_\theta U^\theta(x, y, t)\bigr| \le C\bar\eta h_1^{-p}T \tag{12.2.75}$$

for any $T > 0$ with large enough p. One can improve this estimate and calculate p but this improvement provides no advantage for arguments below.

We want to calculate (12.2.74) after we multiply it by $\bar\chi_T(t)\psi(x)$ and calculate the trace. Let us do it first as T is small enough: applying routine arguments associated with the successive approximation method

and increasing T_1 from $h_1^{\frac12+\delta}$ (*only here,* the subscript 1 at T means that we use the t_1-scale) to ϵ in Sections 4.3 and 7.2 we conclude that

$$\begin{aligned}(12.2.76)\quad \Big|\partial_\theta\Big(\int\Big(F_{t\to\lambda}\bar\chi_T(t)\Gamma_x U^\theta - \sum_{0\le j\le J-1}\kappa_j'^\theta(\lambda\tau)^{\frac{1}{2m}(d-j)-1}\tau\Big)\psi'\,dx-\\ \int_{\partial X}\sum_{1\le j\le J-1}\tilde\kappa_j'^\theta(\lambda\tau)^{\frac{1}{2m}(d-j)-1}\psi'\,dS\Big)\Big|\le C\bar\eta\tau^{\frac{1}{2m}(p-J)}\bar\sigma^{p-J}\end{aligned}$$

where $T=T_1=\epsilon h_1^{-1}$ and p is large enough[10)].

Therefore estimate (12.2.76) holds with $\psi(x')$ (which is γ-admissible) instead of ψ' (which was σ-admissible) and with the additional factor $h_2^{-d'}h_1^{d'}$ in the right-hand expression.

Moreover, applying arguments associated with operators the with operator-valued symbols one can easily obtain from (12.2.75) that

$$(12.2.77)\qquad \big|F_{t\to\lambda}\chi_T(t)\Gamma\big(\psi\partial_\theta U^\theta\big)\big|\le \bar\eta h_2^{-d'}h_1^{-p}T^{-s}$$

for $T\in[T_1,T_2]$ and therefore

$$(12.2.78)\qquad \big|F_{t\to\lambda}\big(\bar\chi_{T_2'}-\bar\chi_{T_1'}\big)\Gamma\big(\psi\partial_\theta U^\theta\big)\big|\le C\bar\eta h_2^{-d'}h_1^{-p}T_1'^{-s}$$

with the same T_1, T_2, T_1', T_2', χ and $\bar\chi$ as in (12.2.65) and (12.2.65).

Therefore estimate (12.2.76) holds with $T=T_2$ instead of $T=T_1$. Integrating over θ from 0 to 1 we see that

(12.2.79) Estimate (12.2.76) holds with $T=T_2$ and with the derivative on θ replaced by the difference operator; so $\partial_\theta f^\theta$ is replaced by (f^1-f^0).

Let us apply the Tauberian technique. Surely, $\Gamma\big(\psi\big(e(.,.,\tau)-e^0(.,.,\tau)\big)\big)$ is not necessarily a monotone function of τ and one should apply the estimate

$$(12.2.80)\quad \Big|\Gamma\Big(\psi\big(e^\theta(.,.,\tau)-e^\theta(.,.,\tau')\big)\Big)\Big|\le \bar\gamma^{d'}\bar\sigma^{d''}\tau^{\frac{d}{2m}}h_2=\bar\gamma^{d'-1}\bar\sigma^{d''}\tau^{\frac{1}{2m}(d-1)}$$

for $|\tau'-\tau|\le h_2$. This estimate was obtained by the Tauberian method (see Sections 4.2 and 4.4) for functions $\Gamma\big(\psi e^\theta(.,.,\tau)\big)$ which are monotone of τ.

[10)] However, considering this estimate with large $J'=J+p$ instead of J and applying estimates (12.2.58) one can reach $p=d$.

This gives the improved estimate (12.2.73):

$$\begin{aligned}(12.2.81)\quad \Big|\Big(\int\Big(\big(e(x,x,\tau)-e^0(x,x,\tau)\big)\\ -\sum_{0\le j\le J-1}\big(\kappa_j(x)-\kappa_j^0(x)\big)\tau^{\frac{1}{2m}(d-j)}\Big)\psi\,dx-\\ \int_{\partial X}\sum_{1\le j\le J-1}\big(\tilde\kappa_j(x)-\tilde\kappa_j^0(x)\big)\tau^{\frac{1}{2m}(d-j)}\psi\,dS\Big)\Big|_{\tau=\tau'}^{\tau=\tau}\Big|\le\\ C\bar\sigma^{d''}\bar\gamma^{d'-1}\tau^{\frac{1}{2m}(d-1)}+C\bar\eta\bar\sigma^{d''-J}\bar\gamma^{d'}\tau^{\frac{1}{2m}(d-J)}.\end{aligned}$$

Recall that $\tau'\in[\epsilon\tau,\tau]$, $\tau\bar\sigma^{2m}\ge\epsilon$ here and terms with $j=d$ contain the factor $\log\tau$ [11)].

Therefore, the following statement has been proven:

Proposition 12.2.15. *(i) In the framework of Theorem 12.2.6 estimate* (12.2.67) *holds.*

(ii) Moreover, let both operators A and A^0 satisfy assumptions of Theorem 12.2.6 and let (12.2.68) *be fulfilled (in $X_{\mathsf{cusp}}\cap|x'-\bar x'|\le\bar\gamma$). Then estimate* (12.2.81) *holds.*

Recall that ψ is γ-admissible and supported in the $\bar\gamma$-neighborhood of $\bar x'$. Let us replace τ and τ' by $2^{-n}\tau$ and $2^{-n-1}\tau$ and sum over $n=0,\dots,n_0=c_0\lfloor\log(\tau\bar\sigma^{2m})\rfloor+c_1$. Applying Proposition 12.2.4 we obtain

Corollary 12.2.16. *(i) In the framework of Theorem 12.2.6 estimate*

$$\begin{aligned}(12.2.82)\quad \mathsf{R}_1:=\Big|\int\Big(\mathsf{tr}(e(x,x,\tau))-\sum_{0\le j\le J-1}\kappa_j(x)\tau^{\frac{1}{2m}(d-j)}\Big)\psi\,dx-\\ \int_{\partial X}\sum_{1\le j\le J-1}\tilde\kappa_j(x)\tau^{\frac{1}{2m}(d-j)})\psi\,dS\Big|\le\\ C\bar\sigma^{d''}\bar\gamma^{d'-1}\tau^{\frac{1}{2m}(d-1)}+C\bar\sigma^{d''-J}\bar\gamma^{d'}\tau^{\frac{1}{2m}(d-J)}\end{aligned}$$

holds for $J<d$; for $J=d$ estimate

$$(12.2.83)\qquad \mathsf{R}_1\le C\bar\sigma^{d''}\bar\gamma^{d'-1}\tau^{\frac{1}{2m}(d-1)}+C\bar\sigma^{-d'}\bar\gamma^{d'}\log(\tau\bar\sigma^{2m})$$

[11)] One can see easily that for Riesz means both terms in the right-hand expression should be modified in the way indicated in [9)].

holds and for $J = d+1$ *the estimate*

$$\mathsf{R}_1 \le C\bar\sigma^{d''}\bar\gamma^{d'-1}\tau^{\frac{1}{2m}(d-1)} + C\bar\sigma^{-d'}\bar\gamma^{d'} \tag{12.2.84}$$

holds, and in the latter case the terms with $j = d$ *are* $\kappa_d \log(\tau\bar\sigma^{2m})$ *and* $\tilde\kappa_d \log(\tau\bar\sigma^{2m})$.

(ii) In the framework of Proposition 12.2.15(ii) estimates

$$\begin{aligned}\mathsf{R}_2 \coloneqq \Big|\Big(\int\Big(\mathsf{tr}(e^\theta(x,x,\tau) - \sum_{0\le j\le J-1}\kappa_j^\theta(x)\tau^{\frac{1}{2m}(d-j)}\Big)\psi\,dx - \\ \int_{\partial X}\sum_{1\le j\le J-1}\tilde\kappa_j^\theta(x)\tau^{\frac{1}{2m}(d-j)}\psi\,dS\Big)\Big|_{\theta=0}^{\theta=1}\Big| \le \\ C\bar\sigma^{d''}\bar\gamma^{d'-1}\tau^{\frac{1}{2m}(d-1)} + C\bar\eta\bar\sigma^{d''-J}\bar\gamma^{d'}\tau^{\frac{1}{2m}(d-J)},\end{aligned} \tag{12.2.85}$$

$$\mathsf{R}_2 \le C\bar\sigma^{d''}\bar\gamma^{d'-1}\tau^{\frac{1}{2m}(d-1)} + C\bar\eta\bar\sigma^{-d'}\bar\gamma^{d'}\log(\tau\bar\sigma^{2m}) \tag{12.2.86}$$

and

$$\mathsf{R}_2 \le C\bar\sigma^{d''}\bar\gamma^{d'-1}\tau^{\frac{1}{2m}(d-1)} + C\bar\eta\bar\sigma^{-d'}\bar\gamma^{d'} \tag{12.2.87}$$

hold for $J < d$, $J = d$, $J = d+1$ *respectively (we remind the reader that terms with* $j = d$ *contain a logarithmic factor).*

Then after summation over a γ-admissible partition of unity in Z we can obtain miscellaneous formulas (later under certain conditions we will calculate the terms associated with the operator A^0):

(i) Let us sum the estimates for R_2 in the domain $\{t_1 \le |x'| \le c_0\tau^{\frac{1}{2m\mu}}\}$ and apply Theorem 12.2.2 in the domain $\{|x'| \le 2t_1\}$. Then we derive the following formula

$$\begin{aligned}\mathsf{N}^-(\tau) = \\ \sum_{j=0,1}\int\phi\big(\frac{x'}{t_1}\big)\kappa_j(x)\tau^{\frac{1}{2m}(d-j)}\,dx + \int\phi\big(\frac{x'}{t_1}\big)\tilde\kappa_1(x)\tau^{\frac{1}{2m}(d-1)}\,dS +\end{aligned} \tag{12.2.88}$$

$$\sum_{j=0,1}\int \bigl(1-\phi(\frac{x'}{t_1})\bigr)\bigl(\kappa_j(x)-\kappa_j^0(x)\bigr)\tau^{\frac{1}{2m}(d-j)}\,dx+$$
$$\int \phi(\frac{x'}{t_1})\bigl(\tilde\kappa_1(x)-\tilde\kappa_1^0(x)\bigr)\tau^{\frac{1}{2m}(d-1)}\,dS+$$
$$\int \bigl(1-\phi(\frac{x'}{t_1})\bigr)\,\mathsf{tr}\bigl(e^0(x,x,\tau)\bigr)\,dx+$$
$$O\bigl(\mathsf{R}^*(\tau)+\tau^{\frac{1}{2m\mu}(d'(\mu+1)-\nu)}\bigr)$$

where $R^*(\tau)$ was introduced by (12.2.41) and here and below $\phi\in \mathscr{C}_0^K(\mathbb{R}^{d'})$, $\phi=1$ in a neighborhood of the origin.

Remark 12.2.17. In formula (12.2.88) we have taken $J=2$ because we gain nothing by increasing J and so we have dropped all the terms with $j\ge 2$.

Further, in (12.2.88) t_1 is either a large enough constant or a slowly increasing function of τ. Moreover, one can skip terms with $j=1$ in the third and fourth items in the right-hand expression in all cases excluding $d'=(d''-1)\mu+\nu$, $\nu<\mu+1$ when the remainder estimate is $O(\tau^{\frac{1}{2m}(d-1)})$, and the contribution of this term is $O\bigl(\tau^{\frac{1}{2m}(d-1)}\log\tau\bigr)$ (and this estimate is the best possible).

Furthermore, for $t_1=\mathsf{const}$ one can skip terms with $j=1$ in the first and second items in the right hand expression. In any case, one can effectively calculate all terms with $j=0,1$.

(ii) Consider the case $d'<1+\mu d''$, $\nu>\mu(1-d'')+d'$. In this case the remainder estimate in (12.2.88) is $O\bigl(\tau^{\frac{1}{2m}(d-1)}\bigr)$.

Moreover, the contribution of the zone $\{t_1\le |x'|\}$ to the remainder estimate is $o\bigl(\tau^{\frac{1}{2m}(d-1)}\bigr)$ for $t_1=t_1(\tau)\to+\infty$ as $\tau\to+\infty$. Then, applying the improved Weyl estimate in the zone $\{|x'|\le t_1\}$ under the standard non-periodicity condition to the Hamiltonian billiards we obtain asymptotics

(12.2.89) $\mathsf{N}^-(\tau)=$
$$\sum_{j=0,1}\int \phi(\frac{x'}{t_1})\kappa_j(x)\tau^{\frac{1}{2m}(d-j)}\,dx+\int \phi(\frac{x'}{t_1})\tilde\kappa_1(x)\tau^{\frac{1}{2m}(d-1)}\,dS+$$
$$\int \bigl(1-\phi(\frac{x'}{t_1})\bigr)\bigl(\kappa_0(x)-\kappa_0^0(x)\bigr)\tau^{\frac{d}{2m}}\,dx+$$
$$\int \bigl(1-\phi(\frac{x'}{t_1})\bigr)\,\mathsf{tr}\bigl(e^0(x,x,\tau)\bigr)\,dx+o\bigl(\tau^{\frac{1}{2m}(d-1)}\bigr)$$

where we have dropped terms with $j = 1$ in the third and fourth items in the right-hand expression of (12.2.89).

(iii) Consider now an opposite case $d' > 1 + \mu d''$, $\nu > \mu + 1$; then the remainder estimate in (12.2.88) is $O\big(\tau^{\frac{1}{2m\mu}(d'-1)(\mu+1)}\big)$. Moreover, in this case one can drop all terms with $j = 1$.

Furthermore, the contribution of the zone $\{|x'| \le t_2\}$, $t_2 = \varepsilon\tau^{\frac{1}{2m\mu}}$ to the remainder estimate is $\varepsilon'\tau^{\frac{1}{2m\mu}(d'-1)(\mu+1)}$ with $\varepsilon' = \varepsilon'(\varepsilon) \to 0$ as $\varepsilon \to 0$.

On the other hand, for every fixed $\varepsilon > 0$ in the zone $t_2(\tau) \le |x'| \le c_0\tau^{\frac{1}{2m\mu}}$ in the context of Theorem 12.2.7 one can improve the remainder estimate. Namely, taking a standard partition of unity in (x', ξ')-space and applying the standard Tauberian arguments we obtain that the contributions of both elements of this partition do not exceed $\varepsilon''\tau^{\frac{1}{2m\mu}(d'-1)(\mu+1)}$ with arbitrarily small $\varepsilon'' > 0$ because the support of the first element is small enough and in the second element one can take $T_2 = (h_2\varepsilon''')^{-1}$ with arbitrarily small $\varepsilon''' > 0$ in the Tauberian arguments.

Remark 12.2.18. The only element of these arguments which is not completely standard is that we apply the propagation results derived from the theory of operators with the operator-valued symbols to Weyl asymptotics.

This approach will be heavily used in forthcoming sections and chapters.

Then we obtain the following asymptotics

$$\begin{aligned}(12.2.90)\quad \mathsf{N}^-(\tau) = \int \phi\big(\frac{x'}{t_1}\big)\kappa_j(x)\tau^{\frac{d}{2m}}\,dx + \\ \int \big(1-\phi\big(\frac{x'}{t_1}\big)\big)\big(\kappa_0(x)-\kappa_0^0(x)\big)\tau^{\frac{d}{2m}}\,dx + \\ \int \big(1-\phi\big(\frac{x'}{t_1}\big)\big)\,\mathsf{tr}\big(e^0(x,x,\tau)\big)\,dx + o\big(\tau^{\frac{1}{2m\mu}(d'-1)(\mu+1)}\big);\end{aligned}$$

here we have dropped all terms with $j = 1$ in (12.2.88). Here as above t_1 is either a large enough constant or a slowly increasing function of τ.

(iv) If one replaces t_1 by $t_1' \ge t_1$ in (12.2.88)–(12.2.90) then the left-hand expression does not change, and in order to estimate the increment of the right-hand expression one can apply one of estimates (12.2.82)–(12.2.84) for

R_1. Then one can easily see that for $t_1' \le \tau^\delta$ with a small enough exponent $\delta > 0$ this increment does not exceed the remainder estimate and therefore in (i)–(iii) one can formulate the condition of slow growth of t_1 more exactly: $t_1 \le \tau^\delta$. We leave the details and the calculation of this exponent $\delta > 0$ to the reader.

(v) One could apply formula (12.2.82) for $|x'| \le 2t_2$ and formula (12.2.85) for $t_2 \le |x'| \le c_0\tau^{\frac{1}{2m\mu}}$ where t_2 is now less than $c_0\tau^{\frac{1}{2m\mu}}$. Therefore the following formula holds:

$$\mathsf{N}^-(\tau) = \sum_{0\le j\le J} \int \phi\bigl(\frac{x'}{t_2}\bigr)\kappa_j(x)\tau^{\frac{1}{2m}(d-j)}\,dx + \sum_{1\le j\le J}\int_{\partial X} \phi\bigl(\frac{x'}{t_2}\bigr)\tilde{\kappa}_j(x)\tau^{\frac{1}{2m}(d-j)}\,dS + \int \bigl(1-\phi\bigl(\frac{x'}{t_2}\bigr)\bigr)e^0(x,x,\tau)\,dx + O(R^* + R_2) \tag{12.2.91}$$

with some remainder estimate expression $R_2 = R_{2,J}(\tau, t_2)$. Taking $t_2 = t_{2,J}(\tau)$ as providing the best possible estimate one can obtain certain asymptotics, maybe with worse remainder estimates. Improvements in the spirit of (ii) and (iii) are possible.

(vi) Let us pick two cases in which the formulas based on (12.2.85)–(12.2.87) and (12.2.82)–(12.2.84) well match and in (12.2.91) we can take $J = 2$ and $t_2 = c_0\tau^{\frac{1}{2m\mu}}$ and the same remainder estimate as in formulas (12.2.88)–(12.2.90). This is the case when $\kappa_j^0 = \kappa_j$ $(j = 0, 1)$, $\tilde{\kappa}_1^0 = \tilde{\kappa}_1$. And these equalities surely hold in the following cases:

(a) A and A^0 differ only in lower order terms. The only coefficient in the Weyl formula depending on lower order terms is κ_1 but even this coefficient surely does not depend on lower order terms because A is a differential operator of even order; moreover, this coefficient vanishes provided the principal symbol a is not only Hermitian but also real-valued (in which case it is symmetric).

(b) The principal symbol of A can be obtained from the principal symbol of A^0 by the transformation $\xi' \to \xi' - T(x, \xi'')$ with scalar function $T(x, \xi'')$ linear in ξ''. In fact, under the completely Weyl procedure this transform surely preserves κ_0, κ_1 and some part of κ_1 which does not depend on the lower order terms; however, the other part surely vanishes.

Simplifying the Answer

Now let us consider the term

$$\int \bigl(1-\phi(\frac{x'}{t})\bigr)\,\mathsf{tr}\bigl(e^0(x,x,\tau)\bigr)\,dx \tag{12.2.92}$$

with $t \ge c_0$ and apply Theorems 12.2.6–12.2.8. Then the remainder estimate is preserved.

Moreover, applying Theorem 12.2.7 one can replace the non-periodicity condition to billiards (which is now unnatural because it is associated with the operator A^0 in the truncated domain X^1) with the condition "$t \to +\infty$ as $\tau \to +\infty$". Let us investigate when the formulae for (12.2.92) associated with A^0 are simpler than in the general case.

(i) Let the principal operator-valued symbol of A^0 [12)] be of the form

$$a^0(x',\xi') = \sigma(x')^{-2m} a_\infty(x'', \sigma(x')\xi'; D'') \tag{12.2.93}$$

in which case the answer is

$$(2\pi)^{-d'} \iint \bigl(1-\phi(\frac{x'}{t})\bigr)\, n_\infty(\xi', \tau\sigma(x')^{2m})\sigma(x')^{d'}\, dx'd\xi' \tag{12.2.94}$$

where $n_\infty(\xi',\tau)$ is the eigenvalue counting function for the operator $a(\xi'; x'', D'')$ in $\mathbb{H} = \mathscr{L}^2(Y,\mathbb{K})$. In this case the final answer uses the eigenvalue counting function for the operator family depending only on ξ'.

Moreover, one can consider the case when a_∞ depends on $x'|x'|^{-1}$.

(ii) On the other hand, let the principal operator-valued symbol of A^0 [12)]

$$a^0(x',\xi') = \sigma(x')^{-2m} F(x', \sigma(x')\xi'; \Lambda(x')) \tag{12.2.95}$$

where $\Lambda = \Lambda(x'; x'', D'')$ is a self-adjoint operator in $\mathbb{H}$ (an elliptic differential operator of order $2m$ with Dirichlet boundary conditions) and $F(x',\xi',\lambda)$ is a monotone increasing function of λ.

Then the answer is

$$(2\pi)^{-d'} \iint \nu\bigl(x', f(x',\xi',\tau\sigma(x')^{2m})\bigr)\sigma(x')^{-d'}\, dx'd\xi' \tag{12.2.96}$$

where $\nu(x',\lambda)$ is the eigenvalue counting function for $\Lambda(x')$ and $\lambda = f(x',\xi',\tau)$ is solution of the equation $F(x',\xi',\lambda) = \tau$.

[12)] After mapping the cusp to the cylinder.

(iii) In particular, let $m = 1$ and let the principal operator-valued symbol of $A^{0\,12)}$ be

$$a^0(x', \xi') = b(x', \xi') + \sigma(x')^{-2}\Lambda(x'; x'', D'') \tag{12.2.97}$$

where $b(x', \xi') = \sum_{jk} b_{jk}(x')\xi'_j\xi'_k$ is a positive scalar quadratic form in ξ'. Then the final answer is

(12.2.98)
$$\frac{1}{2}(2\pi)^{-d'} d'\omega_{d'} \iint \nu(x', \lambda)\sigma(x')^{-d'}\beta(x')^{-\frac{1}{2}}\big(\tau\sigma(x')^2 - \lambda\big)_+^{\frac{1}{2}(d'-2)}\, d\lambda dx'$$

where $\beta(x')$ is the determinant of the matrix of the quadratic form $b(x', .)$.

Main Results

Let us formulate the principal theorems proved in this section. They are due to the results above. Moreover, we need to assume only that A^0 is a symmetric operator given in the cusp and we should not assume that A^0 is self-adjoint. Indeed, otherwise it could be replaced by $\frac{1}{2}(\mathcal{A} + \mathcal{A}^*)$ with

$$\mathcal{A} = \phi\big(\frac{x'}{t_0}\big)A + \big(1 - \phi\big(\frac{x'}{t_0}\big)\big)A^0.$$

Then we obtain the following statements:

Theorem 12.2.19. *(i) Let conditions* (12.2.1)–(12.2.11) *and stabilization conditions* (12.2.68), (12.2.69) *be fulfilled. Further, let microhyperbolicity condition for* $a(x', \xi')$ *be fulfilled*[6)]*.*

Then asymptotics

$$\mathsf{N}^-(\tau) = \mathcal{N}_1(\tau) + O\big(R^*(\tau) + \tau^{\frac{1}{2m\mu}(d'(\mu+1)-\nu)}\big) \tag{12.2.99}$$

holds where $R^*(\tau)$ *is given by* (12.2.41) *and*

(12.2.100) $\mathcal{N}_1(\tau) =$

$$\sum_{j=0,1}\int \phi\big(\frac{x'}{t_1}\big)\kappa_j(x)\tau^{\frac{1}{2m}(d-j)}\, dx + \int_{\partial X}\phi\big(\frac{x'}{t_1}\big)\tilde{\kappa}_1(x)\tau^{\frac{1}{2m}(d-1)}\, dS +$$

$$\sum_{j=0,1}\int \big(1 - \phi\big(\frac{x'}{t_1}\big)\big)\big(\kappa_j(x) - \kappa_j^0(x)\big)\tau^{\frac{1}{2m}(d-j)}\, dx +$$

$$\int_{\partial X}\phi\big(\frac{x'}{t_1}\big)\big(\tilde{\kappa}_1(x) - \tilde{\kappa}_1^0(x)\big)\tau^{\frac{1}{2m}(d-1)}\, dS +$$

$$(2\pi)^{-d'}\iint \big(1 - \phi\big(\frac{x'}{t_1}\big)\big)\boldsymbol{n}^0(x', \xi', \tau)\, dx' d\xi'$$

where n^0 *is the eigenvalue counting function for* $\mathsf{a}^0(x',\xi';x'',D'')$ *in* $\mathbb{H}$, ϕ *is introduced above,* $c_0 \le t_1 \le \tau^\delta$ *with a small enough exponent* $\delta > 0$.

Moreover, (12.2.44) *remains true for* $\mathcal{N}_1(\tau)$.

(ii) Further, let $d' < 1 + \mu d''$, $\nu > d' - \mu(d''-1)$ [13]. *Moreover, let the standard non-periodicity condition be fulfilled for the Hamiltonian billiards of* $\mathsf{a}(x',\xi')$.

Then asymptotics

$$\mathsf{N}^-(\tau) = \mathcal{N}_1(\tau) + o\bigl(\tau^{\frac{d-1}{2m}}\bigr) \tag{12.2.101}$$

holds as $\tau \to +\infty$ *provided* $t_1 = t_1(\tau) \to +\infty$ *(we still assume that* $t_1 \le \tau^\delta$*).*

(iii) On the other hand, let $d' > 1 + \mu d''$, $\nu > \mu + 1$ [14] *Moreover, let stabilization conditions* $(12.2.49)_{1,2}$ *and condition* (12.2.51) *be fulfilled* [15].

Then the asymptotics

$$\mathsf{N}^-(\tau) = \mathcal{N}_1(\tau) + \varkappa_1^*\tau^{\frac{1}{2m\mu}(d'-1)(\mu+1)} + o\bigl(\tau^{\frac{1}{2m\mu}(d'-1)(\mu+1)}\bigr) \tag{12.2.102}$$

holds with the coefficient $\varkappa_1^*$ *given in Remark 4.4.13.*

We leave to the reader the following series of rather easy problems:

Problem 12.2.20. Derive miscellaneous results in the spirit of (12.2.91) (it would be better to take $J = 2$ because we do not know nice formulas for κ_j, $\tilde\kappa_j$ with $j \ge 2$):

(i) Consider cases when σ and η are not purely power functions (for example, if these functions contain logarithmic factors), which is significant for the orders of the principal part of asymptotics and remainder estimate only for $d' \ge \mu d''$, $d' \ge 1 + \mu d''$ respectively.

(ii) Consider slowly shrinking cusps with f.e. with $\sigma \asymp \log^{-\nu}|x'|$.

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[13] Then the remainder estimate in (12.2.99) is $O\bigl(\tau^{\frac{1}{2m}(d-1)}\bigr)$.

[14] Then the remainder estimate in (12.2.99) is $O\bigl(\tau^{\frac{1}{2m\mu}(d'-1)(\mu+1)}\bigr)$.

[15] In these conditions $a_{\infty,\alpha}$ should not necessarily coincide with the coefficients in (12.2.68) and in the definition of $\mathcal{N}_1$ as here we need only "$o(.)$" in the right-hand expression.

(iii) Formulate and prove all results of this section for a domain with more than one cusp (with different characteristics)

$$X = X_0 \cup X_{\mathsf{cusp}(1)} \cup ... \cup X_{\mathsf{cusp}(N)}, \qquad X_{\mathsf{cusp}(j)} \cap X_{\mathsf{cusp}(k)} = \emptyset \text{ as } \ 0 < j < k.$$

This problem is trivial even if the dimensions of these cusps (d'_j) and the exponents of shrinking (μ_j) differ. We only note that in some situations the bounded part of the domain and even some cusps can be ignored in the final answer).

(iv) Derive asymptotics for the Riesz means. We mention only that $R^*_{(\vartheta)}(\tau)$ in this case is defined by (12.2.55) and the additional error due to perturbation is $O\big(R^*_{(\vartheta)}(\tau) + \tau^{\frac{1}{2m\mu}(d'(\mu+1)-\nu)+\vartheta}\big)$ and so the remainder estimate is $O(R^*_{(\vartheta)})$ provided $\nu \ge \min\big((\mu+1)(1+\vartheta), d' - d''\mu + (1+\vartheta)\mu\big)$.

Further, we leave to the reader the following series of problems:

Problem 12.2.21. (i) Generalize to the cases of mixed boundary conditions which are Dirichlet for $|x'| \ge c$, or Dirichlet on the upper boundary of the cusp and Neumann on the lower boundary of the cusp for $|x'| \ge c$ and $d'' = 1$.

Further, consider generalizations for $d'' \ge 2$. Domain must satisfy cone condition which does not contradict shrinking as cone is truncated by the part of the boundary on which Dirichlet condition is imposed.

(ii) Consider more general boundary conditions under assumptions that all the eigenvalues of $\boldsymbol{a}(x', \xi')$ including the lowest ones have the same power growth at infinity. This is rather difficult and interesting modification as we cannot apply results of Chapter 2 directly because $\mathfrak{D}(\boldsymbol{a}(x', \xi')$ depends on (x', ξ').

Probably, the easiest way to overcome this obstacle would be to introduce an unitary operator $Q(x', \xi')$ in $\mathbb{H}$ such that $Q(x', \xi')\, \mathfrak{D}(\boldsymbol{a}(x', \xi')$ does not depend on (x', ξ').

(iii) Derive more precise remainder estimates in the framework of Theorem 12.2.7 as Hamiltonian flow associated with $a(x, \xi)$ is of power or exponential non-periodic type.

(iv) Derive more precise remainder estimate in the frameworks of Theorem 12.2.8 and Example 12.2.9 assuming that the Hamiltonian flow associated with $\zeta(x',\xi')$ is of power or exponential non-periodic type; in particular, consider Schrödinger operator in Euclidean metrics in domains of with $\sigma(x') = |x|^{-\mu}$, $\mu > 0$, $\mu \neq 1$ (with appropriate d' and d'').

(v) In the frameworks of Theorem 12.2.8 and Example 12.2.9 investigate the case when the Hamiltonian flow associated with $\zeta(x',\xi')$ is periodic; in particular, consider Schrödinger operator in Euclidean metrics in domains of with $\sigma(x') = |x|^{-1}$ (with appropriate d' and d'').

(vi) Consider combs and trees as in Examples 11.2.25 and 11.2.26 respectively, but thicker than there.

Remark 12.2.22. If we wish to improve the remainder estimate in the final theorem in the case when the stabilization exponent ν is not large enough then the following algorithm is suggested. In the domain $|x'| \le t$ we apply the previous analysis. So we need to treat

$$\int \psi(x')\,\mathsf{tr}(e^{\theta}(x',\xi',\tau))\,dx'd\xi'$$

with $\theta = 1$ and a γ-admissible function ψ supported in $B(\bar{x}',\bar{\gamma})$ (and later sum on a partition of unity in the zone $|x| \ge t_2$). This expression is smooth with respect to θ because of the microhyperbolicity property.

Then one can apply Taylor's formula with N terms. The remainder estimate is $O\big(\bar{\gamma}^{d'}\bar{\sigma}^{d''}\bar{\eta}^{N}\tau^{d'}\big)$ (for $|\bar{x}''| \le \tau^{\frac{1}{2m\mu}}$) and after summation on the partition of unity we obtain $O\big(R^*(\tau) + R_3(t_2,\tau)\big)$ with R_3 depending on N and ν.

So, the total remainder estimate is $O\big(R^*(\tau) + R_2(\tau,t_2) + R_3(\tau,t_2)\big)$ where R_2 is an upper estimate for R_2. Picking $t_2 = t_2(\tau)$ as providing the best possible remainder estimate we obtain the final formula. However, this formula is rather complicated even in the Case (iii) of the previous Subsubsection 12.2.3.2. Namely, the additional term with $n = 1$ in the Taylor expansion is

$$\mathsf{const}\iint \big(\tau\sigma(x')^{2m} - \lambda\big)^{\frac{d-2}{2}}\, d_\lambda\big(\mathsf{tr}(a'E(\lambda))\big)_+\, dx' \tag{12.2.103}$$

provided $a(x,D) - a^0(x,D) = a'(x,D'')$ where $E(\lambda)$ is the spectral projector for Λ in $\mathbb{H}$, and this term is slightly more complicated for more general perturbations.

However, the next terms (with $n \ge 2$) are very complicated. Still if we know that a_α and σ can be decomposed into cut-off series of positive homogeneous terms (modulo terms decaying fast enough) we can find that these complicated terms in $\mathcal{N}(\tau)$ are of the power (may be multiplied by a logarithm) type.

We leave this task to the reader as well as the generalization to the Riesz means.

It would be very interesting and under certain assumptions not very difficult to

Problem 12.2.23. Consider cusps with Z of more general type than here:

(i) for example, let Z be an open cone in $\mathbb{R}^{d'}$, $Z_0 = Z \times \{0\} \subset \mathbb{R}^d$ with the smooth boundary and

$$\sigma(x') \le \langle x' \rangle^{-\mu} \ \mu > 0, \tag{12.2.104}$$

$$|D^{\beta'}\sigma| \le c\sigma\gamma^{-|\beta'|} \quad \forall \beta' : |\beta'| \le K \tag{12.2.105}$$

with

$$\gamma = \operatorname{dist}(x', \partial Z) \tag{12.2.106}$$

Then provided

$$\int \sigma^{d''}\gamma^{-1}\,dx' \le C|x'|^{-\mu d''+d'-1}. \tag{12.2.107}$$

we can recover the same remainder estimate as for $Z = \mathbb{R}^{d''}$.

(ii) Alternatively $\sigma \asymp \langle x' \rangle^{-\mu}$ but near ∂Z a rounding up of the radius $\asymp \sigma$ is added. Then (12.2.107) holds with the logarithmic factor in the right-hand expression and we can recover almost the same remainder estimate as in (i)–with $R^*_{(\vartheta)}(\tau)$ replaced by $R^*_{(\vartheta)}(\tau)\log\tau$.

(iii) In (i) and (ii) consider quasiconical base Z.

12.3 Operators with Degenerate Potentials Growing at Infinity

12.3.1 Problem Set-up

Let us start from

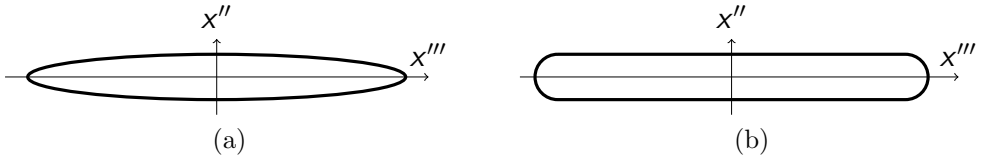


Figure 12.3: As $x' = (x_1, x'')$ we show cross-section with $x_1 = \mathsf{const}$ of the cusp from (i) on (a) and from (ii) on (b)

Example 12.3.1. Consider self-adjoint (matrix) operator on $\mathbb{R}^d$

$$A = \sum_{\alpha:|\alpha|\le 2m} a_\alpha(x) D^\alpha \tag{12.3.1}$$

(possibly in the divergent form) with coefficients such that

$$|D^\beta a_\alpha| \le c\langle x\rangle^{(2m-|\alpha|)\mu-|\beta|} \qquad \forall \beta : |\beta| \le K \tag{12.3.2}$$

with $\mu > 0$.

As usual we assume that operator is elliptic and positive

$$a^0(x,\xi) = \sum_{\alpha:|\alpha|=2m} a_\alpha(x)\xi^\alpha \ge \epsilon_0 |\xi|^{2m} \tag{12.3.3}$$

and that A is a perturbation of operator $A_{(0)}$ with the coefficients $a_{\alpha(0)}(x)$ such that for $j = 0$

$$D^\beta a_{\alpha(j)}|_\Lambda = 0 \qquad \mathsf{as} \qquad |\beta| < (2m - |\alpha|)\nu - j \tag{12.3.4}$$

with $0 < \nu < \mu$ where $\nu \in \mathbb{Z}$ and

(12.3.5) Λ is a conic $\mathscr{C}^K$-manifold of codimension d'', $\zeta(x) = \mathsf{dist}(x, \Lambda)$

and

$$\sum_{\alpha:|\alpha|\le 2m} a_{\alpha,0}(x)\xi^\alpha \ge \epsilon_0 \big(|\xi|^2 + |x|^{2\mu-2\nu}\cdot \zeta(x)^{2\nu}\big)^m. \tag{12.3.6}$$

We will show that "admissible perturbations" are

$$a_\alpha(x) = \sum_{j\le (2m-|\alpha|)\nu} a_{\alpha(j)}(x) \tag{12.3.7}$$

where $a_{\alpha(j)}(x)$ satisfy (12.3.4) and also

$$|D^\beta a_{\alpha(j)}| \le c\langle x\rangle^{(2m-|\alpha|)\mu-|\beta|-(\mu+1)(\nu+1)^{-1}j} \tag{12.3.8}$$

and also condition (12.3.19) below.

Without any loss of the generality we can assume that (at least locally) $\Lambda = \{x : x'' = 0\}$. Then we need to introduce

$$\zeta(x) = \frac{1}{2}|x''|, \qquad \gamma(x) = \frac{1}{2}|x'|, \qquad \rho(x) = \zeta(x)^\nu \cdot |x'|^{\mu-\nu} \tag{12.3.9}$$

scales with respect to x'', x' and ξ respectively.

One can consider more general operators. Assume that $\nu \in (0,\mu)$ is not necessary integer anymore, ζ, γ and ρ are defined by (12.3.9), but ρ, ζ are then redefined by

$$\rho^*(x) = \max\bigl(\rho(x), \sigma_0(x')^{-1}\bigr), \qquad \zeta^*(x) = \max\bigl(\frac{1}{2}|x''|, \sigma_0(x')\bigr) \tag{12.3.10}$$

with

$$\sigma_0(x') \coloneqq c_0\langle x'\rangle^{-(\mu-\nu)/(\nu+1)}. \tag{12.3.11}$$

Assume that

$$|D^\beta a_\alpha| \le C{\rho^*}^{2m-|\alpha|}\gamma^{-|\beta'|}{\zeta^*}^{-|\beta''|}. \tag{12.3.12}$$

Potential surfaces are shown on figure 12.4.

Consider this operator first in the framework of the theory of Section 11.5; after scaling $x_{\mathsf{new}} = x/\sigma$, $\xi_{\mathsf{new}} = \tau^{-1/(2m)}\xi$ we conclude that operator $\tau^{-1}A$ is elliptic on the energy levels 1 and below provided

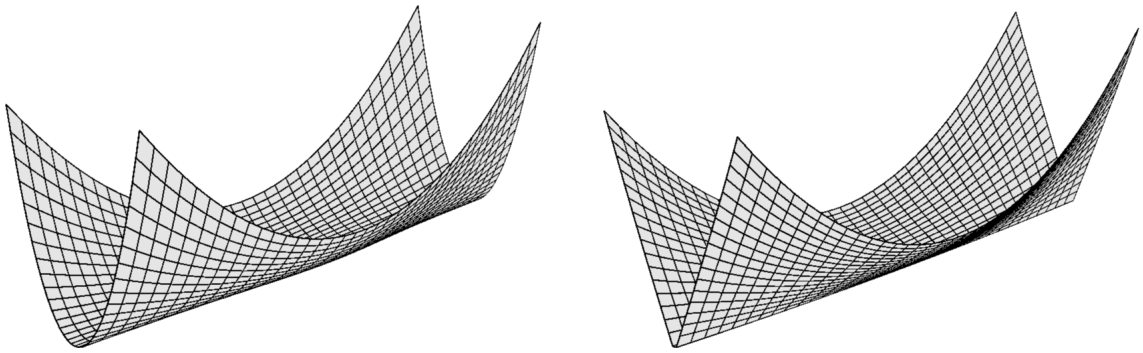
$$|x''| \ge \sigma(x',\tau) \coloneqq c_0|x'|^{-(\mu-\nu)/\nu}\tau^{1/(2m\nu)}, \tag{12.3.13}$$

(we assume that $\tau \gg 1$, $|x'| \gg |x''|$ and $|x'| \gg 1$) and the uncertainty principle $\rho(x)\zeta(x) \ge 1$ is fulfilled provided

$$|x''| \ge \sigma_0(x') \coloneqq c_0\langle x'\rangle^{-(\mu-\nu)/(\nu+1)} \tag{12.3.14}$$

as in (12.3.11).

Then we arrive immediately to the following



(a) Valley with smooth bottom (b) Valley with non-smooth bottom

Figure 12.4: Potential graphs

Proposition 12.3.2. *Let conditions* (12.3.1), (12.3.3), (12.3.6) *and* (12.3.9)–(12.3.12) *be fulfilled at the point* $\bar{x} \in \mathbb{R}^d$.

(i) Then

$$|e(x,y,\tau)| \le C\langle x\rangle^{(\mu-\nu)d/(\nu+1)}\zeta(x)^{-s}\sigma^s(x',\tau) \qquad \forall x,y \in B\bigl(\bar{x}, \frac{1}{2}\zeta(\bar{x})\bigr) \tag{12.3.15}$$

with an arbitrarily large exponent s.

(ii) Moreover, if $U(x,y,t)$ *is the Schwartz kernel of the operator* $\tau^{-1}A$ *then*

$$\bigl|F_{t\to\lambda}\bar{\chi}_{T,T'}(t)U(x,y,t)\bigr| \le C\langle x\rangle^{(\mu-\nu)d/(\nu+1)}\zeta(x)^{-s}\sigma(x',\tau)^s \qquad \forall x,y \in B\bigl(\bar{x}, \frac{1}{2}\zeta(\bar{x})\bigr) \quad \forall T' \in \mathbb{R},\ T = \sigma(\bar{x}) \quad \forall \lambda \le 1+\epsilon \tag{12.3.16}$$

with a small enough constant $\epsilon > 0$.

Remark 12.3.3. Consider "valley" $\mathcal{X}(\tau) = \{x, |x''| \le \sigma(x',\tau)\}$. It consists of the *regular part* where $|x| \le c\tau^{1/(2m\mu)}$ and the *cusp* where $|x| \ge c\tau^{1/(2m\mu)}$; this cusp stretches along $Z = \{x, x'' = 0\}$ and shrinks at infinity (on $|x|$) for any $\tau > 0$.

Let us apply now the theory of operators with the operator-valued symbols (in rather limited manner at this time); to do this we need to analyze operator $\mathsf{a}(x',\xi') = A(x',\xi';x'',D'')$:

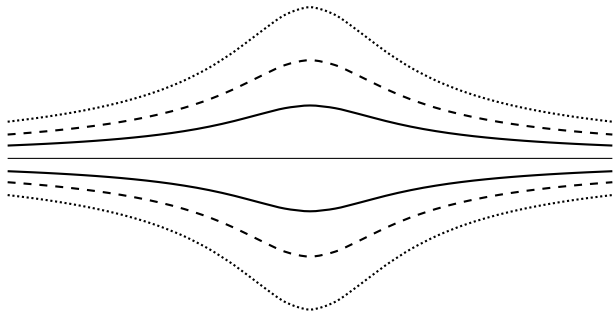


Figure 12.5: Cusps defined for different τ

Proposition 12.3.4. *Let conditions* (12.3.1), (12.3.3), (12.3.6) *and* (12.3.9)–(12.3.12) *be fulfilled. Let* $\lambda_k(x',\xi')$ *be eigenvalues of* $\mathsf{a}(x',\xi')$ *in* $\mathbb{H}=\mathscr{L}^2(\mathbb{R}^{d''})$. *Then*

$$|\lambda_k(x',\xi)| \le c_k\Big(|\xi'|+\frac{1}{\sigma_0(x')}\Big)^{2m} \tag{12.3.17}$$

and

$$\lambda_k(x',\xi') \ge \epsilon_0\Big(|\xi'|+\frac{1}{\sigma_0(x')}\Big)^{2m} \qquad \text{as} \quad k\ge \bar{k}. \tag{12.3.18}$$

where $\sigma_0(x')$ *is defined by* (12.3.11).

Proof. Scaling $x''\mapsto \sigma_0(x')^{-1}x''$, $D''\mapsto \sigma_0(x')D''$ we arrive to elliptic operator with a parameter in $\mathbb{R}^{d''}$ and both statements (12.3.17) and (12.3.18) are well-known. □

Finally we assume that

(12.3.19) Inequality (12.3.18) is fulfilled for all k.

Due to our assumptions $A(x',D';x'',D'')$ is elliptic as an operator with operator valued symbol on the energy level τ and below unless $|\xi'|\le c\tau^{\frac{1}{2m}}$ and $|x'|\le r^*(\tau)$ with

$$r^*(\tau) := c_0|\tau|^{(\nu+1)/2m(\mu-\nu)}. \tag{12.3.20}$$

Proposition 12.3.5. *Let conditions* (11.1.6), (11.1.8) *and* (12.3.1), (12.3.3), (12.3.6), (12.3.9)–(12.3.12) *and* (12.3.19) *be fulfilled. Then estimate*

$$|e(x,y,\tau)| \le C\langle x\rangle^{-s}\langle y\rangle^{-s}\tau^{-s} \qquad \forall x,y : |x| \ge r^*(\tau), |y| \ge r_*(\tau) \tag{12.3.21}$$

holds.

Proof. Our statement is due to Proposition 4.2.8 as we take

$$B_1 = \bigl(\Delta + |x''|^{2\nu}|x''|^{2(\mu-\nu)}\bigr)^m \tag{12.3.22}$$

where

$$\bigl(\Delta + |x''|^{2\nu}|x''|^{2(\mu-\nu)}\bigr) \ge \epsilon_0 |x'|^{2(\mu-\nu)/(\nu+1)} \tag{12.3.23}$$

in the operator sense. Recall that $\mathbb{H} = \mathscr{L}^2(\mathbb{R}^{d''}, \mathbb{K})$, $\mathbb{K} = \mathbb{C}^{\mathsf{D}}$. □

Remark 12.3.6. (i) $\sigma_0(x') \lesssim \sigma(x',\tau)$ if and only if $|x'| \lesssim r^*(\tau)$.

(ii) Note that $\mathsf{mes}\bigl(\mathcal{X}(\tau)\bigr) = O(\tau^{d/(2m\mu)})$ if and only if $\mu d'' > \nu d$. Under this condition the principal part of the Weyl formula is $\asymp \tau^{\frac{1}{2m\mu}d(\mu+1)}$.

(iii) Let us drop condition $\mu d'' > \nu d'$ and let us restrict integration to domain $\{x, |x| \le r^*(\tau)\}$ in the Weyl formula due Proposition 12.3.5. Then the principal part of the Weyl asymptotics (modified in this way) is

$$\asymp \begin{cases} |\tau|^{d(\mu+1)/(2m\mu)} & \text{for } d\dfrac{\nu}{\mu} < d'', \\ |\tau|^{d(\mu+1)/(2m\mu)}|\log|\tau|| & \text{for } d\dfrac{\nu}{\mu} = d'', \\ |\tau|^{d'(\mu+1)/2m(\mu-\nu)} & \text{for } d\dfrac{\nu}{\mu} > d''. \end{cases} \tag{12.3.24}$$

where in this section we can replace $|\tau|$ by τ and $|\log\tau|$ by $\log\tau$.

(iv) Let us consider the remainder estimate in the Weyl formula. We can integrate only over domain $\{x, |x| \le r^*(\tau)\}$ because of Proposition 12.3.5. Then the remainder estimate in the modified Weyl asymptotics is $O\bigl(R(\tau)\bigr)$ with

(12.3.25) $R(\tau) =$

$$\begin{cases} |\tau|^{(d-1)(\mu+1)/(2m\mu)} & \text{for } (d-1)\dfrac{\nu}{\mu} < (d''-1), \\ |\tau|^{(d-1)(\mu+1)/(2m\mu)}|\log|\tau|| & \text{for } (d-1)\dfrac{\nu}{\mu} = (d''-1), \\ |\tau|^{d'(\mu+1)/2m(\mu-\nu)} & \text{for } (d-1)\dfrac{\nu}{\mu} > (d''-1). \end{cases}$$

(v) Let $\mu(d''-1) > \nu(d-1)$. Then under standard stabilization conditions of a_α as $|x| \to \infty$ to positively homogeneous $a_{\infty,\alpha}$ and standard restrictions to the Hamiltonian flow generated by the corresponding Hamiltonian $a_\infty(x,\xi)$ one can obtain the remainder estimate $o(\tau^{\frac{1}{2m\mu}(d-1)(\mu+1)})$.

In this remark the role of the "operator-valued theory" is very limited: it only provides a nice cut-off in the Weyl formulas.

We leave to the reader the following easy

Problem 12.3.7. Calculate the principal part of the asymptotics and remainder estimate for Riesz means in Remark 12.3.6.

12.3.2 Sharp Asymptotics

So, as we are interested in the asymptotics of $\mathsf{N}^-(\tau)$ and remainder estimate (12.3.25) is sharp as $(d-1)\nu/\mu < (d''-1)$, our analysis should be done as

$$\mu(d''-1) \le \nu(d-1). \tag{12.3.26}$$

Here we should treat only the cusp zone

$$\mathcal{X}_{\mathsf{cusp}}(\tau) = \bigl\{c_1\tau^{\frac{1}{2m}} := r_*(\tau) \le |x'| \le r^*(\tau),\ |x''| \le c_2\sigma(x',\tau)\tau^{\delta}\bigr\} \tag{12.3.27}$$

with $r^*(\tau)$ and $\sigma(x',\tau)$ defined by (12.3.20) and (12.3.13) and an arbitrarily small exponent $\delta > 0$.

In fact, estimate (12.3.15) and Proposition 12.3.5 yield that for $|x'| \ge c_1\tau^{\frac{1}{2m\mu}}$ outside of $\mathcal{X}(\tau)$ function $e(x,x,\tau)$ is negligible; on the other hand, we have

Proposition 12.3.8. *(i) Contribution of the regular zone*

$$\mathcal{X}_{\mathsf{reg}}(\tau) = \{x,\ |x'| \le cr_*(\tau)\} \tag{12.3.28}$$

(with arbitrarily large constant c) to the principal part of the Weyl asymptotics and to the remainder estimate are $O\bigl(\tau^{\frac{d(\mu+1)}{2m\mu}}\bigr)$ and $O\bigl(\tau^{\frac{(d-1)(\mu+1)}{2m\mu}}\bigr)$ respectively[16].

(ii) Moreover, under standard stabilization condition (11.5.4) *to quasihomogeneous symbol $a_\infty(x,\xi)$ and the standard non-periodicity condition to the Hamiltonian flow generated by symbol $a_\infty(x,\xi)$ the contribution of the zone $\mathcal{X}_{\mathsf{reg}}(\tau)$ to the remainder estimate is $o\bigl(\tau^{\frac{1}{2m\mu}(d-1)(\mu+1)}\bigr)$.*

Proof. The proof is absolutely standard in the framework of Section 11.5.

To deal with the possible irregularity near $\{x''=0\}$ note that in the zone $\bigl\{x:\ |x'|\le cr_*(\tau),\ |x''|\le \epsilon r_*(\tau)\bigr\}$ with sufficiently small constant $\varepsilon>0$ operator $\tau^{-1}A$ after rescaling is ξ-microhyperbolic on the energy level 1 and $h_{\mathsf{eff}}=\tau^{-1/(2m)}\zeta^{*-1}\le \tau^{-\delta''}$ with $\delta''>0$; so propagation theory of Chapter 2 works. □

Furthermore, the same arguments yield that treating zone $\bigl\{|x'|\le c_0 r^*(\tau)\tau^{-\delta'}\bigr\}$ with an arbitrarily small exponent $\delta'>0$ one can set $\delta=0$ in the definition of $\mathcal{X}(\tau)$; in this case constant c_2 should be large enough.

To treat the cusp zone (12.3.27) let us apply theory of Section 4.4. For this we need to check if operator in question is (x',ξ') microhyperbolic on the energy level τ in (γ,ρ_τ)-scale with $\gamma(x')=\frac12|x'|$ and $\rho_\tau=\tau^{1/2m}$.

Consider first the pilot-model operator with the coefficients

$$a_{\alpha(j)}(x)=\sum_{\beta:|\beta|=(2m-|\alpha|)\nu-j} b_{\alpha,\beta}(x')x''^{\beta} \tag{12.3.29}$$

where

(12.3.30) Coefficients $b_{\alpha,\beta}(x')$ in the decompositions (12.3.29) are positively homogeneous of degrees $(\mu-\nu)\bigl(2m-|\alpha|-j(\nu+1)^{-1}\bigr)$

which is fulfilled as we take terms with $|\beta|=(2m-|\alpha|)\nu-j$ of Taylor decomposition of $a_{\alpha(j)}(x)$ provided

(12.3.31) Coefficients $a_{\alpha(j)}(x)$ are positively homogeneous of degrees $(2m-|\alpha|)-(\mu+1)(\nu+1)^{-1}j$.

[16] Subject to conditions of Section 11.5. For example, as either operator is the Schrödinger operator or its perturbed power or stabilization condition (11.5.4) is fulfilled.

Then rescaling $x'' \mapsto x''\sigma_0(x')$, $D'' \mapsto D''\sigma_0^{-1}(x')$ we arrive to operator

$$\mathsf{a}(x', D') = \sum_{\alpha':|\alpha'|\le 2m} \mathsf{a}_{\alpha'}(x') D'^{\alpha'} \tag{12.3.32}$$

where operator coefficients $\mathsf{a}_{\alpha'}(x') = \mathsf{a}_{\alpha'}(x'; x'', D'')$ are positively homogeneous of degrees $(2m - |\alpha|)(\mu - \nu)(\nu + 1)^{-1}$.

Therefore under assumption (12.3.29)–(12.3.30) microhyperbolicity condition is fulfilled. Then this condition is fulfilled under assumption (12.3.31) as well and therefore we arrive to the Statement (i) of Proposition 12.3.9 below; Statement (ii) follows from the similar arguments.

Proposition 12.3.9. *(i) In the framework of Example 12.3.1 operator* $\mathsf{a}(x', D')$ *is* (x', ξ')*-microhyperbolic under stabilization condition*

$$D^{\beta}\big(\mathsf{a}_{\alpha(j)} - \mathsf{a}^{\infty}_{\alpha(j)}\big) = o\big(|x'|^{(2m-|\alpha|)\mu - |\beta| - (\mu+1)(\nu+1)^{-1}j}\big) \tag{12.3.33}$$
$$\text{as } \ |x'| \to \infty, x'' = 0, |\beta| \le (2m - |\alpha|)\nu + 1$$

(ii) In the general case operator $\mathsf{a}(x', D')$ *is* (x', ξ')*-microhyperbolic under stabilization condition*

$$D^{\beta}_{x'}\big(\mathsf{a}_{\alpha} - \mathsf{a}^{\infty}_{\alpha}\big)(x', \sigma_0(x')y'') = o\big(|x'|^{(2m-|\alpha|)(\mu-\nu)/(\nu+1) - |\beta|}\big) \tag{12.3.34}$$
$$\text{as } \ |x'| \to \infty, |y''| \le c_3, |\beta| \le 1.$$

We leave to the reader the following easy

Problem 12.3.10. Establish other conditions providing (x', ξ')-microhyperbolicity of $\mathsf{a}(x', D')$.

So, assume that microhyperbolicity condition is fulfilled. Then (after all rescalings) one can take

$$B = B_1 = \theta\big(\Delta + \langle x''\rangle^{2\nu}\big)^m, \qquad \theta = \tau^{-1}\gamma^{2m(\mu-\nu)/(\nu+1)} \tag{12.3.35}$$

in $\mathscr{L}^2(\mathbb{R}^{d''})$.

We will need to take the trace norm; one can easily see that

(12.3.36) For $k > d''(\nu + 1)/(2m\nu)$ and $r_*(\tau) \le |x'| \le r^*(\tau)$ the trace norm of B^{-k} does not exceed $\theta^{-d''(\nu+1)/2m\nu}$ which in turn is equal to $\big(\gamma/r^*(\tau)\big)^{-d''(\mu-\nu)/\nu}$.

Then, applying Theorem 4.4.8 as this operator is ξ'-microhyperbolic and Theorem 4.4.9 as this operator is (x',ξ')-microhyperbolic (i.e. in the general case) we obtain that if $\psi=\psi(x')$ is a γ-admissible function supported in $\{|x'-\bar{x}'|\le \frac{1}{2}\bar{\gamma}\}$ with $\bar{\gamma}=\gamma(\bar{x})$ then

$$\Big|\int \psi(x')e(x,x,\tau)\,dx - (2\pi)^{-d'}\iint \psi(x')\boldsymbol{n}(x',\xi',\tau)\,dx'd\xi'\Big|\le \tag{12.3.37}$$
$$C\hbar^{1-d'}\big(\bar{\gamma}/r^*(\tau)\big)^{-d''(\mu-\nu)/\nu} = C\big(\tau^{1/(2m)}\bar{\gamma}\big)^{(d'-1)}\big(\bar{\gamma}/r^*(\tau)\big)^{-d''(\mu-\nu)/\nu}$$

where $\boldsymbol{n}(x',\xi',\tau)$ is the eigenvalue counting function for operator $\boldsymbol{a}(x',\xi')=\boldsymbol{a}(x',\xi';x'',D'')$ which is the operator-valued symbol of A (after a coordinate transform), $\hbar=\tau^{-1/(2m)}\bar{\gamma}^{-1}$.

Let us sum over a γ-admissible partition of unity in the zone

$$\big\{t_1 r_*(\tau)\le |x'|\le r^*(\tau)\big\}; \tag{12.3.38}$$

the right-hand expression then sums to

$$R^*(\tau)=\begin{cases} |\tau|^{(d-1)(\mu+1)/(2m\mu)} & \text{for } (d-1)\dfrac{\nu}{\mu}<d'',\\ |\tau|^{(d-1)(\mu+1)/(2m\mu)}|\log|\tau|| & \text{for } (d-1)\dfrac{\nu}{\mu}=d'',\\ |\tau|^{(d'-1)(\mu+1)/2m(\mu-\nu)} & \text{for } (d-1)\dfrac{\nu}{\mu}>d''.\end{cases} \tag{12.3.39}$$

Let us apply Weyl asymptotics in the regular zone $\{|x'|\le t_1\tau^{1/(2m\mu)}\}$. To ensure (x,ξ)-microhyperbolicity there we assume that the standard stabilization condition (11.5.4) is fulfilled[17].

Then we arrive to the following assertion

Theorem 12.3.11[18]**.** *Let conditions* (12.3.1), (12.3.3), (12.3.6), (12.3.9)–(12.3.12) *and stabilization conditions* (11.5.4)[16),17)], (12.3.34) *be fulfilled.*

Let $\psi\in \mathscr{C}_0^K(\mathbb{R}^{d'})$, $\psi(x')=1$ *as* $|x'|\le c_0$. *Then*

(i) Asymptotics

$$\mathsf{N}^-(\tau)=\mathcal{N}(\tau)+O\big(R^*(\tau)\big) \tag{12.3.40}$$

[17] But for some operators this condition may be weakened or skipped completely. Also the total contribution of the regular zone to the asymptotics may be smaller than the contribution of the cusp to the remainder estimate.

[18] Cf. Theorem 12.2.6.

holds with $\mathcal{N}(\tau)$ *defined by* (12.2.43) *with* $\psi_t(x') = \psi(x'/t)$ *where* $t \asymp r_*(\tau)$ [19] *and with* $R^*(\tau)$ *defined by* (12.3.39).

(ii) Moreover, $\mathcal{N}(\tau)$ *has a magnitude described in* (12.3.24).

Let us note that for $(d-1)\nu < d''\mu$ the main contribution to the remainder estimate is provided by the zone

$$\{x : \ |x'| \le t\tau^{\frac{1}{2m}}\} \tag{12.3.41}$$

with arbitrarily large t (the contribution of the complement does not exceed $\tau^{(d-1)(\mu+1)/(2m\mu)}t^{-\delta}$ with $\delta > 0$); on the other hand, in this zone under the standard non-periodicity condition to the Hamiltonian billiards we have the remainder estimate $o\bigl(\tau^{\frac{1}{2m}(d-1)}\bigr)$. Then applying improved Weyl asymptotics we arrive to the following assertion:

Theorem 12.3.12 [20]**.** *Let the conditions of Theorem 12.3.11 be fulfilled with* $(d-1)\nu < d''\mu$.

Moreover, let the standard stabilization condition to quasihomogeneous symbol $a_\infty(x,\xi)$ *and the standard non-periodicity condition to the Hamiltonian trajectories generated by* $a_\infty(x,\xi)$ *be fulfilled. Then asymptotics*

$$\mathsf{N}^-(\tau) = \mathcal{N}(\tau) + o\bigl(\tau^{\frac{1}{2m}(d-1)(m+1)}\bigr) \tag{12.3.42}$$

holds as $\tau \to +\infty$ *where* $\mathcal{N}(\tau)$ *is defined by* (12.2.46),

$$(2\pi)^{-d}\int \psi\kappa_1\tau^{\frac{1}{2m}(d-1)}\,dx \quad \text{and} \qquad (2\pi)^{1-d}\int_{\partial X} \psi\tilde{\kappa}_1\tau^{\frac{1}{2m}(d-1)}\,dS \tag{12.3.43}$$

are the standard second terms[21] *in the Weyl asymptotics,* $\psi_t(x') = \psi_0(x'/t)$ *where* $t = t(\tau)$ *is an appropriate function such that* $t(\tau)/r_*(\tau) \to +\infty$ *as* $\tau \to +\infty$ [22].

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[19] One can assume that $t = t(\tau)$ and $t(\tau)/r_*(\tau)$ is either bounded or tends slowly to $+\infty$. Recall that $r_*(\tau) = \tau^{1/(2m\mu)}$.

[20] Cf. Theorem 12.2.7.

[21] We will discuss domains with the boundaries later.

[22] If $t(\tau)$ works then every function $t'(\tau) \le t(\tau)$ such that $t'(\tau)/r_*(\tau) \to +\infty$ as $\tau \to +\infty$ also works.

On the other hand, as $d' > 1 + d''\mu$, the main contribution to the remainder estimate in Theorem 12.3.11 is provided by the zone

$$\{\varepsilon r^*(\tau) \le |x'| \le c_0 r^*(\tau)\} \tag{12.3.44}$$

with arbitrarily small $\varepsilon > 0$; contribution of zone $\{|x'| \le 2\varepsilon r^*(\tau)\}$ does not exceed $\varepsilon^\delta R^*(\tau)$.

However, in zone (12.3.44) $\boldsymbol{n}(x', \xi', \tau)$ does not exceed $\nu(\varepsilon) = \varepsilon^M$ and therefore we have effectively here matrix- rather than operator-valued operators.

Applying Theorems 4.4.11 and 4.4.12 and replacing $\boldsymbol{a}$ if necessary by another operator $\boldsymbol{a}_\infty$ in $\mathbb{H}$ with eigenvalues asymptotically close to those of $\boldsymbol{a}$ we obtain the following statement:

Theorem 12.3.13 [23]**.** *Let the conditions of Theorem 12.3.11 be fulfilled with* $(d-1)\nu < d''\mu$.

Let us consider operator-valued symbol $\boldsymbol{a}_\infty(x', \xi'; D'')$ *in* $\mathbb{H}$*; let* $\lambda_j(x', \xi')$ *be its (non-decreasing) eigenvalues*[24]*. Let us consider*

$$\boldsymbol{a}_{\infty,j}(x', \xi') = \boldsymbol{a}_\infty(x', \xi')\boldsymbol{e}_\infty(x', \xi', \zeta_j(x', \xi')) \tag{12.3.45}$$

where $\boldsymbol{e}_\infty(x', \xi', \tau)$ *is the spectral projector for* $\boldsymbol{a}_\infty(x', \xi')$ *and let assume that non-periodicity condition* (12.2.51) *is fulfilled.*

Then asymptotics

$$\mathsf{N}^-(\tau) = \mathcal{N}(\tau) + \bigl(\kappa_1^* + o(1)\bigr)\tau^{(d'-1)(\mu+1)/2m(\mu-\nu)} \tag{12.3.46}$$

holds as $\tau \to +\infty$ *with an appropriate coefficient* κ_1^* *(given in Remark 4.4.13).*

Condition (12.2.51) looks extremely cumbersome and impossible to check. However in the following rather special example this condition becomes much simpler:

Example 12.3.14 [25]. Let $d'' = 1$ and A be a Schrödinger operator stabilizing at infinity to $|\xi^2| + |x_d|^2 \cdot \omega(x')^2$ with ω, positively homogeneous of degree $2(\mu - \nu)$.

---

[23] Cf. Theorem 12.2.8.

[24] Then obviously $\lambda_j(tx', t^k\xi') = t^{2mk}\omega_j(x', \xi')$ with $k = (\mu - \nu)/(\nu + 1)$.

[25] Cf. Example 12.2.9.

Then one needs to check the standard non-periodicity condition for symbols $\lambda_j(x',\xi') = |\xi'|^2 + (2j-1)\omega(x')$ and all of them are equivalent to the same standard non-periodicity condition for a scalar symbol

$$\lambda(x,\xi) = |\xi|^2 + \omega(x')^{2/(\nu+1)} \tag{12.3.47}$$

(on the energy level 1). This conclusion holds as A is a Schrödinger operator stabilizing at infinity to $|\xi^2| + |x_d|^{2\nu}\cdot\omega(x')^2$.

In this case we need to check only check the standard non-periodicity condition for (12.3.47). Note that

(12.3.48) As $d' \ge 2$ the generic trajectory does not hit $0\times\mathbb{R}^{d'}$.

Really, $\mu(\Sigma\cap\{|x'|\le 2\varepsilon\}) = O(\varepsilon^{d'})$ where $\mu = dx'd\xi' : d\lambda$ is the standard measure on the surface $\Sigma = \{(x',\xi') : \lambda(x',\xi') = 1\}$ and any trajectory intersecting $\{|x'|\le\varepsilon\}$ needs at least $\epsilon_0\varepsilon$-time to leave $\{|x'|\le 2\varepsilon\}$; therefore the μ-measure of points such that Ψ_t touches $\{|x'|\le\varepsilon\}$ for some $t : |t|\le T$ does not exceed $CT\varepsilon^{d'-1}$.

In particular, the standard non-periodicity condition for $\lambda(x',\xi')$ is fulfilled as $\sigma\sim|x'|^{-\eta}$, $\eta>0$ and $\eta\ne 1$; recall that $d'\ge 2$.

Finally, let us consider the case $d' = 1 + d''\mu$. Considering diagonal operator with diagonal elements $\lambda_j^{\mathsf{w}}(x',D')$ one can prove easily the remainder estimate $o(\tau^{(d-1)/2m}\log\tau)$ (not subject to any non-periodicity condition). This leads us to conjecture which we were unable to prove

Conjecture 12.3.15 [26]**.** *As $(d-1)\nu = d''\mu$ remainder estimate is in fact $o(\tau^{(d-1)/(2m\mu)}\log\tau)$.*

We leave to the reader the following easy

Problem 12.3.16 [27]*.* Treat Riesz means and derive asymptotics with

$$R^*_{(\vartheta)}(\tau) = \begin{cases} |\tau|^{(d-1-\vartheta)(\mu+1)/(2m\mu)+\vartheta} & \text{for } (d-1-\vartheta)\dfrac{\nu}{\mu} < d'' \\ |\tau|^{(d-1-\vartheta)(\mu+1)/(2m\mu)+\vartheta}|\log|\tau|| & \text{for } (d-1-\vartheta)\dfrac{\nu}{\mu} = d'' \\ |\tau|^{(d'-1-\vartheta)(\mu+1)/2m(\mu-\nu)+\vartheta} & \text{for } (d-1-\vartheta)\dfrac{\nu}{\mu} > d'' \end{cases} \tag{12.3.49}$$

[26] Cf. Conjecture 12.2.10.

[27] Cf. Problem 12.2.11.

and in the analogs of Theorems 12.3.12 and 12.3.13 one should assume that $d' < \mu d'' + \vartheta + 1$, $d' > \mu d'' + \vartheta + 1$ respectively.

12.3.3 Effective Asymptotics

Simplifying the Answer

The main defect of Theorems 12.3.11–12.3.13 again is that the expression for $\mathcal{N}(\tau)$ is not effective and includes the eigenvalue counting function for the *family* of auxiliary operators $A(x', \xi')$. Exactly like in the previous Section 12.2 we would like to derive a formula including the eigenvalue counting function for *one* auxiliary operator or at least for a family of auxiliary operators with a smaller number of parameters; however in contrast to the previous section we want also to simplify these axillary operators.

Again, let us start with a more accurate Weyl analysis. Repeating arguments of the Subsection 12.2.3 one can easily prove in the framework of Example 12.3.1 the following estimate similar to (12.2.67):

$$\Big|\int \psi(x')\big(\mathsf{tr}(e(x,x,\tau)) - \sum_{0\le j\le J-1} \kappa_j(x,\tau)\big)\Big|_{\tau=\tau'}^{\tau=\tau}\, dx\Big| \le C\bar\gamma^{d'-1-\frac{\mu d''}{\nu}}\tau^{\frac{d-1}{2m}+\frac{d''}{2m\nu}} + C\tau^{\frac{d-J}{2m}}\bar\sigma^{d''-J}\bar\gamma^{d'} \tag{12.3.50}$$

where $\psi = \psi(x')$ is γ-admissible and supported in $B\big(\bar x', \frac12\bar\gamma\big)$, $r_*(\tau) \le \bar\gamma = |\bar x'| \le r^*(\tau)$, $\tau' \in [\epsilon\tau, \tau)$, $\bar\sigma = \bar\gamma^{-(\mu-\nu)/\nu}\tau^{1/(2m\nu)}$ and the Weyl coefficients κ_j (which now include also powers of effective semiclassical parameter) satisfy the inequalities

$$\Big|\int \psi(x')\kappa_j(x,\tau)\, dx\Big| \le C\tau^{\frac{d-j}{2m}}\bar\sigma^{d''-j}\bar\gamma^{d'} \tag{12.3.51}$$

and

$$\kappa_0 = (2\pi)^{-d}\int n(x,\xi,\tau)\, d\xi \tag{12.3.52}$$

where as before $n(x,\xi,\tau)$ is an eigenvalue counting function for symbol $A(x,\xi)$.

However in the general case one should add to the right-hand expressions of (12.3.50) and (12.3.51) terms reflecting irregularity as $x'' = 0$; one can

think about last terms in (12.3.50) and (12.3.51) with $\bar\sigma$ replaced by $\bar\sigma_0 = \bar\gamma^{-(\mu-\nu)/(\nu+1)}$ as $J > d''$ or $j > d''$ and with extra factor $\log(\bar\sigma/\bar\sigma_0)$ as $J = d''$ or $j = d''$ but in fact these terms are better.

Thus, as in the previous section, using propagation of singularities for operators with operator-valued symbols we have improved the Weyl asymptotics in the cusp zone. However again we need to take into account many terms in the decomposition and in contrast to the previous section singularity as $x'' = 0$ may prevent us from doing this. To remedy this we consider A as a perturbation of some other operator A^0. So let A^0 be another operator also satisfying all the conditions above with the same exponents.

Let us take $J = 1$ and assume that

$$|D^\beta\big(a_\alpha - a^0_\alpha\big)| \le \sum_{q\in Q} \varepsilon_q(x')|x''|^{q-|\beta|}\rho^{*\,2m-|\alpha|} \qquad \forall \beta : |\beta| \le 1 \tag{12.3.53}$$

with the finite set $Q \subset \mathbb{R}^+ \cup 0$. Note that effectively $(A - A^0)$ contains a factor $|x''|^q$ which due to Proposition 12.3.2 in the estimates translates into factor $\sigma(\gamma,\tau)^q$. Then the following estimate similar to (12.2.82) can be proven in the same way:

$$\mathsf{R}_1 \coloneqq \Big|\int \Big(\big(e(x,x,\tau) - e^0(x,x,\tau)\big) - \psi(x')\big(\kappa_0 - \kappa^0_0\big)\Big)\Big|_{\tau=\tau'}^{\tau=\tau} dx\Big| \le \\ C\bar\gamma^{d'-1-\frac{\mu d''}{\nu}}\tau^{\frac{d-1}{2m}+\frac{d''}{2m\nu}} + C\tau^{\frac{d-1}{2m}}\bar\sigma^{d''-1}\bar\gamma^{d'}\bar\eta \tag{12.3.54}$$

with

$$\Big|\int \psi(x')\big(\kappa_0(x,\tau) - \kappa^0_0(x,\tau)\big)\,dx\Big| \le C\tau^{\frac{d}{2m}}\bar\sigma^{d''}\bar\gamma^{d'}\bar\eta \tag{12.3.55}$$

where A^0 satisfy the same conditions as A, all the objects associated with A^0 have an additional superscript "0" and in comparison with (12.3.50)–(12.3.51) we gain factor

$$\bar\eta = \sum_{q\in Q} \sigma(\bar\gamma,\tau)^q \varepsilon_q(\bar\gamma). \tag{12.3.56}$$

We leave to the reader to prove easily that the logarithmic factor does not appear in the last term (12.3.54) even if $d'' = 1$, $q = 0$; this proof is based on representation of κ_1 and assumption $\nu > 0$.

So we have proved the following assertion:

Proposition 12.3.17. *Let both operators A and A^0 satisfy assumptions of Theorem 12.3.11. Moreover, let condition* (12.3.53) *be fulfilled. Then estimates* (12.3.54)–(12.3.56) *hold.*

Now our arguments basically repeat those of Subsection 12.2.3.

Let us replace τ and τ' by $2^{-n}\tau$ and $2^{-n-1}\tau$ respectively and sum over $n = 0, \dots, n_0 = c_0\lfloor \log(\tau\bar{\sigma}^{2m})\rfloor + c_1$. Applying Proposition 12.3.2 we can get rid of τ' in all these estimates arriving to the statement similar to Corollary 12.2.16. We leave easy details to the reader.

Assume that

(12.3.57) ε_q are γ-admissible with no more than a polynomial growth and $\eta = O(\gamma^{-\varkappa})$ as $\gamma \to \infty$.

Recall that ψ is γ-admissible and supported in the $\bar{\gamma}$-neighborhood of $\bar{x}'$. Then after summation over a γ-admissible partition of unity in Z we can obtain miscellaneous formulae (later under certain conditions we will calculate the terms associated with the operator A^0):

First, we obtain formula (12.2.88); then immediately we arrive to

Proposition 12.3.18. *(i) Let both operators A and A^0 satisfy assumptions of Theorem 12.3.11. Moreover, let condition* (12.3.53) *be fulfilled with $Q = \{0, 1\}$, $\varepsilon_1 = \gamma^{-1}$, $\varepsilon_0 = \sigma_0\gamma^{-1}$. Then estimate* (12.3.54) *holds with the right hand expression $O\bigl(R^*(\tau)\bigr)$.*

(ii) Further, in the framework of Theorem 12.3.12 we obtain formula (12.2.89) *with the remainder estimate $o\bigl(\tau^{(d-1)(\mu+1)/(2m\mu)}\bigr)$ rather than $o\bigl(\tau^{(d-1)/(2m)}\bigr)$.*

(iii) Furthermore, in framework of Theorem 12.3.13 we obtain formula (12.2.90) *with the remainder estimate $o\bigl(\tau^{(d'-1)(\mu+1)/2m(\mu-\nu)}\bigr)$ as $\varepsilon_1 = o(\gamma^{-1})$, $\varepsilon_0 = o(\sigma_0\gamma^{-1})$.*

(iv) In particular, in the framework of Example 12.3.1 in Statements (i)–(ii) one can take

$$a^0_{\alpha(j)}(x) = \sum_{\beta'':|\beta''|=\mu(2m-|\alpha|)-j} \bigl(\partial^{\beta''}_{x''} a_{\alpha(j)}(x)\bigr)_{x''=0} \frac{x''^{\beta''}}{\beta''!} \tag{12.3.58}$$

while in Statement (iii) one should add a correction term $\varkappa'_1\tau^{(d'-1)(\mu+1)/2m(\mu-\nu)}$.

Main Theorems

Let us formulate the principal theorems proved in this section. They are due to the results above. Then we obtain the following statements:

Theorem 12.3.19 [28]**.** *Let conditions* (12.3.1), (12.3.3), (12.3.6), (12.3.9)–(12.3.12), (12.3.19) *and stabilization conditions* (11.5.4), (12.3.34) *and* (12.3.56)–(12.3.57) *be fulfilled.*

(i) Then asymptotics

$$\mathsf{N}^-(\tau) = \mathcal{N}_1(\tau) + O\big(R^*(\tau) + \tau^b\big), \qquad b = \frac{(\mu+1)}{2m(\mu-\nu)}\big(d-1-\frac{(\nu+1)}{\nu}(d''-1)-\varkappa\big) \tag{12.3.59}$$

holds where $R^*(\tau)$ *is given by* (12.3.39) *and* $\mathcal{N}_1(\tau)$ *is defined by* (12.2.100), $\boldsymbol{n}^0(x',\xi',\tau)$ *is the eigenvalue counting function for operator* $\boldsymbol{a}^0(x',\xi';x'',D'')$ *in* $\mathbb{H}$ *and* $c_0 \le t_1\tau^{-1/(2m\mu)} \le \tau^\delta$ *with a small enough exponent* $\delta > 0$.

Moreover, (12.3.24) *remains true for* $\mathcal{N}_1$.

(ii) Further, let $(d-1)\nu < \mu d''$, $\kappa > \nu^{-1}\big(d-1-(\nu+1)(d''-1)\big)$ [29]*. Moreover, let the standard non-periodicity condition be fulfilled for the Hamiltonian flow of* $a(x',\xi')$.

Then asymptotics

$$\mathsf{N}^-(\tau) = \mathcal{N}_1(\tau) + o\big(\tau^{\frac{(d-1)(\mu+1)}{2m\mu}}\big) \tag{12.3.60}$$

holds as $\tau \to +\infty$ *provided* $t_1 \to +\infty$ *(we assume that* $t_1 \le \tau^\delta$*).*

(iii) Furthermore, let $(d-1)\nu < \mu d''$, $\kappa > (d'-1)\mu(\mu+1)^{-1} - (d''-1)\nu^{-1}$ [30]*.*

Moreover, let non-periodicity condition (12.2.51) *be fulfilled. Then the asymptotics*

$$\mathsf{N}^-(\tau) = \mathcal{N}_1(\tau) + \varkappa_1^*\tau^{\frac{1}{2m\mu}(d'-1)(\mu+1)} + o\big(\tau^{\frac{1}{2m(\mu-\nu)}(d'-1)}\big) \tag{12.3.61}$$

holds with the coefficient $\varkappa_1^*$ *given in Remark 4.4.13.*

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[28] Cf. Theorem 12.2.19.

[29] Then the remainder estimate in (12.3.59) is $O\big(\tau^{\frac{1}{2m\mu}(d-1)}\big)$.

[30] Then the remainder estimate in (12.3.59) is $O\big(\tau^{\frac{1}{2m(\mu-\nu)}(d'-1)}\big)$.

Discussion

We leave to the reader the following series of problems:

Problem 12.3.20. (i) Formulate and prove all results of this section for a domain where Λ (the manifold of the degeneration) consists of more than one connected components (with different characteristics) $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_N$, $\Lambda_j \cap \Lambda_k = \emptyset$ as $0 \neq j \neq k \neq 0$. This problem is trivial even if the dimensions of these cusps (d'_k) and the exponents of degenerations (μ_k) differ.

(ii) Consider the case when $X \neq \mathbb{R}^d$ but as $|x| \geq c$ it coincides with the conic domain assuming that

(12.3.62) each connected component of degeneration manifold Λ either does not intersect ∂X or is contained there.

Then coefficients $\tilde{\kappa}_j$ will appear here as well.

(iii) Investigate different boundary conditions.

(iv) In the framework of (ii) get rid of assumption (12.3.62).

(v) Write explicit expressions as $m = 2$, $\nu = 1$.

In this case operator $a_\infty(x', x'; x'', D'')$ is a quadratic Hamiltonian and due to ellipticity it is after metaplectic transformation (see Subsection 1.2.5) it becomes a direct sum of harmonic oscillators $\sum \omega_j(D_j^2 + x_j^2)$ with very well known spectrum and eigenvalue counting function

$$n_\infty(\tau) = \#\{\alpha \in \mathbb{Z}^{+k}, \sum_k \omega_k(2\alpha_k + 1) < \tau\}. \tag{12.3.63}$$

(vi) Conduct analysis similar to one in the cases (iv), (v) of Subsubsection *12.2.3.1 Scheme of Effective Asymptotics*.

(vii) Repeat with trivial modifications arguments of Subsubsection 12.2.3.

(viii) Derive more precise remainder estimates in the framework of Theorem 12.3.12 as Hamiltonian flow associated with $a(x, \xi)$ is of power or exponential non-periodic type.

(ix) Derive more precise remainder estimate in the framework of Theorem 12.3.13 and Example 12.3.14 assuming that the Hamiltonian flow associated with $\zeta(x',\xi')$ is of power or exponential non-periodic type.

(x) Investigate the case in the framework of Theorem 12.3.13 and Example 12.3.14 assuming that the Hamiltonian flow associated with $\zeta(x',\xi')$ is periodic.

(xi) Derive asymptotics for Riesz means.

Obviously Remark 12.2.22 stays.

12.4 Operators with Singular Potentials Decaying at Infinity

12.4.1 Problem Set-up

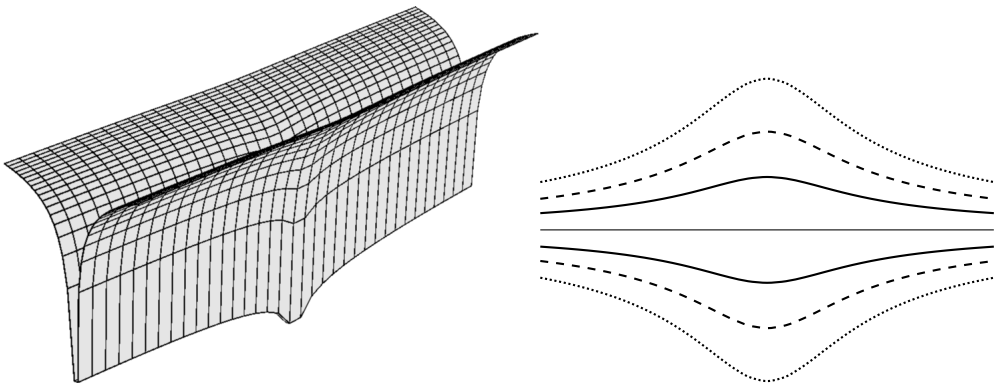
Let us start from the following example:

Example 12.4.1. Consider self-adjoint (matrix) operator on $\mathbb{R}^d$ (12.3.1) (possibly in the divergent form) with the positively homogeneous of degrees $(2m-|\alpha|)\mu$ (as $|x|\ge c$) coefficients a_α. In Section 11.6 we investigated the case when (12.3.2), (12.3.3) hold, $0>\mu>-1$ and established asymptotics for $\mathsf{N}^-(\tau)$ as $\tau\to -0$.

Now we want to consider the case when coefficients are singular at Λ, behaving basically as $\rho^{2m-|\alpha|}$ with $\rho(x)=\zeta(x)^\nu |x|^{\mu-\nu}$, $\zeta(x)=\mathsf{dist}(x,\Lambda)$ where Λ satisfies (12.3.5) and $0>\nu>\mu>-1$.

Without any loss of the generality we can assume that (at least locally) $\Lambda=\{x : x''=0\}$. Then we need to introduce $\zeta(x)$, $\gamma(x)$ and $\rho(x)$ by (12.3.9). However we do not want to have true singularities as $x''=0$: one can manage to investigate them too but it will be a pain with no gain - extra analysis without new effects; so instead we assume that (12.3.12) holds with ρ, ζ redefined as ρ^*, ζ^* by (12.3.10)–(12.3.11).

Remark 12.4.2. (i) We need condition $\nu>\mu$ for the cusp to shrink at infinity. As $0>\nu>-1$ the gorge would be too narrow to generate eigenvalues tending to $-\infty$ even if we did not "regularized" coefficients, departing from Example 12.4.1.



(a) Gorge (b) Cusp

Figure 12.6: Potential graph and level lines

(ii) Our results here are very similar to those of the previous Section 12.3 and could be formally obtained from them by replacing "$\tau \to +\infty$" by "$\tau \to -0$" and assumption $\mu > \nu > 0$ by $0 > \nu > \mu > -1$.

Consider operator first in the framework of the theory of Section 11.6; after scaling $x_{\mathsf{new}} = x\bar{\sigma}$, $\xi_{\mathsf{new}} = |\tau|^{-1/(2m)}\xi$ we conclude that operator $|\tau|^{-1}A$ is elliptic on the energy levels (-1) and below provided

$$|x''| \ge \sigma(x', \tau) \coloneqq c_0 |x'|^{-(\mu-\nu)/\nu} \tau^{\frac{1}{2m\nu}}, \tag{12.4.1}$$

(we assume that $|x'| \gg |x''|$ and $|x'| \gg 1$) and the uncertainty principle $\rho(x)\zeta(x) \ge 1$ is fulfilled provided

$$|x''| \ge \sigma_1(\tau) \coloneqq |\tau|^{-\frac{1}{2m}}. \tag{12.4.2}$$

Then we arrive immediately to the following assertion:

Proposition 12.4.3 [31]**.** *Let conditions* (12.3.1), (12.3.3) *and* (12.3.9)–(12.3.12) *be fulfilled at the point* $\bar{x} \in \mathbb{R}^d$.

(i) Then

$$|e(x, y, \tau)| \le C|\tau|^{\frac{d}{2m}} \zeta(x)^{-s} \sigma_1(\tau)^s \qquad \forall x, y \in B\Big(\bar{x}, \frac{1}{2}\zeta(\bar{x})\Big) \tag{12.4.3}$$

with an arbitrarily large exponent s.

[31] Cf. Proposition 12.3.2.

(ii) Moreover, if $U(x,y,t)$ *is the Schwartz kernel of the operator* $\tau^{-1}A$ *then*

$$\bigl|F_{t\to\lambda}\bar{\chi}_{T,T'}(t)U(x,y,t)\bigr| \le C|\tau|^{d/(2m)}\zeta(x)^{-s}\sigma_1(\tau)^s \tag{12.4.4}$$
$$\forall x,y\in B\bigl(\bar{x},\frac{1}{2}\zeta(\bar{x})\bigr)\quad \forall T'\in\mathbb{R},\ T=\sigma(\bar{x})\quad \forall \lambda\le 1+\epsilon$$

with a small enough constant $\epsilon>0$.

Remark 12.4.4[32]. Consider the "gorge"

$$\mathcal{X}(\tau)=\bigl\{x,|x''|\le\sigma(x',\tau)\bigr\}. \tag{12.4.5}$$

It consists of the *regular part* where $|x|\le c|\tau|^{1/(2m\mu)}$ and the *cusp* where $|x|\ge c|\tau|^{1/(2m\mu)}$; this cusp stretches along $X'=\{x,x''=0\}$ and shrinks at infinity (on $|x|$) for any $\tau>0$.

Let us apply now the theory of the operators with the operator-valued symbols (in rather limited manner at this time); to do this we need to analyze operator $\mathsf{a}(x',\xi')=A(x',\xi';x'',D'')$:

Proposition 12.4.5[33]**.** *Let conditions* (12.3.1), (12.3.3) *and* (12.3.9)–(12.3.12) *be fulfilled. Let* $\lambda_k(x',\xi')$ *be eigenvalues of* $\mathsf{a}(x',\xi')$ *in* $\mathbb{H}=\mathscr{L}^2(\mathbb{R}^{d''})$. *Then*

$$\lambda_k(x',\xi)\ge\epsilon_0|\xi'|^{2m}-\varepsilon_k\sigma_0(x')^{-2m} \tag{12.4.6}$$

where $\sigma_0(x')$ *is defined by* (12.4.2).

Proof. Coincides with the proof of Proposition 12.3.4. □

Due to our assumptions $|\tau|^{-1}\mathsf{a}(x',\xi')$ is elliptic as an operator with an operator-valued symbol on the energy level (-1) unless $|\xi'|\le c\tau^{1/(2m)}$ and $|x'|\le r^*(\tau)$ with $r^*(\tau)$ defined by (12.3.20).

Proposition 12.4.6[34]**.** *Let conditions* (11.1.6), (11.1.8) *and* (12.3.1), (12.3.3) *and* (12.3.9)–(12.3.12) *be fulfilled. Then estimate* (12.3.21) *holds.*

Proof. Coincides with the proof of Proposition 12.3.5. □

[32] Cf. Remark 12.3.3.
[33] Cf. Proposition 12.3.4.
[34] Cf. Proposition 12.3.5.

Remark 12.4.7[35]. (i) Note that $|\tau|^{1/(2m)} \lesssim \sigma(x',\tau)$ if and only if $|x'| \lesssim r^*(\tau)$.

(ii) Note that $\mathsf{mes}\big(\mathcal{X}(\tau)\big) = O(|\tau|^{d/(2m\mu)})$ if and only if $\mu d'' < \nu d$. Under this condition the principal part of the Weyl formula is $\asymp |\tau|^{d(\mu+1)/(2m\mu)}$.

(iii) Let us drop condition $\mu d'' < \nu d'$ and let us restrict integration to the domain $\{x, |x| \le r^*(\tau)\}$ in the Weyl formula due Proposition 12.4.6. Then the principal part of the Weyl asymptotics (modified in this way) has the magnitude of (12.3.24).

(iv) Let us consider the remainder estimate in the Weyl formula. We can integrate only over domain $\{x, |x| \le r^*(\tau)\}$ due to Proposition 12.4.6. Then the remainder estimate in the modified Weyl asymptotics is (12.3.25)

(v) Let $\mu(d''-1) < \nu(d-1)$. Then under standard stabilization conditions of a_α as $|x| \to \infty$ to positively homogeneous $a_{\infty,\alpha}$ and standard restrictions to the Hamiltonian flow generated by the corresponding Hamiltonian $a_\infty(x,\xi)$ one can obtain the remainder estimate $o\big(|\tau|^{(d-1)(\mu+1)/(2m\mu)}\big)$.

Again, in this remark the role of the "operator-valued theory" is very limited: it only provides a nice cut-off in the Weyl formulas.

We leave to the reader the following easy problem:

Problem 12.4.8[36]. Calculate the principal part of the asymptotics and remainder estimate for Riesz means in the Remark 12.4.7.

12.4.2 Sharp Asymptotics

So, as we are interested in the asymptotics of $\mathsf{N}^-(\tau)$, our analysis should be done as

$$\mu(d''-1) \le \nu(d-1). \tag{12.4.7}$$

Here we should treat only the cusp zone

$$\mathcal{X}_{\mathsf{cusp}}(\tau) = \big\{c_1\tau^{\frac{1}{2m}} \coloneqq r_*(\tau) \le |x'| \le r^*(\tau),\ |x''| \le c_2\sigma(x',\tau)|\tau|^{-\delta}\big\} \tag{12.4.8}$$

[35] Cf. Remark 12.3.6.

[36] Cf. Problem 12.3.7.

with $r^*(\tau)$ and $\sigma(x',\tau)$ defined by (12.3.20) and (12.4.1) and an arbitrarily small exponent $\delta > 0$.

Really, estimate (12.4.3) and Proposition 12.4.6 yield that for $|x'| \ge c_1\tau^{1/(2m\mu)}$ outside of $\mathcal{X}$ function $e(x,x,\tau)$ is negligible; on the other hand, we have

Proposition 12.4.9 [37]**.** *(i) Contribution of the regular zone* (12.3.28) *(with arbitrarily large constant c) to the principal part of the Weyl asymptotics and to the remainder estimate are $O\bigl(|\tau|^{d(\mu+1)/(2m\mu)}\bigr)$ and $O\bigl(|\tau|^{(d-1)(\mu+1)/(2m\mu)}\bigr)$ respectively* [38]*.*

(ii) Moreover, under standard stabilization conditions and the standard non-periodicity condition to the Hamiltonian flow the contribution of the zone $\mathcal{X}_{\mathsf{reg}}(\tau)$ to the remainder estimate is $o\bigl(|\tau|^{(d-1)(\mu+1)/(2m\mu)}\bigr)$.

Proof. Coincides with the proof of Proposition 12.3.8. □

Furthermore, the same arguments yield that treating zone $\bigl\{|x'| \le c_0 r^*(\tau)|\tau|^{\delta'}\bigr\}$ with an arbitrarily small exponent $\delta' > 0$ one can set $\delta = 0$ in the definition of $\mathcal{X}(\tau)$; in this case constant c_2 should be large enough.

To treat the cusp zone (12.4.8) let us apply theory of Section 4.4. For this we need to check if operator in question is (x',ξ') microhyperbolic on the energy level τ in (γ,ρ_τ)-scale with $\gamma(x') = \frac{1}{2}|x'|$ and $\rho(x') = \sigma_0(x')^{-1}$.

Proposition 12.4.10 [39]**.** *Operator $a(x',D')$ is (x',ξ')-microhyperbolic under stabilization condition* (12.3.34).

Proof. Coincides with the proof of Proposition 12.3.9. □

We leave to the reader the following easy problem:

Problem 12.4.11 [40]*.* Establish other conditions providing (x',ξ')-microhyperbolicity of $a(x',D')$.

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[37] Cf. Proposition 12.3.8.

[38] Subject to conditions of Section 11.5. For example, as either operator is the Schrödinger operator or its perturbed power or stabilization condition (11.5.4) is fulfilled.

[39] Cf. Proposition 12.3.9.

[40] Cf. Problem 12.3.10.

So, assume that microhyperbolicity condition is fulfilled. Then we can apply Theorem 4.4.8 as this operator is ξ'-microhyperbolic and Theorem 4.4.9 as this operator is (x', ξ')-microhyperbolic (i.e. in the general case) but there is a problem: operators B_1^{-k} is not of the trace norm. So we need to modify our arguments.

First of consider trace only with respect to x' and consider $\mathbb{H} \to \mathbb{H}$ operator norm; then the contribution of γ-element to the remainder does not exceed

$$C\hbar^{1-d'} = C\rho^{d'-1}\gamma^{d'-1} \tag{12.4.9}$$

as $\hbar = \rho^{-1}\gamma^1$. Then due to embedding theorem if we consider just pointwise norm (on (x'', y'')), the the contribution of γ-element to the remainder does not exceed expression (12.4.9) multiplied by $\rho^{d''/(2m)}$, i.e. $C\rho^{d-1}\gamma^{d'-1}$. Then after cut-off as $\zeta(x) \le \sigma(x', \tau)$ leads to the the remainder estimate

$$\int_{\sigma_0}^{\sigma} (\zeta^{\nu}\gamma^{\mu-\nu})^{d-1}\gamma^{d'-1}\zeta^{d''-1}\, d\zeta \tag{12.4.10}$$

on the other hand as $|x''| \ge \sigma(x', \tau)$ the pointwise norm does not exceed

$$C|\tau|^{\frac{d}{2m}}\gamma^{d'-1} \times |x''|^{-s}\sigma(x', \tau)^s$$

where we used the standard ellipticity of our operator here and the contribution of this zone to the remainder estimate does not exceed (12.4.10).

So, the total remainder estimate is

$$\int_{r_*(\tau)}^{r^*(\tau)} \int_{\sigma_0}^{\sigma} (\zeta^{\nu}\gamma^{\mu-\nu})^{d-1}\gamma^{d'-2}\zeta^{d''-1}\, d\zeta\, d\gamma \tag{12.4.11}$$

and one can prove easily that it is $\asymp R^*(\tau)$ defined by (12.3.39).

Let us apply Weyl asymptotics in the zone $\{|x'| \le t_1|\tau|^{1/(2m\mu)}\}$. To ensure (x, ξ)-microhyperbolicity there we assume that the standard stabilization condition (11.5.4) is fulfilled[17].

Then we obtain the following theorem:

Theorem 12.4.12 [41]**.** *Let conditions* (12.3.1), (12.3.3), (12.3.9)–(12.3.12) *and stabilization conditions* (11.5.4) [38),17)], (12.3.34) *be fulfilled.*

Let $\psi \in \mathscr{C}_0^K(\mathbb{R}^{d'})$, $\psi(x') = 1$ *as* $|x'| \le c_0$. *Then*

[41] Cf. Theorems 12.2.6 and 12.3.11.

(i) Asymptotics (12.3.40) *holds as* $\tau \to -0$ *with* $\mathcal{N}(\tau)$ *defined by* (12.2.43) *with* $\psi_t(x') = \psi(x'/t)$ *where* $t \asymp r_*(\tau)$ [19] *and with* $R^*(\tau)$ *defined by* (12.3.39).

(ii) Moreover, $\mathcal{N}(\tau)$ *has a magnitude described in* (12.3.24).

Repeating corresponding arguments of Section 12.3 we arrive to the following theorem:

Theorem 12.4.13 [42]**.** *Let the conditions of Theorem 12.4.12 be fulfilled with* $(d-1)\nu > d''\mu$.

Moreover, let the standard non-periodicity condition to the Hamiltonian trajectories generated by $a_\infty(x,\xi)$ *be fulfilled. Then asymptotics* (12.3.42) *holds as* $\tau \to -0$ *where* $\mathcal{N}(\tau)$ *is defined by* (12.2.46),

$$(12.4.12)\quad (2\pi)^{-d}\int \psi\kappa_1\tau^{\frac{1}{2m}(d-1)}\,dx \quad \text{and} \quad (2\pi)^{1-d}\int_{\partial X}\psi\tilde{\kappa}_1\tau^{\frac{1}{2m}(d-1)}\,dS$$

are the standard second terms [29] *in the Weyl asymptotics,* $\psi_t(x') = \psi_0(x'/t)$ *where* $t = t(\tau)$ *is an appropriate function such that* $t(\tau)/r_*(\tau) \to +\infty$ *as* $\tau \to -0$ [30]

Theorem 12.4.14 [43]**.** *Let the conditions of Theorem 12.4.12 be fulfilled with* $(d-1)\nu < d''\mu$.

Let us consider operator-valued symbol $a_\infty(x',\xi')$ *in* $\mathbb{H}$*; let* $\lambda_j(x',\xi')$ *be its (non-decreasing) eigenvalues*[24]*. Let us consider*

$$(12.4.13)\qquad a_{\infty,j}(x',\xi') = a_\infty(x',\xi')e_\infty(x',\xi',\zeta_j(x',\xi'))$$

where $e_\infty(x',\xi',\tau)$ *is the spectral projector for* $a_\infty(x',\xi')$ *and assume that non-periodicity condition* (12.2.51) *is fulfilled.*

Then asymptotics (12.3.46) *holds as* $\tau \to -0$ *with an appropriate coefficient* κ_1^* *(given in Remark 4.4.13).*

Condition (12.2.51) looks extremely cumbersome and impossible to check. However in the following rather special example this condition becomes much simpler:

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[42] Cf. Theorems 12.2.7 and 12.3.12.

[43] Cf. Theorems 12.2.8 and 12.3.13.

Example 12.4.15[44]. Let $d''=1$ and A be a Schrödinger operator stabilizing at infinity to $|\xi^2|+|x_d|^{2\nu}\cdot\omega(x')^2$ with ω, positively homogeneous of degree $2(\mu-\nu)$.

Then one needs to check the standard non-periodicity condition for symbol In particular, the standard non-periodicity condition for $\lambda(x',\xi')$ is fulfilled as $\sigma^0=|x'|^{-\eta}$, $\eta<0$ and $\eta\neq\frac12$; recall that $d'\geq 2$.

Finally, let us consider the case $d'=1+d''\mu$. Considering diagonal operator with diagonal elements $\lambda_j^{\mathsf{w}}(x',D')$ one can prove easily the remainder estimate $o(\tau^{(d-1)/2m}|\log|\tau||)$ (not subject to any non-periodicity condition). This leads us to conjecture which we were unable to prove

Conjecture 12.4.16[45]**.** *As $(d-1)\nu<d''\mu$ remainder estimate is in fact $o(|\tau|^{(d-1)/(2m\mu)}|\log|\tau||)$.*

We leave to the reader the following easy albeit interesting problem:

Problem 12.4.17[46]. (i) Treat Riesz means and derive asymptotics with $R^*_{(\vartheta)}(\tau)$ defined by (12.3.49).

(ii) Here one needs also to improve results of Section 11.6. To do it one can consider propagation and extend time from $T\asymp|x|^{1-\mu}$ to $T\asymp|\tau|^{(1-\mu)/(m\mu)}$.

12.4.3 Effective Asymptotics

Simplifying the Answer

Again, the main defect of Theorems 12.4.12–12.4.14 again is that the expression for $\mathcal{N}(\tau)$ is not effective and includes the eigenvalue counting function for the *family* of auxiliary operators $a(x',\xi')$. Exactly like in the previous Section 12.3 we would like to derive a formula including the eigenvalue counting function for *one* auxiliary operator or at least for a family of auxiliary operators with a smaller number of parameters; however in contrast to the previous section we want also to simplify these axillary operators.

Let us assume that (12.3.53) holds with the finite set $Q\subset\mathbb{R}^+\cup 0$. Note that effectively $(A-A^0)$ contains a factor $|x''|^q$ which due to Proposition 12.4.3

[44] Cf. Examples 12.2.9 and 12.3.14.
[45] Cf. Conjectures 12.2.10 and 12.3.15.
[46] Cf. Problems 12.2.11 and 12.3.16.

in the estimates translates into factor $\sigma(\gamma,\tau)^q$. Then arguments leading to (12.2.82) (or (12.3.54) imply that

$$\mathsf{R}_1 \le R + \varepsilon_q(\gamma)\int_{\sigma_0}^{\sigma} (\zeta^\nu \gamma^{\mu-\nu})^{d-1}\gamma^{d'}\zeta^{d''-2+q}\,d\zeta \tag{12.4.14}$$

where R_1 is defined by (12.3.54) and the right-hand expression is (12.4.10) modified in the obvious way. Further, one can estimate the left-hand expression of (12.3.55) in the same way.

Recall that A^0 satisfy the same conditions as A, all the objects associated with A^0 have an additional superscript "0".

Then after summation with the respect to partition we conclude that redefined R_1 does not exceed

$$\int_{r_*(\tau)}^{r^*(\tau)}\int_{\sigma_0}^{\sigma} (\zeta^\nu \gamma^{\mu-\nu})^{d-1}\gamma^{d'-1+(\mu-\nu)q/(\nu+1)}\zeta^{d''+q-2}\,d\zeta d\gamma \tag{12.4.15}$$

as $\varepsilon_q(\gamma) = \gamma^{-p+(\mu-\nu)q/(\nu+1)}$. One can see easily that this expression does not exceed $CR^* + C|\tau|^b$ because $(\zeta^\nu\gamma^{\mu-\nu})^{d-1}\gamma^{d'+(\mu-\nu)q/(\nu+1)}\zeta^{d''+q-1}$ calculated as $\gamma = r^*(\tau)$, $\zeta = \sigma_0(r^*(\tau))$ is exactly the second term here.

Now our arguments basically repeat those of Subsections 12.2.3 and 12.3.3. First, we obtain formula (12.2.88); then immediately we arrive to asymptotics similar to (12.3.54)–(12.3.56). Further, in the framework of Theorem 12.4.13 we obtain formula (12.2.89) with the remainder estimate $o(R^*) + O(|\tau|^b)$. Furthermore, in the framework of Theorem 12.4.13 we obtain formula (12.2.90) with the remainder estimate $o(R^*) + O(|\tau|^b)$.

Problem 12.4.18[47]. (i) Consider domains with the boundary (with appropriate boundary conditions), assuming that (12.3.62) holds. Then coefficients $\tilde\kappa_j$ will appear here as well.

(ii) Conduct analysis similar to one in the Cases (iv) and (v) of Subsubsection *12.2.3.1 Scheme of Effective Asymptotics*.

(iii) Repeat with trivial modifications arguments of Subsubsection *12.2.3.2 Scheme of Effective Asymptotics*.

[47] Cf. Problem 12.3.20.

Main tTheorems

Let us formulate the principal theorems proved in this section. They are due to the results above.

Then we obtain the following statements:

Theorem 12.4.19 [48]**.** *Let conditions* (12.3.1), (12.3.3), (12.3.9)–(12.3.12) *and stabilization conditions* (11.5.4), (12.3.34) *and* (12.3.56)–(12.3.57) *be fulfilled.*

(a) Then asymptotics (12.3.59) *holds where* $R^*(\tau)$ *is given by* (12.3.39), $\mathcal{N}_1(\tau)$ *is defined by* (12.2.100), $n^0(x',\xi',\tau)$ *is the eigenvalue counting function for operator* $a^0(x',\xi';x'',D'')$ *in* $\mathbb{H}$ *and* $c_0 \le t_1|\tau|^{-1/(2m\mu)} \le |\tau|^{-\delta}$ *with a small enough exponent* $\delta > 0$.

Moreover, (12.3.24) *remains true for* $\mathcal{N}_1$.

(b) Further, let $(d-1)\nu < \mu d''$ *and* $b > (d-1)\nu$. *Moreover, let the standard non-periodicity condition be fulfilled for the Hamiltonian flow of* $a(x',\xi')$.

Then asymptotics (12.3.60) *holds as* $\tau \to -0$ *provided* $t_1 \to +\infty$ *(we assume that* $t_1 \le |\tau|^{-\delta}$*).*

(c) Furthermore, let $(d-1)\nu < \mu d''$ *and* $R_1^* = o(R^*)$. *Moreover, let non-periodicity condition* (12.2.51) *be fulfilled.*

Then the asymptotics (12.3.61) *holds with the coefficient* $\varkappa_1^*$ *given in Remark 4.4.13.*

We leave to the reader the following easy series of problems:

Problem 12.4.20 [49]. (i) Formulate and prove all results of this section for a domain where Λ (the manifold of the degeneration) consists of more than one connected components (with different characteristics) $\Lambda = \Lambda_1 \cup ... \cup \Lambda_N$, $\Lambda_j \cap \Lambda_k = \emptyset$ as $0 \ne j \ne k \ne 0$. This problem is trivial even if the dimensions of these cusps (d_k') and the exponents of degenerations (μ_k) differ.

(ii) Consider the case when $X \ne \mathbb{R}^d$ but as $|x| \ge c$ it coincides with the conic domain assuming that each component of Λ either does not intersect ∂X or is contained in ∂X. Investigate different boundary conditions.

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[48] Cf. Theorems 12.2.19 and 12.3.19.

[49] Cf. Problem 12.4.20.

(iii) Derive more precise remainder estimates in the framework of Theorem 12.4.13 as Hamiltonian flow associated with $a(x,\xi)$ is of power or exponential non-periodic type.

(iv) Derive more precise remainder estimate in the framework of Theorem 12.4.14 assuming that the Hamiltonian flow associated with $\zeta(x',\xi')$ is of power or exponential non-periodic type.

(v) Derive asymptotics for Riesz means.

Obviously Remark 12.2.22 stays.

12.5 Spectral Asymptotics for Maximally Hypoelliptic Operators

12.5.1 Maximally Hypoelliptic Operators

In this section we consider a class of pseudodifferential operators on a compact closed $\mathscr{C}^{K'}$ manifold X degenerating on a symplectic manifold $\Lambda \subset T^*X \setminus 0$ conical in ξ. The same results hold with no change for operators acting in Hermitian bundles. Moreover, similar results hold after an obvious modification for differential operators on manifolds with boundary provided $\Lambda|_{\partial X} = \emptyset$. We leave these modification to the reader. So, let us assume that

(12.5.1) X is a compact $\mathscr{C}^{K'}$ manifold, $\Lambda \subset T^*X \setminus 0$ is a symplectic manifold conic in ξ, $d = \dim X$, $2d' = \dim \Lambda$, $1 \le d' < d$, $d'' = d - d'$.

Definition 12.5.1. Let $\boldsymbol{\Psi}^{m,\mu}(X,\Lambda,\mathbb{K})$ with $m \in \mathbb{R}$, $\mu \in \mathbb{Z}^+$ denote the class of pseudodifferential operators from $\mathscr{H}^s(X,\mathbb{K})$ to $\mathscr{H}^{s-m}(X,\mathbb{K})$ [50] with symbols in every local chart of the form

$$a = \sum_{0\le n\le K-1} a_{\frac{n}{2}} + a'_{\frac{K}{2}} \tag{12.5.2}$$

[50] With $s \in (-\bar{s},\bar{s})$ and with $\bar{s} = \bar{s}(K',d,m) \to +\infty$ as $K' \to +\infty$ and d,m are fixed. This footnote holds until the end of this section.

where $a_{\frac{n}{2}}$ are $\mathscr{C}^{K'-n}$ symbols positively homogeneous of degrees $(m-\frac{n}{2})$ in ξ and such that

$$a^{(\alpha)}_{\frac{n}{2}(\beta)}(z)=0 \qquad \forall z\in\Lambda \quad \forall \alpha,\beta : |\alpha|+|\beta|+n<\mu \tag{12.5.3}$$

and the remainder $a'_{\frac{K}{2}}$ is assumed to be smooth enough and decrease quickly enough as $|\xi|\to\infty$. Here and below $K'>2K$ and K are large enough.

It is well known (see also Section 1.2) that these classes are microlocally invariant. In an appropriate microlocal symplectic coordinate system on $T^*X\setminus 0$ we have

$$\Lambda=\bigl\{(x,\xi)\in T^*X:\ x''=\xi''=0\bigr\} \tag{12.5.4}$$

where $x=(x';x'')=(x_1,\dots,x_{d'};x_{d'+1},\dots,x_d)$ and similar notation is applied to ξ. Then for the operator $A\in\boldsymbol{\Psi}^{m,\mu}(X,\Lambda,\mathbb{K})$ and for $z\in\Lambda$ let us introduce the operator

$$A_z(x'',D'')\coloneqq \sum_{|\alpha|+|\beta|+n=\mu} \frac{1}{\alpha!\beta!}a^{(\alpha)}_{\frac{n}{2}(\beta)}(z)x''^{\beta}D''^{\alpha} \tag{12.5.5}$$

acting in the space $\mathscr{L}^2(\mathbb{R}^{d''},\mathbb{K})$ [51]. Recall that $f^{(\alpha)}_{\beta}=\partial^{\alpha}_{\xi}\partial^{\beta}_{x}f$.

Definition 12.5.2. This operator $A_z(x'',D'')$ is called the *microlocalization* (of order μ) of the operator A at the point $z\in\Lambda$.

It is well known (see Subsection 1.2.5) that in another microlocal coordinate system with the same form of Λ we obtain the microlocalization $A'_z=Q_z^*A_zQ_z$ where Q_z is a unitary metaplectic (see Definition 1.2.39) operator in $\mathscr{L}^2(\mathbb{R}^{d''})$ also acting in $\mathscr{L}^2(\mathbb{R}^{d''},\mathbb{K})=\mathscr{L}^2(\mathbb{R}^{d''})\otimes\mathbb{K}$.

Definition 12.5.3. One says that the operator $A\in\boldsymbol{\Psi}^{m,\mu}(X,\Lambda,\mathbb{K})$ is *maximally hypoelliptic* if and only if the following two conditions are fulfilled: the inequality

$$\|a_0(x,\xi)v\|\ge\epsilon_0\sigma^{\mu}(x,\xi)|\xi|^m\|v\| \qquad \forall (x,\xi)\in T^*X \quad \forall v\in\mathbb{K} \tag{12.5.6}$$

holds with a constant $\epsilon_0>0$ and with

$$\sigma(x,\xi)=\operatorname{dist}\bigl((x,\xi|\xi|^{-1}),\Lambda\bigr)\asymp |x''|+|\xi''|\cdot|\xi'|^{-1} \tag{12.5.7}$$

where the last relation holds as Λ is given by (12.5.4) and only in the conical vicinity of Λ and

[51] A Hermitian bundle $\mathbb{K}$ should be replaced by its fiber $\mathbb{K}_z$ over z in this definition.

(12.5.8) Operator A_z is injective in $\mathbb{H} = \mathscr{L}^2(\mathbb{R}^{d''}, \mathbb{K})$ at every point $z \in \Lambda$. In this case

$$\|A_z v\|_{\mathbb{H}} \geq \epsilon_0 \|v\|_{\mathbb{H}} \qquad \forall v \in \mathbb{H} \quad \forall z \in \Lambda \cap \{|\xi'| = 1\} \tag{12.5.9}$$

with an appropriate constant $\epsilon_0 > 0$.

Here and below $\|.\|_{\mathbb{K}}$, $\|.\|_{\mathbb{H}}$ and $\|.\|$ are the norms in $\mathbb{K}$, $\mathbb{H}$ and $\mathscr{L}^2(X, \mathbb{K})$ respectively and $\langle ., . \rangle_{\mathbb{K}}$, $\langle ., . \rangle_{\mathbb{H}}$ and $(., .)$ are the corresponding inner products.

It is well known L. Boutet de Monvel [2] that

Theorem 12.5.4. *(i) Every maximally hypoelliptic operator is microlocally hypoelliptic; more precisely, operator $A \in \boldsymbol{\Psi}^{m,\mu}(X, \Lambda, \mathbb{K})$ satisfies*

$$\mathsf{WF}^{s+l}(u) \subset \mathsf{WF}^{s}(Au) \qquad \forall u \in \mathscr{D}'(X, \mathbb{K}) \tag{12.5.10}$$

with $l = m - \frac{\mu}{2}$ if and only if it is maximally hypoelliptic.

(ii) Moreover, in this case (and only in this case) the resolvent of A is a bounded operator from $\mathscr{H}^s(X, \mathbb{K})$ to $\mathscr{H}^{s+l}(X, \mathbb{K})$.

In this section we treat a maximally hypoelliptic operator A with

$$l = m - \frac{\mu}{2} > 0. \tag{12.5.11}$$

Corollary 12.5.5. *If $A \in \boldsymbol{\Psi}^{m,\mu}(X, \Lambda, \mathbb{K})$ and (12.5.11) holds then the resolvent of A is a compact operator in $\mathscr{L}^2(X, \mathbb{K})$ and the spectrum of A is discrete with no accumulation point (provided A is self-adjoint).*

Remark 12.5.6. One can easily prove that this is wrong for $l \leq 0$; moreover, for $l < 0, m > 0$ the continuous spectrum fills either the semiaxis or axis, while for $l = 0, m > 0$ and $l < 0, m = 0$ the spectrum is more complicated: there is an infinite number of accumulation points or points of the infinite multiplicity point spectrum (provided the operator is self-adjoint); for $m < 0$ the structure of the spectrum is transparent.

We leave to the reader the following easy problem

Problem 12.5.7. Prove that

(i) If Weyl symbol of A is Hermitian at every point $z \in T^*X$ then the Weyl symbol of A_z is Hermitian at every point $z'' \in T^*(\mathbb{R}^{d''})$ and therefore A_z is obviously symmetric in $\mathbb{H}$.

(ii) Moreover, under condition (12.5.8) A_z is an elliptic global operator (see Section 1.1.4) and is therefore self-adjoint.

(iii) A symmetric maximally hypoelliptic operator satisfying (12.5.11) is self-adjoint. Moreover, a self-adjoint maximally hypoelliptic operator satisfying (12.5.11) is semibounded from above if and only if both of the following two conditions are fulfilled:

(12.5.12) $a_0(z)$ is a negative definite matrix in $\mathbb{K}$ at every point $z \in (T^*X \setminus 0) \setminus \Lambda$

and

(12.5.13) A_z is a negative definite operator in $\mathbb{H}$ at every point $z \in \Lambda$.

12.5.2 Getting sStarted

So, let $A \in \boldsymbol{\Psi}^{m,\mu}(X, \Lambda, \mathbb{K})$ be a maximally hypoelliptic and self-adjoint operator and let (12.5.11) be fulfilled. Let $\mathsf{N}^{\pm}(\tau)$ be the number of eigenvalues of A lying in the intervals $[0, \tau)$ and $(-\tau, 0]$ respectively. We are interested in the asymptotics of $\mathsf{N}^{+}(\tau)$ as $\tau \to +\infty$. One can obtain the asymptotics of $\mathsf{N}^{-}(\tau)$ by replacing A by $-A$.

First of all, let us rewrite $\mathsf{N}^{\pm}(\tau)$ in the form

$$\mathsf{N}^{\pm}(\tau) = \mathsf{N}_0^{\pm}(\tau) + \mathsf{N}_1^{\pm}(\tau), \qquad \mathsf{N}_j^{\pm}(\tau) = \int \psi_j(x)\, \mathsf{tr}(e(x,x,))\, dx \tag{12.5.14}$$

where $e(x, y, \tau)$ is the Schwartz kernel of the spectral projector of A and ψ_j are classical pseudodifferential operators of order 0,

$$\mathsf{supp}(\psi_0) \cap \Lambda = \emptyset, \qquad \mathsf{supp}(\psi_1) \subset \mathcal{Z} = \big\{(x,\xi) : \ \sigma(x,\xi) < \epsilon\big\} \tag{12.5.15}$$

where $\epsilon > 0$ is an arbitrarily small constant. Recall that $\sigma(x, \xi)$ is defined by (12.5.7). Then

Proposition 12.5.8. *Let A be a self-adjoint operator. Then for $\mathsf{N}_0^{\pm}(\tau)$ the standard Weyl asymptotics holds with principal part*

$$\kappa_0\tau^{\frac{1}{m}(d-1)} + \kappa_{\frac{1}{2}}\tau^{\frac{1}{m}(d-\frac{1}{2})} + \kappa_1\tau^{\frac{1}{m}(d-1)} \tag{12.5.16}$$

and with remainder estimate $O\bigl(\tau^{\frac{1}{m}(d-1)}\bigr)$ and even $o\bigl(\tau^{\frac{1}{m}(d-1)}\bigr)$ under the standard condition non-periodicity condition to the Hamiltonian trajectories of a_0 on the energy level $\tau = 1$. Here $\kappa_{\frac{1}{2}} = 0$ provided $a_{\frac{1}{2}} = 0$.

Therefore we should treat $\mathsf{N}_1^{\pm}(\tau)$ in the same way we treated $\mathsf{N}_{\mathsf{cusp}}^{\pm}(\tau)$ before. First of all we need the following assertion

Proposition 12.5.9. *Let $A \in \boldsymbol{\Psi}^{m,\mu}(X,\Lambda,\mathbb{K})$ be a self-adjoint operator and $\psi_1 \in \boldsymbol{\Psi}^0(X)$ satisfy* (12.5.15).

(i) Let $\phi \in \mathscr{C}_0^{K'}(\mathbb{R}^{d''})$ be supported in $B(0,\epsilon)$ with a small enough constant $\epsilon > 0$. Then estimate

$$\bigl|\bigl(\phi(\tau^{-\frac{1}{m}}D')\psi_1 e\bigr)(x,y,\tau',\tau)\bigr| \le C\tau^{-s} \tag{12.5.17}$$

holds with arbitrarily large s for $\tau > 0$, $\tau' \in [\epsilon_0\tau, \tau]$.

(ii) Let A be maximally hypoelliptic and let $(1-\phi) \in \mathscr{C}_0^{K'}(\mathbb{R}^{d'})$ and ϕ be supported in $\{|\xi'| \ge C_0\}$ with a large enough constant C_0. Then estimate

$$\bigl|\bigl(\phi(\tau^{-\frac{1}{l}}D')\psi_1 e\bigr)(x,y,\tau',\tau)\bigr| \le C\tau^{-s} \tag{12.5.18}$$

holds with arbitrarily large s for $\tau > 0$, $\tau' \in [\epsilon_0\tau, \tau]$.

(iii) Let $\phi \in \mathscr{C}_0^{K'}(\mathbb{R}^{d'})$, $0 \notin \mathsf{supp}(\phi)$ and $\varphi \in \mathscr{C}_0^{K'}(\mathbb{R}^{d''})$. Let $r \ge \tau^{\delta}$, $\epsilon \ge \sigma \ge r^{-\frac{1}{2}}$ and $\rho^{\mu} r^{m} \le \varepsilon\tau$ with an arbitrarily small exponent $\delta > 0$ and with a small enough constant $\varepsilon > 0$. Then estimate

$$\bigl|\bigl(\varphi(\rho^{-1}x'', \rho^{-1}r^{-1}D'')\phi(r^{-1}D')\psi_1 e\bigr)(x,y,\tau',\tau)\bigr| \le C\tau^{-s} \tag{12.5.19}$$

holds with arbitrarily large s for $\tau > 0$, $\tau' \in [\epsilon_0\tau, \tau]$.

Proof. Statement (i) follows from the ellipticity of operator $(\tau - A)$ in the zone $\bigl\{|\xi| \le \varepsilon\tau^{\frac{1}{m}}\bigr\}$ with a small enough constant $\varepsilon > 0$.

Further, Statement (iii) follows from the ellipticity of operator $(\tau - A)$ in the zone $\bigl\{\epsilon r \le |\xi'| \le cr, |x''| + |\xi''|r^{-1} \le c\rho\bigr\}$ with the indicated r and ρ.

On the other hand, Statement (ii) follows from the maximal hypoellipticity of operator $(\tau - A)$ in the zone $\{|\xi'| \ge C_0\tau^{\frac{1}{l}}, \sigma(x,\xi) \le \epsilon\}$. More precisely, estimate

$$\|Au\|_s + \|u\|_s \ge \epsilon_0 \|A'u\|_s \tag{12.5.20}$$

holds where as usual $\|.\|$ means the norm in the Sobolev space $\mathscr{H}^s(X,\mathbb{K})$ and A' means any operator belonging to $\mathbf{\Psi}^{m,\mu}(X,\Lambda,\mathbb{K})$ (Boutet de Monvel [2]). In particular,

$$\|Au\|_s + \|u\|_s \ge \epsilon_0 \|u\|_{s+l} \tag{12.5.21}$$

and estimate (12.5.20) yields Statement (ii). □

Therefore, in what follows we need to treat zone

$$\begin{aligned}\mathcal{U} = \{\sigma(x,\xi) \le \epsilon, \epsilon_0\tau^{\frac{1}{m}} \coloneqq r_*(\tau) \le |\xi'| \le r^*(\tau) \coloneqq C_0\tau^{\frac{1}{l}},\\ \epsilon_0 \le \sigma(x,\xi)|\xi'|^{-\frac{m}{\mu}}\tau^{\frac{1}{\mu}}\}.\end{aligned} \tag{12.5.22}$$

In this zone let us make a $(\epsilon_1, \epsilon_1|\xi'|; \epsilon_1\sigma(x,\xi), \epsilon_1\sigma(x,\xi)|\xi'|)$-admissible partition of unity on $(x',\xi';x'',\xi'')$ with a small enough constant $\epsilon_1 > 0$ and let us treat some element $\bar{q}$ of this partition; let $\bar{r}$ and $\bar{\sigma}$ be the values of $|\xi'|$ and σ calculated at $\bar{z} \in \bar{q}$; then

$$\epsilon_0\tau \le \bar{r}^m\bar{\sigma}\mu \le c_0\tau, \qquad r_*(\tau) \le \bar{r} \le r^*(\tau). \tag{12.5.23}$$

12.5.3 Weyl Asymptotics

Let us start with the completely Weyl theory. For this let us consider an $(\epsilon_1, \epsilon_1\sigma^2 r; \epsilon_1\sigma r, \epsilon_1\sigma)$-admissible partition of unity on $(x',\xi';x'',\xi'')$. Then rescaling $x''_{\mathsf{new}} = x''\bar{\sigma}^{-1}$ we transform A to an h-pseudodifferential operator $\bar{r}^m\bar{\sigma}^\mu\tilde{A}_h$ provided

$$h \coloneqq \bar{r}^{-1}\bar{\sigma}^{-2} \le h_0 \tag{12.5.24}$$

with a small constant h_0.

Then the contribution of this element to the principal part of the asymptotics is 0 and its contribution to the remainder estimate is $O(h^s)$ with an arbitrarily large exponent s provided one of the following two conditions is fulfilled:

(a) a_0 is negative definite on $T^*X \setminus 0$;

(b) $\bar{r}^m \bar{\sigma}^\mu \ge C_0 \tau$.

Then after summation over such elements we conclude that

(12.5.25) The total contribution of all such elements of the partition lying in $\mathcal{U}$ to the principal part of the Weyl asymptotics is 0 and to the remainder estimate is $O\bigl(\tau^{d'l-1}\bigr)$.

Really, it does not exceed

$$\iint_{\mathcal{U}} \sigma^{-2d'-1} r^{-1}\, d\sigma\, dr \asymp \int_{r\le \tau^{l-1}} r^{d'-1}\, dr \asymp \tau^{d'l-1}.$$

In particular, applying Proposition 12.5.9 we arrive to the following two statements

(12.5.26) If A is maximally hypoelliptic and a_0 is negative definite then the contribution of $\complement\mathcal{V}$ to the asymptotics is $O\bigl(\tau^{d'l-1}\bigr)$ where

$$\mathcal{V} = \bigl\{|\xi| \ge c_0, \sigma(x,\xi) \le \epsilon, \sigma^2(x,\xi)|\xi'| \le c_0\bigr\}.$$

and

(12.5.27) In the general case the contribution of zone

$$\mathcal{W} = \bigl\{|\xi| \ge c_0, |\xi'|^m \sigma(x,\xi)^\mu \ge C_0\tau\bigr\}$$

to the asymptotics is $O\bigl(\tau^{d'l-1}\bigr)$ as well.

On the other hand, let us consider zone $\mathcal{V}$. Again due to Proposition 12.5.9 we should consider only elements with $r_*(\tau) \le |\xi'| \le r^*(\tau)$. Applying the previous arguments with σ replaced by $\rho(\xi') = |\xi'|^{-\frac{1}{2}}$ and $h = 1$ we see that

(12.5.28) The total contribution of $\mathcal{V}$ to the asymptotics is $O\bigl(\tau^{d'l-1}\bigr)$ as well.

So we obtain as a corollary the following rather well known estimate

$$\mathsf{N}^+(\tau) = O\bigl(\tau^{\frac{1}{l}d'}\bigr) \qquad \text{as} \quad \tau \to +\infty \tag{12.5.29}$$

for maximally hypoelliptic operators with negative definite symbols.

Let us consider the general case and let us treat the elements of the above partition with $r\sigma^2 \ge C_0$, $r \ge c_0$, $r^m\sigma^\mu \asymp \tau$. Let us note that

(12.5.30) After the coordinate transform $x''_{\mathsf{new}} = x''\bar\sigma^{-1}$ and multiplication by τ^{-1} operator A is microhyperbolic in the direction $\langle x, \partial_\xi\rangle$ provided C_0 is large enough.

Then the contribution of each element to the principal part is given by Weyl expression

$$(2\pi)^{-d}\iint \bar q_1^{\mathsf{w}}(x,\xi) n^+(x,\xi,\tau)\,dxd\xi$$

where $n^+(x,\xi,\tau)$ is the number of eigenvalues of the symbol $a_0(x,\xi)+a_{\frac12}(x,\xi)$ in the interval $[0,\tau)$; one can introduce $n^-(x,\xi,\tau)$ and $n(x,\xi,\tau)$ in the same way.

On the other hand, the contribution of this element to the remainder is $O(h^{1-d}) = O\bigl(\bar r^{d-1}\bar\sigma^{2(d-1)}\bigr)$. Then the contribution of the element of a $(\epsilon_1,\epsilon_1|\xi'|;\epsilon_1\sigma,\epsilon_1\sigma|\xi'|)$-admissible partition to the remainder is

$$O\bigl(\bar r^{d-1}\bar\sigma^{2(d''-1)}\bigr) = O\bigl(\bar r^{d-1-2(d''-1)\frac{m}{\mu}}\tau^{\frac{2}{\mu}(d''-1)}\bigr)$$

and after summation on this partition of unity we obtain the remainder estimate

$$R^{\mathsf{W}}(\tau) = \begin{cases} O\bigl(\epsilon'\tau^{\frac{1}{m}(d-1)}\bigr) & \text{for} \quad (d-1)l > d'm\\ O\bigl(\tau^{\frac{1}{m}(d-1)}\log\tau\bigr) & \text{for} \quad (d-1)l = d'm\\ O\bigl(\tau^{\frac{1}{l}d'}\bigr) & \text{for} \quad (d-1)l < d'm \end{cases} \tag{12.5.31}$$

where $\epsilon' = \epsilon'(\epsilon) \to +0$ as $\epsilon \to +0$.

So (also taking into account estimates for other zones) we arrive to the following theorem

Theorem 12.5.10. *Let $A \in \boldsymbol{\Psi}^{m,\mu}(X,\Lambda,\mathbb{K})$ be a maximally hypoelliptic (self-adjoint) operator, $l > 0$ and suppose a_0 is not a negative definite symbol. Then*

(i) For $(d-1)l > d'm$ *asymptotics*

$$\mathsf{N}^{+}(\tau) = \mathsf{N}^{\mathsf{W}+}(\tau) + O\bigl(\tau^{\frac{d-1}{m}}\bigr) \tag{12.5.32}$$

holds with

$$\mathsf{N}^{\mathsf{W}+}(\tau) = (2\pi)^{-d} \iint n^{+}(x,\xi,\tau)\, dxd\xi. \tag{12.5.33}$$

(ii) Moreover, if $(d-1)l > d'm$ *and if the symbol* a_0 *satisfies the standard non-periodicity condition on the energy level* $\tau = 1$ *then asymptotics*

$$\mathsf{N}^{+}(\tau) = \mathsf{N}^{\mathsf{W}+}(\tau) + \kappa_1' \tau^{\frac{d-1}{m}} + o\bigl(\tau^{\frac{d-1}{m}}\bigr) \tag{12.5.34}$$

hold with the standard coefficient κ_1'.

(iii) Further, for $(d-1)l = d'm$ *asymptotics*

$$\mathsf{N}^{+}(\tau) = \mathsf{N}^{\mathsf{W}+}(\tau) + O\bigl(\tau^{\frac{d-1}{m}} \log \tau\bigr) \tag{12.5.35}$$

holds.

(iv) Furthermore, for $(d-1)l < d'm \le dl$ *asymptotics*

$$\mathsf{N}^{+}(\tau) = \mathsf{N}^{\mathsf{W}+}(\tau) + O\bigl(\tau^{\frac{d'}{l}}\bigr) \tag{12.5.36}$$

holds where for $d'm = dl$ $\mathsf{N}^{\mathsf{W}+}(\tau)$ *is given by cut-off Weyl formula* (12.5.33) *with integration only in the domain* $\bigl\{|\xi'| \ge \tau^{\frac{1}{l}}, \sigma(x,\xi) \ge |\xi'|^{-\frac{1}{2}}\bigr\}$ *and* $\mathsf{N}^{\mathsf{W}+}(\tau) \asymp \tau^{\frac{d'}{l}} \log \tau$.

(v) Finally, for $dl < d'm$ *estimate*

$$\mathsf{N}^{+}(\tau) = O\bigl(\tau^{\frac{d'}{l}}\bigr) \tag{12.5.37}$$

holds. Moreover, the same estimate holds if $a_0(x,\xi)$ *is negative definite.*

(vi) Moreover, in all these asymptotics one can replace $\mathsf{N}^{\mathsf{W}+}(\tau)$ *by*

(12.5.38) $\mathsf{N}_1^{\mathsf{W}+}(\tau) =$

$$\begin{cases} \kappa_0 \tau^{\frac{1}{m}d} + \kappa_{\frac12} \tau^{\frac{1}{m}(d-\frac12)} + \kappa_1'' \tau^{\frac{1}{m}(d-1)} & \text{for } (d-1)l > d'm, \\ \kappa_0 \tau^{\frac{1}{m}d} + \kappa_{\frac12} \tau^{\frac{1}{m}(d-\frac12)} & \text{for } (d-\frac12)l > d'm \ge (d-1)m, \\ \kappa_0 \tau^{\frac{1}{m}d} + \kappa'_{\frac12} \tau^{\frac{1}{m}(d-\frac12)} \log \tau & \text{for } (d-\frac12)l = d'm, \\ \kappa_0 \tau^{\frac{1}{m}d} & \text{for } (d-\frac12)l < d'm < (d-1)m < dl, \\ \kappa'_0 \tau^{\frac{1}{m}d} \log \tau & \text{for } d'm < (d-1)m = dl \end{cases}$$

with positive constants κ_0 *and* κ'_0.

Moreover, $\kappa_{\frac12} = \kappa'_{\frac12} = 0$ *provided* $a_{\frac12} \equiv 0$.

We leave to the reader the following easy problem

Problem 12.5.11. (i) Similarly prove that for Riesz means the remainder estimate is

$$(12.5.39) \quad R^{\mathsf{W}}_{(\vartheta)}(\tau) = \begin{cases} O\bigl(\epsilon' \tau^{\frac{1}{m}(d-1-\vartheta)+\vartheta}\bigr) & \text{for } (d-1-\vartheta)l > d'm, \\ O\bigl(\tau^{\frac{1}{m}(d-1-\vartheta)+\vartheta} \log \tau\bigr) & \text{for } (d-1-\vartheta)l = d'm, \\ O\bigl(\tau^{\frac{1}{l}d'+\vartheta}\bigr) & \text{for } (d-1-\vartheta)l < d'm \end{cases}$$

while the principal part contains terms $\kappa_j \tau^{\frac{1}{m}(d-j)+\vartheta}$ with $j \in \frac12 \mathbb{Z}$; for $(d-j)l = d'm$ one should replace this term by $\kappa'_j \tau^{\frac{1}{m}(d-j)+\vartheta} \log \tau$ while the remainder estimate is $O\bigl(\tau^{\frac{1}{m}(d-j)+\vartheta}\bigr)$ (if $j < 1+\vartheta$).

(ii) Moreover, let $k \notin \mathbb{Z}$ and $a_j \equiv 0$ for all $j \notin \mathbb{Z} : j \le k$. Prove that then $\kappa_k = 0$.

So, if a_0 is not negatively definite in what follows we need only to treat the case $(d-1)l \le d'm$ (or $(d-1-\vartheta)l \le d'm$ if we are interested in Riesz means).

12.5.4 Sharp Asymptotics

Let us treat x'' as an additional variable; however, we do not now introduce the special Hilbert space $\mathbb{H}$. First of all, let us note that standard ellipticity arguments yield the following

Proposition 12.5.12. *Let $q' \in \mathscr{C}_0^{K'}(\mathbb{R}^{2d'})$ and $q'' \in \mathscr{C}_0^{K'}(\mathbb{R}^{2d''})$ be fixed functions vanishing in neighborhoods of $\{\xi' = 0\}$ and $\{x'' = \xi'' = 0\}$ respectively. Moreover, let $\sigma > 0$ be a small enough constant and let $\bar{r} \ge c_\sigma \tau^{\frac{1}{m}}$ where σ and c_σ depend on q', q'' and c_σ is large enough.*

Then for the operators $Q' = q'(x', \bar{r}^{-1}D')$ and $Q'' = q''(\sigma^{-1}x'', \sigma^{-1}\bar{r}^{-1}D'')$ estimate

$$\begin{aligned}(12.5.40)\quad &\bigl|F_{t\to\lambda}\chi_T(t)Q'_x\psi_1 Q''_x U\bigr| \le C\hbar^s\\ &\qquad \forall x, y \in X \quad \forall \lambda : |\lambda - \tau| \le \epsilon\tau \quad \forall T \in [\bar{r}^\delta\tau^{-1}, T_0\bar{r}\tau^{-1}]\end{aligned}$$

holds with an arbitrarily small exponent $\delta > 0$, an arbitrarily large exponent s and with small enough constants $T_0 > 0$ and $\epsilon > 0$; here and below $U(x, y, t)$ is the Schwartz kernel of the operator $\exp(itA)$ and $\hbar = \bar{r}^{-1}$.

This proposition permits us to include x''-mollifying operator $q(x'', \bar{r}^{-1}D'')$ with compactly supported q in our expressions whenever necessary.

Let us note that if $\phi = \phi(x', \hbar D')$ then in the ϵ-neighborhood of the point $(\bar{x}', \bar{\xi}')$ with $|\bar{\xi}'| \asymp \bar{r}$ operator $\bar{r}[A, \phi]$ belongs to $\boldsymbol{\Psi}^{m,\mu}(X, \Lambda, \mathbb{K})$ uniformly in $(\bar{x}', \bar{\xi}')$.

Proposition 12.5.13. *Let $q'_1, q'_2 \in \mathscr{C}_0^{K'}(\mathbb{R}^{2d'})$ be fixed functions and let q' vanish in a neighborhood of $\{\xi' = 0\}$, $q'_2 = 1$ in a neighborhood of $\operatorname{supp}(q'_1)$. Then for the operators $Q'_j = q'_j(x', \bar{r}^{-1}D')$ estimate*

$$\begin{aligned}(12.5.41)\quad &\bigl|F_{t\to\lambda}\psi_1\chi_T(t)Q'_{1x}(I - Q'_{2x})U\bigr| \le C\hbar^s\\ &\qquad \forall x, y \in X \quad \forall \lambda : |\lambda - \tau| \le \epsilon\tau \quad \forall T \in [\bar{r}^\delta\tau^{-1}, T_0\bar{r}\tau^{-1}]\end{aligned}$$

holds with an arbitrarily small exponent $\delta > 0$, an arbitrarily large exponent s and with small enough constants $T_0 > 0$ and $\epsilon > 0$.

Proof. Proof uses the standard energy estimate technique of Chapter 2 (see proof of Theorem 2.1.2 and takes into account estimate (12.5.20) and Proposition 12.5.12.

Easy details are left to the reader. □

This proposition shows us that singularities with respect to (x', ξ') of U propagate with the speed not exceeding $(\bar{r}^{-1}\tau, \tau)$ along (x', ξ') (without rescaling on ξ') while in the zone which is semiclassical with respect to

all the variables singularities propagate along (x'', ξ'') with the speed not exceeding $\bar{\sigma}^{-1}(\bar{r}^{-1}\tau, \tau)$.

Let us consider the $(\epsilon, \epsilon\bar{r})$-neighborhood of $(\bar{x}', \bar{\xi}')$ with $|\bar{\xi}'| = \bar{r}$. Let $\boldsymbol{a}(x', \xi') = A(x', x'', \xi', D'')$ be an operator acting on x'' only. We claim that

Proposition 12.5.14. *For the operator $\boldsymbol{a}(x', \xi)$ the microhyperbolicity estimate*

$$\operatorname{Re} \langle (\langle \bar{\xi}', \partial_{\xi'} \rangle \boldsymbol{a}) v, v \rangle \geq \epsilon_0 |\tau| \cdot \|v\|^2 - c_2 |\tau|^{-1} \|(\tau - \boldsymbol{a}) v\|^2 - C \bar{r}^{-s} \|w\|^2 \tag{12.5.42}$$

holds for $v = \psi_1 w$ where norms and inner products are here considered only on x'', (x', ξ') belongs to this neighborhood, and $\tau > 0$ is such that $r_*(\tau) \leq \bar{r} \leq r^*(\tau)$; here $\epsilon_0 > 0$ is a small enough constant and c_2 and C are large enough constants; none of the constants depends on $(\bar{x}', \bar{\xi}')$ and τ; s is an arbitrarily large exponent.

Proof. It is sufficient to prove this estimate for $\xi' = \bar{\xi}'$. Let us apply Taylor's formula to A; we obtain that

$$\boldsymbol{a}(x', \xi'; x'', D'') \equiv \sum_{|\alpha|+|\beta|+n=\mu} a^{(\alpha)}_{\frac{n}{2}(\beta)}(x', \xi') \frac{1}{\alpha! \beta!} x''^{\beta} D''^{\alpha} \qquad \operatorname{mod} \boldsymbol{\Psi}^{m, \mu+1}$$

where here and below we take $x'' = \xi'' = 0$ in $a^{(\alpha)}_{\frac{n}{2}(\beta)}$. Then for $\xi' = \bar{\xi}'$ we obtain the equality

$$\langle \xi', \partial_{\xi'} \rangle \boldsymbol{a}(x', \xi'; x'', D'') \equiv \sum_{|\alpha|+|\beta|+n=\mu} (m - n - |\alpha|) a^{(\alpha)}_{\frac{n}{2}(\beta)}(x', \xi') \frac{1}{\alpha! \beta!} x''^{\beta} D''\alpha \equiv l\boldsymbol{a} - i[L, \boldsymbol{a}] \qquad \operatorname{mod} \boldsymbol{\Psi}^{m, \mu+1}$$

with $L = \frac{1}{2}\langle x'', D'' \rangle - \frac{1}{4} i d''$. Then estimates (12.5.40)–(12.5.41) together with the assumption that $l > 0$ yield estimate (12.5.42). □

Again applying the analysis of Chapter 2 (see proof of Proposition 2.1.16 and Corollary 2.1.17) we conclude that

Proposition 12.5.15. *In the framework of Proposition 12.5.12 estimate*

$$\bigl|F_{t\to\lambda}\Gamma'_x\psi_1\chi_T(t)Q'_{1x}U\bigr| \le C\bar r^{-s} \tag{12.5.43}$$

holds for x', y', τ and T indicated in this proposition provided χ vanishes in a neighborhood of 0; Γ' means that we set $y'=x'$ and then integrate over x'.

Then we want to apply the successive approximation method but we lack auxiliary spaces $\mathbb{H}$ and $\mathbb{H}_j$. In order to overcome this difficulty we treat two cases:

(a) $\bar r \le r^*(\tau)\tau^{-\delta}$,

(b) $\bar r \ge r_*(\tau)\tau^{\delta}$

with $\delta>0$; these cases overlap and together cover the case $\bar r\in[r_*(\tau), r^*(\tau)]$.

In the former Case (a) we can apply our standard technique with an unperturbed operator $\bar A = a(y,D_x)$ and we obtain estimate

$$\bigl|F_{t\to\lambda}\bar\chi_T(t)\operatorname{tr}(\Gamma Q_x U) - \sum_{0\le j\le J-1} \varkappa'_{\frac{j}{2}}(\tau)\bigr| \le C\tau^{-s} \qquad \forall\lambda : |\lambda-\tau|\le\epsilon\tau \tag{12.5.44}$$

with arbitrary $T\in[\tau^{-1}h^{-\delta'}, T_0\tau^{-1}h^{-1}]$ where $T_0>0$ is a small enough constant, $\delta'>0$ and s are arbitrary and $h=\bar r^{-1}\bar\sigma^{-2}$; in contrast to the previous $\bar\sigma$ is not a small constant but a parameter such that $h\le 1$.

Here $J=J(d,m,\mu,s,\delta,\delta')$ is sufficiently large and $\varkappa'_{\frac{j}{2}}$ are Weyl terms (rather than coefficients) depending on Q and satisfying estimate

$$|\varkappa'_{\frac{j}{2}}| \le C\bigl(\bar r\bar\sigma^2\bigr)^{d-\frac{j}{2}}\tau^{-1} \tag{12.5.45}$$

provided the symbol of Q is supported in the $\epsilon(1,\bar\sigma^2\bar r,\bar\sigma,\bar\sigma\bar r)$-neighborhood of $(\bar x',\bar\xi',\bar x'',\bar\xi'')$, $|\bar\xi'|\asymp\bar r$; here and below we refer to the standard 1-symbol of Q instead of the $\hbar$-symbol.

These estimates (12.5.44) and (12.5.45) are due to the standard micro-hyperbolicity of a_0 in the direction $\langle\xi,\partial_\xi\rangle$ after the coordinate transform $x'_{\text{new}}=x'_{\text{old}}$, $\xi'_{\text{new}}=\xi'_{\text{old}}\bar\sigma^{-2}$, $x''_{\text{new}}=x''_{\text{old}}\bar\sigma^{-1}$, $\xi''_{\text{new}}=\xi''_{\text{old}}\bar\sigma^{-1}$.

It should be noted that in Case (a) in virtue of standard ellipticity arguments we automatically obtain that only points with $h\le\tau^{-\delta''}$ with

sufficiently small exponent $\delta'' > 0$ should be treated. Then for operator Q with symbol supported in the $\epsilon(1, \bar{r}, \bar{\sigma}, \bar{\sigma}\bar{r})$-neighborhood of $(\bar{x}', \bar{\xi}', \bar{x}'', \bar{\xi}'')$ we obtain the same result with another estimate of these terms:

$$\bigl|\varkappa'_{\frac{j}{2}}\bigr| \le C\bar{r}^{d-\frac{j}{2}}\bar{\sigma}^{2(d''-\frac{j}{2})}\tau^{-1}. \tag{12.5.46}$$

Moreover, (12.5.40) yields that the same estimates hold with Q replaced by $\psi_1 Q'$ where $Q' = q'(x', \bar{r}^{-1}D')$ and q' is supported in the ϵ-neighborhood of $(\bar{x}', \bar{\xi}'\bar{r}^{-1})$. Furthermore, estimate (12.5.43) then yields that estimate (12.5.44) holds for arbitrary $T \in [\tau^{-1}h^{-\delta'}, T_0\tau^{-1}\hbar^{-1}]$.

Applying the standard Tauberian technique we obtain that

$$\Bigl|\Bigl(\int \mathsf{tr}(Q'_x\psi_1 e)(x, x, \tau)\,dx - \sum_{0\le j\le J-1} \varkappa_{\frac{j}{2}}(\tau)\Bigr)_{\tau=\tau'}^{\tau=\tau}\Bigr| \le C\bar{r}^{d-1}\bar{\sigma}^{2d''} \asymp C\bar{r}^{d-1-2d''\frac{m}{\mu}}\tau^{\frac{2d''}{\mu}} \tag{12.5.47}$$

with $\tau' \in [\epsilon_0\tau, \tau]$, $\tau > 0$ where $\varkappa_{\frac{j}{2}}(\tau)$ are standard Weyl terms equal to the integrals of $\varkappa'_{j/2}(\tau)$ with respect to τ and therefore

$$\bigl|\varkappa_{\frac{j}{2}}(\tau)\bigr| \le C\bar{r}^{d-\frac{j}{2}}\bar{\sigma}^{2(d''-\frac{j}{2})}. \tag{12.5.48}$$

Let us discuss the terms $\varkappa_{\frac{j}{2}}(\tau)$. Recall that

$$\sum_{0\le j\le J-1} \varkappa'_{\frac{j}{2}} \sim (2\pi)^{-d-1}i^{-1}\,\mathsf{Res}_{\mathbb{R}} \int \bigl((\tau - A)^{-1}Q'\psi_1\bigr)_{\mathsf{w}}(x, \xi)\,dxd\xi \tag{12.5.49}$$

where the subscript "$_{\mathsf{w}}$" means that we take the Weyl symbol of the corresponding operator and $(\tau - A)^{-1}$ is a parametrix of $(\tau - A)$ in the framework of the formal calculus of pseudodifferential operators; we cut this symbol after we have taken a large enough number of $(m, 1)$-quasihomogeneous terms (on (τ, ξ)) of the symbol of $(\tau - A)^{-1}$ and derivatives of these terms on (x, ξ) and therefore derivatives of the symbol of $Q'\psi_1$ on (ξ, x).

Let us note that any differentiation of ψ_1 gives us a term which does not exceed the right-hand expression in (12.5.47); the same is true for any differentiation with respect to x' and therefore with respect to ξ' (and *vice versa*) made in the calculation of $(\tau - A)^{-1}$ and of its product with Q'.

Therefore estimate (12.5.47) remains true if, in the definition of terms coefficients, we replace the right-hand expression of (12.5.49) by

$$(12.5.50) \qquad (2\pi)^{-d-1} i^{-1} \operatorname{Res}_{\mathbb{R}} \int \big((\tau - a)^{-1} \big)_{\mathsf{w}} (x,\xi) Q'(x',\xi') \psi_1(x,\xi)\, dx d\xi$$

where $(\tau - a)^{-1}$ is the parametrix of $(\tau - a)$ in the framework of the formal calculus of pseudodifferential operators with respect to x' and the subscript "w" means the Weyl symbol of the corresponding operator; here $a(x',\xi') = A(x',\xi'; x'', D'')$ but later we will change this definition.

Recall that $\big((\tau - a)^{-1}\big)_{\mathsf{w}}$ is a distribution. However, if we use spherical coordinates $\xi = (r,\varphi)$ with $\varphi \in \mathbb{S}^{d-1}$ and integrate on r (multiplying by a $\mathscr{C}_0^\infty$ function beforehand) we obtain a regular function of (τ, x, φ) provided $(x'', \varphi'') \neq 0$.

Let us introduce the spherical coordinate system $(\zeta, z) \in \mathbb{R}^+ \times \mathbb{S}^{2d''-1}$ in $\mathbb{R}^{2d''} \ni (x'', \varphi'')$; then one can rewrite the term of $\big((\tau - a)^{-1}\big)_{\mathsf{w}}$ which is $(m,1)$-quasihomogeneous of order $(-m - \frac{j}{2})$ in the form

$$\sum_{0 \le k \le J-1} \tau^{-1} b_{jk}(\tau, r, z) \zeta^{k-j} + b'_j(\tau, r, z, \zeta)$$

where b'_j are regular functions. Then a similar representation holds for $\operatorname{Res}_{\mathbb{R}} \big((\tau - a)^{-1}\big)_{\mathsf{w}}$. Let us integrate it with respect to the measure $dx' dz d\varphi' \times \zeta^{2d''-1} d\zeta \times r^{d-1} dr$. Then after integration over τ from τ' to τ and summation over a partition of unity in the zone $\{r_*(\tau) \le r \le \eta\}$ we obtain the following

Proposition 12.5.16. *Let conditions* (12.5.1)–(12.5.3), (12.5.6)–(12.5.9) *and* (12.5.11) *be fulfilled and let*

$$(12.5.51) \qquad Q' = q_2(x', \varepsilon \tau^{-\frac{1}{m}} D') - q_1(x', \eta^{-1} D')$$

with fixed smooth functions q_j supported in $\{|\xi'| \le c\}$ and equal to 1 for $|\xi'| \le \epsilon_0$. Then estimate

$$(12.5.52) \qquad \big| \big(\int \operatorname{tr}(Q'_x \psi_1 e)(x,x,\tau) dx - \mathcal{N}(\tau,\eta) \big)_{\tau=\tau'}^{\tau=\tau} \big| \le C R_1^*(\tau)$$

holds with

$$(12.5.53) \qquad \mathcal{N}(\tau,\eta) = \mathcal{N}_0(\tau) + \mathcal{N}_1(\tau,\eta),$$

$$\mathcal{N}_0(\tau) = \sum_{j=0,1,2} \kappa_{\frac{j}{2}} \tau^{\frac{1}{m}(d-\frac{j}{2})} + \sum_{j,k=0,1,2:(d-\frac{j}{2})l=(d'-\frac{k}{2})m} \kappa'_{\frac{j}{2}} \tau^{\frac{1}{m}(d-\frac{j}{2})} \log \tau, \tag{12.5.54}$$

$$\mathcal{N}_1(\tau,\eta) = \sum_{j,k\le J:(d-\frac{j}{2})l\ne(d'-\frac{k}{2})m} \kappa_{jk} \tau^{\frac{1}{\mu}(2d''+k-j)} \eta^{\frac{2}{\mu}((d'-\frac{k}{2})m-(d-\frac{j}{2})l)} - \sum_{j,k=0,1,2:(d-\frac{j}{2})l=(d'-\frac{k}{2})m} \kappa'_{\frac{j}{2}} m \tau^{\frac{1}{m}(d-\frac{j}{2})} \log \eta, \tag{12.5.55}$$

$$R_1^*(\tau) = \begin{cases} \varepsilon' \tau^{\frac{1}{m}(d-1)} & \text{for } (d-1)l > (d'-1)m \\ C\tau^{\frac{1}{m}(d-1)} \log \tau & \text{for } (d-1)l = (d'-1)m \\ C\tau^{\frac{1}{l}(d'-1)-\delta''} & \text{for } (d-1)l < (d'-1)m \end{cases} \tag{12.5.56}$$

where the coefficients κ_j, κ'_j and κ_{jk} do not depend on τ and η, $J = J(d,m,\delta)$ is large enough, $\delta'' > 0$ and $\varepsilon' \to +0$ as $\varepsilon \to +0$.

Let us now treat Case (b) $\bar r \ge r_*(\tau)\tau^\delta$. In this case in order to obtain the unperturbed operator $\bar A$ we freeze $x' = y'$ in the operator $A_{(x',D')}(x'',D'')$ where $A_{(x',\xi')}(x'',D'')$ is the microlocalization (12.5.4) of $A(x,D)$ at the point (x',ξ') and we take Weyl quantization on x'.

We now treat this operator $\bar A$, and A itself, as pseudodifferential operators on x' with operator-valued symbols in $\mathbb{H} = \mathscr{L}^2(\mathbb{R}^{d''},\mathbb{K})$. Then all the constructions of the propagation of singularities of Chapter 2 are applicable including the proof of microhyperbolicity (we explained this proof above) as well as the successive approximation method.

Moreover, no sandwich procedure is necessary now because we apply a pseudodifferential operator ψ_1 mollifying on x''. So, we obtain that if $Q' = q'(x',\bar r^{-1}D')$ with q' supported in the ϵ-neighborhood of $(\bar x',\bar\xi'\bar r^{-1})$ with $\bar r \asymp |\bar\xi'|$ and $r^*(\tau) \ge \bar r \ge r_*(\tau)\tau^\delta$ then for $T \in [\tau^{-1}\bar r^{\delta'}, T_0\tau^{-1}\bar r]$ with a sufficiently small constant $T_0 > 0$ asymptotics

$$\bigl|F_{t\to\lambda}\chi_T(t)\Gamma\psi_1 Q'_x\bigl(U - \sum_{0\le n\le M} U_n\bigr)\bigr| \le C\tau^{-s} \qquad \forall\lambda : |\lambda-\tau| \le \epsilon\tau \tag{12.5.57}$$

holds where U_n are the terms of the successive approximation method, the number $M = M(d,m,\mu,\delta,s)$ is large enough and

$$\bigl|F_{t\to\lambda}\chi_T(t)\Gamma\psi_1 Q'_x U_n\bigr| \le C\bar r^{d-2d''m\mu^{-1}-\nu(n)}\tau^{2d''\mu^{-1}-1} \tag{12.5.58}$$

for the same values of T, τ with $\nu(n) \to +\infty$ as $n \to +\infty$. Therefore

$$\bigl|F_{t\to\lambda}\chi_T(t)\Gamma\psi_1 Q'_x U\bigr| \le C\bar r^{d-2d''m\mu^{-1}}\tau^{2d''\mu^{-1}-1} \tag{12.5.59}$$

for the same values of T, τ. We leave all the easy details to the reader.

Then applying the standard Tauberian technique we obtain estimate

$$\Bigl|\int \mathsf{tr}(Q'_x\psi_1 e)(x,x,\tau',\tau)\,dx - \sum_{0\le n\le M}\int_{\tau'}^{\tau}\bigl(F_{t\to\lambda}\bar\chi_T(t)\Gamma\psi_1 Q'_x U_{n(q)}\bigr)\,d\lambda\Bigr| \le C\bar r^{d-1-2d''m\mu^{-1}}\tau^{2d'\mu^{-1}} \tag{12.5.60}$$

for $\tau' \in [\epsilon_0\tau, \tau]$ where the conditions on $\bar r$ and T were given above, the subscript "(q)" has no meaning at this time and $\bar\chi \in \mathscr{C}_0^K([-1,1])$ is equal to 1 on $[-\frac12, \frac12]$.

Let us note that if we sum the right-hand expression of estimate (12.5.60) on a partition of unity on the interval for the $\bar r$ indicated above we obtain that the remainder does not exceed $CR_2^*(\tau)$ with

$$R_2^*(\tau) = \begin{cases} \tau^{\frac1m(d-1)-\delta''} & \text{for } (d-1)l > (d'-1)m,\\ \tau^{\frac1m(d-1)-\delta''}\log\tau & \text{for } (d-1)l = (d'-1)m,\\ \varepsilon'\tau^{\frac1l(d'-1)} & \text{for } (d-1)l < (d'-1)m \end{cases} \tag{12.5.61}$$

where $\varepsilon' = 1$ at this moment.

Further, let $(d-1)l < (d'-1)m$ and let us sum only over the partition in the zone $\{\varepsilon r^*(\tau) \ge \bar r \ge r_*\tau^\delta\}$ then $\varepsilon' = \varepsilon'(\varepsilon) \to +0$ as $\varepsilon \to +0$ and C does not depend on ε.

On the other hand, for $\bar r \in [\varepsilon r^*, r^*(\tau)]$ operator $\boldsymbol{a}(x',\xi') = A_{(x',\xi')}(x'', D'')$ has no more than $L(\varepsilon)$ eigenvalues in $[-2\tau, 2\tau]$ and operator $A(x',\xi'; x'', D'')$ is close to this operator in the natural sense.

Therefore for $\bar r$ from this interval and under the standard non-periodicity condition for the generalized Hamiltonian flow generated by operator-valued symbol $\boldsymbol{a}(x',\xi')$ on the energy level $\tau = 1$ (see (12.2.51)) one can replace the right-hand expression in estimate (12.5.60) by $o\bigl(\tau^{\frac1l(d'-1)}\bigr)$; this and the arbitrary choice of $\varepsilon' > 0$ yield that

(12.5.62) For $(d-1)l < (d'-1)m$ under the standard non-periodicity condition for $\boldsymbol{a}(x',\xi')$ on the energy level $\tau = 1$ the remainder estimate is $o\bigl(\tau^{\frac1l(d'-1)}\bigr)$ even for the sum over the partition of unity in the zone $\{r^*(\tau) \ge \bar r \ge r_*\tau^\delta\}$.

12.5.5 Calculations

Let us consider the terms $F_{t\to\lambda}\bar\chi_T(t)\Gamma\psi_1 Q'_x U_{n(q)}$ with $\lambda\in[\epsilon_0\tau,\tau]$ in (12.5.60). Then the standard arguments of Section 4.4 associated with the microhyperbolicity yield that one can replace $\bar\chi_T$ with 1 in these terms; the error caused by this operation is $O(\tau^{-s})$. Moreover, explicit calculation yields that one can replace ψ_1 by 1 in these terms.

On the other hand, one can easily see that for $Q'=q(x',r^*(\tau)^{-1}D')$ with $q'(x',\xi')$ supported in $\{|\xi'|\ge\frac12\}$ and equal to 1 for $|\xi'|\ge 1$ we obtain from Proposition 12.5.9(ii) that the first term in the left-hand expression of (12.5.60) is $O(\tau^{-s})$ and all the other terms modified above (i.e., with $\psi_1=I,\bar\chi=1$) simply vanish (we can write them explicitly and note that the operator $A(x',\xi')$ has no eigenvalue in $[-2\tau,2\tau]$. So for this operator Q' the modified formula (12.5.60) also holds. Therefore we obtain the following

Proposition 12.5.17. *(i) Consider $Q'=q'(x',\eta^{-1}D')$ with q' supported in $\{|\xi'|\ge\frac12\}$ and equal to 1 for $|\xi'|\ge 1$.*

Then for $\eta\in[r_*(\tau)\tau^\delta, r^*(\tau)]$, $\tau'\in[\epsilon_0\tau,\tau]$ asymptotics

$$\int \mathsf{tr}\big(Q'_x\psi_1 e\big)(x,x,\tau',\tau)\,dx = \sum_{0\le n\le M}\int_{\tau'}^{\tau}\big(F_{t\to\lambda}\Gamma Q'_x U_{n(q)}\big)\,d\lambda + O\big(R_2^*(\tau)\big) \tag{12.5.63}$$

holds.

(ii) Moreover, let $(d-1)l<(d'-1)m$ and let the standard non-periodicity condition (12.2.51) *for the generalized Hamiltonian flow generated by operator-valued symbol $A(x',\xi)$ on the energy level $\tau=1$ be fulfilled.*

Then asymptotics (12.5.61) *holds with "$O\big(R_2^*(\tau)\big)$" replaced by "$o\big(R_2^*(\tau)\big)$."*

This proposition is not our final goal. Let us consider all the terms in the expansion in the right-hand expression of (12.5.61). Let us first note that any differentiation on x' (which is accompanied by differentiation on ξ' and *vice versa*) decreases the magnitude of the term in (12.5.60) by an additional factor $\bar r^{-1}$ and therefore

(12.5.64) Asymptotics (12.5.60) remain true even if we skip all the terms containing differentiation on (x',ξ').

Moreover,

(12.5.65) The sharper asymptotics with remainder estimate "$o\big(R_2^*(\tau)\big)$" remains true if we preserve terms with only one differentiation on x' (and one on ξ'). Furthermore, even in this case one can skip all terms with differentiation of the symbol of Q' provided $\eta = o\big(r^*(\tau)\big)$.

In fact, such a term is supported in the zone $\bar{r} \asymp \eta$ and its order coincides with the order of the right-hand expression of (12.5.60), i.e., it is "$o\big(R_2^*(\tau)\big)$" provided $(d-1)l < (d'-1)m$ and $\eta = o\big(r^*(\tau)\big)$.

Therefore one should consider only terms of the form

$$U_{n(q)} = F^{-1}_{\lambda\to t}\, \mathsf{Res}_{\mathbb{R}}\, F_n(x', \xi', \lambda) Q(x', \xi') \tag{12.5.66}$$

where as $\lambda \in \mathbb{C}\setminus\mathbb{R}$ $F_n(x', \xi', \lambda)$ is a product of $(k+1)$ copies of the operator $\big(\lambda - A_{(x',\xi')}(x'', D'')\big)^{-1}$ separated by k operators $B_j(x', \xi', x'', D'')$ of the form

$$B_j = \sum_{\alpha,\beta:|\alpha|+|\beta|\le\mu+q} b_{j\alpha\beta}(x', \xi') x''^{\alpha} D''^{\beta}$$

and $b_{j\alpha\beta}$ are positively homogeneous of degrees $m + \frac{1}{2}(|\alpha| - |\beta| - \mu - q)$ on ξ' and $k \ge n$, $q = 1, 2, \dots$.

Therefore, assigning the index q to the operator B_j we also assign the index q to F_n as the sum of all the indices of factors of this type in the product. Then (12.5.58) holds with $\nu(n) = \delta'' q$.

Further, let us introduce spherical coordinates $\xi' = (r, \varphi)$. Then after the coordinate transform $x''_{\mathsf{new}} = x'' r^{\frac{1}{2}}$, $D''_{\mathsf{new}} = D'' r^{-\frac{1}{2}}$ we rewrite the corresponding term in the decomposition of $F_{t\to\lambda} U_{n(q)}$ in the form

$$\int_0^\infty \mathsf{Res}_{\mathbb{R}}\, F_n(\lambda r^{-l}) Q(r) r^{d'-1-l-\frac{q}{2}}\, dr \tag{12.5.67}$$

where here and below F_n is now calculated for operator $A_{(x',\varphi)}(x'', D'')$ with $\varphi \in \mathbb{S}^{d'-1}$ and we do not indicate the dependence on (x', φ) and integration on these variables. Moreover, one can assume that $Q(r) = \phi(r^{-l}\eta^l)$ where $\phi \in \mathscr{C}_0^K([-1, 1])$ is a fixed function equal to 1 in a neighborhood of 0. Then after substituting $\lambda = z r^l$ we rewrite (12.5.67) in the form

$$\frac{1}{l} \lambda^{\frac{1}{l}(d'-\frac{q}{2})-1} \int_0^\infty \mathsf{Res}_{\mathbb{R}}\, F_n(z) \phi(z\eta^l \lambda^{-1}) z^{-\frac{1}{l}(d'-\frac{q}{2})}\, dz. \tag{12.5.68}$$

For $\omega = \lambda\eta^{-l}$ let us consider expression

$$\Phi_{n(q)}(\omega) = \int_0^\infty \mathsf{Res}_{\mathbb{R}}\, F_n(z)\phi\big(\frac{z}{\omega}\big) z^{\frac{1}{l}(\frac{q}{2}-d')}\, dz. \tag{12.5.69}$$

Proposition 12.5.18. *For $\omega \to +\infty$ asymptotics*

$$\mathsf{Tr}\,\Phi_{n(q)} = \kappa_q + \kappa'_q \log\omega + \sum_{p:\, \frac{1}{l}(d'-\frac{q}{2})\neq \frac{2}{p}(d''-\frac{p}{2})} \kappa'_{qp}\omega^{\frac{2}{\mu}(d''-\frac{p}{2})-\frac{1}{l}(d'-\frac{q}{2})} + O(\omega^{-s}) \tag{12.5.70}$$

holds with arbitrarily large s. Here we sum over $p \in \mathbb{Z}^+$ such that $p \le J(d, m, \mu, s)$ and the term $\kappa'_q \log\omega$ arises if and only if there exists $p \in \mathbb{Z}^+$ with $\frac{1}{l}\big(d' - \frac{q}{2}\big) = \frac{2}{\mu}\big(d'' - \frac{p}{2}\big)$; here we do not indicate the dependence on n.

Proof. Let us differentiate $\Phi_{(q)}(\omega)$:

$$\mathsf{Tr}\,\Phi'_{(q)} = \omega^{-\frac{1}{l}(d'-\frac{q}{2})} \int \mathsf{Tr}\,\mathsf{Res}_{\mathbb{R}}\, F(z)\psi\big(\frac{z}{\omega}\big)\,\frac{dz}{\omega}$$

where $\psi = z_+^{1-\frac{1}{l}(d'-\frac{q}{2})}\phi'(z)$ obviously belongs to $\mathscr{C}_0^K([\epsilon_0, c_0])$; the prime here means the derivative.

One can easily see that modulo $O(\omega^{-s})$ the integral in the right-hand expression is equal to

$$\mathsf{tr}\big(\psi(\omega^{-1}D_t)\bar\chi_T(t)\Gamma' V_k\big)|_{t=0} \tag{12.5.71}$$

with arbitrary $T \ge \omega^{-\delta-1}$ and an arbitrarily small exponent $\delta > 0$; here $\Gamma' v = \int v(x'', x'', .)dx''$ and one defines the functions V_k from the recurrence relations

$$(D_t - a)V_0 = 0, \qquad V_0|_{t=0} = \delta(x'' - y'')I, \tag{12.5.72}_0$$

$$(D_t - a)V_k = B_k V_{k-1}, \qquad V_k|_{t=0} = 0 \quad (k \ge 1), \tag{12.5.72}_1$$

$a(x', \xi)$ here and below means microlocalization, $(k+1)$ is the number of factors $(\omega - a)^{-1}$ in the definition of F and B_j are the other factors in this definition; further, operators $a = a(x'', D'')$ and $B_j = B_j(x'', D'')$ are calculated at (x', ξ').

However, one can apply the standard Weyl successive approximation method of Chapter 4 on x'' with the semiclassical parameter $\omega^{-\mu^{-1}}$; therefore the reader can easily prove (there are no Tauberian arguments!) that expression (12.5.71) is equal to

$$\sum_{p\le J} \kappa''_{qp}\omega^{\frac{1}{\mu}(2d'-p)-1}$$

modulo $O(\omega^{-s})$. Finally, multiplying by $\omega^{-\frac{1}{\mu}(d'-\frac{q}{2})}$ and integrating we obtain (12.5.70). □

Let us set $\eta \le \tau^{\frac{1}{l}(1-\delta)}$ (in which case $\omega \ge \tau^{\delta}$). Then (12.5.71) yields that modulo $O(\lambda^{-s})$ the trace of (12.5.68) is equal to

$$\begin{aligned}(12.5.73)\quad &\bigl(\kappa_q + \kappa'_q \log\lambda - \kappa'_q l\log\eta\bigr)\lambda^{\frac{1}{l}(d'-\frac{q}{2})-1}+\\ &\sum_{p:\frac{1}{l}(d'-\frac{q}{2})\ne\frac{2}{\mu}(d''-\frac{p}{2})} \kappa'_{qp}\lambda^{\frac{1}{\mu}(2d''-p)-1}\eta^{-(2d''-p)\frac{l}{\mu}+(d'-\frac{q}{2})}.\end{aligned}$$

After integration over λ from τ' to τ we obtain terms of expansion (12.5.60). Therefore we have proven the following

Proposition 12.5.19. *(i) For $\eta \in [\tau^{\frac{1}{m}+\delta}, \tau^{\frac{1}{l}-\delta}]$ asymptotics*

$$(12.5.74)\qquad \int \mathsf{tr}\bigl(Q'_x\psi_1 e\bigr)(x,x,\tau',\tau)\,dx = \bar{\mathcal{N}}(\tau,\eta)\bigr|_{\tau=\tau'}^{\tau=\tau} + O\bigl(R_2^*(\tau)\bigr)$$

holds with

$$(12.5.75)\qquad \bar{\mathcal{N}}(\tau,\eta) = \bar{\mathcal{N}}_0(\tau) + \bar{\mathcal{N}}_1(\tau,\eta),$$

$$(12.5.76)\qquad \bar{\mathcal{N}}_0(\tau) = \sum_q \bar{\kappa}_q \tau^{(d'-\frac{q}{2})} + \bar{\kappa}_0 \log\tau,$$

and

$$\begin{aligned}(12.5.77)\quad \bar{\mathcal{N}}_1(\tau,\eta) = \bar{\kappa}'_0\bigl(\log^2\tau - 2l\log\tau\log\eta\bigr)+\\ \sum_{q,p:\frac{1}{l}(d'-\frac{q}{2})=\frac{1}{m}(2d''-p)} \bar{\kappa}''_q\tau^{\frac{1}{l}(d'-\frac{q}{2})}\bigl(\log\tau - l\log\eta\bigr) + \sum_{q\ne 2d'} \bar{\kappa}'''_q\eta^{(d'-\frac{q}{2})}\log\tau+\\ \sum_{q,p:\frac{1}{l}(d'-\frac{q}{2})\ne\frac{1}{\mu}(2d''-p)} \bar{\kappa}'_{qp}\tau^{\frac{1}{\mu}(2d''-p)}\eta^{(d'-\frac{q}{2})-(2d''-p)\frac{l}{\mu}}\end{aligned}$$

and $R_2^*(\tau)$ *given by* (12.5.61). *Here all the coefficients* $\bar{\kappa}^*_{**}$ *do not depend on* τ, τ', η *and we sum over* $p, q \le J = J(d, m, \mu, \delta)$.

(ii) Moreover, for $(d-1)l < (d'-1)m$ *and the standard non-periodicity condition (see* (12.2.51)*) on the Hamiltonian flow for operator-valued symbol* $a(x', \xi')$ *on the energy level* $\tau = 1$ *the sharper asymptotics*

$$\int \mathsf{tr}(Q'_x \psi_1 e)(x, x, \tau', \tau)\, dx = \bar{\mathcal{N}}(\tau, \eta)\big|_{\tau=\tau'}^{\tau=\tau} + o\big(\tau^{\frac{1}{l}(d'-1)}\big) \tag{12.5.78}$$

holds.

Let us combine propositions 12.5.16 and 12.5.19 and note that the final answer *should not* depend on the choice of η (lying in the given interval). Moreover, we know the Weyl asymptotics for $\mathsf{N}_0^+(\tau)$ in (12.5.14). Then we obtain the following statement for $\mathsf{N}(\tau', \tau)$ with $\tau' \in [\epsilon_0 \tau, \tau]$ instead of $\mathsf{N}^+(\tau)$ and with $\mathcal{N}(\tau', \tau) = \mathcal{N}^+(\tau)\big|_{\tau=\tau'}^{\tau=\tau}$ instead of $\mathcal{N}^+(\tau)$:

Theorem 12.5.20. *Let* $A \in \boldsymbol{\Psi}^{m,\mu}(X, \Lambda, \mathbb{K})$ *be a self-adjoint operator satisfying conditions* (12.5.1)–(12.5.9) *and* (12.5.11). *Then*

(i) Asymptotics

$$\mathsf{N}^+(\tau) = \mathcal{N}^+(\tau) + O\big(R^*(\tau)\big) \tag{12.5.79}$$

holds with

$$R^*(\tau) = \begin{cases} \tau^{\frac{1}{m}(d-1)} & \text{for } (d-1)l > (d'-1)m \\ \tau^{\frac{1}{m}(d-1)} \log \tau & \text{for } (d-1)l = (d'-1)m \\ \tau^{\frac{1}{l}(d'-1)} & \text{for } (d-1)l > (d'-1)m \end{cases} \tag{12.5.80}$$

where

$$\mathsf{N}^+(\tau) = \sum_{p=0,1,2} \kappa_{\frac{p}{2}} \tau^{\frac{1}{m}(d-\frac{p}{2})} + \sum_{q=0,1,2} \bar{\kappa}_{\frac{q}{2}} \tau^{\frac{1}{l}(d'-\frac{q}{2})} \tag{12.5.81}$$

provided $(d - \frac{p}{2})l \ne (d' - \frac{q}{2})m$ *for all* p, q *indicated above.*

On the other hand, if $(d - \frac{p}{2})l = (d' - \frac{q}{2})m$ *for some* p, q *then one should replace term* $\kappa_{\frac{p}{2}} \tau^{\frac{1}{m}(d-\frac{p}{2})}$ *by* $\kappa_{\frac{p}{2}} \tau^{\frac{1}{l}(d-\frac{p}{2})} \log \tau$.[52)]

[52)] One can easily see that there is usually at most one logarithmic term; there are two logarithmic terms if $m = 2l$, $d = 2d'$.

(ii) Moreover, let $(d-1)l > (d'-1)m$ *and let the standard non-periodicity condition on the Hamiltonian flow for* $a_0(x,\xi)$ *be fulfilled on the energy level* $\tau = 1$*. Then asymptotics*

$$\mathsf{N}^+(\tau) = \mathcal{N}^+(\tau) + o\bigl(\tau^{\frac{1}{m}(d-1)}\bigr) \tag{12.5.82}$$

holds.

(iii) On the other hand, let $(d-1)l < (d'-1)m$ *and let non-periodicity condition (see* (12.2.51)*) to the Hamiltonian flow for operator-valued symbol* $\mathsf{a}(x',\xi')$ *on the energy level* $\tau = 1$ *be fulfilled. Then asymptotics*

$$\mathsf{N}^+(\tau) = \mathcal{N}^+(\tau) + o\bigl(\tau^{\frac{1}{l}(d'-1)}\bigr) \tag{12.5.83}$$

holds.

Finally, let us replace τ and τ' by $2^{-n}\tau$ and $2^{-n-1}\tau$ and sum over $n = 1, \dots, n_0 \le [\log \tau] + 1$. Then we obtain this theorem in the final form (i.e., with $\tau' = 0$).

Remark 12.5.21. (i) Moreover, one obtains similar statements for Riesz means with

(12.5.84)
$$R^*_{(\vartheta)}(\tau) = \begin{cases} \tau^{\frac{1}{m}(d-1-\vartheta)+\vartheta} & \text{for } (d-1-\vartheta)l > (d'-1-\vartheta)m \\ \tau^{\frac{1}{m}(d-1-\vartheta)+\vartheta}\log\tau & \text{for } (d-1-\vartheta)l = (d'-1-\vartheta)m \\ \tau^{\frac{1}{l}(d'-1-\vartheta)+\vartheta} & \text{for } (d-1-\vartheta)l > (d'-1-\vartheta)m \end{cases}$$

and with principal part

$$\mathcal{N}^+_{(\vartheta)}(\tau) = \sum_{p\le 2+2\vartheta} \kappa_{\frac{p}{2}}\tau^{\frac{1}{m}(d-\frac{p}{2})+\vartheta} + \sum_{q\le 2+2\vartheta} \bar{\kappa}_{\frac{q}{2}}\tau^{\frac{1}{l}(d'-\frac{q}{2})+\vartheta} \tag{12.5.85}$$

provided $(d-\frac{p}{2})l \ne (d'-\frac{q}{2})m$ for all p, q indicated above.

On the other hand, if $(d-\frac{p}{2})l = (d'-\frac{q}{2})m$ for some p, q then one should replace the term $\kappa_{\frac{p}{2}}\tau^{\frac{1}{m}(d-\frac{p}{2})}$ by $\kappa_{\frac{p}{2}}\tau^{\frac{1}{m}(d-\frac{p}{2})}\log\tau$ [53]. Coefficients κ_*, $\bar{\kappa}_*$ depend on $\vartheta \ge 0$.

Surely, some terms are smaller than the remainder estimate and should be removed. One can see that for $\vartheta > \frac{2}{\mu}(ld - md') - 1$ we obtain a Weyl remainder estimate (but the asymptotics contain non-Weyl terms).

[53] So as two terms have equal powers of τ one of them has an extra logarithmic factor.

(ii) Moreover, let $p \notin \mathbb{Z}$ and $a_j \equiv 0$ for all $j \notin \mathbb{Z} : j \le k$. Then $\kappa_p = 0$ which is also true in the case $(d-p)l = (d-q)m$ for some $q \in \frac{1}{2}\mathbb{Z}$: so the logarithmic term does not arise.

12.5.6 Operators with Negative Principal Symbols

Let us consider the case when $a_0(x,\xi)$ is a negative definite symbol. However, we assume that $\boldsymbol{a}(x',\xi') = A_{(x',\xi')}(x'',D'')$ is not a negative definite operator (otherwise A is semibounded from above). However in this case $\boldsymbol{a}(x',\xi')$ has only finite number of positive eigenvalues and thus its positive part is essentially a matrix.

Our goal is to prove the following

Theorem 12.5.22. *Let the conditions of Theorem 12.5.20 be fulfilled and let the condition*

(12.5.86) $a_0(x,\xi)$ *is a negative definite symbol on* $T^*X \setminus 0 \setminus \Lambda$

also be fulfilled.

Then

(i) Asymptotics

$$\mathsf{N}^+(\tau) = \mathcal{N}^+(\tau) + O\bigl(\tau^{\frac{1}{l}(d'-1)}\bigr) \tag{12.5.87}$$

holds with

$$\mathcal{N}^+(\tau) = \bar{\kappa}_0 \tau^{\frac{1}{m}d'} + \bar{\kappa}_{\frac{1}{2}} \tau^{\frac{1}{l}(d'-\frac{1}{2})} + \bar{\kappa}_1 \tau^{\frac{1}{l}(d'-1)}. \tag{12.5.88}$$

(ii) Moreover, let the non-periodicity condition (see (12.2.51)*) for Hamiltonian flow for operator-valued symbol* $\boldsymbol{a}(x',\xi')$ *on the energy level* $\tau = 1$ *be fulfilled. Then asymptotics*

$$\mathsf{N}^+(\tau) = \mathcal{N}^+(\tau) + o\bigl(\tau^{\frac{1}{l}(d'-1)}\bigr) \tag{12.5.89}$$

holds.

Proof. First of all, (12.5.57) and the explicit formula for U_n yield that for $\chi \in \mathscr{C}_0^K(\mathbb{R})$ vanishing in a neighborhood of 0 one can replace (12.5.59) by estimate

$$\bigl|F_{t\to\lambda}\chi_T(t)\Gamma\psi_1 Q'_x U\bigr| \le C\bar{r}^{d-\frac{2}{\mu}d''m}T^{\frac{2}{\mu}d''-1-s}T^{-sl} \tag{12.5.90}$$

for $T\in[T_0\tau^{-1}, T_0\tau^{-1}\bar{r}]$, $\lambda\in[\epsilon_0\tau, c\tau]$. Therefore the same estimate holds for

$$\bigl|F_{t\to\lambda}\bigl(\bar{\chi}_{T'}(t)-\bar{\chi}_T(t)\bigr)\Gamma\psi_1 Q'_x U\bigr|$$

for $T, T'\in[T_0\tau^{-1}, T_0\tau^{-1}\bar{r}]$, $T<T'$ and $\bar{\chi}\in\mathscr{C}_0^K(\mathbb{R})$ equal to 1 in a neighborhood of 0.

On the other hand, for $T\in[T_0, \tau^{-1}h^{-\delta}, T_0\tau^{-1}h^{-1}]$ with $h=\bar{r}^{-\frac{2}{\mu}}T^{\frac{2}{\mu}}$, the standard Weyl formula (which is derived by setting $\bar{A}=a_0(y,D)$) holds modulo terms which do not exceed the right-hand expression in (12.5.90).

Condition (12.5.74) also yields that all the coefficients in this formula vanish. Taking $T=T_0\tau^{-1}\bar{h}'^{-1}$ we replace (12.5.59) by the estimate

$$\bigl|F_{t\to\lambda}\bar{\chi}_T(t)\Gamma\psi_1 Q'_x U\bigr| \le C\bar{r}^{d-\frac{2}{\mu}d''m+s}T^{\frac{2}{\mu}d''-1-s}T^{-sl} \tag{12.5.91}$$

with $T\in[T_0, \tau^{-1}\tau^{-1}\bar{h}'^{-\delta}, T_0\tau^{-1}\bar{r}]$, $\lambda\in[\epsilon_0\tau, c\tau]$ and with another still arbitrarily large s. Here we have omitted the prime on T.

Therefore one can replace $\bar{\chi}$ with a Hörmander function χ and apply the routine Tauberian technique to obtain that the left hand expression in (12.5.60) now does not exceed

$$C\bar{r}^{d-1-\frac{2}{\mu}d''m+sl}T^{\frac{2}{\mu}d''-s}.$$

Then after summation over a partition of unity in the zone $\bigl\{r_*(\tau)\le|\xi'|\le\varepsilon r^*(\tau)\bigr\}$ we obtain remainder estimate $\varepsilon'\tau^{\frac{1}{l}(d'-1)}$ with $\varepsilon'=\varepsilon'(\varepsilon)\to+0$ as $\varepsilon\to+0$. Therefore under condition (12.5.86) Proposition 12.5.17 holds with the remainder estimate $O\bigl(\tau^{\frac{1}{l}(d'-1)}\bigr)$; moreover it holds with the remainder estimate $o\bigl(\tau^{\frac{1}{l}(d'-1)}\bigr)$ under standard non-periodicity condition to the Hamiltonian flow for matrix-valued symbol $a(x',\xi')$ on the energy level $\tau=1$.

On the other hand, Proposition 12.5.9 yields that the remainder estimate in Proposition 12.5.16 is now $O(\tau^{-s})$. Therefore for $\mathsf{N}(\tau',\tau)$ with $\tau'\in[\epsilon_0,\tau]$ the remainder estimate is $O\bigl(\tau^{\frac{1}{l}(d'-1)}\bigr)$ [54]. This estimate yields the same remainder estimate for $\mathsf{N}^+(\tau)$ for $d'\ge 2$.

For $d'=1$ standard arguments with τ, τ' replaced by $2^{-n}\tau$, $2^{-n-1}\tau$, etc., yield the remainder estimate $O\bigl(\log\tau\bigr)$ but one can also easily obtain the

[54] And even $o\bigl(\tau^{\frac{1}{l}(d'-1)}\bigr)$ under the standard non-periodicity condition.

remainder estimate $O(1)$. In fact, one has the remainder estimate $O(1)$ for $\mathsf{N}(\tau',\tau)$ and the remainder estimate $O(\tau^{-s})$ for

$$\int \phi\bigl(\frac{\lambda}{\tau}\bigr)\, d_\lambda \mathsf{N}^+(\lambda) d\lambda$$

with $\phi \in \mathscr{C}_0^K([\epsilon_0, c])$ (this estimate is the routine estimate with mollification, the deduction of which requires neither the Tauberian technique nor condition (12.5.86)). Replacing $\phi(\lambda)$ by $\bigl(\bar\phi(\tau) - \bar\phi(2\tau)\bigr)$ with $\bar\phi \in \mathscr{C}^K(\mathbb{R})$ supported in $[-\infty, 1]$ and equal to 1 on $[-\infty, \frac{1}{2}]$ and replacing τ by $2^{-n}\tau$, etc., we obtain the remainder estimate $O(1)$ for

$$\int \bar\phi\bigl(\frac{\lambda}{\tau}\bigr)\, d_\lambda \mathsf{N}^+(\lambda) d\lambda,$$

and this remainder estimate and the remainder estimate for $\mathsf{N}(\tau',\tau)$ obviously yield the remainder estimate $O(1)$ for $\mathsf{N}^+(\tau)$. So, both remainder estimates in Theorem 12.5.22 have been proven.

Moreover, in Proposition 12.5.16 we obtain $\mathcal{N}(\tau,\eta) = 0$ because of condition (12.5.86) (recall that standard Weyl asymptotics were applied here) and in Proposition 12.5.18 one should replace (12.5.70) by the asymptotics

$$\mathsf{Tr}\, \Phi_{n(q)}(\omega) = \kappa_q + O(\omega^{-s}). \tag{12.5.92}$$

Therefore expression (12.5.73) is reduced to

$$\kappa_q \tau^{\frac{1}{l}(d'-\frac{q}{2})-1}$$

and $\bar{\mathcal{N}}(\tau,\eta) = 0$.

Thus for $d' \ge 2$ we obtain expression (12.5.88) exactly for $\mathcal{N}^+(\tau)$, but for $d' = 1$ an additional term $\bar\kappa_1' \log \tau$ arises (this term is less than the remainder estimate for $d' \ge 2$). Then we apply the Remark 12.5.23 below. □

Remark 12.5.23. By explicit calculation prove that $\bar\kappa_1' = 0$ for $d' = 1$. Note that the same calculation was done in the analysis of non-semibounded boundary value problems for operators with positive principal symbols in Subsection 7.3.3 (one can apply these arguments without any modification).

Remark 12.5.24. The remainder estimate for Riesz means is $O\bigl(\tau^{\frac{1}{l}(d-1-\vartheta)+\vartheta}\bigr)$ and even $o\bigl(\tau^{\frac{1}{l}(d-1-\vartheta)+\vartheta}\bigr)$ under the standard condition to the Hamiltonian flow.

12.5.7 Coefficients in Spectral Asymptotics

Let us discuss the coefficients κ_p and $\bar{\kappa}_q$ in the asymptotics of Theorems 12.5.20, 12.5.22 and Remarks 12.5.21, 12.5.24. First of all the following obvious remark holds (we discuss coefficients for $\vartheta = 0$; the coefficients of $\tau^{k+\vartheta}$ for $\vartheta > 0$ have the extra factor $k(k+\vartheta)^{-1}$ compared with the coefficient of τ^k for $\vartheta = 0$).

Remark 12.5.25. (i) If $p\,(= 0, \frac{1}{2}, 1, \dots)$ is such that $l(d-p) > d'm$ then for κ_p the standard Weyl formula holds, the integral in which surely converges; in particular,

$$\kappa_0 = (2\pi)^{-d} \int n_0^+(x,\xi,1)\, dxd\xi \tag{12.5.93}$$

where $n_0^+(x,\xi,1)$ is the number of eigenvalues of $a_0(x,\xi)$ lying in $[0,1]$. Therefore $\kappa_0 > 0$ provided $a_0(x,\xi)$ is not negative definite.

(ii) On the other hand, for $p\,(= 0, \frac{1}{2}, 1, \dots)$ let equality $(d-p)l = d'm$ hold. Then the integral providing the standard Weyl coefficient of $\tau^{\frac{1}{m}(d-p)}$ diverges logarithmically; namely, modulo convergent integrals we have

$$\int \omega(x',\xi';\varphi)\, dx'd\xi'd\varphi \frac{d\zeta}{\zeta}$$

with $\varphi = \varrho^{-1}\cdot(x'', \xi''|\xi'|^{-1}) \in \mathbb{S}^{2d''-1}$, $\varrho = \bigl(|x''|^2 + |\xi''|^2|\xi'|^{-2}\bigr)^{\frac{1}{2}}$; then the coefficient $\bar{\kappa}_p$ at $\tau^{(\frac{1}{m}d-p)}\log\tau$ is given by formula

$$\bar{\kappa}_p = \frac{1}{2l}\int \omega(x',\xi',\varphi)\, dx'd\xi'd\varphi.$$

In particular, for $dl = d'm$ the coefficient $\bar{\kappa}_0$ at $\tau^{\frac{d}{m}}\log\tau$ is

$$\bar{\kappa}_0 = \frac{1}{2l(2\pi)^d}\int_{\Lambda\times\mathbb{S}^{2d''-1}} \tilde{n}^+(x',\xi',\varphi,1)\, dx'd\xi'd\varphi \tag{12.5.94}$$

where $\tilde{n}^+(x',\xi',\varphi,1)$ is the number of eigenvalues of the symbol

$$\tilde{a}_0(x',\xi',\varphi) = \sum_{|\alpha|+|\beta|=\mu} \frac{1}{\alpha!\beta!} a_{0(\beta)}^{(\alpha)}(x',\xi') x''^{\beta}\xi''^{\alpha} \tag{12.5.95}$$

again with $\varphi = \varrho^{-1}\cdot(x'', \xi''|\xi''|^{-1})(x,\xi) \in \mathbb{S}^{2d''-1}$.

(iii) Either let $q = (0, \frac{1}{2}, 1, \dots)$ be such that $(d' - q)m > dl$ or let $a_0(x, \xi)$ be a negative definite symbol. Then the coefficient $\bar{\kappa}_q$ is given by a convergent integral

$$\frac{1}{2d' - q} \int_0^\infty \operatorname{Tr} \operatorname{Res}_{\mathbb{R}} F_{(q)}(z) z^{-\frac{1}{l}(d'-q)} dz$$

where $F_{(q)}$ is the sum of all the symbols with the given index q. In particular, if either $d'm > dl$ or $a_0(x, \xi)$ is a negative definite symbol then

$$\bar{\kappa}_0 = (2\pi)^{-d'} \int_\Lambda \boldsymbol{n}^+(x', \xi', 1)\, dx' d\xi' = \frac{1}{(2\pi)^d d} \int_{\Lambda \cap S^* X} \omega(z)\, dz \tag{12.5.96}$$

where $\omega(z) = \sum_j \eta_j^{-\frac{d}{l}}$, η_j are all the positive eigenvalues of $A_{(x', \xi')}$ and dz is the natural density on $\Lambda \cap S^* X$ provided a Riemannian metric is introduced on X.

(iv) $a_{\frac{1}{2}} = 0$ yields that $\kappa_{\frac{1}{2}} = 0$. See also problem 12.5.11(ii).

(v) Moreover, let μ be even. Let us again treat the zone $\{|\xi'| \le r^*(\tau)\tau^{-\delta}\}$. Let us consider a term in the expansion of $\varkappa_{\frac{j}{2}}(x, \xi)$ in terms which are homogeneous with respect to $(x'', \xi''|\xi'|^{-1})$. Obviously, if this term is odd with respect to $(x'', \xi''|\xi'|^{-1})$ then its integral vanishes. Therefore a term with a logarithmic factor appears only if the equality $\left(d - \frac{j}{2}\right)l = \left(d' - \frac{k}{2}\right)m$ holds with $j \equiv k \mod 2$. So, if μ is even then a term $\kappa_{\frac{j}{2}} \tau^{\frac{1}{m}(d - \frac{j}{2})} \log \tau$ appears only if $\left(d - \frac{j}{2}\right)l = \left(d' - \frac{k}{2}\right)m$ and $j \equiv k \,(\mod 2)$.

(vi) Let $a_{\frac{1}{2}} \equiv 0$. Then in the previous analysis one should only treat even j (and therefore only even k). On the other hand, analysis in the zone $\{|\xi'| \ge r_*(\tau)\tau^{\delta}\}$ proves that $\bar{\kappa}_{\frac{q}{2}} = 0$ for odd q. So, if $a_{\frac{1}{2}} \equiv 0$ and μ is even then in (12.5.81), (12.5.88) one should treat only $p = 0, 2$ and $q = 0, 2$ in the calculation of powers and in the criterion for terms with a logarithmic factor. Similarly, for $\vartheta > 0$ in this case ($a_{\frac{n}{2}} \equiv 0$ for odd n and μ is even) one should consider only even p, q in the criterion for terms with a logarithmic factor.

(vii) On the other hand, one can easily see that for an operator with the symbol $\begin{pmatrix} \lambda & \mathcal{L} \\ \mathcal{L}^* & \lambda \end{pmatrix}$ with $\mathcal{L} \in \boldsymbol{\Psi}^{1,1}$ and $\lambda \in \mathbb{C}$ the previous statement for $\bar{\kappa}_q$ is wrong.

(viii) Finally, if $a_{\frac{1}{2}} \equiv 0$, μ and m are even and a_0 and a_1 are homogeneous (not only positive homogeneous) of degrees m and $(m-1)$ in ξ respectively then in (12.5.81) one should treat only $p=0$ in the calculation of powers and in the criterion for terms with a logarithmic factor.

We leave to the reader the following set of problems

Problem 12.5.26. (i) Consider the asymptotics of Riesz means.

(ii) Consider manifolds with boundary. In this case one should assume that every connected component of Λ either does not intersect $T^*X|_{\partial X}$ or is contained in this manifold. In the second case one should consider $A_{(x',\xi')}(x'', D'')$ as an operator in the half-space of $\mathbb{R}^{d''}$ and the problem of boundary conditions arises.

Moreover, there are more variants for the leading term of the asymptotics because we know that the condition that the operator is semi-bounded on a manifold with boundary consists of two conditions, and if the first condition is fulfilled and the second condition is violated then the first term on the asymptotics vanishes in the non-degenerate case.

(iii) The other problem left to the reader is to return to Sections 12.2–12.4 and obtain (under appropriate conditions) the non-Weyl term in the asymptotics in a simpler form (as we did here, and in the form used here).

12.6 Riesz Means Asymptotics for Operators Singular at a Point

12.6.1 Problem Set-up

In this section we consider h-differential operator $A = A_h$ with Weyl symbol

$$A(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha, \tag{12.6.1}$$

Hermitian and such that in $B(0,1)$

$$|D^\beta a_\alpha| \le c|x|^{\mu(m-|\alpha|)-|\beta|} \qquad \forall \alpha : |\alpha| \le m \quad \forall \beta : |\beta| \le K \tag{12.6.2}$$

with

$$\mu > \max\bigl(-1, -\frac{d}{m}\bigr). \tag{12.6.3}$$

Moreover, we assume that

$$\langle a(x,\xi)v, v\rangle \ge \epsilon_0|\xi|^m\|v\|^2 \qquad \forall (x,\xi) \in T^*B(0,1) \quad \forall v \in \mathbb{K} \tag{12.6.4}$$

where $a(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ is the senior symbol of A.

Let us consider the operator A in the domain $X \subset \mathbb{R}^d$ assuming that

$$B(0,1) \subset X, \quad d \ge 2 \tag{12.6.5}$$

and

(12.6.6) A is a self-adjoint operator with $\mathfrak{D}(A) \supset \mathscr{C}_0^m\bigl(B(0,1), \mathbb{K}\bigr)$.

We are interested in semiclassical asymptotics of the Riesz mean

$$\int \psi(x)\,\mathsf{tr}\bigl(e_{(\vartheta)}(x,x,0)\bigr)\,dx \tag{12.6.7}$$

where $\psi \in \mathscr{C}_0^K\bigl(B(0,\frac{1}{2})\bigr)$ and $e_{(\vartheta)}(.,.,\tau)$ is the Riesz mean of order ϑ of the Schwartz kernel $e(.,.,\tau)$ of the spectral projector A, $\vartheta > 0$:

$$e_{(\vartheta)}(.,.,\tau) = \frac{1}{\Gamma(\vartheta+1)}\int_0^\tau (\tau-\lambda)^\vartheta\, d_\lambda e(.,.,\lambda) \tag{12.6.8}$$

where $\Gamma(.)$ is an Euler' Γ-function.

In Section 11.1 we proved that under an appropriate condition associated with microhyperbolicity (namely, condition (12.6.34) below) asymptotics

$$\int \psi(x)\,\mathsf{tr}\bigl(e_{(\vartheta)}(x,x,0)\bigr)\,dx = \sum_{0\le j\le J} h^{-d+j}\int \psi(x)\kappa_{(\vartheta)j}(x)\,dx + O\bigl(R_{(\vartheta)}(\tau)\bigr) \tag{12.6.9}$$

holds with

$$R_{(\vartheta)}(\tau) = \begin{cases} h^{-d+\vartheta+1} & \text{for } \vartheta < \bar\vartheta_p,\\ h^{-d+\vartheta+1}\log h & \text{for } \vartheta = \bar\vartheta_p,\\ h^l & \text{for } \vartheta > \bar\vartheta_p \end{cases} \tag{12.6.10}$$

where here and below

$$\bar\vartheta_p = \begin{cases} \big((d-1)(\mu+1)+p\big)(-m\mu+\mu+1)^{-1} & \text{for } m\mu<\mu+1,\\ +\infty & \text{for } m\mu\ge\mu+1,\end{cases} \tag{12.6.11}$$

and

$$l = l_0 = m\mu\vartheta(\mu+1)^{-1}, \qquad l_p = (m\mu\vartheta+p)(\mu+1)^{-1}, \tag{12.6.12}$$

$\kappa_{(\vartheta)j}$ are standard Weyl coefficients and

$$J = \begin{cases} \lfloor\vartheta\rfloor+1 & \text{for } \vartheta\le\bar\vartheta_p,\\ d+\lfloor l\rfloor & \text{for } \vartheta\ge\bar\vartheta_p\end{cases} \tag{12.6.13}$$

excluding the special case

$$J=\vartheta \qquad \text{for } \vartheta=\bar\vartheta_p\in\mathbb{N}. \tag{12.6.13}'$$

Further, for $\vartheta>\bar\vartheta_p$ and $l\in\mathbb{N}$ one should cut the corresponding term near the origin, i.e., replace $\psi(x)$ by $\psi(x)\phi(xr_*(h)^{-1})$ in the term with $j=d+l$ of the right-hand expression of (12.6.9) where here and below

$$r_*(h) = h^{1/(\mu+1)}, \tag{12.6.14}$$

$(1-\phi)\in \mathscr{C}_0^K(\mathbb{R}^d)$ and ϕ vanishes in a neighborhood of 0.

Furthermore, we proved that if $\vartheta<\bar\vartheta_p$ and the standard non-periodicity condition is fulfilled on the energy level $\tau=0$ then one can replace $O(h^{-d+\vartheta+1})$ by $o(h^{-d+\vartheta+1})$ in asymptotics (12.6.9).

The principal part of the asymptotics is $O(h^{-d})$ for $m\mu\vartheta>-d(\mu+1)$ and $O(h^{-d}\log h)$ for $m\mu\vartheta+p=-d(\mu+1)$ and $O(h^l)$ for $m\mu\vartheta+p<-d(\mu+1)$ [55]; so in the last case we have only an estimate instead of an asymptotics.

Remark 12.6.1. All these statements hold for $p=0$ but also remain true with $l=l_p$ for $p\ge 0$ provided

$$\operatorname{supp}(\psi)\subset B(0,\frac{1}{2}), \tag{12.6.15}$$

$$|D^\alpha\psi|\le c|x|^{p-|\alpha|} \qquad \forall\alpha:|\alpha|\le K \tag{12.6.16}$$

instead of the previous assumptions. One can see easily that $\vartheta_p>p$.

[55] There are equivalence relations under obvious conditions.

We leave to the reader the following easy problem:

Problem 12.6.2. Consider case $-d < p < 0$, $m\mu < \mu+1$; consider separately $m\mu < \mu+1$ and $m\mu \ge \mu+1$.

12.6.2 The Scott Correction Term

Our goal is to treat the case $\vartheta \ge \bar{\vartheta}_p$ and to improve this remainder estimate. This better estimate, obtained under appropriate conditions, requires an additional non-Weyl term ωh^l in the asymptotics. Physicists call this term the *Scott correction term*. Later we will obtain more than one such term for large ϑ. In this analysis, without any loss of the generality we can treat

$$\int x^\alpha \psi(x)\, \mathsf{tr}\big(e_{(\vartheta)}(x,x,0)\big)\, dx$$

with $|\alpha| = p$ and $\psi \in \mathscr{C}_0^K(\mathbb{R}^d)$ as

$$\bar{\vartheta}_p < +\infty \qquad (\iff m\mu < \mu+1). \tag{12.6.17}$$

Instead we consider more general case

(12.6.18) $\psi \in \mathscr{C}^K\big(\mathbb{R}^d \setminus 0\big)$ is positively homogeneous of degree p in $B(0,\frac{1}{2})$ and $\mathsf{supp}(\psi) \subset B(0,1)$.

First of all we should treat the positive homogeneous case:

Proposition 12.6.3. *Let conditions* (12.6.2)–(12.6.5) *be fulfilled. Moreover, let $X = \mathbb{R}^d$, A be a self-adjoint operator with $\mathfrak{D}(A) \supset \mathscr{C}_0^m(\mathbb{R}^d, \mathbb{K})$ and let us assume that*

(12.6.19) *The coefficients a_α are positively homogeneous of degrees $(m-|\alpha|)\mu$ in x.*

Finally, let us assume that

(12.6.20) *$A(x,\xi)$ is microhyperbolic in (x,ξ) on the energy level $\tau = 0$ for $x \ne 0$* [56].

[56] Surely, the constants in the microhyperbolicity condition depend on $|x|$.

Then for any $\beta \in \mathbb{N}^d$ *and for* $W \in \mathscr{C}^K\big(\mathbb{R}^d \setminus 0, \mathsf{Hom}(\mathbb{K},\mathbb{K})\big)$ *positively homogeneous of degree* $(p-\mu|\beta|)$ *in* x *equality*

$$\int W(x)\Big(\big((hD_x)^\beta e_{(\vartheta)}\big)(x,x,0) - \phi\big(\frac{x}{r_*}\big)\sum_{0\le j\le J}\varkappa_{(\vartheta)\beta,j}(x)h^{-d+j}\Big)\,dx = \omega h^l \tag{12.6.21}$$

holds with the coefficients $\kappa_{(\vartheta)\beta,j}(x)$ [57] *and a certain constant coefficient* $\omega \in \mathsf{Hom}(\mathbb{K},\mathbb{K})$ *which depends on* W,β,ϕ [58] *and* $J \ge [l]+d$ *and the integral in the left-hand expression converges at infinity* [59].

Moreover, in terms with $j < l+d$ *one can replace* ϕ *by* 1 [60]*; in particular, if either* $l \notin \mathbb{N}$ *or* $l \in \mathbb{N}$ *but* $\kappa_{(\vartheta)\beta,l+d} \equiv 0$ *then one can set* $J = \lfloor l \rfloor + d$ *and replace* ϕ *by* 1 *in all the terms.*

Proof. Let us first note that in the context of the proposition

(12.6.22) Coefficients $\kappa_{(\vartheta)\beta,j}(x)$ are positively homogeneous in x of degrees $\big(d\mu + m\mu\vartheta - j(\mu+1) - |\beta|\big)$.

Moreover, let us note that after rescaling $x_{\mathsf{new}} = x_{\mathsf{old}}/r_*(h)$ with $r_*(h) = h^{-1/(\mu+1)}$ operator $A^{\mathsf{w}}(x,hD)$ is transformed into operator $h^{m\mu(\mu+1)^{-1}}A^{\mathsf{w}}(x,D)$ and therefore

$$e_{(\vartheta)}(x,y,0;h) = h^{(m\mu\vartheta+d)/(\mu+1)}e_{(\vartheta)}(xh^{-1/(\mu+1)}, yh^{-1/(\mu+1)},0;1) \tag{12.6.23}$$

due to the definition of $e_{(\vartheta)}(.,.,.,.)$; here the fourth variable is the semiclassical parameter h.

This equality and (12.6.22) yield that the left-hand expression in (12.6.21) is positively homogeneous of degree l on h provided it converges; therefore we should prove only that the integral in (12.6.21) converges at infinity.

However, Theorem 4.4.6 yields that if $\psi' \in \mathscr{C}_0^K\big(\mathbb{R}^d, \mathsf{Hom}(\mathbb{K},\mathbb{K})\big)$ vanishes in a neighborhood of 0 then the estimate

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[57] Here we do not take the trace and therefore the coefficients κ_* and ω are matrix-valued in this statement.

[58] Recall that ϕ is a fixed function such that $(1-\phi) \in \mathscr{C}_0^K(\mathbb{R}^d)$, $\phi = 1$ near 0.

[59] It surely converges at 0 due to the cutting function ϕ.

[60] Surely, the constant ω changes.

$$\Big|\int \psi'\big(\frac{x}{r}\big)\Big(\big((hD_x)^{\beta} e_{(\vartheta)}\big)(x,x,0) - \phi\big(\frac{x}{r_*}\big)\sum_{0\le j\le J} \kappa_{(\vartheta)\beta,j}(x) h^{-d+j}\Big)\,dx\Big| \le Ch^{-d+J+1} r^{l(\mu+1)(l-J-1)}$$

holds uniformly on $r \ge r_* = h^{(\mu+1)^{-1}}$; this estimate immediately yields that the integral in the left-hand expression of (12.6.21) converges for $J \ge \lfloor l \rfloor + d$. □

12.6.3 Improved Asymptotics: the First Attempt

Our first goal is to improve the remainder estimate $O(h^l)$ for $\vartheta > \bar{\vartheta}_p$ in the general situation. To do this we will pass from one operator to another to reduce the problem.

Proposition 12.6.4. *Let the operators A and A' satisfy conditions* (12.6.2)–(12.6.6) *and coincide in $B(0,1)$. Let $\psi \in \mathscr{C}_0^K\big(B(0,\frac{1}{2})\big)$, $\psi \ge 0$. Moreover, let estimate*

$$\Big|\int \psi(x) e(x,x,\tau+h,\tau)\,dx\Big| \le Ch^{1-d} \qquad \forall \tau : |\tau| \le \epsilon_0 \tag{12.6.24}$$

hold. Then the same estimate holds for A' as well and, moreover, estimate

$$\Big|\int \psi(x)\big(e_{(\vartheta)}(x,x,0) - e'_{(\vartheta)}(x,x,0)\big)\,dx\Big| \le Ch^{1-d+\vartheta} \tag{12.6.25}$$

holds where here and below $e(.,.,.)$ with a superscript is associated with the operator A with the same superscript.

Proof. Proof follows from the same analysis as in Chapter 9. We leave the easy details to the reader. □

Therefore the remainder estimate which will not be better than $O\big(h^{1-d+\vartheta}\big)$ is preserved if we change the operator A outside of $B(0,\epsilon_0)$ with some fixed $\epsilon_0 > 0$.

(12.6.26) Let us pick an operator A' coinciding with A in $B(0,\epsilon_0)$ and such that the negative spectrum of A' is discrete for every $h \in (0,h_0]$ (we can even pick an operator with negative spectrum, finite for every $h \in (0,h_0]$).

Let us now assume that $\vartheta \ge 1$. In fact the arguments below are really only good for $\vartheta = 1$ but we will later modify them to work in the case $\vartheta < 1$ and to provide a really good estimate for $\vartheta > 1$.

Proposition 12.6.5. *For $\theta \in [0,1]$ let the operator $A^\theta = A^0 + \theta B$ satisfy conditions* (12.6.2)–(12.6.6)*, let the negative spectrum of A be discrete and let B be supported in $B(0, \frac{1}{2})$. Then equality*

$$\int \mathsf{tr}\big(e^1_{(\vartheta)}(x,x,\tau) - e^0_{(\vartheta)}(x,x,\tau)\big)\,dx = - \int_0^1 d\theta \int \mathsf{tr}\big(B_x e^\theta_{(\vartheta-1)}\big)(x,x,\tau)\,dx \tag{12.6.27}$$

holds for $\tau \le 0$ and $\vartheta \ge 1$.

Proof. Let $E^\theta(\tau)$ be the spectral projector of A^θ. Let us differentiate the equality

$$\partial_\tau \mathsf{Tr}\big(E^\theta(\tau)\big) = \mathsf{Tr}\big(\mathsf{Res}_{\mathbb{R}}\big(\tau - A^\theta\big)^{-1}\big), \tag{12.6.28}$$

which holds for $\tau < 0$ in the distribution sense, with respect to θ. We obtain

$$\partial_\tau \partial_\theta \mathsf{Tr}\big(E^\theta(\tau)\big) = -\partial_\tau^2 \mathsf{Tr}\big(BE^\theta(\tau)\big).$$

This equality and the semiboundedness of A^θ from below yield that

$$\partial_\eta \mathsf{Tr}\big(E^\theta(\tau)\big) = -\partial_\tau \mathsf{Tr}\big(BE^\theta(\tau)\big). \tag{12.6.29}$$

Integrating over θ from 0 to 1 and convolving with $\tau_+^{(\vartheta-1)}/\Gamma(\theta)$ we obtain after integration by part (12.6.27) for $\tau < 0$. Semi-continuity from the left also yields (12.6.27) for $\tau = 0$. □

Let us assume that the coefficients b_α of the operator $B = A - A^0$ satisfy estimates

$$|D^\beta b_\alpha| \le c|x|^{\mu(m-|\alpha|)+\nu-|\beta|} \qquad \forall \alpha : |\alpha| \le m \quad \forall \beta : |\beta| \le K \tag{12.6.30}$$

with $\nu > 0$ and

$$\mathsf{supp}(b_\alpha) \subset B(0,r), \qquad 1 \ge r \ge r_*(h) = h^{1/(\mu+1)} \tag{12.6.31}$$

where r is a parameter as well as h.

Let us apply the Weyl approximation to the right-hand expression in (12.6.27); this means that we should replace the integrand by the convolution of

$$(2\pi)^{-d}\int \operatorname{Res}_{\mathbb{R}} \operatorname{tr}\bigl(B(\tau - A)^{-1}\bigr)_{\mathsf{w}}(x,\xi)\,d\xi \tag{12.6.32}$$

with $\tau_+^{\vartheta-1}/\Gamma(\vartheta)$; recall that the subscript w means the Weyl symbol and all calculations are done in the context of the formal calculus of pseudodifferential operators.

In this situation let us reverse the deduction of (12.6.27). We then obtain the routine answer

$$\sum_{0\le j\le \lfloor\vartheta\rfloor} h^{-d+j}\int \bigl(\kappa^1_{(\vartheta)j}(x) - \kappa^0_{(\vartheta)j}(x)\bigr)\,dx \tag{12.6.33}$$

where $\kappa^\theta_{(\vartheta)j}$ are the Weyl coefficients for the operator A^θ. However, we gain a remainder estimate: calculating the remainder estimate in our Weyl approximation we obtain

Proposition 12.6.6. *Let the operator $A^\theta = A + \theta B$ satisfy conditions* (12.6.2)–(12.6.6), (12.6.30), (12.6.31) *uniformly on $\theta \in [0,1]$.*

In addition, let the negative spectrum of A^θ be discrete and let us assume that

(12.6.34) *Symbol $A(xr, \xi r^\mu) r^{-m\mu}$ is microhyperbolic in (x,ξ) on the energy level 0 uniformly with respect to $x \in \mathbb{S}^{d-1}$, $\xi \in \mathbb{R}^d$, $r \in (0,1)$.*

Then estimate

$$\Bigl|\int \operatorname{tr}\bigl(e^1_{(\vartheta)}(x,x,0) - e^0_{(\vartheta)}(x,x,0)\bigr) - \sum_{0\le j\le\lfloor\vartheta\rfloor+1} \bigl(\varkappa^1_{(\vartheta)j}(x) - \varkappa^0_{(\vartheta)j}(x)\bigr)\phi\bigl(\frac{x}{r_*}\bigr) h^{-d+j}\,dx\Bigr| \le C R_1^*(r,h) \tag{12.6.35}$$

holds with

$$R_1^*(r,h) = \begin{cases} h^{-d+\vartheta} r^{m\mu\vartheta + (d-\vartheta)(\mu+1)+\nu} & \text{for } \vartheta < \bar\vartheta_\nu,\\ h^{-d+\vartheta}\,|\log r| & \text{for } \vartheta = \bar\vartheta_\nu,\\ h^{l'} & \text{for } \vartheta > \bar\vartheta_\nu \end{cases} \tag{12.6.36}$$

where according to (12.6.11) *and* (12.6.12)

$$\bar{\vartheta}_\nu = \big(d(\mu+1)+\nu\big)(\mu+1-m\mu)^{-1} > \bar{\vartheta}_0, \tag{12.6.37}$$

and

$$l' = l_\nu = (m\mu\vartheta+\nu)(\mu+1)^{-1} > l. \tag{12.6.38}$$

Recall that $r_*(h) = h^{1/(\mu+1)}$, ϕ *is a fixed function with* $(1-\phi) \in \mathscr{C}_0^K(\mathbb{R}^d)$, *and* ϕ *vanishes in a neighborhood of* 0.

Moreover, for $\vartheta < \bar{\vartheta}_\nu$ *one can replace* ϕ *by* 1 *in all the terms; one can also do this for* $\vartheta = \bar{\vartheta}_\nu$, *skipping terms with* $j = \vartheta$; *furthermore, for* $\vartheta > \bar{\vartheta}_\nu$ *one can skip terms with* $j > l' + d$ *and one can replace* ϕ *by* 1 *in terms with* $j < l' + d$.

Proof. Proof follows the scheme we indicated but with the following details:

We apply the Weyl approximation only in the semiclassical zone, i.e., for $r \ge r_*$. In the ball $B(0, r_*)$ we apply a routine estimate of Chapter 9 and obtain that the contribution of this ball to the asymptotics does not exceed $h^{l'}$ with l' given by (12.6.38).

Surely we only treated the case of a bounded function B in Chapter 9. However, it follows from (12.6.3) that on $\mathscr{C}_0^m\big(B(0,2r_*),\mathbb{K}\big)$ operator A satisfies inequalities

$$A \ge -C_0 h^l, \qquad A + C_0 h^l \ge h^{l-2mq/(m+1)} L^* L \tag{12.6.39}$$

for any operator L of the same type as A but with m replaced by $\frac{q}{2}$, $q \le m$; then the routine estimates of Chapter 9 yield the estimate

$$\mathsf{Tr}\big(ME(-\infty,0)\big) \le h^{mq/(m+1)} \tag{12.6.40}$$

for $M = L^*L$ with M supported in $B(0, \frac{3}{2}r_*)$ and therefore the same estimate (12.6.40) holds for every operator M supported in $B(0, r_*)$ of the same type as A but with m replaced by q. □

Furthermore, let $X = \mathbb{R}^d$ and let conditions (12.6.2), (12.6.4) be fulfilled in $\mathbb{R}^d$ instead of $B(0,1)$. Applying the arguments of Chapter 9 one can easily prove estimate

$$\Big|\int \big(1-\psi\big(\frac{x}{r_*}\big)\big)\big(e^1_{(\vartheta)}(x,x,0) - e^0_{(\vartheta)}(x,x,0)\big)\,dx\Big| \le Ch^{-d+\vartheta+1} \tag{12.6.41}$$

provided $\psi = 1$ in $B(0, \frac{1}{3})$ [61] and $r \le \epsilon r^*$ where $r^* > 0$ is an arbitrarily small constant and $\epsilon > 0$ is sufficiently small.

Moreover, the local spectral asymptotics of Chapter 4 applied in the zone $\{1 \ge |x| \ge \frac{1}{4} r\}$ yield that

Proposition 12.6.7. *Let microhyperbolicity condition* (12.6.34) *be fulfilled for operator* $A = A^0$ [62]. *Then estimate*

$$\Big|\int \mathsf{tr}\big(\psi\big(\frac{x}{r^*}\big) - \psi\big(\frac{x}{r}\big)\big)\big(e^{\theta}_{(\vartheta)}(x,x,0) - \sum_{0\le j\le \lfloor \vartheta\rfloor +1} \kappa^{\theta}_{(\vartheta)j} h^{-d+j}\big)\, dx\Big| \le C R^*_2(r,h) \tag{12.6.42}$$

holds with

$$R^*_2(r,h) = \begin{cases} h^{-d+\vartheta+1} r^{m\mu\vartheta + (d-\vartheta-1)(\mu+1)} & \text{for } \vartheta > \bar{\vartheta}_\nu, \\ h^{-d+\vartheta+1} \log r & \text{for } \vartheta = \bar{\vartheta}_\nu. \end{cases} \tag{12.6.43}$$

Then (12.6.35)–(12.6.36), (12.6.41) and (12.6.42)–(12.6.43) yield estimate

$$\Big|\int \psi\big(\frac{x}{r^*}\big)\, \mathsf{tr}\Big(e^1_{(\vartheta)}(x,x,0) - e^0_{(\vartheta)}(x,x,0) - \sum_{0\le j\le \lfloor \vartheta\rfloor +1} \big(\kappa^1_{(\vartheta)j}(x) - \kappa^0_{(\vartheta)j}(x)\big)\phi\big(\frac{x}{r_*}\big) h^{-d+j}\Big)\, dx\Big| \le C R^*_1(r,h) + C R^*_2(r,h) \tag{12.6.44}$$

with R^*_1, R^*_2 given by (12.6.36) and (12.6.43).

Further, Proposition 12.6.4 yields that one can skip the assumptions that $X = \mathbb{R}^d$, that condition (12.6.2) is fulfilled in X instead of $B(0,1)$ and that the negative spectrum of A is discrete. In fact, it is easy to construct an appropriate extension of A to $\mathbb{R}^d$.

Furthermore, let $\bar{A}^0$ coincide with A^0 in $B(0, \frac{1}{2} r)$ and be self-adjoint. Then the arguments of Chapter 9 easily yield that

$$\Big|\int \psi\big(\frac{x}{r}\big)\, \mathsf{tr}\big(e^0_{(\vartheta)}(x,x,0) - \bar{e}^0_{(\vartheta)}(x,x,0)\big)\, dx\Big| \le C R^*_1. \tag{12.6.45}$$

---

[61] This condition is also assumed to be fulfilled below.

[62] Then it is also fulfilled for $A = A^\eta$ for $r \le r^*$.

Finally, in this case $\kappa^0_{(\vartheta)j}(x) = \bar\kappa^0_{(\vartheta)j}(x)$ in a neighborhood of $\mathsf{supp}(\psi(x/r))$. Therefore, one can skip condition (12.6.31) on the support of B in the deduction of estimate (12.6.44).

Therefore we conclude that

(12.6.46) Estimate (12.6.44) remains true under the following weakened assumptions: A^0 is an operator in $\mathbb{R}^d$ satisfying conditions of Proposition 12.6.3, A^1 is an operator in the domain $X \supset B(0,1)$ satisfying conditions (12.6.6) and $b_\alpha = a_\alpha - a^0_\alpha$ satisfying (12.6.30).

Here and below we skip the superscript "1".

Again applying estimates (12.6.42)–(12.6.43) we obtain that

(12.6.47) Estimate (12.6.44) remains true under weakened assumptions if we replace r by r_* in its left-hand expression. Moreover, dividing the left-hand expression into two parts (associated with A, $\kappa_{(\vartheta)j}$ and with A^0, $\kappa^0_{(\vartheta)j}$) we see that in the second part one can replace r by arbitrary $r' > r$; moreover, one can even pick $r' = +\infty$, i.e., replace ψ by 1 here.

Minimizing the right-hand expression by $r \in [r_*, \epsilon r^*]$ (the left-hand expression surely does not depend on r and the optimal value is $r \asymp \bar r = h^{1/(\mu+\nu+1)}$) we obtain estimate

$$
\begin{aligned}
(12.6.48)\quad & \Big|\int \psi\big(\frac{x}{r^*}\big)\,\mathsf{tr}\Big(e_{(\vartheta)}(x,x,0) - \sum_{0\le j\le \lfloor\vartheta\rfloor+1} \kappa_{(\vartheta)j}\phi\big(\frac{x}{r_*}\big)h^{-d+j}\Big)\,dx - \\
& \int \psi\big(\frac{x}{r_2}\big)\,\mathsf{tr}\Big(e^0_{(\vartheta)}(x,x,0) - \sum_{0\le j\le \lfloor\vartheta\rfloor+1} \kappa^0_{(\vartheta)j}\phi\big(\frac{x}{r_*}\big)h^{-d+j}\Big)\,dx\Big| \le CR^*
\end{aligned}
$$

with

$$
(12.6.49)\qquad R^* = R^*(h) = \begin{cases} h^{-d+\vartheta+1}|\log h| & \text{for } \vartheta = \bar\vartheta_0,\\ h^{l''} & \text{for } \bar\vartheta_0 < \vartheta < \bar\vartheta_\nu,\\ h^{l'}|\log h| & \text{for } \vartheta = \bar\vartheta_\nu,\\ h^{l'} & \text{for } \vartheta > \bar\vartheta_\nu \end{cases}
$$

where

$$
(12.6.50)\qquad l'' = \big(m\mu\vartheta - (d-\vartheta-1)\nu\big)(\mu+\nu+1)^{-1}.
$$

One can easily see that $l < l'' < l'$ for $\bar{\vartheta}_0 < \vartheta \le \bar{\vartheta}_\nu$, $l'' = l = -d + \vartheta + 1$ for $\vartheta = \bar{\vartheta}_0$, $l'' = l'$ for $\vartheta = \bar{\vartheta}_\nu$.

In virtue of Proposition 12.6.3 with $J = \lfloor \vartheta \rfloor + 1$ the second line in the left-hand expression of (12.6.48) is equal to $\omega h^{l''}$; picking up $r_2 = +\infty$ we arrive to asymptotics

$$\int \psi(x)\,\mathsf{tr}\big(e_{(\vartheta)}(x,x,0)\big)\,dx = \sum_{0\le j\le \lfloor \vartheta\rfloor+1} h^{-d+j}\int \psi(x)\kappa_{(\vartheta)j}(x)\phi\big(\frac{x}{r_*}\big)\,dx + \omega h^{l''} + R^* \tag{12.6.51}$$

with R^* given by (12.6.49) and $\psi \in \mathscr{C}_0^\infty$, $\psi = 1$ in the vicinity of 0.

Therefore, we have proven the following

Theorem 12.6.8. *Let the operator A satisfy conditions* (12.6.2)–(12.6.6). *Furthermore, let $b_\alpha = a_\alpha - a_\alpha^0$ satisfy* (12.6.30) *with (matrix-valued) coefficients a_α^0 which are positively homogeneous of degrees $(m - |\alpha|)\mu$. Let $\vartheta \ge \max(\bar{\vartheta}_0, 1)$.*

Then asymptotics (12.6.51) *holds with Weyl coefficients $\kappa_{(\vartheta)j}(x)$ and some (non-Weyl) coefficient ω and with the remainder estimate given by* (12.6.49).

Recall that $r_* = h^{(\mu+1)^{-1}}$ and that in the terms with $j < \vartheta + 1$ one can replace ϕ by 1 (changing ω).

Let us note that here we gained nothing for $\vartheta = \bar{\vartheta}_0$.

12.6.4 Adjustment for $\vartheta < 1$

Let us consider the case $\vartheta < 1$ when one should modify the above arguments. Let us pick $\tau = 0$. Moreover, let us consider a partition unity on $\mathbb{R}$ corresponding to the scaling function $\frac{1}{2}(|\tau| + L)$ where we choose parameter L later. Then $\psi(x)e_\vartheta(x,x,0)$ is equal to a sum of terms of the form

$$\vartheta \iint \lambda_-^{\vartheta-1}\varphi\big(\frac{\lambda}{\ell}\big)\psi(x)e(x,x,\lambda)\,d\lambda dx \tag{12.6.52}$$

with <u>either</u> $\varphi = \varphi_0 \in \mathscr{C}_0^K([-1,1])$ and $\ell = L$ <u>or</u> $\varphi = \varphi_1 \in \mathscr{C}_0^K([\frac{1}{2},1])$ and $\ell \ge L$.

We apply this decomposition (12.3.54) *only* with $\psi(x/r)$ instead of ψ where we will choose $r \in [r_*, 1]$ later. Let us apply the arguments of the

previous subsection which are not associated with Proposition 12.6.5 (and therefore do not require the assumption $\vartheta \ge 1$). In particular, let us apply estimates (12.6.41) and (12.6.42) [63]. We then obtain that one needs to evaluate

$$\frac{1}{\Gamma(\vartheta)} \iint \lambda_-^{\vartheta-1} \varphi_0(\frac{\lambda}{\ell}) \psi(\frac{x}{r}) \operatorname{tr}\bigl(e^{\theta}(x,x,\lambda)\bigr)\, d\lambda dx \tag{12.6.53}$$

and

$$\frac{1}{\Gamma(\vartheta)} \iint \lambda_-^{\vartheta-1} \varphi_1\bigl(\frac{\lambda}{\ell}\bigr) \operatorname{tr}\bigl(e(x,x,\lambda) - e^{0}(x,x,\lambda)\bigr)\, d\lambda dx \tag{12.6.54}$$

where $\theta = 0, 1$, the error $O(R_2^*)$ (which is $= O\bigl(h^{-d+1+\vartheta} r^{m\mu\vartheta+(d-\vartheta-1)(\mu+1)}\bigr)$ in this case) is already accounted for, R_2^* is given by (12.6.43) and without any loss of the generality one can assume that conditions (12.6.2), (12.6.4) are fulfilled in $\mathbb{R}^d$.

Applying standard Weyl asymptotics to (12.6.53) one gets the remainder estimate $O\bigl(L^{\vartheta} h_{\mathrm{eff}}^{1-d}\bigr)$. On the other hand, one can apply the arguments of the proof of Proposition 12.6.5 to (12.6.54) and represent it in the form

$$\int_0^1 dt \iint \bigl(\partial_\tau^2 \lambda_-^{\vartheta} \varphi_1\bigl(\frac{\lambda}{\ell}\bigr)\bigr) \operatorname{tr}\bigl(B_x e^{\eta}\bigr)(x,x,\lambda)\, d\lambda dx. \tag{12.6.55}$$

Let us apply Weyl asymptotics to this expression. We obtain then the remainder estimate $O\bigl(r^{m\mu+\nu} \ell^{\vartheta-1} h_{\mathrm{eff}}^{1-d}\bigr)$ provided

$$m\mu + \nu + (d-1)(\mu+1) > 0. \tag{12.6.56}$$

After summation on the partition of unity we obtain that the total contribution of the terms of (12.6.54) type to the remainder is $O\bigl(r^{m\mu+\nu} L^{\vartheta-1} h_{\mathrm{eff}}^{1-d}\bigr)$ while the contribution of (12.6.53) is $O\bigl(L^{\vartheta} h_{\mathrm{eff}}^{1-d}\bigr)$. Minimizing the sum with respect to L (the optimal value is $L \asymp r^{m\mu+\nu}$) we obtain

$$O\bigl(h^{1-d} r^{(m\mu+1)\vartheta+(\mu+1)(d-1)}\bigr). \tag{12.6.57}$$

Recall that there is also an error $O\bigl(h^{-d+1+\vartheta} r^{m\mu\vartheta+(d-\vartheta-1)(\mu+1)}\bigr)$. Minimizing their sum over r (the optimal value is again $\asymp \bar{r} = h^{1/(\mu+\nu+1)}$) we obtain the final remainder estimate $O\bigl(h^{l''}\bigr)$ where l'' is given by (12.6.50).

So, the following statement has been proven:

---

[63] In order to get these with the convolutions, including the cut-off function φ, one should apply Theorem 4.4.1.

Theorem 12.6.9. *Let all the conditions of Theorem 12.6.8 be fulfilled excluding $\vartheta \geq 1$. Let $\bar{\vartheta}_0 < \vartheta < 1$ and condition* (12.6.56) *be fulfilled (which means exactly that $\bar{\vartheta}_\nu > 1$).*

Then asymptotics (12.6.51) *remains true with the Weyl coefficients $\kappa_{\vartheta j}(x)$ and some (non-Weyl) coefficient ω and with the remainder estimate given by* (12.6.49).

Remark 12.6.10. Surely, even if condition (12.6.56) is violated one can get semiclassical asymptotics for (12.6.55) but the remainder estimate will be worse and the final remainder estimate will be also worse. However, this condition is only technical, and we are going to get rid of it without deterioration of the remainder estimate in the general approach.

12.6.5 The General Case

Let us pick up $\bar{r} = h^{(1+\mu+\nu)^{-1}}$ and then apply the same arguments as before in the zone $\{\bar{r} \leq |x| \leq 1\}$.

Then

(12.6.58) The contribution of this zone $\{\bar{r} \leq |x| \leq 1\}$ to the remainder is $O\bigl(R_2^*(\bar{r})\bigr)$ as before.

Let us consider $B(0, \bar{r})$ and apply the same partition of unity here. Thus one should consider the same expression

$$\frac{1}{\Gamma(\vartheta+1)} \int \tau_-^{\vartheta} \mathsf{Tr}\bigl(\psi\, d_\tau E(\tau)\bigr) \tag{12.6.59}$$

as before. We skip all the subscripts at ψ. Let us define $r = \frac{1}{2}(|x| + r_*)$ at some point of $\mathsf{supp}(\psi)$ and let us pick up

$$L = L(r) = hr^{m\mu+\nu-\mu-1} \tag{12.6.60}$$

for this element. Let us consider the same partition on $\mathbb{R}$ as before and let us replace $\mathsf{Tr}\,\psi\, d_\tau E(\tau)$ with its Weyl expression (with cut-off for $|x| \leq r_*$). Then the contribution of the element with $\ell \leq 2L$ to the remainder does not exceed $CR(r,h)$ with

$$R(r,h) = h^{-d+1} r^{(d-1)(\mu+1)} L^{\vartheta} = h^{-d+1+\vartheta} r^{(m\mu+\nu)\vartheta+(d-1-\vartheta)(\mu+1)}. \tag{12.6.61}$$

Summation (12.6.61) over the partition of unity on x results in the same expression with $r = r_*$ or $r = \bar{r}$ as $(m\mu+\nu)\vartheta + (d-1-\vartheta)(\mu+1) \lessgtr 0$ with an extra factor $\log(\bar{r}/r_*)$ in the case of equality; so we get

$$R^*(h) = \begin{cases} h^{l''} & \text{for } (m\mu+\nu)\vartheta + (d-1)(\mu+1) > 0 \\ h^{l'} \log h & \text{for } (m\mu+\nu)\vartheta + (d-1)(\mu+1) = 0 \\ h^{l'} & \text{for } (m\mu+\nu)\vartheta + (d-1)(\mu+1) < 0 \end{cases} \tag{12.6.62}$$

where

$$l'' = \big((m\mu+\nu)\vartheta - (d-1)\nu\big)(\mu+\nu+1)^{-1}, \tag{12.6.63}$$

$$l' = (m\mu+\nu)\vartheta(\mu+1)^{-1}. \tag{12.6.64}$$

Moreover, the contribution of $B(0,3r_*)$ to the remainder does not exceed $Ch^{l''}$.

Let us consider elements with $\ell \geq L$. Then one can apply the same arguments as before with k differentiations instead of 1 and easily get the following generalization of (12.6.27):

$$\begin{aligned} &\int \varphi(\tau)\tau_-^{\vartheta}\, d_\tau E^1(\tau) = \\ &\sum_{0\leq j\leq k-1} \int \varphi_{\vartheta,j}(\lambda_1,\dots,\lambda_{j+1}) E^0(\lambda_1) B E^0(\lambda_2)\cdots B E^0(\lambda_{j+1})\, d\lambda_1\cdots d\lambda_{j+1} + \\ &\int_0^1 (k+1)(1-\theta)^k\, d\theta \times \\ &\qquad \int \varphi_{\vartheta,k+1}(\lambda_1,\dots,\lambda_{k+1}) E^\theta(\lambda_1) B E^\theta(\lambda_2)\cdots B E^\theta(\lambda_{k+1})\, d\lambda_1\cdots d\lambda_{k+1} \end{aligned} \tag{12.6.65}$$

where $E^\theta(\lambda)$ is the spectral projector of A^θ and

$$\varphi_{\vartheta,j} = (-1)^{j+1} \int \delta(\tau-\lambda_1)\cdots\delta(\tau-\lambda_j)\Big(\partial_\tau^{j+1}\big(\tau_-^{\vartheta}\varphi(\tau)\big)\Big)\, d\tau, \tag{12.6.66}$$

understood as a distribution on $(\lambda_1,\dots,\lambda_j)$, is in fact a smooth function such that

$$\operatorname{supp}(\varphi_{j,\vartheta}) \subset [\frac{1}{2}\ell, 2\ell]^j, \qquad |D_\lambda^\beta \varphi_{\vartheta,j}| \leq c\ell^{\vartheta+1-2j-|\beta|}. \tag{12.6.67}$$

Let us multiply by ψ and take the trace. Our goal is to evaluate the result.

First of all we are going to evaluate the elements with $r \geq 2r_*$. For this purpose let us return to Chapters 4 and 7. Under the standard assumptions of those chapters let us consider the following problems:

$$(12.6.68)_0 \qquad \big(hD_t - A\big)U_0(t) = 0, \qquad U_0(0) = I$$

$$(12.6.68)_j \qquad \big(hD_t - A\big)U_j(t) = B_j U_{j-1}(t), \qquad U_j(0) = 0, \qquad j \geq 1$$

where the operators B_j are also regular h-pseudodifferential operators of order 0. Then the results of Chapters 2 and 3 concerning the oscillatory front set of $U = U_0$ obviously remain true for U_j as well.

In particular, we get[64)]

(12.6.69) Under the microhyperbolicity condition for the operator A on the energy level 0 (with the boundary conditions, if there is a boundary), $\varphi(hD_t)\chi_T(t)\big(\Gamma\psi U_j\big)$ is negligible for $\varphi \in \mathscr{C}_0^K$ supported in a small neighborhood of 0, $\chi \in \mathscr{C}_0^K\big([-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]\big)$ and a sufficiently small constant $T > 0$.

Applying rescaling-decomposition arguments one can prove easily that[65)]

(12.6.70) Estimate (2.1.58) remains true for $U = U_j$ as well

On the other hand, one can apply the successive approximation method in the interval $|t| \leq h^{\frac{1}{2}+\delta}$. Let us note that for $B_1 = B_2 = \cdots = B_j = B$,

$$\varphi(hD_t)\Gamma\,\mathsf{tr}\big(\psi U_j(.,t)\big)\big|_{t=0}$$

is exactly the term to be evaluated (for $j = k+1$ one should integrate over the parameter θ, and A depends on this parameter.

Thus for $r \asymp 1$ the remainder estimate (after we replace the whole thing by its Weyl expression) is $Ch^{1-d}\ell^{\vartheta}\big(1 + \ell/h\big)^{-s}$ where we have taken into account condition (12.6.67) on φ.

Remark 12.6.11. The procedure to get the Weyl approximation is obvious: one replaces $d_\tau E(\tau)$ by the formal parametrix $(\tau - A)^{-1}$ of the operator A (with the corresponding boundary conditions), then takes $A = A^\theta = A_0 + \theta B$ and calculates the j-th derivative with respect to θ. Afterward one applies $\mathsf{Res}_{\mathbb{R}}$ and formally calculates the trace of this operator.

---

64) Cf. Corollary 2.1.17 and Theorem 3.3.5.

65) Cf. Theorems 2.1.19 and 3.3.6.

Rescaling, we obtain that for $r \ge r_*$ the contribution to the remainder estimate is

$$C\ell^{\vartheta-j}r^{(m\mu+\nu)j}\bigl(\frac{h}{r^{\mu+1}}\bigr)^{1-d+\vartheta-j}\bigl(1+\frac{\ell r^{\mu+1-m\mu}}{h}\bigr)^{-s}. \tag{12.6.71}$$

For $j > \vartheta$ we obtain that the sum with respect to $\ell \ge L(r)$ does not exceed this expression with $\ell = L(r)$ which is equal to $CR(r,h)$ with $R(r,h)$ given by (12.6.61) and after summation over the partition on r we get $CR^*(h)$ with $R^*(h)$ given by (12.6.62). *Let us pick* $k > \vartheta$.

In order to treat $j \le \vartheta$ let us replace $\sum_{r\le \bar r}\sum_{\ell\ge L(r)}$ by $\sum_{\ell\ge L^*}\sum_{r\le r(L)}$ where $L^* = L(r_*) = h^{m\mu(\mu+1)^{-1}}$ and

$$r(\ell) = \bigl(\ell/h\bigr)^{1/(m\mu+\nu-\mu-1)} \tag{12.6.72}$$

inverse function to $L(r)$ defined by (12.6.60). Let us now consider the sum on r only.

(i) For

$$f(j) := m\mu\vartheta + (\mu+1)(d-1-\vartheta) + (\mu+\nu+1)j > 0 \tag{12.6.73}$$

we replace the corresponding term in the partition of the right-hand expressions of (12.6.65) by its Weyl approximations. Then the error will not exceed $CR'(\ell)$ with $R'(\ell) = R\bigl(r(\ell)\bigr)$. After summation on $\ell \ge L^*$ one again obtains $CR^*(h)$.

(ii) Observe that for $f(j) = 0$ one gets the same answer with the additional factor $|\log h|$ in the right-hand expression. Thus, we need to replace $R^*(h)$ by

$$R^{**}(h) = \begin{cases} h^{l'''} & \text{for } (m\mu+\nu)\vartheta + (d-1)(\mu+1) > 0,\\ & \qquad \vartheta - (\mu+1)(d-1)(\mu+\nu+1)^{-1} \notin \mathbb{Z}^+,\\ h^{l'''}\,|\log h| & \text{for } (m\mu+\nu)\vartheta + (d-1)(\mu+1) > 0,\\ & \qquad \vartheta - (\mu+1)(d-1)(\mu+\nu+1)^{-1} \in \mathbb{Z}^+,\\ h^{l'}\,|\log h| & \text{for } (m\mu+\nu)\vartheta + (d-1)(\mu+1) = 0,\\ h^{l'} & \text{for } (m\mu+\nu)\vartheta + (d-1)(\mu+1) < 0. \end{cases} \tag{12.6.74}$$

(iii) Let us consider the terms with $f(j) < 0$. We replace them by their Weyl expressions as well but treat these terms in a different way. First, we consider the difference between the terms and their Weyl approximations on each element of the partition of unity, but we *do not estimate it by* (12.6.71).

Instead we sum this difference with respect to $r \le r(\ell)$, then we replace $\sum_{r\le r(\ell)}$ by $\sum_{r<\infty} - \sum_{r>r(\ell)}$ and only then for $r > r(\ell)$ let us apply the above remainder estimate (12.6.71) and get the same answer as before. However we also get $\sum_{r<\infty}$.

One can easily see that the contribution of $B(0, r_*) \times [0, 3L^*]$ to the original expression does not exceed $Ch^{l''}$.

Later we will prove that

(12.6.75) The contribution of $B(0, 2r_*)$ to terms with $f(j) > 0$ does not exceed $Ch^{l''}$ as well.

Then modulo $O\big(R^{**}(h)\big)$ our original expression is equal to

$$\mathsf{Weyl} + \sum_{j:f(j)<0} c_j \iint \tau_-^{\vartheta-1} \operatorname{tr}\big(\mathcal{K}_{(B(\tau-A^0)^{-1})^j}(x,x) - \mathsf{Weyl}_j(x,\tau)\big)\, dxd\tau \tag{12.6.76}$$

where the first Weyl is some combination of the Weyl expressions and Weyl_j is the Weyl approximation for $\mathcal{K}_{(B(\tau-A^0)^{-1})^j}(x,x)$ where $\mathcal{K}_S$ denotes the Schwartz kernel of operator S and c_j are some numerical coefficients; these terms are coming from $\sum_{r<\infty}$.

Recall that all these Weyl expressions contain the factor $\big(1-\phi(x/r_*)\big)$ cutting them in $B(0, r_*)$.

Now, let us assume for the sake of simplicity that

(12.6.77) Operator B is quasihomogeneous; i.e., $B(\zeta x, \zeta^\mu \xi) = \zeta^{m\mu+\nu} B(x,\xi)$ $\forall x, \xi, \zeta > 0$.

Later we will get rid of this condition with the possible deterioration of the remainder estimate, and it is not necessary from the very beginning if $f(1) \ge 0$.

Under this condition (12.6.77) let us consider terms with $f(j) < 0$ and with integration on $x \in \mathbb{R}^d$. Then same homogeneity arguments as in Proposition 12.6.3 imply that

(12.6.78) The terms with $f(j) < 0$ in (12.6.76) are equal to $\omega_{(\vartheta)j} h^{(m\mu\vartheta+\nu j)(\mu+1)^{-1}}$ with unknown numerical coefficients $\omega_{(\vartheta)j}$.

We will call them *Scott correction terms*. Note that

$$f(j) < 0 \iff (m\mu\vartheta + \nu j)(\mu+1)^{-1} < \min(l', l''). \tag{12.6.79}$$

Further, one can easily see that modulo $O\big(R^{**}(h)\big)$ the first Weyl is equal to

$$\begin{aligned} &\sum_{p<d+\min(l',l'')} \kappa_{(\vartheta)p} h^{-d+p} + \\ &\sum_{p<d+\min(l',l''),j:-d+p=(m\mu\vartheta+\nu j)/(\mu+1)} \kappa'_{(\vartheta)p} h^{-d+p} |\log h| \end{aligned} \tag{12.6.80}$$

plus the sum of the "superficial" terms $c_{(\vartheta)p,j} h^{-d+p+\nu j} \bar{r}^{\alpha_{p,j}}$ with exponents $\alpha_{p,j} := (\mu+1)(d-p-\nu j) \neq 0$ and $c_{(\vartheta)p,j} h^{-d+p+\nu j} \log \bar{r}$ with $\alpha_{p,j} = 0$.

Here $\kappa_{(\vartheta)p}$ are standard Weyl coefficients (one can easily see that the integrals defining them converge (at 0) for $p < d + m\mu\vartheta 1/(\mu+1)$. Taking into account the fact that one can pick any $\bar{r} \asymp h^{(\mu+\nu+1)^{-1}}$ and the final answer should not depend on $\bar{r}$ we see that all these coefficients $c_{(\vartheta)p,j}$ should vanish and only (12.6.80) survives.

Therefore (12.6.76) modulo $O\big(R^{**}(h)\big)$ is equal to (12.6.80) plus sum of the terms from (12.6.78):

$$\begin{aligned} &\sum_{p<d+\min(l',l'')} \kappa_{(\vartheta)p} h^{-d+p} + \\ &\sum_{p<d+\min(l',l''),j:-d+p=(m\mu\vartheta+\nu j)/(\mu+1)} \kappa'_{(\vartheta)p} h^{-d+p} |\log h| + \\ &\sum_{j:(m\mu\vartheta+\nu j)/(\mu+1)<\min(l',l'')} \omega_{(\vartheta)j} h^{(m\mu\vartheta+\nu j)/(\mu+1)} \end{aligned} \tag{12.6.81}$$

where $\omega_j h^{(m\mu\vartheta+\nu j)/(\mu+1)}$ are (multiple) Scott correction terms. More precisely, defining

$$\bar{\vartheta}_J := \max\Big(\big((\mu+1)(d-1) + (\mu+\nu+1)J\big)(\mu+1-m\mu)^{-1}, J\Big) \tag{12.6.82}$$

we see that

(12.6.83) For $\bar\vartheta_{J-1} < \vartheta \le \bar\vartheta_J$ there are exactly J Scott correction terms ($j = 1, \dots$)

To prove pending claim (12.6.75) let us now consider an element of the partition supported in $B(0, 2r_*)$. Let us replace every factor B by $B_1 = \bar\psi B$ where the cut-off function $\bar\psi$ is supported in $B(0, 4r_*)$ and equal to 1 in $B(0, r_*)$. We will explain how to get rid of this cut-off function later.

Let us note that

(12.6.84) For $\tau \le 0$ the operator norm of $\bar\psi B E^\theta(\tau)$ does not exceed $r_*^{m\mu+\nu}$ while the trace norm of $\bar\psi E^\theta(\tau)$ does not exceed C [66)].

One can prove this easily for $h = r_* = 1$ and then extend to the general case by rescaling (recall that $r_*^{\mu+1} = h$).

Then the trace norm of the operator

$$\int \varphi_{\vartheta,k+1}(\lambda_1, \dots, \lambda_j)\psi E^\theta(\lambda_1) B_1 E^\theta(\lambda_2) \cdots B_1 E^\theta(\lambda_{k+1})\, d\lambda_1 \cdots d\lambda_{k+1}$$

does not exceed $\ell^{\vartheta-k} r_*^{(m\mu+\nu)k}$ and after summation on the partition we get the same expression with $\ell = L = r_*^{m\mu+\nu}$ provided $k > \vartheta$. Namely, we get $r_*^{(m\mu+\nu)\vartheta} = h^{(m\mu+\nu)\vartheta/(\mu+1)}$.

Finally, we should consider similar expressions with some copies of B replaced by $\bar B$ and other copies replaced by $\psi' B$ where ψ' is supported outside of $B(0, 3r_*)$. Taking partitions one can replace the i-th copy of ψ' by ψ_i supported in $\{\frac{1}{2} r_i \le |x| \le 2r_i\}$. Combining the above arguments one can estimate this term through $r_*^{(m\mu+\nu)\vartheta} \prod_i \big(r_*/r_i\big)^s$ and the total contribution will not exceed $r_*^{(m\mu+\nu)\vartheta}$ again.

Thus, all the estimates are done. Therefore the first statement of the following theorem has been proven:

Theorem 12.6.12. *Let the conditions of Theorem 12.6.8 be fulfilled.*

[66)] One can add the factor $\big(|\tau| r_*^{-m\mu} + 1\big)^{-s}$ with arbitrarily large s to the right side of these estimates but it is not important here.

(i) Moreover, let condition (12.6.77) *be fulfilled. Then asymptotics*

$$\int \psi(x) e_{(\vartheta)}(x,x,\tau)\,dx = \sum_{p:p<d+\min(l',l'')} \kappa_{(\vartheta)p} h^{-d+p} + \sum_{p<d+\min(l',l''),j:-d+p=(m\mu\vartheta+\nu j)/(\mu+1)} \kappa'_{(\vartheta)p} h^{-d+p}|\log h| + \sum_{j:(m\mu\vartheta+\nu j)/(\mu+1)<\min(l',l'')} \omega_{(\vartheta)j} h^{(m\mu\vartheta+\nu j)/(\mu+1)} + O\big(R^{**}(h)\big) \tag{12.6.85}$$

holds with $R^{**}(h)$ *given by* (12.6.74).

(ii) Moreover, for $\vartheta \le \bar{\vartheta}_1$ *one can drop condition* (12.6.77).

(iii) Let us assume that

$$\big|D^\beta\big(a_\alpha - a^0_\alpha - b^0_\alpha\big)\big| \le c|x|^{(m-|\alpha|)\mu+\nu'-|\beta|} \tag{12.6.86}$$

where a^0_α, b^0_α *are positively homogeneous of degrees* $(m-|\alpha|)\mu$, $(m-|\alpha|)\mu+\nu$ *respectively,* $\nu' > \nu$.

Then asymptotics (12.6.85) *holds with* $R^{**}(h)$ *replaced by* $R^{**}(h)+R^{*\prime}(h)$ *where* $R^{*\prime}(h)$ *is given by* ((12.6.49) *with* ν' *instead of* ν.

The proof of Statement (iii) is simple: we first make the transition from A to A^0+B^0 (then the remainder estimate $O\big(R^*(h)\big)$ arises) and then apply Statement (i).

We believe that it would not be difficult to

Problem 12.6.13. Generalize this theorem to the case when

$$D^\beta\big(b_\alpha - b^1_\alpha - ... - b^s_\alpha\big) = O\big(|x|^{(m-\alpha)\mu+\nu_{s+1}-|\beta|}\big)$$

and b^k_α are positively homogeneous of degrees $(m-|\alpha|)\mu+\nu_k$.

12.6.6 Propagation of Singularities

Now our goal is to improve the remainder estimate, which requires some condition of a global nature on the symbol $A^0(x,\xi)$. To formulate the

condition the reader must recall the definitions of generalized bicharacteristics and generalized δ-bicharacteristics (see Definition 2.2.8 or similar notion in Chapter 3); otherwise we should assume that the eigenvalues of $A^0(x,\xi)$ which are close to 0 have constant multiplicities. Moreover, the reader should remember quasiconical sets and the quasiconical hull of a set; these notions are associated with the rescaling family $\Phi_\zeta : (x,\xi) \to (\zeta x, \zeta^\mu \xi)$ with $\zeta > 0$.

Definition 12.6.14. We say that the *escape condition* is fulfilled if there exist quasiconical sets Λ^+ and Λ^- closed in $(\mathbb{R}^d \setminus 0) \times \mathbb{R}^d$ and numbers $T_0 > 0$, $\epsilon_1 > 0$ such that

$$\Lambda^+ \cup \Lambda^- = \Sigma^0 = \bigl\{(x,\xi) \in (\mathbb{R}^d \setminus 0) \times \mathbb{R}^d : \det A^0(x,\xi) = 0\bigr\} \tag{12.6.87}$$

and for every point $(\bar x, \bar\xi)$ belonging to the ϵ_1-neighborhood of $\Lambda^\pm$ intersected with $\bigl\{(x,\xi) : |x| = 1\bigr\} \cap \Sigma^0$ all the generalized bicharacteristics of the symbol $-A^0(x,\xi)$ passing through $(\bar x, \bar\xi)$ in the direction of $\pm t > 0$ are infinitely long with respect to t, and along them

$$(x(t),\xi(t)) \in \Lambda^\pm, \quad |x(t)| \ge 2|\bar x| \qquad \forall t : \pm t \ge T_0; \tag{12.6.88}$$

these conditions should be fulfilled for each sign "+" and "−" separately.

We discuss this condition below; here we make certain corollaries and remarks concerning it.

Remark 12.6.15. (i) Condition (12.6.4) yields that $\Sigma^0 \cap \{|x| = 1\}$ is compact.

(ii) Under condition (12.6.88) for each generalized bicharacteristics $(x(t),\xi(t))$ with $(x(0),\xi(0)) \in \Lambda^\pm_{\epsilon_1}$ for $\pm t > 0$ inequalities

$$\epsilon_0 \le |x(t)| \cdot \bigl(|x(0)| + |t|^\sigma\bigr)^{-1} \le C_0 \tag{12.6.89}$$

hold with

$$\sigma^{-1} := \sigma' := 1 + \mu - m\mu > 0 \tag{12.6.90}$$

(otherwise $\bar\vartheta_0 = +\infty$) and with appropriate constants ϵ_0, C_0 which do not depend on t and $(x(0),\xi(0))$; here and below $\Lambda^\pm_\varepsilon$ denotes the intersection of the quasiconical hull of ε-neighborhood of $\Lambda^\pm \cap \{|x| + 1\}$ with Σ^0.

In fact, let us pick t_n: $t_0 = 0$, $t_{n+1} = t_n + T_0 r_n^{\sigma'}$ with $r_n = |x(t_n)|$. Then condition (12.6.88) and the quasihomogeneity of A^0 and the quasiconic nature of Λ^+ yield that $2r_n \le r_{n+1} \le c_0 r_n$ (we have taken $(x(0),\xi(0)) \in \Lambda^+$ for notational simplicity).

Therefore $r_n \to +\infty$ as $n \to +\infty$. Then these inequalities and the definition of t_{n+1} obviously yield the inequality $r_{n+1} \le c_1 t_{n+1}^{\sigma}$ (and hence $t_n \to +\infty$ as $n \to +\infty$) and the inequality $t_{n+1} r_{n+1}^{-\sigma'} \le c_2 + (1-\epsilon) t_n r_n^{\sigma'}$ with $\epsilon > 0$. This inequality yields by induction that $t_n r_n^{-\sigma'} \le c_3$ with a sufficiently large constant c_3. So, both inequalities (12.6.89) are fulfilled for $t = t_n$ with $n = 0, 1, ...$ and therefore these inequalities are fulfilled for $t \in (t_n, t_{n+1}]$ as well.

(iii) Let us replace the symbol $A^0(x,\xi)$ with another symbol $A^1(x,\xi)$ also of the form (12.6.1) such that

$$\bigl|D^{\beta}(a^1_{\alpha} - a^0_{\alpha})\bigr| \le \varepsilon |x|^{\mu(m-|\alpha|)-|\beta|} \qquad \forall \alpha : |\alpha| \le m \quad \forall \beta : |\beta| \le K \tag{12.6.91}$$

with a small enough constant $\varepsilon > 0$. We *do not* assume that $A^1(x,\xi)$ is a quasihomogeneous symbol. Then

$$|\xi| \le c_3 |x|^{\mu} \qquad \text{on} \quad \Sigma^j \qquad j = 0, 1 \tag{12.6.92}$$

and the microhyperbolicity of A^0 on the energy level 0 yields that

$$\forall (x,\xi) \in \Sigma^0 \quad \exists (x',\xi') \in \Sigma^1 : \quad |x - x'| \le c_3 \varepsilon |x| \ \text{ and } \ |\xi - \xi'| \le c_4 \varepsilon |x|^{\mu} \tag{12.6.93}$$

and *vice versa* (one can permute Σ^0 and Σ^1 in this assertion), and c_4 does not depend on ε.

Let us introduce the sets

$$\Lambda_1^{\pm} = \bigl\{(x,\xi) \in \Sigma^1 : \exists (x',\xi') \in \Lambda^{\pm} : |x - x'| \le \epsilon |x| \ \text{ and } \ |\xi - \xi'| \le \epsilon |x|^{\mu}\bigr\} \tag{12.6.94}$$

with sufficiently small $\epsilon > 0$. Moreover, let us introduce the sets $\Lambda_2^{\pm}$ in the same way but with ϵ replaced 2ϵ.

Then the properties of the generalized bicharacteristics (see Lemma 2.2.9) and condition (12.6.88) yield that if $\epsilon > 0$ is a small enough constant and if $\varepsilon = \varepsilon(\epsilon) > 0$ is also small enough then every generalized bicharacteristics $(x(t), (\xi(t))$ of the symbol $A^1(x,\xi)$ passing through the point $(\bar{x}, \bar{\xi}) \in \Lambda_2^{\pm}$ in the direction of positive $\pm t$ can be extended to the time interval $\pm t \in [0,\, T_1 |\bar{x}|^{\sigma'}]$, and for $\pm t \in [2T_0 |\bar{x}|^{\sigma'},\, T_1 |\bar{x}|^{\sigma'}]$ along this bicharacteristics

$$(x(t), \xi(t)) \in \Lambda_1^{\pm}, \qquad |x(t)| \ge \frac{3}{2} |\bar{x}|; \tag{12.6.88}'$$

$T_1 > 2T_0$ is arbitrarily large and ϵ and ε depend on T_1.

Then the same arguments as in Statement (i) yield that along these trajectories inequalities (12.6.91) are fulfilled with constants c_0, ϵ_0 which do not depend on T_1. Then the arbitrary choice of T_1 yields that this statement remains true even for $T_1 = +\infty$.

(iv) The same analysis remains true if we assume that (12.6.91) is fulfilled in $B(0,r)$ instead of $\mathbb{R}^d$; surely, we should treat the trajectory only as long as it remains in $B(0,r)$. In this case (12.6.91) yields that all the statements remain true in the time interval $\pm t \in [0, \epsilon_2 r^{\sigma'}]$ with a small enough constant $\epsilon_2 > 0$ and $|\bar{x}| \le \frac{1}{2}r$.

(v) Therefore in the analysis of the trajectories on the energy level τ instead of $\tau = 0$ all the Statements (i)–(iii) remain true with the following modifications:

(a) For $\mu < 0$ one should replace r with $\min\bigl(r, \epsilon_2|\tau|^{(m\mu)^{-1}}\bigr)$, i.e., one should treat the starting points $(\bar{x}, \bar{\xi})$ with $|\bar{x}| \le \min\bigl(r, \epsilon_2|\tau|^{(m\mu)^{-1}}\bigr)$ and the time interval $\pm t \in \bigl[0, \epsilon_2 \min(r^{\sigma'}, |\tau|^{(m\mu\sigma)^{-1}})\bigr]$.

(b) For $\mu > 0$ one should leave both r and the time interval $\pm t \in [0, \epsilon_2 r^{\sigma'}]$ unchanged and one should treat the starting points $(\bar{x}, \bar{\xi})$ with $c_2|\tau|^{(m\mu)^{-1}} \le |\bar{x}| \le r$.

(c) rem-12-5-15-vc For $\mu = 0$ one should leave r and the time interval $\pm t \in [0, \epsilon_2 r^{\sigma'}]$ and the condition $|\bar{x}| \le r$ unchanged and one should only assume that $|\tau| \le \epsilon_2$ where $\epsilon_2 > 0$ is a small enough constant.

Remark 12.6.16. Further, let us consider a more complicated flow. Namely, let us assume that at the moments $\pm t_n = T_1^n |\bar{x}|^{\sigma'}$ with $n = 1, 2, \ldots$ the point $(x(t), \xi(t))$ makes random "jumps"; namely it jumps to an arbitrary point in its εr_n-neighborhood intersected with the set Ω; $\Omega = \Pi_{\tau_0}$ for $\mu \le 0$ and $\Omega = \Pi_{\tau_n}$ with $\tau_n = (r_n/r_*)^{m\mu}\tau_0$ for $\mu > 0$ where $\Pi_\zeta = \cup_{\tau:|\tau|\le\zeta} \Sigma_\tau^1$ and we assume that $\tau_0 > 0$.

So, we assume that the "jump" and the flow itself preserve the energy levels only approximately. This exotic flow will be useful later. We claim that for sufficiently small $\varepsilon > 0$ and for sufficiently large $T_1 > 0$, Statements (i)–(v) of Remark 12.6.15 remain true.

In fact, let $r_n = |x(\pm t_n \pm 0)|$. Then Remark 12.6.15 and the conditions on the jump at the end of a segment of the trajectory yield that if

$$0 < \epsilon_0 \le r_n t_n^{-\sigma} \le c_0 \tag{12.6.95}_n$$

then for sufficiently large $T_1 > 0$ depending on c_0, ϵ_0 estimates

$$r_n + \epsilon_1 t_{n+1}^{\sigma} \le r_{n+1} \le r_n + c_1 t_{n+1}^{\sigma}$$

hold with c_1, ϵ_1 which do not depend on c_0, ϵ_0. Then under an appropriate choice of the constants c_0, $\epsilon_0 > 0$ condition $(12.6.95)_n$ yields $(12.6.95)_{n+1}$ with the same constants c_0, ϵ_0; then induction on n yields (12.6.91).

Moreover, the energy level after a jump remains admissible.

Let us prove some assertions concerning the long-time propagation of singularities:

Proposition 12.6.17. *Let the operator A satisfy* (12.6.2)–(12.6.6). *Let $b_\alpha = a_\alpha - a_\alpha^0$ satisfy* (12.6.30) *and let a_α^0 be positively homogeneous functions of degrees $(m - |\alpha|)\mu$. Let symbol A^0 satisfy conditions* (12.6.20) *and escape condition* (12.6.87)–(12.6.88).

Further, let $r \in [r_* h^{-\delta}, \varepsilon_0]$ with $r_* = h^{(\mu+1)^{-1}}$, an arbitrarily small exponent $\delta > 0$ and a sufficiently small constant $\varepsilon_0 > 0$. Let $h' = hr^{-\mu-1}$, $\gamma = h'^{\rho}$ with an exponent $\rho \in (0, \frac{1}{2})$. Let $\tau_0 = \epsilon_3 r^{m\mu}\gamma$ for $\mu < 0$ and $\tau_0 = \epsilon_3 r^{m\mu}$ for $\mu \ge 0$.

Furthermore, let $Q \in \boldsymbol{\Psi}_{h,\gamma r^\mu,\gamma r}$ be an operator with the symbol supported in the intersection of the $\varepsilon(r, r^\mu)$-neighborhood (in (x,ξ)) of the set $\Lambda^{\pm} \cap \{\epsilon_0 r \le |x| \le c_0 r\}$ and Π_{τ_0}.

Finally, let

$$T'' = \begin{cases} \epsilon_3 \min(1, r^{\sigma'}\gamma^{(m\mu\sigma)^{-1}}) & \text{for } \mu < 0, \\ \epsilon_3 & \text{for } 0 \le \mu < (m-1)^{-1} \end{cases} \tag{12.6.96}$$

and $T' = c_3 r^{\sigma'}$ with small enough constants $\epsilon_3 > 0$ and $\varepsilon > 0$ and a large enough constant c_3. Let $\bar{T} \in [T', T'']$.

Then there exists an operator $\bar{Q} = \bar{Q}_{\bar{T}} \in \boldsymbol{\Psi}_{h,\bar{\gamma}\bar{r}^\mu,\bar{\gamma}\bar{r}}$ with the symbol supported in the $\varepsilon(\bar{r}, \bar{r}^\mu)$-neighborhood (in (x,ξ)) of the set $\Lambda^{\pm} \cap \{\bar{\epsilon}_0 r \le |x| \le \bar{c}_0 r\}$ with $\bar{r} = \bar{T}^{\sigma'}$, $\bar{h}' = h\bar{r}^{-\mu-1}$, $\bar{\gamma} = \bar{h}'^{\bar{\rho}}$ such that the equality

$$U(\pm\bar{T})Q \equiv \bar{Q}U(\pm\bar{T})Q \tag{12.6.97}$$

holds for the operator $U(t) = \exp(ih^{-1}tA)$ *modulo operators with operator norm less than* Ch^s *with arbitrarily large* s.

Moreover, constants $\bar\epsilon_0$, $\bar c_0$ *do not depend on* $\varepsilon > 0$ *(but the constant* c_3 *does) and one can choose the exponent* $\bar\rho > \rho$ *arbitrarily close to* ρ, *and the statement that the operator belongs to the operator class is uniform with respect to the parameters* $h, r, \bar T$.

Proof. Let us consider the sign "+" for definiteness. Let us pick $Q_0 = Q$ and define moments t_n according to Remark 12.6.16 with T_1 depending on ε. Let $r_n = t_n^\sigma$. Let us introduce γ_n, h'_n with r replaced by r_n and with ρ replaced by $\bar\rho$ as was done above. We want to construct operators $Q_n \in \mathbf{\Psi}_{h,\gamma_n r_n^\mu,\gamma_n r_n}$ with symbols supported in the $\varepsilon(r_n, r_n^\mu)$-neighborhoods of the sets $\Lambda \cap \{\bar\epsilon_0 r_n \le |x| \le \bar c_0 r_n\}$ intersected with Π_{τ_n} (where we define τ_n later) such that for all n with $t_n \le T''$ equalities

$$U(t_{n+1} - t_n)Q_n \equiv Q_{n+1}U(t_{n+1} - t_n)Q_n \tag{12.6.98}$$

are fulfilled, and all statements that operators belong to operator classes and all equivalence relations are uniform in h, r, n; here $t_0 = 0$.

Let us rescale $x_{\mathsf{new}} = x_{\mathsf{old}} r_n^{-1}$, $\xi_{\mathsf{new}} = \xi_{\mathsf{old}} r_n^{-\mu}$, $t_{\mathsf{new}} = t_{\mathsf{old}} r_n^{-\sigma'}$ and set $h_n = h r_n^{-\mu-1}$. Then we get a routine problem of propagation of singularities in bounded time in the domain $\{\epsilon \le |x_{\mathsf{new}}| \le c, |\xi_{\mathsf{new}}| \le c\}$.

This problem was treated in Sections 2.1, 2.2 and therefore Theorem 2.2.10 and the Remark 12.6.16 yield the following procedure to construct Q_n: let Q_n be constructed and let M_n be the support of its symbol. Let us consider all the generalized bicharacteristics passing through M_n at the moment $t = 0$; let M'_{n+1} be the set of all the ends of these trajectories at the moment $t = (t_{n+1} - t_n)$.

Then we can choose Q_n from the class indicated above, with symbol equal to 1 in the $\frac12\varepsilon(r_{n+1}, r_{n+1}^\mu)$-neighborhood of M'_{n+1} intersected with $\Pi_{\tau_{n+\frac12}}$ and supported in the $\varepsilon(r_{n+1}, r_{n+1}^\mu)$-neighborhood of M'_{n+1} intersected with $\Pi_{\tau_{n+1}}$ where $\tau_k = \tau_0(1 + kh^{\delta'})$ with a small enough exponent $\delta' > 0$ for $\mu \le 0$ and $\tau_k = \epsilon' r_k^{m\mu}$ with a small enough constant $\epsilon' > 0$ for $\mu > 0$ [67]. Obviously, $t_k \le T''$ yields $k \le C_0|\log h|$ and therefore $\tau_k \le 2\tau_0$ for $\mu \le 0$.

Then (12.6.98) yields that for n with $t_{n+1} \le T''$

$$U(t_{n+1})Q_0 \equiv Q_{n+1}U(t_{n+1} - t_n)Q_n(t_n - t_{n-1})Q_{n-1} \cdots Q_1 U(t_1)Q_0$$

[67] A neighborhood of $\Pi_{\tau_{n+\frac12}}$ may be cut out by standard elliptic arguments.

and we immediately obtain (12.6.97) for $\bar{T} = t_{n+1}$. However, if $t_n < \bar{T} < t_{n+1}$ then one can always redefine t_n and take $t_{n+1} = \bar{T}$ in our analysis. □

So, in our analysis for $\mu \ge 0$ we can take small enough constants $\tau_0 > 0$ and $T'' > 0$. For $\mu < 0$ the situation is more subtle: we took the smallest $\tau_0 = r^{m\mu}\gamma$ admissible by the uncertainty principle (for coordinates and momenta) and the condition $T'' \le \epsilon_3 r^{\sigma'}\gamma^{(m\mu\sigma)^{-1}}$ means that $\tau_0 \le |x|^{m\mu}$ along the trajectory. In order to avoid this restriction we must use some subtle arguments.

Now we want to extend the time interval:

Proposition 12.6.18. *Let the conditions of Proposition 12.5.17 be fulfilled and let* $\varphi \in \mathscr{C}_0^K(\mathbb{R}^d)$ *be supported in* $\{1 \le |x| \le 2\}$, $\chi \in \mathscr{C}_0^K([\frac14, 1])$ [68]. *Then for the Schwartz kernel* $u(x,y,t)$ *of the operator* $U(t) = \exp(ith^{-1}A)$ *estimate*

$$\bigl|F_{t\to h^{-1}\tau}\operatorname{tr}\Bigl(\chi_T(t)\int \varphi\bigl(\frac{x}{r}\bigr)u(x,x,t)\,dx\Bigr)\bigr| \le Ch^s \qquad \forall \tau : |\tau| \le \frac{1}{2}\tau_0 \quad \forall T \in [3T', T^*] \tag{12.6.99}$$

holds where $\tau^* = \min(\frac12\tau_0, \epsilon')$, $T^* > 0$ *and* $\epsilon' > 0$ *are small enough constants,* $T' = c_1 r^{\sigma'}$, c_1 *is a large enough constant and* $\chi \in \mathscr{C}_0^K([-1,1])$ *is arbitrary. Recall that* $r \in (r^*, \varepsilon_0)$.

Remark 12.6.19. (i) Therefore one can pick $\tau^* = \epsilon'$ for $\mu \le 0$ and $\tau^* = \epsilon' r^{m\mu}$ for $\mu > 0$.

(ii) This proposition and Corollary 12.6.20 (see below) remain true if one replaces $u(x,y,t)$ by $B_{j+1}u_j(x,y,t)$ where u_j is a solution of the recurrence problem $(12.6.68)_0$–$(12.6.68)_j$.

Proof of Proposition 12.6.18. Let us make a partition of unity in a neighborhood of $\operatorname{supp}(\varphi_r)$ where $\varphi_r(x) = \varphi(x/r)$: $\varphi_r = Q^+ + Q^- + Q'$ where $Q^+, Q^-, Q' \in \boldsymbol{\Psi}_{h,\gamma r^\mu,\gamma r}$ with symbols of $Q^\pm$ supported in the $\varepsilon(r, r^\mu)$-neighborhoods of $\Lambda^\pm \cap \{\epsilon_0 r \le |x| \le c_0 r\}$ intersected with Π_{τ_0} and with symbol of Q' supported in $\{\epsilon_0 r \le |x| \le c_0 r\} \setminus \Pi_{\frac23\tau_0}$. Then

$$U(t)\varphi_r = U(t)Q^+ + U(t)Q^- + U(t)Q' \tag{12.6.100}$$

[68] Then this proposition remains true for $\chi \in \mathscr{C}_0^K([-1,-\frac14]\cup[\frac14,1])$.

and standard elliptic arguments yield that for $V(t) = U(t)Q'$ estimate

$$\|F_{t\to h^{-1}\tau}\chi_T(t)V(t)\| \le Ch^s \quad \forall T \in [3T', T^*] \quad \tau : \forall \tau : |\tau| \le \tau^* \tag{12.6.101}$$

holds where the norm in the left-hand expression is the operator norm and we use the definition of Q', τ_0, T' and the inequalities $r \ge h^{(\mu+1)^{-1}-\delta}$ and $\tau_0 T \ge h^{1-\delta'}$ due to $\mu < (m-1)^{-1}$.

We denote equivalence modulo operator-valued functions satisfying (12.6.101) by "$\cong$." So,

$$U(\pm t)\varphi_r \cong U(\pm t)Q^+ + U(\pm t)Q^-. \tag{12.6.102}$$

Then applying $F_{t\to h^{-1}\tau}\chi_T(t)\,\mathsf{Tr}$ we obtain that for $T \in [3T', T'']$, both terms in the right-hand expression of (12.6.102) will result in $O(h^s)$ in the final answer; this follows from Proposition 12.6.17, and for the second term one should pass to the adjoint operator and then permute Q^{-*} and $U(-t)$ because of the presence of "Tr".

Thus, for $\mu \ge 0$ the proposition is proven and for $\mu < 0$ it is proven only under the additional restriction $T < T''$.

Let $\mu < 0$ and $T > T''$; then $T'' < T^*$. Let us take $T_1 = \epsilon T''$ with sufficiently small $\epsilon > 0$ and let us construct the operators $\bar{Q}^+$ and $\bar{Q}^-$ associated with $Q = Q_0^+ = Q^+$ and with $Q = Q_0^- = Q^-$ respectively according to Proposition 12.6.17; $\bar{T} = T_1$. Then (12.6.97) yields that

$$U\big(\pm t\big)Q_0^{\pm} \cong U\big(\pm(t-T_1)\big)\bar{Q}^{\pm}U\big(\pm T_1\big)Q_0^{\pm}. \tag{12.6.103)$_1$}$$

Let us represent $\bar{Q}^{\pm}$ in the form $\bar{Q}^{\pm} = Q_1^{\pm} + Q_1^{\pm\prime}$ where $Q_\bullet \in \boldsymbol{\Psi}_{h,\gamma_1 r^{*\mu},\gamma_1 r^*}$ and the symbols of $Q_1^{\pm}$ and $Q_1^{\pm\prime}$ are supported in the $\varepsilon(r^*, r^{*\mu})$-neighborhoods of $\Lambda^{\pm}\cap\{\bar{\epsilon}_1 r^* \le |x| \le \bar{c}_1 r^*\}$ intersected with Π_{τ_1} and in $\{\bar{\epsilon}_1 r^* \le |x| \le \bar{c}_1 r^*\}\setminus\Pi_{\frac{2}{3}\tau_1}$ respectively where $r_n = T_n^\sigma$, $\hbar_n = hr_n^{-\mu-1}$, $\gamma_n = \hbar_n^{\rho_n}$, $\rho < \rho_1 < \ldots < \rho_n$ now and one can choose ρ_n arbitrarily close to ρ; now the subscript "1" replaces the "bar."

Elliptic arguments again yield that $U(t-T_1)Q^{\pm\prime} \cong 0$ and the presence of the factor $U(T_1)Q^{\pm}$ is not essential because here we do not apply "Tr". Therefore (12.6.96)$_1$ remains true with $\bar{Q}_1^{\pm}$ replaced by $Q_1^{\pm}$ where we replace τ_0 by τ_1 in the definition of "$\cong$." Repeating this process we obtain that

$$\begin{aligned}(12.6.103)_n \quad & U\big(\pm t\big)Q_0^{\pm} \cong \\ & U\big(\pm(t-T_n)\big)Q_n^{\pm}U\big(\pm(T_n-T_{n-1})\big)Q_{n-1}^{\pm}\cdots Q_1^{\pm}U\big(\pm T_1\big)Q_0^{\pm}\end{aligned}$$

where we replace τ_0 by τ_n in the definition of "$\cong$."

Therefore again applying Proposition 12.6.17 we obtain that

$$U(\pm t)Q_0^{\pm} \cong \bar{\varphi}_{r_n} U(\pm t) Q_0^{\pm}$$

provided $T \in [\bar{\epsilon}_n, T_{n+1}]$, $\bar{\varphi} = 1$ in $\{|x| \ge \epsilon_4\}$ with a sufficiently small constant $\epsilon_4 > 0$ and χ is supported in $[\frac{1}{4}, 1]$. Hence, applying the operator $F_{t\to h^{-1}\tau}\chi_T(t)\,\mathsf{Tr}$ we obtain relations

$$F_{t\to h^{-1}\tau}\chi_T(t)\,\mathsf{Tr}\, U(t)Q^{\pm} = O(h^s)$$

where in the "$-$" case one should pass to the adjoint operator and move Q_0^{-*} to the right (and replace Q_0^{-*} by Q_0^{-} in the analysis).

Therefore, it remains to prove that in the case $\mu < 0$ we can pass from $r_* \ge h^{1/(\mu+1)-\delta}$ to $r_n > \varepsilon_0$ with a small enough constant ε_0 in a finite number of steps depending only on the exponents. But this assertion obviously follows from the equality

$$hr_n^{-\mu-1} = \mathsf{const}_n \bigl(hr_{n-1}^{-\mu-1}\bigr)^{1-\rho_{n-1}/(m\mu)}$$

which is due to (12.6.96) for $r_n < \epsilon$; here $1 - \rho_{n-1}(m\mu)^{-1} > 1 - \rho(m\mu)^{-1} > 1$ and $hr_*^{-\mu-1} \le h^{\delta''}$ with $\delta'' > 0$ and therefore for appropriate n the inequality $hr_n^{-\mu-1} \le Ch$ holds. □

12.6.7 Improved Remainder Estimates

Applying Proposition 12.6.18 we obtain Statement (i) of the following assertion:

Corollary 12.6.20. *(i) In the framework of Proposition 12.6.18 estimate*

$$\bigl|F_{t\to h^{-1}\tau}\bigl(\bar{\chi}_{T_2}(t) - \bar{\chi}_{T_1}(t)\bigr)\int \varphi\bigl(\tfrac{x}{r}\bigr)\Gamma u(x,x,)\,dx\bigr| \le Ch^s \qquad \forall \tau : |\tau| \le \tau^* \tag{12.6.104}$$

holds with a sufficiently small constant $T_2 > 0$ and $T_1 = C_0 r^{\theta'}$. Recall that $\bar{\chi} \in \mathscr{C}_0^K([-1,1])$ and $\bar{\chi} = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$.

(ii) Moreover, $T_1 = \varepsilon r^{\sigma'}$ works with an arbitrarily small constant $\varepsilon > 0$.

Proof. To prove Statement (ii) note that in the time interval $|t| \le T_1$ Theorem 2.2.10 is applicable with no modification and escape condition (12.6.87)–(12.6.88) obviously forbids periodic generalized trajectories for the symbol A^0 on the energy level 0. So these trajectories are also forbidden for A under the restrictions $|x| = r$, $T \le Lr^{\sigma'}$, $r \le r_*$ with arbitrarily large L and sufficiently small $r_* = r_*(L) > 0$; moreover, one can consider energy levels τ with $|\tau| \le \tau^*$. □

Now we can get very precise remainder estimates:

Theorem 12.6.21. *Let conditions of Theorem 12.6.8 and escape condition* (12.6.87)–(12.6.88) *be fulfilled.*

(i) Moreover, let condition (12.6.77) *be fulfilled. Then asymptotics*

$$\mathsf{R} := \Big| \int \psi(x) e_{(\vartheta)}(x,x,\tau)\,dx - \sum_{p} \kappa_{(\vartheta)p} h^{-d+p} - \sum_{p,j:\,-d+p=(m\mu\vartheta+\nu j)/(\mu+1)} \kappa'_{(\vartheta)p} h^{-d+p} |\log h| - \sum_{j} \omega_{(\vartheta)j} h^{(m\mu\vartheta+\nu j)/(\mu+1)} \Big| = O\big(R^{\#}(h)\big) \tag{12.6.105}$$

holds with

$$R^{\#}(h) = h^{\min(-d+1+\vartheta,\, l')}; \tag{12.6.106}$$

recall that $l' = (m\mu+\nu)\vartheta(\mu+1)^{-1}$.

Moreover, for $\vartheta \le 1$ *one can skip condition* (12.6.77) *unless* $\vartheta = 1$ *and* $(m\mu+\nu) = (-d+p)(\mu+1)$ *with* $p = 0, 1$.

(ii) Furthermore, let conditions of Statement (i) be fulfilled and

$$\vartheta(m\mu+\nu-\mu-1) + (d-1)(\mu+1) > 0 \tag{12.6.107}$$

(then $l' > -d+1+\vartheta$ *and Statement 12.6.21 yields the remainder estimate* $O(h^{-d+1+\vartheta})$*) and let the standard non-periodicity condition for the operator* A *be fulfilled.*

Then one can replace the remainder estimate $O\big(h^{-d+\vartheta+1}\big)$ *with* $o\big(h^{-d+\vartheta+1}\big)$.

(iii) Replacing condition (12.6.77) *with* (12.6.86) *one should replace* $R^{\#}(h)$ *by* $R^{\#}(h)+R^{*\prime}(h)$ *with* $R^{*\prime}(h)$ *defined by* (12.6.62) *with* ν' *instead of* ν.

Proof. Let us first consider the zone $\{|x|\ge r'_*=r_*h^{-\delta}\}$. Let us change the coordinate system $x'=xr^{-1}$ and set $h'=hr^{-\mu-1}$, $t'=tr^{-\sigma}$. Then

$$F_{t\to h^{-1}\tau}\chi_T(t)\int \varphi\bigl(\frac{x}{r}\bigr)u(x,x,t)\,dx \tag{12.6.108}$$

transforms to

$$r^{\sigma}F_{t'\to h'^{-1}\tau'}\chi_{T'}(t)\int \varphi(x')u'(x',x',t')\,dx'$$

where $T'=Tr^{-\sigma}$, $\tau'=\tau r^{-m\mu}$ (then $|\tau|\le\tau_0 \implies |\tau'|\le\epsilon'$), $u'(.,.,t')$ is the Schwartz kernel of the operator $\exp(ith'^{-1}A')$, $A'=r^{-m\mu}A$ in the new coordinate system and then for $T=T_1=\epsilon_0 r^{\sigma'}$ with small enough constants ϵ_0, ϵ_1 the results of Chapter 4 yield the standard Weyl formula for expression (12.6.108) with $T=T_1$ modulo $O(h^s)$. Therefore this formula also holds for expression (12.6.108) with $T=T_2$.

Recall that in Proposition 12.6.17 $\gamma=h'^{\rho}$ with an exponent $\rho\in(0,\frac12)$ and τ_0 is defined there. Then, for sufficiently small $\rho>0$ we obtain that $\tau^*\ge h^{1-\delta''}$ with $\delta''=(\mu+1-m\mu)(\mu+1)^{-1}>0$ and therefore taking $T=T_2$ and applying standard Tauberian arguments we obtain estimate

$$\Bigl|\int \varphi\bigl(\frac{x}{r}\bigr)\bigl(e_{(\vartheta)}(x,x,0)-\sum_{0\le p\le\bar p}\kappa_{(\vartheta)p}(x)h^{-d+p}\bigr)\,dx\Bigr|\le \tag{12.6.109}$$
$$Ch^{-d+\vartheta+1}r^{m\mu\vartheta+(d-\vartheta-1)(\mu+1)}\bigl(\frac{T_1}{T_2}\bigr)^{\vartheta+1}=Ch^{-d+\vartheta+1}r^{d(\mu+1)-m\mu}.$$

Here the right-hand expression has an additional factor

$$\bigl(\frac{T_1}{T_2}\bigr)^{\vartheta+1}\asymp r^{(\vartheta+1)\theta'}=r^{(\vartheta+1)(\mu+1-m\mu)}$$

if we compare it with the right-hand expression obtained in Subsection 12.6.1, $\bar p=\bar p(d,m,\mu,\vartheta,s,\delta)$ is large enough.

Since $d(\mu+1)-m\mu>0$ we immediately obtain that

(12.6.110) The contribution of the zone $\{r_3\le|x|\le\varepsilon\}$ to the remainder does not exceed $\varepsilon' h^{1-d+\vartheta}$ with $\varepsilon'\to0$ as $\varepsilon\to0$.

Let us apply the arguments of Subsection 12.6.3 with $r'_* = h^{(\mu+1)^{-1}-\delta}$, $L = L(r^*) = h^{(m\mu+\nu)(\mu+1)^{-1}}$ and sufficiently small $\delta > 0$ depending on $d, m, \mu, \vartheta < \bar{\vartheta}_1, \nu$.

(a) Let us first assume that condition (12.6.107) is fulfilled. Then the terms with $m\mu\vartheta+\nu j > (-d+1+\vartheta)(\mu+1)$ are not essential (their total contribution is less than $\varepsilon' h^{-d+1+\vartheta}$).

So, we only need to consider terms with

$$m\mu\vartheta + \nu j \le (-d+1+\vartheta)(\mu+1) \implies j < \vartheta.$$

In order to handle these terms by the same method as before we should estimate

$$\iint \tau_-^{\vartheta-1} \operatorname{tr}\bigl(\mathcal{K}_{(B(\tau-A^0)^{-1})^j} - \mathsf{Weyl}_j\bigr)\, dxd\tau. \tag{12.6.111}$$

We do this exactly as before but due to escape condition (12.6.87)–(12.6.88) an extra factor $T^{-1-\vartheta+j}$ appears in the right-hand expression where one can take $T(x) \asymp h_{\mathsf{eff}}^{-s}$ with arbitrarily large s; recall that $h_{\mathsf{eff}} = hr^{-\mu-1}$. One can easily see that for an appropriate s the corresponding integrals converge and are negligible.

Thus, under condition (12.6.107) Statement (i) has been proven. Then Statement (ii) follows by standard arguments (recall that we have also proven that the contribution of $B(0,\varepsilon)$ to the remainder estimate does not exceed $\varepsilon' h^{1-d+\vartheta}$). The proof of Statement (iii) is obvious.

(b) Let us now assume that condition (12.6.107) is violated. Then applying the same arguments as before we obtain the required estimates for the contributions of $B(0,2r_*)$ and $B(0,1)\setminus B(0,r'_*)$. Further, in $B(0,2r'_*)\setminus B(0,r_*)$ the only source of trouble is the presence of Weyl approximation errors of the form $h^{l'}\bigl(r/r_*\bigr)^{\zeta}$ with exponents $\zeta \ge 0$.

To handle them let us apply the above propagation results. In order to do this let us rescale $x' = \gamma^{-1}x$, then one obtains $r' = \gamma^{-1}$ and $h' = h\gamma^{-1-\mu}$ instead of r and h. To fulfill conditions $1 \ge r' \ge h'^{1/(\mu+1)-\delta'}$ let us pick $\gamma = r\bigl(r/r_*\bigr)^{s}$ with large s. Then for arbitrarily large s and sufficiently small δ and δ' depending on s one can set $T = \bigl(r/r_*\bigr)^{s}$ which means that we should multiply the right-hand expression of the remainder estimate by T^{-1} and obtain $h^{l'}\bigl(r/r_*\bigr)^{\zeta}$ with $\zeta < 0$. Then the contribution of such terms does not exceed $Ch^{l'}$. □

12.6.8 Final Remarks

Let us discuss escape condition (12.6.87)–(12.6.88) and some generalizations of our results.

Remark 12.6.22. (i) Escape condition (12.6.87)–(12.6.88) is fulfilled for $A^0 = \big(|\xi|^2 - V(x)\big)^{\frac{m}{2}}$ with $V(x)$ positively homogeneous of degree 2μ, $\mu > -1$ and $V(x) \asymp |x|^{2\mu}$. Really, one needs to consider only $m = 2$. Then for $\varphi = \langle x, \xi\rangle$, on the energy level 0

$$\{A^0, \varphi\} = 2\big(|\xi|^2 + \mu V(x)\big) = 2(\mu+1)|\xi|^2 \asymp |x|^{2\mu}$$

due to positive homogeneity of $V(x)$ and therefore one can apply Theorem 12.6.21.

(ii) For $\mu = -\frac{1}{2}$, $d = 3$, $m = 2$, $V(x) = |x|^{-\frac{1}{2}}$ and $\vartheta = 1$ the numerical value of the coefficient ω_0 is well known (see V. Ivrii and I. M. Sigal [1], for example).

Remark 12.6.23. (i) One can easily generalize Theorems 12.6.8, 12.6.9 and 12.6.12 to the case when X is a conical domain (in a neighborhood of 0) and Dirichlet boundary condition are given on ∂X.

(ii) In particular, consider domain X with conical points, operator with smooth coefficients in the vicinity of $\bar{X}$, $h = 1$ and asymptotics of $\mathsf{N}^-_{(\vartheta)}(\tau)$ as $\tau \to +\infty$. Then if the domain was smooth the remainder estimate would be $O\big(\tau^{\vartheta + (d-1-\vartheta)/m}\big)$.

Applying the generalization we mentioned (with $\mu = 0$, $\nu = 1$ and with h replaced by $\tau^{-1/m}$) we arrive to the same remainder estimate and with Scott correction terms $\omega_{(\vartheta)j}\tau^{\vartheta - j/m}$ and the main (with $j = 0$) is above remainder estimate as $\vartheta > (d-1)$; as $\vartheta = (d-1)$ singularity manifests itself through cut Weyl term $\kappa'_{(\vartheta)\vartheta}\tau^{\vartheta}|\log\tau|$ and if the remainder estimate is "o" rather than "O" through Scott correction term as well.

Remark 12.6.24. (i) Moreover, one could extend Theorem 12.6.21 under nice propagation of singularities results. Namely, escape condition has the following important corollaries:

(12.6.112) For $t = T_0$ and $(x,\xi) \in \Lambda^{\pm}_{\epsilon_1} \cap \{|x| = 1\}$ we conclude that $K^{\pm}_t(x,\xi) \subset \Lambda^{\pm} \cap \{|x| \ge 2\}$ where $K^{\pm}_t$ mean propagation of singularities

"cones" on the energy level 0 for A^0 (see Sections 2.2 and 3.3). Moreover, this property is stable with respect to small perturbations of A^0

and

(12.6.113) $(x,\xi) \notin K_t^{\pm}(x,\xi)$ for $t > 0$. Moreover, this property is also stable.

Exactly these conditions along with a stability condition apply to boundary-value problems. By the geometric interpretation of Section 3.3 one can replace $K_t(x,\xi)$ by a union of generalized bicharacteristics (in which case stability is achieved automatically).

(ii) Let us consider $\mu = 0$ and $A^0(x,\xi)$ which is $(1,0)$-homogeneous of degree m in (x,ξ), and the energy level $\tau = 1$ instead of $\tau = 0$. We obtain this problem when we treat the asymptotics of Riesz means when the spectral parameter τ tends to $+\infty$.

In order to prove escape condition note that one can take the function $\phi = \langle x,\xi\rangle$ in the analysis of propagation of singularities in Section 3.1. For $x \in \partial X$ this function is constant on the layers $\{(x,\xi + n\lambda), \lambda \in \mathbb{R}\}$ where n is a normal to ∂X at x and we assume that X is a conical set; moreover, the microhyperbolicity conditions are fulfilled with $\ell = H_\phi$. Therefore, for $(x,\xi) \in \Sigma_{1,f}$, $(y,\eta) \in K^{\pm}(x,\xi)$ (see definitions of $\Sigma_{\tau,b}$, $\Sigma_{\tau,f}$ in Chapter 3) the inequality $\pm\langle y,\eta\rangle \ge \pm\langle x,\xi\rangle + \epsilon_0 t$ holds.

Therefore condition (12.6.113) is fulfilled. Moreover, ellipticity yields that ξ is bounded on Σ_1 and $\langle x,\xi\rangle |x|^{-1}$ is bounded on $\Sigma_{1,b}$. Therefore picking $\Lambda^{\pm} = \{\pm\langle x,\xi\rangle \ge -\epsilon_1|x|\}$ with sufficiently small ϵ_1 we obtain $K_t^{\pm}(x,\xi) \subset \{(y,\eta), |y| \ge \epsilon(|x| + t)\}$ for $t \ge 0$ and therefore condition (12.6.112) is also fulfilled.

Note that condition (12.6.107) is now "$(\nu - 1)\vartheta > 1 - d$". Moreover, the non-Weyl terms are $\omega_{(\vartheta)j}\tau^{\vartheta - \frac{1}{m}\nu j}$ while the Weyl terms are $\kappa_{(\vartheta)p}\tau^{\vartheta + \frac{1}{m}(d-p)}$ and a logarithmic factor appears as soon as these exponents coincide or one of them vanishes. For $\nu = 1$ condition (12.6.107) is fulfilled for all ϑ and there are many terms with logarithms.

(iii) All the arguments in Statements (i) and (ii) apply to other boundary conditions provided that for fixed h the problem is elliptic in the classical sense and the operator is semibounded from below.

Remark 12.6.25. Let us assume that $X = \mathbb{R}^d$ (or X is a cone). Further, let stabilization condition (12.6.30) be fulfilled as $|x| \to \infty$ as well but with the operator A^∞ and exponents $\mu' \in (-1, 0), \nu' < 0$ instead of A^0 and μ, ν. Furthermore, let us assume that escape condition is fulfilled for this operator A^∞ as well with the sets $\Lambda^\pm_\infty$ instead of $\Lambda^\pm$.

Let us define the sets $\Lambda^\pm_1$ according to (12.6.94); we define these sets for $|x| \le \epsilon$ and $|x| \ge R$ with $\epsilon > 0$ very small and R very large. Finally, let us assume that there exists t_0 such that if $(x(0), \xi(0))$ lies in the (small) set $\Lambda^\pm_1 \cap \{|x| \le \epsilon\}$ then $(x(\pm t_0), \xi(\pm t_0)) \in \Lambda^\pm_1 \cap \{|x| \ge R\}$.

Under these assumptions we control the propagation of singularities for very large T ($\asymp h^{-s}_{\mathsf{eff}}$ with an arbitrarily large s) and therefore

(12.6.114) In the framework of Theorem 12.6.21 and the assumptions of this remark the remainder estimate is $O\bigl(h^{l'}\bigr)$ with $l' = (m\mu + \nu)\vartheta(\mu+1)^{-1}$ even if $l' > -d + 1 + \vartheta$. Moreover, for $(\mu+1)d + m\mu\vartheta < 0$ one can take $\psi = 1$.

As an example one can consider $X = \mathbb{R}^d$ and the Schrödinger operator with potential $W(|x|)$ where $W(r)$ is positively homogeneous of degrees 2μ and $2\mu'$ for $|x| \le \epsilon$ and $|x| \ge R$ respectively, $-1 < \mu' \le \mu < 0$, and $W(r) < 0$, $\frac{d}{dr} r^2 W(r) < 0$ $\forall r$. Moreover, one can perturb this Schrödinger operator slightly and get rid of its spherical symmetry and homogeneity.

The following problem seems to be very interesting and challenging:

Problem 12.6.26. (i) Construct the similar theory as singularity is on manifolds Z of dimension d' rather than 0.

One may suspect that Scott correction terms should be in the form

$$\omega_{(\vartheta)d',j,k} h^{-d'+k+(m\mu\vartheta+\nu j)(\mu+1)^{-1}} \qquad j, k \in \mathbb{Z}^+. \tag{12.6.115}$$

(ii) In particular, consider the case of the regular operator in domain with $(d-2)$-dimensional edges (i.e. when ∂X consists of smooth manifolds which intersect only pairwise along submanifolds of dimension $(d-2)$).

(iii) Allow edges of any codimension. Then as $h = 1$ and we consider $\mathsf{N}^-_{(\vartheta)}(\tau)$ as $\tau \to +\infty$, in accordance to (12.6.115) we should expect Scott terms associated with edges of dimension d' to be $\omega'_{(\vartheta),d',k} \tau^{\vartheta+(d'-k)/m}$ and the final answer should contain their sum with respect to d', k.

In Part (i) propagation along Z seems to be a difficult obstacle and in Part (iii) additional difficulties appear from "collision" of different manifolds Z_k.

12.7 Sharp Spectral Asymptotics for Neumann Laplacian in Thin Cusps

12.7.1 Problem Set-up

In this section we consider the Neumann Laplacian and similar operators in domains with ultrathin cusps (with $\exp(-|x|^{\mu+1})$-like width ($\mu > 0$)) and recover eigenvalue asymptotics with accurate remainder estimate.

It is known that in some classes of domains the spectral theories of the Neumann and Dirichlet Laplacians could be very different. In particular, in domains with cusps the spectrum of Dirichlet Laplacian is always discrete and, moreover, Weyl formula holds as cusp is not very thick while for Neumann Laplacian the spectrum is essential unless cusp is very thin and the Weyl formula holds even for thinner cusps.

As in Section 12.2 we considered this operator in the domain $X \subset \mathbb{R}^d$ such of (12.2.7)-type i.e.

$$X = X_0 \cup X_{\mathsf{cusp}},\ \partial X \in \mathscr{C}^K,\ X_0 \Subset \mathbb{R}^d, \tag{12.7.1}$$
$$X_{\mathsf{cusp}} = \bigl\{x = (x', x'') : x' \in Z,\ x'' \in \sigma(x')Y\bigr\}, \quad Y \Subset \mathbb{R}^{d''},\ \partial Y \in \mathscr{C}^K$$

We consider here only second-order $\mathsf{D} \times \mathsf{D}$-matrix operators

$$A \coloneqq \sum_{j,k} D_j a_{jk} D_k + \sum_j (a_j D_j + D_j a_j^\dagger) + a_0 \tag{12.7.2}$$

which are defined by a quadratic form

$$Q(u) = \int \Bigl(\sum_{j,k} \langle a_{jk} D_k u, D_j u\rangle + 2\,\mathsf{Re}\langle a_k D_k u, u\rangle + \langle a_0 u, u\rangle\Bigr)\,dx \tag{12.7.3}$$

where we assume that this operator is self-adjoint

$$a_{kj} = a_{jk}^\dagger, \qquad a_0 = a_0^\dagger, \qquad a_b = a_b^\dagger \tag{12.7.4}$$

in $\mathscr{L}^2(X, \mathbb{C}^{\mathsf{D}})$ and (uniformly) elliptic

$$C_0|\xi|^2 \ge a(x,\xi) := \sum a_{jk}\xi_j\xi_k \ge \epsilon_0|\xi|^2 \qquad \forall x,\xi. \tag{12.7.5}$$

We supply quadratic form with no boundary conditions which leads to the *Neumann boundary conditions*

$$Bu := \Big(\sum_j n_j\big(\sum_k a_{jk}D_k + a_j^\dagger\big) + ia_b\Big)u\big|_{\partial X} = 0 \tag{12.7.6}$$

with $a_b = 0$ where (n_j) is an interior unit normal to ∂X.

More generally we consider quadratic form

$$Q(u) = \int_X \Big(\sum_{j,k}\langle a_{jk}D_k u, D_j u\rangle + 2\,\mathsf{Re}\langle a_k D_k u, u\rangle + \langle a_0 u, u\rangle\Big)\,dx + \int_{\partial X}\langle a_b u, u\rangle\,dS \tag{12.7.7}$$

where dS is a Lebesgue measure on ∂X. Then a_b in (12.7.6) does not necessary vanish.

Let us recall analysis of Section 12.3.1. Let us introduce new variables $x''_{\mathsf{new}} = x''_{\mathsf{old}}\sigma(x')^{-1}$ transforming a cusp X_{cusp} into a cylinder $Z \times Y$ as on Figure 12.1; then $dx_{\mathsf{old}} = \sigma(x')^d dx''_{\mathsf{new}}$ and to preserve $\mathscr{L}^2(X, \mathbb{C}^{\mathsf{D}})$ we also plug $u_{\mathsf{old}} = \sigma(x')^{-\frac{d''}{2}}u_{\mathsf{new}}$ resulting in the quadratic form

$$Q_{\mathsf{new}}(u) := \int_{Z\times Y}\Big(\sum_{j,k}\langle a_{jk}P_k u, P_j u\rangle + 2\,\mathsf{Re}\sum_k\langle a_k P_k u, u\rangle + \langle a_0 u, u\rangle\Big)\,dx'dx'' + \int_{Z\times\partial Y}\sigma^{-1}(x')\langle a_b u, u\rangle\,dx'dS'' \tag{12.7.8}$$

where

$$P_j = \begin{cases} D_j - b_j(x')L & \text{as } j = 1,\dots,d',\\ \sigma^{-1}(x')D_j & \text{as } j = d'+1,\dots,d\end{cases} \tag{12.7.9}$$

with

$$b_j(x') = \partial_{x_j}\log\sigma(x') \tag{12.7.10}$$

and

$$L = \langle x'', D''\rangle - \frac{1}{2}d''i = \frac{1}{2}\big(\langle x'', D''\rangle + \langle D'', Dx''\rangle\big). \tag{12.7.11}$$

This is another version of (12.2.23). Sure, we changed coordinates in all coefficients.

Our first question is *if all the spectrum of operator discrete?* The problem now is that condition (12.3.14) fails for D lowest eigenvalues of $\boldsymbol{a}$ corresponding to constant $w(x'')$. Therefore, at least heuristically, to answer this question we need to restrict our quadratic form to the space of such functions (which coincides with $\mathcal{L}^2(Z, \mathbb{C}^{\mathsf{D}})$)

$$Q'(v) \coloneqq \int_Z \Big(\sum_{j,k}\langle a'_{jk}P'_k v, P'_j v\rangle + 2\,\mathrm{Re}\sum_k \langle a'_k P'_k v, v\rangle + \langle a'_0 v, v\rangle\Big)\,dx' \tag{12.7.12}$$

where summation is taken over $j, k = 1, \dots, d'$,

$$a'_{jk}(x') = \int_Y a_{jk}\,dx'', \qquad a'_k(x') = \int_Y a_k\,dx'', \tag{12.7.13$_{1-3}$}$$

$$a'_0(x') = \int_Y a_0\,dx'' + \int_{\partial Y} a_b(x)\,dS'',$$

$$P'_j = D_j + \frac{i}{2}d''b_j, \qquad P'^{*}_j = D_j - \frac{i}{2}d''b_j \tag{12.7.14}$$

and corresponding operator is

$$A' = \sum_{j,k} P'^{*}_j a'_{jk} P'_k + \sum_k \big(P'^{*}_k a'^{\dagger}_k + a'_k P'_k\big) + a'_0 = \tag{12.7.15}$$

$$\sum_{j,k} D_j a'_{jk} D_k + \sum_k \big(D_k a''^{\dagger}_k + a''_k D_k\big) + a''_0$$

with

$$a''_0 = a'_0 + \frac{1}{4}d''^2\sum_{jk} a_{jk}b_j b_k + \frac{i}{2}d''\sum_k b_k\big(a'_j - a'^{\dagger}_j\big), \tag{12.7.16}$$

$$a''_k = a'_k - \frac{i}{2}d''\sum_j b_j a_{jk}. \tag{12.7.17}$$

Assuming that the lower order coefficients a_j, a_0 and a_b are not important (we will ensure this imposing conditions) we conclude that in order to ensure that the spectrum of A' and thus of A is discrete we should assume that a_0'' grows at infinity i.e.

$$|\nabla \log \sigma| \to \infty \qquad \text{as} \quad |x'| \to \infty. \tag{12.7.18}$$

This is definitely not the case for cusps described in Section 12.7.1.

The spectral asymptotics for operator A' may play an important role in this section. To fall into conditions of Section 11.5 let us assume that

$$-\log \sigma \asymp |x'|^{\mu+1}, \tag{12.7.19}$$

$$|\nabla \log \sigma| \asymp |x'|^{\mu} \tag{12.7.20}$$

and

$$|\nabla^\alpha \log \sigma| \le c|x|^{\mu+1-|\alpha|} \qquad \forall \alpha : |\alpha| \le K \tag{12.7.21}$$

with $\mu > 0$. Further, let us assume that

$$|\nabla^\alpha a_{jk}''| \le c|x'|^{-|\alpha'|}, \qquad |\nabla^\alpha a_j''| \le c|x'|^{-|\alpha'|-1}, \qquad |\nabla^\alpha a_0''| \le c|x'|^{-|\alpha'|-2} \tag{12.7.22}_{1-3}$$

and

$$a_0(x', \xi') \ge \epsilon(|\xi'|^2 + |x'|^{2\mu}) \qquad \text{as} \quad |x'| \ge c \tag{12.7.23}$$

with

$$a_0(x', \xi') = \sum_{j,k} a_{jk}'' \xi_j' \xi_k' + \sum_j (a_j'' + a''^{\dagger}) \xi_j' + a_0''. \tag{12.7.24}$$

Let us also assume that

$$|\nabla^\alpha a_{jk}| \le c\gamma^{-|\alpha'|}\sigma^{-|\alpha''|}, \qquad |\nabla^\alpha a_j| \le c\gamma^{-|\alpha'|-1}\sigma^{-|\alpha''|}, \tag{12.7.25}_{1-2}$$

$$|\nabla^\alpha a_0| \le c\gamma^{-|\alpha'|-2}\sigma^{-|\alpha''|} \qquad |\nabla^\alpha a_b| \le c\gamma^{-|\alpha'|-1}\sigma^{-|\alpha''|} \tag{12.7.25}_{3-4}$$

with $\gamma(x') = |x'|^{-\mu}$; these and all the other inequalities are supposed to hold as $|\alpha| \le K = K(d, m)$, $|x'| \ge c$.

12.7.2 Weyl Part of Asymptotics

Let us consider the "core domain" and the "near cusp". Let us introduce the scaling function $\epsilon\sigma(x')$. Let $\psi(x)$ be a ($\sigma(x')$-admissible) function supported in

$$\mathcal{X}(\tau) = \{x : \ \sigma(x) \ge \epsilon\tau^{-\frac{1}{2}}\} \tag{12.7.26}$$

and equal 1 in $\mathcal{X}(\tau/2)$. Then the theory of Section 11.2 and and 11.3 implies that

$$|\int \psi(x)\big(e(x,x,\lambda) - \mathsf{Weyl}(x)\big)\,dx| \le C\tau^{\frac{d-1}{2}} \int_{\mathcal{X}(\tau)} \sigma^{-1}\,dx \tag{12.7.27}$$

where $\mathsf{Weyl}(x)$ is the standard Weyl expression for $A(x,D)$ and we should assume that a_j, a_0 and a_b do not grow very fast as $|x'| \to \infty$.

In particular, if

$$\int_X \sigma(x')^{-1}\,dx \asymp \int_Y \sigma(x')^{d''-1}\,dx' \tag{12.7.28}$$

converges which is the case as $d'' \ge 2$ due to superfast decay of $\sigma(x')$, the right-hand expression of (12.7.27) is $O\big(\tau^{\frac{1}{2}(d-1)}\big)$.

Furthermore, the same arguments imply that then one can replace $\int \psi(x)\mathsf{Weyl}(x)\,dx$ by $\int \mathsf{Weyl}(x)\,dx$ and by $\varkappa_0\tau^{\frac{d}{2}}$ in the very end.

Moreover, in this case under standard non-periodicity assumption to Hamiltonian billiard trajectories generated by $a(x,\xi)$ the remainder estimate is $o\big(\tau^{\frac{1}{2}(d-1)}\big)$ but one should include in the left-hand expression lower order and boundary terms so the final expression is $\varkappa_0\tau^{\frac{1}{2}d} + \varkappa_1\tau^{\frac{1}{2}(d-1)}$.

On the other hand, as $d'' = 1$ no matter how fast $\sigma(x')$ decays, integral (12.7.28) does not converge and the right-hand expression of (12.7.27) is of magnitude $\tau^{\frac{1}{2}(d-1)}r_*^{d'}(\tau)$ where

$$r_*(\tau) \asymp (\log\tau)^{1/(\mu+1)} \tag{12.7.29}$$

is defined from $|x'| \le r_*(\tau) \implies \sigma(x') \le \tau^{-\frac{1}{2}}$.

In Subsection 12.7.4 we will improve it by the long-range propagation arguments. So far we proved

Proposition 12.7.1. *Let operator A satisfy assumptions of Theorem 11.2.7 with $m = 2$ and σ satisfy* (12.7.19), (12.7.21).

Then[69)]

(i) Asymptotics

$$\int \psi_\tau(x')e(x,x,\tau)\,dx = \varkappa_0\tau^{\frac{1}{2}d} + O\big(R(\tau)\big) \tag{12.7.30}$$

holds where

$$R(\tau) = \begin{cases} \tau^{\frac{1}{2}(d-1)} & \text{as } \ d''\ge 2,\\ \tau^{\frac{1}{2}(d-1)}\,|\log\tau|^{(d-1)/(\mu+1)} & \text{as } \ d''=1, \end{cases} \tag{12.7.31}$$

coefficients $\varkappa_0$ and $\varkappa_1$ are standard Weyl coefficients.

(ii) Let $d''\ge 2$ and the standard non-periodicity condition be fulfilled to Hamiltonian billiard trajectories generated by $a(x,\xi)$. Then

$$\int \psi_\tau(x')e(x,x,\tau)\,dx = \varkappa_0\tau^{\frac{1}{2}d} + \varkappa_1\tau^{\frac{1}{2}(d-1)} + o\big(\tau^{\frac{1}{2}(d-1)}\big). \tag{12.7.32}$$

12.7.3 Cusp pPart of Asymptotics

Now we need to calculate the contribution of the remaining part of the cusp

$$\mathcal{X}'(\tau) := \big\{x, \sigma(x') \le 2\epsilon\tau^{-\frac{1}{2}}\big\}. \tag{12.7.33}$$

We would like to transform our cusp to a cylinder and apply the theory of operators with the operator-valued symbols. According to Subsection 12.7.1 after change of coordinates $x'_{\text{new}} = x'_{\text{old}}$, $x''_{\text{new}} = \sigma(x')^{-1}x''_{\text{old}}$ we arrive to operator A_{new} corresponding to quadratic form Q_{new} defined by (12.7.8)

$$A_{\text{new}}(x,D) = \sum_{j,k} P_j^* a_{jk} P_k + \sum_k \big(a_k P_k + P_k^* a_k^\dagger\big) + a_0 \tag{12.7.34}$$

with P_j defined by (12.7.9). Then "no boundary condition" to quadratic form translates to boundary condition

$$B_{\text{new}}u := \Big(\sum_j \nu_j\big(\sum_k a_{jk}P_k + a_j^\dagger\big) + i\sigma^{-1}a_b\Big)u\big|_{\partial X} = 0 \tag{12.7.35}$$

with

[69)] No matter if we consider Dirichlet or Neumann boundary conditions.

(12.7.36) $\nu_j = n_j$ for $j = d'+1, \dots, d$, $\nu_j = -(\partial_{x_j}\sigma_j)\langle x'', n''\rangle$ for $j = 1, \dots, d'$ as $n'' = (n_{d'+1}, \dots, n_d)$ is a unit normal to ∂Y.

However in contrast to the Dirichlet boundary problem now the boundary operator contains differentiation with respect to x' and we cannot apply directly theory of Chapter 4. Instead we decompose $\mathbb{H} = \mathscr{L}^2(Y, \mathbb{C}^{\mathsf{D}})$ into $\mathbb{H}_0 \oplus \mathbb{H}_1$ where $\mathbb{H}_0$ consists of constants and $\mathbb{H}_1 = \mathbb{H}_0^{\perp}$. Then

$$(12.7.37)\quad \boldsymbol{a}(x', \xi') = \begin{pmatrix} \boldsymbol{a}_{00}(x', \xi') & \boldsymbol{a}_{01}(x', \xi') \\ \boldsymbol{a}_{10}(x', \xi') & \boldsymbol{a}_{11}(x', \xi') \end{pmatrix}, \qquad \boldsymbol{b}(x', \xi') = \begin{pmatrix} \boldsymbol{b}_0(x', \xi') \\ \boldsymbol{a}_1(x', \xi') \end{pmatrix}$$

and the problem $(hD_t - A)U = 0$, $BU = 0$ (we skip subscript "new") becomes a system

$$(12.7.38)_1 \qquad \big(hD_t - \boldsymbol{a}_{00}\big) U_{0j} - \boldsymbol{a}_{01} U_{1j} = 0,$$

$$(12.7.38)_2 \qquad -\boldsymbol{a}_{10} U_{0j} + \big(hD_t - \boldsymbol{a}_{11}\big) U_{1j} = 0,$$

$$(12.7.39) \qquad \big(\boldsymbol{b}_0 u_0 + \boldsymbol{b}_1 u_1\big)\big|_{\partial Y} = 0.$$

We will use $(12.7.38)_2$–(12.7.39) to express U_{1j} through U_{0j} and to plug it into $(12.7.38)_1$.

One can see easily that the first step is possible in the zone in question provided

$$(12.7.40)\quad \langle \boldsymbol{a}(x', \xi'; x'', D'')v, v\rangle \ge \epsilon_0\big(|\xi'|^2 + \sigma(x')^{-2}\big)\|v\|^2 + \|D''v\|^2 \qquad \forall v \in \mathbb{H}_1$$

which is ensured by

$$(12.7.41)_{2-4}\quad |a_j(x)| \le c\rho(x'), \qquad |a_0(x)| \le c\rho(x')^2, \qquad |a_b(x)| \le c\sigma(x')\rho(x')^2$$

with $\rho(x') = |x'|^{\mu}$ and we get that $\boldsymbol{a}_{01}U_{1j} = \boldsymbol{r}(x', hD_t, hD')U_{0j}$ with h-pseudodifferential operator $\boldsymbol{r}(x', hD_t, hD')$ with $h = \tau^{-\frac{1}{2}}$.

Further, one can see easily that $\boldsymbol{r}(x', hD_t, hD')$ decays as $\tau \to +\infty$ provided

$$(12.7.41)_1 \qquad |(a_{jk}(x) - \bar{a}_{jk}(x')| \le c\sigma(x'),$$

Then one arrives to the following proposition:

Proposition 12.7.2. *Assume that* (12.7.41)$_{1-4}$, (12.7.19), (12.7.21) *and*

$$c^{-1}\bigl(|\xi'|^2+\rho(x')^2\bigr)\le \boldsymbol{a}_0(x',\xi')\le c\bigl(|\xi'|^2+\rho(x')^2\bigr) \tag{12.7.42}$$

are fulfilled where $\boldsymbol{a}_0=A'$ *is defined by* (12.7.15).

Then contribution of zone (12.7.33) *to the asymptotics is given by the Weyl approximation for* d'*-dimensional operator* $\boldsymbol{a}_0^{\mathsf{w}}(x',D')$

$$\mathsf{N}_{\mathsf{cusp}}(\lambda)=(2\pi)^{-d'}\int \boldsymbol{n}(x',\xi')\,dx'd\xi', \tag{12.7.43}$$

$\boldsymbol{n}(x',\xi')$ *is an eigenvalue counting function for symbol* $\boldsymbol{a}_0(x',\xi)$ *and the contribution of this zone to the remainder estimate is* $O\bigl(R'(\tau)+\tau^{d'/2}\bigr)$ *where* $R'(\tau)$ *is the corresponding remainder estimate*

(i) $R'(\tau)=O\bigl(\tau^{d'(\mu+1)/2\mu}\bigr)$ *in the general case.*

(ii) $R'(\tau)=O\bigl(\tau^{(d'-1)(\mu+1)/2\mu}\bigr)$ *provided* <u>*either*</u> $\boldsymbol{a}_0$ *is* (x',ξ')*-microhyperbolic on the energy level* τ <u>*or*</u> $\mathsf{D}=1$ *and* $d'\ge 2$.

(iii) $R'(\tau)=O\bigl(\tau^{\delta}\bigr)$ *provided* $\mathsf{D}=1$ *and* $d'=1$.

Proof. One needs to prove only that extra term $\boldsymbol{r}$ does not contribute much to the principal part of the asymptotics. One can see easily that its norm does not exceed $C\tau\sigma$ and then one can see easily that its contribution to the principal part is $O(\tau^{d'/2})$. □

Further, we arrive to

Proposition 12.7.3. *Let assumptions of Proposition 12.7.2(ii) be fulfilled. Furthermore, assume that* $d'\ge 2$ *and* $\boldsymbol{a}_0(x',\mathsf{x}')$ *stabilizes to quasihomogeneous symbol* $\boldsymbol{a}_{0,\infty}(x',\xi')$ *and the standard non-periodicity condition is fulfilled for the Hamiltonian flow generated by* $\boldsymbol{a}_{0,\infty}(x',\xi)$.

Then the contribution of zone (12.7.33) *to the asymptotics is given by improved two-term Weyl approximation* $\mathsf{N}^*_{\mathsf{cusp}}(\tau)$ *for* $\boldsymbol{a}_0$ *and the contribution of this zone to the remainder estimate is* $o\bigl(R'(\tau)\bigr)+O\bigl(\tau^{d'/2}\bigr)$.

12.7.4 Main Results. I

Combining Propositions 12.7.1–12.7.3 we arrive to the following theorem:

Theorem 12.7.4. *Let conditions* (12.7.1)–(12.7.5) *be fulfilled and let* σ *satisfy* (12.7.19), (12.7.21).

Further, let conditions $(12.7.41)_{1-4}$ *and* (12.7.42) *be fulfilled where* $\mathsf{a}_0 = A'$ *is defined by* (12.7.15).

Then

(i) For $d'' \ge 2$ *asymptotics*

$$\mathsf{N}(\tau) = \varkappa_0 \tau^{\frac{d}{2}} + \mathsf{N}_{\mathsf{cusp}}(\tau) + O\bigl(\tau^{\frac{d-1}{2}}\bigr) + O\bigl(R'(\tau)\bigr) \tag{12.7.44}$$

holds where $\varkappa_0$ *and* $\varkappa_1$ *(see below) are standard Weyl coefficients for operator* A *and* $\mathsf{N}_{\mathsf{cusp}}(\tau)$ *is the Weyl approximation for* d'*-dimensional operator* $\mathsf{a}_0^{\mathsf{w}}(x', D')$ *given by* (12.7.43) *and* $R'(\tau)$ *is defined in Proposition 12.7.2.*

(ii) For $d'' = 1$ *the following estimate holds*

$$\mathsf{N}(\tau) = \varkappa_0 \tau^{\frac{d}{2}} + \mathsf{N}_{\mathsf{cusp}}(\tau) + O\bigl(\tau^{\frac{d-1}{2}} (\log \tau)^{\frac{d-1}{\mu+1}}\bigr) + O\bigl(R'(\tau)\bigr). \tag{12.7.45}$$

(iii) Moreover, if $d'' \ge 2$ *and the Hamiltonian billiard flow generated by* $\mathsf{a}(x,\xi)$ *on* T^*X *satisfies the standard non-periodicity condition then one can replace* $O(\tau^{(d-1)/2})$ *by* $\bigl(\varkappa_1 + o(1)\bigr)\tau^{\frac{1}{2}(d-1)}$.

(iv) On the other hand, if $d' \ge 2$ *if* $\mathsf{a}_0(x', \xi')$ *stabilizes to quasihomogeneous symbol* $\mathsf{a}_{0,\infty}(x', \xi')$ *and the Hamiltonian flow generated by* $\mathsf{a}_{0,\infty}(x', \xi')$ *on* T^*Z *satisfies the standard non-periodicity condition then one can replace* $\mathsf{N}_{\mathsf{cusp}}(\tau)$ *by improved two-term Weyl approximation* $\mathsf{N}^*_{\mathsf{cusp}}(\tau)$ *and* $O(\tau^{(d-1)(\mu+1)/2\mu})$ *by* $o(\tau^{(d-1)(\mu+1)/2\mu})$.

(v) Skipping conditions ensuring that the Weyl zone contribution to the remainder estimate is $O(\tau^{(d-1)/2})$ *we should replace* $\mathsf{N}^{\mathsf{W}}(\tau)$ *by* $O(\tau^{d/2})$.

12.7.5 Improved Estimate as $d'' = 1$

In this case $d'' = 1$ the semiclassical part of the cusp (also called *near cusp*, cf. (12.7.26) where we use the same notation)

$$\mathcal{X}(\tau) = \{x : |x'| \ge R, \sigma(x') \ge \epsilon \tau^{-\frac{1}{2}}\} \tag{12.7.46}$$

contributes a bit more (by the logarithmic factor) then usual to the remainder estimate as it contributes $\mathsf{N}^{\mathsf{W}}(\tau)$ to the main part. This is important only as $(d-1)/2 \ge (d-2)(\mu+1)/(2\mu)$ i.e.

$$\mu \ge d-2. \tag{12.7.47}$$

So we need to consider

$$\int \mathsf{tr}(e(x,x,\tau))\psi_1(x')\,dx \tag{12.7.48}$$

where $\psi_1(x')$ is supported in $\{x' : |x'| \ge R, \sigma(x') \ge \epsilon\tau^{-\frac{1}{2}}\}$ and equals 1 in $\{x' : |x'| \ge 2R, \sigma(x') \ge 2\epsilon\tau^{-\frac{1}{2}}\}$ with constant R.

To improve this remainder estimate we need (x',ξ')-microhyperbolicity for operator $\boldsymbol{a}(x',\xi') = A(x',\xi';x'',D_{x''})$ under boundary condition $\boldsymbol{b}(x',\xi)v=0$ with $\boldsymbol{b} = B(x',\xi';x'',D_{x''})$. This condition is obviously defined provided "mixed" coefficients a_{jk} (i.e. with $j=1,\dots,d'$, $k=d'+1,\dots,d$) decay as $|x'|\to\infty$ because according to $(12.7.41)_{2-4}$ lower order terms are not large enough to influence it. In the general case we do not have the notion of (x',ξ')-microhyperbolicity in Chapter 2 but we essentially have it in Chapter 3 (see Definition 3.1.4.

Let us assume that condition (12.7.20) is fulfilled.

Then the error does not exceed

$$C\tau^{(d-1)/2}\int_{\mathcal{X}(\tau)} \gamma(x')^{-1}\,dx \asymp C\tau^{(d-1)/2}\int_{\mathcal{X}(\tau)} \gamma(x')^{-1}\sigma(x')\,dx' \tag{12.7.49}$$

where $\gamma(x') \asymp |x'|^{-\mu}$ is the scale with respect to x'. This integral does not exceed $\varepsilon(R)$ (with $\varepsilon(R)=o(1)$ as $r\to\infty$) if taken over $\{|x'|\ge R\}$.

Further, if we consider

$$\tau^{(d-j)/2}\int_{\mathcal{X}(\tau)} \sigma(x')^{-j}\,dx \asymp \tau^{(d-j)/2}\int_{\mathcal{X}(\tau)} \sigma(x')^{1-j}\,dx' \tag{12.7.50}$$

we conclude that it is $o(\tau^{(d-1)/2})$ as $\mu > d-2$ and is $O(\tau^{(d-1)/2})$ as $\mu = d-2$. Therefore we need to consider only terms with $j=0,1$ unless $\mu = d-2$ and we are interested in the remainder estimate $o(\tau^{(d-1)/2})$ in which case we need to consider also $j\ge 2$.

Furthermore,

$$\int_{\mathcal{X}'(\tau)} \tau^{d/2}\,dx \asymp \int_{\mathcal{X}'(\tau)} \tau^{d/2}\sigma(x')^1\,dx' \tag{12.7.51}$$

is $o(\tau^{(d-1)/2})$ as $\mu > d-2$ and $O(\tau^{(d-2)/2})$ as $\mu = d-2$; here and below $\mathcal{X}'(\tau) = \{x : \sigma(x') \le \epsilon\tau^{-\frac{1}{2}}$ is a *far cusp*.

To provide (x', ξ')-microhyperbolicity we need the following proposition:

Proposition 12.7.5. *Let conditions of Theorem 12.7.4 be fulfilled. Further, let condition* (12.7.20) *be fulfilled and*

$$|\nabla^{\alpha}\big(a_{jk}(x) - a_{jk,\infty}(x')\big)| \le c|x'|^{-\varsigma-|\alpha'|}\sigma(x')^{-|\alpha''|} \tag{12.7.52}$$

with $\varsigma > 0$, $ajk, \infty(x')$ *positively homogeneous of degree* 0. *Then operator* $A(x', \xi'; x'', D_{x''})$ *with* $B(x', \xi'; x'', D_{x''})v = 0$ *is uniformly* (x', ξ')*-microhyperbolic.*

Proof. Proof is trivial as one needs to consider operator with variables (ξ', σ) where σ is now independent variable and to prove that it is (ξ', σ)-microhyperbolic. □

Now we arrive to Statements (i) and (ii) of the following theorem:

Theorem 12.7.6. *Let* $d'' = 1$ *and conditions* (12.7.47), (12.7.1)–(12.7.5) *be fulfilled and let* σ *satisfy* (12.7.19)–(12.7.21)*; let* (12.7.52) *be fulfilled.*

Further, let conditions $(12.7.41)_{1-4}$ *and* (12.7.42) *be fulfilled where* $a_0 = A'$ *is defined by* (12.7.15).

Then

(i) Asymptotics

$$\mathsf{N}(\tau) = \varkappa_0\tau^{\frac{d}{2}} + \nu(\tau) + \mathsf{N}^{\mathsf{W}}_{\mathsf{cusp}}(\tau) + O\big(\tau^{\frac{d-1}{2}}\big) \tag{12.7.53}$$

holds with

$$\nu(\tau) = \tau^{(d-1)/2}\int \big(1 - \psi'_{\tau}(x)\big)\kappa_1(x)\,dx \tag{12.7.54}$$

with $\mathsf{supp}(\psi'_{\tau}) \subset \mathcal{X}'(\tau)$.

(ii) Moreover, if $\mu > d-2$ *and the standard non-periodicity condition is fulfilled for Hamiltonian billiard flow generated by symbol* $a(x, \xi)$ *one can replace* $O\big(\tau^{\frac{d-1}{2}}\big)$ *by* $o(\tau^{\frac{d-1}{2}})$.

(iii) Let $\mu = d-2$ and the standard non-periodicity condition be fulfilled. Further, let $d \ge 3$, $a_0(x',\xi')$ stabilize to quasihomogeneous symbol $a_{0,\infty}(x',\xi')$ and the Hamiltonian flow generated by $a_{0,\infty}(x',\xi')$ on T^*Z satisfy the standard non-periodicity condition.

Then asymptotics

$$\mathsf{N}(\tau) = \varkappa_0 \tau^{\frac{d}{2}} + \nu_1(\tau) + \mathsf{N}^{\mathsf{W}}_{\mathsf{cusp}}(\tau) + \varkappa_1 \tau^{\frac{d-1}{2}} + o\bigl(\tau^{\frac{d-1}{2}}\bigr) \tag{12.7.55}$$

holds with

$$\nu_1(\tau) = \nu(\tau) - \tau^{\frac{d}{2}} \int \kappa_0(x)\psi'_\tau(x)\,dx. \tag{12.7.56}$$

(iv) Under stabilization condition alone one can replace $\nu(\tau)$ and $\nu_1(\tau)$ by

$$\varkappa'_1 \tau^{\frac{d-1}{2}} |\log \tau|^{(d-\mu-2)/(\mu+1)} \tag{12.7.57}$$

and

$$\tau^{\frac{d-1}{2}} \bigl(\varkappa'_1 |\log \tau|^{(d-\mu-2)/(\mu+1)} + \varkappa''_1\bigr) \tag{12.7.58}$$

respectively.

Proof. To prove Statement (iii) one needs just to notice that $|\kappa_j| \ll c\sigma^{-j}$ as $j \ge 2$ because only terms which do not satisfy this inequality could come from the analysis of 1-dimensional operator $A(x',\xi';D'')$ with boundary condition $B(x',\xi';x'',D_{x''})v = 0$ on each side of the segment Y and one can prove easily by formal calculations that in this case $\kappa_j = 0$ as $j \ge 2$.

Proof of Statement (iv) is trivial. □

Remark 12.7.7. All the above results hold for manifolds with the compact boundary and with cusps, like $Z \times Y$ with the metrics $dx'^2 + \sigma(x')^2 dx''^2$ where now Y is a compact closed manifold. In particular we can consider surfaces of revolutions.

Remark 12.7.8. Let $\mathsf{D} = 1$. Then we can get rid of derivatives with respect to x' in the boundary conditions. One can see easily that one can introduce variables $x'_{\mathsf{new}} = x'_{\mathsf{old}} - F(x)$ with

$$|\nabla^\alpha F| \le c|x'| \cdot \sigma(x')\gamma(x')^{-1-|\alpha'|} \tag{12.7.59}$$

such that in the new coordinates the quadratic form is of the same type with the same inequalities and $\tilde{a}_{jk} = 0$ as $j \le d', k \ge d''+1$ and $x'' \in \partial Y$ and we replaced $\sigma(x')$ by $\sigma(x'_{\mathsf{new}})$ which leads to the negligible perturbations.

12.7.6 Maxwell System

Reminder

We are interested in eigenvalue asymptotics for Maxwell operator A in $X \subset \mathbb{R}^d$. Namely, let $\mathfrak{H} = \mathscr{L}^2(X, \Lambda^k) \oplus \mathscr{L}^2(X, \Lambda^{k+1})$ where Λ^k is a space of exterior forms of a degree $k = 0, \dots, d$ [70], and let for

$$\phi = \sum_I \phi_I dx_I, \quad dx_I = dx_{i_1} \wedge \dots dx_{i_k} : \tag{12.7.60}$$

$$I = (i_1, \dots, i_k), 1 \le i_1 < \dots < i_k \le d$$

we define

$$\|\phi\|^2 = \sum_I \|\phi_I\|^2_{\mathscr{L}^2(X)} \qquad \text{and} \qquad \|\begin{pmatrix}\phi\\ \psi\end{pmatrix}\|^2 = \|\phi\|^2 + \|\psi\|^2. \tag{12.7.61}$$

Let us consider an operator

$$A = \begin{pmatrix} 0 & \beta^\dagger \mathsf{d}^* \alpha^\dagger \\ \alpha \mathsf{d} \beta & 0 \end{pmatrix} \tag{12.7.62}$$

with domain

$$\mathfrak{D}(A) = \{\begin{pmatrix}\phi\\ \psi\end{pmatrix} \in \mathfrak{H}, Au \in \mathfrak{H}, \iota_{\partial X}\phi = 0\} \tag{12.7.63}$$

where $\mathsf{d} = i\langle dx, D\rangle\wedge : \mathscr{C}^\infty(X, \Lambda^k) \to \mathscr{C}^\infty(X, \Lambda^{k+1})$ is the operator of the exterior differentiation and $\mathsf{d}^* : \mathscr{C}^\infty(X, \Lambda^{k+1}) \to \mathscr{C}^\infty(X, \Lambda^k)$ is the formally adjoint operator[71]; for ϕ in form (12.7.60) we have

$$\mathsf{d}\phi = \sum_j (iD_j\phi_I) dx_j \wedge dx_I, \qquad \mathsf{d}^*\phi = \sum_{1\le p\le k} (iD_{i_p}\phi_I)(-1)^p dx_{I\setminus i_p}, \tag{12.7.64}$$

(12.7.65) $\alpha(x) \in \mathcal{L}(\Lambda^{k+1}, \Lambda^{k+1})$; and $\beta(x) \in \mathcal{L}(\Lambda^k, \Lambda^k)$ are nondegenerate matrices smoothly depending on x and constant close to infinity (or quickly stabilizing to constant),

---

[70] At a given point, $\dim \Lambda^k = \frac{d!}{k!(d-k)!}$ and we consider complex rather than real spaces.

[71] We can define action of $\mathcal{D}$, $\mathcal{D}^*$ and $\mathcal{A}$ at distributions as well.

$\iota_Y : \mathscr{C}^\infty(X, \Lambda^k(X)) \to \mathscr{C}^\infty(X, \Lambda^k(Y))$ is the restriction of the exterior form to submanifold Y [72].

There is no problem with self-adjoint expansion of such an operator defined on functions with compact support first provided ∂X, α and β are smooth enough. Furthermore, we assume that

(12.7.66) α and β are Hermitian positive matrices

(otherwise one can reach it by means of the unitary transformation $\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ with unitary matrices $T_1(x)$ and $T_2(x)$).

Remark 12.7.9. (i) Let $\begin{pmatrix} \phi \\ \psi \end{pmatrix}$ be an eigenfunction with eigenvalue $\lambda \neq 0$. Then automatically

$$\mathsf{d}\alpha^{-1}\psi = \mathsf{d}^*\beta^{-1}\phi = 0 \tag{12.7.67}$$

and further,

$$\iota_{\partial X}\alpha^{-1}\psi = 0. \tag{12.7.68}$$

(ii) Further, $\begin{pmatrix} \phi \\ -\psi \end{pmatrix}$ is an eigenfunction with an eigenvalue $(-\tau)$.

Then we conclude that

Remark 12.7.10. (i) The number of eigenvalues of operator A belonging to $(0, \tau)$ is

$$\mathsf{N}_k(\tau) = \mathsf{N}^-_{\Delta_{k,0}}(\tau^2) - \mathsf{N}^-_{\Delta_{k,0}}(0+0) \tag{12.7.69}$$

where $\mathsf{N}^-_{\Delta_{k,0}}(\lambda)$ is the number of the eigenvalues of operator $\Delta_{k,0}$ generated by a quadratic form

$$Q(\phi) = \|\alpha\mathsf{d}\beta\phi\|^2 + \|\alpha_1^{-1}\mathsf{d}^*\beta^{-1}\phi\|^2 \tag{12.7.70}$$

on the space

$$\mathbb{H}_{k,0} = \{\phi \in \mathscr{L}^2(X, \Lambda^k), \mathsf{d}^*\beta^{-1}\phi = 0\} \tag{12.7.71}$$

[72] One can see easily that if Y is smooth of codimension 1 and $\phi \in \mathscr{L}^2(X, \Lambda^K)$, $\mathsf{d}\phi \in \mathscr{L}^2$ then $\iota_Y\phi$ is well-defined; if $Y = \{x_1 = 0\}$ in appropriate coordinates and ϕ is of the form (12.7.60) then $\iota_Y\phi = \sum_{I \not\ni 1} \phi_I|_Y dx_I$.

with domain

$$\mathfrak{D}(Q) = \{\phi \in \mathbb{H}_{k,0},\, Q(\phi) < \infty, \iota_{\partial X}\beta\phi = 0\} \tag{12.7.72}$$

which are less than λ; later in our conditions it will be a finite number; $\alpha_1 \in \mathcal{L}(\Lambda^{k-1}, \Lambda^{k-1})$ is a nondegenerate matrix (usually constant close to infinity).

(ii) Furthermore, considering the same form $Q(u)$ on the space

$$\mathbb{H}_k = \{\phi \in \mathscr{L}^2(X, \Lambda^k)\} \tag{12.7.73}$$

with domain

$$\mathfrak{D}(Q) = \{\phi \in \mathbb{H}_k \colon Q(\phi) < \infty,\ \iota_{\partial X}\beta\phi = 0\} \tag{12.7.74}$$

we get an operator Δ_k; one can check easily that for $\tau > 0$ eigenspaces $\mathbb{H}_k(\tau)$ and $\mathbb{H}_{k,0}(\tau)$ of $\Delta_{k,0}$ and Δ_k satisfy

$$\mathbb{H}_k(\tau) \subset \mathbb{H}_{k,0}(\tau), \tag{12.7.75}$$

$$\mathbb{H}_k(\tau) \ominus \mathbb{H}_{k,0}(\tau) = \beta^{-1}\mathsf{d}\alpha_1^{-1}\big(\alpha_1^{-1}\mathsf{d}^*\beta^{-1}\mathbb{H}_k(\tau)\big) \tag{12.7.76}$$

and that $\alpha_1^{-1}\mathsf{d}^*\beta^{-1}\mathbb{H}_k(\tau)$ is an eigenspace of the operator $\Delta_{k-1,0}$ generated by a quadratic form $\|\beta^{-1}\mathsf{d}\alpha_1^{-1}\phi'\|^2 + \dots$ on the space

$$\mathbb{H}_{k-1,0} = \{\phi \in \mathscr{L}^2(X, \Lambda^{k-1}), \mathsf{d}^*\alpha_1\phi = 0, \iota_{\partial X}\alpha_1^{-1}\phi = 0\}. \tag{12.7.77}$$

(iii) Therefore

$$\mathsf{N}(\tau) = \mathsf{N}^-_{\Delta_k}(\tau^2) - \mathsf{N}^-_{\Delta_k}(0+0) - \mathsf{N}^-_{\Delta_{k-1,0}}(\tau^2) + \mathsf{N}^-_{\Delta_{k-1,0}}(0+0) \tag{12.7.78}$$

(iv) This reduction is not necessarily correct for non-smooth problems unless we are able to prove that $\mathfrak{D}(\Delta_k) \subset \mathscr{H}^2_{\mathsf{loc}}(X)$ which is not true even for $\alpha = \beta = I$ and $X = W \oplus \mathbb{R}^{d-2}$ with a sector W with an angle between π and 2π.

(v) Note that for $k = d-1$, $\alpha = I$, $\beta = I$ and $\psi = u dx_1 \wedge \dots \wedge dx_d$ we obtain exactly eigenvalue problem for the Neumann Laplacian.

Remark 12.7.11. (i) These formulae lead us to standard (known) asymptotics for compact domains with smooth boundaries:

$$\mathsf{N}(\tau^2) = \kappa_0 \tau^d + O(\tau^{d-1}) \tag{12.7.79}$$

(here and below we omit indices k and may be 0) and even

$$\mathsf{N}(\tau^2) = \kappa_0 \tau^d + \kappa_1 \tau^{d-1} + o(\tau^{d-1}) \tag{12.7.80}$$

provided

$$\det\Big(\tau^2 - \beta \mathsf{d}(\xi)^\dagger \alpha^2 \mathsf{d}(\xi)\beta\Big) = \tau^r \big(\tau^2 - g(x,\xi)\big)^s \tag{12.7.81}$$

where $\mathsf{d}(\xi) = i\langle dx, \xi\rangle\wedge$ is a principal symbol of d, g is a metrics on X and standard billiard condition holds.

(ii) The same results hold for other types of compact domains: as long as formula (12.7.78) holds, the irregularity of the boundary should be of the type described in Sections 11.2 and 11.3 and inner cone condition should be fulfilled.

Thin Cusps. Heuristics

Let us consider domains with cusps; we consider one cusp for simplicity. So we consider domains of the type described in Section 12.2.

Look first for operators Δ_k. As we already know, boundary conditions are very important for the Laplace operator: if we have the Dirichlet boundary condition then the spectrum of operator is discrete (provided cusp shrinks at infinity).

On the other hand for operator with a Neumann boundary condition spectrum is discrete only if cusp is very thin ($\log \sigma \asymp |x'|^{\mu+1}$ with $\mu > 0$) and for such operators asymptotics with sharp remainder estimate are derived in Subsections 12.7.1–12.7.5.

So basically we should determine first if for operator Δ_k condition is "Dirichlet"-like or "Neumann"-like at infinity; then for "Neumann"-like cusp assume that it is ultra-thin and write the cusp contribution; for "Dirichlet"-like cusp we need to describe non-Weyl contribution.

Let us consider the space

$$\Phi_k = \big\{\phi \in \mathscr{C}^\infty(Y, \Lambda^k(X)), \quad \mathsf{d}''\beta\phi = \mathsf{d}''^*\beta^{-1}\phi = 0, \iota_{Z\times\partial Y}\beta\phi = 0\big\} \tag{12.7.82}$$

(so we consider full forms with coefficients depending on x'' only); one can see easily that $\mathsf{dim}\,\Phi_k$ does not depend on the choice of matrix β and

$$\mathsf{dim}\,\Phi_k = \sum_j \frac{d'!}{j!(d'-j)!}\,\mathsf{dim}\,\Phi''_{k-j} \tag{12.7.83}$$

where

$$\Phi''_j = \bigl\{\phi \in \mathscr{C}^\infty(Y, \Lambda(Y)^j), \mathsf{d}''\phi = \mathsf{d}''^*\phi = 0, \iota_{\partial Y}\phi = 0\bigr\}. \tag{12.7.84}$$

From the point of view of operator Δ_k the cusp is "Dirichlet"-like iff $\mathsf{dim}\,\Phi_k = 0$; we will discuss this case later and assume that $\mathsf{dim}\,\Phi_k \geq 1$. Let us consider operator

$$\ell_k(x', D') = \boldsymbol{P}_k\beta M_k(x', D')^*\alpha^2 M_k(x', D')\beta + \beta^{-1}M_{k-1}(x', D')\alpha_1^{-2}M_{k-1}(x', D')^*\beta^{-1}\phi \tag{12.7.85}$$

acting from $\mathscr{C}^\infty(Z, \Phi_k)$ to $\mathscr{C}^\infty(Z, \Phi_k)$ with

$$M(x', D') = \mathsf{d}' + \frac{d''}{2}(D'\log\sigma)\wedge \tag{12.7.86}$$

acting from : $\mathscr{C}^\infty(Z, \Phi_k)$ to $\mathscr{C}^\infty(Z, \Lambda(Y)^{k+1})$, $M(x', D')^*$ a formally adjoint operator and $\boldsymbol{P}_k$ orthogonal projector from $\mathscr{L}^2(Y, \Lambda^k)$ to Φ_k.

Then the cusp term for operator Δ_k will be

$$\mathsf{N}_{k,\mathsf{cusp}}(\tau) = (2\pi)^{-d'}\iint \boldsymbol{n}_k(x', \xi', \tau^2)\,dx'd\xi' \tag{12.7.87}$$

where $\boldsymbol{n}_k(x', \xi', \tau)$ is the eigenvalue counting function for finite-dimensional[73] symbol $\ell_k(x', \xi')$ of an operator $\ell_k(x', D')$.

Applying then formula (12.7.78) we conclude that the cusp contribution for operator $\Delta_{k,0}$ is given by the formula (12.7.87) with $\boldsymbol{n}_k(x', \xi', \tau^2)$ replaced by $\boldsymbol{n}_{k,0}(x', \xi', \tau^2)$ which is the eigenvalue counting function for $\mathcal{L}(\Phi_{k,0}, \Phi_{k,0})$-valued symbol $\ell_{k,0}(x', \xi') = \boldsymbol{P}_{k,0}(x', \xi')\ell_{k,0}(x', \xi')$; here

$$\Phi_{k,0}(x', \xi') = \bigl\{\phi \in \Phi_k, M_{k-1}(x', \xi')^*\beta^{-1}\phi = 0\bigr\} \tag{12.7.88}$$

and $\boldsymbol{P}_{k,0}(x', \xi')$ is the orthogonal projection onto $\Phi_{k,0}(x', \xi')$.

[73] $\mathcal{L}(\Phi_k, \Phi_k)$-valued.

One can see easily that

$$\dim \Phi_{k,0} = \dim \Phi_k - \dim \Phi_{k-1,0}, \qquad \dim \Phi_{0,0} = 0 \tag{12.7.89}$$

and (12.7.83), (12.7.89) yield that

$$\dim \Phi_{k,0} = \sum_j \frac{(d'-1)!}{j!(d'-1-j)!} \dim \Phi''_{k-j}. \tag{12.7.90}$$

Furthermore, for $\alpha = I$, $\beta = I$ near infinity

$$\mathsf{N}_{k,\mathsf{cusp}}(\tau^2) = (\dim \Phi_k) N_{\mathsf{cusp}}(\tau^2), \tag{12.7.91}$$

$$\mathsf{N}_{k,0,\mathsf{cusp}}(\tau^2) = (\dim \Phi_{k,0}) N_{\mathsf{cusp}}(\tau^2) \tag{12.7.92}$$

where $\mathsf{N}_{\mathsf{cusp}}(\tau^2)$ is a cusp contribution to the asymptotics for Neumann Laplacian on $\mathscr{L}^2(X, \mathbb{C})$.

One can expect that Δ_k or $\Delta_{k,0}$ are "Neumann-like" iff $\dim \Phi_k \neq 0$ or $\dim \Phi_{k,0} \neq 0$ respectively.

Thin cusps. Results

Theorem 12.7.12 [74]**.** *Let* <u>*either*</u> $\Delta := \Delta_k$ *and* $\dim \Phi_k \geq 1$ <u>*or*</u> $\Delta = \Delta_{k,0}$ *and* $\dim \Phi_{k,0} \geq 1$. *Let*

$$\alpha = I, \ \beta = I \qquad \text{as} \ \ |x| \geq C \tag{12.7.93}$$

and let conditions (12.7.19)–(12.7.21) *be fulfilled with* $\mu > 0$. *Then*

(i) For $d'' \geq 2$ *the following asymptotics holds*

$$\mathsf{N}(\tau) = \varkappa_0 \tau^d + \mathsf{N}_{\mathsf{cusp}}(\tau^2) + O\bigl(\tau^{d-1}\bigr) + O\bigl(\tau^q\bigr) \tag{12.7.94}$$

where $\varkappa_0$ *and* $\varkappa_1$ *(see below) are standard Weyl coefficients,* $\mathsf{N}_{\mathsf{cusp}}(\tau) \asymp \tau^q$ *is defined by* (12.7.87) *for Neumann Laplacian,* $p = d'(\mu+1)/\mu$, $q = (d'-1)(\mu+1)/\mu$ *and we skip operator-related indices* k *and may be* 0.

(ii) For $d'' = 1$ *the following asymptotics holds*

$$\mathsf{N}(\tau) = \varkappa_0 \tau^d + \mathsf{N}_{\mathsf{cusp}}(\tau^2) + O\bigl(\tau^{d-1}(\log \tau)^{(d-1)/(\mu+1)}\bigr) + O(\tau^q). \tag{12.7.95}$$

[74] Cf. Theorem 12.7.4.

(iii) Moreover, if $d'' \ge 2$, (12.7.81) *holds,* $\Delta = \Delta_{k,0}$ *(and similar condition for* $\Delta = \Delta_k$*) and standard Hamiltonian condition is fulfilled, then one can replace* $O(\tau^{d-1})$ *in asymptotics* (12.7.94) *by* $(\varkappa_1 + o(1))\tau^{d-1}$.

(iv) Moreover, as $d'' \ge 2$ *and stabilization and non-periodicity condition of Theorem 12.7.4(iv) are fulfilled then one can replace* $O(\tau^q)$ *by* $\big(\varkappa_1^* + o(1)\big)\tau^q$.

Theorem 12.7.13 [75]**.** *Let* $d'' = 1$, $\mu \ge d-2$ *and conditions* (12.7.93) *and* (12.7.19)–(12.7.21) *be fulfilled and let* $\log \sigma$ *be positively homogeneous of degree* $(\mu+1)$. *Then*

(i) Asymptotics

$$\mathsf{N}(\tau) = c_0\tau^d + \nu(\tau^2) + N_{\mathsf{cusp}}(\tau^2) + O\big(\lambda^{fracd-12}\big) \tag{12.7.96}$$

holds with

$$\nu(\tau^2) = \varkappa_1'\tau^{d-1}(\log\tau)^{(d-1)/(\mu+1)} + O(\tau^{d-1}) \tag{12.7.97}$$

the Weyl expression for second order term in domain $\{\sigma(x')\tau \le 1\}$.

(ii) Moreover, if $\mu > d-2$, (12.7.81) *holds,* $\Delta = \Delta_{k,0}$ *(and similar condition for* $\Delta = \Delta_k$*) and standard Hamiltonian condition is fulfilled, then one can replace* $O(\tau^{d-1})$ *in asymptotics* (12.7.96) *by* $\big(\kappa_1 + o(1)\big)\tau^{d-1}$.

(iii) Further, as $d'' = 1$, $\mu = d-2$ *and stabilization and non-periodicity condition of Theorem 12.7.6(iii) are fulfilled then one can replace* $\nu(\tau) + O(\tau^q)$ *by* $\nu_1(\tau) + o(\tau^q)$.

Remark 12.7.14. One can weaken condition (12.7.93). Moreover, same asymptotics holds for manifolds and for manifolds with compact boundary one can skip condition (12.7.93).

Thick Cusps. Sketch

In the case of the "Dirichlet"-like' cusp condition of being ultra-thin is no longer necessary and in this case we leave to the reader rather interesting and challenging

Problem 12.7.15. Derive asymptotics with the sharp remainder estimates exactly in the same type as in Section 12.2.

[75] Cf. Theorem 12.7.6.

The only one real difficulty is in the case of operator $\Delta_{k,0}$ as $\dim \Phi_{k,0} = 0$ and $\dim \Phi_k \ge 1$ but one can overcome it by taking α_1 fast growing rather than constant at infinity.

Another difficulty is that we need to study operators with the operator-valued symbols with domain depending on (x', ξ') but it already was discussed in Problem 12.2.21.

12.7.7 Generalizations

Higher Order Operators–Discussion

Remark 12.7.16. As we mentioned in Problem 12.2.21 results of Section 12.2 should hold for appropriate boundary value problem such that all eigenvalues $\lambda_k(x', \xi')$ of $\mathsf{a}(x', \xi')$ satisfy (12.3.14) (actually this is condition only for few lowest eigenvalues).

Let us recall that $\mathsf{a}(x', \xi')$ is an operator obtained by replacing in $A(x, D)$ $D' \mapsto (x', \xi')$ and $(x'', D'') \mapsto (\sigma(x')x'', \sigma(x')^{-1}D'')$ and considered on Y where Y is a fixed domain in $\mathbb{R}^{d''}$ defining the shape of the cusp cross-section.

Now we want to consider the case when condition (12.3.14) fails for few lowest eigenvalues. This failure can be of very different nature: it may fail just for some x' or fail mildly in the sense that $\lambda_1(x', \xi') \ge (|\xi'| + \sigma(x')^{-\theta})^{2m}$; and the failure is basically very unstable.

Consider operator

$$\sum_{\alpha,\beta:|\alpha|=m,|\beta|=m} D^\alpha a_{\alpha,\beta}(x) D^\beta \tag{12.7.98}$$

defined by the corresponding quadratic form

$$\int \sum_{\alpha,\beta:|\alpha=m,|\beta|=m} \langle a_{\alpha\beta}(x) D^\beta u, D^\alpha u\rangle \, dx \tag{12.7.99}$$

with the boundary condition

$$D^\alpha u|_{\partial X} = 0 \qquad \forall \alpha : |\alpha| \le n-1. \tag{12.7.100}$$

Remark 12.7.17. Note that different quadratic forms may lead to the same differential expression but to different boundary conditions f.e. consider

$$Q_1(u) = \|D_1^2 u\|^2 + \|D_2^2 u\|^2 + 2\|D_1 D_2 u\|^2 \tag{12.7.101}$$

and

$$Q_2(u) = \|D_1^2 u\|^2 + \|D_2^2 u\|^2 + 2\,\mathsf{Re}(D_1^2 u, D_2^2 u) \tag{12.7.102}$$

leading to operator Δ^2 but with the different boundary conditions (as $n = 0$ in (12.7.100)) and only (12.7.101) leads to elliptic problem. However as $n = 1$ these forms coincide and lead to elliptic problem.

Assume that quadratic form

$$q(x)(\mathbf{v}) = \sum_{\alpha,\beta:|\alpha=m,|\beta|=m} \langle a_{\alpha\beta} v_\beta, v_\alpha\rangle \tag{12.7.103}$$

satisfies

(12.7.104) $q(x)(\mathbf{v}) \asymp \|\mathbf{v}\|^2$ uniformly with respect to x' and $\mathbf{v} = (v_\alpha)_{|\alpha|=m}$.

Only form (12.7.101) satisfies this condition.

Then we need to consider $u(x'; x'')$ which are polynomial of degree $(m-1)$ with respect to x'' and check if such polynomials satisfying (12.7.100) must be identically 0. The existence of non-zero such polynomials is rather unstable. So if originally Y depended mildly on x' and we arrive to constant Y by a change of variables x'' then the resulting operator will not be of form (12.7.98) (lower order term come into play) and violation of assumption (12.3.14) for the lowest λ_k could be mild.

Let us consider quadratic forms after rescaling (which means that originally these forms contained lower order terms)

$$Q_1(u) = \|D_1^2 u\|^2 + \|\sigma^{-2} D_2^2 u\|^2 + 2\|\sigma^{-1} D_1 D_2 u\|^2 + (Wu, u) \tag{12.7.105}$$

and

$$Q_2(u) = \|D_1^2 u\|^2 + \|\sigma^{-2} D_2^2 u\|^2 + (Wu, u) \tag{12.7.106}$$

with $\sigma = \sigma(x_1)$ rapidly decaying at infinity and $W = W(x_1)$ polynomially growing at infinity.

Obviously only form (12.7.105) satisfies (12.7.104) after rescaling back; however it does not prevent problem associated with (12.7.106) from being uniformly elliptic.

Consider quadratic forms with D_1 replaced by ξ_1 and integration over x_2 only. Both forms behave the same way on $v = \mathsf{const}$; so for both forms $\lambda_1(x_1, \xi_1) = \xi_1^4 + W(x_1)$. However on $v = \mathsf{const} x_2$ we get $\lambda_2(x_1, \xi_1) = \xi_1^4 + c^2\xi_1^2\sigma^{-2} + W(x_1)$ and $\lambda_2(x_1, \xi_1) = \lambda_1(x_1, \xi_1) = \xi_1^4 + W(x_1)$ for forms (12.7.105) and (12.7.106) respectively.

We leave to the reader the following series of the problems

Problem 12.7.18. (i) Develop theory covering (12.7.105); in particular explore if $\lambda_2(x', \xi')$ produces any noticeable contribution to $\mathsf{N}_{\mathsf{cusp}}(\tau)$.

For this purpose one needs to construct the theory similar to one of Section 11.5 but for matrix operators with eigenvalues of the symbol having different growth as (x', ξ') escaping to infinity.

(ii) Develop theory covering (12.7.106). This theory seems to be rather straightforward and easy.

More General cCusps

One can consider more general cusps. We believe that the first problem is rather challenging:

Problem 12.7.19. Consider cusps as in Problem 12.2.23(i) and (ii) albeit with "superexponential" as in this section rather than power decay as in Section 12.2 and with the Neumann-like rather than the Dirichlet-like boundary conditions.

The next problem seems to be rather easy:

Problem 12.7.20. Analyze the case when a_0 is a Schrödinger operator with a potential having valleys as in Section 12.3.

It will be the case as $|\nabla \log \sigma| \asymp |x'|^\nu |x'''|^\varsigma$ where $x''' = (x_{d'''+1}, \dots, x_{d'})$ and $1 \le d''' \le d'$, $\nu > 0$, $\varsigma > 0$; then $|\log \sigma| \asymp |x'|^\nu |x'''|^{\varsigma+1}$ and thus $\sigma \cong \exp\bigl(-|x'|^\nu |x'''|^{\varsigma+1}\bigr)$.

Domains with Spikes

We consider domains with the *spikes* which are "cusps of the finite length" (see Figure 11.2). Now Z is a compact domain with a smooth boundary and

we can assume locally that $Z = \{x_1 > 0\}$ and $X = \{x : x' \in Z, x'' \in \sigma(x')Y\}$ with compact domain Y with the smooth boundary ∂Y and $\sigma(x_1) \le cx_1$,

$$|D^\alpha \nu(x')| \le c\sigma(x')|x_1|^{-|\nu|}. \tag{12.7.107}$$

We also assume that

$$|D^\nu a_{\alpha\beta}(x')| \le c|x_1|^{-|\nu'|+\mu(2m-|\alpha|-|\beta|)}\sigma(x')^{-|\nu''|} \tag{12.7.108}$$

with $\mu > \max(-1, -d/2m)$.

We consider quadratic form with the boundary terms satisfying conditions

$$|D^\alpha a'_{\alpha\beta}(x')| \le c|x_1|^{-|\nu'|+\mu(2m-2-|\alpha|-|\beta|)}\sigma(x')^{-|\nu''|}. \tag{12.7.109}$$

The following problem seems to be rather easy:

Problem 12.7.21. Using arguments of this section prove that for such domains under Neumann (or reduced order Dirichlet leading to mixed boundary conditions) standard asymptotics as in the domains with the smooth boundaries hold (despite cone condition does not necessarily hold).

More precisely, after coordinates $x''_{\mathsf{new}} = x''_{\mathsf{old}}/\sigma(x')$ transforming spike into finite cylinder $Z \times Y$ singular at $x_1 = 0$ but with singularities not violating positivity. Moreover the possible degeneration of the lowest eigenvalues of $A(x'', \xi''; x'', D'')$ is compensated by lower order term $W(x')u$ with $W \ge \epsilon|x_1|^{-2m}$.

12.7.8 Operators with Degenerate Potentials

While not covered by this section, the following topic is related:

Example 12.7.22. Consider Schrödinger operator

$$A = D_1^2 + |D''|^2 + V(x), \qquad V(x) = e^{2|x_1|^{\mu+1}}|x''|^2 - e^{|x_1|^{\mu+1}}k \tag{12.7.110}$$

with $k = d''$. Without degeneration such operators were considered in Section 11.5; one can see easily that even with this degeneration but with $k < d''$ such operators would be still covered by the general theory of that section.

However now the lower eigenvalue of $|D''|^2 + V(x)$ is 0. Then changing variables $x''_{\mathsf{new}} = x''_{\mathsf{old}} \exp(|x_1|^{\mu+1})$ we arrive to the operator not much different from

$$A = D_1^2 + cx_1^{2\mu} + e^{x_1^{\mu+1}}B, \qquad \text{with } \ B = |D''|^2 + |x''|^2 - d''. \tag{12.7.111}$$

Problem 12.7.23. Develop the theory, covering Example 12.7.22.

12.8 Periodic Operators

In this section we consider number $\mathsf{N}(\tau)$ of eigenvalues of operator $A_t(x,D) = A(x,D) - tW(x,x)$ crossing level E as t runs from 0 to τ, $\tau \to \infty$. Here A is periodic matrix operator, matrix W is positive, periodic with respect to the first copy of x and decaying as second copy of x goes to infinity, E either belongs to a spectral gap of A or is one its ends.

12.8.1 Periodic Operators

Let us consider in $\mathbb{R}^d$ $\mathsf{D}\times\mathsf{D}$-matrix operator $A(x,x,D) = A^{\mathsf{w}}(x,x,D)$ with the Weyl symbol

$$A(z,y,\xi) = \sum_{|\sigma|\le m} a_\sigma(z,y)\xi^\sigma \tag{12.8.1}$$

where

(12.8.2) Operator $A(z,y,D)$ is z-periodic with period lattice

$$\Gamma = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \dots \mathbb{Z}e_d \tag{12.8.3}$$

and transformation matrices $\{T_1,\dots,T_d\}$, which means that

$$A(z+e_j,y,D) = T_j^\dagger A(z,y,D)\,T_j \qquad \forall j = 1,\dots,d \tag{12.8.4}$$

with unitary commuting matrices T_j.

Let $T^n = T_1^{n_1}\cdots T_d^{n_d}$ for $n = n_1e_1 + \cdots + n_de_d \in \Gamma$.

Further, let us assume that

$$|\nabla_z^\alpha \nabla_y^\beta \bigl(a_\sigma(z,y) - a_\sigma^\infty(z)\bigr)| \le C\langle y\rangle^{-\delta-|\beta|} \qquad \forall \alpha : |\alpha| \le K\ \forall |\beta| : |\beta| \le K \tag{12.8.5}$$

$$|\nabla_z^\alpha a_\sigma^\infty(z)| \le C \qquad \forall \alpha : |\alpha| \le K \tag{12.8.6}$$

with $\delta > 0$ and large enough $K = K(m,d,\mu)$ where

(12.8.7) Operator $A^\infty(z, D_z)$ is z-periodic with period lattice (12.8.3) and transformation matrices $\{T_1, \dots, T_d\}$.

Furthermore, as usual we assume that A is elliptic:

$$|A(x,\xi)v| \ge \epsilon_0\big(|\xi|^m - C\big)|v| \qquad \forall x, \xi \in \mathbb{R}^d \ \forall v \in \mathbb{C}^D \tag{12.8.8}$$

and that

$$A(x,\xi)^\dagger = A(x,\xi) \qquad \forall (x,\xi). \tag{12.8.9}$$

To describe the spectrum of operator A^∞ (and the essential spectrum of A) we need the following definition:

Definition 12.8.1. Function $v(x)$ is *quasiperiodic with quasimomentum* ξ *and transformation matrices* $\{T\} = \{T_1, \dots, T_d\}$ if

$$u(x+n) = T^{-n}u(x)e^{i\langle n,\xi\rangle} \qquad \forall n \in \Gamma. \tag{12.8.10}$$

We denote by $\mathbb{H}_{\xi,\{T\}}$ the space of such functions with an inner product as in $\mathscr{L}^2(\mathcal{Q}, \mathbb{C}^d)$ but multiplied by $(2\pi)^{-d}\operatorname{vol}(\mathcal{Q}')$ where $\mathcal{Q} = [0,1]e_1 \oplus \dots [0,1]e_d$ is an *elementary cell*, $\mathcal{Q}' = [0,1]e_1' \oplus \dots \oplus [0,1]e_d'$ is a *dual elementary cell* and $\langle e_j', e_k\rangle = 2\pi\delta_{jk} \ \ \forall j,k$.

Then we arrive to the following proposition:

Proposition 12.8.2. *Both operators A and A^∞ are self-adjoint in $\mathscr{L}^2(\mathbb{R}^d, \mathbb{C}^{\mathsf{D}})$ and both essential spectra of A and A^∞ and spectrum of A^∞ coincide with*

$$\mathcal{E} := \bigcup_k \mathcal{E}_k, \qquad \mathcal{E}_k := \bigcup_{\xi\in\mathcal{Q}'} \lambda_k(\xi) \tag{12.8.11}$$

where $\lambda_k(\xi)$ are eigenvalues of operator A^∞ restricted to $\mathbb{H}_{\xi,\{T\}}$.

Remark 12.8.3. (i) Proposition 12.8.2 as well as Statements (i)–(iv) below are well-known and their proofs are easy.

(ii) k runs $\mathcal{K}$ where $\mathcal{K} = \{1,2,3,\dots\}$ if and only if A^∞ is semi-bounded from below, $\mathcal{K} = \{0,-1,-2,\dots\}$ if and only if A^∞ is semi-bounded from above and $\mathcal{K} = \mathbb{Z}$ otherwise.

(iii) Spectrum of A^∞ restricted to $\mathbb{H}_{\xi,\{T\}}$ is discrete and consists of eigenvalues $\lambda_k(\xi)$.

(iv) Functions $\lambda_k(\xi)$ are periodic with respect to the *dual lattice* $\Gamma' = \mathbb{Z}e'_1 \oplus \mathbb{Z}e_2 \oplus ... \mathbb{Z}e'_d$ and thus we can consider them as defined on $\mathcal{Q}'$.

(v) Without any loss of the generality we assume that

$$\lambda_k(\xi) \ge \lambda_{k-1}(\xi) \qquad \forall k. \tag{12.8.12}$$

So, $\mathsf{Spec}(A^\infty)$ has is *zone spectrum* with possible overlapping of *spectral zones* (*spectral bands*)

$$\mathcal{E}_k = [\min_\xi \lambda_k(\xi), \max_\xi \lambda_k(\xi)] \tag{12.8.13}$$

separated by *spectral gaps*

$$\mathcal{G}_k := (\max_\xi \lambda_k(\xi), \min_\xi \lambda_{k+1}(\xi)) \tag{12.8.14}$$
$$\text{provided} \quad \max_\xi \lambda_k(\xi) < \min_\xi \lambda_{k+1}(\xi).$$

If operator A^∞ is semibounded from below (above) we introduce also the spectral gap $\mathcal{G}_0 := (-\infty, \mathsf{min}_\xi\, \lambda_1(\xi))$ or $\mathcal{G}_0 := (\mathsf{max}_\xi\, \lambda_0(\xi), \infty)$ respectively.

It follows from Proposition 12.8.2 that

Corollary 12.8.4. $\mathsf{Spec}(A)$ *intersected with the spectra gap* $\mathcal{G}_k$ *is discrete and may be accumulating to one or both ends of it.*

12.8.2 Statement of the Problem

Let us pick some energy level which is in the spectral gap: $E \in \mathcal{G}_k$.

Later we consider also cases when E is either lower end or upper end of the spectral gap: $[E, E+\epsilon) \cap \mathsf{Spec}(A^\infty) = \{E\}$ or $(E-\epsilon, E] \cap \mathsf{Spec}(A^\infty) = \{E\}$.

Let $W(x) = W(x, x)$ where

(12.8.15) Matrices $W(z, y)$ and $W^\infty(z, y)$ are Hermitian and z-periodic with the period lattice Γ and transformation matrices $\{T\} = \{T_1, ..., T_d\}$ with respect to x,

and

$$|\nabla_z^\alpha \nabla_y^\beta W(z, y)| \le C\langle y\rangle^{-m\mu-|\beta|} \tag{12.8.16}$$
$$\forall \alpha : |\alpha| \le K \ \forall |\beta| : |\beta| \le K, \qquad \mu > 0$$

$$|\nabla_z^\alpha \nabla_y^\beta \big(W(z,y) - W^\infty(z,y)\big)| = o(|y|^{-m\mu-|\beta|}) \quad \text{as } |y| \to \infty \quad \forall \beta : |\beta| = 1 \tag{12.8.17}$$

$$\langle W(z,y)v, v\rangle \ge C|v|^2\langle y\rangle^{-m\mu} \qquad \forall x, y \in \mathbb{R}^d \ \forall v \in \mathbb{C}^d \tag{12.8.18}$$

where

(12.8.19) $W^\infty(z,y)$ is positive homogeneous with respect to y of degree $-m\mu$ and satisfies condition (12.8.18) as well.

In the the framework of the previous assumptions let us consider operator $A_t = A - tW$ with $t > 0$. Then

Remark 12.8.5. (i) Let E reside within spectral gap; then for each t only finite number of eigenvalues of A_t belong to $(E-\epsilon, E+\epsilon)$;

(ii) Further, if E is the lower end of the spectral gap then under appropriate conditions for each t only finite number of eigenvalues of A_t belong to $(E, E+\epsilon)$;

(iii) Similarly, if E is the upper end of the spectral gap then under appropriate conditions for each t only finite number of eigenvalues of A_t belong to $(E-\epsilon, E)$;

(iv) All the eigenvalues are monotone decreasing functions of t.

Let $\mathsf{N}_E(\tau)$ be a number of the eigenvalues of $A - tW$ (counting multiplicities) passing through the *observation point* E in in the framework of (i) (reaching E from above, leaving E to below respectively in the frameworks of (ii) and (iii)) as t changes from $0+$ to $\tau-$.

We are interested in asymptotics of $\mathsf{N}_E(\tau)$ as $\tau \to +\infty$. There is also a lite version of the problem when there is no periodic variable at all; then formally $\mathcal{Q} = \{0\}$ and $\mathcal{Q}' = \mathbb{R}^d$.

12.8.3 Reformulation of the Problem

We start from the following proposition:

Proposition 12.8.6. *In the framework of our assumptions*

(i) Equality

$$\mathsf{N}_E(t) = \mathsf{N}(L_E; [0,t)) \tag{12.8.20}$$

holds where $\mathsf{N}(L; [0,t))$ *is a number of eigenvalues of the operator*

$$L = L_E = W^{-\frac{1}{2}}(A-E)W^{-\frac{1}{2}} \tag{12.8.21}$$

belonging to $[0,t)$.

(ii) L_E *is a self-adjoint operator in* $\mathscr{L}^2(\mathbb{R}^d, \mathbb{C}^{\mathsf{D}})$.

Proof. Both statement are obvious for $A = A^\infty$ and

$$L_E^\infty = W^{-\frac{1}{2}}(A^\infty - E)W^{-\frac{1}{2}}$$

because $(A^\infty - E)$ is invertible if $E \notin \mathsf{Spec}(A^\infty)$ and this is true for perturbed operator L_E since L_E^∞-bound of the operator

$$L'_E = W^{-\frac{1}{2}}(A - A^\infty)W^{-\frac{1}{2}}$$

is 0 which means that

$$\|L'_E u\| \le \epsilon \|L_E^\infty u\| + C_\epsilon \|u\| \qquad \forall u\ \forall \epsilon > 0.$$

Furthermore,

$$\mathsf{Spec}_{\mathsf{ess}}(L_E) = \emptyset. \tag{12.8.22}$$

Remark 12.8.7. Moreover, even if E is the extreme of a spectral gap, conditions of our theorems will assure that L_E^∞ is self-adjoint unbounded operator with compact inverse and therefore all above statements (save that $(A^\infty - E)$ is invertible) remain true.

12.8.4 Reduction to Operators with Operator-Valued Symbols

One can try to apply arguments of Chapters 9 and 10 directly but they lead to sharp remainder estimates only if $\rho^{d-1} \in \mathscr{L}^1$ even in the framework of more relaxed Theorem 12.8.12 below. We need to involve operator-valued theory of Chapter 4) and we introduce Gelfand' transformation:

Definition 12.8.8. Let Γ and $\{T\}$ be introduced above. Then

$$\mathcal{F}u(\xi,x) = (2\pi)^{\frac{d}{2}}(\mathsf{vol}(\mathcal{Q}'))^{-1}\sum_{n\in\Gamma} T^n e^{-i\langle x-n,\xi\rangle}u(x-n). \tag{12.8.23}$$

is a *Gelfand' transformation* of u.

Proposition 12.8.9. *(i) As defined for x in $\mathbb{R}^d$ $\mathcal{F}(\xi,x)$ is x-quasiperiodic with quasimomentum ξ and transformation matrices $\{T\}$.*

(ii) $\mathcal{F}$ is an unitary operator from $\mathscr{L}^2(\mathbb{R}^d,\mathbb{C}^{\mathsf{D}})$ to $\mathscr{L}^2(\mathcal{Q}',\mathbb{H}_{\xi,\{T\}})$.

(iii) Inverse Gelfand's transform is defined by

$$u(x) = (2\pi)^{-\frac{d}{2}}\int_{\mathcal{Q}'} e^{i\langle x,\xi\rangle}(\mathcal{F}u)(\xi,x)\,d\xi. \tag{12.8.24}$$

Proof. Statement (i) is obvious.

Plugging (12.8.23) into the right-hand expression of (12.8.24) and using the first of equalities

$$\int_{\mathcal{Q}'} e^{i\langle n,\xi\rangle}\,d\xi = c\delta_{n0}, \qquad \sum_{n\in\Gamma} e^{i\langle n,\xi\rangle} = c\sum_{k\in\Gamma'}\delta(\xi-k), \qquad \text{with } \ c=\mathsf{vol}(\mathcal{Q}') \tag{12.8.25}_{1,2}$$

we get $u(x)$; similarly, plugging (12.8.24) into the right-hand expression of (12.8.23) with $\mathcal{F}u(\xi,x)$ replaced by $v(\xi,x)$ which is x-quasiperiodic with quasimomentum ξ and transformation matrices $\{T\}$ and using the $(12.8.25)_2$ we get $v(\xi,x)$. This proves Statement (iii).

Finally, calculating $\int_{\mathcal{Q}'}\|\mathcal{F}(\xi,x)\|^2_{\mathbb{H}_{\xi,\{T\}}}\,d\xi$ and using $(12.8.25)_1$ again we get $\sum_{n\in\Gamma}\int_{\mathcal{Q}}\|u(x+n)\|^2\,dx = \|u\|^2_{\mathscr{L}^2(\mathbb{R}^d,\mathbb{C}^{\mathsf{D}})}$ which proves Statement (ii). □

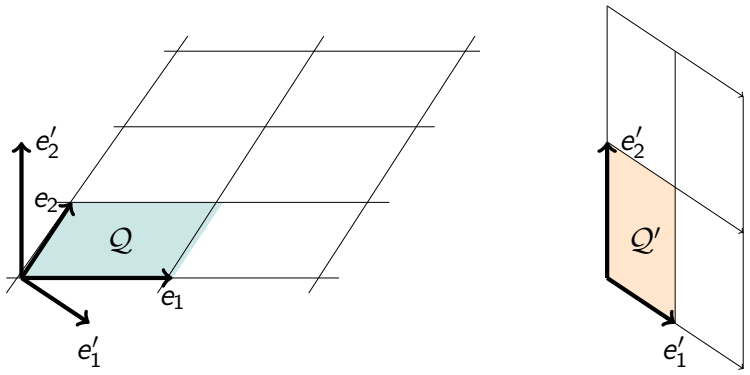
Remark 12.8.10. (i) Operator

$$u\to \mathcal{F}'u(x,\xi) \coloneqq (2\pi)^{\frac{d}{2}}(\mathsf{vol}(\mathcal{Q}'))^{-1}\sum_{n\in\Gamma} T^n e^{i\langle n,\xi\rangle}u(x-n) \tag{12.8.26}$$

with

$$u(x) = (2\pi)^{-\frac{d}{2}}\int_{\mathcal{Q}'}(\mathcal{F}'u)(\xi,x)\,d\xi \tag{12.8.27}$$

has the similar properties albeit transforms $\mathscr{L}^2(\mathbb{R}^d,\mathbb{C}^{\mathsf{D}})$ into $\mathscr{L}^2(\mathcal{Q}',\mathbb{H}_{0,\{T\}})$.



(a) Lattice (b) Dual lattice

Figure 12.7: Lattice, elementary cell $\mathcal{Q}$, dual lattice and dual elementary cell $\mathcal{Q}'$.

(ii) Consider $T = I$, replace Γ by $\varepsilon\Gamma$ and calculate limits as $\varepsilon \to +0$. Then $\mathcal{F}(\xi, x) \to (2\pi)^{d/2} F_{x\to\xi} u$.

Proposition 12.8.11. *Operator $\mathcal{F}$ transforms our operators in the following way:*

$$\mathcal{F} A^{\mathsf{w}} \mathcal{F}^* = A^{\mathsf{w}}(x - D_\xi, x, \xi + D_x), \tag{12.8.28}$$

$$\mathcal{F} A^{\infty\,\mathsf{w}} \mathcal{F}^* = A^{\infty\,\mathsf{w}}(x, \xi + D_x), \tag{12.8.29}$$

$$\mathcal{F} W \mathcal{F}^* = W^{\mathsf{w}}(x - D_\xi, x). \tag{12.8.30}$$

Proof. Proof is easy by direct calculations from (12.8.24). The only thing to notice is that we can treat the fast copy of x in $\mathbb{R}^d$ in different ways than the slow one using periodicity with respect to fast x. □

12.8.5 Main Theorem: E inside Spectral Gap

Theorem 12.8.12. *Let conditions* (12.8.2), (12.8.15), (12.8.5)–(12.8.9) *and* (12.8.16)–(12.8.18) *be fulfilled and let*

$$E \notin \mathsf{Spec}(A^\infty). \tag{12.8.31}$$

(i) Further, let us assume that there is an infinite number of $\lambda_k(\xi)$ exceeding E (or equivalently A is not semibounded from above). Then the following

asymptotics holds as $\tau \to +\infty$*:*

$$|\mathsf{N}(\tau) - \mathcal{N}(\tau)| \le R(\tau) = \begin{cases} \tau^{(d-1)/m} & \text{for } \mu > 1, \\ \tau^{(d-1)/m} \log \tau & \text{for } \mu = 1, \\ \tau^{(d-1)/(m\mu)} & \text{for } \mu < 1, \end{cases} \tag{12.8.32}$$

with

$$\mathcal{N}(\tau) = \int_{\mathbb{R}^d} \int_{\mathcal{Q}'} \boldsymbol{n}(y, \xi, \tau)\, d\xi dy \tag{12.8.33}$$

where $\boldsymbol{n}(y, \xi, \tau)$ *is the number of eigenvalues of operator* $A(x, y, D_x) - tW(x, y)$ *crossing* E *as* t *runs from* $0+$ *to* $\tau - 0$*; this operator depends on* y *and is restricted to* $\mathbb{H}_{\xi,\{T\}}$*.*

Moreover,

$$\mathcal{N}(\tau) \asymp \begin{cases} \tau^{d/m} & \text{for } \mu > 1, \\ \tau^{d/m} \log \tau & \text{for } \mu = 1, \\ \tau^{d/(m\mu)}, & \text{for } \mu < 1. \end{cases} \tag{12.8.34}$$

(ii) Further, if $\mu > 1$ *and the standard non-periodicity condition for symbol*

$$L_m(x, \xi) = A_m W^{-1}(x), \qquad A_m(x, \xi) = \sum_{\sigma : |\sigma| = m} a_\sigma(x) \xi^\sigma \tag{12.8.35}$$

is fulfilled then asymptotics

$$\mathsf{N}(\tau) = \mathcal{N}(\tau) + \big(\kappa_1 + o(1)\big)\tau^{(d-1)/m} \tag{12.8.36}$$

holds.

(iii) Finally, let us assume that there is finite number of λ_k *exceeding* E *(or equivalently* A *is semibounded from above). Then*

$$|\mathsf{N}(\tau) - \mathcal{N}(\tau)| \le C\tau^{(d-1)/(m\mu)} \tag{12.8.37}$$

and

$$\mathsf{N}(\tau) \asymp \tau^{d(m\mu)}. \tag{12.8.38}$$

To prove Theorem 12.8.12 let us consider operator $L = L_E(x, D)$ defined by (12.8.21) and the Schwartz kernel $e_L(x, x, \tau)$ of its spectral projector. Then Theorem 12.8.12 follows from Proposition 12.8.13 below as we pick up $\rho(y) = \langle y \rangle^{-\mu}$ and $\gamma_*(\tau) = \tau^{1/(m\mu)}$ and consider γ-admissible partition of unity with $\gamma = \frac{1}{2}\langle x \rangle$.

Proposition 12.8.13. *Let conditions* (12.8.2), (12.8.15), (12.8.4)–(12.8.9) *and* (12.8.16)–(12.8.18) *be fulfilled.*

(i) Let ψ be γ-admissible function supported in $B(\bar{x}, \frac{1}{2}\gamma)$ with $\gamma \le C\gamma_*(\tau)$. Then

$$\mathsf{R}_1 := \Big| \int \psi(x)\Big(e_L(x,x,\tau) - e_L(x,x,\frac{1}{2}\tau) - (2\pi)^{-d}\int \big(\nu_{L_m}(x,\xi,\tau) - \nu_{L_m}(x,\xi,\frac{1}{2}\tau)\big)\,d\xi\Big)\,dx\Big| \le C\tau^{(d-1)/m}\rho^{d-1}\gamma^d \tag{12.8.39}$$

where $\nu_{L_m}(x,\xi,\tau)$ is the eigenvalue counting function for symbol $L_m(x,\xi)$ defined by (12.8.35).

(ii) Further, for fixed r under non-periodicity condition for symbol $L_m(x,\xi)$

$$\mathsf{R}_1 := \Big| \int \psi(x)\Big(e_L(x,x,\tau) - e_L(x,x,\frac{1}{2}\tau) - (2\pi)^{-d}\int \big(\nu_{L_m}(x,\xi,\tau) - \nu_{L_m}(x,\xi,\frac{1}{2}\tau)\big)\,d\xi - \kappa_1(x)\tau^{(d-1)/m}\Big)\,dx\Big| = o\big(\tau^{(d-1)/m}\rho^{d-1}\gamma^d\big). \tag{12.8.40}$$

(iii) Furthermore, for $2 < \gamma \le C\gamma_*(\tau)$

$$\Big| \int \psi(x)\Big(e_L(x,x,\tau) - e_L(x,x,\frac{1}{2}\tau) - (2\pi)^{-d}\int \big(\boldsymbol{n}_M(x,\xi,\tau) - \boldsymbol{n}_M(x,\xi,\frac{1}{2}\tau)\big)\,d\xi\Big)\,dx\Big| \le C\tau^{(d-1)m}\mathsf{r}^{(1-\mu)(d-1)}\rho^{d-1}\gamma^{d-1} \tag{12.8.41}$$

where

$\boldsymbol{n}_M(y,\xi,\tau)$ is the number of eigenvalues between 0 and τ of operator (12.8.42)

$$M(y,\xi;z,D_z) = W^{-\frac{1}{2}}(z,y)\big(A(z,y,\xi+D_z) - E\big)W^{-\frac{1}{2}}(z,y) \tag{12.8.43}$$

restricted to the space $\mathbb{K}_{\xi,\{T\}}$ for fixed "slow" argument y.

(iv) Let $E \notin \mathsf{Spec}(A^{\infty})$. *Then for* $\gamma \ge C_0\gamma_*(\tau)$ *with large enough constant* C_0

$$|\int \psi(x)\big(e_L(x,x,\tau) - e_L(x,x,\frac{1}{2}\tau)\big)\,dx| \le C\tau^{-s}\gamma^{-s} \tag{12.8.44}$$

with arbitrarily large s.

(v) Finally, let A *be semibounded from above. Then estimate* (12.8.44) *also holds for* $\gamma \le \epsilon_0\gamma_*(\tau)$ *with small enough constant* ϵ_0.

Proof. (a) Statements (i) and (ii) follow from the standard local spectral asymptotics.

(b) Statement (iii) follows from the spectral asymptotics for operator $M(D_\xi, \xi; x, D_x)$ with the operator-valued symbol $M(y, \xi; z, D_z)$.

Let us apply Theorems 4.4.5, 4.4.6, 4.4.8 and 4.4.9) with $B = (I + \Delta_z)^{m/2}$ in $\mathbb{H}$; due to Remark 12.8.10(i) we can pass from $\mathbb{H}_\xi$ to $\mathbb{H}_0$.

Remark 12.8.14. Here and during all remaining part of the section ξ plays the role of the spatial variable, $z \in \mathcal{Q}$ is an "inner variable" defining $\mathcal{K} = \mathscr{L}^2(\mathcal{Q}, \mathbb{C}^{\mathsf{D}})$ and $y \to D_\xi$ is a dual to ξ variable, so $(y, \xi) \in T^*\mathcal{Q}'$.

To prove Statement (iii) we need to establish that

(12.8.45) $M(y, \xi; z, D_z)$ is (y, ξ)-microhyperbolic on the energy level $\tau \gg 1$

which is due to (12.8.5) and (12.8.17).

(c) Finally, Statements (iv) and (v) and follow from the fact that zones in question are classically forbidden for operators with operator-valued symbols introduced in Part (b) of this proof and uses Theorem 4.4.2. □

Corollary 12.8.15. *(i) In the framework of Theorem 12.8.12(i) with* $\mu > d(d-1)^{-1}$ *the standard Weyl asymptotics holds with the remainder estimate* $O(\tau^{(d-1)/m})$ *and even* $o(\tau^{(d-1)/m})$ *under standard non-periodicity condition as in Theorem 12.8.12(ii).*

(ii) In the framework of Theorem 12.8.12(i) with $\mu > 1$ *and*

$$|\nabla_y^\beta \big(W(z,y) - W^\infty(z,y)\big)| = o(|y|^{-\delta-m\mu-|\beta|}) \quad \text{as } |y| \to \infty \ \forall \beta : |\beta| = 1 \tag{12.8.46}$$

with $\delta > \delta_0 = 1 - (d-1)(\mu-1)$ *asymptotics* (12.8.32) *holds with*

$$\mathcal{N}(\tau) = \int_{\mathbb{R}^d}\int_{\mathcal{Q}'} \boldsymbol{n}^\infty(y,\xi,\tau)\,d\xi dy + (2\pi)^{-d}\int_{\mathbb{R}^{2d}} \Big(\nu(x,\xi,\tau) - \nu(x,\xi,0) - \nu^\infty(x,\xi,\tau) + \nu^\infty(x,\xi,0)\Big)\,dxd\xi \tag{12.8.47}$$

where $\nu^\infty(x,\xi,\tau)$ *is defined in the same way as* $\nu(x,\xi,\tau)$ *but for* $A^\infty(x,\xi)$ *and* $W^\infty(x,x)$ *and* $\boldsymbol{n}^\infty(y,\xi,\lambda)$ *is defined in the same way as* $\boldsymbol{n}(y,\xi,\tau)$ *but for operators* A^∞ *and* W^∞.

(iii) In the framework of Theorem 12.8.12(i) with $\mu < (d-1)d^{-1}$ *and* (12.8.46), (12.8.17) *with* $\delta > (1-\mu)(d-1)(d-d\mu-1)^{-1}$ *asymptotics* (12.8.32) *holds with*

$$\mathcal{N}(\tau) = \int_{\mathbb{R}^d}\int_{\mathcal{Q}'} \boldsymbol{n}^\infty(y,\xi,\tau)\,d\xi dy. \tag{12.8.48}$$

(iv) In the framework of Theorem 12.8.12(ii) and (12.8.46), (12.8.17) *with* $\delta > 1$ *asymptotics* (12.8.32) *holds with* $\mathcal{N}$ *defined by* (12.8.48).

Proof. Statement (iii) is due to Proposition 12.8.13(i) and (ii).

Note that if $\gamma^{-\delta} < h = \tau^{-1/m}\gamma^{\mu-1}$ then one can replace in Proposition 12.8.13(iii) $\boldsymbol{n}$ by $\boldsymbol{n}^\infty$. Combining with second part of Proposition 12.8.13(iv) we get Statement (iv) and with Proposition 12.8.13(i) we get Statement (iii).

To prove Statement (ii) note that if $\tau^{d/m}\gamma^{-\delta+d(1-\mu)} < \tau^{(d-1)/m}$ then one can replace in Proposition 12.8.13(iii) $\boldsymbol{n}$ by $\boldsymbol{n}^\infty$. On the hand, one can employ the theory of operators with operator-valued symbols for propagation of singularities only. This yields standard Weyl asymptotics as in Proposition 12.8.13(i) but with the remainder estimate

$$C\big(\tau^{(d-2)/m}\gamma^{(1-\mu)(d-2)+2} + \tau^{(d-1)/m}\gamma^{(1-\mu)(d-1)}\big)$$

where the first part comes from the third term in complete Weyl asymptotics. One can find γ satisfying first condition and making last expression less than $O(\tau^{(d-1)/m})$ iff $\delta > \delta_0$. □

Now let us assume that either $\mu < 1$ or A is semi-bounded from above. Then the main contribution to the remainder is delivered by zone

$$\{\epsilon\gamma_*(\tau) \le |x| \le C_0\gamma_*(\tau)\}. \tag{12.8.49}$$

To improve remainder estimate $O\big(\tau^{(d-1)/(m\mu)}\big)$ we need to consider Hamiltonian flow generated by operator-valued symbol

$$M^\infty(D_\xi, \xi; z, D_z) := W^{\infty-\frac12}(-D_\xi, z)\big(A^{\infty,\mathsf{w}}(x, \xi + D_z) - E\big)W^{\infty-\frac12}(-D_\xi, z) \tag{12.8.50}$$

which leads immediately to the following theorem:

Theorem 12.8.16. *Let in the framework of Theorem 12.8.12 either $\mu < 1$ or A be semi-bounded from above. Further let the standard non-periodicity condition be fulfilled for operator-valued symbol* (12.8.50).

Then asymptotics

$$\mathsf{N}(\tau) = \mathcal{N}(\tau) + \big(\varkappa_1^* + o(1)\big)\tau^{(d-1)/(m\mu)} \tag{12.8.51}$$

holds with an appropriate coefficient $\varkappa_1^*$.

Now let us consider the case $\bar\mu = 1$ but W contains extra-logarithmic factor.

Theorem 12.8.17 [76]**.** *(i) Let all the conditions of Theorem 12.8.12 be fulfilled excluding* (12.8.16)–(12.8.18) *which are replaced by*

$$|\nabla_z^\alpha\nabla_y^\beta W(z,y)| \le C\rho(y)^m\langle y\rangle^{-|\beta|} \qquad \forall\alpha : |\alpha| \le K\ \forall\beta : |\beta| \le K, \tag{12.8.16}'$$

$$|\nabla_y^\beta\big(W(z,y) - W^\infty(z,y)\big)| = o\big(\rho(y)^m\langle y\rangle^{-|\beta|}\big) \quad \text{as } |y| \to \infty \quad \forall\beta : |\beta| = 1, \tag{12.8.17}'$$

$$\langle W(z,y)v, v\rangle \ge C\rho(y)^m|v|^2 \qquad \forall x, y \in \mathbb{R}^d\ \forall v \in \mathbb{C}^d \tag{12.8.18}'$$

[76] Cf. Theorem 12.8.12.

with

$$\rho(y) = |y|^{-\mu}(\log|y|)^{-\mu_1}(\log\log|y|)^{-\mu_2}\cdots(\underbrace{\log\dots\log}_{l\ \text{times}}|y|)^{-\mu_l} \qquad (|y|>C) \tag{12.8.52}$$

with

$$\mu > 0. \tag{12.8.53}$$

Then Statements (i) and (ii) of Theorem 12.8.12 remain true as soon as

$$\int \rho^{d-1}\gamma^{d-2}\,d\gamma < \infty. \tag{12.8.54}$$

On the other hand, if this assumption is violated, Statement (i) of Theorem 12.8.12 modified in the obvious way remains true (modification affects remainder estimate which is now

$$R(\tau) = \tau^{(d-1)/m}\int_{\gamma<\gamma_*(\tau)} \rho^{d-1}\gamma^{d-2}\,d\gamma \tag{12.8.55}$$

with $\gamma_*(\tau)$ *defined from*

$$\tau^{\frac{1}{m}}\rho(\gamma_*(\tau)) = 1. \tag{12.8.56}$$

Moreover, as soon as

$$\int \rho^{d}\gamma^{d-1}\,d\gamma < \infty \tag{12.8.57}$$

magnitude of the principal part does not change; if this assumption is violated, the principal part will be

$$\mathcal{N}(\tau) \asymp \int_{\gamma<\gamma_*(\tau)} \rho^{d}\gamma^{d-1}\,d\gamma. \tag{12.8.58}$$

(ii) Further, Statement (iii) of Theorem 12.8.12 remains true after modification of the remainder estimate and the the principal part (which are now

$$R_0(\tau) = \gamma_*(\tau)^{d-1} \tag{12.8.59}$$

and

$$\mathcal{N}(\tau) \asymp \gamma_*(\tau)^{d} \tag{12.8.60}$$

respectively).

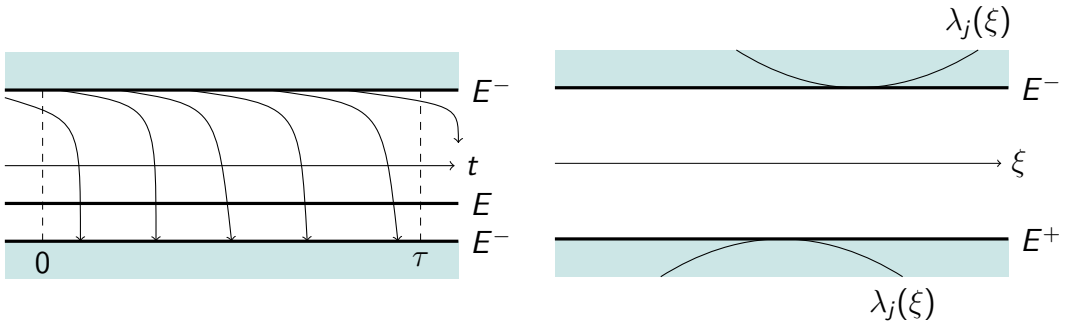
12.8.6 E on the Bottom of Spectral Gap

Now let us consider the case when E is a boundary of the spectral gap: either that either

$$[E, E+\epsilon) \cap \mathsf{Spec}(A^\infty) = \{E\} \tag{12.8.61}^+$$

or

$$(E-\epsilon, E] \cap \mathsf{Spec}(A^\infty) = \{E\}. \tag{12.8.61}^-$$



(a) Observation points (E inside spectral gap, E^+ on the bottom and E^- on the top) and eigenvalues of $A - tW$;

(b) Illustration to $(12.8.62)^\pm$

Figure 12.8: Spectral gap, observation points and behavior of $\lambda_j(\xi)$.

In the former case

$$\exists j : \lambda_k(\xi) \le E \quad \forall k \le j, \qquad \lambda_k(\xi) \ge E+\epsilon \quad \forall k > j \qquad \forall \xi, \tag{12.8.62}^+$$

while in the latter

$$\exists j : \lambda_k(\xi) \le E \quad \forall k \le j, \qquad \lambda_k(\xi) \ge E+\epsilon \quad \forall k > j \qquad \forall \xi \tag{12.8.62}^-$$

In virtue of Proposition 12.8.13 we need to consider only an *external zone*

$$X_{\mathsf{ext}} := \{x : \ |x| \ge \gamma_*(\tau)\} \tag{12.8.63}$$

as an *inner zone* $X_{\mathsf{inn}} := \{x : \ |x| \ge \gamma_*(\tau)\}$ is already covered.

However situation is really different under assumptions $(12.8.61)^+$ and $(12.8.61)^-$. In this subsection we consider only the much easier former case.

Theorem 12.8.18. *Let assumptions of one of Theorems 12.8.12, 12.8.16 or 12.8.17 be fulfilled, except assumption* (12.8.31) $E \notin \mathsf{Spec}(A^\infty)$, *which is replaced by* $(12.8.61)^+$. *Also let us assume that* $A = A^\infty$.

Then all assertions of the corresponding theorem remain true.

Proof. The Gelfand's transform transforms A^∞ into $A^\infty(\xi)$ which in turn due to $(12.8.62)^+$ one transforms by unitary operator $V(\xi)$ in $\mathbb{H}_{\xi,\{T\}}$ into the block-diagonal form

$$V(\xi)\big(A^\infty(\xi) - E\big)V^*(\xi) = \begin{pmatrix} \Lambda^-(\xi) & 0 \\ 0 & \Lambda^+(\xi) \end{pmatrix}, \tag{12.8.64}$$

$$V(\xi)WV^*(\xi) = \begin{pmatrix} W^{--} & W^{-+} \\ W^{+-} & W^{++} \end{pmatrix} \tag{12.8.65}$$

with

$$\Lambda^-(\xi) \le 0, \qquad \Lambda^+(\xi) \ge \epsilon. \tag{12.8.66}$$

and $W^{\cdots}$ ξ-pseudodifferential operators.

As $|y| \ge \gamma_*(\tau)$ [77] we can invert $\big(\Lambda^+(\xi) - E - tW^{++}\big)$ which is elliptic operator as $t \ge 0$, plug into the second equation and derive that the system is elliptic and negative.

Then under condition $(12.8.61)^+$ contribution of external zone to the remainder is negligible. We do not need the uncertainty principle assumption here because we have a define sign. □

12.8.7 E on the Top of Spectral Gap

Now let us consider the most complicated case when condition $(12.8.61)^-$ is fulfilled.

Let us assume that

(12.8.67) $\lambda_k(\xi) = E$ implies that $\lambda_k(\xi)$ is the simple eigenvalue of $A^\infty(x, D_x)$ on $\mathbb{H}_{\xi,\{T\}}$ [78]

---

[77] With a large enough constant in its definition, so $\rho^m\tau \le \epsilon$ there.

[78] Then in virtue of $(12.8.62)^+$ there exists j such that only for $k = j$

$$\Sigma_k := \{\xi :\ \lambda_k(\xi) = E\} \ne \emptyset. \tag{12.8.68}$$

In what follows $\Sigma := \Sigma_j$.

and

(12.8.69) $\lambda_j(\xi) = E$ implies that $\mathsf{Hess}\, \lambda_j(\xi)$ is non-degenerate.

Then

(12.8.70) $\Lambda = \{\xi :\ \lambda_j(\xi) = E\} = \{\bar{\xi}_0^{(i)}, i = 1, \dots \bar{i}\}$ with $\bar{i} < \infty$ and

$$\epsilon\omega(\xi) \le \mp\big(\lambda_j(\xi) - E\big) \le C\omega^2(\xi) \tag{12.8.71}$$

with $\omega(\xi) \coloneqq \mathsf{min}_i\, |\xi - \bar{\xi}^{(i)}|$.

Considering operator $A-E-tW$ near $\bar{\xi}^{(i)} \in \Sigma$ and assuming for simplicity that $\bar{\xi}^{(i)} = 0$ we observe that it is similar to $\Delta - tW(y)$ "projected" to the null-space of operator $(A - E)$ and unless W decays faster than $|y|^{-2}$ it may happen that this operator has a sequence of eigenvalues accumulating to -0. To avoid this we assume that

(12.8.72) Either $\mu' \coloneqq m\mu/2 > 1$, or $\mu' = 1$ and $\mu_1 = \dots = \mu_{l-1} = 0$ and $\mu_l > 0$ $(k = 1, \dots, n-1)$.

So, we need to consider X_{ext} defined by (12.8.63) and only as $\omega(\xi) \le \epsilon$. In this zone we can make reduction similar to (12.8.64)–(12.8.66)

$$V(\xi)\big(A^{\infty}(\xi) - E\big)V^*(\xi) = \begin{pmatrix} \lambda(\xi) & 0 \\ 0 & \Lambda'(\xi) \end{pmatrix}, \tag{12.8.73}$$

$$V(\xi)WV^*(\xi) = \begin{pmatrix} W^{00} & W^{0\prime} \\ W^{\prime 0} & W^{\prime\prime} \end{pmatrix} \tag{12.8.74}$$

with

$$|\lambda(\xi)| \le \epsilon, \qquad |\big(\Lambda'(\xi) - E\big)^{-1}| \le c_0 \tag{12.8.75}$$

where $\lambda(\xi) = \lambda_j(\xi) - E$.

Due to our methods we can consider in the zone X_{ext} a different operator, namely,

$$\lambda(D) - \tau V \tag{12.8.76}$$

with

$$V = W^{00} - \tau W^{0\prime}\big(\Lambda'(D) - \tau - W^{\prime\prime}\big)^{-1} W^{0\prime} \tag{12.8.77}$$

and problem is to find the number of negative eigenvalues of operator (12.8.76); this is basically the very special case of our original problem (one we mentioned as the "lite" problem: when A is not a periodic but a constant coefficients operator.

Due to (12.8.68) this is a semiclassical problem with an effective semiclassical parameter

$$\hbar = (\tau\rho^m)^{\frac{1}{2}}\gamma \tag{12.8.78}$$

and then $\hbar \ll 1 \iff |x| \le \gamma^*(\tau)$ with $\gamma^*\tau$ defined from equation

$$\tau\rho^m\gamma^2 = 1. \tag{12.8.79}$$

In the normal Weyl theory contribution of the zone

$$X'_{\mathsf{ext}} := \{x : \gamma_*(\tau) \le |x| \le \gamma^*(\tau)\} \tag{12.8.80}$$

to the principal part of asymptotics is

$$\mathcal{N}'_{\mathsf{ext}} := (2\pi)^{-d} \iint_{\{y\in X'_{\mathsf{ext}}, \lambda(y,\xi)-\tau V(y,\xi,\tau)<0\}} \psi_{\mathsf{ext}}(y)\, dyd\xi \tag{12.8.81}$$

and has a magnitude

$$\tau^{\frac{d}{2}} \int_{\{\gamma_*(\tau)\le|x|\le\gamma^*(\tau)\}} (\rho^m\gamma^2)^{d/2}\, \gamma^{-d}\, dx \tag{12.8.82}$$

while contribution to the remainder does not exceed

$$R'_{\mathsf{ext}} = C\tau^{\frac{1}{2}(d-1)} \int_{\{\gamma_*(\tau)\le|x|\le\gamma^*(\tau)\}} (\rho^m\gamma^2)^{(d-1)/2}\, \gamma^{-d}\, dx. \tag{12.8.83}$$

In this subsection we assume that $\mu' > 1$ i.e. $\mu > m/2$. Then contribution of the zone in question to the remainder does not exceed $C(\tau\rho^m\gamma^2)^{(d-1)/2}$ calculated as $\gamma = \gamma_*(\tau)$ and in turn it is equal to $C\gamma_*(\tau)^{d-1}$ which is no more than the contribution of X_{inn}.

It clears remainder estimate "O" obtained before and even "o" as $\mu \ge 1$; however as $\mu < 1$ we get here "o" as well due to increasing old $\gamma_*(\tau)$ to $C_0(\tau)\gamma_*(\tau)$ with slowly growing $C_0(\tau)$.

Further, to estimate contribution of the zone $X''_{\mathsf{ext}} := \{x : |x| \ge a = \gamma^*(\tau)\}$ to the remainder one needs to consider the quadratic form $\|Du\|^2 - \tau(Vu, u)$

under the Dirichlet boundary condition on $\partial X''_{\mathsf{ext}}$ and estimate properly the dimension of its negative subspace $\mathsf{N}''_{\mathsf{ext}}(\tau)$, applying then arguments of Chapter 9.

As $d \geq 3$

$$\mathsf{N}''_{\mathsf{ext}}(\tau) \leq \tau^{\frac{d}{2}} \int_{X''_{\mathsf{ext}}} \rho^d \, dx \tag{12.8.84}$$

due to Rozenblioum estimate (9.A.11). Note that the right hand expression does not exceed C as $\mu' > 1$.

As $d = 2$ however we should be a bit more subtle and refer to Rozenblioum–Solomyak estimate (see Theorem 12.A.1) which says that in our assumptions Rozenblioum' estimate (9.A.11) with $q = 1$ still is applicable. Then we arrive to the following assertion:

(12.8.85) As $\mu' > 1$ a contribution of the zone

$$X''_{\mathsf{ext}} = \{x : |x| \geq \gamma^*(\tau)\} \tag{12.8.86}$$

to both main part of the asymptotics and the remainder does not exceed C.

Therefore as $\mu' = \mu m/2 > 1$ we arrive to the same remainder estimate as before but we need also include into asymptotics term (12.8.81) of magnitude $O(\tau^{-d/m\mu})$ [79] Obviously as the result here is given by Weyl formula the final answer is also given by Weyl formula and we arrive to the following theorem:

Theorem 12.8.19 [80]**.** *Let assumptions of one of Theorems 12.8.12, 12.8.16 or 12.8.17 be fulfilled, except assumption* (12.8.31) $E \notin \mathsf{Spec}(A^\infty)$*, which is replaced by assumption* $(12.8.61)^-$ *and conditions* (12.8.67)–(12.8.79)*. Also let us assume that* $A = A^\infty$ *and* $m\mu > 2$*.*

Then all assertions of the corresponding theorem remain true.

Situation may drastically change as $m\mu = 2$. In this case to ensure that $\rho^{dm/2}$ and $\rho^{(d-1)m/2}\gamma^{-1}$ belong to $\mathscr{L}^1$ we need to assume that either $\mu'_1 = m\mu_1/2 > 1$ or $\mu'_1 = 1$ and then the same condition recurrently goes to $\mu'_2 = m\mu_2/2$.

This implies the following theorem:

---

[79] As $\rho = \gamma^{-\mu}$; otherwise extra logarithmic factors come into game.

[80] Cf. Theorem 12.8.18.

Theorem 12.8.20. *Let assumptions of one of Theorems 12.8.12, 12.8.16 or 12.8.17, except assumption* (12.8.31) $E \notin \mathsf{Spec}(A^\infty)$ *which is replaced by assumption* $(12.8.61)^-$, *and conditions* (12.8.67)–(12.8.79). *Also let us assume that* $A = A^\infty$ *and* $m\mu = 2$.

(i) Let

$$m\mu_1(d-1) = 2,\ \dots,\ m\mu_{l-1}(d-1) = 2,\ m\mu_l(d-1) > 2. \tag{12.8.87}$$

Then all assertions of the corresponding theorem in question.

(ii) Let

$$m\mu_1(d-1) = 2,\ \dots,\ m\mu_{k-1}(d-1) = 2,\ m\mu_k(d-1) \le 2. \tag{12.8.88}$$

Then all assertions of the corresponding theorem remain true with the remainder estimate

$$R(\tau) = C\tau^{1/m\mu_1} \times \begin{cases} 1 & \text{as } \ k=1, m\mu_1(d-1) < 2, \\ |\log \tau| & \text{as } \ k=1, m\mu_1(d-1) = 2, \\ |\log_{k-1} \tau|^{1-m\mu_k(d-1)/2} & \text{as } \ k>1, m\mu_k(d-1) < 2, \\ |\log_k \tau| & \text{as } \ k>1, m\mu_k(d-1) = 2, \end{cases} \tag{12.8.89}$$

where $\log_l \tau = \underbrace{\log \dots \log}_{l \text{ times}} \tau$.

(iii) Meanwhile the main part of asymptotics has the magnitude as in the theorem in question provided $m\mu_1 d > 2$.

Proof. Observe that as $m\mu = -2$ we have $\rho^{m(d-1)/2}\gamma^{-1} \in \mathscr{L}^1$ if and only if (12.8.87) holds; otherwise $\tau^{(d-1)/2}\int_{X'_{\mathsf{ext}}} \rho^{m(d-1)/2}\gamma^{-1}\,dx$ is given by (12.8.89) and is larger than that what we got before. So, Statements (i) and (ii) are proven.

Note that $\rho^{md/2} \in \mathscr{L}^1$ as $m\mu_1 d > 2$. It implies Statement (iii). □

12.8.8 E on the Top of Spectral Gap. II. Improved Asymptotics

Now we want to consider case of $\rho^{m(d-1)/2}|x|^{-1} \notin \mathscr{L}^1$ and improve the remainder estimate (12.8.89) adding some correction; this correction will be even larger than the previous type of asymptotics as $\rho^{md/2} \notin \mathscr{L}^1$.

Let us consider first

$$A = \Delta - tW(x), \qquad W(x) = |x|^{-2} V(\log|x|) \tag{12.8.90}$$

where

$$|D^\alpha V(z)| \le C\varrho(z)^2 |z|^{-|\alpha|} \qquad \forall \alpha : |\alpha| \le K, \tag{12.8.91}$$

$$|\partial_z| V(z) \asymp \varrho^2(z)|z|^{-1} \qquad \text{as} \quad |z| \ge C \tag{12.8.92}$$

$$\varrho(z) = |z|^{-\nu_\pm} \qquad \text{as} \quad \pm z \ge C, \nu_\pm > 0. \tag{12.8.93}$$

As $V(z)$ decays fast enough as $z \to \pm\infty$ non-degeneracy condition (12.8.92) will not be needed.

Then introducing spherical coordinates $(r, \theta) \in \mathbb{R}^+ \times \mathbb{S}^{d-1}$, separating variables and then substituting $z = \log r$ we arrive to the equivalent problem for the operator

$$A' = D_z^2 + \lambda + \frac{1}{4}(d-2)^2 - tV(z) \tag{12.8.94}$$

where

$$\lambda \in \mathsf{Spec}(\Lambda) \text{ and } \Lambda \text{ is Laplace-Beltrami operator on } \mathbb{S}^{d-1}. \tag{12.8.95}$$

To this operator we can apply Weyl asymptotics:

$$\mathsf{N}_\lambda^-(\tau) = \mathsf{N}_\lambda^{\mathsf{W}}(\tau) + O(1) \tag{12.8.96}$$

with

$$\mathsf{N}_\lambda^{\mathsf{W}}(\tau) = \frac{1}{\pi} \int \Big(\tau V(z) - \lambda - \frac{1}{4}(d-2)^2\Big)_+^{\frac{1}{2}}\, dz, \tag{12.8.97}$$

as $\lambda + \frac{1}{4}(d-2)^2 \ge \epsilon_0 > 0$ and after summation over eigenvalues of Λ we arrive to the asymptotics with the principal part

$$\mathcal{N}(\tau) := \frac{1}{\pi} \sum_{\lambda \in \mathsf{Spec}(\Lambda)} \int \big(\tau V(z) - \lambda - \frac{1}{4}(d-1)^2\big)_+^{\frac{1}{2}}\, dz = \tag{12.8.98}$$

$$\frac{1}{\pi} \sum_n P(n,d) \int \big(\tau V(z) - (n + \frac{1}{2}d - 1)^2\big)_+^{\frac{1}{2}}\, dz$$

where $P(n,d)$ is the multiplicity of the eigenvalue $n(n+d-2)$ of Λ and with the remainder estimate

$$\sum_{\lambda\in \mathsf{Spec}(\Lambda)\le \tau} 1 \asymp \tau^{(d-1)/2}.$$

Obviously

$$\mathcal{N}(\tau) \asymp \begin{cases} \tau^{d/2} & \text{as } -d\nu > 1,\\ \tau^{d/2}|\log\tau| & \text{as } -d\nu = 1,\\ \tau^{1/2\nu} & \text{as } -d\nu < 1, \end{cases} \tag{12.8.99}$$

where $\nu = \min(\nu_+,\nu_-)$.

Formula (12.8.99) holds as either $d\ge 3$ or $\lambda\in\mathsf{Spec}(\Lambda)$ is not 0.

So, as $d=2$ we need to investigate $\lambda=0$ separately. In this case we arrive to the operator $D_z^2 - \tau V(z)$ and to avoid the infinite negative spectrum we must assume that $V(z) = o(|z|^{-2})$ as $|z|\to\infty$.

Recall that as $d=2$ the critical value of the previous subsection is exactly $\alpha=1$. Thus we now are interested in $V(z) = |z|^{-2}V_1(\log z)$ with $V_1(z_1)$ decaying at infinity as $|z_1|^{-2}$ or slower. We consider only $z\ge 0$; the second semi-axis is treated in the same way.

Then plugging into quadratic form $u = e^{-z_1/2}u_1$, $z = e^{z_1}$ we arrive to quadratic form

$$\|D_z u\|^2 - t(z^{-2}V_1(\log z)u,u) = \|D_{z_1}u_1\|^2 + \frac{1}{4}\|u_1\|^2 - t(V_1(z_1)u_1,u_1) \tag{12.8.100}$$

and a $V_1(z_1)\to 0$ as $|z_1|\to\infty$ we guarantee that there is only finite number negative eigenvalues. Moreover, as $V(z_1)\asymp |z_1|^{-2\beta_1}$ with $\beta_1\le 1$ the remainder estimate is $O(\log\tau)$ and the main part is given by the Weyl formula

$$\mathsf{N}_0^{\mathsf{W}}(\tau) = \frac{2}{\pi}\int \Big(\tau V_1(z_1) - \frac{1}{4}\Big)^{\frac{1}{2}}\, dz_1 \tag{12.8.101}$$

While remainder estimate is below one we received earlier, the main part is of magnitude $\tau^{(\beta_1+1)/(2\beta_1)}$ and it is above of what we got earlier.

So, we have proven the following theorem:

Theorem 12.8.21. *(i) Let* $d \ge 3$, $W(x) = |x|^{-2}V(\log|x|)$ *where* $V(z)$ *is a function of the type described above.*

Then asymptotics

$$\mathsf{N}(\tau) = \mathcal{N}(\tau) + O(\tau^{\frac{1}{2}(d-1)}) \qquad \text{as } \tau \to \infty \tag{12.8.102}$$

holds with the principal part (12.8.98) *of magnitude* (12.8.99).

(ii) Let $d = 2$, $W(x) = |x|^{-2}|\log|x||^{-2}V_1(\log|\log|x||, \pm)$ *where* $V(z)$ *is a function of the type described above and* $\pm$ *is responsible for different behavior as* $|x| \to \infty$ *and* $|x| \to 0$.

Then asymptotics

$$\mathsf{N}(\tau) = \mathcal{N}^{\#}(\tau) + O(\tau^{\frac{1}{2}(d-1)}) \qquad \text{as } \tau \to \infty \tag{12.8.103}$$

holds where superscript $^{\#}$ *means that the term with* $\lambda = 0$ *is replaced by* (12.8.101) *of magnitude* $\tau^{(\beta+1)/(2\beta)}$.

We leave to the reader the following series of interesting problems and probably not very difficult problems:

Problem 12.8.22. (i) Prove that the similar results hold as $W(x) = W(|x|)$ is replaced by $W(x) = W_r(|x|)w(\theta)$ with smooth disjoint from 0 function $w(\theta)$.

Investigate if we need condition $w(\theta) \ne 0$ or one can replace it by a condition $|w(\theta)| + |\nabla_\theta w(\theta)| \asymp 1$ or drop it completely (depending on dimension)?

(ii) Prove that the similar results hold as $W(x)$ fast enough stabilizes to $W^\infty(x) = W_r(|x|)w(\theta)$ described above; then there will be a mixture of Weyl terms associated with $(W - W^\infty)$ and non-Weyl (with respect to θ) terms depending only on W^∞.

(iii) Extend Part (ii) to the case when Δ is replaced by $\lambda^{\mathsf{w}}(x, D)$ with $\lambda(x, \xi) = g(x, \xi)$ where $g(x, \xi)$ is non-degenerate quadratic form with respect to ξ stabilizing to $g^\infty(\theta, \xi)$ as $|x| \to \infty$.

(iv) As $W(x)$ has no singularity at 0 extend Part (iii) to the case when $\lambda^{\mathsf{w}}(x, D)$ is a pseudodifferential operator with the symbol $\lambda(x, \xi) = g(x, \xi) + O(|\xi|^3)$ uniformly with respect to x with $g(x, \xi)$ described in Part (iii).

Problem 12.8.23. Using Problem 12.8.22 improve results of the previous Subsection 12.8.7 by using non-Weyl asymptotics of the type (12.8.102), (12.8.103) in the external zone.

12.8.9 Case of Positive $(A-E)$

Let us consider case when operator $(A-E)$ is positive definite albeit not necessarily disjoint from 0 but W is indefinite. So, E is either inside the spectral gap $\mathcal{G}_0$ or on its top.

Note that eigenvalues of $(A-tW)$ can cross E only in the negative direction. Let $\mathsf{N}_E([0,\tau])$ be a number of eigenvalues crossing 0 as t changes from 0 to τ. Then

$$\mathsf{N}_E([0,\tau]) = \mathsf{N}^-(A-E-\tau W). \tag{12.8.104}$$

Problem 12.8.24. (i) Prove that the corresponding results of the previous subsections remain true in these settings as well.

(ii) Generalize these results to the case when $W=W(x,D)$ is an operator of lower order than A.

12.8.10 Generalizations and Modifications

Let us consider the operators treated in M.S. Birman, A. Laptev and T. Suslina [1] and in M.S. Birman and T. Suslina [1–3] but without remainder estimates:

Problem 12.8.25. (i) Prove that the same asymptotics hold for operators not in $\mathbb{R}^d$ but in a domain X^∞ and with the boundary condition $Bu|_{\partial X}=0$ satisfying the following periodicity conditions:

$$X^\infty + \Gamma = X^\infty \tag{12.8.105}$$

with the lattice Γ (see Figure 12.9(a)),

(12.8.106) There exist functions ϕ_j $(j=1,\dots,d')$ such that

$$\phi_j(x+e_k)-\phi_j(x)=\delta_{jk} \qquad \forall j,k=1,\dots,d' \tag{12.8.107}$$

and

$$B^\infty e^{i\langle\phi(x),\xi\rangle}u|_{\partial X^\infty} = B^\infty u|_{\partial X^\infty} \qquad \forall \xi \tag{12.8.108}$$

where $\langle \phi(x), \xi\rangle = \sum_j \phi_j(x)\xi_j$.

In this case $\mathcal{Q}$ should be replaced by $\mathcal{Q}\cap X$ with the previous definition of the elementary cell $\mathcal{Q}$ and $\mathcal{Q}'$ remains intact. In this case the auxiliary space $\mathbb{H}_{\xi,\{T\}}$ is subspace of $\mathscr{L}^2_{\mathsf{loc}}(\pi^{-1}X^\infty, \mathbb{C}^{\mathsf{D}})$ satisfying condition of quasiperiodicity with the quasimomentum ξ and transformation matrices $\{T\}$; $\mathbb{H}_{\xi,\{T\}}$ is identified with $\mathscr{L}^2(\pi^{-1}\mathcal{Q}\cap X^\infty, \mathbb{C}^{\mathsf{D}})$. Here $d' = d$ and $\pi = I$.

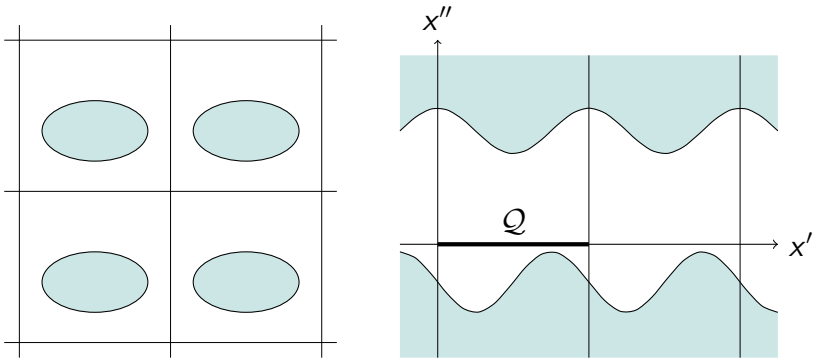
(ii) Generalize these results to the case when X and B coincide with X^∞ and B^∞ as $|x| \ge c$.

Problem 12.8.26. Prove that the similar asymptotics hold for operators in waveguides i.e. domains bounded in $x'' = (x_{d'+1}, \dots, x_d)$ and satisfying above conditions with $\Gamma \subset Z = \mathbb{R}^{d'} \ni x' = (x_1, \dots, x_{d'})$ (see Figure 12.9(b)) and matrices $T_1, \dots, T_{d'}$; here $d' < d$.

Now $\mathcal{Q} \subset Z$ is an elementary cell, $\pi : X \ni (x', x'') \to x' \in Z$ is a projection. Then one should expect $\mathsf{N}(\tau) \asymp \mathsf{max}\big(\tau^{d'/(m\mu)}, \tau^{d/m}\big)$ and $R(\tau) \asymp \mathsf{max}\big(\tau^{(d'-1)/(m\mu)}, \tau^{(d-1)/m}\big)$ as E is inside the spectral gap or on its bottom, whatever power is larger and the logarithmic factor appears as powers coincide; similar results hold as E is on the top of the spectral gap.

In this case $d' = 1$ should be also treated.

Note that in the previous two Problems 12.8.25 and 12.8.26 X is not necessarily a connected domain.



(a) Periodic domain (b) Waveguide

Figure 12.9: Periodic domain in full dimension ($d' = d$) and "waveguide" i.e. periodic domain in reduced dimension ($d' < d$)

Problem 12.8.27. Prove that the same asymptotics hold for operators with Weyl symbol $a(x, \xi - \alpha x)$ where α is a constant matrix such that

$$\alpha n \in \Gamma' \qquad \forall n \in \Gamma. \tag{12.8.109}$$

In this case $\mathbb{H}_{\xi,\{T\}}$ is the space of functions such that

$$u(x+n) = T^{-n} u(x) e^{i\langle n, \xi + \alpha x\rangle} u(x) \qquad \forall n \in \Gamma; \tag{12.8.110}$$

without condition (12.8.109) this is impossible equality.

Condition (12.8.109) needs to be satisfied only for skew-symmetric part of α as we can get rid of symmetric part by the gauge transformation. So, condition is

$$\frac{1}{2\pi} \int_{\mathcal{Q}_{jk}} d\omega \in \mathbb{Z} \qquad \forall j \neq k \tag{12.8.111}$$

where $\omega = \sum_j (\alpha x)_j dx_j$ and d denotes an exterior differential.

Therefore the problem is admissible as αx is replaced by non-linear vector-potential $V(x)$ both in operator and in condition (12.8.111): now $\omega = \sum_j V_j(x) dx_j$.

As an example one can consider Schrödinger operators with periodic magnetic field with "admissible" but not necessary vanishing flows over elementary cell.

Remark 12.8.28. As condition (12.8.111) is violated and some of the left-hand expressions are irrational operator seems to be beyond reach of the modern methods.

12.A Appendices

12.A.1 Variational Estimates as $m = 1$, $d = 1, 2$

In the $m = 1, d = 2$ case Rozenblioum estimate (9.A.11) is not often good enough as it contains $q > 1$ which often does not work (f.e. if potential is $|x|^{-2} |\log |x||^{-2\alpha}$). However there are other estimates albeit not of the same level of the generality and elegance, which work better.

First of all as $d = 1, 2$ replacing Δ by $\Delta + \epsilon |x|^{-2}$ with arbitrarily small constant ϵ saves the day restoring Rozenblioum estimate (9.A.11) with $q = 1$ (A. Laptev [2]). However consider unmodified Δ.

Let us start from $d = 1$ and half-line $\mathbb{R}^+ = \{x > 0\}$,

$$Q(u) = \int_0^\infty \bigl(|\nabla u|^2 - V(x)|u|^2\bigr)\,dx \tag{12.A.1}$$

with $V(x) \ge 0$.

In this case G. Rozenblioum-M.Z. Solomyak [2] paper provides estimate

$$\mathsf{N}^-(Q) \le C \int_1^\infty \Bigl(t^{-1} \int_t^{2t} V(x)\,dx\Bigr)^{\frac{1}{2}}\,dt \tag{12.A.2}$$

which is the estimate following (5.3) in the mentioned paper modified in the obviously equivalent way. Then for potentials V such that

$$|x| \asymp |y| \implies V(x) \asymp V(y) \tag{12.A.3}$$

the right-hand expression of (12.A.2) is equivalent to the right-hand expression of

$$\mathsf{N}^-(Q) \le C \int_{\mathbb{R}^+} V(x)^{\frac{1}{2}}\,dx \tag{12.A.4}$$

which is Rozenblioum estimate with formal $q = \frac{1}{2}$.

Consider now $d = 2$ and a potential satisfying (12.A.3). This is the same as considering radial potential $V(r)$ with $r = |x|$. Then we can go to polar coordinates (θ, r) and split u into two parts: one is radial $u_0(r)$ and another orthogonal $u_1(r,\theta) = \sum n \ne 0 u_n(r)e^{in\theta}$.

Then $\mathsf{N}^-(Q) = N_0^-(Q) + \mathsf{N}_2^-(Q)$ and due to A. Laptev [2] for $\mathsf{N}_1^-(Q)$ Rozenblioum estimate holds:

$$\mathsf{N}_1^-(Q) \le C \int_{\mathbb{R}^2} V(x)\,dx \asymp C \int_0^\infty V(r) r\,dr. \tag{12.A.5}$$

So, we need to consider only radial component. Then we arrive to quadratic form

$$\int_0^\infty r\bigl(|D_r|^2 - V(r)|u|^2\bigr)\,dr = \int_{-\infty}^\infty \bigl(|u|^2 - V(e^z)e^{2z}\bigr)\,dz \tag{12.A.6}$$

where we changed variables $r = e^z$ and skipped index "0". So we arrive to one-dimensional problem with "effective potential" $F(z) = e^{2z}V(e^z)$. This effective potential does not necessary satisfies condition (12.A.3) even if original potential V satisfied it. However assume that $F(z)$ satisfies this condition, or equivalently, that

(12.A.7) $V(r) = r^{-2}F(\log r)$ where $F(z)$ satisfies (12.A.3).

Then in virtue of (9.A.4)

$$\mathsf{N}_0^-(Q) \le C\int_{-\infty}^{\infty} e^{2z}V(e^z)\,dz = C\int_0^{\infty} rV(r)\,dr. \tag{12.A.8}$$

Thus we arrive to

Theorem 12.A.1. *(i) As $m = 1$, $d = 1$ and V satisfies* (12.A.3) *Rozenblioum' estimate* (9.A.11) *holds with $p = \infty$, $q = \frac{1}{2}$.*

(ii) As $m = 1$, $d = 2$ and V satisfies (12.A.7) *Rozenblioum' estimate* (9.A.11) *holds with $p = \infty$, $q = 1$.*

Comments

There are few papers devoted to the non-classical asymptotics in the domains with thick cusps (we are aware only about the existence of these papers); however, there are either no accurate remainder estimates or even no remainder estimate at all. The principal part of asymptotics in these papers, naturally, coincides with the one derived in Section 12.2.

The theory of operators with the strong degeneration of the potential at infinity also is not very well developed. I am aware of the paper M. Z. Solomyak [1] in which there is also no remainder estimate and again the principal part coincides with the one derived in Section 12.3.

There seem to be no papers[81)] at all related to Section 12.4.

The list of papers devoted to the operators with the degenerations is much larger. We cannot list them all. The following papers are devoted to operators of the type we consider in Section 12.5: A. Menikoff and J. Sjöstrand [1–3], C. Iwasaki and N. Iwasaki [1], A. Mohamed [1, 2] etc. Remainder estimates were not obtained in them. A special case was treated in R. Melrose [5] and the remainder estimate was accurate in the case when it was Weyl and was not accurate otherwise.

There are many papers[81)] dealing with the Scott correction term in mathematical physics. The reader can find a survey in V. Ivrii, V. and I.M. Sigal [1]. However, the method was completely different from the one

[81)] Prior to our analysis.

we employ in Section 12.6 and only the spherically symmetric case was treated.

In V. Jakšić, S. Molčanov and B. Simon [1] the eigenvalue asymptotics for Neumann operators in domains with ultra-thin cusps was derived without remainder estimate. This paper inspired our research in Section 12.7. As cusp decay rate is exponential rather than superexponential the spectrum is continuous and the scattering theory was developed by many authors.

Finally, problems for periodic operators as in Section 12.8 were treated in papers of M. Sh. Birman [5, 8–10], M. Sh. Birman and -A. Laptev [2, 3], M.S. Birman, A. Laptev and T. Suslina [1], and M.S. Birman and T. Suslina [1–3]. These papers inspired our research.

Remark. We understand that these commentaries are completely inadequate and we are completely open and will be extremely grateful for suggestions.

Microlocal Analysis, Sharp Spectral Asymptotics and Applications

Volume III. Magnetic Schrödinger and Dirac Operators. 1

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Part VI

Smooth Theory in Dimensions 2 and 3

Chapter 13

Schrödinger Operator. Standard Theory in Dimensions 2 and 3

13.1 Introduction

In this chapter we analyze 2- and 3-dimensional Schrödinger operators with the strong magnetic field; now we have not only a small parameter h but a large parameter μ (a coupling constant with the magnetic field or simply a magnetic parameter); moreover, a natural condition $\mu \le h^{-1}$ arises.

In Section 13.2, which has a preliminary character, we obtain a formula for $e(x,x,\tau)$ for operators in $\mathbb{R}^d$ with constant g^{jk}, V and F_{jk} (which is the tensor intensity of the magnetic field) and we also prove some preliminary assertions.

In Section 13.3 we obtain microlocal canonical forms for $d = 2, 3$ which are the basis for the advanced analysis.

In Section 13.4 we derive (quasi)Weyl and related asymptotics for $d = 2, 3$ in the case of a weak magnetic field ($\mu \le h^{\delta-1}$ with arbitrarily small exponent $\delta > 0$).

Further, in Section 13.5 we derive non-Weyl asymptotics in the case of a noderate magnetic field ($\mu \ge h^{-\delta}$); these formulae permit us to justify the (quasi)Weyl asymptotics under weaker restrictions than those of Section 13.4.

All these results are essentially different in the cases $d = 2$ and $d = 3$. For example, in $\mathbb{R}^d$ for constant g^{jk}, V and $(F_{jk}) \ne 0$ the spectrum of

the Schrödinger operator for $d = 2$ is pure point of infinite multiplicity while it is absolutely continuous for $d = 3$. The remainder estimate in Sections 13.4 and 13.5 in the case $d = 2$ under appropriate non-degeneracy assumptions is $O(\mu^{-1}h^{-1})$ or even better (the best possible is $O(1)$); so it is better than in the standard case (without magnetic field); however, without such assumptions the remainder estimate is as bad as $O(\mu h^{-1})$ which can be derived from the results in the standard case.

On the other hand, for $d = 3$ the remainder estimate is $O\big(h^{-2} + \mu h^{-1-\delta}\big)$ without any assumptions and $O(h^{-2})$ under very weak non-degeneracy assumption. Thus, as either $\mu \le h^{-1+\delta}$ or mentioned non-degeneracy condition is fulfilled we obtain the same estimate $O(h^{-2})$ as in the non-magnetic case. So in the case $d = 2$ the remainder estimate is way more more sensitive to additional restrictions.

In Section 13.6 we consider strong magnetic field $\mu \ge \epsilon h^{-1}$ and slightly modify our results; we also consider magnetic Schrödinger-Pauli operators which provide us with non-trivial results even as $\mu \ge Ch^{-1}$. Then the principal part of asymptotics is of magnitude μh^{1-d} rather than h^{-d} and remainder estimate is $O(\mu h^{-1})$ (or a bit worse, depending on non-degeneracy assumptions) as $d = 3$ and $O(1)$ (or worse, depending on non-degeneracy assumptions) as $d = 2$. Further, in the same section for $d = 2$ t we treat spectral gaps not containing the semiclassical approximation of the spectrum and we obtain complete spectral asymptotics (i.e., $O(h^s)$ with arbitrary s). Under appropriate assumptions these gaps may appear even if $\mu \ge h^{-\delta}$.

Furthermore, in Section 13.7 under additional restrictions of a global nature we treat the long-term propagation of singularities and justify the classical mechanics answer: in the 3-dimensional case singularities propagate along magnetic lines while in the 2-dimension case they propagate along magnetic drift lines (with average velocities $\asymp 1$ and $\asymp \nu\mu^{-1}$ where parameter $\nu \in (0, 1]$ appears in assumptions). This provides us with slightly improved remainder estimates under conditions on these lines including conditions on the growth of the Jacobi matrix elements, avoidance of the boundary and irrevocability of the movement.

Also in this section we consider the case of 3-dimensional Schrödinger operator when classical movement along magnetic lines is trapped by potential and then we can trace very long-time movement (up to $T \asymp \mu$) and under appropriate non-degeneracy assumptions including assumption that the total drift is present (magnetic lines may drift in 3-dimensional case with the speed $\nu\mu^{-1}$) we recover supersharp remainder estimate $O(\mu^{-1}h^{-2})$.

Finally, in Section 13.8 we apply the method of partition of unity and rescaling plus some additional arguments and we weaken the conditions on the assertions proven earlier: while condition $V \asymp 1$ is removed for $d = 2.3$, conditions $F \asymp 1$ and $B(0,1) \subset X$ are removed for $d = 3$ only because for $d = 2$ much more subtle arguments allow to derive much sharper asymptotics and we will do it in two next chapters.

13.2 Preliminary Analysis

13.2.1 Settings

In this chapter we treat the Schrödinger operator

$$A = \sum_{j,k} P_j g^{jk} P_k + V, \qquad \text{with } P_j = hD_j - \mu V_j \tag{13.2.1}$$

where

(13.2.2) $g^{jk} = g^{kj}$, V_j and V are real-valued functions, $h \in (0,1]$, $\mu \in [1,\infty)$.

and in $B(0,1) \subset X$ the following conditions are fulfilled:

$$|D^\alpha g^{jk}| \le c, \quad |D^\alpha V_j| \le c, \quad |D^\alpha V| \le c \qquad \forall \alpha : |\alpha| \le K, \tag{13.2.3}_{1-3}$$

$$\epsilon_0 \le \sum_{j,k} g^{jk}\eta_j\eta_k \cdot |\eta|^{-2} \le c \quad \forall \eta \in \mathbb{R}^d \setminus 0 \quad \forall x \in B(0,1) \tag{13.2.4}$$

and unless opposite is specified, we always assume that

$$X \supset B(0,1). \tag{13.2.5}$$

Let us note that

(13.2.6) $e(x,x,\tau)$ does not change if we replace A by the unitarily equivalent operator $e^{-i\mu h^{-1}\phi} A e^{i\mu h^{-1}\phi}$ with an arbitrary real-valued function $\phi(x)$ which means precisely that we replace V_j by $V_j + \partial_{x_j}\phi$ and $e(x,y,\tau)$ by $e(x,y,\tau)e^{-i\mu h^{-1}(\phi(y)-\phi(x))}$ (*gauge transformation*).

Therefore one can replace condition $(13.2.3)_2$ by

$$|D^\alpha F_{jk}| \le c \quad \forall \alpha : |\alpha| \le K-1 \tag{13.2.3}'_2$$

where

$$F_{jk} = \partial_k V_j - \partial_j V_k \tag{13.2.7}$$

is the *tensor intensity* of the magnetic field.

13.2.2 Toy-Model Case

Calculations

Let us start with the toy-model case in which $X = \mathbb{R}^d$, g^{jk}, V and F_{jk} are constant:

Theorem 13.2.1. *Let $X = \mathbb{R}^d$, g^{jk}, V and F_{jk} be real constants and let condition* (13.2.4) *be fulfilled.*

(i) Then

$$e(x,x,\tau) = h^{-d}\mathcal{N}_x^{\mathsf{MW}}(\tau,\mu h) \tag{13.2.8}$$

where

$$\mathcal{N}^{\mathsf{MW}}(\tau,\mu h) \coloneqq (2\pi)^{r-d}\varpi_{d-2r}\sqrt{g}\times \sum_{\alpha\in\mathbb{Z}^{+r}} \Big(\tau - \sum_{1\le j\le r}(2\alpha_j+1)\mu h f_j - V\Big)_+^{\frac{d-2r}{2}} f_1\cdots f_r(\mu h)^r, \tag{13.2.9}$$

(13.2.10) $2r = \mathsf{rank}(F_{jk})$, $f_j > 0$ *and* $\pm i f_j$ $(j = 1,\dots,r)$ *are all the eigenvalues (counting multiplicities) of the matrix* $(F_j^l) = (g^{lj})(F_{jk})$, $g = \det(g^{jk})^{-1}$.

(ii) In particular, for $2r = d$ the spectrum of the operator A is pure point of infinite multiplicity and is given by the formula

$$\mathsf{Spec}(A) = \Big\{ \sum_{1\le j\le r}(2\alpha_j+1)\mu h f_j + V, \quad \alpha\in\mathbb{Z}^{+r}\Big\} \tag{13.2.11}$$

and in order to have equality (13.2.8) *at every point $\tau\in\mathbb{R}$ we take function τ_+^0 semicontinuous from the left in* (13.2.9).

(iii) On the other hand, if $2r < d$ then the spectrum of A is absolutely continuous and is given by the formula

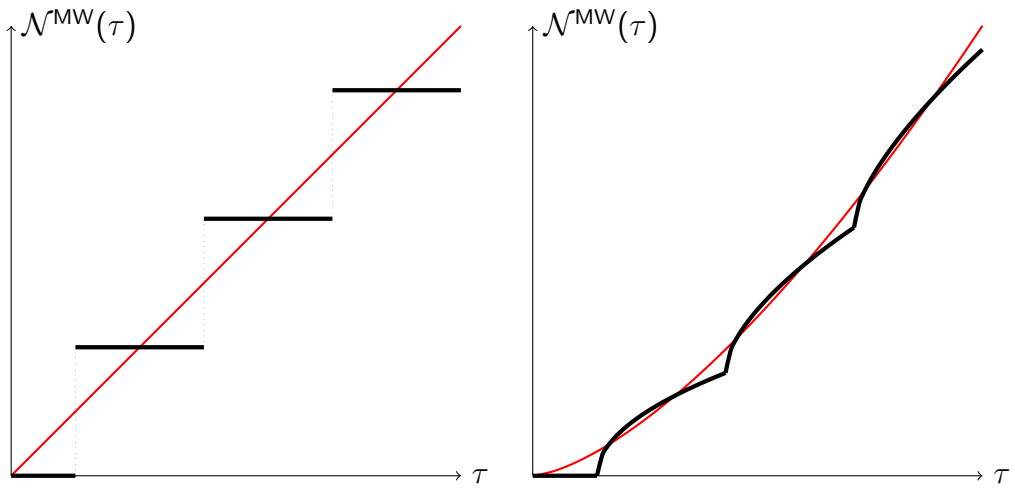
$$(13.2.11)' \qquad \mathsf{Spec}(A) = [\mu h(f_1 + ... + f_r) + V, \infty)\,.$$

Definition 13.2.2. (i) $h^{-d}\mathcal{N}^{\mathsf{MW}}(\tau, \mu h)$ is called *Magnetic Weyl expression.*

(ii) The numbers

$$(13.2.12) \qquad \lambda(\alpha) := \sum_{1\le j\le r} (2\alpha_j + 1)\mu h f_j + V$$

with $\alpha \in \mathbb{Z}^{+r}$ are called *Landau levels.*



(a) $d = 2$ (b) $d = 3$

Figure 13.1: Graph of $\mathcal{N}^{\mathsf{MW}}(\tau)$; red line shows graph of $\mathcal{N}^{\mathsf{W}}(\tau)$

Corollary 13.2.3. *In the framework of Theorem 13.2.1*

(i) Let $d = 2$ and $F_{12} \ne 0$. Then the spectrum of A is pure point of infinite multiplicity, is equal to $\{(2j + 1)\mu hF,\ j \in \mathbb{Z}^+\}$ and

$$(13.2.9)_{(2)} \quad \mathcal{N}^{\mathsf{MW}}(\tau, \mu h) = \#\{j \in \mathbb{Z}^+ : (2j + 1)\mu hF + V \le \tau\} \cdot (2\pi)^{-1}\sqrt{g}(\mu h)F,$$

where

$$F = \frac{1}{\sqrt{g}}|F_{21}| \tag{13.2.13)$_{(2)}}$$

is the scalar intensity of the magnetic field.

(ii) Let $d = 3$ and $(F_{jk}) \neq 0$. Then the spectrum of A is absolutely continuous, is equal to $[\mu hF, \infty]$ and

$$\mathcal{N}^{\mathsf{MW}}(\tau, \mu h) = \sum_{0\le j<\infty} \big(\tau - (2j+1)\mu hF - V\big)_+^{\frac{1}{2}} \mu hF \cdot (2\pi)^{-2}\sqrt{g} \tag{13.2.9)$_{(3)}}$$

where

$$F = \big(\sum_{j,k} g_{jk}F^jF^k\big)^{\frac{1}{2}} = \frac{1}{2}\big(\sum_{j,k,l,m} g^{jk}g^{lm}F_{jl}F_{km}\big)^{\frac{1}{2}} \tag{13.2.13)$_{(3)}}$$

is the scalar intensity of the magnetic field,

$$F^j = \sum_{k,l} \frac{1}{2}\varepsilon^{jkl}F_{kl} \tag{13.2.14}$$

is the vector intensity of the magnetic field;

(13.2.15) *(ε^{jkl}) is the absolutely skew-symmetric tensor with $\varepsilon^{123} = 1/\sqrt{g}$, $(g_{jk}) = (g^{jk})^{-1}$.*

Proof of Theorem 13.2.1. One can see easily that under a coordinate change $\mathcal{N}^{\mathsf{MW}}$ changes as a density; we know that $e(x, x, \tau)$ also changes as a density and therefore we can make a linear coordinate change by which we can arrange $g^{jk} = \delta_{jk}$; then by an orthogonal coordinate change we can reduce (F_{jk}) to the canonical form for skew-symmetric matrices: $F_{2l-1,2l} = f_l$, $F_{2l,2l-1} = -f_l$ for $l = 1, \dots, r$ and all the other elements of F_{jk} vanish.

Further, without any loss of the generality one can assume that V_j are linear functions, $V_{2l} = -f_l x_{2l-1}$, $l = 1, \dots, r$ and all the other components of the vector potential vanish; we can obtain this by a gauge transform.

Furthermore, without any loss of the generality we can assume that $V = 0$ (otherwise we replace V by 0 and τ by $(\tau - V)$), that $h = 1$ (otherwise we

divide A and τ by h^2, i.e., we replace h, μ, τ, V by 1, μh^{-1}, τh^{-2} and Vh^{-2} respectively) and that $\mu = 1$ (otherwise we include μ in V_j); the statements of our theorem is invariant with respect to all these transformations.

So we need to consider a toy-model operator

$$A = \sum_{1\le j\le r} \Bigl(D_{2j-1}^2 + (D_{2j} + x_{2j-1}f_j)^2\Bigr) + \sum_{2r+1\le j\le d} D_j^2. \tag{13.2.16}$$

Let $x' = (x_1, x_3, \dots, x_{2r-1})$, $x'' = (x_2, x_4, \dots, x_{2r})$ and $x''' = (x_{2r+1}, \dots, x_d)$. Let us make Fourier transformation $x'' \to \xi'', x''' \to \xi'''$; then A will be transformed into operator

$$A_I = \sum_{1\le j\le r} \Bigl(D_{2j-1}^2 + (x_{2j-1}f_j + \xi_{2j})^2\Bigr) + |\xi'''|^2. \tag{13.2.17}$$

Further, let us change coordinates $\tilde{x}_{2j-1} = \frac{1}{\sqrt{f_j}}(x_{2j-1}f_j + \xi_{2j})$; then we obtain operator

$$A_{II} = \sum_{1\le j\le r} f_j(D_{2j-1}^2 + x_{2j-1}^2) + |\xi'''|^2. \tag{13.2.18}$$

The operators $(D_{2j-1}^2 + x_{2j-1}^2)$ are harmonic oscillators with eigenvalues $(2k+1)$, $k \in \mathbb{Z}^+$ and with the corresponding eigenfunctions υ_k which are Hermite functions[1)]. Then the Schwartz kernel of the spectral projector of A_{II} is

$$e_{II}(x', \xi'', \xi'''; y', \eta'', \eta'''; \tau) = \sum_{\alpha} \upsilon_\alpha(x')\upsilon_\alpha(y')\delta(\xi'' - \eta'')\delta(\xi''' - \eta''') \tag{13.2.19}$$

where

$$\upsilon_\alpha(x') := \prod_{1\le j\le r} \upsilon_{\alpha_j}(x_{2j-1}) \tag{13.2.20}$$

and we sum in (13.2.19) with respect to $\alpha \in \mathbb{Z}^{+r}$ such that

$$\sum_{1\le j\le r} (2\alpha_j + 1)f_j + |\xi'''|^2 < \tau. \tag{13.2.21}$$

[1)] We take orthonormal and real-valued Hermite functions.

Then the Schwartz kernel of the spectral projector of A_l is

$$e_l(x',\xi'',\xi''';y',\eta'',\eta''';\tau) = \sum_{\alpha} \upsilon_\alpha(\tilde{x}')\upsilon_\alpha(\tilde{y}')(f_1\cdots f_r)^{\frac{1}{2}}\delta(\xi''-\eta'')\delta(\xi'''-\eta''') \tag{13.2.22}$$

with

$$\tilde{x}_{2j-1} = \frac{1}{\sqrt{f_j}}(x_{2j-1}f_j + \xi_{2j}) \tag{13.2.23}$$

and therefore

$$e(x,y,\tau) = (2\pi)^{r-d}\sum_{\alpha}\iint \upsilon_\alpha(\tilde{x}')\upsilon_\alpha(\tilde{y}')(f_1\cdots f_r)^{\frac{1}{2}}\times e^{i\langle x''-y'',\xi''\rangle+\langle x'''-y''',\xi'''\rangle}\,dx''d\xi''' \tag{13.2.24}$$

where we sum with respect to $\alpha\in\mathbb{Z}^{+r}$ and integrate with respect to $\xi''\in\mathbb{R}^r$ and $\xi'''\in\mathbb{R}^{d-2r}$ such that (13.2.21) holds. Let us set $x=y$; going from integration with respect to ξ'' to integration with respect to $\tilde{x}'$ (see (13.2.23) we obtain that

$$\int \upsilon_\alpha(\tilde{x}')^2\,d\xi'' = (f_1\cdots f_r)^{\frac{1}{2}}$$

and therefore

$$e(x,x,\tau) = (2\pi)^{r-d}\sum_{\alpha\in(\mathbb{Z}^+)^r}\int d\xi'''\cdot f_1\cdots f_r$$

where we sum with respect to $\alpha\in(\mathbb{Z}^+)^r$ and integrate with respect to $\xi''\in\mathbb{R}^r$ and $\xi'''\in\mathbb{R}^{d-2r}$ such that (13.2.21) holds; this formula implies (13.2.9).

Statement (i) is proven; Statements (ii) and (iii) follow immediately from it. □

Magnetic Weyl Expression vs (Standard) Weyl Expression

Let us compare magnetic Weyl expression $\mathcal{N}^{\mathsf{MW}}(\tau,\hbar)$ defined by (13.2.9) and the standard Weyl expression

$$\mathcal{N}^{\mathsf{W}}(\tau) \coloneqq (2\pi)^{-d}\varpi_d\sqrt{g}\big(\tau-V\big)_+^{\frac{d}{2}} \tag{13.2.25}$$

(which is also formula (13.2.9) with $r = 0$).

Without any loss of the generality one can assume that $V = 0$. Let us notice that the right-hand expression in (13.2.9) can be treated as a Riemann integral sum for the r-dimensional integral of the function

$$\varphi(z) = (2\pi)^{r-d}2^{-r}\varpi_{d-2r}\sqrt{g}\big(\tau - f_1z_1 - ... - f_rz_r\big)_+^{\frac{d-2r}{2}} f_1\cdots f_r \tag{13.2.26}$$

with respect to $z \in \mathbb{R}^{+r}$ with step $2\hbar$ on each variable z_j, $\hbar = \mu h$. Therefore as $\hbar \to 0$, $\mathcal{N}^{\mathsf{MW}}(\tau, \hbar)$ tends to the corresponding integral which is equal to (13.2.25).

The corresponding integral formula is accurate to the second order but the deduction that $\mathcal{N}^{\mathsf{MW}}(\tau, \hbar) = \mathcal{N}^{\mathsf{W}}(\tau) + O(\hbar^2)$ is not necessarily true because the function φ is not necessarily smooth enough.

So we must treat this formula with more care if needed and with better precision if possible. First of all note[2)] that for one-dimensional smooth function φ

$$\Phi_1 := 2\hbar \sum_{0\le\alpha\le N-1} \varphi\big((2\alpha+1)\hbar\big) \sim \sum_{n\ge 0} c_n\hbar^{2n}\int_0^a \varphi^{(2n)}(z)\,dz \sim \int_0^a \varphi(z)\,dz + \sum_{n\ge 1} c_n\hbar^{2n}\varphi^{(2n-1)}(z)\Big|_{z=0}^{z=a} \tag{13.2.27}$$

with absolute constants c_k, $c_0 = 1$ and $c_1 = -\frac{1}{6}$, $a = 2N\hbar$.

Then after easy calculations we arrive to the *formal decomposition*

$$\mathcal{N}^{\mathsf{MW}}(\tau, \hbar) \sim \sum_{n\ge 0} \kappa_n\hbar^{2n}\tau^{\frac{d}{2}-2n} \tag{13.2.28}$$

with

$$\kappa_0 = \mathcal{N}^{\mathsf{W}}(1), \qquad \kappa_1 = -\frac{1}{6}\big(f_1^2 + ... + f_r^2\big)\big(\partial_\tau^2\mathcal{N}^{\mathsf{W}}(\tau)\big)\big|_{\tau=1}. \tag{13.2.29}$$

Let us find out an actual error in (13.2.28).

Proposition 13.2.4. *Formula* (13.2.28) *holds with an error* $O(\tau^{d/2-l}\hbar^l)$, $l = \frac{1}{2}(d-2r)+1$ *and one should take summation* $n \le N-1$ *with* $N = \lceil \frac{1}{4}(d-2r+2)\rceil$.

[2)] This well-known formula of numerical integration is an easy first year Calculus exercise.

Proof. Without any loss of the generality one can assume that $\tau \asymp 1$ (and even $\tau = 1$). Note that if

$$\rho(z) := \tau - \bigl(f_1 z_1 + ... + f_r z_r\bigr) > \hbar \tag{13.2.30}$$

for $z = (z_1, ..., z_r)$, $z_j = (2\alpha_j + 1)\hbar$ then both inequalities $\epsilon_0 \rho(z) \le \rho(z') \le c\rho(z)$ hold for all points z, z' belonging to the box

$$\Omega_\alpha := \bigl\{z : (2\alpha_j - 1)\hbar \le z_j \le (2\alpha_j)\hbar \quad \forall j\bigr\}$$

and hence the error of our integral formula in this box does not exceed

$$C\hbar^{2N} \int_{\Omega_\alpha} \rho(z)^{\frac{1}{2}(d-2r)-2N}\, dz.$$

After summation over all such boxes we arrive to the same expression with an integral over zone described by (13.2.30) which is $O(\hbar^{2N})$ and $O(\hbar^l)$ as $\frac{1}{2}(d-2r) - 2N > -1$ and $\frac{1}{2}(d-2r) - 2N < -1$ respectively.

As $\frac{1}{2}(d-2r) - 2N = -1$ we get $O(\hbar^{2N}|\log\hbar|)$. However we simply increase N and notice that the newly acquired term simply vanishes. So, due to the choice of N we get $O(\hbar^l)$.

On the other hand, in zone where (13.2.30) is violated $|\varphi(z)| \le C\hbar^{\frac{1}{2}(d-2r)}$ and the total volume of this zone is $O(\hbar)$; then contribution of this zone to asymptotics is $O(\hbar^l)$ as well. □

One can ask if an error is actually less than $O(\hbar^l)$.

Remark 13.2.5. (i) As $d = 2$, $r = 1$ $\mathcal{N}^{\mathsf{MW}}(\tau, \hbar)$ is a jump function and an uniform with respect to τ error cannot be better than $O(\hbar)$. The same is true as $d = 2r$, $r \ge 2$ and $f_1, ..., f_r$ are commensurable and may be not true otherwise.

(ii) As $d = 3$, $r = 1$ one cannot improve error estimate $O(\hbar^{\frac{3}{2}})$ either. Really, formula $(13.2.9)_{(3)}$ yields that as $\tau = (2m+1)\hbar f \asymp 1$, $m \in \mathbb{Z}^+$, $\delta > 0$

$$\mathcal{N}^{\mathsf{MW}}(\tau + \delta\hbar) \ge \mathcal{N}^{\mathsf{MW}}(\tau, \hbar) + c_1\sqrt{\tau}\delta\hbar + c_2\hbar^{\frac{3}{2}}\sqrt{\delta} + O(\hbar^{\frac{3}{2}}\delta)$$

with positive constants c_1 and c_2; on the other hand

$$\mathcal{N}^{\mathsf{W}}(\tau + \delta\hbar) = \mathcal{N}^{\mathsf{W}}(\tau) + c_1\sqrt{\tau}\delta\hbar + O(\hbar^2\delta^2)$$

and therefore the error estimate $O(\hbar^{\frac{3}{2}})$ is the best possible in either τ or $\tau + \delta\hbar$.

We leave to the reader

Problem 13.2.6. (i) Examine the general case $d > 2r$ as *either* $r = 1$ *or* $r \ge 2$ and $f_1, \dots, f_r$ are commensurable.

(ii) Examine the Riesz means.

(iii) Prove the similar results[3)] for *modified Magnetic Weyl expression* which will appear later for magnetic Schrödinger-Pauli and magnetic Dirac operators:

$$(13.2.9)' \quad \mathcal{N}^{\mathsf{MW}}(\tau, \mu h) := (2\pi)^{r-d}\varpi_{d-2r}\sqrt{g}\times \sum_{\alpha\in\mathbb{Z}^{+r}} 2^{-\#\{j:\alpha_j=0\}}\Big(\tau - \sum_{1\le j\le r} 2\alpha_j\mu h f_j - V\Big)_+^{\frac{d-2r}{2}} f_1\cdots f_r(\mu h)^r.$$

Dynamics

Consider classical dynamics for the toy-model operator; without any loss of the generality one can assume that it is in the form (13.2.16) with f_j replaced by μf_j and we assume only that $fj \ne 0$.

(i) As $d = 2$, $r = 1$ we get $\pi f^{-1}\mu^{-1}$-periodic movements[4)]

$$(13.2.31) \qquad x_1 = \bar{x}_1 + \mu^{-1}a\cos(2\mu ft), \qquad x_2 = \bar{x}_2 + \mu^{-1}a\sin(2\mu ft),$$

$$(13.2.32) \qquad \xi_1 = -af\sin(2\mu ft), \qquad \xi_2 = -\mu f\bar{x}_1$$

where (13.2.31) is gauge invariant and describes circular movement, $\bar{x}$ is the center of the circle, $\mu^{-1}a$ its radius. Note that (13.2.32) are not gauge invariant and as one transforms terms $\xi_1^2 + (\xi_2 + \mu f x_1)^2$ to more symmetric form

$$(13.2.33) \qquad \big(\xi_1 - \frac{1}{2}\mu f x_2\big)^2 + \big(\xi_2 + \frac{1}{2}\mu f x_1\big)^2$$

we arrive to

$$(13.2.32)' \quad \xi_1 = \frac{1}{2}\mu f\bar{x}_2 - \frac{1}{2}af\sin(2\mu ft), \qquad \xi_2 = -\frac{1}{2}\mu f\bar{x}_1 + \frac{1}{2}af\cos(2\mu ft).$$

[3)] With $-\frac{1}{6}$ replaced by $\frac{1}{3}$.

[4)] Meanwhile quantum dynamics is only $2\pi f^{-1}\mu^{-1}$-periodic.

Note that

$$p_1 := \xi_1 - \frac{1}{2}\mu f x_2, \qquad p_2 := \xi_2 + \frac{1}{2}\mu f x_1 \tag{13.2.34}$$

then evolve as

$$p_1 = -af\sin(2\mu f t), \qquad p_2 = af\cos(2\mu f t) \tag{13.2.35}$$

independently from gauge transform and energy is $\tau = a^2 f^2$.

Meanwhile *angular momentum with respect to* $\bar{x}$ is

$$M := (x_1 - \bar{x}_1)p_2 - (x_2 - \bar{x}_2)p_1 = \mu^{-1}a^2 f. \tag{13.2.36}$$

(ii) As $d = 2r$, $r \ge 2$ we have the same equations for x_{2j-1}, x_{2j}, ξ_{2j-1}, ξ_{2j}, p_{2j-1}, p_{2j} with a and f replaced by f_j and p_j respectively, $j = 1, \dots, r$. Then we conclude that the dynamics is periodic if and only if $f_1, \dots, f_r$ are commensurable[5)] and quasiperiodic otherwise. While it does not affect the nature of the spectrum, it affects its distribution. In this case energy is $\tau = \sum_j a_j^2 f_j^2$.

(iii) As $d > 2r$ in addition to *magnetic coordinates* $x_1, \dots, x_{2r}$ *free coordinates* $x_{2r+1}, \dots, x_d$ appear with evolution

$$\xi_k = \bar{\xi}_k, \qquad x_k = \bar{x}_k + \bar{\xi}_k t \qquad k = 2r+1, \dots, d. \tag{13.2.37}$$

So dynamics is no more periodic at all (with the possible exception as $\bar{\xi}_{2r+1} = \dots \bar{\xi}_d = 0$). In this case energy is

$$\tau = \sum_{1 \le j \le r} a_j^2 f_j^2 + \sum_{2r+1 \le k \le d} \xi_k^2 \tag{13.2.38}$$

Lines described by (13.2.37) (with $x_k = \mathsf{const}$ as $k = 1, \dots, 2r$) are called *magnetic lines*. Their directions form $\mathsf{Ker}\, F$ where $F = (F_{jk})$ is a matrix intensity of the magnetic field. In particular, as $d = 3$, $r = 1$ magnetic lines are integral curves of the vector intensity of the magnetic field (which is constant in our case).

[5)] If $f_j \in \mathbb{Z}f$ for all $j = 1, \dots, r$ then the classical dynamics is $\pi\mu^{-1}f^{-1}$-periodic while quantum dynamics $\pi\mu^{-1}f^{-1}$-periodic as r is even and only $2\pi\mu^{-1}f^{-1}$-periodic as r is odd.

(iv) Let us perturb the toy-model operator by a constant electric field i.e. by the linear potential $V(x) = -\langle \boldsymbol{e}, x\rangle$. As each pair of magnetic variables and each free variable are affected separately, we consider only case $d = 2$, $r = 1$.

Then we have

$$\frac{d}{dt}x_1 = 2p_1, \qquad \frac{d}{dt}x_2 = 2p_2 \tag{13.2.39}$$

$$\frac{d}{dt}p_1 = -2\mu f p_2 + e_1, \qquad \frac{d}{dt}p_2 = 2\mu f p_1 + e_2 \tag{13.2.40}$$

and the second pair of equations implies

$$p_1 = -\frac{1}{2}\mu^{-1}f^{-1}e_2 - af\sin(2\mu ft), \qquad p_2 = \frac{1}{2}\mu^{-1}f^{-1}e_1 + af\cos(2\mu ft) \tag{13.2.35)*}$$

while the first pair of equations then implies

$$x_1 = \bar{x}_1 + \mu^{-1}a\cos(2\mu ft) - \mu^{-1}f^{-1}e_2 t, \qquad x_2 = \bar{x}_2 + \mu^{-1}a\sin(2\mu ft) + \mu^{-1}f^{-1}e_1 t. \tag{13.2.31)*}$$

Therefore we have an additional *drift* with the velocity $\mu^{-1}f^{-1}\boldsymbol{e}^{\perp} = \mu^{-1}F^{-1}\boldsymbol{e}$ with $\boldsymbol{e}^{\perp} = (-e_2, e_1)$ and F^{-1} inverse matrix to F.

Therefore, as $d = 2r$ magnetic drift has velocity $-\mu^{-1}F^{-1}\nabla V$. As $d > 2r$ there is a magnetic drift in magnetic variables and a movement with Hamiltonian $(|\xi_k|^2 - e_k x_k)$ in each of the free variables.

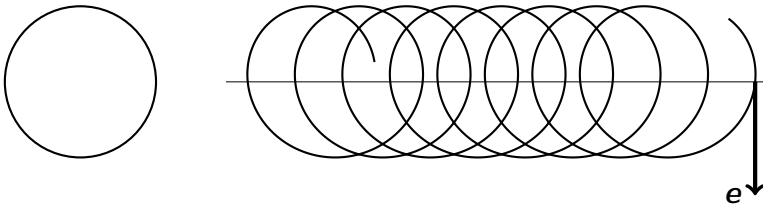
13.2.3 Rough Estimates

Functional Analytic Arguments

Let us now turn to functional analytic arguments.

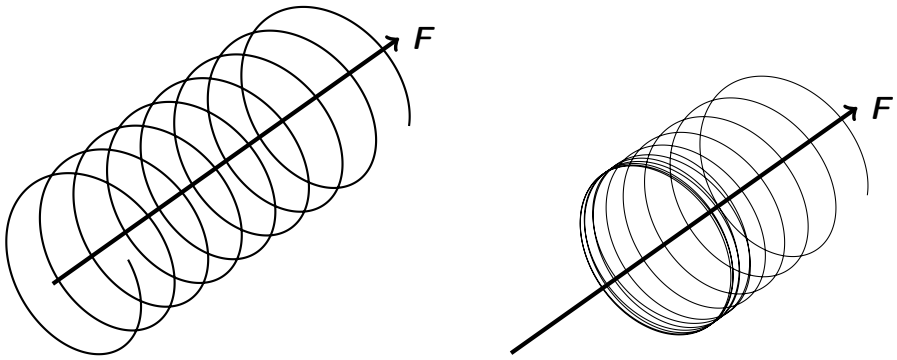
Theorem 13.2.7. *Let conditions* (13.2.1)–(13.2.4) *be fulfilled with* $K > \frac{d}{2}$. *Let* $h \in (0, 1]$, $\mu \geq 1$. *Assume that operator* A *is self-adjoint and that* $u \in \mathscr{C}_0^2(B(0,1))$ *belongs to* $\mathfrak{D}(A)$ *if and only if* $u|_{\partial X} = 0$ [6]. *Then* $U(x, y, t)$,

[6] In this theorem we do not assume that condition (13.2.5) is fulfilled.



(a) Periodic (b) Drift

Figure 13.2: Trajectories for 2-dimensional model operator:
(a) Unperturbed (b) Perturbed by a constant electric field



(a) Helix (b) Deformed helix

Figure 13.3: Trajectories for 3-dimensional model operator:
(a) Unperturbed (b) Perturbed by a constant electric field parallel to magnetic lines: non-uniform movement along magnetic lines; drift is negligible in comparison

the Schwartz kernel of $\exp(ith^{-1}A)$*, satisfies*

$$(13.2.41)\quad |F_{t\to h^{-1}\tau}\chi_{T,T'}(t)U(x,y,t)| \le C'Th^s\mu^{-s}(|\tau|+1)^{-s}$$
$$\forall x,y\in X\cap B(0,1-\epsilon)\quad \forall \tau \le \inf_{X\cap B(0,1)}(V+\mu h\,\mathrm{Tr}^+(gF))-\epsilon(1+\mu h)$$

where

$$(13.2.42)\quad \mathrm{Tr}^+(gF) := \textstyle\sum_{1\le j\le r} f_j \text{ and } \pm if_j \text{ are all the eigenvalues of } ((gF)^k_j),$$
$f_j > 0$ [7)]

[7)] Note that $(gF)\sim g^{\frac12}Fg^{\frac12}$ which is skew-symmetric matrix.

and s, $\epsilon > 0$, $T > \epsilon_0 h^{1-\delta}$, $\delta > 0$, T' *are arbitrary and* $C' = C'(d, c, \delta, \epsilon, s)$.

Proof. Let us note that the proof of Proposition 4.2.6 obviously yields estimate

$$\|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)U\| \le C' T \mu^L h^{-L}(|\tau|+1)^L \qquad \forall x \in X \cap B(0, 1-\epsilon) \quad \forall \tau \in \mathbb{R} \tag{13.2.43}$$

where $L = d/2$ and in the left-hand expression the $\mathscr{L}^2$-norm with respect to y is calculated, also the same estimate with permuted x and y and also estimate

$$|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)U(x,y,t)| \le C' T \mu^L h^{-L}(|\tau|+1)^L \qquad \forall x, y \in X \cap B(0, 1-\epsilon) \quad \forall \tau \in \mathbb{R} \tag{13.2.44}$$

with $L = d$; moreover, this proof yields similar estimates with $F_{t\to h^{-1}\tau}\chi_{T,T'}(t)U$ replaced by $e(x, y, -\infty, \tau)$.

The key in the forthcoming arguments is

Lemma 13.2.8. *If the conditions of Theorem 13.2.7 are fulfilled with* $K = 2$ *then*

$$\epsilon \sum_j \|P_j w\|^2 + \epsilon(\mu h + |\tau| + 1)\|w\|^2 \le C'\|(\tau - A)w\| \cdot \|w\| + C' h \|w\|^2 \tag{13.2.45}$$

$$\forall \tau \le \inf_{X\cap B(0,1)} (V + \mu h \,\mathsf{Tr}^+(gF)) - \epsilon(\mu h + 1)$$

$$\forall w \in \mathscr{C}^2(X \cap B(0,1)) : \ \mathsf{supp}(w) \subset B(0, 1-\epsilon),\ w|_{\partial X} = 0.$$

This lemma will be proven later; now let us finish the proof of Theorem 13.2.7. Let

$$v \in \mathfrak{D}(A), \qquad u = \exp(ith^{-1}A)v \tag{13.2.46}$$

Then equation and boundary condition

$$(hD_t - A)u = 0 \tag{13.2.47}$$

$$u|_{\partial X \cap B(0,1)} = 0, \tag{13.2.48}$$

Lemma 13.2.8 and the standard elliptic arguments with induction with respect to s yield estimate

$$\sum_j \|P_j \hat{w}\| + \|\hat{w}\| \le C' T \mu^{-s} h^s (|\tau|+1)^{-s} \|v\|$$

for $\hat{w} = F_{t\to h^{-1}\tau}\big(\chi_{T,T'}(t)\psi u\big)$ where $\psi \in \mathscr{C}_0^\infty(B(0,1-\epsilon))$ is arbitrary fixed function and constant C' also depends on ψ; at every step the support of ψ is decreased but remains arbitrary.

Then treating our equation and boundary condition (13.2.47)–(13.2.48) again we conclude by induction with respect to $m = 1, \dots, K$ that

$$\sum_{|\alpha|\le m} \|D^\alpha \psi \hat{w}\| \le C' T \mu^{-s} h^s (|\tau|+1)^{-s} \|v\| \tag{13.2.49}$$

where s decreases at every step but remains arbitrarily large. Then the embedding inequalities yield that the same estimate holds for $|\hat{w}(x,\tau)|$ with arbitrary $x \in X \cap B(0,1-\epsilon)$; recalling definition of $\hat{w}$ and that in (13.2.46) v is arbitrary we conclude that

$$\|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)U(x,.,t)\| \le C' T \mu^{-s} h^s (|\tau|+1)^{-s}$$

for every $x \in X \cap B(0,1-\epsilon)$ and for the indicated τ where in the left-hand expression the $\mathscr{L}^2$-norm with respect to y is calculated. The previous arguments with y instead of x and with $u(y,t) = U(x,y,t)$ with fixed x, the dual to (13.2.47) equation and estimate (13.2.44) prove theorem.

Proof of Lemma 13.2.8. Note that it is sufficient to prove that if all the assumptions of the lemma excluding the conditions on V are fulfilled and if

$$\inf(V + \mu h \operatorname{Tr}^+(gF)) \ge \epsilon(\mu h + 1) \tag{13.2.50}$$

then

$$(Aw, w) \ge C^{-1}\epsilon\Big(\sum_j \|P_j w\|^2 + (\mu h+1)\|w\|^2\Big) - Ch\|w\|^2. \tag{13.2.51}$$

Also note that it is sufficient to prove this inequality under the additional assumption that w is supported in $B(\bar{x},\varepsilon)$ with an arbitrarily small constant $\varepsilon > 0$. In fact, let ψ_ν^2 be an ε-admissible partition of unity; then

$$|(Aw,w) - \sum_\nu (A\psi_\nu w, \psi_\nu w)|$$

and

$$|\|P_j w\|^2 - \sum_\nu \|P_j \psi_\nu w\|^2|$$

do not exceed $Ch\varepsilon^{-1}\|w\|^2$ and therefore estimate (13.2.51) for w follows from the same estimate for $\psi_\nu w$.

Without any loss of the generality one can assume that $g^{jk} = \delta_{jk}$ at $\bar{x}$. Let $G = (G^{jk})$ be a real skew-symmetric matrix with operator norm (in $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$) equal to 1. Then

$$\sum_j \|P_j w\|^2 \geq |\sum_{j,k} (G^{jk} P_j w, P_k w)| = \frac{1}{2}|\sum_{j,k} (G^{jk}[P_k, P_j]w, w)| = \frac{1}{2}\mu h|\sum_{j,k} (G^{jk} F_{kj} w, w)|.$$

Note that one can choose G of the indicated type such $\sum_{j,k} G^{jk} F_{jk} = 2\,\mathsf{Tr}^+(gF)$; this is easy to do in the coordinate system in which (F_{kj}) is in the canonical form for skew-symmetric matrices, in which case $G^{jk} = -\operatorname{sgn}(F_{jk})$ is an appropriate matrix. Then the previous inequality implies that

$$\sum_j \|P_j w\|^2 \geq \mu h(\mathsf{Tr}^+(gF)w, w) - C\varepsilon\mu h\|w\|^2;$$

since $g^{jk} = \delta_{jk}$ we conclude that

$$(Aw, w) \geq \big((\mu h\, \mathsf{Tr}^+(gF) + V)w, w\big) - C\varepsilon\mu h\|w\|^2.$$

This inequality and (13.2.50) yield (13.2.51). □

Theorem 13.2.7 and the standard Tauberian arguments (see the proof of Proposition 4.2.6 again) yield

Theorem 13.2.9. *Let assumptions of Theorem 13.2.7 be fulfilled. Then*

$$|e(x, y, \tau)| \leq C'\mu^{-s} h^s (|\tau| + 1)^{-s} \quad \forall x, y \in X \cap B(0, 1-\epsilon) \quad \forall \tau \leq \inf_{(X \cap B(0,1))} (V + \mu h F) - \epsilon(\mu h + 1) \tag{13.2.52}$$

with the same s, ϵ and C' as in Theorem 13.2.7.

Remark 13.2.10. (i) Condition on τ in (13.2.41), (13.2.52) can be weakened for $d = 2, 3$ (see Sections 13.5, 13.6).

(ii) Now assumption

$$\mathsf{Tr}^+(gF) \geq \epsilon_0 \tag{13.2.53}$$

and the condition $\tau \leq c$ on the energy level yield that one can assume that $\mu h \leq C_0$. Hence without any loss of the generality one can assume that

$$\mu h \leq c_0. \tag{13.2.54}$$

(iii) However it would not be the case for Schrödinger-Pauli and Dirac operators. Moreover, as we will see in Subsection 13.8.2 and Chapter 14 that for degenerating magnetic field condition (13.2.54) is too restrictive and needs to be replaced by a weaker one.

Propagation of sSingularities

Let us now prove that singularities propagate with finite speed with respect to x at energy levels $\tau \leq c$.

Theorem 13.2.11. *Let assumptions of Theorem 13.2.7 be fulfilled. Then*

$$|F_{t\to h^{-1}\tau}\chi_T(t)U(x,y,t)| \leq C'h^s \tag{13.2.55}$$
$$\forall T \geq h^{1-\delta} \quad \forall x,y \in X \cap B(0, 1 - C_0T) : |x-y| \geq C_0T \quad \forall \tau \leq c$$

where $C_0 = C_0(c, \epsilon_0)$, $C' = C'(d, c, \epsilon_0, \delta, s)$, s *and* $\delta > 0$ *are arbitrary.*

Proof. Let us note that it is sufficient to prove this estimate only for $T \geq \epsilon$ with an arbitrarily small constant $\epsilon > 0$ and with C' also depending on ϵ. The generalization is made by the rescaling method (see the proof of Theorem 2.1.19).

Moreover, elliptic arguments (see the proof of Theorem 13.2.7) yield that it is sufficient to prove that if $\bar{x}, \bar{y} \in B(0, 1 - C_0T)$, $|\bar{x} - \bar{y}| \geq C_0T$, $\zeta \in \mathscr{L}^2(X \cap B(\bar{y}, \epsilon_1))$ with a small enough constant $\epsilon_1 > 0$ and $u = U(t)\zeta$ then

$$\|F_{t\to h^{-1}\tau}\chi_T(t)u\|_{X\cap B(\bar{x},\epsilon_1)} \leq C'h^s\|\zeta\| \qquad \forall T \in [\epsilon, T_0] \quad \forall \tau \leq c. \tag{13.2.56}$$

In order to prove this estimate we apply the method of the proof of Theorem 2.1.2.

Let $u^{\pm} = \theta(\pm t)u$, $v^{\pm} = \Phi(hD_t)\bar{\chi}(t)u^{\pm}$ where $\Phi \in \mathscr{C}^{\infty}(\mathbb{R})$ is fixed and supported in $[-\infty, 2c]$, $\Phi(\tau) = 1$ for $\tau \le \frac{3}{2}c$, $\bar{\chi} \in \mathscr{C}_0^{\infty}([-2T, 2T])$ and $\bar{\chi} = 1$ at $[-\frac{3}{2}T, \frac{3}{2}T]$. Then

$$(hD_t - A)v^{\pm} = \mp ih\Phi(hD_t)\delta(t)\zeta(x) \quad \text{for} \quad |t| \le T \tag{13.2.57}$$

modulo terms with $\mathscr{L}^2$-norm not exceeding $C'h^s$ where here and in what follows C' depends on all the auxiliary functions and constants.

Let $\phi \in \mathscr{C}^{\infty}(B(0,1))$ be real-valued and satisfy

$$|\nabla_x \phi| \le \varepsilon, \tag{13.2.58}$$

and

$$\phi < 0 \quad \text{provided either} \quad |x| \ge 1 - 2\epsilon_1 \text{ or } |x - y| \le 2\epsilon_1 \tag{13.2.59}$$

where $\varepsilon > 0$ will be chosen later.

Let $\chi_0 \in \mathscr{C}^{\infty}(\mathbb{R})$ be supported in $[-\infty, 1]$, $\chi_0(t) = 1$ for $t \le 0$ and $\chi_0 = \chi_1^2$, $\chi' = d\chi_0/dt = -\chi_2^2$ with real-valued functions $\chi_1, \chi_2 \in \mathscr{C}^{\infty}(\mathbb{R})$. Then for $\alpha = \alpha(c, \epsilon_0, \epsilon_1)$ function $\psi(x,t) = \chi_0(\alpha(t - \phi(x))$ belongs to $\mathscr{C}^{\infty}(B(0,1) \times \mathbb{R})$ and

$$\operatorname{supp}(\psi) \cap \{t \ge 0\} \subset \{|x| \le 1 - \epsilon_1, |x - y| \ge \epsilon_1\}.$$

Finally, let $\chi(t) = \chi_0(\gamma t)$ and $\chi_3(t) = \chi_0(-\gamma_1 t - 1)$. Then modulo terms with absolute value not exceeding $C'h^{2s}\|\zeta\|^2$

$$\begin{aligned} 0 \equiv 2\operatorname{Re}\big(i\Phi(hD_t)\delta(t)\zeta(x), e^{-2t}\chi\chi_3\psi v^+\big) \equiv \\ -h^{-1}\operatorname{Re} i([hD_t - A, e^{-2t}\chi\chi_3\psi]v^+, v^+) \equiv \\ \operatorname{Re}(2\chi\chi_3\psi e^{-2t}v^+, v^+) - \operatorname{Re}(\chi'\chi_3\psi e^{-2t}v^+, v^+) - \\ \operatorname{Re} i(h^{-1}[hD_t - A, \psi]e^{-2t}\chi\chi_3 v^+, v^+) \ge \\ 2\|(\chi\chi_3\psi)^{\frac{1}{2}}e^{-t}v^+\|^2 - \operatorname{Re} i(h^{-1}[hD_t - A, \psi]e^{-2t}\chi\chi_3 v^+, v^+) \end{aligned} \tag{13.2.60}$$

where $(.,.)$ and $\|.\|$ denote the inner product and the norm in $\mathscr{L}^2(X \times \mathbb{R})$.

Let us consider the second term in the right-hand expression; it is equal to

$$\alpha \operatorname{Re}\big((I + B)w^+, w^+\big) + h(\Psi v^+, v^+)$$

where $B = -2\sum_{j,k} g^{jk}(D_j\phi)P_k$, $w^+ = e^{-t}\psi_2(\chi\chi_3)^{\frac12}v^+$, $\psi_2 = \chi_2(\alpha(t-\phi))$ and $\Psi_{\dots}$ here and in what follows denotes a $\mathscr{C}^\infty$-function supported in $\mathsf{supp}(\chi\chi_3\psi)$.

Then assumption (13.2.58) yields that

$$
\begin{aligned}
(13.2.61)\quad |(Bw^+, w^+)| &\le C_0\varepsilon\bigl(\|w^+\|^2 + (g^{jk}P_jw^+, P_kw^+)\bigr) \le \\
&C_0\varepsilon\bigl(\|w^+\|^2 + (Aw^+, w^+)\bigr) \le \\
&C_0\varepsilon\bigl(\|w^+\|^2 + (Av^+, e^{-2t}\chi\chi_3\psi_2^2v^+)\bigr) + h(\Psi'v^+, v^+)
\end{aligned}
$$

where we change C_0 if necessary.

On the other hand, (13.2.57), the definition of ψ_2 and (13.2.59) yield that

$$
\begin{aligned}
(13.2.62)\quad (Av^+, e^{-2t}\chi\chi_3\psi_2^2v^+) &\equiv (hD_tv^+, e^{-2t}\chi\chi_3\psi_2^2v^+) \equiv \\
&\bigl(hD_t\Phi(hD_t)(\chi\chi_3)^{\frac12}\psi_2u^+, \Phi(hD_t)(\chi\chi_3)^{\frac12}\psi_2u^+\bigr) + \\
&\sum_{n\ge1} h^n(\Phi_n(hD_t)\Psi_nu^+, u^+)
\end{aligned}
$$

where $\Phi_n \in \mathscr{C}_0^\infty([c, 2c])$ and $\Psi_n \in \mathscr{C}^\infty$ are supported in $\mathsf{supp}(\chi\chi_3\psi)$. The first term in the right-hand expression of (13.2.62) does not exceed

$$
\|\Phi(hD_t)(\chi\chi_3)^{1/2}\psi_2u^+\| \equiv 2c\|w^+\|^2 + \sum_{n\ge1} h^n(\Phi_n(hD_t)\Psi_nu^+, u^+)
$$

where we change Φ_n and Ψ_n if necessary. Then (13.2.61) and (13.2.62) yield that

$$
|(Bw^+, w^+)| \le \frac14 C_0\varepsilon\|w^+\|^2 + \sum_{n\ge1} h^n(\Phi_n(hD_t)\Psi_nu^+, u^+)
$$

and then (13.2.60) yields that for $\varepsilon \le 1/(3C_0)$

$$
\|(\chi\chi_3\psi)^{\frac12}e^{-t}v^+\|^2 \le \sum_{n\ge1} h^n(\Phi_n(hD_t)\Psi_nu^+, u^+) + C'h^{2s}\|\zeta\|^2.
$$

Moreover, one can allow Φ and Ψ to be smooth functions here (nevertheless these functions were not smooth when they appeared the first time).

Using this inequality and applying induction with respect to s (where the supports of χ_0 and Φ are decreased at each step) we can easily prove that if ϕ satisfies (13.2.58) and (13.2.59) with $\varepsilon = 1/(3C_0)$ then for $T \in [\epsilon, T_0]$

$$
\|\Phi(hD_t)\chi_T(t)\chi_1(\alpha(t-\phi))u^+\| \le C'h^s\|\zeta\|.
$$

Since we can arbitrarily choose ϕ satisfying (13.2.58) and (13.2.59) with $\varepsilon = 1/(3C_0)$ we obtain (13.2.56). □

Remark 13.2.12. (i) In this chapter for $d = 2, 3$ using microlocal canonical form we will essentially improve Theorems 13.2.7 and 13.2.11 (assuming that the coefficients are smooth enough).

(ii) Later we will improve these theorems in the higher dimensions.

Let us conclude this section with the following assertion on the support of singularities:

Theorem 13.2.13. *Let conditions* (13.2.1)–(13.2.5) *be fulfilled with* $K = K(d, s)$. *Let* $q \in \mathsf{S}^0_{h,\rho,1,K}$, $\mu h \le 1$, $\rho \ge 1$,

$$\rho \ge \mu^{\frac{1}{2}} h^{\frac{1}{2}-\delta} \tag{13.2.63}$$

with $\delta > 0$ *and let* q *be supported in* $\{|\xi| \ge C_0\rho\}$. *Let*

$$Q = q(x, hD_1 - \mu V_1, \dots, hD_d - \mu V_d)$$

be an operator with symbol $Q(x,\xi) = q(x, h\xi_1 - \mu V_1(x), \dots, h\xi_d - \mu V_d(x))$.

Further, let $\psi \in \mathscr{C}_0^K(B(0, 1-\epsilon))$. *Then*

$$|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)Q\psi U| \le C'MTh^s(|\tau|+1)^{-s} \qquad \forall x, y \in X \cap B(0, 1-\epsilon) \quad \forall \tau \le c \tag{13.2.64}$$

where $T \ge h^{1-\delta}$, T', $\epsilon > 0$, $\delta > 0$ *and* s *are arbitrary,* $C_0 = C_0(c, \epsilon_0)$, $K = K(d, \delta, s)$, $C' = C'(d, c, \epsilon_0, \epsilon, \delta, s)$ *and* $M = \|\psi\| \cdot \|\!|q|\!\|$.

Proof. The proof is standard elliptic, based on the equation (13.2.47), estimates (13.2.41), (13.2.43) and the ellipticity of the operator $(hD_t - A)$ as $\tau \le c$ and $\sum_j |\xi_j - \mu V_j|^2 \ge C_0^2$.

It should be mentioned that the condition (13.2.63) is necessary to obtain $Q(x,\xi) \in \mathsf{S}^0_{h,\rho_1,\rho_2,K}$ with $\rho_1 \ge 1$, $\rho_2 \in (0, 1]$, $\rho_1\rho_2 \ge h^{-\delta'}$ with $\delta' > 0$; this condition is essential only for $\mu \in [h^{\delta-1}, h^{-1}]$. □

13.3 Microlocal Canonical Form

In this section for $d = 2, 3$ we construct the microlocal canonical form for operator (13.2.1) satisfying conditions (13.2.2)–(13.2.4) and

$$F \geq \epsilon_0 \tag{13.3.1}$$

in the ball $B(0,1) \subset X$ where F is the scalar intensity of the magnetic field.

13.3.1 Symplectic Canonical Form

Let us multiply operator (13.2.1) by μ^{-2} and then note that $\mu^{-2}A$ is a $\mu^{-1}h$-differential operator

$$\mu^{-2}A = \sum_{j,k} (\mu^{-1}hD_j - V_j) g^{jk} (\mu^{-1}hD_k - V_k) + \mu^{-2}V) \tag{13.3.2}$$

with principal symbol

$$a_\varepsilon(x,\xi) = a(x,\xi) + \varepsilon V, \qquad \varepsilon = \mu^{-2}, \tag{13.3.3}$$

with

$$a(x,\xi) = \sum_{j,k} g^{jk} (\xi_j - V_j)(\xi_k - V_k). \tag{13.3.4}$$

Symplectic Canonical Form: $\varepsilon = 0$

Let us first treat the symbol $a(x,\xi)$; conditions (13.2.1)–(13.2.4) yield that $a \geq 0$ and the stationary set of a is

$$\Lambda = \{(x,\xi) : \ da(x,\xi) = 0\} = \{(x,\xi) : \ a(x,\xi) = 0\} = \{(x,\xi) : \ \xi_j = V_j(x) \quad \forall j = 1, \dots, d\} \tag{13.3.5}$$

which is a d-dimensional $\mathscr{C}^K$ manifold. Moreover, (13.3.1) yields that

(13.3.6) For $d = 2$, Λ is a symplectic manifold

and

(13.3.7) For $d = 3$ the restriction to Λ of the symplectic form has constant rank 2 and therefore there exists a unique (modulo a scalar factor) vector field on Λ which is tangent and symplectically orthogonal to Λ at the same time.

This field is the Hamiltonian field generated by

$$F(x,\xi) := \sum_j F^j(\xi_j - V_j) \tag{13.3.8}$$

where F^j are the components of the vector intensity of the magnetic field.

For $d \geq 4$ conditions (13.2.1)–(13.2.4) and (13.3.1) do not completely fix the structure of Λ.

Proposition 13.3.1. *Let conditions* (13.2.1)–(13.2.4) *and* (13.3.1) *be fulfilled. Then*

(i) For $d = 2$ there exists a symplectomorphism Ψ^{-1} of a neighborhood $\mathcal{W}$ of every point $z \in \Lambda$ to a neighborhood $\mathcal{U}$ of

$$\Lambda^0 = \{(x,\xi) : \ x_2 = \xi_2 = 0, (x_1,\xi_1) \in \bar{\Omega}\} \tag{13.3.9}$$

where Ω is a domain in $\mathbb{R}^2$ such that the inverse symplectomorphism Ψ has the following properties:

$$\Psi(\Lambda^0) = \Lambda \cap \mathcal{U}, \tag{13.3.10}$$

$$a \circ \Psi \equiv \sum_{1 \leq n \leq N-1} b_n(x_1,\xi_1)(x_2^2 + \xi_2^2)^n \qquad \mathrm{mod}\ O_{2N-1}(\Lambda^0) \tag{13.3.11}$$

where $O_M(\Lambda)$ is the set of functions vanishing together with all their derivatives up to order $(M-1)$ on Λ; moreover,

$$b_1 = F \circ \bar{\Psi}, \tag{13.3.12}$$

and

$$\bar{\Psi}^{-1*}(dx' \wedge d\xi_1) = F\sqrt{g}\,dx \tag{13.3.13}$$

where $\bar{\Psi} : \Lambda^0 \to \Lambda$ is the restriction of Ψ to Λ^0 and $\bar{\Psi}^*$ is the induced map of densities, and $x' = x_1$, $dx' = dx_1$ for $d = 2$.

(ii) For $d = 3$ there exists a symplectomorphism Ψ^{-1} of a neighborhood $\mathcal{W}$ of every non-closed Hamiltonian trajectory Γ of the finite length generated by the Hamiltonian $F(x,\xi)$ and lying on Λ to a neighborhood of

$$\Lambda^0 = \{(x,\xi) : \ x_3 = \xi_3 = 0, (x_1, x_2, \xi_1) \in \bar{\Omega}\} \tag{13.3.14}$$

where Ω is a domain in $\mathbb{R}^3$ such that the inverse symplectomorphism Ψ satisfies (13.3.10),

$$a \circ \Psi \equiv \xi_2^2 + \sum_{1\le n\le N-1} b_n(x', \xi_1)(x_3^2 + \xi_3^2)^n \qquad \mathsf{mod}\ O_{2N-1}(\Lambda^0), \tag{13.3.15}$$

and also both (13.3.12) *and* (13.3.13) *where $x' = (x_1, x_2)$, $dx' = dx_1 \wedge dx_2$ for $d = 3$.*

(iii) In (i) for $F = \mathsf{const}$

$$b_2 = \frac{1}{8} F \kappa \circ \bar{\Psi} \tag{13.3.16}$$

where κ is the scalar curvature corresponding to the metrics $g^{jk}F^{-1}$.

Remark 13.3.2. (i) In this proposition N is arbitrary and $K = K(N)$ in condition (13.2.3); moreover, $b_n \in \mathscr{C}^N$. Therefore (13.3.11) and (13.3.15) hold with O_{2N} instead of O_{2N-1}.

(ii) It follows from the proof that for $d = 3$ one can also take another canonical form

$$(\xi_2^2 + x_3^2 + \xi_3^2) \sum_{1\le n\le N-1} b_n(x', \xi_1)(x_3^2 + \xi_3^2)^{n-1} \qquad \mathsf{mod}\ O_{2N-1}(\Lambda^0). \tag{13.3.15}'$$

This remark will be essential in the case of the strong magnetic field (see Section 13.6).

Proof of Proposition 13.3.1. (i) Let us first treat the simpler case $d = 2$. Since Λ is a 2-dimensional symplectic manifold, $a \ge 0$ and a has a zero on

Λ of precisely the second order then there exist functions $w_j(x,\xi)$ $(j=1,2)$ such that

$$w_j = 0 \quad \text{at} \quad \Lambda\cap\mathcal{W} \qquad (\forall j = 1,\dots,d), \tag{13.3.17}$$

$$\{w_1, w_2\} = 1 \qquad \text{at} \quad \Lambda\cap\mathcal{W}, \tag{13.3.18}$$

$$a \equiv b_1(w_1^2 + w_2^2) \qquad \operatorname{mod} O_3(\Lambda\cap\mathcal{W}). \tag{13.3.19}$$

Moreover, it is easy to see that here one can take $\mathcal{W} = B(0,1)\times\mathbb{R}^2$.

Further, one can replace (13.3.18) by

$$\{w_1, w_2\} \equiv 1 \qquad \operatorname{mod} O_M(\Lambda\cap\mathcal{W}) \tag{13.3.18}'_M$$

with arbitrary M (in which case $K = K(M)$).

In fact, if $(13.3.18)'_M$ holds then replacing w_2 by $w_2' = w_2 + \mathcal{P}(w;x,\xi)$ where $\mathcal{P}$ is a homogeneous polynomial of order $(M+1)$ with respect to $w = (w_1, w_2)$ and noting that

$$\{w_1, \mathcal{P}\} \equiv \partial_{w_2}\mathcal{P} \qquad \operatorname{mod} O_{M+1}(\Lambda\cap\mathcal{W}) \tag{13.3.20}$$

we obtain that for appropriate $\mathcal{P}$ condition $(13.3.18)'_{M+1}$ will be fulfilled.

Furthermore, one can replace $(13.3.18)'_M$ by

$$\{w_1, w_2\} = 1 \qquad \text{in} \quad \mathcal{W}. \tag{13.3.18}''$$

In fact, one can reach this equality by replacing w_2 with $w_2' \equiv w_2 \mod O_M(\Lambda)$ because $\nabla^{\#}w_1 = \{w_1, .\}$ is a non-vanishing tangent to Λ vector field in $\mathcal{W}$.

But if (13.3.17) and $(13.3.18)''$ are fulfilled then there exists a symplectomorphism $\Psi_{\frac{3}{2}}$ of a neighborhood of Λ^0 onto $\mathcal{W}$ (where $\mathcal{W}$ is decreased if necessary) such that $w_1\circ\Psi_{\frac{3}{2}} = \xi_2$ and $w_2\circ\Psi_{\frac{3}{2}} = x_2$. This yields (13.3.11) with $\Psi = \Psi_{\frac{3}{2}}$ and $N = 2$.

Let us calculate the non-zero eigenvalues of skew-Hessians of $a(x,\xi)$ and $x_2^2+\xi_2^2$ at points of Λ and Λ^0 respectively; we obtain $\pm 2iF$ and $\pm 2i$ respectively; this yields (13.3.12) and (13.3.13) [8)].

Let us now prove that

[8)] We recall that symplectomorphisms preserve the spectra of skew-Hessians at stationary points and the canonical density $dx\wedge d\xi$.

(13.3.21) If (13.3.11) is fulfilled with some $N \geq 2$ then there exists a symplectomorphism Ψ_N of a neighborhood of Λ^0 onto a neighborhood of Λ^0 such that $\Psi_N \equiv I \mod O_{2N-2}(\Lambda^0)$ and (13.3.11) holds for Ψ replaced by $\Psi \circ \Psi_N$ with N replaced by $(N + \frac{1}{2})$.

Then for $\Psi = \Psi_{\frac{3}{2}} \circ \Psi_2 \circ \cdots \circ \Psi_{N-\frac{1}{2}}$ we have equality (13.3.11) with arbitrarily chosen N.

To prove (13.3.21) let us construct this symplectomorphism Ψ_N as an exponential map $\Psi_N = \exp(\nabla^\# S)$ where $\nabla^\# S$ is the Hamiltonian field generated by $S \in O_{2N-1}(\Lambda^0)$. Then for any smooth enough function $f(x, \xi)$

$$f \circ \Psi_N \sim f + \{S, f\} + \frac{1}{2}\{S, \{S, f\}\} + ... \tag{13.3.22}$$

where the order of the zero at Λ^0 increases when the number of the term increases. In particular,

$$x_2 \circ \Psi_N \equiv x_2 + \{S, x_2\}, \quad \xi_2 \circ \Psi_N \equiv \xi_2 + \{S, \xi_2\}, \quad x_1 \circ \Psi_N \equiv x_1, \quad \xi_1 \equiv \xi_1 \mod O_{2N-1}(\Lambda^0) \tag{13.3.23}$$

and

$$a \circ \Psi \circ \Psi_N \equiv a \circ \Psi_N + b_1\{S, x_2^2 + \xi_2^2\} \mod O_{2N}(\Lambda^0). \tag{13.3.24}$$

Therefore the existence of the appropriate generating function S follows from

Lemma 13.3.3. *Let $d = 2$. Then for every $M \in \mathbb{Z}^+ \setminus 0$ and for every function $Q \in O_M(\Lambda^0)$ there exist functions $S \in O_M(\Lambda^0)$ and ω such that*

$$Q \equiv \{S, x_2^2 + \xi_2^2\} + \omega(x_2^2 + \xi_2^2)^{\frac{M}{2}} \mod O_{M+1}(\Lambda^0).$$

Moreover, for odd M, $\omega = 0$ and S $(\mod O_{M+1}(\Lambda^0))$ is defined uniquely, while for even M, S $(\mod O_{M+1}(\Lambda^0))$ is defined uniquely modulo the term $\omega'(x_2^2 + \xi_2^2)^{\frac{M}{2}}$ and ω $(\mod O_1(\Lambda^0))$ is defined uniquely.

Proof. The proof of this lemma is obvious. □

(ii) Let us now treat the case $d = 3$. Then the symplectic structure of Λ and the properties of a yield that there exist functions $w_j(x, \xi)$ $(j = 1, 2, 3)$ such that (13.3.1) holds and

(13.3.25) dw_j $(j = 1, 2, 3)$ are linearly independent on $\Lambda \cap \mathcal{W}$,

(13.3.26) $$\{w_1, w_2\} = 1, \quad \{w_1, w_3\} = \{w_2, w_3\} = 0 \quad \text{at} \quad \Lambda \cap \mathcal{W},$$

and

(13.3.27) $$a \equiv w_3^2 + b_1(w_1^2 + w_2^2) \qquad \mathsf{mod}\ O_3(\Lambda^0 \cap \mathcal{W});$$

then on $\Lambda \cap \mathcal{W}$, $\sum_j F^j d(\xi_j - V_j)$ and dw_3 differ by a scalar factor. Here one can take $\mathcal{W} = B(0, 1) \times \mathbb{R}^3$.

We assert that one can replace (13.3.26) by

(13.3.26)′ $$\{w_1, w_2\} \equiv 1, \quad \{w_1, w_3\} \equiv \{w_2, w_3\} \equiv 0 \qquad \mathsf{mod}\ O_2(\Lambda \cap \mathcal{W}).$$

In fact, (13.3.26) means precisely that

$$\{w_j, w_k\} \equiv \varepsilon_{jk} + \sum_l \alpha_{jkl} w_l \qquad \mathsf{mod}\ O_2(\Lambda \cap \mathcal{W})$$

with $\varepsilon_{12} = 1$, $\varepsilon_{13} = \varepsilon_{23} = 0$; the Jacobi identity yields that $\alpha_{132} = -\alpha_{231}$ on $\Lambda \cap \mathcal{W}$. It is easy to reach (13.3.26)′ by replacing w_j by $w_j + \mathcal{P}_j(w; x, \xi)$ with appropriate homogeneous second degree polynomials $\mathcal{P}$ of $w = (w_1, w_2, w_3)$; here we use the equalities

$$\{w_1, \mathcal{P}\} \equiv \partial_{w_2}\mathcal{P}, \quad \{w_2, \mathcal{P}\} \equiv -\partial_{w_1}\mathcal{P}, \{w_3, \mathcal{P}\} \equiv 0 \qquad \mathsf{mod}\ O_2(\Lambda \cap \mathcal{W})$$

and the equality $\alpha_{132} = -\alpha_{231}$ on $\Lambda \cap \mathcal{W}$.

Moreover, one can replace (13.3.26)′ by

(13.3.26)″ $$\{w_1, w_2\} = 1, \quad \{w_1, w_3\} = \{w_2, w_3\} = 0 \qquad \text{in} \quad \mathcal{W}.$$

In fact, since $\nabla^{\#} w_3$ is a non-vanishing vector field in $\mathcal{W}$ tangent to Λ, Γ is the non-closed trajectory of $\nabla^{\#} w_3$ and $\mathcal{W}$ is a narrow neighborhood of Γ then the second of these equalities will be fulfilled if we replace w_1 by an appropriate function $w_1' \equiv w_1 \ \mathsf{mod}\ O_2(\Lambda)$.

Further, the first equality in (13.3.26)′ remains true and the third equality remains true automatically. Then the vector fields $\nabla^{\#} w_1$ and $\nabla^{\#} w_3$ commute and are linearly independent and it is easy to reach the first and third equalities in (13.3.26)′ by replacing w_2 with an appropriate function $w_2' \equiv w_2$ $\mathsf{mod}\ O_2(\Lambda)$.

Furthermore, the same arguments with vector fields yield that there exist smooth functions w_4, w_5, w_6 in $\mathcal{W}$ such that

(13.3.28) $\{w_j, w_k\} = 1$ for $j = k-1$ and even k; otherwise $\{w_j, w_k\} = 0$ as $j \le k$.

and therefore there exists a symplectomorphism $\Psi_{\frac{3}{2}}$ such that

$$w_1 \circ \Psi_{\frac{3}{2}} = \xi_3, \quad w_2 \circ \Psi_{\frac{3}{2}} = x_3, \quad w_3 \circ \Psi_{\frac{3}{2}} = \xi_2; \tag{13.3.29}$$

these equalities yield (13.3.15) with $\Psi = \Psi_{\frac{3}{2}}$ and $N = 2$.

Calculating the non-vanishing eigenvalues of the skew-Hessians of a and $x_3^2 + \xi_3^2$ on Λ and Λ^0 respectively we obtain $\pm 2iF$ and $\pm 2i$ respectively and this yields (13.3.12) and (13.3.13).

Let us prove that

(13.3.30) If (13.3.15) is fulfilled with $N \ge 2$ then there exists a symplectomorphism Ψ_N of a neighborhood of Λ^0 onto a neighborhood of Λ^0 such that $\Psi_N \equiv I \mod O_{2N-2(\Lambda^0)}$ and (13.3.15) holds with Ψ replaced by $\Psi \circ \Psi_N$ and N replaced by $(N + \frac{1}{2})$ in the sum in the right-hand expression and in $R'_n \in O_{2N-2n-1}(\Lambda^0)$.

(a) Let us first treat the most delicate case $N = 2$. Let

$$\Psi' = \exp(\nabla^{\#} S'), \qquad S' = \alpha(x', \xi_1)\xi_2^2 + \beta(x', \xi_1)(x_3^2 + \xi_3^2).$$

Then formula (13.3.22) is wrong but one can see easily that

$$(x_3^2+\xi_3^2)\circ\Psi' = (\xi_3^2+\xi_3^2), \quad x_1\circ\Psi' \equiv x_1, \quad \xi_1\circ\Psi' \equiv \xi_1, \quad x_2\circ\Psi' \equiv x_2+\partial_{\xi_2}S' \mod O_2(\Lambda^0)$$

and $\xi_2 \equiv \xi_2 - \partial_{x_2} S' \mod O_3(\Lambda^0)$; then

$$\xi_3 \circ \Psi' \equiv \xi_2 \cos(2\beta) - x_3 \sin(2\beta), \quad x_3 \circ \Psi' \equiv x_3 \cos(2\beta) + \xi_3 \sin(2\beta) \mod O_2(\Lambda^0).$$

Therefore

$$a \circ \Psi_{\frac{3}{2}} \circ \Psi' \equiv a \circ \Psi_{\frac{3}{2}} + \{S', a^0\}_2 + \sum_{1 \le k \le 3} T_k(x', \xi_1; x_3, \xi_3)\xi_2^{3-k} \mod O_4(\Lambda^0) \tag{13.3.31}$$

where

$$a^0 = \xi_2^2 + b_1(x_3^2 + \xi_3^2), \tag{13.3.32}$$

$$\{f, g\}_2 := \partial_{\xi_2} f \cdot \partial_{x_2} g - \partial_{x_2} f \cdot \partial_{\xi_2} g \tag{13.3.33}$$

and T_k are homogeneous k-th order polynomials of (x_3, ξ_3) with coefficients smoothly depending on (x', ξ_1); moreover,

(13.3.34) The trace of the matrix of the quadratic form T_2 vanishes.

Then (13.3.31)–(13.3.33) yield that for appropriate functions α and β

$$a \circ \Psi_{\frac{3}{2}} \circ \Psi' \equiv a^0 + \sum_{1 \le k \le 3} T'_k(x', \xi_1; x_3, \xi_3)\xi_2^{3-k} \qquad \mod O_4(\Lambda^0)$$

where T'_k are homogeneous k-th order polynomials of (x_3, ξ_3) with coefficients smoothly depending on (x', ξ_1); moreover, (13.3.33) holds.

Let $S'' \in O_3(\Lambda^0)$ and $\Psi'' = \nabla^\# S''$. Then (13.3.22) yields that

$$a \circ \Psi_{\frac{3}{2}} \circ \Psi' \circ \Psi'' \equiv a^0 + b_1\{S'', x_3^2 + \xi_3^2\} + \sum_{1 \le k \le 3} T'_k(x', \xi_1; x_3, \xi_3)\xi_2^{3-k} \qquad \mod O_4(\Lambda^0)$$

and one can see easily that for appropriate S''

$$a \circ \Psi_{\frac{3}{2}} \circ \Psi' \circ \Psi'' \equiv a^0 \qquad \mod O_4(\Lambda^0).$$

This completes the proof of (13.3.30) with $N = 2$.

(b) Let us consider the case $N \ge \frac{5}{2}$. Let us look for Ψ_N in the form $\Psi_N = \exp(\nabla^\# S)$, $S = S' + S''$ where

$$S' = \sum_{0 \le k \le \lfloor N \rfloor} \alpha_k(x', \xi_1)(x_3^2 + \xi_3^2)^k \xi_2^{2N-2k-2} \tag{13.3.35}$$

belongs to $O_{2N-2}(\Lambda^0)$ and $S'' \in O_{2N-1}(\Lambda^0)$. Then (13.3.19) yields that

$$a \circ \Psi \circ \Psi_N \equiv a \circ \Psi + \{S', a^0\}_2 + b_1\{S'', x_3^2 + \xi_3^2\} \qquad \mod O_{2N}(\Lambda^0)$$

and the existence of the appropriate generating function S follows from the following

Lemma 13.3.4. *Let $d = 3$, $a^0 = \xi_2^2 + b_1(x_3^2 + \xi_3^2)$, $b_1 = b_1(x', \xi_1) > 0$. Then for every $M \in \mathbb{Z}^+$ and for every function $Q \in O_M(\Lambda^0)$ there exists S' of the form $(13.3.35)_N$ with $N = \frac{1}{2}(M+1)$, $S'' \in O_M(\Lambda^0)$ and b ($b = 0$ for odd M) such that*

$$Q \equiv \{S', a^0\}_2 + b_1\{S'', x_3^2 + \xi_3^2\} + b(x_3^2 + \xi_3^2)^{\frac{M}{2}} \qquad \mathrm{mod}\ O_{M+1}(\Lambda^0).$$

Proof. The proof of this lemma is obvious. □

Thus (13.3.30) is proven in the full scope and it implies Statement (ii).

(iii) Without any loss of the generality one can assume that $F = 1$; then $b_1 = 1$. Let us introduce a local conformal coordinate system[9)].

Then $g^{11} = g^{22} = \rho^2$, $g^{12} = 0$ and $a(x, \xi) = s^2 + t^2$, $\{s, t\} = \pm 1$ at Λ where $s = \rho(\xi_1 - V_1)$, $t = \rho(\xi_2 - V_2)$. Permuting x_1 and x_2, s and t if necessary we obtain that $\{s, t\} = 1$ at Λ. Then

$$\{s, t\} = 1 + \beta s - \alpha t, \quad \alpha = -\partial_1\rho, \quad \beta = -\partial_2\rho. \tag{13.3.36}$$

Therefore $w_1 = s$, $w_2 = t$ fulfill the initial relations (13.3.17)–(13.3.19).

Let us take $w_1' = s + ut$, $w_2' = t - us$ with $u \in O_1(\Lambda)$; then

$$a \equiv w_1'^2 + w_2'^2 \qquad \mathrm{mod}\ O_4(\Lambda)$$

and

$$\{w_1', w_2'\} \equiv \{s, t\} - s\{s, u\} - t\{t, u\} \qquad \mathrm{mod}\ O_2(\Lambda).$$

Then (13.3.12) yields that $\{w_1', w_2'\} \equiv 1 \mod O_2(\Lambda)$ for $u = \alpha s + \beta t$. More detailed calculation then gives that

$$\begin{aligned}\{w_1', w_2'\} \equiv 1 + s^2\big(2\alpha^2 - \beta^2 - \{s, \alpha\}\big) + \\ st\big(6\alpha\beta - \{t, \alpha\} - \{s, \beta\}\big) + t^2\big(2\beta^2 - \alpha^2 - \{t, \beta\}\big) \\ \mathrm{mod}\ O_3(\Lambda).\end{aligned} \tag{13.3.37}$$

Let us construct functions w_1'' and w_2'' coinciding modulo $O_3(\Lambda)$ with w_1' and w_2' and such that

$$\{w_1'', w_2''\} \equiv 1 \qquad \mathrm{mod}\ O_3(\Lambda), \tag{13.3.38}$$

[9)] See f.e. B. A. Dubrovin, A. T. Fomenko and S. P. Novikov [1].

and

$$a \equiv w_1''^2 + w_2''^2 + \omega\big(w_1''^2 + w_2''^2\big)^2 \qquad \mathrm{mod}\ O_5(\Lambda), \tag{13.3.39}$$

and let us calculate the coefficient ω. In order to fulfill (13.3.39) let us note that

$$a = \big(w_1'^2 + w_2'^2\big)\big(1+u^2\big)^{-1} \equiv \big(w_1'^2 + w_2'^2\big)\big(1-u^2\big) \qquad \mathrm{mod}\ O_5(\Lambda)$$

and let us set

$$w_1'' = s + ut - \frac{1}{2}(\omega(s^2+t^2) + u^2) + vt,$$
$$w_2'' = t - us - \frac{1}{2}(\omega(s^2+t^2) + u^2) - vs$$

with $v \in O_2(\Lambda)$. Then (13.3.13) yields that

$$\{w_1'', w_2''\} \equiv 1 + s^2\big(-\beta^2 - \{s,\alpha\} - 2\omega\big) + st\big(2\alpha\beta - \{t,\alpha\} - \{s,\beta\}\big) + t^2\big(-\alpha^2 - \{t,\beta\} - 2\omega\big) - s\{s,v\} - t\{t,v\} \qquad \mathrm{mod}\ O_3(\Lambda). \tag{13.3.40}$$

Then Lemma 13.3.3 yields that (13.3.38) holds for appropriate $v \in O_2(\Lambda)$ if and only if the sum of coefficients at s^2 and t^2 in (13.3.40) vanishes, i.e.,

$$\omega = \frac{1}{4}\big(-\{s,\alpha\} - \{t,\beta\} - \alpha^2 - \beta^2\big).$$

Equalities (13.3.36) and the definitions of s and t then yield that

$$\omega = \frac{1}{4}(\rho\Delta\rho - (\nabla\rho)^2) = \frac{1}{4}\rho^2\Delta\log\rho \tag{13.3.41}$$

where Δ is the two-dimensional Laplace operator and therefore the formula for the scalar curvature in a conformal coordinate system[9)] yields that $\omega = \frac{1}{8}\kappa$.

This equality yields (13.3.16) if we take a symplectomorphism Ψ such that $w_1'' \circ \Psi = \xi_2$, $w_2'' \circ \Psi \equiv x_2 \ \mathrm{mod}\ O_4(\Lambda^0)$); moreover, (13.3.11) is fulfilled with $N = 3$. Taking this symplectomorphism as the initial point of the construction used in Part (i) we obtain (13.3.11) for arbitrary N; moreover, (13.3.16) is also fulfilled.

On the other hand, if Φ is an arbitrary symplectomorphism of a neighborhood of Λ_1^0 onto a neighborhood of Λ_2^0 where in the definition of Λ_j^0 domains Ω_j are taken instead of Ω ($j = 1, 2$) and if Φ preserves the decomposition (13.3.11) then $\Phi = \Phi' \circ \Phi''$ where $\Phi'(x_1, \xi_1, x_2, \xi_2) = (\bar{\Phi}'(x_1, \xi_1), x_2, \xi_2)$, $\bar{\Phi}' : \Lambda_1^0 \to \Lambda_2^0$ is a symplectomorphism and Φ'' is also a symplectomorphism and $\Phi''|_{\Lambda_1^0} = I$.

Then $\Phi'' = \exp(\nabla^{\#} S)$ with $S \in O_2(\Lambda_1^0)$. Since Φ' obviously preserves the decomposition (13.3.11) the same also is true for Φ''; then the arguments of Part (i) yield that

$$S \equiv \sum_{1 \le n \le N-1} \phi_n(x_1, \xi_1)(x_2^2 + \xi_2^2)^n \qquad \mathsf{mod}\ O_{2N-1}(\Lambda_1^0).$$

One can see easily that all the coefficients b_n are invariant with respect to Φ' (i.e., $b_n \to b_n \circ \bar{\Phi}'$) and that Φ'' transforms coefficient b_2 to $b_2 + \{\phi_1, b_1\}$ (so it is also invariant provided b_1 is constant, i.e., provided F is constant).

Therefore (13.3.16) holds for every symplectomorphism Ψ reducing $a(x, \xi)$ to its canonical form. Statement (iii) of Proposition 13.3.1 is proven. □

Problem 13.3.5. Generalize Proposition 13.3.1 to higher dimensions. First of all one needs to assume that

(13.3.42) F has constant rank $2r$, or equivalently restriction of symplectic form to Λ has constant rank $2r$.

Then

(13.3.43) Λ stratifies into $(d - 2r)$ dimensional leafs and restriction of symplectic form to each leaf is 0.

(i) Assumption (13.3.42) with $r = 1$ is sufficient for reduction. On the other hand, as $r \ge 2$ for reduction modulo $O_{M+1}(\Lambda)$ one needs *non-resonance condition*

$$(13.3.44)_M \qquad |\sum_j f_j(x)\alpha_j| \ge \epsilon_M \qquad \forall \alpha \in \mathbb{Z}^r : \sum_j |\alpha_j| < M + 1.$$

In particular, as $M = 2$ this condition is equivalent to $|f_j - f_k| \ge \epsilon\ \forall j \ne k$ and for $r = 2$ we have $(13.3.44)_\infty$ equivalent to f_1/f_2 is irrational.

(ii) Explore more complicated case when A has eigenvalues $\pm if_j$ of constant multiplicities r_j, $j = 1, \dots, m$. Even as modified (in an obvious) non-resonance conditions are fulfilled canonical form becomes more complicated and includes polynomials of all "basic" quadratic forms commuting with a.

Form: $\varepsilon \neq 0$

Let us now treat the case $\varepsilon \neq 0$.

Theorem 13.3.6. *Let conditions* (13.2.1)–(13.2.4) *and* (13.3.1) *be fulfilled. Then*

(i) Let $d = 2$. Then for every $\varepsilon \in [-\epsilon_0, \epsilon_0]$ [10] *there exists a symplectomorphism Ψ_ε^{-1} of a neighborhood $\mathcal{W}$ of every point $z \in \Lambda$ to a neighborhood of Λ^0 (defined by* (13.3.9)*), depending smoothly on ε, coinciding with Ψ^{-1} (defined in Proposition 13.3.1(i)) as $\varepsilon = 0$* [11] *and such that the inverse symplectomorphism Ψ has the following properties:*

$$a_\varepsilon \circ \Psi_\varepsilon = \sum_{m+n\le N-1} b_{mn}(x_1, \xi_1)\varepsilon^m (x_2^2 + \xi_2^2)^n + \sum_{0\le n\le N} \varepsilon^n R_n(x, \xi, \varepsilon) \tag{13.3.45}$$

where the coefficients b_{mn} and R_n are smooth functions of all their arguments; moreover

$$b_{00} = 0, \qquad b_{01} = F \circ \bar{\Psi}_0, q \qquad b_{10} = V \circ \bar{\Psi}_0 \tag{13.3.46}_{1-3}$$

and $R_n \in O_{2N-2n}(\Lambda^0)$.

(ii) Let $d = 3$. Then for every $\varepsilon \in [-\epsilon_0, \epsilon_0]$ [10] *there exists a symplectomorphism Ψ_ε^{-1} of a neighborhood $\mathcal{W}$ of every non-closed Hamiltonian trajectory Γ of finite length generated by the Hamiltonian $F(x, \xi) = \sum_j F^j(\xi_j - V_j)$ and lying on Λ to a neighborhood of Λ^0 = (defined by* (13.3.14)*), depending smoothly on ε, coinciding with Ψ^{-1} (defined in Proposition 13.3.1(ii)) as $\varepsilon = 0$* [10] *and such that the inverse symplectomorphism Ψ_ε has the following properties:*

$$a_\varepsilon \circ \Psi_\varepsilon \equiv \xi_2^2 + \sum_{1\le m+n\le N-1} b_{mn}(x', \xi_1)\varepsilon^m (x_3^2 + \xi_3^2)^n + \sum_{0\le n\le N} \varepsilon^n R_n(x, \xi, \varepsilon) \tag{13.3.47}$$

[10] Where $\epsilon_0 > 0$ is a small enough constant.

[11] Therefore we will denote Ψ, etc as Ψ_0 etc.

where the coefficients b_{mn} *and* R_n *are smooth functions. Moreover,* $(13.3.46)_{1-3}$ *hold and* $R_n \in O_{2N-2n}(\Lambda^0)$.

(iii) If $d = 2$ *and* $F = \mathrm{const}$ *then*

$$b_{02} = \frac{1}{8} F\kappa \circ \bar{\Psi}_0 \tag{13.3.48}_1$$

$$b_{11} = \big(\frac{1}{4}\sum_{j,k} g^{jk} F^{-1} \partial_j \partial_k V + \sum_j k^j \partial_j V\big) \circ \bar{\Psi}_0, \tag{13.3.48}_2$$

and

$$b_{20} = \big(\sum_j l^j \partial_j V\big) \circ \bar{\Psi}_0 \tag{13.3.48}_3$$

where the coefficients k^j *and* l^j *depend on the method of reduction and can be chosen arbitrarily at every point separately.*

Proof. If we take the symplectomorphism Ψ constructed in Proposition 13.3.1 instead of Ψ_ε then (13.3.45) or (13.3.47) will be fulfilled modulo $\varepsilon O_1(\Lambda^0)$ and the coefficients in these decompositions will satisfy $(13.3.46)_{1-3}$.

Therefore it is sufficient to prove that

(13.3.49) If (13.3.45) or (13.3.47) are fulfilled modulo $\varepsilon^L O_1(\Lambda^0)$ with $L \geq 1$ then there also exists a symplectomorphism $\Psi_{\varepsilon,L}$ of a neighborhood of Λ^0 onto a neighborhood of Λ^0, depending smoothly on ε and such that

$$\Psi_\varepsilon \equiv I \mod O(\varepsilon^L) \tag{13.3.50}$$

and (13.3.45) or (13.3.47) is fulfilled modulo $\varepsilon^{L+1} O_1(\Lambda^0)$ for $\Psi\varepsilon$ replaced by $\Psi_\varepsilon \circ \Psi_{\varepsilon,L}$.

Note that equalities $(13.3.48)_{1-3}$ automatically remain true at each step.

To prove (13.3.49) we assert first that

(13.3.51) In the framework of (13.3.49) there exists a symplectomorphism also satisfying (13.3.50) and such that $a_\varepsilon \circ \Psi_\varepsilon \circ \Psi_{\varepsilon,L,0}$ is equal to the right-hand expression of (13.3.45) or (13.3.47) modulo $\varepsilon^L O_2(\Lambda^0) + O(\varepsilon^{L+1})$.

In fact, for $d = 2$ the term preventing us from having the last equality is $\varepsilon^L(\alpha\xi_2 + \beta x_2)$ with $\alpha = \alpha(x_1, \xi_1)$ and $\beta = \beta(x_1, \xi_1)$. Let us set

$$w_1 = \xi_2 + \alpha_1 \varepsilon^L,\ w_2 = x_2 + \beta_1 \varepsilon^L,\ w_3 = \xi_1,\ w_4 = x_1$$

with $\alpha_1 = \alpha/(2b_1)$, $\beta_1 = \beta/(2b_1)$. Then the necessary commutation relations are fulfilled modulo $O(\varepsilon^L)$. Furthermore, perturbing w_j with $j = 1, 2$ by functions of the class $\varepsilon^L O_1(\Lambda^0)$ we obtain a commutation relation between w_1 and w_2 fulfilled modulo $\varepsilon^L O_1(\Lambda^0) + O(\varepsilon^{L+1})$.

Then there exist functions w'_j fulfilling the same commutation relations exactly and such that $w'_j \equiv w_j \mod (\varepsilon^L O_1(\Lambda^0) + O(\varepsilon^{L+1}))$ for $j = 1, 2$ and $w'_j \equiv w_j \mod O(\varepsilon^L)$ for $j = 3, 4$. Then symplectomorphism $\Psi_{\varepsilon,L,0}$ transforming (w'_1, w'_2, w'_3, w'_4) into (ξ_2, x_2, ξ_1, x_1) is obviously the required symplectomorphism.

Similarly, for $d = 3$ the term preventing us from having the equality claimed in (13.3.51) is $\varepsilon^L(\alpha\xi_3 + \beta x_3 + \gamma\xi_2)$ with $\alpha = \alpha(x', \xi_1)$, $\beta = \beta(x', \xi_1)$ and $\gamma = \gamma(x', \xi_1)$. Let us set

$$w_1 = \xi_3 + \alpha_1\varepsilon^L,\ w_2 = x_3 + \beta_1\varepsilon^L,\ w_3 = \xi_2 + \gamma_1\varepsilon^L$$

with $\alpha_1 = \alpha/(2b_1)$, $\beta_1 = \beta/(2b_1)$, $\gamma_1 = \gamma/(2b_1)$ and, moreover, let us set $w_4 = x_2$, $w_5 = \xi_1$, $w_6 = x_1$. Then all the remaining arguments used for $d = 2$ remain true with the obvious modifications.

So, (13.3.51) has been proven. Therefore to prove (13.3.49) it is sufficient to prove that

(13.3.52) If (13.3.45) or (13.3.47) is fulfilled modulo $\varepsilon^L O_M(\Lambda^0) + O(\varepsilon^{L+1})$ with $2 \le M \le 2(N - L) - 1$ then there exists a symplectomorphism

$$\Psi_{\varepsilon,L,M} \equiv I \mod (\varepsilon^L O_{M-1}(\Lambda^0) + O(\varepsilon^{L+1}))$$

such that for Ψ_ε replaced by $\Psi_\varepsilon \circ \Psi_{\varepsilon,L,M}$ the same equality is fulfilled modulo $\varepsilon^L O_{M+1}(\Lambda^0) + O(\varepsilon^{L+1})$.

Repeating the arguments used in the Parts (i) or (i) of the proof of Proposition 13.3.1 as $d = 2, 3$ respectively one can construct easily this symplectomorphism in the form $\Psi_{\varepsilon,L,M} = \exp(\varepsilon^L \nabla^\# S)$ where

(i) $S \in O_M(\Lambda^0)$ for $d = 2$ and

(ii) $S = S' + S''$ where S' is of the form $(13.3.35)_N$ with $\frac{1}{2}(M + 1)$ instead of N and $S'' \in O_M(\Lambda^0)$ for $d = 3$.

We only note that (13.3.22) is replaced by formula

$$f \circ \Psi_{\varepsilon,L,M} \sim f + \varepsilon^L\{S, f\} + \frac{1}{2}\varepsilon^{2L}\{S, \{S, f\}\} + \dots$$

which is always applicable. So Statements (i) and (ii) of Theorem 13.3.6 are proven.

Let us prove Statement (iii). Note first that $(13.3.48)_1$ follows from (13.3.16). Without any loss of generality one can assume in the proof of $(13.3.48)_{2,3}$ that $F = 1$. Let us again consider a conformal coordinate system. Let Ψ be the symplectomorphism constructed in Proposition 13.3.1(iii); let us consider the decomposition of $W = V \circ \Psi$ in powers of x_2, ξ_2:

$$W \sim \sum_{\gamma \in \mathbb{Z}^{+2}} v_\gamma(x_1, \xi_1) x_2^{\gamma_1} \xi_2^{\gamma_2}.$$

Let us return to the very beginning of the proof of Statement (i). Recall that we took $w_1 = \xi_2 + \alpha_1\varepsilon$, $w_2 = x_2 + \beta_1\varepsilon$, $w_3 = \xi_1$ and $w_4 = x_1$ and noted that the commutation relations were fulfilled modulo $O(\varepsilon)$. It is easy to see that one can change $w_1, \dots, w_4$ by functions of the class $\varepsilon O_1(\Lambda^0)$ such that for new $w_1, \dots, w_4$ all the commutation relations (not only between w_1 and w_2) will be fulfilled modulo $\varepsilon O_1(\Lambda^0) + O(\varepsilon^2)$ and therefore $w_j' \equiv w_j$ $\mathsf{mod}\ \varepsilon O_1(\Lambda^0) + O(\varepsilon^2)$ for the appropriate method of reduction. Therefore for the symplectomorphism constructed at this step

$$a_\varepsilon \circ \Psi_\varepsilon \equiv a \circ \Psi + \varepsilon Q(x_1, \xi_1; x_2, \xi_2) - \varepsilon^2(\alpha_1^2 + \alpha_2^2) \quad \mathsf{mod}\ \big(\varepsilon O_2(\Lambda^0) + \varepsilon^2 O_1(\Lambda^0) + O(\varepsilon^3)\big)$$

with

$$Q(x_1, \xi_1; x_2, \xi_2) = v_{(20)} x_2^2 + 2v_{(21)} x_2\xi_2 + v_{(02)}\xi_2^2;$$

symplectomorphisms of the remaining part of the proof of Statement (i) kill the traceless part of the quadratic form Q and preserve its trace part. Since $\alpha_1 = \frac{1}{2}v_{(01)}^2$, $\beta_1 = \frac{1}{2}v_{(10)}$ we conclude that

$$(13.3.53)_{1,2} \qquad b_{11} = \frac{1}{2}(v_{(20)} + v_{(02)}), \quad b_{20} = -\frac{1}{4}(v_{(10)}^2 + v_{(01)}^2)$$

and we only need to calculate $v_{(10)}$, $v_{(01)}$, $v_{(20)}$, $v_{(02)}$. But

$$v_{(10)} = -\{W, \xi_2\} \circ \pi = -\{V, w_1\} \circ \Psi \circ \pi$$

where here and below $w_1 = \xi_2 \circ \Psi^{-1}$, $w_2 = x_2 \circ \Psi^{-1}$, $\pi(x, \xi) = (x_1, \xi_1, 0, 0)$; since $w_1 \equiv s$, $w_2 \equiv t$ $\mathsf{mod}\ O_2(\Lambda^0)$ (see the Part (iii) of the proof of Proposition 13.3.1) then

$$v_{(10)} = \{s, V\} \circ \bar\Psi = \rho\partial_1 V \circ \bar\Psi$$

and similarly $v_{(01)} = -\rho\partial_2 V \circ \bar{\Psi}$; these equalities yield $(13.3.48)_3$ with $l^j = -\frac{1}{4}\rho^2\partial_j V$.

Moreover,

$$v_{(20)} = \frac{1}{2}\{\{W, \xi_2\}, \xi_2\} \circ \pi = \frac{1}{2}\{\{V, w_1\}, w_1\} \circ \Psi \circ \pi$$

and a similar equality is true for $v_{(02)}$. Let us recall that $w_1 \equiv s + ut$, $w_2 \equiv t - us \mod O_3(\Lambda)$ with $u = \alpha s + \beta t$; then (13.3.36) and (13.3.53) yield that

$$b_{11} = \frac{1}{4}(\rho^2 \Delta V) \circ \bar{\Psi};$$

this equality yields $(13.3.48)_2$ with $k^j = 0$.

Finally, if Φ_ε is an arbitrary symplectomorphism of a neighborhood of Λ_1^0 onto a neighborhood of Λ_2^0, smoothly depending on ε, mapping Λ_1^0 onto Λ_2^0 and preserving decomposition (13.3.25), then repeating the corresponding arguments of the proof of Proposition 13.3.1(i) with obvious modifications we obtain that

$$\Phi_\varepsilon \equiv \Phi' \circ \Phi''_\varepsilon \qquad \mod \sum_n \varepsilon^n O_{2N-2n-2}(\Lambda_1^0)$$

where Φ' is a symplectomorphism preserving (x_2, ξ_2) (as in the end of the proof of Proposition 13.3.1(iii)) and $\Phi''_\varepsilon = \exp(\nabla^\# S_\varepsilon)$,

$$S_\varepsilon = \sum_{1 \le m+n \le N-1} \phi_{mn}(x_1, \xi_1)\varepsilon^m (x_2^2 + \xi_2^2)^n$$

with arbitrary functions ϕ_{mn}; all the coefficients b_{mn} are invariant with respect to Φ'; on the other hand, Φ''_ε transforms b_{20} and b_{11} to $b_{20} + \{\phi_{10}, b_{10}\}$ and $b_{11} + \{\phi_{01}, b_{10}\}$ respectively. This and $(13.3.46)_3$ yield $(13.3.48)_{2,3}$ and the possibility of choosing k^j and l^j arbitrarily at every fixed point. □

At the end of this subsection let us get rid of the condition that F is constant in Statement (iii):

Proposition 13.3.7. *Let $d = 2$ and Ψ_ε be the symplectomorphism constructed in Theorem 13.3.6(i). Then*

$$a_\varepsilon \circ \Psi_\varepsilon = \sum_{m+n \le N-1} b'_{mn}(x_1, \xi_1)\varepsilon^m \bar{a}_\varepsilon^n + \sum_{0 \le n \le N} \varepsilon^n R'_n(x, \xi, \varepsilon) \tag{13.3.54}$$

where b'_{mn}, R'_n are smooth functions and $R'_n \in O_{2N-2n-1}(\Lambda^0)$,

$$\bar{a}_\varepsilon := x_2^2 + \xi_2^2 + \varepsilon b(x_1, \xi_1), \tag{13.3.55}$$

$$b = v \circ \bar{\Psi}_0, \quad v = \frac{V}{F}, \tag{13.3.56}$$

$$b'_{00} = b'_{10} = 0, \quad b'_{01} = F \circ \bar{\Psi}_0 \tag{13.3.57}$$

and

$$b'_{20} = (\frac{1}{8} F \kappa v^2 - \frac{1}{4} \sum_{j,k} g^{jk} v \partial_j \partial_k v + \sum_j m^j \partial_j v) \circ \bar{\Psi}_0, \tag{13.3.58}$$

and m^j depend on the method of reduction and can be chosen arbitrarily at every fixed point.

Proof. Formulae (13.3.54)–(13.3.57) obviously follow from Theorem 13.3.6(i); formula (13.3.58) for constant F follows from Statement (iii) of the same theorem and $m^j = l^j - k^j v$ here.

Let us prove (13.3.58) in the general case; the proof of assertion (i) of Theorem 13.3.6(i) yields that it is sufficient to do the reduction with $N = 3$. Let Ψ'_ε be a symplectomorphism, reducing $F^{-1} a_\varepsilon$; then modulo terms negligible for $N = 3$

$$(F^{-1} a_\varepsilon) \circ \Psi'_\varepsilon \equiv \bar{a}_\varepsilon + \gamma(x_1, \xi_1)\varepsilon^2 + f(x_1, \xi_1)\bar{a}_\varepsilon$$

with $\gamma = b_{20} - b_{11} b + b_{02} b^2$ and $f \in O_2(\Lambda^0) + O(\varepsilon)$. Theorem 13.3.6(iii) yields that γ coincides with the right-hand expression of (13.3.58) multiplied by $F^{-1} \circ \bar{\Psi}_0$. Then

$$a_\varepsilon \circ \Psi'_\varepsilon = (F \circ \Psi'_\varepsilon) \cdot (\bar{a}_\varepsilon + \gamma\varepsilon^2 + f \bar{a}_\varepsilon) \equiv (b'_{01} + f')\bar{a}_\varepsilon + b'_{20}\varepsilon^2$$

where b'_{10} and b'_{20} are given by (13.3.56) and (13.3.57) and $f' \in O_1(\Lambda^0) + O(\varepsilon)$.

One can easily construct a symplectomorphism $\Psi''_\varepsilon = \exp(\nabla^\# S_\varepsilon)$ with $S_\varepsilon = (\varepsilon S' + S'')\bar{a}_\varepsilon$, where $S' \in O_1(\Lambda^0)$ and $S'' \in O_2(\Lambda^0)$ are chosen consequently, such that

$$a_\varepsilon \circ \Psi''_\varepsilon \equiv (b'_{01} + f'')\bar{a}_\varepsilon + b'_{20}\varepsilon^2$$

with $f'' = \phi_1(x_1, \xi_1)\varepsilon + \phi_2(x_1, \xi_1)\bar{a}_\varepsilon$; the proposition is proven. □

13.3.2 Operator Reduction

Let us proceed from symplectomorphisms to Fourier integral operators. First of all let us consider the $\hbar$-differential operator

$$A'_{\hbar,\varepsilon} = \sum_{j,k} (\hbar D_j - V_j) g^{jk} (\hbar D_k - V_k) + \varepsilon V, \tag{13.3.59}$$

depending on a small parameter ε. In what follows[12)]

$$\hbar = \mu^{-1} h \qquad \text{and} \qquad \varepsilon = \mu^{-2}. \tag{13.3.60}$$

Theorem 13.3.8. *Let conditions* (13.2.1)–(13.2.5) *and* (13.3.1) *be fulfilled. Then for* $d = 2, 3$ *and small enough* ε *there exists an* $\hbar$*-Fourier integral operator* $\mathcal{F}_{\hbar,\varepsilon}$*, depending smoothly on* ε *and corresponding to the symplectomorphism* Ψ_ε *constructed in Theorem 13.3.6, such that modulo* $\hbar$*-pseudodifferential operators depending smoothly on* ε *with symbols vanishing in a neighborhood*[13)] *of* Λ^0

$$\mathcal{F}^*_{\hbar,\varepsilon} \mathcal{F}_{\hbar,\varepsilon} \equiv I, \tag{13.3.61}$$

$$\mathcal{F}^*_{\hbar,\varepsilon} A'_{\hbar,\varepsilon} \mathcal{F}_{\hbar,\varepsilon} \equiv \mathcal{A}'_{\hbar,\varepsilon} \tag{13.3.62}$$

and modulo $\hbar$*-pseudodifferential operators depending smoothly on* ε *with symbols vanishing in a neighborhood* $\mathcal{W}$ *of any point*[14)] $z \in \Lambda$ *(for* $d = 2$*) or any non-closed Hamiltonian trajectory*[14)] Γ *of finite length generated by* $F(x, \xi)$ *(for* $d = 3$*)*

$$\mathcal{F}_{\hbar,\varepsilon} \mathcal{F}^*_{\hbar,\varepsilon} \equiv I, \tag{13.3.63}$$

$$\mathcal{F}_{\hbar,\varepsilon} \mathcal{A}'_{\hbar,\varepsilon} \mathcal{F}^*_{\hbar,\varepsilon} \equiv A'_{\hbar,\varepsilon} \tag{13.3.64}$$

where for $d = 2$

$$\mathcal{A}'_{\hbar,\varepsilon} = \sum_{0 \le n \le N-1} B^{\mathrm{w}}_n (x_1, \hbar D_1, \varepsilon, \hbar) \mathcal{L}_0^n + R \tag{13.3.65}$$

and for $d = 3$

$$\mathcal{A}'_{\hbar,\varepsilon} = \hbar^2 D_2^2 + \sum_{0 \le n \le N-1} B^{\mathrm{w}}_n (x', \hbar D_1, \varepsilon, \hbar) \mathcal{L}_0^n + R \tag{13.3.66}$$

[12)] One needs to distinguish between $\hbar = \mu h$ and $\hbar = \mu^{-1} h$.

[13)] All neighborhoods mentioned in this theorem do not depend on $(\varepsilon, \hbar)$.

[14)] Fixed beforehand.

with

$$\mathcal{L}_0 = x_d^2 + \hbar^2 D_d^2, \tag{13.3.67}$$

and

$$R = \sum_{m+n\le N} R_{mn}(x, \hbar D, \varepsilon, \hbar)\varepsilon^m \hbar^n \mathcal{L}_0^{N-m-n}, \tag{13.3.68}$$

B_n^{w} *and* R_{mn} *are* $\hbar$*-pseudodifferential operators depending smoothly on* ε*, the Weyl symbols of* B_n *are real-valued and*

$$B_0(.,.,\varepsilon,\hbar) = \varepsilon V \circ \bar{\Psi}_0 + O(\varepsilon^2 + \hbar^2), \tag{13.3.69}_1$$

and

$$B_1(.,.,0,0) = F \circ \bar{\Psi}_0. \tag{13.3.69}_2$$

Definition 13.3.9. We call (13.3.65) and (13.3.66) with $\mathcal{L}_0$ defined by (13.3.67)–(13.3.68) *microlocal canonical form* for two- and three-dimensional Schrödinger operators with magnetic field.

We call (13.3.45) and (13.3.47) *symplectic canonical form* and we refer to any of (13.3.45), (13.3.47), (13.3.65) and (13.3.66) as *Birkhoff normal form*.

Proof of Theorem 13.3.8. Let $\mathcal{F}'_{\hbar,\varepsilon}$ be a compactly supported $\hbar$-Fourier integral operator corresponding to the symplectomorphism Ψ_ε, depending smoothly on ε and satisfying (13.3.61); then (13.3.63) is fulfilled automatically.

Theorem 13.3.6 yields that operator $\mathcal{B}'_{\hbar,\varepsilon} = {\mathcal{F}'_{\hbar,\varepsilon}}^* A'_{\hbar,\varepsilon}\mathcal{F}'_{\hbar,\varepsilon}$ differs from operator $\mathcal{A}'_{\hbar,\varepsilon}$ of the form (13.3.65) or (13.3.66) by a term $\hbar r(x, \hbar D, \varepsilon, \hbar)$ where r is an $\hbar$-pseudodifferential operator; all operators here and in what follows depend smoothly on ε.

Since $A'_{\hbar,\varepsilon}$ is formally self-adjoint the same is true for $\mathcal{B}'_{\hbar,\varepsilon}$, i.e., its Weyl symbol is real-valued.

Further, one can see easily that the subprincipal symbol of $A'_{\hbar,\varepsilon}$ is equal to 0; therefore the subprincipal symbol of $B'_{\hbar,\varepsilon}$ is a linear combination of the first-order derivatives of its principal symbol. Hence r is an $\hbar$-pseudodifferential operator and $r(.,.,0,0) = 0$ at Λ^0. This yields $(13.3.69)_2$ automatically in what follows.

It is sufficient to prove that there exists an $\hbar$-pseudodifferential operator $Q_{\hbar,\varepsilon}$ such that

$$Q^*_{\hbar,\varepsilon} Q_{\hbar,\varepsilon} \equiv I, \qquad Q_{\hbar,\varepsilon} Q^*_{\hbar,\varepsilon} \equiv I, \tag{13.3.70}$$

and

$$Q^*_{\hbar,\varepsilon}\mathcal{B}'_{\hbar,\varepsilon}Q_{\hbar,\varepsilon} \equiv \mathcal{A}'_{\hbar,\varepsilon}, \qquad Q_{\hbar,\varepsilon}\mathcal{A}'_{\hbar,\varepsilon}Q^*_{\hbar,\varepsilon} \equiv \mathcal{B}'_{\hbar,\varepsilon}, \tag{13.3.71}$$

modulo $\hbar$-pseudodifferential operators with symbols vanishing in a neighborhood of Λ^0 where $\mathcal{A}'_{\hbar,\varepsilon}$ is an operator of the form (13.3.65) or (13.3.66).

Then $\mathcal{F}_{\hbar,\varepsilon} = \mathcal{F}'_{\hbar,\varepsilon}Q_{\hbar,\varepsilon}$ satisfies (13.3.61)–(13.3.64). Obviously, we need to fulfill only first equalities in (13.3.70) and (13.3.71); the second equalities follow from them automatically.

In order to construct this operator $Q_{\hbar,\varepsilon}$ it is sufficient to prove that

(13.3.72) Let operator $\mathcal{B}'_{\hbar,\varepsilon}$ differ from an operator of the form (13.3.65) or (13.3.66) by a term $\hbar^J\varepsilon^L r^{\mathrm{w}}(x,\hbar D,\varepsilon,\hbar)$ where r is an $\hbar$-pseudodifferential operator with real-valued symbol such that $r(.,.,0,0) \in O_M(\Lambda^0)$ and $J \ge 1$, $M \ge 1$, $L \ge 0$.

Then there exists an $\hbar$-pseudodifferential operator $q_{\hbar,\varepsilon}$ satisfying (13.3.70) such that $q^*_{\hbar,\varepsilon}\mathcal{B}'_{\hbar,\varepsilon}q_{\hbar,\varepsilon}$ differs from an operator of the form (13.3.65) or (13.3.66) by a term $\hbar^J\varepsilon^L r'^{\mathrm{w}}(x,\hbar D,\varepsilon,\hbar)$ where r' is an $\hbar$-pseudodifferential operator (automatically with real-valued symbol) such that $r'(.,.,0,0) \in O_{M+1}(\Lambda^0)$.

We start with $M = 1$ because the term $\hbar^J\varepsilon^L r^{\mathrm{w}}(x',\hbar D_1,\varepsilon,\hbar)$ can be included in decomposition (13.3.65) or (13.3.66).

It is easy to construct $q_{\hbar,\varepsilon}$ by means of simplifications of the arguments of the proof of Proposition 13.3.1: we search for $q_{\hbar,\varepsilon}$ of the form $\exp(i\hbar^{J-1}\varepsilon^L S^{\mathrm{w}}(x,\hbar D))$ with a real-valued function S; here

(i) $S \in O_M(\Lambda^0)$ for $d = 2$ and

(ii) $S = S' + S''$ where $S' \in O_{M-1}(\Lambda^0)$ is of the form $(13.3.35)_N$ with $N = \frac{1}{2}(M+1)$ and $S'' \in O_M(\Lambda^0)$ for $d = 3$.

Then for $J = 1$, $q_{\hbar,\varepsilon}$ is an operator with principal symbol $\exp(i\varepsilon^L S(x,\xi))$ and for $J \ge 2$ $q_{\hbar,\varepsilon}$ differs from I by an operator with principal symbol $i\hbar^{J-1}\varepsilon^L S(x,\xi)$. □

For $d = 2$ one can slightly improve this theorem:

Proposition 13.3.10. *Let conditions* (13.2.1)–(13.2.5) *and* (13.3.1) *be fulfilled and let* $\mathcal{A}'_{\hbar,\varepsilon}$ *be a reduced operator of the form* (13.3.65). *Then one can rewrite it in the form*

$$\mathcal{A}'_{\hbar,\varepsilon} \equiv \sum_{1\le n\le N-1} b_n(x_1,\hbar D_1,\varepsilon,\hbar)\mathcal{L}^n + R' \tag{13.3.73}$$

where R' is an operator of the form (13.3.68)*, b_n are now not necessarily real-valued but still $b_1(.,.,0,0) = F \circ \bar{\Psi}_0$ as in* $(13.3.69)_2$*,*

$$\mathcal{L} = x_2^2 + \hbar^2 D_2^2 + \varepsilon B^{\mathrm{w}}(x_1, \hbar D_1, \varepsilon, \hbar), \tag{13.3.74}$$

$$B(.,.,\varepsilon,\hbar) = v \circ \bar{\Psi}_0 + \varepsilon\omega_1 + \hbar\omega_2 + O(\varepsilon^2) + O(\hbar^2\varepsilon^{-1}) + O(\varepsilon\hbar), \tag{13.3.75}$$

$$\omega_1 = \big(\frac{1}{8}\kappa v^2 - \frac{1}{4}\sum_{j,k} g^{jk}F^{-1}v\partial_j\partial_k v + \sum_j m^j\partial_j v\big) \circ \bar{\Psi}_0, \tag{13.3.76}_1$$

and

$$\omega_2 = \big(\sum_j r^j\partial_j v\big) \circ \bar{\Psi}_0. \tag{13.3.76}_2$$

Let us recall that $v = VF^{-1}$ and κ is the scalar curvature corresponding to the metrics $F^{-1}g^{jk}$; here r^j and m^j can be chosen arbitrarily at every fixed point.

Proof. All of the proposition excluding formulas $(13.3.76)_{1,2}$ follows immediately from Theorem 13.3.8; $(13.3.76)_1$ follows from Proposition 13.3.7 and $(13.3.76)_2$ follows from the fact that the subprincipal symbol of $\mathcal{A}'_{\hbar,\varepsilon}$ is a linear combination of the first-order derivatives of its principal symbol.

The possibility of choosing m^j arbitrarily was proven and the possibility of choosing r^j arbitrarily then follows if we multiply $\mathcal{A}'_{\hbar,\varepsilon}$ by q^* and $q = \exp(iS(x_1, \hbar D_1))$ on the left and right respectively. □

Remark 13.3.11. (i) No part of the reduction process (from Proposition 13.3.1 to Proposition 13.3.10) is uniquely determined; however, it can be taken stable in the natural sense; surely there are natural limitations on $k^j, \dots, r^j$: $|k^j| \le c$, etc.

(ii) One can see easily that (as in, e.g., the construction of Theorem 13.2.1) if g^{jk}, F_{jk} and V are constant then (13.3.65) and (13.3.66) are replaced respectively by

$$\mathcal{A}'_{\hbar,\varepsilon} \equiv F(x_2^2 + \hbar^2 D_2^2) + \varepsilon V, \tag{13.3.65}'$$

and

$$\mathcal{A}'_{\hbar,\varepsilon} \equiv \hbar^2 D_2^2 + F(x_3^2 + \hbar^2 D_3^2) + \varepsilon V. \tag{13.3.66}'$$

(iii) This fact and stability yields that if

$$(13.3.77)_{1-3} \quad |D^\alpha(g^{jk}-\bar g^{jk})|\le c\bar\nu, \quad |D^\alpha(F_{jk}-\bar F_{jk})|\le c\bar\nu, \quad |D^\alpha(V-\bar V)|\le c\bar\nu \qquad \forall \alpha : |\alpha|\le K \quad \forall x\in B(0,1)$$

with constant $\bar g^{jk}$, $\bar F_{jk}$, $\bar V$ then in (13.3.65), (13.3.66) and (13.3.67) operators $\bar\nu^{-1}(B_0-\bar V)$, $\bar\nu^{-1}(B_1-\bar F)$ and $\bar\nu^{-1}B_n$ with $n\ge 2$ are $\hbar$-pseudodifferential operators uniformly with respect to $\bar\nu\in(0,1]$.

Further, in (13.3.73), (13.3.74) the same conclusion is true for operators $\bar\nu^{-1}(b_1-\bar F)$, $\bar\nu^{-1}(B-\bar V\bar F^{-1})$ and $\bar\nu^{-1}b_n$ with $n\ge 2$.

Furthermore, in (13.3.75) one can replace $O(\varepsilon^2)$, etc., by $O(\varepsilon^2\bar\nu)$, etc.

Remark 13.3.12. In what follows we assume without any loss of the generality and without special mention that the symbol of the $\hbar$-pseudodifferential operator $\mathcal{F}^*_{\hbar,\varepsilon}\mathcal{F}_{\hbar,\varepsilon}$ is supported in a small enough neighborhood of Λ^0.

13.3.3 Reduced Problems

Let us consider what we have obtained in the investigation of U, etc. Let $\mu\ge 1$, $h\in(0,1]$, $\mu h\le 1$ and $\hbar=\mu^{-1}h$, $\varepsilon=\mu^{-2}$. Let operator $\mathcal{F}=\mathcal{F}_{\hbar,\varepsilon}$ be the operator constructed in Theorem 13.3.8 and let

$$(13.3.78) \qquad \mathsf{U}=\mathcal{F}^*_x U\, {}^t\mathcal{F}^*_y.$$

Then

$$(13.3.79) \qquad (hD_t-\mathcal{A})\mathsf{U}\equiv 0,$$

and

$$(13.3.80) \qquad \mathsf{U}|_{t=0}\equiv \delta(x-y)$$

modulo functions f and f' respectively such that if Q is a $\mu^{-1}h$-pseudodifferential operator with symbol q supported in a small enough neighborhood of Λ^0 such that

$$(13.3.81) \qquad \|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)Q_x f\|\le C'Th^s$$

and

$$(13.3.82) \qquad \|Q_x f'\|\le C'h^s$$

for arbitrary s [15)], $T \ge h^{1-\delta}$, T' and $\delta > 0$ are arbitrary, C' depends on $c, \epsilon_0, N, s, \delta$ and contains factors $\|\!|q|\!\|_{\mathsf{S}_{\mu^{-1}h}}$ and $\|\chi\|_{\mathscr{C}^K}$ (only in (13.3.81)).

Further,

$$\mathcal{A} = \mathcal{A}_{\mu,h} \equiv \mu^2 \mathcal{A}'_{\mu,h} \tag{13.3.83}$$

where $\mathcal{A}'_{\mu,h}$ is given by (13.3.65) or (13.3.73) as $d = 2$, and $\mathcal{A}'_{\mu,h}$ is given by (13.3.66) as $d = 3$, $\hbar = \mu^{-1}h$, $\varepsilon = \mu^{-2}$.

Moreover, U satisfies the dual equation and the properties of U yield similar properties of U:

$$\mathsf{U}(x, y, t) = \mathsf{U}^\dagger(y, x, -t), \tag{13.3.84}$$

and

$$|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)\mathsf{U}| \le C'\mu T h^{-d} \tag{13.3.85}$$

provided $T \ge h^{1-\delta}$ where C' is the same constant as above and one can assume without loss of generality that

$$\mathsf{supp}(\mathsf{U}) \subset B(0, c_1) \times B(0, c_1) \times \mathbb{R}. \tag{13.3.86}$$

Finally, the definition of U, (13.3.61), (13.3.62) and Theorem 13.2.13 yield that

$$|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)\big(U - \mathcal{F}_x \mathsf{U}^t \mathcal{F}_y\big)| \le C' T h^s \qquad \forall x, y \in \mathcal{W}_x \quad \forall \tau \le 4c \tag{13.3.87}$$

where $\mathcal{W}_x$ is the x-projection of $\mathcal{W}$.

13.4 Weak Magnetic Field

In this section we obtain spectral asymptotics under the initial assumption that the magnetic field is not very strong, i.e.,

$$\mu \in [\mu_0, h^{\delta-1}] \tag{13.4.1}$$

where here and in what follows μ_0 is a large enough constant and the exponent $\delta > 0$ is arbitrarily small. Under this condition the rescaled

[15)] Provided $K = K(N, s, \delta)$ in condition (13.2.3).

operator (as $x \mapsto \mu^{-1}x$, $hD \mapsto \mu hD$) is $\hbar$-pseudodifferential operator with semi-classical parameter $\hbar \le h^{\delta}$. Recall that $\hbar = \mu h$ appeared in section 13.2 twice: in the calculations for the model case in Subsection 13.2.2 and as the result of rescaling in Subsubsection 13.2.3.

Other restriction to μ will depend on properties of the coefficients[16] and appear later. The results obtained here make it possible to assume in the next sections that $\mu \ge h^{-\delta}$; in this case the remainder R in decompositions (13.3.65), (13.3.66) as well as all the terms containing large negative degrees of μ will be negligible.

In this section the microlocal canonical form is applied only to obtain a more refined results on the propagation of singularities; the calculations are taken from Subsection 4.3.3 and are made for the original (non-reduced to the canonical form) operator; a more accurate analysis of the propagation of singularities permits us to increase $T \asymp \mu^{-1}$ [17] in the Tauberian arguments. In the following sections the microlocal canonical form will also be used for calculations. We always assume that all the assumptions used in Section 13.3 are fulfilled.

13.4.1 Analysis of Singularities

Let us start with the analysis of singularities. First of all let us analyze their location, starting from rather crude "elliptical" result:

Proposition 13.4.1. *Let* $d = 2, 3$ *and let conditions* (13.3.1), (13.4.1) *be fulfilled. Consider symbol* $q \in \mathsf{S}_{\mu^{-1}h,\mu^{-1},\mu^{-1},K}(T^*\mathbb{R}^d)$ *and operator* $Q = \mathsf{Op}(q) = q(x, \mu^{-1}hD, \mu^{-1}h)$*; let us assume that*

$$(13.4.2)_{(2)} \quad \mathsf{supp}(q) \cap \{\tau_1 \le \bar{B}(x_1,\xi_1) + \bar{B}_1(x_1,\xi_1)(x_2^2+\xi_2^2)\mu^2 \le \tau_2\} = \emptyset,$$

$$(13.4.2)_{(3)} \quad \mathsf{supp}(q) \cap \{\tau_1 \le \bar{B}_0(x',\xi_1) + \bar{B}_1(x',\xi_1)(x_3^2+\xi_3^2)\mu^2 + \mu^2\xi_2^2 \le \tau_2\} = \emptyset$$

for $d = 2, 3$ *respectively where the bar means that the corresponding symbol is calculated for* $\hbar = \varepsilon = 0$ *and where* $-\infty \le \tau_1 < \tau_2 \le \infty$.

[16] Mainly $(V-\tau)F^{-1}$ where τ is an energy level in question; under assumption $|\nabla((V-\tau)F^{-1})| \asymp 1$ condition (13.4.1) would suffice.

[17] Usually to $T \asymp \mu$ or $T \asymp 1$ as $d = 2, 3$ respectively.

Then

$$|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)Q_x\mathsf{U}| \le C'Th^s\big(1+\min(\tau-\tau_1,\tau_2-\tau)\big)^{-s} \quad \forall x,y\in\mathbb{R}^d \quad \forall \tau\in[\tau_1+\epsilon,\tau_2-\epsilon] \tag{13.4.3}$$

where $s,\epsilon>0,\delta>0, T\ge h^{1-\delta}$ *and* T' *are arbitrary,* $K=K(\delta,s)$, $N=N(\delta,s)$ *in* (13.3.65), (13.3.66), $\mu_0=\mu_0(c,\epsilon)$ *in* (13.4.1) *and* $C'=C'(c,\epsilon,\delta,s)$ *also includes factors* $\|\!|q|\!\|$ *and* $\|\chi\|$.

This estimate remains true for Q_x *replaced by* ${}^tQ_y$ *(written as usual to the right from* $\boldsymbol{U}$*.*

Proof. Let us consider equation (13.3.79). The principal symbol of the $\hbar$-pseudodifferential operator $\mathcal{A}$ is

$$\bar{B}_0+\bar{B}_1(x_2^2+\xi_2^2)\mu^2+O\big((x_2^2+\xi_2^2)^2\mu^2+\mu^{-2}\big) \tag{13.4.4$_{(2)}$}$$

for $d=2$ and

$$\bar{B}_0+\bar{B}_1(x_3^2+\xi_3^2)\mu^2+\mu^2\xi_2^2+O\big((x_3^2+\xi_3^2)^2\mu^2+\mu^{-2}\big) \tag{13.4.4$_{(3)}$}$$

for $d=3$. Therefore operator $(hD_t-\mathcal{A})$ is elliptic under condition (13.4.2)$_{(2)}$ or (13.4.2)$_{(3)}$ in the zone

$$\{(x,\xi):\ \lambda(x,\xi)\le\epsilon', \tau_1+\epsilon\le\tau\le\tau_2-\epsilon\}$$

where $\epsilon'>0$ is a small enough constant and $x'=x_1$, $\lambda(x,\xi)=x_2^2+\xi_2^2$ for $d=2$ and $x'=(x_1,x_2)$, $\lambda(x,\xi)=x_3^2+\xi_3^2+\xi_2^2$ for $d=3$.

On the other hand, for any $\epsilon'>0$ and $\mu\ge\mu_0(c,\delta,\epsilon,\epsilon')$ this operator is automatically elliptic in the zone

$$\{(x,\xi):\ \lambda(x,\xi)\ge\epsilon', \tau_1+\epsilon\le\tau\le\tau_2-\epsilon\}$$

(recall that $\bar{B}_1\ge\epsilon_0$ in virtue of (13.3.12) and condition (13.3.1)).

Therefore, the standard elliptic arguments, estimate (13.3.85) and Remark 13.3.12 conclude the proof of this proposition. □

Remark 13.4.2. (i) Let us note that for $\tau_1=-\infty$, $\tau_2\le c$ and q supported in $\{(x,\xi):\ \lambda(x,\xi)\ge C_0\mu^{-2}\}$ with $C_0=C_0(c)$ condition (13.4.2) is fulfilled automatically; therefore Proposition 13.4.1 asserts in particular that

(13.4.5) U is negligible outside of the zone $\{(x,\xi):\ \lambda(x,\xi)\le C_0\mu^{-2}\}$

and hence in decompositions (13.3.65), (13.3.66) and (13.3.73) $R=O(\mu^{2-2N})$ in the heuristic sense.

Replacing U by $Q_x\mathsf{U}\equiv\mathsf{U}$ with $q=1$ for $\lambda(x,\xi)\le C_0\mu^{-2}$ and supported in $\{(x,\xi):\ \lambda(x,\xi)\le 2C_0\mu^{-2}\}$ we reach $R=O(\mu^{2-2N})$ in the sense that $R\mathsf{U}\equiv R'\mathsf{U}$ with $\mu^{2N-2}R'\in \boldsymbol{\Psi}_{\mu^{-1}h,\mu^{-1},\mu^{-1},K}$.

(ii) Moreover, Remark 13.3.11(ii) yields that $\bar\nu^{-1}\mu^{2N-2}R'\in \boldsymbol{\Psi}_{\mu^{-1}h,\mu^{-1},\mu^{-1},K}$ under conditions $(13.3.77)_{1-3}$.

Let us proceed to an analysis of propagation of singularities; here we obtain the principal difference between $d=2$ and $d=3$:

Theorem 13.4.3. *Let* $d=2,3$ *and conditions* (13.3.1), (13.4.1) *be fulfilled with* $\mu_0=\mu_0(c,\delta)$.

Let $Q'=\mathsf{Op}(q')$, $Q''=\mathsf{Op}(q'')$, $q',q''\in \mathsf{S}_{\boldsymbol{h},\rho,\gamma,K}(T^*\mathbb{R}^d)$ *where*

(13.4.6) $\boldsymbol{h}=(h_1,\dots,h_d)$, $\boldsymbol{\gamma}=(\gamma_1,\dots,\gamma_d)$, $\boldsymbol{\rho}=(\rho_1,\dots,\rho_d)$,
$h_1=\mu^{-1}h$, $h_2=h_3=h$, $\rho_j,\gamma_j\in(0,1]$, $\rho_j\gamma_j\ge h_jh^{-\delta}\quad\forall j$, $\gamma_d\le\mu^{-1}$

and quantization is taken with the vector semiclassical parameter $\boldsymbol{h}$.

(i) Let $d=3$ *and one of the following conditions*

$(13.4.7)_1$ $|x_1-y_1|\ge C_0\mu^{-1}T+\gamma_1$,

$(13.4.7)_2$ $|\xi_1-\eta_1|\ge C_0\mu^{-1}T+\rho_1$,

$(13.4.7)_3$ $|x_2-y_2|\ge C_0T+\gamma_2$,

$(13.4.7)_4$ $|\xi_2-\eta_2|\ge C_0T+\rho_2$

be fulfilled for all $(x,\xi)\in\mathsf{supp}(q')$ *and for all* $(y,\eta)\in\mathsf{supp}(q'')$ *where*

$(13.4.8)_{(3)}$ $T\in[h^{1-\delta},T_0]$,

$C_0=C_0(c)$, $T_0=T_0(c,\delta)>0$. *Then*

(13.4.9) $|F_{t\to h^{-1}\tau}\chi_T(t)Q'_x\mathsf{U}\,{}^tQ''_y|\le C'h^s\qquad\forall x,y\in\mathbb{R}^d\quad\forall\tau\le c.$

(ii) Let $d = 2$ *and one of conditions* $(13.4.7)_{1,2}$ *be fulfilled with*

$$T \in [h^{1-\delta}, \mu T_0]. \tag{13.4.8}_{(2)}$$

Then with C_0, T_0 *the same as before, estimate* (13.4.9) *remains true.*

(iii) Furthermore, if $\rho_d = \mu^{\frac{1}{2}} h^{\frac{1}{2}(1-\delta)}$, $\gamma_d = \mu^{-\frac{1}{2}} h^{\frac{1}{2}(1-\delta)}$ *and if condition*

$$|\mu^2 x_d^2 + \xi_d^2 - \mu^2 y_d^2 - \eta_d^2| \ge C\mu^{-l} + \rho_d \tag{13.4.10}$$

is fulfilled for all $(x, \xi) \in \mathsf{supp}(q')$ *and for all* $(y, \eta) \in \mathsf{supp}(q'')$ *with* $C = C(c, \delta, l)$ *then for* T *satisfying* $(13.4.8)_{(d)}$ *estimate* (13.4.9) *holds.*

We will prove this theorem together with Theorem 13.4.4 below. Statements (i) and (ii) mean that the velocity of propagation of singularities with respect to (x_1, ξ_1) does not exceed $C_0\mu^{-1}$ and for $d = 3$ the velocity with respect to (x_2, ξ_2) does not exceed C_0 and Statement (iii) means that expression $(\mu^2 x_d^2 + \xi_d^2)$ is almost constant for large μ (all these assertions hold for energy levels not exceeding c and for quantization corresponding to the vector $\boldsymbol{h}$) indicated in (13.4.6).

Let us emphasize that for $d = 2$ and $d = 3$ the restrictions on T from above are different, which will be essential for the Tauberian arguments.

For $d = 2, 3$ a more accurate theorem will be proven in Section 6.5. For $d = 2$ we have the more accurate

Theorem 13.4.4. *Let* $d = 2$ *and* $q', \dots, Q''$ *be as in Theorem 13.4.3. Let us assume that conditions* $(13.3.77)_{1-3}$ *are fulfilled and that*

$$\nu^{-1}(B - \beta) \in \boldsymbol{\Psi}_{\mu^{-1}h,K}(T^*\mathbb{R}^d) \tag{13.4.11}$$

where B *is the operator introduced in* (13.3.74) *and* $\beta = \mathsf{const}$.

Furthermore, let us assume that

$$1 \ge \bar\nu \ge \mu^{-l'}, \qquad \bar\nu \ge \nu \ge \mu^{-l}\bar\nu \tag{13.4.12}_{1,2}$$

and in what follows N, K, μ_0, C_0 *and* C *also depend on* l *and* K' *and* C' *also depend on* l, l'. *Then*

(i) Let one of the following conditions

$$|x_1 - y_1| \ge C_0\nu\mu^{-1}T + \gamma_1, \tag{13.4.7}'_1$$

$$|\xi_1 - \eta_1| \ge C_0\nu\mu^{-1}T + \rho_1 \tag{13.4.7}'_2$$

be fulfilled for all $(x,\xi)\in \mathsf{supp}(q')$ *and for all* $(y,\eta)\in \mathsf{supp}(q'')$ *with*

$$T\in [\bar\nu\nu^{-1}h^{1-\delta}, \mu\nu^{-1}T_0], \tag{13.4.8$'_{(2)}$}$$

and with $T_0 = T_0(c,\delta,l) > 0$. *Then estimate* (13.4.9) *remains true for* $|\tau|\le c\nu\bar\nu^{-1}$.

(ii) Moreover, if $\rho_2 = \mu^{\frac12}h^{\frac12(1-\delta)}$, $\gamma_2 = \mu^{-\frac12}h^{\frac12(1-\delta)}$ *and if the condition*

$$|\mu^2x_2^2+\xi_2^2-\mu^2y_2^2-\eta_2^2|\ge C\mu^{-l}\bar\nu+\rho_2 \tag{13.4.10$'$}$$

is fulfilled for all $(x,\xi)\in \mathsf{supp}(q')$ *and for all* $(y,\eta)\in \mathsf{supp}(q'')$ *then for the* T *and* τ *indicated in (i) estimate* (13.4.9) *holds.*

Let us note that for $\bar\nu=\nu$ condition (13.4.11) is fulfilled automatically by Remark 13.3.11(ii).

Proof of Theorems 13.4.3 and 13.4.4. The proof is similar to that of Theorem 2.1.2.

(i) Theorem 13.4.3(i) obviously follows from Theorem 13.4.4; let us prove the latter theorem. Dividing equation (13.3.79) by ν and introducing $t'=\mu\nu^{-1}t$ in place of t we obtain the following equation

$$\bigl(h_1D_{t'}-M(x_1,h_1D_1,h_1,\mu^{-2})\mathcal{B}-R'\bigr)\boldsymbol{U}\equiv 0 \tag{13.4.13}$$

with

$$\mathcal{B}=\nu^{-1}\mu^2\mathcal{L}=\nu^{-1}(\mu^2x_2^2+h^2D_2^2)+\nu^{-1}B^{\mathsf{w}} \tag{13.4.14}$$

and with

$$M=\sum_{0\le n\le N-2} b_{n+1}(x_1,h_1D_1,h_1,\mu^{-2})\mathcal{L}^n. \tag{13.4.15}$$

Moreover, Remark 13.4.2 yields that we can assume without any loss of the generality that $\mu^NR'\in \boldsymbol{\Psi}_{\boldsymbol{h},\rho,\gamma,K}$; condition (13.4.12) justifies all these transformations.

Let us note that the operator M is elliptic in the indicated zone and therefore equation (13.4.13) yields that

$$\mathcal{B}\mathsf{U}\equiv M'\bigl(h_1D_{t'}-R'\bigr)\mathsf{U}, \tag{13.4.16}$$

where M' is a parametrix of M.

Further, let us note that if function $\omega(x', \xi', \lambda)$ satisfies inequalities

$$|D^{\alpha}_{x'} D^{\beta}_{\xi'} D^{\sigma}_{\lambda} \omega| \le c_1 \boldsymbol{\rho}'^{-\beta} \boldsymbol{\gamma}'^{-\alpha} \bar{\rho}^{-\sigma} \qquad \forall \alpha, \beta, \sigma : |\alpha| + |\beta| + \sigma \le K_0 + K \tag{13.4.17}$$

with $\boldsymbol{\rho}' = (\rho_1)$, $\boldsymbol{\gamma}' = (\gamma_1)$ for $d = 2$ [18] and with $\bar{\rho} = \mu^{\frac{1}{2}} h^{\frac{1}{2}(1-\delta)}$ and if condition (13.4.6) is fulfilled then the symbol $\psi(x, \xi) = \omega(x_1, \xi_1, \mu^2 x_2^2 + \xi_2^2)$ belongs to $\mathsf{S}_{\boldsymbol{h}, \boldsymbol{\rho}, \boldsymbol{\gamma}, K}$ in the zone $\{\mu^2 x_2^2 + \xi_2^2 \le C_0\}$ where $\boldsymbol{\rho} = (\rho_1, \rho_2)$, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$, ρ_1 and γ_1 were introduced earlier and $\rho_2 = \bar{\rho}$, $\gamma_2 = \mu^{-1}\bar{\rho}$ for $d = 2$.

Moreover, the absolute values of the Poisson brackets $\{\psi, \mathcal{B}\}$ and $\{\psi, M\}$ do not exceed $C_0 |\nabla_{x_1, \xi_1} \omega|$ and the absolute value of the Poisson bracket $\{\psi, R'\}$ does not exceed $C_0 \mu^{-l} |\nabla_{x_1, \xi_1, \lambda} \omega|$ with arbitrarily large l [19].

Taking into account these observations we can apply the arguments used in the proof of Theorem 2.1.2 to

$$\phi(x, \xi, t') = \psi(x, \xi) + t' = \omega(x_1, \xi_1, \mu^2 x_2^2 + \xi_2^2) + t' \tag{13.4.18}_{(2)}$$

provided conditions (13.4.17) and (13.4.6) with $j = 1$ are fulfilled and

$$|\nabla_{x_1, \xi_1} \omega| \le C_0^{-1}, \qquad |\partial_{\lambda} \omega| \le C^{-1} \mu^{l}. \tag{13.4.19}_{(2)}$$

(ii) Theorem 13.4.3(ii) is proven in the same way but now we neither divide by ν, nor introduce the variable t' and apply a parametrix M' of M; we need only apply the arguments used in the proof of Theorem 2.1.2 to symbol

$$\phi(x, \xi, t) = \omega(x', \xi', \mu^2 x_3^2 + \xi_3^2) + t \tag{13.4.18}_{(3)}$$

where ω satisfies (13.4.17) with $\boldsymbol{\rho}' = (\rho_1, \rho_2)$, $\boldsymbol{\gamma}' = (\gamma_1, \gamma_2)$ and

$$|\nabla_{x_1, \xi_1} \omega| \le C_0^{-1} \mu, \quad |\nabla_{x_2, \xi_2} \omega| \le C_0^{-1}, \quad |\partial_{\lambda} \omega| \le C^{-1} \mu^{l}, \tag{13.4.19}_{(3)}$$

and condition (13.4.6) with $j = 1, 2$ is fulfilled. □

Theorems 13.4.3, 13.4.4 and 13.2.13 and formula (13.3.87) immediately imply the following

[18] And with $\boldsymbol{\rho}' = (\rho_1, \rho_2)$, $\boldsymbol{\gamma}' = (\gamma_1, \rho_2)$ for $d = 3$ below.

[19] After the calculation of the gradients one should substitute $\lambda = \mu^2 x_2^2 + \xi_2^2$.

Theorem 13.4.5. *Let conditions* (13.2.1)–(13.2.5), (13.3.1) *and* (13.4.1) *be fulfilled.*

(i) Further, let $d = 2$ *and condition* $(13.4.8)_{(2)}$ *be fulfilled. Then*

$$|F_{t\to h^{-1}\tau}\chi_T(t)U(x,y,t)| \le C'h^s \quad \forall x,y \in B(0,\frac{3}{4}) : |x-y| \ge C_0\mu^{-1}(T+1) \quad \forall \tau \le c. \tag{13.4.20}$$

(ii) Let $d = 2$ *and conditions* $(13.3.77)_{1-3}$, (13.4.11), (13.4.12), $(13.4.8)'_{(2)}$ *be fulfilled. Then the same estimate* (13.4.20) *holds for all* $x, y \in B(0, \frac{3}{4})$ *and* τ *such that*

$$|x-y| \ge C_0\mu^{-1}(\nu T+1) \qquad \text{and} \qquad |\tau| \le c\nu. \tag{13.4.21}$$

(iii) On the other hand, let $d = 3$ *and condition* $(13.4.8)_{(3)}$ *be fulfilled. Then estimate* (13.4.20) *holds for all* $x, y \in B(0, \frac{3}{4}), \tau \le c$ *such that*

$$\operatorname{dist}(x, \Gamma(y)) \ge C_0\mu^{-1} \tag{13.4.22}$$

where $\Gamma(y)$ *is a magnetic line passing through* y *(more precisely, its segment lying in* $B(0,1)$*).*

Remark 13.4.6. (i) Thus for $d = 2$ the average (in time intervals greater than ϵ_0) spatial velocity of propagation of singularities does not exceed $C_0\mu^{-1}$ which permits us to control U on a time interval of length $T_0\mu$ (under the assumptions of Statement 13.4.5(ii) this length increases to $\mu\nu^{-1}T_0$).

(ii) On the other hand, for $d = 3$ singularities propagate almost along magnetic lines. Moreover, let us recall that the average (on time intervals greater than $h^{1-\delta}$) spatial velocity of propagation of singularities does not exceed C_0 (see Theorem 13.2.11); this is important for $d = 3$ and for $d = 2$ for short time intervals.

13.4.2 Refined Estimates

Let us proceed with a more refined analysis. It will be convenient for us to use symbols of the form

(13.4.23) $q = q_1 q_2 \in \mathsf{S}_{\boldsymbol{h},\boldsymbol{\rho},\boldsymbol{\gamma},K}(T^*\mathbb{R}^d)$ with factors $q_1 \in \mathsf{S}_{\boldsymbol{h}',\boldsymbol{\gamma},\boldsymbol{\gamma},K}(T^*\mathbb{R}^{d-1})$ and $q_2 = \varphi(\mu^2 x_d^2 + \xi_d^2) \in \mathsf{S}_{h,\bar\rho,\mu^{-1}\bar\rho,K}(T^*\mathbb{R})$ where $\boldsymbol{\rho} = (\gamma,\gamma,\mu\bar\rho)$, $\boldsymbol{\gamma} = (\gamma,\gamma,\bar\rho)$, $\bar\rho \ge \mu^{\frac12} h^{\frac12(1-\delta)}$, φ satisfies (13.4.17), i.e.,

$$|D_\lambda^\sigma \varphi(\lambda)| \le C\bar\rho^{-\sigma} \qquad \text{for } \ \sigma : |\sigma| \le K$$

and $\mathsf{diam}(\mathsf{supp}(\varphi)) \le c\bar\rho$, $\mathsf{diam}(\mathsf{supp}(q_1)) \le c\gamma$.

Restrictions on γ depending on d will be given later; let us recall that $\boldsymbol{h}$ and $\boldsymbol{h}'$ are defined in (13.4.6).

Elliptic Estimates

Standard elliptic methods combined with the rough estimate (13.3.85) easily imply

Proposition 13.4.7. *Let conditions* (13.2.1)–(13.2.5), (13.3.1) *and* (13.4.1) *be fulfilled.*

(i) Let $d = 2$ *and conditions* $(13.3.77)_{1-3}$, (13.4.11) *and* (13.4.12) *be fulfilled and* $Q = \mathsf{Op}(q)$, q *of the form* (13.4.23). *Further, assume that*

$$\gamma \ge \mu^{-\frac12} h^{\frac12(1-\delta)} + C\mu^{-l}, \tag{13.4.24}$$

$$|\lambda + B(x_1,\xi_1,\mu^{-2},0)| \ge \zeta \ge \epsilon_0(\bar\rho + \nu\gamma^2), \tag{13.4.25}$$

and

$$|\nabla B(x_1,\xi_1,\mu^{-2},0)| \le c\nu\gamma \quad \forall (x_1,\xi_1) \in \mathsf{supp}(q_1)\ \forall \lambda \in \mathsf{supp}(\varphi). \tag{13.4.26}$$

Then

$$|F_{t\to h^{-1}\tau}\chi_{T,T'}(t) Q_x \mathsf{U}| \le C' T h^s \qquad \forall \tau : |\tau| \le \epsilon\zeta \quad \forall x,y \in \mathbb{R}^d \tag{13.4.27}$$

where $T \ge h^{1-\delta}\zeta^{-1}$, T' *and* s *are arbitrary, and here and in what follows* $\epsilon = \epsilon(c, l, \epsilon_0, \delta) > 0$ *is a small enough constant.*

(ii) Let $d = 3$, $Q = \mathsf{Op}(q)$ *where* q *of the form* (13.4.23). *Further, assume that*

$$\gamma \ge h^{\frac12(1-\delta)} + C\mu^{-l}, \tag{13.4.24}$$

$$|\xi_2^2 + \mathcal{B}(x',\xi_1,\lambda,\mu^{-2},0)| \ge \zeta \ge \epsilon_0(\bar\rho + \gamma^2) \tag{13.4.25}$$

and

$$(13.4.26)_{(3)} \quad |\xi_2| + |\nabla\mathcal{B}(x',\xi_1,\lambda,\mu^{-2},0)| \le c\gamma \qquad \forall (x',\xi') \in \mathsf{supp}(q_1) \quad \forall \lambda \in \mathsf{supp}(\varphi)$$

where here and below

$$\mathcal{B} = \sum_{0\le n\le N-1} B_n(x',\xi_1,.,.)\lambda^n \tag{13.4.28}$$

and B_n are taken from decomposition (13.3.66)*.*

Then estimate (13.4.27) *holds.*

Propagation of Singularities

Let us now treat propagation of singularities. Our final goal is to prove that if we cut propagator $U(x,y,t)$, then take a trace, the result will be negligible as $T_* \le |t| \le T^*$; later *in this section* we request $T_* \le \epsilon\mu^{-1}$ as on interval $|t| \le \epsilon\mu^{-1}$ the standard theory works (f.e. in virtue of rescaling).

Proposition 13.4.8. *Let conditions* (13.2.1)–(13.2.5)*,* (13.3.1) *and* (13.4.1) *be fulfilled.*

(i) Further, let $d = 2$, conditions $(13.3.77)_{1-3}$*,* (13.4.11) *and* (13.4.12) *be fulfilled*[20]*, $Q' = \mathsf{Op}(q')$, $Q'' = \mathsf{Op}(q'')$, q' and q'' of the form* (13.4.23) *and γ satisfy* $(13.4.24)_{(2)}$*.*

Moreover, let us assume that $(13.4.26)_{(2)}$ *is fulfilled on* $\mathsf{supp}(q')$ *and that*

$$(13.4.29)_{(2)} \quad |x_1 - y_1| + |\xi_1 - \eta_1| \ge C_0\nu\mu^{-1}\gamma T + \gamma \qquad \forall (x_1,\xi_1) \in \mathsf{supp}\, q' \quad \forall (y_1,\eta_1) \in \mathsf{supp}(q'').$$

Then estimate (13.4.9) *holds for all $x, y \in \mathbb{R}^2$ and all τ : $|\tau| \le c\nu\gamma$ provided*

$$(13.4.30)_{(2)} \qquad T \in [h^{1-\delta}\frac{1}{\gamma\nu}, \mu\frac{T_0}{\nu}]$$

with a small constant $T_0 > 0$.

[20] Recall that these conditions are always fulfilled with $\bar\nu = \nu = 1$.

(ii) On the other hand, let $d = 3$, $Q' = \mathsf{Op}(q')$, $Q'' = \mathsf{Op}(q'')$, q' *and* q'' *of the form* (13.4.23) *and* γ *satisfy* $(13.4.24)_{(3)}$.

Moreover, let us assume that $(13.4.26)_{(3)}$ *is fulfilled on* $\mathsf{supp}(q')$ *and that*

$$(13.4.29)_{(3)} \quad \mu(|x_1 - y_1| + |\xi_1 - \eta_1|) + |x_2 - y_2| + |\xi_2 - \eta_2| \geq C_0\gamma T + \gamma \qquad \forall (x', \xi') \in \mathsf{supp}(q') \quad \forall (y', \eta') \in \mathsf{supp}(q'').$$

Then estimate (13.4.9) *holds for all* $x, y \in \mathbb{R}^3$ *and all* $\tau : \ |\tau| \leq c\gamma$ *provided*

$$(13.4.30)_{(3)} \qquad T \in [h^{1-\delta}\gamma^{-1}, T_0].$$

with a small constant $T_0 > 0$.

Proof. The proof is very similar to that of Propositions 2.1.12 and 2.1.13; moreover, we take into account the remarks to the proofs of Theorems 13.4.3 and 13.4.4. The easy details are left to the readers. □

Moreover, the same arguments and arguments used in the proofs of Corollaries 2.1.14 and 2.1.17 and Proposition 2.1.16 and and the same remarks yield the following assertion similar to Corollary 2.1.17:

Proposition 13.4.9. *Let conditions* (13.2.1)–(13.2.5), (13.3.1) *and* (13.4.1) *be fulfilled.*

(i) Further, let $d = 2$, *conditions* $(13.3.77)_{1-3}$, (13.4.11) *and* (13.4.12) *be fulfilled*[20], $Q' = \mathsf{Op}(q')$, $Q'' = \mathsf{Op}(q'')$, q' *and* q'' *of the form* (13.4.23) *and* γ *satisfy* $(13.4.24)_{(2)}$.

Moreover, let us assume that on $\mathsf{supp}(q')$ *condition* $(13.4.26)_{(2)}$ *is fulfilled and*

$$(13.4.31)_{(2)} \qquad |\nabla B| \geq \epsilon_0\nu\gamma \quad \textit{on} \quad \mathsf{supp}(q').$$

Finally, let $\chi, \bar{\chi} \in \mathscr{C}_0^K([-1,1])$, $\bar{\chi} = 1$ *on* $[-\frac{1}{2}, \frac{1}{2}]$ *and let* T', T *satisfy* $(13.4.30)_{(2)}$ *and*

$$(13.4.32) \qquad T_* := \frac{h^{1-\delta}}{\nu\gamma^2} \leq T' \leq T.$$

Then

$$(13.4.33) \quad |F_{t\to h^{-1}\tau}\chi_T(t)\bigl(1 - \bar{\chi}_{T'}\bigr)\Gamma Q'_x U\, {}^tQ''_y| \leq C'h^s \qquad \forall \tau : |\tau| \leq \epsilon\frac{\nu\gamma^2}{\bar{\nu}}.$$

(ii) On the other hand, let $d = 3$, $Q' = \mathsf{Op}(q')$, $Q'' = \mathsf{Op}(q'')$, q' *and* q'' *of the form* (13.4.23) *and* γ *satisfy* $(13.4.24)_{(3)}$.

Moreover, let us assume that on $\mathsf{supp}(q')$ *condition* $(13.4.26)_{(3)}$ *is fulfilled and*

$$(13.4.31)_{(3)} \qquad |\xi_2| + |\nabla \mathcal{B}| \ge \epsilon_0 \gamma \quad \textit{on} \quad \mathsf{supp}(q').$$

Finally, let $\chi, \bar{\chi} \in \mathscr{C}_0^K([-1,1])$, $\bar{\chi} = 1$ *on* $[-\frac{1}{2}, \frac{1}{2}]$ *and let* T', T *satisfy conditions* $(13.4.30)_{(3)}$ *and* (13.4.32) *where*

(13.4.34) $\bar{\nu} = \nu = 1$ *automatically for* $d = 3$ *in this section.*

Then estimate (13.4.33) *holds.*

Remark 13.4.10 (Remark to the proof). (i) For $d = 3$ we treat four cases, namely when one of the following conditions holds on $\mathsf{supp}(q')$:

$$(13.4.35)_1 \qquad |\partial_{x_1} \mathcal{B}| \ge \epsilon_0 \nu \gamma,$$

$$(13.4.35)_2 \qquad |\partial_{\xi_1} \mathcal{B}| \ge \epsilon_0 \nu \gamma,$$

$$(13.4.35)_3 \qquad |\partial_{x_2} \mathcal{B}| \ge \epsilon_0 \gamma,$$

$$(13.4.35)_4 \qquad |\xi_2| \ge \epsilon_0 \gamma$$

with $\nu = 1$.

Moreover, our arguments repeat those of the proof of Proposition 2.1.16 with no change if one of conditions $(13.4.35)_{3,4}$ is fulfilled; the same is true if one of conditions $(13.4.35)_{1,2}$ is fulfilled but in the latter case we also need to change coordinates: $x_{1\mathsf{new}} = \mu x_1$.

Moreover, in the auxiliary partition of unity (see proof of Proposition 2.1.16) we use standard h- (but not $\boldsymbol{h}$-) pseudodifferential operators.

(ii) For $d = 2$ we treat only two cases $(13.4.35)_{1,2}$ and our arguments repeat those of the proof of Proposition 2.1.16.

Remark 13.4.11. (i) Proposition 13.4.7 yields that in estimates (13.4.9) and (13.4.33) in Propositions 13.4.8 and 13.4.9 we can replace Q', Q'' by Q_1', Q_1'' respectively where $Q_1' = \mathsf{Op}(q_1')$, $Q_1'' = \mathsf{Op}(q_1'')$, q_1' and q_1'' are taken from decomposition (13.4.23).

In this case we need to check conditions $(13.4.26)_{(3)}$ and $(13.4.31)_{(3)}$ with λ violating condition $(13.4.25)_{(3)}$ with $\zeta = \epsilon(\bar{\rho}+\gamma)$; the concrete choice of λ is not essential.

(ii) For $d = 2$ the corresponding conditions do not include λ at all.

Estimates. I

On the other hand, propagation with respect to $(\mu x_d, \xi_d)$ has a speed $\asymp 1$ provided $(\mu^2 x_d^2 + \xi_d^2) \asymp 1$ and the corresponding trajectories are not periodic as $|t| \le \epsilon\mu^{-1}$. Then we immediately arrive to

Proposition 13.4.12. *Let* $d = 2, 3$, $Q' = \mathsf{Op}(q')$, $Q'' = \mathsf{Op}(q'')$, q' *and* q'' *of the form* (13.4.23),

$$\gamma \ge (\mu h)^{\frac{1}{2}-\delta} \tag{13.4.36}$$

and

$$(\mu^2 x_d^2 + \xi_d^2) \ge \epsilon_0 \qquad \textit{on} \quad \mathsf{supp}(q). \tag{13.4.37}$$

Then

$$|F_{t\to h^{-1}\tau}\bar{\chi}_{T'}(t)\Gamma Q'_x \mathsf{U}\, {}^t Q''_y| \le Ch^{1-d}\gamma^{2(d-1)} \qquad \forall \tau : |\tau| \le c, \tag{13.4.38}$$

as

$$h\gamma^{-2} \le T' \le \epsilon\mu^{-1}. \tag{13.4.39}$$

Proof. Proof is the standard one: as usual in virtue of propagation we establish that

(13.4.40) The left-hand expression of (13.4.38) with $\bar{\chi}$ replaced by χ does not exceed $Ch^{1-d}\gamma^{2(d-1)}h^s T'^{-s}$ with arbitrarily large exponent s under assumptions (13.4.37), (13.4.39)

and also establish crude

(13.4.41) The left-hand expression of (13.4.38) does not exceed $Ch^{-d}\gamma^{2d}T'$.

Summation over partition of unity with respect to t concludes the proof. □

Now, combining results of Propositions 13.4.9 and 13.4.12 we arrive to

Corollary 13.4.13. *Let assumptions of Proposition 13.4.12 and either of Proposition 13.4.9(i) or of Proposition 13.4.9(ii) be fulfilled.*

Then estimate (13.4.38) *holds for all* τ *satisfying* (13.4.33) *and* $h\gamma^{-2} \le T' \le T^*$ *with* T^* *defined by* $(13.4.30)_{(2)}$ *or* $(13.4.30)_{(3)}$ *respectively.*

Recall that $e(x,y,\tau)$ is the Schwartz kernel of the spectral projector. Consider $\mathcal{E}(x,y,\tau) = \mathcal{F}^*_x e'(.,.,\tau)\, {}^t\mathcal{F}_y$ its transformation by $\mathcal{F} = \mathcal{F}_{\mu^{-2},\mu^{-1}h}$ introduced in Theorem 13.3.8.

Consider its Tauberian approximation

$$\mathcal{E}^{\mathsf{T}}(x,y,\tau) \coloneqq h^{-1}\int_{-\infty}^{\tau} F_{t\to h^{-1}\tau'}\bigl(\bar{\chi}_{T'}(t)U\bigr)\,d\tau' \tag{13.4.42}$$

with

$$h^{1-\delta}\gamma^{-2} \le T' \le \epsilon\mu^{-1} \tag{13.4.43}$$

where the second inequality in (13.4.43) will be needed only in the next subsection.

Then applying Tauberian arguments we arrive to

Proposition 13.4.14. *Let assumptions of Proposition 13.4.12 and either of Proposition 13.4.9(i) or of Proposition 13.4.9(ii) be fulfilled.*

Then the following estimate holds:

$$|\Gamma Q'_x\Bigl(\mathcal{F}^*_x e(.,.,\tau)\,{}^t\mathcal{F}_y - \mathcal{E}^{\mathsf{T}}(x,y,\tau)\Bigr)\,{}^tQ''_y| \le CT^{*-1}h^{1-d}\gamma^{2(d-1)} \tag{13.4.44}$$

and, moreover, the following Riesz means estimate holds:

$$|\tau_+^{\vartheta-1} * \Gamma Q'_x\Bigl(\mathcal{F}^*_x e'(.,.,\tau)\,{}^t\,\mathcal{F}_y - \mathcal{E}^{\mathsf{T}}(x,y,\tau)\Bigr)\,{}^tQ''_y| \le CT^{*-1-\vartheta}h^{1-d+\vartheta}\gamma^{2(d-1)} \tag{13.4.44}$$

where $\tau = 0$ *(or satisfies* (13.4.33)*) and* T^* *is defined by* $(13.4.30)_{(2)}$ *or* $(13.4.30)_{(3)}$ *respectively.*

On the other hand, applying Tauberian arguments we arrive to

Proposition 13.4.15. *Let assumptions of Proposition 13.4.12 be fulfilled.*

Then estimates (13.4.44) *and* $(13.4.44)_\vartheta$ *hold with* T^* *replaced by* $T_* = \epsilon\mu^{-1}$.

Remark 13.4.16. (i) In propositions above we do not need cut-off with respect to $(\mu^2 x_d^2 + \xi_d^2)$ which automatically is selected to break ellipticity on level 0 of operator $\mathcal{B}$. However we need it to fulfill (13.4.37) to cover interval $|t| \le \epsilon\mu$; however as $d = 3$ the same effect will be achieved by $|\xi_2| \ge \epsilon$ so we can replace (13.4.37) by

$$(\mu^2 x_3^2 + \xi_3^2 + \xi_2^2) \ge \epsilon_0 \qquad \text{on} \quad \mathsf{supp}(q). \tag{13.4.37}_{(3)}$$

(ii) Note that condition (13.4.37) or $(13.4.37)_{(3)}$ is fulfilled as ellipticity assumption $(13.4.25)_{(2)}$ or $(13.4.25)_{(3)}$ respectively as $d = 2, 3$ is broken if and only if

$$V \le -\epsilon_0 \qquad \forall x \in B(0,1). \tag{13.4.45}$$

(iii) Furthermore to cover interval $|t| \le \epsilon\mu^{-1}$ we may assume that

$$|V| + |\nabla V| \ge \epsilon_0 \qquad \forall x \in B(0,1). \tag{13.4.46}$$

Then using γ-admissible partition of unity we immediately arrive to

Proposition 13.4.17. *Let condition* (13.4.46) *be fulfilled. Further, let* <u>*either*</u> $d = 2$ *and conditions* $(13.3.77)_{1-3}$ *and* (13.4.11) *be fulfilled with*

$$\nu \ge \mu h^{1-\delta} \tag{13.4.47}$$

<u>*or*</u> *let* $d = 3$.

Furthermore, let $\gamma = \gamma(x', \xi')$ *be a scaling function such that*

(13.4.48) *As* $\gamma \ge \bar{\gamma} := (\mu h \nu^{-1})^{\frac{1}{2}} h^{-\delta}$ *and* $\lambda \in \mathbb{R}$ *either ellipticity condition* $|\mathcal{B}| \ge \epsilon\nu\gamma^2$ *or nondegeneracy condition* $(13.4.31)_{(d)}$ *be fulfilled.*

Let $Q', Q'' \in \boldsymbol{\Psi}_h$ *be fixed operators. Then*

(i) As $d = 2$ *the left-hand expression of* $(13.4.44)_\vartheta$ *does not exceed*

$$C\mu^{-1-\vartheta} h^{-1+\vartheta} \nu^{-1-\vartheta} + C\mu^{1+\vartheta} h^{-1+\vartheta}\, \mathsf{mes}_2(\Omega_2) \tag{13.4.49}_\vartheta$$

with

$$\Omega_{2d-2} := \{(x', \xi') \in \mathsf{supp}(q) \colon \gamma(x', \xi') \le C_0 \bar{\gamma}\} \tag{13.4.50}$$

(ii) As $d = 3$ the left-hand expression of $(13.4.44)_\vartheta$ *does not exceed*

$$Ch^{-2+\vartheta} + C\mu^{1+\vartheta}h^{-2+\vartheta}\,\mathsf{mes}_4(\Omega_4). \tag{13.4.51}_\vartheta$$

Finally, we can get rid of transformation to canonical form in the expression we estimate (albeit not yet in the non-degeneracy conditions):

Remark 13.4.18. Consider Tauberian approximation to $e(x,y,\tau)$

$$e^{\mathsf{T}}(x,y,\tau) \coloneqq h^{-1}\int_{-\infty}^{\tau} F_{t\to h^{-1}\tau'}\bigl(\bar{\chi}_{T'}(t)U\bigr)\,d\tau' \tag{13.4.52}$$

with T' satisfying (13.4.43).

Then estimates $(13.4.49)_\vartheta$ (as $d = 2$) and $(13.4.51)_\vartheta$ (as $d = 3$) hold for

$$\mathsf{R}^{\mathsf{T}} \coloneqq |\Gamma\psi(x)\bigl(e(.,.,\tau) - e^{\mathsf{T}}(x,y,\tau)\bigr)| \tag{13.4.53}$$

(as $\vartheta = 0$) and

$$\mathsf{R}^{\mathsf{T}}_{(\vartheta)} \coloneqq |\tau_+^{\vartheta-1} * \Gamma\psi(x)\bigl(e(.,.,\tau) - e^{\mathsf{T}}(x,y,\tau)\bigr)| \tag{13.4.53}_\vartheta$$

(as $\vartheta > 0$) where $\tau = 0$; now all conditions should be checked in the vicinity of $\bar{\Psi}_0(\mathsf{supp}(\psi))$.

Estimates. II

Let us discuss the right-hand expression of our estimates. So far "remainder" means R^{T} as $\vartheta = 0$ and $\mathsf{R}^{\mathsf{T}}_{(\vartheta)}$ as $\vartheta > 0$; we will deal with the left-hand expression in the next subsection.

Remark 13.4.19. Let $Q' = Q'' = I$.

(i) Note that $\Omega_d = \emptyset$ provided

$$|\nabla \frac{V}{F}| \ge \epsilon_0 \tag{13.4.54}$$

and therefore for $\vartheta \ge 0$ remainder is $O\bigl(\mu^{-1-\vartheta}h^{-1+\vartheta}\bigr)$ (as $d = 2$) and $O\bigl(h^{-2+\vartheta}\bigr)$ (as $d = 3$)

Furthermore, in this case[21)] no partition is needed and condition (13.4.46) is fulfilled automatically, and one does not need condition (13.4.45).

[21)] And so far only in this case; in Subsection 13.8.1 we get rid of condition (13.4.45) in general.

(ii) On the other hand, let $d = 2$ and conditions $(13.3.77)_{1-3}$ are fulfilled and

$$(13.3.77)^*_3 \qquad |D^\alpha(\frac{V}{F})| \le c\nu \qquad \forall \alpha : 1 \le |\alpha| \le K$$

then $\Omega_2 = \emptyset$ as

$$(13.4.54)'_3 \qquad |\nabla \frac{V}{F}| \ge \epsilon_0 \nu$$

and

$$(13.4.55) \qquad \nu \ge \max\bigl(\mu h^{1-\delta}, C(\epsilon_0)\mu^{-2}\bar{\nu}\bigr).$$

Therefore for $\vartheta \ge 0$ remainder is $O\bigl(\mu^{-1-\vartheta} h^{-1+\vartheta} \nu^{1+\vartheta}\bigr)$.

(iii) As $d = 3$ automatically $\mathsf{mes}_4(\Omega_4) \le C(\mu h)^{\frac{1}{2}} h^{-\delta}$. Therefore estimate $O(h^{-2+\vartheta})$ is achieved as $\mu \le h^{-1/(3+2\vartheta)+\delta}$. In particular, as $\vartheta = 0$ remainder estimate $O(h^{-2})$ is achieved as $\mu \le h^{-\frac{1}{3}+\delta}$.

(iv) On the contrary, as $d = 2$ one always needs some nondegeneracy condition. Otherwise remainder estimate $O(\mu^{1+\vartheta} h^{-1+\vartheta})$ could not be improved.

(v) In particular, under non-degeneracy condition

$(13.4.56)_r$ $|\nabla(V/F)| \le \epsilon\nu$ implies that $\mathsf{Hess}(V/F)$ has r eigenvalues by absolute value larger than ϵ

$\mathsf{mes}_2(\Omega_2) \le C(\mu h/\nu)^{r/2} h^{-\delta}$. In particular, as $\vartheta = 0$, $\nu = 1$ we derive remainder estimate $O(\mu^{-1}h^{-1})$ as $\mu \le h^{-r/(r+4)+\delta}$.

(vi) Under non-degeneracy condition

$$(13.4.57)_m \qquad \sum_{1\le|\alpha|\le m} |D^\alpha(\frac{V}{F})| \ge \epsilon\nu$$

$\mathsf{mes}_2(\Omega_2) \le (\mu h/\nu)^\sigma h^{-\delta}$ with $\sigma = \sigma(m)$. Therefore, under these conditions we arrive to remainder estimate $O(\mu^{-1-\vartheta} h^{-1+\vartheta} \nu^{1+\vartheta})$ as $\mu \le h^{-\sigma'}$, $\nu \ge h^{\sigma'}$ with $\sigma' = \sigma'(m, \vartheta)$.

13.4.3 Calculations

Let us proceed from estimates to calculations. In this subsection $d \ge 2$ is arbitrary and the crucial statement is the following:

Proposition 13.4.20. *Let conditions* (13.2.1)–(13.2.5) *and* (13.4.1) *be fulfilled and let* $Q' = \mathsf{Op}(q')$, $Q'' = \mathsf{Op}(q'')$, $q', q'' \in \mathsf{S}_{h,\gamma,\gamma,K}$,

$$\gamma \ge (\mu h)^{\frac{1}{2}-\delta} \tag{13.4.58}$$

and $\psi', \psi'' \in \mathscr{C}_0^K(B(0,\frac{3}{4}))$, $\chi \in \mathscr{C}_0^\infty([-1,1])$. *Then*

(i) For $T \in [h^{1-\delta}, (\mu^{-1}h)^{\frac{1}{2}+\delta}]$ *the following estimate holds:*

$$\begin{aligned} &|F_{t\to h^{-1}\tau}\chi_T(t)\big(Q'_x\psi'(x)U\psi''(y)\,{}^tQ''_y\big)(x,x,t)- \\ &\sum_{m+n\le M} Th^{-1}\int \hat\chi\big(\frac{(\tau-\tau')T}{h}\big)\varkappa'_{mn}(x,\tau')\mu^m h^{1-d+m+n}\,d\tau'| \le \\ &\qquad C'h^{1-d}(\mu h)^s \qquad\qquad \forall x\in\mathbb{R}^d \quad \forall \tau : |\tau|\le c \end{aligned} \tag{13.4.59}$$

where κ_{mn} *depend also on* Q', Q''.

Further, for $L \in [(\mu h)^{1-\delta}, 1]$, $T' \ge h^{1-\delta}L^{-1}$ *the following estimate holds:*

$$\begin{aligned} &\Big|\int \phi\big(\frac{\tau}{L}\big)\big(F_{t\to h^{-1}\tau}\bar\chi_T(t)(Q'_x\psi'(x)U)(x,x,t)- \\ &\sum_{m+n\le M} \varkappa'_{mn}(x,\tau)\mu^m h^{1-d+m+n}\big)\,d\tau\Big| \le C'h^{1-d}(\mu h)^s \qquad \forall x\in\mathbb{R}^d \end{aligned} \tag{13.4.60}$$

where $\varkappa_{mn} \in \mathscr{C}^s(B(0,1), \mathscr{S}'(\mathbb{R}))$ *are supported in* $\{\tau \ge V(x)\}$, $\varkappa'_{mn} = \partial_\tau \varkappa_{mn}$ *and we assume that* $\chi, \bar\chi \in \mathscr{C}_0^K([-1,1])$ *are fixed,* $\bar\chi = 1$ *in* $[-\frac{1}{2}, \frac{1}{2}]$ *and* $\phi \in \mathscr{C}_0^K([-c,c])$ *is also fixed,* $\delta > 0$ *and* s *are arbitrary and* $K = K(d,\delta,s)$, $M = M(d,\delta,s)$.

Furthermore, terms in both decompositions become smaller as $(m+n)$ *increases; in particular, for* $\gamma = L = 1$ *in* (13.4.60) *one can take* $s = (M+1)$.

(ii) Moreover, if condition (13.4.45) *is fulfilled then for* $|\tau| \le \frac{1}{2}\epsilon_0$ *estimate* (13.4.59) *holds with every* $T \in [h^{1-\delta}, \mu^{-1}T_0]$ *where* $T_0 = T_0(d,c,\epsilon_0,\delta) > 0$ *is a small enough constant.*

In particular, if $\mathsf{diam}(\mathsf{supp}(q')) \le c\gamma$ *then under condition* (13.4.45) *the following estimate holds:*

$$|F_{t\to h^{-1}\tau}\chi_T(t)(Q'_x\psi'(x)U\psi''(y)\,{}^tQ''_y)(x,x,t)| \le C'h^{1-d}\gamma^{d-2} \qquad \forall x\in\mathbb{R}^d \quad \forall \tau : |\tau| \le \frac{1}{2}\epsilon_0 \tag{13.4.61}$$

where T *is as before.*

(iii) On the other hand, if condition (13.4.46) *is fulfilled then for* $\gamma = 1$ *estimate*

$$\Big|\int F_{t\to h^{-1}\tau}\chi_T(t)\big((Q'_x\psi'(x)U\psi''(y)\,{}^tQ''_y)(x,x,t) - \sum_{m+n\le M} Th^{-1}\int \hat\chi(\frac{(\tau-\tau')T}{h})\varkappa'_{mn}(x,\tau')\mu^m h^{1-d+m+n}d\tau'\big)\,dx\Big| \le C'h^{1-d}(\mu h)^s \qquad \forall \tau : |\tau|\le\epsilon \tag{13.4.62}$$

(which is due to (13.4.59)*) holds for* $\epsilon = \epsilon(d,c,\epsilon_0) > 0$ *with the same* T *as in Statement (ii).*

(iv) Furthermore,

$$\varkappa_{00} = (2\pi)^{-d}\int_{\{a(x,\xi)<\tau\}} Q'_0(x,\xi)q''_0(x,\xi)\psi'(x)\psi''(x)\,d\xi \tag{13.4.63}$$

where

$$a(x,\xi) = \sum_{j,k} g^{jk}(x)\big(\xi_j - \mu V_j(x)\big)\big(\xi_k - \mu V_k(x)\big) + V \tag{13.4.64}$$

and Q'_0, Q''_0 *denote the principal symbols of* Q', Q'' *in* (13.4.63).

(v) Finally, for $Q' = Q'' = I$

$$\varkappa_{00} = (2\pi)^{-d}\varpi_d(\tau - V)_+^{d/2}\sqrt{g}\psi'\psi'' \tag{13.4.65}$$

and

(13.4.66) $\varkappa_{mn} = 0$ *unless both* m *are* n *is even.*

Proof. (i) Let us rescale $x_{\mathsf{new}} = \mu(x - \bar{x})$; we then obtain the case in which h and μ are replaced by $h_{\mathsf{new}} = \mu h$ and $\mu_{\mathsf{new}} = 1$ respectively. In this case Statement (i) is due to Proposition 4.3.7; we obtain series with respect to powers of μh and h also (because the basic operator, cutting operators and functions depend on μ^{-1} (which is a coefficient of some copies of x). One can obtain the same results simply repeating constructions of Section 4.3 (the method of successive approximations) in which case no rescaling is necessary.

(ii) Moreover, an analysis based on Appendix 4.A.1 yields that under condition (13.4.45) and $\gamma = 1$ the coefficients $\varkappa'_{mn}$ are bounded functions, even regular with respect to $\tau \in [-\epsilon, \epsilon]$; this yields Statement (ii).

Similarly, the same arguments imply that under condition (13.4.46) and $\gamma = 1$ the coefficients $\varkappa'_{mn}$ integrated with respect to $x \in B(0,1)$ are bounded functions, even regular with respect to $\tau \in [-\epsilon, \epsilon]$; this yields Statement (iii).

(iii) Formulae (13.4.63) and (13.4.65) for $\varkappa_{00}$ are well known (see Chapter 4).

(iv) Let us note that replacing h and μ by $-h$ and $-\mu$ respectively we preserve the basic operator and therefore we also preserve $e(x,x,\tau)$; on the other hand, the functions

$$\int \phi(\frac{\tau}{L})(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)U(x,x,t))\,d\tau \tag{13.4.67}$$

and

$$h\int \phi(\frac{\tau}{L})\,d_\tau e(x,x,\tau) \tag{13.4.68}$$

coincide modulo $O(h^s)$ and hence for $Q' = Q'' = I$ all the terms in our decompositions with odd $(m-n)$ vanish. Really, the term with $m = n = 0$ does not vanish; obviously the factor h^{1-d} arises as $|h|^{1-d}$ through the Jacobian) [22)].

[22)] Alternatively, one can notice that in the method of successive approximation each factor h unaccompanied by μ comes from factor $(x_j - y_j)$ and leads to differentiation with respect to ξ_j which changes the parity of the term with respect to ξ. Meanwhile each factor μh comes from factor factor $(x_j - y_j)\xi_k$ and leads to differentiation with respect to ξ_j and multiplication by ξ_k and does not change the parity of the term. After integration with respect to ξ all odd terms disappear as $q' = q'' = 1$ (or are even with respect to ξ).

(v) Moreover, let us note that replacing μ by $-\mu$ we replace the basic operator by its complex conjugate and therefore we also preserve $e(x,x,\tau)$ in this case; since $e(x,x,\tau)$ is real-valued, all terms with odd n in our decompositions vanish for $Q'=Q''=I$. Statement (v) is also proven. □

Let us treat the terms with $n=0$ for $Q'=Q''=I$. The standard calculations of the method of successive approximations yield that for $x=\bar{x}$ these coefficients depend only on $g^{jk}(\bar{x})$, $\partial_k V_j(\bar{x})$ and $V(\bar{x})$ and therefore calculating these coefficients one can assume without any loss of the generality that g^{jk} and V are constants and V_j are linear functions.

Moreover, the same calculations yield that if g^{jk} and V are constants and V_j are linear functions then as $Q'=''=I$, $\psi'=\psi''=1$ only terms with $n=0$ will appear in our decompositions and since (13.4.67) and (13.4.68) coincide modulo $O(h^s)$ we obtain from Theorem 13.2.1 that

Proposition 13.4.21. *(i) In the framework of Proposition 13.4.20*

$$\mathcal{N}^{\mathsf{MW}}(x,\tau,\mu h) \sim \sum_{m\in\mathbb{Z}^+} \varkappa_{m0}(x)(\mu h)^m \tag{13.4.69}$$

in the weak sense[23] *where* $\mathcal{N}^{\mathsf{MW}}(x,\tau,\mu h)$ *defined by* (13.2.8) *with* $g^{jk}=g^{jk}(x)$, $V=V(x)$ *and* $F_{jk}=F_{jk}(x)$.

(ii) Thus in all statements and estimates of Proposition 13.4.20 one can replace sum of terms with $m\le M$, $n=0$ *by* $h^{-d}\mathcal{N}^{\mathsf{MW}}(x,\tau,\mu h)$.

We noted earlier that $\mathcal{N}^{\mathsf{MW}}(\tau,\mu h)$ can be treated as a Riemann sum for the function $\varphi(z)$ defined by (13.2.26):

$$\varphi(z)=(2\pi)^{r-d}2^{-r}\varpi_{d-2r}\bigl(\tau-f_1z_1-\cdots-f_rz_r\bigr)_+^{\frac{d-2r}{2}}f_1\cdots f_r\sqrt{g}$$

but this function was not smooth; however after multiplication by $\phi(\tau)$ and integration on $\mathbb{R}_\tau$ the resulting function will be smooth and hence we can obtain a complete decomposition with respect to powers of μh. Therefore we obtain

[23] I.e. this asymptotic decomposition will be true if we multiply it by $\phi\in\mathscr{C}_0^\infty(\mathbb{R})$ and integrate over $\mathbb{R}$; surely we proved this only for equality (13.4.69) differentiated with respect to τ but (13.4.69) is also true without differentiation because the supports of both sides are semi-bounded from below with respect to τ).

Proposition 13.4.22. *For* $Q' = Q'' = I$ *and* $\psi' = \psi'' = 1$

$$\varkappa_{m0} = \partial_\tau (\tau - V)_+^{\frac{d}{2}} \mathcal{P}_m(f_1^2, \dots, f_r^2)\sqrt{g} \tag{13.4.70}$$

where $\mathcal{P}_m$ *are homogeneous symmetric polynomials of degrees* $\frac{m}{2}$ *also depending on* d *and* r*. In particular,*

$$\varkappa_{20} = -c\partial_\tau^2 (\tau - V)_+^{\frac{d}{2}} F^2 \sqrt{g} \tag{13.4.71}$$

where $c = \frac{1}{3}(2\pi)^{r-d} 2^{-r-1} \varpi_{d-2r}$ *and*

$$F = \bigl(f_1^2 + \dots + f_r^2\bigr)^{\frac{1}{2}} = \frac{1}{2}\Bigl(\sum_{j,k,l,m} g^{jk} g^{lm} F_{jl} F_{km}\Bigr)^{\frac{1}{2}} \tag{13.4.72}$$

is the scalar intensity of the magnetic field.

Remark 13.4.23. Let us note that if Q' is an operator with Weyl symbol $q'(x,\xi)$ even with respect to ξ and h then Q is preserved if we replace h by $-h$ and for Q complex and Hermitian conjugation coincide. Therefore for this operator Q' and for $Q'' = I$, $\psi' = \psi'' = 1$ in a neighborhood of $\mathsf{supp}(q')$, $\varkappa_{mn}$ vanish unless m and n are even. Se also footnote [22)].

13.4.4 Synthesis

Let us plug the results of the previous subsection into Propositions 13.4.9, 13.4.12, 13.4.14, 13.4.15, 13.4.17 and Corollary 13.4.13 and take into account Remark 13.4.19.

Then we obtain the following

Proposition 13.4.24. *Let conditions* (13.2.1)–(13.2.5)*,* (13.3.1) *and* (13.4.1) *be fulfilled. Then*

(i) Under the condition (13.4.54) *estimate* (13.4.62) *holds with* $T = \mu T_0$ *for* $d = 2$ *and with* $T = T_0$ *for* $d = 3$*.*

(ii) Further, let $d = 2$*, conditions* $(13.3.77)_{1-3}$*,* (13.4.11)*,* (13.4.12) *and* (13.4.45) *be fulfilled.*

Let $\tilde{Q}' = \mathsf{Op}(\tilde{q}')$*,* $\tilde{Q}'' = \mathsf{Op}(\tilde{q}'')$*,* $\tilde{q}', \tilde{q}'' \in \mathsf{S}_{h_1,\gamma,\gamma,K}(T^*\mathbb{R})$*,*

$$1 \ge \gamma \ge \bar{\gamma} := \bigl(\frac{\mu h}{\nu}\bigr)^{\frac{1}{2}} h^{-\delta}, \tag{13.4.73}$$

condition $(13.4.31)_{(2)}$ *be fulfilled on* $\mathsf{supp}(\tilde{q}'\tilde{q}'')$ *and* $Q' = \mathcal{F}\tilde{Q}'\mathcal{F}^*$, $Q'' = \mathcal{F}\tilde{Q}''\mathcal{F}^*$.

Then estimate (13.4.62) *holds for* $\tau : |\tau| \le \epsilon\nu\gamma^2$ *with* $T = \mu T_0/\nu$ [24)]

(iii) On the other hand, let $d = 3$, $\tilde{Q}' = \mathsf{Op}(\tilde{q}')$, $\tilde{Q}'' = \mathsf{Op}(\tilde{q}'')$, $\tilde{q}', \tilde{q}'' \in \mathsf{S}_{\boldsymbol{h}',\gamma,\gamma,K}(T^*\mathbb{R}^2)$, *let conditions* (13.4.45) *and* (13.4.73) *be fulfilled with* $\nu = 1$ *and let condition* $(13.4.31)_{(3)}$ *be fulfilled on* $\mathsf{supp}(\tilde{q}'\tilde{q}'')$ *and* $Q' = \mathcal{F}\tilde{Q}'\mathcal{F}^*$, $Q'' = \mathcal{F}\tilde{Q}''\mathcal{F}^*$.

Then estimate (13.4.62) *holds with* $T = T_0$.

Combining this proposition with the standard Tauberian arguments we immediately arrive to the following two statements:

Theorem 13.4.25. *Let conditions* (13.2.1)–(13.2.5), (13.3.1), (13.4.1) *and* (13.4.54) *be fulfilled. Then*

(i) For $d = 2$ *estimates*

$$(13.4.74)_{(2)} \quad \mathsf{R}^{\mathsf{W}\prime} := \Big| \int \psi(x)\Big(e(x,x,-\infty,\tau) - h^{-d}\sum_{m\le M} \varkappa_{m0}(x,\tau)(\mu h)^m\Big)\,dx\Big| \le C\mu^{-1}h^{-1} + C'h$$

and

$$(13.4.75)_{(2)} \quad \mathsf{R}^{\mathsf{W}\prime}_{(\vartheta)} = \Big| \int \tau_+^{\vartheta-1} * \psi(x)\Big(e(x,x,-\infty,\tau) - h^{-d}\sum_{m\le M,n\le N} \varkappa_{mn}(x,\tau)(\mu h)^m h^n\Big)\,dx\Big| \le C(\mu h)^{-1+\vartheta} + C'h^s$$

hold for $|\tau| \le \epsilon$, $\vartheta > 0$ *where* M *and* N *depend on* ϑ *and restriction to* μ *here and below.*

(ii) for $d = 3$ *estimates*

$$(13.4.74)_{(3)} \qquad \mathsf{R}^{\mathsf{W}\prime} \le Ch^{-2} + C',$$

[24)] Thus T does not depend on γ satisfying (13.4.73), $(13.4.31)_{(2)}$.

and

$$(13.4.75)_{(3)} \qquad \mathsf{R}^{\mathsf{W}\prime}_{(\vartheta)} \le Ch^{-2+\vartheta} + C'h^{s},$$

hold for $|\tau| \le \epsilon$, $\vartheta > 0$.

Remark 13.4.26. (i) We dropped terms with $n \ge 1$ for $\vartheta = 0$ because they do not exceed the remainder estimate due to Proposition 13.4.20(v). Similarly we can drop some terms as $\vartheta > 0$.

(ii) On the other hand, all the derivatives of ψ could be moved to other factors and in virtue of conditions (13.4.45) integration the coefficients $\varkappa_{mn}$ by ξ makes these terms smooth.

Proposition 13.4.27. *Suppose conditions* (13.2.1)–(13.2.5), (13.3.1), (13.4.1) *and* (13.4.45) *are fulfilled.*

Let $\tilde{Q}' = \mathsf{Op}(\tilde{q}')$, $\tilde{q}' \in \mathsf{S}_{\boldsymbol{h}',\gamma,\gamma,K}(T^*\mathbb{R}^{d-1})$ *where* γ *satisfies* (13.4.73), $\mathsf{diam}(\mathsf{supp}(\tilde{q}')) \le c\gamma$ *and* $Q' = \mathcal{F}\tilde{Q}'\mathcal{F}^*$. *Then*

(i) Let $d = 2$ *and conditions* $(13.3.77)_{1-3}$, (13.4.11) *and* (13.4.12) *be fulfilled.*

Moreover, let condition $(13.4.31)_{(2)}$ *be fulfilled on* $\mathsf{supp}(\tilde{q}')$. *Then the estimates*

$$(13.4.76)_{(2)} \quad \mathsf{R}^{\mathsf{W}\prime\prime} := \Big| \int \Big((Q'_x\psi(x)e)(x,x,-\infty,0) - h^{-d} \sum_{m\le M, n\le N} \varkappa_{mn}(x,0)(\mu h)^m h^n \Big)\, dx \Big| \le C\nu\gamma^2\mu^{-1}h^{-1} + C'h^{s},$$

and

$$(13.4.77)_{(2)} \quad \mathsf{R}^{\mathsf{W}\prime\prime}_{(\vartheta)} := \Big| \int \tau_+^{\vartheta-1} * \Big((Q'_x\psi(x)e)(x,x,-\infty,\tau) - h^{-d} \sum_{m\le M, n\le N} \varkappa_{Q'\psi,mn}(x,\tau)(\mu h)^m h^n \Big) dx \Big| \le C\nu^{1+\vartheta}\gamma^2(\mu h)^{-1+\vartheta} + C'h^{s}$$

hold for $\tau = 0$.

(ii) Let $d = 3$, $\nu = 1$ *and condition* $(13.4.31)_{(3)}$ *be fulfilled on* $\mathsf{supp}(\tilde{q}'\tilde{q}'')$. *Then the estimates*

$$(13.4.76)_{(3)} \qquad \mathsf{R}^{\mathsf{W}\prime\prime} \le C\gamma^4 h^{-2} + C'h^{s},$$

and

$$\mathsf{R}^{\mathsf{W}\prime\prime}_{(\vartheta)} \le C\gamma^4 h^{-2+\vartheta} + C'h^s, \tag{13.4.77}_{(3)}$$

hold for $\tau = 0$.

Taking in account zone where $(13.4.31)_{(2)}$ or $(13.4.31)_{(3)}$ is violated we again arrive to the following

Proposition 13.4.28. *Suppose conditions* (13.2.1)–(13.2.5), (13.3.1), (13.4.1) *and* (13.4.45) *be fulfilled. Then*

(i) Let $d = 2$ *and let conditions* $(13.3.77)_{1-3}$, (13.4.11) *and* (13.4.12) *be fulfilled. Then estimate*

$$\mathsf{R}^{\mathsf{W}\prime}_{(\vartheta)} \le C\nu^{1+\vartheta}(\mu h)^{-1+\vartheta} + C\mu^{1+\vartheta}h^{-1+\vartheta}\,\mathsf{mes}_2(\Omega_2) + C'h^s \tag{13.4.78}_{(2)}$$

holds for $\tau = 0$, $\vartheta \ge 0$*; recall that* Ω_2 *was defined as a set violating* $(13.4.31)_{(2)}$.

(ii) For $d = 3$ *the estimate*

$$\mathsf{R}^{\mathsf{W}\prime}_{(\vartheta)} \le Ch^{-2+\vartheta} + C\mu^{1+\vartheta}h^{-2}(\mu h)^{\frac{1}{2}-\delta}\,\mathsf{mes}_3(\Omega_3) + C'h^s \tag{13.4.78}_{(3)}$$

holds for $\tau = 0$, $\vartheta \ge 0$ *where* Ω_3 *is* (x', ξ_1)*-projection of* Ω_4. *Recall that* Ω_4 *was defined as a set violating both* $(13.4.31)_{(3)}$ *and ellipticity.*

We leave it to the reader the following

Problem 13.4.29. Treat the remainder estimates after convolution with the function $\varphi(\tau L^{-1})$ where φ is either a smooth function or satisfies (4.2.75). In this case an additional factor $\big(h(LT)^{-1}\big)^{\vartheta}$ arises in the right-hand expression in $(13.4.74)_{(d)}$, $(13.4.76)_{(d)}$ with corresponding T while the right-hand expression in estimates $(13.4.78)_{(d)}$ will be $h^{1-d}\gamma^{2d-2}\min\big(\mu h/L)^{\vartheta}, 1\big)$ and we assume that $L \ge hT^{-1}$. This yields the obvious modifications of the further estimates.

13.4.5 Miscellaneous Modifications of $\mathsf{R}^{\mathsf{W}\prime}$

At the end of this section let us consider various modifications of $\mathsf{R}^{\mathsf{W}\prime}$. So $\vartheta = 0$ here. The case $\vartheta > 0$ is left to the reader, except that we want to note that for $\mu \le h^{-\sigma}$ with $\sigma = \sigma(\vartheta) > 0$ all the arguments below remain true.

First of all, for $d = 3$, $\mu \le h^{-\frac{1}{2}}$ and for $d = 2$, $\mu \le h^{-\frac{1}{3}}$ in estimate $(13.4.74)_{(d)}$ one can obviously take $M = 0$ and therefore replace $\mathsf{R}^{\mathsf{W}\prime}$ by genuine Weyl expression

$$\mathsf{R}^{\mathsf{W}} = |\int \psi(x)\Big(e(x,x,-\infty,0) - (2\pi)^{-d}\varpi_d h^{-d} V_-^{\frac{d}{2}}\sqrt{g}\Big)\,dx|. \tag{13.4.79}$$

Moreover,

(13.4.80) For $d = 2$ and condition (13.4.45) formula (13.4.70) yields that $\varkappa_{m0} = 0$ for $m \ge 1$ in which case $\mathsf{R}^{\mathsf{W}\prime}$ and R^{W} coincide.

Let us discuss the possibility of replacing $\mathsf{R}^{\mathsf{W}\prime}$ by

$$\mathsf{R}^{\mathsf{MW}} = |\int \psi(x)\Big(e(x,x,-\infty,0) - h^{-d}\mathcal{N}^{\mathsf{MW}}(x,0,\mu h)\Big)\,dx|. \tag{13.4.81}$$

Let us recall that equality (13.4.69) holds only in the weak sense. However, if condition (13.4.54) is fulfilled then after multiplication by $\psi'(x)$ and integration over $\mathbb{R}^d_x$ the function F given by (13.4.72) in the proof of Proposition 13.4.22 will be smooth with respect to (τ, z) and therefore the equality

$$\int \psi'(x)\mathcal{N}^{\mathsf{MW}}(x,\tau,\mu h)\,dx \equiv \sum_{0\le m\le M}\int \psi'(x)\varkappa_{m0}(x,\tau)(\mu h)^m\,dx \qquad \mathsf{mod}\ O(h^s) \tag{13.4.82}$$

holds in the strong sense for $|\tau| \le \epsilon$ and $M = M(d,s)$. Therefore

(13.4.83) Theorem 13.4.25 remains true for $\mathsf{R}^{\mathsf{W}\prime}$ replaced by R^{MW}.

Moreover, one can see easily that if $\psi' \in \mathscr{C}_0^K(B(0,1))$ with $\mathsf{diam}(\mathsf{supp}\,\psi') \le c\gamma$,

$$|D^\alpha \psi'| \le c\gamma^{-|\alpha|} \qquad \forall \alpha : |\alpha| \le K, \tag{13.4.84}$$

and if conditions (13.4.45),

$$|D^\alpha\big(\frac{V}{F}\big)| \le c\nu \qquad \forall \alpha : 1 \le |\alpha| \le K, \tag{13.4.85}$$

and

$$c\nu\gamma \ge |\nabla \frac{V}{F}| \ge \epsilon_0 \nu\gamma \qquad \text{on} \quad \operatorname{supp}\psi' \tag{13.4.86}$$

are fulfilled then

$$|D_z^\alpha \int \psi' \varphi(x,z,\tau)dx| \le C'\gamma^{d-2|\alpha|}\nu^{-|\alpha|} \tag{13.4.87}$$
$$\forall \alpha : |\alpha| \le K' \quad \forall \tau : |\tau| \le \epsilon\nu\gamma$$

for arbitrary K' (K depends on K') and therefore (13.4.84) holds.

On the other hand, if condition (13.4.86) is violated then the difference between the left-hand and right-hand expressions in (13.4.84) does not exceed $C\gamma^d\mu h$. Summing with respect to a partition of unity we then obtain that the estimate

$$|\mathsf{R}^{\mathsf{MW}} - \mathsf{R}^{\mathsf{W}\prime}| \le C\mu h^{-1} \operatorname{mes}\bigl(\{x\colon |\nabla \frac{V}{F}| \le \bar\gamma\}\bigr) + C'h \tag{13.4.88}_{(2)}$$

holds with $\bar\gamma = (\mu h\nu)^{\frac12-\delta}$.

The same (with the factor h^{-2} instead of h^{-1}) is true for $d = 3$, but we could rewriting the function $\varphi(x,z,\tau)$ in the form

$$\varphi(x,z,\tau) = \operatorname{const} \int \theta(\tau - V - Fz - \eta^2)\, d\eta\sqrt{g} \tag{13.4.89}$$

with $\eta \in \mathbb{R}$, $z \in \mathbb{R}$ and making an appropriate partition of unity with respect to (x,η) with condition (13.4.86) replaced by the condition

$$c\gamma \ge |\eta| + |\nabla \frac{V}{F}| \ge c^{-1}\gamma \qquad \text{at} \quad \operatorname{supp}\psi' \tag{13.4.86}'$$

we obtain the estimates

$$|\mathsf{R}^{\mathsf{MW}} - \mathsf{R}^{\mathsf{W}\prime}| \le \tag{13.4.88}_{(3)}$$
$$C\mu h^{-2} \operatorname{mes}_4\bigl(\{(x,\eta) : |\eta| + |\nabla \frac{V}{F}| \le \bar\gamma\}\bigr) + C'h \le$$
$$C'\mu^{\frac32} h^{-\frac32-\delta} \operatorname{mes}_3\bigl(\{x, |\nabla \frac{V}{F}| \le \bar\gamma\}\bigr) + C'h$$

with $\bar\gamma = (\mu h)^{\frac12-\delta}$.

Therefore Theorem 13.4.25, Propositions 13.4.22, 13.4.24, 13.4.27 and 13.4.28, Remarks 13.4.23 and 13.4.26 and formulae $(13.4.88)_{(2)}$ and $(13.4.88)_{(3)}$ yield the following

Theorem 13.4.30. *(i) In the framework of Theorem 13.4.25 estimates* $(13.4.74)_{(2)}$ *and* $(13.4.74)_{(3)}$ *hold for* R^{MW} *as well.*

(ii) In the framework of Proposition 13.4.28(i) with the additional assumption (13.4.85) *estimate*

$$(13.4.90)_{(2)} \quad \mathsf{R}^{\mathsf{MW}}_{(\vartheta)} \le C\nu^{1+\vartheta}\mu^{-1-\vartheta}h^{-1+\vartheta}+ C\mu^{1+\vartheta}h^{-1+\vartheta}\Big(\mathsf{mes}\big(\{x : |\nabla\frac{V}{F}| \le \bar\gamma\}\big) + \mathsf{mes}_2(\Omega_2)\Big) + C'h^s$$

holds with $\bar\gamma = (\mu h/\nu)^{\frac12-\delta}$ *but now we need to include in* $\mathsf{R}^{\mathsf{MW}}_{(\vartheta)}$ *also terms with* $n \ge 1$.

(iii) Then under non-degeneracy condition $(13.4.57)_m$

$$(13.4.91)_{(2)} \qquad \mathsf{R}^{\mathsf{MW}}_{(\vartheta)} \le C\nu^{1+\vartheta}\mu^{-1-\vartheta}h^{-1+\vartheta}$$

as $d = 2$, $\mu \le h^{-\delta}$.

(iv) Without any non-degeneracy condition

$$(13.4.91)_{(3)} \qquad \mathsf{R}^{\mathsf{MW}}_{(\vartheta)} \le Ch^{-2+\vartheta}$$

as $d = 3$, $\mu \le h^{-1/(2\theta+3)+\delta}$.

(v) In the frameworks of Propositions 13.4.27(i) and (ii) estimates $(13.4.74)_{(2)}$, $(13.4.74)_{(3)}$, *etc respectively hold for* R^{MW} *also.*

Remark 13.4.31. (i) We will prove that in many cases we can choose $\bar\gamma = (\mu h)^{\frac12}$ or $\bar\gamma = (\mu h/\nu)^{\frac12}$ and improve both estimates and restrictions to μ.

(ii) Our latest estimates invoke measures of two sets: one in x related to V/F and another in (x', ξ_1) related to $\mathcal{B}$. Under assumption $\nu\gamma \ge C\bar\nu\mu^{-2}$ they are the same.

Remark 13.4.32. One could apply successive approximations method with unperturbed operator

$$\bar A = \sum_{j,k} \bar P_j g^{jk}(y)\bar P_k + V(\bar y) \tag{13.4.92}$$

with

$$\bar P_j = hD_{x_j} - \mu A_j(y) - \mu\sum_k (\partial_k A_j)(y)\cdot(x_k - y_k). \tag{13.4.93}$$

We will follow this route in forthcoming chapters.

13.5 Moderate Magnetic Field

13.5.1 Decomposition of the Fundamental Solution and Propagation of Singularities

In this section we will always assume that $d = 2, 3$ and

$$h^{-\delta} \le \mu \le h^{-1} \qquad (\delta > 0). \tag{13.5.1}$$

Then the remainder term R in decomposition (13.3.65), (13.3.66) or (13.3.68) is negligible by Theorem 13.2.13 and Remark 13.4.2, and we skip it.

But then x_d and D_d are included in the operator $\mathcal{A}$ only through the harmonic oscillator $\mathcal{L}_0 = x_d^2 + \mu^{-2}h^2D_d^2$. Decomposing all functions of x_d into series with respect to eigenfunctions of $\mathcal{L}_0$ we decompose $\mathcal{A}$ into a direct sum of $\boldsymbol{h}'$-pseudodifferential operators with respect to x' which we treat by the methods of Chapter 4.

In particular, all of the analysis of Section 13.4 can be repeated with the weakened restrictions on γ and ζ.

Let us recall that

(13.5.2) Operator $\mathcal{L}_0$ has simple eigenvalues $(2j+1)\mu^{-1}h$ with eigenfunctions

$$\Upsilon_j(x_d) = (\frac{\mu}{h})^{\frac{1}{4}} \upsilon_j(\sqrt{\frac{\mu}{h}}x_d) \tag{13.5.3}$$

which are orthonormal in $\mathscr{L}^2(\mathbb{R})$ where υ_j are real-valued orthonormal Hermite functions and here and in what follows $j \in \mathbb{Z}^+$.

Theorem 13.5.1. *Let condition* (13.5.1) *be fulfilled.*

(i) For some $j \in \mathbb{Z}^+$ let

$$(2j+1)\mu h\bar{B}_1 + \bar{B}_0' \ge \bar{\tau} + \epsilon_0 \qquad \forall (x', \xi_1) \in \Omega \tag{13.5.4}$$

with $\epsilon_0 > 0$, $\bar{\tau} \le c$ where

$$\bar{B}_1 := B_1|_{\varepsilon=h_1=0}, \qquad \bar{B}_0' := (\varepsilon^{-1}B_1)|_{\varepsilon=h_1=0}, \tag{13.5.5}$$

Then estimate

$$|F_{t\to h^{-1}\tau}\chi_{T,T'}(t) \int Q_x'U(x,y,t)\Upsilon_j(x_d)\,dx_d| \le C'Th^s(1+j\mu h)^{-s}(1+\bar{\tau}-\tau)^{-s} \tag{13.5.6}$$

$$\forall x' \in \mathbb{R}^{d-1} \quad \forall y \in \mathbb{R}^d \quad \forall \tau \le \bar{\tau} \quad \forall T \ge h^{1-\delta}$$

holds with $Q' = I$ *where here and in what follows* $C' = C'(c, \epsilon_0, \delta, s)$, $N = N(\delta, s)$, $K = K(\delta, s)$ *and* $\delta > 0$ *and* s *are arbitrary.*

(ii) On the other hand, let $d = 3$, $Q' = \mathsf{Op}(q')$ *with* $q' \in \mathsf{S}_{\boldsymbol{h}',K}(T^*\mathbb{R}^2)$, $\mathsf{supp}(1 - q') \subset \{|\xi_2| \le c\}$ *and for some* $j \in \mathbb{Z}^+$ *let*

$$\xi_2^2 + (2j+1)\mu h\bar{B}_1 + \bar{B}_0' \ge \bar{\tau} + \epsilon_0 \qquad \forall (x', \xi') \in \mathsf{supp}\, q' \tag{13.5.7}$$

with the same constants ϵ_0 *and* $\bar{\tau}$*; then estimate* (13.5.6) *remains true.*

(iii) The following decomposition holds

$$\mathsf{U} \equiv \sum_{j,k \le J} \mathsf{U}_{j,k}(x', y', t)\Upsilon_j(x_d)\Upsilon_k(y_d) \tag{13.5.8}$$

modulo functions f *satisfying* (13.3.81) *with arbitrary* $q \in \mathsf{S}_{\boldsymbol{h}',K}(T^*\mathbb{R}^{d-1})$ *supported in* Ω *and* $Q = \mathsf{Op}(q)$*; here* $\bar{\tau} \le c$ *is arbitrary and*

$$J = J(\bar{\tau}) := \mathsf{min}\{j \in \mathbb{Z}^+ : (2j+1)\mu h\bar{B}_1 + \bar{B}_0' \ge \bar{\tau} + \epsilon_0 \quad \forall (x', \xi_1) \in \Omega\}. \tag{13.5.9}$$

(iv) Furthermore,

$$\mathsf{U}_{j,k}(x', y', t) = \mathsf{U}_{k,j}^{\dagger}(y', x', -t), \tag{13.5.10}$$

estimate

$$|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)\mathsf{U}_{j,k}| \le CT\mu h^{1-d}(1 + |\tau|)^{d-1} \quad \forall x', y' \in \mathbb{R}^{d-1} \quad \forall j, k \quad \forall \tau \quad \forall T \ge h^{1-\delta} \tag{13.5.11}$$

holds and $\mathsf{U}_{j,k}$ *satisfy Cauchy problems*

$$\big(hD_t - \mathcal{A}_{jx}\big)\mathsf{U}_{j,k} \equiv 0, \tag{13.5.12}$$

$$\mathsf{U}_{j,k}|_{t=0} \equiv \delta_{jk}\delta(x' - y') \tag{13.5.13}$$

modulo functions f *and* f' *respectively satisfying* (13.3.81) *and* (13.3.82) *with arbitrary* $q \in \mathsf{S}_{\boldsymbol{h}',K}$ *supported in* Ω *(*$d = 2$*) and in* $\Omega \times \{|\xi_2| \le c\}$ *(*$d = 3$*) where* δ_{jk} *is the Kronecker symbol and*

$$\mathcal{A}_j = \sum_{0 \le n \le N-1} B_n(x_1, h_1D_1, \mu^{-2}, h_1)\mu^{2-2n}\big((2j+1)\mu h\big)^n \tag{13.5.14}$$

for $d = 2$

and

$$\mathcal{A}_j = h^2 D_2^2 + \sum_{0\le n\le N-1} B_n(x', h_1 D_1, \mu^{-2}, h_1)\mu^{2-2n}\big((2j+1)\mu h\big)^n \quad \text{for } d=3; \tag{13.5.15}$$

let us recall that $h_1 = \hbar = \mu^{-1}h$.

(v) $\mathsf{U}_{j,k}$ *also satisfy dual problems*

$$\mathsf{U}_{j,k}\big(hD_t - {}^t\mathcal{A}_{ky}\big) \equiv 0, \tag{13.5.12}$$

and (13.5.13) *modulo functions* f *and* f' *respectively satisfying* (13.3.81)† *and* (13.3.82)† *dual*[25] *to* (13.3.81) *and* (13.3.82).

Proof. First of all (13.5.10) and (13.5.11) follow from (13.3.84) and (13.3.85) respectively if we set

$$\mathsf{U}_{j,k} = \iint \mathsf{U}(x,y,t)\Upsilon_j(x_d)\Upsilon_k(y_d)\,dx_d dy_d. \tag{13.5.16}$$

Consider $q' \in \mathsf{S}_{\boldsymbol{h},\rho,\gamma,K}(T^*\mathbb{R}^d)$ with $\rho_j, \gamma_j \in (0,1]$, $\rho_j\gamma_j \ge h_j h^{-\delta}$ $\forall j \le d-1$, $\rho_d = \mu^{\frac12} h^{\frac12-\delta'}$, $\gamma_d = \mu^{-\frac12} h^{\frac12-\delta'}$, $\delta' > 0$. One can prove easily by standard elliptic arguments that if

$$\operatorname{supp} q' \cap \{\mu^2 x_2^2 + \xi_2^2 \le h^{-\delta''}\} = \emptyset \quad \text{for } d=2, \tag{13.5.17}_{(2)}$$

$$\operatorname{supp} q' \cap \{\mu^2 x_3^2 + \xi_3^2 \le h^{-\delta''},\ \xi_2^2 \le C_0\} = \emptyset \quad \text{for } d=3 \tag{13.5.17}_{(3)}$$

with $\delta'' > 0$ then for $f = \mathsf{U}$, $q = Q'$ estimate (13.3.81) holds; in this deduction we also cut U (see Remark 13.3.12).

This yields that for $j \ge J' = \mu^{-1}h^{-1-\delta''}$ estimate (13.5.6) holds and therefore decomposition (13.5.8) with J' instead of J holds.

Then formulae (13.3.79), (13.3.80) and (13.3.65), (13.3.66) yield (13.5.12), (13.5.13) and (13.5.14), (13.5.15) respectively.

Furthermore, (13.5.12), (13.5.11) and the standard elliptic arguments yield the implications (13.5.4) $\implies$ (13.5.6) and (13.5.7) $\implies$ (13.5.6); therefore in decomposition (13.5.8) one can replace J' by J given by formula (13.5.9).

Finally, (13.5.10) and Statement (iii) imply Statement (v) and that one can skip $k \ge J$ in summation as well. □

[25] Where here and below "dual" means that $Q_x f$ is replaced by $f\,{}^tQ_y$ and j and k are permuted everywhere save $\boldsymbol{U}_{j,k}$.

We can now apply all the methods of Section 2.1, Chapter 4 and Section 13.4 to the analysis of $\mathsf{U}_{j,k}$; moreover, when we apply the methods of Section 13.4 we can weaken restrictions on γ and ζ associated with the uncertainty principle for $\boldsymbol{h}$-pseudodifferential operators because we now have only $\boldsymbol{h}'$-pseudodifferential operators and therefore condition (13.4.6) should be replaced by

$$\boldsymbol{h}' = (h_1, \dots, h_{d-1}),\ \boldsymbol{\gamma}' = (\gamma_1, \dots, \gamma_{d-1}),\ \boldsymbol{\rho}' = (\rho_1, \dots, \rho_{d-1}), \quad h_1 = \mu^{-1}h,\ h_2 = h,\ \rho_j, \gamma_j \in (0,1],\ \rho_j\gamma_j \ge h_j h^{-\delta} \quad \forall j = 1, \dots, d-1. \tag{13.5.18}$$

Proposition 13.5.2. *Let $d = 2$, $Q' = \mathsf{Op}(q')$, $Q'' = \mathsf{Op}(q'')$ with $q', q'' \in \mathsf{S}_{h_1,\rho_1,\gamma_1,K}(T^*\mathbb{R})$ and let condition* (13.5.18) *be fulfilled. Then*

(i) If

$$\operatorname{supp} q' \cap \bigl\{\tau_1 \le (2j+1)\mu h + \bar{B}(x_1,\xi_1) \le \tau_2\bigr\} = \emptyset \tag{13.5.19}$$

with $-\infty \le \tau_1 < \tau_2 \le \infty$ [26] *then estimate*

$$|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)Q'_x\mathsf{U}_{j,k}| \le C'Th^s\bigl(1+\min(\tau-\tau_1,\tau_2-\tau)\bigr)^{-s} \quad \forall x', y' \in \mathbb{R}^{d-1} \quad \forall \tau \in [\tau_1+\epsilon, \tau_2-\epsilon] \quad \forall T \ge h^{1-\delta} \tag{13.5.20}$$

and the holds (and due to (13.5.10) *dual estimate holds).*

(ii) For $h^{1-\delta} \le T \le \mu T_0$ estimate

$$|F_{t\to h^{-1}\tau}\chi_T(t)\mathsf{U}_{j,k}| \le C'h^s \qquad \forall x', y' \in \mathbb{R}^{d-1} \quad \forall \tau \le c \quad \forall j \ne k \tag{13.5.21}$$

holds.

(iii) If $h^{1-\delta} \le T \le \mu T_0$ and one of the conditions $(13.4.7)_{1,2}$ *is fulfilled then estimate*

$$|F_{t\to h^{-1}\tau}\chi_T(t)Q'_x\mathsf{U}_{j,k}{}^tQ''_y| \le C'h^s \qquad \forall x', y' \in \mathbb{R}^{d-1} \quad \forall \tau \le c \tag{13.5.22}$$

holds.

[26] Here and in what follows B is taken from (13.3.74) and as in (13.5.5) $\bar{B} = B|_{\varepsilon=h_1=0}$.

(iv) Moreover, if conditions $(13.3.77)_{1-3}$, (13.4.11) *and* (13.4.12) *are fulfilled then in Statement (i) we can replace* ϵ *by* $\epsilon\nu$ *and in Statements (ii) and (iii) we can take* $T \in [h^{1-\delta}\nu^{-1}, \mu T_0\nu^{-1}]$, $|\tau| \le c\nu$ *and replace conditions* $(13.4.7)_{1,2}$ *by* $(13.4.7)'_2$.

Proposition 13.5.3. *Let* $d = 3$, $Q' = \mathsf{Op}(q')$, $Q'' = \mathsf{Op}(q'')$ *with* $q', q'' \in \mathsf{S}_{\boldsymbol{h}',\boldsymbol{\rho}',\gamma',K}(T^*\mathbb{R}^2)$ *and let condition* (13.5.18) *be fulfilled. Then*

(i) If

$$\text{(13.5.23)}\quad \operatorname{supp} q' \cap \bigl\{\tau_1 \le \xi_2^2 + (2j+1)\mu h\bar{B})_1(x',\xi_1) + \bar{B}'_0(x',\xi_1) \le \tau_2\bigr\} = \emptyset$$

with $-\infty \le \tau_1 < \tau_2 \le \infty$ *then estimate* (13.5.20) *holds.*

(ii) For $h^{1-\delta} \le T \le T_0$ *estimate* (13.5.21) *holds.*

(iii) If $h^{1-\delta} \le T \le T_0$ *and one of conditions* $(13.4.7)_{1-4}$ *is fulfilled then estimate* (13.5.22) *holds.*

Proof of Propositions 13.5.2 and 13.5.3. The proof of both Statements (i) is based on the standard elliptic arguments, and Statements (ii) and (iii) follow from the theory of propagation of singularities (see Section 2.1 and proof of Theorem 13.4.3) □

13.5.2 Calculations

Let us proceed from estimates to calculations[27)]. First, we need to replace U by its transformation $\boldsymbol{U}$ (and in the end, by $\boldsymbol{U}_{j,k}$ and we want to get the expression of the same type as before.

Proposition 13.5.4. *Let* $d = 2$ *and* $Q' = \mathsf{Op}(q')$, $Q'' = \mathsf{Op}(q'')$ *with* $q', q'' \in \mathsf{S}_{h_1,\rho_1,\gamma_1,K}(T^*\mathbb{R})$. *Let condition* (13.5.18) *be fulfilled and let* $\tilde{Q}' = \mathcal{F}Q'\mathcal{F}^*$, $\tilde{Q}'' = \mathcal{F}Q''\mathcal{F}^*$. *Further, let* $\psi', \psi'' \in \mathscr{C}_0^K(B(0, \frac{1}{2}))$ *be fixed functions. Then*

[27)] More refined calculations will be made in the next subsection.

(i) The following equality holds modulo $O(h^s)$

$$(13.5.24)_{(2)}\quad F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\big(\tilde{Q}'_x\psi'(x)U\psi''(y)\,{}^t\tilde{Q}''_y\big)\equiv$$
$$\sum_{j\le J}\sum_{r+m+p\le N} F_{t\to h^{-1}\tau}\chi_T(t)\Gamma'\big(Q'_x\mathsf{M}_{rmp}(x_1,y_1,h_1D_{x_1},h_1D_{y_1})\mathsf{U}_{j,j}\,{}^tQ''_y\big)\times$$
$$((2j+1)\mu h)^r\,\mu^{-2(m+r)-p}h^p$$

for $\tau\le\epsilon_1$, $T\in[h^{1-\delta},\mu T_0]$ *where* J *is defined by* (13.5.9) *with* $\bar\tau=0$ *and* $\epsilon=2\epsilon_1$, Γ' *is the operator of restriction to the diagonal and integration acting with respect to the variables* x' *and* y', M_{rmp} *are operators with symbols*

$$(13.5.25)_{(2)}\quad \mathsf{M}_{rmp}(x_1,y_1,\xi_1,\eta_1)=\sum_{|\alpha|+|\beta|\le 2p+2r+m}\mathsf{m}_{rmp\alpha\beta}(x_1,y_1,\xi_1,\eta_1)\times$$
$$(D^\alpha\psi')\circ\bar\Psi_0(x_1,\xi_1)\times(D^\beta\psi'')\circ\bar\Psi_0(y_1,\eta_1),$$

$\mathsf{m}_{rmp\alpha\beta}$ *are smooth symbols independent of* ψ',ψ'', $\mathsf{m}_{rmp\alpha\beta}=1$ *for* $m=p=r=\alpha=\beta=0$ *and* $\mathsf{m}_{rmp\alpha\beta}=0$ *for* $m+p+r>0$, $\alpha=\beta=0$.

(ii) Moreover, if conditions $(13.3.77)_{1-3}$, (13.4.11) *and* (13.4.12) *are fulfilled then equality* $(13.5.24)_{(2)}$ *holds for* $|\tau|\le\epsilon_1\nu$, $T\in[h^{1-\delta}\nu^{-1},\mu T_0\nu^{-1}]$.

(iii) Further, if $\psi''=1$ *in a neighborhood of* $\operatorname{supp}\psi'$ *then one can take* M_{rmp} *depending only on* (x_1,ξ_1) *and* $\mathsf{m}_{\dots}=0$ *for* $\beta\ne 0$.

(iv) In particular, if $F=\mathsf{const}$ *and we apply the reduction method described in Proposition 13.3.1(iii) then in the conformal coordinate system*

$$(13.5.26)_1\qquad \mathsf{M}_{010}=-\frac{1}{2}(\rho^2\nabla\psi'\cdot\nabla v)\circ\bar\Psi_0,$$

and

$$(13.5.26)_2\qquad \mathsf{M}_{100}=\frac{1}{4}(\rho^2\Delta\psi')\circ\bar\Psi_0.$$

(v) Furthermore, if conditions $(13.3.77)_{1-3}$ *are fulfilled then*

$$(13.5.27)\qquad |D^\sigma\mathsf{m}_{rmp\alpha\beta}|\le C\bar\nu$$

for all r,m,p,α,β *indicated above and for all* $\sigma:1\le\sigma\le K_1$ *with arbitrary* K_1 *(*K *depends on* K_1*).*

Proposition 13.5.5. *Let $d = 3$ and $Q' = \mathsf{Op}(q')$, $Q'' = \mathsf{Op}(q'')$ with $q', q'' \in \mathsf{S}_{\boldsymbol{h}',\rho',\gamma',K}(T^*\mathbb{R}^2)$. Let condition* (13.5.18) *be fulfilled and let $\tilde{Q}' = \mathcal{F}Q'\mathcal{F}^*$, $\tilde{Q}'' = \mathcal{F}Q''\mathcal{F}^*$ and $\psi', \psi'' \in \mathscr{C}_0^K(B(0, \frac{1}{2}))$ be fixed functions. Then*

(i) The following equality holds modulo $O(h^s)$:

$$(13.5.24)_{(3)} \quad F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\big(\tilde{Q}'_x\psi'(x)U\psi''(y)\,{}^t\tilde{Q}''_y\big) \equiv$$
$$\sum_{j\le J}\ \sum_{r+m+p+k'+k''\le N} F_{t\to h^{-1}\tau}\chi_T(t)\Gamma'\big(Q'_x\mathsf{M}_{rmpk'k''}(x', y', \boldsymbol{h}'D_{x'}, \boldsymbol{h}'D_{y'})\times$$
$$(hD_{x_2})^{k'}(hD_{y_2})^{k''}\mathsf{U}_{j,j}\,{}^tQ''_y\big)\cdot((2j+1)\mu h)^r\,\mu^{-2(m+r)-p-k'-k''}h^p$$

for $\tau \le \epsilon_1$, $T \in [h^{1-\delta}, T_0]$ with the same J and Γ' as before, $\mathsf{M}_{rmpk'k''}$ operators with symbols

$$(13.5.25)_{(3)} \quad \mathsf{M}_{rmpk'k''}(x', y', \xi', \xi'') =$$
$$\sum_{|\alpha|+|\beta|\le 2p+2r+m+k'+k''} \mathsf{m}_{rmpk'k''\alpha\beta}(x', y', \xi_1, \eta_1)\times$$
$$(D^\alpha\psi')\circ\bar{\Psi}_0(x', \xi_1)\times(D^\beta\psi'')\circ\bar{\Psi}_0(y', \eta_1),$$

where $\mathsf{m}_{rmpk'k''\alpha\beta}$ are smooth symbols independent of ψ', ψ'' and $\mathsf{m}_{rmpk'k''\alpha\beta} = 1$ for $m = p = r = k' = k'' = \alpha = \beta = 0$ and $\mathsf{m}_{rmpk'k''\alpha\beta} = 0$ for $m + p + r + k' + k'' > 0$, $\alpha = \beta = 0$.

(ii) Further, if $\psi'' = 1$ in a neighborhood of $\operatorname{supp}\psi'$ then one can take M_{rmp} depending only on (x_1, ξ_1) and $\mathsf{m}_{rmpk'k''\alpha\beta} = 0$ for $(\beta, k') \ne 0$.

(iii) Statement (v) of Proposition 13.5.4 remains true. moreover, it is true for $k' + k'' > 0$ and $\sigma = 0$.

Proof of Propositions 13.5.4 and 13.5.5. Let us first consider the operators $\mathcal{Q}'(x, \mu^{-1}hD, \mu^{-2}, \mu^{-1}h) = \mathcal{F}^*\psi'\mathcal{F}$ and $\mathcal{Q}''$ (which is defined in a similar way and has similar properties).

We know that

$$(13.5.28) \qquad \mathcal{Q}' \sim \sum_p \mathcal{Q}'_p(x, \mu^{-1}hD, \mu^{-2})(\mu^{-1}h)^p,$$

where $\mathcal{Q}'_0 = \psi'\circ\Psi_{\mu^{-2}}$ and $\mathcal{Q}'_p$ is a linear combination of $D^\alpha\psi'\circ\Psi_{\mu^{-2}}$ with $1 \le |\alpha| \le 2p$ for $p \ge 1$.

Let us decompose $\mathcal{Q}'_p$ with respect to powers of μ^{-2}. Moreover, let us decompose $\mathcal{Q}'_p$ with respect to powers of x_d and $\mu^{-1}hD_d$; for $d = 3$ let us also decompose $\mathcal{Q}'_p$ with respect to powers of $\mu^{-1}hD_2$. Then Remark 13.4.2 and condition (13.5.1) yield that we need to take only a finite number of terms provided we apply the operator $\mathcal{Q}'_x$ or ${}^t\mathcal{Q}''_y$ to U and then we apply operator $F_{t\to h^{-1}\tau}\chi_T(t)$ with $\tau \le c$.

Therefore for $d = 3$

$$\mathcal{Q}' \sim \sum_{m,\sigma,k,p} \mathcal{Q}'_{mp\sigma k}(x', \mu^{-1}hD_1)\mu^{-2m-p-k}h^p x_3^{\sigma_1}(\mu^{-1}hD_3)^{\sigma_2}(hD_2)^k \tag{13.5.29}$$

where $\mathcal{Q}'_{0000} = \psi \circ \bar{\Psi}_0$ and $\mathcal{Q}'_{mp\sigma k}$ is a linear combination of $D^\alpha\psi' \circ \bar{\Psi}_0$ with $1 \le |\alpha| \le m + 2p + |\sigma| + k$ for $m + 2p + |\sigma| + k \ge 1$.

It will be convenient for us to pass from this formula to the formula

$$\mathcal{Q}' \sim \sum_{m,\sigma,k,p} \mathcal{Q}'_{mp\sigma k}(x', \mu^{-1}hD_1)\mu^{-2m-p-k}h^p(\mu^{-1}h)^{\frac{1}{2}|\sigma|}Z_+^{\sigma_1}Z_-^{\sigma_2}(hD_2)^k \tag{13.5.30}$$

with operators[28)]

$$Z_\pm = \sqrt{\frac{\mu}{h}}\bigl(x_d \pm i\mu^{-1}hD_d\bigr) \tag{13.5.31}$$

and with coefficients $\mathcal{Q}'_{mp\sigma k}$ of the same nature as before; then $[Z_+, Z_-] = 2$. A similar but simpler formula holds for $d = 2$.

Let us apply operators $F_{t\to h^{-1}\tau}\mathcal{Q}'_x$ and ${}^t\mathcal{Q}''_y$ to U and let us take decomposition (13.5.8) for U; let us take into account that Proposition 13.5.2 yields that only terms with $j = k \le J$ are not necessarily negligible in this decomposition with the T and τ indicated in Proposition 13.5.4(i),(ii) for $d = 2$ and in Proposition 13.5.5(i) for $d = 3$.

Furthermore, let us apply equalities

$$(z + \partial_z)v_j = \sqrt{2j}v_{j-1}, \qquad (z - \partial_z)v_j = \sqrt{2(j+1)}v_{j+1}, \tag{13.5.32}_{1-2}$$

for our Hermite functions.

We obtain that in the resulting formula for the left-hand expression in $(13.5.24)_{(d)}$ terms with multiindices σ' and σ'' from the decompositions of

[28)] Often called annihilation and creation operators in quantum mechanics.

$\mathcal{Q}'$ and $\mathcal{Q}''$ are negligible unless $\sigma'_1 - \sigma'_2 = \sigma''_1 - \sigma''_2$; therefore $|\sigma'| + |\sigma''|$ is even.

Moreover, the inner product in $\mathscr{L}^2(\mathbb{R})$ of functions $Z_+^{\sigma'_1} Z_-^{\sigma'_2} \Upsilon_j$ and $Z_+^{\sigma''_1} Z_-^{\sigma''_2} \Upsilon_j$ is a $\frac{1}{2}(|\sigma'| + |\sigma''|)$-degree polynomial of $(2j+1)$; this yields $(13.5.24)_{(d)}$ for the indicated T and τ. The same calculations also yield $(13.5.25)_{(d)}$.

Therefore Proposition 13.5.4(i),(ii) and Proposition 13.5.5(i) are proven. Further, the same arguments yield Propositions 13.5.4(ii) and 13.5.5(ii).

Furthermore, the same arguments yield Propositions 13.5.4(v) and 13.5.5(iii). In fact, if $d = 3$, $g^{jk} = \delta_{jk}$, $F^1 = F^3 = 0$ and $F^2 = \mathsf{const}$, $V = \mathsf{const}$ then in the framework of the proof of Theorem 13.2.1

$$\mathcal{Q}' = \psi'(x_1 - \mu^{-1} h\ell D_3, x_2, x_3 - \mu^{-1} h\ell D_1)$$

with constant ℓ; a similar equality holds for $d = 2$, $g^{jk} = \delta_{jk}$ and $F = \mathsf{const}$, $V = \mathsf{const}$. Therefore in these cases $\mathsf{m}_{\dots} = \mathsf{const}$; moreover, $\mathsf{m}_{\dots} = 0$ for $d = 3$ and $k' + k'' \ge 1$. Then the stability of all the procedures yields Propositions 13.5.4(v) and 13.5.5(iii).

Finally, in order to prove Proposition 13.5.4(iv) it is sufficient to note that under indicated method of reduction

$$\mathsf{M}_{100} = \frac{1}{2}\Big(\{x_2, \{x_2, \mathcal{Q}'\}\} + \{\xi_2, \{\xi_2, \mathcal{Q}'\}\}\Big)|_{\Lambda^0, \varepsilon=0} = \frac{1}{2}\Big(\{w''_2, \{w''_2, \psi'\}\} + \{w''_1, \{w''_1, \psi'\}\}\Big)\Big|_{\Lambda, \varepsilon=0} \circ \bar{\Psi}_0, \tag{13.5.33}$$

and

$$\mathsf{M}_{010} = \partial_\varepsilon \psi'|_{\Lambda, \varepsilon=0} \circ \bar{\Psi}_0 \tag{13.5.34}$$

where w''_1, w''_2 are defined in the proof of Proposition 13.3.1(iii); the remaining arguments are the same as in the proof of Theorem 13.3.6(iii). □

13.5.3 Estimates

Now we are able to apply the results of Chapters 4 and 5 in their complete power.

Estimates in the Zone $\{\gamma \ge \bar{\gamma}\}$

Let us recall that in the weak magnetic field theory there was a zone where gradient of the principal symbol exceeded $\bar{\gamma} = (\mu h)^{\frac{1}{2}} h^{-\delta}$ or $\nu\bar{\gamma}$ with

$\bar\gamma = (\mu h/\nu)^{\frac12} h^{-\delta}$ as $d = 2$ and some conditions are fulfilled. Our next step is to recover similar results albeit with $\delta = 0$.

Proposition 13.5.6. *Let either* $d = 2$ *or* $d = 3$ *and let* $q', q'', Q', Q'', \tilde Q', \tilde Q''$ *be the same symbols and operators as in Proposition 13.5.4 or Proposition 13.5.5 respectively with* $\rho_j = \gamma_j = \gamma \ge \bar\gamma$ *and*

$$\bar\gamma = \bigl(\mu h \nu^{-1}\bigr)^{\frac12}. \tag{13.5.35}$$

Let $\operatorname{diam}(\operatorname{supp} q') \le c\gamma$, $\operatorname{diam}(\operatorname{supp} q'') \le c\gamma$ *and suppose that on* $\operatorname{supp}(q')$ *conditions* $(13.4.26)_{(d)}$ *and* $(13.4.31)_{(d)}$ [29] *be fulfilled where* $\nu = 1$ *for* $d = 3$.

Furthermore, let conditions $(13.3.77)_{1-3}$, (13.4.11) *and* (13.4.12) *be fulfilled as* $d = 2$ *and* $\nu < 1$. *Then the estimate*

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma \tilde Q'_x \psi'(x) U \psi''(y)\, {}^t\tilde Q''_y| \le Ch^{1-d}\gamma^{2d-2} + C'h^s \qquad \forall \tau : |\tau| \le \epsilon\nu\gamma \tag{13.5.36}$$

holds with arbitrary $T \in [h^{1-\delta}\nu^{-1}\gamma^{-2}, \mu T_0 \nu^{-1}]$ *for* $d = 2$, $T \in [h^{1-\delta}\gamma^{-2}, T_0]$ *for* $d = 3$*; recall that* $T_0 > 0$ *is a small constant.*

Proof. The arguments of Subsections 5.2.1 and s 5.2.2 yield that

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma' Q'_x \mathsf{U}_{j,j}\, {}^t Q''_y| \le C\mu h^{2-d}\gamma^{2d-4}\nu^{-1} + C'h^s \tag{13.5.37}$$

for the T and τ indicated above and for all $j \in \mathbb{Z}^+$. In fact, we should replace d by $(d-1)$ in all the corresponding estimates and remember that the semiclassical parameter with respect to (x_1, ξ_1) is $h_1 = \mu^{-1}h$; however the semiclassical parameter for t is h or even $h\nu^{-1}$ so we need to rescale $t : t_{\mathsf{new}} = \mu^{-1}\nu t_{\mathsf{old}}$ which brings a factor $\mu\nu^{-1}$ to the right-hand expressions.

Further, for $d = 3$ we should add another factor μ in the right-hand expressions because of the additional $\mu^{-1}\gamma$-admissible partition of unity with respect to ξ_1 or x_1 (which is necessary in order to be in the context of this subsection when its arguments are applied).

Furthermore, if condition

$$|-\bar\tau + (2j+1)\mu h + B(x_1, \xi_1, \mu^{-2})| \ge \epsilon_0 \nu\gamma^2, \tag{13.5.38}_{(2)}$$

[29] Let us recall that in these conditions for $d = 3$ we take λ violating condition $(13.4.25)_{(3)}$ with ζ replaced by $\epsilon\gamma^2$.

or

$$|-\bar\tau+\xi_2^2+\mathcal{B}(x',\xi_1,(2j+1)\mu h,\mu^{-2},0)|\ge\epsilon_0\gamma^2, \tag{13.5.38}_{(3)}$$

is fulfilled on $\mathsf{supp}(q')$ for $d=2,3$ respectively (compare these conditions with $(13.4.25)_{(2,3)}$) then for $|\bar\tau-\tau|\le\epsilon\nu\gamma^2$ and for indicated T the left-hand expression in (13.5.37) does not exceed $C'h^s$.

Finally, for every τ from the indicated interval the quantity of numbers j violating condition $(13.5.38)_{(d)}$ does not exceed $C_0\nu\gamma^2(\mu h)^{-1}$ because $C_0\nu\gamma^2(\mu h)^{-1}\gtrsim 1$ iff $\gamma\gtrsim\bar\gamma$ as $\bar\gamma$ is defined by (13.5.36).

This fact, estimate (13.5.37) and Propositions 13.5.4 and 13.5.6 yield that the left-hand expression of (13.5.36) does not exceed

$$C(\mu^{-1}h)^{d-2}\gamma^{2d-4}\times\mu\nu^{-1}\times\mu^{d-2}\times\nu\gamma^2(\mu h)^{-1} \tag{13.5.39}$$

which is exactly the right-hand expression of it. □

Now applying the standard Tauberian arguments of Chapter 4 we immediately obtain that under the assumptions of Proposition 13.5.6 estimates

$$\begin{aligned}\mathsf{R}^{\mathsf{T}} := |\Gamma\big(\tilde Q_x'\psi'(x)e\psi''(y)\,{}^t\tilde Q_y''\big)(-\infty,\tau)-\\ h^{-1}\int_{-\infty}^{\tau}\big(F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma\big(\tilde Q_x'\psi'(x)U\psi''(y)\,{}^t\tilde Q_y''\big)\big)\,d\tau|\le R_{1(0)}\end{aligned} \tag{13.5.40}$$

and

$$\mathsf{R}^{\mathsf{T}}_{(\vartheta)}\le R_{1(\vartheta)} \tag{13.5.40}_{(\vartheta)}$$

hold for $\vartheta>0$, $|\tau|\le\epsilon\nu\gamma$ where for $\vartheta>0$ $\mathsf{R}^{\mathsf{T}}_{(\vartheta)}$ is the convolution of "R^{T} without absolute value" with $\tau_+^{\vartheta-1}$ and

$$R_{1(\vartheta)}=\begin{cases}Ch^{\vartheta-1}\gamma^2\big(\dfrac{\nu}{\mu}\big)^{\vartheta+1}+C'h^s & \text{for } d=2,\\ Ch^{-2+\vartheta}\gamma^4+C'h^s & \text{for } d=3,\end{cases} \tag{13.5.41}$$

for $\vartheta\ge 0$, $T=\mu\nu^{-1}T_0$ for $d=2$, $T=T_0$ for $d=3$ and $\bar\chi=1$ on $[-\frac12,\frac12]$. While we can do better than this, it would have no implication for the final result.

We can apply Proposition 13.5.4 or 13.5.5 to reduce the calculation of the second term in the middle expression in (13.5.40) to the calculation of $F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma' Q'_x \mathsf{M}\mathsf{U}_{j,j}\,{}^t Q''_y$ where actually we have the sum of such expressions with different Q', Q'' and M.

Further, we can calculate these expressions by means of the methods of Section 4.3. The fact that $\mathsf{M} = \mathsf{M}(x', y', \boldsymbol{h}' D_{x'}, \boldsymbol{h}' D_{y'})$ is an an obstacle. We conclude that

$$h^{-1}\int_{-\infty}^{\tau} \bigl(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(\tilde{Q}'_x\psi'(x)U\psi''(y)\,{}^t\tilde{Q}''_y)\bigr)\,d\tau \equiv \sum_{j\le J}\sum_{m,p,l\le N} \mu^{1-2m-2p-l}h^{1+l-d}\bigl((2j+1)\mu h\bigr)^m \Theta_{j,mpl}(\tau) \tag{13.5.42}$$

with $\Theta_{j,mpl}$ defined below and thus arrive to

Proposition 13.5.7. *(i) Under the assumptions of Proposition 13.5.6 the following estimates hold:*

$$\mathsf{R}^{\mathsf{MW}}_{\infty} := |\Gamma(\tilde{Q}'_x\psi'(x)e)(-\infty,\tau) - \sum_{j\le J}\sum_{k,p,l\le N} \mu^{1-2m-2p-l}h^{1+l-d}\bigl((2j+1)\mu h\bigr)^m \Theta_{j,kpl}(\tau)| \le R_{1(0)}, \tag{13.5.43}$$

$$\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)} \le R_{1(\vartheta)} \tag{13.5.43$_{(\vartheta)}$}$$

where for $\vartheta > 0$ $\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)}$ *is the convolution of "*$\mathsf{R}^{\mathsf{MW}}_{\infty}$ *without absolute value" with* $\tau_+^{\vartheta-1}$ *and*

$$\Theta_{j,kpl}(\tau) := \int \theta\bigl(\tau - \mathcal{A}_j(x',\xi',\mu^{-2},\mu^{-1}h,h^2)\bigr) f_{kpl}(x',\xi')\,dx'd\xi' \tag{13.5.44}$$

with symbols f_{kpl} *such that*

$$\operatorname{supp} f_{kpl} \subset \operatorname{supp} q', \tag{13.5.45}$$

$$|D^{\alpha}_{x',\xi'} f_{kpl}| \le C\gamma^{-|\alpha|-2l} \qquad \forall \alpha : |\alpha| \le K', \tag{13.5.46}$$

where K' *is arbitrary and*

$$f_{000} = (2\pi)^{1-d} q'(x',\xi')(\psi'\circ\bar{\Psi}_0)(x',\xi_1). \tag{13.5.47}$$

(ii) In particular, for $d = 2$ *and* $\tau = 0$ *we can replace in* (13.5.44) *symbol* $\mathcal{A}_j$ *by*

$$\bar{\mathcal{A}}_j := \bigl((2j+1)\mu h + B(x_1, \xi_1, \mu^{-2}, \mu^{-1}h, h^2)\bigr) \tag{13.5.48}$$

Using map $\bar{\Psi}$ one can rewrite integral defining $\Theta_{...}$ as integral with respect to x as $d = 2$ and as integral with respect to (x, η) with $\eta = \langle \boldsymbol{F}^0, \xi\rangle$ as $d = 3$. In comparison wit $\bar{\Theta}_{...}$ introduced in the similar way arguments of θ contain many terms we consider as perturbation which explains subscript "∞" in our notations.

Our next goal is to estimate $\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)}$ with $Q' = I$. In this case for $d = 3$, $\mu \ge h^{-1+\delta}$, $\vartheta = 0$ the desired estimate will be more precise under conditions $(13.3.77)_{1-3}$, (13.4.85), and (13.4.86).

In order to use this condition let us prove

Proposition 13.5.8. *Let* $d = 3$ *and conditions* (13.4.85), (13.4.85) *and* (13.4.86) *be fulfilled with* $\nu \ge \mathsf{max}(\mu h, \mu^{-2})$.

Let $Q' = \mathsf{Op}(q')$, $q' \in \mathsf{S}_{\boldsymbol{h}',\boldsymbol{\rho}',\boldsymbol{\gamma}',K}(T^*\mathbb{R}^2)$ *with* $\rho_1 = \gamma_1 = \gamma_2 = \bigl(\mu h/\nu\bigr)^{\frac{1}{2}}$, $\rho_2 = (\mu h)^{\frac{1}{2}}$. *Let* $\mathsf{diam}(\mathsf{supp}\, q') \le c\gamma_1$ *and let the following conditions be fulfilled on* $\mathsf{supp}(q')$*:*

$$|\xi_2| \le c\rho_2, \tag{13.5.49}$$

$$|\mathcal{A}_j| \le \epsilon_0 \mu h \implies \epsilon_0 \bigl(\mu h \nu\bigr)^{\frac{1}{2}} \le |\nabla_{x',\xi_1} \mathcal{A}_j| \le c\bigl(\mu h \nu\bigr)^{\frac{1}{2}}. \tag{13.5.50}$$

Then

$$\mathsf{R}^{\mathsf{MW}}_{\infty} \le C\mu h^{-1}\gamma_1^4 + C' h^s \qquad \forall \tau : |\tau| \le \epsilon \mu h. \tag{13.5.51}$$

Proof. First of all, let us apply an $\epsilon(\boldsymbol{\rho}', \boldsymbol{\gamma}')$-admissible partition of unity on $\mathsf{supp}(q')$; we conclude that without any loss of the generality one can assume that condition $|\mathcal{A}_j| \ge \epsilon\mu h$ can be violated on $\mathsf{supp}(q')$ for only one number $j = \bar{\jmath}$. But then the standard elliptic arguments yield estimates

$$|F_{t\to h^{-1}\tau}\chi_T(t) Q'_x \mathsf{U}_{j,k}| \le C' h^s \qquad \forall j \ne \bar{\jmath},$$

and

$$|F_{t\to h^{-1}\tau}\chi_T(t) \mathsf{U}_{j,k}\, {}^{t}Q'_y| \le C' h^s \qquad \forall k \ne \bar{\jmath}$$

with arbitrary $T \in [\mu^{-1}h^{-\delta}, h^{-l}]$, $\tau : |\tau| \le \epsilon\mu h$ and therefore for these T and τ estimate $(13.5.24)_{(3)}$ holds.

Let us multiply the operator $\mathcal{A}_j$ with $j = \bar{j}$ by a parametrix of $\bar{B}_1$ and by ν^{-1}; we then obtain a $(\mu^{-1}h, h\nu^{-\frac{1}{2}})$-pseudodifferential operator with respect to x' and we can apply the arguments of Chapter 5; in the estimates we obtain an additional factor $\mu\nu^{-\frac{1}{2}}$ in the right-hand expressions because we need an additional $\gamma_1\mu^{-1}\nu^{\frac{1}{2}}$-partition of unity with respect to ξ_1 or x_1. □

Let us note that under the assumptions of Proposition 13.5.8 (excluding condition (13.5.50)) we can replace ν by an arbitrary $\nu' \in [\nu, 1]$ and therefore we can make an appropriate partition of unity in the zone

$$\{(x', \xi_1) : \ \max\big(\mu h, \ (\mu h\nu)^{\frac{1}{2}}\big) \le |\nabla_{x',\xi_1}\mathcal{A}_j| \le (\mu h)^{\frac{1}{2}}\} \tag{13.5.52}$$

where $j \in \mathbb{Z}^+$ is defined in order to satisfy condition $|\mathcal{A}_j| \le \epsilon\mu h$ (in which case the choice of j is unique).

Let us note that for $d = 2$ Proposition 13.4.7 leaves only zone

$$\{(x_1, \xi_1) : \ |\nabla B| \le \epsilon_0(\mu h\nu)^{\frac{1}{2}}\} \tag{13.5.53}$$

uncovered, and for $d = 3$ Proposition 13.4.8 leaves only zone

$$\{(x', \xi') : \ |\xi_2| \le \epsilon_0(\mu h)^{\frac{1}{2}}, |\nabla_{x',\xi_1}\mathcal{A}_j| \le \epsilon_0(\mu h\nu)^{\frac{1}{2}}\}$$

uncovered where $j \in \mathbb{Z}^+$ is always taken by the indicated method.

Moreover, for $d = 3$, $\nu < 1$, both these propositions together leave only zone

$$\{(x', \xi') : \ |\xi_2| \le \epsilon_0(\mu h)^{\frac{1}{2}}, |\nabla_{x',\xi_1}\mathcal{A}_j| \le \epsilon_0(\mu h\nu)^{\frac{1}{2}}\} \tag{13.5.54}$$

uncovered.

Let us sum with respect to a partition of unity the remainder estimates obtained in Propositions 13.5.7 and 13.5.8. We arrive to

$$\int R_{1(\vartheta)}\gamma^{-d-1}\, d\gamma \tag{13.5.55}$$

with $R_{1(\vartheta)} = R_{1(\vartheta)}(\gamma)$ defined by (13.5.41).

As $d = 3$ essential term in $R_{1(\vartheta)}(\gamma)$ is $Ch^{-2+\vartheta}\gamma^4$ and (13.5.55) is $Ch^{-2+\vartheta}$ (modulo negligible). Therefore we arrive to

Proposition 13.5.9. *Let $d = 3$.*

(i) Suppose conditions (13.4.85), (13.5.47) *with $\bar{\nu} = 1$ and $\nu \ge \mu h$ are fulfilled and let $Q'' = \mathsf{Op}(q'')$ where $q'' \in \mathsf{S}_{\boldsymbol{h}',\boldsymbol{\rho}',\bar{\gamma},K}(T^*\mathbb{R}^2)$ with $\rho_1 = \bar{\gamma}$ with $\bar{\gamma}$ defined by* 13.5.35) *$\rho_2 = (\mu h)^{\frac{1}{2}}$ and*

$$
\begin{aligned}
(13.5.56)\quad & \operatorname{supp} q'' \subset \bigl\{(x',\xi') : \ |\xi_2| \le c(\mu h)^{\frac{1}{2}}, |\nabla_{x',\xi_1}\mathcal{A}_j| \le c(\mu h)^{\frac{1}{2}}\bigr\},\\
& \qquad q'' = 1 \quad \text{at} \quad \bigl\{(x',\xi') : \ |\xi_2| \le \epsilon_0(\mu h)^{\frac{1}{2}}, |\nabla_{x',\xi_1}\mathcal{A}_j| \le \epsilon_0(\mu h\nu)^{\frac{1}{2}}\bigr\}
\end{aligned}
$$

where j is defined for given (x',ξ') by the method indicated above. Then for $Q' = I - Q''$, $\vartheta = 0$ estimate

$$
(13.5.57)_0 \qquad \mathsf{R}^{\mathsf{MW}}_{\infty} \le Ch^{-2} + C'h^{s}
$$

holds.

(ii) Furthermore, for $\nu = \mu h$ we obtain $\bar{\gamma} = 1$ and condition (13.5.56) *means exactly that*

$$
\begin{aligned}
(13.5.56)' \quad & \operatorname{supp} q'' \subset \bigl\{(x',\xi') : \ |\xi_2| \le c(\mu h)^{\frac{1}{2}}\bigr\},\\
& \qquad q'' = 1 \quad \text{at} \quad \bigl\{(x',\xi') : \ |\xi_2| \le \epsilon_0(\mu h)^{\frac{1}{2}}\bigr\}.
\end{aligned}
$$

(iii) Finally, let $Q'' = \mathsf{Op}(q'')$ where $q'' \in \mathsf{S}_{\boldsymbol{h}',\boldsymbol{\rho}',\bar{\gamma},K}(T^*\mathbb{R}^2)$ with $\rho_1 = \rho_2 = \bar{\gamma} = (\mu h)^{\frac{1}{2}}$. Let condition (13.5.56) *with $\nu = 1$ be fulfilled, $Q' = I - Q''$. Then estimate*

$$
(13.5.57)_{(\vartheta)} \qquad \mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)} \le Ch^{-2+\vartheta} + C'h^{s}
$$

holds for $\vartheta > 0$.

As $d = 2$ situation is more subtle as essential term in $R_{1(\vartheta)}(\gamma)$ is $Ch^{\vartheta-1}\nu^{\vartheta+1}\mu^{-\vartheta-1}\gamma^2$ and (13.5.55) is $Ch^{\vartheta-1}\nu^{\vartheta+1}\mu^{-\vartheta-1}(|\log\bar{\gamma}| + 1)$ (modulo negligible). However we will need some non-degeneracy condition anyway and one easily can get rid of logarithmic factor under weak non-degeneracy condition

$$
(13.5.58)_m \qquad \sum_{1\le|\alpha|\le m} |D^{\alpha}B| \ge \epsilon_0\nu \qquad \forall (x_1,\xi_1).
$$

Proposition 13.5.10. *Let $d = 2$.*

(i) Let conditions $(13.3.77)_{1-3}$, (13.5.12), (13.5.13) [30] *and* (13.4.85) *be fulfilled.*

Let $Q'' = \mathsf{Op}(q'')$ where $q'' \in \mathsf{S}_{h_1,\bar\gamma,\bar\gamma,K}(T^*\mathbb{R})$ with $\bar\gamma$ defined by (13.5.47). *Moreover, let us assume that*

$$\mathsf{supp}\, q'' \subset \bigl\{(x_1,\xi_1):\ |\nabla B| \le c(\mu h\nu)^{\frac12}\bigr\}, \qquad q''=1 \quad \text{at}\ \ \bigl\{(x_1,\xi_1):\ |\nabla B| \le \epsilon_0(\mu h\nu)^{\frac12}\bigr\}. \tag{13.5.59}$$

Then for $Q' = I - Q''$, $\vartheta \ge 0$ estimate

$$\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)} \le C\nu^{1+\vartheta}\mu^{-\vartheta-1}h^{-1+\vartheta}\bigl(|\log(\mu h/\nu)|+1\bigr) + C'h^s \tag{13.5.60}$$

holds.

(ii) Under non-degeneracy condition $(13.5.58)_m$ *estimate*

$$\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)} \le C\nu^{1+\vartheta}\mu^{-\vartheta-1}h^{-1+\vartheta} + C'h^s \tag{13.5.61}$$

holds.

Now, after summation with respect to partitions we conclude that

Corollary 13.5.11. *Let all the above conditions be fulfilled and let $\mu h \le \nu \le 1$, $\vartheta \ge 0$.*

(i) Let $d = 2$. Then contributions of zone, complimentary to (13.5.59), *to $\mathsf{R}^{\mathsf{T}}_{(\vartheta)}$ and $\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)}$ do not exceed $C\mu^{-1-\vartheta}h^{-1+\vartheta}\nu^{1+\vartheta}(|\log(\mu h/\nu)|+1)$ and under non-degeneracy condition* $(13.5.58)_m$ *these contributions do not exceed $C\mu^{-1-\vartheta}h^{-1+\vartheta}\nu^{1+\vartheta}$.*

(ii) Let $d = 3$. Then contributions of zone, complimentary to (13.5.56), *to $\mathsf{R}^{\mathsf{T}}_{(\vartheta)}$ and $\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)}$ do not exceed $Ch^{-2+\vartheta}$.*

Estimates in the Zone $\{\gamma \le \bar\gamma\}$

Now we need to consider contributions of zones (13.5.59) as $d = 2$ and (13.5.56) as $d = 3$ to $\mathsf{R}^{\mathsf{T}}_{(\vartheta)}$ and thus to $\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)}$.

[30] For $\nu \le \mu h$ this statement is empty.

Remark 13.5.12. We can apply the same arguments as before with the exception that the number of indices affected does not exceed C (or even 1) now instead of $C\nu\gamma^2(\mu h)^{-1}$ and therefore estimate (13.5.39) of the contribution of one partition element to the remainder loses the last factor and becomes

$$C(\mu^{-1}h)^{d-2}\gamma^{2d-4}\times \mu\nu^{-1}\times \mu^{d-2}. \tag{13.5.39$'$}$$

However it makes a difference as now power of γ is $(2d-4)$ and $(d-4)$ after division by γ^d it becomes negative. Therefore we need to apply more subtle arguments.

Let us consider $d=2$ first. Note that if q'' is $\bar\gamma$-admissible symbol with $\mathsf{diam}(\mathsf{supp}\, q'')\le \epsilon\bar\gamma$ then all operators $\mathcal{A}_j$ with $j\ne \bar\jmath$ are elliptic and thus contributions of all indices $j\ne\bar\jmath$ to R^{T} and $\mathsf{R}^{\mathsf{MW}}_{\infty}$ are negligible.

Then we can apply arguments of Section 5.2. We arrive to

Proposition 13.5.13. *Let $d=2$ and let all above conditions be fulfilled. Then*

(i) Let $\vartheta=0$ and a weak non-degeneracy condition $(13.5.58)_m$ *be fulfilled.*

Then contributions of zone (13.5.59) *to* R^{T} *and* $\mathsf{R}^{\mathsf{MW}}_{\infty}$ *do not exceed $Ch^{-\delta}+C'h^s$ with arbitrarily small exponent $\delta>0$.*

(ii) Let $\vartheta=0$ and a strong non-degeneracy condition

$$|\nabla B|\le \epsilon_0\nu \implies |\det \mathsf{Hess}\, B|\ge \epsilon_0\nu^2. \tag{13.5.62}$$

be fulfilled. Then contributions of zone (13.5.59) *to* R^{T} *and* $\mathsf{R}^{\mathsf{MW}}_{\infty}$ *do not exceed $C+C'h^s$.*

(iii) Let $\vartheta>0$. Then contributions of zone (13.5.59) *to* $\mathsf{R}^{\mathsf{T}}_{(\vartheta)}$ *and* $\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)}$ *do not exceed $C\nu^{\vartheta}\mu^{-\vartheta}h^{\vartheta}$.*

Proof. As we mentioned, proof follows up arguments of Section 5.2: Statements 13.5.13 and (ii) follow them up to Subsection 5.2.4 and Statement (iii) follows them up to Subsection 5.2.1 or Subsection 5.2.2 as $\det \mathsf{Hess}\, B \gtrless 0$ respectively.

We leave details to the reader. □

Remark 13.5.14. The same arguments allow us to get rid of logarithmic factor and derive estimate (13.5.61) for contribution of zone complimentary to (13.5.59) as $\vartheta > 0$ even without $(13.5.58)_m$.

Consider $d = 3$ now. Note that if q'' is $\bar{\gamma}$-admissible symbol with $\mathsf{diam}(\mathsf{supp}\, q'') \le \epsilon\bar{\gamma}$ and $\mathsf{supp}\, q'' \subset \{|\xi_2| \le \epsilon(\mu h)^{\frac{1}{2}}\}$ then all operators $\mathcal{A}_j$ with $j \ne \bar{\jmath}$ are elliptic and thus contributions of all indices $j \ne \bar{\jmath}$ to R^{T} and $\mathsf{R}^{\mathsf{MW}}_{\infty}$ are negligible.

Again following arguments of Section 5.2 we arrive to

Proposition 13.5.15. *Let $d = 3$ and all above conditions be fulfilled. Then*

(i) Let $\vartheta = 0$. Then contributions of zone (13.5.56) *to R^{T} and $\mathsf{R}^{\mathsf{MW}}_{\infty}$ do not exceed $C\mu h^{-1-\delta} + C'h^s$ with arbitrarily small exponent $\delta > 0$.*

(ii) Let $\vartheta = 0$ and weak non-degeneracy condition

$$(13.5.63)_m \qquad \sum_{|\alpha|\le m} |D^{\alpha}\mathcal{B}_j| \ge \epsilon_0\nu \qquad \forall(x', \xi_1)$$

be fulfilled. Then contributions of zone (13.5.56) *to R^{T} and $\mathsf{R}^{\mathsf{MW}}_{\infty}$ do not exceed $C\mu h^{-1} + C'h^s$.*

(iii) Let $\vartheta > 0$. Then contribution of zone (13.5.56) *to $\mathsf{R}^{\mathsf{T}}_{(\vartheta)}$ and $\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)}$ do not exceed $Ch^{-2+\vartheta} + C'h^s$.*

Proof. As we mentioned, proof follows up arguments of Section 5.2 up to Subsection 5.2.3.

Factor μh^{-1} appears as $\hbar^{-1}$. Again we leave all easy details to the reader. □

Synthesis

Combining Corollary 13.5.11(i) with Remark 13.5.14 and Proposition 13.5.13 we arrive to the following

Theorem 13.5.16. *Let $d = 2$, $Q' = I$, $\tau = 0$. Let conditions* $(13.3.77)_{1-3}$, (13.4.11) *and* (13.4.12) *be fulfilled. Let $\nu \ge C'\bar{\nu}\mu^{-2}$. Then*

(i) Under weak non-degeneracy condition $(13.4.57)_m$ *estimate*

$$\mathsf{R}^{\mathsf{MW}}_{\infty} \le C\frac{\nu}{\mu h} + Ch^{-\delta} + C'h^s \tag{13.5.64}$$

holds.

(ii) Under strong non-degeneracy condition

$$|\nabla \frac{V}{F}| \le \epsilon_0\nu \implies |\det \mathsf{Hess}\, \frac{V}{F}| \ge \epsilon_0\nu^2 \tag{13.5.65}$$

estimate

$$\mathsf{R}^{\mathsf{MW}}_{\infty} \le C\frac{\nu}{\mu h} + C + C'h^s \tag{13.5.66}$$

holds.

(iii) As $\vartheta > 0$ *estimate*

$$\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)} \le C\nu^{1+\vartheta}\mu^{-1-\vartheta}h^{-1+\vartheta} + C\nu^{\vartheta}\mu^{-\vartheta}h^{\vartheta} + C'h^s \tag{13.5.67}$$

holds.

Combining Corollary 13.5.11(ii) and Proposition 13.5.15 we arrive to the following

Theorem 13.5.17. *Let* $d = 2$, $Q' = I$, $\tau = 0$. *Then*

(i) Under non-degeneracy condition $(13.4.57)_m$ *estimate*

$$\mathsf{R}^{\mathsf{MW}}_{\infty} \le Ch^{-2} + C'h^s \tag{13.5.68}$$

holds.

(ii) In the general case estimate

$$\mathsf{R}^{\mathsf{MW}}_{\infty} \le Ch^{-2} + C\mu h^{-1-\delta} + C'h^s \tag{13.5.69}$$

holds with arbitrarily small exponent $\delta > 0$*; in particular, as* $\mu \le h^{-1+\delta}$ *estimate* (13.5.68) *holds.*

(iii) As $\vartheta > 0$ *estimate*

$$\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)} \le Ch^{-2+\vartheta} + C'h^s \tag{13.5.70}$$

holds.

Synthesis: Estimates with Mollifications

Finally, Propositions 13.5.4 and 13.5.5 and the same arguments of Section 5.2 yield the following

Proposition 13.5.18. *Let φ satisfy condition* (4.2.73)–(4.2.75) *with $\vartheta(z)$ replaced by z_+^ϑ with $\vartheta > 0$.*

(i) In the framework of Theorem 13.5.16 estimate

$$\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)L} := \Big| \int \phi\big(\frac{\tau-\tau'}{L}\big) d_{\tau'}\big((\Gamma\psi(x)e)(-\infty,\tau') - \sum_{j\le J} \sum_{m,p,l,n\le N, n\le 2l} \mu^{1-2(m+p)-l} h^{l-1} ((2j+1)\mu h)^m \Theta^{(n)}_{j,mpln}(\tau')\big)\Big| \le CR_1\big(\frac{h\nu}{\mu L}\big)^\vartheta \qquad \forall \tau : |\tau| \le c\nu \quad \forall L \in [\nu h\mu^{-1}, c] \tag{13.5.71}$$

holds where $\Theta_{j,mpln}(\tau)$ are given by (13.5.43) *with f_{mpln} instead of f_{mpl} with $f_{mpln} \in \mathscr{C}^{K'}$ and $f_{mpl0} = f_{mpl}$, the superscript (n) means the n-th order derivative with respect to the spectral parameter, and here and below R_1 is the right-hand expression in the corresponding estimate for $\mathsf{R}^{\mathsf{MW}}_\infty$ (in the general case or under some additional condition).*

(ii) In the framework of Theorem 13.5.17 estimate

$$\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)L} \le C' R_1\big(\frac{h}{L}\big)^\vartheta \qquad \forall \tau \le c \quad \forall L \in [h, c] \tag{13.5.72}$$

holds.

(iii) In particular, for $\phi \in \mathscr{C}_0^K(\mathbb{R})$ and $L \ge h^{1-\delta}\nu\mu^{-1}$, $L \ge h^{1-\delta}$ estimate $\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)L} = O(h^s)$ holds without any additional conditions.

Remark 13.5.19. (i) For odd integers ϑ one should also include in $\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)}$ an extra term possibly regularized.

(ii) For $d = 3$ the extra term appears only if the condition

$$|\frac{V}{F} + (2j+1)\mu h| + |\nabla \frac{V}{F}| \le \epsilon_0 \implies \mathsf{Hess}\,\frac{V}{F} \text{ has an eigenvalue } f \ge \epsilon_0 \tag{13.5.73}_+$$

is violated in which case it does not exceed $C\mu h^{-1+\vartheta}|\log h|$ (so it is negligible for $\mu \le h|\log h|^{-1}$ anyway).

(iii) For $d=2$ the extra term appears only if the condition

$$(13.5.74)_+ \qquad |\frac{V}{F}+(2j+1)\mu h|+|\nabla\frac{V}{F}|\le \epsilon_0\nu \implies \det \mathsf{Hess}\,\frac{V}{F}\ge \epsilon_0\nu^2$$

is violated in which case it does not exceed $C\nu\vartheta\mu^{-\vartheta}h^{\vartheta}|\log \mu^{-1}h|$ (so it is negligible for $\nu \ge \mu h|\log \mu^{-1}h|$ anyway).

(iv) The reader can easily prove that one can replace the eigenvalues of the skew-Hessian of the reduced symbol by $\pm\sqrt{-\zeta_k}$ where ζ_k are the eigenvalues of the matrix (V^j_l) with $V^j_l=\sum_k g^{jk}\partial_{kl}(V/F)$.

13.5.4 Effective Asymptotics

General Scheme

The asymptotics obtained in Theorems 13.5.16 and 13.5.17 are optimal in a reasonable sense, but the answer is given by ansatz in the right hand expression of (13.5.42)

$$(13.5.75) \qquad \sum_{j\le J}\sum_{m,p,l\le N}\mu^{1-2m-2p-l}h^{1+l-d}\big((2j+1)\mu h\big)^m\Theta_{j,mpl}(\tau)$$

with

$$(13.5.44) \quad \Theta_{j,mpl}(\tau):=\int \theta\big(\tau-\mathcal{A}_j(x',\xi',\mu^{-2},\mu^{-1}h,h^2)\big)f_{mpl}(x',\xi')\,dx'd\xi'$$

and we cannot effectively calculate all the terms of the decompositions of $\mathcal{A}_j$ and amplitudes $f_{\dots}$. Therefore our purpose through the end of this section is to obtain simpler (though more special and maybe not so optimal) asymptotics and estimates.

The good candidate is the standard Magnetic Weyl expression and our main goal is to estimate

$$(13.5.76) \qquad \mathsf{R}^{\mathsf{MW}}=|\int \psi(x)\Big(e(x,x,-\infty,\tau)-h^{-d}\mathcal{N}^{\mathsf{MW}}(x,\tau,\mu h)\Big)\,dx|$$

and

$$(13.5.77) \quad \mathsf{R}^{\mathsf{MW}}_{(\vartheta)}= |\iint (\tau-\tau')^{\vartheta}_+\psi(x)\,d_{\tau'}\Big(e(x,x,-\infty,\tau')-h^{-d}\mathcal{N}^{\mathsf{MW}}(x,\tau',\mu h)\Big)\,dx|$$

where $\mathcal{N}^{\mathsf{MW}}(x,\tau,\mu h)$ is given by $(13.2.9)_{(2)}$ and $(13.2.9)_{(3)}$ for $d=2,3$ respectively.

We deduce them from Theorems 13.5.16 and 13.5.17, Proposition 13.5.18 and the results of Section 13.4 by means of a certain universal approach which will be described right now.

Let us skip all the terms with $m+p+l\ge 1$ in formula (13.5.75) and replace $\mathcal{A}_j(x',\xi',\mu^{-2},\mu^{-1}h,h^2)$ by $\bar{\mathcal{A}}_j(x',\xi')=\mathcal{A}_j(x',\xi',0,0)$ in (13.5.44).

Furthermore, let us integrate with respect to ξ_2 for $d=3$. So we obtain an integral over Λ^0 in both cases $d=2,3$.

Then going from an integral with respect to $(x',\xi_1)\in\Lambda^0$ to an integral with respect to $x\in\Lambda$ (by means of the symplectomorphism $\bar\Psi_0$) and taking into account formulas (13.5.47) with $\psi'=\psi$, $Q'=I$ and

$$\bar{\mathcal{A}}_j=\bigl((2j+1)\mu hF+V\bigr)\circ\bar\Psi_0 \tag{13.5.78}$$

(which follows from Proposition 13.3.1 and the definition of $\mathcal{A}_j$) we obtain that expression (13.5.75) becomes precisely

$$h^{-d}\int \psi(x)\mathcal{N}^{\mathsf{MW}}(x,\tau,\mu h)\,dx. \tag{13.5.79}$$

Therefore we need to consider an error (13.5.75)–(13.5.79)

$$\mathcal{J}:=\sum_{j\le J}\mu h^{1-d}\Bigl(-\bar\Theta_{j,000}(\tau)+\sum_{r+m+p+l\le N} h^{l+r}\mu^{-2(m+p)-l}\bigl((2j+1)\mu h\bigr)^m\Theta_{j,mplr}(\tau)\Bigr) \tag{13.5.80}$$

where here and in what follows the functions $\bar\Theta_{j,\dots}(\tau)$ are given by (13.5.44) with $\bar{\mathcal{A}}_j$ instead of $\mathcal{A}_j$.

Let us note that if we introduce $z=(2j+1)\mu h$ and formally apply formula $(13.A.3)_1$ (replacing the sum with respect to j by an integral with respect to z divided by $2\mu h$) then under condition (13.4.46) we obtain for $|\tau|\le\epsilon$ that

$$\mathcal{J}=h^{-d}G_0(\tau,\mu^{-1})+h^{1-d}G_1(\tau,\mu^{-1})+O(h^{2-d}) \tag{13.5.81}$$

with functions $G_k(\tau,\zeta)$ smooth in the neighborhood of $(0,0)$; here we have integrated with respect to $z\in(0,C_0)$ and for $z\in(\frac13 C_0,3C_0)$ the integrand

vanishes. Here condition (13.4.46) provides the smoothness in a neighborhood of $(0,0)$ of the functions $\Theta_{\dots}(z,\tau)$ and $\bar\Theta_{\dots}(z,\tau)$ obtained from $\Theta_{j,\dots}(\tau)$ and $\bar\Theta_{j,\dots}(\tau)$ respectively by the substitution $z=(2j+1)\mu h$.

Therefore the following estimate holds:

$$|\int \psi(x)\Big(e(x,x,-\infty,\tau)-h^{-d}\mathcal{N}^{\mathsf{MW}}(x,\tau,\mu h)\Big)\,dx- h^{-d}G_0(\tau,\mu^{-1})-h^{1-d}G_1(\tau,\mu^{-1})|\le \mathsf{R}^{\mathsf{MW}}_{\infty}+\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}+C'h^{2-d} \tag{13.5.82}$$

where

(13.5.83) $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}$ is an error obtained when we replaced the sum with respect to j by the integral with respect to z (divided by $2\mu h$) in the definition of $\mathcal{J}$.

A similar estimate holds for $\vartheta>0$.

Moreover, if we perform the same procedure but with preliminary mollification with respect to τ with $L=1$ then Proposition 13.5.18 yields estimate

$$|\int \phi(\tau)\Big(\int \psi(x)\Big(e(x,x,-\infty,\tau)-h^{-d}\mathcal{N}^{\mathsf{MW}}(x,\tau,\mu h)\Big)\,dx- h^{-d}G_0(\tau,\mu^{-1})-h^{1-d}G_1(\tau,\mu^{-1})\Big)\,d\tau|\le C'h^{2-d} \tag{13.5.84}$$

with the same functions G_0 and G_1; here application of formula $(13.\mathrm{A}.3)_1$ is justified rather than merely formal because

$$|D_z^k\int \phi(\tau)\Theta_{\dots}(z,\tau)\,d\tau|\le C,$$

$$|D_z^k\int \phi(\tau)\big(\Theta_{\dots}(z,\tau)-\bar\Theta_{\dots}(z,\tau)\big)\,d\tau|\le C\mu^{-2}$$

for all $k\le K'$ and $z\in[0,C_0]$ provided ϕ is smooth enough.

On the other hand, Proposition 13.4.20 and formula (13.4.69) (in the weak sense) yield that estimate

$$|\mathsf{R}^{\mathsf{MW}}_{L}|:=|\iint \phi(\tau)\Big(e(x,x,-\infty,\tau)-h^{-d}\mathcal{N}^{\mathsf{MW}}(x,\tau,\mu h)\Big)\,dxd\tau|\le C'h^{2-d} \tag{13.5.85}$$

holds for $1 \le \mu \le h^{\delta-1}$.

Comparing (13.5.84) and (13.5.85) for $h^{-\delta} \le \mu \le h^{\delta-1}$ we obtain that the functions

$$\int \phi(\tau) G_j(\tau, \zeta) d\tau$$

with $j = 1, 2$ have a zero of arbitrarily high order at $\zeta = 0$; since mollifying function ϕ is arbitrary and $G_j(\tau, \zeta)$ are smooth near $(0, 0)$ we obtain that the functions $G_j(z, \tau)$ have a zero of arbitrarily high order at $\zeta = 0$ provided $|\tau| \le \epsilon$.

Then (13.5.82) and (13.5.84) yield respectively

Proposition 13.5.20. *Let condition* (13.4.46) *be fulfilled. Then for* $h^{-\delta} \le \mu \le h^{-1}$ *estimates* (13.5.85) *and*

$$|\mathsf{R}^{\mathsf{MW}}| \le |\mathsf{R}^{\mathsf{MW}}_{\infty}| + |\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}| + C' h^{2-d} \qquad \forall \tau : |\tau| \le \epsilon \tag{13.5.86}$$

hold.

Remark 13.5.21. A similar estimates hold for $\vartheta > 0$ but for $\vartheta > 1$ one should add the correction terms

$$-\kappa_2(\tau, \mu h) h^{2-d} - \kappa_4(\tau, \mu h) h^{4-d} - \dots, \tag{13.5.87}$$

and for odd integers ϑ the extra term described in Remark 13.5.19 should be included as well.

So in order to estimate R^{MW} we should estimate $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}$.

Implementation: Strongest Non-Degeneracy Condition

First of all let us note that under strongest non-degeneracy condition (13.4.54) the following inequalities hold:

$$|D_z^k \Theta(z, \tau)| \le C', \qquad |D_z^k (\Theta(z, \tau) - \bar{\Theta}(z, \tau))| \le C' \mu^{-2} \qquad \forall \tau : |\tau| \le \epsilon \quad \forall z \in [0, C_0] \quad \forall k \le K'. \tag{13.5.88}$$

Therefore $|\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}| \le C' h^{2-d}$ in this case; so we obtain

Theorem 13.5.22. *Let conditions* (13.2.1)–(13.2.5), (13.3.1), (13.4.46) *and* (13.4.54) *be fulfilled. Let* $\vartheta > 0$. *Then for* $d = 2, 3$ *respectively the following estimates hold:*

$$|\mathsf{R}^{\mathsf{MW}}_{(\vartheta)}| \le C\mu^{-1-\vartheta}h^{\vartheta-1} + C + C'h^s, \tag{13.5.89}_{(2)}$$

$$|\mathsf{R}^{\mathsf{MW}}_{(\vartheta)}| \le Ch^{-2+\vartheta} + Ch^{-1} + C' \tag{13.5.89}_{(3)}$$

Proof. Case $\mu \ge h^{-\delta}$ is covered by above arguments and Theorem 13.4.25 permitted us to get rid of condition (13.5.1). □

Implementation: $d = 3$

In what follows we assume that condition (13.4.45) is fulfilled; we will get rid of this restriction in Section 13.8. Let us first treat the case $d = 3$ which is simpler as the target remainder estimate is relatively larger and there is additional integration over ξ_2. We will return to this case later when studying supersharp asymptotics.

Theorem 13.5.23. *Let* $d = 3$, $\vartheta = 0$, $\tau = 0$ *and let conditions* (13.2.1)–(13.2.5), (13.4.45) *and* (13.3.1) *be fulfilled. Then*

(i) Estimate (13.5.68) *holds for* R^{MW} *under under weak non-degeneracy condition* $(13.4.57)_m$.

(ii) Estimate (13.5.69) *holds for* R^{MW} *in the general case.*

Proof. Recall equality

$$\mathcal{B}_j = \Big(V + (2j+1)\mu hF\Big) \circ \bar{\Psi}_0 + O(\mu^{-2}). \tag{13.5.90}$$

We want to estimate $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}$ and $\mathsf{R}^{\mathsf{MW}}_{(\vartheta)}\mathsf{corr}$; for this purpose let us introduce scaling function

$$\gamma_1 = \gamma_1(x', \xi') = \min_j(|\bar{\mathcal{A}}_j| + |\nabla\bar{\mathcal{A}}_j|^2)^{\frac{1}{2}} \tag{13.5.91}$$

and let us introduce scaling function γ_2 in the same way with $\mathcal{A}_j$ instead of $\bar{\mathcal{A}}_j$. Let us treat various zones and various γ_1- and γ_2-admissible partitions of unity in those zones:

(a) Zone $\{(x',\xi') : \gamma_1 \ge \max(C_0\mu^{-1}, \sqrt{\mu h})\}$; in this zone $\gamma_1 \asymp \gamma_2$. One can easily see that if $\Theta(z)$ and $\bar\Theta(z)$ are functions associated with the same element of the partition of unity ($\tau = 0$ and m, p, l are omitted here for the sake of simplicity) then

$$|D_z^k \bar\Theta(z)| \le C_k \gamma_1^{2d-2-2k} + C' h^s \qquad \forall k \le K' \tag{13.5.92)$_1$}$$

and the same estimate with γ_1 replaced by γ_2 (and this replacement is not essential in this zone) holds for $\Theta(z)$.

Further, let us apply formula

$$\Theta(z) - \bar\Theta(z) = \int_0^1 \partial_t \Theta(z,t)\,dt$$

with $\Theta(t,z)$ defined by (13.5.44) with $\tau = 0$ and with $\mathcal{A}_j t + \bar{\mathcal{A}}_j(1-t)$ instead of $\mathcal{A}_j$; we then easily obtain estimate

$$|D_z^k(\Theta(z) - \bar\Theta(z))| \le C_k \mu^{-2} \gamma_1^{2d-4-2k} + C' h^s \qquad \forall k \le K'. \tag{13.5.92)$_2$}$$

Furthermore, let us note that outside of an interval of length $C_0\gamma_1^2$ the functions $\Theta(z)$ and $\bar\Theta(z)$ are constant and that for $\vartheta = 0$ the integrands are polynomial with respect to z; since the integral formula (13.A.3)$_1$ is exact on polynomials of arbitrary fixed degree we obtain in the error estimate an additional factor $C_0\gamma_1^2$.

So the contribution to $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}$ of this element of the partition does not exceed

$$C_k h^{-d} \mu^{-2} \gamma_1^{2d-4-2k} (\mu h)^k \gamma_1^2 + C' h^s. \tag{13.5.93}$$

On the other hand, in the zone $\{(x',\xi') : 2C_0\mu^{-1} \ge \gamma_1\}$ a similar inequality[31)] holds for γ_2; in this zone we treat $\bar\Theta$ and Θ separately; we do not apply (13.5.92)$_2$ in this case and so we take γ_1- and γ_2-admissible partitions of unity for $\bar\Theta$ and Θ respectively. We consider the contribution of $\bar\Theta$; the contribution of Θ is treated similarly:

[31)] I.e. $C_1\mu^{-1} \ge \gamma$.

(b) Let us treat subzone $\{(x_1,\xi_1) : 2C_0\mu^{-1} \ge \gamma_1 \ge \sqrt{\mu h}\}$. In this case we apply $(13.5.92)_1$ and obtain that the contribution to $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}$ of such an element of the partition does not exceed

$$C_k h^{-d}\gamma_1^{2d-2-2k}(\mu h)^k\gamma_1^2 + C'h^s \tag{13.5.94}$$

if we treat $\bar{\Theta}$ only; a similar estimate with γ_2 instead of γ_1 (for the corresponding element) holds if we treat Θ only.

(c) Let us treat subzone $\{(x_1,\xi_1) : 2C_0\mu^{-1} \le \gamma_1 \le \sqrt{\mu h}\}$. In this case we treat the integrals and Riemann sums separately; however since the integral formulas are exact on polynomials we must integrate over an interval of length not exceeding $C_0\mu h$ and we must take less than C_0 terms in the Riemann sum. This remark and estimates $(13.5.92)_{1,2}$ yield that the contribution of this element of the partition does not exceed

$$Ch^{-d}\mu^{-2}\gamma_1^{2d-4}\mu h + C'h^s; \tag{13.5.95}$$

let us recall that in this case $\gamma_1 \asymp \gamma_2$ and we take the same partition for $\bar{\Theta}$ and Θ.

(d) Let us consider subzone $\{(x_1,\xi_1) : 2\min(\sqrt{\mu h}, C_0\mu^{-1}) \ge \gamma_1\}$; then a similar inequality holds for γ_2. In this zone we apply the same arguments as in (c) but we do not use $(13.5.92)_2$ and consider $\bar{\Theta}$ and Θ separately. Then the contribution to R^{MW} of this element does not exceed

$$Ch^{-d}\gamma_1^{2d-2}\mu h + C'h^s \tag{13.5.96}$$

if we treat $\bar{\Theta}$ only; a similar estimate (for the corresponding element) holds if we treat Θ only.

Summing with respect to partitions, using (13.5.93)–(13.5.96) and observing that the estimates of the contribution of the common element of two (sub)zones calculated by different recipes are equivalent we obtain that

$$\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}} \le \sum_{\iota=1,2} Ch^{-d}\int \min\Big(\mu^{-2}\big(\frac{\mu h}{\gamma_\iota^2}\big)^k, \mu h\big(\frac{\mu h}{\gamma_\iota^2}\big)^k, \mu^{-1}h\gamma_\iota^{-2}, \mu h\Big)\,dx'd\xi' + C'h^{-1}. \tag{13.5.97}$$

The right-hand expression in this estimate can only increase if we replace γ_ι by functions which are smaller. Then for $d = 3$ we can replace both γ_1

and γ_2 by $|\xi_2|$ and then separately treating the cases $\mu \le h^{-\frac{1}{3}}$ and $\mu \ge h^{-\frac{1}{3}}$ we obtain that in both cases $|\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}| \le Ch^{-2} + C'h^s$; we will get more precise estimates below.

So our theorem is proven under condition (13.5.1) (but Theorem 13.4.30(iv) yields that we can skip this condition). □

Obviously, the same arguments hold for $\vartheta > 0$ but we obtain estimate (13.5.70) with the additional term Ch^{-2} in the right-hand expression which is obviously bad. In order to get rid of this term let us consider mollification with the function $\phi(\tau/L)$ with a smooth function ϕ. Obviously in this case all the arguments of the proof remain true with γ_ι replaced by $\gamma_\iota + L^{\frac{1}{2}}$.

Then, summing over a partition of non-smooth φ via $\phi(\tau/L)$, with smooth ϕ for $L \ge \mu h$ and with non-smooth ϕ for $L = \mu h$ (as we did in Section 4.4), we obtain estimates

$$\mathsf{R}^{\mathsf{MW}}_{(\vartheta)\mathsf{corr}} \le C\mu h^{1-d} \int_{\{\gamma \ge \sqrt{\mu h}\}} (\mu h)^{\vartheta} \big(\frac{\mu h}{\gamma^2}\big)^s \, dx' d\xi' + C\mu h^{1-d} \int_{\{\gamma \le \sqrt{\mu h}\}} (\mu h)^{\vartheta} \, dx' d\xi' \tag{13.5.98}$$

provided $\mu^3 h \le 1$, $\vartheta \le 1$ and

$$\mathsf{R}^{\mathsf{MW}}_{(\vartheta)\mathsf{corr}} \le C\mu^{-2} h^{-d} \int_{\{\gamma \ge \sqrt{\mu h}\}} (\mu h)^{\vartheta} \big(\frac{\mu h}{\gamma^2}\big)^s \, dx' d\xi' + C\mu^{-2} h^{-d} \int_{\{\mu^{-1} \le \gamma \le \sqrt{\mu h}\}} \gamma^{-2} (\mu h)^{\vartheta} \, dx' d\xi' + C\mu h^{1-d} \int_{\{\gamma \le \mu^{-1}\}} (\mu h)^{\vartheta} \, dx' d\xi' \tag{13.5.99}$$

provided $\mu^3 h \ge 1$, $\vartheta \le 1$.

These estimates yield the estimate

$$\mathsf{R}^{\mathsf{MW}}_{(\vartheta)\mathsf{corr}} \le R_{2(\vartheta)} \coloneqq \begin{cases} C\mu^{\vartheta+\frac{3}{2}} h^{\vartheta-\frac{3}{2}} & \text{for } \ \mu \le h^{-\frac{1}{3}}, \\ C\mu^{-\frac{3}{2}+\vartheta} h^{-\frac{5}{2}+\vartheta} & \text{for } \ \mu \ge h^{-\frac{1}{3}} \end{cases} \tag{13.5.100}$$

in the general case and the estimate

$$\mathsf{R}^{\mathsf{MW}}_{(\vartheta)\mathsf{corr}} \le Ch^{-2+\vartheta} \tag{13.5.101}$$

under intermediate non-degeneracy condition

$$|\frac{V}{F} + (2j+1)\mu h| + |\nabla \frac{V}{F}| + |\operatorname{Hess} \frac{V}{F}| \ge \epsilon_0. \tag{13.5.102}$$

Normally to achieve such estimate we need to consider many terms but the largest extra term does not exceed Ch^{-1}. Therefore, the following statement is proven:

Theorem 13.5.24. *Let $d = 3$, $\vartheta > 0$, $\tau = 0$. Then*

(i) Estimate

$$\mathsf{R}^{\mathsf{MW}}_{(\vartheta)} \le Ch^{-2+\vartheta} + Ch^{-1} \tag{13.5.103}$$

holds provided either $\mu \ge h^{-1/(3-2\vartheta)}$ or $\mu \le h^{-1/(3+2\vartheta)}$ or condition (13.5.102) *is fulfilled.*

(ii) In the general case for $h^{-1/(3+2\vartheta)} \le \mu \le h^{-1/(3-2\vartheta)}$ estimate

$$|\mathsf{R}^{\mathsf{MW}}_{(\vartheta)}| \le Ch^{-2+\vartheta} + Ch^{-1} + R_{3(\vartheta)} \tag{13.5.104}$$

holds where $R_{3(\vartheta)}$ is the right-hand expression of (13.5.98), (13.5.99) *for $\mu \le h^{-\frac{1}{3}}$ and $\mu \ge h^{-\frac{1}{3}}$ respectively.*

Moreover, $R_{3(\vartheta)} \le R_{2(\vartheta)}$ defined by (13.5.100).

Recall that for $\vartheta = 1$, $\mathsf{R}^{\mathsf{MW}}_{(\vartheta)}$ contains an additional regularized term by Remark 13.5.19.

We leave to the reader the following rather easy problem:

Problem 13.5.25. (i) Improve estimates for $\mathsf{R}^{\mathsf{MW}}_{(\vartheta)\mathsf{corr}}$ under more restrictive conditions (taking into account the term $\kappa_2 h^{-1}$ with $\kappa_2 = \kappa_2(\tau, \mu h)$).

(ii) Consider the case $\vartheta > 1$ when this term should always be taken into account.

(iii) Improve estimates for $\mathsf{R}^{\mathsf{MW}}_{(\vartheta)\mathsf{corr}}$ as $\bar{\nu} < 1$.

(iv) Improve estimates for $\mathsf{R}^{\mathsf{MW}}_{(\vartheta)\mathsf{corr}}$ from $O(h^{-2+\vartheta})$ to $o(h^{-2+\vartheta})$ calculating $\mathcal{A}_j$ modulo $O\big(\mu^{-4} + \mu^{-2}\gamma^2\big)$ instead of $O(\mu^{-2})$ as we did. In this case in $\mathsf{R}^{\mathsf{MW}}_{(\vartheta)}$ an integrand $\mathcal{N}^{\mathsf{MW}}(x, \tau', \mu h)$ should be replaced by $\mathcal{N}^{\mathsf{MW}}((x, \tau' - \rho(x)\mu^{-2}, \mu h)$ and additional terms of the type $\mathsf{m}_j(x)\mu^{-2j}h^{-3}$ appear.

Implementation: $d = 2$

Obviously, our arguments apply to the case $d = 2$ as well. In particular, estimate (13.5.97) remains true for $d = 2$, $\nu = 1$, $\vartheta = 0$ as well but we want to improve it for $\mu \ge C_0 h^{-\frac{1}{3}}$. Namely, we want to estimate the contribution of the (sub)zone

$$\{x : \ \epsilon_0(\mu h)^{\frac{1}{2}} \ge \gamma_1 \ge C_1\mu^{-1}\} \tag{13.5.105}$$

more accurately where C_1 depends on $\epsilon > 0$.

More precisely, if $|v + (2j+1)\mu h| \ge \epsilon\gamma_1^2$ on the element of the partition of unity (with $j \in \mathbb{Z}^+$ for which this absolute value is minimal) then Θ_j and $\bar\Theta_j$ coincide for all $j \le J$ and therefore the contributions to the error of Riemann sums vanish for $m = p = l = 0$ and since $\Theta(z)$ and $\bar\Theta(z)$ coincide outside of intervals of the length $C_0\mu^{-2}$, the contribution of $\big(\Theta(z) - \bar\Theta(z)\big)$ to the error does not exceed

$$Ch^{1-d}\gamma_1^{2d-4}\mu^{-1} + C'h^s. \tag{13.5.106}$$

The contributions of all the other terms do not exceed the previous estimate in zone (c) in the proof of Theorem 13.5.24 with the additional factor μ^{-2} and so they are less than this estimate.

Now applying estimate (13.5.97) with this improvement and Theorem 13.5.16 we immediately obtain that under condition (13.5.1) (which could be skipped by Remark 13.4.19) the following theorem holds:

Theorem 13.5.26. *Let $d = 2$, $\vartheta = 0$, $\tau = 0$ and let conditions* (13.2.1)–(13.2.5)*,* (13.3.1)*,* (13.4.45) *and* (13.5.1) *be fulfilled. Then*

(i) For $\mu^3 h \le C_0$ the following estimate holds:

$$\mathsf{R}^{\mathsf{MW}} \le \frac{C}{\mu h} + \frac{C'}{\mu h}\int_{\{\gamma \ge C_0\mu^{-1}\}} \big(\frac{\mu h}{\gamma}\big)^s \frac{dx}{\gamma^2} + C\frac{\mu}{h}\int_{\{\gamma \le C_0\mu^{-1}\}} dx + C'. \tag{13.5.107}$$

Moreover, under condition (13.5.73) *estimate*

$$\mathsf{R}^{\mathsf{MW}} \le \frac{C}{\mu h} + C\mu^{\frac{3}{2}}h^{-\frac{1}{2}} \tag{13.5.108}$$

holds and under condition

$$|\frac{V}{F} + (2j+1)\mu h| + |\nabla\frac{V}{F}| + |\det \mathsf{Hess}\, \frac{V}{F}| \ge \epsilon_0 \tag{13.5.109}$$

estimate $(13.5.89)_{(2)}$ *holds.*

(ii) For $\mu^3 h \ge C_0$ the following estimate holds:

$$\mathsf{R}^{\mathsf{MW}} \le \frac{C}{\mu h} + \frac{C'}{\mu h}\int_{\{\gamma\ge\sqrt{\mu h}\}} \big(\frac{\mu h}{\gamma^2}\big)^s \frac{dx}{\gamma^2} + \frac{C}{\mu h}\int_{\{\sqrt{\mu h}\ge\gamma\ge C_1\mu^{-1}, |v+(2j+1)\mu h|\le\epsilon\mu h\}} \frac{dx}{\gamma^2} + C\frac{\mu}{h}\int_{\{\gamma\le C_0\mu^{-1}\}} dx + C'. \tag{13.5.110}$$

Moreover, under condition (13.5.73) *estimate*

$$|\mathsf{R}^{\mathsf{MW}}| \le Ch^{-1} + C' \tag{13.5.111}$$

holds and under the condition

$$|\frac{V}{F} + (2j+1)\mu h| + |\nabla\frac{V}{F}| \le \epsilon_0 \implies \det \mathsf{Hess}\, \frac{V}{F} \ge \epsilon_0 \tag{13.5.109}_+$$

estimate $(13.5.89)_{(2)}$ *holds; furthermore, under condition* (13.5.109) *the estimates*

$$\mathsf{R}^{\mathsf{MW}} \le \frac{C}{\mu h}\big(1 + |\log\frac{\hat\tau}{\mu h}|\big) + C' \le \frac{C}{\mu h}(1+\log\mu^3 h) + C' \tag{13.5.112}$$

hold where $\hat\tau = (\min_x \gamma^2(x) + \mu^{-2})$.

Remark 13.5.27. (i) Theorem 13.4.30 yields that in the general case and under conditions (13.5.73) and (13.5.109) the corresponding estimates have a place even if condition (13.5.1) is violated.

(ii) We leave the case $\vartheta > 0$ to the reader: in this case even condition $(13.5.109)_+$ does not provide estimate $(13.5.89)_{(2)}$ for $\mathsf{R}^{\mathsf{MW}}_{(\vartheta)}$ for μ close to $h^{\frac{1}{3}}$.

Implementation: $d = 2$, II

In order to improve estimate (13.5.112) we need more sophisticated arguments which are applicable in other situations as well. So let us assume that $\mu^3 h \ge C_0$ and condition (13.5.109) holds but $(13.5.109)_+$ does not. Without any loss of the generality one can assume that $v = V/F$ has only one stationary point $\bar x$ in $B(0,1)$ (in which case this point is a saddle point) and that $\tau = 0$.

Let us replace $\mathcal{A}_j(x_1,\xi_1,\dots)$ in the argument of the Heaviside function by $(2j+1)\mu h + B(x,\xi,\dots)$. In our previous arguments we dropped $O(\mu^{-2})$ here;

let us now preserve this term but drop $O(\mu^{-4})$, $O(h^2)$ and $O(\mu^{-2}\gamma)$; equalities $(13.3.76)_{1,2}$ yield that this is equivalent to replacing B by $v\circ\bar\Psi_0+\bar\omega_1\mu^{-2}$ where $\bar\omega_1$ is the value of the coefficient ω_1 at $\bar x$,

$$\bar\omega_1=\bigl(\frac{1}{8}\kappa v^2-\frac{1}{4}\sum_{j,k}g^{jk}F^{-1}\partial_j\partial_k v\bigr)(\bar x). \tag{13.5.113}$$

Moreover, we skip all terms with $m+p+l\ge 1$ as before. If in the difference (between original and final expressions) we replace the sum with respect to j by the corresponding integral then the previous arguments yield that the error is $O\bigl(\mu^{-1}h^{-1}\mu h|\log(\hat\tau\mu^{-1}h^{-1})|\bigr)$ and therefore Theorem 13.5.16(ii) yields estimate

$$|\int \psi(x)\Bigl(e(x,x,-\infty,0)-h^{-2}\mathcal{N}^{\mathsf{MW}}(x,-\bar\rho\mu^{-2},\mu h)\Bigr)\,dx- h^{-2}G(\mu^{-2})|\le \frac{C}{\mu h}+C\mu h|\log\frac{\hat\tau}{\mu h}|+C' \tag{13.5.114}$$

with $\bar\rho=\bar\omega_1 F(\bar x)$; this estimate holds for $h^{-\delta}\le\mu\le h^{-1}$ and moreover, for $\mu\le h^{-\frac{1}{3}}$ this estimate holds without the second term in the right-hand expression.

However, we know that under condition (13.4.45) and for $\mu\le h^{\delta-\frac{1}{3}}$, $|\tau|\le\epsilon$

$$\int \psi(x)\Bigl(\mathcal{N}^{\mathsf{MW}}(x,\tau,\mu h)-\frac{1}{4}(\tau-V)\sqrt{g}\Bigr)\,dx=O(h^s)$$

and, moreover, we know that in this case Weyl asymptotics holds with remainder estimate $O(\mu^{-1}h^{-1})$ (see Theorem 13.4.30(iii)). Then comparing (13.5.114) and the Weyl asymptotics we deduce that the function $G(\zeta)-\mathsf{m}\zeta$ has at $\zeta=0$ a zero of arbitrarily high order with coefficient $\mathsf{m}=\frac{1}{4}\int\psi\sqrt{g}\,dx$. Therefore the following estimate holds:

$$\mathsf{R}_1^{\mathsf{MW}}:= |\int \psi(x)\Bigl(e(x,x,-\infty,0)-h^{-2}\mathcal{N}^{\mathsf{MW}}(x,-\bar\rho\mu^{-2},\mu h)-\bar\rho\sqrt{g}\mu^{-2}h^{-2}\Bigr)\,dx|\le \frac{C}{\mu h}+C\mu h|\log\frac{\hat\tau}{\mu h}|+C' \tag{13.5.115}$$

where $\bar\rho = \bar\omega_1 F(\bar x)$ and $\bar\omega_1$ is given by (13.5.113). This estimate is effective. Further, it is worse than corresponding estimate for $\mathsf{R}^{\mathsf{MW}}_\infty$ only if

$$(\mu h)^2 |\log \frac{\hat\tau}{\mu h}| \ge 1 \implies \mu \ge h^{-1} |\log h|^{-\frac12}$$

i.e. in the case of intermediate magnetic field; however, we will transform this estimate slightly.

Here subscript "1" in our notations indicates that only one perturbation term is present in the arguments of θ.

The same arguments with the Riemann sum replaced by an integral and with the same error estimate as in the proof of Theorem 13.5.22 yield that if $\psi_1 \in \mathscr{C}^K$ coincides with ψ for $|x-\bar x| \ge \frac12\gamma$, vanishes in $B(\bar x, \frac12\gamma)$ and satisfies the inequalities $|D^\alpha \psi_1| \le c\gamma^{-|\alpha|}$ and if $\gamma = \sqrt{\mu h}$, $\mu^3 h \ge 1$ then

$$\Big|\int \psi_1(x)\Big(\mathcal{N}^{\mathsf{MW}}(x, -\bar\rho\mu^{-2}, \mu h) - \mathcal{N}^{\mathsf{MW}}(x,0,\mu h) + \frac{1}{4\pi}\bar\rho\mu^{-2}\sqrt{g}\Big)\,dx\Big| \le C\mu^{-1}h + C'h^2.$$

Let us consider a similar expression for ψ_1 supported in $B(\bar x, \gamma)$, $\psi_1 = \psi$ in $B(\bar x, \frac12\gamma)$; it is easy to prove that in this case

$$\Big|\int \psi_1(x)\Big(\mathcal{N}^{\mathsf{MW}}(x, -\bar\rho\mu^{-2}, \mu h) - \mathcal{N}^{\mathsf{MW}}(x,0,\mu h) + \frac{1}{4\pi}\bar\rho\mu^{-2}\sqrt{g}\Big)\,dx - \sigma(\bar x)\mu^{-1}h\Big| \le C\mu^{-1}h + C'h^2$$

where

$$\sigma(\bar x) = \frac{\mu^2}{\pi}\big(f(\bar\tau') - f(\bar\tau)\big) F\sqrt{g}\,|\det \mathsf{Hess}\, v|^{-\frac12}\psi\big|_{x=\bar x}, \tag{13.5.116}$$

$$f(\tau) = \tau \log\big(\frac{|\tau|}{\mu h}\big), \tag{13.5.117}$$

$$\bar\tau = v(\bar x) + (2\bar\jmath + 1)\mu h, \quad \bar\tau' = \bar\tau + \bar\rho\mu^{-2}, \quad \bar\rho = \bar\omega_1 F(\bar x) \tag{13.5.118}$$

and for $j = \bar\jmath$ the quantity $|v(\bar x + (2j+1)\mu h|$ is minimal on $\mathbb{Z}^+$.

We leave this proof to the reader; let us note only that it is convenient to use the coordinate system in which $\bar x = 0$, $v = x_1x_2|\det \mathsf{Hess}\, v(\bar x)|^{-\frac12}$ in this proof.

Passing from the case with one critical point to general one we obtain the following

Theorem 13.5.28. *Let $d = 2$ and let conditions* (13.2.1)–(13.2.5), (13.3.1), (13.4.45) *and* (13.5.73) *be fulfilled.*

Then for $\mu^3 h \ge 1$ the following estimate holds:

$$\mathsf{R}^{\mathsf{MW}\prime} = \Big| \int \psi(x)\Big(e(x,x,-\infty,0) - h^{-2}\mathcal{N}^{\mathsf{MW}}(x,0,\mu h)\Big)\,dx - \sum_{x_k\in\mathfrak{M}} \frac{1}{\mu h}\sigma(x_k)\Big| \le \frac{C}{\mu h} + C\mu h|\log\frac{\hat\tau}{\mu h}| + C' \tag{13.5.119}$$

where $\mathfrak{M}$ is the set of saddle points of $v = V/F$ and at every point $x_k \in \mathfrak{M}$, $\sigma(x_k)$ is given by formulae (13.5.116)–(13.5.118), (13.5.113) *with x_k instead of $\bar x$.*

Remark 13.5.29. (i) Here and in estimates (13.5.112) and (13.5.115) we can redefine $\hat\tau = \min_{x_k\in\mathfrak{M}} |\tau(x_k)| + \mu^{-2}$.

(ii) Similarly one can treat the case of $\vartheta \in [0,1]$. In this case we replace $\mathcal{A}_j$ by $\bar{\mathcal{A}}'_j = \mathcal{A}_j + O\big(\mu^{-4}\mu^{-2}\gamma^2\big)$ and note that by a shift one can get rid of the term which is linear with respect to x in $\bar{\mathcal{A}}'_j$ (which is $O(\mu^{-2}\gamma)$). One can easily prove the required estimate for $\mathsf{R}^{\mathsf{MW}}_{1(\vartheta)}$.

However, we have failed to derive an estimate for something similar to $\mathsf{R}^{\mathsf{MW}\prime}_{(\vartheta)}$ (with $\vartheta > 0$).

(iii) Moreover, the same approach works when condition (13.5.73) is fulfilled; the Morse theorem yields that without any loss of the generality one can assume that

$$v(x) = \bar v(x_1) \pm x_2^2. \tag{13.5.120}_{\pm}$$

We leave to the reader the detailed proofs of the following two statements (using the same arguments as we did before):

Theorem 13.5.30. *Under the assumptions of Theorem 13.5.28 estimate* (13.5.60) *holds for $\mathsf{R}^{\mathsf{MW}}_{1(\vartheta)}$ with $\vartheta \in [0,1]$.*

Theorem 13.5.31. *Let $d = 2$, $\vartheta = 0$ and let conditions* (13.2.1)–(13.2.5), (13.3.1) *and* $(13.5.120)_{\pm}$ *be fulfilled. Then*

(i) For $h^{-\frac15} \le \mu \le h^{-\frac13}$ the following estimate holds:

$$\mathsf{R}_1^{\mathsf{MW}} \le C\mu^{-1}h^{-1} + C' \tag{13.5.121}$$

where $\mathsf{R}_1^{\mathsf{MW}}$ is given by (13.5.114)*, ω_1 is given by* (13.5.113) *and in this theorem the bar means that the corresponding function is calculated at $x_2 = 0$.*

(ii) For $\mu \ge h^{-\frac13}$ the following estimate holds:

$$\mathsf{R}_1^{\mathsf{MW}} \le C\mu^{-1}h^{-1} + C\mu^{\frac12}h^{-\frac12} + C' \tag{13.5.122}$$

and moreover the following estimate holds as well:

$$\mathsf{R}_1^{\mathsf{MW}} \le \frac{C}{\mu h} + C\int_{\{\sqrt{\mu^{-1}h}\le\gamma_1\le\sqrt{\mu h}\}} \frac{dx_1}{\gamma_1} + C\mu^{\frac12}h^{-\frac12}\int_{\{\gamma_1\le\sqrt{\mu^{-1}h}\}} dx_1 + C' \tag{13.5.123}$$

where

$$\gamma_1 = \gamma_1(x_1) = \min_{j\in\mathbb{Z}^+}\bigl(|\partial_1\bar{w}|^2 + |\bar{w} + (2j+1)\mu h|\bigr)^{\frac12}$$

and $\bar{w} = \bar{v} + \mu^{-2}\bar{\omega}_1$.

Implementation: $d = 2$, III

Let us now treat the case $d = 2$ and $\nu < 1$. In this case the optimal estimate (13.5.60) for R^{T} and $\mathsf{R}_\infty^{\mathsf{MW}}$ is better than in the case $\nu = 1$ but the transition to estimates for R^{MW} and similar effective estimates can make it essentially worse.

In this case it is important whether $\nu \ge \mu h$ or $\nu \le \mu h$; in the second case the transition from summation to integration is completely useless in the analysis of the term with $m = p = l = 0$ but in this case the replacement of $\mathcal{A}_j$ by $\bar{\mathcal{A}}_J$ changes only terms with less than C_0 numbers j. Taking into account this observation we prove the following two theorems (the second theorem improves the first one):

Theorem 13.5.32. *Let $d = 2$, $\vartheta = 0$ and let conditions* (13.2.1)–(13.2.5)*,* (13.3.1)*,* $(13.3.77)_{1-3}$*,* $(13.4.12)_{1,2}$*,* (13.3.58)*,* (13.5.1)*,* (13.4.45) *and*

$$|\nabla\frac{V}{F}| \ge \epsilon_0\nu \ge C_0\bar{\nu}\mu^{-2} + Ch^{1-\delta} \tag{13.5.124}$$

be fulfilled. Then

(i) For $\nu \ge \mu h$ the following estimate holds:

$$\mathsf{R}^{\mathsf{MW}} \le C\frac{\nu}{\mu h} + C'\bar\nu\nu^{-2}\big(\frac{\mu h}{\nu}\big)^s + C' \qquad \forall \tau : |\tau| \le \epsilon\nu \tag{13.5.125}$$

with arbitrary s and with $\epsilon = \epsilon(c, \epsilon_0) > 0$.

(ii) For $\nu \le \mu h$ the following estimate holds:

$$\mathsf{R}^{\mathsf{MW}} \le C\frac{\bar\nu}{\mu h\nu} + C' \qquad \forall \tau : |\tau| \le \epsilon\nu \tag{13.5.126}$$

with $\epsilon = \epsilon(c, \epsilon_0) > 0$.

Theorem 13.5.33. *Let $d = 2$, $\vartheta = 0$ and let conditions* (13.2.1)–(13.2.5), (13.3.1), $(13.3.77)_{1-3}$, (13.5.1) *and* (13.4.45) *be fulfilled. Let us assume that*

$$|D^\alpha\big(\frac{V}{F} + \omega_1\mu^{-2}\big)| \le c\nu \qquad \textit{in} \quad B(0,1) \quad \forall \alpha : 1 \le |\alpha| \le K, \tag{13.5.127}$$

with

$$\nu \ge C_0\bar\nu(\mu^{-4} + h^2), \tag{13.5.128}$$

and

$$\min_j |\frac{V}{F} + (2j+1)\mu h + \omega_1\mu^{-2}| + |\nabla\big(\frac{V}{F}v + \omega_1\mu^{-2}\big)| \ge \epsilon_0\nu \qquad \textit{in} \quad B(0,1) \tag{13.5.129}$$

where

$$\omega_1 = \frac{1}{8}\kappa v^2 - \frac{1}{4}v\mathcal{L}v - \sum_{j,k}\frac{1}{4}g^{jk}\partial_j v\cdot\partial_k v, \qquad v = \frac{V}{F}, \tag{13.5.130}$$

and

$$\mathcal{L}v = \frac{1}{F\sqrt{g}}\sum_{j,k}\partial_j(g^{jk}\sqrt{g}\partial_k v), \tag{13.5.131}$$

(i.e., $\mathcal{L}$ is the Laplace-Beltrami operator in the metric $F^{-1}g^{jk}$). Then

(i) For $\mu h \le \nu$ and $\tau = 0$ the following estimate holds

$$\mathsf{R}_1^{\mathsf{MW}} := |\int\Big(\psi(x)e(x,x,-\infty,\tau) - h^{-2}\mathcal{N}_\psi^{\mathsf{MW}\prime}(x,\mu h,\mu^{-2})\Big)\,dx| \le C\frac{\nu}{\mu h} + C'\bar\nu\mu^{-2}\nu^{-2}(\frac{\mu h}{\nu})^s + C' \tag{13.5.132}$$

where

$$\mathcal{N}_{\psi}^{\mathsf{MW}\prime} = \frac{1}{2\pi}\Big(\sum_{j\in\mathbb{Z}^+} \theta\big(-v-(2j+1)\mu h-\omega_1\mu^{-2}\big)(\psi+\psi'\mu^{-2})\mu h + \frac{1}{4}\mu^{-2}(\omega_1\psi+v\psi') + \frac{1}{4}\mu^{-4}\omega_1\psi'\Big)F\sqrt{g}, \tag{13.5.133}$$

$$\psi' = -\frac{1}{4}v\mathcal{L}\psi - \frac{1}{2}\sum_{j,k} vg^{jk}F^{-2}\partial_j F\cdot\partial_k\psi. \tag{13.5.134}$$

(ii) For $\mu h \ge \nu$ and $\tau = 0$ the following estimate holds:

$$\mathsf{R}_1^{\mathsf{MW}} \le C'\big(1+\bar\nu\mu^{-3}h^{-1}\nu^{-1}+\bar\nu\mu h\nu^{-1}\big). \tag{13.5.135}$$

Proof. We will prove Theorem 13.5.33; the proof of Theorem 13.5.32 is similar but simpler.

In the argument of the Heaviside function in the terms with $m+p\le 1$, $l=0$ let us skip terms which are $O(\mu^{-4}\bar\nu)$, $O(h^2\bar\nu)$ and $O(\mu^{-1}h\nu)$; then repeating the previous arguments we obtain that modulo a term with an absolute value not exceeding the right-hand expression of estimate (13.5.132) for $\mu h\le\nu$ (or the right-hand expression of estimate (13.5.135) for $\mu h\ge\nu$) the correction is of the form $h^{-2}G_0(\mu^{-2})+\mu^{-1}h^{-1}G_1(\mu^{-2})$.

The same arguments give the same answer if we drop all terms with $m+p+2l\ge 2$. Let us treat terms with $m+p=1$, $l=0$; they are of the form

$$\frac{1}{2\pi h^2}\int\sum_j \theta\big(-v-(2j+1)\mu h-\omega_1\mu^{-2}\big)\times \big(\psi'+(v+(2j+1)\mu h+\omega_1\mu^{-2})\psi''\big)\mu^{-2}F\sqrt{g}\,dx.$$

Here we have replaced the integration with respect to (x_1,ξ_1) with the integration with respect to x by means of the symplectomorphism $\bar\Psi_0$. Then similar arguments show that if we skip terms which are $O(\nu)$ in ψ' and if we skip ψ'' completely then the correction has the same form with the same error estimate; in the second case we used the fact that the function $f(z)$ (which is integrated with respect to z) satisfies the estimate

$$|D_z^k f|\le C\nu^{1-k}\qquad \forall k\le K'$$

on the "bad" interval of length $C_0\nu$ (if $\nu \ge \mu h$); moreover, on the "bad" interval of length $C_0\mu h$ the estimate $|f| \le C\nu$ holds for $\nu \le \mu h$. Thus the expression

$$\mathsf{R}_1^{\mathsf{MW}\prime} := \Big| \int \psi(x)\Big(e(x,x,-\infty,0) - \frac{1}{2\pi h^2}\sum_{j\in\mathbb{Z}^+} \theta\big(-v-(2j+1)\mu h - \omega_1\mu^{-2}\big)(\psi+\psi'\mu^{-2})\mu h\Big)\,dx - h^{-2}G_0(\mu^{-2}) - \mu^{-1}h^{-1}G_1(\mu^{-2})\Big|$$

does not exceed the right-hand expression of (13.5.132) or (13.5.135) for $\mu h \ge \nu$ and $\mu h \le \nu$ respectively.

Furthermore, the same estimate remains true if we replace $\tau = 0$ by $\tau \in [-\epsilon\nu, \epsilon\nu]$ (with a small constant $\epsilon > 0$) and if we simultaneously replace V by $(V-\tau)$ in the argument of the Heaviside function and in the definition of ω_1 and ψ'; moreover, in this case the functions G_0 and G_1 depend smoothly on τ.

Finally, if we have mollified (inside the absolute value) with respect to τ with a small constant $L > 0$ and with ϕ supported in $[-1,1]$ we would obtain the remainder estimate C'.

But then Proposition 13.4.20 and formula (13.4.69) yield that for $1 \le \mu \le h^{\delta-1}$ the same estimate (with mollification) remains true with

$$\bar{G}_0(\zeta) = \frac{1}{4}\zeta(\omega_1 + v\psi') + \frac{1}{4}\zeta^2\omega_1\psi', \quad \bar{G}_1(\zeta) = 0 \tag{13.5.136}$$

instead of G_0 and G_1 respectively; here $\tau \in [-\epsilon, \epsilon]$ enters into the definition of these functions through v, ω_1 and ψ'.

Comparing these two estimates for $h^{-\delta} \le \mu \le h^{\delta-1}$ we conclude that the functions $\big(G_0 - \bar{G}_0\big)$ and G_1 have zeros of arbitrarily high order at $\zeta = 0$, $\tau \in [-\epsilon, \epsilon]$; we first make this conclusion for functions mollified with respect to τ and then we use the arbitrary choice of mollifying function and the smoothness of the functions G_0, $\bar{G}_0$, G_1 (see proof of Theorem 13.5.22).

But then we can replace G_0 and G_1 by $\bar{G}_0$ and $\bar{G}_1 = 0$ in the definition of $\mathsf{R}_1^{\mathsf{MW}\prime}$ (without mollification); so the theorem is almost proven: we only need to calculate ω_1 and ψ'.

In order to calculate ω_1 and ψ' let us note that in the conformal coordinate system for $F = 1$ and for the method of reduction used in Proposi-

tion 13.3.1(iii)

$$\omega_1 = \frac{1}{8}\kappa v^2 - \frac{1}{4}v\rho^2\Delta v - \frac{1}{4}\rho^2(\nabla v)^2,$$

i.e., ω_1 is given by (13.5.130); moreover, in this case

$$\psi' \equiv -\frac{1}{4}v\rho^2\Delta\psi \qquad \mathsf{mod}\ |\nabla v|,$$

i.e., ψ' is given by (13.5.134).

Furthermore, let us note that in the transition from $F = 1$ to arbitrary F (see the procedure used in Theorem 13.3.6) the arguments in the Heaviside functions do not change and the trace of the matrix of quadratic form

$$(S_1 w_1 + S_2 w_2)\big\{(\{\psi, w_2\}w_1 - \{\psi, w_1\}w_2), w_1^2 + w_2^2\big\} \qquad \mathsf{mod}\ O(\nu)$$

is added to ψ'; here S_1 and S_2 are chosen from the equation

$$F\{S_1 w_1 + S_2 w_2, w_1^2 + w_2^2\} + \{F, w_2\}w_1 - \{F, w_1\}w_2 = 0 \qquad \text{at } \Lambda.$$

Then explicit calculation gives that this quantity is equal to $-\frac{1}{2}v\rho^2\nabla F \cdot \nabla\psi$ and therefore ψ' is again given by (13.5.134).

Since formulae (13.5.130) and (13.5.134) are invariant with respect to coordinate transformations we conclude that ω_1 and ψ' are given by these formulas in an arbitrary coordinate system. □

Remark 13.5.34. (i) In a method of reduction different from the method used in Proposition 13.3.1(iii) and Theorem 13.3.6 the term $O(\nu)$ is added to ω_1 and the term $O(1)$ is added to ψ'.

We have not managed to work with these additional terms and to preserve estimate (13.5.132) (or (13.5.135)); namely in this estimate the term $C'\nu^{-1}(\mu h\nu^{-1})^s$ (or $C'\mu^{-1}h^{-1}$ respectively) should be added to the right-hand expression.

(ii) On the other hand, item $O(\nu^2)$ in ω_1 could be skipped; therefore term $-\frac{1}{4}\sum_{j,k} g^{jk}\partial_j v \cdot \partial_k v$ could also be skipped.

Problem 13.5.35. Investigate the case when

$$|\nabla(v + \omega_1\mu^{-2})| + \min_j |(2j+1)\mu h + v + \omega_1\mu^{-2}| \le \epsilon_0\nu \implies \det \mathsf{Hess}(v + \omega_1\mu^{-2}) \ge \epsilon_0\nu^2 \qquad \text{in } B(0,1) \tag{13.5.137}$$

or when this condition is fulfilled for the absolute value of the Hessian determinant.

Let us formulate without proof only a simpler statement:

Theorem 13.5.36. *Let $d = 2$, $\vartheta = 0$ and let conditions* (13.2.1)–(13.2.5), (13.3.1), $(13.3.77)_{1-3}$, (13.5.1), $(13.4.12)_{1,2}$, (13.5.47) *and* (13.4.45) *be fulfilled.*

Let us assume that

$$|\frac{V}{F} + (2j+1)\mu h| + |\nabla \frac{V}{F}| \le \epsilon_0 \nu \implies |\det \mathsf{Hess}\, \frac{V}{F}| \ge \epsilon_0 \nu^2 \tag{13.5.138}$$

Then

(i) For $\nu \ge \mu h$, $\bar\nu \ge \mu^3 h$ the following estimate holds:

$$|\mathsf{R}^{\mathsf{MW}}| \le C\big(\frac{\nu}{\mu h} + \frac{\bar\nu \mu^2}{\nu}\big) + C'. \tag{13.5.139}$$

(ii) For $\nu \ge \mu h$, $\bar\nu \le \mu^3 h$ the following estimate holds:

$$|\mathsf{R}^{\mathsf{MW}}| \le C' \frac{\bar\nu}{\mu h \nu}\big(1 + \log \frac{\mu^3 h}{\bar\nu}\big) + C'. \tag{13.5.140}$$

Moreover, under condition $(13.5.135)_+$ *estimate* (13.5.126) *remains true in this case.*

(iii) For $\nu \le \mu h$ (and then $\bar\nu \le \mu^3 h$ automatically) the following estimate holds:

$$|\mathsf{R}^{\mathsf{MW}}| \le C \frac{\bar\nu}{\mu h \nu}\big(1 + \log \frac{\nu \mu^2}{\bar\nu} + 1\big) + C'. \tag{13.5.141}$$

Moreover, under condition $(13.5.74)_+$ *estimate* (13.5.126) *holds.*

13.6 Strong Magnetic Field

In this section we consider the case of *strong magnetic field*

$$\mu \ge \epsilon h^{-1} \tag{13.6.1}$$

covering also the case of *intermediate magnetic field*

$$\epsilon h^{-1} \le \mu \le C h^{-1} \tag{13.6.2}$$

and strong magnetic field proper $\mu \ge Ch^{-1}$. In the latter case to avoid trivial results as the lower Landau level becomes $\asymp \mu h$ we either consider *Magnetic Schrödinger-Pauli operator*

$$A = \sum_{j,k} P_j g^{jk} P_k + V - \mu h F, \qquad \text{with } P_j = hD_j - \mu V_j \tag{13.6.3}$$

where F is a scalar intensity of magnetic field[32] thus subtracting μhF from V or subtract $\mu h\mathfrak{z}F$ from V (alternatively setting $\tau = \mu h\mathfrak{z}F$ rather than $\tau = 0$ if $F = \mathsf{const}$). Here and below with $\mathfrak{z} = \mathsf{const}$.

In all these cases we can divide by μh thus reducing everything to the case $\mu h \asymp 1$ but $\nu \le 1$.

The behavior is very different for $d = 2$ and $d = 3$: as $d = 2$ we get spectral gaps; this also happens in the case $\nu \le \mu h$ even as $\mu h \le 1$; so basically we can say even in this case that magnetic field is strong.

We will prove that as $d = 2$, $\mu h \ge 1$ and $\tau \le c\mu h$ then the main part of asymptotics is $\asymp \mu h^{-1}$ and the remainder estimate could be as good as $O(\mu^{-s})$ (with arbitrarily large exponent s) as τ belongs to the spectral gap and $O(1)$ otherwise.

For $d = 3$ we get no new spectral gaps in addition to one from $-\infty$ to $\mathsf{inf}(V + \mu hF)$ but it may happen that one-dimensional Schrödinger operator will have effective potential $V_* \asymp \nu \le 1$ or, in other words, it will have semiclassical parameter $h_* = h\nu^{-\frac{1}{2}}$ instead of h.

We will prove that as $d = 3$, $\mu h \ge 1$ and $\tau \le c$ then for Magnetic Schrödinger-Pauli operator (13.6.3) the main part of asymptotics is $\asymp \mu h^{-2}$ and the remainder estimate could be as good as $O(\mu^{-s})$ (with arbitrarily large exponent s) as τ belongs to the (lower) spectral gap and $O(\mu h^{-1})$ otherwise.

In what follows we discuss also alternative forms of our main results.

13.6.1 Existence of Spectral Gaps

First of all, using by the standard elliptic methods (for the reduced operator) one can prove easily the following two propositions:

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[32] We consider genuine Magnetic Schrödinger-Pauli operator in Chapter 17; now we consider just "half" of it.

Proposition 13.6.1. *Let $d = 2$ and let conditions* (13.2.1)–(13.2.5), (13.3.1), $(13.3.77)_{1-3}$,

$$\mu \ge h^{-\delta} \tag{13.6.4}$$

and $(13.4.12)_1$ *be fulfilled.*

Moreover, let us assume that for some $j \in \mathbb{Z}^+$ and $\varsigma = \pm 1$

$$\varsigma\big((2j+1)\mu h + B(x_1, \xi_1, \mu^{-2}, \mu^{-1}h, h^2)\big) \ge \zeta^2 \ge C_0\bar{\nu}\mu^{-2}h^2 \qquad \forall (x_1, \xi_1) \in \Omega. \tag{13.6.5}$$

(i) Then

$$|F_{t\to h^{-1}\tau}\chi_T(t)Q_x\mathsf{U}_{jk}| \le C'T\mu h^{-1}\big(\frac{\mu^{-1}h\sqrt{\bar{\nu}}}{\zeta}\big)^s(1+|\tau|)^{-s} \qquad \forall x, y, \in \mathbb{R}^d \quad \forall T \ge h\zeta^{-2} \quad \forall \tau : \varsigma\tau \le \epsilon\zeta^2. \tag{13.6.6}$$

(ii) In particular, if condition (13.6.5) *holds for $j = 0$ and $\varsigma = 1$ (in which case it holds for $\varsigma = 1$ for all $j \in \mathbb{Z}^+$) then*

$$|F_{t\to h^{-1}\tau}\chi_T(t)U| \le C'T\mu h^{-1}\big(\frac{\mu^{-1}h\sqrt{\bar{\nu}}}{\zeta}\big)^s(1+|\tau|)^{-s} \qquad \forall x, y, \in B(0, \frac{1}{2}) \quad \forall T \ge h\zeta^{-2} \ge h \quad \forall \tau : \varsigma\tau \le \epsilon\zeta^2. \tag{13.6.7}$$

Proposition 13.6.2. *Let $d = 3$ and let conditions* (13.2.1)–(13.2.5), (13.3.1), $(13.3.77)_{1-3}$, (13.6.4) *and* $(13.4.12)_1$ *be fulfilled.*

Moreover, let us assume that

$$\mathcal{B}_0(x', \xi_1, \mu^{-2}, \mu^{-1}h, h^2) \ge \zeta^2 \ge C_0\bar{\nu}\mu^{-2}h^2 \qquad \forall (x', \xi_1) \in \Omega. \tag{13.6.8}$$

Then

$$|F_{t\to h^{-1}\tau}\chi_T(t)U| \le C'T\mu h^{-2}\big(\frac{\mu^{-1}h\sqrt{\bar{\nu}}}{\zeta}\big)^s \qquad \forall x, y, \in B(0, \frac{1}{2}) \quad \forall T \ge h\zeta^{-2} \quad \forall \tau : \tau \le \epsilon\zeta^2. \tag{13.6.9}$$

Proof. In Proposition 13.6.2 taking in account that hD_2 enters into equation in a very special way making ellipticity "stronger" we conclude that effective semiclassical parameter is still $\mu^{-1}h$. □

By the standard Tauberian methods, Propositions 13.6.1 and 13.6.2 imply the following two statements:

Proposition 13.6.3. *Let* $d = 2$ *and let conditions* (13.2.1)–(13.2.5), (13.3.1), $(13.3.77)_{1-3}$, (13.6.1), $(13.4.12)_1$ *and* (13.6.5) *be fulfilled. Then as* $\zeta \ge \mu^{-1}h^{1-\delta}$

(i) For $j = k$ *the following estimate holds:*

$$|\mathsf{e}_{Q,jk}(\tau_2) - \mathsf{e}_{Q,jk}(\tau_1)| \le C'\mu h^s \big(1 + \min(|\tau_1|, |\tau_2|)\big)^{-s} \qquad \forall \tau_1, \tau_2 : \varsigma\tau_1 \le \epsilon\zeta^2, \varsigma\tau_2 \le \epsilon\zeta^2 \quad \forall x_1, y_1 \in \mathbb{R} \tag{13.6.10}$$

where

$$\mathsf{e}_{Q,jk}(x_1, y_1, \tau) \coloneqq \iint Q_x \mathcal{F}^*_x e(x, y, -\infty, \tau)\, {}^t\mathcal{F}^*_y \Upsilon_j(x_2)\Upsilon_k(y_2)\, dx_2 dy_2 \tag{13.6.11}$$

and the functions Υ_j *are introduced by* (13.5.3).

(ii) Therefore estimate (13.6.10) *holds for* $j \ne k$ *as well.*

(iii) Therefore making a γ*-admissible partition of unity in* Ω *with* $\gamma = \mu^{-\frac{1}{2}} h^{\frac{1}{2}-\delta}$ *and applying a similar theorem with* x *and* y *permuted we obtain that estimate* (13.6.10) *holds for* $j \ne k$ *even if condition* (13.6.5) *is violated but* $\mu^{\frac{3}{2}} h^{\frac{1}{2}+\delta} \ge \bar{\nu}$.

In particular, this estimate holds for $\mu \ge h^{-\frac{1}{3}-\delta}$.

(iv) Therefore if condition (13.6.5) *with* $\varsigma = 1$ *is fulfilled then*

$$|\mathsf{e}_{Q,jk}(x_1, y_1, \tau)| \le C''\mu h^s (1 + |\tau|)^{-s} \quad \forall \tau \le \epsilon\zeta^2 \quad \forall x_1, y_1 \in \mathbb{R} \quad \forall k. \tag{13.6.12}$$

(v) On the other hand, if condition (13.6.5) *with* $\varsigma = -1$ *is fulfilled then estimate*

$$|\mathsf{e}_{Q,jk}(x_1, y_1, 0) - \int \phi(\tau)\, d_\tau \mathsf{e}_{Q,jk}(\tau)| \le C'\mu h^s \qquad \forall x_1, y_1 \in \mathbb{R} \quad \forall k \tag{13.6.13}$$

holds where ϕ is an arbitrary fixed function, equal to 0 for $\tau \le 0$ and equal to 1 for $\tau \ge 2c$.

(vi) Therefore if

$$|(2j+1)\mu h + B(x_1, \xi_1, \mu^{-2}, \mu^{-1}h, h^2)| \ge \zeta^2 \ge \bar{\nu}\mu^{-2}h^{2-\delta} \quad \forall (x_1, \xi_1) \in \Omega \quad \forall j \in \mathbb{Z}^+ \tag{13.6.14}$$

then estimate

$$|\int \psi(x)\Big(e(x,x,-\infty,0) - \int_{-\infty}^{\infty} h^{-1}\phi(\tau)\big(F_{t\to h^{-1}\tau}\chi_T(t)U^J(x,y,t)\big)\,d\tau\Big)\,dx| \le C'\mu h^s \tag{13.6.15}$$

holds for $T \in [h^{1-\delta}, 1]$ where

$$U^J = \mathcal{F}_x \sum_{j,k \le J-1} \mathsf{U}_{jk}(x_1, y_1, t)\Upsilon_j(x_2)\Upsilon_k(y_2)\,{}^t\mathcal{F}_y \tag{13.6.16}$$

with

$$J = \max\{j \in \mathbb{Z}^+ : (2j-1)\mu h + B < 0 \qquad \forall (x_1, \xi_1)\}. \tag{13.6.17}$$

In particular, $J = 0$ if and only if condition (13.6.5) *with $\varsigma = 1$ and $j = 0$ is fulfilled.*

(vii) Therefore if (13.6.5) *holds with $\varsigma = 1$, $j = 0$ and $\zeta \ge C\mu^{-1}h^{1-\delta}$ then*

$$|e(x,y,-\infty,\tau)| \le C'\mu h^s(1+|\tau|)^{-s} \quad \forall x, y \in B(0, \frac{1}{2}) \quad \forall \tau \le \epsilon\zeta^2. \tag{13.6.18}$$

Proposition 13.6.4. *In the framework of Proposition 13.6.2 (i.e. $d = 3$) with $\zeta \ge \mu^{-1}h^{1-\delta}$ estimate* (13.6.18) *holds.*

Remark 13.6.5. (i) One can prove that Statement (iii) of Proposition 13.6.3 remains true for $d = 3$ as well provided $\zeta \ge h^{1-\delta}$.

(ii) The construction of Section 13.3 yields that the conditions of Proposition 13.6.2 are fulfilled provided $d = 3$ and

$$V + \mu hF \ge \zeta^2 \ge C_0(\mu^{-1}h + h^2) \qquad \text{in} \quad B(0,1). \tag{13.6.19}$$

In this way one can prove that the assertion of Proposition 13.6.3(ii) remains true provided

$$V + \mu hF \ge \zeta^2 \ge C_0 h^2. \tag{13.6.20}$$

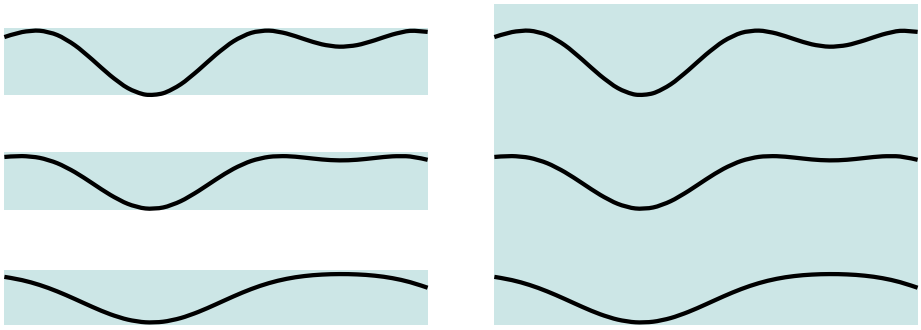
Moreover, one can weaken this condition. However, we will use another method which is simpler (see Remark 13.6.10refrem-13-5-10-ii below).

(iii) Moreover, the same construction yields that if $d = 2$ and conditions $(13.3.77)_{1-3}$, $(13.4.12)_1$ are fulfilled then conditions of Proposition 13.6.3(i) are fulfilled provided

$$V+\mu hF+\frac{1}{4}\sum_{j,k} g^{jk}\partial_j\partial_k\big(\frac{V}{F}\big)\mu^{-1}h \ge \zeta^2 \ge C_0\bar{\nu}h^2 \qquad \forall x \in B(0,1). \tag{13.6.21}$$

(iv) As $\zeta \ge C\mu^{\delta-1}h$ [33)] one can replace in all the estimates above and below h^s by μ^{-s}.

In particular, it is always the case in Proposition 13.6.3(ii) and (iii).



(a) 2D (b) 3D

Figure 13.4: Landau levels (depending on x), spectral bands (filled) and spectral gaps (white); there also is a spectral gap $(-\infty, \min_x(V + \mu hF))$

[33)] Which is different from $\zeta \ge C\mu^{-1}h^{1-\delta}$ in the case of $\mu \gg h^{-s}$ for any s.

13.6.2 Calculations in Spectral Gaps as $d = 2$

Let us pass from the analysis of the lower ("under-bottom") spectral gap to the other gaps. So, let $d = 2$ and let us assume that conditions (13.5.1) and (13.6.14) are fulfilled and $J \ge 1$. In this case Proposition 13.5.2(ii) yields that in decomposition (13.6.16) all the terms with $j \ne k$ are negligible.

Thus, repeating the decomposition of Section 13.5 in a very rough form one can replace estimate (13.5.71) by

$$\Big|\int e(x,x,-\infty,0)\psi(x)\,dx- \int \phi(\tau)\sum_{j<J}\sum_{m,p,l,n\le N}\mu^{1-2(m+p)-l}h^{l-1}\big((2j+1)\mu h\big)^m\Theta^{(n)}_{j,mpln}(\tau)\,d\tau\Big| \le C'h^s$$

provided $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^+)$ and $\int_0^\infty \phi d\tau = 1$.

Recall that the functions $\Theta_{\dots}$ are given by (13.5.44). Let us note that in this decomposition one can replace $\theta(\dots)$ by 1 in which case all terms with $n \ge 1$ disappear and we obtain

$$\Big|\int e(x,x,-\infty,0)\psi(x)\,dx- \sum_{j<J}\sum_{m,p,l,\le N}\mu^{1-2(m+p)-l}h^{l-1}\big((2j+1)\mu h\big)^m\int f_{mpl}\,dx_1 d\xi_1\Big| \le C'h^s.$$

Further, let us note that in this case the only source of terms with $l+m+p \ge 1$ is the transformation of ψ: we introduce the $\mu^{-1}h$-pseudodifferential operator $\mathcal{F}^*\psi\mathcal{F} = q(x,\mu^{-1}hD,\mu^{-2},\mu^{-1}h)$ which we then decompose in powers of x_2, $\mu^{-1}hD_2$ and μ^{-2}, $\mu^{-1}h$.

Therefore, f_{mpl} with $m+p+l \ge 1$ are linear combinations of $D^\alpha\psi\circ\bar\Psi$ with $1 \le |\alpha| \le \frac12(p+l)+m$ and only terms with $p+l \in 2\mathbb{Z}^+$ arise.

Therefore we obtain the following estimate

$$\Big|\int \Big(e(x,x,-\infty,0)\psi(x)- (2\pi)^{-1}\sum_{k+m+p\le N} J^{k+1}\mu^{1-2m-k-p}h^{k+p-1}(\mathcal{L}_{kmp}\psi)F\sqrt{g}\Big)\,dx\Big| \le C'h^s \tag{13.6.22}$$

where $N = N(s,\delta)$, J is defined by (13.6.17), $\mathcal{L}_{kmp}$ are $2(k+m+p)$-th order linear partial differential operators with the coefficients depending

on $D^\alpha g_{..}$, $D^\alpha F$, $D^\alpha V$ with $|\alpha| \le 2(k+m+p)$ and $\mathcal{L}_{000} = I$, $\mathcal{L}_{kmp}1 = 0$ for $k+m+p > 0$. In this deduction we have also used Proposition 13.5.4 and formula (13.3.13). Moreover, Proposition 13.5.4(v) yields that

$$|\mathcal{L}_{kmp}(F\sqrt{g})| \le C'\bar{\nu} \qquad \forall k, m, p : 1 \le k+m+p \le K. \tag{13.6.23}$$

Therefore we have proven

Theorem 13.6.6. *Let $d = 2$. Let us assume that conditions* (13.2.1)–(13.2.5), $(13.3.77)_{1-3}$, $(13.4.12)_1$, (13.5.1) *and* (13.6.14) *are fulfilled.*

Then estimates (13.6.22) *and*

$$|e(x, y, \tau_1, \tau_2)| \le C'h^s \qquad \forall x, y \in B(0, \frac{1}{2}) \quad \forall \tau_1, \tau_2 \in [-\epsilon\zeta^2, \epsilon\zeta^2] \tag{13.6.24}$$

hold.

Theorem 13.6.7. *Let $d = 2$. Let us assume that conditions* (13.2.1)–(13.2.5), $(13.3.77)_{1-3}$, $(13.4.12)_1$, (13.5.1) *are fulfilled.*

Furthermore, let

$$|\frac{V}{F} + (2j+1)\mu h| \ge \zeta^2 \ge C_0\mu^{-2}\bar{\nu} \qquad \forall x \in B(0,1). \tag{13.6.25}$$

Then estimates (13.6.22) *and* (13.6.24) *hold with*

$$J = \max\{j \in \mathbb{Z}^+ : V + (2j-1)\mu hF \le 0 \quad \forall x \in B(0,1)\} \tag{13.6.26}$$

and $J = 0$ if the right-hand set is empty.

Remark 13.6.8. Under condition (13.6.25) one can apply the method of successive approximations to the operator which is reduced to canonical form modulo $O(\nu(|x_2| + |\xi_2|) + \mu^{-1}h)$ provided

$$\zeta^2 \ge \nu\mu^{-1}h^{-\delta'} \tag{13.6.27}$$

with arbitrarily small δ'. Then one can take $\mathcal{F}$ and q which do not depend on μ^{-2} explicitly. Then calculations easily show that all the terms with $m \ge 1$ vanish in (13.6.22).

We leave the details to the reader (see also the proof of Proposition 17.6.9 below. However, one can easily see that condition (13.6.25) with $j \ge 1$ yields condition (13.4.85) with $\nu = (\mu h)^{1-\delta'}$ with arbitrarily small $\delta' > 0$, and this ν fulfils the above inequality.

Let us note that if $|D^\alpha f| \le c\bar\nu \quad \forall \alpha : |\alpha| \le 2$ and $f \ge \zeta^2$ in $B(0,1)$ then $|\nabla f| \le c_1(\bar\nu f)^{\frac12}$ and

$$f + t\sum_k m^k \partial_k f \ge \zeta^2(1-\epsilon) - C_2 t^2 \bar\nu$$

where $C_2 = C_2(c,\epsilon)$, $\epsilon > 0$ is arbitrary and we assume that $|\boldsymbol{m}| \le c$, $\boldsymbol{m} = (m^1, m^2)$; therefore the term $\sum_k m^k \partial_k (\frac{V}{F}) \circ \bar\Psi_0$ in the expression for ω_1 is not essential and therefore Theorem 13.6.6 yields the following theorem

Theorem 13.6.9. *Let $d = 2$ and suppose conditions* (13.2.1)–(13.2.5), $(13.3.77)_{1-3}$, $(13.4.12)_1$ *and* (13.5.1) *are fulfilled.*

Furthermore, let

$$|\frac{V}{F} + (2j+1)\mu h + \mu^{-2}\omega_1| \ge \zeta^2 \ge C_0\bar\nu(\mu^{-4} + h^2) \qquad \forall x \in B(0,1) \quad \forall j \in \mathbb{Z}^+ \tag{13.6.28}$$

where now

$$\omega_1 = \frac18 \kappa v^2 - \frac14 v\mathcal{L}v, \qquad v = \frac{V}{F}; \tag{13.6.29}$$

let us recall that κ and $\mathcal{L}$ are the scalar curvature and the Laplace-Beltrami operator in the metric $\frac{1}{F} g^{jk}$.

Then estimates (13.6.22) *and* (13.6.24) *hold with*

$$J = \max\{j \in \mathbb{Z}^+ : v + (2j-1)\mu h + \mu^{-2}\omega_1 \le 0 \quad \forall x \in B(0,1)\} \tag{13.6.30}$$

and $J = 0$ if the right-hand set is empty.

13.6.3 Final Remarks as $d = 2$

Let us make a few final remarks for $d = 2$.

Remark 13.6.10. (i) If there is a spectral gap described by (13.6.28) with $J \ge 1$ and $\bar\nu \ge \mu h$ then the following inequality holds:

$$|D^\alpha v| \le C\bar\nu\mu^{-2} + C'\bar\nu\big(\frac{\mu h}{\bar\nu}\big)^{1-\delta} \qquad \forall \alpha : |\alpha| \le K_0 \quad \forall x \in B(0, \frac{9}{10}) \tag{13.6.31}$$

where $C' = C'(c, \delta)$ and $\delta > 0$ is arbitrary and C depends only on K_0 and on a constant c for which the basic inequalities hold with K replaced by $K_0 + 2$.

Therefore for

$$\zeta^2 \geq C_0\bar{\nu}\mu^{-4} + C'\mu^{-1}h\big(\frac{\bar{\nu}}{\mu h}\big)^{\delta} \tag{13.6.32}$$

in the definition of the spectral gap (13.6.28) one can set $\omega_1 = \frac{1}{8}\kappa v^2$. Conversely if condition (13.6.28) holds with $\omega_1 = \frac{1}{8}\kappa v^2$ then this condition also holds with ω_1 given by (13.6.29).

(ii) On the other hand, for

$$\zeta^2 \geq C_0\bar{\nu}\big(\mu^{-4} + \mu^{-1}h\big) \tag{13.6.33}$$

the last term in the left-hand expression of (13.6.21) can be dropped.

Therefore for the restrictions on ζ indicated in (i) one can always set $\omega_1 = \frac{1}{8}\kappa v^2$.

Remark 13.6.11. Direct calculation shows that $\int(\mathcal{L}_{kmp}\psi)F\sqrt{g}\,dx$ with $p = 0$ and $k + m = 1$ does not necessarily vanish.

Remark 13.6.12. (i) If

$$\zeta \geq (\mu^{-1}h)^{\frac{4}{5}}\bar{\nu}^{-\frac{1}{2}} \tag{13.6.34}$$

then in the proof of Proposition 13.6.1 we can replace the Gårding inequality with the Gårding-Melin inequality (1.1.131); this leads us to the condition

$$\begin{aligned}(13.6.5)'\quad & \varsigma\big((2j+1)\mu h + B(x_1, \xi_1, \mu^{-2}, \mu^{-1}h, h^2)\big) + \\ & \frac{1}{2}\,\mathsf{Tr}^+(\mathsf{Hess}^{\#}\, B(x_1, \xi_1, \mu^{-2}, 0, 0)) \geq \zeta^2 \geq C_0\bar{\nu}(\mu^{-2}h^{2-\delta} + \mu^{-\frac{8}{5}}h^{\frac{8}{5}})\end{aligned}$$

instead of (13.6.5) where $\mathsf{Hess}^{\#}$ is the skew-Hessian of B and $\mathsf{Tr}^+ M$ is the semi-sum of absolute values of the purely imaginary eigenvalues of the matrix M.

Since for $d = 2$, $\mathsf{Hess}^{\#} B$ has only two eigenvalues η and $-\eta$ then

$$\mathsf{Tr}^+(\mathsf{Hess}^{\#} B) = (-\frac{1}{2}\,\mathsf{Tr}(\mathsf{Hess}^{\#} B)^2)_+^{\frac{1}{2}} = \frac{1}{\sqrt{2}}(\det(\mathsf{Hess}\, B))_+^{\frac{1}{2}}. \tag{13.6.35}$$

Formula (13.3.13) yields that the latter expression is equal to $2\omega_3 \circ \bar{\Psi}_0 + O(\mu^{-2}\bar{\nu})$ where

$$\omega_3 = \frac{1}{F\sqrt{8g}}(\det \mathsf{Hess}\, v)_+^{\frac{1}{2}} \tag{13.6.36}$$

and therefore (13.6.28) should here be replaced by

(13.6.37) For every $j \in \mathbb{Z}^+$ one of the following two conditions holds:

$$\pm\big(v + (2j+1)\mu h + \mu^{-2}\omega_1\big) + \mu^{-1}h\omega_3 \geq \zeta^2 \geq C_0\big(\mu^{-4} + h^2\big)$$

(in which case $\zeta^2 \geq C(\mu^{-2}h^{2-\delta} + \mu^{-\frac{8}{5}}h^{\frac{8}{5}})$ automatically) and condition (13.6.21) should be replaced here by

$$v + \mu h + \mu^{-1}h\omega_4 \geq \zeta^2 \geq C_0\bar{\nu}\big(h^2 + \mu^{-\frac{8}{5}}h^{\frac{8}{5}}\big) \tag{13.6.38}$$

where

$$\omega_4 = \frac{1}{4}\sum_{j,k} g^{jk}F^{-1}\partial_j\partial_k v + \frac{1}{F\sqrt{8g}}(\det \mathsf{Hess}\, v)_+^{\frac{1}{2}}. \tag{13.6.39}$$

(ii) One can see easily that for $d = 2, 3$ and conditions (13.2.1)–(13.2.4) and (13.3.1) operator A can be written in the form

$$A = \mathcal{M}_1^*\mathcal{M}_1 + V + \mu hF + h^2V' \tag{13.6.40}$$

and

$$A = \mathcal{M}_2^*\mathcal{M}_2 + \mathcal{M}_1^*\mathcal{M}_1 + V + \mu hF + h^2V' \tag{13.6.41}$$

respectively with $\mathcal{M}_k = \sum_j \rho_{jk}P_j$ with complex-valued (real-valued for $k = 2$) coefficients ρ_{kj} and with

$$|D^\alpha V'| \leq C' \qquad \forall \alpha : |\alpha| \leq K' - 2. \tag{13.6.42}$$

(iii) This easily yields that the assertion of Proposition 13.6.3(ii) holds for $d = 2, 3$ provided conditions (13.2.1)–(13.2.4), (13.3.1) and (13.6.20) are fulfilled in $B(0,1) \cap \bar{X}$ even if $B(0,1) \cap \partial X \neq \emptyset$ but all the conditions of Theorem 13.2.9 (excluding the restriction on τ in (13.2.52)) are fulfilled and the Dirichlet boundary condition is fulfilled on $B(0,1) \cap \partial X$.

(iv) However, this is not true for the Neumann boundary condition.

Remark 13.6.13. (i) Assume that $g^{jk} = \mathsf{const}$, $F = \mathsf{const}$ or conditions $(13.3.77)_{1,2}$ are fulfilled with really small $\bar{\nu}$; we leave details to the reader). Then the term largest uncontrolled term h^2 in the canonical form either disappears or is tamed by factor $\bar{\nu}$.

In this case we do not even need to assume that h is small parameter; it can be even large parameter as long as $\hbar = \mu^{-1}h$ remains a small parameter.

(ii) As $\mu h \ge 1$ the worst perturbation in the canonical form is $\asymp h^2$ and it comes from the "kinetic part" (not from V/F). In this case we need either to consider $\mathsf{R}^{\mathsf{MW}}_1$ instead of R^{MW} where in comparison with R^{MW} arguments of θ contain an extra term $\omega'_1 h^2$ or to assume that $\mu \le h^{-2}$ or to impose conditions $(13.3.77)_{1-2}$ with $\bar{\nu} \le C\mu^{-1}h^{-2}$.

Conclusion: As $d = 2$ and one Landau level is separated from other levels by spectral gaps (one spectral gap if we are discussing the lowest Landau level) its contribution to asymptotics is $\asymp \mu h^{-1}$. The remainder estimate as $\mu h \ge 1$ could be as good as $O(\mu^{-s})$ if τ belongs to the spectral gap and as good as $O(1)$ otherwise (as τ is close to Landau level).

13.6.4 Case $d = 3$

As $d = 3$ we really need to consider only case $\mu h \asymp 1$, $\nu < 1$.

Theorem 13.6.14. *Let* $d = 3$, $\mu h \le 1$ *and suppose conditions* (13.2.1)–(13.2.5) *and*

$$|D^{\alpha}\bigl(\frac{V}{F} + \mu h\bigr)| \le c\nu \qquad \forall \alpha : |\alpha| \le K \tag{13.6.43}$$

be fulfilled with $\nu \ge h^2$ *and* $K = K(m, \delta)$. *Then*

(i) Under non-degeneracy condition

$$\sum_{\alpha:|\alpha|\le m} |D^{\alpha}\bigl(\frac{V}{F} + (2j+1)\mu h\bigr)| \ge \epsilon\nu \qquad \forall j \in \mathbb{Z}^+ \tag{13.6.44}_m$$

estimate

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{-2}\nu + C\mu h^{-1} \tag{13.6.45}$$

holds.

(ii) In the general case estimate

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{-2}\nu + C'\mu h^{-1-\delta}\nu^{\frac{1}{2}\delta} \tag{13.6.46}$$

holds.

(iii) Meanwhile $\mathsf{N}^-(0) = O\big(\mu h^{-2}\nu^{\frac{1}{2}} + h^{-3}\nu^{\frac{3}{2}}\big)$ *and as* $(V + \mu hF) < -\epsilon\nu$ *at least in one point* $\mathsf{N}^-(0) \asymp \big(\mu h^{-2}\nu^{\frac{1}{2}} + h^{-3}\nu^{\frac{3}{2}}\big)$.

Proof. We proved it albeit with summation with respect to $\alpha > 0$ in non-degeneracy condition $(13.6.44)_m$ as $\nu = 1$. However we actually need non-degeneracy condition $(13.6.44)_m$ and we dropped $\alpha = 0$ as it used to be $\mu h \le \epsilon$.

As $\mu h \lesssim \nu$ we can reduce it to the case $\nu = 1$ dividing operator by ν; then $\mu_{\mathsf{new}} = \mu\nu^{-\frac{1}{2}}$, $h_{\mathsf{new}} = h\nu^{-\frac{1}{2}}$ and $\nu_{\mathsf{new}} = 1$.

As $\nu \le \epsilon\mu h$ we need to consider in the canonical form only $j = 0$ and then dividing canonical form by ν leads us to the same theory as before albeit while $h_1 = \mu^{-1}h$ as it used to be, $h_2 = h\nu^{-\frac{1}{2}}$. □

Alternatively we can reformulate it as

Theorem 13.6.15. *Let* $d = 3$, $\mu h \ge 1$ *and suppose conditions* (13.2.1)–(13.2.5) *and* (13.6.43) *be fulfilled with* $\nu \ge h^2$ *and* $K = K(m, \delta)$.

Then for Magnetic Schrödinger-Pauli operator (13.6.3) *all statements of Theorem 13.6.14 remain true albeit one should replace* $(V + \mu hF)$ *by* V *everywhere. In particular*

$$\mathcal{N}^{\mathsf{MW}}(\tau, \mu h) = \sum_{0\le j<\infty} \big(\tau - 2j\mu hF\big)_+^{\frac{1}{2}} \mu hF \cdot (2\pi)^{-2}\sqrt{g}. \tag{13.2.9}'_{(3)}$$

Proof. As $\mu h \ge 1$ we divide operator by μh thus getting $h_{\mathsf{new}} = (\mu^{-1}h)^{\frac{1}{2}}$, $\mu_{\mathsf{new}} = h_{\mathsf{new}}^{-1}$, $\nu_{\mathsf{new}} = \nu(\mu h)^{-1}$ with $h_{\mathsf{new}}\nu_{\mathsf{new}}^{-\frac{1}{2}} = h\nu^{-\frac{1}{2}}$, $\mu_{\mathsf{new}}h_{\mathsf{new}}^{-1} = \mu h^{-1}$. □

Conclusion: As $d = 3$ and $\mu h \ge 1$ for the Magnetic Schrödinger-Pauli operator (13.6.3) main part of the asymptotics is of magnitude μh^{-2} (or $\mu h^{-2}\nu^{\frac{1}{2}}$ as $\nu < 1$) and the remainder estimate is $O(\mu h^{-1})$ or slightly worse.

13.7 Propagation of Singularities and Sharper aAsymptotics

In this section we consider the two- and three-dimensional cases and under some assumptions of a global nature we obtain slightly more precise estimates than in the previous sections; with the except of the last subsection this improvement will be by no more than an additional factor $\varsigma \in [\mu^{-\delta'}, 1]$ with a small enough exponent $\delta' > 0$ in the right-hand expression of the estimates.

These results are not very important from the point of view of applications but they seem to be interesting enough by themselves. Here we need to revise all the arguments of Sections 13.3–13.6.

We start with the two-dimensional case when singularities propagate along magnetic drift lines and then proceed to the three-dimensional case when singularities propagate along magnetic lines which could however have "side-drift".

In the last Subsection 13.7.3 we consider supersharp asymptotics as $d = 3$ with the remainder estimate $O(\mu^{-1}h^{-2})$ as $\mu h \le 1$ and $O(h^{-1})$ as $\mu h \ge 1$.

13.7.1 Case $d = 2$

Classical Dynamics and Heuristics

Let us start from the classical dynamics. We assume that potential is truly generic i.e. no special conditions with ν are assumed. To make things simpler we assume at the moment that $g^{jk} = \delta_{jk}$ and $F = \mathsf{const}$.

Looking at the canonical form we conclude that there is fast circular movement in $(\mu x_2, \xi_2)$ and there is a slow drift movement in $(x_1, \mu^{-1}\xi_1)$. More precisely, consider first two terms of the canonical form:

$$\omega_0(x_1, \mu^{-1}\xi_1) + \omega_1(x_1, \mu^{-1}\xi_1)(\mu^2 x_2^2 + \xi_2) + O(\mu^{-2}) \tag{13.7.1}$$

and we understand that along $(\mu x_2, \xi_2)$ there is a circular movement with period $T(\mu) = \pi\mu^{-1}\omega_1^{-1} + O(\mu^{-3})$ and along $(x_1, \mu^{-1}\xi_1)$ there is a drift with the velocity $\mu^{-1}\nabla^{\#}(\omega_0 + \omega_1 k)$ with Hamiltonian calculated with respect to (x_1, ξ_1) only and with $k = (\tau - \omega_0)\omega_1$ plugged after (as we are interested at energy level 0).

Going back with symplectomorphism $\bar{\Psi}$ we conclude that modulo $O(\mu^{-3})$

$$T(\mu) = \pi\mu^{-1}F^{-1} \tag{13.7.2}$$

and the drift speed is

$$v_{\mathsf{drift}} = \mu^{-1}\bigl(\nabla\bigl(\frac{V-\tau}{F}\bigr)\bigr)^{\perp} \tag{13.7.3}$$

which defines *magnetic drift lines* which are level lines of $(V-\tau)F^{-1}$ [34].

Remark 13.7.1. Note that in an invariant form

$$F = \frac{1}{\sqrt{g}}|F_{12}|, \quad F_{12} = \partial_1 V_2 - \partial_2 V_1, \quad g = (g^{11}g^{22} - g^{12}g^{21})^{-1} \tag{13.7.4}$$

and magnetic drift is described by

$$\frac{dx_1}{dt} = \frac{1}{\sqrt{g}}\partial_2\frac{V}{f}, \qquad \frac{dx_2}{dt} = -\frac{1}{\sqrt{g}}\frac{V}{f}. \tag{13.7.5}$$

where

$$f = \frac{1}{\sqrt{g}}F_{12} \tag{13.7.6}$$

is a *pseudo-scalar*[35].

Remark 13.7.2. Magnetic drift lines are terminated in the critical points of $(V-\tau)F^{-1}$ [36].

If we are interested in the time $T \gg \mu$ we need to follow magnetic drift line either close to termination point, or to large distance, or assume that it is closed smooth line. However we need to remember that Hamiltonian trajectories follow drift lines albeit do not coincide with them!

Example 13.7.3. Let $g^{jk} = \delta_{jk}$, $F = 1$.

(i) As $V = x_2$ magnetic drift lines are just straight lines $x_2 = \mathsf{const}$ with no termination points and Hamiltonian trajectories are cycloids of Figure 13.2(b).

---

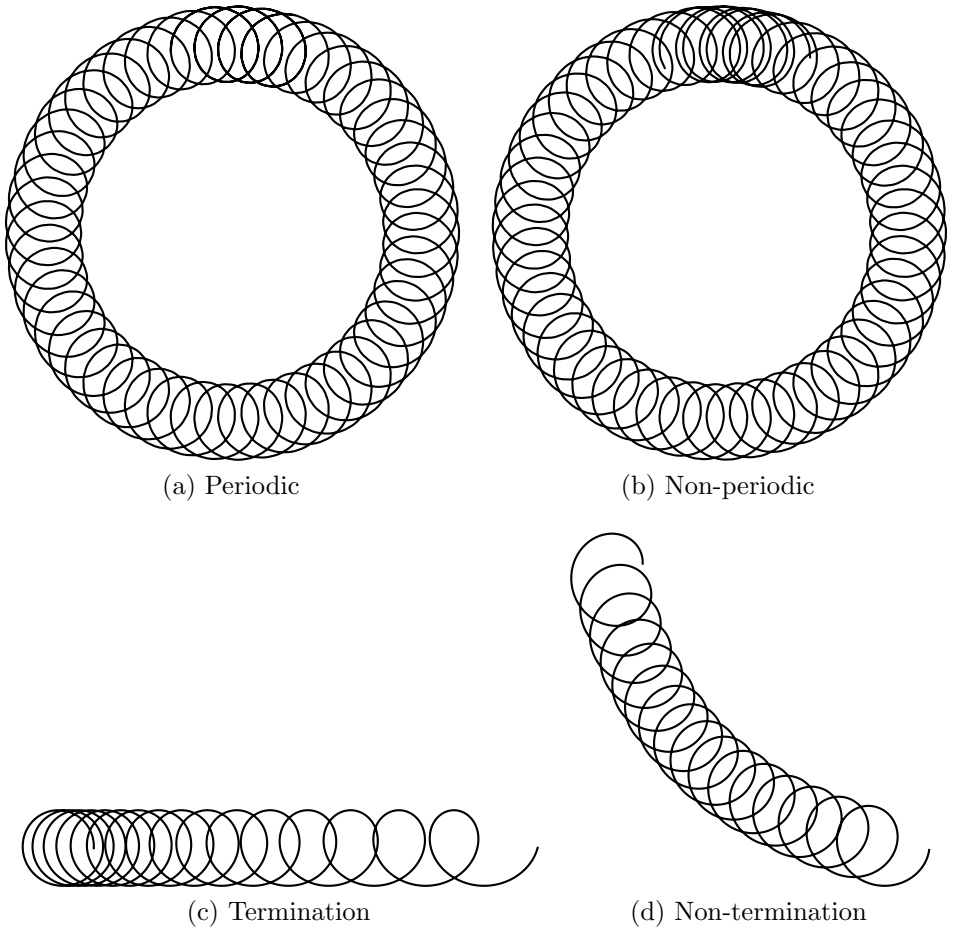
[34] In the special case $F = \mathsf{const}$ magnetic drift lines are level lines of V.

[35] I.e. it changes sign as we change orientation of the coordinate system.

[36] In the special case $F = \mathsf{const}$ those are critical points of V.

(ii) Consider $V = \pm(x_1^2 + x_2^2)$. Then magnetic drift lines are circles $x_1^2 + x_2^2 = \mathsf{const}$. On the other hand, Hamiltonian trajectories are cycloids with frequencies $2\mu(1 + O(\mu^{-2}))$ and $2\mu^{-1} + (1 + O(\mu^{-2}))$ and they are periodic only as $\mu = \mu_k = k^{\frac{1}{2}}(1 + O(k^{-1}))$ with $k \in \mathbb{Q}^+$. See Figure 13.5(a),(b).

(iii) Consider $V = x_1x_2$. Then magnetic drift lines are hyperbolas $x_1x_2 = \mathsf{const}$. On the other hand, Hamiltonian trajectories are shown on Figure 13.5(c),(d), one which terminates in $x = 0$ and another which bypasses it.



(a) Periodic (b) Non-periodic

(c) Termination (d) Non-termination

Figure 13.5: Trajectories for 2-dimensional model operator to Examples 13.7.3(ii) (in (a),(b)) and 13.7.3(iii) (in (c),(d))

Our conjecture is that we can derive remainder estimate $o\bigl(\mu^{-1}h^{-1}+1\bigr)$ in all the cases when drift lines are not closed; if drift lines are closed we can derive this remainder estimate for μ except belonging to rather thin exceptional set.

One should notice that magnetic drift lines escaping to infinity are rather dangerous as then V does not grow which may indicate presence of the essential spectrum.

Propagation of Singularities

Theorem 13.7.4. *Let $d=2$ and ψ_1,ψ_2 satisfy* (13.4.84) *with $\gamma=\mu^{-1}$.*

Let us assume that along trajectories of (13.7.5) *starting in the intersection of the $C_0\gamma$-vicinity of $\operatorname{supp}\psi_1$ with $\{-V/(\mu hF)\in\mathbb{Z}^+\}$ (where here and below C_0 is a large enough constant, $L=L(I)$ is large enough and $\delta=\delta(I)>0$ is small enough) the following conditions are fulfilled*

$$(13.7.7)_{1-3}\quad |D^\alpha g^{jk}|\le c\zeta^{-I},\quad |D^\alpha f|\le c\zeta^{-I},\quad |D^\alpha V|\le c\zeta^{-I}\qquad \forall \alpha:|\alpha|\le K,$$

$$F\ge \epsilon_0\zeta^I, \tag{13.7.8}$$

$$\sum_{j,k} g^{jk}\eta_j\eta_k\ge \epsilon_0\zeta^I|\eta|^2\qquad \forall \eta\in\mathbb{R}^3, \tag{13.7.9}$$

$$\operatorname{dist}(\phi_t(x),\partial X)\ge \epsilon_0\zeta^I, \tag{13.7.10}$$

$$\operatorname{dist}(\phi_t(x),x)\ge \epsilon_0\zeta^I\min(|t|,1), \tag{13.7.11}$$

$$|D_x\phi_t(x)|\le c\zeta^{-I} \tag{13.7.12}$$

where ϕ_t is a magnetic drift flow defined by system (13.7.5), *$D_x\phi_t(x)$ denotes the Jacobi matrix of $\phi_t(x)$ and $\zeta\ge h^\delta$, $0\le\pm t\le\pm T'$,*

$$|T'|\le T\le \zeta^{-1}\nu^{-1} \tag{13.7.13}$$

$\delta>0$ is a sufficiently small exponent and $\nu=1$ in (i).

Moreover, let us assume that for $t=T$ the distance between these trajectories and $\operatorname{supp}\psi_2$ is larger than $C_0\mu^{-1}\zeta^{-L}$.

Then

(i) The following estimate holds:

$$|D_x^\alpha D_y^\beta F_{t\to h^{-1}\tau}\chi_{\mu T,\mu T'}(t)\psi_1(x)U\psi_2(y)| \le Ch^s \quad \forall \tau : \mu T|\tau| \le h^{1-\delta} \quad \forall \alpha,\beta : |\alpha| \le s, |\beta| \le s. \tag{13.7.14}$$

(ii) Moreover, if conditions

$$|D^\alpha g^{jk}| \le c\nu\mu\zeta^{-l}, \quad |D^\alpha F| \le c\nu\mu\zeta^{-l}, \quad |D^\alpha \frac{V}{F}| \le c\nu\zeta^{-l} \quad \forall \alpha : 1 \le |\alpha| \le K, \tag{13.7.7}'_{1-3}$$

hold along this trajectory then one can replace $T = \zeta^{-1}$ by $T = \nu^{-1}\zeta^{-1}$.

(iii) As $\mu h \ge 1$ both Statements (i) and (ii) remain true for operator with V replaced by $(V - k\mu hF)$ with $k = \mathsf{const}$.

Proof. We leave the details proof to the reader. The main and actually only obstacle is the absence of the global reduction. However, reduction can be done locally and then one can break interval $[-\mu T, \mu T]$ on the subintervals of the lengths $\asymp \mu\zeta^l$. □

Remark 13.7.5. Recall that as $\mu h \ge \nu$ the spectral gaps may appear and in this case we need to apply our results only outside of those gaps.

We will need however a more delicate theorem to tackle periodic trajectories of the magnetic drift flow. Recall that these lines are level lines of VF^{-1} and therefore if some such line is closed and non-degenerate, all lines close to it have the same property and therefore such lines lines are always coming "en masse".

Theorem 13.7.6. *Assume that the level lines*

$$\ell(\lambda) = \{x \in X : VF^{-1} = \lambda\} \tag{13.7.15}$$

are closed and non-degenerate (i.e. $|\nabla(VF^{-1})| \ge \epsilon$). Let $T(\lambda)$ be a corresponding period

$$T(\lambda) = \frac{1}{2}\int_{\ell(\lambda)} \frac{|dx|_g}{|\nabla(VF^{-1})|_{g^{-1}}} \tag{13.7.16}$$

where both lengths are measured according in the corresponding metrics:

$$|dx|_g = \Big(\sum_{j,k} g_{jk}\,dx_j\,dx_k\Big)^{\frac12}, \qquad |\nabla v|_{g^{-1}} = \Big(\sum_{j,k} g^{jk}\partial_j v\partial_k v\Big)^{\frac12}. \tag{13.7.17}$$

Let T' be an arbitrarily large constant. Let ψ_1, ψ_2 be supported in ε-vicinity of $\bar{x}$ with $\bar{\lambda} = (VF^{-1})(\bar{x})$ and $\varepsilon = \varepsilon(T')$. Then

(i) Let $|T' - nT(\bar{\lambda})| \ge C_0 T$ for all $n \in \mathbb{Z}$ and $T = C_0\varepsilon$. Then (13.7.14) *holds. Here C_0 does not depend on T' and ε.*

(ii) Let $|T' - nT(\bar{\lambda})| \le C_0 T'$ for some $n = \bar{n} =\in \mathbb{Z}$. Let now ψ_1, ψ_2 be supported in γ-vicinity of $\bar{x}$ with $\gamma = \mu^{-\frac12} h^{\frac12-\delta}$ and $\mu \ge Ch^{1-2\delta}$. Then

$$|\Gamma F_{t\to h^{-1}\tau}\chi_{\mu T,\mu T'}(t)\psi_1(x)U\psi_2(y)| \le Ch^s \qquad \forall \tau : \mu T|\tau| \le h^{1-\delta} \tag{13.7.18}$$

unless

$$|T' - nT(\bar{\lambda})| \le C_0\gamma, \qquad |n\rho(\bar{\lambda}, \mu^{-1})\mu^2 - m| \le C\mu\gamma \tag{13.7.19}$$

with some $m \in \mathbb{Z}$ where

$$\rho(\lambda, \mu^{-1}) \sim \sum_{j\ge 0} \rho_j(\lambda)\mu^{-j}, \tag{13.7.20}$$

$$\rho_0(\lambda) = \frac{1}{\pi}\int_{\ell(\lambda)} \frac{F|dx|_g}{|\nabla(VF^{-1})|_{g^{-1}}} \tag{13.7.21}$$

Proof. Details we leave to the reader as well, just note that during one period $\mu T(\lambda)$ fast movement makes $\mu^2\rho(\lambda, \mu^{-1})$ rotations and for it minimal distance in uncertainty principle is $\mu\gamma$ (could be improved). □

Spectral Asymptotics

Applying standard Tauberian arguments one can easily prove the following

Theorem 13.7.7. *Let $d = 2$ and let conditions* (13.2.1)–(13.2.5), (13.3.1) *and* (13.4.3) *be fulfilled.*

Further, let conditions (13.4.85), $(13.3.77)_{1-3}$ *and* $(13.4.12)_{1,2}$ *be fulfilled and*

$$\min_j |V + (2j+1)\mu hF| + |\nabla\big(V + (2j+1)\mu hF\big)| \ge \epsilon_0\nu. \tag{13.7.22}$$

Furthermore, let either $\mu \ge h^{-\delta}$ or $\nu \ge \mu h^{1-\delta}$ *and condition* (13.4.46) *be fulfilled.*

Finally, let $\Xi_1, \dots, \Xi_n$ *be closed subsets in* $B(0,1)$ *and let* $1 \le T_1 \le \dots \le T_n \le \mu^{\delta'}$ *with a small enough exponent* $\delta' = \delta'(\delta, l) > 0$. *Let us assume that through every point* x *belonging to the* γ_i*-neighborhood of* Ξ_i *(with* $\gamma_i = \mu^{-1}\zeta_i^{-L}$, $\zeta_i = T_i^{-1}$ *where* $L = L(\delta, l)$ *is large enough) there passes a trajectory of the flow* $\phi_t(x)$ *given by the system* (13.7.5) *with* $\phi_0(x) = x$, $\pm t \in [0, \nu^{-1}T_i]$ *along which conditions* $(13.7.7)'_{1-3}$, (13.7.8)–(13.7.9),

$$(13.7.10)' \qquad \operatorname{dist}(\phi_t(x), \partial X) \ge \gamma_i,$$

$$(13.7.11)' \qquad \operatorname{dist}(\phi_t(x), x) \ge \gamma_i \min(|t|, 1),$$

(13.7.12) *and* (13.7.12) *are fulfilled.*

(i) Then estimate

$$(13.7.23) \quad \mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)} \le C\Big(\frac{\mu h}{\nu} + 1\Big) \sum_{0 \le i \le n} \sum_j \frac{h^{-1+\vartheta}\nu^{1+\vartheta}\mu^{-1-\vartheta}}{T_i^{1+\vartheta}} \int_{\Xi_i \cap \{V_j = 0\}} dx : dV_j + C' h^s$$

holds where $V_j = V + (2j+1)\mu h$ *and* $\Xi_0 = B(0,1) \setminus (\Xi_1 \cup \dots \cup \Xi_n)$.

(ii) Moreover, for $\nu = 1$, $\vartheta = 0$, *the same estimate holds for* R^{MW} *if* either $\mu h \ll 1$ *or one includes the term* $-\kappa_2$ *into* R^{MW}.

(iii) As $\mu h \ge 1$ *Statements (i) and remain (ii) true for operator with* V *replaced by* $(V - k\mu h F)$ *with* $k = \mathsf{const}$ [37].

Corollary 13.7.8. *Let* $d = 2$ *and let conditions* (13.2.1)–(13.2.5), (13.3.1) *and* (13.4.3) *be fulfilled. Further, let condition* (13.7.22) *be fulfilled with* $\nu = 1$.

Finally, let Λ *be closed subset of measure* 0 *in* $B(0,1)$ *and let us assume that through every point* x *belonging to the* $B(0,1)\setminus\Lambda$ *there passes a trajectory of the flow* $\phi_t(x)$ *given by the system* (13.7.5) *with* $\phi_0(x) = x$, $\pm t \in [0, \infty)$ *along which conditions* (13.7.7)–(13.7.12) *holds with* $\zeta = \zeta(x,t)$.

(i) Then $\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)} = o(h^{-1+\vartheta}\mu^{-1-\vartheta})$ *as* $h \to +0$, $\mu \to +\infty$.

[37] Sure, we need to plug this into $\mathcal{N}^{\mathsf{MW}}$ as well.

(ii) Moreover, for $\vartheta = 0$, the same estimate holds for R^{MW} if either $\mu h \ll 1$ or one includes the term $-\kappa_2$ in R^{MW}.

(iii) As $\mu h \ge 1$ Statements (i) and (ii) remain true for operator with V replaced by $(V - k\mu hF)$ with $k = \mathsf{const}$ [37].

Assume now that 0 is (an only) non-degenerate critical point of VF^{-1} in $B(0,1)$. Then introducing $\gamma(x) = \frac{1}{2}|x|$ we conclude that in $B(\bar{x}, \gamma(\bar{x})) \mapsto B(0,1)$ after rescaling $x \mapsto (x - \bar{x})\gamma(\bar{x})^{-1}$ conditions of theorem are fulfilled with $\mu_{\mathsf{new}} = \mu\gamma$, $h_{\mathsf{new}} = h\gamma^{-1}$, $\nu_{\mathsf{new}} = \gamma$, and therefore, as $\gamma \ge \bar{\gamma} := (\mu^{-1}h)^{\frac{1}{2}}$ its contribution to R^{T} does not exceed

$$C\Big(\frac{\nu_{\mathsf{new}}}{\mu_{\mathsf{new}} h_{\mathsf{new}}} + 1\Big) = C\Big(\frac{\gamma}{\mu h} + 1\Big)$$

and therefore the contribution of all such elements with $C\mu^{-1} \le |x| \le \varepsilon$ does not exceed

$$C \int \Big(\frac{\gamma}{\mu h} + 1\Big) \frac{\gamma \, d\gamma}{\gamma^2} \asymp C\Big(\frac{\varepsilon}{\mu h} + |\log \mu|\Big). \tag{13.7.24}$$

One can prove easily that contribution of $B(0, \bar{\gamma})$ is less than this.

Further, if 0 is a saddle point we can play with escape and replace $|\log \mu|$ by 1 and if 0 is an extremum there is no escape in points close to 0.

Thus we arrive to

Corollary 13.7.9. *In the framework of Corollary 13.7.8 let us allow non-degenerate critical points of VF^{-1} and escape condition is fulfilled in all other points.*

(i) Then as $h \to 0$, $\mu \to \infty$, $\mu h \to 0$ $\mathsf{R}^{\mathsf{MW}}_{\infty} = o(\mu^{-1}h^{-1})$.

(ii) Further, as either $\mu \le h^{-\frac{1}{3}+\delta}$ or $\mu \ge h^{-\frac{1}{3}-\delta}$ the same is true for R^{MW}. Otherwise one needs to replace R^{MW} by $\mathsf{R}^{\mathsf{MW}}_1$.

More delicate but also due to standard Tauberian estimates and Theorem 13.7.6 is the following theorem:

Theorem 13.7.10. *Let conditions of Theorem 13.7.6 be fulfilled as original points of flow ϕ_t belongs to $B(0,1)$. Then*

(i) Assume that

$$\mathsf{mes}\{x : \rho_0(\lambda(x)) = k\} = 0 \qquad \forall k \in \mathbb{R}^+. \tag{13.7.25}$$

Then Statements (i) and (ii) of Corollary 13.7.8 hold as $h \to 0$, $\mu \to \infty$, $\mu \le h^{\delta-1}$.

(ii) Assume that

$$\rho(\lambda, \mu^{-1}) = \rho(\mu^{-1}). \tag{13.7.26}$$

Then Statements (i) and (ii) of Corollary 13.7.8 hold as $h \to 0$, $\mu \to \infty$, $\mu \le h^{2\delta-1}$ *provided* μ *satisfies also*

$$\mathsf{dist}\big(\mu^2\rho(\mu^{-1})n, \mathbb{Z}\big) \ge \epsilon_n \mu^{\frac{1}{2}} h^{\frac{1}{2}-\delta} \qquad \forall n \in \mathbb{Z}. \tag{13.7.27}$$

Example 13.7.11. In the framework of Example 13.7.3(ii) we have a quadratic Hamiltonian

$$\mathsf{a}(x, \xi) = (\xi_1 - \frac{1}{2}\mu x_2)^2 + (\xi_2 + \frac{1}{2}\mu x_1)^2 + \varsigma(x_1^2 + x_2^2) \tag{13.7.28}$$

and eigenvalues of $\frac{1}{2}\,\mathsf{Hess}^{\#}\,\mathsf{a}$ are purely imaginary $\pm i\omega_1$, $\pm i\omega_2$ with

$$\omega_{1,2}^2 = \frac{1}{2}(\mu^2 + 2\varsigma) \pm \Big(\frac{1}{2}(\mu^2 + 2\varsigma)^2 - \varsigma^2\Big)^{\frac{1}{2}}. \tag{13.7.29}$$

Note that

$$\omega_1^2 = \mu^2 + 2\varsigma - \varsigma^2\mu^{-2} + O(\mu^{-4}), \qquad \omega_2^2 = \varsigma^2\mu^{-2} + O(\mu^{-4}). \tag{13.7.30}$$

Then (13.7.30) implies classical dynamics as on Figure 13.5(a) and (b) with $\rho = \mu^{-2}\omega_1/\omega_2 = |\varsigma|^{-1} + O(\mu^{-2})$ but also that the eigenvalues are

$$\tau_{m,n} = (2m+1)\omega_1 h + (2n+1)\omega_2 h \qquad m, n \in \mathbb{Z}^+. \tag{13.7.31}$$

Therefore we get two imposed lattices with steps $2\omega_1 h$ and $2\omega_2 h$.

13.7.2 Case $d = 3$

Now we consider 3-dimensional case.

Preliminaries and Improved Reduction

It is well known from physics that the classical three-dimensional particles in a strong magnetic field move almost along magnetic lines. So it is not surprising that in our analysis magnetic lines are also important. Let us recall that a magnetic line of the magnetic field $\boldsymbol{F} = (F^1, F^2, F^3)$ is an integral curve $x = x(t)$ of the field $F^{-1}\boldsymbol{F}$. Thus, $x(t)$ is given by the system

$$\frac{dx_j}{dt} = \frac{2}{F}F^j \qquad (j = 1, 2, 3) \tag{13.7.32}$$

where

$$F = \big(\sum_{j,k} g_{jk}F^jF^k\big)^{\frac{1}{2}} = \frac{1}{2}\big(\sum_{j,k,l,m} g^{jk}g^{lm}F_{jl}F_{km}\big)^{\frac{1}{2}} \tag{13.7.33}$$

is the scalar intensity; let us introduce the corresponding flow Φ_t on X.

It follows from the definition of $\boldsymbol{F}$ that this field is solenoidal, i.e.,

$$\sum_j \partial_j(F^j\sqrt{g}) = 0 \tag{13.7.34}$$

and therefore

(13.7.35) *Magnetic flow* Φ_t preserves the density $\sqrt{g}dx$ on X.

Moreover,

(13.7.36) Magnetic flow ϕ_t can be lifted to the Hamiltonian flow $\tilde{\Phi}_t$ on Λ which is given by the Hamiltonian

$$\frac{1}{F}\sum_j F^j\big(\xi_j - \mu V_j(x)\big). \tag{13.7.37}$$

Remark 13.7.12. We could define the flow Φ_t by the system

$$\frac{dx_j}{dt} = \frac{2}{F}\omega F^j \quad (j = 1, 2, 3) \tag{13.7.32$'$}$$

with functions ω and ω^{-1} regular along the trajectory; then $\tilde{\Phi}_t$ would be given by the Hamiltonian

$$\frac{\omega}{F}\sum_j F^j\big(\xi_j - \mu V_j(x)\big) \tag{13.7.37$'$}$$

and the invariant density would be $\omega^{-1}F\sqrt{g}dx$.

The proof of Proposition 13.3.1 and the analysis which follows in Section 13.3 easily yield the following statement (the detailed proof of which we leave to the reader):

Theorem 13.7.13. *Let conditions* (13.2.1)–(13.2.5) *and* (13.3.1) *be fulfilled.*

Let $\hat{x} \in B(0, \frac{1}{2})$ *and for all* $x \in B(0, \zeta^l)$ *let the flow* Φ_t *be defined by system* (13.7.32) *with* $t \in [-T_1, T_2]$ *where* $\zeta \in [\mu^{-\delta'}, 1]$, $\delta' > 0$ *is small enough,* $0 \le T_i \le \zeta^{-1}$ *and* $T_1 T_2 = 0$.

Moreover, let us assume that the following conditions are fulfilled along this flow:

$$(13.7.38)_{1-3} \quad |D^\alpha g^{jk}| \le c\zeta^{-l}, \quad |D^\alpha F^j| \le c\zeta^{-l}, \quad |D^\alpha V| \le c\zeta^{-l} \qquad \forall \alpha : |\alpha| \le K,$$

$$F \ge \epsilon_0 \zeta^l, \tag{13.7.39}$$

$$\sum_{j,k} g^{jk}\eta_j\eta_k \ge \epsilon_0 \zeta^l |\eta|^2 \qquad \forall \eta \in \mathbb{R}^3, \tag{13.7.40}$$

$$\operatorname{dist}(\Phi_t(x), \partial X) \ge \epsilon_0 \zeta^l, \tag{13.7.41}$$

$$\operatorname{dist}(\Phi_t(x), x) \ge \epsilon_0 \zeta^l \min(|t|, 1), \tag{13.7.42}$$

$$|D_x \Phi_t(x)| \le c\zeta^{-l} \tag{13.7.43}$$

where $D_x\Phi_t(x)$ *denotes the Jacobi matrix of* $\Phi_t(x)$. *Then*

(i) All the reductions of Section 13.3 can be done in the subdomain

$$\bigcup_{t \in [-T_1, T_2]} \Phi_t\big(B(\hat{x}, \epsilon\zeta^l)\big) \tag{13.7.44}$$

of X *where* $\epsilon = \epsilon(l, c, \epsilon_0) > 0$ *is a small enough constant, and then all the symplectomorphisms and symbols satisfy the estimates*

$$(13.7.45)_{1-2} \quad |D^\alpha \Psi_{\dots}(x, \xi)| \le C'\zeta^{-L}, \quad |D^\alpha b_{\dots}(x, \xi)| \le C'\zeta^{-L} \qquad \forall \alpha : |\alpha| \le K'$$

with arbitrary N *and* K' *(and then* K, L, δ' *depend on* N, K', l*); moreover, for* $x \in \Phi_t(B(\hat{x}, \epsilon\zeta^l))$ *with* $|t| \le \epsilon$ *both these inequalities hold with* $L = 0$.

(ii) Moreover, the same statement remains true for the flow Φ_t *defined by* $(13.7.32)'$ *provided that on domain* (13.7.44)

$$|D^\alpha \omega| \le c_1 \zeta^{-l} \qquad \forall \alpha : |\alpha| \le K, \tag{13.7.45}_3$$

and

$$\omega \ge \epsilon_1^{-1} \zeta^l. \tag{13.7.46}$$

(iii) Finally, this statement remains true without condition (13.7.42) *if we introduce coordinates* (x_2, x') *in the pipe* (13.7.44) *transforming it into a cylinder (thus removing self-intersections).*

Remark 13.7.14. Even from the heuristic point of view there is an essential difference:

(i) Magnetic lines depend only on F while magnetic drift lines depend on V as well. In order to the get magnetic drift lines on the energy level τ one should replace V by $V - \tau$. Then, for $F \ne \mathsf{const}$ the magnetic drift lines depend on the energy level as well.

(ii) As soon as magnetic lines are defined, the movement along them is defined by the quantum number j and the energy level τ. Moreover, the direction of the movement depends on (13.7.37). In contrast, in the two-dimensional case the choice of j is automatic and the movement is defined only by the energy level which should be fixed. System (13.7.5) is a dynamical system while (13.7.32) only describes geometrical shape.

(iii) The movement along magnetic drift lines is given in scaled time; the real time is μt. According to the theory of Sections 2.2 and 2.4, t should not be too large. Moreover, under condition (13.4.85) it is sufficient to assume that νt is not too large (because we can divide operator by ν and then the scaled time will be νt).

(iv) Vf^{-1} is constant along magnetic drift lines and for big enough μ (e.g., $\mu \ge h^{-\frac{1}{2}}$) one need only consider lines with $-V/(\mu hF) \in 2\mathbb{Z}^+ + 1$.

Weak Magnetic Field

In this subsection we assuming that (13.4.1) holds improve certain statements of Section 6.4. For this goal let us first assume that the reduction of the

Section 13.3 is done and let us take $(\bar{x}', \bar{\xi}_1) \in \Omega$ [38]. Let us introduce sets

$$\Lambda^0_{(\bar{x}_1,\bar{\xi}_1)} = \{(x', \xi') \in \Omega \times \mathbb{R}, x_1 = \bar{x}_1, \xi_1 = \bar{\xi}_1\}, \tag{13.7.47}$$

1-dimensional Hamiltonians

$$H_{(\bar{x}_1,\bar{\xi}_1)r}(x_2, \xi_2) = \xi_2^2 + \bar{B}'_0(\bar{x}_1, x_2, \bar{\xi}_1) + \bar{B}_1(\bar{x}_1, x_2, \bar{\xi}_1)r \tag{13.7.48}$$

and the corresponding Hamiltonian flows $\Psi_{t,r}$ on $\Lambda^0_{(\bar{x}_1,\bar{\xi}_1)}$. Here $(\bar{x}_1, \bar{\xi}_1, r)$ is considered constant when we define the Hamiltonian field.

Theorem 13.7.15. *Let $q', q'' \in S_{\boldsymbol{h},\rho,\gamma,K}(T^*\mathbb{R}^3)$ and $\boldsymbol{h}, \boldsymbol{\rho}, \boldsymbol{\gamma}$ satisfy* (13.4.6)*, $\rho_1 = \rho_2 = \gamma_1 = \gamma_2 = \gamma$ and $\rho_3 = \mu^{\frac{1}{2}} h^{\frac{1}{2}(1-\delta)}$, $\gamma_3 = \mu^{-\frac{1}{2}} h^{\frac{1}{2}(1-\delta)}$.*

Further, let $\epsilon_0 \le T^* \le \zeta^{-1}$ and let $\delta' = \delta'(\delta, s, l)$ in the condition $\zeta \ge \mu^{-\delta'}$ to ζ.

Furthermore, let us assume that

$(13.7.49)_\pm$ *If the point (x, ξ) belongs to the $(\boldsymbol{\rho}, \boldsymbol{\gamma})$-neighborhood of $\operatorname{supp}(q')$ and if $|r - (\mu^2 x_3^2 + \xi_3^2)| \le C\mu^{-l} + \rho_3$ then for all t: $\pm t \in [0, 2T^*]$*

$$\Psi_{t,r}(x', \xi') \in \Omega \times \mathbb{R}$$

and along $\Psi_{t,r}(x', \xi')$ conditions (13.7.38)–(13.7.43) *and the following conditions are fulfilled:*

$$(13.7.50)_{1,2} \quad |D^\alpha B_n| \le c\zeta^{-l}, \quad |D^\alpha R| \le c\mu^{-N}\zeta^{-l} \qquad \forall \alpha : |\alpha| \le K \quad \forall n \le N$$

where B_n and R are taken from (13.3.66)*, $N = N(\delta, s, l)$, $K = K(\delta, s, l)$ and*

$$|D_{(x',\xi')}\Psi_{t,r}(x', \xi')| \le c\zeta^{-l} \tag{13.7.51}$$

where $D_Z\Psi$ means the Jacobi matrix of Ψ at the point Z. Then

(i) Let one of the following conditions be fulfilled:

$$(13.7.52)_1 \qquad |x_1 - y_1| \ge C_0 T^* \mu^{-1}\zeta^{-L} + \gamma,$$

$$(13.7.52)_2 \qquad |\xi_1 - \eta_1| \ge C_0 T^* \mu^{-1}\zeta^{-L} + \gamma,$$

$$(13.7.52)_3 \qquad |x_2 - y_2| \ge C_0 T^* \zeta^{-L},$$

$$(13.7.52)_4 \qquad |\xi_2 - \eta_2| \ge C_0 T^* \zeta^{-L},$$

[38] Recall that $\Lambda^0 = \{(x, \xi),\ (x', \xi_1) \in \Omega, \xi_2 = \xi_3 = x_3 = 0\}$.

for all $(x,\xi) \in \operatorname{supp} q'$ *and for all* $(y,\eta) \in \operatorname{supp} q''$ *or let condition* (13.4.10) *be fulfilled where* $L = L(\delta, s, l)$ *is large enough here.*

Then for all $x, y \in \mathbb{R}^3$, $\tau \le c$ *estimate* (13.4.9) *with* $\chi_T(t)$ *replaced by* $\chi_{T^*,\pm T^*}(t)$ *holds; recall that* χ *is supported in* $[-\frac{1}{2}, \frac{1}{2}]$, $Q' = \operatorname{Op}(q')$, $Q'' = \operatorname{Op}(q'')$.

(ii) Let $\gamma \ge \mu^{\frac{1}{2}} h^{\frac{1}{2}(1-\delta)}$. *Moreover, let us assume that if* $(x,\xi) \in \operatorname{supp} q'$ *and* $|r - (\mu^2 x_3^2 + \xi_3^2)| \le C_0 \mu^{-l} + \rho_3$ *and*

$$|-\bar\tau + \xi_2^2 + \bar B_0'(x',\xi_1) + \bar B_1(x',\xi_1) r| \le C\rho_3 + C\zeta^{-L}\mu^{-2} = C\bar\gamma \tag{13.7.53}$$

then

$$\begin{aligned} &\operatorname{dist}(\Phi_{t,r}(x',\xi'),(y',\eta')) \ge C\bar\gamma \\ &\qquad \forall (y,\eta) \in \operatorname{supp} q'' : |\mu^2 y_3^2 + \eta_3^2 - r| \le C_0\mu^{-l} + \rho_3 \quad \forall t : \pm t \in [T', T''] \end{aligned} \tag{13.7.54}$$

where $0 \le T' < T'' \le 2T^*$. *Then estimate* (13.4.9) *with* $\chi_T(t)$ *replaced by* $\chi_{T,\pm T'''}(t)$ *with* $T = \frac{1}{2}(T'' - T')$, $T''' = \frac{1}{2}(T'' + T')$ *holds for all* $x, y \in \mathbb{R}^3$, $\tau \in [\bar\tau - \epsilon\bar\gamma, \bar\tau + \epsilon\bar\gamma]$ *where* $\epsilon > 0$ *is a small enough constant.*

Proof. Let us apply Theorem 2.4.12. In order to do this we need to compare the Hamiltonian flow generated by the three-dimensional Hamiltonian $\mathcal{A}(x,\xi)$ with the flow $\Psi_{t,r}$. Let us introduce variables

$$Z = (x_1, \mu^{-1}\xi_1, x_2, \xi_2, r, \mu^{-2}x_3, \mu^{-2}\xi_3)$$

with $r = \mu^2 x_3^2 + \xi_3^2$, which will be treated as an "independent" variable in what follows.

In these variables the flow $\Psi_{t,r}$ is described by the system $dZ/dt = f(Z)$ and the Hamiltonian flow generated by $\mathcal{A}(x,\xi)$ is described by the system $dZ/dt = \tilde f(Z)$, and the first, second, sixth and seventh components of f vanish, and for $(x',\xi_1) \in \Omega$ with $\xi_2^2 + \xi_3^2 + \mu^2 x_3^2 \le C_0\zeta^{-l}$ estimate

$$|D^\alpha(f(Z) - \tilde f(Z))| \le C\mu^{-1}\zeta^{-L} \qquad \forall \alpha : |\alpha| \le K \tag{13.7.55}$$

holds with appropriate L.

Let $Z(t)$ and $\tilde Z(t)$ be trajectories corresponding to these two systems originating from the same point. We assert that

$$|Z(t) - \tilde Z(t)| < C\mu^{-1}\zeta^{-L'} \qquad \forall t : \pm t \in [0, 2T^*] \tag{13.7.56}$$

with appropriate $L' = L'(\delta, s, l)$.

In fact, if this statement is wrong then (13.7.56) holds for $\pm t \in [0, T)$ with $T < 2T^*$ and for $\pm t = T$ there is equality in (13.7.56). But in this case (13.7.55) yields that for $\pm t \in [0, T]$

$$|\frac{d}{dt}(Z - \tilde{Z})(t) - (D_{Z(t)}f)(Z - \tilde{Z})(t)| \le C\mu^{-1}\zeta^{-L}$$

where $(D_Z f)(Z')$ means the differential of f at the point Z applied to Z'. Therefore (13.7.51) yields that

$$|\frac{d}{dt}\mathcal{J}^{-1}(Z - \tilde{Z})(t)| \le C\mu^{-1}\zeta^{-2L}$$

where $\mathcal{J}$ is the 7×7 Jacobi matrix corresponding to the system $dZ/dt = f(Z)$; its elements are equal to δ_{jk} for $j \ne 3, 4$, to 0 for $j = 3, 4$ and $k = 6, 7$ and the upper left 4×5-submatrix of $\mathcal{J}$ is equal to $D_{(x',\xi',r)}\Phi_{t,r}(x', \xi')$; the determinant of $\mathcal{J}$ is equal to 1.

Therefore (13.7.51) yields that the norms of $\mathcal{J}$ and $\mathcal{J}^{-1}$ do not exceed $C\zeta^{-2L}$. Since $\mathcal{J}(0) = I$, $Z(0) = \tilde{Z}(0)$ and $T \le \zeta^{-1}$ it follows that

$$|\mathcal{J}^{-1}(Z - \tilde{Z})(t)| \le C\mu^{-1}\zeta^{-3L}$$

and therefore $|(Z - \tilde{Z})(t)| \le C\mu^{-1}\zeta^{-5L}$. But then for $\zeta \le \zeta_0 = \mathsf{const}$ condition (13.7.56) with $L' = 9L$ cannot be violated and for $\zeta \in [\zeta_0, 1]$ this condition is provided by the appropriate choice of C_0. So condition (13.7.56) is fulfilled and for the sake of simplicity one can assume that $L' = L$.

Let us now verify conditions of Theorem 2.4.12 on Jacobi matrix. Let us vary the initial data in which case we obtain the system

$$\frac{d}{dt}\mathfrak{d}\tilde{Z} = (D_{\tilde{Z}}\tilde{f})\mathfrak{d}\tilde{Z}$$

where $\mathfrak{d}$ denotes variation; this equation and (13.7.56) yield that

$$\frac{d}{dt}\mathfrak{d}\tilde{Z} = (D_Z\tilde{f})\mathfrak{d}\tilde{Z} + O(\mu^{-1}\zeta^{-2L}|\mathfrak{d}\tilde{Z}|)$$

and therefore

$$|\frac{d}{dt}\mathcal{J}(t)^{-1}\mathfrak{d}\tilde{Z}| \le C\mu^{-1}\zeta^{-3L}|\mathcal{J}(t)^{-1}\mathfrak{d}\tilde{Z}|.$$

This inequality yields that $|\mathcal{J}(t)^{-1}\mathfrak{d}\tilde{Z}| \le C|\mathfrak{d}\tilde{Z}(0)|$ provided $\delta' > 0$ is small enough in the condition on ζ. Then

$$|\frac{d}{dt}\mathcal{J}(t)^{-1}\mathfrak{d}\tilde{Z}| \le C\mu^{-1}\zeta^{-3L}|\mathfrak{d}\tilde{Z}(0)|$$

and therefore

$$|\mathfrak{d}\tilde{Z}(t) - \mathcal{J}(t)\mathfrak{d}\tilde{Z}(0)| \le C\mu^{-1}\zeta^{-5L}|\mathfrak{d}\tilde{Z}(0)|.$$

This latter inequality obviously yields that

$$|D_{(x(0),\xi(0))}(x'(t),\xi(t))| \le C\zeta^{-5L}$$

and

$$|D_{(x(0),\xi(0))}(x_3(t),\xi_3(t))| \le C\mu\zeta^{-5L}.$$

Therefore for $\mu \in [\mu_0, h^{-\delta'}]$ with a small enough exponent $\delta' > 0$ our theorem follows from Theorem 2.4.12.

On the other hand, for $\mu \in [h^{-\delta}, h^{\delta-1}]$ this theorem is due to Proposition 13.7.21 which will be proven independently below and which is also due to Theorem 2.4.12; excluding (x_3,ξ_3) from the analysis we will be able to check conditions only for part of the Jacobi matrix. □

Remark 13.7.16. One can replace condition (13.7.51) in Theorem 13.7.15 by condition

$$|\xi_2| \ge \epsilon\zeta^{l} \qquad \text{along } \Psi_{t,r}(x',\xi'). \tag{13.7.57}$$

In order to prove this let us first of all note that the definition of $\Psi_{t,r}$ yields that in $\Psi_{t,r}(x',\xi')$ only (x_2,ξ_2)-components need to be checked. Moreover, the fact that $H_{(\bar{x}_1,\bar{\xi}_1)r}(x_2,\xi_2)$ is constant along trajectories, $(13.7.50)_{1,2}$ and (13.7.57) yield that one need check only x_2-component (we change l if necessary).

Let $z = (x_1,\xi_1,r)$ and $W(z,x_2) = \bar{B}_0'(z,x_2) + \bar{B}_1(z,x_2)r$. Then

$$\pm 2t = \int_{x_2(0)}^{x_2(t)} (E - W(z,s))^{-\frac{1}{2}}\,ds \tag{13.7.58}$$

with $E = \xi_2^2(0) + W(z,x_2(0))$. Moreover, (13.7.57) yields that

$$E - W(z,s) = \xi_2^2 \ge \epsilon_0\zeta^{2l}.$$

Let us treat $x_2(t)$ as an implicit function of $E, z, x_2(0)$ and t (where E is a function of $\xi_2(0)$, $x_2(0)$ and r but the derivatives of E with respect to these variables are easily estimated). Then (13.7.58) and (13.7.57) yield that the absolute values of the first order partial derivatives of t with respect to E and z do not exceed $C\zeta^{-2l}|t|$ and that the absolute values of the first order partial derivatives of t with respect to $x_2(t)$ and $x_2(0)$ do not exceed $C\zeta^{-l}$ and are greater than $\epsilon\zeta^l$ (we have again used (13.7.50)).

This yields that the absolute values of the first order partial derivatives of $x_2(t)$ with respect to $x_2(0), z, \xi_2(0)$ do not exceed $C\zeta^{-5l}$.

Remark 13.7.17. (i) Condition (13.7.57) provides that the movement is irrevocable:

$$\mathsf{dist}(\Phi_{t,r}(x',\xi'),(x',\xi')) \ge \epsilon\zeta^l \min(|t|,1) \qquad \forall t : \pm t \in [0, 2T^*]. \tag{13.7.59}$$

(ii) One can rewrite condition (13.7.57) in the form

$$|\sum_j F^j(\xi_j - \mu V_j)| \ge \epsilon\zeta^l \tag{13.7.60}$$

or

$$|\tau - V - rF| \ge \epsilon\zeta^l \tag{13.7.61}$$

along the trajectory in the original coordinates.

(iii) Moreover, returning to the original coordinates one can write the trajectory equation in the form

$$\frac{dx_j}{dt} = \frac{2}{F}(\tau - V - rF)^{\frac{1}{2}}F^j \qquad (j = 1,2,3) \tag{13.7.32}''$$

where we reverse the time if necessary.

(iv) Finally, returning to $U(t)$ one can pick $Q' = \psi'$, $Q'' = Q'''\psi''$ where $Q''' = \mathcal{F}\tilde{Q}'''\mathcal{F}^*$ and r runs through all the possible values of $x_3^2 + \mu^2\xi_3^2$ on $\mathsf{supp}(\tilde{q}'''\psi'')$ where small and capital letters denote symbols and corresponding operators.

Alternatively, r runs through all the possible values of

$$F^{-1}\big(\tau - V - F^{-2}\big(\sum_j F^j(\xi_j - \mu V_j)\big)^2\big) \tag{13.7.62}$$

on $\mathsf{supp}\, q'''$.

Therefore Theorem 13.7.15 yields that if its conditions and condition (13.7.57) are fulfilled then estimate (13.4.33) holds with $T = T^*$ and with $T' \in [h^{1-\delta}, T^*]$ for all $\tau \in [-\epsilon'\zeta^l, \epsilon'\zeta^l]$. Therefore we can apply Tauberian arguments with $T = T^*$ instead of $T = T_0$.

Let us now assume that condition (13.4.46) is fulfilled. Then the analysis of Section 13.4 yields that for $\mu \le h^{\delta-\frac{1}{3}}$ the main contribution to R is given by the zone $\{(x,\xi) : |\xi_2| \ge c\zeta\}$.

More precisely, if we calculate R' for the operator $Q' = \mathcal{F}\tilde{Q}'\mathcal{F}^*$ with $\tilde{Q}' = \mathsf{Op}(\tilde{q}')$, $\tilde{q}' \in \mathsf{S}_{\boldsymbol{h}',\zeta,\zeta,K}(T^*\mathbb{R}^2)$ supported in $\{|\xi_2| \le 2c\zeta\}$ and for the function ψ_1 supported in $B(0, \frac{1}{2})$ and satisfying (13.4.84) then the estimate

$$\mathsf{R}' \le C\zeta\gamma^3 h^{-2} + C'h^s \tag{13.7.63}$$

holds provided $\mathsf{diam}(\mathsf{supp}\,\psi_1) \le c\gamma$.

Let us estimate the contribution of the zone $\{(x,\xi), |\xi_2| \ge c\zeta\}$ to R by means of Tauberian arguments. Namely, let us estimate R' for operator $Q' = \mathcal{F}\tilde{Q}'\mathcal{F}^*$ with $\tilde{Q}' = \mathsf{Op}(\tilde{q}')$, $\tilde{q}' \in \mathsf{S}_{\boldsymbol{h}',\zeta,\zeta,K}(T^*\mathbb{R}^2)$ supported in $\{|\xi_2| \ge c\zeta\}$ and for the function ψ_1 supported in $B(0, \frac{1}{2})$ and satisfying (13.4.84) with $\gamma \ge \mu^{-1}$ and $\mathsf{diam}(\mathsf{supp}\,\psi_1) \le c\gamma$.

Then under conditions on the flows $\Phi_{\bar{t}}$ and $\Psi_{t,r}$ estimate (13.7.63) holds; here $\Phi_{\bar{t}}$ is given by (13.7.32)$''$ with $(\pm)\bar{t} \in [0, \bar{T}]$ and $\bar{T} \le \zeta^{-l}$, conditions $(13.7.45)_{1,2}$, $(13.7.49)_{\pm}$ are assumed to be fulfilled and the signs $(\pm)$ of $\bar{t}$ and $\pm$ of t are independent; moreover, one should take $r \in [0, r^*]$ where r^* is the minimal value of $r \in \mathbb{R}$ for which the inequality

$$V + Fr + c^2\zeta^2 \ge 0 \quad \text{at} \quad B(\hat{x}, \zeta^l) \tag{13.7.64}$$

is fulfilled (we now pick $\tau = 0$).

One can easily see that all the conditions excluding irrevocability should be checked for $r = 0$ only. Thus, let us consider the flow defined by (13.7.32)$''$ with $r = 0$, $\tau = 0$:

$$\frac{dx_j}{dt} = \frac{2}{F}\sqrt{-V}F^j \qquad (j = 1,2,3). \tag{13.7.32}'''$$

Let Φ_t be the corresponding flow.

Then the irrevocability for $r \in [r^*, r]$ is due to the following condition

$$v(\Phi_t(x)) \le v(x) + c^2\zeta^2 \qquad \forall t : \pm t \in [0, T] \tag{13.7.65}$$

with $v = VF^{-1}$.

Then our arguments yield estimate (13.7.63) in this case also; thus it is true for $Q = I$ as well. Summing with respect to a partition of unity in $B(0, \frac{2}{3})$ we obtain the principal statement of this step:

Theorem 13.7.18. *Let conditions* (13.2.1)–(13.2.5), (13.3.1), (13.4.1) *and* (13.4.46) *be fulfilled.*

Let $\Xi_1, \dots, \Xi_n$ be closed subsets of $B(0,1)$ and let $1 \le T_1 \le \dots \le T_n \le \mu^{\delta'}$ with a small enough exponent $\delta' = \delta'(\delta, I) > 0$.

Let us assume that through every point x belonging to the γ_i-neighborhood of Ξ_i (with

$$\gamma_i = (\mu^{\frac{1}{2}} h^{\frac{1}{2}(1-\delta)} + C_0\mu^{-1})\zeta_i^{-L},$$

$\zeta_i = T_i^{-1}$ where $L = L(\delta, I)$ is large enough) there passes a trajectory of the flow $\Phi_t(x)$ given by the system (13.7.32)‴ *with $\Phi_0(x) = x$, $\pm t \in [0, T_i]$ and along this trajectory conditions* (13.7.38)–(13.7.40), (13.7.43), (13.7.65),

$$V \le -\epsilon\zeta^2, \tag{13.7.66}$$

(13.7.38)$_{1-3}$, (13.7.43) *and*

$$\operatorname{dist}(\Phi_t(x), \partial X) \ge \gamma_i, \tag{13.7.41}'$$

$$\operatorname{dist}(\Phi_t(x), x) \ge \gamma_i \min(|t|, 1), \tag{13.7.42}'$$

are fulfilled with $\zeta = \zeta_i$. Then

(i) For $\mu \le h^{\delta - \frac{1}{3}}$ estimate

$$\mathsf{R}^{\mathsf{MW}} \le \sum_{0 \le i \le n} C T_i^{-1} h^{-2} \operatorname{mes}(\Xi)_i + C' \tag{13.7.67}$$

holds with $T_0 = 1$, $\Xi_0 = B(0,1) \cap \{V \le 0\} \setminus (\Xi_1 \cup \dots \cup \Xi_n)$.

(ii) Under condition (13.4.54) *this estimate holds for $\mu \le h^{\delta - 1}$.*

Remark 13.7.19. (i) In the theorem we have replaced $\gamma_i = \zeta_i^l$ which follows from the previous analysis by $\gamma_i = \mu^{-1}\zeta_i^{-L}$.

One can do this for the following reasons. Let us note that despite the absence of condition (13.7.42) which prevents the global[39] reduction of our

[39] I.e., in $\bigcup_t \Phi_t(B(\hat{x}, \zeta^{\frac{1}{2}})$.

operator to canonical form, we may still reduce the operator to canonical form locally[40] provided $|T_2 - T_1| \le \zeta^{L'}$ with appropriate exponent L').

Moreover, the absence of condition (13.7.42) does not prevent us from making the global reduction in the canonical coordinates described in Theorem 13.7.13(iii). But then the singularities of $U\psi''(y)\,{}^tQ''_y$ (intersected with $\{|\tau| \le \epsilon\gamma\}$) lie in the pipe consisting of the lines described below starting in the γ-vicinity of $\operatorname{supp}\psi''$.

Moreover, for $t \in [T_1, T_2]$ with $|T_2 - T_1| \le \zeta^{L'}$ these singularities lie in the part of the pipe where reduction is possible. This permits us to skip condition (13.7.42) and prove estimate

$$|F_{t\to h^{-1}\tau}\chi_{T,\pm T'}(t)Q'_x\psi'(x)U\psi''_y\,{}^tQ''_y| \le C'h^s \qquad \forall \tau \in [-\epsilon\gamma\zeta^L, \epsilon\gamma\zeta^L]$$

provided $Q' = \operatorname{Op}(q')$, $Q'' = \operatorname{Op}(q'')$, $q', q'' \in \mathscr{S}_{h,\gamma,\gamma,K}(T^*\mathbb{R}^3)$, ψ', ψ'' satisfy (13.4.84), $\operatorname{diam}(\operatorname{supp}\psi') \le c\gamma$, $\operatorname{dist}(\operatorname{supp}\psi', \partial X) \ge \epsilon_0\gamma$ (and the same is true for ψ''),

$$\operatorname{supp} q' \subset \iota\{|\omega| \ge c\zeta^l\} \tag{13.7.68}$$

where $\iota : T^*X \to X \times \mathbb{R}$ is the natural projection $\iota(x,\xi) = (x,\omega)$ with $\omega = \frac{1}{F}\sum_j F^j(\xi_j - \mu V_j)$, and the following condition is fulfilled:

(13.7.69) If we consider trajectories of the system

$$\frac{dx_j}{dt} = 2\omega\frac{1}{F}F^j \quad (j = 1,2,3), \tag{13.7.70}_1$$

$$F\frac{d\omega}{dt} = -(\omega^2 F^j\partial_j F + F^j\partial_j\frac{V}{F}) \tag{13.7.70}_2$$

passing through $\iota(\operatorname{supp} q')$, then along these trajectories $\pm\omega \ge \zeta^l$ and for $t \in [T' - T, T' + T]$

$$\operatorname{dist}((x(t),\omega(t),\iota(y,\eta)) \ge \gamma \qquad \forall (y,\eta) \in \operatorname{supp} q'',$$

or if similar conditions with x, y, q', q'', t replaced by $y, x, q'', q', -t$ respectively are fulfilled. System $(13.7.70)_{1,2}$ is an alternative form of $(13.7.32)''$.

Results of this type permit us to apply Tauberian arguments.

[40] I.e., in $\bigcup_t \Phi_{t\in[T_1,T_2]}(B(\hat{x}, \zeta^{\frac{1}{2}L'})$.

(ii) Let us replace condition (13.7.41) by (13.7.41)$'$.

In order to do this let us apply the previous arguments but in each "small" time interval let us add the following idea: let us "move" the boundary ∂X by the distance ζ^l and let us treat the new fundamental solution U^0; our previous arguments yield that $Q'_x\psi'(x)U^0$ is negligible provided $y \in X$ and $\mathsf{dist}(y, \partial X) \ge \mu^{-1}\zeta^{-L'}$ where X means the "old" X; it is then easy to obtain that $Q'_x\psi'(x)U \equiv Q'_x\psi'(x)U^0$, i.e., the analysis of propagation of singularities along trajectories satisfying (13.7.41)$'$ can be reduced to the analysis of propagation of singularities of U^0 (along trajectories satisfying (13.7.41) in X_{new}).

(iii) The connections between $\mathsf{R}^{\mathsf{W}\prime}$ and R^{W} (see Section 13.4) and between R and R^{MW} (see Section 13.5) yield that in Theorem 13.7.18 estimate (13.7.67) remains true for both R^{W} and R^{MW} as $\mu \le h^{\delta-\frac{1}{3}}$; moreover, under condition (13.4.54) this estimate holds for R^{MW} as $\mu \le h^{\delta-1}$.

We leave to the reader the following easy problem

Problem 13.7.20. (i) Formulate the statement similar to Corollary 13.7.8.

(ii) Consider the case $\vartheta > 0$. Recall that in this case the contribution of the zone $\{(x,\xi) : |\xi_2| \le c\zeta\}$ is negligible without a condition on VF^{-1} only if $\mu \le h^{-\delta''}$ with $\delta' = \delta''(\vartheta)$.

Moderate Magnetic Field

Let us assume now that condition (13.5.1) is fulfilled; we want to improve the arguments of Section 13.5. In order to do this let us introduce the one-dimensional Hamiltonian (13.7.48) with $r = (2j+1)\mu h$; let us denote it by $H_{(x_1,\xi_1)j}(x_2,\xi_2)$ and let us denote the corresponding Hamiltonian flow by $\Phi_{t,j}(x_2,\xi_2)$. Let us improve Proposition 13.5.3 (in what follows we will improve it again):

Proposition 13.7.21. *Let* $j \in \mathbb{Z}^+$, $Q' = \mathsf{Op}(q')$, $Q'' = \mathsf{Op}(q'')$ *where* $q', q'' \in \mathsf{S}_{\boldsymbol{h}',\gamma,\gamma,K}(T^*\mathbb{R}^2)$ *with* $\gamma \ge h^{\frac{1}{2}(1-\delta)}$ *and* $\boldsymbol{h}'$ *defined in* (13.5.18). *Let us assume*

$(13.7.71)_{\pm}$ [41] *If* (x',ξ') *belongs to* γ*-neighborhood of* $\mathsf{supp}(q')$ *then for all* $t : \pm t \in [0, 2T^*]$ $\Phi_{t,j}(x',\xi')$ *belongs to* $\Omega \times \mathbb{R}$

[41] Compare with $(13.7.49)_{\pm}$.

and also that along $\Phi_{t,j}(x',\xi')$ *conditions* (13.7.38)–(13.7.40), (13.7.43), (13.7.50)–(13.7.51) *are fulfilled.*

Further, let $\zeta \in [\mu^{-\delta'},1]$ *with* $\delta'=\delta'(\delta,l,s)>0$. *Then*

(i) Let <u>*either*</u> *one of conditions* $(13.7.52)_{1-4}$ *be fulfilled* <u>*or*</u> *let* $j\neq k$. *Then estimate* (13.5.22) *holds with* $\chi_T(t)$ *replaced by* $\chi_{T^*,\pm T^*}$.

(ii) Furthermore, assume that for all $(x',\xi')\in \operatorname{supp} q'$, $(y',\eta')\in \operatorname{supp} q''$

$$(13.7.72)_{\pm} \qquad \operatorname{dist}\bigl(\Phi_{t,j}(x',\xi'),(y',\eta')\bigr)\geq C\bar\gamma \qquad \forall t:\pm t\in[T',T''],$$

with

$$(13.7.73) \qquad \bar\gamma = h^{\frac12(1-\delta)}+C_0\zeta^{-L}\mu^{-2},\quad 0\leq T'<T''\leq 2T^*.$$

Then estimate (13.5.22) *holds with* $\chi_T(t)$ *replaced by* $\chi_{T,\pm T'''}(t)$ *where* $x,y\in\mathbb{R}^3$, $\tau\in[\bar\tau-\epsilon\bar\gamma,\bar\tau+\epsilon\bar\gamma]$, $T=\frac12(T''-T')$, $T'''=\frac12(T''+T')$.

Proof. This proof repeats the proof of Theorem 13.7.15 but now the variables (x_3,ξ_3,r) are absent and the proof is simplified. Moreover, for $h^{-\delta}\leq\mu\leq h^{\delta-1}$ Theorem 13.7.15 obviously follows from this proposition. □

Remark 13.7.22. (i) Obviously, one need only check conditions $(13.7.71)_{\pm}$ and $(13.7.72)_{\pm}$ for (x',ξ') such that

$$(13.7.74) \qquad |H_{(x_1,\xi_1),j}(x_2,\xi_2)-\bar\tau|\leq C\bar\gamma$$

and for (y',η') satisfying the same restriction.

(ii) One should remember the relationship $r=(2j+1)\mu h$ between $r\in\mathbb{R}^+$ from the previous subsection and $j\in\mathbb{Z}^+$ used here.

Proposition 13.7.23. *Let the conditions of Proposition 13.5.5 be fulfilled and let* $\rho_1=\rho_2=\gamma_1=\gamma_2\geq h^{\frac12(1-\delta)}$.

Let us assume that for every $j\leq J$ *conditions* $(13.7.71)_{\pm}$ *are fulfilled and along the trajectory conditions* (13.7.38)–(13.7.40), (13.7.43), (13.7.50), (13.7.51) *are fulfilled*[42)].

Then formula $(13.5.24)_{(3)}$ *remains true for* $\tau\in[\bar\tau-\epsilon\bar\gamma,\bar\tau+\epsilon\bar\gamma]$ *and* $T\in[1,2T^*]$.

[42)] We take into account remark 13.7.22).

Proof. Let us recall that in Proposition 13.5.5 the restriction $T \le T_0$ appeared only in order to get rid of terms with $j \ne k$. But now Theorem 13.7.18(i) permits us to do this for $T \le 2T^*$. In fact, applying an admissible partition of unity we see that we should prove $(13.5.24)_{(3)}$ under additional condition $\mathsf{diam}(\mathsf{supp}\, q' \cup \mathsf{supp}\, q'') \le c\gamma$. □

Let us assume that the magnetic field is not very strong, i.e., that condition (13.4.1) is fulfilled. Then by Section 13.5 the contribution to R_1 of the zone $\{|\xi_2| \le \zeta^L\}$ with $\zeta \le \mu^{-\delta'}$ does not exceed $Ch^{-2}\zeta$ where $\delta > 0$ is arbitrarily small, L is large enough and $\delta' = \delta'(\delta, L) > 0$ is small enough.

So, only the zone $\{|\xi_2| \ge \zeta^L\}$ should be treated and in this zone all the arguments of the previous subsection remain true. Then we obtain the following proposition:

Proposition 13.7.24. *Let the conditions of Theorem 13.7.18 be fulfilled as well as condition* (13.5.1)*. Then estimate* (13.7.67) *holds for* $\mathsf{R}^{\mathsf{MW}}_{\infty}$ *and* $\mathsf{R}^{\mathsf{MW}}_{1}$ *without any additional assumptions.*

However, the transition from $\mathsf{R}^{\mathsf{MW}}_{1}$ to R^{MW} is not very smooth: one can easily prove that $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}$ (defined by (13.5.83)) does not exceed

$$R' := Ch^{-2} \int_{\{\gamma \ge \mu^{-1}\}} \Big(\frac{\mu h}{\gamma^2}\Big)^s \, dx + Ch^{-2}(\mu^3 h)^s \, \mathsf{mes}(\{\gamma \le \mu^{-1}\}) \tag{13.7.75}$$

for $\mu^3 h \le 1$ and does not exceed

$$\begin{aligned} R' := C\mu^{-\frac{3}{2}} h^{-\frac{5}{2}} \int_{\{\gamma \ge \sqrt{\mu h}\}} \Big(\frac{\mu h}{\gamma^2}\Big)^s \, dx + \\ C\mu^{-1}h^{-2} \int_{\{\mu^{-1} \le \gamma \le \sqrt{\mu h}\}} \gamma^{-1} \, dx + Ch^{-2} \, \mathsf{mes}(\{\gamma \le \mu^{-1}\}) \end{aligned} \tag{13.7.76}$$

for $\mu^3 h \ge 1$; recall that

$$\gamma = |\nabla \frac{V}{F}| + \min_j |\frac{V}{F} + (2j+1)\mu h|^{\frac{1}{2}}. \tag{13.7.77}$$

Therefore we obtain the following proposition:

Proposition 13.7.25. *Let the conditions of Theorem 13.7.18 be fulfilled as well as condition* (13.5.1).

(i) Then estimate

$$\mathsf{R}^{\mathsf{MW}} \le \sum_{0\le i\le n} CT_i^{-1}h^{-2}\,\mathsf{mes}(\Xi_i) + R' + C' \tag{13.7.78}$$

holds with R' given by (13.7.75) *and* (13.7.76) *for $\mu^3 h \le 1$ and $\mu^3 h \ge 1$ respectively.*

(ii) In particular, one can skip the second term under non-degeneracy condition $(13.5.58)_m$ *with $\nu = 1$.*

Corollary 13.7.26 [43]**.** *Let conditions* (13.2.1)–(13.2.5), (13.3.1), (13.4.1) *and* (13.4.46) *be fulfilled.*

Finally, let Λ be closed subset of measure 0 in $B(0,1)$ and let us assume that through every point x belonging to the $B(0,1)\setminus\Lambda$ there passes an infinitely long in one direction trajectory of the flow $\Psi_{r,t}(x)$ along which conditions of Theorem 13.7.18 are fulfilled with $\zeta = \zeta(x,t)$ [44]

(i) As $h \to 0$, $\mu \to \infty$ and <u>either</u> $\mu^3 h = o(1)$ <u>or</u>

$$\mathsf{mes}(\{x : \nabla \frac{V}{F} = 0\}) = 0 \tag{13.7.80}$$

asymptotics

$$\mathsf{R}^{\mathsf{MW}} = o(h^{-2}) \tag{13.7.81}$$

holds.

(ii) Without additional assumptions of (i) this us true for R^{MW} modified in the following way: magnetic Weyl expression is written for $V(x) + \kappa(x)\mu^{-2}$ and $-\kappa_1 h^{-3}\mu^{-2}$ is subtracted.

Problem 13.7.27. We again leave the case $\vartheta > 0$ to the reader.

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[43] Cf. Corollary 13.7.8.

[44] Note that then

$$\frac{V}{F}(\Phi_t(x)) \le \frac{V}{F}(x) \qquad \forall t : \pm t > 0 \tag{13.7.79}$$

Now let us assume that the magnetic field is transitional, i.e.,

$$h^{1-\delta} \le \mu \le h^{-1} \tag{13.7.82}$$

with an arbitrarily small exponent $\delta > 0$.

Recall that both R^{T} and R^{MW} do not exceed $C(h^{-2} + \mu h^{-1}(h/\sqrt{\nu})^{-\delta}$ and in order the second (uncontrollable) term did not exceed $Ch^{-2}\zeta$ (where ζ is the measure of the smallness) one should assume that condition (13.4.85) with

$$\nu \le ch^2(\zeta/\mu h)^L \tag{13.7.83}$$

with arbitrarily large exponent L is fulfilled.

Alternatively both R^{T} and R^{MW} do not exceed Ch^{-2} under <u>either</u> non-degeneracy condition $(13.5.58)_m$ with $\nu = 1$ <u>or</u> (13.4.85) and non-degeneracy condition $(13.5.58)_m$ with $\nu < 1$. Then the "matching" part of the remainder which corresponds to a single number j in the definition of Landau levels is $C\mu h^{-1}$ and we assume that

$$\zeta \ge c\mu h. \tag{13.7.84}$$

Then in both cases (general with (13.7.83) and non-degenerate with (13.7.84)) results of Section 13.5 yield that the contributions of the zone $\{(x', \xi') : |\xi_2| \le 2c\zeta\}$ to R^{T} and $\mathsf{R}^{\mathsf{MW}}_{\infty}$ do not exceed $C\zeta h^{-2}$.

Then all the above arguments remain true and we easily get the following proposition:

Proposition 13.7.28. *Let all the conditions of Proposition 13.7.25 excluding condition* (13.4.1) *be fulfilled. Further, let condition* (13.7.84) *be fulfilled*[45)].

Furthermore, let condition (13.4.85) *be fulfilled where <u>either</u> ν satisfies* (13.4.82) *or non-degeneracy condition* $(13.5.58)_m$ *is fulfilled.*

(i) Then estimate (13.7.78) *remains true for* $\mathsf{R}^{\mathsf{MW}}_1$.

(ii) Moreover if non-degeneracy condition is fulfilled with $\nu = 1$ then estimate (13.7.78) *remains true for* R^{MW}.

Proof. Easy details are left to the reader. We just note that without non-degeneracy condition error resulting from dropping term $O(\mu^{-2})$ while going from $\mathsf{R}^{\mathsf{MW}}_1$ to R^{MW} could be as bad as $O(\mu^{-1} \times \mu h^{-2}) = O(h^{-2})$.

We leave to the reader to weaken assumption $\nu = 1$ in Statement (ii).□

[45)] All conditions of this proposition should be fulfilled for all $\zeta = T_i^{-1}$, $i = 1, \dots, n$.

Strong Magnetic Field

Now we are going to consider case

$$\mu h = 1. \tag{13.7.85}$$

We recall that the case of $\mu h \ge \nu$ and the case of the Schrödinger-Pauli operator with $\mu h \ge 1$ can be reduced to this one.

Then in our decompositions $j \le \bar{\jmath}$. Without any loss of the generality one can assume that

$$V + (2\bar{\jmath} + 3)\mu h \ge \epsilon_0 \tag{13.7.86}_1$$

and

$$V + (2\bar{\jmath} - 1)\mu h \le -\epsilon_0 \tag{13.7.86}_2$$

in $B(0,1)$ (otherwise we can arrange this by an ϵ-partition of unity).

If $\bar{\jmath} \ge 2$ then the contribution of terms with $j = 0, \dots, \bar{\jmath} - 1$ can be estimated according to the above results. One can easily see that only trajectories with $j = \bar{\jmath} - 1$ need to be treated.

Let us consider the contribution of $j = \bar{\jmath}$ and for the sake of simplicity in the following statements let us assume that $\bar{\jmath} = 1$. Moreover, let us assume that

$$|D^\alpha V_{\bar{\jmath}}| \le c\nu \qquad \forall \alpha : |\alpha| \le K \tag{13.7.87}$$

and

$$|V_{\bar{\jmath}}| + |\nabla V_{\bar{\jmath}}| \ge \epsilon_0 \nu \tag{13.7.88}$$

with $\nu \ge h^2$, $V_j = V + (2j+1)\mu h F$. Then in order to improve the results of Section 13.5 one should treat the trajectories of the system

$$\frac{dx_i}{dt} = \frac{1}{F\sqrt{\nu}}(-V_{\bar{\jmath}})^{\frac{1}{2}} F^i \qquad (i = 1,2,3). \tag{13.7.32}^{IV}$$

One should assume that $\zeta_i^{-1} = T_i \le {h'}^{-\delta'}$ with $h' = \left(h/\sqrt{\nu}\right)$ and that along these trajectories all the conditions (13.7.38), (13.7.43), $(13.7.41)'$, $(13.7.42)'$,

$$|D^\alpha V_{\bar{\jmath}}| \le c\nu\zeta^{-l}, \tag{13.7.38}'_3$$

and

$$V_{\bar{\jmath}} \ge \epsilon_0 \zeta^l \tag{13.7.65}'$$

are fulfilled.

Proposition 13.7.29. *(i) Under all these assumptions estimate*

$$\mathsf{R}^{\mathsf{MW}} \le \sum_{0\le i\le n} Ch^{-2}T_i^{-1}\,\mathsf{mes}(\Xi_i) + C'h^{-2}h'^{s} \tag{13.7.78}'$$

holds.

(ii) Under all these assumptions with $\nu = 1$, $\mu h \ge 1$ for Schrödinger-Pauli operator (with properly modified main part) estimate

$$\mathsf{R}^{\mathsf{MW}} \le \sum_{0\le i\le n} C\mu h^{-1}T_i^{-1}\,\mathsf{mes}(\Xi_i) + C'\mu h^{s} \tag{13.7.78}''$$

holds.

Proof. We leave details of the proof and the possible corollaries and generalizations to the reader. □

13.7.3 Supersharp Estimates as $d = 3$

Framework

Now instead of escape conditions we assume that all trajectories of $\Psi_{r,t}$ are trapped either because all magnetic lines are periodic or by the growing potential. We discuss only the latter case but the former could be analyzed in the same way. Idea is that magnetic lines drift with a speed $O(\mu^{-1})$ and because of this drift we can increase T^* from $\asymp 1$ to $\asymp \mu$.

So, let us consider coordinate system in which $F^1 = F^2 = 0$, $F^3 \ne 0$ and

$$\frac{V}{F} > 0 \qquad \text{as } |x'| < 1,\ |x_3| \ge C_0 \tag{13.7.89}$$

where $x' = (x_1, x_2)$.

Then V/F reaches its minimum $z^0(x')$ with respect to x_3 at $[-C_0, C_0]$; we assume that this minimum is unique and non-degenerate we also assume that it is unique and moreover

$$\epsilon_0 \le \big(x_3 - z^0(x')\big)^{-1}\partial_{x_3}\frac{V}{F} \le C_0. \tag{13.7.90}$$

We also assume first that $|\nabla(V/F)| \ge \epsilon$. Then we can extend time $T_* = h^{1-\delta}$ to $T^* = \epsilon$ but now we will try to go beyond using canonical form. First, we

have a fast magnetron movement in (x_3, ξ_3), then an oscillatory movement in (x_2, ξ_2) and finally a slow magnetic drift movement in (x_1, ξ_1).

Let us denote by $T(x, r)$ the oscillation time with respect to (x_2, ξ_2) on energy level 0. Then the drift velocity is $\mu^{-1}\nabla_1^{\#}(b_0 + b_1 r)$ with $b_0 = V \circ \bar{\Psi}$, $b_1 = F \circ \bar{\Psi}$ where $\nabla_1^{\#}$ means that we consider Hamiltonian field only with respect to (x_1, ξ_1).

Therefore (x_1, ξ_1)-shift will be

$$\mu^{-1} \int_0^{T(x_1,\xi_1,r)} \nabla_1^{\#}(b_0 + b_1 r)\, dt = \mu^{-1} \int_{z^-}^{z^+} \frac{\nabla_1^{\#}(b_0 + b_1 r)}{\sqrt{-b_0 - b_1 r}}\, dx_2 \tag{13.7.91}$$

which in turn is equal to $-2\mu^{-1}\nabla_1^{\#}\eta + O(\mu^{-2})$ with

$$\eta(x_1, \xi_1, r) = \int_{z^-}^{z^+} \bigl(-b_0 - b_1 r\bigr)^{\frac{1}{2}}\, dx_2 \tag{13.7.92}$$

where $z^{\pm} = z^{\pm}(x_1, \xi_1, r)$ are roots of

$$b_0(x_1, \xi_1, z) + b_1(x_1, \xi_1, z) r = 0, \tag{13.7.93}$$

$z^- < z^0 < z^+$, and

(13.7.94) We use the same notation for functions of (x_1, ξ_1) and for functions of $x' = (x_1, x_2)$ as $(x_1, x_2) = \bar{\Psi}(x_1, \xi_1)$;

$$r \le \bar{r}(x_1, \xi_1) = -\frac{b_0}{b_1}(x_1, \xi_1, z^0(x_1, \xi_1)) \tag{13.7.95}$$

and

$$T(x_1, \xi_1, r) = \int_{z^-}^{z^+} \frac{dx_2}{\sqrt{-b_0 - b_1 r}} \tag{13.7.96}$$

is a period of oscillations. Note that due to (13.7.90)

(13.7.97) $|\nabla_1^{\#}\eta| \asymp 1$ as $z^{\pm}$ are close to $z^0(x_1, \xi_1)$

and therefore shift is observable: $\mu^{-1} \gg \mu^{-1}h$. The same is true for any r provided $|\nabla\eta| \asymp 1$ for all $r \le \bar{r}(x_1, \xi_1)$.

Note that we can rewrite this condition

$$|\nabla_{x'}\eta(x', r)| \ge \epsilon_0 \qquad \forall x', r \le \bar{r}(x') \tag{13.7.98}$$

with

$$\eta(x',r)=\int_{\ell(x')}\bigl(-V-Fr\bigr)^{\frac12}\,ds \tag{13.7.99}$$

where x' coordinates on the surface transversal to magnetic lines and ds is an element of length along such lines where $\ell(x',r)$ indicates the segment of the magnetic line $\Phi_t(x')$ where $V+rF<0$. Our "new" η is connected with "old" η in the obvious way.

Therefore $T^*\asymp 1$ is upgraded to $T^*\asymp \mu$.

Case $\mu h\le 1$

The fact that the standard under condition (13.7.101) below $T^*\asymp 1$ can be replaced by $T^*\asymp\mu$ without adding new singularities implies that the standard remainder estimate $\mathsf{R}^{\mathsf{T}}\le Ch^{-2}$ and $\mathsf{R}^{\mathsf{MW}}_{\infty}\le Ch^{-2}$ could be upgraded to $\mathsf{R}^{\mathsf{T}}\le C\mu^{-1}h^{-2}$ and $\mathsf{R}^{\mathsf{MW}}_{\infty}\le C\mu^{-1}h^{-2}$ respectively as $\mu h\le 1$. So we arrive to

Theorem 13.7.30. *Let conditions* (13.2.1)–(13.2.5), (13.3.1), (13.4.46) *and* (13.4.54) *be fulfilled.*

Further, let conditions (13.7.89), (13.7.90) *and* (13.7.98) *be fulfilled. Finally let non-degeneracy condition*

$$|\nabla\bigl(\frac{V}{F}\bigr)|\le\epsilon \implies |\det \mathsf{Hess}\bigl(\frac{V}{F}\bigr)|\ge\epsilon \tag{13.7.100}$$

be fulfilled. Then as $\mu h\le 1$ estimate

$$\mathsf{R}^{\mathsf{MW}}\le C\mu^{-1}h^{-2} \tag{13.7.101}$$

holds.

Proof. We leave to the reader an easy exercise to prove that all extra terms in $\mathsf{R}^{\mathsf{MW}}_{\infty}$ could be dropped so $\mathsf{R}^{\mathsf{MW}}_{\infty}$ becomes R^{MW}. □

This theorem is illustrated by Example 13.7.34(i) below. On the other hand, assume that η has a critical point. Let us introduce γ-admissible partition of unity with respect to (x_1,ξ_1) with

$$\gamma=\gamma(x_1,\xi_1,r)\coloneqq \mathsf{max}\bigl(\epsilon|\nabla\eta|, C\bar\gamma\bigr). \tag{13.7.102}$$

Then the shift during period is of magnitude $\mu^{-1}\gamma$ and uncertainty principle requires $\mu^{-1}\gamma\times\gamma\asymp\mu^{-1}h$ or $\gamma\asymp h^{\frac12}$. Therefore selecting $\bar\gamma = h^{\frac12-\delta}$ we conclude that the contribution of the zone $\{\gamma\ge\bar\gamma\}$ to R^{MW} does not exceed $C\mu^{-1}h^{-2}$. Meanwhile contribution of the zone $\{\gamma\le\bar\gamma\}$ does not exceed $Ch^{-2}\,\mathsf{mes}(\{x:\ |\nabla\eta(x)|\le\bar\gamma\})$ which is $Ch^{-2}\bar\gamma^2$ under assumption (13.7.107) below. Then we arrive to the remainder estimate

$$\mathsf{R}^{\mathsf{MW}}\le C\mu^{-1}h^{-2}+Ch^{\delta-1} \tag{13.7.103}$$

with an arbitrarily small exponent $\delta>0$.

One can improve it by using logarithmic uncertainty principle; then we can select $\bar\gamma = C(h|\log h|)^{\frac12}$ leading us to the remainder estimate

$$\mathsf{R}^{\mathsf{MW}}\le C\mu^{-1}h^{-2}+Ch^{-1}|\log h| \tag{13.7.104}$$

However we can do better than this. Let us select $\bar\gamma = Ch^{\frac12}$. The following problem is easily accessible by our methods and we leave it to the reader:

Problem 13.7.31. (i) Prove that as $Q'=Q'(x_1,\mu^{-1}hD_1,j)$, $Q''=Q''(x_1,\mu^{-1}hD_1,j)$ are γ-admissible elements with $\gamma\ge C_0\bar\gamma$ then

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma Q'_{x'}\mathsf{U}_{j,j}\,{}^tQ''_y|\le C\mu h^{-1}\gamma^2\,T\big(\frac{\bar\gamma}{T\gamma}\big)^s\Big) \tag{13.7.105}$$

(ii) Then prove that the contribution of γ-element to R^{T} with $T=\epsilon$ does not exceed

$$Ch^{-2}\gamma^2\Big(\mu^{-1}+\big(\frac{\bar\gamma}{\gamma}\big)^s\Big). \tag{13.7.106}$$

Then we arrive to the following

Theorem 13.7.32. *Let all conditions of Theorem 13.7.30 except* (13.7.98) *be fulfilled. Then under condition*

$$|\nabla_{x'}\eta(x',r)|\le\epsilon\implies|\det\mathsf{Hess}_{x'}\,\eta(x',r)|\ge\epsilon\qquad\forall x',r\le\bar r(x') \tag{13.7.107}$$

as $\mu h\le 1$ *estimate* (13.7.101) *holds.*

Remark 13.7.33. (i) As $\epsilon_0h^{-1}\le\mu\le h^{-1}$ we need to check conditions (13.7.98), (13.7.107) only $r=(2j+1)\mu h$; also condition (13.7.100) should be checked only as $|V-(2j+1)\mu hF|\le\epsilon$.

(ii) Estimate $o(h^{-2})$ holds for $\mu h \le 1$ under weak non-degeneracy condition

$$\mathsf{mes}(\{x' : \nabla\eta(x', r) = 0\}) = 0 \qquad \text{as} \quad r = (2j+1)\mu h \quad \forall j. \tag{13.7.108}$$

(iii) Estimate $O\big(\mu^{-1}h^{-2} + h^{-\frac{3}{2}}\big)$ holds for $\mu h \le 1$ under intermediate non-degeneracy condition

$$|\nabla_{x'}\eta(x', r)| \le \epsilon \implies |\mathsf{Hess}_{x'}\,\eta(x', r)| \ge \epsilon \qquad \forall x', r = (2j+1)\mu h, j \in \mathbb{Z}^+. \tag{13.7.109}$$

Example 13.7.34. Consider

$$a = (\xi_1 - \frac{1}{2}\mu x_2)^2 + (\xi_2 + \frac{1}{2}\mu x_1)^2 + V(x). \tag{13.7.110}$$

(i) As

$$V = k^2x_3^2 + lx_1 - \tau \qquad (\tau > 0) \tag{13.7.111}$$

$k > 0$ ensures trapping and $l \ne 0$ ensures condition (13.7.98).

(ii) As

$$V = k^2x_3^2 + m_1x_1^2 + m_2x_2^2 - \tau \qquad (\tau > 0) \tag{13.7.112}$$

$k > 0$ ensures trapping and $l_1 \ne 0$, $l_2 \ne 0$ ensures condition (13.7.107).

(iii) In (ii) assume that $m_1 = m_2 = l^2 > 0$; then we have at least two oscillatory movements (with respect to (x_2, ξ_2) and slow oscillatory drift movement with respect to (x_1, ξ_1)). Then we can increase T^* to $T^* \gg \mu$ and under appropriate non-commensurability condition we can get remainder estimate $o(\mu^{-1}h^{-2})$ instead of $O(\mu^{-1}h^{-2})$.

As $\mu h \to 0$ the third periodic movement (with respect to (x_3, ξ_3) in the canonical form) plays role and non-commensurability means that at least one of two commensurability conditions should be violated. This is fulfilled automatically for non-commensurable k and l while for commensurable k and l we get one non-commensurability condition to μ. Then we get remainder estimate $o(\mu^{-1}h^{-2})$ instead of $O(\mu^{-1}h^{-2})$.

(iv) Let

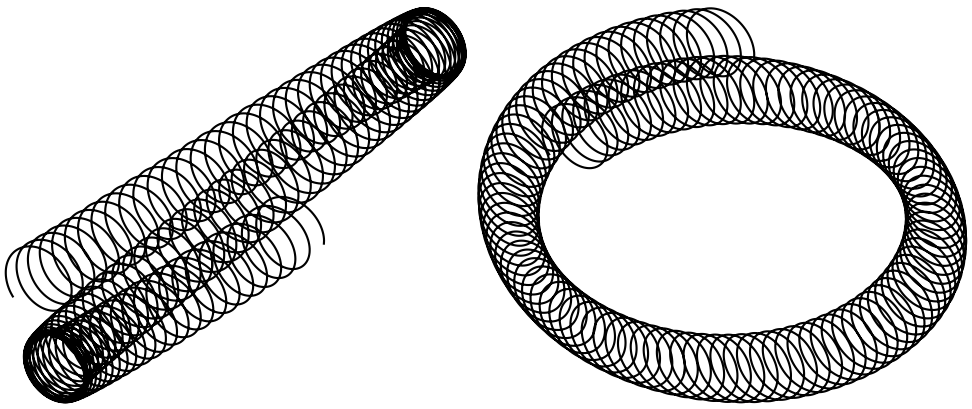
$$V = k^2 x_3^2 - \tau \qquad (\tau > 0). \tag{13.7.113}$$

Then there is no drift but while fast rotations have frequency 2μ, normal oscillation have frequency $2k$ and unless they are commensurable we get non-periodic movement and as $\mu h \to 0$ we get remainder estimate $o(h^{-2})$ under appropriate assumption to μ as $\mu h \to +0$.

Problem 13.7.35. (i) Investigate case when potential traps only trajectories with $r \ge \bar{r}$ and other trajectories escape. One should expect remainder estimate $\mathsf{R}^{\mathsf{MW}} = o(h^{-2})$.

(ii) Investigate the case when all magnetic lines are periodic and V/F is constant along them.

(iii) Investigate the case when all magnetic lines are periodic and V/F is not constant along them (then trajectories of $\Psi_{r,t}$ with $r \ge \bar{r}$ are trapped and other trajectories are periodic).



(a) Oscillation and drift (b) Periodic and drift

Figure 13.6: Trajectories for 3-dimensional model operator:
(a) Operator (13.7.110)–(13.7.111) (b) Operator (13.7.114)

Example 13.7.36.

$$a = \xi_1^2 + \xi_2^2 + (\xi_3 - \mu\alpha(|x'|))^2 + w(|x'|) - \tau \tag{13.7.114}$$

Then $\boldsymbol{F} = (-x_2, x_1, 0) f(|x'|)|x'|^{-1}$, $F = f(|x'|)$, $f(\rho) = \partial_\rho \alpha > 0$ and and all magnetic lines are circles.

(i) Then drift speed is $\partial_\rho \eta(\rho)$ with $\eta = (-w(\rho) - r|f(\rho)|)$ and it is directed along x_3. Then under appropriate assumptions ensuring (13.7.98) or (13.7.100) one can derive corresponding asymptotics.

(ii) As $w(\rho) = -\tau$ and $f(\rho) = \mathsf{const}$ there is no drift and again one can derive asymptotics $o(h^{-2})$ under non-commensurability condition as $\mu h \to 0$.

Schrödinger-Pauli Operator as $\mu h \ge 1$

Consider now Schrödinger-Pauli operator (13.6.3) as $\mu h \ge 1$. We assume that modified condition (13.7.100)

$$\min_j |\frac{V}{F} - 2j\mu h| + |\nabla(\frac{V}{F})| \le \epsilon \implies |\det \mathsf{Hess}(\frac{V}{F})| \ge \epsilon \tag{13.7.100)$'}$$

is fulfilled. Then exactly the same arguments as in the proofs of Theorems 13.7.30 and 13.7.32 lead us to the following theorem:

Theorem 13.7.37. *Consider Schrödinger-Pauli operator* (13.6.3) *with* $\mu h \ge 1$. *Let conditions* (13.2.1)–(13.2.5), (13.3.1), (13.4.46) *and* (13.4.54) *be fulfilled.*

Further, let conditions (13.7.89), (13.7.90) *and* (13.7.100)$'$ *be fulfilled. Then*

(i) Under assumption

$$|\nabla_{x'} \eta(x', r)| \ge \epsilon_0 \qquad \forall x', r \le \bar{r}(x') \qquad \forall x', r = 2j\mu h,\ j \in \mathbb{Z}^+ \tag{13.7.98)$'}$$

estimate

$$\mathsf{R}_1^{\mathsf{MW}} \le Ch^{-1} + C'_s \mu h^s \tag{13.7.115}$$

where $\mathsf{R}_1^{\mathsf{MW}}$ *is due to the perturbation term* $\asymp h^2$ *in the canonical form*[46)].

[46)] See Remark 13.6.13(ii). We should not care about this before as the sharpest remainder estimate was $O(\mu h^{-1+\delta})$.

(ii) Estimate

$$\mathsf{R}^{\mathsf{MW}} \le C\mu \tag{13.7.116}$$

holds under condition

$$|\nabla_{x'}\eta(x',r)| \le \epsilon \implies |\det \mathsf{Hess}_{x'}\,\eta(x',r)| \ge \epsilon \qquad \forall x', r = 2j\mu h,\ j \in \mathbb{Z}^+ \tag{13.7.107}'$$

Proof. We leave easy details to the reader. □

Remark 13.7.38. (i) Estimate $\mathsf{R}^{\mathsf{MW}} = o(\mu h^{-1})$ holds under weak non-degeneracy condition

$$\mathsf{mes}(\{x' : \nabla\eta(x',r) = 0\}) = 0 \qquad \text{as} \quad r = 2j\mu h \ \ \forall j. \tag{13.7.108}'$$

(ii) Estimate $\mathsf{R}^{\mathsf{MW}} = O(\mu h^{-\frac{1}{2}})$ holds under intermediate non-degeneracy condition

$$|\nabla_{x'}\eta(x',r)| \le \epsilon \implies |\mathsf{Hess}_{x'}\,\eta(x',r)| \ge \epsilon \qquad \forall x', r = 2j\mu h, j \in \mathbb{Z}^+. \tag{13.7.109}'$$

Schrödinger-Pauli Operator as $\mu h \ge 1$ reloaded

While most likely one can improve (13.7.115) to

$$\mathsf{R}^{\mathsf{MW}}_1 \le Ch^{-1} + C'\mu \exp(\epsilon h^{-1}) \tag{13.7.117}$$

we still want to improve it and (13.7.116) further. To do this we consider operator in the reduced form; as $\mu h \ge C$ we need to consider only operator $\mathcal{A}_0(x_1, \hbar D_1; x_2, hD_2)$ with $\hbar = \mu^{-1}h$ and we consider it as $\hbar$-pseudodifferential operator with an operator-valued symbol $\boldsymbol{a}(x_1, \xi_1; x_2, hD_2)$ acting in the axillary space $\mathbb{H} = \mathscr{L}(\mathbb{R}_{x_2})$. Then under condition (13.7.98) it is microhyperbolic which leads us to Statements (i) and (ii) of the following theorem:

Theorem 13.7.39. *(i) Let conditions of Theorem 13.7.37(i) be fulfilled. Then as $h \in (\mu^{-1}, h_0)$ where h_0 is small enough constant*

$$\mathsf{R}^{\mathsf{MW}}_2 := \Bigl|\int \Bigl(e(x,x,0) - (2\pi)^{-1}\mu h^{-1}\boldsymbol{e}(x_1,x_2;x_3,x_3;0)\Bigr)\psi(x)\,dx\Bigr| \le Ch^{-1} \tag{13.7.118}$$

where $\mathbf{e}(x';x_3,y_3;\tau)$ *is the Schwartz kernel of the spectral projector* $\mathbf{e}(x';\tau)$ *of operator* $\mathbf{a}(x')=\mathbf{a}(x_1,x_2;x_3,hD_3)$ *and as usual we identify objects depending on* x *and on* (x_1,ξ_1,x_2) *through* $\bar{\Psi}$.

(ii) In particular, for $\psi=\psi(x')$

$$\mathsf{R}_2^{\mathsf{MW}}=|\int\Big(\int e(x,x,0)\,dx_3-(2\pi)^{-1}\mu h^{-1}\mathbf{n}(x_1,x_2;0)\Big)\psi(x')\,dx'| \tag{13.7.119}$$

where $\mathbf{n}(x';\tau)=\mathsf{Tr}_{x_3}\,\mathbf{e}(x';\tau)$ *is an eigenvalue counting function of operator* $\mathbf{a}(x')$.

Proof. Proof is standard: as the propagation speed with respect to (x_1,ξ_1) is $\asymp\mu^{-1}$ we can take $T^*\asymp\mu$ and $T_*\asymp\mu^{\delta}h$ with an arbitrarily small exponent $\delta>0$ and then we can launch successive approximations with unperturbed operator frozen as $x_1=y_1$; then $(x_1-y_1)=O(\mu^{-1}T_*)$.

We leave easy details to the reader. Note only that we need $h\in(\mu^{-1},h_0)$ because actually $\eta=\eta(x_1,\xi_1,h)$ and to have $|\nabla\eta|\asymp 1$ we need the above assumption. □

Assume now that condition (13.7.107) is fulfilled instead.

Theorem 13.7.40. *Let conditions of Theorem 13.7.37(ii) be fulfilled. Let* $h\in(\mu^{-1},h_0)$. *Then estimate*

$$\mathsf{R}_2^{\mathsf{MW}}\le Ch^{-1}|\log\mu| \tag{13.7.120}$$

with an arbitrarily small exponent holds.

Proof. Proof is standard. Consider $\eta=\eta(x_1,\xi_1,h)$ which as $h\in(\mu^{-1},h_0)$ has non-degenerate critical point.

Then propagation speed with respect to (x_1,ξ_1) speed is $\asymp\mu^{-1}\gamma$ we can take $T^*\asymp\mu$ and $T_*\asymp|\log\mu|h/\gamma^2$ and then we can launch successive approximations with unperturbed operator frozen as $x_1=y_1$; then $(x_1-y_1)=O(\mu^{-1}\gamma T_*)$. Here we must assume that $\gamma\ge\bar{\gamma}=\mu^{-\frac{1}{2}}|\log\mu|$ and contribution to the remainder of the zone $\{\gamma\le\bar{\gamma}\}$ does not exceed $C\mu h^{-1}\bar{\gamma}^2$.

We leave easy details to the reader. □

Example 13.7.41. (i) Example 13.7.34(i) and (ii) illustrates Theorems 13.7.39 and 13.7.40.

(ii) In particular, consider Example 13.7.34(ii). Then (as $\psi = 1$) the number of eigenvalues below 0 equals to the number of the lattice points $\{(2i+1)l\mu^{-1}h, (2j+1)kh\}$ in the triangle $\{z_1 \ge 0, z_2 \ge 0, z+1+z_2 < \tau\}$. Transition to "Weyl" expression with respect to (x_1, ξ_1) means that we replace summation with respect to i by integration thus making an error $O(1)$ for each j and the total error $O(h^{-1})$, resulting in the expression $\mathsf{const} \cdot \mu h^{-1} \sum_j \big(\tau - (2j+1)h\big)_+$.

13.8 Certain gGeneralizations

In this section we discuss what happens to the results of the previous sections (Sections 13.4–13.7) in the case in which condition "$V \le -\epsilon$" is violated.

We also consider the case when one of the conditions "$F \ge \epsilon_0$" and "$B(0,1) \subset X$" is violated but, at least as $d = 2$ our results here are far from optimal and are preliminary. Much more delicate analysis of the next Chapters 14 and 15 will lead to much more sharp results.

We also discuss the Weyl asymptotics following from the non-Weyl asymptotics of Sections 13.5 and 13.7.

13.8.1 Getting Rid of the Condition "$v \le -\epsilon$"

Let us treat the case in which condition "$v \le -\epsilon$" is violated. Recall that $v = V/F$.

Formal Rescaling

Until further notice we assume that

$$\nu = 1. \tag{13.8.1}$$

Remark 13.8.1. (i) Recall that in the case of a weak magnetic field we need this condition (13.4.45) only to calculate $F_{t\to h^{-1}\tau}\big(\bar{\chi}_T(t)\psi U\big)$ as $T \in [h^{1-\delta}, \epsilon\mu^{-1}]$. However the same result could be achieved under assumption

$$|v| + |\nabla v| \ge \epsilon_0. \tag{13.8.2}$$

(ii) In the case of the moderate magnetic field this condition was needed only to reference to the case of weak magnetic field.

(iii) In the case of the strong magnetic field such comparison was not needed and thus even assumption (13.8.2) was not needed.

Consider *formal rescaling technique* applied to estimate $\mu^{-\sigma}h^{1-d+\vartheta}$. As we rescale $x \mapsto x/\gamma$ and divide by ρ^2 i.e. $\xi \mapsto \xi/\rho$ we have $h \mapsto h/(\rho\gamma)$ and $\mu \mapsto \mu\gamma/\rho$ and therefore we estimate contribution of $B(x,\gamma(x))$ to the remainder by

$$C\Big(\frac{\mu\gamma}{\rho}\Big)^{-\sigma}\Big(\frac{h}{\gamma\rho}\Big)^{1-d+\vartheta} = C\mu^{-\sigma}h^{1-d+\vartheta}\gamma^{d}\times\gamma^{-1-\vartheta-\sigma}\rho^{\sigma-1+d+\vartheta} \tag{13.8.3}$$

and as $\rho=\gamma$ the second factor in the right-hand expression is $\gamma^{d-2}\le 1$.

This means that taking

$$\gamma=\rho=\max\big(\epsilon(|v|+|\nabla v|^2)^{\frac12}\bar\gamma\big), \tag{13.8.4}$$

which instantly allows us to drop condition (13.8.2). Here we take

$$\bar\gamma=(\mu h)^{\frac12} \tag{13.8.5}$$

as $\mu h\mapsto \mu h/\gamma^2$ and as $\gamma=\bar\gamma$ we get $\mu h\mapsto 1$.

We could even start rescaling from

$$\gamma=\rho^2=\max\big(\epsilon|v|,\bar\gamma\big), \tag{13.8.6}$$

with $\bar\gamma$ defined by (13.8.5) but then the second factor in the right-hand expression of (13.8.3) would be $\gamma^{\frac12(d-3-\sigma-\vartheta)}$ and after integration with respect to x under condition (13.8.2) we would get $O(1)$ as $d>1+\sigma+\vartheta$; so as $d=3$, $\vartheta=0$, $\sigma\le 1$ everything would be fine. Then rescaling again with γ defined by (13.8.4) we would drop condition (13.8.3).

Justification: $d=3$

Consider $d=3$. Then

(13.8.7) We need no justification for the results of Sections 13.4 and 13.5 with $\mu\le h^{\delta-1}$ as $\vartheta=0$ and with $\mu\le h^{-1}$ as $\vartheta>0$.

Really, in this cases estimate $O(h^{-2+\vartheta})$ was proven under no non-degeneracy conditions.

(13.8.8) We need no justification for the results of Section 13.5 with $\mu \in [h^{\delta-1}, 1]$ as $\vartheta = 0$.

Really, in this case we do not need to compare with weak magnetic field at all: removing $O(\mu^{-2})$ terms in the argument of $\mathcal{N}^{\mathsf{MW}}$ leads to the error not exceeding

$$C\big(\mu^{-2}h^{-3} + \mu^{-1} \times \mu h^{-2}\big) \tag{13.8.9}$$

which does not exceed Ch^{-2} as $\mu \ge h^{-\frac{1}{2}}$; this error estimate does not require any additional assumption. So, for $\vartheta = 0$ condition "$|v| \asymp 1$" was needed only for $\mu \le h^{-\frac{1}{2}}$.

Furthermore, as contribution of zone $\{x, \gamma < \varepsilon\}$ to the remainder according to formal scaling does not exceed $C\varepsilon = o(1)$ we conclude that

(13.8.10) Remainder estimate $o(h^{-2})$ remains true for R^{T} or $\mathsf{R}^{\mathsf{MW}}_1$. Further, under non-degeneracy condition $(13.4.57)_m$ remainder estimate $o(h^{-2})$ remains true for R^{T} or $\mathsf{R}^{\mathsf{MW}}_1$.

Finally, consider supersharp results of Subsection 13.7.3. As $\gamma \ge C\bar{\gamma}$ with

$$\bar{\gamma} := (\mu h)^{\frac{1}{2}-\delta'} \tag{13.8.11}$$

now, consider two cases

(a) $|\nabla v| \gtrsim \gamma$ on element in question. Then after rescaling $|\nabla v| \asymp 1$ and we can apply weak magnetic field approach as $\mu_{\mathsf{new}} =\le h_{\mathsf{new}}^{\delta''-1}$. Recall that $\mu_{\mathsf{new}} = \mu$ and $h_{\mathsf{new}} = h\gamma^{-2}$ we rewrite this condition as $\gamma \ge \bar{\gamma}$ with $\bar{\gamma}$ defined by (13.8.11) and $\mu \le h^{1-\delta}$.

(b) $|v| \gtrsim \gamma^2$ and $|\nabla v| \le \epsilon\gamma$ on element in question. Then after rescaling $|v| \asymp 1$, $\nabla v| \le C_0\epsilon$ and elements of $\mathsf{Hess}\, v$ are preserved. So condition (13.7.100) survives. Applying weak magnetic field approach we conclude that contribution of this particular element to the remainder does not exceed

$$C\mu_{\mathsf{new}}^{-1}h_{\mathsf{new}}^{-2} + C\mu_{\mathsf{new}}(\mu_{\mathsf{new}}h_{\mathsf{new}})^2 h_{\mathsf{new}}^{-2-\delta} = C\mu^{-1}h^{-2}\gamma^4 + C\mu^3 h^{-\delta}\gamma^{2\delta}.$$

dividing by γ^3 and integrating and using (13.7.100) again we conclude that the total error does not exceed $C\mu^{-1}h^{-2} + C\mu^3 h^{-\delta}$ and the second term

does not exceed the first one provided $\mu \le h^{\delta-\frac{1}{2}}$. So, in this case our dream estimate is proven.

Let us apply a moderate magnetic field approach as $\mu \ge h^{\delta-\frac{1}{2}}$. Note that removing $O(\mu^{-4}+h^2)$ terms in the argument of $\mathcal{N}^{\mathsf{MW}}$ leads to the error not exceeding $C(\mu^{-4}+h^2)h^{-3}$ due to (13.7.100) again and this does not exceed $C\mu^{-1}h^{-2}$ as $\mu \ge h^{-\frac{1}{3}}$. So the only term $O(\mu^{-2})$ in the argument of $\mathcal{N}^{\mathsf{MW}}$ should be treated separately.

But the same condition (13.7.100) implies that removing this term in the argument of $\mathcal{N}^{\mathsf{MW}}$ and in ψ (transformed, the latter error never exceeded the former one) and replacing the sum by an integral leads to correction term $C\kappa\mu^{-2}h^{-3}$ plus the same error as before plus $O(\mu^{-1}h^{-2})$ error. As zones $\mu \ge h^{-\frac{1}{3}}$ and $\mu \le h^{\delta-\frac{1}{2}}$ overlap we conclude that κ must be equal to 0 which implies that we can use unmodified $\mathcal{N}^{\mathsf{MW}}$ with $O(\mu^{-1}h^{-2})$ error.

We used the fact that extra analysis of Subsection 13.7.3 was needed to extend $|t| \le \epsilon$ to $|t| \le \epsilon\mu$ and did not use assumption (13.8.2) at all.

So, we arrive to

Theorem 13.8.2. *Let* $d = 3$ *(and not necessary* $\nu = 1$*). Then estimates for* R^{MW} *and* $\mathsf{R}^{\mathsf{MW}}_1$ *derived in Sections 13.4–13.7 remain true without condition "*$|v| \asymp 1$*"*[47]*.*

Justification: $d = 2$, $\nu = 1$

Consider case $d = 2$.

Theorem 13.8.3. *Let* $d = 2$*. Theorem 13.5.26 remains true without condition "*$|v| \asymp 1$*"*[47]*.*

Proof. (i) Let us introduce a function ρ by means of (13.8.4) with $\bar{\rho} = \epsilon_0\sqrt{\mu h}$ and let us make a ρ-admissible partition of unity, division and rescaling. We leave "γ" to denote function used in Theorem 13.5.26.

Let us treat various elements of this partition as in the previous subsubsection. Then

(a) Let $\rho \ge C\bar{\rho}$ and $|\nabla v| \ge \epsilon\rho$. Then after rescaling we get into the situation of "$|\nabla v| \asymp 1$" and the contribution to R^{MW} of this element does not exceed

[47] Or even condition (13.8.2).

$C\mu_{\mathsf{new}}^{-1}h_{\mathsf{new}}^{-1} = C\rho^2\mu^{-1}h^{-1}$ and the total contribution to R^{MW} of such elements does not exceed $C\mu^{-1}h^{-1}$.

(b) Let $\rho \ge C\bar{\rho}$ and $-v \ge \epsilon\rho^2$. Then after rescaling we get into the situation of Theorem 13.5.26. Let us treat various subvariants:

(b)$_1$ Let $\mu^{\frac{3}{2}}h^{\frac{1}{2}} \le C_0\gamma$. Then we get into the situation of Theorem 13.5.26(i). Substituting $\mu_{\mathsf{new}} = \mu$, $h_{\mathsf{new}} = h\rho^{-2}$ and $\gamma_{\mathsf{new}} = \gamma\rho^{-1}$ in place of μ, h, γ in (13.5.107) we obtain that the contribution to R^{MW} of this element does not exceed

$$C\rho^2\mu^{-1}h^{-1} + C'\mu^{-1}h^{-1}\int_{\{\gamma\ge C_0\mu^{-1}\rho\}}\bigl(\frac{\mu h}{\gamma^2}\bigr)^s\gamma^{-2}\,dx+ C\mu h^{-1}\int_{\{\gamma\le C_0\mu^{-1}\rho\}} dx \tag{13.8.12}$$

where we have taken into account that $dx_{\mathsf{new}} = \rho^{-2}dx$ and where here and in what follows we integrate only over one element of the covering if we discuss a separate element of the partition (we do not indicate this in what follows).

Obviously this expression (13.8.12) does not decrease if the domains in the second and third terms are replaced by $\{\gamma \ge C_0\mu^{-1}\}$ and $\{\gamma \le C_0\mu^{-1}\}$ respectively. Then the total contribution of such elements of the partition does not exceed the right-hand expression of (13.5.107).

(b)$_2$ Let $\mu^{\frac{3}{2}}h^{\frac{1}{2}} \ge C_0\rho$. Then we get into the situation of Theorem 13.5.23(ii). According to (13.5.110) the contribution to R^{MW} of such an element does not exceed

$$C\rho^2\mu^{-1}h^{-1} + C'\mu^{-1}h^{-1}\int_{\{\gamma\ge\sqrt{\mu h}\}}\bigl(\frac{\mu h}{\gamma^2}\bigr)^s\gamma^{-2}\,dx+ C\mu^{-1}h^{-1}\int_{\{\sqrt{\mu h}\ge\gamma\ge C_1\mu^{-1}\rho, |v+(2j+1)\mu h|\le\epsilon_2\mu h\}}\gamma^{-2}\,dx+ C\mu h^{-1}\int_{\{\gamma\le C_1\mu^{-1}\rho\}} dx. \tag{13.8.13}$$

If $\mu^{\frac{3}{2}}h^{\frac{1}{2}} \ge C_0$ here then expression (13.8.13) does not decrease if in the restrictions on γ from below in the third term and in the restrictions on γ from above in the fourth term we replace ρ by 1; then summing with respect to the partition of unity we obtain that the total contribution of all such terms does not exceed the right-hand expression of (13.5.110).

On the other hand, if $\mu^{\frac{3}{2}}h^{\frac{1}{2}} \le C_0$ then expression (13.8.13) does not decrease if we replace the domain of integration in the second term by $\{\gamma \ge C_0\mu^{-1}\}$ and we replace the integral over the remaining part of the original domain and two other integrals (in the third and fourth terms) by

$$C\mu h^{-1}\int_{\{\gamma\le C_0\mu^{-1}\}} dx.$$

Then the total contribution of all such terms does not exceed the right-hand expression of (13.5.107).

(c) Finally, Theorem 13.2.9 yields that the contribution of the zone $\{\rho \le 2\bar\rho\}$ does not exceed $C'h'^s\bar\rho^{-2}$. Therefore Statement (i) of Theorem 13.5.26 has been proven (without condition (13.4.45)) [47].

(ii) Let us now assume that condition (13.5.73) is fulfilled. Then without any loss of the generality we can assume that $v = \pm x_1^2 + w(x_2)$. Let us introduce function $\rho'(x_2)$ by formula (13.8.4) with v replaced by w.

Then applying a $\frac{1}{2}|x_1|$-admissible partition of unity and referring to the case "$|\nabla v| \asymp 1$" we obtain that

(13.8.14) The contribution to R^{MW} of the zone $\{|x_1| \ge \frac{1}{2}C_0\rho'\}$ does not exceed $C\mu^{-1}h^{-1} + C'$.

Let us consider zone $\{|x_1| \le C_0\rho'\}$ and ρ'-admissible partition of unity there. Then repeating the previous arguments and referring to Theorem 13.5.26 we obtain that the contribution to R^{MW} of the separate element does not exceed

$$(13.8.15)_{1,2} \qquad C\rho'^2\mu^{-1}h^{-1} + C\mu^{\frac{3}{2}}h^{-\frac{1}{2}}\rho' \qquad \text{or} \qquad C'\rho'^2h^{-1}$$

respectively for $\rho' \ge \mu^{\frac{3}{2}}h^{\frac{1}{2}}$ or $\rho' \le \mu^{\frac{3}{2}}h^{\frac{1}{2}}$.

Note that if $\mu^{\frac{3}{2}}h^{\frac{1}{2}} \le 1$ then elements of both types can occur and one can easily see that the total contribution does not exceed the right-hand expression of (13.5.108).

On the other hand, if $\mu^{\frac{3}{2}}h^{\frac{1}{2}} \ge 1$ then only elements of the second type can occur and one can easily see that the total contribution does not exceed the right-hand expression of (13.5.111).

Therefore Statement (ii) of Theorem 13.5.26 has been proven without condition (13.4.45) [47].

(iii) Let us now assume that condition (13.5.109) is fulfilled. Without any loss of the generality one can assume that $x = 0$ is the unique critical point of v. Let us set

$$r = \begin{cases} (-v(0))^{\frac{1}{2}} & \text{as} \quad v(0) \le -\epsilon_0 \mu h, \\ (\epsilon_0 \mu h)^{\frac{1}{2}} & \text{as} \quad v(0) \ge -\epsilon_0 \mu h. \end{cases} \tag{13.8.16}$$

Then the previous arguments which refer to the case "$|\nabla v| \asymp 1$" give that the contribution to R^{MW} of the zone $\{|x| \ge \epsilon_0 r\}$ does not exceed $C\mu^{-1}h^{-1} + C'$.

Moreover, referring to Theorem 13.5.24 we conclude that

(13.8.17) If $v(0) \le -\epsilon_0 \mu h$ then contribution to R^{MW} of the zone $\{|x| \le 2\epsilon_0 r\}$ does not exceed $C\mu^{-1}h^{-1}$ provided either condition $(13.5.109)_+$ is fulfilled or $\mu^3 h \le r^2$.

Really, in the second case $\mu^3_{\mathsf{new}} h_{\mathsf{new}} \le 1$.

On the other hand, if condition $(13.5.109)_+$ is violated and $\mu^3 h \ge r^2$ then this contribution does not exceed $Cr^2\mu^{-1}h^{-1}(|\log \mu^3 h r^{-2}| + 1)$ again due to Theorem 13.5.26; this expression only increases if we replace r by $C_0 \min(\mu^{\frac{3}{2}} h^{\frac{1}{2}}, 1)$.

We then obtain that the contribution in question does not exceed $C\mu^{-1}h^{-1}$ for $\mu^3 h \le 1$ and $C\mu^{-1}h^{-1}(|\log \mu^3 h| + 1)$ for $\mu^3 h \ge 1$. Moreover, if condition $(13.5.109)_+$ is violated, $\mu^3 h \ge 1$ and

$$\hat{\lambda} = \min_j (|v(0) + (2j+1)\mu h| \ge \mu^{-2}$$

then estimate (13.5.112) yields that the indicated contribution does not exceed $Cr^2\mu^{-1}h^{-1}(|\log \hat{\lambda}\mu^{-1}h^{-1}| + 1)$ and the last expression is maximal for $r = 1$.

Finally for $v(0) \ge -\epsilon_0 \mu h$ we can apply Theorem 13.2.9. In this case we obtain estimate $(13.5.89)_{(2)}$ provided either $\mu^3 h \le 1$ or condition $(13.5.109)_+$ is fulfilled and estimate (13.5.112) otherwise. □

Theorem 13.8.4. *Theorem 13.5.28 remains true without condition* (13.4.45) [47].

Proof. The arguments used in the proof of the previous proposition provide the estimate

$$\mathsf{R}^{\mathsf{MW}\prime} \le C\mu^{-1}h^{-1} + C|\log \frac{\hat\lambda}{\mu h}|$$

which does not coincide with estimate (13.5.119) only if $\mu h \ge |\log \frac{\hat\lambda}{\mu h}|^{-1}$ ($\ge |\log h|^{-1}$). So for $\mu \le h^{\delta-1}$ this proposition is proven.

In order to prove it for $\mu \in [h^{\delta-1}, h^{-1}]$ let us recall that we had the estimate

$$\mathsf{R}^{\mathsf{MW}}_{\infty} \le C\mu^{-1}h^{-1}$$

without condition (13.4.45) which appeared only when we replaced Riemann sums with integrals because we wanted to obtain a regular function as an integrand. However, if we apply the integral formula $(13.A.3)_1$ of the first order of accuracy we have no need of this condition. On the other hand, for $\mu \ge h^{-\frac12}$ we need only an integral formula of the first order of accuracy and therefore for $\mu \le h^{-\frac12}$ we obtained (13.5.119) without condition (13.4.45) from the beginning. □

We leave to the reader the proof of the following theorem:

Theorem 13.8.5. *Theorem 13.5.30 remains true without condition* (13.4.45) [47].

Summing all the achieved results we obtain

Theorem 13.8.6. *As* $d = 2$ *and* $\nu = 1$ *results of Sections 13.4–13.7 remain true without condition "*$|v| \asymp 1$*"* [47].

Justification: $d = 2$, $\nu < 1$

Let us note that one can assume that $\nu = 1$. As this is important only for $d = 2$ we consider $d = 2$ only. In fact, if $\nu < 1$ then one of the following conditions is fulfilled:

(a) $v \ge \epsilon\nu$ in $B(0,1)$;

(b) $v \le -\epsilon\nu$ in $B(0,1)$;

(c) $|v| \le c\nu$ in $B(0,1)$.

In cases (a) and (b) dividing the operator A by $\varrho = |v(0)|$ and introducing

$$h' = h\varrho^{-\frac12}, \qquad \mu' = \mu\varrho^{-\frac12}, \qquad \nu' = \nu\varrho^{-1} \tag{13.8.18}$$

we get into the situation of conditions

(a)′ $v' \geq \epsilon$,

(b)′ $v' \leq -\epsilon$

respectively. Moreover, Theorem 13.2.9 yields estimate (13.2.52) for $\tau \leq 0$ in case (a) unconditionally and in cases (b) and (c) as $\mu h \geq C_0 r$. Thus we can assume that $\mu h \leq C_0 \varrho$ and we should discuss the results of Sections 13.4, 13.5 and 13.7. One can see easily that

$$\frac{\nu}{\mu h} = \frac{\nu'}{\mu' h'}, \quad \frac{\nu'}{\mu'} = \frac{\nu \varrho^{\frac{1}{2}}}{\mu}$$

and that the conditions $\mu \geq h^{-\delta}$, $\nu \geq C_0 \mu^{-2} \bar{\nu}$ are also fulfilled for h', μ', ν' and $\bar{\nu}' = \bar{\nu}$.

Moreover, one can see easily that the conditions of all our statements remain true after this reduction and that the estimates become stronger as h, μ, ν are replaced by h', μ', ν' respectively. Therefore in cases (a) and (b) the condition $v \leq -\epsilon$ in the statements indicated above is not necessary.

Let us treat case (c). Dividing the operator A by $\varrho = \nu$ and introducing h', μ' and ν by (13.8.18) we get into the situation of the case $\nu' = 1$; recall that $\nu \geq C_0^{-1} \mu h$ and therefore $\mu' h' \leq C_0$. One can check easily that after this reduction all the conditions and estimates are also preserved.

So we can assume in our previous statements that $\nu = 1$. Let us note that a similar reduction is applicable to Theorem 13.4.30 provided either $\nu \geq \mu h^{1-\delta'}$ or $|v(0)| \geq \mu h^{1-\delta'}$ with arbitrarily small $\delta' > 0$.

13.8.2 Getting Rid of the Condition "$F \geq \epsilon$"

Let us now treat the case when condition (13.3.1) is violated. Then the restriction $\mu h \leq 1$ is no longer natural and assumed; however, cases $\mu h \leq 1$ and $\mu h \geq 1$ differ in the order of the principal part of the asymptotics.

Case $d = 3$

Let us start with the case $d = 3$.

Theorem 13.8.7. *Let* $d = 3$ *and let conditions* (13.2.3)–(13.2.5) *be fulfilled. Then*

(i) Let $\vartheta = 0$, $\mu \le h^{-1+\delta}$ and let

$$\sum_{1\le j\le 3}\sum_{\alpha:|\alpha|\le m} |D^\alpha F^j| \ge \epsilon_0 \tag{13.8.19}_m$$

with a constant $\epsilon_0 > 0$. Then estimate

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{-2} + C' \tag{13.8.20}$$

holds.

(ii) For $0 < \vartheta \le 1$, $\mu \le h^{-2}$ estimates

$$\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)} \le Ch^{-2+\vartheta}\mu^{\frac{\vartheta}{2}} \tag{13.8.21}$$

and

$$\mathsf{R}^{\mathsf{MW}}_{(\vartheta)} \le Ch^{-2+\vartheta}\mu^{\frac{\vartheta}{2}} + C\min\bigl(\mu^{\frac{3}{2}+\vartheta}h^{-\frac{3}{2}+\vartheta}, \mu^{\frac{2}{5}\vartheta}h^{-2+\frac{4}{5}\vartheta}\bigr) \tag{13.8.22}$$

hold.

(iii) In the general case for $\vartheta = 0$, $\mu \le h^{-L}$ estimate

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{-2}\mu^{\delta} \tag{13.8.23}$$

holds with arbitrarily small $\delta > 0$.

Proof, Part I. Let us assume that under condition $(13.8.19)_{m-1}$ this theorem is proven. Let us introduce the scaling function

$$\gamma(x) = \epsilon\sum_j \Bigl(\sum_{|\alpha|\le m-1} |D^\alpha F_j|^{\frac{m}{m-|\alpha|}}\Bigr)^{\frac{1}{m}} + \bar\gamma, \qquad \bar\gamma = \mu^{-\frac{1}{m+1}} + h. \tag{13.8.24}_m$$

Then $|\nabla\gamma| \le \frac{1}{2}$ for sufficiently small constant ϵ. Let us introduce a γ-admissible partition and consider its separate element. After the standard rescaling we obtain that conditions (13.2.3)–(13.2.5) are fulfilled with $h' = h\gamma^{-1}$ and $\mu' = \mu\gamma^{m+1}$ instead of h and μ respectively.

(a) Moreover, for $\gamma \ge 2\bar\gamma$ condition $(13.8.19)_{m-1}$ is fulfilled as well and so we can apply an induction assumption. Thus, if $0 < \vartheta \le 1$ the contribution of this element to $\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)}$ does not exceed $Ch^{-2+\vartheta}\gamma^{2-\vartheta}$, $Ch^{-2+\vartheta}\mu^{\frac{\vartheta}{2}}\gamma^{2+\frac{\vartheta}{2}(m-1)}$

for $m=1$, $m\ge 2$ respectively. Here case $m=1$ is different as we appeal to the case $m=0$ and then the remainder estimate is $Ch^{-2+\vartheta}$.

Moreover, the contribution of this element to $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}(\vartheta)}$ does not exceed

$$C\min\bigl(\mu^{\frac{3}{2}+\vartheta}h^{-\frac{3}{2}+\vartheta}\gamma^{(\frac{3}{2}+\vartheta)(m+1)+(-\frac{3}{2}+\vartheta)}, \mu^{-\frac{3}{2}+\vartheta}h^{-\frac{5}{2}+\vartheta}\gamma^{(-\frac{3}{2}+\vartheta)(m+1)+(-\frac{5}{2}+\vartheta)}\bigr),$$

$$C\min\bigl(\mu^{\frac{3}{2}+\vartheta}h^{-\frac{3}{2}+\vartheta}\gamma^{(\frac{3}{2}+\vartheta)(m+1)+(-\frac{3}{2}+\vartheta)}, \mu^{\frac{2}{5}\vartheta}h^{-2+\frac{4}{5}\vartheta}\gamma^{\frac{2}{5}\vartheta(m+1)+(2-\frac{4}{5}\vartheta)}\bigr)$$

for $m=1$, $m\ge 2$ respectively. Therefore, the total contribution of such elements of the partition does not exceed these expressions multiplied by γ^{-3} and integrated over x.

Due to condition $(13.8.19)_m$ in an appropriate coordinate system $\gamma\ge \epsilon|x_1|$ and therefore the total contribution of such elements to $\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)}$ does not exceed $Ch^{-2+\vartheta}\gamma^{-\vartheta}$ calculated as $\gamma=\bar\gamma$ if $m=1$ and $Ch^{-2+\vartheta}\mu^{\frac{\vartheta}{2}}\gamma^{\frac{\vartheta}{2}(m-1)}$ calculated as $\gamma=1$ and in both cases we get $Ch^{-2+\vartheta}\mu^{\frac{\vartheta}{2}}$.

The same arguments imply that the total contribution to $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}(\vartheta)}$ does not exceed

$$C\min\bigl(\mu^{\frac{3}{2}+\vartheta}h^{-\frac{3}{2}+\vartheta}, \mu^{\frac{2}{5}\vartheta}h^{-2+\frac{4}{5}\vartheta}\bigr).$$

The same arguments apply to $\vartheta=0$ as well provided $\mu\le h^{-1+\delta}$. However, at each step of the induction there appears an additional factor $\log\mu$ in the total contribution to R^{MW}.

Moreover, the same arguments are applicable for $h^{-1+\delta}\le\mu\le h^{-1}$ but in this case the remainder estimate is $O(h^{-2}\mu^{\delta})$ and this estimate does not change under the induction transition.

(b) Let us consider elements with $\gamma\le 2\bar\gamma$. On these elements $\mu'=1$ and their contributions do not exceed $Ch^{-2+\vartheta}\bar\gamma^{2-\vartheta}$. Due to condition $(13.8.19)_m$ their number does not exceed $C\bar\gamma^{-2}$ and their total contribution to the remainder estimate does not exceed $Ch^{-2+\vartheta}\mu^{\frac{\vartheta}{2}}$ for $0\le\vartheta\le 1$.

Thus Statement (i) is proven albeit with an additional factor $\log^m\mu$ and Statement (ii) is proven under the additional condition (13.8.19). In order to get rid of this condition one can apply the same arguments with large m without reference to condition $(13.8.19)_m$. Then keeping in mind that $\gamma\ge\mu^{\delta}$ with $\delta=\frac{1}{m+1}$ we obtain Statements (ii) and (ii). □

Proof, Part II. To get rid of logarithmic factor in Statement (i) we apply more subtle arguments.

First of all, applying the same arguments as in Theorem 5.2.10 one can see that the total contribution of the complement of the set

$$\mathcal{X} = \{x, \exists \boldsymbol{e}, \boldsymbol{f}, \bar{x} : \mu\gamma^{m+1} \ge 1, |D^{\alpha}(\boldsymbol{F} - \boldsymbol{f}\ell(x)^m)| \le \gamma^{m-|\alpha|+\delta'} \\ \forall \alpha : |\alpha| \le m, -V \ge \gamma^{\delta_1}\}$$

to R^{MW} does not exceed Ch^{-2} where $\ell(x) = \langle \boldsymbol{e}, x - \bar{x}\rangle$, $\gamma(x) = |\ell(x)|$, $\delta' > 0$ is sufficiently small and $\delta_1 > 0$ is arbitrarily small.

Let us consider $\gamma^{1-\delta''}$-vicinity of the point $\bar{x}$. Without any loss of the generality one can assume that $g^{jk} = \delta_{jk}$ in $\bar{x}$ and therefore $|g^{jk} - \delta_{jk}| \le \gamma^{\delta}$ in this vicinity. Then condition $\langle \nabla, \boldsymbol{F}\sqrt{g}\rangle = 0$ yields that without any loss of the generality one can assume that $\langle \boldsymbol{e}, \boldsymbol{f}\rangle = 0$ and therefore in an appropriate coordinate system this "approximate" field is $(x_2^m, 0, 0)$.

Considering magnetic lines of length $\gamma^{1-\delta'''}$ with sufficiently small $\delta''' > 0$ one can easily see that they are non-closed and that all the assumptions of Theorem 13.7.18 are fulfilled with $T = \min\bigl(\gamma^{-\delta'''}, (\mu\gamma^{m+1})^{\delta'''}\bigr)$. Then the contribution of the γ-vicinity of $x \in \mathcal{X}$ to the remainder estimate does not exceed

$$Ch^{-2}\gamma^2(\gamma^{\delta'''} + (\mu\gamma^{m+1})^{-\delta'''})$$

and therefore the total contribution of $\mathcal{X}$ is $O(h^{-2})$ as well.

We leave all the details to the reader. □

Remark 13.8.8. (i) Assume that $\mu h \gg 1$. Then the principal part of the asymptotics does not exceed $Ch^{-d}\,\mathsf{mes}(\{x, \mu hF + V \le t\})$ with $t = \epsilon$ and is larger than the same expression with $t = -\epsilon$. More precisely, under condition $(13.8.19)_1$ the principal part of asymptotics is $\asymp (\mu h)^{-1}h^{-d}$ and under condition $(13.8.19)_m$ the principal part of asymptotics is $O\bigl((\mu h)^{-\delta}h^{-d}\bigr)$ with $\delta = \delta_m > 0$.

(ii) Surely, one can improve the remainder estimate for (13.8.20) for $\vartheta = 0$, $\mu \ge h^{-1+\delta}$ under an appropriate condition non-degeneracy condition (as will be done for $d = 2$; see condition $(13.8.25)_m$). However, for $d = 3$ and degenerate $\boldsymbol{F}$ the function F is not necessarily smooth.

(iii) Consider Schrödinger-Pauli operator with $\mu h \ge 1$. Then contribution of zone $\mu' h' \le 1$ to the remainder is estimated exactly as before.

However as $\mu' h' \ge 1$ i.e. $\gamma \ge (\mu h)^{1/m}$ contribution of $B(x, \gamma(x))$ to the remainder does not exceed $C\mu' h'^{-1-\delta} = C\mu h^{-1-\delta}\gamma^{m+2+\delta}$; dividing by γ^3 and integrating we arrive to $C\mu h^{-1-\delta}$. Surely condition $\mu \le h^{-L}$ is not natural anymore.

Case $d = 2$

As $d = 2$ the function F is smooth and we can write non-degeneracy condition assuming that $V \asymp 1$:

$$\sum_{1\le|\alpha|\le m} |D^\alpha \frac{F}{V}| \ge \epsilon_0. \tag{13.8.25}_m$$

However we postpone analysis until Chapter 14 when we prove remainder estimate $O(\mu^{-\frac{1}{2}} h^{-1})$ as $(13.8.25)_1$ holds (instead of $O(h^{-1})$ which one can easily prove using current technique).

13.8.3 Getting Rid of the cCondition "$B(0,1) \subset X$"

Finally, let us treat the case in which $B(0,1)$ is not contained in X but $\partial X \cap B(0,1)$ is not very singular. We assume that $F \asymp 1$, $V \asymp 1$.

Case $d = 3$

Consider point $\bar{x} \in X$, $\ell(\bar{x}) \ge C\mu^{-1}$ with

$$\ell(x) = \epsilon \mathsf{dist}(x, \partial X). \tag{13.8.26}$$

Then for time $T(x) = \ell(x)$ classical and quantum movement started at $B(\bar{x}, \ell(\bar{x}))$ are not affected (modulo negligible) by the boundary and contribution of $B(\bar{x}, \ell(\bar{x}))$ to the remainder does not exceed $CT(\bar{x})^{-1}\ell(\bar{x})^3 h^{-2} = C\ell(\bar{x})^2 h^{-2}$ as long as $\mu \le h^{\delta-1}$. The same result could be achieved by rescaling $x \mapsto x\ell^{-1}$, $h \mapsto h\ell^{-1}$, $\mu \mapsto \mu\ell$.

Then the total contribution of zone

$$X_{\mathsf{inn}} := \{x : \mathsf{dist}(x, \partial X) \ge C\mu^{-1}\} \tag{13.8.27}$$

to the remainder does not exceed

$$Ch^{-2} \int_{\mathsf{dist}(x,\partial X)\ge C\mu^{-1}} T(x)^{-1}\, dx \tag{13.8.28}$$

and the contribution of zone

$$X_{\mathsf{bound}} := \{x : \mathsf{dist}(x, \partial X) \le C\mu^{-1}\} \tag{13.8.29}$$

does not exceed $C\mu h^{-2}\,\mathsf{mes}(X_{\mathsf{bound}})$. So the total remainder estimate is

$$Ch^{-2}\int_{X_{\mathsf{inn}}} T(x)^{-1}\,dx + C\mu h^{-2}\,\mathsf{mes}(X_{\mathsf{bound}}). \tag{13.8.30}$$

As ∂X is regular enough the first term is $O(h^{-2}\log h)$ while the second term is $O(h^{-2})$.

So, we need to reconsider contribution of zone X_{bound} and for this purpose we have a Seeley' method but we need to modify it first to tackle degenerations.

Let

$$T_{\pm}(x) = \mathsf{max}\{t :\ \mathsf{dist}(\Phi_{t'}(x), \partial X) \ge \frac{1}{2}C\mu \ \ \forall t' \in (0, \pm t)\}, \tag{13.8.31}$$

$$T^*(x) = \mathsf{max}\big(T_+(x), T_-(x)\big), \qquad T(x) = \mathsf{min}\big(T^*(x), \ell(x)^{1-\delta}\big) \tag{13.8.32}$$

where Φ_t is magnetic flow.

Theorem 13.8.9. *Let $d = 3$, $\vartheta = 0$. Then under Dirichlet boundary condition*[48)]

(i) As $\mu \le h^{\delta'-1}$ estimate

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{-2}\int_{X_{\mathsf{inn}}} T(x)^{-1}\,dx + C\mu h^{-2}\,\mathsf{mes}(X_{\mathsf{bound}}). \tag{13.8.34}$$

holds.

(ii) Further, let in the framework of (i)

$$\int_{X\cap B(0,1)} T(x)^{-1}dx < \infty \tag{13.8.35}$$

which is true under rather minimal regularity conditions to ∂X. Then $\mathsf{R}^{\mathsf{MW}} \le Ch^{-2}$.

[48)] Or Neumann boundary condition i.e.

$$\sum n_j g^{jk} P_k u|_{\partial X} = 0 \tag{13.8.33}$$

where $\boldsymbol{n} = (n_1, \dots, n_d)$ is a normal to ∂X, as domain satisfies cone condition.

(iii) Finally, estimate $\mathsf{R}^{\mathsf{MW}} \le Ch^{-2} + C\mu h^{-1-\delta}$ *holds.*

Proof. Obviously, we need to consider only X_{bound}.

First, both $T_{\pm}(x) \gtrsim \ell(x)$ and then Statement (i) is proven with $T(x) = \ell(x)$ which however leads to logarithmical divergence of integral.

Proof is almost trivial but one needs to take in account that for some r magnetic dynamics $\Psi_{t,r}$ moving in the chosen direction may be almost instantly turned back by potential v. However as $|\langle \boldsymbol{F}(x), \xi\rangle| \ge \ell(x)^{1-\delta}$ this dynamics would move in the chosen direction for time greater than $\ell(x)^{1-\delta}$. Then contribution of this zone to the remainder does not exceed the first term in (13.8.30).

On the other hand, one can see easily that as $\mu \le h^{-\delta'}$ the contribution of the alternative zone to the remainder does not exceed the first term in (13.8.30) with $T(x) = \ell(x)^{1-\delta}$. So, as $\mu \le h^{-\delta'}$ theorem is proven.

As $\mu \ge h^{-\delta'}$ we can consider different magnetic numbers j and notice that dynamics moves in the chosen direction at least time $(\mu h|j - \bar{\jmath}(x)|)^{\frac{1}{2}}$ as $j \ne \bar{\jmath}(x)$. Then repeating arguments of Section 13.5 one can prove easily that contribution of $j \ne \bar{\jmath}(x)$ to the remainder does not exceed the first term in (13.8.30) and the contribution of $j = \bar{\jmath}(x)$ does not exceed $C\mu h^{-1-\delta''}$. Therefore as $\mu \le h^{\delta'-1}$ estimate we arrive to the announced estimate. Also as $\mu \ge h^{\delta'-1}$ we arrive to estimate $O(h^{-2-\delta})$. So, Statements (ii) and (iii) are proven. □

We leave to the reader rather challenging

Problem 13.8.10. (i) Prove that under non-degeneracy conditions $(13.4.57)_m$ with $\nu = 1$ and may be

$$\sum_{1\le|\alpha|\le m} |\nabla^{\alpha}_{\partial X}\bigl(\frac{V}{F}\bigr)| \ge \epsilon \tag{13.8.36}_m$$

where $\nabla_{\partial X}$ mean gradient along the boundary estimate (13.8.34) holds even as $\mu \in [h^{\delta'-1}, h^{-1}]$.

(ii) Investigate Schrödinger-Pauli operator as $\mu h \ge 1$.

However, as $\vartheta > 0$ situation deteriorates and the same arguments imply

Theorem 13.8.11. *Let* $d = 3$, $\vartheta > 0$. *Then under Dirichlet boundary condition*[48)]

(i) Then

$$\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)} \le Ch^{\vartheta-2}\int_{X_{\mathsf{inn}}} T(x)^{-1-\vartheta}\,dx + C\mu^{1+\vartheta}h^{-2}\,\mathsf{mes}(X_{\mathsf{bound}}). \tag{13.8.37}$$

(ii) As

$$\mathsf{mes}\big(\{x\in X\,\mathsf{dist}(x,\partial X)<\rho\}\cap B(0,1)\big) = O(\rho) \tag{13.8.38}$$

estimate

$$\mathsf{R}^{\mathsf{MW}}_{\infty(\vartheta)} \le C\mu^{\vartheta}h^{-2} \tag{13.8.39}$$

holds.

Proof. Proof is trivial with $T(x)=\ell(x)$. □

Proof. The simple rescaling technique works for estimate of X_{inn}. To estimate contribution of X_{bound} we just make $\mu^{-\frac{1}{2}}$-partition and rescale; the contribution of each such element to the remainder does not exceed C. □

Case $d=3$, Improved Remainder Estimates

Magnetic billiards must follow magnetic lines; but magnetic lines are not billiards and all of them bounce back from the boundary going the same path in the opposite direction. So escape condition in the bounded domain fails for sure but these magnetic lines drift.

Consider what happens as Hamiltonian trajectory reflects from the boundary. We need to analyze only the following

Example 13.8.12. Let

$$a(x,\xi) = \frac{1}{2}\big(x_1^2+(\xi_2-\mu x_1)^2+\xi_3^2\big) \tag{13.8.40}$$

and $X=\{x : x_3 > kx_1\}$ i.e. we assume that magnetic lines are transversal to ∂X.

Then before reflection movement is described by

$$\begin{aligned} x_1=\mu^{-1}\rho\sin(\mu t)+\bar{x}_1,\quad x_2=\mu^{-1}\rho\cos(\mu t)+\bar{x}_2,\quad x_3=\sigma t+\bar{x}_3,\\ \xi_1=\rho\cos(\mu t),\quad \xi_2=\mu\bar{x}_1,\quad \xi_3=\sigma \end{aligned} \tag{13.8.41}$$

with constant $\bar{x}$, $\rho \geq 0$, σ and after reflection movement is described by (13.8.41)$'$ with parameters $\bar{x}$, ρ', σ' and with t replaced by t'.

One can prove easily that

$$\rho^2 + \sigma^2 = \rho'^2 + \sigma'^2 \tag{13.8.42}$$

and

$$\bar{x}_1' = \bar{x}_1, \quad , \bar{x}_2' = \bar{x}_2 - 2\mu^{-1} l k, \quad \sigma' = \sigma - 2l \tag{13.8.43}$$

with $l \in [\sigma - \rho|k|, \sigma + \rho|k|]$ the cosine of the incidence angle.

So, if $k = 0$ i.e. $\boldsymbol{F}$ is orthogonal to ∂X nothing really happens: $\bar{x}' = \bar{x}$, $\rho' = \rho$ and $\sigma' = -\sigma$.

However if $k \neq 0$ i.e. $\boldsymbol{F}$ is not orthogonal to ∂X, $\bar{x}$ jumps by $-2\mu^{-1} l k$ in the direction $\boldsymbol{F}^{\perp} \cap T\partial X$ and also energy between movement along magnetic line and winding redistributes thus magnetic number j which has sense only away from the boundary, jumps by $O((\mu h)^{-1} k)$.

Therefore even calculation of the total jump seems to be a difficult problem; most likely there is no consistent movement but just a wobbling.

Microlocal implications especially as μ is close to h^{-1} seem to be really unclear. This leads to an interesting and very challenging

Problem 13.8.13. Repeat analysis of Subsection 13.7.3 and recover remainder estimate $O(\mu^{-1} h^{-2})$ as $\mu h \leq 1$ under appropriate assumptions:

(i) Either magnetic drift of trajectories is larger than the possible effect of shifting

(ii) Or movement along magnetic line is bounded by ∂X only on one side and by a growing potential on the other; then jumps are going only in one direction and drift direction is disjoint from it.

(iii) Consider Schrödinger-Pauli operator as $\mu h \geq 1$ and recover remainder estimate $O(\mu h^{-1})$ under similar assumptions.

Another challenging problem is

Problem 13.8.14. In the generic case as $\mu h \to 0$ prove the remainder estimate $o(h^{-2})$.

Case $d = 2$

One can prove that under usual non-degeneration assumptions the contribution of inner zone to the remainder estimate does not exceed

$$C\mu^{-1}h^{-1}\int_{X_{\mathsf{inn}}} T(x)^{-1}\,dx \tag{13.8.44}$$

as $\mu \le h^{-\frac{1}{2}}$ with X_{inn} defined by (13.8.27) and $T(x)$ defined by (13.8.31)–(13.8.32) with Φ_t replaced by drift flow ϕ_t. Under assumption (13.8.33) it would be $O(\mu^{-1}h^{-1})$.

However contribution of X_{bound} is estimated by $C\mu h^{-1}\,\mathsf{mes}(X_{\mathsf{bound}})$ and it is $\asymp h^{-1}$. It could be improved to $O(\mu^{-1}h^{-1})$ under certain assumptions; we will do it in the dedicated Chapter 15.

13.A Appendices

13.A.1 Two Integral Formulae

Let us define numbers A_k and B_k with $k \in \mathbb{Z}^+$ such that

$$\sum_{0\le k\le m-1} A_k \frac{(2m+1)!}{(2m-2k-1)!} = 1, \tag{13.A.1}_1$$

$$\sum_{0\le k\le m-1} B_k \frac{(2m+1)!}{(2m-2k-1)!} = -2m; \tag{13.A.1}_2$$

in particular, $A_0 = \frac{1}{6}$, $B_0 = -\frac{1}{3}$.

Then for $m \in \mathbb{Z}^+$ the formulas

$$I(c-h, c+h) \sim 2hf(c) + \sum_{0\le k\le m-1} A_k h^{2k+2} f^{(2k+1)}(x)\big|_{x=c-h}^{x=c+h}, \tag{13.A.2}_1$$

$$I(c-h, c+h) \sim h(f(c-h)+f(c+h)) + \sum_{0\le k\le m-1} B_k h^{2k+2} f^{(2k+1)}(x)\big|_{x=c-h}^{x=c+h} \tag{13.A.2}_2$$

with $I(a,b) = \int_a^b f(x)dx$ are exact provided $f(x)$ is a polynomial of degree not exceeding $2m+1$. This is easy to prove for $c = 0$ by first observing that these formulas are exact for all odd functions; then we translate. Applying the Taylor formula we therefore obtain that the error $\mathcal{R}'_m$ of formula $(13.A.2)_j$ (i.e., the absolute value of the difference between its left- and right-hand expressions) does not exceed

$$C_m h^{2m+1+i} \max_{c-h\le x\le c+h} |f^{(2m+i)}(x)|$$

with $i = 0, 1, 2$.

Let us treat the segment $[a,b]$ of length $L = b-a$; let us divide it into N subsegments of length $2h = \frac{L}{N}$. Let us consider the integral formulas

$$(13.A.3)_1 \quad I(c-h, c+h) \sim \sum_{0\le j\le N-1} f(a+(2j+1)h)\cdot 2h + \sum_{0\le k\le m-1} A_k h^{2k+2} f^{(2k+1)}(x)|_{x=a}^{x=b},$$

$$(13.A.3)_2 \quad I(c-h, c+h) \sim (f(a)+f(b))h + \sum_{1\le j\le N-1} f(a+2jh)\cdot 2h + \sum_{0\le k\le m-1} B_k h^{2k+2} f^{(2k+1)}(x)|_{x=a}^{x=b}$$

which follow from $(13.A.2)_{1,2}$ by the substitution $c = a+(2j+1)h$ and summation with respect to $j = 0, \dots, N-1$.

Then the previous statements imply that these formulas are exact on all polynomials of degree not exceeding $2m+1$ and their errors satisfy the estimates

$$(13.A.4) \quad \mathcal{R}_m \le C_m L h^{2m+i+1} \sum_{0\le j\le N-1} \max_{a+2jh\le x\le a+2(j+1)h} |f^{(2m+i)}(x)| \le C_m L h^{2m+i} \max_{a\le x\le b} |f^{(2m+i)}(x)|$$

with $i = 0, 1, 2$.

In particular, for $m = 0$ we have formulas of second order accuracy.

13.A.2 Trajectories of the Classical Particles in a Magnetic Field

Let us treat the movement of a classical particle of mass $m = \frac{1}{2}$ and energy τ in various situations. We assume that $g^{jk} = \delta_{jk}$.

Case $V = 0, \boldsymbol{F} = \text{const}$

Let there be no electric field (i.e., $V = 0$) and let the magnetic field be constant: $F_{12} = \mu$ (and $F_{13} = F_{23} = 0$ in the three-dimensional case). Then the equations

$$\frac{dp_1}{dt} = -2\mu p_2 - \partial_{x_1} V, \quad \frac{dp_2}{dt} = 2\mu p_1 - \partial_{x_2} V, \quad \frac{dp_3}{dt} = -\partial_{x_3} V, \qquad \frac{dx_j}{dt} = 2p_j \qquad (j = 1, 2, 3) \tag{13.A.5}$$

yield that a two-dimensional particle moves along the circle

$$x_1 = \bar{x}_1 + \frac{a}{\mu}\cos 2\mu t, \quad x_2 = \bar{x}_2 + \frac{a}{\mu}\sin 2\mu t \tag{13.A.6}$$

with $a^2 = \tau$ where here and in what follows we replace t by $t - t_0$ if necessary. The corresponding drift line degenerates into a point.

A three-dimensional particle moves along the spiral

$$x_1 = \bar{x}_1 + \frac{a}{\mu}\cos 2\mu t, \quad x_2 = \bar{x}_2 + \frac{a}{\mu}\sin 2\mu t, \quad x_3 = \bar{x}_3 + 2bt \tag{13.A.7}$$

with $a^2 + b^2 = \tau$. The magnetic lines are parallel to the x_3-axis.

Case $\nabla V = \text{const}, \boldsymbol{F} = \text{const}$

Let us assume that electric field is constant (and the magnetic field as the same as before until otherwise indicated). Then for $d = 2$ one can assume without any loss of the generality that $V = Ex_2$ where E is the scalar intensity of the electric field; then the particle moves along the (epi-, hypo-) cycloid

$$x_1 = \bar{x}_1 + \frac{a}{\mu}\cos 2\mu t + \frac{E}{\mu}t, \quad x_2 = \bar{x}_2 + \frac{a}{\mu}\sin 2\mu t \tag{13.A.8}$$

with

$$a^2 + \frac{1}{4\mu^2}E^2 + E\bar{x}_2 = \tau. \tag{13.A.9}$$

So the average speed of the particle is equal to $\mu^{-1}E$ and is orthogonal to electric field. The corresponding drift lines are parallel to the x_1-axis.

For $d = 3$ one can assume without loss of generality that $V = E_1x_1 + E_3x_3$ in which case

$$x_1 = \bar{x}_1 + \frac{a}{\mu}\cos 2\mu t, \quad x_2 = \bar{x}_2 + \frac{a}{\mu}\sin 2\mu t - \frac{E_1}{\mu}t, \quad x_3 = \bar{x}_3 + 2bt - E_3t^2 \tag{13.A.10}$$

with $a^2 + \frac{1}{4\mu^2}E_1^2 + E_1\bar{x}_1 + E_3\bar{x}_3 + \frac{1}{4}b^2 = \tau$. The corresponding magnetic lines are parabolas (for $E_3 \neq 0$) or straight lines (for $E_3 = 0$).

Case $V = \frac{1}{4}k^2|x|^2$, $\boldsymbol{F} = \mathrm{const}$

Let $V = \frac{1}{4}k^2|x|^2$ with $k > 0$. Then the two-dimensional particle moves along the cycloidal curve

$$\begin{cases} x_1 = \dfrac{A_1}{\omega_1}\cos\omega_1 t + \dfrac{A_2}{\omega_2}\cos(\omega_2 t + \varphi), \\ x_2 = \dfrac{A_1}{\omega_1}\sin\omega_1 t + \dfrac{A_2}{\omega_2}\sin(\omega_2 t + \varphi) \end{cases} \tag{13.A.11}$$

with

$$\omega_{1,2} = (k^2 + \mu^2)^{\frac{1}{2}} \mp \mu \tag{13.A.12}$$

and

$$A_1^2(1 + \frac{k^2}{\omega_1^2}) + A_2^2(1 + \frac{k^2}{\omega_2^2}) = 4\tau. \tag{13.A.13}$$

It is obvious that for $k \in (0, 1]$ and $\mu \gg 1$ we obtain $\omega_1 \asymp k^2/\mu$ and $\omega_2 \asymp \mu$. The corresponding drift lines are circles.

In the three-dimensional case one should to add the third equation

$$x_3 = A_3\cos(kt + \psi) \tag{13.A.14}$$

and the energy equation is now

$$A_1^2(1+\frac{k^2}{\omega_1^2})+A_2^2(1+\frac{k^2}{\omega_2^2})+k^2A_3^2=4\tau. \tag{13.A.15}$$

The corresponding magnetic lines are parallel to the x_3-axis.

Let us note that in the case 13.A.2.2 *Case* $\nabla V = \mathsf{const}, \boldsymbol{F} = \mathsf{const}$ Hamiltonian trajectories are non-periodic and in the case 13.A.2.3 *Case* $V=\frac{1}{4}k^2|x|^2, \boldsymbol{F}=\mathsf{const}$ generic Hamiltonian trajectories are non-periodic; less generic trajectories (with commensurable ω_1 and ω_2 for $d=2$ and with commensurable ω_1, ω_2 and k for $d=3$) are long-periodic (with period $\ge \epsilon\mu/k^2$). On the other hand, in the case 13.A.2.1 *Case* $V=0, \boldsymbol{F}=\mathsf{const}$ Hamiltonian trajectories are short-periodic (with period $2\tau/\mu$).

Case $V=\frac{1}{4}k^2x_3^2, \boldsymbol{F}=\mathsf{const}$

One can easily prove that in this case generic trajectories are non-periodic and less generic trajectories (with commensurable k and μ) are periodic (with period $\ge \epsilon\,\mathsf{max}(1/k, 1/\mu)$).

Comments

The spectral theory of the Schrödinger operator with magnetic field became rather popular lately but majority of the authors are concentrating on other problems.

Relevant papers, especially those of Y. Colin de Verdiere, H. Tamura, G. Raikov, G. Rozenblum and A. Sobolev will be mentioned in the forthcoming chapters.

The same normal forms as we use here can also be used in the analysis of Schrödinger magnetic operators at the bottom; see f.e. N. Raymond and S. Vu-Ngoc [1], B. Helffer, Y. Kordyukov, N. Raymond and S. Vu-Ngoc [1] and few earlier papers.

Chapter 14

2D-Schrödinger Operator with the Strong Degenerating Magnetic Field

In this chapter we consider 2D-Schrödinger operator (13.2.1) with the strong magnetic field like in the previous chapter albeit now magnetic intensity F degenerates along smooth line and the degeneration is precisely of order $\nu - 1$ with $\nu \geq 2$. Surely, we are most interested in the generic case $\nu = 2$.

We start from very short Section 14.1 where we apply results of the previous Chapter rescaled to understand shortcomings of this approach.

Then in Section 14.2 we consider classical dynamics which is useful both for understanding and proofs of our main results. We will notice the obvious difference between cases of odd and even ν.

In Section 14.3 we consider corresponding quantum (semiclassical) dynamics and the Tauberian remainder estimates. It appears that there will be several zones: a *classically forbidden zone* $\{|x_1| \geq \bar{\gamma}_1 = C(\mu h)^{1/(\nu-1)}\}$ [1) 2)], a *regular zone* $\mathcal{X}_{\mathsf{reg}} = \{\bar{\gamma}_0 = C\mu^{-1/\nu} \leq |x_1| \leq \bar{\gamma}_1\}$ and a *degeneration zone* $\mathcal{X}_{\mathsf{deg}} = \{|x_1| \leq \bar{\gamma}_0\}$ divided into a *non-periodic zone* and a rather thin *periodic zone* with the analysis in the latter one the most difficult and interesting. The width (in the phase space) of the periodic zone is defined by the uncertainty principle and it increases as μ grows.

In Section 14.4 we instead of implicit main part of asymptotics derived in Section 14.3 provide much more explicit answer $h^{-2}\mathcal{N}^{\mathsf{MW}} + h^{-1}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ where $h^{-2}\mathcal{N}^{\mathsf{MW}}$ is the standard Magnetic Weyl expression of the previous chapter and $h^{-1}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ is a *correction term*. All the analysis in Sections 14.3 and 14.4

[1)] Only as $Ch^{-1} \leq \mu \leq Ch^{-\nu}$; otherwise $\bar{\gamma}_1$ is artificially set to ϵ.

[2)] Here and below coordinate x_1 is defined so that $\Sigma = \{x_1 = 0\}$.

is under assumption $\mu \le h^{-\nu+\delta}$.

Section 14.5 is devoted to the similar analysis in the case of a *very trong magnetic field* $h^{-\nu+\delta} \le \mu \le \epsilon h^{-\nu}$ which is in some sense easier and in some sense more difficult than the previous one because $\hbar = \mu^{1/\nu} h$ is not very small anymore.

Let us recall that in the previous Chapter 13 we called the case $\mu \gtrsim h^{-1}$ *strong magnetic field* and the difference between this case and the case of $\mu \lesssim h^{-1}$ remains crucial. However now due to degeneration our focus shifts to larger μ.

Then in Section 14.6 we summarize our results as main theorems and generalize them for vanishing V and discuss some other generalizations.

In Section 14.7 we modify potential V replacing it by $V - \mu h_3 F$ and establish our results in this case as well.

With this new V we consider in Section 14.8 a case of *superstrong magnetic field* $\mu h^{\nu} \ge C$ which is not trivial anymore.

Finally, Appendix 14.A is devoted to the analysis of eigenvalues of some auxiliary 1D-operator.

14.1 Preliminary Analysis

14.1.1 Set-up

So, consider operator (13.2.1) ($d = 2$) satisfying conditions (13.2.2)–(13.2.5) and (13.4.45) ($V \le -\epsilon_0$) [3] but we do not assume that F is disjoint from 0; however we are mainly interested in the case of the generic degeneration

$$|F_{12}| + |\nabla F_{12}| \ge \epsilon_0. \tag{14.1.1}$$

Remark 14.1.1. It is well-known (see f.e. V. Arnold [2]) that the generic 2-form in dimension 2 may degenerate in a very special way: locally it can be reduced to form

$$\omega = x_1 dx_1 \wedge dx_2 \tag{14.1.2}$$

matching to 14.1.1).

[3] Condition (13.4.45) will be dropped in the very end by rescaling arguments.

Then $\Sigma = \{x : F_{12}(x) = 0\}$ is a smooth curve. We call Σ *degeneration line*. One can introduce a coordinate x_1 as a Riemannian distance from x to Σ; then $g^{11} = 1$. One can always change x_2 to keep $g^{12} = 0$ and then

$$A = P_1^2 + P_2 g^{-1} P_2 + V, \qquad P_j = D_j - \mu V_j. \tag{14.1.3}$$

Furthermore, without any loss of the generality one can assume locally that

$$V_1 = 0, \qquad V_2 \asymp |x_1|^\nu \tag{14.1.4}$$

with $\nu = 2$; one can achieve this by the gauge transformation. Actually instead we consider operator (14.1.3) with positive integer $\nu \ge 2$ and in the classical dynamics we even take any real $\nu \ge 2$.

As usual we are interested in an asymptotics as $h \to +0$, $\mu \to +\infty$ of

$$\int e(x,x,0)\psi(x)\,dx$$

where $e(x,y,\tau)$ is the Schwartz kernel of the spectral projector $E(\tau)$ of operator A. We will prove later that $e(x,x,\tau)$ will be $O(\mu^{-s})$ as $\mu \ge C_0 h^{-\nu}$ and therefore we will assume that

$$\mu \le C_0 h^{-\nu}. \tag{14.1.5}$$

The natural answer coming from the non-degenerate case is $\int h^{-2}\mathcal{N}^{\mathsf{MW}}(x,0)\psi(x)\,dx$ where $\mathcal{N}^{\mathsf{MW}}$ is defined by $(13.2.9)_{(2)}$ i.e.

$$\mathcal{N}^{\mathsf{MW}} = \frac{1}{2\pi}\sum_{n\ge 0} \theta\big(\tau - V - (2n+1)\mu hF\big)\mu hF\sqrt{g}$$

which implies that there are actually two different cases:

Remark 14.1.2. There are two cases: the case of the strong magnetic field in the sense of the previous chapter

$$1 \le \mu\epsilon \le h^{-1} \implies \mathcal{N}^{\mathsf{MW}} = O(1), \tag{14.1.6}$$

and the case of the superstrong magnetic field in the sense of the previous chapter

$$\epsilon h^{-1} \le \mu \le C_0 h^{-\nu} \implies \mathcal{N}^{\mathsf{MW}} = O\big((\mu h)^{-1/(\nu-1)}\big); \tag{14.1.7}$$

in the latter case the contribution to the answer will be given by the strip

$$\big\{x,\ |x_1| \le \bar{\gamma}_1 = C_0(\mu h)^{-1/(\nu-1)}\big\}. \tag{14.1.8}$$

The standard rescaling procedure applied to results of Chapter 13 implies

$$\mathsf{R}^{\mathsf{MW}} := \Big| \int \Big(e(x,x,0) - h^{-2}\mathcal{N}^{\mathsf{MW}}(x,0)\Big)\psi(x)\,dx\Big| \le Ch^{-1} \tag{14.1.9}$$

(see Subsection 14.1.2) but an aim of this chapter is much better estimate, up to $O(\mu^{-1/\nu}h^{-1})$. So we are going to improve remainder estimate (14.1.9).

14.1.2 Zones and Rescaling: the First Blood

Let us consider case (14.1.6) $B(x,\gamma(x))$ with $\gamma(x) = \frac{1}{2}\mathsf{dist}(x,\Sigma)$ and rescale $x \mapsto x/\gamma$, $h \mapsto h_{\mathsf{new}} = h\gamma^{-1}$, $\mu \mapsto \mu_{\mathsf{new}} = \mu\gamma^{\nu}$ where we select

$$\bar{\gamma} := \mu^{1/\nu} \tag{14.1.10}$$

to keep $\mu_{\mathsf{new}} \ge 1$. As $\gamma \le \epsilon$ non-degeneracy condition $|\nabla V/F| \ge \epsilon$ is obviously fulfilled[4)] and therefore due to the results of the previous chapter contribution of $B(x,\gamma(x))$ to the remainder does not exceed

$$C\mu_{\mathsf{new}}^{-1}h_{\mathsf{new}}^{-1} = C\mu^{-1}h^{-1}\gamma^{1-\nu} \tag{14.1.11}$$

and then the contribution of *regular zone*

$$\mathcal{X}_{\mathsf{reg}} := \{x,\ \gamma(x) \ge \bar{\gamma}\} \tag{14.1.12}$$

to the remainder does not exceed

$$C\int_{\mathcal{X}_{\mathsf{reg}}} \mu^{-1}h^{-1}\gamma^{1-\nu} \times \gamma^{-2}\,dx \asymp C\mu^{-1}h^{-1}\bar{\gamma}^{-\nu} \asymp Ch^{-1} \tag{14.1.13}$$

exactly as we promised. Meanwhile as $\gamma(x) \le \bar{\gamma}$ we reset $\gamma = \bar{\gamma}$ and then $\mu_{\mathsf{new}} = 1$ and the contribution of $B(x,\bar{\gamma})$ to the remainder does not exceed (14.1.11) with $h_{\mathsf{new}} = h\bar{\gamma}^{-1}$ and $\mu_{\mathsf{new}} = 1$ (which is $Ch^{-1}\bar{\gamma}$) and then contribution of *degeneration zone*

$$\mathcal{X}_{\mathsf{deg}} := \{x,\ \gamma(x) \le \bar{\gamma}\} \tag{14.1.14}$$

to the remainder does not exceed Ch^{-1}.

Therefore we need to improve estimates of contributions of both zones: $\mathcal{X}_{\mathsf{reg}}$ and $\mathcal{X}_{\mathsf{deg}}$.

[4)] As we replace F by $\gamma^{1-\nu}F$.

14.2 Classical Dynamics

In this section we consider classical dynamics on energy level 0 described by Hamiltonian

$$a = \sum_{j,k} g^{jk}(x) p_j p_k - V, \qquad p_j = \xi_j - \mu V_j \tag{14.2.1}$$

where all conditions of Subsection 14.1.1 are assumed to be fulfilled.

14.2.1 Toy-mModel

Let us consider first a model

$$a = \xi_1^2 + \bigl(\eta - \frac{1}{\nu}|x_1|^{\nu}\bigr)^2 - 1 \tag{14.2.2}$$

with $\nu \geq 2$, corresponding to $\mu = 1$, $V = -1$, $\eta = \xi_2$.

Proposition 14.2.1. *For symbol* (14.2.2)

(i) Along trajectories $\eta = \mathsf{const}$, *and*

(a) If $\eta > 1$ *then on the energy level* 0 *these trajectories oscillate* <u>*either*</u> *between* $x_1 = b_1(\eta)$ *and* $x_1 = b_2(\eta)$ <u>*or*</u> *between* $x_1 = -b_1(\eta)$ *and* $x_1 = -b_2(\eta)$ *with*

$$b_1(\eta) = \bigl((\eta - 1)\nu\bigr)^{1/\nu}, \quad b_2(\eta) = \bigl((\eta + 1)\nu\bigr)^{1/\nu}. \tag{14.2.3}$$

(b) If $\eta = 1$ *then these trajectories oscillate between* $b_1(1) = 0$ *and* $b_2(1) = 1$ *or between* -1 *and* 0*; there is also an exceptional trajectory* $x_1 = \xi_1 = 0$, $x_2 = 2t + \mathsf{const}$.

(c) If $-1 < \eta < 1$ *then these trajectories oscillate between* $x_1 = b_1(\eta) := -b_2(\eta)$ *and* $x_1 = b_2(\eta)$ *with* $b_2(\eta)$ *defined by* (14.2.3).

(d) If $\eta = -1$ *then the energy level* 0 *degenerates into* $\{x_1 = \xi_1 = 0\}$ *and this trajectory is* $x_1 = \xi_1 = 0$, $x_2 = -2t + \mathsf{const}$.

(e) energy level 0 is empty as $\eta < -1$.

(ii) Along each of these trajectories phase coordinates (x_1, ξ_1) *are periodic with period*

$$T(\eta) = \int_{b_1\eta)}^{b_2\eta)} \frac{1}{\sqrt{1-(\eta-|y|^\nu/\nu)^2}}\, dy \tag{14.2.4}$$

and $x_2(t) = v(\eta)t + \tilde{x}_2(t)$ *where* $v(\eta) = I(\eta)/T(\eta)$,

$$I(\eta) = 2\int_{b_1\eta)}^{b_2\eta)} \frac{\eta-|y|^\nu/\nu}{\sqrt{1-(\eta-|y|^\nu/\nu)^2}}\, dy, \tag{14.2.5}$$

where $b_2(\eta)$ *defined by* (14.2.3) *and* $b_1(\eta) := -b_2(\eta)$ *is redefined as* $\eta < 1$ *and* $\tilde{x}_2(t)$ *is* $T(\eta)$*-periodic function.*

Proof. Proof is obvious consequence of the fact that evolution in (x_1, ξ_1) is described by 1-dimensional Hamiltonian (14.2.2) while $\xi_2 = \eta = \mathsf{const}$. Potential

$$U = U(x, \eta) = \bigl(\eta - \frac{1}{\nu}|x|_1^\nu\bigr)^2 - 1 \tag{14.2.6}$$

has two wells as $\eta > 0$ and one well as $\eta \le 0$; as $k = 0$ it has a flat bottom. Note that the bottoms of the potential are in $\pm b(\eta)$, $b(\eta) = (\eta\nu)^{1/\nu}$.

Further, as $\eta > 1$ we have $U(0;\eta) > 0$ and thus the dynamics lives in one well oscillating either between $b_1(\eta)$ and $b_2(\eta)$ or between $-b_2(\eta)$ and $-b_1(\eta)$. As $\eta < 1$ we have either one-well potential or $U(0;\eta) < 0$ so dynamics oscillates between $b_1(\eta) = -b_2(\eta)$ and $b_2(\eta)$. As $\eta = 1$ dynamics moves between $b_1(1) = 0$ (as $t \to \pm\infty$) and $\pm b_2(1)$ (as long as $\nu \ge 2$). □

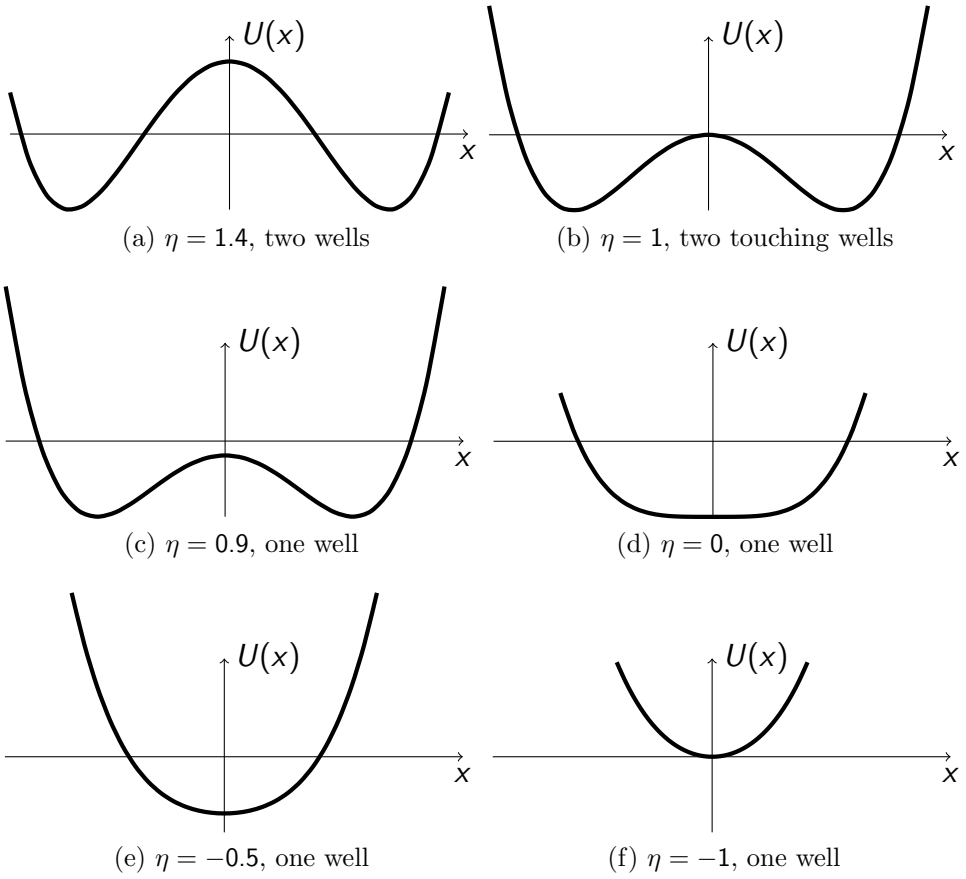
Proposition 14.2.2. *(i)* $v(\eta) > 0$ *and both* $T(\eta)$ *and* $v(\eta)$ *are decreasing functions of* $\eta > 1$.

(ii) The width of the wells is $\sim 2b(\eta)^{(1-\nu)/\nu}$ *and* $T(\eta) \sim \pi b(\eta)^{1-\nu}$, $v(\eta) \sim (\nu-1)b(\eta)^{-\nu}$ *as* $\eta \to +\infty$, $b(\eta) = (\eta\nu)^{1/\nu}$.

(iii) $T(\eta) \sim 2\bigl|\log|\eta-1|\bigr|$ *(for* $\nu = 2$*),* $T(\eta) \sim \mathsf{const}|\eta-1|^{(2-\nu)/2\nu}$ *(for* $\nu > 2$*) and* $v(\eta) \sim 1$ *as* $\eta \to 1+0$.

Proof. Consider $\eta > 1$. Changing $z = y^\nu/\nu - \eta$ we arrive to

$$T(\eta) = \int_{-1}^{1} \bigl((\eta+z)\nu\bigr)^{\frac{1}{\nu}-1}(1-z^2)^{-\frac{1}{2}}\, dz, \tag{14.2.7}$$



(a) $\eta = 1.4$, two wells (b) $\eta = 1$, two touching wells

(c) $\eta = 0.9$, one well (d) $\eta = 0$, one well

(e) $\eta = -0.5$, one well (f) $\eta = -1$, one well

Figure 14.1: Graphs of $U(x;\eta)$, 1-st toy-model (14.2.2) with $\nu = 2$

and

$$I(\eta) = 2\int_{-1}^{1} z\big((\eta+z)\nu\big)^{\frac{1}{\nu}-1}(1-z^2)^{-\frac{1}{2}}\,dz = \int_{-1}^{1} z\Big(\big((\eta-z)\nu\big)^{\frac{1}{\nu}-1} - \big((\eta+z)\nu\big)^{\frac{1}{\nu}-1}\Big)(1-z^2)^{-\frac{1}{2}}\,dz, \tag{14.2.8}$$

which implies that $I(\eta) > 0$ as $\eta > 1$.

Asymptotics $T(\eta) \sim \pi(\eta\nu)^{(1-\nu)/\nu}$, $I(\eta) \sim \pi(\nu-1)(\eta\nu)^{(2\nu-1)/2}$ as $\eta \to +\infty$ follow from (14.2.7) and (14.2.8) as well.

Also (14.2.7) and (14.2.8) imply that

$$\frac{\partial T}{\partial \eta} = -(\nu-1)\int_{-1}^{1} \big((\eta+z)\nu\big)^{\frac{1}{\nu}-2}(1-z^2)^{-\frac{1}{2}}\,dz, \tag{14.2.9}$$

$$\frac{\partial I}{\partial \eta} = -2(\nu-1)\int_{-1}^{1} z\big((\eta+z)\nu\big)^{\frac{1}{\nu}-2}(1-z^2)^{-\frac{1}{2}}\,dz, \tag{14.2.10}$$

$$\frac{\partial v}{\partial \eta} = -\nu(\nu-1)T^{-2}\times \int_{-1}^{1}\int_{-1}^{1}(z-z')^2\big((\eta+z)\nu\big)^{\frac{1}{\nu}-2}\big((\eta+z')\nu\big)^{\frac{1}{\nu}-2}(1-z^2)^{-\frac{1}{2}}\,dzdz' \tag{14.2.11}$$

implying monotonicity properties.

Asymptotics as $\eta \to 1+0$ follow directly from (14.2.4),(14.2.5). □

Remark 14.2.3. In the strong magnetic field the magnetic drift is described by

$$\frac{dx}{dt} = -\mathcal{L}VF^{-1} = (\partial_{x_2}F, -\partial_{x_1}F) = (0, (\nu-1)x_1^{-\nu}) \tag{14.2.12}$$

as $V=-1$ and this matches to asymptotics of v as $\xi_2 \to +\infty$.

As $-1<\eta<1$ one can rewrite

$$T(\eta) = 2\int_0^{b_2}\Big(1-(\eta-y^\nu/\nu)^2\Big)^{-\frac{1}{2}}\,dy = 2(\eta+1)^{\frac{1}{\nu}-\frac{1}{2}}\nu^{\frac{1}{\nu}}\int_0^1 (1-y^\nu)^{-\frac{1}{2}}\Big(2-(\eta+1)(1-y^\nu)\Big)^{-\frac{1}{2}}\,dy = 2(\eta+1)^{\frac{1}{\nu}-1}\nu^{\frac{1}{\nu}}\int_0^1 (1-y^\nu)^{-\frac{1}{2}}\varphi(u,y^\nu)\,dy \tag{14.2.13}$$

and

$$I(\eta) = 4\int_0^{b_2}(\eta-y^\nu/\nu)\Big(1-(\eta-y^\nu/\nu)^2\Big)^{-\frac{1}{2}}\,dy = 4\big((\eta+1)\nu\big)^{\frac{1}{\nu}}\int_0^1 (1-y^\nu)^{-\frac{1}{2}}\varphi_1(u,y^\nu)\,dy \tag{14.2.14}$$

with $u = \eta/(\eta+1) \in (-\infty, \frac{1}{2})$,

$$\varphi(u,z) = (1-2u+z)^{-\frac{1}{2}}, \qquad \varphi_1(u,z) = (u-z)(1-2u+z)^{-\frac{1}{2}}.$$

Let $T_1(u)$, $I_1(u)$ denote integrals in the right-hand expressions of (14.2.13) and (14.2.14) respectively. Then

$$\frac{\partial T_1}{\partial u} > 0, \qquad \frac{\partial I_1}{\partial u} = (1-u)\frac{\partial T_1}{\partial u} > 0$$

because $\frac{\partial \varphi}{\partial u} = (1-2u+z)^{-\frac{3}{2}}$, $\frac{\partial \varphi_1}{\partial u} = (1-u)(1-2u+z)^{-\frac{3}{2}}$.

Then

$$\frac{\partial}{\partial u}\frac{I_1}{T_1} = T_1^{-2}\frac{\partial T_1}{\partial u}\big((1-u)T_1 - I_1\big) = T_1^{-2}\frac{\partial T_1}{\partial u} \times \int_0^1 (1-y^\nu)^{-\frac{1}{2}}\big(1-2u+y^\nu\big)^{\frac{1}{2}}\,dy > 1$$

due to Cauchy inequality.

Since $\frac{1}{2}v(\eta) = (\eta+1)I_1(u)/T_1(u)$ and $(\eta+1)\frac{du}{d\eta} = (1-u)$

$$\frac{1}{2}\frac{\partial v}{\partial \eta} = (1-u)\frac{\partial}{\partial u}\frac{I_1}{T_1} + \frac{I_1}{T_1} > \frac{1}{T_1}\big((1-u)T_1 + I_1\big) = \frac{1}{T_1}\int_0^1 (1-y^\nu)^{\frac{1}{2}}\big(1-2u+y^\nu\big)^{-\frac{1}{2}}\,dy > 0.$$

Proposition 14.2.4. *(i) There exists $\eta^* = \eta^*_\nu \in (0,1)$ such that*

$$v(\eta)(\eta-\eta^*)^{-1} > 0 \tag{14.2.15}$$

and it is both bounded and disjoint from 0 as $\eta \in (-1+\epsilon, 1)$.

(ii) $v(\eta)$ is monotone increasing at $(-1,1)$.

(iii) $T(\eta) \sim \mathsf{const}(\eta+1)^{1/\nu}$ and $v(\eta) \sim -1$ as $\eta \to -1+0$.

(iv) $T(\eta) \sim 4\big|\log|\eta-1|\big|$ (for $\nu = 2$), $T(\eta) \sim 2\mathsf{const}|\eta-1|^{(2-\nu)/2\nu}$ (for $\nu > 2$) and $v(\eta) \sim 1$ as $\eta \to 1+0$.

Proof. Statement (ii) is already proven. Statements (iii) and (iv) are easy and Statement (i) follows from Statements (ii)–(iii) and from $v(0) < 0$. □

Remark 14.2.5. (i) Calculating derivative one can prove easily that $T(\eta)$ is monotone decreasing function of η for $\nu = 2$; we are not sure if this is the case for larger ν.

(ii) $dx_2/dt > 0$ iff $|x_1| < (\eta\nu)^{1/\nu}$; in particular $dx_2/dt < 0$ on the whole trajectory iff $-1 < \eta < 0$.

(iii) One can prove easily that

$$\frac{\partial v}{\partial \eta} \sim \mp \mathsf{const} |\eta - 1|^{-\frac{1}{\nu}-\frac{1}{2}} |\log |\eta - 1||^{-2\delta_{\nu 2}} \qquad \text{as} \quad \eta \to 1 \pm 0. \tag{14.2.16}$$

(iv) In particular, $x_2(t)$ is periodic iff $\eta = \eta^*$.

(v) Since

$$\frac{\partial I_1}{\partial \nu} = \int_0^1 (u-1)^2 (1-y^\nu)^{-\frac{3}{2}} (1-2u+y^\nu)^{-\frac{3}{2}} y^\nu (-\log y)\, dy > 0$$

we conclude that η^*_ν monotonically decreases; one can see easily that $\eta^*_\nu \to +0$ as $\nu \to +\infty$.

(vi) Maple (see Subappendix 14.A.5) experiments show that $\eta^*_2 \approx 0.65$ (but not $2/3$).

The following series of Figures 14.2(a)–(l) show (x_1, x_2)-trajectories at level 0 for different values of η as $\nu = 2$. In particular, Figure 14.2(g) shows periodic curve.

Proposition 14.2.6. *There exist functions $Z = Z(x_1, \xi_1)$, $\alpha = \alpha(x_1, \xi_1, \eta)$ and $\beta = \beta(x_1, \eta)$, such that*

$$\{a, x_2 - Z\} = \eta - \{a, Z\} = \alpha(\eta - \eta^*) + \beta a(x_1, \xi_1, \eta). \tag{14.2.17}$$

Further, Z is odd and α, β are even with respect to each of variables x_1, ξ_1.

Proof. Consider now periodic curve as $\eta_=\eta^*$ with the "center" at 0. It is described by equations in (x, ξ)-phase space

$$\eta = \eta^*, \quad a(x_1, \xi_1, \eta) = 0, \quad x_2 = Z(x_1, \xi_1) \tag{14.2.18}$$

with smooth odd with respect to each x_1, ξ_1 function Z. We cannot express it via x_1 or ξ_1 alone. Then

$$\{a, x_2 - Z\} = \eta - \{a, Z\} = 0 \qquad \text{as} \quad \eta = \eta^*, \ a(x_1, \xi_1, \eta) = 0.$$

Note that da and $d\eta$ are linearly independent unless $\xi_1 = 0$ and $(\eta - |x_1|^\nu/\nu)x_1 = 0$ which for $a = 0$ means that $\xi_2 = \pm 1$. Therefore we arrive to (14.2.17). □

Remark 14.2.7. For even integer ν all these functions are analytic.

14.2.2 Another Toy-Model

Consider now another model

$$a = \xi_1^2 + \bigl(\eta - \frac{1}{\nu}|x_1|^\nu \operatorname{sgn}(x)\bigr)^2 - 1 \tag{14.2.19}$$

with $\nu \geq 2$, corresponding to $\mu = 1$, $V = -1$. Again we are looking at energy level 0. Consider again the graph of potential.

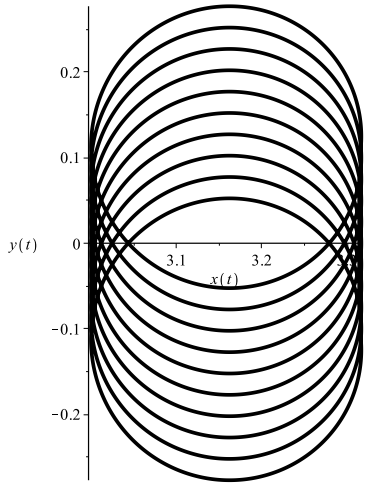
In the similar way movement in (x_1, ξ_1) is described by Hamiltonian with one-well potential and if $\eta \geq 1$ this particle moves only in $\{x_1 \geq 0\}$ and the evolution coincides with given in Figures 14.2(a)–(d) (where we considered only one component anyway) but for $-1 < \eta < 1$ picture changes; it is not central symmetric as on Figures 14.2(e)–(f) and oscillates between $b_1(\eta)$ and $b_2(\eta)$ where $b_2(\eta)$ is still defined by (14.2.3) and $b_1(\eta) := -b_2(-\eta)$ is redefined again.

As $\eta = 0$ movement is periodic, as $\eta < 0$ dynamics is central reflected to one with $-\eta$ (so we arrive to Figures 14.4(g)–(e) and then to Figures 14.2(d)–(a) rotated by π).

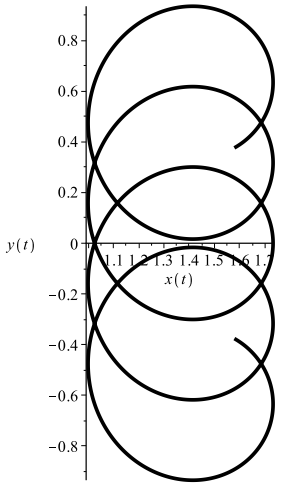
Then

$$T(\eta) = \frac{1}{2}\bigl(T_+(\eta) + T_+(-\eta)\bigr), \qquad I(\eta) = \frac{1}{2}\bigl(I_+(\eta) - I_+(-\eta)\bigr) \tag{14.2.20}$$

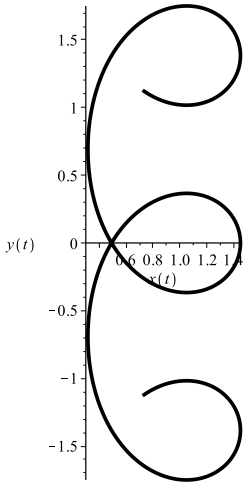
where $T_+(\eta)$, $I_+(\eta)$ are exactly what we introduced as $T(\eta)$ and $I(\eta)$ for Hamiltonian (14.2.2) by (14.2.13) and (14.2.14) respectively. However due to monotonicity of $|y|^\nu \operatorname{sgn}(y)$ we can apply idea of Proposition 14.2.2:



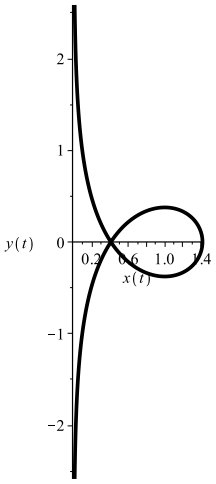
(a) $\eta = 10$, $w \approx 0.34$



(b) $\eta = 2$, $w \approx 0.73$

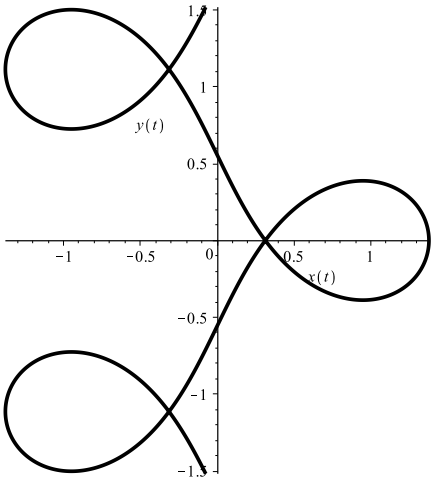


(c) $\eta = 1.1$, $w \approx 1.13$

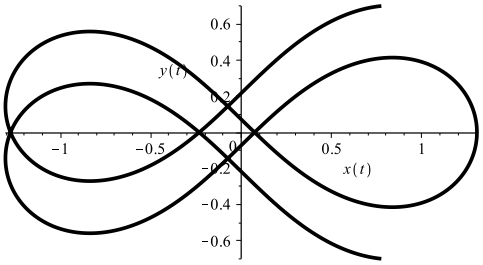


(d) $\eta = 1$, $w \approx 1.13$, first critical case

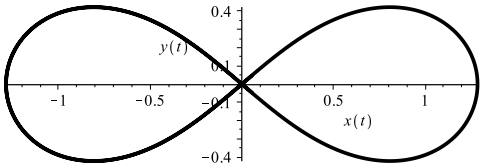
Figure 14.2: Drift up; and rotation clockwise, each figure has its mirror with respect to x_2-axis with drift up and rotation counter-clockwise as F changes sign; $w = \sqrt{\eta + 1} - \sqrt{\eta - 1}$ is the width. Aspect ratio is preserved albeit scaling differs.



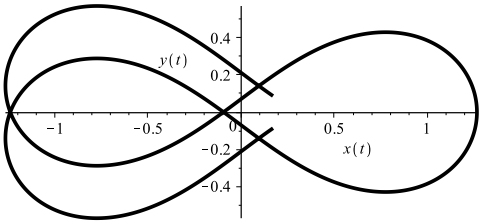
(e) $\eta = 0.9$, $w \approx 4.32$



(f) $\eta = 0.7$, $w \approx 3.78$

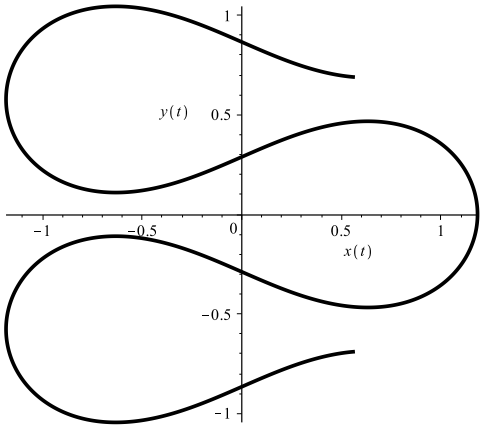


(g) $\eta \approx 0.65$, $w \approx 3.65$, second critical case

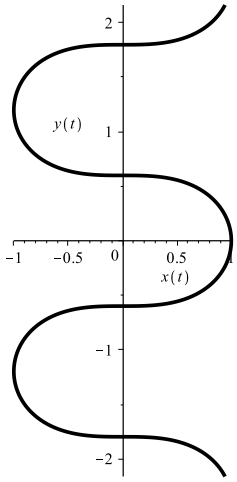


(h) $\eta = 0.6$, $w \approx 3.52$

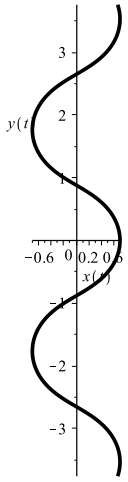
Figure 14.2: (continued) As η decreases drift slows down, (g) movement is periodic ($v(\eta^*) = 0$) and on (h) drift is down; $w = 2\sqrt{\eta + 1}$ is the width.



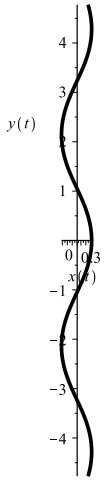
(i) $\eta = 0.4$, $w \approx 2.28$



(j) $\eta = 0$, $w = 2$

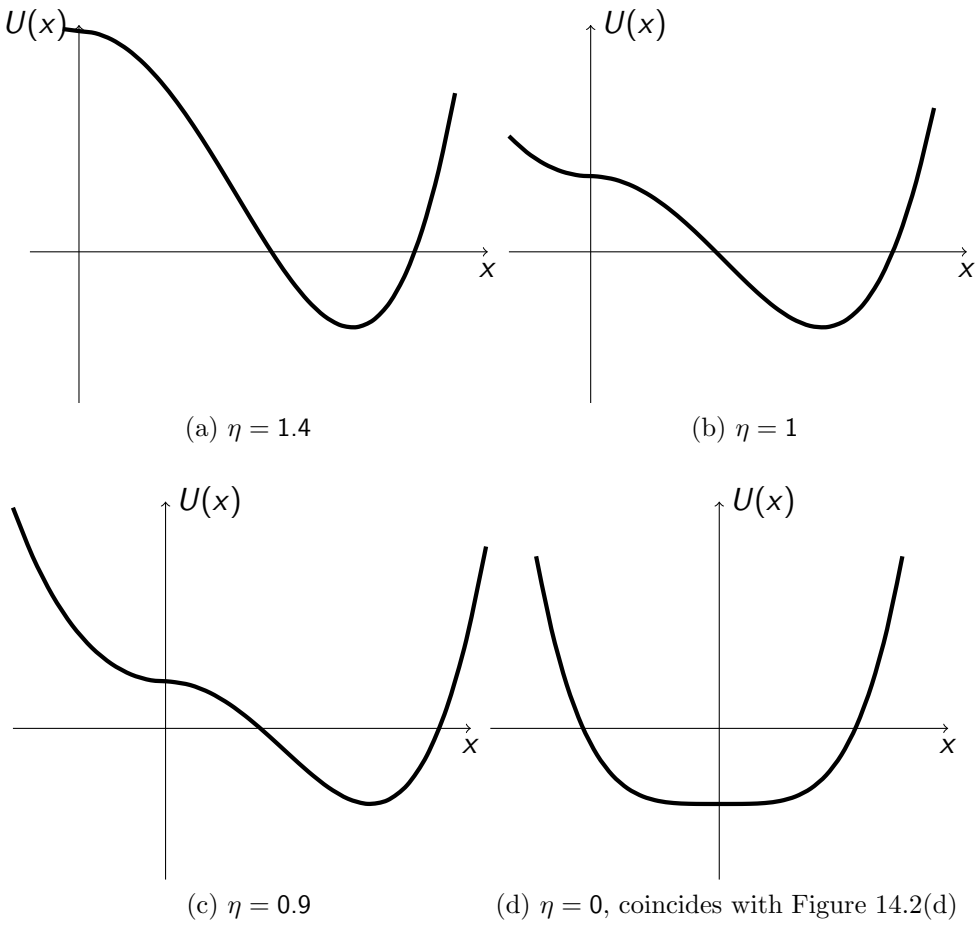


(k) $\eta = -0.5$, $w \approx 1.41$



(l) $\eta = -0.9$, $w \approx 0.63$

Figure 14.2: (continued) As η decays further drift down accelerates and width decays.



(a) $\eta = 1.4$ (b) $\eta = 1$

(c) $\eta = 0.9$ (d) $\eta = 0$, coincides with Figure 14.2(d)

Figure 14.3: Graphs of $U(x;\eta)$, 2-nd toy-model (14.2.19) with $\nu = 2$. As $\eta < 0$ graph coincides with mirror image of $U(x;-\eta)$

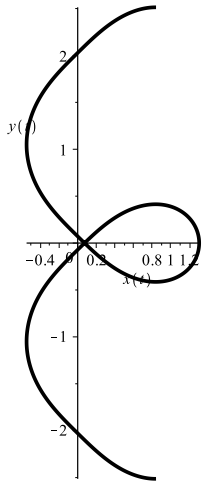
introducing $z = |y|^\nu \operatorname{sgn}(y) - \eta$ we arrive to

$$T(\eta) = \int_{-1}^{1} |(\eta+z)\nu|^{\frac{1}{\nu}-1}(1-z^2)^{-\frac{1}{2}}\,dz, \tag{14.2.21}$$

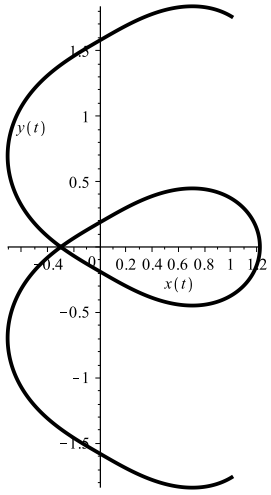
$$I(\eta) = -2\int_{-1}^{1} z|(\eta+z)\nu|^{\frac{1}{\nu}-1}(1-z^2)^{-\frac{1}{2}}\,dz = 2\int_{0}^{1} z\Big(|(\eta-z)\nu|^{\frac{1}{\nu}-1} - |(\eta+z)\nu|^{\frac{1}{\nu}-1}\Big)(1-z^2)^{-\frac{1}{2}}\,dz \tag{14.2.22}$$

and obviously $\epsilon_0 I(\eta)/\eta \le c_0$ as $\eta \ne 0$.

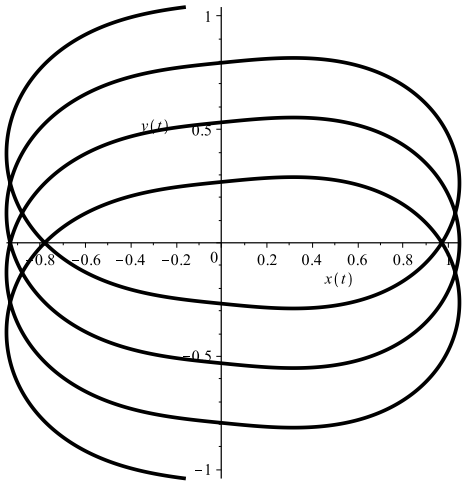
Thus we arrive to



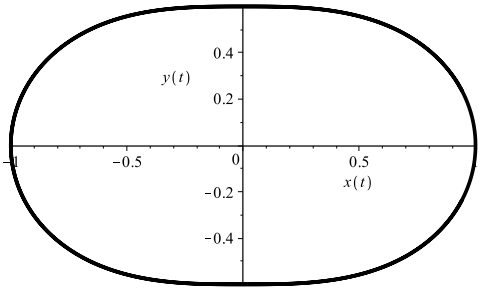
(e) $\eta = 0.9$, $w \approx 2.4$



(f) $\eta = 0.5$, $w \approx 2.73$



(g) $\eta = 0.1$, $w \approx 2.82$



(h) $\eta = 0$, $w \approx 2.83$, second critical case

Figure 14.4: Drift is up and movement clockwise. As η decays from 1 to 0 drift up slows down; (h) corresponds to periodic case; we skip $\eta \geq 1$ which repeat those at Figures 14.2(a)–(d), $\eta \leq -1$ which will be mirror-symmetric to them with drift down and also clockwise and $-1 < \eta < 0$ as central-symmetric to given there with drift down and also clockwise.

Proposition 14.2.8. *Consider on level* 0 *trajectories of Hamiltonian* (14.2.19) *with* $\eta \in (-1,1)$. *Along them*

(i) Phase coordinates (x_1,ξ_1) *are* $T(\eta)$*-periodic and* x_1 *oscillates between* $b_1 = -((1-\eta)\nu)^{1/\nu}$ *and* $b_2 = ((1+\eta)\nu)^{1/\nu}$.

(ii) $x_2(t) = \tilde{x}_2(t) + v(\eta)t$ *with* $T(\eta)$*-periodic* $\tilde{x}_2(t)$ *and* $v(\eta) = I(\eta)/T(\eta)$.

(iii) $\epsilon_0 I(\eta)/\eta \le c_0$ *as* $\eta \ne 0$.

(iv) Therefore the only periodic trajectory is with $\eta = 0$.

(v) Proposition 14.2.6 holds with $\eta^* = 0$ *but* Z *is even and* α, β *are odd with respect to* ξ_1.

(vi) For odd integer $\nu \ge 3$, Z, α, β *are analytic.*

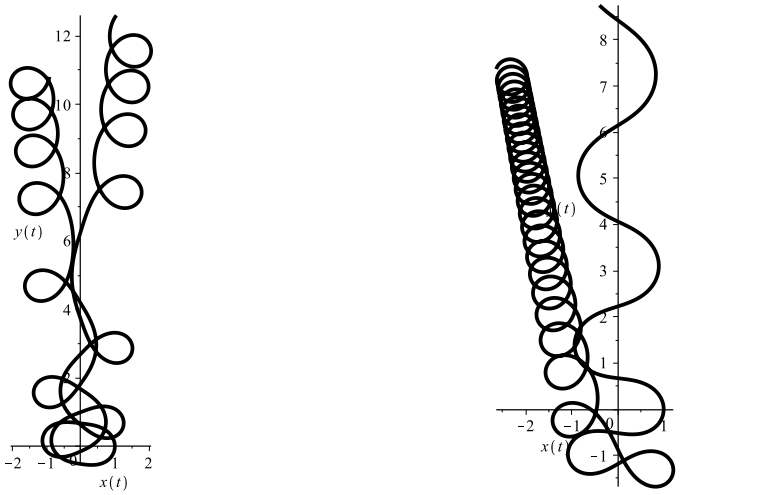
14.2.3 Toy-Models Perturbed

The periodic trajectories of our toy-models are very fragile and one can destroy them easily. Consider *heuristically* the same Hamiltonians albeit with $V = -\alpha x_1$ with $\alpha > 0$ instead of $V = -1$. Then $d\xi_2/dt = -\partial_{x_2} V = \alpha$ and thus $\xi_2 = c_0 + \alpha t$. Then the averaged movement along $y = x_2$ is described by

$$\frac{d}{dt}y = (-V)^{\frac{1}{2}} v\big(\xi_2(t)(-V)^{-\frac{1}{2}}\big) \sim (\alpha y)^{\frac{1}{2}} v\big(\alpha t(\alpha y)^{-\frac{1}{2}}\big)$$
$$\Longleftrightarrow \quad \frac{d}{dt}z \sim 2v\big(tz^{-1}\big)$$

with $z = 2\sqrt{y/\alpha}$. This "equation" has a solution $z = \beta_\pm^{-1} t$ as $\pm t > 0$ as $\beta_\pm v(\beta_\pm) = 2$ and $\pm\beta_\pm > 0$.

In the second toy-model this equation has two solutions $\beta_\pm$ with $-\beta_- = \beta_+ > 2$ and therefore on them "in average" $x_1 \sim (\nu\xi_2 V^{-1/2})^{1/\nu} V^{1/2\nu} \sim \rho y^{1/\nu}$. In the first toy-model there is just one solution $\beta_- = -2$. These observations explain the evolutions in vertical direction on Figures 14.5(a)–(b).



(a) First toy-model perturbed (b) Second toy-model perturbed

Figure 14.5: Both toy-models are perturbed by a weak electric field directed up ($V = \alpha x_2$)

14.2.4 General Case

Remarks

We are going to consider

$$a = \xi_1^2 + \sigma(x)^2\bigl(\xi_2 - \mu\phi(x)\varrho_\nu(x_1)/\nu\bigr)^2 + V(x) \tag{14.2.23}$$

with coefficients

$$\sigma \ge \epsilon_0, \quad \phi \ge \epsilon_0, \quad -V \ge \epsilon_0 \tag{14.2.24}$$

and all of them are smooth and with $\varrho_\nu = |x_1|^\nu$ or $\varrho_\nu = |x_1|^\nu \operatorname{sgn} x_1$, $\nu \ge 2$.

Let

$$p_1 = \xi_1, \quad p_2 = \xi_2 - \mu\phi(x)\varrho_\nu(x_1)/\nu; \tag{14.2.25}$$

then

$$\{p_1, p_2\} = -\mu\partial_{x_1}\bigl(\phi(x)\varrho_\nu(x_1)/\nu\bigr) = \\ -\mu\phi(x)\varrho_{\nu-2}(x_1)x_1\bigl(1 + O(x_1)\bigr). \tag{14.2.26}$$

We want to consider trajectories on level $\{a = 0\}$ (and thus on any level $\{a = \tau\}$) and thus we can multiply a by any smooth factor disjoint from 0; then we can assume that

$$\sigma|_\Sigma = \phi|_\Sigma = 1. \tag{14.2.27}$$

Really, starting from (14.2.23) we can multiply a by smooth function ω^2 and redefine x_1 according to the new metrics; then we redefine x_2 to get rid of the mixed term. Then (14.2.23) will still hold but with $V \mapsto \omega^2 V$, $\sigma|_\Sigma \mapsto (\sigma\omega^2)|_\Sigma$, $\phi|_\Sigma \mapsto (\phi\omega^\nu)|_\Sigma$ and picking up $\omega = (\sigma\phi)^{-1/(\nu+1)}$ we arrive to the case $(\sigma\phi)|_\Sigma = 1$ with

$$V \mapsto V^* = (\sigma\phi)^{-2/(\nu+1)} V. \tag{14.2.28}$$

Finally, replacing x_2 by $\rho(x_2)$ we can achieve $\sigma|_\Sigma = 1$ and thus (14.2.27).

From now on

$$W(x_2) \coloneqq -V^*(0, x_2) = \big((\sigma\phi)^{-2/(\nu+1)} V\big)(0, x_2) \tag{14.2.29}$$

(i. e. $W(x_2) = -V(0, x_2)$ under assumption (14.2.27)).

Regular Zone

Proposition 14.2.9. *For a general Hamiltonian* (14.2.23) *in* $B(0, 1)$ *satisfying* (14.2.24)

(i) There exists $b = b(x, p_1, p_2)$*, such that as* $|a| \le \epsilon_0$ *and* $|f| \ge \bar\gamma = C\mu^{-1/\nu}$

$$C_0^{-1} \le \frac{b}{f} \le C_0, \tag{14.2.30}$$

$$|\{a, b\}| \le C_0 \mu^{-1} |f|^{-1}; \tag{14.2.31}$$

here and below $f \in \mathscr{C}^\infty$*,* $f = 0$*,* $\nabla f \ne 0$ *on* Σ*.*

(ii) Trajectories starting at $t = 0$ *from* $(\bar x, \bar\xi) \in B(0, \frac12) \cap \{x, \pm f(x) = \gamma\}$ *with* $\gamma \ge \bar\gamma$ *for time* $|t| \le T^* = \epsilon_0 \mu f(\bar x)^2$ *remain in* $B(0, 1)$ *and along them* $\epsilon \le \pm f(x(t))/\gamma \le C$*.*

(iii) Trajectories starting at $t = 0$ *from* $(\bar x, \bar\xi) \in B(0, \frac12) \cap \{x, |f(x)| \le \gamma = \bar\gamma\}$ *for time* $|t| \le T^* = \epsilon$ *remain in* $B(0, 1)$ *and along them* $|f(x(t))| \le C_1\bar\gamma$*.*

Proof. Let us consider a trajectory starting at $(\bar{x},\bar{\xi})$ with $\bar{x}\in B(0,\frac{1}{2})$, $a\le c$, $\gamma=|\bar{x}_1|\ge\bar{\gamma}=C_0\mu^{-1/\nu}$. Consider this trajectory as long as it is contained in $B(0,1)$ but not further than $T^*=\epsilon\mu\gamma^{\nu}$ [5)].

Along this trajectory

$$\frac{1}{2}\frac{d\xi_2}{dt}=\frac{1}{2}\{a,\xi_2\}=\sigma^2(x)p_2\phi_{x_2}(x)\times\mu\varrho_\nu(x_1)/\nu+O(1). \tag{14.2.32}$$

Then for $X:=\xi_2-\beta(x)p_1$

$$\frac{1}{2}\frac{dX}{dt}=\frac{1}{2}\{a,X\}=\sigma^2p_2\mu\varrho_\nu(x_1)\big(\frac{1}{\nu}\phi_{x_2}(x)-\phi(x)\beta x_1^{-1}\big)+O(1) \tag{14.2.33}$$

(as $a=0$) and therefore $\frac{d}{dt}X=O(1)$ for

$$X=\xi_2-\beta(x)p_1,\qquad \beta=\frac{1}{\nu}x_1\phi_{x_2}\phi^{-1}=\frac{1}{\nu}x_1\partial_{x_2}\log|\phi(x)|. \tag{14.2.34}$$

Therefore $|X(x)-X(\bar{x})|\le CT\le C\epsilon\mu\gamma^{\nu}$, which together with inequality $|p_2|\le c$ imply that $|\xi_2-\bar{\xi}_2|\le C\epsilon\mu\gamma^{\nu}$; for small enough constant $\epsilon>0$ we also get $\epsilon_1\mu\gamma^{\nu}\le|\xi_2|\le c_1\mu\gamma^{\nu}$ (recall that $\mu\gamma^{\nu}\ge C$) and thus $\epsilon_1^{-1}\gamma\le|x_1|\le c_1\gamma$.

Thus this trajectory remains in the strip described above and since the drift speed according to Chapter 13 does not exceed $C_0(\mu\gamma^{\nu})^{-1}$ there we are insured that this trajectory remains in $B(0,1)$ for time T. This proves Statement (ii).

On the other hand, if a trajectory starts in $(\bar{x},\bar{\xi})$ with $|\bar{x}_1|\le\bar{\gamma}$, it remains in $B(0,1)$ for time $T=\epsilon$ since the speed does not exceed C_0 and due to above arguments it cannot get into zone $\{x_1\ge C\bar{\gamma}\}$. This proves Statement (iii).

To prove Statement (i) consider $b=(\mu^{-1}X)^{1/\nu}$; then as $|x_1|\ge\bar{\gamma}$ obviously $b\asymp|x_1|$ and furthermore $|\{a,b\}|\le C(\mu^{-1}|\xi_2|)^{1/\nu}|\xi_2|^{-1}\le c\mu^{-1}|x_1|^{1-\nu}$ which is exactly (14.2.31). □

Obviously $\frac{d}{dt}x_2=2\sigma^2p_2$ and for $Y=x_2-\mu^{-1}\varrho_{-\nu}(x_1)x_1\beta'p_1$ we have

$$\frac{dY}{dt}=2\sigma^2\big(1-\beta'\phi\big)p_2+2(\nu-1)\beta'\mu^{-1}\varrho_{-\nu}(x_1)p_1^2+O(\mu^{-1}|x_1|^{1-\nu}).$$

So, let $\beta'=\phi^{-1}$; then the first term vanishes and

$$\frac{dY}{dt}=(\nu-1)\beta'\mu^{-1}\varrho_{-\nu}(x_1)(a-V)+(\nu-1)\beta'\mu^{-1}\varrho_{-\nu}(x_1)(p_1^2-\sigma^2p_2^2)+O(\mu^{-1}|x_1|^{1-\nu})$$

[5)] Then $T^*\asymp 1$ as $|\bar{x}_1|\asymp\bar{\gamma}$ and $T^*\asymp\mu$ as $|\bar{x}_1|\asymp 1$.

as $a - V = p_1^2 + \sigma^2 p_2^2$ and redefining

$$Y = x_2 - \mu^{-1}\varrho_{-\nu}(x_1)x_1\beta' p_1 - \mu^{-2}\varrho_{-2\nu}(x_1)x_1 p_1 p_2 \tag{14.2.35}$$

we arrive to

$$\frac{dY}{dt} = -(\nu-1)\beta'\mu^{-1}\varrho_{-\nu}(x_1) + O\big(\mu^{-2}|x_1|^{-2\nu} + \mu^{-1}|x_1|^{1-\nu}\big), \tag{14.2.36}$$

and finally we arrive to

Proposition 14.2.10. *Along trajectories in the* regular zone $\{\epsilon \ge |x_1| \ge \bar\gamma\}$

$$\epsilon_0(\mu\gamma^\nu)^{-1}|t| \le \pm\varrho_0(\bar x)\big(Y(t) - Y(0)\big) \le C(\mu\gamma^\nu)^{-1}|t| \tag{14.2.37}$$

as $0 \le \pm t \le \epsilon\mu\gamma^\nu$.

Degeneration Zone

Now we need to consider trajectories lying in the degeneration zone $\mathcal{X}_{\mathsf{deg}} = \{x : |x_1| \le C\bar\gamma\}$. Here function X is still defined but Y is not and from the analysis of the toy-models one can guess that the important role is played by function $X(x,p)|V(0,x_2)|^{-\frac12}$.

Proposition 14.2.11. *Let conditions* (14.2.24) *and* (14.2.27) *be fulfilled. Further let conditions* $|\xi_2(0)| \le C$ *and*

$$\varsigma\big(\xi_2 - \eta^*|V|^{\frac12}\big) \ge \epsilon_1, \qquad \varsigma = \pm 1 \tag{14.2.38}$$

be satisfied as $t = 0$. *Then*

(i) For $|t| \le T = \epsilon$ *this latter condition is satisfied (with* ϵ_1 *replaced by* $\frac12\epsilon_1$*).*

(ii) Moreover

$$\epsilon_2|t| \le \pm\varsigma\big(x_2(t) - x_2(0)\big) \le c_2|t| \qquad \text{as} \qquad c\bar\gamma \le \pm t \le T \tag{14.2.39}$$

and

$$\epsilon_2|t| \le |x(t) - x(0)| + |p(t) - p(0)| \le c_2|t| \qquad \text{as} \qquad |t| \le c\bar\gamma. \tag{14.2.40}$$

Proof. Proof of Statement (i) is obvious; (14.2.39) and (14.2.40) then follow from comparison of the general and toy-model system. □

So there are no T-periodic trajectories with $0 < T \le \epsilon$ unless

$$\rho = |\xi_2 - \eta^* W^{\frac{1}{2}}| \le \epsilon_1. \tag{14.2.41}$$

Note that now

$$\frac{d\xi_2}{dt} = W_{x_2} + O(\bar\gamma) \tag{14.2.42}$$

which immediately yields the following proposition:

Proposition 14.2.12. *Let conditions* (14.2.24), (14.2.27), (14.2.41) *be fulfilled and*

$$\varsigma \partial_{x_2} W \ge \epsilon_0 \qquad \textit{with} \quad \varsigma = \pm 1. \tag{14.2.43}$$

Then trajectories on level 0 *satisfy*

$$\pm\varsigma\big(\xi_2(t) - \xi_2(0)\big) \ge \epsilon|t| \qquad \textit{as} \quad 0 \le \pm T \le \epsilon. \tag{14.2.44}$$

In what follows we will consider more precisely trajectories satisfying (14.2.41) and also $|W_{x_2}| \le \epsilon_0$.

We still want to consider such trajectories as long as they are in a certain vicinity of the original point $x(0)$. For the toy-models this vicinity is $B(0,1)$ and $T^* = \epsilon\rho^{-1}$ as $|\xi_2 - \eta^*| = \rho$ but for other Hamiltonians both vicinity and time may be smaller.

Proposition 14.2.13. *For any of two toy-models with coupling constant* $\mu \ge 1$ *and potential* $V = -1 \mp \zeta x_2$, $0 \le \zeta \le \epsilon$

(i) Trajectory starting at $(\bar x, \bar\xi)$ *satisfying* (14.2.41) *remains in* $B(0,1)$ *and satisfies* (14.2.41) *with* $2\epsilon_1$ *as long* $|t| \le T^* = \epsilon \min\big(\zeta^{-\frac{1}{2}}, \rho^{-1}\big)$.

(ii) Along it

$$|x_2(t) - x_2(0)| \le C(\zeta|t|^2 + \rho|t|) + C\bar\gamma, \qquad \xi_2(t) - \xi_2(0) = \pm\frac{1}{2}\zeta t \tag{14.2.45}$$

and in at least one time direction

$$|x_2(t) - x_2(0)| \ge \epsilon(\zeta|t|^2 + \rho|t|) - C\bar\gamma. \tag{14.2.46}$$

Proposition 14.2.14. *For a general Hamiltonian* (14.2.23), *satisfying conditions* (14.2.24), (14.2.27) *and*

$$|\partial_{x_2} W| \le \zeta \tag{14.2.47}$$

(i) Statement (i) of Proposition 14.2.13 holds as

$$\rho \ge \bar\gamma, \qquad \zeta \ge \bar\gamma \tag{14.2.48}$$

and $|\xi_2 - \eta^* W^{\frac12}| \le \rho$.

(ii) Assuming (14.2.48)*, if*

$$\left|\partial_{x_2} W\right| \ge \epsilon_1 \zeta \tag{14.2.49}$$

then

$$|\xi_2(t) - \xi_2(0)| \ge \epsilon\zeta|t|. \tag{14.2.50}$$

(iii) Assuming (14.2.48)*, if*

$$\left|\xi_2(0) - \eta^* W(x(0))^{\frac12}\right| \ge \epsilon_1\rho \tag{14.2.51}$$

then in at least one time direction

$$|x_2(t) - x_2(0)| \ge \epsilon\rho|t|. \tag{14.2.52}$$

Proofs of Propositions 14.2.13 and 14.2.14. Easy proofs are left to the reader.□

Hamiltonian mMaps

Now we need to analyze more precisely what happens in the periodic zone $\mathcal{X}_{\mathsf{per}} = \{x,\ |x_1| \le c\bar\gamma, |\xi_2 - \eta^* W^{\frac12}| \le \rho\}$ where ρ will be specified in the next section. Let us consider toy-models first.

If on each energy level τ, $|\tau| \le \epsilon$ all the trajectories were periodic with period $T = T(\tau)$ then replacing Hamiltonian a by $g(a)$ with

$$g(\tau) = \frac{1}{\bar T}\int^{\tau} T(\tau)\, d\tau \tag{14.2.53}$$

we would get $\bar T$-periodic Hamiltonian flow.

However it is not the case:

(14.2.54) For both toy-model Hamiltonians on each energy level τ the only periodic trajectory is one with

$$\xi_2 = \xi_2(\tau) = \eta^*(1+\tau)^{\frac12} \tag{14.2.55}$$

and a period is

$$T(\tau) = \bar T(1+\tau)^{-\frac{1}{2\nu}(\nu-1)} \tag{14.2.56}$$

Still defining $g(\tau)$ accordingly let us consider the Hamiltonian $g(a)$.

Proposition 14.2.15. *For a toy-model symbol a given by* (14.2.2) *or* (14.2.19)

$$e^{\bar{T}H_{g(a)}} = e^{H_b}, \tag{14.2.57}$$

$$g(\tau) = \kappa^{-1}(1+\tau)^{\kappa} - \kappa^{-1}, \qquad \kappa = \frac{\nu+1}{2\nu}, \tag{14.2.58}$$

$$b = \bar{b}(\xi_2, a) = (1+a)^{\kappa}\Big(\frac{\xi_2}{(1+a)^{\frac{1}{2}}} - \eta^*\Big)^2 \omega\Big(\frac{\xi_2}{(1+a)^{\frac{1}{2}}}\Big) \tag{14.2.59}$$

where $\bar{T}$ is an elementary period and

$$\omega(\eta^*) = \frac{1}{2}\partial_\eta I(\eta)\big|_{\eta=\eta^*}. \tag{14.2.60}$$

Proof. As a is a toy-model Hamiltonian then taking $\xi_2 = \eta^*$, $a = 0$ we would conclude that the Hamiltonian trajectories are periodic with period $\bar{T}$. As $a = \tau$ disjoint from -1 we have both $\xi_2^*(\tau)$ and $T(\tau)$ defined by (14.2.55)–(14.2.56).

To get period which is independent on energy level one must replace a by $g(a)$ with $g(\tau)$ defined by (14.2.53); then $e^{\bar{T}H_{g(a)}} = I$ as $\xi_2 = \xi_2^*(a)$; period is $\bar{T}$ rather than 1 since we define $g(a)$ without factor $\bar{T}$.

As $\xi_2 \neq \xi_2^*(a)$ periodicity is broken. Since H_a and H_{ξ_2} near $\{a = 0\}$ are linearly independent we conclude that $e^{\bar{T}H_{g(a)}} = I + 2(\xi_2 - \xi_2^*(a))\omega_1(x_1, \xi_1, \xi_2)$. However since it is a symplectic map it must be of the form e^{H_b} with

$$b = (\xi_2 - \xi_2^*(a))^2\omega_2(x_1, \xi_1, \xi_2) + \lambda(x_1, \xi_1).$$

We know that as $\xi_2 = \xi_2^*(a)$ we have $e^{H_b} = e^{H_\lambda} = I$ and since λ is "small" we conclude that $\lambda = \mathsf{const}$ (and thus we can take it 0).

Since e^{H_b} commutes with e^{H_a} we conclude that $\omega = \omega(\xi_2, a)$ and due to homogeneity properties $\omega_2 = \omega\big(\xi_2(1+a)^{-\frac{1}{2}}\big)(1+a)^{\kappa-1}$. Then x_2-shift as $a = 0$ and $\xi_2 \approx \eta^*$ is $2\omega(k^*)(\xi_2 - \eta^*) + O\big((\xi_2 - \eta^*)^2\big)$ which implies (14.2.60). □

If we consider a toy-model but with $\mu \neq 1$ one needs to replace (14.2.57) by

$$e^{\bar{\gamma}\bar{T}g(a)} = e^{\bar{\gamma}H_b}. \tag{14.2.61}$$

Consider now trajectories residing in $B(0, C_0\bar\gamma)$ and corresponding to these Hamiltonians but with potential $V = -1 - \zeta x_2$. One can see easily that (14.2.61) will remain true with $b = \bar b - \frac{1}{2}\bar T W + b'$, $W := -V(0, x_2)$ and b' satisfying (14.2.63) below. Then for the general Hamiltonians one can prove easily the following proposition:

Proposition 14.2.16. *For a general Hamiltonian* a *defined by* (14.2.23) *with* σ, ϕ *satisfying* (14.2.27) *and with* W, $W(0) = 1$, *satisfying in* $C_0\bar\gamma$*-vicinity of* 0

$$|\partial_{x_2} W| \le \zeta \tag{14.2.62}$$

with $\zeta \ge C\bar\gamma$, *equality* (14.2.61) *holds with symbol* $b = \bar b - \frac{1}{2}\bar T W(x_2) + b'$ *where* $\bar b = \bar b(\xi_2, a)$ *is defined by* (14.2.59), *and*

$$|\partial_x^\alpha b'| \le C\big(\zeta(\zeta+\rho) + \bar\gamma\big)\bar\gamma^{1-|\alpha|}, \qquad |\alpha| \le 1. \tag{14.2.63}$$

14.3 Quantum Dynamics

14.3.1 Preliminary Notes. Forbidden Zone

Starting from this section $\nu \ge 2$ is an integer and magnetic potentials

$$V_1 = 0, \quad V_2 = \phi(x)\nu^{-1}\varrho_\nu(x_1), \qquad \varrho_\nu(x_1) := x_1^\nu. \tag{14.3.1}$$

Thus we have different toy-models for even and odd ν.

In this subsection we consider $\mu \ge h^{-1}$ and *forbidden zone* is defined by (14.3.2) below. Also we consider ξ_2-localization which is useful when μ is close to $h^{-\nu}$.

Proposition 14.3.1. *Let* $\mu \ge h^{-1}$ *and*

$$\mu h F + V \ge \epsilon \tag{14.3.2}$$

in $B(\bar x, \gamma)$ *with*

$$\gamma \ge \max\big(h, \bar\gamma\big), \qquad \bar\gamma := C\mu^{-\frac{1}{\nu}}. \tag{14.3.3}$$

Then

$$|e(x, x, 0)| \le C\gamma^{-2}\big(\mu^{-1}h\gamma^{-\nu-1}\big)^s \qquad \text{in } \ B(\bar x, \gamma). \tag{14.3.4}$$

Proof. Proof follows from results of the previous Chapter 13 just by rescaling. □

Corollary 14.3.2. *In particular, as*

$$h^{-1} \le \mu \le C_0 h^{-\nu} \tag{14.3.5}$$

the total contribution to $\int e(x,x,0)\psi(x)\,dx$ *of the* classically forbidden zone *defined by* (14.3.2) *does not exceed* $C\bar{\gamma}_1^{-1}(\mu h^{\nu})^s$. *Thus we need to consider only* classically allowed zone $\{x,\ \mu hF + V \le \epsilon\}$ *and there*

$$\gamma \le \bar{\gamma}_1 := C_0(\mu h)^{-1/\nu-1)}. \tag{14.3.6}$$

This result is not very useful as $\mu \ge h^{-\nu+\delta}$. However using logarithmic uncertainty principles and related arguments one can prove easily

Proposition 14.3.3. *Let conditions* (14.3.2), (14.3.5) *be fulfilled and*

$$\gamma \ge \max\bigl(Ch|\log h|, \bar{\gamma}\bigr). \tag{14.3.7}$$

with $C = C_s$. *Then*

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\psi u| \le CTh^s \qquad \forall \tau \le \epsilon \quad \forall T \ge Ch|\log h| \tag{14.3.8}$$

and therefore $e(x,x,0) \le Ch^s$ *in* $B(\bar{x},\gamma)$.

In particular, (14.3.7) is fulfilled automatically as $|x_1| \ge \bar{\gamma}_1$ provided

$$\mu \le \epsilon(h|\log h|)^{-\nu} \tag{14.3.9}$$

while the *superstrong magnetic field* case when (14.3.9) is violated requires some special consideration based on the analysis operators with operator-valued symbols. We need the following axillary proposition:

Proposition 14.3.4. *Let* $\lambda = \lambda_n(\eta)$ *be eigenvalues of* 1*-dimensional Schrödinger operator*

$$L(\eta) := D_1^2 + \bigl(\eta - \mu\nu^{-1}\varrho_\nu(x_1)\bigr)^2 \tag{14.3.10}$$

in $\mathscr{L}^2(\mathbb{R})$*; here we assume only that* $2 \le \nu \in \mathbb{R}$ *and either* $\varrho_\nu(x_1) = |x_1|^\nu$ *or* $\varrho_\nu(x_1) = |x_1|^\nu \cdot \operatorname{sgn}(x_1)$. *Then*

(i) $\lambda_n(\eta) > 0$.

(ii) Assume that either $\eta \ge 0$ *or* $\varrho_\nu = |x_1|^\nu \operatorname{sgn}(x_1)$. *Then for* $n \le \epsilon|\eta|^{(\nu+1)/\nu}$

$$\lambda_n(\eta) \asymp |\eta|^{(nu-1)/\nu} n, \tag{14.3.11}$$

$$0 < \partial_\eta \lambda_n(\eta) \cdot \operatorname{sgn}(\eta) \asymp |\eta|^{-1/\nu} n. \tag{14.3.12}$$

(iii) Assume that $\eta < 0$ *and* $\varrho_\nu = |x_1|^\nu$. *Then*

$$\lambda_n(\eta) \asymp \eta^2 + n^{2\nu/(\nu+1)}. \tag{14.3.13}$$

Proof. Proof of Statement (i) is obvious; Statements (ii) and (iii) easily follow from the semiclassical character of the spectrum as $|\eta| + n \gg 1$, $n \gg 1$ respectively. Easy details are left to the reader. More detailed analysis is left for Appendix 14.A. □

Furthermore, using operators with operator-valued symbols one can prove easily the following proposition:

Proposition 14.3.5. *Let* $\mu \ge Ch^{-\nu}$. *Then*

$$|e(x,x,\tau)| \le C\mu^{-s} \qquad \text{for } \tau \le \epsilon\big(\mu h^\nu\big)^{2/(\nu+1)}. \tag{14.3.14}$$

So, in what follows can assume that $\mu \le Ch^{-\nu}$. It will not be a case starting from Section 14.7 when we modify potential. As usual in what follows $u = u(x,y,t)$ is the propagator, i.e. the Schwartz kernel of operator $e^{ih^{-1}tA}$.

Proposition 14.3.6. *Let* $\mu \le Ch^{-\nu}$ *and let* $\varphi \in \mathscr{C}^\infty(\mathbb{R})$, $\varphi(t) = 1$ *as* $t \ge 2$, $\varphi(t) = 0$ *as* $t \le 1$. *Then*

(i) The following inequality holds

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\varphi\big(\pm\epsilon(\mu h^\nu)^{1/(\nu-1)}hD_{x_2}\big)\psi u| \le C(T+1)h^s \qquad \forall \tau \le \epsilon \quad \forall T \ge Ch|\log h| \tag{14.3.15}$$

and therefore

$$|\varphi\big(\pm\epsilon(\mu h^\nu)^{1/(\nu-1)}hD_2\big)e(x,y,0)| \le Ch^s; \tag{14.3.16}$$

here and below $x, y \in B(0, 1-\epsilon)$.

(ii) Furthermore, for even ν

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\varphi\big(-\epsilon hD_2\big)\psi u| \le C(T+1)h^s \qquad \forall \tau \le \epsilon \quad \forall T \ge Ch|\log h| \tag{14.3.17}$$

and therefore

$$|\varphi\big(-\epsilon hD_2\big)e(x,y,0)| \le Ch^s. \tag{14.3.18}$$

Proofs of Propositions 14.3.5 and 14.3.6. Easy proofs are left to the reader. Inequalities (14.3.15) and (14.3.17) are easily proven by the standard elliptic methods. Then inequalities (14.3.16) and (14.3.18) are proven by the standard Tauberian methods. □

Thus in what follows we localized u with respect to hD_{x_2} (and thus with respect to hD_{y_2} as well due to symmetry). This localization is more precise than x_1, y_1 localization as $\epsilon(h|\log h|)^{-\nu} \le \mu \le Ch^{-\nu}$; as $\mu \le \epsilon(h|\log h|)^{-\nu}$ both localizations are equivalent.

14.3.2 Regular Zone

In this subsection we consider *regular zone* $\mathcal{X}_{\mathsf{reg}} = \{C_0\bar{\gamma} \le |x_1| \le \bar{\gamma}_1\}$ with

$$\bar{\gamma} := \mu^{-1/\nu} \le \bar{\gamma}_1 := \min\big(\epsilon, C(\mu h)^{-1/(\nu-1)}\big) \tag{14.3.19}$$

which mathes to

$$\mathcal{X}_{\mathsf{reg}} = \{C_0 \le |\xi_2| \le \min\big(\epsilon\mu, (\mu h^\nu)^{-1/(\nu-1)}\big)\}. \tag{14.3.19}'$$

This definition works fine under assumption (14.3.9) but under assumption

$$\epsilon(h|\log h|)^{-\nu} \le \mu \le C_0 h^{-\nu} \tag{14.3.20}$$

it should be modified to

$$\mathcal{X}_{\mathsf{reg}} = \{C_0 \le |\xi_2| \le C(\mu h^\nu)^{-1/(\nu-1)}\} \tag{14.3.21}$$

while restriction to x_1 becomes

$$|x_1| \le Ch|\log h| \tag{14.3.21}'$$

with an extra logarithm due to logarithmical uncertainty principle.

Remark 14.3.7. For even ν we need to add restriction to sign: $\xi_2 > 0$, i.e. $\xi_2 \ge C_0$.

Our first statement is that in the quantum dynamic in this zone the magnitude of x_1 (and ξ_2) persists.

Proposition 14.3.8. *(i) Let* $\bar{x} \in B(0, \frac{1}{2}) \cap \{x, \pm x_1 \ge C\bar{\gamma}\}$, $\gamma = \epsilon|\bar{x}_1|$ *satisfy* (14.3.7) *and* $\psi \in \mathscr{C}_0^\infty(B(\bar{x}, \gamma))$ *be a rescaling of the standard function. Let* ψ_1 *be* γ*-admissible and supported in* $\{1/(2C_0) \le x_1/\bar{x}_1 \le 2C_0\}$ *and equal* 1 *in* $\{1/C_0 \le x_1/\bar{x}_1 \le C_0\gamma\}$. *Let* $T^* = \epsilon\mu\gamma^\nu$. *Then for* $T = T^*$

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)(1-\psi_{1x})u\psi_y| \le Ch^s. \tag{14.3.22}$$

(ii) Let $\bar{\xi}_2 = \pm\rho$, $\rho \ge C$ *and* $\varphi \in \mathscr{C}_0^\infty([\frac{3}{4}, \frac{3}{4}])$ *and* $\varphi_1 \in \mathscr{C}_0^\infty$, $\varphi_1 = 1$ *on* $[\frac{1}{2}, \frac{3}{2}]$ *be the standard functions. Let* $T^* = \epsilon\rho$. *Then for* $T = T^*$

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Big(1-\varphi_1\big(\frac{1}{\bar{\xi}_2}hD_x\big)\Big)u\,{}^t\varphi\big(\frac{1}{\bar{\xi}_2}hD_y\big)| \le Ch^s. \tag{14.3.23}$$

Proof. Note that both statements are equivalent as $\rho \asymp \mu\gamma^\nu$; so we will prove Statement (ii). As $|x_1| \ge \bar{\gamma}$ we constructed by (14.2.34) symbol which after rescaling becomes $X = \xi_2 - \mu^{-1}x_1^{1-\nu}\beta(x)p_1$ such that

$$|\{a, X\} - \alpha a| \le C_0 \tag{14.3.24}$$

Then using arguments of the proof of Theorem 2.3.2 with symbol

$$\chi(\varsigma Ct \pm X) \tag{14.3.25}$$

with the same standard function χ as in this proof and $\varsigma = \pm 1$, one can easily prove Statement (ii) as long as symbol (14.3.25) is quantizable; this latter condition is equivalent to (14.3.7).

Surely one needs to check that in time T^* we stay in $B(0,1)$ but it will be done in the next Proposition 14.3.9.

If (14.3.7) is violated (and thus (14.3.9) is violated as well) we can quantize with respect to (x_2, ξ_2) but not with respect to (x_1, ξ_1) unless $x_1 \ge Ch|\log h|$ and instead we note that

$$\{a, \xi_2\} = \alpha a + \sigma^2\big(\xi_2 - \phi\varrho_\nu/\nu\big) + O(1)$$

and then $X = \omega\xi_2$ with $\omega = \phi(0, x_2)^{-1}$ satisfies (14.3.24) and therefore the same arguments of the proof of Theorem 2.3.2 remain valid but we consider h-pseudodifferential operators with operator-valued symbols (in the axillary space $\mathbb{K} = \mathscr{L}^2(\mathbb{R}^1)$). □

The following proposition estimates by $C_0\mu^{-1}\gamma^{-\nu}$ and by $C\rho^{-1}$ x_2-speed of the propagation from above; the similar result will hold for degeneration zone as well.

Proposition 14.3.9. *(i) In the framework of Proposition 14.3.8(i) let ψ be supported in ℓ-vicinity of $\bar{y}$ and $\psi_2 = \psi_2(x_2)$ satisfy*

$$\psi_2 = 1 \qquad \text{in} \quad \{|y_2 - \bar{y}_2| \le C_0(\mu\gamma^{\nu})^{-1}T + C_0\ell\} \tag{14.3.26}$$

with $\ell = C_0\mu^{-1}\gamma^{1-\nu}$.

Then for $T_* C_0 h|\log h| \le T \le T^*$

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)(1-\psi_{2x})u\psi_y| \le Ch^s \qquad \forall \tau \in (-\epsilon,\epsilon); \tag{14.3.27}$$

(ii) In the framework of Proposition 14.3.8(ii) let $\psi = \psi(x_2)$ be supported in ℓ-vicinity of $\bar{y}_2$, $\varphi = \varphi(\xi_2)$ in $\epsilon\rho$-vicinity of $\bar{\xi}_2$ and ψ_2 satisfy (14.3.25)*. Let*

$$\ell\rho \ge Ch|\log h|. \tag{14.3.28}$$

Then for $T_* \le T \le T^*$

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)(1-\psi_{2x})u\psi_y\, {}^t\varphi(hD_{y_2})| \le Ch^s \qquad \forall \tau \in (-\epsilon,\epsilon). \tag{14.3.29}$$

Proof. Both statements are equivalent under condition (14.3.7).

(i) Assume first that (14.3.7) holds. Then symbol

$$Y = x_2 - \phi^{-1}\mu^{-1}x_1^{1-\nu}\xi_1 \tag{14.3.30}$$

satisfies

$$|\{a, Y\}| \le C_0\rho^{-1} \tag{14.3.31}$$

with $\rho = \mu\gamma^{\nu}$ and symbols

$$\chi\big(C_0\varsigma\frac{t}{T} \pm \frac{Y}{\rho^{-1}T}\big) \tag{14.3.32}$$

($\varsigma = \pm 1$) are quantizable as long as

$$T \ge T_0 = C\big(\gamma h|\log h|\big)^{\frac{1}{2}} \tag{14.3.33}$$

and then the standard energy estimates method (see proof of Theorem 2.3.2) implies (14.3.28) with ψ and ψ_2 replaced by h-pseudodifferential operators Q and Q_2 with symbol of Q supported in the first domain and symbol of Q_2 equals 1 in the second domain:

$$\{(y,\eta) : |y_1 - \bar{y}_1| \le \epsilon\gamma, |Y(y,\eta_1) - Y(\bar{y},\bar{\xi}_1)| \le \ell\},$$
$$\{(x,\xi) : |x_1 - \bar{y}_1| \le 2\epsilon\gamma, |Y(x,\xi_1) - Y(\bar{y},\bar{\xi}_1)| \le \ell + C\mu^{-1}\gamma^{-\nu}T\},$$

as $T_0 \le T \le T^* = C\mu\gamma^\nu$ and

$$\ell_0 = C\big(\gamma h|\log h|\big)^{\frac{1}{2}}\rho^{-1} \le \ell \le \epsilon. \tag{14.3.34}$$

Original estimate (14.3.27) follows from this because $|Y(x,\xi_1) - x_2| \le C_0\ell$.

As (14.3.27) is proven for $T = T_0$ it is valid also for $T \in [T_*, T_0]$.

(ii) Assume now that condition (14.3.7) is violated. Then x_1 is localized only in $Ch|\log h|$-vicinity of 0 rather than in γ-vicinity of it and x_1 is not separated from 0. However a naive idea that $\mu\phi x_1^\nu/\nu$ should be replaced by ξ_2 and thus we should define

$$Y = x_2 - \nu^{-1}x_1\xi_1\xi_2^{-1} \tag{14.3.35}$$

instead of (14.3.30) works; (14.3.31) and (14.3.32) are preserved, also (14.3.33) becomes $T^* = Ch|\log h|$ and (14.3.34) becomes (14.3.28). □

The following proposition estimates by $\epsilon_0\mu^{-1}\gamma^{-\nu}$ (and almost equivalently) by $\epsilon_0\rho^{-1}$ x_2-speed of the propagation from below; this result would fail in the degeneration zone unless far from periodic trajectories.

Proposition 14.3.10. *(i) In the framework of Proposition 14.3.8(i) let ψ be supported in γ-vicinity of $\bar{y}$ and $\psi_2 = \psi_2(x_2)$ satisfy*

$$\psi_2 = 1 \qquad \text{in } \ \{|y_2 - \bar{y}_2| \ge \epsilon_0(\mu\gamma^\nu)^{-1}T - C_0\ell\} \tag{14.3.36}$$

with $\ell = \mu^{-1}\gamma^{1-\nu}$.

Then for $T_0' = C_0\gamma \le T \le T^*$ inequality (14.3.28) *holds.*

(ii) In the framework of Proposition 14.3.8(ii) let $\psi = \psi(x_2)$ be supported in ℓ-vicinity of $\bar{y}_2$, $\varphi = \varphi(\xi_2)$ in $\epsilon\rho$-vicinity of $\bar{\xi}_2$ and ψ_2 satisfy (14.3.35). *Let ℓ satisfy* (14.3.29).

Then for $T_* = C_0h|\log h| \le T \le T^*$ inequality (14.3.30) *holds.*

Proof. One needs to apply the standard arguments with

$$\chi\big(C_0\varsigma\frac{Y}{\rho^{-1}T}\pm\frac{t}{T}\big) \tag{14.3.37}$$

assuming that inequality (14.3.32) is reversed to

$$|\{a, Y\}| \ge \epsilon_0\rho^{-1}. \tag{14.3.38}$$

In the framework of (14.3.7) one needs to modify Y in the spirit of (14.2.35) which would be equivalent to the following modification in the framework of Statement (ii). Namely, note that as $\gamma \ll 1$, $\rho \gg 1$

$$\{a, Y\} = \xi_2^{-1}\big(-\nu^{-1}p_1^2 + p_2^2\big) + o(\rho^{-1}) = \\ \frac{1}{2}(1+\nu^{-1})\xi_2^{-1}(p_2^2 - p_1^2) + \frac{1}{2}(1-\nu^{-1})\xi_2^{-1}(p_2^2 + p_1^2) + o(\rho^{-1})$$

and if we redefine Y as

$$Y = x_2 - \nu^{-1}x_1\xi_1\xi_2^{-1} + \frac{1}{2}\nu^{-1}(1+\nu^{-1})x_1\xi_1 p_2\xi_2^{-2} \tag{14.3.39}$$

we arrive to

$$\{a, Y\} = \frac{1}{2}(1-\nu^{-1})\xi_2^{-1}(p_2^2 + p_1^2) + o(\rho^{-1}) = \\ \frac{1}{2}(1-\nu^{-1})\xi_2^{-1}(a - V) + o(\rho^{-1}) \tag{14.3.40}$$

which implies (14.3.39) and also symbols (14.3.38) are quantizable. □

From the proof of Proposition 14.3.10 immediately follows that (14.3.29) holds with $\psi(x_2)$, $\psi_2(x_2)$ replaced by $\psi(Y(x,\xi))^{\mathsf{w}}$, $\psi_2(Y(x,\xi))^{\mathsf{w}}$ with assumption $\ell = C_0\mu^{-1}\gamma^{1-\nu}$ replaced by (14.3.28) i.e.

$$\ell = C_0\mu^{-1}\gamma^{-\nu}h|\log h|. \tag{14.3.28)$'}$$

Then we immediately get the following corollary:

Corollary 14.3.11. *(i) In the framework of Proposition 14.3.8(i)*

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\big(u\psi\big)| \le Ch^s \qquad \forall\tau\in(-\epsilon,\epsilon); \tag{14.3.41}$$

as $T_* = Ch|\log h| \le T \le T^*$ *and thus*

$$|F_{t\to h^{-1}\tau}\big(\bar\chi_{T^*}(t) - \bar\chi_{T_*}(t)\big)\Gamma\big(u\psi\big)| \le Ch^s \tag{14.3.42}$$

where we recall that $\bar\chi$ *is supported in* $[-1,1]$ *and equal* 1 *in* $[-\frac12,\frac12]$.

(ii) In the framework of Proposition 14.3.8(ii) (14.3.41) *and* (14.3.42) *hold with* $\psi(x_1, x_2)$ *replaced by* $\psi(x_2)\varphi(hD_2)$ *with* φ *supported in* $\{|\xi_2| \ge C\}$.

Assume now that condition (14.3.7) is fulfilled. Then we arrive to

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(u\psi)| \le C\gamma h^{-1} \qquad \forall T \in [T_*, T^*] \tag{14.3.43}$$

as $\psi = \psi'(x_1)\psi'(x_2)$ is an element of γ-admissible partition with respect to x_1; really this estimate holds with $T \le T_0 = \gamma T^*$ due to the results of Chapter 13 in non-degenerate case rescaled: $x \mapsto x/\gamma$, $h \mapsto h' = h/\gamma$, $t \mapsto t/\gamma$ and $\mu \mapsto \mu' = \mu\gamma^\nu$ with $\mu' h' \le 1$ and contribution of interval $[T_0, T^*]$ is negligible.

Then contribution of each element of γ-admissible (sub)partition with respect to x contributes $C(h/\gamma)^{-1} \times \gamma$ with extra-factor γ coming from dt in Fourier transform $F_{t\to h^{-1}\tau}$. After summation with respect to all such subelements with fixed magnitude of γ we get (14.3.43),

After (14.3.43) is established we can apply Tauberian arguments and conclude immediately that the contribution of ψ to the Tauberian remainder does not exceed

$$CT^{*-1} \times h^{-1}\gamma = C(\mu\gamma^\nu)^{-1}h^{-1}\gamma = C\mu^{-1}\gamma^{1-\nu} \tag{14.3.44}$$

while the principal part is given by the standard Tauberian expression

$$h^{-1}\int_{-\infty}^{0} \Bigl(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma\bigl(u\psi\bigr)\Bigr)\,d\tau \tag{14.3.45}$$

with arbitrary $T \in\in [T_*, T^*]$.

Then summation of (14.3.44) over zone $\{|x_1| \ge \gamma\}$ results in $C\mu^{-1}h^{-1}\gamma^{1-\nu}$ coinciding with $C\bar{\gamma}h^{-1}$ as $\gamma = \bar{\gamma}$ and we arrive to

Proposition 14.3.12. *Under condition* (14.3.9) *contribution of the whole regular zone* $\mathcal{X}_{\mathsf{reg}} = \{x,\ |x_1| \ge C_0\bar{\gamma}\}$ *to the Tauberian remainder*

$$\mathsf{R}^{\mathsf{T}} = |\Gamma(e\psi) - h^{-1}\int_{-\infty}^{0} \Bigl(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(u\psi)\Bigr)\,d\tau| \tag{14.3.46}$$

does not exceed $C\bar{\gamma}h^{-1}$ *where* $\psi = 1 - \phi\bigl(x_1/(C_0\bar{\gamma})\bigr)\psi'(x_1)\psi''(x_2)$ *with all functions regular.*

The similar results as condition (14.3.9) violated will be derived later in Sections 14.5 and 14.6. We will replace (14.3.45) by more explicit expression in Section 14.4. Basically it will be the answer prescribed by the non-vanishing magnetic field theory but for μ large enough some correction will be needed.

14.3.3 Degeneration Zone

General Analysis

Recall that the degeneration zone is $\mathcal{X}_{\mathsf{deg}} = \{|\xi_2| \le C_0\}$ or under condition (14.3.9) equivalently (around energy level 0) $\mathcal{X}_{\mathsf{deg}} = \{|x_1| \le C_0\bar\gamma\}$; under condition (14.3.20) we would have $\{|x_1| \le C_0 h|\log h|\}$ in the microlocal sense in this zone.

Let us start from the confinement of the propagation in the degeneration zone and a finite speed of propagation; proofs repeating those of the corresponding propositions of the previous subsection are left to the reader. Just to remark that in this zone we study ξ_2 and x_2 directly without going to X and Y.

Proposition 14.3.13. *(cf. Proposition 14.3.8)*

(i) Let condition (14.3.9) *be fulfilled. Let* $\bar x \in B(0,\frac12) \cap \{x, |x_1| \le \bar\gamma\}$ *and* $\psi \in \mathscr{C}_0^\infty(B(\bar x, \bar\gamma))$ *be a rescaling of the standard function. Let* ψ_1 *be* $\bar\gamma$*-admissible, supported in* $\{|x_1| \le 3C_0\bar\gamma\}$ *and equal* 1 *in* $\{|x_1| \le 2C_0\bar\gamma\}$*. Let* $T^* = \epsilon$*. Then estimate* (14.3.22) *holds with* $T = T^*$*.*

(ii) Without condition (14.3.9) *let* $\psi \in \mathscr{C}_0^\infty\big((\frac12,\frac12)\big)$*,* $\varphi \in \mathscr{C}_0^\infty\big((-C_0, C_0)\big)$*,* $\varphi_1 \in \mathscr{C}_0^\infty\big((-3C_0, 3C_0)\big)$ *and equal* 1 *in* $(-2C_0, 2C_0)$*. Then estimate* (14.3.23) *holds with* $\bar\xi_2 = 1$ *and* $T = T^*$*.*

Proposition 14.3.14. *(cf. Proposition 14.3.9)*

(i) In the framework of Proposition 14.3.13(i) let $\psi_1 = \psi(x_2)$*,*

$$\psi_1 = 1 \qquad \text{in} \quad \{|y_2 - \bar y_2| \le C_0 T\}.$$

Then for $T_0 = C_0\bar\gamma \le T \le T_1$ *inequality*

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)(1-\psi_{1x})uQ_y| \le Ch^s \qquad \forall \tau \in (-\epsilon,\epsilon); \tag{14.3.47}$$

holds with $Q_y = \psi_y$*.*

(ii) Without condition (14.3.9) *this estimate holds with* $Q_y = \psi(y_2)\varphi(-hD_{y_2})$ *where* $\varphi \in \mathscr{C}_0^\infty\big((-C_0, C_0)\big)$.

Now we are studying the finite speed with respect to ξ_2 and the proof repeats those of Proposition 14.3.9:

Proposition 14.3.15. *Let* $\psi = \psi(x_2) \in \mathscr{C}_0^\infty\big((-\frac{1}{2}, \frac{1}{2})\big)$, $\varphi \in \mathscr{C}_0^\infty\big(B(\bar\xi_2, \rho)\big)$ *and* $\varphi_1 \in \mathscr{C}_0^\infty\big(B(\bar\xi_2, 3\rho)\big)$ *be equal to* 1 *in* $B(\bar\xi_2, 3\rho)$ *be standard functions rescaled. Let*

$$\rho \ge C_1 h|\log h| + C_1\bar\gamma \tag{14.3.48}$$

Then for $T_0 = C_2\bar\gamma \le T \le T^*(\rho) = \epsilon\rho$ *inequality*

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)\big(1-\varphi_1(hD_{x_2})\big)u\psi_y\varphi(-hD_{y_2})| \le Ch^s \quad \forall \tau \in (-\epsilon, \epsilon); \tag{14.3.49}$$

holds.

We will prove more specific statements later. Let us prove that as ξ_2 is disjoint from $\eta^* W^{\frac{1}{2}}$ there is a drift with the velocity $\asymp (\xi_2 - \eta^* W^{\frac{1}{2}})$ (with the correct sign). Let $\varphi = \varphi(\xi_2) \in \mathscr{C}_0^\infty\big(B(\bar\xi_2, \rho)\big)$ and $\psi = \psi(x_2) \in \mathscr{C}_0^\infty(B(\bar y_2, \ell)$, $\ell = \epsilon_1\rho$ be standard functions rescaled. Assuming that

$$\rho \ge C(h|\log h|)^{\frac{1}{2}} + C_0\bar\gamma \tag{14.3.50}$$

we conclude from Propositions 14.3.14 and 14.3.15 that

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)\big(1-\varphi_1(hD_{x_2})\psi_2(x_2)\big)u\psi(y_2)\varphi(-hD_{y_2})| \le Ch^s \qquad \forall \tau \in (-\epsilon, \epsilon)$$

as long as $T \le \epsilon\rho$.

Proposition 14.3.16. *Let conditions* (14.3.9) *and* (14.2.27) *be fulfilled. Let* $\psi = \psi(x_2) \in \mathscr{C}_0^\infty\big(B(\bar y_2, \ell)\big)$ *and* $\varphi \in \mathscr{C}_0^\infty\big(B(\bar\xi_2, \rho)\big)$ *be also standard function rescaled, with* $\rho \le C_0$,

$$\bar\xi_2 - \eta^* W(\bar y_2)^{\frac{1}{2}} = \pm\rho. \tag{14.3.51}$$

Further, let $\psi_1 = \psi_1(x_2)$ *be also a standard function rescaled and equal* 1 *in* $\{|x_2 - \bar y_2| \ge \epsilon_0\rho T - C_0\ell\}$.

Then estimate (14.3.47) *holds with* $Q_y = \psi(y_2)\varphi(-hD_{y_2})$ *as* $|\tau| \le \epsilon\rho$, $\ell \ge \bar\gamma$,

$$\rho\ell \ge Ch|\log h|, \tag{14.3.52}$$

$$T \ge C_0\frac{\ell}{\rho}. \tag{14.3.53}$$

Proof. Without any loss of the generality we can assume that $W(\bar{y}_2) = 1$. Let us consider

$$Y = x_2 - \bar{\gamma} Z\big(\frac{x_1}{\bar{\gamma}}, \xi_1\big) \tag{14.3.54}$$

where Z is a function introduced in Proposition 14.2.6 and pick up $\bar{\gamma} = \mu^{-1/\nu}$ exactly. Then

$$|\{a, Y\} - \beta(\xi_2 - \eta^*) - \alpha a| \le \epsilon' \rho$$

with a constant ϵ' which one can make arbitrarily small.

Then we can apply standard arguments with

$$\chi\Big(C_0 \varsigma \frac{1}{\ell}\big(Y(x,\xi) - Y(y,\eta)\big) \pm \frac{t}{T}\Big) \tag{14.3.55}$$

as long as $T\rho \ge C\ell$; symbol (14.3.55) is obviously quantizable. □

Further, it follows from this proof that estimate (14.3.47) holds with $Q_y = \psi(Y)^{\mathsf{w}} \phi(-hD_2)$ and ψ_1 replaced by $\psi_1(Y)^{\mathsf{w}}$ without assumption $\ell \ge \bar{\gamma}$; this estimate immediately implies

Corollary 14.3.17. *In the framework of Proposition 14.3.16 estimate* (14.3.59) *(see below) holds with the standard function ψ and $|\tau| \le \epsilon\rho$ as long as*

$$\rho \ge C_0 (h|\log h|)^{\frac{1}{2}}, \tag{14.3.56}$$

$$T_* = T_*(\rho) := \frac{C_0 h|\log h|}{\rho^2} \le T \le T^*(\rho) = \epsilon\rho. \tag{14.3.57}$$

Note that $T_*(\rho) \le \epsilon\bar{\gamma}$ as long as

$$\rho \ge \bar{\rho}_1 := C\big(\frac{h}{\bar{\gamma}}\big)^{\frac{1}{2}} \tag{14.3.58}$$

and we arrive to

Proposition 14.3.18. *Let conditions* (14.3.9), (14.2.27) *and* (14.3.58) *be fulfilled. Then*

(i) Estimates

$$|F_{t\to h^{-1}\tau} \chi_T(t) \Gamma\big(\varphi(hD_2)\psi u\big)| \le Ch^s \tag{14.3.59}$$

and

$$|F_{t\to h^{-1}\tau} \big(\bar{\chi}_{T_1}(t) - \bar{\chi}_{\bar{T}_0}(t)\big) \Gamma\big(\varphi(hD_2)\psi u\big)| \le Ch^s \tag{14.3.60}$$

hold with $|\tau| \le \epsilon\rho$ and $T_* = Ch|\log h| \le T \le T^*(\rho) = \epsilon\rho$.

(ii) Furthermore

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(\varphi(hD_2)\psi u)| \le Ch^{-1}\bar{\gamma}\rho \tag{14.3.61}$$

as $T \le T_*(\rho)$.

Proof. We need to cover $T \le \epsilon\bar{\gamma}$ only. After rescaling $x \mapsto x/\bar{\gamma}$, $t \mapsto t/\bar{\gamma}$, $h \mapsto \hbar = h/\bar{\gamma}$ we find ourselves in the framework of the standard propagation and therefore estimate (14.3.41) with h replaced by $\hbar$ would hold as long as $C_0\hbar|\log\hbar| \le T/\bar{\gamma} \le \epsilon$.

Then if $\hbar \le h^{\delta}$ or equivalently $\mu \le h^{\delta-\nu}$ with arbitrarily small exponent $\delta > 0$ we arrive to original (14.3.41) with indicated T, τ.

Further, these arguments imply estimate (14.3.61) with no such restriction.

For $h^{\delta-\nu} \le \mu \le \epsilon(h|\log h|)^{-\nu}$ we instead can make a fine $(\epsilon'\bar{\gamma}, \epsilon')$-subpartition with respect to (x_1, ξ_1) and then depending on subelement apply standard arguments with one of

$$\chi\Big(C_0\varsigma\frac{1}{T}(x_1-y_1)\pm\frac{t}{T}\Big), \qquad \chi\Big(C_0\varsigma\frac{1}{T}(\xi_1-\eta_1)\pm\frac{t}{T}\Big) \tag{14.3.62}$$

which are quantizable. Easy details are left to the reader. □

Then applying the standard Tauberian arguments we conclude that

(14.3.63) Contribution of this element to the Tauberian remainder does not exceed $Ch^{-1}\bar{\gamma}\rho/T^*(\rho) \asymp Ch^{-1}\bar{\gamma}$.

Then after summation over partition in $\rho \ge \epsilon_1$ we get Tauberian remainder estimate $Ch^{-1}\bar{\gamma}$; we will cover smaller values of ρ in the next subsubsection. Thus we arrive to

Proposition 14.3.19. *Under conditions* (14.3.9) *and* (14.2.27) *contribution of zone*

$$\big\{|x_1| \le \bar{\gamma}, |\xi_2 - \eta^* W(x_2)^{\frac{1}{2}}| \ge \epsilon_0\big\}$$

to the Tauberian remainder does not exceed $C_\epsilon\mu^{-1/\nu}h^{-1}$.

Near Periodic Sub-Zone

Now we need to consider *near-periodic* zone

$$\bigl\{|x_1| \le \bar{\gamma}, |\xi_2 - \eta^* W(x_2)^{\frac{1}{2}}| \le \epsilon_0\bigr\}.$$

Let us investigate this case under condition (14.2.27), assuming first that W_{x_2} is disjoint from 0. We consider a bit more general case: namely $W_{x_2} \asymp \zeta$ with a large enough parameter ζ. We consider first propagation assuming that $|W_{x_2}| \le \zeta$.

Proposition 14.3.20. *Let conditions* (14.3.9) *and* (14.2.27) *be fulfilled and*

$$|\partial_{x_2} W| \le \zeta \tag{14.3.64}$$

with

$$\zeta \ge C_0 \bar{\gamma}. \tag{14.3.65}$$

Let $\psi = \psi(x_2) \in \mathscr{C}_0^\infty(B(0, \frac{1}{2}))$ be a standard function and $\varphi = \varphi(\xi_2)$ be a standard function rescaled supported in $\{|\xi_2 - \eta^* W^{\frac{1}{2}}(0)| \le \rho\}$ [6] *with*

$$\rho \ge C_0 \zeta. \tag{14.3.66}$$

Further, let $\varphi_1 = \varphi_1(\xi_2)$ be also a standard function rescaled, equal 1 as $|\xi_2 - \eta^* W^{\frac{1}{2}}(x_2)| \le 3\rho$. Then

(i) Estimate (14.3.49) *holds as $|\tau| \le \epsilon\zeta$ and $Ch\zeta^{-1}|\log h| \le T \le \epsilon$.*

(ii) In addition, if

$$\rho \ge \bar{\rho}_1 := (C\bar{\gamma}^{-1} h |\log h|)^{\frac{1}{2}} + C\bar{\gamma}. \tag{14.3.67}$$

estimate (14.3.49) *holds as $|\tau| \le \epsilon\zeta$ and*

$$T_* := Ch\zeta^{-1}|\log h| \le T \le \mathsf{T}^* := \epsilon \min\bigl(\frac{1}{\rho}, \frac{\rho}{\zeta}\bigr). \tag{14.3.68}$$

Proof. We leave to the reader a standard proof based on the axillary symbol

$$\chi\Bigl(C_0\varsigma \frac{t}{T} \pm \frac{1}{\rho}(\xi_2 - \eta_2)\Bigr).$$

[6] Then under condition (14.3.66) $|\xi_2 - \eta^* W^{\frac{1}{2}}(x_2)| \le 2\rho$ for all x_2.

(which under our assumptions is quantizable) that the speed of propagation with respect to ξ_2 does not exceed $C\zeta$. This implies Statement (i).

To prove Statement (ii) one needs to show in addition that the "averaged" speed of propagation with respect to x_2 does not exceed ρ; under condition (14.3.68) one can easily do it for $t = \bar{T}W(0)^{\frac{1}{2}}$ by means of the standard microlocal analysis. Recall that $\bar{T} = T(\eta^*)$ is a period.

We leave details to the reader. More delicate analysis of Section 14.3.4 will also imply results of this subsection. □

Proposition 14.3.20 immediately implies

Corollary 14.3.21. *Under conditions* (14.3.20), (14.2.27),(14.3.64)–(14.3.67) *propagation remains confined to*

$$\bigl\{|x_2| \le \frac{3}{4}, |\xi_2 - \eta^* W(0)^{\frac{1}{2}}| \le \rho + \zeta + \zeta T\bigr\}$$

as $T \le T^* := \epsilon \min\bigl(\rho^{-1}, \zeta^{-\frac{1}{2}}\bigr)$.

Now let us estimate from below the propagation speed with respect to ξ_2.

Proposition 14.3.22. *Let conditions* (14.3.9), (14.2.27), (14.3.67) *be fulfilled and*

$$\epsilon_0\zeta \le |\partial_{x_2} W| \le \zeta \tag{14.3.69}$$

with $\zeta \ge C_0\bar{\gamma}$. *Let* $\varphi = \varphi(\xi_2)$ *be a standard function rescaled supported in* ρ*-vicinity of* $\bar{\eta}_2 = \eta^* W(0)^{\frac{1}{2}}$. *Then estimate* (14.3.49) *holds as* $|\tau| \le \epsilon\zeta$,

$$T_* := C\rho\zeta^{-1} \le T \le T^* := \epsilon \min\bigl(\rho^{-1}, \zeta^{-\frac{1}{2}}\bigr), \tag{14.3.70}$$

$\varphi_1 = 1 - \varphi$.

Proof. We leave to the reader a standard proof based on the axillary symbol

$$\chi\Bigl(C_0\varsigma\frac{1}{\rho}(\xi_2 - \eta_2) \pm \frac{t}{T}\Bigr)$$

which under our assumptions is quantizable. □

Then in the framework of this proposition estimate (14.3.59) holds. Note that $T_* \le \epsilon\bar{\gamma}$ as

$$\zeta \ge C\bar{\gamma}^{-1}h|\log h| \tag{14.3.71}$$

and due to the same arguments as in Proposition 14.3.18 we arrive to

Proposition 14.3.23. *Let conditions of Proposition 14.3.22 and condition* (14.3.71) *be fulfilled. Then estimates* (14.3.59), (14.3.60) *and* (14.3.61) *hold as* $T_* = Ch|\log h| \le T \le T^*$ *with* T^* *defined by* (14.3.70).

Then due to the standard Tauberian arguments we arrive immediately to

Proposition 14.3.24. *Under conditions* (14.3.9), (14.2.27) *and* (14.3.71) *the contribution of the degeneration zone to the Tauberian remainder* R^{T} *does not exceed* $C\mu^{-1/\nu}h^{-1}$.

In particular $\zeta \asymp 1$ satisfies (14.3.71) under condition (14.3.9). Case of smaller $\zeta \le Ch\bar{\gamma}^{-1}|\log h|$ will be considered in Subsection 14.3.4.

Near Periodic Sub-Zone. End

Let us finish analysis in the *near-periodic* but not the *periodic* zone. To do this we are going to analyze elements of (x_2, ξ_2) partitions on which $|\xi_2 - \eta^* W^{1/2}| \le C\bar{\rho}_1$ but $|\partial_{x_2} W| \gtrsim \zeta\ell^{-1}$ branding the rest as a *periodic zone*.

The following results would be the results of the previous subsection rescaled but some conditions are more relaxed because after rescaling $x \mapsto x/\ell$, $\bar{\gamma} \mapsto \bar{\gamma}/\ell$, $\zeta \mapsto \zeta\ell$ our new symbols have smaller derivatives (with a factor ℓ) with respect to x_1 than it was assumed there.

Proposition 14.3.25. *(cf. Proposition 14.3.20 and Corollary 14.3.21) Let conditions* (14.3.9), (14.2.27) *and* (14.3.67) *be fulfilled. Consider point* $(\bar{y}_2, \bar{\eta}_2)$ *and assume that*

$$|\bar{\eta}_2 - \eta^* W^{\frac{1}{2}}(0, \bar{y}_2)| \le \rho, \tag{14.3.72}$$

$$|\partial^\alpha_{x_2} W| \le C_\alpha \zeta \ell^{1-|\alpha|} \qquad \text{as} \quad |x_2 - \bar{y}_2| \le \ell \qquad \forall \alpha : |\alpha| \le K, \tag{14.3.73}$$

$$\ell\rho \ge Ch|\log h|, \tag{14.3.74}$$

$$\ell\zeta \le \epsilon\rho, \tag{14.3.75}$$

$$\ell \ge C_0\bar{\gamma}, \quad \rho \ge C_0\bar{\gamma}, \quad \zeta \ge C_0\bar{\gamma} + C_0 h^{1-\delta}. \tag{14.3.76}$$

Then for $|\tau| \le \rho$,

$$|t| \le T^* := \epsilon \min\big(\frac{\ell}{\rho}, \frac{\rho}{\zeta}\big) \tag{14.3.77}$$

quantum evolution starting from $\{|y_2 - \bar{y}_2| \le \frac{1}{3}\ell, |\eta_2 - \bar{\eta}_2| \le \frac{1}{3}\rho\}$ *is confined to* $\{|y_2 - \bar{y}_2| \le \ell, |\eta_2 - \bar{\eta}_2| \le \frac{1}{2}\rho\}$.

Proposition 14.3.26 [7]**.** *Let conditions of Proposition 14.3.25 be fulfilled.*

(i) If

$$|\bar{\eta}_2 - \eta^* W^{\frac{1}{2}}(0, \bar{y}_2)| \le \rho, \tag{14.3.78}$$

then estimate (14.3.59) *holds for* $|\tau| \le \rho$,

$$T_0 := C\frac{h|\log h|}{\rho^2} \le T \le T_1; \tag{14.3.79}$$

(ii) If condition (14.3.69) *is fulfilled as* $|x_2 - \bar{y}_2| \le \ell$ *then estimate* (14.3.59) *holds for* $|\tau| \le \rho$,

$$T_* := C\frac{h|\log h|}{\zeta\ell} \le T \le T^*. \tag{14.3.80}$$

Proofs of Propositions 14.3.25 and 14.3.26. Easy proofs are left to the reader.□

Now, given point $(\bar{y}_2, \bar{\eta}_2)$ let us define

$$\ell = \rho = \zeta = \varrho(\bar{y}_2, \bar{\eta}_2) \qquad \text{with} \tag{14.3.81}$$

$$\varrho(x_2, \xi_2) := \epsilon\Big(|\xi_2 - \eta^* W(x_2)^{\frac{1}{2}}| + |\partial_{x_2} W(x_2)|\Big).$$

Note that (14.3.74), (14.3.76) become equivalent to (14.3.67) and thus Propositions 14.3.25 and 14.3.26 hold with $T^* = \epsilon$ under condition (14.3.67). Also note that $T_* \le \epsilon\bar{\gamma}$ is equivalent to condition (14.3.67).

Then using the same arguments as above we arrive to (14.3.59) with $T_* = Ch|\log h| \le T \le T^*$ and also to estimate

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma\big(\varphi(hD_2)\psi u\big)| \le Ch^{-1}\rho\bar{\gamma}\ell \qquad \forall \tau : |\tau| \le \epsilon\rho \tag{14.3.82}$$

and $T_* \le T \le T^*$.

Therefore due to the standard Tauberian arguments

[7] Cf. Proposition 14.3.22.

(14.3.83) Contribution of (ℓ,ρ)-vicinity of $(\bar{y}_2,\bar{\eta}_2)$ to the Tauberian remainder does not exceed the right-hand expression of (14.3.82) multiplied by $CT^{*\,-1}$, i.e. $Ch^{-1}\bar{\gamma}\rho\ell$;

recall that $\ell=\rho$ here.

Then for given ρ the contribution of zone $\{(x_2,\xi_2),\frac{1}{2}\rho\le\varrho(x_2,\xi_2)\le\rho\}$ to the Tauberian remainder does not exceed $Ch^{-1}\rho\bar{\gamma}$. Finally, summation over ϱ results in $Ch^{-1}\bar{\gamma}$. This yields Statement (i) of the following proposition:

Proposition 14.3.27. *Under conditions* (14.3.9), (14.2.27)

(i) Contribution of zone $\{(x_2,\xi_2):\varrho(x_2,\xi_2)\ge\bar{\rho}_1\}$ *to the Tauberian remainder* (14.3.46) *with any* $T\in[T_*,\epsilon]$ *does not exceed* $C\mu^{-1/\nu}h^{-1}$.

(ii) The total Tauberian remainder (14.3.46) *does not exceed*

$$C\mu^{-\frac{1}{\nu}}h^{-1}+C\bar{\rho}_1h^{-1}; \tag{14.3.84}$$

(iii) In particular, as $\bar{\rho}_1\le\bar{\gamma}$ *which is equivalent to*

$$\mu\le C(h|\log h|)^{-\frac{\nu}{3}} \tag{14.3.85}$$

the total Tauberian remainder (14.3.46) *does not exceed* $C\mu^{-1/\nu}h^{-1}$.

Proof. To prove Statement (ii) one needs to estimate the contribution of *periodic zone* $\{\varrho\le C\bar{\rho}_1\}$. One can take $T^*=\epsilon\bar{\gamma}$ there and thus its contribution does not exceed $Ch^{-1}\bar{\rho}_1\bar{\gamma}\times\bar{\gamma}^{-1}=Ch^{-1}\bar{\rho}_1$.

Statement (iii) follows trivially. □

14.3.4 Periodic Zone

Toy-Model

Therefore one needs to consider contribution of the *periodic zone*

$$\mathcal{X}_{\mathsf{per}}=\{(x_2,\xi_2),\varrho(x_2,\xi_2)\le\bar{\rho}_1=C(\bar{\gamma}^{-1}h|\log h|)^{\frac{1}{2}}\} \tag{14.3.86}$$

as (14.3.85) is violated i.e. as

$$C(h|\log h|)^{-\frac{\nu}{3}}\le\mu\le\epsilon(h|\log h|)^{-\nu}. \tag{14.3.87}$$

Assume that the statements of Proposition 14.3.26 remain valid[8] as

$$\bar{\rho}_0 = C(h|\log h|)^{\frac{1}{2}} + C\bar{\gamma} \le \varrho \le \bar{\rho}_1. \tag{14.3.88}$$

Then for particular partition element inequality

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma\big(\varphi(hD_2)\psi(x_2)u\big)| \le Ch^{-2}\varrho^2\bar{\gamma} \times \frac{h|\log h|}{\varrho^2} = Ch^{-1}\bar{\gamma}|\log h| \tag{14.3.89}$$

holds as $|\tau| \le \epsilon\rho^2$, $T_* = Ch\rho^{-2}|\log h| \le T \le T_* = \epsilon$ where the second factor in the middle expression of (14.3.89) is just T_*.

Inequality (14.3.89) leads to the estimate of the contribution of the given element to the Tauberian remainder, equal to the right-hand expression of (14.3.89) (since $T^* \asymp 1$). Then the total contribution of the zone $\{(x_2,\xi_2) : \frac{1}{2}\rho \le \varrho(x_2,\xi_2) \le \rho\}$ is equal to $Ch^{-1}\bar{\gamma}\rho^{-1}|\log h|$. This estimate is not only is not good enough for the sharp remainder estimate but is much worse than estimates we had before[9].

To improve the estimates above let us start from the toy-model operator. However let us consider first 1-dimensional Schrödinger operator

$$A(\eta) = \hbar^2 D_1^2 + \big(\eta - x_1^\nu/\nu\big)^2 - 1 \tag{14.3.90}$$

with $\hbar = h/\bar{\gamma}$.

Proposition 14.3.28. *As $\hbar \ll 1$*

$$\operatorname{Spec}\big(A(\eta)\big) \cap [-\epsilon,\epsilon] = \check{g}\Big(\frac{\pi}{T(\eta)}(2n+1)\hbar;\eta,\hbar\Big) + O(\hbar^2), \quad n \in \mathbb{Z}^+ \tag{14.3.91}$$

and consists of simple eigenvalues where $\check{g}(\tau;\eta,\hbar)$ is an analytic function of all its arguments as $|\tau| < \epsilon, \hbar < \epsilon, |\eta - \eta^*| < \epsilon$, inverse (with respect to τ) to

$$g(\tau;\eta,\hbar) := \int T\big(\eta(1+\tau)^{-\frac{1}{2}}\big)(1+\tau)^{(1-\nu)/2\nu}\,d\tau - \hbar + O(\hbar^2) \tag{14.3.92}$$

and $T(\eta)$ is an elementary period of Section 14.2 calculated as $\mu = 1$, $\tau = 0$.

In particular,

$$e^{i\hbar^{-1}T(\eta)g(A(\eta);\eta,\hbar)}\bar{\chi}_\epsilon(A(\eta)) = \bar{\chi}_\epsilon(A(\eta)). \tag{14.3.93}$$

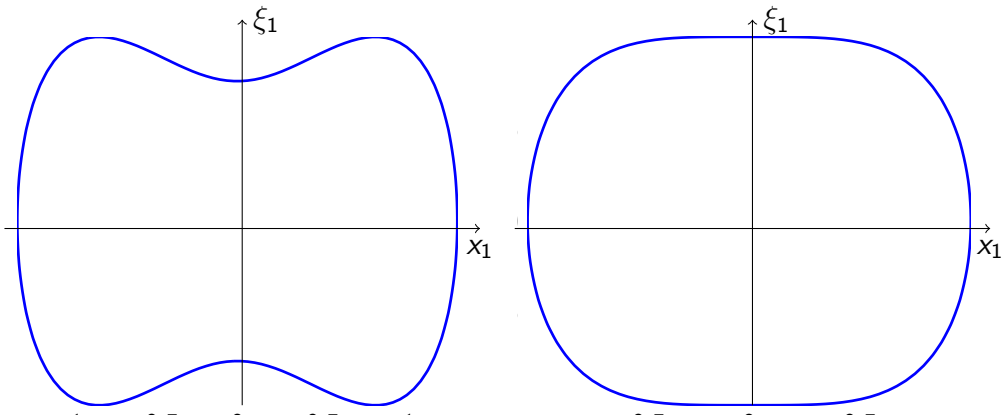
[8] Actually they do but with a twist.

[9] One can increase T^* and decrease $\bar{\rho}_0$ in the toy-model cases improving this estimate but (14.3.89) should be improved as well.

Proof. Note that for $\mathbf{a}(\eta)$ Hamiltonian flow on the energy level τ is periodic with period

$$T(\tau,\eta) = T\bigl(\eta(1+\tau)^{-1/2}\bigr)(1+\tau)^{(1-\nu)/2\nu}. \tag{14.3.94}$$

Proof now follows from Bohr-Sommerfeld formula. To derive asymptotics with $O(\hbar^2)$ error one should note that subprincipal symbol of $\mathbf{a}(\eta)$ is 0 and Maslov' index ι_M of the trajectory in (x_1,ξ_1) space is 2.



(a) First toy-model, $\nu=2$, $\eta=\eta^*=0.65$ (b) Second toy-model, $\nu=2$, $\eta=\eta^*=0$

Figure 14.6: Lagrangian manifolds for toy-model operators

Easy details are left to the reader.

Proposition 14.3.29. *Let condition* (14.3.9) *be fulfilled and let* $Q=\varphi(hD_2)$ *be a partition element supported in* $B(\eta^*,\rho)$. *Let us assume for simplicity that* $W(0)=1$. *Let* A *be the toy-model operator:*

$$A=\bar{A}:=\hbar^2D_1^2+(hD_2-\mu x_1^\nu/\nu)^2-1 \tag{14.3.95}$$

Assume that $\bar\gamma\ge Ch|\log h|$. *Then modulo negligible operator*

$$e^{ih^{-1}\bar\gamma T^*\mathcal{A}}Q\bar\chi_\epsilon(\mathcal{A})\equiv e^{ih^{-1}\bar\gamma B}Q\bar\chi_\epsilon(\mathcal{A}) \tag{14.3.96}$$

as

$$\mathcal{A}=g(A,\eta^*,\hbar),\qquad T^*=T(\eta^*), \tag{14.3.97}$$

$B = \beta(hD_2, A, \hbar)$ *such that*

$$\beta(\xi_2, \tau, \hbar) = \kappa(\xi_2, \hbar)\big(\xi_2 - \eta^*_\hbar\big)^2 + \kappa_1(\hbar) + O(|\tau|) \tag{14.3.98}$$

and function $\beta(\xi_2, \tau, 0)$ *coincides with* b *defined by* (14.2.59). *Further,* $\eta^*_\hbar = \eta^* + O(\hbar)$, $\kappa_1 = O(\hbar)$ *are analytic functions as* $\hbar \ll 1$.

Proof. Proof immediately follows from Propositions 14.2.15and 14.3.28. □

Remark 14.3.30. From now in the analysis of periodic zone we set $\eta^* = \eta^*_\hbar$ and respectively $\rho = |\xi_2 - \eta^*_\hbar|$ or $\rho = |\xi_2 - \eta^*_\hbar W^{1/2}|$

Now formula (14.3.96) implies that

$$e^{ih^{-1}t\mathcal{A}} Q\bar{\chi}_{\epsilon\rho}(A) \equiv e^{ih^{-1}t''\mathcal{A}} e^{ih^{-1}t'B} Q\bar{\chi}_{\epsilon\rho}(A), \tag{14.3.99}$$
$$\text{with} \quad t' = \bar{\gamma}\lfloor \frac{t}{\bar{\gamma}} \rfloor, \quad t'' = t - t' \qquad \text{as} \quad |t| \le T^* = \epsilon\rho^{-1}.$$

Here important thing is that on the partition element in question operator $e^{ih^{-1}t'B}$ is a proper Fourier integral operator as long as

$$|t'| \le T^{*\prime} := \frac{\bar{\gamma}}{\rho^2} \tag{14.3.100}$$

which is greater than $T'_* = C\rho^{-2}h|\log h|$. Then repeating corresponding analysis of Subsubsections 14.3.3 and 14.3.3 one can prove easily the following proposition:

Proposition 14.3.31. *Let* ρ *be defined as in Remark 14.3.30. Then statements of Proposition 14.3.20(ii), Corollary 14.3.21 and Proposition 14.3.26 remains valid with condition* (14.3.67) *replaced by condition* (14.3.88).

These extended statements allow us to prove the following proposition:

Proposition 14.3.32. *In the framework of Proposition 14.3.29 assume that* $\operatorname{supp}(\varphi) \cap B(\eta^*_\hbar, \frac{1}{2}\rho) = \emptyset$ *and that* ψ *is a fixed admissible function. Then*

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma\big(\varphi(hD_2)\psi(x_2)u\big)| \le C\rho^{-1}|\log h|^2. \tag{14.3.101}$$

Proof. Due to Proposition 14.3.26 extended by Proposition 14.3.31 it is sufficient to consider $T = T_*$ only.

Consider the left-hand expression of (14.3.101) with $\bar\chi_T(t)$ replaced by $\chi_n(t) = \bar\chi_{\bar\gamma}(t - n\bar T\bar\gamma)$, $|n| \le T_0/\bar T\bar\gamma$ with $\bar T = T(\eta^*)$. Note that then $e^{ih^{-1}\bar\gamma nB}$ is h-pseudodifferential operator with the symbol which will be regular after rescaling $x_1 \mapsto x_1/(\bar\gamma|\log h|)$, $\xi_1 \mapsto \xi_1/|\log h|$, $\xi_2 \mapsto \xi_2/|\log h|$. Therefore

$$|F_{t\to h^{-1}\tau}\chi_n(t)\Gamma\big(\varphi(hD_2)\psi(x_2)u\big)| = |F_{t\to h^{-1}\tau}\chi_0(t)\Gamma\big(Q_n u\big)| \tag{14.3.102}$$

with $Q_n = e^{ih^{-1}\bar\gamma nB}\varphi(hD_2)\psi(x_2)$ and therefore expression (14.3.102) does not exceed $C\rho\bar\gamma h^{-2} \times h|\log h| = C\rho\bar\gamma h^{-1}|\log h|$ due to the standard theory.

After summation by $n : |n| \le CT_0/\bar\gamma$ we get $C\rho\bar\gamma h^{-1}|\log h| \times h\rho^{-2}\bar\gamma^{-1}|\log h|$ which is exactly the right-hand expression of (14.3.101). □

We believe that factor $|\log h|^2$ is superficial and one can easily get rid of it: one factor $|\log h|$ is due to an inaccuracy in the estimate of (14.3.101) modified and another due to the lost factor $\big(1 + n\bar\gamma/T'_*\big)^{-s}$ in the same estimate (with $T'_* = C\rho^{-2}h$). However even such weakened estimate is sufficient for our needs unless μ is really close to $h^{-\nu}$ and the latter special case will be considered separately.

Really, after (14.3.101) is proven, Tauberian theorem implies that the contribution of this partition element to the Tauberian remainder does not exceed $C\rho^{-1}|\log h|^2 \times T^{*-1} = C|\log h|^2$ and thus the total contribution of zone $\{|\xi_2 - \eta^*_\hbar| \ge C\bar\gamma\}$ to the Tauberian remainder does not exceed this expression integrated over $\rho^{-1}\,d\rho$ resulting in $C|\log h|^3$ while contribution of zone $\{|\xi_2 - \eta^*_\hbar| \le C\bar\gamma\}$ does not exceed $C\bar\gamma h^{-1}$ as we already know. Thus we arrive to

Proposition 14.3.33. *For operator coinciding with the toy-model* (14.3.95) *in* $B(0,1)$ *the Tauberian remainder*[10)] *does not exceed* $C\bar\gamma h^{-1}$ *as* $\mu \le Ch^{-\nu}|\log h|^{-2\nu}$.

Another Example

Consider now the toy-model (14.3.95) perturbed by potential $W(x_2)$ satisfying

$$\epsilon_0\zeta \le |\partial_{x_2}W| \le \zeta \qquad \forall x. \tag{14.3.103}$$

[10)] Which still means R^{T} given by (14.3.46).

Then we can apply the same arguments as above for partition elements with $\rho^2 \ge \zeta$ thus covering completely case $\zeta \le C\bar{\gamma}^2$. On the other hand, as $\zeta \ge \bar{\gamma}^2$ we replace ρ by $\varrho = \mathsf{max}(\zeta^{\frac{1}{2}}, \rho)$ and on partition elements with $\varrho = \zeta^{\frac{1}{2}}$ we again can apply the same arguments with $T^* = \epsilon\zeta^{-\frac{1}{2}}$, $T'_* = \epsilon h\zeta^{-1}|\log h|$.

So, we arrive to

Proposition 14.3.34. *For operator coinciding in $B(0,1)$ with the toy-model perturbed by potential $W(x_2)$ satisfying* (14.3.96) *Tauberian remainder does not exceed $O(\bar{\gamma}h^{-1})$ as long as $\zeta^{\frac{1}{2}} + \bar{\gamma} \ge Ch|\log h|^3$ i.e. either $\mu \le Ch^{-\nu}|\log h|^{-3\nu}$ or $\zeta \ge Ch^2|\log h|^6$.*

General Case

Consider now the general case. Let $\varphi(\xi_2)$ and $\psi(x_2)$ are two ρ- and ℓ-admissible functions supported in $\frac{1}{3}\rho$- and $\frac{1}{3}\ell$-vicinities of $\bar{\xi}_2$ and $\bar{x}_2$ respectively. Let us assume that conditions (14.3.72)–(14.3.76) are fulfilled.

Then due to Proposition 14.3.26 extended by Proposition 14.3.31 as

$$|t| \le T^* := \epsilon\,\mathsf{min}\big(\frac{\ell}{\rho}, \frac{\rho}{\zeta}\big) \asymp \epsilon\rho\ell\big(\rho^2 + \ell\zeta\big)^{-1} \tag{14.3.104}$$

propagation started from $\mathsf{supp}\,\varphi \times \mathsf{supp}\,\psi$ is confined to $(\frac{1}{3}\rho, \frac{1}{3}\ell)$-vicinity of it.

Now we can easily generalize Proposition 14.3.29:

Proposition 14.3.35. *Let conditions* (14.3.9), (14.2.27) *and* (14.3.72)–(14.3.76) *be fulfilled. Assume for simplicity that $W(0) = 1$, Then*

(i) Equality (14.3.96) *holds as $\mathcal{A}$ is defined by* (14.3.97).

(ii) Therefore equality (14.3.99) *holds.*

Proof. Proof repeats one of Proposition 14.3.29. We leave easy details to the reader.

Proposition 14.3.36. *Let conditions* (14.3.9), (14.2.27) *and* (14.3.72)–(14.3.76) *be fulfilled. Assume for simplicity that $W(0) = 1$. Then estimate*

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma\big(\varphi(hD_2)\psi(x_2)u\big)| \le CT\rho\ell h^{-1}|\log h| \tag{14.3.105}$$

holds as

$$\bar{\gamma} \le T \le T'_* := C_1 h|\log h|(\rho^2 + \zeta\ell)^{-1}. \tag{14.3.106}$$

Proof. Let us note that for $|t'| \le T'_*$ operator $e^{ih^{-1}t'B}$ remains legitimate h-pseudodifferential operator and therefore contribution to the left-hand expression of (14.3.105) of the time interval $[t_1, t_2]$ with $|t_j| \le T'_*$, $|t_2 - t_1| = \epsilon\bar{\gamma}$ does not exceed $C\rho\ell\bar{\gamma}h^{-1}|\log h|$ and therefore the contribution of $\asymp T/\bar{\gamma}$ of such intervals does not exceed the right-hand expression of (14.3.105). □

It immediately implies

Corollary 14.3.37. *Let in the framework of the previous propositions either*

$$|\xi_2 - \eta^*_\hbar W(x_2)^{\frac{1}{2}}| \ge \rho, \tag{14.3.107}$$

or

$$|\partial_{x_2} W| \ge \epsilon_0\zeta \tag{14.3.108}$$

in ℓ-vicinity of $\bar{x}_2$. Then

(i) The left-hand expression of (14.3.105) *with $T \le T_1$ does not exceed $CT'_0\rho\ell h^{-1}|\log h|^2$.*

(ii) Contribution of the partition element $\varphi(hD_2)\psi(x_2)$ to the Tauberian remainder does not exceed $C|\log h|^2$.

Proof. Statement (i) immediately follows from Proposition 14.3.26 extended and Proposition 14.3.36.

Statement (ii) then follows from the Tauberian arguments: contribution of this element to the Tauberian remainder does not exceed $Ch^{-1}\rho\ell|\log h| \times T'_*/T^* = C|\log h|^2$ since $T'_*/T^* = Ch|\log h|/(\rho\ell)$. □

Assume now that

$$|\partial_{x_2} W| + |\partial^2_{x_2} W| \ge \epsilon_0. \tag{14.3.109}$$

Let us introduce function ϱ by (14.3.81) and let us consider ϱ-admissible partition in (x_2, ξ_2) and apply the same arguments with $\rho = \zeta = \ell = \varrho$; then on each partition element either $|\xi_2 - \eta^*_\hbar W(x_2)^{\frac{1}{2}}| \ge \epsilon_0\varrho$ or $|\partial_{x_2} W(x_2)| \ge \epsilon\varrho$ and previous propositions work as long as

$$\varrho \ge C\bar{\gamma}, \qquad \varrho \ge C(h|\log h|)^{\frac{1}{2}} \tag{14.3.110}$$

where the second inequality is equivalent to logarithmic uncertainty principle $C\ell\rho \ge Ch|\log h|$.

We know that the contribution of each such element to the Tauberian remainder does not exceed $C|\log h|^2$ and therefore the total contribution of all such elements does not exceed $C\log h|^2 I$ with $I = \int \varrho^{-2}\,dx_2 d\xi_2 \asymp C|\log h|$ due to condition (14.3.109).

On the other hand, as $\varrho \le \bar{\varrho} = C\bar{\gamma} + C(h|\log h|)^{\frac{1}{2}}$ let us redefine ϱ as $\bar{\varrho}$. Then contribution of each such partition element to the Tauberian remainder does not exceed $C\bar{\varrho}^2 h^{-1} = C\bar{\gamma}^2 h^{-1} + C|\log h|^2$ and again due to condition (14.3.109) there is no more than C of such elements. Thus we arrive to

Proposition 14.3.38. *For magnetic Schrödinger operator satisfying in* $B(0,1)$ *conditions* (14.2.27) *and* (14.3.109) *the Tauberian remainder is* $O(\bar{\gamma}h^{-1})$ *as long as* $\mu \le Ch^{-\nu}|\log h|^{-3\nu}$.

On the other hand, exactly the same approach but without any non-degeneracy condition (14.3.109) results in the remainder estimate

$$C\bar{\varrho}^{-1}|\log h|^2 + C\bar{\varrho}h^{-1}$$

where the first term is an upper estimate of $C|\log h|^2 I$ with integral over $\{\varrho \ge \bar{\varrho}\}$ and the second term is the contribution of elements with $\varrho \asymp \bar{\varrho}$. Picking up $\bar{\varrho} = C\max(\bar{\gamma}, h^{\frac{1}{2}}|\log h|)$ we arrive to the remainder estimate

$$C\bar{\gamma}h^{-1} + Ch^{-\frac{1}{2}}|\log h|$$

which is our target $C\bar{\gamma}h^{-1}$ as long as $\bar{\gamma} \ge h^{\frac{1}{2}}|\log h|$ i.e. $\mu \le Ch^{-\nu/2}|\log h|^{-\nu}$. Thus we arrive to

Proposition 14.3.39. *For magnetic Schrödinger operator satisfying in* $B(0,1)$ *condition* (14.2.27)

(i) The Tauberian remainder is $O(\bar{\gamma}h^{-1})$ *as long as* $\mu \le h^{-\nu/2}|\log h|^{-\nu}$.

(ii) The Tauberian remainder is $O(h^{-\frac{1}{2}}|\log h|)$ *as long as* $h^{-\nu/2}|\log h|^{-\nu} \le \mu \le Ch^{-\nu}|\log h|^{-\nu}$.

General Case. II

Finally we are going to prove the most general proposition:

Proposition 14.3.40. *(i) Under conditions* (14.3.9), (14.2.27) *and*

$$\sum_{1\le k\le m} |\partial_{x_2}^k W| \ge \epsilon_0. \tag{14.3.111}_m$$

the Tauberian remainder does not exceed $C\bigl(|\log h|^{m+1} + \bar\gamma h^{-1}\bigr)$ *and thus the remainder estimate* $O(\bar\gamma h^{-1}) = O\bigl(\mu^{-1/\nu} h^{-1}\bigr)$ *holds as long as* $\mu \le Ch^{-\nu}|\log h|^{-(m+1)\nu}$.

(ii) In the general case[11] *the Tauberian remainder does not exceed* $C\bigl(\bar\gamma^{-\delta} + \bar\gamma h^{-1}\bigr)$ *and thus the remainder estimate* $O\bigl(\bigl(\bar\gamma h^{-1}\bigr)$ *holds as long as* $\mu \le Ch^{-\nu+\delta_1}$ *with arbitrarily small exponents* $\delta, \delta_1 > 0$.

Proof. Let us consider the scaling function

$$\ell(x_2) \coloneqq \epsilon_1\Bigl(\sum_{1\le k\le m-1} |\partial_{x_2}^k W(x_2)|^{\frac{m}{m-k}}\Bigr)^{\frac{1}{m}} + \bar\ell \tag{14.3.112}_m$$

with small enough constant $\epsilon_1 > 0$ such that $|\partial_{x_2}\ell| \le \frac{1}{2}$. Let us consider ℓ-admissible covering of $(-1, 1)$.

Then

$$|\partial_{x_2}^k W| \le C_0 \ell(x_2)^{m-k} \qquad \forall k : k = 1, \dots, m-1; \tag{14.3.113}_m$$

further, on each element with $\ell \ge 2\bar\ell$ one of these inequalities could be reversed with C_0 replaced by ϵ_0:

$$|\partial_{x_2}^k W| \ge \epsilon_0 \ell(x_2)^{m-k} \qquad k = 1, \dots, m-1. \tag{14.3.114}_k$$

Furthermore,

(14.3.116) Under condition $(14.3.111)_m$ there is no more than C_0 of ℓ-elements with $\ell \in (l, 2l)$ for each l and thus no more than $C|\log \bar\ell|$ elements in total

while

(14.3.117) In the general case there is no more than $C_0 l^{-1}$ of ℓ-elements with $\ell \in (l, 2l)$ for each l and thus no more than $C\bar\ell^{-1}$ elements in total.

[11] I.e. without non-degeneracy condition $(14.3.111)_m$.

On each ℓ-element oscillation of $W(x_2)$ does not exceed $C\ell^m$ and thus we can ponder condition

$$|\xi_2 - \eta^*_\hbar W(\bar{y})^{\frac{1}{2}}| \asymp \rho$$

with $\rho \ge C\ell^m$ and $\bar{y}$ the center of this element. As this condition is fulfilled and also $\rho\ell \ge Ch|\log h|$, $\rho \ge h^{1-\delta}$ we can apply Corollary 14.3.37(ii) and conclude that the contribution of this element to the remainder estimate does not exceed $C|\log h|^2$.

Now we need to sum contributions of all such elements. To do it in the most efficient way we redefine

$$(14.3.118)_m \quad \ell(x_2,\xi_2) := \epsilon_1\Big(|\xi_2 - \eta^*_\hbar W^{\frac{1}{2}}(x_2)| + \sum_{1\le k\le m-1} |\partial^k_{x_2} W(x_2)|^{\frac{m}{m-k}}\Big)^{\frac{1}{m}} + \bar{\ell}.$$

Then for each given ρ the total contribution of all elements of this type would not exceed $C|\log h|^2$ and $C\rho^{-1/m}|\log h|^2$ under condition $(14.3.111)_m$ and without it respectively; after integration with respect to $\rho^{-1}d\rho$ we get $C|\log h|^3$ and $C\bar{\ell}^{-1}|\log h|^2$ respectively and both these expressions do not exceed claimed in Statements 14.3.40 and (ii) remainder estimates[12] as we pick up

$$\bar{\ell} = C_1\bar{\gamma}^{1/(m-1)} + C_1(h|\log h|)^{1/(m+1)}. \tag{14.3.119}$$

On the other hand, the contribution of each element with $\ell \asymp \bar{\ell}$ does not exceed $C\bar{\ell}^{m+1}h^{-1}|\log h|^2$ and the total contribution of such elements does not exceed $C\bar{\ell}^{m+1}|\log h|^2h^{-1}$ and $C\bar{\ell}^m|\log h|^2h^{-1}$ under condition $(14.3.111)_m$ and without it respectively and these expressions do not exceed remainder estimates announced in Statements 14.3.40 and (ii) respectively.

Thus we need to consider only elements with $\rho \le C\ell^m$ and $\ell \ge 2\bar{\ell}$ with ℓ given by $(14.3.112)_m$ rather than $(14.3.118)_m$. Let us consider one such element $(\bar{y} - \frac{1}{2}\ell, \bar{y} + \frac{1}{2}\ell)$; after rescaling

$$x \mapsto (x-\bar{y})/\ell,\ W(x_2) - W(\bar{y}) \mapsto \big(W(x_2) - W(\bar{y})\big)/\ell^m,\ \rho \mapsto \rho\ell^{-m}$$

we find ourselves in the framework of condition $(14.3.111)_{m-1}$.

[12] Provided m is large enough in (ii) which we assume.

So in the rescaled coordinates let us define scaling function ℓ_1' by $(14.3.118)_{m-1}$ and redefine W as $\ell^{-m}W$. However, as we return to the original scale this new rescaling function $\ell_1 = \ell_1'\ell$ will be defined by $(14.3.118)_{m-1}$ with original W.

Now we can repeat the same arguments as before with $\bar{\ell}_1$ defined by $(14.3.119)_{m-1}$. However now in the total partition there is no more than C_1 of ℓ_1 elements with $\ell_1 \in (l, 2l)$ if original condition $(14.3.111)_m$ was fulfilled and no more than $C_1 l^{-1}$ of them otherwise because as we mentioned after rescaling of the elements in question $(14.3.111)_{m-1}$ always is fulfilled.

Then we eliminate elements of ℓ_1 partition with $\ell_1 \le 2\bar{\ell}_1$ and with $\rho \ge C_1\ell_1^{m-1}$ estimating their total contribution by $C|\log h|^{m+1} + C\bar{\gamma}h^{-1}$ and $C|\log h|^{m+1} + C\bar{\ell}_1^{m+1}\bar{\ell}^{-1}h^{-1}$ in the frameworks of Statements 14.3.40 and (ii) respectively which do not exceed remainder estimates announced there.

We will continue this process as long as our scaling function ℓ_k is defined just by $(14.3.118)_{m-k}$ i.e. until $k = m-2$. Then all elements remaining are eliminated by Corollary 14.3.37(ii) under condition $|\partial_{x_2} W| \asymp \zeta$. □

Thus we have covered the case

$$\mu \le h^{-\nu}|\log h|^{-\nu N} \tag{14.3.120}$$

leaving the case of the strong magnetic field $h^{-\nu}|\log h|^{-\nu N} \le \mu \le Ch^{-\nu}$ for Section 14.5.

14.4 Calculations

So far we recovered good remainder estimates but the Tauberian expression (14.3.45) we derived was implicit. In this section we am going to rewrite it in more explicit form as either

$$h^{-2}\int \mathcal{N}^{\mathsf{MW}}(x,0)\psi(x)\,dx \tag{14.4.1}$$

which is of magnitude $h^{-2}\big(1+\mu h\big)^{-1/(\nu-1)}$ or the same expression but with a smaller correction term. Again we consider only the case (14.3.120) leaving the case of the strong magnetic field for Section 14.5.

Remark 14.4.1. Without any loss of the generality one can assume that $\psi = \psi_2(x_2)\psi_1(x_1)$ with fixed admissible functions ψ_j; moreover, one can take

$\psi_1 = 1$ as $\mu \ge Ch^{-1}$. Really, it follows from the analysis of Subsection 14.1 that for fixed ψ vanishing as $x_1 = 0$ the Magnetic Weyl remainder (with the principal part is given by (14.4.1)) does not exceed

$$C \int_{\{\gamma \ge \bar\gamma\}} \mu^{-1} h^{-1} \gamma^{1-\nu} \times \gamma^{-1}\, dx + Ch^{-1}\bar\gamma = Ch^{-1}\bar\gamma = C\mu^{-1/\nu} h^{-1}$$

where the first and second terms are contribution of the regular and degeneration zones respectively and in comparison with (14.1.13) and Ch^{-1} they have an extra factor γ if $\psi = 0$ as $x_1 = 0$.

14.4.1 Regular Zone

Recall that the regular zone is $\mathcal{X}_{\mathsf{reg}} = \{C\bar\gamma \le |x_1| \le \bar\gamma_1\}$ and in this subsection γ ranges from $\bar\gamma$ to $\bar\gamma_1$ as well. Let us also recall that according to Subsection 14.3.2 we can take here $T = Ch|\log h|$ in (14.4.1).

Proposition 14.4.2. *Let condition* (14.3.9) *be fulfilled. Then*

$$|h^{-1} \int_{-\infty}^{0} \Big(F_{t\to h^{-1}\tau} \bar\chi_T(t) \Gamma\big(\psi(1-\psi_1)u\big)\Big)\, d\tau - h^{-2} \int \mathcal{N}^{\mathsf{MW}}(x,0)\big(\psi(1-\psi_1)\big)\, dx| \le C(\mu h + 1)^{1/(\nu-1)} \tag{14.4.2}$$

for $T = Ch|\log h|$, *fixed function* ψ *and* $\psi_1(x_1) = \bar\psi_1(x_1/\bar\gamma)$ *with a fixed admissible function* $\bar\psi$, *supported in* $(-2,2)$ *and equal* 1 *in* $(-1,1)$.

In particular we get $C\bar\gamma h^{-1}$*-estimate as* $\mu \le Ch^{-\nu^2/(2\nu-1)}$.

Proof. It follows from argument of Section 13.4 that for 2-dimensional Schrödinger operator with non-degenerate magnetic field, binding parameter μ and semiclassical parameter h under non-degeneracy condition $|\nabla V/F| \ge \epsilon_0$ the following asymptotic expansion holds

$$h^{-1} \int_{-\infty}^{0} \Big(F_{t\to h^{-1}\tau} \bar\chi_T(t) \Gamma\big(\psi u\big)\Big)\, d\tau \sim \sum_{m,n\ge 0} \kappa_{mn} h^{-2+2n+2m} \mu^{2m} \tag{14.4.3}$$

as $\mu \le h^{\delta-1}$ where

$$\sum_{m\ge 0} \kappa_{m0} h^{-2} (\mu h)^{2m} \sim h^{-2} \int \mathcal{N}^{\mathsf{MW}}(x,0)\psi(x)\, dx \tag{14.4.4}$$

and therefore one can replace (14.4.3) by (14.4.4) with an error $\kappa_{10}+O((\mu h)^2)$.

It also follows that in the same settings but with $h^{\delta-1} \le \mu \le Ch^{-1}$ another asymptotic expansion holds

$$h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma\big(\psi u\big)\Big)\,d\tau \sim \sum_{m,n\ge 0}\kappa'_{m,n}h^{-2+2n+2m}+\sum_{m,n\ge 0}\kappa''_{m,n}\mu^{-2-2m}h^{-2+2n} \tag{14.4.5}$$

and comparison of this expression with (14.4.3) where they overlap implies that $\kappa''_{**}=0$ and an error in question is given again by $\kappa_{10}+O((\mu h)^2)$.

Therefore in our setting the contribution (to this error) of any element of γ-partition in the regular zone will be of magnitude 1. To calculate the total contribution of the regular zone one must integrate the latter expression over $\gamma^{-2}d\gamma$ resulting in $O(\bar{\gamma}^{-1})$ which however is not as good as $O(\bar{\gamma}_1^{-1})$ with $\bar{\gamma}_1=\min\big(1,(\mu h)^{-1/(\nu-1)}\big)$ which we claimed.

On the other hand, substitution $\mu\mapsto\mu\gamma^{\nu}$, $h\mapsto h/\gamma$ into $O((\mu h)^2)$ and integration produces $O(\bar{\gamma}_1^{-1})$ which is exactly what we claimed and one needs just to calculate explicitly the correction appearing from κ_{10}-term.

In the non-degenerate settings this latter correction does not depend on μ and therefore one can calculate it as $\mu=0$. But this is non-magnetic case and then due to the standard theory and condition $|V|\ge\epsilon_0$

$$h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\big(\bar{\chi}_T(t)u\big)\Big)\,d\tau\sim\sum_{n\ge 0}\varkappa_n(x)h^{-2+2n}$$

(without integration with respect to dx) with $\varkappa_0(x)=\mathcal{N}^{\mathsf{W}}(x,0)$ and therefore the total contribution of this correction term is $O(1)$. □

Now we am going to derive a bit sharper remainder estimate; to do this we must appeal to the spectral projector of the corresponding 1-dimensional Schrödinger operator:

Proposition 14.4.3. *Let condition* (14.2.27) *be fulfilled and let* ψ_1 *be an even function,* $\psi=\psi(x_2)$. *Then as* $h^{-\nu^2/(2\nu-1)}\le\mu\le\epsilon(h|\log h|)^{-\nu}$, $T=\epsilon\bar{\gamma}$ *the following estimate holds:*

$$|h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma\big(\psi(1-\psi_1)u\big)\Big)\,d\tau- \tag{14.4.6}$$

$$h^{-2}\int \Big(\mathcal{N}^{\mathsf{MW}}(x,0)-\mathcal{N}_0^{\mathsf{MW}}(x,0)\Big)\psi(x_2)(1-\psi_1(x_1))\,dx-$$
$$(2\pi h)^{-1}\int e_0(x_1,x_1;x_2,\xi_2,0,\hbar)\psi(x_2)\big(1-\bar\psi_1(x_1)\big)\,dxd\xi_2|\le$$
$$Ch^{-1}\bar\gamma + C|\log h|^K$$

where $e_0(x_1,y_1;x_2,\xi_2,\tau,\hbar)$ *is the Schwartz kernel of the spectral projector* $e_0(x_2,\xi_2,\tau,\hbar)$ *for 1-dimensional Schrödinger operator*

$$a_0(x_2,\xi_2,\hbar)=\hbar^2D_1^2+(\xi_2-x_1^\nu/\nu)^2-W(x_2) \tag{14.4.7}$$

with $\hbar=h/\bar\gamma$, *while*

$$\mathcal{N}_0^{\mathsf{MW}}(x,0)=\frac{1}{2\pi}\sum_{n\ge 0}\theta\Big(\tau+W(x_2)-(2n+1)\mu h x_1^{\nu-1}\Big)\mu h x_1^{\nu-1} \tag{14.4.8}$$

is Magnetic Weyl approximation (without factor h^{-2}*) for corresponding toy-model operator.*

Proof. (a) To calculate $u(x,y,t)$ let us apply first the method of successive approximations with unperturbed operator $\bar A$ obtained from A by freezing $x_2=y_2$ in V, σ, ϕ:

$$\bar A=h^2D_1^2+\bar\sigma(x_1)^2\big(\xi_2-\mu\bar\phi(x_1)x_1^\nu)/\nu\big)^2+\bar V(x_1) \tag{14.4.9}$$

with $\bar\sigma=\sigma(x_1,y_2)$, $\bar\phi=\phi(x_1,y_2)$, $\bar V=V(x_1,y_2)$.

Then the first term of the Tauberian expression

$$h^{-1}\int_{-\infty}^0\Big(F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma\big(\psi(1-\psi_1)u\big)\Big)\,d\tau \tag{14.4.10}$$

is also given by formula (14.4.10) but for operator $\bar A$ instead of A and it is $O\big(h^{-3}\bar\gamma_1T_*\big)=O\big(h^{-2}\bar\gamma_1|\log h|\big)$ [13].

The estimate for the second term gains the factor $CT_*^2/h\asymp Ch|\log h|^2$ provided $\phi=1$ identically or factor

$$CT_*^2\big(1+\mu\bar\gamma_1^{\nu+1}h\big)/h\asymp C\big(h+\bar\gamma_1^2\big)|\log h|^2$$

[13] Actually factor $|\log h|$ is superficial and the first term is of magnitude $h^{-2}\bar\gamma_1$; further, all logarithmic factors below are superficial as effectively one can take in estimates $T_*=h$ rather than $T_*=Ch|\log h|$.

in the general case (recall that then $\phi = 1$ as $x_1 = 0$); therefore the second term does not exceed $Ch^{-2}\bar{\gamma}_1\big(h + \bar{\gamma}_1^2\big)|\log h|^3$.

Finally, in this two-term approximation the remainder gains one more such factor and thus does not exceed $Ch^{-2}\bar{\gamma}_1(h^2+\bar{\gamma}_1^4)|\log h|^5$. One can check easily that this is less than $Ch^{-1}\bar{\gamma}$ unless $\nu = 2$ and $\mu \le h^{-\nu^2/(2\nu-1)}|\log h|^K$ in which *special case* there will be an extra $|\log h|^K$ factor (rather superficial); then we consider three-term approximation and one can see easily that the remainder estimate will be $Ch^{-1}\bar{\gamma}$.

So we are left only with expression (14.4.10) for

$$u \mapsto \bar{u} = -ih\sum_{\varsigma=\pm} \bar{G}^{\varsigma}\delta(t)\delta(x_2-y_2)\delta(x_1-y_1) \tag{14.4.11}$$

where here and below "$\mapsto$" means "replaced by" and two extra terms given by (14.4.10) again for

$$u \mapsto \bar{u}_1 = -ih\sum_{\varsigma=\pm} \bar{G}^{\varsigma}A_1\bar{G}^{\varsigma}\delta(t)\delta(x_2-y_2)\delta(x_1-y_1) \tag{14.4.12}$$

and

$$u \mapsto \bar{u}_2 = -ih\sum_{\varsigma=\pm} \bar{G}^{\varsigma}A_1\bar{G}^{\varsigma}A_1\bar{G}^{\varsigma}\delta(t)\delta(x_2-y_2)\delta(x_1-y_1) \tag{14.4.13}$$

where $\bar{G}^{\pm}$ are corresponding parametrices and $A_1 = A - \bar{A}$.

Also, note that replacing in (14.4.12) A_1 by $\bar{A}_1 = (x_2-y_2)\cdot B_1$ with $B_1 = (\partial_{x_2}A)|_{x_2=y_2}$ one gets a proper remainder estimate with the exception of the special case in which in addition to terms due to (14.4.11)–(14.4.13) one must consider also an extra term given by (14.4.10) for

$$u \mapsto \bar{u}_2' = -ih\sum_{\varsigma=\pm} \bar{G}^{\varsigma}A_2\bar{G}^{\varsigma}\delta(t)\delta(x_2-y_2)\delta(x_1-y_1) \tag{14.4.14}$$

with $\bar{A}_2 = \frac{1}{2}(x_2-y_2)^2\cdot(\partial_{x_2}^2A)|_{x_2=y_2}$.

(b) We will tackle terms due to (14.4.12)–(14.4.14) later; now let us concentrate on the main term due to (14.4.11). After this term is derived with $T = Ch|\log h|$ one can increase T to T_1 where $T^* = \epsilon\mu\gamma^{\nu}$; however arguments of Subsection 14.3.2 applied to $\bar{A}$ (whose coefficients do not depend on x_2) imply that now one can take even $T = +\infty$.

With $T = +\infty$ one can rewrite this term (before integration by ψ and integration over x_2) as

$$(2\pi h)^{-1} \int \boldsymbol{e}(x_1, x_1; x_2, \xi_2, 0, \hbar)\bigl(1 - \bar{\psi}_1(x_1)\bigr)\, dx_1 d\xi_2 \tag{14.4.15}$$

where $\boldsymbol{e}(x_1, y_1; x_2, \xi_2, 0, \hbar)$ is the Schwartz kernel of 1-dimensional Schrödinger operator $\boldsymbol{a}(x_2, \xi_2, \hbar)$ obtained from $\bar{A}$ by change of variables $x_1 \mapsto \bar{\gamma} x_1$ (and thus $\mu \mapsto 1$) and $hD_2 \mapsto \xi_2$ while $\boldsymbol{e}^{\mathsf{W}}(x_1; x_2, \xi_2, \hbar)$ would be Weyl approximation for its restriction to diagonal.

However we would like to refer to a canonical operator $\boldsymbol{a}_0$ rather than to more general operator $\boldsymbol{a}$ and thus few extra steps are needed. We will delay them slightly.

(c) Consider now the second term corresponding to (14.4.12); recall that it does not exceed $Ch\gamma|\log h|^2$ with the superficial logarithmic factor factor. We claim that

(14.4.16) The total contribution of the second term is 0 modulo $C\mu^{-1/\nu}h^{-1} + |\log h|^K$.

Really, note first that one can assume that $\mu \le h^{-\nu}|\log h|^{-K}$, otherwise the total contribution $Ch^{-1}\bar{\gamma}_1|\log h|^2$ is less than announced. Let us rewrite the term in question as

$$\begin{aligned} u \mapsto \bar{u}_1 = &- ih \sum_{\varsigma=\pm} \bar{G}^{\varsigma} B_1 \bar{G}^{\varsigma} [\bar{A}, x_2 - y_2] \bar{G}^{\varsigma} \delta(t)\delta(x_2 - y_2)\delta(x_1 - y_1) \\ &- ih \sum_{\varsigma=\pm} \bar{G}^{\varsigma} [x_2 - y_2, B_1] \bar{G}^{\varsigma} \delta(t)\delta(x_2 - y_2)\delta(x_1 - y_1). \end{aligned} \tag{14.4.17}$$

Let us apply to (14.4.17) the method of successive approximations with an unperturbed operator frozen as $x_1 = y_1$; then each next term will acquire factor $C\mu\gamma^{\nu-1}h|\log h| \asymp (\gamma/\bar{\gamma}_1)^{\nu-1}|\log h|$ and therefore the second term in the successive approximation decomposition does not exceed $Ch^{-1}\bar{\gamma}|\log h|^{-K_1}$ as long as $\gamma \le \bar{\gamma}|\log h|^{K_2}$.

Meanwhile, the leading term in the successive approximation decomposition is just a standard Weyl expression without any axillary operators; this expression is odd with respect to $(\xi_2 - \mu x_1^{\nu}/\nu)$ and thus it produces 0 after integration with respect to ξ_2.

On the other hand, as $\gamma \ge C\bar{\gamma}|\log h|^{K_2}$ one can apply successive approximation method with unperturbed operator $\hat{A}$ which is obtained from $\bar{A}$ by transformation $\mu x_1^\nu/\nu \mapsto \xi_2 + \mu z_1^{\nu-1}(x_1 - z_1)$ with $z_1 = (\mu^{-1}\nu\xi_2)^{1/\nu}$ and the same is true for commutators $[\bar{A}, x_2 - y_2]$ and $[B_1, x_2 - y_2]$.

Then since both x_1 and y_1 are confined to an interval of the length $\beta = (\mu\gamma^{\nu-1})^{-1}$, each next term acquires factor $\beta\gamma^{-1}|\log h| \asymp \mu^{-1}h|\log h|\gamma^{-\nu}$ and therefore the second term does not exceed $C\mu^{-1}h^{-1}|\log h|^2\gamma^{1-\nu}$ and its total contribution does not exceed what is announced as $\gamma \ge \bar{\gamma}|\log h|^{K_2}$.

Meanwhile the leading term corresponds to the harmonic oscillator and obviously sums to 0 as $x_1 = y_1$ runs through $\mathbb{R}$.

So we will keep $T = Ch|\log h|$ for a while. In the analysis below we are going to consider expression (14.4.10) with Γ replaced by Γ' which contains integration with respect to x_1 but not x_2; then one can assume without any loss of the generality that

(14.4.18) V, σ, ϕ do not depend on x_2.

It is convenient to have $\phi = 1$; one can always reach it now changing $x_1 \mapsto x_1 + O(x_1^2)$ and paying the price: now

$$\bar{A} = \sigma_1(x_1)^2(hD_1)^2 + \sigma(x_1)^2\bigl(hD_2 - \mu x_1^\nu/\nu\bigr)^2 + V(x_1) \tag{14.4.19}$$

with $\sigma = \sigma_1 = 1$ as $x_1 = 0$.

Let us apply the method of successive approximations again this time using as unperturbed operator $\bar{A}_0$ obtained from $\bar{A}$ by freezing $x_1 = 0$ in V, σ and σ_1:

$$\bar{A}_0 = (hD_1)^2 + \bigl(hD_2 - \mu x_1^\nu/\nu\bigr)^2 + V(0). \tag{14.4.20}$$

In these successive approximations the first term is still $O\bigl(h^{-2}\bar{\gamma}_1|\log h|\bigr)$ and an estimate for each subsequent term is gaining an extra factor $CT_*\bar{\gamma}_1/h \asymp \bar{\gamma}_1|\log h|$; therefore $(n{+}1)$-th term does not exceed $Ch^{-2}\bar{\gamma}_1^n|\log h|^n$ which is small for large n since $\bar{\gamma}_1 \le h^{(\nu-1)/(2\nu-1)}$ due to our assumption $\mu \ge h^{-\nu^2/(2\nu-1)}$. Note that we can write

$$A \sim \sum_{k=0}^{K} x_1^k \bar{A}_k \qquad \text{with} \quad \bar{A}_k = \bar{A}_k(x_1, x_2, hD_1, hD_2; h, \mu) \quad \text{as} \quad k \ge 1,$$

$\bar{A}_0$ is already defined by (14.4.20). Plugging it into successive approximation formula we get many terms but the first term results in

$$(2\pi h)^{-1}\int e_0(x_1,x_1;x_2,\xi_2,0,\hbar)\bigl(1-\bar{\psi}_1(x_1)\bigr)\,dx_1d\xi_2 \tag{14.4.21}$$

(after finally one takes $T=+\infty$ instead of $T=Ch|\log h|$) and the second term will be one for the approximation $A=\bar{A}_0+x_1\bar{A}_1$.

But then the contribution to (14.4.15) of this refined second term would be 0 because (before integration over $\psi\,dx_2$) it is equal to

$$\int \mathsf{Tr}_1\bigl(b(\xi_2)\bar{\psi}_1\bigr)\,d\xi_2 \tag{14.4.22}$$

where $b(\xi_2)$ is an operator in $\mathbb{H}=\mathscr{L}^2(\mathbb{R})$, Tr_1 is the trace in this auxiliary space and $\mathcal{T}b(\xi_2)\mathcal{T}=-b(\xi_2')$ with $\xi_2'=(-1)^\nu\xi_2$, $(\mathcal{T}v)(x_1)=v(-x_1)$.

(d) Now let us consider terms in this successive approximation with $n\ge 3$; they will be still too large just to be skipped. Let us apply to them the successive approximations method taking as unperturbed operator $\hat{A}_0$ with x_1^ν/ν replaced by $y_1^\nu/\nu+y_1^{\nu-1}(x-x_1)$:

$$\hat{A}_0=(hD_1)^2+\bigl(hD_2-\mu\bigl(y_1^\nu/\nu+y_1^{\nu-1}(x-x_1)\bigr)\bigr)^2+V(0). \tag{14.4.23}$$

One can see easily that each successive term of this approximation is gaining factor $\mu\gamma^{\nu-2}T^3/h\asymp\mu\gamma^{\nu-2}h^2|\log h|^3$. Further, contributions of the second terms to (14.4.15) will be equal 0 again after easy calculations. Finally, the remainder will be then less than

$$Ch^{-2}\bar{\gamma}_1^3\times(\mu h^2\bar{\gamma}_1^{\nu-2})^2|\log h|^K\asymp C\bar{\gamma}_1|\log h|^K\le 1$$

where the first factor is the estimate of the 3-rd term in the previous approximations and the second factor is what we gained now. So we can leave only the first terms in these last approximations.

Now when we sum all these terms with $n\ge 3$, we get $h^{-2}\bigl(\mathcal{N}^{\mathsf{MW}}-\mathcal{N}_0^{\mathsf{MW}}\bigr)$ (multiplied by $\psi(1-\psi_1)$ and integrated in the end of the day) where $h^{-2}\mathcal{N}_0^{\mathsf{MW}}$ is what we would get for the first term if we ran last round of successive approximations for it; and this is exactly $h^{-2}\mathcal{N}_0^{\mathsf{MW}}(x_1,x_2,\tau,\mu,h)$.

However since we did not do it we have the same expression but with a correction term

$$(2\pi h)^{-1}\int e_0(x_1/\bar{\gamma};\xi_2,\tau,\hbar)\,d\xi_2-h^{-2}\mathcal{N}_0^{\mathsf{MW}}(x_1,x_2,\tau,\mu,h). \tag{14.4.24}$$

(e) Consider now the second term given by (14.4.10) with u replaced by $\bar{u}_1$ defined by (14.4.12). This term does not exceed $Ch^{-1}\gamma|\log h|^2$ (with a superficial logarithmic factor). Again applying the successive approximations of (ii) one can see that replacing $\bar{G}^\varsigma$ by $\bar{G}_0^\varsigma$ brings an extra factor $\gamma|\log h|^K$ and therefore all the terms but the first one do not exceed $Ch^{-1}\bar{\gamma}_1^2|\log h|^K$ which is less than the right-hand expression of of (14.4.6) (unless in the framework of the special case). Similarly one can replace A_1 by

$$\bar{A}_1 = -2\mu(x_2-y_2)\beta(y_2)x_1^{\nu+1}\bigl(hD_2 - \mu x_1^\nu/\nu\bigr) - (x_2-y_2)w(y_2).$$

One can even consider the original second term; recall that it does not exceed $Ch^{-1}\gamma|\log h|$.

(f) Now there is a special case when one needs to consider extra terms due to (14.4.13), (14.4.14) and the second term in approximation of (iv); all these terms do not exceed $Ch^{-1}\bar{\gamma}_1^2|\log h|^K$ and one needs only to get rid of this logarithmic factor.

Replacing $\bar{G}^\zeta$, $\bar{G}_0^\zeta$ by $\hat{G}_0^\zeta$ linked to $\hat{A}_0$ defined by (14.4.23) we will get errors less than the right-hand expression of (14.4.6). On the other hand, calculations after this substitution show that these terms do not exceed $Ch^{-1}\bar{\gamma}_1^2$ (so there is no logarithmic factor) which even in the special case does not exceed $Ch^{-1}\bar{\gamma}$. Easy but tedious calculations we leave to the reader. □

Proposition 14.4.4. *(i) Let condition* (14.2.27) *be fulfilled and let* ψ_1 *be an even function,* $\psi = \psi(x_2)$. *Then as* $Ch^{-1} \le \mu \le h^{-\nu^2/(2\nu-1)}$

$$(14.4.25)\quad |(2\pi h)^{-1}\int e_0(x_1,x_1;x_2,\xi_2,0,\hbar)\bigl(1-\bar{\psi}_1(x_1)\bigr)\,dx_1d\xi_2 - h^{-2}\int \mathcal{N}_0^{\mathsf{MW}}(x_1,x_2,\xi_2,0,\hbar)\bigl(1-\psi_1(x_1)\bigr)\,dx_1| \le C\bar{\gamma}h^{-1}.$$

(ii) Therefore estimate (14.4.6) *holds for* $Ch^{-1} \le \mu \le h^{-\nu^2/(2\nu-1)}$ *as well.*

Proof. Statement (i) instantly follows from Proposition 14.4.2 for $A = \bar{A}_0$ and Statement (ii) follows from Propositions 14.4.2 and (i).

14.4.2 Degeneration Zone

Singularity at $t = 0$

Now we need to consider an degeneration zone $\mathcal{X}_{\mathsf{deg}} = \{|x_1| \le \bar\gamma\}$. In this subsubsection we consider Tauberian expression

$$h^{-1}\int_{-\infty}^{0} \Bigl(F_{t\to h^{-1}\tau}\bar\chi_T \Gamma\bigl(\psi\psi_1 u\bigr)\Bigr)\,d\tau \tag{14.4.26}$$

with $T = Ch|\log h|$ even if this formula does not necessarily represent $\Gamma(\psi\psi_1 e)|_{\tau=0}$ with a desired precision, leaving the necessary correction for the rest of the section. In the end of this subsection we will list cases (due to Section 14.3) when provides the desired precision.

Running an absolutely standard method of successive approximations (i.e. taking as unperturbed operator with $x = y$) one can easily recover Weyl asymptotics with $C\bar\gamma h^{-2}\times \hbar^2 = C\bar\gamma^{-1}$ error with $\hbar = h\bar\gamma^{-1}$ while the contribution of the whole degeneration zone to the asymptotics is of magnitude $h^{-2}\bar\gamma$ [14].

This $C\bar\gamma^{-1}$ error does not exceed $Ch^{-1}\bar\gamma$ as long as $\bar\gamma \ge \epsilon h^{\frac12}$ which means exactly that

$$\mu \le Ch^{-\nu/2}. \tag{14.4.27}$$

This is not extremely restrictive condition, much weaker than condition $\mu \le (h|\log h|)^{-\nu/3}$ (i. e. $\bar\gamma \ge (h|\log h|)^{\frac13}$) arising as $h^{-1}(\hbar|\log h|)^{\frac12} \le h^{-1}\bar\gamma$ to provide remainder estimate $Ch^{-1}\bar\gamma$ for the spectral asymptotics in the general case. Moreover, (14.4.27) is even weaker a bit than condition $\mu \le (h|\log h|)^{-\nu/2}$ (i.e. $\bar\gamma \ge (h|\log h|)^{\frac12}$) arising as $h^{-1}(\hbar|\log h|) \le h^{-1}\bar\gamma$ to provide remainder estimate $Ch^{-1}\bar\gamma$ for the spectral asymptotics under condition (14.3.109).

And in the degeneration zone we can write a standard Weyl approximation or magnetic Weyl approximation to our sole discretion[15].

Observe that

(14.4.28) Both Weyl and magnetic Weyl expressions are actually more precise in the degeneration zone than we have written before, giving an error not exceeding $Ch^{-2}\bar\gamma\times\hbar^4$.

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[14] It is easy to get rid of superficial logarithmic factors in this case.

[15] Which would not be the case in the outer zone where we must use magnetic Weyl at least as $\mu \ge h^{-1}$.

Really, when we write the complete decomposition in powers of μ and h fixing $\bar\gamma \asymp \bar\mu^{-1/\nu}$ but considering $\mu \le \bar\mu$ as variable, we realize that terms with $\mu^{2k}h^{-2+2k}$ all are in the magnetic Weyl approximation, terms without μ are produced by Weyl expression for non-singular operator and thus are estimated by $Ch^{-2+2k}\bar\gamma$ rather than by $Ch^{-2}\hbar^{2k}\bar\gamma$ and unaccounted terms contain factor $\mu^{2k}h^{2l}$ with $l \ge k \ge 1$ and thus an error term does not exceed $Ch^{-2}\hbar^4\bar\gamma = Ch^2\bar\gamma^{-3}$.

Note that $Ch^2\bar\gamma^{-3} \le Ch^{-1}\bar\gamma$ as $\bar\gamma \ge h^{3/4}$ or equivalently $\mu \le Ch^{-3\nu/4}$. Further, in virtue of the above arguments one can rewrite expression (14.4.26) as

$$\int \frac{T}{h}\hat{\bar\chi}\bigl(-\frac{T}{h}\tau\bigr)\int h^{-2}\mathcal{N}^{\mathsf{MW}}(x,0)\psi_1(x)\psi(x)\,dxd\tau$$

which can be rewritten as expression (14.4.29) below with $O(1)$ error. Therefore we arrive to

Proposition 14.4.5 [16]**.** *With an error $Ch^2\bar\gamma^{-3}$ one can rewrite Tauberian expression* (14.4.26) *with $T = Ch|\log h|$ as*

$$h^{-2}\int \mathcal{N}^{\mathsf{MW}}(x,0)\psi_1(x)\psi(x)\,dx. \tag{14.4.29}$$

Therefore due to Subsection 14.3 we arrive to

Corollary 14.4.6. *(i) In the general case the contribution of the degeneration zone $\mathcal{X}_{\mathsf{deg}}$ to the remainder with the principal part* (14.4.29) *does not exceed*

$$Ch^{-1}\bar\gamma + Ch^{-1}(h|\log h|/\bar\gamma)^{\frac12}; \tag{14.4.30}$$

in particular as $\mu \le (h|\log h|)^{-\nu/3}$ the remainder estimate is $O(h^{-1}\bar\gamma)$.

(ii) Under conditions (14.2.27) *and* (14.3.109) *the contribution of the degeneration zone to the remainder with the principal part* (14.4.29) *does not exceed*

$$Ch^{-1}\bar\gamma + C|\log h|\bar\gamma^{-1}; \tag{14.4.31}$$

in particular as $\mu \le (h|\log h|)^{-\nu/2}$ this remainder estimate is $O(h^{-1}\bar\gamma)$.

[16] Cf. Proposition 14.4.2.

(iii) Under conditions (14.2.27) *and* (14.3.103) *with*

$$\zeta \ge Ch|\log h|/\bar\gamma + C\bar\gamma \tag{14.4.32}$$

the contribution of the degeneration zone to the remainder with the principal part (14.4.29) *does not exceed*

$$Ch^{-1}\bar\gamma + Ch^{2}\bar\gamma^{-3}; \tag{14.4.33}$$

in particular as $\mu \le h^{-3/4\nu}$ *this remainder estimate is* $Ch^{-1}\bar\gamma$.

Then combining with the result of the regular zone analysis we arrive to the following theorem:

Theorem 14.4.7. *Let condition* (14.3.9) *be fulfilled. Then*

(i) In the general case R^{MW} *defined by* (14.1.9) *does not exceed* (14.4.30)*; in particular,* $\mathsf{R}^{\mathsf{MW}} = O(Ch^{-1}\bar\gamma)$ *as* $\mu \le C(h|\log h|)^{-\nu/3}$.

(ii) Under conditions (14.2.27) *and* (14.3.109) R^{MW} *does not exceed* (14.4.31)*; in particular,* $\mathsf{R}^{\mathsf{MW}} = O(Ch^{-1}\bar\gamma)$ *as* $\mu \le C(h|\log h|)^{-\nu/2}$.

(iii) Under conditions (14.2.27) *and* (14.3.103) *with* ζ *satisfying* (14.4.32) R^{MW} *does not exceed* (14.4.33)*; in particular,* $\mathsf{R}^{\mathsf{MW}} = O(Ch^{-1}\bar\gamma)$ *as* $\mu \le Ch^{-\nu^2/(2\nu-1)}$.

On the other hand, repeating (almost all) arguments of the proof of Proposition 14.4.3 one can prove easily

Proposition 14.4.8. *In the framework of Proposition 14.4.3*

$$\begin{aligned} |h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\bar\chi_\tau(t)\Gamma\big(\psi\psi_1 u\big)\Big)\,d\tau - \\ h^{-2}\int\Big(\mathcal{N}^{\mathsf{MW}}(x,0)-\mathcal{N}_0^{\mathsf{MW}}(x,0)\Big)\psi(x_2)\psi_1(x_1)\,dx - \\ (2\pi h)^{-1}\int \frac{T}{h}\hat{\bar\chi}\big(-\frac{T}{h}\tau\big)\big)\int e_0(x_1,x_1;x_2,\xi_2,\tau,\hbar)\psi(x_2)\bar\psi(x_1)\,dx d\xi_2\tau| \le \\ Ch^{-1}\bar\gamma + C|\log h|^{K} \end{aligned} \tag{14.4.34}$$

Combining with the results of the previous subsection we arrive to

Theorem 14.4.9. *Let conditions* (14.2.27) *and* (14.3.103) *be fulfilled with* ζ *satisfying* (14.4.32) *and let* $\psi = \psi(x_2)$.

Then for $h^{-\nu^2/(2\nu-1)} \le \mu \le h^{-\nu}|\log h|^{-K}$

$$|\int \Big(e(x,x,0) - h^{-2}\mathcal{N}^{\mathsf{MW}}(x,0)\Big)\psi(x)\,dx - h^{-1}\int \mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x_2,0)\psi(x_2)\,dx_2| \le Ch^{-1}\bar\gamma + C|\log h|^K \tag{14.4.35}$$

with correction term

$$h^{-1}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}} := (2\pi h)^{-1}\int e_0(x_1,x_1;x_2,\xi_2,\tau,\hbar)\,dx_1 d\xi_2 - h^{-2}\int \mathcal{N}^{\mathsf{MW}}_0(x,\tau)\,dx_1. \tag{14.4.36}$$

Extending Time Interval

In Section 14.3 we derived asymptotics with the principal part which is the sum of

$$\sum_m h^{-1}\int_{-\infty}^0 \Big(F_{t\to h^{-1}\tau}\bar\chi_{T_m}(t)\Gamma(\psi_1\psi Q_m u)\Big)\,d\tau \tag{14.4.37}$$

where Q_m are elements of partition (in hD_2 and may be x_2) and $T_m = Ch|\log h|/\varrho_m^2$. We can rewrite (14.4.37) as the sum of (14.4.26) with $T = \bar T = \epsilon\bar\gamma$ and

$$\sum_m h^{-1}\int_{-\infty}^0 \Big(F_{t\to h^{-1}\tau}\big(\bar\chi_{T_m}(t) - \bar\chi_{\bar T}(t)\big)\Gamma(\psi_1\psi Q_m u)\Big)\,d\tau. \tag{14.4.38}$$

Consider now expression (14.4.26) with $T = \epsilon\bar\gamma$. Also due to propagation results of Section 14.3 one can replace there $T = \epsilon\bar\gamma$ by $T = Ch|\log h|$; then all the results of the previous subsection are applicable. It includes (14.4.34) as well where one can replace back $T = Ch|\log h|$ by $T = \epsilon\bar\gamma$ in the first and/or the last term of the left-hand expression.

Moreover, again due to propagation results of Section 14.3, all terms in (14.4.38) with $\varrho_m \ge C\bar\rho_1$ are negligible and therefore one can rewrite (modulo negligible) expression (14.4.37) as the sum of expressions (14.4.26) with $T = Ch|\log h|$ and (14.4.38) with $\bar T = \epsilon\bar\gamma$ with summation with respect to m such that $\varrho_m \le C\bar\rho_1$. Therefore we arrive to the following proposition:

Proposition 14.4.10. *(i) With an error not exceeding* $Ch^{-1}\bar{\gamma} + C\bar{\gamma}_1^{-1}$ *one can rewrite expression* (14.4.37) *as the sum of expressions* (14.4.29) *and* (14.4.38) *with* $\bar{T} = \epsilon\bar{\gamma}$ *and* Q_m *partition of unity in the periodic zone only.*

(ii) In particular, as $\mu \le Ch^{-\nu^2/(2\nu-1)}$ *this error does not exceed* $Ch^{-1}\bar{\gamma}$.

As $\mu \ge Ch^{-\nu^2/(2\nu-1)}$ we need a bit more subtle analysis:

Proposition 14.4.11. *(i) With an error not exceeding* $C\bar{\gamma}h^{-1} + C|\log h|^K$ *one can rewrite expression* (14.4.37) *as the sum of the following three expressions:*

$$(14.4.39)\quad h^{-2}\int \Big(\mathcal{N}^{\mathsf{MW}}(x,0) - \mathcal{N}_0^{\mathsf{MW}}(x,0)\Big)\psi(x_2)\psi_1(x_1)\,dx + (2\pi h)^{-1}\int e_0(x_1,x_1;x_2,\xi_2,0,\hbar)\psi(x_2)\bar{\psi}(x_1)\,dxd\xi_2$$

where the first term is actually less than the remainder estimate,

$$(14.4.40)\quad (2\pi h)^{-1}\int \Big(\frac{T}{h}\hat{\bar{\chi}}(-\frac{T}{h}\tau)) - \delta(\tau)\Big)\times \int e_0(x_1,x_1;x_2,\xi_2,\tau,\hbar)\psi(x_2)\bar{\psi}(x_1)\,dxd\xi_2\,d\tau$$

with $T = \bar{T} = \epsilon\bar{\gamma}$, *Dirac* δ*-function* <u>*and*</u> *the third expression is* (14.4.38) *where now* $\{Q_m\}$ *is a partition of unity in the periodic zone only.*

(ii) Further, one can rewrite (14.4.40) *as the sum*

$$(14.4.41)\quad \sum_m (2\pi h)^{-1}\int \Big(\frac{T}{h}\hat{\bar{\chi}}(-\frac{T}{h}\tau)) - \delta(\tau)\Big)\times \int e_0(x_1,x_1;x_2,\xi_2,\tau,\hbar)\psi(x_2)\bar{\psi}(x_1)Q_m(x_2,\xi_2)\,dxd\xi_2\,d\tau$$

where now $\{Q_m\}$ *is a partition of unity in the periodic zone only.*

Proof. Proposition immediately follows from Proposition 14.4.8 and propagation results of Section 14.3 with the only exception that in (14.4.41)

summation is taken over all m, and one needs to prove that the corresponding terms are negligible in the non-periodic zone where condition

$$\bigl(|\xi_2 - W^{\frac{1}{2}}\eta^*_\hbar| + |\partial_{x_2} W|\bigr) \asymp \rho \le C\bar{\rho}_1 \tag{14.4.42}$$

is violated.

Note that the expression in question is the sum of

$$(2\pi T)^{-1}\hat{\chi}\bigl(-\frac{h}{T}\tau\bigr)\int e_0(x_1,x_1;x_2,\xi_2,\tau,\hbar)\times \psi(x_2)\bar{\psi}(x_1)Q_m(x_2,\xi_2)\,dxd\xi_2\,d\tau = \int h^{-1}\int_{-\infty}^{0}\Bigl(F_{t\to h^{-1}\tau}\chi_T(t)\bigl(\Gamma'\psi_1(x_1)Q_m(y_2,hD_2)\bar{u}_0\bigr)\Bigr)\psi(y_2)\,dy_2d\tau \tag{14.4.43}$$

with T running from $\epsilon\bar{\gamma}$ to $+\infty$ and $\bar{u}_0$ defined for operator $\bar{A}_0$.

If $|\xi_2 - W^{\frac{1}{2}}\eta^*_\hbar| \ge \rho$ on partition element Q_m then expression (14.4.43) does not exceed $Ch^s(T+1)^{-s}$ as $T \ge C_0 h|\log h|/\rho^2$ just due to propagation results of Section 14.3 applied to $\bar{u}_0$ and $\bar{A}_0$ and since for $C_0h|\log h|/\rho^2 \le \epsilon\bar{\gamma}$ as $\rho \ge \bar{\rho}_1$, these elements are covered.

On the other hand, if $|W_{x_2}| \asymp \rho$ on the partition element in question, one can notice that $\bar{A}_0$ depends on y_2 via $-W(y_2)$ only and introducing new variable $y_2' = -W(y_2)$ we get mollification with respect to the spectral parameter and thus expression (14.4.43) does not exceed $Ch^s(T+1)^{-s}$ as well.

We leave easy details to the reader. □

14.4.3 Periodic and Near-Periodic Orbits

Toy-Model

Thus we need to analyze periodic zone $\mathcal{X}_{\mathsf{per}}$ more accurately defining it by (14.4.42) rather than by

$$\rho := |\xi_2 - W^{\frac{1}{2}}\eta^*_\hbar| \le C\bar{\rho}_1 \tag{14.4.44}$$

and only in the case when magnetic field is strong enough to prevent Corollary 14.4.6 from delivering a sharp remainder estimate.

Namely, we need to consider term

$$h^{-1}\int_{-\infty}^{0}\Bigl(F_{t\to h^{-1}\tau}\bigl(\bar{\chi}_T(t) - \bar{\chi}_{\bar{T}_*}(t)\bigr)\Gamma(\psi_1\psi Qu)\Bigr)\,d\tau \tag{14.4.45}$$

with partition element $Q = Q(x_2, hD_2)$, $T = T^*$ where T_* and T^* are defined in two last subsubsections of Section 14.3.4 and $\bar{T}_* = C_0 h|\log h|$. Here due to propagation results of these subsections we can take any $T \in [T_*, T^*]$ to our discretion, in particular $T = T_*$.

Let consider a toy-model first; more precisely, let us assume that operator under consideration coincides with a toy-model in $B(0,1)$. Then $T_* = C\rho^{-2}h|\log h|$ and $T^* = \epsilon\rho$ with $|\xi_2 - \eta^*_\hbar| \asymp \rho$ on $\operatorname{supp} Q$.

Then expression for correction (14.4.45) will be the same (modulo negligible) if one replaces u by $\bar{u}$ constructed for operator coinciding with toy-model in $\mathbb{R}^2$. But then one can replace $T = \epsilon\rho^{-1}$ by $T = \infty$. Also one can remove restriction from below on $\rho : \rho \ge \bar{\gamma}$.

Also in this case one can replace $\bar{\psi}_1$ by 1 since contribution to (14.4.45) of the regular zone is negligible due to propagation results there and contribution of the forbidden zone is negligible as well. Further, one does not need to integrate over $\psi(x_2)dx_2$ in (14.4.45) anymore.

After all these modifications there is no need to consider a partition and then correction (14.4.45) is transformed into

$$h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\big(1-\bar{\chi}_{\bar{T}_*}(t)\big)\Gamma'\bar{u}\Big)\,d\tau \tag{14.4.46}$$

where we recall that $\Gamma' v = \int v(x_1,x_2;x_1,x_2)\,dx_1$; in our case $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ does not depend on x_2.

Obviously

$$h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\Gamma'\bar{u}\Big)\,d\tau = (2\pi h)^{-1}\int \boldsymbol{n}_0(\xi_2,\hbar)\,d\xi_2 \tag{14.4.47}$$

where

(14.4.48) $\boldsymbol{n}_0(\xi_2,\hbar)$ is the number of negative eigenvalues of operator $\boldsymbol{a}_0(\xi_2,\hbar)$ with $W = 1$.

On the other hand, we know that modulo remainder estimate under weak magnetic field approach, i.e. $O\big(\bar{\gamma}h^{-1} + h^{-1}\hbar^{\frac{1}{2}}\big)$

$$h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\bar{\chi}_{\bar{T}_*}(t)\Gamma'\bar{u}\Big)\,d\tau \equiv h^{-2}\int \mathcal{N}_0^{\mathsf{MW}}(x,0)\,dx_1. \tag{14.4.49}$$

Really, it follows from our earlier arguments that contribution of zone $\gamma \le \bar{\gamma}$ and therefore $\mu_{\mathsf{eff}} h_{\mathsf{eff}} \le 1$, contribution of γ-admissible element to the error does not exceed $C\gamma^{-1} \times (\gamma/\bar{\gamma})^s$. Therefore

$$h^{-1}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}} \equiv (2\pi h)^{-1} \int \boldsymbol{n}_0(\xi_2, \hbar)\, d\xi_2 - h^{-2} \int \mathcal{N}^{\mathsf{MW}}_0(x, 0)\, dx_1 \tag{14.4.50}$$

also modulo $O(\bar{\gamma}_1^1)$.

The error in (14.4.50) does not exceed $Ch^{-1}\bar{\gamma}$ even if $\bar{\gamma}^{-1} \ge \bar{\gamma} h^{-1}$ because in this case we already made an error replacing the left-hand expression of (14.4.49) by its right-hand-expression in the analysis of Subsection 14.4.1 and now we just compensated it by adding the skipped term. One can consider (14.4.50) as a definition of the correction term for the model operator. Thus we arrive to the following proposition:

Proposition 14.4.12. *For an operator, coinciding with the toy-toy-model in* $(-1, 1) \times \mathbb{R}$*,*

$$\int e(x, x, 0)\, dx_1 \equiv h^{-2} \int \mathcal{N}^{\mathsf{MW}}_0\, dx_1 + h^{-1}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}} \mod O\bigl(\bar{\gamma}h^{-1} + h^{-\delta}\bigr) \tag{14.4.51}$$

where $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ *is defined by* (14.4.50).

Expression (14.4.50), multiplied by $\psi(x_2)dx_2$ and integrated will be used in two next subsubsections in the general case as well. However to get more explicit even if less precise expression one can replace (14.4.46) by

$$h^{-1} \int_{-\infty}^{0} \Bigl(F_{t\to h^{-1}\tau}\bigl(1 - \bar{\chi}_{\bar{T}_*}(t)\bigr)\Gamma'\phi(hD_2)\bar{u}\Bigr)\, d\tau \tag{14.4.52}$$

with ϕ supported in $[-2C_0, 2C_0]$ and equal 1 in $[-C_0, C_0]$, making the same error as in (14.4.51).

Using equalities

$$h^{-1} \int_{-\infty}^{0} \Bigl(F_{t\to h^{-1}\tau}\Gamma'\phi(hD_2)\bar{u}\Bigr)\, d\tau = (2\pi h)^{-1} \int \boldsymbol{n}_0(\xi_2, \hbar)\phi(\xi_2)\, d\xi_2 \tag{14.4.53}$$

and

$$h^{-1} \int_{-\infty}^{0} \Bigl(F_{t\to h^{-1}\tau}\bar{\chi}_{\bar{T}_*}\Gamma'\phi(hD_2)\bar{u}\Bigr)\, d\tau \equiv (2\pi h)^{-1} \int n_0^{\mathsf{W}}(\xi_2, \hbar)\phi(\xi_2)\, d\xi_2 \tag{14.4.54}$$

with the second equality modulo $O(h^{-1}\hbar^2) = O(h\bar{\gamma}^{-2})$ with

$$n_0^{\mathsf{W}}(\xi_2,\hbar) = (\pi\hbar)^{-1}\int \Big(1 - \big(\xi_2 - x_1^{\nu}/\nu\big)^2\Big)_+^{\frac{1}{2}}\, dx_1 \tag{14.4.55}$$

the Weyl approximation of $\boldsymbol{n}_0(\xi_2,\hbar)$, we arrive to

Proposition 14.4.13. *For an operator, coinciding with toy-model in* $(-1,1)\times\mathbb{R}$

$$h^{-2}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}} \equiv (2\pi h)^{-1}\int \Big(\boldsymbol{n}_0(\xi_2,\hbar) - n_0^{\mathsf{W}}(\xi_2,\hbar)\Big)\phi(\xi_2)\, d\xi_2 \qquad \mathsf{mod}\ O\big(h\bar{\gamma}^{-2}\big). \tag{14.4.56}$$

To calculate expression (14.4.56) more explicitly we need to calculate eigenvalues $\lambda_n(\xi_2,\hbar)$ of operator $\boldsymbol{a}_0(\xi_2,\hbar)$ and we are interested in those eigenvalues which are close to 0 as ξ_2 is close to $\eta^*_\hbar$.

These eigenvalues are defined modulo $O(\hbar^2)$ from *Bohr-Sommerfeld condition*

$$\frac{1}{2\pi\hbar}S(\xi_2,\tau) + \frac{1}{4}\iota_M\big(\mathcal{L}(\xi_2,\tau)\big) \in \mathbb{Z} \qquad \text{as}\ \ \tau = \lambda_n \tag{14.4.57}$$

where

$$S(\xi_2,\tau) = \oint_{\mathcal{L}(\xi_2,\tau)} \xi_1\, dx_1 \tag{14.4.58}$$

and $\iota_M\big(\mathcal{L}(\xi_2,\tau)\big)$ is Maslov' index of the closed trajectory $\mathcal{L}(\xi_2,\tau)$ (on 2-dimensional phase plane) on the energy level τ; $\iota_M\big(\mathcal{L}(\xi_2,\tau)\big) = 2$ as $\tau \approx 0$.

Since $\partial_\tau S(\xi_2,\tau) = T(\xi_2,\tau)$ due to Hamiltonian mechanics, the spacing between two consecutive eigenvalues is $2\pi\hbar/T(\xi_2,\tau) + O(\hbar^2)$. Further, all these eigenvalues $\lambda_n(\xi_2,\hbar)$ are uniformly analytic functions of ξ_2, $|\xi_2| \ll 1$, $\hbar \ll 1$ as $|\lambda_n| \ll 1$ and also $\partial^2_{\xi_2}\lambda_n \ge \epsilon_0$. This yields immediately

Proposition 14.4.14. *As* $\mu \le \epsilon(h|\log h|)^{-\nu}$

(i) $|\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}| \le C\hbar^{\frac{1}{2}}$*; in particular,* $|\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}| \le C\bar{\gamma}$ *as* $\mu \le Ch^{-\nu/3}$.

(ii) One can calculate $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ *with an error not exceeding* $C\hbar^{(k+1)/2}$ *replacing* $\lambda_n(\xi_2,\hbar)$ *by its approximation modulo* $O\big(\rho^{2k+2}+\hbar^{k+1}\big)$ *with* $k = 1,2,\dots$ *where here and below* $\rho = \xi_2 - \eta^*_\hbar$.

(iii) In particular, one can calculate $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ *with an error not exceeding* $C\hbar$ *(which does not exceed* $C\bar{\gamma}$ *as* $\mu \le Ch^{-\nu/2}$*) replacing* $\lambda_n(\xi_2,\hbar)$ *by its approximation modulo* $O\big(\rho^4+\hbar^2\big)$.

(iv) And one can calculate $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ *with an error not exceeding* $C\hbar^{3/2}$ *(which does not exceed* $C\big(\bar{\gamma}+h\bar{\gamma}_1^{-1}\big)$ *as* $\nu\ge 3$*) replacing* $\lambda_n(\xi_2,\hbar)$ *by its approximation modulo* $O\big(\rho^6+\hbar^3\big)$.

(v) And as $\nu=2$ *one can calculate* $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ *with an error not exceeding* $C\hbar^2$ *(which does not exceed* $Ch\bar{\gamma}_1^{-1}$*)) replacing* $\lambda_n(\xi_2,\hbar)$ *by its approximation modulo* $O\big(\rho^8+\hbar^8\big)$.

To exploit Proposition 14.4.14(iii)one can replace $\lambda_n(\xi_2,\hbar)$ by its approximate value

$$\lambda_n(\xi_2,\hbar)\equiv \bar{\lambda}(\xi_2)+\frac{(2n+1)\pi\hbar}{T(\xi_2)}. \tag{14.4.59}$$

Here $\bar{\lambda}(\xi_2)$ is a "classical eigenvalue" corresponding to $n=0$ and Maslov index 0 (instead of 2) and one can find it from (14.4.57) as $\tau=0$; then

$$\bar{\lambda}(\xi_2)=-S(\xi_2,\tau)/\big(\partial_\tau S(\xi_2,\tau)\big)\big|_{\tau=0}. \tag{14.4.60}$$

Recall that

$$\big(\partial_\tau S\big)\big|_{\tau=0}=T(\xi_2),\qquad \partial_{\xi_2}S|_{\tau=0}=I(\xi_2) \tag{14.4.61}$$

with $I(\xi_2)$ defined by (14.2.5) for even ν and similar formula for odd ν; further,

$$I(\eta^*)=0,\qquad \partial_{\xi_2}I(\eta^*)=\kappa>0. \tag{14.4.62}$$

Then

$$\begin{aligned}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}\equiv (2\pi)^{-1}\int\Big(\frac{1}{2\pi\hbar}S(\xi_2)-\Big\lfloor\frac{1}{2\pi\hbar}S(\xi_2)+\frac{1}{2}\Big\rfloor\Big)\,d\xi_2\equiv\\ (2\pi)^{-1}\int\Big(\frac{1}{2\pi\hbar}\big(S_0+\frac{1}{2}\kappa\eta^2\big)-\Big\lfloor\frac{1}{2\pi\hbar}\big(S_0+\frac{1}{2}\kappa\eta^2\big)+\frac{1}{2}\Big\rfloor\Big)\,d\eta=\\ (2\pi)^{-\frac{3}{2}}\hbar^{\frac{1}{2}}\kappa^{-\frac{1}{2}}G\big(\frac{S_0}{2\pi\hbar}\big)\qquad \mathsf{mod}\ O(\hbar)\end{aligned} \tag{14.4.63}$$

with function $G(t)$ defined by

$$G(t) \coloneqq \int_{\mathbb{R}} \Big(t + \frac{1}{2}\eta^2 - \big\lfloor t + \frac{1}{2}\eta^2 + \frac{1}{2}\big\rfloor\Big)\, d\eta \tag{14.4.64}$$

with the converging integral in the right-hand expression.

One can prove easily that

$$G \not\equiv 0, \quad G(t+1) = G(t), \qquad \int_0^1 G(t)\, dt = 0, \qquad G \in \mathscr{C}^{\frac{1}{2}}. \tag{14.4.65}$$

To exploit Proposition 14.4.14(ii) with $k \ge 2$ one can use more precise version of (14.4.59)

$$\lambda_n(\xi_2, \hbar) \equiv \bar{\lambda}(\xi_2) + \frac{(2n+1)\pi\hbar}{T(\xi_2)} + \sum_{1\le j\le k} \varkappa_j(\xi_2)\hbar^{j+1}. \tag{14.4.66}$$

Then modulo $O(\hbar(k+1)/2)$ (14.4.63) holds albeit with $S_0 + \frac{1}{2}\kappa\eta^2$ corrected by $\sum_{j,l} S_{j,l}\hbar^j\eta^l$ with $2j + l \ge 3$. Changing η we arrive to the middle line of (14.4.63) modified:

$$(2\pi)^{-1} \int \Big(\frac{1}{2\pi\hbar}\big(S(\hbar) + \frac{1}{2}\kappa\eta^2\big) - \Big\lfloor \frac{1}{2\pi\hbar}\big(S(\hbar) + \frac{1}{2}\kappa\eta^2\big) + \frac{1}{2}\Big\rfloor\Big)\times \Big(1 + \sum_{j,l} \kappa'_{jl}\,\hbar^j\eta^l\Big)\, d\eta \tag{14.4.67}$$

with

$$S(\hbar) \equiv S_0 + S_1\hbar + S_2\hbar^2 + \dots \tag{14.4.68}$$

and with some constants S_j, κ'_{jm} (and κ_{jm} below); obviously odd powers of η in (14.4.67) result in 0; plugging $l = 2m$ and integrating by parts m times we arrive to

$$\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}} \sim (2\pi)^{-\frac{3}{2}}\hbar^{\frac{1}{2}}\kappa^{-\frac{1}{2}}\Big(G\big(S(\hbar)\big) + \sum_{j,m} \kappa_{jm}\,\hbar^{j+m} G_m\big(S(\hbar)\big)\Big) \tag{14.4.63)*}$$

where

(14.4.69) $G_m \in \mathscr{C}^{m+\frac{1}{2}}$ satisfy (14.4.65), $\partial_t G_m(t) = G_{m-1}(t)$, $G_0(t) = G(t)$ and therefore

$$G_m(t) = \int_0^t G_{m-1}(t')\, dt' - \int_0^1 (1-t')G_{m-1}(t')\, dt'. \tag{14.4.70}$$

Thus we arrive to

Proposition 14.4.15. *For operator coinciding in $B(0,1)$ with the toy-model with potential $W=1$ and fixed ψ, ψ_1 ($\psi_1=1$ as $|x_1|\le\epsilon$) in asymptotics* (14.4.55)

(i) Modulo $O\big(h^{-1}\hbar^{1/2}\big)$ one can skip correction term $h^{-1}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$.

(ii) In particular, uncorrected asymptotics has sharp remainder estimate $O(\bar\gamma h^{-1})$ if and only if $\hbar^{\frac12}\le C\bar\gamma$ i.e. $\mu\le Ch^{-\nu/3}$.

(iii) Further, one can define $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ by (14.4.63)–(14.4.64) *without spoiling remainder estimate $O(\bar\gamma h^{-1})$ as $Ch^{-\nu/3}\le\mu\le Ch^{-\nu/2}$*

(iv) Furthermore, one can define $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ by (14.4.63)*, (14.4.68)–(14.4.70) *without spoiling remainder estimate $O(\bar\gamma h^{-1})$ as $Ch^{-\nu/3}\le\mu\le Ch^{\delta-\nu}$ with an arbitrarily small exponent $\delta>0$ (with the number of terms in* (14.4.63)* *and* (14.4.68) *defined by δ).*

Remark 14.4.16. (i) One can then generalize Proposition 14.4.15 immediately to a toy-model with potential $W=\mathsf{const}$ instead of $W=1$ by replacing $h\mapsto hW^{-1/2}$, $\mu\mapsto\mu W^{-1/2}$ and therefore $\hbar\mapsto\hbar W^{-(\nu+1)/2\nu}$ just modifying (14.4.63) to

$$\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}\equiv(2\pi)^{-\frac32}\hbar^{\frac12}\kappa^{-\frac12}W^{(\nu-1)/(4\nu)}G\Big(\frac{S_0W^{(\nu+1)/(2\nu)}}{2\pi\hbar}\Big) \tag{14.4.71}$$

and similarly modifying (14.4.63)*.

(ii) One should expect the similar correction for the general operators as well but one can see easily that under condition $|\partial_{x_2}W|\ge\epsilon$ integrated correction term

$$h^{-1}\int\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x)\psi\,dx_2 \tag{14.4.72}$$

is negligible, and under condition (14.3.109) it should not exceed $Ch^{-1}\hbar$ (and be of this magnitude if there are critical points of W) with integrated versions of (14.4.63) valid modulo $O(h^{-1}\hbar^{\frac32})$.

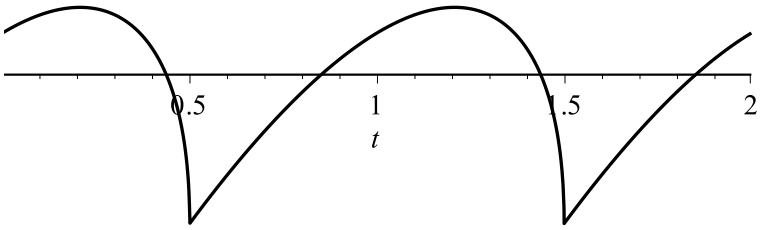

Figure 14.7: Plot of $G(t)$.

Generic Settings

Now let us consider degeneration zone in the framework of two last subsubsections of Subsection 14.3.4 when we managed to prove sharp remainder estimate $O(\bar{\gamma}h^{-1})$ as $\mu \le h^{\delta-\nu}$ producing the final result as the sum of expressions (14.4.37) taken over partitions $\psi = \psi_m(x_2)$ and $Q = Q_m(hD_2)$ of unity. Recall that ψ_m is ℓ-admissible and Q_m is ρ-admissible with ℓ, ρ satisfying some conditions the most important of which are (14.3.74)–(14.3.76); here ℓ denotes the scaling function associated with the finest of subpartitions and we assume that on this element

$$|\partial_{x_2} W| \le C_0\zeta, \qquad |\xi_2 - \eta^*_\hbar W^{\frac{1}{2}}| \le C_0\rho \tag{14.4.73}$$

with one of these inequalities being reversible (with ϵ_0 instead of C_0); then one can take in (14.4.37) any T ranging from

$$T_* = Ch|\log h| \min\bigl(\frac{1}{\rho^2}, \frac{1}{\ell\zeta}\bigr) = C\epsilon^{-1}h|\log h| \times T_1 \tag{14.4.74}$$

to T^*.

There are also exceptional elements with $T = \bar{T}_* = Ch|\log h|$ but they will be treated easily in the same manner.

Therefore we need to consider a main part–expression (14.4.37) with $T = \bar{T}_* = Ch|\log h|$ and also correction terms (14.4.45) with arbitrarily $T \in [T_*, T^*]$ depending on the partition element; index m indicating partition element we are skipping.

One can replace then $\bigl(\bar{\chi}_T(t) - \bar{\chi}_{\bar{T}}(t)\bigr)$ by the sum of $\chi_T(t)$ with T running from $\bar{T}_*$ to T_* and with χ supported in $(-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1)$ thus

replacing expression (14.4.45) with the sum of terms

$$h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\chi_T(t)\Gamma(\psi_1\psi Qu)\Big)\,d\tau = \tag{14.4.75}$$
$$iT^{-1}F_{t\to h^{-1}\tau}\check{\chi}_T(t)\Gamma(\psi_1\psi Qu)$$

with $\check{\chi}(t)=t^{-1}\chi(t)$ and $T\in[\bar{T},T_0]$.

Further, one needs to consider only T ranging from $\bar{T}'=\epsilon\bar{\gamma}$ to T_* because terms (14.4.75) with T ranging from $\bar{T}$ to $\bar{T}'$ are negligible; therefore partition elements with $T_*\le\bar{T}'$ [17] will be eliminated.

Remark 14.4.17. (i) Obviously expression (14.4.75) does not exceed $Ch^{-2}\bar{\gamma}\rho\ell$ and thus expression (14.4.45) does not exceed $Ch^{-2}\bar{\gamma}\rho\ell|\log h|$.

(ii) Further, one can decompose expression (14.4.75) in the same way as in the proof of Proposition 14.3.32. Therefore this expression does not exceed $Ch^{-2}\bar{\gamma}\rho\ell\times h|\log h|/\bar{\gamma}=Ch^{-1}\rho\ell\times|\log h|$ and thus expression (14.4.45) does not exceed $Ch^{-1}\rho\ell|\log h|^2$.

Even if this is the remainder estimate obtained by taking $T=\epsilon\bar{\gamma}$ in the Tauberian arguments (with a superficial logarithmic factor), it provides a solid ground for the further estimates.

Let us consider regular elements and apply successive approximation method freezing in unperturbed operator $x_2=y_2$ as usual. However to estimate approximation terms we will not commute (x_2-y_2) with $G^{\pm},\bar{G}^{\pm}$ but simply remember that $|x_2-y_2|\le C\rho T_*+C\bar{\gamma}$ on the partition element in question in propagation as $T\le T_*$; this is due to Proposition 14.3.26 extended according to Proposition 14.3.35.

In this subsection we consider the case when condition (14.3.109) is fulfilled leaving the most general case for the next one.

Then $\rho=\zeta=\ell$ and $T_*=C\rho^{-2}h|\log h|$ and hence $|x_2-y_2|\le C\rho T_*+\bar{\gamma}$. Therefore the norm of perturbation $R_1=(x_2-y_2)\partial_{x_2}A+O\big(|x_2-y_2|^2\big)$ does not exceed

$$C\zeta(\rho T_*+\bar{\gamma})+C(\rho T_*+\bar{\gamma})^2\asymp r:=\big(h|\log h|+\rho\bar{\gamma}+\rho^{-2}h^2|\log h|^2+\bar{\gamma}^2\big).$$

[17] Or equivalently with $(\zeta\ell+\rho^2)\bar{\gamma}\ge Ch|\log h|$.

Therefore while the first term of what is obtained when one plugs approximations into (14.4.75) does not exceed $Ch^{-1}\rho\ell|\log h| = Ch^{-1}\rho^2|\log h|$, each next term acquires a factor

$$r \times Th^{-1} \asymp \bigl(h|\log h| + \rho\bar{\gamma} + \rho^{-2}h^2|\log h|^2 + \bar{\gamma}^2\bigr) \times \rho^{-2}|\log h|; \tag{14.4.76}$$

to keep this factor less than 1 one needs to consider

$$\rho \ge \rho_0^* := Ch^{\frac{1}{2}}|\log h|. \tag{14.4.77}$$

Without any loss of the generality one can assume that $\rho_0^* \ge C\bar{\gamma}$ i.e.

$$\mu \ge h^{-\nu/2}|\log h|^{-\nu} \tag{14.4.78}$$

because under condition (14.3.109) and $\mu \le h^{-\nu/2}|\log h|^{-\nu}$ remainder estimate $O\bigl(h^{-1}\bar{\gamma}\bigr)$ is already proven without any correction terms.

Under conditions (14.4.77) and (14.4.78) expression (14.4.76) is of magnitude of $\bigl(\rho^{-2}h|\log h|^2 + \rho^{-1}\bar{\gamma}|\log h|\bigr)$.

Now the second term of approximations does not exceed

$$Ch^{-1}\rho^2|\log h| \times \bigl(\rho^{-2}h|\log h|^2 + \rho^{-1}\bar{\gamma}|\log h|\bigr) \asymp |\log h|^3 + Ch^{-1}\rho\bar{\gamma}|\log h|.$$

Finally, summation with respect to all partition results in $C|\log h|^K + Ch^{-1}\bar{\gamma}\bar{\rho}_1|\log h|$ which does not exceed $Ch^{-1}\bar{\gamma}$ under condition (14.3.120).

On the other hand, contribution of the *periodic core* $\{\rho \le \rho_0^*\}$ to the Tauberian remainder due to the Tauberian arguments with $T = \epsilon\bar{\gamma}$ does not exceed $Ch^{-1}\rho_0^{*\,2} \le C|\log h|^K$.

So, one needs to consider only the first term of the approximation:

(14.4.79) Under conditions (14.3.109), (14.4.78) and $\mu \le h^{-\nu}|\log h|^{-K}$ remainder estimate $Ch^{-1}\bar{\gamma}$ still holds with u replaced in the principal part (14.4.37) by the first term of this successive approximation procedure.

Thus we can take $T_* = \infty$ thus arriving to

$$\begin{aligned} h^{-1}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr},Q} = \int \Bigl(& \bigl((2\pi h)^{-1} e(x_1, x_1, 0; x_2, \xi_2, \hbar, \mu) - \\ & h^{-1}\int_{-\infty}^{0} F_{t\to h^{-1}\tau}\bar{\chi}\bar{\tau}(t)\Gamma_x(u)\,d\tau\Bigr)\psi(x_2)\bar{\psi}_1(x)\varphi(\xi_2)\,dx_1dx_2d\xi_2 \end{aligned} \tag{14.4.80}$$

where $\boldsymbol{e} = \boldsymbol{e}(x_1, y_1, \tau; x_2, \xi_2, \hbar, \mu)$ is a Schwartz kernel of the spectral projector, associated with one-dimensional Schrödinger operator $\boldsymbol{a} = \boldsymbol{a}(x_2, \xi_2, \hbar, \mu)$ which is obtained from A by replacement $hD_2 \mapsto \xi_2$ and subsequent change of variables $x_1 \mapsto x_1/\bar{\gamma}$ but not setting $\sigma = \phi = 1$, $V = -W$; finally $\boldsymbol{u} = \boldsymbol{u}(x_1, y_1, t; x_2, \xi_2, \hbar, \mu)$ is a corresponding propagator, i.e. Schwartz kernel of $e^{i\hbar^{-1}t\boldsymbol{a}}$. Here and below $Q = \psi(x_2)\varphi(\xi_2)$ denotes partition element.

However after summation over x_1-partition remembering that as $|x_1| \asymp C$ on $\mathsf{supp}\,\bar{\psi}_1$ we get negligible terms, we can replace $\bar{\psi}_1$ by 1 thus arriving to

$$h^{-1}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr},Q} = \int \Big((2\pi h)^{-1}\boldsymbol{n}(0; x_2, \xi_2, \hbar, \mu) - h^{-1}\int_{-\infty}^{0} F_{t\to h^{-1}\tau}\bar{\chi}_{\bar{T}}(t)\Gamma'(\boldsymbol{u})\,d\tau\Big)\psi(x_2)\varphi(\xi_2)\,dx_2 d\xi_2 \tag{14.4.81}$$

with $\boldsymbol{n}$ eigenvalue counting function associated with the same operator $\boldsymbol{a}$; recall that Γ' includes integration with respect to x_1.

Without the last term formula (14.4.80) delivers the correct principal part of asymptotics which after summation over all partitions (including non-periodic and outer zones) is

$$(2\pi h)^{-1}\int \boldsymbol{e}(x_1, x_1, 0; x_2, \xi_2, \hbar, \mu)\psi(x_2)\psi_1(x_1)\,dx_1 dx_2 d\xi_2 \tag{14.4.82}$$

which as we show in the next subsubsection is a correct answer without condition (14.3.109) as well.

However, we would like to replace the reference to operator $\boldsymbol{a}$ by the reference to operator $\boldsymbol{a}_0$.

Recall that

$$h^{-1}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr},Q} = h^{-1}\int \Big(\int_{-\infty}^{0} F_{t\to h^{-1}\tau}\big(\bar{\chi}_{T}(t) - \bar{\chi}_{\bar{T}}(t)\big)\Gamma'(\bar{U})\,d\tau\Big)\varphi(\xi_2)\psi(x_2)\,dx_2 d\xi_2 \tag{14.4.83}$$

and one can rewrite it as the sum of partition with respect to t:

$$T^{-1}\int \Big(\int_{-\infty}^{0} F_{t\to h^{-1}\tau}\check{\chi}(t)\Gamma'(\bar{U})\,d\tau\Big)\varphi(\xi_2)\psi(x_2)\,dx_2 d\xi_2. \tag{14.4.84}$$

Consider operator $\boldsymbol{a}$ as a perturbation of $\boldsymbol{a}_0$ and applying method of successive approximations; recall that contribution of ρ-element to the correction

term is of magnitude $h^{-1}\rho^2$ under condition (14.3.109) and we need to consider only elements with $\rho \le \hbar^{\frac12}|\log h|^{\frac12}$ and we want a remainder $O(\bar\gamma h^{-1})$ so we need to consider $\bar\gamma^{1/2} \le \rho \le \hbar^{\frac12}|\log h|^{\frac12}$ and only as $\bar\gamma \le h^{\frac12}|\log h|^{\frac12}$.

Then each successive approximation term acquires factor $CT_* h^{-1}\sigma = C\sigma\rho^{-2}$ where σ is the bound of $(\boldsymbol{a}-\boldsymbol{a}_0)$ and so the next term does not exceed $Ch^{-1}\rho^2 \times \sigma\rho^{-2} = Ch^{-1}\sigma$ and summation over partition results in estimate $O(h^{-1}\sigma|\log h|)$. To achieve $O(h^{-1}\bar\gamma)$ we need to have $\sigma \le C\bar\gamma|\log h|^{-1}$.

While it is not generally the case we obviously can get rid of all the terms except containing just one extra factor x_1 and now we need to think about the latter. Further, next terms of successive approximations acquire factor $\bar\gamma\rho^{-2}$ again and the third term does not exceed $Ch^{-1}\bar\gamma^2\rho^{-2}$ and summation over partition as $\{\rho \ge \bar\gamma^{\frac12}$ results in $O(\bar\gamma h^{-1})$ so only the second term needs to be calculated.

It will contain however $\mathsf{Res}_{\mathbb{R}}\, \mathsf{Tr}\big((\boldsymbol{a}_0-\tau)^{-1}x_1(\boldsymbol{a}_0-\tau)^{-1}\big)$ and as operator inside is odd with respect to $x_1 \mapsto -x_1$ as ν is even this trace will be 0. On the other hand, as ν is odd the operator inside is odd with respect to $(x_1,\xi_2) \mapsto (-x_1,-\xi_2)$ transformations so trace will be odd with respect to ξ_2 and if symbol of $Q = Q(x_2, hD_2)$ is even with respect to ξ_2 the integral also will be 0.

Thus the second term in successive approximations is simply 0. Therefore we have proven the following proposition:

Proposition 14.4.18. *Let condition* (14.3.109) *be fulfilled. Then as* $h^{-\nu/2} \le \mu \le h^{-\nu}|\log h|^{-K}$

(i) With an error not exceeding $C\rho\bar\gamma h^{-1}$ *one can replace in the right-hand expressions of* (14.4.83) $\bar U$ *by* $\bar U_0$ *associated with operator* $\boldsymbol{a}_0(x_2,\xi_2,\hbar)$*; therefore in* (14.4.81) *one can replace* $\boldsymbol{n}$ *by* $\boldsymbol{n}_0$ *and simultaneously* $\bar U$ *by* $\bar U_0$*.*

(ii) $|h^{-1}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}| \le C\bar\gamma^{-1}$ *and this estimate cannot be improved unless* $W(x_2)$ *has no critical points at all. In particular, as* $\mu \le Ch^{-\nu/2}$ *and only then one can skip* $h^{-1}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ *without deteriorating remainder estimate* $C\bar\gamma h^{-1}$*.*

Therefore we under non-degeneracy condition we arrive to

Proposition 14.4.19. *Let non-degeneracy condition* (14.3.109) *be fulfilled. Then as* $h^{-\nu/2} \le \mu \le h^{-\nu}|\log h|^{-K}$

(i) Asymptotics

$$\int e(x,x,0)\psi(x)\,dx \equiv h^{-2}\int \mathcal{N}^{\mathsf{MW}}(x)\psi(x)\,dx + h^{-1}\int \mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x_2)\psi(x_2)\,dx_2 \quad \mathsf{mod}\ O\bigl(h^{-1}\bar{\gamma}\bigr) \tag{14.4.85}$$

holds with $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ *defined by* (14.4.50).

(ii) Moreover, with $O\bigl(h^{-1}\hbar^{3/2}\bigr)$*-error one can replace* $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ *by expression* (14.4.71) *and with* $O\bigl(h^{-1}\hbar^{K}\bigr)$*-error one can replace* $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ *by expressions* $(14.4.63)^*$ *modified in the same way.*

General Settings. II

Let us skip non-degeneracy condition (14.3.109). Then we need to consider $\rho \ge \bar{\gamma}$ as contribution of zone $\{\rho \le \bar{\gamma}\}$ to the Tauberian remainder and correction is $O(\bar{\gamma}h^{-1})$.

Next let us note that contribution of ρ-element to the correction does not exceed $C\rho h^{-1}$ and an error when substituting $\boldsymbol{a}$ by $\boldsymbol{a}_0$ does not exceed $C\rho h^{-1}\times T_*\sigma h^{-1} = C\rho^{-1}h^{-1}\sigma$ and the total error does not exceed $C\bar{\gamma}^{-1}\sigma h^{-1} = O(\bar{\gamma}h^{-1})$ as $\sigma = O(\bar{\gamma}^2)$.

Therefore we can shed of all terms $O(x_1^2)$ in metrics and in $V(x)$ anytime we want and we need to consider operator

$$\boldsymbol{a}(\eta,\bar{\gamma}) = (1+\alpha_1)\hbar^2 D_1^2 + (1+\alpha_2\bar{\gamma}x_1)(\eta - x_1^{\nu}/\nu)^2 - W + \alpha_3\bar{\gamma}x_1 \tag{14.4.86}$$

with $\alpha_j = \alpha_j(x_2)$. Without any loss of the generality we can rescale and make $W = 1$ (while $\hbar \mapsto \hbar W^{-1/(2\nu+2)}$).

Let us consider $T(\eta)$ as a period of Hamiltonian flow associated with $\boldsymbol{a}_0(\eta)$ and operator $\exp(i\hbar^{-1}T(\eta)\boldsymbol{a}(\eta,\bar{\gamma}))$ as a perturbation for $\exp(i\hbar^{-1}T(\eta)\boldsymbol{a}_0(\eta))$. Recall that $\bar{\gamma}\le \hbar^{\frac{1}{2}}$. Unless $\hbar$ is close to 1, semiclassical approach works the best. Let $T = T(\eta,\bar{\gamma})$ be a period associated with a perturbed operator.

Then we can apply Bohr-Sommerfeld formula and arrive to formulae (14.4.71) and rescalings (to $W\mapsto 1$) of $(14.4.63)^*$ albeit with $S_0(\bar{\gamma})$ instead of S_0 and with coefficients also depending on $\bar{\gamma}$. The dependence coefficients of $\bar{\gamma}$ is not essential as replacing there $\bar{\gamma}$ by 0 leads to an error $O\bigl(h^{-1}\hbar^{\frac{1}{2}}\bar{\gamma}\bigr)$.

What about argument of G and related functions? Let us note that

$S_0(\bar{\gamma})$ is an even function of $\bar{\gamma}$. (14.4.87)

Really, potential well is always symmetric for even ν and potential well is symmetric as $\eta = \eta^* = 0$ for odd ν.

Therefore replacing $\bar\gamma$ by 0 leads to an error $O(\bar\gamma^2\hbar^{-1})$ in an argument (recall that $\bar\gamma^2\hbar^{-1} \le \epsilon$ as otherwise correction term is $O(\bar\gamma h^{-1})$) and since $G \in \mathscr{C}^{\frac12}$ the total error is $O\big(h^{-1}\hbar^{\frac12} \times (\bar\gamma^2\hbar^{-1})^{\frac12}\big) = O(\bar\gamma h^{-1})$.

Therefore we conclude that

(14.4.88) As $\hbar \le h^\delta$ (i.e. $\mu \le h^{-\nu+\delta_1}$)) one can replace in the correction term $\boldsymbol{a}$ by $\boldsymbol{a}_0$ with an error $O(\bar\gamma h^{-1})$.

Assume now that $\hbar \ge h^\delta$ (i.e. $\mu \ge h^{-\nu+\delta_1}$)). To cover this case we need the following proposition:

Proposition 14.4.20. *(i) There exists $\eta^*_\hbar$ smoothly depending on $\hbar \in (0,\epsilon)$ such that as $\rho = |\eta - \eta^*_\hbar| \le \epsilon$, $\bar\gamma \le \epsilon$ the spacing between neighboring eigenvalues $\lambda_n(\eta,\bar\gamma)$ of $\boldsymbol{a}(\eta,\bar\gamma)$ is of magnitude $\hbar$ and therefore interval $[-c\rho^2, c\rho^2]$ contains no more than $C\rho^2\hbar^{-1}+1]$ of them.*

(ii) Moreover as $\rho \ge C\bar\gamma$ these eigenvalues satisfy

$$|\nabla_\eta \lambda_n| \asymp \rho. \tag{14.4.89}$$

(iii) As ν is even

$$|\lambda_n(\eta,\bar\gamma) - \lambda^0_n(\eta)| \le C\bar\gamma^2\hbar^{-1} \tag{14.4.90}$$

where λ^0_n are corresponding eigenvalues of $\boldsymbol{a}_0$.

(iv) As ν is odd

$$|\lambda_n(\eta,\bar\gamma) - \lambda^0_n(\eta) - \beta(\eta,\bar\gamma)\eta\bar\gamma| \le C\bar\gamma^2\hbar^{-1} \tag{14.4.91}$$

where

$$|\beta(\eta,\bar\gamma)| \le C \tag{14.4.92}$$

and $\eta^*_\hbar = 0$.

Proof. Statements (i) and (ii) obviously follow from the semi-classical theory of 1-dimensional operators.

Observe that $\boldsymbol{a}(\eta) = \boldsymbol{a}_0(\eta) + \bar\gamma\boldsymbol{a}'(\eta)$ with $\boldsymbol{a}'(\eta)$ bounded by $C\bar\gamma\big(\boldsymbol{a}(\eta) + C_0\big)$. Therefore due to the Rellich decomposition to prove Statements (iii) and (iv) one needs just to find $(\partial_{\bar\gamma}\lambda_n)(\eta,0)$.

As ν is even eigenfunctions Υ_n of $\boldsymbol{a}_0$ are even or odd functions alternatively and therefore $\langle \boldsymbol{a}'\Upsilon_n, \Upsilon_n\rangle = 0$; recall that

$$\boldsymbol{a}' = \alpha_1 x_1 D_1^2 + \alpha_2 x_1(\eta - x_1^\nu/\nu)^2 + \alpha_3 x_1. \tag{14.4.93}$$

So, (14.4.90) holds.

Consider odd ν. Then $\eta^*_\hbar = 0$ just due to symmetry and the above arguments are applicable for $\eta = 0$ which implies (14.4.91). □

Recall that according to (14.4.48) the correction term is expressed through eigenvalue counting function

$$\boldsymbol{n}(x_2, \eta, \tau, \bar\gamma, \hbar) = \sum_n \theta\big(\tau - \lambda_n(\eta, \bar\gamma)\big) \tag{14.4.94}$$

and setting here $\bar\gamma = 0$ leads for even ν to an error not exceeding

$$Ch^{-1}\int_{\{\rho\ge\bar\gamma\hbar^{-1/2}\}} \big(\frac{\rho^2}{\hbar}+1\big)\cdot\frac{\bar\gamma^2}{\hbar}\cdot\frac{1}{\rho^2}\,d\rho + Ch^{-1}\int_{\{\rho\ge\bar\gamma\hbar^{-1/2}\}} d\rho \asymp Ch^{-1}\bar\gamma\hbar^{-\frac12} \tag{14.4.95}$$

which is $O(h^{-1-\delta})$ in our settings (as $\hbar \ge h^\delta$); for odd ν we get instead

$$Ch^{-1}\int_{\{\rho\ge\bar\gamma\hbar^{-1/2}\}} \big(\frac{\rho^2}{\hbar}+1\big)\cdot\big(\frac{\bar\gamma^2}{\hbar}+\bar\gamma\rho\big)\cdot\frac{1}{\rho^2}\,d\rho + Ch^{-1}\int_{\{\rho\ge\bar\gamma\hbar^{-1/2}\}} d\rho \asymp Ch^{-1}\bar\gamma|\log h|\hbar^{-\frac12} \tag{14.4.96}$$

which is $O(h^{-\delta})$ in our settings. So we arrive to

(14.4.97) As $\hbar \ge h^\delta$ (i.e. $\mu \ge h^{-\nu+\delta_1}$)) one can replace in the correction term $\boldsymbol{a}$ by $\boldsymbol{a}_0$ with an error $O(h^{-\delta})$.

This is exactly the Tauberian remainder estimate we derived in the general settings without non-degeneracy conditions.

Assume now that non-degeneracy condition $(14.3.111)_m$ is fulfilled. Then the Tauberian remainder estimate is $O(\bar\gamma h^{-1})$ and therefore we need to

improve our arguments. Consider (ρ,ℓ) partition as we did in the previous section while deriving Tauberian estimates.

In virtue of our above arguments contribution of zone $\{x:\ell\le \bar\ell\}$ to an error does not exceed $Ch^{-1}\bar\gamma\hbar^{-\frac12}|\log h|\times \mathsf{mes}(\{x:\ell\le\bar\ell\})$ where factor $|\log h|$ appears for odd ν only. However under assumption $(14.3.111)_m$ $\mathsf{mes}(\{x:\ell\le\bar\ell\})=O(\bar\ell^\sigma$ with $\sigma=\sigma(m)$ and selecting $\bar\ell=h^{\delta_2}$ with arbitrarily small $\delta_2>0$ and then selecting $\delta=\delta(\delta_2)>0$ we get $O(1)$.

Consider now elements with $\ell\ge\bar\ell$. If we know that on such element $|W_{x_2}|\ge\zeta$ the same is true for $|\partial_{x_2}\lambda_n|$ and we can replace in the left-hand expressions of (14.4.94) and (14.4.95) factor ρ^{-2} by $(\rho^2+\zeta)^{-1}$. Then summation with respect to ρ would result in $Ch^{-1}\bar\gamma^2\zeta^{-1}$.

The similar arguments work for odd ν with the only exception of the term linear with respect to $\bar\gamma$. However this term is just 0 as $\lambda_n(\eta)$ are even with respect to η and this copy of $\bar\gamma$ comes with cofactor η.

Now summation with respect to partition of unity would result in the proper estimate as well and we can run induction with respect to m exactly in the standard manner of rescaling technique. We leave details to the reader.

Thus we arrive to

(14.4.98) As $\hbar\ge h^\delta$ (i.e. $\mu\ge h^{-\nu+\delta_1}$)) one can replace in the correction term $a(\eta)$ by $a_0(\eta)$ with an error $O(\bar\gamma h^{-1})$ under non-degeneracy condition $(14.3.111)_m$.

Combining with (14.4.88) and (14.4.97) we arrive to

Proposition 14.4.21. *Let $\mu\le Ch^{-\nu}|\log h|^{-K}$. Then*

(i) Under non-degeneracy condition $(14.3.111)_m$ *asymptotics* (14.4.85) *holds with the correction term defined by* (14.4.50) *and the remainder estimate* $O\big(\bar\gamma h^{-1}\big)$.

(ii) In the general case asymptotics (14.4.85) *holds with the correction term defined by* (14.4.50) *and the remainder estimate* $O\big(\bar\gamma h^{-1}+h^{-\delta}\big)$.

Thus all our main results are established as $\mu\le h^{-\nu}|\log h|^{-K}$ albeit in the case $\mu\ge h^{\delta-\nu}$ we rely upon expression (14.4.82) as main answer which has not been established yet.

14.5 Very Strong Magnetic Field

In this section we assume that μ is close to $h^{-\nu}$; more precisely we assume that

$$h^{\delta-\nu} \le \mu \le \epsilon h^{-\nu} \tag{14.5.1}$$

with a very small exponent $\delta > 0$ and constant ϵ and we leave the case $\mu \ge \epsilon h^{-\nu}$ for the Section 14.8. Under assumption (14.5.1) parameter $\hbar = h/\bar\gamma$ is small but not very small: $h^\delta \le \hbar \ll 1$. Moreover, as $\hbar \ge \epsilon |\log h|^{-1}$ variable x_1 is no more microlocal.

We are going to consider operator in the framework of pseudodifferential operators theory with operator valued symbols in the axillary space $\mathbb{H} = \mathscr{L}^2(\mathbb{R})$ with inner product $\langle .,. \rangle$ and norm $\|.\|$.

14.5.1 Regular Zone

Propagation theory in the regular zone is basically done: as $\mu \le \epsilon h^{-\nu} |\log h|^{-\nu}$ we already proved that the propagation trace $F_{t\to h^{-1}\tau}\chi_T(t)\Gamma(\psi Qu)$ is negligible as $T \in [T_*, T^*]$ with $T_* = Ch|\log h|$ and $T^* = \epsilon'\rho$, where $\psi = \psi(x_2)$ and $Q = Q(hD_2)$ with $\rho \le |\eta| \le 2\rho$ on the support of Q and $C \le \rho \le C\bigl(\mu h^\nu\bigr)^{-1/(\nu-1)}$.

Further, as $\rho \ge C\bigl(\mu h^\nu\bigr)^{-1/(\nu-1)}$ the standard ellipticity arguments imply that the propagation trace is negligible even as $\chi_T(t)$ is replaced by $\bar\chi_T(t)$.

Also we proved that as $\epsilon h^{-\nu}|\log h|^{-\nu} \le \mu \le Ch^{-\nu}$ propagation trace $F_{t\to h^{-1}\tau}\chi_T(t)\Gamma(\psi Qu)$ is negligible as $T \in [T_*, T^*]$ with the same T_*, T^* and $C \le \rho \le C|\log h|^\nu$ while ellipticity arguments work as $\rho \ge C\mu h^\nu |\log h|^\nu$.

But then one can prove estimate

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma(\psi Qu)| \le Ch^{-1}\bar\gamma\rho^{1/\nu}|\log h| \tag{14.5.2}$$

which is "almost perfect": the problem here lies only with the logarithmic factor.

To improve estimate (14.5.2) let us launch the successive approximation method, fixing in the coefficients of unperturbed operator $x_2 = y_2$ as usual and also fixing $x_1 = 0$ there but not in $\mu x_1^\nu/\nu$. Then one can estimate easily the final contribution of the first term by the same $Ch^{-1}\bar\gamma\rho^{1/\nu}|\log h|$ as before and also estimate by C contribution of all other terms. Therefore

in the estimate part only the first term should be reconsidered. But then one needs to calculate

$$h^{-1}\int \Big(F_{t\to h^{-1}\tau}\big(\Gamma' U(t;\eta)\big)\Big)\, Q(\eta)\, dx_2 d\eta \tag{14.5.3}$$

where

$$U = U(x_1, y_1, t; x_2, \eta) = \sum_n e^{ih^{-1}\lambda_n(x_2,\eta,\hbar)t}\Upsilon_n(x_1; x_2, \eta, \hbar)\Upsilon_n(y_1; x_2, \eta, \hbar) \tag{14.5.4}$$

is a propagator for 1-dimensional operator which after change of variable $x_1 \mapsto x_1/\bar{\gamma}$ is just $\boldsymbol{a}_0 = \boldsymbol{a}_0(x_2, \eta, \hbar)$.

In the arguments below let us skip integration over x_2 and also x_2 in the notations. So, $\lambda_n(\eta, \hbar)$ are just eigenvalues of $\boldsymbol{a}_0(\eta, \hbar)$ and $\Upsilon_n(x_1; \eta, \hbar)$ are orthonormalized eigenfunctions. Plugging (14.5.4) into (14.5.3) we arrive to expression

$$\sum_n h^{-1} T \int \widehat{\chi}\Big(\frac{T}{h}\big(\tau - \lambda_n(\eta, \hbar)\big)\Big)\, Q(\eta)\, d\eta \tag{14.5.5}$$

and in the case of even ν let us consider separately sums with respect to eigenvalues corresponding to eigenfunctions which are odd and even with respect to x_1.

Proposition 14.5.1. *As $\hbar \ll 1$ eigenvalues $\lambda_n(\eta, \hbar)$ of $\boldsymbol{a}_0(\eta, \hbar)$ have the following properties as $|\eta| \ge C_0$:*

(i) $\lambda_n(\eta, \hbar) \ge \Delta(\eta, \hbar) := \epsilon_0 \hbar |\eta|^{(\nu-1)/\nu} - C_1$.

(ii) For even ν and $\eta \le -\epsilon$ all eigenvalues are larger than $C_0\epsilon^2$.

(iii) Spacing between two consecutive eigenvalues[18] *$\lambda_n(\eta, \hbar)$ and $\lambda_{n+1}(\eta, \hbar)$ belonging to interval $(-\epsilon_0, \epsilon_0)$ is not less than $\Delta(\eta, \hbar)$.*

(iv) Also for these eigenvalues of Statement (iii)

$$\epsilon_0 \le \eta \partial_\eta \lambda_n(\eta, \hbar) \le C_1. \tag{14.5.6}$$

[18] As ν is even we consider two series of eigenvalues separately: those with even with respect to x_1 eigenfunctions and those with odd eigenfunctions.

Proof. Proofs of Statements (i)–(iii) follow from the Bohr-Sommerfeld theory with the semiclassical parameter $h' = \hbar\rho^{(\nu-1)/\nu}$ when it is small enough. Note that zone $\{|x_1| \le \epsilon\bar{\gamma}\rho^{1/\nu}\}$ is classically forbidden with the small parameter $h'' = \hbar/\rho$.

The easiest way to prove (14.5.6) is to observe that the propagation results of Subsection 14.3.2 basically are equivalent to these inequalities as $\mu \le \epsilon h^{-\nu}|\log h|^{-\nu}$; however this condition could be replaced just by $\hbar \le \epsilon'$ because $e_0(x_1, y_1, ., .)$ depends on $\hbar$ and η rather than μ and h separately.□

Let us consider expression (14.5.5). Since $\widehat{\bar{\chi}}$ is fast-decaying at infinity, each term in (14.5.5) does not exceed

$$h^{-1}T\int \bigl(1+T|\tau-\lambda_n(\eta)|h^{-1}\bigr)^{-s}\,d\eta \asymp \rho$$

due to (14.5.6); on the other hand, the number of terms in (14.5.5) does not exceed $C\Delta^{-1} \asymp \hbar^{-1}\rho^{(1-\nu)/\nu}$. Therefore we arrive to

Proposition 14.5.2. *As $\hbar \ll 1$ and $\rho \ge C$ expression* (14.5.5) *does not exceed $C\hbar^{-1}\rho^{1/\nu}$.*

Thus, we got rid of the logarithmic factor in (14.5.2). Then dividing by $T^* = \epsilon\rho$ we conclude that the contribution of this partition element Q to the remainder estimate does not exceed $C\hbar^{-1}\rho^{(1-\nu)/\nu}$. After integration over $d\rho/\rho$ we conclude that the contribution of the whole regular zone to the Tauberian remainder does not exceed $C\hbar^{-1} = Ch^{-1}\bar{\gamma}$. Thus we arrive to the following proposition:

Proposition 14.5.3. *Under condition* (14.5.1) *contribution of the regular zone $\mathcal{X}_{\mathsf{reg}} = \{|\eta| \ge C\}$ to the Tauberian remainder does not exceed $Ch^{-1}\bar{\gamma}$.*

14.5.2 Degeneration Zone

Non-Periodic Zone

So far we have no results in this zone $\mathcal{X}_{\mathsf{deg}} = \{|\eta| \le C_0\}$ as $\mu \ge \epsilon h^{-\nu}|\log h|^{-\nu}$. Thus we cannot even refer to operator $a_0(\eta, \hbar)$ directly. However let us start from the properties of this operator $a_0 = a_0(\eta, \hbar)$ with potential $V = -1$. Let us first to move through $\eta = 1$ or $\eta = \pm 1$ for even and odd ν respectively.

Proposition 14.5.4. *Let* $\hbar \le \epsilon$ *and*

$$1-\epsilon \le |\eta| \le C_0 \tag{14.5.7}$$

with small enough constant $\epsilon > 0$*. Then operator* $\boldsymbol{a}_0(\eta,\hbar)$ *is microhyperbolic with respect to* η *on energy level* 0*; more precisely*

$$\eta\langle \partial_\eta \boldsymbol{a}_0(\eta,\hbar)v, v\rangle \ge \epsilon\|v\|^2 - C\hbar^{-r}\|\boldsymbol{a}_0(\eta,\hbar)v\|^2 \qquad \forall v \in \mathfrak{D}(\boldsymbol{a}_0(\eta,\hbar)) \tag{14.5.8}$$

with large enough exponent r*.*

Proof. Let us consider $\eta \ge 1-\epsilon$ in the both cases; case $\eta \le -1+\epsilon$ and odd ν is treated then due to the symmetry. It is sufficient to prove (14.5.8) for $v \in \big(\boldsymbol{e}(\eta,\hbar;\epsilon\hbar^r) - \boldsymbol{e}(\eta,\hbar;-\epsilon\hbar^r)\big)\mathbb{H}$ where $\boldsymbol{e}(\eta,\hbar;\lambda,\hbar)$ is the spectral projector of $\boldsymbol{a}_0(\eta,\hbar)$.

Then the standard WKB approach implies that for any v described above with $\|v\| = 1$ and $\varepsilon > 0$ one can approximate v by WKB solution as $x_1 > \varepsilon$. Namely, $\|v-w\|_{\{x_1\ge\epsilon_1\}} \le C\hbar^{r-2}$ where w is a WKB solution constructed for $\lambda = 0$, $C = C(\epsilon_1)$, and $\|w\|_{\{x_1\ge\epsilon_1\}} \le 2$. Then

$$|w|^2 = \varkappa\Big(\frac{1}{\sqrt{1-(\eta - x_1^\nu/\nu)^2}} - o(1)\Big) \tag{14.5.9}$$

with $0 \le \varkappa \le c$.

Then, from explicit the WKB calculations it follows that

$$\langle \partial_\eta \boldsymbol{a}_0(\eta,\hbar)w, w\rangle_{\{x_1\ge\varepsilon\}} = \operatorname{const}\Big(\int_\varepsilon^{b(\xi_2)} \frac{(\eta - x_1^\nu/\nu)\,dx_1}{\sqrt{1-(\eta - x_1^\nu/\nu)^2}} - o(1)\Big) \ge 3\epsilon\|w\|^2_{\{x_1\ge\varepsilon\}}$$

and therefore

$$\langle \partial_\eta \boldsymbol{a}_0(\eta,\hbar)v, v\rangle_{\{x_1\ge\varepsilon\}} \ge 2\epsilon\|v\|^2_{\{x_1\ge\varepsilon\}} - C\hbar^{r-2}. \tag{14.5.10}$$

The last inequality holds for domain $\{x_1 \le -\varepsilon\}$ as well because it is either classically forbidden (for odd ν) or just due to symmetry (for even ν). Furthermore (14.5.10) obviously holds for domain $\{|x_1| \le \varepsilon\}$.

After summation we arrive to (14.5.10) with integration over $\mathbb{R} \ni x_1$. Easy details linked to WKB calculations we leave to the reader. □

Proposition 14.5.5. *Let $\hbar \le \hbar_0(\epsilon_0)$. Consider eigenvalues $\lambda_n(\eta,\hbar)$ of $\mathsf{a}_0(\eta,\hbar)$. Further, let ν be even. Then*

(i) As $1+\epsilon_0 \le \eta \le \mathsf{C}_0$ the spacing between consecutive eigenvalues[19] *$\lambda_n(\eta,\hbar)$ and $\lambda_{n+1}(\eta,\hbar)$ (since ν is even we separate two series of eigenvalues; see footnote* [18]*.) is at least $\epsilon\hbar$ where here and below ϵ is a small positive constant.*

(ii) As $\epsilon_0 \le \eta \le 1-\epsilon_0$ spacing between consecutive eigenvalues $\lambda_n(\eta,\hbar)$ and $\lambda_{n+1}(\eta,\hbar)$ (here we do not separate two series of eigenvalues anymore) is at least $\epsilon\hbar$.

(iii) As $|\eta| \le \epsilon_0$ spacing between consecutive eigenvalues $\lambda_n(\eta,\hbar)$ and $\lambda_{n+1}(\eta,\hbar)$ is at least $\epsilon\hbar^{2\nu/(\nu+1)}$.

Proposition 14.5.6. *Let $\hbar \le \hbar_0(\epsilon_0)$. Consider eigenvalues $\lambda_n(\eta,\hbar)$ of $\mathsf{a}_0(\eta,\hbar)$. Further, let ν be odd. Then*

(i) As $1+\epsilon_0 \le |\eta| \le \mathsf{C}_0$ spacing between consecutive eigenvalues $\lambda_n(\eta,\hbar)$ and $\lambda_{n+1}(\eta,\hbar)$ is at least $\epsilon\hbar$.

(ii) As $|\eta| \le 1-\epsilon_0$ spacing between consecutive eigenvalues $\lambda_n(\eta,\hbar)$ and $\lambda_{n+1}(\eta,\hbar)$ is at least $\epsilon\hbar$.

Proof of Propositions 14.5.5 and 14.5.6. All Statements except (iii) follow easily from the standard WKB method.

Moreover, Statement (iii) follows from the standard WKB method and rescaling. Easy details are left to the reader. □

Here we do not mention case $1-\epsilon_0 \le \eta \le 1+\epsilon_0$ as ν is odd and $1-\epsilon_0 \le |\eta| \le 1+\epsilon_0$ as ν is even but we do not need them because they are covered by Proposition 14.5.4.

Therefore, the spacing between consecutive eigenvalues is rather large– larger than $\epsilon\hbar^2$ (as $\hbar^2 \ge h^{-\delta}$) and therefore their derivatives of order α could be estimated by $\mathsf{C}_\alpha\hbar^{-2\alpha}$. Further, the first order derivative is bounded:

$$|\partial_{\eta,\hbar}\lambda_n(\eta_2)| \le \mathsf{C}_0. \tag{14.5.11}$$

[19] In this proposition we consider only eigenvalues belonging to $[-\epsilon,\epsilon]$.

Then decomposing $u(x,y,t)$ into $\Upsilon_n(x_1/\bar\gamma; hD_2, \hbar)$ [20] which are functions of x_1 but also h-pseudodifferential operators with respect to x_2

$$u(x,y,t) \equiv \sum_{n,l} \bar\gamma^{-1}\Upsilon_n\big(\frac{x_1}{\bar\gamma}; hD_{x_2}, \hbar\big)\, u_{nl}(x_2,y_2,t)\, {}^t\Upsilon_l\big(\frac{y_1}{\bar\gamma}; hD_{y_2}, \hbar\big) \tag{14.5.12}$$

one can rewrite the basic equation $(hD_t - A)u = 0$ as a system

$$\Big(hD_t - \Lambda_n(x_2, hD_2, \hbar)\Big)u_{nl} \equiv \sum_k \mathcal{B}_{kl}u_{kl}, \tag{14.5.13}$$

with

$$\Lambda_n(x_2, hD_2, \hbar) = \lambda_n(hD_2, \hbar) - W(x_2) \tag{14.5.14}$$

and with operators $\mathcal{B}_{kl}$ estimated by $Ch\hbar^{-K}$ where we increase K if needed.

Let us consider $\epsilon\hbar^2$-sized intervals with respect to x_2 and τ. Note that on each such interval at most one of operators $hD_t - \Lambda_n(x_2, hD_2, \hbar)$ fails to be elliptic there (with the ellipticity constant $\epsilon\hbar^2$).

There is one exception though: if ν is even and $\eta \ge 1-\epsilon$ "odd" and "even" eigenvalues[21] could be pretty close to one another and then there are at most two such operators with numbers n and $(n+1)$ (we cannot separate eigenvalues anymore; n corresponds to even eigenfunctions and $(n+1)$ to odd ones).

Let n be such exceptional number. Then one can rewrite system (14.5.13) either as a single equation

$$\Big(hD_t - \Lambda_n(x_2, hD_2, \hbar) - \mathcal{B}'_n\Big)u_{nl} \equiv 0 \tag{14.5.15}$$

where u_{kn} with $k \ne n$ are expressed via u_{nl} and $\mathcal{B}'_n \equiv \mathcal{B}_{nn}$ modulo operators not exceeding $h^2\hbar^{-K}$, or as a 2×2-system with Λ_n replaced by $\begin{pmatrix}\Lambda_n & 0\\ 0 & \Lambda_{n+1}\end{pmatrix}$ and with 2×2 matrix operator $\mathcal{B}'_n$.

Proposition 14.5.7. *In the framework of Proposition 14.5.5 or 14.5.6*

$$\pm\partial_\eta\lambda_n(\eta,\hbar) \ge \epsilon_1 \tag{14.5.16}$$

as long as $\pm(\eta-\eta^*) \ge \epsilon_0$, $\hbar \le \epsilon_1$ *and* $|\lambda_n(\eta,\hbar)| \le \epsilon_1$.

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[20] Where $\Upsilon_n(x_1,\eta,\hbar)$ are eigenfunctions of $\mathbf{a}_0(\xi_2,\hbar)$.

[21] I.e. eigenvalues corresponding to eigenfunctions even or odd with respect to x_1.

Proof. Proof immediately follows from Bohr-Sommerfeld approximation for $\lambda_n(\eta,\hbar)$. Again we leave easy details for the reader. □

Remark 14.5.8. So far Propositions 14.5.5–14.5.7 were proven for $W=1$ but they obviously hold for any $W\ge\epsilon$ with the critical values $\pm 1, \eta^*$ replaced by $\pm W^{1/2}$, $\eta^* W^{1/2}$ respectively.

Then one can easily recover all the results similar to those of Subsection 14.3.3 and thus one can set $T_1=\epsilon$, $T_0=C\hbar^{-r}h|\log h|$ in the degeneration but not periodic zone

$$\mathcal{X}_{\mathsf{deg}}\setminus\mathcal{X}_{\mathsf{per}}=\{\epsilon_0\le|\eta-\eta^* W^{\frac12}|\le C_0\}. \tag{14.5.17}$$

These propagation results imply immediately estimate

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma(Q\psi u)|\le C\hbar^{-r}h^{-1-\delta K}\bar\gamma|\log h|\le C\bar\gamma h^{-1-\delta_1}.$$

However one can engage the successive approximation method as in Subsection 14.5.1 (with $\rho=1$ now); then one needs to consider only the first term of it and following arguments of Subsection 14.5.1 one can see easily that in fact

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma(Q\psi u)|\le C\bar\gamma h^{-1-\delta_1}. \tag{14.5.18}$$

We leave easy details to the reader. Thus estimate immediately leads to

Proposition 14.5.9. *For fixed ϵ_0 contribution of zone $\mathcal{X}_{\mathsf{deg}}\setminus\mathcal{X}_{\mathsf{per}}$ defined by* (14.5.17) *to the Tauberian remainder does not exceed $C\bar\gamma h^{-1}$ as $\mu\le\epsilon h^{-\nu}$ with a small enough constant $\epsilon=\epsilon(\epsilon_0)$.*

Periodic Zone

Now we need to treat periodic zone $\mathcal{X}_{\mathsf{per}}:=\{|\eta-\eta^* W^{\frac12}|\le\epsilon_0\}$. However now we have just one equation rather than 2×2-system due to Proposition 14.5.5 and the construction after it.

We need the following

Proposition 14.5.10. *Let $\mathsf{a}_0(\eta,\hbar)$ be a toy-model operator with the potential $V=-1$ and $\hbar\le\epsilon=\epsilon(\epsilon_0)$. Then*

$$\partial_\eta^2\lambda_n(\eta,\hbar)\ge\epsilon_1\qquad\forall\eta:|\eta-\eta^*|\le\epsilon_0\quad\forall n:|\lambda_n(\xi_2,\hbar)|\le\epsilon. \tag{14.5.19}$$

Proof. Proof follows from Bohr-Sommerfeld approximation and Propositions 14.2.4 and 14.2.8 for even and odd ν respectively. □

Now one can apply our standard rescaling technique as in Section 14.4 (but in a marginally simpler form) with unperturbed operator $\lambda_n(hD_2,\hbar) - W(x_2)$ leading to

Proposition 14.5.11. *Contribution of zone* $\mathcal{X}_{\mathsf{per}} = \{|\eta - \eta^* W^{1/2}| \le \epsilon_0\}$ *to the Tauberian remainder does not exceed* $C\bar{\gamma}h^{-1}$ *as non-degeneracy condition* $(14.3.111)_m$ *is fulfilled and* $C\bar{\gamma}h^{-1-\delta}$ *otherwise with an arbitrarily small exponent* $\delta > 0$.

14.5.3 Calculations

Now let us apply our standard successive approximation technique as in Section 14.4 with an unperturbed operator $a_0(hD_2,\hbar) - W(y_2)$ in the regular and degenerate non-periodic zones and with an unperturbed operator $\lambda_n(hD_2,\hbar) - W(y_2)$ in the periodic zone leading exactly to

Proposition 14.5.12. *Under condition* (14.5.1) *with the Tauberian remainder equal to* $C\bar{\gamma}h^{-1}$ *as non-degeneracy condition* $(14.3.111)_m$ *is fulfilled and* $C\bar{\gamma}h^{-1-\delta'}$ *otherwise with arbitrarily small exponent* $\delta' > 0$ *the following asymptotics holds*

$$\int e(x,x,0)\psi(x_2)\,dx \equiv (2\pi h)^{-1}\int n_0(x_2,\xi_2,0)\psi(x_2)\,dx_2 d\xi_2. \tag{14.5.20}$$

One can see easily that $h^{-2}\mathcal{N}^{\mathsf{MW}} \equiv h^{-2}\mathcal{N}_0^{\mathsf{MW}} \mod O(1)$ under condition (14.5.1); thus this proposition finalizes the proof of Theorem 14.6.1 below under condition (14.5.1).

14.6 Main Theorem and Generalizations

14.6.1 Main Theorem

Thus we arrive to the main theorem of this part of the Chapter:

Theorem 14.6.1. *As* $d = 2$ *consider magnetic Schrödinger operator* (14.1.1) *with sufficiently smooth coefficients. Let conditions* (14.1.4) *and* (13.4.45) *be*

fulfilled and let ψ be supported in the small enough neighborhood of $\{x_1 = 0\}$. Then

(i) As $1 \le \mu \le \mu_1^* := C_0 h^{-\nu/3}$ estimate

$$\mathsf{R}^{\mathsf{MW}} := \Big| \int \Big(e(x,x,0) - h^{-2} \mathcal{N}^{\mathsf{MW}}(x,0) \Big) \psi(x)\, dx \Big| \le C \mu^{-1/\nu} h^{-1} \tag{14.6.1}$$

holds.

(ii) As $\mu_1^* \le \mu \le \mu_2^* := \epsilon h^{-\nu}$ estimate

$$\mathsf{R}^{\mathsf{MW}}_* := \Big| \int \Big(e(x,x,0) - h^{-2} \mathcal{N}^{\mathsf{MW}}(x,0) \Big) \psi(x)\, dx - h^{-1} \int \mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x_2,0) \psi(0,x_2)\, dx_2 \Big| \le C \mu^{-1/\nu} h^{-1} + C h^{-\delta} \tag{14.6.2}$$

holds with arbitrarily small exponent $\delta > 0$ and with the correction term $h^{-1}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ defined by (14.4.50) *in the terms of the eigenvalue counting function $\boldsymbol{n}_0$ of the axillary 1-dimensional Schrödinger operator* (14.4.7) *with semiclassical parameter $\hbar = \mu^{1/\nu} h$.*

(iii) Under non-degeneracy condition $(14.3.111)_m$ *for $W(x_2)$ defined by* (14.2.28)–(14.2.29) *as $\mu \le \epsilon h^{-\nu}$ with small enough constant $\epsilon = \epsilon(\epsilon_0, m)$ (where ϵ_0 is a constant in* $(14.3.111)_m$*)one can skip the last term in the right-hand expression of* (14.6.2)*.*

Remark 14.6.2. In the framework of Theorem 14.6.1 (may be without condition (13.4.45)) the main part of asymptotics is $O(h^{-2})$ as $\mu \le h^{-1}$ and $O(h^{-2}(\mu h)^{1/(\nu-1)})$ as $h^{-1} \le \mu \le \epsilon h^{-\nu}$.

Moreover, under condition (13.4.45) as $\mu \le \epsilon h^{-\nu}$ it has exactly indicated magnitudes: it is $\asymp h^{-2}$ as $\mu \le h^{-1}$ and $\asymp h^{-2}(\mu h)^{1/(\nu-1)}$ as $h^{-1} \le \mu \le \epsilon h^{-\nu}$.

Remark 14.6.3. (i) Generally correction term (integrated) is $O(h^{-1}\hbar^{\frac{1}{2}})$ and in the general case this estimate cannot be made smaller; however under some nondegeneracy assumptions one can improve this estimate and thus push μ_1^* up: if W has no critical points integrated corrected term is $O(h^{-1}\hbar^K)$ with arbitrarily large K; if W has no degenerate critical points integrated corrected term is $O(h^{-1}\hbar)$.

(ii) One can replace $\boldsymbol{n}_0$ by Bohr-Sommerfeld approximation for it, providing formula $(14.4.63)^*$ modulo $O(h^{-1}\hbar^K)$.

14.6.2 Generalization: Vanishing V

We can also generalize all our results to the case of V vanishing. However we need some strong non-degeneracy condition to VF^{-1} as x_1 is small but fixed. Before it was automatic, but now we need to impose it directly, so we assume non-degeneracy condition (14.6.3).

Theorem 14.6.4. *Theorem 14.6.1 remains true if condition* (13.4.45) *is replaced by non-degeneracy condition*

$$|V| + |\partial_{x_2} V| \ge \epsilon_0. \tag{14.6.3}$$

Proof. Consider ℓ-admissible partition with $\ell = \epsilon|V|$.

(a) First, consider elements on which $\ell \le \epsilon_1 |x_1|$. Then rescaling $x \mapsto x/\ell$ and diving by ℓ we arrive to the situation of non-vanishing magnetic field $\mu_{\mathsf{eff}} = \mu\gamma^{\nu-1}\ell^{1/2}$ with $F_{\mathsf{eff}} = F\gamma^{-\nu}$ and $V_{\mathsf{eff}} = V\ell^{-1}$ with a parameter $\gamma \asymp |x_1|$.

One can see easily that after rescaling $V_{\mathsf{eff}}/F_{\mathsf{eff}} = (V/F)\cdot(\gamma^\nu/\ell) \asymp 1$ and also $\ell|\nabla V_{\mathsf{eff}}/F_{\mathsf{eff}}| \asymp 1$.

To apply "non-vanishing" theory with $h_{\mathsf{eff}} = h\ell^{-3/2}$ and μ_{eff} one must assume that $h_{\mathsf{eff}} \le 1$, $\mu_{\mathsf{eff}} \ge 1$ or equivalently

$$\ell \ge \bar{\ell} := C_0 \max\bigl(h^{\frac{2}{3}}, (\mu\gamma^{\nu-1})^{-2}\bigr). \tag{14.6.4}$$

Then the contribution of such element to the remainder does not exceed $C\mu_{\mathsf{eff}}^{-1}h_{\mathsf{eff}}^{-1} = C\mu^{-1}h^{-1}\ell\gamma^{1-\nu}$.

After summation over partition with fixed γ one gets due to condition (14.6.3) the same expression but with $\ell = \gamma$ i.e. $C\mu^{-1}h^{-1}\gamma^{2-\nu}$. One must assume that condition (14.6.4) is fulfilled for this top ℓ, which is equivalent to

$$\gamma \ge L := C_0 \max\bigl(h^{\frac{2}{3}}, \mu^{-2/(2\nu-1)}\bigr). \tag{14.6.5}$$

Then summation over x_1-partition satisfying this condition results in $\mu^{-1}h^{-1}L^{2-\nu}$ as $\nu > 2$ and $\mu^{-1}h^{-1}|\log L|$ as $\nu = 2$ which in both cases as one can check easily does not exceed $C\mu^{-1/\nu}h^{-1}$.

(b)

(c) One needs also to consider elements satisfying (14.6.5) but not (14.6.4). Let us unify such elements, so in fact we redefine ℓ setting it $\ell = \epsilon|V| + \bar{\ell}$. Then on elements with $\ell \asymp \bar{\ell}$ either $\mu_{\mathsf{eff}} \asymp 1$ or $h_{\mathsf{eff}} \asymp 1$. In the former case the condition of a potential being disjoint from 0 is not needed due to the theory of non-magnetic Schrödinger operator while in the latter case element in question is forbidden as $\mu_{\mathsf{eff}} \ge C_1$ and the condition of a potential being disjoint from 0 is not needed again.

The contribution to the remainder of of each of these elements is either Ch_{eff}^{-1} (as $\mu_{\mathsf{eff}} \asymp 1$) or $C\mu_{\mathsf{eff}}^{-s}$ (as $h_{\mathsf{eff}} \asymp 1$) and in any case it does not exceed $C\mu_{\mathsf{eff}}^{-1}h_{\mathsf{eff}}$ again. Repeating summation procedure of Part (a) of the proof one can see that the total contribution of such elements does not exceed $C\mu^{-1/\nu}h^{-1}$.

So one needs to consider two other types of elements.

(d) Consider first elements on which

$$|x_1| \le C\ell, \qquad \ell = \epsilon|W|. \tag{14.6.6}$$

Then rescaling $x \mapsto x/\ell$ and dividing by ℓ brings us to the frames of our main theory with $\mu_{\mathsf{eff}} = \mu\ell^{\nu-1/2}$, $h_{\mathsf{eff}} = h\ell^{-3/2}$ and one must assume that

$$\ell \ge C_0 \max\bigl(h^{\frac{2}{3}}, \mu^{-2/(2\nu-1)}\bigr). \tag{14.6.7}$$

Then the contribution of each element to the remainder does not exceed $C\mu_{\mathsf{eff}}^{-1/\nu}h_{\mathsf{eff}}^{-1} = C\mu^{-1/\nu}h^{-1}\ell^{(\nu+1)/2}$ and the sum over partition results in $C\mu^{-1/\nu}h^{-1}$. The exceptional elements will be covered in the Part (e) of the proof.

(e) So, we are left with the analysis of the zone $\{|x_1| + |W| \le L\}$ with L defined by (14.6.5). Due to condition (14.6.3) without any loss of the generality this zone could be replaced by $\{|x| \le L\}$. Rescaling $x \mapsto x/L$ and dividing by L we arrive to the same situation as we started but with $h_{\mathsf{eff}} = hL^{-3/2}$, $\mu_{\mathsf{eff}} = \mu L^{\nu-\frac{1}{2}}$ instead of h, μ i.e. with either $h_{\mathsf{eff}} \le 1$, $\mu_{\mathsf{eff}} \asymp 1$ or with $h_{\mathsf{eff}} \asymp 1$, $\mu_{\mathsf{eff}} \ge 1$.

In the former case we just refer to the classical theory: contribution of this zone to the remainder does not exceed $Ch_{\mathsf{eff}}^{-1} = Ch^{-1}L^{3/2} \le Ch^{-1}\mu^{-1/\nu}$. In the latter case contribution of this zone to the whole asymptotics does not exceed $C\mu_{\mathsf{eff}}^{-s}$ as this zone is "forbidden"; one can prove it easily going to the axillary space $\mathscr{L}^2(\mathbb{R}_{x_1})$. □

14.7 Strong Magnetic Field Reloaded

14.7.1 Statement of the Problem

Let us recall that for the non-degenerate magnetic field as $\mu \ge h^{-1}$ we also considered operator "(14.1.3)$-\mu h\mathfrak{z}F$":

$$A = \sum_{j,k} P_j g^{jk}(x) P_k - V - \mu h\mathfrak{z}F \tag{14.7.1}$$

with $\mathbb{R} \ni \mathfrak{z} > 0$ and F scalar intensity of the magnetic field.

Remark 14.7.1. (i) In Section 13.6 we assumed that $\mathfrak{z} \ge 1$ as for $\mathfrak{z} < 1$ and $\mu h \ge C_{\mathfrak{z}}$ we still would get negligible $e(x,x,0)$ but it not a case anymore.

(ii) For even ν we need to modify definition of F because usual definition $F = |F_{12}|g^{-\frac{1}{2}}$ would lead to non-smooth F. So let us redefine

$$F \coloneqq F_{12}g^{-\frac{1}{2}}, \qquad F_{12} = \partial_{x_1}V_2 - \partial_{x_2}V_1 \tag{14.7.2}$$

$g = \det(g^{jk})^{-1}$. So for odd ν we still have $F \ge 0$ while for even ν now F has the same sign as x_1; our old F is just $|F|$.

Then as $\mathfrak{z} \in (2\mathbb{Z}^+ + 1)$ zone

$$\mathcal{X}'' \coloneqq \{|x_1| \ge \bar{\gamma}_1 \coloneqq C(\mu h)^{1/(\nu-1)},\ F > 0\} \tag{14.7.3}$$

is not forbidden anymore[22)] at least as $\mu \le h^{-\nu}|\log h|^{-\nu}$ (even stronger magnetic field we consider in the next sections).

On the other hand, as $\mathfrak{z} \ge 1$ but $\mathfrak{z} \notin (2\mathbb{Z}^+ + 1)$ this zone will be forbidden and lacunar but not the lowest lacunar zone. In both cases the contribution of this zone (with possible exception of case $\mathfrak{z} = 1$ when the sign of V is crucial) contribution of this formerly forbidden zone to the main part of asymptotics is $\asymp \mu h^{-1}$.

To make a meaningful statement about remainder we need either to assume that either $\mathfrak{z} \notin (2\mathbb{Z}^+ + 1)$ or some non-degeneracy condition is fulfilled and we adopt generic condition (14.6.3) where as before we are interested in the asymptotics of $\int e(x,x,0)\psi(x)\,dx$ where and $\psi \in \mathscr{C}_0^\infty(B(0,\frac{1}{2}))$ is supported in the small but fixed vicinity of $\{x_1 = 0\}$.

[22)] For even ν zone $\{x_1 \le -C(\mu h)^{1/(\nu-1)}\}$ is still forbidden.

Actually instead of (14.6.3) we assume that on $\mathsf{supp}\,\psi$ one of the following three conditions is fulfilled

$$\pm W \ge \epsilon_0, \tag{14.7.4}_{\pm}$$

$$|\partial_{x_2} W| \ge \epsilon_0. \tag{14.7.5}$$

In this section we analyze the case $\mu \le \epsilon h^{-\nu}$. Analysis in the zone $\mathcal{X}' := \{|x_1| \le 2\bar{\gamma}_1\}$ remains basically the same and the main attention is paid here to the formally forbidden zone (14.7.3).

As $\mu \ge \epsilon h^{-\nu}$ this separation to zones is no more reasonable and will be modified.

Further, in Section 14.8 we analyze the case of $\mu \ge \epsilon h^{-\nu}$.

14.7.2 Simple Rescaling

As in Subsection 14.1.1 the simple rescaling arguments help us to get the easy but not sharp results.

As we mentioned, in this section we assume that $\mu \le Ch^{-\nu}$ [23]. Rescaling arguments in $\mathcal{X}' = \{|x_1| \le 2\bar{\gamma}_1\}$ work exactly in the same manner as in Subsection 14.1.1 leading to the asymptotics of $\int e(x,x,0)\psi'(x)\,dx$ with the principal part $h^{-2}\int \mathcal{N}^{\mathsf{MW}}(x,0)\psi'(x)\,dx$ and the remainder estimate $O(h^{-1})$ where $\psi'(x)$ and $\psi''(x)$ are cut-off functions supported in zones $\mathcal{X}'$ and $\mathcal{X}''$ (defined as above) respectively; one can take $\psi'(x) = \psi(x)\psi_0(x_1/\bar{\gamma}_1)$ and $\psi'' = \psi - \psi'$ where $\psi_0 \in \mathscr{C}_0^\infty$ is supported in $(-1,1)$ and equals 1 in $[-\frac{1}{2},\frac{1}{2}]$.

Consider previously forbidden zone $\mathcal{X}''$.

(14.7.6) If $\mathfrak{z} \notin (2\mathbb{Z}^+ + 1)$ then contribution of $\mathcal{X}''$ to the remainder does not exceed $C\mu(\mu h^\nu)^s$ with arbitrarily large exponent s while contribution to the principal part is

$$h^{-2}\int \mathcal{N}^{\mathsf{MW}}(x)\psi''(x)\,dx = \frac{1}{4\pi}\mu h^{-1} I \int \psi''|F|\sqrt{g}\,dx \tag{14.7.7}$$

with $I = \lfloor(\mathfrak{z}-1)/2\rfloor$; recall that ψ'' is supported in $\mathcal{X}''$ defined by (14.7.3) and for even ν we discard forbidden zone mentioned in footnote [22].

[23] We weaken our assumption temporarily.

Really, contribution of element $B(x,\gamma)\subset \mathcal{X}''$ to the remainder does not exceed then $C\mu_{\text{eff}}^{-s-1}= C(\mu\gamma^{\nu})^{-s-1}$ and the total contribution does not exceed

$$C\int_{\{\gamma\ge\bar\gamma_1\}} (\mu\gamma^{\nu})^{-s-1}\gamma^{-2}\,d\gamma \asymp C(\mu\gamma_1^{\nu})^{-s-1}\gamma_1^{-1}.$$

Let $\mathfrak{z}\in (2\mathbb{Z}^+ +1)$. Then

(14.7.8) Under condition (14.6.3) the contribution of the previously forbidden zone $\mathcal{X}''$ to the remainder does not exceed Ch^{-1}.

Really, contribution of element $B(x,\gamma)\subset \mathcal{X}''$ to the remainder does not exceed then C and the total contribution does not exceed

$$C\int_{\{\bar\gamma_1\le\gamma\le 1\}} \gamma^{-2}\,d\gamma \asymp C\bar\gamma_1^{-1}$$

which is $O(h^{-1})$ due to assumption $\mu\le Ch^{-\nu}$.

Meanwhile

(14.7.9) Under condition $(14.7.4)_{\pm}$ contribution of $\mathcal{X}''$ to the principal part is given by (14.7.7) where under assumption $(14.7.4)_{-}$ one should replace I by $(I-1)$.

Actually then one can improve the remainder estimate.

Without assumption $(14.7.4)_{\pm}$ the above formula remains true for the contribution of the subzone $\mathcal{X}''\cap\{|W|\ge C\gamma\}$; for the contribution of the subzone $\mathcal{X}''\cap\{|W|\le C\gamma\}$ one needs to use the left-hand expression of (14.7.7) but also to take into account correction term (see Section 13.6) $\sum_m \kappa_m\mu_{\text{eff}}h_{\text{eff}}^{1+2m}$ for the case $\mu_{\text{eff}}h_{\text{eff}}\ge 1$, $h_{\text{eff}}\le 1$ where in the rescaling and division arguments $\mu_{\text{eff}}=\mu\gamma^{\nu-\frac{1}{2}}$, $h_{\text{eff}}=h\gamma^{-\frac{3}{2}}$ and the number of balls is $O(1)$ for each γ. Then the total contribution of this correction terms is $O\big(\mu h\big)$ as $\nu\ge 3$ and $O\big(\mu h|\log h|\big)$ as $\nu=2$.

Replacing ψ by $x_1\psi$ in the above arguments one gains factor γ in each integrand; then the total contribution of the zone $\mathcal{X}'$ to the remainder does not exceed

$$C\int \mu^{-1}h^{-1}\gamma^{1-\nu}\times\gamma\times\gamma^{-2}\,d\gamma \asymp C\mu^{-1/\nu}h^{-1}$$

which is exactly what we want. On the other hand, the contribution of zone $\mathcal{X}''$ to the remainder becomes $O(\int\gamma^{-1}\,d\gamma)=O(|\log h|)$ which is what we

want as $\mu \le C(h|\log h|)^{-\nu}$ only. To fix it under condition $(14.7.4)_{\pm}$ one can notice that zone $\mathcal{X}''$ is the spectral gap and therefore the contribution of the individual ball to the remainder estimate is $O(\gamma h_{\mathsf{eff}}^{s})$ with $h_{\mathsf{eff}} = h\gamma^{-1}$ rather than $O(1)$ and therefore the total contribution of zone $\mathcal{X}''$ to the remainder estimate is $O(1)$.

As before, under condition (14.6.3) only these arguments are applicable in the subzone $\mathcal{X}'' \cap \{|W| \ge C\gamma\}$ with $h_{\mathsf{eff}} = h/(\gamma|W|^{1/2})$ as long as $h_{\mathsf{eff}} \le 1$. This leads to $O(1)$ estimate of the contribution of the subzone

$$\mathcal{X}'' \cap \{|W| \ge C\gamma, |W|^{1/2}\gamma \ge h\}$$

into the remainder. One can see easily that the integral of γ^{-1} taken over subzones $\mathcal{X}'' \cap \{|W| \ge C\gamma, |W|^{1/2}\gamma \le h\}$ and $\mathcal{X}'' \cap \{|W| \le C\gamma\}$ is $O(1)$ as well. Thus rescaling arguments provide remainder estimate $O\big(\mu^{-1/\nu}+1\big)$ if ψ contains an extra factor x_1 and under condition (14.6.3) alone correction terms are taken into account.

Therefore

(14.7.10) As $\mu \le Ch^{-\nu}$ in what follows one can assume without any loss of the generality that $\psi(x) = \psi_1(x_1)\psi_2(x_2)$.

Remark 14.7.2. As $\mu \ge Ch^{\nu}$ simple rescaling arguments work as $\{|x_1| \ge Ch\}$ providing $O(h^{-1})$ contribution of this zone to the remainder while the contribution of zone $\{|x_1| \le Ch\}$ will be $O(\mu h^{\nu-1})$. The main part of the asymptotics will be the same as above.

Moreover, these arguments work as $\{|x_1| \ge Ch\}$ providing $O(1)$ contribution of this zone to the remainder estimate as ψ is replaced by $x_1\psi$ while the contribution of zone $\{|x_1| \le Ch\}$ will be $O(\mu h^{\nu})$.

In the next section we will improve these latter results.

14.7.3 Estimates

Regular Zone

In Section 14.3 and subsections 14.5.1–14.5.2 various properties of operator A were proven in the regular and degeneration zones $\mathcal{X}_{\mathsf{reg}} = \{\bar\gamma \le |x_1| \le 2\bar\gamma_1\}$ and $\mathcal{X}_{\mathsf{deg}} = \{|x_1| \le 2\bar\gamma\}$ with $\bar\gamma := C\mu^{-1/\nu}$ as long as $\bar\gamma \le \bar\gamma_1$ i.e. $\mu \le \epsilon h^{-\nu}$.

These properties were proven first in Section 14.3 under assumption

$$C \le \mu \le \epsilon(h|\log h|)^{-\nu} \tag{14.7.11}$$

using standard microlocal analysis with logarithmic uncertainty principle and then in Subsections 14.5.1–14.5.1 under assumption

$$\epsilon(h|\log h|)^{-\nu} \le \mu \le \epsilon h^{-\nu} \tag{14.7.12}$$

they were proven by applying microlocal analysis for h-pseudodifferential operators with respect to x_2 with operator-valued symbols which are operators in the auxiliary space $\mathbb{H} = \mathscr{L}^2(\mathbb{R}_{x_1})$. Recall that in te case (14.7.12) localization was done with respect to ξ_2 rather x_1.

Therefore in both cases (14.7.11) and (14.7.12) in the *redefined regular zone*

$$\mathcal{X}_{\mathsf{reg}} = \{\bar\gamma \le |x_1| \le \bar\gamma_1' = \epsilon\bar\gamma_1\} \tag{14.7.13}$$

(with the small constant $\epsilon > 0$) all these arguments remain true leading us eventually to three following statements; we leave the proofs of Propositions 14.7.3 and 14.7.5 and Corollary 14.7.4 to the reader.

Proposition 14.7.3. *Let conditions* (14.1.4) *and* $((14.7.4)_+$ *be fulfilled. Let* $\psi = \psi(x_2)$ *be supported in* $B(0, \frac12)$ *and let* $\varphi = \varphi(\xi_2)$ *be supported in the strip*

$$\mathcal{Y}_\gamma := \{\mu\gamma^\nu \le |\xi_2| \le 2\mu\gamma^\nu\} \tag{14.7.14}$$

with $C_1\bar\gamma \le \gamma \le \epsilon_1\bar\gamma_1$. *Then*

(i) As $\mu \le \epsilon h^{-\nu}$ *estimates*

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma(Qe)| \le Ch^s \tag{14.7.15}$$

and

$$\mathsf{R}^{\mathsf{T}} := |\Gamma\big(Qe\big) - h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma(Qe)\Big)\,d\tau| \le C\mu^{-1}\gamma^{1-\nu}h^{-1} \tag{14.7.16}$$

hold with $Qe = \varphi(hD_2)\big(e\psi\big)$, $e = e(x,y,\tau)$, $|\tau| \le \epsilon$, $T \in [T_0, T_1]$, $T_0 = Ch|\log h|$, $T_1 = \epsilon\mu^{-1}\gamma^{-\nu}$.

(ii) Moreover, under condition (14.7.11) *Statement (i) holds with* $Q = \psi_1\psi$, $\psi_1 = \psi_1(x_1)$ *supported in* $\mathcal{X}_{\mathsf{reg},\gamma} \coloneqq \{\gamma \le |x_1| \le 2\gamma\}$.

Corollary 14.7.4. *Let conditions* (14.1.4) *and* $(14.7.4)_+$ *be fulfilled. Let* $\psi = \psi(x_2)$ *be supported in* $B(0, \frac{1}{2})$ *and* $\varphi = \varphi(\xi_2)$ *be supported in the regular zone defined in the terms of* ξ_2

$$\mathcal{Y}_{\mathsf{reg}} = \bigl\{C_0 \le |\xi_2| \le \epsilon\bigl(\mu h^\nu\bigr)^{-1/(\nu-1)}\bigr\}. \tag{14.7.17}$$

Then

(i) As $\mu \le \epsilon h^{-\nu}$ *estimate*

$$\mathsf{R}^{\mathsf{T}} \le C\mu^{-1/\nu}h^{-1} \tag{14.7.18}$$

holds.

(ii) Moreover, under condition (14.7.11) *Statement (i) holds with* $Q = \psi_1\psi$, $\psi_1 = \psi_1(x_1)$ *supported in* $\mathcal{X}_{\mathsf{reg}}$.

On the other hand, under condition $(14.7.4)_-$ the whole zone $\mathcal{X}' \coloneqq \mathcal{X}_{\mathsf{deg}} \cup \mathcal{X}_{\mathsf{reg}}$ will be forbidden leading us to the following statement not having analogues in Sections 14.1–14.6:

Proposition 14.7.5. *Let conditions* (14.1.4) *and* $(14.7.4)_-$ *be fulfilled. Let* $\psi = \psi(x)$, $\psi_1 = \psi_1(x_1)$ *be supported in* $B(0, \frac{1}{2})$ *and* $\mathcal{X}'$ *respectively and let* $\varphi = \varphi(\xi_2)$ *be supported in the zone*

$$\mathcal{Y}' = \bigl\{|\xi_2| \le \epsilon\bigl(\mu h^\nu\bigr)^{-1/(\nu-1)}\bigr\}. \tag{14.7.19}$$

Then

(i) $|Qe| \le Ch^s$ *with* $Qe = \varphi(hD_2)\bigl(e\psi\bigr)$, $e = e(x, y, \tau)$, $|\tau| \le \epsilon$ *as* $\mu \le \epsilon h^{-\nu}$.

(ii) Moreover, under condition (14.7.11) *Statement (i) holds with* $Q = \psi_1\psi$, $\psi_1 = \psi_1(x_1)$ *supported in* $\mathcal{X}'$.

Therefore as $\mu \le \epsilon h^{-\nu}$ and condition $(14.7.4)_-$ is fulfilled one needs to discuss the contribution of the degeneration zone $\mathcal{X}_{\mathsf{deg}} = \{|x_1| \le \bar\gamma\}$ or equivalently $\mathcal{Y}_{\mathsf{deg}} = \{|\xi_2| \le C_0\}$ [24] to the remainder.

[24] These two definitions are essentially equivalent under condition (14.7.11) but in the case (14.7.12) one needs always use definition in the framework of ξ_2.

Furthermore one needs to consider the contribution of the previously forbidden zone $\mathcal{X}'' = \{|x_1| \ge \bar{\gamma}_1'\}$ or equivalently $\mathcal{Y}'' = \{|\xi_2| \ge \epsilon\big(\mu h^\nu\big)^{-1/(\nu-1)}\}$ [24] to the remainder.

Degeneration Zone

The degeneration zone is analyzed exactly as in Section 14.3 and Subsections 14.5.1–14.5.2 leading us eventually to the following proposition:

Proposition 14.7.6. *Let conditions* (14.1.4) *and* $(14.7.4)_+$ *be fulfilled. Let* $\psi = \psi(x_2)$ *and* $\psi_1 = \psi_1(x_1)$ *be supported in* $B(0, \frac{1}{2})$ *and* $\mathcal{X}_{\mathsf{deg}}$ *respectively and let* $\varphi = \varphi(\xi_2)$ *be supported in* $\mathcal{Y}_{\mathsf{deg}} = \{|\xi_2| \le C_0\}$. *Then all the results of Section 14.3 and Subsections 14.5.1–14.5.2 remain true; in particular*

(i) As $\mu \le Ch^{\delta-\nu}$

$$\mathsf{R}^{\mathsf{T}\prime} := |\Gamma\big(Qe\big) - h^{-1}\sum_j \int_{-\infty}^{0} \Big(F_{t\to h^{-1}\tau}\bar{\chi}_{T_j}(t)\Gamma\big(Q_j e\big)\Big)\, d\tau| \le C\mu^{-1/\nu}h^{-1} \tag{14.7.20}$$

with $Qe = \varphi(hD_2)\big(e\psi\big)$, $e = e(x, y, \tau)$, $Q = \sum_j Q_j$ *and* $|\tau| \le \epsilon$ *where partition* Q_j *and* $Ch|\log h| \le T_j$ *are defined immediately after formula* (14.4.37).

(ii) Moreover, under nondegeneracy condition $(14.3.111)_m$ $\mathsf{R}^{\mathsf{T}\prime}$ *does not exceed* $C\mu^{-1/\nu}h^{-1}$ *as* $\mu \le \epsilon h^{-\nu}$.

(iii) On the other hand, in the general case $\mathsf{R}^{\mathsf{T}\prime}$ *does not exceed* $C\mu^{-1/\nu}h^{-1} + Ch^{-\delta}$ *as* $\mu \le \epsilon h^{-\nu}$.

(iv) Furthermore, under condition (14.7.11) *all Statements (i)–(iii) hold with* $Q = \psi_1\psi$.

Proof. We leave the easy proof to the reader. □

Remark 14.7.7. In the framework of Proposition 14.7.6 estimate (14.7.15) holds for $Q = Q_m$ and $T \in [T_m, T_m']$ with T_m' defined by (14.3.104) (it was denoted by T_1 then).

Extended Formerly Forbidden Zone

Now we need to consider the extended formerly forbidden zone

$$\mathcal{Y}'' = \{|\xi_2| \ge \epsilon\mu\bar{\gamma}_1^{\nu}\}$$

consisting of the *transitional zone*

$$\mathcal{X}_{\mathsf{trans}} := \{\epsilon\bar{\gamma}_1 \le |x_1| \le 2C\bar{\gamma}_1\} \tag{14.7.21}$$

and the *formerly forbidden zone*

$$\mathcal{X}_{\mathsf{ff}} := \{|x_1| \ge C\bar{\gamma}_1\}. \tag{14.7.22}$$

Let us apply the theory of operators with operator-valued symbols. However, as $\mu \le \epsilon(h|\log h|)^{-\nu}$ one can apply a usual microlocal analysis with logarithmic uncertainty principle.

So, let us consider A as h-pseudodifferential operator $\mathcal{A}(x_2, hD_2)$ with operator-valued symbol $\mathcal{A}(x_2,\xi_2)$. However, before doing this one can assume without any loss of the generality that $g^{11}=1$, $g^{12}=0$ and therefore

$$\boldsymbol{a}(x_2,\xi_2) = h^2D_1^2 + \sigma^2(x)\big(\xi_2 - \mu V_2(x)\big)^2 - \mathfrak{z}\mu hF + V(x), \qquad V_2 = \phi(x)\frac{1}{\nu}x_1^{\nu} \tag{14.7.23}$$

with $\phi(x)=1$ as $x_1=0$; then $f=\sigma\phi$.

Further, for given x_2 by change of variable x_1 one can transform $\boldsymbol{a}(x_2,\xi_2)$ unitarily to the similar operator with $\phi=1$ and with

$$\sigma = 1 \qquad \text{as } x_1 = 0; \tag{14.7.24}$$

but this new operator is multiplied from the left and the right by $\alpha(x)$. So operator $\boldsymbol{a}(x_2,\xi_2)$ is unitary equivalent to

$$\boldsymbol{a}'(x_2,\xi_2) = \alpha(x)\Big(h^2D_1^2 + \sigma^2(x)\big(\xi_2 - \mu\frac{1}{\nu}x_1^{\nu}\big)^2 - \mathfrak{z})\mu h\sigma(x)x_1^{\nu-1} - W'(x)\Big)\alpha(x). \tag{14.7.25}$$

Proposition 14.7.8. *Let condition* (14.1.4) *be fulfilled and* $\mu \le \epsilon h^{-\nu}$. *Then*

(i) As $\mathfrak{z} \notin (2\mathbb{Z}^+ + 1)$ $\mathcal{X}_{\mathsf{ff}}$ *is a lacunar zone and its contribution to the Tauberian remainder* R^{T} *is* $O(\mu^{-s})$.

(ii) On the other hand, as $\mathfrak{z} \in (2\mathbb{Z}^+ + 1)$ *and condition* $(14.7.4)_\pm$ *is fulfilled* $\mathcal{X}_{\mathsf{ff}}$ *also is a lacunar zone and its contribution to the Tauberian remainder* R^{T} *also is* $O(\mu^{-s})$.

In both cases R^{T} *is defined by* (14.7.16) *with* $Qe = \varphi(hD_2)(e\psi)$, φ *supported in the zone* $\mathcal{Y}_{\mathsf{ff}}$, $T \ge T_0 = Ch|\log h|$.

Proof. Really, Proposition 14.A.5(i) implies that under condition $(14.7.4)_\pm$ zone $\mathcal{Y}_{\mathsf{ff}}$ is forbidden on energy levels $|\tau| \le \epsilon$ as long as $\mu \le \epsilon h^{-\nu}$; namely

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)(Qu)(x,y,t)| \le CTh^s \qquad \forall \tau : |\tau| \le \epsilon \tag{14.7.26}$$

as $Q\psi = \varphi(hD_2)(u\psi)$ with φ supported in the zone $\mathcal{Y}_{\mathsf{ff}}$ and therefore its contribution to the remainder R^{T} defined by (14.7.16) is negligible as well.□

Consider now transitional zone $\mathcal{Y}_{\mathsf{trans}}$. Proposition 14.A.5(ii) implies that under condition $(14.7.4)_\pm$ in this zone operator $\boldsymbol{a}(\xi_2)$ is microhyperbolic with respect to ξ_2 and we immediately arrive to the following proposition:

Proposition 14.7.9. *Let conditions* (14.1.4) *and* $(14.7.4)_\pm$ *be fulfilled and* $\mu \le \epsilon h^{-\nu}$.

Then estimate $\mathsf{R}^{\mathsf{T}} \le C$ *holds as* R^{T} *is defined by* (14.7.16) *with* $Qe = \varphi(hD_2)(e\psi)$, φ *supported in the zone* $\mathcal{Y}_{\mathsf{trans}}$, $T \in [T_0, T_1]$, $T_0 = Ch|\log h|$, $T_1 = \epsilon(\mu h^\nu)^{-1/(\nu-1)}$.

We can be completely happy with this result unless condition $(14.7.4)_-$ is fulfilled as in this case all other zones are lacunar and we wonder if the same is true for $\mathcal{Y}_{\mathsf{trans}}$. Theory of Section 13.6 tells us that this is the case provided

$$|(2n+1-\mathfrak{z})\mu hF + V| \ge \epsilon \qquad \forall n \in \mathbb{Z}^+ \tag{14.7.27}$$

which is obviously impossible as $n = 1$ save trivial under condition $(14.7.4)_-$ case $\mathfrak{z} \le 1$.

Synthesis

So, we arrive to

Proposition 14.7.10. *Under assumptions* (14.1.4) *and* $(14.7.4)_\pm$ *consider Tauberian remainder* R^{T} *with* $\varphi = 1$ *as* $\mu \le \epsilon h^{-\nu}$.

Then under condition $(14.7.4)_+$ *the main contribution to* R^{T} *is delivered by* $\mathcal{X}_{\mathsf{reg}} \cup \mathcal{X}_{\mathsf{deg}}$ *and it is* $O(\mu^{-1/\nu}h^{-1} + h^{-\delta})$ *in the general case and* $O(\mu^{-1/\nu}h^{-1})$ *under non-degeneracy condition* $(14.3.111)_m$.

Remark 14.7.11. The analysis of all zones under condition (14.7.5) will be done in Subsection 14.7.5.

14.7.4 Calculations

Calculations. I

After estimates were derived in the previous subsection under assumption $Ch^{-1} \le \mu \le \epsilon h^{-\nu}$ and condition $(14.7.4)_{\pm}$ calculations in zone $\mathcal{Y}_{\mathsf{deg}} \cup \mathcal{Y}_{\mathsf{reg}} \cup \mathcal{Y}_{\mathsf{trans}}$ are done exactly as in Section 14.4.

On the other hand, calculations in zone $\mathcal{Y}_{\mathsf{ff}}$ as $\mu \le \epsilon h^{-\nu}$ are rather obvious under assumptions $Ch^{-1} \le \mu \le \epsilon h^{-\nu}$ and $(14.7.4)_{\pm}$. Therefore we arrive to the following intermediate estimate

$$|\int \Big(\big(\varphi(hD_2)e\big)(x,x,0) - (2\pi h)^{-1} \int e(x_1,x_1;x_2,\xi_2,0)\varphi(\xi_2)\,d\xi_2\Big)\psi_2(x_2)\,dx| \le R \tag{14.7.28}$$

where R is an estimate already derived in the corresponding conditions (also see below) and $\varphi \in \mathscr{C}_0^\infty(-\epsilon',\epsilon')$ with sufficiently small constant ϵ'.

Then the same estimate holds with $\psi(x_2)$ replaced by $\psi(x)$ such that $\psi(x) = \psi_2(x_2)$ as $|x_1| \le C_1\epsilon'$ because this transition leads to a negligible error. We take ψ also satisfying $\psi(x) = 0$ as $|x_1| \ge 2C_1\epsilon'$. Then in the latter estimate we can replace φ by 1. Really, then the error would be

$$|\int \Big(\big((1-\varphi(hD_2))e\big)(x,x,0) - (2\pi h)^{-1} \int e(x_1,x_1;x_2,\xi_2,0)(1-\varphi(\xi_2))\,d\xi_2\Big)\psi(x)\,dx| \tag{14.7.29}$$

and replacing ψ by ψ' equal to ψ as $|x_1| \ge 2C_2^{-1}\epsilon'$ and equal to 0 as $|x_1| \le C_2^{-1}\epsilon'$ leads to a negligible error. However, to expression (14.7.29) modified this way one can apply the theory of operators with non-degenerating magnetic field and then to estimate expression (14.7.29) by C.

Thus we derived estimate (14.7.28) with φ replaced by 1 and $\psi_2(x_2)$ replaced by some "special" function $\psi(x)$. Then due to rescaling arguments like before (14.7.10 the same estimate holds for a general function $\psi(x)$ supported in $\{|x_1| \le 2C_1\epsilon'\}$. Thus we arrive to

Proposition 14.7.12. *Let conditions* (14.1.4) *and* $(14.7.4)_+$ *be fulfilled. Then*

(i) As <u>*either*</u> $\mu \le h^{\delta-\nu}$ <u>*or*</u> *condition* $(14.3.111)_m$ *is fulfilled and* $\mu \le \epsilon h^{-\nu}$ *the following estimate holds*

$$\mathsf{R}_I := \Big| \int \Big(e(x,x,0) - (2\pi h)^{-1} \int \mathbf{e}(x_1,x_1;x_2,\xi_2,0)\, d\xi_2 \Big) \psi(x)\, dx \Big| \le C\mu^{-\frac{1}{\nu}} h^{-1} \tag{14.7.30}$$

where here and below $\mathbf{e}(x_1,y_1;x_2,\xi_2,\tau)$ *is the Schwartz kernel of the spectral projector* $\mathbf{e}(x_2,\xi_2,\tau)$ *of operator* $\mathbf{a}(x_2,\xi_2)$ *defined by* (14.7.23) *and* $\delta > 0$ *is an arbitrarily small exponent.*

(ii) In the general case with $\mu \le \epsilon h^{-\nu}$ *estimate*

$$\mathsf{R}_I \le C\mu^{-\frac{1}{\nu}} h^{-1} + Ch^{-\delta} \tag{14.7.31}$$

holds.

Recall remind that in both statements of Proposition 14.7.12 the principal part of asymptotics has magnitude $\asymp \mu h^{-1}$ (as $\mu \ge h^{-1}$). Therefore if due to condition $(14.7.4)_-$ we remove $\mathcal{Y}_{\mathsf{deg}} \cup \mathcal{Y}_{\mathsf{reg}}$ from the analysis we arrive to the better remainder estimate:

Proposition 14.7.13. *Let conditions* (14.1.4) *and* $(14.7.4)_-$ *be fulfilled. Further, let* $\mathfrak{z} > 1$ [25]*.*

Then for $Ch^{-1} \le \mu \le \epsilon h^{-\nu}$ *estimate* $\mathsf{R}_I \le C$ *holds while the principal part of asymptotics has magnitude* $\asymp \mu h^{-1}$*.*

Calculations. II

Transition to the auxiliary operator $\mathbf{a}_0$ without increasing error estimates could be done easily in zone $\mathcal{Y}_{\mathsf{reg}}$ exactly as it was done in the proof of Propositions 14.4.3 and 14.4.4 while arguments of the proof of Proposition 14.4.8 etc work in zone $\mathcal{Y}_{\mathsf{deg}}$.

[25] Otherwise the principal part of asymptotics would be 0 and remainder estimate would be $O(\mu^{-s})$.

On the other hand, this transition in zone $\mathcal{Y}_{\mathsf{ff}}$ is obvious under condition $(14.7.4)_{\pm}$, and we arrive to two theorems below as $\mu \le h^{-\nu}|\log h|^{-K}$ and function ψ is "special" in the sense of the previous subsection. Then the same arguments as there extend theorem to the general function ψ.

Furthermore, under condition $(14.7.4)_{\pm}$ the case $h^{-\nu}|\log h|^{-K} \le \mu \le \epsilon h^{-\nu}$ is analyzed exactly as in Section 14.5 leading to the extension of these theorems to $\mu \le \epsilon h^{-\nu}$:

Theorem 14.7.14. *Let conditions* (14.1.4) *and* $(14.7.4)_{+}$ *be fulfilled. Then*

(i) As <u>*either*</u> $\mu \le h^{\delta-\nu}$ <u>*or*</u> *condition* $(14.3.111)_m$ *is fulfilled and* $\mu \le \epsilon h^{-\nu}$

$$\mathsf{R}^* := \Big| \int \Big(e(x,x,0) - h^{-2}\mathcal{N}^{\mathsf{MW}}(x,0)\Big)\psi(x)\,dx - \int h^{-2}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x_2,0)\psi(0,x_2)\,dx_2\Big| \tag{14.7.32}$$

does not exceed $C\mu^{-1/\nu}h^{-1}$ *where*

$$h^{-1}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x,\tau) := (2\pi h)^{-1}\int e_0(x_1,x_1;x_2,\xi_2,\tau,\hbar)\,dx_1\,d\xi_2 - \int \tilde{\mathcal{N}}^{\mathsf{MW}}_0(x,\tau)\,dx_1, \tag{14.7.33}$$

and here and below $e_0(x_1,y_1;x_2,\xi_2,\tau)$ *is the Schwartz kernel of the spectral projector* $e_0(x_2,\xi_2,\tau)$ *of operator* $a_0(x_2,\xi_2)$ *defined by* (14.7.23) *and with* α,ϕ,σ,W *restricted to* $\{x_1=0\}$ *and* $h^{-2}\mathcal{N}^{\mathsf{MW}}_0$ *is Magnetic Weyl approximation for this operator.*

(ii) In the general case with $\mu \le \epsilon h^{-\nu}$ *estimate* $\mathsf{R}^* \le C\mu^{-1/\nu}h^{-1} + Ch^{-\delta}$ *holds.*

Similarly, Proposition 14.7.13 is upgraded to the following theorem:

Theorem 14.7.15. *Let conditions* (14.1.4) *and* $(14.7.4)_{-}$ *be fulfilled. Further, let* $\mathfrak{z} > 1$ [25]*.*

Then as $Ch^{-1} \le \mu \le \epsilon h^{-\nu}$ *estimate* $\mathsf{R}^* \le C$ *holds while the principal part of asymptotics has magnitude* $\asymp \mu h^{-1}$*.*

Remark 14.7.16. Obviously the same approximate expressions (14.4.63), $(14.4.63)^*$, hold for the part of $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ "associated" with $\mathcal{Y}_{\mathsf{deg}}$.

14.7.5 Analysis under Condition (14.7.5)

Estimates

We start from the remainder estimate in zone $\mathcal{X}_{\mathsf{deg}} \cup \mathcal{X}_{\mathsf{reg}} \cup \mathcal{X}_{\mathsf{trans}}$ which is trivial:

Proposition 14.7.17. *Let conditions* (14.1.4), (14.7.24) *and* (14.7.5) *be fulfilled. Then*

(i) Estimate (14.7.16) *holds with* $Qe = \varphi(hD_2)(e\psi)$, φ *supported in the strip* $\mathcal{Y}_\gamma$ *with the same restrictions to* γ *and the same* T_0, T_1 *as in Proposition 14.7.3(i).*

(ii) Furthermore, the same estimate holds as φ *is supported in zone* $\mathcal{Y}_{\mathsf{deg}}$ *and* $\gamma = \bar\gamma_0 = \mu^{-1/\nu}$.

(iii) Therefore R^{T} *defined by* (14.7.16) *does not exceed* $C\mu^{-1/\nu}h^{-1}$ *as* φ *is supported in zone* $\mathcal{X}_{\mathsf{deg}} \cup \mathcal{X}_{\mathsf{reg}} \cup \mathcal{X}_{\mathsf{trans}}$ *and* $T = T_0$.

Now let us analyze zone $\mathcal{Y}_{\mathsf{ff}}$ under condition (14.7.5):

Proposition 14.7.18. *Let conditions* (14.1.4), (14.7.24) *and* (14.7.5) *be fulfilled. Then as* $\mathfrak{z} \in (2\mathbb{Z}^+ + 1)$ [26] *estimate* $\mathsf{R}^{\mathsf{T}} \le C$ *holds as* R^{T} *is defined by* (14.7.16) *with* $Qe = \varphi(hD_2)(e\psi)$, φ *supported in the zone* $\mathcal{Y}_{\mathsf{ff}}$.

Proof. (a) Let us note first that estimate

$$|F_{t\to h^{-1}\tau}\bigl(\bar\chi_{T^*}(t) - \bar\chi_{\bar T_*}(t)\bigr)(Qu)(x,y,t)| \le Ch^s \qquad \forall \tau : |\tau| \le \epsilon \tag{14.7.34}$$

holds with $T^* = \epsilon\mu^{-1}\gamma^{-\nu}$, $\bar T_* = Ch|\log h|$ as $Qu = \varphi(hD_2)(u\psi)$, φ supported in the strip $\mathcal{Y}_{(\gamma)} = \{\mu\gamma^\nu \le |\xi_2| \le 2\gamma^\nu\}$ with $\gamma \ge C\bar\gamma_1$.

Really, us consider a partial trace $\Gamma'(Qu)$ (with respect to x_1). Due to Proposition 14.A.5 the propagation speed with respect to x_2 does not exceed $C|\xi_2|^{-1} \asymp C(\mu\gamma^\nu)^{-1}$ and the propagation speed with respect to ξ_2 does not exceed C [27]; moreover, under condition (14.7.5) this propagation speed with respect to ξ_2 is greater than ϵ.

[26] Recall that for $\mathfrak{z} \in (2\mathbb{Z}^+ + 1)$ contribution of $\mathcal{Y}_{\mathsf{ff}}$ to the remainder would be negligible.

[27] Under some assumptions this would be equivalent to the estimate of the the average propagation speed with respect to x_1 of Qu by $C\gamma(\mu\gamma^\nu)^{-1}$; further one can estimate average propagation speed with respect to x_2 of Qu by $C(\mu\gamma^\nu)^{-1}$ as well.

On the other hand, an obvious estimate

$$|F_{t\to h^{-1}\tau}\bar{\chi}_{T_*}(t)\Gamma(Qu)(t)| \le C\mu\gamma^\nu h^{-1}\times T_* = C\mu\gamma^\nu|\log h| \tag{14.7.35}$$

holds where the first factor is $\mu_{\mathsf{eff}}h_{\mathsf{eff}}^{-1}\gamma^{-1}$; furthermore, due to (14.7.34) this estimate holds for the left-hand expression with T_* replaced by T^*.

Therefore the contribution of the strip $\mathcal{Y}_\gamma$ to the Tauberian remainder does not exceed

$$C\mu\gamma^\nu|\log h|\times T^{*\,-1} = C|\log h| \tag{14.7.36}$$

and therefore the total contribution of $\mathcal{Y}_{\mathsf{ff}}$ to the Tauberian remainder does not exceed $C|\log h|\int \gamma^{-1}\,d\gamma \asymp C|\log h|^2$.

This estimate is as good as we need for $\mu \le Ch^{-\nu}|\log h|^{-2\nu}$. However for $Ch^{-\nu}|\log h|^{-2\nu} \le \mu \le \epsilon h^{-\nu}$ we would like to improve it getting rid of two logarithmic factors.

(b) Getting rid of one of these two factors is easy: rescaling $t\mapsto t/T$, $(x_j-y_j)\mapsto (x_j-y_j)/T$, $\mu\mapsto \mu T$, $h\mapsto h/T$ estimates for Schrödinger operator with strong non-degenerate magnetic field of Chapter 13 (with arbitrary parameters μ and h such that $\mu h\ge C$) we arrive to two following inequalities

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma(Qu)| \le C\mu\big(\frac{h}{T}\big)^s \tag{14.7.37}$$

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(Qu)| \le C\mu \tag{14.7.38}$$

as $h\le T\le 1$, $|\tau|\le\epsilon$ under condition $|W|+|\nabla W|\ge\epsilon_0$.

Then using our standard scaling $x_1\mapsto x_1/\gamma$, $x_2\mapsto (x_2-y_2)/\gamma$, $\mu\mapsto\mu_{\mathsf{eff}}=\mu\gamma^\nu$, $h\mapsto h_{\mathsf{eff}}=h/\gamma$ and $T\mapsto T/\gamma$ we arrive to estimate (14.7.35) without logarithmic factor

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(Qu)(t)| \le C\mu\gamma^\nu \tag{14.7.35}^*$$

as $|\tau|\le\epsilon$, $T/\gamma\le\epsilon\mu\gamma^\nu \iff T\le T_1^*=\epsilon\mu\gamma^{\nu+1}$. Further, in virtue of (14.7.34) this estimate holds as $h\le T\le T^*=\epsilon\mu\gamma^\nu$ provided $T_1^*\ge Ch$ i.e. $\gamma\ge\bar{\gamma}_1$.

Then the contribution of the strip $\mathcal{Y}_\gamma$ to the remainder R^{T} is C and therefore the total estimate is $C|\log h|$.

(c) To get rid of the second logarithmic factor we need to further increase T^* in the previous arguments and for this purpose we need for each γ to make x_2-partition of $\mathcal{Y}_\gamma$ of the size

$$\ell = \epsilon|W(0,x_2)| + \bar{\ell}, \qquad \bar{\ell} \ge C\gamma. \tag{14.7.39}$$

Consider first elements $\mathcal{U}_{\gamma,\ell}$ with $\ell \ge C\bar{\ell}$. For every such element on levels τ with $|\tau| \le \epsilon\ell$ after rescaling

$$x_2 \mapsto x_2\ell^{-1}, \quad h \mapsto h' = h\ell^{-\frac{3}{2}}, \quad t \mapsto t\ell^{-1}, \quad \mu \mapsto \mu' = \mu\ell^{\frac{1}{2}} \tag{14.7.40}$$

we are in the elliptic situation.

Therefore contribution of each such element to the Tauberian remainder R^{T} does not exceed $C\mu'(h')^s$ and therefore the total contribution of such elements is negligible as $\bar{\ell} = h^\delta$.

So we need to consider only elements $\mathcal{U}'_\gamma = \mathcal{U}_{\gamma,\ell}$ with $\ell \asymp \bar{\ell} = h^\delta$. For such elements after rescaling (14.7.40) we can apply estimate (14.7.35)*; then scaling back we get the same estimate (14.7.35)* again but with $Q = \psi'(x_2)\varphi(hD_2)$ supported in $\mathcal{U}'_\gamma$, $|\tau| \le \epsilon\ell$ and $Ch|\log h|\ell^{-1} \le T \le T^* = \epsilon\mu\gamma^{\nu+1}$ [28]. Furthermore, applying (14.7.34) we can increase T^* to $\epsilon\mu\gamma^\nu$.

So far we gained nothing: the estimate we proved alone would bring us the same final remainder estimate $C|\log h|$ as before but now we can further increase T^* and thus reduce the remainder estimate.

Namely, let us consider propagation in the time direction in which $|\xi_2|$ increases. If only propagation with respect to ξ_2 was considered, until time $\epsilon_3\mu$ it would be confined to the zone

$$\bigl\{\epsilon_0 \le |\xi_2|\bigl(\mu\gamma^\nu + |t|\bigr)^{-1} \le C\bigr\} \subset \bigl\{\frac{1}{2}\mu\gamma^\nu \le |\xi_2| \le \epsilon_1\mu\bigr\}$$

and thus to $\{|x_1| \le \epsilon_3\}$.

However let us note that the propagation speed with respect to x_2 does not exceed $C\ell/|\xi_2|$ as $\ell \ge C|W| + \bar{\ell}$. Therefore one can prove easily that propagation, which started in the zone $\{|x_2| \le \frac{1}{2}, |V| \le h^\delta\}$ as we

[28] It is consistent with the fact that support of ψ' is of the length ℓ but now $\bar{T}_* = Ch|\log h|/\ell$.

have assumed, until time $T_2^* = \mu\gamma^\nu h^{-\delta_1}$ is confined to a bit larger zone $\{|x_2| \le \frac{3}{4}, |W| \le h^{\delta/2}\}$ of the same type.

Therefore estimate (14.7.35)* holds with $Ch^{1-2\delta} \le T \le T_2^*$. Then due to the Tauberian approach contribution of each partition element $\mathcal{U}'_\gamma$ to the Tauberian remainder R^{T} does not exceed $C\mu\gamma^\nu T_2^{*\,-1} = Ch^{\delta_1}$ and the contribution of the whole strip $\mathcal{Y}_\gamma$ does not exceed Ch^{δ_1} as well and of the whole zone $\mathcal{Y}_{\mathsf{ff}}$ does not exceed Ch^{δ_2}.

Clearly, at some moment we increased slightly T_* but after summation over partition was done we can (using negligibility of the trace on $[Ch|\log h|, h^{1-\delta}]$ time interval on energy levels $|\tau| \le \epsilon$) return to original $\bar{T}_*$.

(d) If $\mathfrak{z} \notin (2\mathbb{Z}^+ + 1)$ then $\mathcal{Y}_{\mathsf{ff}}$ remains a forbidden zone and estimate (14.7.34) holds also for $\bar{\chi}_T(t)$ with $T = \bar{T}_*$, $T = T^*$ rather than their difference and the contribution of $\mathcal{Y}_{\mathsf{ff}}$ to the Tauberian remainder is negligible. □

Calculations

Calculations in zone $\mathcal{Y}_{\mathsf{deg}} \cup \mathcal{Y}_{\mathsf{reg}} \cup \mathcal{Y}_{\mathsf{trans}}$ are exactly as in Section 14.4. However one should be more careful with calculations in zone $\mathcal{Y}_{\mathsf{ff}}$ as $\mathfrak{z} \in (2\mathbb{Z}^+ + 1)$.

Let us recall that according to sections 13.3, 13.6 in the nondegenerate case with $\mu h \ge C$ the operator in question is reduced to one-dimensional $\mu^{-1}h$-pseudodifferential operator $B(x_2, \mu^{-1}hD_2, h^2)$ [29] with the "main symbol" $B(x_2, \xi_2, 0) = -W \circ \Psi$ and therefore the contribution of the partition element to the final answer will be given as in Section 13.6 by magnetic Weyl expression $\int h^{-2}\mathcal{N}^{\mathsf{MW}}(x,0)\psi(x)\,dx$ plus correction terms $\mu h^{1+2m}\int \varkappa_{l,m}(x)\psi(x)\,dx$ with $m = 0, 1, ...$

After rescaling $\mu \mapsto \mu\gamma^\nu$, $h \mapsto h/\gamma$, $dx \mapsto \gamma^{-2}dx$ these terms are transformed into

$$\mu h^{1+2m} \int \varkappa_{l,m}(x,\gamma)\psi(x)\gamma^{\nu-2m-3}\,dx \tag{14.7.41}$$

integrated over zone $\{x\colon \bar{\gamma}_1 \le \gamma(x) \le \epsilon\}$.

One can see easily that if there was an extra factor γ one would be able to rewrite this expression (14.7.41) modulo $O(1)$ as the similar expression

[29] Where x_2 is not our original x_2.

with integration over $\{x, \gamma(x) \le \epsilon\}$ as $2m+2 < \nu$ [30)] or to simply skip it as $2m+2 > \nu$ or to get a term which is $O\big(\mu h^{\nu} |\log h|\big)$ as $2m+2 = \nu$.

To gain this extra factor one needs to consider the difference of expressions $\int e(x,x,0)\psi(x)\,dx$ for two operators with coefficients $g^{jk}(x)$, $f(x)$, $V(x)$ coinciding as $x_1 = 0$. As this second operator it is natural to pick up the simplest one i.e.

$$A_0 = h^2 D_1^2 + \big(hD_2 - \mu x_1^{\nu}/\nu\big)^2 - (2l+1)\mu h x_1^{\nu-1} - W(x_2) \tag{14.7.42}$$

Therefore we arrive to the following proposition:

Proposition 14.7.19. *Under condition* (14.7.5) *estimate*

$$\Big|\int \Big(e(x,x,0) - e_0(x,x,0) - h^{-2}\mathcal{N}^{\mathsf{MW}}(x,0) + h^{-2}\mathcal{N}_0^{\mathsf{MW}}(x,0)\Big)\psi(x)\,dx - \sum \kappa'_{l,m}\mu h^{1+2m}\Big| \le C\mu^{-1/\nu}h^{-1} \tag{14.7.43}$$

holds as $\mu \le h^{-\nu}|\log h|^{-K}$ *where* e_0 *and* $\mathcal{N}_0^{\mathsf{MW}}$ *are defined for operator* A_0.

(14.7.44) Now in what follows we can consider operator A_0 instead of A.

Then we can apply the standard method of successive approximations with unperturbed operator $a(y_2, hD_2)$ and plug the results of successive approximations into expression

$$h^{-1}\int_{-\epsilon}^{0} \Big(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(Qu)\Big)\,d\tau \tag{14.7.45}$$

which calculates exactly contribution of the "problematic" eigenvalue λ_l as $\mathfrak{z} = (2l+1)$ of the corresponding one-dimensional Schrödinger operator; recall that $T = \bar{T} = Ch|\log h|$.

Thus while the main part of asymptotics is estimated by $C\mu h^{-2}\gamma^{\nu}T = C\mu h^{-1}\gamma^{\nu}|\log h|$, each next term seemingly acquires factor

$$Ch^{-1}\big(\mu h\gamma^{\nu-1}\big)^{1/2}T^2 \asymp Ch\big(\mu h\gamma^{\nu-1}\big)^{1/2}|\log h|^2; \tag{14.7.46}$$

since the propagation speed with respect to x_2 is estimated by $C_0(\mu h\gamma^{\nu-1})^{1/2}$ such factor could be larger than 1.

[30)] Thus resulting in exactly expression $\kappa_{l,m}\mu h^{1+2m}$ as in non-degenerate case.

In fact however, $C_0(\mu h\gamma^{\nu-1})^{1/2}$ is the estimate for the instant propagation speed only. Using instead the mentioned reduction to a one-dimensional $\mu^{-1}h$-pseudodifferential operator one can find that the propagation speed with respect to x_2 is estimated by $C_0\mu^{-1}$ if magnetic field is non-degenerate and then in the canonical coordinates for time $T = \bar{T}_*$ the shift of (x_2', ξ_2') will be estimated by $C_0\bigl(\mu^{-}h|\log h|\bigr)^{1/2}$ which is the smallest distance allowed by the logarithmic uncertainty principle[31)] and this would persist if one returns back to the original $(x_2, \mu^{-1}\xi_2)$; so one would be able to estimate $(x_2 - y_2)$ on the time interval in question by $C_0\bigl(\mu^{-1}h|\log h|\bigr)^{1/2}$.

In the degenerate case described here one must replace μ, h by $\mu\gamma^{\nu}$, h/γ respectively and then multiply by γ thus producing final estimate for $|x_2 - y_2|$

$$\varrho := C\bigl(\mu^{-1}h\gamma^{1-\nu}|\log h|\bigr)^{1/2} \asymp Ch\bar{\gamma}_1^{\frac{1}{2}(\nu-1)}\gamma^{-\frac{1}{2}(\nu-1)}|\log h|^{\frac{1}{2}} \tag{14.7.47}$$

and therefore each next term acquires factor $\varrho|\log h|$. Then m-th term of the final answer is estimated by

$$C\mu h^{-1}\varrho^{m-1}|\log h|^{K} \asymp C\mu h^{m-2}\gamma^{\nu-\frac{1}{2}(\nu-1)(m-1)}\bar{\gamma}_1^{\frac{1}{2}(\nu-1)(m-1)}|\log h|^{K}. \tag{14.7.48}$$

After integration over $\gamma^{-1}\,d\gamma$ with $\bar{\gamma}_1 \le \gamma \le \epsilon$ expression (14.7.48) results in $C\mu h^{m-2}\bar{\gamma}_1^{\nu}|\log h|^{K}$ as $\nu - \frac{1}{2}(\nu-1)(m-1) \le 0$ or in $C(\mu^{-1}h)^{(m-3)/2}|\log h|^{K}$ otherwise. One can check easily that in either case the answer is $O(|\log h|^{K})$ as $m \ge 3$ and only terms with $m = 1, 2$ should be considered more carefully under condition (14.7.11).

On the other hand, the main term appears as (14.7.45) with u replaced by $\bar{u}$ and modulo negligible one can rewrite it with any $T \ge \bar{T}_*$, in particular with $T = \infty$ which leads to

$$(2\pi h)^{-1}\int e_0(x_1, x_1, 0; x_2, \xi_2)\psi(x_1)\varphi(\xi_2)\,dx_2\,d\xi_2 \tag{14.7.49}$$

where we recall that $e_0(x_1, y_1, 0; x_2, \xi_2)$ is the Schwartz kernel of the spectral projector of one-dimensional Schrödinger operator $a_0(x_2, \xi_2)$.

Let us consider terms with $m = 2$ i.e. expression (14.7.45) with u replaced by $\bar{u}_1$; similarly to the above analysis one can estimate contribution

[31)] Since $\mu^{-1}h$-Fourier integral operators are involved later one needs the same distance in each (x, ξ) direction.

of $O\big((x_2-y_2)^2\big)$ terms in the perturbation $\boldsymbol{a}(x_2, hD_2)-\boldsymbol{a}(y_2, hD_2)$ by $C|\log h|^K$. Therefore one should consider only $\boldsymbol{a}(x_2, hD_2) - \boldsymbol{a}(y_2, hD_2) = (x_2 - y_2)B_1(y_2)$ in which case $\bar{u}_1$ is defined by (14.4.12) without the last term since B_1 commutes with $(x_2 - y_2)$:

$$u \mapsto \bar{u}_1 = -ih\sum_{\varsigma=\pm} \varsigma \bar{G}^{\varsigma} B_1 \bar{G}^{\varsigma}[\bar{A}, x_2 - y_2]\bar{G}^{\varsigma}\delta(t)\delta(x_2 - y_2)\delta(x_1 - y_1). \tag{14.7.50}$$

One needs to multiply this by $h^{-1}\psi$, integrate with respect to τ and apply Γ to it. Obviously since for odd ν operators $\bar{G}^{\varsigma}$ and $[\bar{A}, x_2 - y_2]$ are even and odd respectively as $x_1 \mapsto -x_1$, $\xi_2 \mapsto -\xi_2$ the answer would be 0 if ψ is even with respect to x_1.

To cover the case of even ν and general ψ let us note that B_1 commutes with $\bar{G}^{\varsigma}$ considered as operators in the auxiliary space $\mathscr{L}^2(\mathbb{R}^1_{x_1})$. Then if $\bar{G}^{\varsigma}$ commuted with ψ, taking trace and integrating with respect to τ would result in

$$\mathsf{const}\cdot \partial_{\xi_2} B_1 \sum_{\varsigma=\pm} \varsigma\, \mathsf{Tr}\big(\bar{G}^{\varsigma}\psi\big)$$

which after integration over ξ_2 results in 0.

However $\bar{G}^{\varsigma}$ does not commute with ψ, so instead of 0 one gets

$$\mathsf{const}\cdot B_1 \sum_{\varsigma=\pm} \mathsf{Tr}\,\varsigma\Big(\bar{G}^{\varsigma}\big(\partial_{\xi_2}\bar{G}^{\varsigma}\big)\bar{G}^{\varsigma}[\bar{A}, \psi]\Big)$$

and to this expression one can apply the same type of transformations and calculations as in the proof of Proposition 14.7.19 resulting in the expression $\sum_m \kappa_{l,m}\mu h^{1+2m}$ where coefficients $\kappa_{l,m}$ are changed as needed.

Therefore combining with the results for zone $\mathcal{Y}_{\mathsf{deg}} \cup \mathcal{Y}_{\mathsf{reg}} \cup \mathcal{Y}_{\mathsf{trans}}$ we arrive to

Proposition 14.7.20. *For a toy-model operator*

$$|\int \Big(e_0(x, x, 0) - (2\pi h)^{-1}\int \boldsymbol{e}_0(x_1, x_1, 0; x_2, \xi_2)\, d\xi_2\Big)\psi(x)\, dx - \sum \kappa_m \mu h^{1+2m}| \le C\mu^{-1/\nu}h^{-1} \tag{14.7.51}$$

as $\mu \le Ch^{-\nu}|\log h|^{-K}$.

Further, combining this with Proposition 14.7.18 we get estimate (14.7.52) as $\mu \le h^{-\nu}|\log h|^{-K}$:

Theorem 14.7.21. *Under condition* (14.7.5) *estimate*

$$(14.7.52)\quad \Big|\int \Big(e(x,x,0) - (2\pi h)^{-1}\int \mathsf{e}_0(x_1,x_1,0;x_2,\xi_2)\,d\xi_2 - h^{-2}\mathcal{N}^{\mathsf{MW}}(x,0) + h^{-2}\mathcal{N}_0^{\mathsf{MW}}(x,0)\Big)\psi(x)\,dx - \sum \kappa_{l,m}\mu h^{1+2m}\Big| \le C\mu^{-1/\nu}h^{-1}$$

holds as $\mu \le \epsilon h^{-\nu}$.

Proof. To finish the proof of this theorem one needs to cover the case $h^{-\nu}|\log h|^{-K} \le \mu \le \epsilon h^{-\nu}$, getting rid of the term $|\log h|^K$ in the error estimates.

The first problematic error comes from the correction terms in Proposition 14.7.19, namely from the terms of the type $\mu h^{1+2m}\int \varkappa_{l,m}(x_2)\gamma^{\nu-2m-3+k}\,dx$ with $k \ge 1$, $\nu - 2m - 3 + k = -1$ and this error term is $O(1)$ unless $k = 1$, $\nu = 2m+1$ in which case it it is $\kappa'_l\mu h^\nu|\log h|$.

This is possible only for odd ν in which case operator a_0 is even with respect to $x_1 \mapsto -x_1$, $\xi_2 \mapsto -\xi_2$ but perturbation contains exactly one factor x_1 and therefore it is odd and after integration with respect to x_1, ξ_2 this correction term results in 0 if ψ is even with respect to x_1.

Further, one needs to consider terms corresponding to $m = 3$ in the successive approximations leading to Proposition 14.7.20 and there one can replace $\mathsf{a}_0(x_2,\xi_2) - \mathsf{a}_0(y_2,\xi_2)$ by $B_1(x_2-y_2)$, and also terms corresponding to $m = 2$ in the same successive approximations and there one can replace $\mathsf{a}_0(x_2,\xi_2) - \mathsf{a}_0(y_2,\xi_2)$ by $B_2(x_2-y_2)^2$.

To calculate the contribution of such terms one can apply the same approach as in the proof of Proposition 14.7.19 and the contribution of γ-admissible partition element with respect to x_1 will be

$$\sum_m \mu h^{1+2m}\int \varkappa_{l,m,k}(x_2)\psi(x)\gamma^{\nu-2m-3+k}\,dx$$

with $k \ge 0$; however since this expression should be $O(|\log h|^K)$ all the terms but those with $\nu \le 2m+1$, $k \ge 1$ should vanish; further, the total contribution of all remaining terms save those with $\nu = 2m+1$ and $k = 1$ is $O(1)$, which leaves us with no "bad" terms for even ν and with one "bad" term $\kappa'_l\mu h^\nu \log h$ for odd ν, $m = (\nu-1)/2$. However, parity considerations with respect to x_1 show that this term should vanish as well. □

Remark 14.7.22. (i) All the coefficients $\varkappa_{I,*}$ and $\kappa_{I,*}$ vanish for $I = 0$.

(ii) Obviously the same approximate expressions (14.4.63), (14.4.63)* hold for part $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ "associated" with $\mathcal{Y}_{\mathsf{deg}}$.

14.8 Superstrong Magnetic Field

14.8.1 Preliminary Analysis

In this section we consider the case of *superstrong magnetic field*

$$\mu \geq \epsilon_0 h^{-\nu} \tag{14.8.1}$$

which in turn consists of two subcases: transitional superstrong magnetic field

$$\epsilon_0 h^{-\nu} \leq \mu \leq C_0 h^{-\nu} \tag{14.8.2}$$

and genuine superstrong magnetic field

$$\mu \geq C_0 h^{-\nu}. \tag{14.8.3}$$

In the latter case the previous definition of distances does not make sense anymore and we need to consider 1D-pseudodifferential operator with operator valued symbol, which is a perturbed operator

$$h^2 D_1^2 + \bigl(hD_2 - \mu x_1^\nu/\nu\bigr)^2 - \mu h x_1^{\nu-1}\mathfrak{z} - W(x); \tag{14.8.4}$$

rescaling $x_1 \mapsto x_1/\hbar$ with

$$\hbar = (\mu^{-1}h)^{1/(\nu+1)} \tag{14.8.5}$$

we arrive to a perturbed operator

$$(\mu h^\nu)^{2/\nu+1}\Bigl(D_1^2 + \bigl(\hbar D_2 - x_1^\nu/\nu\bigr)^2 - \mu h x_1^{\nu-1}\mathfrak{z}\Bigr) - W(\hbar x_1, x_2). \tag{14.8.6}$$

In virtue of Appendix 14.A this latter operator could be diagonalized to a perturbed operator

$$(\mu h^\nu)^{2/\nu+1}\lambda_n(\hbar D_2) - W(\hbar x_1, x_2) \tag{14.8.7}$$

where

(14.8.8) $\lambda_n(\eta) = \lambda_n(\eta;\mathfrak{z})$ ($n = 0, 1, ...$) are eigenvalues of operator

$$L(\eta;\mathfrak{z}) = D_1^2 + \bigl(\eta - x_1^\nu/\nu\bigr)^2 - \mu h x_1^{\nu-1}\mathfrak{z} \tag{14.8.9}$$

and all of these operators save C are elliptic; in the case of (14.8.3) only operators with $n = 0, ..., \lfloor \frac{1}{2}(\mathfrak{z}-1)\rfloor$ may be not elliptic. In particular, we conclude that

Proposition 14.8.1. *(i) For* $\mathfrak{z} \in [0,1)$ *case* (14.8.3) *is trivial with the main part of asymptotics equal* 0 *and the remainder estimate* $O(\mu^{-s})$.

(ii) For $\mathfrak{z} \in [0,1]$ *under assumption* $(14.7.4)_-$ *case* (14.8.1) *is trivial.*

14.8.2 Transitional Superstrong Magnetic Field

Let us consider case (14.8.2) when factor $(\mu h^\nu)^{2/(\nu+1)}$ in front of λ_n is of magnitude 1. Note that $\lambda_n(\eta)$ stabilize as $\eta \to \pm\infty$ either to $\pm\infty$ or to 0. In the former case we are in the framework of elliptic theory; in the latter case we are in the framework of elliptic theory under conditions $(14.7.4)_\pm$ and in both cases contribution of zones $\pm\eta \ge C$ to the remainder are negligible:

(14.8.10) Let condition (14.8.2) be fulfilled. Further, let either $\mathfrak{z} \notin (2\mathbb{Z}^+ + 1)$ or condition $(14.7.4)_\pm$ be fulfilled. Then all operators (14.8.7) and thus operator $\mathsf{a}(x_2,\eta)$ are elliptic in the zone $\mathcal{Y}_{\mathsf{ff}}$ and the contribution of $\mathcal{Y}_{\mathsf{ff}}$ to the Tauberian remainder estimate is negligible.

On the other hand, if $\mathfrak{z} = (2l+1)$ with $l \in \mathbb{Z}^+$ then all operators (14.8.7) with $n \ne l$ (and even with $n = l$ but with $-\eta \gg 1$ as ν is even) are elliptic but operator with $n = l$ is not. However under condition (14.7.5) we are in the framework of microhyperbolic theory everywhere:

Proposition 14.8.2. *Similarly, let* $\mathfrak{z} \in (2\mathbb{Z}^+ + 1)$. *Then under conditions* (14.8.2) *and* (14.7.5) *operator* $\mathsf{a}(x,\eta)$ *is* x_2*-microhyperbolic everywhere and the total remainder estimate is* $O(1)$.

Proof. Really, contributions of zones $\{|\eta| \le C\}$ and $\{\eta \asymp \rho\}$ are obviously $O(1)$ due to microhyperbolicity and remainder estimate $O(|\log\mu|)$ is trivial. To recover remainder estimate $O(1)$ let us notice that in propagation in the direction of increasing $|\eta|$ in zone $\{|\eta| \ge C\}$ one can take $T^*(\eta) \asymp |\eta|^{1+\delta}$ rather than $T^*(\eta) \asymp |\eta|$ and then $\int T^{*-1}(\eta)\,d\eta \le C$. We leave easy details to the reader. □

In the absence of condition (14.7.5) consider zone $\{|\eta| \le C\}$. We have a real problem as we do not know how for η with $|\eta| \le C$ $\lambda_n(\eta)$ depends on η (albeit we know for $|\eta| \gg 1$). It contrasts to the case of $\nu = 1$ but operator on half-line when this behavior is well established (see Appendix 15.A). In our present case we may only have some conjectures.

However we do know that $\lambda_n(\eta, \mathfrak{z})$ are analytic and non-constant except of the case of odd ν, $\mathfrak{z} = 1$ and $n = 0$ and therefore with this singular exception we conclude that condition

$$\sum_{1\le k\le m} |\partial_\eta^k \lambda_n(\eta)| \ge \epsilon_0 \qquad n \le C, |\eta| \le C \tag{14.8.11}_r$$

for some $r = r(\nu, C)$.

Then in virtue of results of Subsection 5.2.3 we arrive to the remainder estimate $O(1)$ under assumption

$$|(\mu h^\nu)^{2/(\nu+1)}\lambda_n(\eta, \mathfrak{z}) - W(x_2)| + |\partial_{x_2} W(x_2)| + |\partial_{x_2}^2 W(x_2)| \ge \epsilon_1 \qquad \forall n \quad \forall \eta, x_2. \tag{14.8.12}$$

Symmetrically, the remainder estimate is also $O(1)$ under assumptions $(14.8.11)_2$ and $(14.3.111)_m$ for some m.

On the other hand in virtue of results of Subsection 5.2.4 without this assumption the remainder is $O(h^{-\delta})$ with arbitrarily small exponent $\delta > 0$.

Note, that under condition

$$|(\mu h^\nu)^{2/(\nu+1)}\lambda_n(\eta, \mathfrak{z}) - W(x_2)| \ge \epsilon_1 \qquad \forall n \quad \forall \eta, x_2 \tag{14.8.13}$$

we are in the framework of elliptic theory; in particular, we are in the framework of elliptic theory in the case of odd ν, $\mathfrak{z} = 1$ and $n = 0$ provided $(14.7.4)_+$ is fulfilled (while this case is trivial under condition $(14.7.4)_-$) .

Meanwhile the principal part of asymptotics is expected to be of magnitude $O(\mu^{-1}h)$ but it would be less, only $O(\mu^{-1/(\nu+1)}h^{1/\nu+1})$ if only zone $\{|\eta| \le C\}$ contributes there.

So, we arrive to two propositions below:

Proposition 14.8.3. *Let condition* (14.8.2) *be fulfilled. Then under condition* (14.8.13) *Tauberian remainder is* $O(\mu^{-s})$.

Proposition 14.8.4. *Let condition* (14.8.2) *be fulfilled. Assume that* <u>*either*</u> $\mathfrak{z} \notin (2\mathbb{Z}^+ + 1)$ <u>*or*</u> $\mathfrak{z} \in (2\mathbb{Z}^+ + 1)$ *and one of conditions* $(14.7.4)_\pm$ *is fulfilled. Then*

(i) Tauberian remainder is $O(\mu^{\delta})$.

(ii) Under assumption (14.8.12) *Tauberian remainder is* $O(1)$.

(iii) Under assumptions $(14.8.11)_2$ *and* $(14.3.111)_m$ *for some* m *Tauberian remainder is* $O(1)$ *as well.*

Calculations in this case also do not differ from those in Section 14.5 except that in zone $\{|\eta| \le C\}$ in absence of strong non-degeneracy condition one needs to have $\lambda_n(x_2,\eta)$ the full Weyl symbols of diagonalized operators. So we arrive to the following theorem:

Theorem 14.8.5. *Let condition* (14.8.2) *be fulfilled. Further, let* <u>*either*</u> $\mathfrak{z} \notin (2\mathbb{Z}^+ + 1)$ <u>*or*</u> $\mathfrak{z} \in (2\mathbb{Z}^+ + 1)$ *and one of condition* $(14.7.4)_{\pm}$, (14.7.5) *be fulfilled. Then for* R_I *defined by* (14.7.30) *for complete symbol* $a(x_2,\xi_2)$ *of the reduced operator estimates of Propositions 14.8.3 and 14.8.4 hold under additional assumptions of these propositions.*

According to Chapter 5 we can ignore $O(\hbar)$ perturbations without deteriorating (much) remainder estimate if either (14.7.5) is fulfilled, or both (14.8.12) and $(14.8.11)_2$ are fulfilled and in the latter case remainder is $O(1)$ if $\lambda_{k,\eta\eta}$ and $-W_{x_2x_2}$ have the same signs in the critical points; otherwise remainder is $O(|\log\hbar|)$. Finally, under either condition (14.8.12), $(14.8.11)_2$ remainder is $O(\hbar^{-1})$. So we arrive to another theorem:

Theorem 14.8.6. *Let condition* (14.8.2) *be fulfilled. Further, let* <u>*either*</u> $\mathfrak{z} \notin (2\mathbb{Z}^+ + 1)$ <u>*or*</u> $\mathfrak{z} \in (2\mathbb{Z}^+ + 1)$ *and one of conditions* $(14.7.4)_{\pm}$, (14.7.5) *be fulfilled. Then for* R^* *defined by* (14.7.32) *the following estimates hold:*

(i) $O(\mu^{-s})$ *under condition* (14.8.13).

(ii) $O(1)$ *as* <u>*either*</u> *condition* (14.7.5) <u>*or*</u> *both conditions* (14.8.12) *and* $(14.8.11)_2$ *are fulfilled and in such points* $\lambda_{k,\eta\eta}$ *and* $-W_{x_2x_2}$ *have the same signs.*

(iii) $O(\log\mu)$ *as both conditions* (14.8.12) *and* $(14.8.11)_2$ *are fulfilled.*

(iv) $O(\hbar^{-\frac{1}{2}}) = O(\mu^{1/(2(\nu+1))}h^{-1/(2(\nu+1))})$ *as one of conditions* (14.8.12) *and* $(14.8.11)_2$ *is fulfilled.*

14.8.3 Genuine Superstrong Magnetic Field

Now we consider the previously forbidden case (14.8.3) with sufficiently large constant C_0. In this case all zones should be redefined as mentioned in Subsection 14.8.1.

Recall that now λ_n acquire a large factor $(\mu h^\nu)^{2/(\nu+1)}$ or, alternatively that W acquires a small factor $(\mu h^\nu)^{-2/(\nu+1)}$.

Degeneration Zone

Degeneration zone $\{|\eta| \le C\}$ can be treated approximately as in the case of condition (14.8.2). We refer here to methods of Section 15.4 [32] which could be applied without any modifications.

Then we arrive to the following proposition:

Proposition 14.8.7. *Let* (14.8.3) *be fulfilled. Then contribution of zone* $\{|\eta| \le C\}$ *to the Tauberian remainder and* R_I *are*

(i) $O(\mu^{-s})$ *provided*

$$|\lambda_n(\eta)| \ge \epsilon_0 \qquad \forall n, |\eta| \le C. \tag{14.8.14}$$

(ii) $O(1)$ *provided one of conditions* $(14.8.11)_1$, (14.7.5) *is fulfilled.*

(iii) $O(|\log \mu|)$ *if both* $(14.3.111)_2$ *and* $(14.8.11)_2$ *are fulfilled; further, if both* $\lambda_{n,\eta\eta}(\eta)$ *and* $-W_{x_2x_2}(x_2)$ *have the same signs as*

$$|\lambda_n(\eta)| + |\lambda_{n,\eta}(\eta)| + |W_{x_2}(x_2)| \le \epsilon.$$

this estimate is actually $O(1)$.

Recall that as ν is odd, $\mathfrak{z} = 1$ then λ_0 is identically 0 and as $n = 0$ no other above assumption but (14.7.5) could be fulfilled. However then assumption $(14.7.4)_\pm$ provides ellipticity as $n = 0$; meanwhile then $\lambda_n \ge \epsilon$ for all $n \ge 1$ and we arrive to the following proposition:

Proposition 14.8.8. *Let conditions* (14.8.3) *and* $(14.7.4)_\pm$ *be fulfilled,* ν *be odd and* $\mathfrak{z} = 1$. *Then Tauberian remainder and* R_I *are* $O(\mu^{-s})$*; meanwhile the principal part of asymptotics is* 0 *under condition* $(14.7.4)_-$.

[32] Chapter 15 was written before this chapter and is independent from it with the singular exception of Subsection 15.5.3.

Corollary 14.8.9. *(i) As* $\mathfrak{z} \notin (2\mathbb{Z}^+ + 1)$ *all statements of Proposition 14.8.7 hold for total* R^*.

(ii) As $\mathfrak{z} = (2l+1)$, $l \in \mathbb{Z}^+$ *in what follows we need to investigate only* $n = l$.

(iii) As $\mathfrak{z} = 1$ *and* ν *is odd we need in what follows to investigate only* $n = 0$ *and only if* $(14.7.4)_\pm$ *are violated.*

Regular Zone: Case $\mathfrak{z} \neq 1$

As $|\eta| \geq C$ let us consider first eigenvalues $\lambda_n(x_2, \xi_2, \mu, h)$ of operator $a_0(x_2, \xi_2, \mu, h)$. Then in virtue of Proposition 14.A.3

(14.8.15) As $\mathfrak{z} \neq 2n+1$ and $|\eta| \geq C_0$,

$$\lambda_n(x_2, \eta, \mu, h) \asymp (2n+1-\mathfrak{z})|\eta|^{(\nu-1)/\nu}; \tag{14.8.16}$$

and signs of the left and right-hand expressions coincide[33)]

and in virtue of Proposition 14.A.5

(14.8.17) As $\mathfrak{z} = 2n+1$

$$\lambda_n(x_2, \xi_2, \mu, h) \sim \alpha_n \eta^{-2/\nu} \qquad \text{with} \quad \alpha_n > 0. \tag{14.8.18}$$

Therefore we conclude that

(14.8.19) As $\mathfrak{z} \notin (2\mathbb{Z}^+ + 1)$ zone $\{|\eta| \geq C_0\}$ is elliptic and its contribution to the Tauberian remainder is $O(\mu^{-s})$

and

(14.8.20) As $\mathfrak{z} = 2n+1$ with $n \geq 1$ zone $\{C_0 \leq |\eta| \leq \epsilon(\mu h^\nu)^{\nu/(\nu+1)}\}$ is elliptic and its contribution to the Tauberian remainder is $O(\mu^{-s})$.

On the other hand,

[33)] Recall that for even ν zone $\{\eta \leq -C_0\}$ is always elliptic and only zone $\{\eta \geq C_0\}$ needs to be considered now.

(14.8.21) Let $\mathfrak{z} = 2n+1$, $n \ge 0$ and condition $(14.7.4)_\pm$ be fulfilled. Then zone $\{|\eta| \ge C(\mu h^\nu)^{\nu/(\nu+1)}\}$ is elliptic as well and its contribution to the Tauberian remainder estimate is $O(\mu^{-s})$.

Therefore as $\mathfrak{z} = 2n+1$ with $n \ge 1$ and condition $(14.7.4)_\pm$ is fulfilled, one needs to analyze only zone

$$\mathcal{Y}_{\mathsf{trans}} := \{\epsilon\rho_1 \le |\eta| \le C\rho_1\}, \qquad \rho_1 = (\mu h^\nu)^{\nu/(\nu+1)} \tag{14.8.22}$$

and for even ν we need to consider here only $\eta > 0$.

In the zone $\mathcal{Y}_{\mathsf{trans}}$ propagation speed with respect to x_2 is exactly of magnitude[34)]

$$\lambda_{n,\eta} \asymp (\mu h^\nu)^{2/(\nu+1)}\rho^{-2/\nu-2} \asymp \rho^{-1}$$

(with $\rho = \rho_1$) due to Proposition 14.A.5 again[35)] and the propagation speed with respect to η is $O(1)$ and therefore one can take

$$T_* = C\hbar|\log\hbar|, \qquad T^* = \epsilon_1\rho_1 \tag{14.8.23}$$

and for $T \in [T_*, T^*]$ propagation on the energy levels $\tau \in [-\epsilon, \epsilon]$ which started in $B(0, \frac{1}{2})$ does not leave $B(0,1)$ but the shift with respect to x_2 is $\asymp \rho^{-1}T$ and it satisfies logarithmic uncertainty principle and thus the spectral trace is negligible.

Remark 14.8.10. These arguments are easily extended to the case when $\boldsymbol{a}$ does not coincide with $\boldsymbol{a}_0$ since perturbation will change speed by $O(\hbar)$ and original zone $\{|x_1| \le \epsilon\}$ becomes $\{|\eta| \le \epsilon\hbar^{-1}\}$.

Therefore[34)] expression

$$|F_{t\to\hbar^{-1}\tau}\bar{\chi}_T(t)(Qu)| \tag{14.8.24}$$

does not exceed $C\hbar^{-1}\rho T_* = C\rho|\log\mu|$ where Q is a partition element corresponding to $\mathcal{Y}_{\mathsf{trans}}$, $|\tau| \le \epsilon$.

Then due to the Tauberian arguments the contribution of this zone to the remainder is $O(\hbar^{-1}T_*/T^*) = O(\log\mu)$. One can get rid of this superficial logarithmic factor both in the estimate of (14.8.24) and in the remainder estimate; standard details we leave to the reader. Thus we arrive to the following proposition:

[34)] After rescaling $t_{\mathsf{new}} = \hbar h^{-1}t_{\mathsf{old}} = (\mu h^\nu)^{-1/(\nu+1)}t_{\mathsf{old}}$. Recall that $\hbar = (\mu^{-1}h)^{1/(\nu+1)}$.

[35)] At least if either $\boldsymbol{a} = \boldsymbol{a}_0$.

Proposition 14.8.11. *Let* $\mathfrak{z} = 2n+1$ *with* $n \ge 1$ *and conditions* $(14.7.4)_\pm$ *and* (14.8.3) *be fulfilled. Then as* Q *is supported in the zone* $\mathcal{Y}_{\mathsf{trans}}$ *expression* (14.8.24) *does not exceed* $C\rho_1$ *and the contribution of* $\mathcal{Y}_{\mathsf{trans}}$ *to the Tauberian remainder is* $O(1)$.

On the other hand we would like to note that eigenvalue $\lambda_l(\eta)$ tends to $+0$ as $|\eta| \to +\infty$ and therefore we conclude that

Proposition 14.8.12. *Let condition* (14.8.3) *be fulfilled. Let* $1 \ne \mathfrak{z} \in (2\mathbb{Z}^+ + 1)1$ *and condition* $(14.7.4)_-$ *be fulfilled. Then zone* $\mathcal{Y}_{\mathsf{trans}}$ *is elliptic and its contribution to the remainder is* $O(\mu^{-s})$,

Regular Zone: Case $\mathfrak{z} = 1$

Let us consider the special case $\mathfrak{z} = 1$; recall that then only eigenvalue $\lambda_0(\eta)$ should be considered and that condition $(14.7.4)_-$ leads then to the asymptotics with the principal part 0 and remainder estimate $O(h^s)$ and therefore is excluded from the further consideration.

Further, as ν is odd $\lambda_0 = 0$ identically, condition $(14.7.4)_+$ provides ellipticity everywhere. Thus we arrive to

Proposition 14.8.13. *Let* $\mathfrak{z} = 1$, ν *be odd and conditions* $(14.7.4)_-$ *and* (14.8.3) *be fulfilled. Then the remainder estimate is* $O(\mu^{-s})$ *while the principal part is given by* (14.7.45).

On the other hand, if $\mathfrak{z} = 1$, ν is even and condition $(14.7.4)_-$ is fulfilled then due to Proposition 14.A.6 $\lambda_0(\eta)$ decays exponentially and ellipticity is violated only in the strip

$$\mathcal{Y}_{\mathsf{trans}} = \{\epsilon_1\Delta \le |\eta - \sigma\rho_0| \le C\Delta\}, \quad \text{with } \sigma \asymp |\log\rho_0|^{\nu/(\nu+1)} \text{ and } \Delta = \rho_0 |\log\rho_0|^{-1/(\nu+1)} \tag{14.8.25}$$

where as before $\rho_0 = (\mu h^\nu)^{1/(\nu+1)}$. In this strip propagation speed with respect to x_2 is $\asymp \Delta^{-1}$ and again

$$T_* = C\hbar\Delta|\log\hbar|, \qquad T^* = \epsilon\Delta \tag{14.8.26}$$

and expression (14.8.24) does not exceed $C\hbar^{-1}\Delta T_* = C\Delta|\log h|$ and therefore the Tauberian remainder is $O(|\log h|)$. Further, by the standard arguments one can get rid of the superficial logarithmic factors. Thus we arrive to

Proposition 14.8.14. *Let* $\mathfrak{z} = 1$, ν *be even and conditions* $(14.7.4)_+$ *and* (14.8.3) *be fulfilled. Then the contribution of* $\mathcal{Y}_{\mathsf{trans}}$ *to the remainder is* $O(1)$ *while the principal part is given by* (14.7.45) *with* T_0, T_1 *defined by* (14.8.26).

We leave details to the reader who can also look to Section 15.4 [32] for more detailed analysis as there all eigenvalues tend to their limits exponentially.

Regular Zone: Case of Condition (14.7.5)

Now we want to derive estimates under condition (14.7.5) instead of conditions $(14.7.4)_\pm$. Without condition $(14.7.4)_\pm$ some zones cease to be elliptic and should be reexamined:

(14.8.27) As $\mathfrak{z} = 2l+1$, $l \ge 1$ these zones are $\mathcal{Y} = \{\eta \ge \rho_1\}$ with ρ_1 defined by (14.8.22);

(14.8.28) As $\mathfrak{z} = 1$, ν even this zone is $\{|\eta| \ge C\rho_0 |\log \rho_0|^{\nu/(\nu+1)}\}$ with ρ_0 defined by (14.8.25);

(14.8.29) As $\mathfrak{z} = 1$, ν odd this zone is $\{|\eta| \le \epsilon\hbar^{-1}\}$.

Since condition (14.7.5) provides $T_* = C\hbar|\log\hbar|$ anyway the standard partition-rescaling arguments in all these zones imply that their total contribution to the remainder are $O(\log\mu)$; however additional arguments of the proof of Proposition 14.7.18 allow us to reduce it to $O(1)$. Therefore we arrive to

Proposition 14.8.15. *Let conditions* (14.8.3) *and* (14.7.5) *be fulfilled. Then the remainder estimate is* $O(1)$ *while the principal part of the asymptotics is given by* (14.7.45) *for different zones with any* $T \in [T_*, T^*]$, $T_* = C\hbar|\log\hbar|$ *and* T^* *defined as in Propositions 14.8.11–14.8.14.*

14.8.4 Calculations and Main Results

In this subsection we derive in more explicit form the principal parts of asymptotics.

Degeneration Zone and Case $\mathfrak{z} \notin (2\mathbb{Z}^+ + 1)$

Consider degeneration zone $\{|\eta| \le C_0\}$. Then the standard method of successive approximations allows us to replace Tauberian expression with

$$(2\pi\hbar)^{-1} \int \mathsf{e}(x_2, \eta, 0)\psi(x_2)\varphi(\eta)\, dx_2 d\eta \tag{14.8.30}$$

with the same error as in Proposition 14.8.7. In similar way we can shed from operator $\boldsymbol{a}$ all terms which are $O(\hbar)$. We leave details to the reader; they are no different from those in Subsubsection 15.4.4.2 *Boundary Zone*[32].

This covers completely degenerate cases when ellipticity condition is fulfilled as $\{|\eta| \ge C_0\}$ and we arrive to the following statement except the case of condition $(14.7.4)_+$:

Theorem 14.8.16. *Let condition* (14.8.3) *be fulfilled. Assume also that* <u>*either*</u> $\mathfrak{z} \notin (2\mathbb{Z}^+ + 1)$ <u>*or*</u> $\mathfrak{z} \in (2\mathbb{Z}^+ + 1)$ *and one of condition* $(14.7.4)_\pm$ *is fulfilled. Then* R^* *defined by* (14.7.32) *is*

(i) $O(\mu^{-s})$ *under condition* (14.8.14).

(ii) $O(1)$ *provided one of conditions* $(14.8.11)_1$, (14.7.5) *is fulfilled.*

(iii) $O(|\log \mu|)$ *if both* $(14.3.111)_2$ *and* $(14.8.11)_2$ *are fulfilled; further, if both* $\lambda_{n,\eta\eta}(\eta)$ *and* $-W_{x_2x_2}(x_2)$ *have the same signs as*

$$|\lambda_n(\eta)| + |\lambda_{n,\eta}(\eta)| + |W_{x_2}(x_2)| \le \epsilon.$$

this estimate is actually $O(1)$.

We also arrive to

Theorem 14.8.17. *Let conditions* (14.8.3) *and* $(14.7.4)_\pm$ *be fulfilled,* ν *be odd and* $\mathfrak{z} = 1$. *Then Tauberian remainder and* R_I *are* $O(\mu^{-s})$*; meanwhile the principal part of asymptotics is* 0 *under condition* $(14.7.4)_-$.

Calculations under Conditions $(14.7.4)_\pm$

Consider now case not covered by the previous subsubsection, however we assume that $(14.7.4)_+$ is fulfilled. Then for $\mathfrak{z} = (2n+1)$ we need to consider only zone $\mathcal{Y}_{\mathsf{trans}}$ defined by (14.8.22) or (14.8.25) respectively as $n \ge 1$ and $n = 0$. Details are similar to those in Subsubsection 15.4.4.3 *Transitional Zone*[32] but simpler; we leave them to the reader. This finishes justification of Theorem 14.8.16.

Calculations under Condition (14.7.5)

In this case we need to reconsider only zones $\{|\eta| \ge C\rho_1\}$ and $\{|\eta| \ge C\rho_0\}$ with ρ_1 or ρ_0 defined by (14.8.22) or (14.8.25) respectively as $n \ge 1$ and $n = 0$. Details are similar to those in Subsubsection 15.4.4.1 *Inner Zone*[32)] but simpler; we leave them to the reader. Then we arrive to

Theorem 14.8.18. *Let conditions* (14.8.3) *and* (14.7.5) *be fulfilled. Then* R^* *defined by* (14.7.32) *is* $O(1)$.

14.A Eigenvalues of 1D-Operators

General Observations

In this Appendix $\lambda_n(\eta, \mathfrak{z})$ ($n = 0, 1, \dots$) denote eigenvalues of one-dimensional toy-model Schrödinger operators with $\mu = h = 1$

$$L(\eta, \mathfrak{z}) \coloneqq a_0(\eta, \mathfrak{z}, 1, 1) = D^2 + (\eta - x^\nu/\nu)^2 - \mathfrak{z}x^{\nu-1} \tag{14.A.1}$$

with $\nu = 2, 3, \dots$.

If there were parameters μ, h

$$a_0(\eta, \mathfrak{z}, \mu, h) = h^2D^2 + (\eta - \mu x^\nu/\nu)^2 - \mathfrak{z}\mu h x^{\nu-1} \tag{14.A.2}$$

then scaling $x \mapsto x\gamma$, $D \mapsto \gamma^{-1}D$ with $\gamma = (\mu^{-1}h)^{-1/(\nu+1)}$ we get operator $(\mu h^\nu)^{2(\nu+1)}L(\eta', \mathfrak{z})$ with $\eta' = (\mu h^\nu)^{-1/(\nu+1)}$.

We are interested in operator (14.A.1) with fixed $\mathfrak{z} \in \mathbb{R}$ and variable η. One can prove easily the following statement:

Proposition 14.A.1. *Let* $\mathfrak{z} \in \mathbb{R}$. *Then*

(i) As $|\eta| \le C_0$ *the spacing between two consecutive eigenvalues* $\lambda_n(\eta, \mathfrak{z})$ *and* $\lambda_{n+1}(\eta, \mathfrak{z})$ *with* $n \le c_0$ *is* $\asymp 1$.

(ii) For operator $L(\eta, \mathfrak{z})$ *with odd* ν $\lambda_n(-\eta, \mathfrak{z}) = \lambda_n(\eta, \mathfrak{z})$.

(iii) For operator $L(\eta, \mathfrak{z})$ *with even* ν *and* $\eta \le 0$ $\lambda_n(\eta, \mathfrak{z}) \ge (1 - \epsilon)\eta^2 - C_1$ $\forall n = 0, 1, \dots$.

However, the case of even ν and $\eta \to -\infty$ is rather exceptional.

14.A.1 Asymptotics as $\eta \to +\infty$

Remark 14.A.2. Observe that

(i) As ν is odd operators in question have for large η one potential well with the bottom near $x = \ell = (\nu\eta)^{1/\nu}$; in the case of $\mathfrak{z} = 0$ bottom is exactly in $x = \ell$.

(ii) As ν is even and $\eta \ge C_0$ there will be two distant potential wells with the bottoms near $x = \pm\ell = \pm(\nu\eta)^{1/\nu}$; in the case of $\mathfrak{z} = 0$ these wells are symmetric and bottoms are exactly at $\eta = \pm\ell$.

To derive asymptotics of $\lambda_n(\eta, \mathfrak{z})$ as $\eta \to +\infty$ (and thus in the case of odd ν also as $\eta \to -\infty$) let us consider these wells (separately as ν is even).

Let us plug $\eta = \ell^\nu/\nu$ with $\ell \gg 1$ where in the case even ν this is the only scenario and in the case of odd ν analysis of scenario $\xi_2 = -\ell^\nu/\nu$ is done by the symmetry. Then after shift $x \mapsto x + \ell$ operator $L(\eta, \mathfrak{z})$ is transformed into operator

$$D^2 + x^2\Big(\ell^{\nu-1} + \frac{1}{2}(\nu-1)x\ell^{\nu-2} + \frac{1}{6}(\nu-1)(\nu-2)x^2\ell^{\nu-3} + \dots\Big)^2 - \mathfrak{z}\Big(\ell^{\nu-1} + (\nu-1)x\ell^{\nu-2} + \frac{1}{2}(\nu-1)(\nu-2)x^2\ell^{\nu-3} + \dots\Big)$$

and after rescaling $x \mapsto x\ell^{(1-\nu)/2}$ this operator is transformed into $\ell^{\nu-1}\boldsymbol{b}_\varepsilon$ where

$$\boldsymbol{b}_\varepsilon = D^2 + x^2\Big(1 + \frac{1}{2}(\nu-1)x\varepsilon + \frac{1}{6}(\nu-1)(\nu-2)x^2\varepsilon^2 + \dots\Big)^2 - \mathfrak{z}\Big(1 + (\nu-1)x\varepsilon + \frac{1}{2}(\nu-1)(\nu-2)x^2\varepsilon^2 + \dots\Big)$$

with $\varepsilon = \ell^{-(\nu+1)/2}$. Then

$$\boldsymbol{b}_\varepsilon = \underbrace{D^2 + x^2 - \mathfrak{z}}_{\boldsymbol{h}_0} + \varepsilon \underbrace{(\nu-1)\Big(x^3 - \mathfrak{z}x\Big)}_{\boldsymbol{h}_1} + \varepsilon^2 \underbrace{(\nu-1)\Big((\frac{7}{12}\nu - \frac{11}{12})x^4 - \frac{1}{2}\mathfrak{z}(\nu-2)x^2\Big)}_{\boldsymbol{h}_2} + O(\varepsilon^3) \tag{14.A.3}$$

which immediately implies

Proposition 14.A.3. *(i) As ν is odd for $n \le c_0$, $|\eta| \gg 1$*

$$\lambda_n(\eta, \mathfrak{z}) = (2n+1-\mathfrak{z})(\eta\nu)^{(\nu-1)/\nu} + O(\eta^{-2/\nu}); \tag{14.A.4}$$

(ii) As ν is even then for $|\eta| \gg 1$, $\eta > 0$ there are two series of eigenvalues

$$\lambda_{\pm,n}(\eta, \mathfrak{z}) = (2n+1\mp\mathfrak{z})(\eta\nu)^{(\nu-1)/\nu} + O(\eta^{-2/\nu}). \tag{14.A.5}$$

Proof. Indeed, the standard theory just gives

$$\lambda_n(\eta, \mathfrak{z}) = \big(2n+1-\mathfrak{z}+O(\varepsilon^2)\big)\ell^{\nu-1}$$

in both Statements (i) and (ii) as we plug $x \mapsto x+\ell$ and also

$$\lambda_n(\eta, \mathfrak{z}) = \big(2n+1+\mathfrak{z}+O(\varepsilon^2)\big)\ell^{\nu-1}$$

in Statement (ii) only as we plug $x \mapsto x-\ell$. We eliminate linear terms by $O(\varepsilon)$ shift with respect to x. □

This leaves us with

Corollary 14.A.4. *(i) As ν is odd and $\eta \ge C_0$ the spacing between eigenvalues with $n \le c_0$ is $\asymp (1+|\eta|)^{(\nu-1)/\nu}$; further, eigenvalues with $2n+1 \lessgtr \mathfrak{z}$ are $\asymp \eta^{(\nu-1)/\nu}$ and negative (positive respectively).*

(ii) As ν is even the spacing between eigenvalues in the same series with $n \le c_0$ is $\asymp (1+|\eta|)^{(\nu-1)/\nu}$; further, eigenvalues $\lambda_{\pm,n}(\eta, \mathfrak{z})$ with $2n+1 \lessgtr \pm\mathfrak{z}$ are $\asymp \eta^{(\nu-1)/\nu}$ and negative (positive respectively).

This completely describes eigenvalue $\lambda_n(\eta, \mathfrak{z})$ as $\mathfrak{z} \ge 0$ and $\mathfrak{z} \ne 2n+1$.

To describe case $2n+1 = \mathfrak{z}$ let us denote by Ω_ε and U_ε its eigenvalue close to 0 and the corresponding eigenfunction. Then

$$\Omega_\varepsilon = \omega_1\varepsilon + \omega_2\varepsilon^2 + ... \qquad \text{and} \qquad U_\varepsilon = u_0 + u_1\varepsilon + u_2\varepsilon^2 + ... \tag{14.A.6}$$

where obviously $u_0 = v_l$ is a Hermite function, $\omega_1 = \omega_3 = ... = 0$ and

$$\boldsymbol{h}_0 u_1 + \boldsymbol{h}_1 u_0 = 0 \qquad \boldsymbol{h}_0 u_2 + \boldsymbol{h}_1 u + \boldsymbol{h}_2 u_0 = \omega_2 u_0. \tag{14.A.7}$$

Therefore

$$\omega_2 = \langle \boldsymbol{h}_1 u + \boldsymbol{h}_2 u_0, u_0\rangle = -\langle u, \boldsymbol{h}_0 u\rangle + \langle \boldsymbol{h}_2 u_0, u_0\rangle. \tag{14.A.8}$$

It is known that $(x-iD)\upsilon_k = (2k+2)^{1/2}\upsilon_{k+1}$, $(x+iD)\upsilon_k = (2k)^{1/2}\upsilon_{k-1}$ and therefore

$$x\upsilon_n = \tfrac{1}{2}\Big((2n+2)^{1/2}\upsilon_{n+1} + (2n)^{1/2}\upsilon_{n-1}\Big),$$

$$x^2\upsilon_n = \tfrac{1}{4}\Big((2n+2)^{1/2}(2n+4)^{1/2}\upsilon_{n+2} + (4n+2)\upsilon_n + (2n)^{1/2}(2n-2)^{1/2}\upsilon_{n-2}\Big),$$

$$\big(x^2-2n-1\big)\upsilon_n = \tfrac{1}{4}\Big((2n+2)^{1/2}(2n+4)^{1/2}\upsilon_{n+2} - 2(2n+1)\upsilon_n + (2n)^{1/2}(2n-2)^{1/2}\upsilon_{n-2}\Big),$$

$$x\big(x^2-2n-1\big)\upsilon_n = \tfrac{1}{8}\Big((2n+2)^{1/2}(2n+4)^{1/2}(2n+6)^{1/2}\upsilon_{n+3} - (2n+2)^{1/2}(2n-2)\upsilon_{n+1} - (2n)^{1/2}(2n+4)\upsilon_{n-1} + (2n)^{1/2}(2n-2)^{1/2}(2n-4)^{1/2}\upsilon_{n-3}\Big),$$

which imply

$$\langle \boldsymbol{h}_0 u, u\rangle = \frac{1}{64}(\nu-1)^2\Big(\frac{1}{6}(2n+2)(2n+4)(2n+6) + \frac{1}{2}(2n+2)(2n-2)^2 - \frac{1}{2}(2n)(2n+4)^2 - \frac{1}{6}(2n)(2n-2)(2n-4)\Big) = \frac{1}{16}(\nu-1)^2\big(-2n^2-2n+3\big).$$

On the other hand

$$\langle \boldsymbol{h}_2 u_0, u_0\rangle = (\nu-1)\Big((\frac{7}{12}\nu - \frac{11}{12})\|x^2u_0\|^2 - \frac{1}{2}(\nu-2)(2n+1)\|xu_0\|^2\Big) = (\nu-1)(\frac{7}{12}\nu-\frac{11}{12})\cdot\frac{1}{16}\Big((2n+2)(2n+4) + (4n+2)^2 + (2n)(2ln-2)\Big) - \frac{1}{4}(\nu-1)(\nu-2)\cdot(2n+1)^2 = (\nu-1)(7\nu-11)\cdot\frac{1}{16}\Big(2n^2+2n+1\Big) - \frac{1}{4}(\nu-1)(\nu-2)(2n+1)^2$$

and

$$\omega_2 = \frac{1}{16}(\nu-1)\Big((7\nu-11)\big(2n^2+2n+1\big) - 4(\nu-2)\big(4n^2+4n+1\big) - (\nu-1)\big(-2n^2-2n+3\big)\Big) = \frac{1}{2}(\nu-1)n(n+1),$$

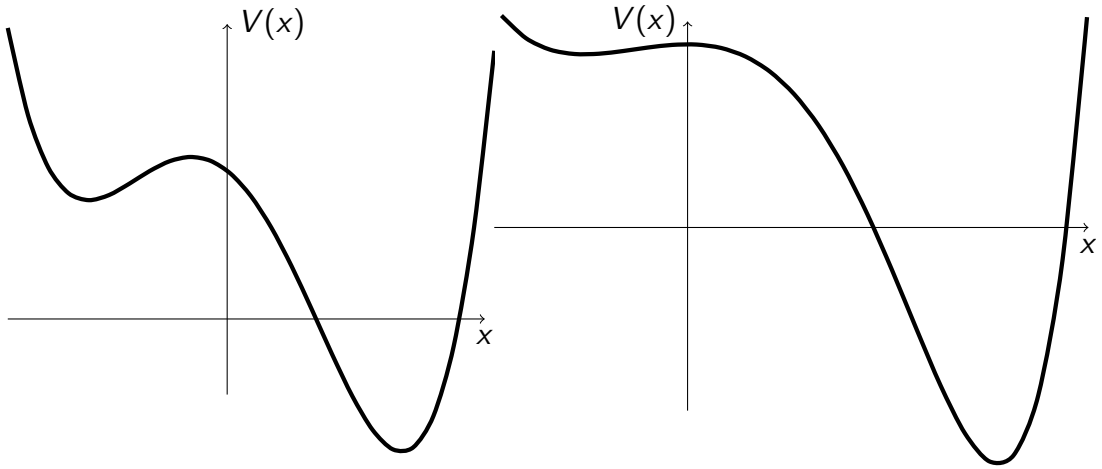
Therefore $\Omega_\varepsilon = \omega_2\varepsilon^2 + O(\varepsilon^4)$ as $\varepsilon \to 0$ (because $\omega_3 = 0$ as well) which implies the following statement with $\kappa = \omega_2\nu^{-2/\nu}$:

Proposition 14.A.5. *(i) Let* ν *be odd,* $\mathfrak{z} = 2n+1$, $n \geq 1$. *Then as* $|\eta| \to +\infty$

$$\lambda_n(\eta, \mathfrak{z}) = \kappa \eta^{-2/\nu} + O\big(\eta^{-(\nu+3)/\nu}\big) \tag{14.A.9}$$

with $\kappa > 0$.

(ii) Let ν *be even,* $\mathfrak{z} = 2n+1$, $n \geq 1$. *Then as* $\eta \to +\infty$ (14.A.9) *holds for* $\lambda_{+,n}(\eta, 2n+1)$.



(a) ν even (b) ν odd

Figure 14.8: Graphs of $V(x) = V(x; \eta, \mathfrak{z})$, with $\mathfrak{z} > 0$; one see two wells but the right one is much deeper.

14.A.2 Case of $\mathfrak{z} = 1$

Here cases of odd and even ν differ drastically. Note first that

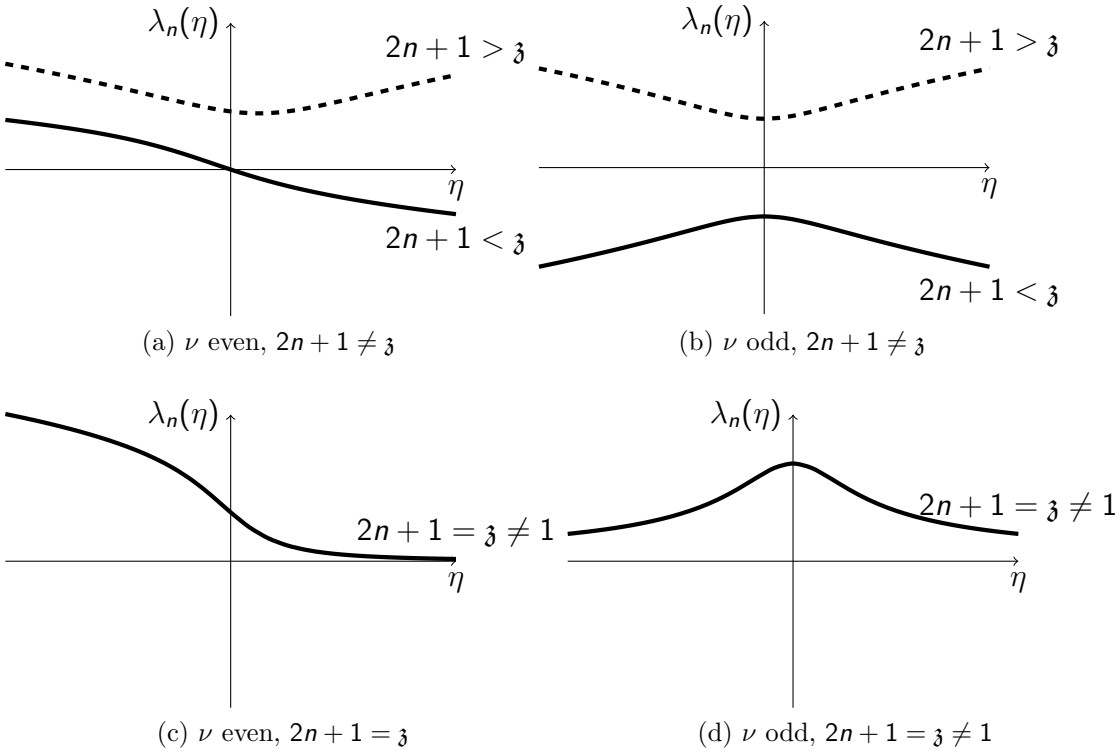
$$L(\eta, 1) = \big(iD + \eta - x^\nu/\nu\big)\big(-iD + \eta - x^\nu/\nu\big) = \boldsymbol{b}(\eta)^* \boldsymbol{b}(\eta) \tag{14.A.10}$$

with

$$\boldsymbol{b} = -\partial + \eta + x^\nu/\nu \tag{14.A.11}$$

and as ν is odd operator $L(\eta, 1)$ has the bottom eigenvalue $\lambda_0(\eta, 1)$ with eigenfunction defined from equation $\boldsymbol{b}v = \big(-\partial + \xi_2 - x^\nu/\nu\big)v = 0$ i.e.

$$v(x, \eta) = \exp(\eta x - x^{\nu+1}/\nu(\nu+1)) \tag{14.A.12}$$



(a) ν even, $2n+1 \neq \mathfrak{z}$ (b) ν odd, $2n+1 \neq \mathfrak{z}$

(c) ν even, $2n+1 = \mathfrak{z}$ (d) ν odd, $2n+1 = \mathfrak{z} \neq 1$

Figure 14.9: Sketches of graphs of $\lambda_n(\eta, \mathfrak{z})$: not the actual calculations.

and therefore $\lambda_0(\eta, 1)$ is identically 0.

Thus we arrive to Statement (i) of the following proposition:

Proposition 14.A.6. *(i) For odd ν the bottom eigenvalue of $L(\eta, 1)$ is identically 0.*

(ii) For even ν the bottom eigenvalue of $L(\eta, 1)$ is $\lambda_0(\eta, 1)$ such that

$$\lambda_0(\eta, 1) \sim 2(\nu\eta)^{(\nu-1)/\nu} \exp(-\kappa\eta^{(\nu+1)/\nu}), \qquad \kappa = 4\nu(\nu+1)/\nu/(\nu+1), \tag{14.A.13}$$

$$\partial_\eta^k \lambda_0(\eta, 1) \sim 2(-\kappa(\nu+1)/\nu)^k \eta^{k/\nu} \lambda_0(\eta, 1) \qquad k = 1, 2, ... \tag{14.A.14}$$

Proof. We need to consider the case of even ν only. While one can prove this statement based on construction of quazimodes (see B. Helffer–M. Klein–F. Nier [1] or A. Bovier–M. Eckhoff–V. Gayrard–M. Klein [1].

We, using basically the same idea will give a simplified "in-house" proof because we are interested in a single bottom eigenvalue $\lambda_0(\eta,1)$. While the same decomposition (14.A.10)–(14.A.11) holds for even ν but then operator $\boldsymbol{b}(\eta)$ has no null-space as $v(x,\eta)$ grows as $x\to-\infty$ and we conclude that $\lambda_0(\eta,1)>0$.

Here we skip in notations $\mathfrak{z}=1$. Let us modify operator and decomposition and define $M(\eta)$ by (14.A.10) but with

$$\boldsymbol{b}(\eta)=-\partial+(\eta-x^{\nu}/\nu)\,\mathsf{sgn}(x+\ell),\qquad \ell=(\nu\eta)^{1/\nu}; \tag{14.A.15}$$

then

$$M(\eta)=L(\eta)+2x^{\nu-1}\theta(-x-\ell) \tag{14.A.16}$$

and

$$V(x,\eta)=\begin{cases}\exp\bigl(\eta x-x^{\nu+1}/\nu(\nu+1)\bigr) & x>-\ell,\\ \exp\bigl(-\eta x+x^{\nu+1}/\nu(\nu+1)\bigr) & x<-\ell\end{cases} \tag{14.A.17}$$

annuls $\boldsymbol{b}(\eta)$. Note that $V(x,\eta)$ reaches its single maximum $\exp(\ell^{\nu+1}/(\nu+1))$ as $x=\ell$ and then as we define

$$v(x,\eta)=\pi^{-\frac14}\ell^{(\nu-1)/4}\exp\bigl(-s(x)\bigr) \tag{14.A.18}$$

with

$$s(x)=\begin{cases} S(x) & \text{as } x>-\ell,\\ -S(x)+2S(-\ell) & \text{as } x\le -\ell\end{cases}, \tag{14.A.19}$$

$$S(x)=-(x-\ell)\eta+(x^{\nu+1}-\ell^{\nu+1})/\nu(\nu+1), \tag{14.A.20}$$

we get $\|v\|\sim 1$. Here we took an extra factor $\sqrt{S''(\ell)/\pi}$ to ensure this as $S(\ell)=S'(\ell)=0$, $S''(\ell)=\ell^{\nu-1}$.

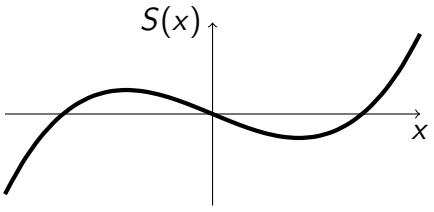


Figure 14.10: $S(x)$

Meanwhile

$$(L(\eta)v,v)=2\int_{-\infty}^{-\ell}|x|^{\nu-1}|v|^2\,dx\sim \ell^{\nu-1}\exp\bigl(-2S(-\ell)\bigr)= \tag{14.A.21}$$

$$\ell^{\nu-1}\exp\bigl(-4\ell^{\nu+1}/(\nu+1)\bigr)=:\varepsilon(\ell)$$

and

$$\|L(\eta_0)v\|^2 \sim 2\ell^{2(\nu-1)}\exp\bigl(-4\ell^{\nu+1}/(\nu+1)\bigr) = 2\ell^{\nu-1}\varepsilon(\ell) \tag{14.A.22}$$

as $S(-\ell) = 2\ell^{\nu+1}/(\nu+1)$, $S'(-\ell) = 0$, $S''(-\ell) = -S''(\ell)$.

Therefore

$$\lambda_0(\eta) \le \varepsilon(\ell)\bigl(1+o(1)\bigr). \tag{14.A.23}$$

So, upper estimate for $\lambda_0(\eta)$ has been proven.

Let u_n be all orthonormal eigenfunctions. As $\lambda_n \gtrsim \ell^{\nu-1}$ for $n \ge 1$ we conclude from

$$\lambda_n(\eta)(v,u_n) = (L(\eta)v,u_n) \le \|L(\eta)v\| \le c\ell^{(\nu-1)/2}\varepsilon^{1/2}$$

that

$$|(v,u_n)| \le c\ell^{(1-\nu)/2}\varepsilon^{1/2} \qquad \forall n \ge 1, \qquad (v,u_0) = 1 + O(\ell^{1-\nu}\varepsilon) \tag{14.A.24}$$

as $\|v\| = 1 + O(\ell^{1-\nu}\varepsilon)$, $\|u_0\| = 1$.

On the other hand

$$\lambda_0(eta) = \|(-\partial+\eta-x^\nu/\nu)u_0\|^2 \ge \|(-\partial+\eta-x^\nu/\nu)u_0\|^2_{x<y} = \|\partial u_0\|^2_{x<y} + \int_{-\infty}^{y}\bigl((\eta-x^\nu/\nu)^2 - x^{\nu-1}\bigr)|u_0|^2\,dx - (\eta-y^\nu/\nu)|u(y)|^2 \tag{14.A.25}$$

and picking $y = \ell - z$, $0 < z \asymp \gamma \ll \ell$ we conclude that

$$|u(-\ell-z)|^2 \le c\ell^{1-\nu}\gamma^{-1}\lambda_0 \asymp \lambda_0 \tag{14.A.26}$$

as $\gamma \asymp \ell^{1-\nu}$. Then scaling $x \mapsto (x+\ell)/\gamma$ with we find that in the new scale in equation (multiplied by γ^2) all large parameters disappear and then as this estimate (14.A.26) holds for $\partial_z u$ (after rescaling) as well and therefore

(14.A.27) Estimate (14.A.26) with c replaced by $C(t)$ holds for all $z : |z| \le t\gamma$.

Then $\|u\|_{x<-\ell+t\gamma} \le C(t)\lambda_0^{1/2}$ and taking into account (14.A.24) we conclude that $\|v\|_{x<-\ell+t\gamma} \le C(t)\lambda_0^{1/2} + C_0\varepsilon^{1/2}$. However the left-hand expression is $C_1(t)\varepsilon^{1/2}$ with $C_1(t) \ge 2C_0$ large for large enough $t > 0$. Then $\varepsilon \le C\lambda_0$ and lower estimate for λ_0 has been proven.

Now, we need to prove exact asymptotics. Obviously

$$0 = (L(\eta)v, u)_{x>-\ell} = (v, M(\eta)u)_{x>-\ell} + (uv_x - vu_x)|_{x=-\ell} = \lambda_0(\eta)(u, v)_{x>-\ell} - v(-\ell)\partial u(-\ell)$$

because $v_x(-\ell) = 0$ and therefore

$$\lambda_0(\ell) = v(-\ell, \ell)u_x(-\ell, \ell) + O\big(\lambda_0 \varepsilon^{1-\delta}\big) \tag{14.A.28}$$

where we consider ℓ rather than η as a main variable.

Let $v_1 = \pi^{-\frac{1}{4}} \ell^{(\nu-1)/4} \exp\big(-S(x)\big)$ and v_2 another element of null-space of $L(\eta)$ satisfying $v_2' v_1 - v_2 v_1' = 1$ and decaying as $x \to -\infty$:

$$v_2 = v_1 \int_{-\infty}^{x} v_1^{-2}(y)\, dy. \tag{14.A.29}$$

Then $L(\eta)u_0 = \lambda_0 u_0$ implies that

$$u_0 = \alpha v_2 - \lambda v_1 \int_{-\infty}^{x} v_2 u\, dy + \lambda v_2 \int_{-\infty}^{x} v_1 u\, dy$$

and by method of successive approximations we conclude that modulo much smaller terms it is just αv_2 and in order to be consistent with u_0 as $x > -\ell$

$$\alpha v_2(-\ell) \equiv v_1(-\ell), \qquad \alpha v_2'(-\ell) = u'(-\ell) \equiv \lambda_0 / v_1(-\ell)$$

as $v_1'(-\ell) = 0$, $v_2'(-\ell) = 1/v_1(-\ell)$ we conclude that

$$\lambda_0(\eta) \equiv v_1(-\ell)/v_2(-\ell) = \Big(\int_{-\infty}^{-\ell} v_1(x)^{-2}\, dx\Big)^{-1} = \operatorname{const}\big(\ell^{(1-\nu)/2} \int_{-\infty}^{-\ell} e^{2S(x)}\, dx\big)^{-1} \sim \ell^{1-\nu} e^{-2S(-\ell)}. \tag{14.A.30}$$

exactly as we claimed. □

Remark 14.A.7. In B. Helffer–M. Klein–F. Nier [1] full asymptotics was derived in much more general settings. In our setting our arguments lead also to full asymptotics.

14.A.3 Generalizations

Consider more general operator

$$L(\eta,\Lambda)=(1+\alpha_1 x+\beta_1^2x^2)D^2+(1+\alpha_2x+\beta_2^2x^2)(\eta-x^\nu/\nu)^2-\mathfrak{z}(1+\alpha_3x)x^{\nu-1} \tag{14.A.31}$$

with $\nu=2,3,...$ and $\beta_j>\alpha_j^2/4$ and also

$$a(\eta,\Lambda,\mu,h)=(1+\alpha_1x+\beta_1^2x^2)h^2D^2+(1+\alpha_2x+\beta_2^2x^2)(\eta-\mu x^\nu/\nu)^2-\mathfrak{z}\mu h(1+\alpha_3x)x^{\nu-1}; \tag{14.A.32}$$

then the same scaling as Subsection 14.A reduces latter to the former.

Proposition 14.A.8. *(i) For operator* (14.A.31) *statements of Propositions 14.A.1, 14.A.3, 14.A.5, 14.A.6 and Corollary 14.A.4 remain true.*

(ii) For operator (14.A.31) *with* $(2\mathbb{Z}^+ +1)\ni\mathfrak{z}\neq 1$ *as* $\eta\to+\infty$ *(and thus as* $\eta\to-\infty$ *and* ν *is odd)*

$$\partial_{\alpha_j}\lambda_l(\eta)\big|_{\alpha=\beta=0}=\kappa_j\eta+O(\eta^{-1/\nu}) \tag{14.A.33}$$

with $\kappa_1=\kappa_2=-\kappa_3/2$, $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, $\beta=(\beta_1,\beta_2,\beta_3)$ *and furthermore*

$$\sum_{1\le j\le 3}\partial_{\alpha_j}\lambda_l(\eta)\big|_{\alpha=\beta=0}=\kappa_4\eta^{1/\nu}\lambda_l+O(\eta^{-2/\nu}). \tag{14.A.34}$$

Proof. (i) Proofs are rather easy and we leave them to the reader.

(ii) After obvious transformations

$$\partial_{\alpha_j}\lambda_l(\eta)\big|_{\alpha=\beta=0}=\gamma^{\nu-1}\langle \boldsymbol{k}_jU_\varepsilon,U_\varepsilon\rangle$$

with

$$\begin{aligned}
&\boldsymbol{k}_1=(\gamma+\varepsilon x)D^2,\\
&\boldsymbol{k}_2=x^2\Big(1+\frac{1}{2}(\nu-1)x\varepsilon+\frac{1}{6}(\nu-1)(\nu-2)x^2\varepsilon^2+...\Big)^2\\
&\boldsymbol{k}_3=-\Lambda\Big(1+(\nu-1)x\varepsilon+\frac{1}{2}(\nu-1)(\nu-2)x^2\varepsilon^2+...\Big)
\end{aligned}$$

and therefore

$$\langle \mathbf{k}_j U_\varepsilon, U_\varepsilon\rangle = \gamma\langle \mathbf{k}'_j u_0, u_0\rangle + O\big(\varepsilon^2\gamma\big)$$

with $\mathbf{k}'_1 = D^2$, $\mathbf{k}'_2 = x^2$, $\mathbf{k}'_3 = -\Lambda$ which implies (14.A.33).

Known equalities $\langle x^2 v_l, v_l\rangle = \langle D^2 v_l, v_l\rangle = \Lambda/2$ imply that $\kappa_1 = \kappa_2 = -\kappa_3/2$. Further, $\sum_{1\le j\le 3}\langle \mathbf{k}_j U_\varepsilon, U_\varepsilon\rangle = \gamma\lambda_l + O(\varepsilon^2)$ which implies (14.A.34). □

Now we consider operator

$$L(y,\eta) \coloneqq \beta\Big(\alpha h^2 D^2\alpha + \alpha^{-2}(\eta - \mu x^\nu/\nu)^2 - \Lambda\mu h x^{\nu-1}\Big)\beta \tag{14.A.35}$$

with

$$\alpha = \alpha(x,y),\quad \beta = \beta(x,y),\quad \alpha(0,y) = 1,\quad c_0^{-1} \le \beta \le c_0. \tag{14.A.36}$$

Again we can reduce the general case to one with $\mu = h = 1$ to the former and to prove

Proposition 14.A.9. *For operator* (14.A.35) *statements of Propositions 14.A.1, 14.A.3, 14.A.5, 14.A.6 and Corollary 14.A.4 remain true.*

Remark 14.A.10. Instead of operator (14.A.1) with integer ν we consider operators

$$L(\eta, \mathfrak{z}) = D^2 + (\eta - |x|^\nu/\nu)^2 - \mathfrak{z} x^{\nu-1}\operatorname{sgn}(x) \tag{14.A.37}$$

and

$$L(\eta, \mathfrak{z}) = D^2 + (\eta - |x|^\nu \operatorname{sgn}(x)/\nu)^2 - \mathfrak{z} x^{\nu-1} \tag{14.A.38}$$

with $1 \le \nu \in \mathbb{R}$; these operators are similar to those with even and odd integer ν respectively.

14.A.4 Open Problems

The following problem seems to be really challenging:

Problem 14.A.11. Figure 14.9 shows plots of $\lambda_n(\eta;\mathfrak{z})$ schematically. Prove that these graphs are correct; in particular, prove the impossibility of

$$\lambda_n(\eta;\mathfrak{z}) = \lambda_{n,\eta}(\eta;\mathfrak{z}) = 0, \tag{14.A.39}$$

and

$$\lambda_{n,\eta}(\eta;\mathfrak{z}) = \lambda_{n,\eta\eta}(\eta;\mathfrak{z}) = 0. \tag{14.A.40}$$

Problem 14.A.12. Using Matlab make real calculations and construct plots of $\lambda_n(\eta; \mathfrak{z})$ for some values of $\mathfrak{z}$, n.

14.A.5 Maple for Trajectories

```
>with(DEtools):
k:= ;
b:=(1+k)^(1/2);
phaseportrait([D(x)(t)=z(t),
    D(z)(t)=2*x(t)*(k-x(t)^2),
    D(y)(t)=(k-x(t)^2)],
[x(t),y(t),z(t)], t=-5..5, [[x(0)=b,y(0)=0,z(0)=0]],
stepsize=.005,scene=[x(t),y(t)],
    scaling=constrained,linecolor=black);

>with(DEtools):
k:= ;
b:=(1+k)^(1/2);
phaseportrait([D(x)(t)=z(t),
    D(z)(t)=2*abs(x(t))*(k-x(t)*abs(x(t))),
    D(y)(t)=(k-x(t)*abs(x(t)))],
[x(t),y(t),z(t)], t=-5..5, [[x(0)=b,y(0)=0,z(0)=0]],
stepsize=.005,scene=[x(t),y(t)],
    scaling=constrained,linecolor=black);
```

Chapter 15

2D-Schrödinger Operator with the Strong Magnetic Field near the Boundary

Our goal is to derive spectral asymptotics of 2-dimensional Schrödinger operator near the boundary. So we basically want to generalize results of Chapter 13 as $d = 2$ to the case of $\operatorname{supp}\psi \cap \partial X \neq \emptyset$. We assume that condition (13.3.1) is fulfilled: $F \asymp 1$.

However it is not a simple generalization as propagation near the boundary is completely different from one inside the domain. While classical dynamics inside is a normal speed cyclotron movement combined with a slow (with the speed $O(\mu^{-1})$) magnetic drift, it is not the case near boundary: when cyclotron hits the boundary it reflects from it and we arrive to a normal speed (with the speed $O(1)$) hop movement along the boundary.

The really difficult part is that hop movement is not separated from cyclotron plus magnetic drift movement: first, as we move away from the boundary the former is replaced by the latter; second, during some hop the hop movement can be torn away from the boundary and become cyclotron plus magnetic drift movement and v.v.: cyclotron plus magnetic drift movement can collide with the boundary and become hop movement.

The main goal is to investigate the generic case as $\nabla VF^{-1} \neq 0$ and $\nabla_{\partial X} VF^{-1} = 0 \implies \nabla^2_{\partial X} VF^{-1} \neq 0$ on ∂X.

Plan of the Chapter

Section 15.1 is preliminary: first, we consider a classical dynamics, described above, in details. Then we consider a toy-model operator in the half-plane and derive precise formula for it.

In Section 15.2 we consider a weak magnetic field case $\mu h \ll 1$ when classical dynamics defines everything. Then hop movement breaks periodic cyclotron movement which allows us to prove that the contribution of this zone to the remainder is $O(\mu^{-1}h^{-1})$ where factor μ^{-1} is the width of the boundary zone and 1 is the time for which we typically follow classical dynamics. Recall that the contribution of inner zone to the remainder (under appropriate non-degeneracy conditions) also is $O(\mu^{-1}h^{-1})$ albeit there factor μ^{-1} comes from time $\asymp \mu$ for which we typically follow a classical dynamics. Sure, there is a transitional zone between boundary and inner zones but as magnetic field is not strong, it is very thin.

In Section 15.3 we study a moderate magnetic field case. In Subsections 15.3.1–15.3.3 we establish Tauberian remainder estimates under different non-degeneracy assumptions.

In Subsection 15.3.4 we study propagation of singularities in the transitional zone and find that under Dirichlet boundary condition we are able to prove better results than under Neumann boundary condition. This difference is not technical as it is the first manifestation of the fact that as magnetic field grows stronger the classical dynamics loses its value as predictor of the propagation and in the case of Neumann boundary condition some singularities propagate along the boundary in the direction opposite to hops; this is not related to a magnetic drift but rather to a different behavior of the eigenvalues of the toy-model operator.

In Subsection 15.3.5 in the case of the intermediate magnetic field we pass from estimates to calculations and derive our final results.

In Section 15.4 we consider the cases of the strong magnetic field ($\mu h \asymp 1$ and $\mu h \gg 1$ respectively; in the latter case we need to take potential $-\mu h \mathfrak{z} F + V$ rather than V to prevent pushing bottom of the spectrum to high. Here $\mathfrak{z} \ge 1$ under Dirichlet boundary condition but could be smaller under Neumann boundary condition; then spectral asymptotics are concentrated near boundary). As we know from Chapter 13 cyclotrons are no more observable due to uncertainty principle but magnetic drift preserves its sense. The same remains true near boundary: we do not see hops but we observe propagation along the boundary and it can be torn away from

the boundary and became magnetic drift and v.v.

Finally, in Section 15.5 we consider generalizations: first, we get rid of condition $V \asymp 1$ and then we get rid of condition $F \asymp 1$. In the latter case as we know from Chapter 14 our remainder estimate *must* deteriorate to $O(\mu^{-\frac{1}{2}}h^{-1})$.

Appendix 15.A is devoted to auxiliary 1-dimensional harmonic oscillator $-\partial_x^2 + x^2$ considered on $(-\infty, \eta]$ with Dirichlet or Neumann boundary condition as $x = \eta$. We study its eigenvalues $\lambda_{\mathsf{D},n}(\eta)$ and $\lambda_{\mathsf{N},n}(\eta)$ reproducing results of B. Helffer, C. Bolley and M. Dauge. These properties are crucial for our formulae and for analysis of Section 15.4.

15.1 Preliminary Discussion

15.1.1 Inner and Boundary Zones

Let us consider 2-dimensional Magnetic Schrödinger operator near the boundary. Let as usual $\gamma(x) = \frac{1}{2}\mathsf{dist}(x, \partial X)$ and consider disk (i.e. 2-dimensional ball) $B(y, \gamma(y))$. Scaling it $x \mapsto (x-y)\gamma^{-1}$ to the unit disk we get

$$h \mapsto h_{\mathsf{new}} = h\gamma^{-1}, \quad \mu \mapsto \mu_{\mathsf{new}} = \mu\gamma, \quad \nu \mapsto \nu_{\mathsf{new}} = \nu\gamma, \quad \bar{\nu} \mapsto \bar{\nu}_{\mathsf{new}} = \bar{\nu}\gamma \tag{15.1.1}$$

where recall ν and $\bar{\nu}$ are introduced in (13.4.11) and $(13.3.77)_{1-3}$. We assume that originally $\nu = \bar{\nu} = 1$. Then according to Chapter 13 under reasonable conditions contribution of $B(y, \gamma(y))$ to the remainder $\mathsf{R}^{\mathsf{MW}}_{\infty}$ does not exceed

$$C\nu_{\mathsf{new}}\mu_{\mathsf{new}}^{-1}h_{\mathsf{new}}^{-1} = C\mu^{-1}h^{-1}\gamma \tag{15.1.2}$$

as $\gamma \ge \mu h$ and dividing by γ^2 and integrating we conclude that the contribution of the *inner zone*

$$X_{\mathsf{inn}} := \{x, \gamma(x) \ge \bar{\gamma} := C_0\mu^{-1}\} \tag{15.1.3}$$

to $\mathsf{R}^{\mathsf{MW}}_{\infty}$ does not exceed

$$C\mu^{-1}h^{-1}\int \gamma^{-1}\,dx \asymp C\mu^{-1}h^{-1}\log\mu \tag{15.1.4}$$

as $\mu \le h^{-\frac{1}{2}}$.

Remark 15.1.1. We *must* take C_0 large enough in (15.1.2) to ensure that $\mu_{\text{new}} \ge \mu_0$ where μ_0 is large enough constant.

One can get rid of logarithmic factor in (15.1.4) using Seeley' approach. One also should consider case $\mu \ge h^{-\frac{1}{2}}$ but then zone $\{\mu h \ge \gamma \ge C_0\mu^{-1}\}$ appears and its contribution to remainder is estimated differently, as

$$C\int \gamma(x)^{-1}\, T(x)^{-1}\, dx. \tag{15.1.5}$$

We will postpone this until real proofs.

In the *boundary zone*

$$X_{\text{bound}} := \{x, \gamma(x) \le 2\bar{\gamma}\} \tag{15.1.6}$$

one can apply standard Weyl remainder estimate after rescaling. Then contribution of $B(y, \bar{\gamma})$ to R^{W} does not exceed $Ch_{\text{new}}^{-1} = Ch^{-1}\bar{\gamma}$ and the total contribution of X_{bound} to the remainder does not exceed $Ch^{-1}\bar{\gamma} \times \bar{\gamma}^{-1} = Ch^{-1}$. This is much worse than the contribution of the inner zone we derived and needs to be fixed. To do it we need to extend dynamics from time $T \asymp \bar{\gamma} \asymp \mu^{-1}$ ($T \asymp 1$ after rescaling) to $T \asymp 1$ ($T \asymp \mu$ after rescaling) or even more in the moderate, intermediate and strong magnetic field cases.

15.1.2 Classical Dynamics near the Boundary

To understand the role of the boundary, consider classical dynamics. Let us consider first a half-plane $X = \mathbb{R}^2_+ = \{x,\ x_1 > 0\}$, $g^{jk} = \delta_{jk}$, $F = 1$ and $V = 0$. Let us write the toy-model operator in the form

$$\bar{A} := h^2 D_1^2 + (hD_2 - \mu x_1)^2, \tag{15.1.7}$$

as $F_{12} := \partial_2 V_1 - \partial_1 V_2 = -1$ according to (13.2.7).

Then we have a Hamiltonian circular trajectory

$$\begin{aligned} x_1 = \mu^{-1}\bar{\xi}_2 + a\mu^{-1}\cos(2\mu t), \qquad & x_2 = \bar{x}_2 - a\mu^{-1}\sin(2\mu t), \\ & \xi_1 = -a\sin(2\mu t), \qquad \xi_2 = \bar{\xi}_2 \end{aligned} \tag{15.1.8}$$

where $a = \tau^{\frac{1}{2}}$ and τ is an energy level. So we got circular counter-clock-wise trajectories of the radius $\tau^{\frac{1}{2}}$ centered at $\bar{x}$ with $\bar{x}_1 = \mu^{-1}\bar{\xi}_2$ and depending on $\bar{\xi}_2$ these trajectories behave differently:

(a) As $\bar{\xi}_2 \geq \tau^{\frac{1}{2}}$ trajectory does not intersect ∂X or just touches it and remains circular.

(b) As $\bar{\xi}_2 < \tau^{\frac{1}{2}}$ trajectory reflects from ∂X and we get a "hop"-movement:

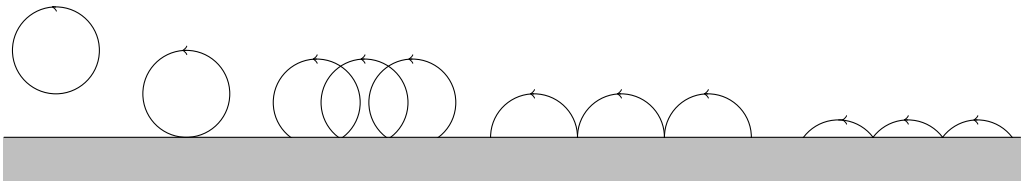
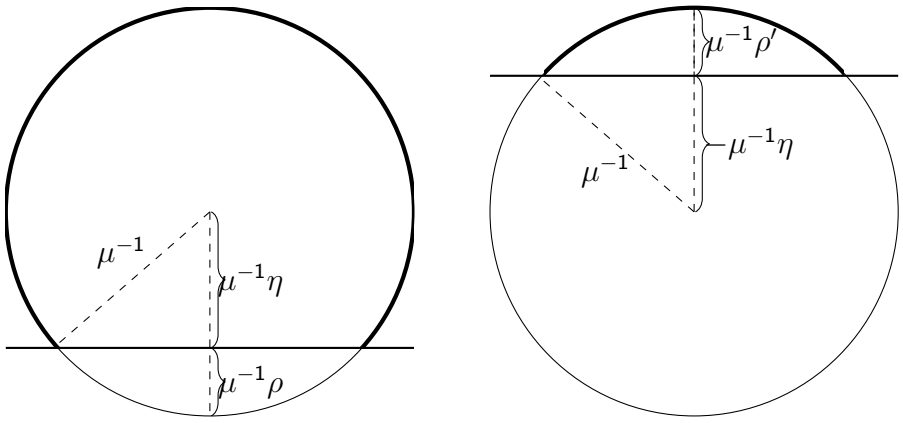


Figure 15.1: Different classical trajectories in half-plane for a toy-model operator

(c) As $\bar{\xi}_2 \searrow -\tau^{\frac{1}{2}}$ trajectory stays closer and closer to ∂X and becomes a kind of gliding ray in the limit. And we are interested only at zone $\{\xi_2 \geq -\tau^{\frac{1}{2}}\}$.

So, trajectories described in (b) are not periodic even for the toy-model operator.



(a) $\xi_2 > 0$ (b) $\xi_2 < 0$

Figure 15.2: Calculating the length of the hop; $\rho = 1 - \eta$, $\rho' = 1 + \eta$

One can calculate easily that

(15.1.9) The *length of the hop* (i.e. the distance between hop's start and end points) is $2a\mu^{-1}(1-\eta^2)^{\frac12}$ with $\eta = \xi_2/a$ while the length of the arc is $2a\mu^{-1}(\pi - \arccos\eta)$ and thus the time of the hop is $\mu^{-1}(\pi - \arccos\eta)$.

Therefore

(15.1.10) As $|\eta|<1$ *the average hop-speed* along x_2 is $\frac{\delta x_2}{\delta t} = -2av(\eta)$ with

$$v(\eta) \coloneqq \frac{(1-\eta^2)^{\frac12}}{\pi - \arccos(\eta)} \tag{15.1.11}$$

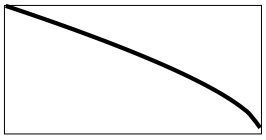


Figure 15.3: $v(\eta)$

This is a plot of $v(\eta)$. One can see easily that $v(\eta)$ is defined on $(-1,1)$ where it decays from 0 to 1; $\eta = 1$ is a threshold between circular and hop-movement and $\eta = -1$ corresponds to gliding rays.

As we cannot get remainder estimate better than $O(\mu h^{-1})$ for a toy-model operator we need to consider a perturbation by a potential:

$$A \coloneqq h^2D_1^2 + (hD_2 - \mu x_1)^2 + V(x); \tag{15.1.12}$$

a classical dynamics for a general operator (13.2.1) in dimension $d=2$ will be not different in our assumptions.

Then one should use the same classification as before with

$$\eta \coloneqq \frac{\xi_2}{W_0(x)^{\frac12}}, \qquad W = (\tau - V(x))F(x)^{-1},\ W_0 = W|_{\partial X} \tag{15.1.13}$$

where *so far* $F(x)=1$.

Actually the last statement is not completely true in the *transitional zone* (where now we consider zones in (x,ξ)-space)

$$\mathcal{X}_{\mathsf{trans}} = \bigl\{(x,\xi):\ |\xi_2 - W_0(x)^{\frac12}| \le 2\bar\rho\bigr\} \tag{15.1.14}$$

or equivalently $\{|\eta - 1| \le C_0\bar\rho\}$ where *so far*

$$\bar\rho = C_0\mu^{-1} \tag{15.1.15}$$

but later it may be increased due to uncertainty principle.

So, let us introduce an *inner zone*

$$\mathcal{X}_{\mathsf{inn}} = \{(x,\xi):\ \xi_2 - W_0(x)^{\frac12} \ge \bar\rho\} \tag{15.1.16}$$

and a *boundary zone*

$$\mathcal{X}_{\mathsf{bound}} = \{(x,\xi):\ \xi_2 - W_0(x)^{\frac12} \le -\bar\rho\}. \tag{15.1.17}$$

There is no need to consider *gliding zone*

$$\mathcal{X}_{\mathsf{glid}} = \{(x,\xi):\ \xi_2 + W_0(x)^{\frac12} \le \bar\rho\} \tag{15.1.18}$$

separately from $\mathcal{X}_{\mathsf{bound}}$.

Recall that inside the domain potential causes magnetic drift

$$\frac{d}{dt}x_1 = \mu^{-1}\partial_{x_2}W, \qquad \frac{d}{dt}x_2 = -\mu^{-1}\partial_{x_1}W. \tag{15.1.19}_{1,2}$$

Let us analyze what happens near the boundary. Note first that

(15.1.20) Billiards do not branch as $\mu \ge \mu_0$ where μ_0 is large enough.

Really, one can prove easily that

(15.1.21) With respect to Hamiltonian trajectories ∂X is strongly concave in the gliding zone and strongly convex in the transitional zone (and domain X has opposite property) as $\mu \ge \mu_0$.

Recall that (according to Figure 15.2) $\rho = 1-\eta$, $\rho' = 1+\eta$, $\eta = \xi_2 W_0^{-1}$. Then along Hamiltonian trajectories

$$\frac{d\rho'}{dt} = \frac{d}{dt}\big(\xi_2 W_0^{-\frac12}\big) = W_{x_2}W_0^{-\frac12} - \frac12\xi_2 W_0^{-\frac32}W_{0x_2}\frac{dx_2}{dt} \equiv$$
$$W_{0x_2}W_0^{-\frac12}\Big(1-\frac12\xi_2 W_0^{-1}\frac{dx_2}{dt}\Big) \qquad \mathsf{mod}\ O\big(\mu^{-1}\rho'\big)$$

as $x_1 = O(\mu^{-1}\rho')$ and therefore for one hop

$$\frac{\delta\rho'}{\delta x_2} \equiv W_{0x_2}W_0^{-\frac12}\Big(1-\frac12\xi_2 W_0^{-1}\frac{\delta x_2}{\delta t}\Big)$$

with W_0, W_{0x_2}, ξ_2 calculated in the middle of it

$$\equiv W_{0x_2}W_0^{-\frac12}\Big(1+\eta v(\eta)\Big)v(\eta)^{-1} \equiv \frac23\rho' W_{0x_2}W_0^{-\frac12} \qquad \mathsf{mod}\ O\big(\rho'(\mu^{-1}+\rho')\big)$$

as $\eta = \rho'-1$ and $v(\eta) = 1-\frac13\rho'\ \ \mathsf{mod}\ \rho'^2$.

Therefore

(15.1.22) Along trajectories of the length ≤ 1 $\rho' \exp(-\frac{4}{3} W_0^{\frac{1}{2}})$ remains constant modulo $O\big(\rho'(\mu^{-1}+\rho')\big)$.

Remark 15.1.2. The similar statement would be completely wrong for ρ because as $\rho \approx 0$ $\frac{\delta \rho}{\delta t} \approx -W_{0x_2} W_0^{-\frac{1}{2}}$ and therefore as $W_{0x_2} < 0$ hop-trajectories[1)] will be torn out of the boundary and begin magnetic drift movement. Meanwhile as $W_{0x_2} > 0$ trajectories drifting in the inner zone may collide with the boundary and begin hop-movement[1)].

In other words hops move away from the boundary (to the boundary) in the direction along the boundary, in which W_0 decreases (increases).

Example 15.1.3. Meanwhile W_{x_1} has more subtle effect. As hop-speed is larger than $C_0 \mu^{-1}$ (i.e. in $\mathcal{X}_{\mathsf{bound}}$) magnetic drift with respect to x_2 has no qualitative effect. However there are no hops in $\mathcal{X}_{\mathsf{inn}}$. Therefore as $W_{x_1} \asymp 1$ we have two rather different cases:

(i) $W_{x_1} > 0$. Then according to $(15.1.19)_2$ magnetic drift is to the left, in the same direction as hops. Then all dynamics is to the left[1)]. In particular as $W_{x_2} > 0$ hop-trajectories are torn from the boundary and begin drift movement (see Figure 15.4(a)) while as $W_{x_2} < 0$ drift-trajectories collide with the boundary and begin hop-movement (see Figure 15.4(b)).

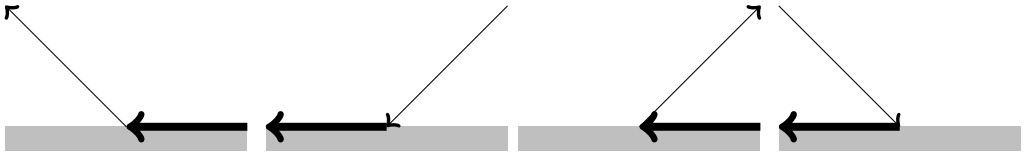
(ii) $W_{x_1} < 0$. Then according to $(15.1.19)_2$ magnetic drift is to the right, in the opposite direction to the hops. So direction of dynamics (with respect to x_2) in $\mathcal{X}_{\mathsf{inn}}$ is opposite to the hop-movement. In particular as $W_{x_2} \le -\epsilon_0$ hop-trajectories are torn from the boundary and begin drift movement (see Figure 15.4(c)) while as $W_{x_2} \ge \epsilon_0$ drift-trajectories collide with the boundary and begin hop-movement (see Figure 15.4(d)).

This is consistent with the fact that drift trajectories are level curves of W.

Example 15.1.4. Assume now that W_{x_2} vanishes at some point but $W_{x_2x_2} \ne 0$. Then repeating analysis of the previous example we arrive to the following four pictures:

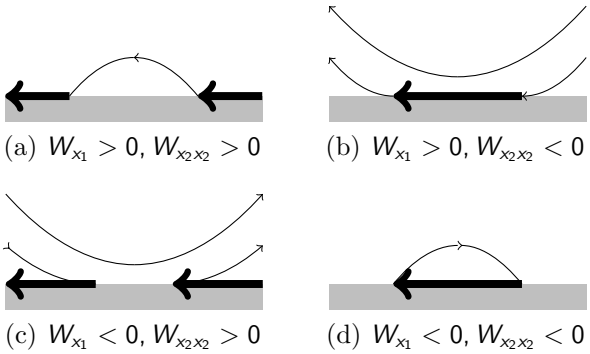
Again this is consistent with the fact that drift trajectories are level curves of W: in cases shown at Figures (a), (d) and (b)–(c) point $\bar{x}$ is a local minimum, maximum and minimax respectively.

[1)] In the positive time direction (and negative x_2-direction).



(a) $W_{x_1} > 0, W_{x_2} > 0$ (b) $W_{x_1} > 0, W_{x_2} < 0$ (c) $W_{x_1} < 0, W_{x_2} > 0$ (d) $W_{x_1} < 0, W_{x_2} < 0$

Figure 15.4: To Example 15.1.3. Bold lines show hop movement and thin lines show drift movement which is along level lines of W.



(a) $W_{x_1} > 0, W_{x_2x_2} > 0$ (b) $W_{x_1} > 0, W_{x_2x_2} < 0$

(c) $W_{x_1} < 0, W_{x_2x_2} > 0$ (d) $W_{x_1} < 0, W_{x_2x_2} < 0$

Figure 15.5: To Example 15.1.4. Bold lines show hop movement and thin lines show drift movement which is along level lines of W.

However the following observation basically remains true:

(15.1.23) The speed of magnetic drift is $O(\mu^{-1})$ (the typical speed is $\asymp \mu^{-1}$) while the speed of hop-movement is $O(1)$ (and the typical speed is $\asymp 1$).

15.1.3 Spectrum of the Toy-Model Operator

Consider toy-model operator (15.1.7) as $x_1 > 0$ with the Dirichlet or Neumann boundary condition $u|_{x_1=0}$ or $D_1 u|_{x_1=0}$. Making h-Fourier transform we arrive to 1-dimensional operator

$$h^2 D_1^2 + (\xi_2 - \mu x_1)^2 \tag{15.1.24}$$

which after transformations $x_1 \mapsto \mu x_1$ or $x_1 \mapsto \hbar^{-\frac{1}{2}} x_1$ with $\hbar := \mu^{-1} h$ becomes

$$\hbar^2 D_1^2 + (\xi_2 - x_1)^2 \qquad \text{with} \quad \hbar = \mu h \tag{15.1.25}$$

or

$$L(\eta) \coloneqq \hbar\big(D_1^2 + (\eta - x_1)^2\big) \qquad \text{with } \eta = \hbar^{-\frac{1}{2}}\xi_2 \tag{15.1.26}$$

at $\mathbb{R}^+ \ni x_1$ respectively, again with the Dirichlet or Neumann boundary condition at $x_1 = 0$. As in the previous chapters one needs to distinguish between $\hbar = \mu h$ and $\hbar = \mu^{-1} h$.

Obviously

(15.1.27) Each of these operators has a simple discrete spectrum, let $0 \le \lambda_{*,0}(\eta) < \lambda_{*,1}(\eta) < \lambda_{*,2}(\eta) < \dots$ be eigenvalues of operator $L_*(\eta)$ defined by (15.1.26) as $\hbar = 1$ where $*$ means either D or N and as usual D and N denote Dirichlet and Neumann respectively.

Further

(15.1.28) Let $\upsilon_{*,j}(x_1, \eta)$ be real-valued orthonormal eigenfunctions of operator $L_*(\eta)$ (as $\hbar = 1$) corresponding to eigenvalues $\lambda_{*,j}(\eta)$.

We will analyze them in details later (see Appendix 15.A), so far let us notice only that $\lambda_{*,j}(\eta) \to +\infty$ as $\eta \to -\infty$ and as $\lambda_{*,j}(\eta)$ are analytic we conclude that

Proposition 15.1.5. *(i) Spectrum of the toy-model operator* (15.1.7) *in* $\{x_1 > 0\}$ *with Dirichlet or Neumann boundary conditions is absolutely continuous and occupies* $[\hbar \cdot \inf_\eta \lambda_{*,0}(\eta), +\infty)$.

(ii) Schwartz kernel of its spectral projector is

$$e(x, y, \tau, \mu, h) = \tag{15.1.29}$$

$$(2\pi)^{-1}\hbar^{-1} \sum_{j \ge 0} \int e^{i\hbar^{-\frac{1}{2}}(x_2 - y_2)\eta} \theta\big(\tau - \hbar\lambda_{*,j}(\eta)\big) \upsilon_{*,j}(\hbar^{-\frac{1}{2}} x_1, \eta) \upsilon_{*,j}(\hbar^{-\frac{1}{2}} y_1, \eta)\, d\eta.$$

Setting $x = y$, subtracting $h^{-2}\mathcal{N}^{\mathsf{MW}}(\tau, \hbar)$ with $\mathcal{N}^{\mathsf{MW}}(\tau, \hbar)$ defined by $(13.2.9)_{(2)}$ and integrating with respect to x_1 we arrive to the following *formal*

(at least for now) equality

$$(15.1.30)\quad \int_0^\infty \Big(e(x,x,\tau,\mu,h) - h^{-2}\mathcal{N}^{\mathsf{MW}}(\tau,\hbar)\Big)\,dx_1 = (2\pi)^{-1}\mu h^{-1}\int_0^\infty \sum_{j\ge 0}\Big(\int \theta\big(\tau-\hbar\lambda_{*,j}(\eta)\big)\upsilon^2_{*,j}(\hbar^{-\frac12}x_1,\eta)\,d\eta - \theta\big(\tau-(2j+1)\hbar\big)\Big)\,dx_1 = h^{-1}\mathcal{N}^{\mathsf{MW}}_{*,\mathsf{bound}}(\tau,\mu h)$$

with

$$(15.1.31)\quad \mathcal{N}^{\mathsf{MW}}_{*,\mathsf{bound}}(\tau,\hbar) := (2\pi)^{-1}\int_0^\infty \sum_{j\ge 0}\Big(\int \theta\big(\tau-\hbar\lambda_{*,j}(\eta)\big)\upsilon^2_{*,j}(x_1,\eta)\,d\eta - \theta\big(\tau-(2j+1)\hbar\big)\Big)\hbar^{\frac12}\,dx_1$$

where in the last transition we rescaled x_1.

Recall that $\bar\lambda_j = (2j+1)$ are eigenvalues of operator (15.1.26) on the whole line $\mathbb{R}\ni x_1$ as $\hbar = 1$.

The following properties of $\lambda_{*,j}(\eta)$ are useful to know

In virtue of Proposition 15.A.1 $\tau > (2j+1)\hbar \iff \tau > \lambda_{\mathsf{D}j}(\eta)$ as $0<\eta$ is large enough and thus

$$(15.1.32)_{\mathsf{D}}\qquad \theta\big(\tau-\lambda_{\mathsf{D}j}(\eta)\hbar\big) = \theta\big(\tau-(2j+1)\hbar\big) \impliedby 0<\eta\gg 1$$

However $\tau\ge (2j+1)\hbar \iff \tau > \lambda_{\mathsf{N}j}(\eta)$ as $0<\eta$ is large enough and in this case

$$(15.1.32)_{\mathsf{N}}\qquad \theta\big(\tau-\lambda_{\mathsf{N}j}(\eta)\hbar\big) = \theta\big(\tau-(2j+1)\hbar+\big) \impliedby 0<\eta\gg 1$$

Recall that $\theta(\tau)$ is semi-continuous from the left and therefore $\theta(0)=0$. Meanwhile $\theta(\tau+)$ is semi-continuous from the right and therefore $\theta(0)=1$. Thus for Dirichlet boundary problem formulae (15.1.30)–(15.1.31) should be retained

$$(15.1.30)_{\mathsf{D}}\quad \int_0^\infty \Big(e_{\mathsf{D}}(x,x,\tau,\hbar) - h^{-2}\mathcal{N}^{\mathsf{MW}}(\tau,\hbar)\Big)\,dx_1 = h^{-1}\mathcal{N}^{\mathsf{MW}}_{\mathsf{D},\mathsf{bound}}(\tau,\mu h)$$

with

$$(15.1.31)_{\mathsf{D}}\quad \mathcal{N}^{\mathsf{MW}}_{\mathsf{D},\mathsf{bound}}(\tau,\hbar) := (2\pi)^{-1}\int_0^\infty \sum_{j\ge 0}\Big(\int \theta\big(\tau-\hbar\lambda_{\mathsf{D},j}(\eta)\big)\upsilon^2_{\mathsf{D},j}(x_1,\eta)\,d\eta - \theta\big(\tau-(2j+1)\hbar\big)\Big)\hbar^{\frac12}\,dx_1$$

but for Neumann boundary problem they need to be adjusted to

$$(15.1.30)_{\mathsf{N}} \quad \int_0^\infty \Big(e_{\mathsf{N}}(x,x,\tau,\hbar) - h^{-2}\mathcal{N}^{\mathsf{MW}}(\tau+,\hbar)\Big)\,dx_1 = h^{-1}\mathcal{N}^{\mathsf{MW}}_{\mathsf{N,bound}}(\tau,\mu h)$$

with

$$(15.1.31)_{\mathsf{N}} \quad \mathcal{N}^{\mathsf{MW}}_{\mathsf{N,bound}}(\tau,\hbar) := (2\pi)^{-1}\int_0^\infty \sum_{j\ge 0}\Big(\int \theta\big(\tau-\hbar\lambda_{\mathsf{N},j}(\eta)\big)\upsilon^2_{\mathsf{N},j}(x_1,\eta)\,d\eta - \theta\big(\tau-(2j+1)\hbar+\big)\Big)\hbar^{\frac12}\,dx_1.$$

Proposition 15.1.6. *(i) Equalities* $(15.1.30)_{\mathsf{D}}$–$(15.1.31)_{\mathsf{D}}$ *and* $(15.1.30)_{\mathsf{N}}$–$(15.1.31)_{\mathsf{N}}$ *hold and all integrals converge.*

(ii) Alternatively

$$(15.1.33)_{\mathsf{D}} \quad \mathcal{N}^{\mathsf{MW}}_{\mathsf{D,bound}}(\tau,\hbar) = (2\pi)^{-1}\sum_{j\ge 0}\int \Big(\theta\big(\tau-\hbar\lambda_{\mathsf{D},j}(\eta)\big) - \theta\big(\tau-(2j+1)\hbar\big)\theta(\eta)\Big)\hbar^{\frac12}\,d\eta$$

and

$$(15.1.33)_{\mathsf{N}} \quad \mathcal{N}^{\mathsf{MW}}_{\mathsf{N,bound}}(\tau,\hbar) = (2\pi)^{-1}\sum_{j\ge 0}\int \Big(\theta\big(\tau-\hbar\lambda_{\mathsf{N},j}(\eta)\big) - \theta\big(\tau-(2j+1)\hbar+\big)\theta(\eta)\Big)\hbar^{\frac12}\,d\eta$$

Proof. One can prove easily that

$$(15.1.34) \qquad |x_1-\eta| \ge \epsilon(x_1+|\eta|) \implies |\upsilon_{*j}(x_1,\eta)| \le C_j(1+x_1+|\eta|)^{-s}.$$

By definition

$$(15.1.35) \qquad \int_0^\infty \upsilon^2_{*j}(x_1,\eta)\,dx_1 = 1$$

and based on these two facts and $(15.1.32)_{\mathsf{D,N}}$ one can prove (i) easily.

Inserting η in the second term in the big parenthesis in $(15.1.31)_{\mathsf{D,N}}$ through factor $1 = \int \delta(x_1-\eta)\,d\eta$, changing order of integration and calculating integral with respect to x_1 we prove (ii). □

Remark 15.1.7. From forthcoming weak magnetic field arguments it follows that in the weak sense

$$\mathcal{N}^{\mathsf{MW}}_{*,\mathsf{bound}}(\tau,\hbar) \sim \sum_{n\ge 0} \kappa_{*n}(\tau)\hbar^{2n} \tag{15.1.36}$$

with $\kappa_{*0} = \mp(4\pi)^{-1}\tau_+^{\frac{1}{2}}$ as $* = \mathsf{D},\mathsf{N}$ respectively.

15.2 Weak Magnetic Field

15.2.1 Precanonical Form

We will consider a general magnetic Schrödinger operator (13.2.1) in $X \subset \mathbb{R}^2$

$$A = \sum_{1\le j,k\le 2} P_j g^{jk} P_k + V, \qquad \text{with } P_j = hD_j - \mu V_j \tag{15.2.1}$$

satisfying in $B(0,1) \subset X$ assumptions (13.2.2)–(13.2.4) but in contrast to (13.2.5) we assume that

$$X \cap B(0,1) = \{x_1 > 0\} \cap B(0,1). \tag{15.2.2}$$

Then in contrast to Chapter 13 we cannot unleash a full power of Fourier integral operators as we must preserve boundary, but we can assume that x_1 is small, in fact as small as $\mu^{-1}P_j$ are: not exceeding $C\,\mathsf{max}\big(\mu^{-1}, \mu^{-\frac{1}{2}}h^{\frac{1}{2}-\delta}\big)$.

Without any loss of the generality one can assume that

$$g^{11} = F, \qquad g^{12} = 0, \qquad \big(g^{22} - F\big)\big|_{x_1=0} = 0. \tag{15.2.3}_{1-3}$$

To achieve subsequently $(15.2.3)_{1-3}$ we just reintroduce $x_1 \coloneqq \mathsf{dist}_{g/F}(x, \partial X)$ in the given metrics $(g^{jk}F^{-1})$, then reintroduce $x_2 \coloneqq \alpha(x) + \beta(x_2)$ with an appropriate function $\alpha(x)$ and arbitrary function $\beta(x_2)$ and finally chose $\beta(x_2)$.

Remark 15.2.1. In this construction F is any arbitrary positive function but we select it to be a scalar intensity of the magnetic field. We cannot however choose $\partial_{x_1} g^{22}|_{x_1=0}$ as this is defined by the curvature of the metrics $(g^{jk}F^{-1})$.

Then

$$F_{12}|_{x_1=0} = -1 \tag{15.2.4}$$

(actually it may be 1 but then we change $x_2 \mapsto -x_2$) and without any loss of the generality one can assume that $V_1 = 0$ and $V_2 = x_1 + O(x_1^2)$; we can always reach it by a gauge transformation.

So, changing V by $O(h^2)$

$$\begin{aligned} A = (hD_1)F(x)(hD_1)+ \\ \bigl(hD_2 - \mu x_1 - \mu x_1^2 b(x)\bigr)g^{22}(x)\bigl(hD_2 - \mu x_1 - \mu x_1^2 b(x)\bigr) + V(x) \end{aligned} \tag{15.2.5}$$

with $V(x)$ satisfying $(15.2.3)_3$. One can easily generalize to such operator results of the previous section.

15.2.2 Propagation of Singularities

No-Critical Point Case

In the case of a *weak magnetic field*

$$\mu_0 \le \mu \le h^{\delta-1} \tag{15.2.6}$$

we know that inside the domain the shift during the first winding is microlocally observable under condition (13.4.54) i.e. as

$$|\nabla VF^{-1}| \ge \epsilon_0. \tag{15.2.7}$$

We are going to prove that near boundary the same is true as

$$|\nabla_{\partial X} VF^{-1}| \ge \epsilon_0 \tag{15.2.8}$$

where $\nabla_{\partial X}$ is a derivative along boundary (i.e. ∂_{x_2}).

To prove this assertion one needs just to prove the following proposition:

Proposition 15.2.2. *(i) The propagation speed with respect to (x, ξ_2) does not exceed C.*

(ii) Under condition (15.2.8) *for operator in the form* (15.2.5) *propagation speed with respect to ξ_2 is of magnitude 1 as $x_1 \le C\mu^{-1}$. Namely, for time t shift with respect to ξ_2 is of magnitude $|t|$ and has the same sign as tW_{x_2} with $W = (\tau - V)F^{-1}$.*

Proof. Proof follows the very standard way of the proof propagation of singularities as in Theorems 2.1.2 and 3.1.2. Here we are using an auxiliary function $\varphi = \varphi(x, \xi_2, t)$ and not invoking actually reflections from the boundary.

Then proof of Statement (i) is then completely trivial.

To prove Statement (ii) note that modulo $O(x_1)$

$$\{a, \xi_2\} \equiv \{F, \xi_2\}(a - V) + \{V, \xi_2\} - 2\big(\xi_2 - \mu x_1 - \mu x_1^2 b\big) F \mu x_1^2 \{b, \xi_2\}$$

and the last term is $O(\mu^{-1})$ as $|a| = O(1)$; so

$$\{a, \xi_2\} \equiv \{F, \xi_2\}(a - \tau) + F^2\{(V - \tau)F^{-1}, \xi_2\} + O(\mu^{-1}) \tag{15.2.9}$$

and as $(a - \tau)$ is small everything is defined by the second term in the right hand expression.

Further details of this very simple proof are left to the reader. □

Then we immediately arrive to

Corollary 15.2.3. *Let $U = U(x, y, t)$ be a Schwartz kernel of $e^{-ih^{-1}tA}$. Let $\psi(x) = \psi'(x_2)\psi''(\mu x_1)$ with fixed functions $\psi', \psi'' \in \mathscr{C}_0^\infty$.*

Finally, let condition (15.2.8) *be fulfilled on* $\operatorname{supp}\psi'$. *Then under condition* (15.2.6) *both*

$$F_{t\to h^{-1}\tau}\chi_T(t)\Gamma(U\psi) \tag{15.2.10}$$

and

$$F_{t\to h^{-1}\tau}\big(\bar\chi_T(t) - \bar\chi_{T'}(t)\big)\Gamma(U\psi) \tag{15.2.11}$$

are negligible as $\epsilon_1\mu^{-1} \le T' < T \le \epsilon_2$ with arbitrarily small constant ϵ_1 and small enough constant ϵ_2 [2)].

Meanwhile rescaling $x \mapsto x\mu$, $h \mapsto \mu h$ we immediately arrive to

Proposition 15.2.4. *Let condition* (15.2.6) *be fulfilled and let condition* (13.4.45) *i.e.*

$$V \le -\epsilon_0 \tag{15.2.12}$$

[2)] Recall that as usual $\chi \in \mathscr{C}_0^\infty([-1, -\frac12] \cap [\frac12, 1])$, $\bar\chi \in \mathscr{C}_0^\infty([-1, 1])$ and equals 1 at $[-\frac12, \frac12]$.

be fulfilled on $\mathsf{supp}\,\psi'$. *Let* ψ *be specified in Corollary 15.2.3.*

Then as $|\tau| \le \epsilon$ *and* $T = \epsilon\mu^{-1}$

$$F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(U\psi) \equiv \sum_{n\ge 0, m\ge 0} \partial_\tau \kappa_{nm}(\tau)\mu^{2n-1}h^{2n+2m-1} + \sum_{n\ge 0, m\ge 0} \partial_\tau \kappa'_{nm}(\tau)\mu^{2n}h^{2n+2m} \tag{15.2.13}$$

where $\kappa_{nm}(\tau)$ *and* $\kappa'_{nm}(\tau)$ *are smooth coefficients.*

Combining Proposition 15.2.4 and Corollary 15.2.3 we immediately conclude that

Corollary 15.2.5. *Under conditions* (15.2.6), (15.2.8) *and* (15.2.12) *decomposition* (15.2.13) *holds with* $T = \epsilon_2$.

Therefore applying Tauberian theorem we arrive to estimate (15.2.14) below as ψ specified in Corollary 15.2.3.

Theorem 15.2.6. *Let conditions* (13.2.1)–(13.2.4), (15.2.2), (15.2.6), (15.2.8) *and* (15.2.12) *be fulfilled on* $\mathsf{supp}\,\psi$ *where* $\psi(x) \in \mathscr{C}_0^\infty(\mathbb{R}^2)$. *Then*

$$\mathsf{R}^{\mathsf{W}}_\infty := \Big|\int_X \Big(e(x,x,0) - \sum_{n\ge 0, m\ge 0} \kappa_{nm}(0)\mu^{2n}h^{2n+2m-2}\Big)\psi(x)\,dx - \int_{\partial X}\sum_{n\ge 0, m\ge 0} \kappa'_{nm}(0)\mu^{2n}h^{2n+2m-1}\psi(x)\,ds_g\Big| \le C\mu^{-1}h^{-1} \tag{15.2.14}$$

where ds_g *is a measure on the boundary corresponding to metrics* g.

Proof. So, as $\psi = \psi'(x_2)\psi''(\mu x_1)$ estimate (15.2.14) is proven; thus contribution of zone $\{x,\ x_1 \le 2C_0\mu^{-1}\}$ to the remainder is $O(\mu^{-1}h^{-1})$. This is rather a broad zone as C_0 is arbitrarily large and in zone $\{x,\ x_1 \ge C_0\mu\}$ analysis is almost as if there was no boundary. We need to use the time direction in which ξ_2 increases and therefore trajectories drift inside X. Still it is not quite as without boundary as we are forced to use cut-off functions which are scaled with respect to x_1.

However, this is technicality and we overcome it in the following way: first note that such scaling functions are admissible as long as microlocal uncertainty principle $\mu^{-1}\times\mu^{-2} \ge \mu^{-1}h^{1-\delta}$ holds as we reduced our operator

to microlocal canonical form inside domain. Here the first factor μ^{-1} is due to the scale. So, as $\mu \le h^{\delta-\frac{1}{2}}$ we can appeal to theory of Chapter 13.

However we can change variables $x_{1\mathsf{new}} = x_1 + bx_1^2$ so that

$$\xi_1 = \big(1 + O(x_1)\big)\xi_{1\mathsf{new}}, \quad \xi_2 - \mu x_1 - \mu b x_1^2 = \xi_{2\mathsf{new}} - \mu x_{1\mathsf{new}} + O(x_1^2)\xi_1$$

and then bad term does not appear at all; in old coordinates it would amount to replacing ξ_2 by $\xi_2 + O(x_1^2\xi_1)$.

This proves that in the inner zone we can take $T \asymp \mu$ in the correct time direction and most importantly, we do not scale with respect to x_2 so microlocal uncertainty principle would be $1 \times \mu^{-2} \ge \mu^{-1}h^{1-\delta}$ i.e. in our framework $\mu \le h^{\delta-1}$. Formula (15.2.14) is proven. □

Finally we need to pass to magnetic Weyl formula:

Theorem 15.2.7. *In the framework of Theorem 15.2.6*

$$\mathsf{R}^{\mathsf{MW}} := \Big| \int_X \Big(e(x,x,0) - h^{-2}\mathcal{N}^{\mathsf{MW}}(x,0,\mu h)\Big)\psi(x)\,dx - h^{-1}\int_{\partial X} \mathcal{N}^{\mathsf{MW}}_{*,\mathsf{bound}}(x,0,\mu h)\psi(x)\,ds_g\Big| \le C\mu^{-1}h^{-1} \tag{15.2.15}$$

where $\mathcal{N}^{\mathsf{MW}}_{*,\mathsf{bound}}$ *is introduced by* $(15.1.31)_{\mathsf{D}}$ *or* $(15.1.31)_{\mathsf{N}}$ *for Dirichlet or Neumann boundary condition respectively with* $\hbar = \mu h F(x)$ *and* τ *replaced by* $-V(x)$.

Proof. To pass from (15.2.14) to (15.2.15) let us recall that coefficients are obtained by successive approximation method and notice that as ψ is fixed (non-scaled) function running successive approximation method to derive (15.2.14) leads to an extra factor h rather than μh if we differentiate ψ or g^{jk} or V or differentiate twice V_j; however exactly one such differentiation leads to 0 in the final calculations, so we conclude that at least 2 "losing" differentiations must be there, extra factor is $O(h^2)$ and the term is $O(1)$.

Thus to derive $\mathsf{R}^{\mathsf{W}}_{\infty}$ modulo $O(1)$ we should not differentiate g^{jk}, V, ψ at all and to differentiate V_j only once. However it means exactly considering a toy-model operator $A = A_y$ at each point y.

The microhyperbolicity with respect to x_2 implies that with integration (with the scale 1) over x_2 we can replace $\mathcal{N}^{\mathsf{MW}}$ by its decomposition with respect to powers of $\hbar = \mu h$ which justifies the transition from (15.2.14) to (15.2.15).

Analysis in the Boundary Zone

In what follows "formula" and "remainder" mean *Tauberian formula with* $T = \epsilon\mu^{-1}$ and the corresponding *Tauberian remainder* R^{T} until we will pass to Weyl and magnetic Weyl formula and remainder.

To cover the case when condition (15.2.8) is violated (with an extreme case $W := VF^{-1}|_{\partial X} = \mathsf{const}$) let us consider first boundary zone $\mathcal{X}_{\mathsf{bound}}$ defined by (15.1.17) where we chose a small parameter $\bar{\rho}$ later. To do so let us consider a stripe

$$\mathcal{X}_{\mathsf{bound},\rho} = \mathcal{X}_{\mathsf{bound}} \cap \{\rho \le -\xi_2 + \big(VF^{-1}\big)^{\frac{1}{2}} \le 2\rho\}. \tag{15.2.16}$$

Then the length of the hop is $\asymeq \mu^{-1}\rho^{\frac{1}{2}}$ with the possible perturbation $O(\mu^{-2})$ due to the magnetic drift (so $\rho \gg \mu^{-2}$ would suffice, but we request $\rho \ge \mu^{-1}$ anyway).

Now, weak magnetic field approach would mean that uncertainty principle is fulfilled after the first hop: $\rho \times \mu^{-1}\rho^{\frac{1}{2}} \ge h^{1-\delta}$ or equivalently $\rho \ge \mu^{\frac{2}{3}} h^{\frac{2}{3}-\delta}$. Combining with restriction $\rho \ge \mu^{-1}$ we arrive to

$$\rho \ge \bar{\rho} := \mathsf{max}\big(C_0\mu^{-1}, \mu^{\frac{2}{3}} h^{\frac{2}{3}-\delta}\big) \tag{15.2.17}$$

and our goal is to prove that under this the contribution of $\mathcal{X}_{\mathsf{bound}}$ to the remainder does not exceed $C\mu^{-1}h^{-1}$.

To achieve this goal let note first that

$$\int_{\mathcal{X}_{\mathsf{bound},\rho}\cap\{x_2=\mathsf{const}\}} dx_1 d\xi : da \le C\rho \tag{15.2.18}$$

which enables us to estimate

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(u\,{}^tQ_y\psi)| \le C\mu^{-1}h^{-1}\rho\gamma \tag{15.2.19}$$

as $|\tau| \le \epsilon$, $Q = Q(x_2, hD_2) \in \mathsf{S}_{h,\rho,\gamma}(\mathbb{R})$ and with the symbol supported in $\mathcal{X}_{\mathsf{bound},\rho} \cap \{|x_2 - \bar{x}_2| \le \gamma\}$, $\rho \ge C\,\mathsf{max}(\gamma, \mu^{-1})$, $\psi = \psi''(\mu x_1)$ and $T = T_* = \epsilon\mu^{-1}$.

Remark 15.2.8. Note that factor μ^{-1} appears here and in all similar estimates because width $\asymeq \rho$ with respect to ξ_2 matches to the width $\asymeq \mu^{-1}\rho$ with respect to x_1 on energy levels below c.

What we need is to investigate propagation until time $T = T^*(\rho)$ and prove that

$$|F_{t\to h^{-1}\tau}\big(\bar{\chi}_{T^*}(t) - \bar{\chi}_{T_*}(t)\big)\Gamma(u\,{}^tQ_y\psi)| \le C'h^s \tag{15.2.20}$$

with

$$\int T^{*\,-1}(\rho)\,d\rho \le C \tag{15.2.21}$$

then as the main part is given by Tauberian formula with $T = T_*$, remainder does not exceed

$$C\mu^{-1}h^{-1}\int T^{*\,-1}(\rho)\,d\rho dx_2 \le C\mu^{-1}h^{-1}.$$

In what follows $T^*(\rho) = \rho^{1-\delta'}$ with arbitrarily small exponent $\delta' > 0$.

To prove (15.2.20) with $T^*(\rho) = \rho^{1-\delta'}$ note first that

(15.2.22) If $|\partial_{x_2}VF^{-1}| \le c\rho^{\delta'}$ then propagation in *any* time direction remains in the zone

$$\big\{\frac{1}{2}\rho \le -\xi_2 + (VF^{-1})^{\frac{1}{2}} \le \rho'\big\} \cap \big\{|x_2 - \bar{x}_2| \le 2\gamma\big\} \tag{15.2.23}$$

for time $T = T^*(\rho)$

and

(15.2.24) If $|\partial_{x_2}VF^{-1}| \ge \frac{1}{2}c\rho^{\delta'}$ then propagation in *an appropriate*[3)] time direction remains in (15.2.23) for time $T = T^*(\rho)$;

in both cases $\gamma = C\mu^{-1}T^*(\rho') \le \epsilon\rho$ for sure where in the former case $\rho' = 3\rho$ and in the latter case $\rho' = \rho^{1-\delta''}$.

After assertions (15.2.22) and (15.2.24) are proven, we need to prove that singularities really propagate. Figure 15.2(a) shows that as $VF^{-1} = 1$ and the movement is strictly circular the length of the hop is exactly $2\mu^{-1}\big(1 - (1-\rho)^2\big)^{\frac{1}{2}}$ while in general case it will be of the same magnitude $\mu^{-1}\rho^{\frac{1}{2}}$ as $\rho \ll 1$. But obviously in the most critical zone movement in x_2 is

[3)] In which $(VF^{-1})(0, x_2)$ decays; as rotation is counter-clock-wise we take $\pm t > 0$ as $\pm(VF^{-1})(0, x_2) > 0$.

not monotone. However for sure x_2 increases with each hop as we bound ourselves with $x_1 \le \epsilon' \mu^{-1}\rho$ with small enough constant ϵ'.

The main problem are trajectories which are almost tangent to the boundary. There are two kind of them: with $\rho' \ll 1$ and with $\rho \ll 1$. Trajectories of of the first kind are not actually difficult (see proof of Proposition 15.2.11). So we start from trajectories of the second kind.

Proposition 15.2.9. *Consider zone* $\mathcal{Z} := \{\xi_2 \ge -\epsilon\}$. *Consider point* $(\bar{x}, \bar{\xi}_2) \in \mathcal{Z}$ *and its* $(\boldsymbol{\gamma}, \sigma)$*-vicinity* Ω *with* $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ *and*[4)]

$$\gamma_1 = \sigma = \hbar^{\frac{1}{2}-\delta}, \qquad \text{as } \ \bar{x}_1 \ge \hbar^{\frac{1}{2}-\delta}, \tag{15.2.25}$$

$$\gamma_1 = \sigma = \bigl(\hbar\gamma^{-\frac{3}{2}}\bigr)^{\frac{1}{2}-\delta}\gamma, \qquad \text{as } \ \bar{x}_1 \asymp \gamma \le \hbar^{\frac{1}{2}-\delta}, \tag{15.2.26}$$

$$\gamma_2 = \hbar^{\frac{1}{3}-\delta} \tag{15.2.27}$$

and operator $Q = Q(x, hD_2) \in \mathsf{S}_{h,\gamma_1,\gamma_2,\sigma}$ *with symbol supported in* Ω.

Further, consider point $(\bar{y}, \bar{\eta}_2) \in \mathcal{Z}$ *and its* $(\boldsymbol{\gamma}, \sigma)$*-vicinity* Ω' *with* $\boldsymbol{\gamma}$ *reintroduced for this point according to* (15.2.25)–(15.2.27) *and also operator* $Q' = Q'(x, hD_2) \in \mathsf{S}_{h,\gamma_1,\gamma_2,\sigma}$ *with symbol supported in* Ω'.

Let $T \asymp \mu^{-1}$ *and*

$$|x_1 - y_1|\gamma_1^{-1} + |x_2 - y_2|\gamma_2^{-1} + |\xi_2 - \eta_2|\sigma^{-1} \ge \epsilon \qquad \forall x \in \Psi_{t,\tau}(\Omega),\ y \in \Omega' \qquad \forall t \in \operatorname{supp}\chi_T \tag{15.2.28}$$

where $\Psi_{t,\tau}$ *is a Hamiltonian flow with reflections and* ξ_1 *is defined so that* $a(x,\xi) = \tau$. *Then*

$$F_{t\to h^{-1}\tau}\chi_T(t) Q' U\, {}^t Q' \tag{15.2.29}$$

is negligible.

Proof. As we assume $\bar{\xi}_2 \ge -\epsilon_0$ we do not need to consider "short and low" hops as on Figure 15.2(b) and the time of the hop is $\asymp \mu^{-1}$.

Obviously as $\mu \le h^{-\delta}$ one can apply results of Sections 2.4 and 3.5 and justify our final conclusion.

Let us consider larger μ. To do so we need to understand how small vicinity of Hamiltonian billiard flow with reflections we must take to contain propagation.

To do so consider propagation in the different zones, step by step, and we do consider $\sigma, \gamma_1, \gamma_2$ not necessarily defined by (15.2.25)–(15.2.27).

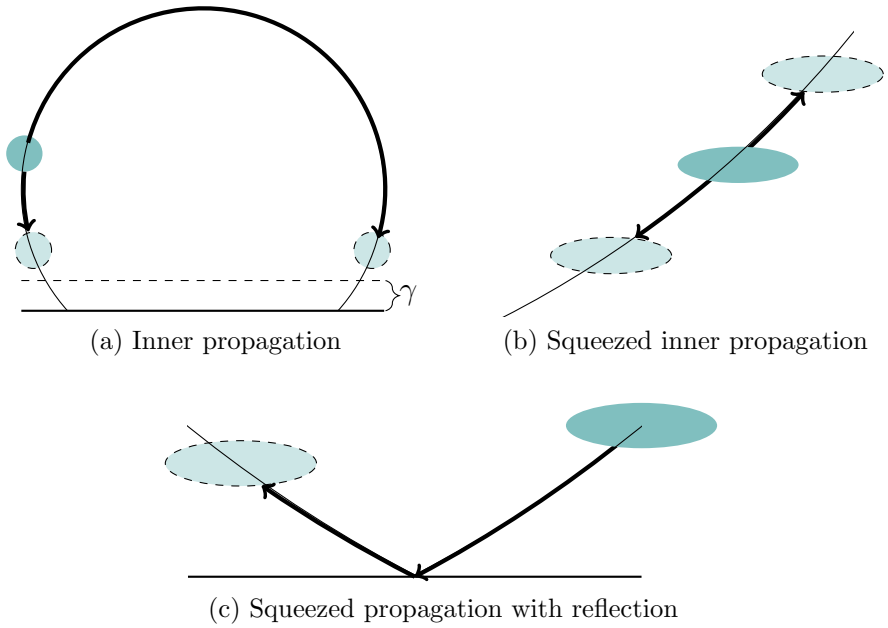
[4)] After rescaling $x \mapsto \mu x$.

(a) First of all, as $x_1 \ge \gamma \ge \hbar^{\frac{1}{2}-\delta}$ one can take any $\gamma_j \ge \gamma$ and $\rho_j \ge \gamma$. So, as $x_1 \ge \hbar^{\frac{1}{2}-\delta}$ we can take $\gamma_1 = \hbar^{\frac{1}{2}-\delta}$ and other scales different but larger. Thus,

(15.2.30) Let $\bar{x}_1 \ge \gamma \ge \hbar^{\frac{1}{2}-\delta}$ and let $\gamma_j \ge \gamma$, $\sigma \ge \gamma$. Let t be such that $\Psi_{t'}(\operatorname{supp} Q)$ does not intersect γ-vicinity of ∂X as $0 < \pm t' < \pm t$ (if $\pm t > 0$).

Then (15.2.29) is negligible, provided $\Psi_{t,\tau}(\operatorname{supp} Q)$ does not intersect (γ,σ)-vicinity of $\operatorname{supp} Q'$.

We refer to this as *inner propagation* (see Figure 15.6(a)).



(a) Inner propagation

(b) Squeezed inner propagation

(c) Squeezed propagation with reflection

Figure 15.6: Inner, squeezed inner (and similar mirror symmetric), squeezed with reflection propagations

(b) Consider now zone $C\rho \le x_1 \asymp \gamma \le \hbar^{\frac{1}{2}-\delta}$. Let us scale $x_1 \mapsto x_1\gamma^{-1}$, $x_2 \mapsto x_2\gamma^{-\frac{1}{2}}$ and $\hbar \mapsto \hbar' := \hbar\gamma^{-\frac{3}{2}}$. As after original rescaling $x \mapsto \mu x$ we had that derivatives of all coefficients were less than $C\mu^{-1}$ we conclude that after this rescaling they are less then $C\mu^{-1}\gamma^{\frac{1}{2}}$ and after division by γ they are still bounded.

More precisely: we recalled that "up to perturbation" operator was

$$\bar{A} = (\hbar D_1)^2 - (\hbar D_2 - x_1)^2 - 1 \tag{15.2.31}$$

with $|\xi_2 - 1| \le \rho$; so in the zone in question rescaling as described is justified. And therefore respectively we need to scale $\xi_1 \mapsto \xi_1 \gamma^{-\frac{1}{2}}$ and $\xi_2 - 1 \mapsto \xi_2 \gamma^{-1}$.

Now we have rather regular situation and can take $\hbar'^{\frac{1}{2}-\delta}$ vicinity. In other words, we can take

$$\gamma_1, \sigma \ge \hbar'^{\frac{1}{2}-\delta}\gamma = \left(\hbar\gamma^{-\frac{3}{2}}\right)^{\frac{1}{2}-\delta}\gamma, \tag{15.2.32}$$

$$\gamma_2 \ge \hbar'^{\frac{1}{2}-\delta}\gamma^{\frac{1}{2}} = \left(\hbar\gamma^{-\frac{3}{2}}\right)^{\frac{1}{2}-\delta}\gamma^{\frac{1}{2}} \tag{15.2.33}$$

and we can replace these right hand expressions by $\hbar^{\frac{1}{2}-\delta}\gamma^{\frac{1}{4}}$ and $\hbar^{\frac{1}{2}-\delta}\gamma^{-\frac{1}{4}}$ respectively and the latter does not exceed its value as $\gamma = \bar{\rho}$ and it is $\hbar^{\frac{1}{3}-\delta}$. This is *squeezed inner propagation* (see Figure 15.6(b)).

Furthermore, we can apply long-range propagation of Section 2.4 and replace condition $x_1 \asymp \gamma$ by a weaker condition $\gamma^{1+\delta'} \le x_1 \le \gamma^{1-\delta'}$ with $\delta' > 0$ small enough. But then we need no more than $C(\delta')$ jumps to reach from $\gamma \asymp \hbar^{\frac{1}{2}-\delta}$ to $\gamma \asymp \hbar^{23-\delta}$ or inversely.

(c) Finally, as $x_1 \le C\rho$ we can apply the same scaling with $\gamma = \rho$ and note that after rescaling trajectories meet the boundary under angle disjoint from 0. So we have a standard reflection situation. Scaling back we arrive to the same conclusion as in Part (b). This is *squeezed propagation with reflection* (see Figure 15.6(c)). □

Repeating this procedure $N \asymp \mu T$ times we arrive to

Corollary 15.2.10. *Consider $\mu^{-1} \le T \le \epsilon\rho$. Then conclusion of Proposition 15.2.9 remains true if we redefine $\sigma := \sigma_{\mathsf{old}} N$, $\gamma_j := \gamma_{j,\mathsf{old}} N$ with $N \asymp \mu T$ number of rotations.*

Now we can prove our main claim

Proposition 15.2.11. *Estimate* (15.2.20) *holds with $T_* = \epsilon\mu^{-1}$ and $T^*(\rho) = \rho^{1-\delta'}$.*

Proof. We consider a pseudodifferential partition of unity. Let $Q' = Q'(x_2, hD_2) \in \mathsf{S}_{h,\rho,\gamma}$ with symbol supported in $\epsilon(\gamma,\rho)$-vicinity of $(\bar{x}_2, \bar{\xi}_2)$ and $\gamma = \mu^{-1}\rho^{\frac{1}{2}}$ satisfy (15.2.17).

(a) Consider first $\bar{\xi}_2 \le -\epsilon_0$.

Note first that

(15.2.34) As $\xi_2 \le -\epsilon$ both operator and boundary value problem are ξ_2-microhyperbolic as $x_1 \ge 0$.

Really, $ih^{-1}[A, x_2] = hD_2 - \mu x_1 \le -\epsilon$.

Then due to results of Chapter 3 $U\,{}^tQ_y$ is negligible as $\pm t > 0$ and $\pm(x_2 - \bar{x}_2 + \epsilon\gamma) < \pm\epsilon_0 t$, which implies (15.2.20).

(b) Consider now $\bar{\xi}_2 \ge -\epsilon_0$. Note that Ψ_t after exactly one turn moves x_2 to the left by at least $\epsilon\rho^{\frac{1}{2}}$; we assume that $\rho \ge C_0\mu^{-1}$ to counter possible perturbations. Further, as $\rho \ge (\mu h)^{\frac{2}{3}-\delta}$ it exceeds γ_2 expansion due to uncertainty principle. This justifies conclusion of the proposition as $T^* = \epsilon_0\rho$ as propagation speed with respect to ξ_2, x_2 does not exceed C_0.

(c) To increase $T^*(\rho)$ notice that we can take $T^*(\rho) = \rho^{1-\delta'}$ unless $\ell \ge \rho^{\delta'}$ where

$$\ell := \epsilon_0 |\nabla_{\partial X} V F^{-1}| \tag{15.2.35}$$

as the speed of propagation with respect to ρ does not exceed ℓ and dynamics remains in the same (ℓ, ρ)-element with respect to (x_2, ξ_2).

Meanwhile as $\ell \ge \rho^{\delta'}$ we can take time direction in which ρ increases and then take $T^*(\rho) = \rho^{1-\delta'}$ anyway.

Uncertainty principle $\rho\ell \ge h^{1-\delta}$ is obviously satisfied in both cases. □

Remark 15.2.12. Surely one can improve arguments of (iii) and we will do it studying strong magnetic field. However at this moment this leads to no improvements in the remainder estimate.

Then immediately we arrive to

Corollary 15.2.13. *Contribution of zone* $\mathcal{X}_{\mathsf{bound}}$ *defined by* (15.1.17), (15.2.17) *to the remainder does not exceed* $C\mu^{-1}h^{-1}$.

Proof. Contribution of $\mathcal{X}_{\mathsf{bound},\rho}$ to the remainder does not exceed

$$C\mu^{-1}\rho h^{-1} \times T^{*,-1} \asymp C\mu^{-1}h^{-1}\rho^{\delta'} \tag{15.2.36}$$

and summation over $\rho \ge \bar{\rho}$ (i.e. integration with respect to $\rho^{-1}d\rho$) results in $O(\mu^{-1}h^{-1})$. □

Analysis in the Transitional Zone

Consider now *transitional zone* defined by (15.1.14). Obviously we get a rather rough estimate

Proposition 15.2.14. *Contribution of the transitional zone* (15.1.14) *to the remainder does not exceed*

$$C\bar{\rho}h^{-1} \asymp C\mu^{-1}h^{-1} + C(\mu h)^{\frac{2}{3}}h^{-1-\delta}. \tag{15.2.37}$$

In particular, as $\mu \le h^{-\frac{2}{5}+\delta}$ *it does not exceed* $C\mu^{-1}h^{-1}$.

Now we want to improve this estimate under some non-degeneracy condition invoking $VF^{-1}|_{\partial X}$.

Let us introduce ℓ-admissible partition with ℓ defined by (15.2.35).

Proposition 15.2.15. *On* ℓ*-element with*

$$\ell \ge \bar{\ell} := \mathsf{max}\big(C\mu^{-1}, \mu^{\frac{1}{2}}h^{\frac{1}{2}-\delta}\big) \tag{15.2.38}$$

expressions (15.2.10) *and* (15.2.11) *are negligible as* $T_* \le T \le T^*$ *with* $T_* = \epsilon\mu^{-1}$ *and* $T^*(\ell) = \ell^{1-\delta'}$.

Proof. As in Subsubsection *15.2.2.1 No-Critical Point Case* consider shift with respect to ξ_2 and it will be $\asymp \ell T$. So, uncertainty principle requests $\ell T \times \ell \ge h^{1-\delta}$ which is exactly our restriction to ℓ as $T = \epsilon\mu^{-1}$. Meanwhile $x_1 \le C_0\mu^{-1} + C\mu^{-1}\ell T$ and it remains less than $C\mu^{-1}$ as $T \le 1$. On the other hand shift with respect to x_2 does not exceed $C\mu^{-1} + C\rho^{\frac{1}{2}}T$ in virtue of Proposition 15.2.17 below and it remains less than $\epsilon\ell$ as $T = \epsilon\ell^{1-\delta'}$ unless $\rho \ge \ell^{\delta'}$ which is impossible in the transitional zone.

We surely need to keep $\rho\ell \ge h^{1-\delta}$ but one can check easily that $\bar{\rho}\bar{\ell} \ge \mu^{-\frac{1}{2}}h^{\frac{1}{2}-\delta} \ge h^{1-\delta}$. □

Corollary 15.2.16. *(i) Contribution of the part of the transitional zone* (15.1.14) *where* (15.2.38) *is fulfilled to the remainder does not exceed* $C\mu^{-1}h^{-1}$.

(ii) Contribution of the transitional zone (15.1.14) *to the remainder does not exceed*

$$C\mu^{-1}h^{-1} + C(\mu h)^{\frac{2}{3}}h^{-1-\delta}\,\mathsf{mes}_{\partial X}\big(\{x \in \partial X,\ |\nabla_{\partial X}V| \le C(\mu h)^{\frac{1}{2}-\delta}\}\big)h^{-1-\delta}, \tag{15.2.39}$$

(iii) In particular, it does not exceed

$$C\mu^{-1}h^{-1} + C(\mu h)^{\frac{7}{6}}h^{-1-\delta}, \tag{15.2.40}$$

provided

$$|\nabla_{\partial X} VF^{-1}| \le \epsilon \implies \pm\nabla^2_{\partial X} VF^{-1} \ge \epsilon \tag{15.2.41}^{\pm}$$

and expression (15.2.41) *does not exceed* $C\mu^{-1}h^{-1}$ *as* $\mu \le h^{-\frac{7}{13}+\delta}$.

While either sign in $(15.2.41)^{\pm}$ assures remainder estimate (15.2.40), in the future we will need to distinguish between different signs in this condition as dynamics will be different (see Example 15.1.4).

We finish this subsubsection by

Proposition 15.2.17. *An "average" propagation speed with respect to* x_2 *in* $\mathcal{X}_{\mathsf{trans}}$ *does not exceed* $C\bar{\rho}^{\frac{1}{2}}$.

Proof. Proof coincides with one of Proposition 15.2.9 and Corollary 15.2.10.

Surely trajectories in the squeezed reflection zone are not transversal to boundary anymore after rescaling but we do not need it as instead of $\hbar'^{\frac{1}{2}-\delta}$-vicinity we can take ϵ-vicinity and appeal f.e. to results of Section 3.4.

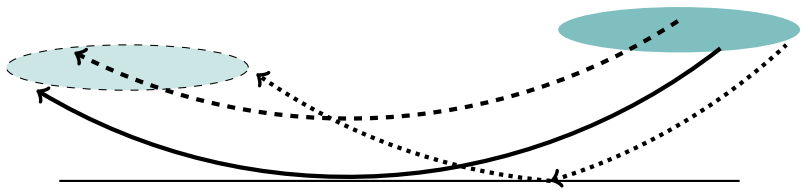


Figure 15.7: Squeezed propagation with tangency, reflections or miss the boundary

Analysis in the Inner Zone

Consider now *inner zone* defined by (15.1.16). As we know, in this zone we must assume some non-degeneracy condition to get remainder estimate better than $O(\mu h^{-1})$. In Chapter 13 we used few of such conditions rendering

different remainder estimates; all of them boiled down to the remainder estimate $O(\mu^{-1}h^{-1})$ under different restrictions to μ.

Consider first propagation of singularities.

Proposition 15.2.18. *Let $Q \in \mathsf{S}_{h,\rho_1,\rho_2,\gamma_1,\gamma_2}$ and let Ω be $(\gamma_1,\gamma_2,\rho_1,\rho_2)$-vicinity (with respect to (x,ξ)) of its symbol. Then long as trajectories started from Ω remain in $\mathcal{X}_{\mathsf{inn}}$ defined by* (15.1.16), (15.2.17)

$$U\,{}^tQ_y \equiv U^0\,{}^tQ_y \tag{15.2.42}$$

where U^0 is a propagator for operator A considered in X^0 the fixed vicinity of $\bar{X}$ (i.e. in the whole plane effectively).

Proof. Proof is trivial as

$$\rho \ge \bar{\rho} \coloneqq \mathsf{max}\big(C\mu^{-1}, \mu^{\frac{1}{2}}h^{\frac{1}{2}-\delta}\big) \tag{15.2.43}$$

as then we can apply Fourier integral operators as described in Chapter 13.

As $\bar{\rho}$ is defined by (15.2.17) one needs to apply the same arguments as in the proof of Proposition 15.2.9. Albeit in that proof trajectories "comfortably met" ∂X and now they "comfortably miss" it. □

Remark 15.2.19. This proposition however does not imply that results of Section 13.4 are automatically valid in our case as in that section we used partition elements which could be rather large in some directions (sometimes as large as $\asymp 1$ which could be replaced by $h^{\delta'}$ as we apply microlocal standard uncertainty principle rather than logarithmic uncertainty principle). Now boundary prevents us from doing this.

Still, however, as $\mu \le h^{-\delta}$ with sufficiently small exponent $\delta > 0$ these arguments of Section 13.4 remain automatically valid and leaving easy details to the reader arriving to the following proposition:

Proposition 15.2.20. *Under non-degeneracy condition* $(13.4.57)_m$ *with* $\nu = 1$ *i.e.*

$$\sum_{1\le|\alpha|\le m} |D^\alpha(VF^{-1})| \ge \epsilon \tag{15.2.44}_m$$

a contribution of the inner zone $\mathcal{X}_{\mathsf{inn}}$ to the remainder does not exceed $C\mu^{-1}h^{-1}$ as $\mu \le h^{-\delta}$ with sufficiently small exponent $\delta = \delta(m) > 0$.

Combining with results in inner and transitional zones where the same contributions were derived even without non-degeneracy condition we conclude that

Theorem 15.2.21. *Let conditions* (13.2.1)–(13.2.4), (15.2.2) *and* (15.2.12) *and non-degeneracy condition* $(15.2.44)_m$ *be fulfilled on* $\mathsf{supp}\,\psi$ *where* $\psi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$.

Then the remainder does not exceed $C\mu^{-1}h^{-1}$ *as* $\mu \le h^{-\delta}$ *with sufficiently small exponent* $\delta = \delta(m) > 0$.

Proof. As in the Chapter 13 (see $(13.4.49)_\vartheta$) the contribution of $\mathcal{X}_{\mathsf{inn}}$ to the remainder does not exceed

$$C\mu^{-1}h^{-1}\int_{\mathcal{X}_{\mathsf{inn}}} T^{-1}\,dx_2 d\xi_2 + C\mu h^{-1}\,\mathsf{mes}_2(\Omega_2) \tag{15.2.45}$$

with the same definition (13.4.50) of Ω_2; here $T = \epsilon\mu^{\frac12}\rho^{\frac12}$ and $d\xi_2 = d\rho$ and integral is bounded as $\rho \le c\mu$. □

In almost all other cases we however need to take Remark 15.2.19 into account. Still there is an exception: arguments linked to evolution with respect to ξ_2 work in a bit larger zone, namely $\{(x,\xi):\ \xi_2 \le C_0\}$ and thus in zone $\{x,\ x_1 \le C_1\mu^{-1}\}$ (which is a definition of X_{bound}).

No-Critical Point Case. Assume temporarily that condition (15.2.8) is fulfilled. Then, as we mentioned, Theorem 15.2.6 and Proposition 15.2.15 cover this zone. So we need to consider zone

$$\{x:\ C_0\mu^{-1} \le x_1 \le \epsilon\}. \tag{15.2.46}$$

However under condition (15.2.8) we can take $\asymp 1$ scale with respect to x_2 and consider drift in the time direction in which ξ_2 increases; then we remain in the zone in question for time $T^* = \epsilon'\mu$. Meanwhile under this condition shift with respect to ξ_2 is observable as $|t| \ge T_* = \epsilon\mu^{-1}$. This leads us to to conclusion that

(15.2.47) Under assumption (15.2.8) contribution of this (15.2.46) zone to the Tauberian remainder does not exceed $C\mu^{-1}h^{-1}$

(i.e. the same as the contribution of zone $\{x:\ x_1 \le C_0\mu^{-1}\}$ albeit factor μ^{-1} now comes as $T^{*\,-1}$ rather than as the width of the zone).

So we arrive to

Theorem 15.2.22. *Let conditions* (13.2.1)–(13.2.4), (15.2.2) *and* (15.2.12) *and non-degeneracy condition* (15.2.8) *be fulfilled on* $\operatorname{supp}\psi$ *where* $\psi(x) \in \mathscr{C}_0^\infty(\mathbb{R}^2)$.

Then contribution of zone $\{x:\ \operatorname{dist}(x,\partial X)\le \epsilon_1\}$ *to the remainder does not exceed* $C\mu^{-1}h^{-1}$ *as* $\mu \le h^{\delta-1}$ *with arbitrarily small exponent* $\delta > 0$.

Non-Degenerate Critical Point Case. Consider now case when non-degeneracy condition (15.2.41) is fulfilled. Then obviously contribution of zone $\{x_1 \le C_0\mu^{-1}\}$ to the remainder does not exceed

$$C\mu^{-1}h^{-1} + C(\mu h)^{\frac{1}{2}-\delta}h^{-1} \tag{15.2.48}$$

where the second term comes from zone

$$\big\{x:\ x_1 \le C_0\mu^{-1},\ |W_{x_2}| \le C_0 \max\big(\mu^{-1}, (\mu h)^{\frac{1}{2}-\delta}\big)\big\} \tag{15.2.49}$$

and this is not as good as (15.2.40) because $\bar\rho \ll 1$.

Now in the zone

$$\big\{x:\ C_0\mu^{-1} \le x_1 \le \epsilon_1,\ |W_{x_2}| \ge C_0 \max\big(\mu^{-1}, (\mu h)^{\frac{1}{2}-\delta}\big)\big\} \tag{15.2.50}$$

we can use $T^* = \epsilon\mu\ell$ with $\ell = \epsilon|W_{x_2}|$ and we can even take $T^* = \ell^{1-\delta'}$; then contribution of this zone to the remainder does not exceed $C\mu^{-1}h^{-1}$.

Meanwhile contribution of zone

$$\big\{x:\ C_0\mu^{-1} \le x_1 \le \epsilon_1,\ |W_{x_2}| \le C_0 \max\big(\mu^{-1}, (\mu h)^{\frac{1}{2}-\delta}\big)\big\} \tag{15.2.51}$$

to the Tauberian remainder under this condition does not exceed

$$C\mu^{-1}h^{-1} + C\mu(\mu h)^{\frac{1}{2}-\delta}h^{-1}. \tag{15.2.52}$$

So we arrive to

Theorem 15.2.23. *Let conditions* (13.2.1)–(13.2.4), (15.2.2) *and* (15.2.12) *and non-degeneracy condition* (15.2.41) *be fulfilled on* $\operatorname{supp}\psi$ *where* $\psi(x) \in \mathscr{C}_0^\infty(\mathbb{R}^2)$.

Then contribution of zone $\{x:\ \operatorname{dist}(x,\partial X)\le \epsilon_1\}$ *to the remainder does not exceed* (15.2.52)*; in particular, it does not exceed* $C\mu^{-1}h^{-1}$ *as* $\mu \le h^{\delta-\frac{1}{5}}$.

This is exactly the same estimate as was derived in Chapter 13 without boundary under condition

$$|\nabla VF^{-1}| + |\nabla^2 VF^{-1}| \ge \epsilon. \tag{15.2.53}$$

Now let us consider derivatives of VF^{-1} with respect to x_1 which leads to the drift in the direction x_2.

Our goal is to prove

Theorem 15.2.24. *(i) Let conditions* (13.2.1)–(13.2.4), (15.2.2) *and* (15.2.12) *and non-degeneracy condition* (15.2.7) *be fulfilled on* $\mathsf{supp}\,\psi$ *where* $\psi(x) \in \mathscr{C}_0^\infty(\mathbb{R}^2)$. *Then the remainder does not exceed*

$$C\mu^{-1}h^{-1} + C\mu^2 h^{-\delta} \tag{15.2.54}$$

as $\mu \le h^{\delta-\frac12}$; *in particular remainder does not exceed* $C\mu^{-1}h^{-1}$ *as* $\mu \le h^{\delta-\frac13}$.

(ii) Additionally assume that condition (15.2.41) *is also fulfilled on* $\mathsf{supp}\,\psi$. *Then the remainder does not exceed*

$$C\mu^{-1}h^{-1} + C\mu^{\frac52}h^{\frac12-\delta} \tag{15.2.55}$$

as $\mu \le h^{\delta-\frac12}$; *in particular remainder does not exceed* $C\mu^{-1}h^{-1}$ *as* $\mu \le h^{\delta-\frac37}$.

Proof. (i) First assume that

$$\rho \ge \bar\rho := \mathsf{max}\bigl(C\mu^{-1}, \mu^{\frac12}h^{\frac12-\delta}\bigr). \tag{15.2.56}$$

Then along the whole trajectory $x_1 \ge \mu^{-1}\rho \ge \mu^{-\frac12}h^{\frac12-\delta}$ and using $\mu^{-1}h$-Fourier integral operators we can reduce operator to the canonical form of Section 13.3. We can then notice that the shift with respect to "new" x_1 is $\mu^{-1}Tk$ where $k \asymp |\nabla VF^{-1}|$ but the scale with respect to the dual variable is $\mu^{-1}\rho$ and we need to write uncertainty principle

$$\mu^{-1}kT \times \mu^{-1}\rho \ge \mu^{-1}h^{1-\delta} \tag{15.2.57}$$

or plugging $T \asymp \mu^{-1}$ and $k \asymp 1$ we get

$$\rho \ge \bar\rho := \mathsf{max}\bigl(C\mu^{-1}, \mu^2 h^{1-\delta}\bigr) \tag{15.2.58}$$

and this restriction is stronger than (15.2.56) which in turn is stronger than (15.2.17).

Therefore

(15.2.59) Under condition (15.1.8) contribution to the remainder of $\mathcal{X}_{\mathsf{inn}}$ defined by (15.1.16), (15.2.58) does not exceed $C\mu^{-1}h^{-1}$.

Meanwhile with this choice of $\bar{\rho}$ contribution to the remainder of $\mathcal{X}_{\mathsf{trans}}$ defined by (15.1.14) does not exceed $C\bar{\rho}h^{-1}$ which is exactly the second term in (15.2.54).

Statement (i) is proven.

(ii) To prove Statement (ii) we need to note that in $\mathcal{X}_{\mathsf{trans}}$ shift with respect to ξ_2 is observable as $T \asymp \mu^{-1}$ and $|W_{x_2}| \asymp \ell$ satisfies (15.2.38). Then the contributions to the remainder of all zones save $\{\mathcal{X}_{\mathsf{trans}},\ |W_{x_2}| \le \bar{\ell}\}$ are $O(\mu^{-1}h^{-1})$ and under condition (15.2.41) the contribution of this zone does not exceed $C\bar{\rho}\bar{\ell}h^{-1}$ which is exactly the second term in (15.2.55). □

Remark 15.2.25. (i) Conditions (15.1.8) and (15.2.41) are fulfilled generically at the boundary.

(ii) Obviously in the setting of Theorem 15.2.24 the inner zone provides the worst contribution to the remainder and we will need to improve it in the framework of condition (15.2.24) using the strong magnetic field approach there.

15.2.3 From Tauberian to Magnetic Weyl Formula

Now our goal is to pass from Tauberian with $T = \epsilon\mu^{-1}$ to magnetic Weyl formula and estimate remainder R^{MW}. We also consider extended Weyl formula and estimate remainder $\mathsf{R}^{\mathsf{W}}_{\infty}$.

Theorem 15.2.26. *Let $\psi \in \mathscr{C}^{\infty}(\bar{X})$ be a fixed function with a compact support contained in the small vicinity of ∂X and let conditions* (13.2.1)–(13.2.4), (13.3.1) *and* (13.4.45) *be fulfilled there. Further, let condition* (15.2.6) *be fulfilled.*

Then

(i) Under condition (15.2.8) *both $\mathsf{R}^{\mathsf{W}}_{\infty}$ defined by* (15.2.14) *and R^{MW} defined by* (15.2.15) *do not exceed $C\mu^{-1}h^{-1}$ as $\mu \le h^{\delta-1}$.*

(ii) Under condition $(15.2.44)_m$ *both $\mathsf{R}^{\mathsf{W}}_{\infty}$ and R^{MW} do not exceed $C\mu^{-1}h^{-1}$ as $\mu \le h^{-\delta}$.*

(iii) Under condition (15.2.41) *both* $\mathsf{R}^{\mathsf{W}}_{\infty}$ *and* R^{MW} *do not exceed* (15.2.52) *as* $\mu \le h^{\delta-1}$*; in particular remainder does not exceed* $C\mu^{-1}h^{-1}$ *as* $\mu \le h^{\delta-\frac{1}{5}}$.

(iv) Under condition (15.2.7) *both* $\mathsf{R}^{\mathsf{W}}_{\infty}$ *and* R^{MW} *do not exceed* (15.2.54) *as* $\mu \le h^{\delta-1}$*; in particular remainder does not exceed* $C\mu^{-1}h^{-1}$ *as* $\mu \le h^{\delta-\frac{1}{3}}$.

(v) Under conditions (15.2.7) *and* (15.2.41) *both* $\mathsf{R}^{\mathsf{W}}_{\infty}$ *and* R^{MW} *do not exceed* (15.2.55) *as* $\mu \le h^{\delta-1}$*; in particular remainder does not exceed* $C\mu^{-1}h^{-1}$ *as* $\mu \le h^{\delta-\frac{3}{7}}$.

Proof. (a) To go from R^{T} to $\mathsf{R}^{\mathsf{W}}_{\infty}$ is easy: using condition (13.4.45) (i.e. $|V| \asymp 1$) we can apply the standard results of Chapters 4 and 7 after rescaling $x \mapsto \mu x$.

Going from $\mathsf{R}^{\mathsf{W}}_{\infty}$ to R^{MW} is more subtle. Note first that we can drop all terms with $m > 0$ in (15.2.14). Therefore only surviving terms are those with $h^{-2}(\mu h)^{2n}$ with integration over X and $h^{-1}(\mu h)^{2n}$ with integration over ∂X.

(b) Consider first terms with integration over X.

Note first that under condition (15.2.8) their sum is equal to

$$\int_X \mathcal{N}^{\mathsf{MW}}(x, \mu h, \tau)\psi(x)\,dx \tag{15.2.60}$$

modulo $O\big(h^{-2}(\mu h)^s\big)$; actually we go in the opposite direction: from expression (15.2.60) to its decomposition into powers of $\hbar = \mu h$.

Consider expression (15.2.60) under condition (15.1.8); again we can replace $\theta(\tau - V - (2j+1)F\mu h)$ by derivative with respect to x_1 (of high order and with smooth coefficient) of smooth $\mathscr{C}^s$-function; so integrating by parts we again arrive to the sum of the terms in question plus the similar integral over the boundary; however the latter contains at least one extra integration with respect to τ so we get

$$(\mu h)^{2n}h^{-2}\int \big(\tau - V - (2j+1)F\mu h\big)^r_+ \phi_{n,r}(x)\,ds_g \tag{15.2.61}$$

where $r \ge 1$. If we replace in the latter term summation with respect to j by integration the error will not exceed $C(\mu h)^{r+1}h^{-2}$ i.e. $C\mu^2$ which is less than (15.2.54).

On the other hand, under condition (15.2.41) the error in question will not exceed $C(\mu h)^{r+1\frac{3}{2}}h^{-2}$ which is less than (15.2.55). So, in Statements (iv) and (v) we also can replace terms with integration over dx into $\mathcal{N}^{\mathsf{MW}}$, may be changing κ'_{n0}.

The same arguments albeit without integration with respect to x_1 work under condition (15.2.41) alone; however we gain only factor $(\mu h)^{\frac{1}{2}}$.

The similar arguments work in the framework of Statement (ii) as well.

(c) Now Statement (i) becomes the most complicated as many terms should be taken into account. We apply the following trick: consider the same operator albeit we replace D_2 everywhere by D_3:

$$\bar{A} := \sum_{j,k} \bar{P}_j g^{jk}(x_1,x_2)\bar{P}_k + V(x_1,x_2), \qquad \bar{P}_1 = hD_1,\ \bar{P}_2 = hD_3 - \mu V_2(x_1,x_2) \tag{15.2.62}$$

Then it will affect in $\mathsf{R}^{\mathsf{W}}_{\infty}$ only terms with $m \ge 1$ we do not care about. But the problem remains microhyperbolic in the variable x_2 and therefore everything works as it should. We need to consider then shifts with respect to ξ_2 only and therefore only averaging with respect to x_2 is needed.

Note then that what we get instead of $\int e(x,x,\tau)\psi\,dx$ is

$$(2\pi h)^{-1}\int e(x_1,x_1;x_2,\xi_3,\tau)\psi\,dx_1\,dx_2 d\xi_3 \tag{15.2.63}$$

where $e(x_1,y_1;x_2,\xi_2,\tau)$ is a Schwartz kernel for spectral projector for 1-dimensional operator. Here integration with respect to x_3 is not needed. Neither is needed integration with respect to ξ_2 as we pass from Tauberian expression with $T = \epsilon\mu^{-1}$ for $e(...)$ to $e(...)$ itself. Finally we change in (15.2.63) ξ_3 by ξ_2.

Also we can replace everywhere (save V_2) x_1 by 0 while V_2 we replace by

$$\bar{V}_2 = V(0,x_2) + (\partial_{x_1}V_2)(0,x_2)x_1; \tag{15.2.64}$$

it will not affect essential boundary terms. But then boundary terms in $\mathsf{R}^{\mathsf{W}}_{\infty}$ together must match to a boundary term in R^{MW} . This proves Statement (i) completely.

(d) Exactly the same arguments work for Statements (ii)–(v); we do not use condition (15.2.7) at all and in $\mathcal{X}_{\mathsf{bound}}$ defined in terms of (ξ_3, x_2) we consider shifts after hops with respect to x_3, so we do not need to integrate over x_3. Without condition (15.2.21) we use the trivial $O(\bar{\rho}h^{-1})$ estimate for contribution of $\mathcal{X}_{\mathsf{trans}}$.

Under condition (15.2.21) in $\mathcal{X}_{\mathsf{trans}}$ we consider shifts with respect ξ_2 and again integration over x_3 and ξ_2 is not needed. □

15.3 Moderate Magnetic Field

In this section we consider a case of the moderate magnetic field when the results of the previous section are not as sharp as we want (so the remainder is not $O(\mu^{-1}h^{-1})$). As under different assumption it happens under different restrictions to μ, we consider separately different cases.

15.3.1 Most Non-Degenerate Case

If condition (15.2.8) is fulfilled, we need to consider only the case of the *intermediate magnetic field*

$$h^{\delta-1} \le \mu \le h^{-1} \tag{15.3.1}$$

and then operator is x_2-microhyperbolic. Therefore as $\psi = \psi'\psi''_\mu$ with $\psi' = \psi'(x_2) \in \mathscr{C}_0^\infty$ and $\psi''_\gamma = \psi''(x_1\gamma^{-1})$ [5)] with ψ''_γ supported in $|x_1| \le C_0\gamma$, $\gamma = \mu^{-1}$,

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\psi u| \le C\mu^{-1}h^{-2}T\bigl(\frac{h}{T}\bigr)^s, \tag{15.3.2}$$

and then

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma\psi u| \le C\mu^{-1}h^{-1} \tag{15.3.3}$$

as $h \le T \le T^* = \epsilon_0$ where μ^{-1} comes as a measure of X_{bound} and therefore

$$\mathsf{R}^{\mathsf{T}} \le C\mu^{-1}h^{-1}. \tag{15.3.4}$$

[5)] Averaging with respect to x_1 is not needed at all.

Further advancing method of successive approximations with unperturbed operator[6)]

$$\bar{A} \coloneqq \sum_{j,k} \bar{P}_j g^{jk}(y) \bar{P}_j + V(y), \tag{15.3.5}$$

$$\bar{P}_j = hD_j - \mu V_j(y) - \mu \sum_k (x_j - y_j)(\partial_k V_j)(y) \tag{15.3.6}$$

we see that the first term results in expression (15.2.63) of the magnitude $C\mu^{-1}h^{-2}$ while any next term acquires factor h in the corresponding power and thus does not exceed the remainder estimate.

So, under conditions (15.2.8) and (15.3.1) and indicated ψ the remainder does not exceed $C\mu^{-1}h^{-1}$ while the principal part is given by the Tauberian expression for the first term in the successive approximation method.

On the other hand, if we take $\psi'' \in \mathscr{C}_0^\infty(\mathbb{R}^+)$ (supported in $(\frac{1}{2}, 1)$) and $\gamma \ge \mu^{-1}$ we can take

$$T^* \asymp \min\bigl((h^{-1}\gamma)^{1+\sigma}, \mu\gamma^{1-\sigma}\bigr). \tag{15.3.7}$$

Really, we could take $T^* \asymp \gamma^{1-\sigma}$ as

$$\gamma \ge \bar{\gamma} \coloneqq C_0\mu^{-1} + C_0\mu^{-\frac{1}{2}}h^{\frac{1}{2}-\delta} \tag{15.3.8}$$

which completely covers the case $\mu \le h^{\delta-1}$. Otherwise we can arrive to this case scaling $x \mapsto x\zeta$, $\mu \mapsto \mu\zeta$, $h \mapsto h\zeta^{-1}$ with $\zeta = \min\bigl(1, (\mu^{-1}h)^{\frac{1}{2}}, (\mu^{-1}h\gamma^{-2})^s\bigr)$ with large s.

Therefore the contribution of the strip $\{x\colon x_1 \asymp \gamma\}$ to the remainder does not exceed

$$C\gamma h^{-1}\Bigl((h^{-1}\gamma)^{-1-\sigma} + \mu^{-1}\gamma^{-1+\sigma}\Bigr)$$

and hence contribution of $\{x\colon x_1 \le \epsilon\}$ does not exceed this expression integrated over $\gamma^{-1}d\gamma$ resulting in

$$Ch^{-1}\Bigl((h^{-1}\gamma)^{-1-\sigma}\gamma\big|_{\gamma=\mu^{-1}} + \mu^{-1}\gamma^\sigma|_{\gamma=1}\Bigr) \asymp C\mu^{-1}h^{-1}.$$

So, we can take ψ fixed function rather than scaled with respect to x_1.

However let us partition it into functions supported in $\{x\colon x_1 \le 2\bar{\gamma}\}$ and in $\{x\colon x_1 \ge \bar{\gamma}\}$. Then such expression in the latter case is not affected

[6)] In comparison with (15.2.63), (15.2.64) we freeze at y not $(0, y_2)$ at this moment.

by the presence of the boundary resulting in the same expression but with approximation term calculated for the whole space.

However in the former case the presence of the boundary should be taken into account. Let us use again the method of successive approximation but use as unperturbed operator one with x_1 set to 0 everywhere save in the linear part of magnetic field; thus unperturbed operator is

$$\bar{A} := h^2 D_1^2 + (\mu x_1 - hD_2)^2 - W(0, x_2). \tag{15.3.9}$$

Again as the main part of asymptotics is of magnitude $\bar{\gamma} h^{-2}$ and each next term acquires factor $\bar{\gamma}$, so only first two terms need to be considered.

So, let us consider the second term; we claim that calculating this term one does not need to take a boundary into account. Really, as perturbation vanishes at the boundary one needs to kill x_1 before restricting to the boundary, but it can be done only by commutator and then factor h rather than $\bar{\gamma}$ appears. It is not enough but if we plug instead of $\psi''_{\bar{\gamma}}$ function ψ''_{γ} with $C_0\mu^{-1} \le \gamma \le \bar{\gamma}$ and $\operatorname{supp} \psi''$ disjoint from 0 we acquire factor $(\mu\gamma)^{-s}$ and then contribution to the error is $C\gamma h^{-1}(\mu\gamma)^{-s}$ and it boils down to $C\mu^{-1}h^{-1}$ after summation with respect to γ.

Then we get the final answer as the sum of two terms: one is for operator (15.3.5) albeit with calculation (before taking Γ) in the whole plane i.e.

$$h^{-2} \int_X \mathcal{N}^{\mathsf{MW}}(x, \mu h)\psi(x)\, dx \tag{15.3.10}$$

and the second one for operator (15.3.9) but in half-plane and subtracting the same expression for the same operator albeit in the whole plane we arrive to

$$\int_X \Big(\int \boldsymbol{e}_1(x_1, x_1; x_2, \xi_2, 0)\, d\xi_2 - h^{-2}\mathcal{N}_0^{\mathsf{MW}}(x, \mu h)\Big)\psi(x)\, dx \tag{15.3.11}$$

where $\boldsymbol{e}_1(x_1, y_1; x_2, \xi_2, 0)$ is the Schwartz kernel of one-dimensional operator

$$h^2 D_1^2 + (\mu x_1 - \xi_2)^2 - W(0, x_2) \tag{15.3.12}$$

and $\mathcal{N}_0^{\mathsf{MW}}$ refers to $\mathcal{N}^{\mathsf{MW}}$ calculated for the same operator (15.3.9).

We will transform operators (15.3.9), (15.3.12) and silently (without changing notations) transform $\boldsymbol{e}(...)$ and $\mathcal{N}_0^{\mathsf{MW}}(...)$.

Changing $x_1 \mapsto x_1\zeta$ and $\xi_2 \mapsto \xi_2\zeta\mu$ we acquire factor $\mu\zeta$ and get instead of the first term in (15.3.11)

$$(2\pi h)^{-1}\zeta\mu \int \mathbf{e}_1(x_1, x_1; x_2, \xi_2, 0)\psi'(x_2)\psi''(x_1\zeta)\, dx_1\, dx_2 d\xi_2 \tag{15.3.13}$$

meanwhile transforming operator into

$$h^2\zeta^{-2}\Big(D_1^2 + h^{-2}\mu^2\zeta^4(x_1-\xi_2)^2 - W(0,x_2)h^{-2}\zeta^{-2}\Big). \tag{15.3.14}$$

One can drop a factor in the front of operator and select $\zeta = \mu^{-\frac{1}{2}}h^{\frac{1}{2}}$ thus resulting in the answer

$$(2\pi)^{-1}\mu^{\frac{1}{2}}h^{-\frac{1}{2}} \int \mathbf{e}_1(x_1, x_1; x_2, \xi_2, 0)\psi'(x_2)\psi''(x_1\mu^{\frac{1}{2}}h^{-\frac{1}{2}})\, dx_1\, dx_2 d\xi_2 \tag{15.3.15}$$

and in operator

$$D_1^2 + (x_1-\xi_2)^2 - \mu^{-1}h^{-1}W(0,x_2). \tag{15.3.16}$$

However we need to subtract from (15.3.15) also transformed the second term in (15.3.11). We can then tend $\zeta \to +\infty$ (the error will be negligible) and the total difference will tend to

$$h^{-1}\int_{\partial X} \mathcal{N}^{\mathsf{MW}}_{*,\mathsf{bound}}(x_2, \mu h)\psi(x)\, ds_g. \tag{15.3.17}$$

So, the final answer is

$$h^{-2}\int_X \mathcal{N}^{\mathsf{MW}}(x, \mu h)\psi(x)\, dx + h^{-1}\int_{\partial X} \mathcal{N}^{\mathsf{MW}}_{*,\mathsf{bound}}(x_2, \mu h)\psi(x)\, ds_g \tag{15.3.18}$$

with $* = \mathsf{D}, \mathsf{N}$ and we arrive to the following theorem:

Theorem 15.3.1. *Let $\psi \in \mathscr{C}^\infty(\bar{X})$ be a fixed function with a compact support contained in the small vicinity of ∂X and let conditions* (13.2.1)–(13.2.4), (13.3.1) *and non-degeneracy condition* (15.2.8) *be fulfilled on* $\mathsf{supp}\,\psi$.

Then as $\mu \le h^{-1}$ formula (15.3.18) *gives* N *with an error* $\mathsf{R}^{\mathsf{MW}} = O(\mu^{-1}h^{-1})$.

15.3.2 Generic Case. Analysis in Inner Zone

Now we are interested to improve results of the previous section in the generic case i.e. when both conditions (15.2.7) and (15.2.41) are fulfilled. Then in virtue of Theorem 15.2.26(v) we can assume that

$$\mu \ge h^{\delta-\frac{3}{7}}. \tag{15.3.19}$$

We start from the simpler analysis in $\mathcal{X}_{\mathsf{inn}}$. In this case as $\rho \ge \bar{\rho} = (\mu h)^{\frac{2}{3}} h^{-\delta}$ we need to consider operator without boundary condition; however presence of the boundary as we remember manifests itself through uncertainty principle; we should take $T_* = h^{1-\delta'}\ell^{-2}$ or $T_* = \mu h^{1-\delta'}\rho^{-1}$ whatever is smaller and *at this moment* we are interested only in the zone where $T_* \ge \epsilon\mu^{-1}$.

Actually we can take here effectively even $\delta' = 0$ in the following sense: note that

$$|F_{t\to h^{-1}\tau}\phi_T(t)\Gamma\big(U\,{}^tQ_y\big)| \le C\mu^{-1}h^{-1}\rho\ell \times \big(\mu T+1\big) \tag{15.3.20}$$

as $\phi \in \mathscr{C}_0^\infty([-1,1])$, $T \ge h$ and

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\big(U\,{}^tQ_y\big)| \le C\mu^{-1}h^{-1}\rho\ell \times \big(\mu T+1\big)\big(\frac{T}{T_*}\big)^{-s} \tag{15.3.21}$$

as $\chi \in \mathscr{C}_0^\infty([-1,-\frac{1}{2}]\cup[\frac{1}{2},1])$ and $T_* \le T \le T^*$ with unspecified at this moment T^* and

$$T_* = \mathsf{min}\big(\mu h\rho^{-1}, \ell^{-2}\big)h = h\times \begin{cases} \mu\rho^{-1} & \text{as } \ \rho > \mu\ell^2,\\ \ell^{-2} & \text{as } \ \rho < \mu\ell^2.\end{cases} \tag{15.3.22}$$

Recall that $Q = Q(x_2, hD_2)$ is an (ℓ,ρ) admissible element with $\rho\ell \ge h^{1-\delta}$.

Really for one winding (i.e. $T = \epsilon\mu^{-1}$) estimate (15.3.20) is obvious, and we need to take in account $N \asymp \mu T+1$ windings.

Estimates (15.3.20) and (15.3.21) imply that

$$|F_{t\to h^{-1}\tau}\phi_T(t)\Gamma\big(U\,{}^tQ_y\big)| \le C\mu^{-1}h^{-1}\rho\ell \times \big(\mu T_*+1\big) \tag{15.3.23}$$

as $\phi \in \mathscr{C}_0^\infty([-1,1])$ and $T_* \le T \le T^*$.

Remark 15.3.2. Therefore from the point of view of the remainder estimate, rather than the final formula we need to take in account $N^* \asymp (\mu T_*+1)$ windings but in the main part of asymptotics still $(\mu T_* h^{-\delta'}+1)$ windings should be taken in account.

Let us discuss T^*. As average propagation speed with respect to x does not exceed $C_0\mu^{-1}$ and propagation speed with respect to ξ_2 does not exceed $C_0\ell$, our dynamics remains in the same (ℓ,ρ)-element for time

$$T^* = \epsilon \min\bigl(\mu\ell, \rho\ell^{-1}\bigr) \tag{15.3.24}$$

and $T_* : T^* \asymp h\rho^{-1}\ell^{-1}$.

Therefore contribution of (ℓ,ρ)-element to the Tauberian remainder with $T \ge \bigl(T_* h^{-\delta'} + \epsilon\mu^{-1}\bigr)$ does not exceed

$$C\mu^{-1}h^{-1}\rho\ell\bigl(\mu T_* + 1\bigr) \times T^{*\,-1}. \tag{15.3.25}$$

Note that in zone $\ell \le \mu^{-\frac{1}{2}}\rho^{\frac{1}{2}}$ we can reset $\ell = \ell(\rho) := \mu^{-\frac{1}{2}}\rho^{\frac{1}{2}}$ thus covering the whole zone with a fixed magnitude of ρ by a single element. Also note that $\ell(\rho)\rho = \mu^{-\frac{1}{2}}\rho^{\frac{3}{2}} \ge \mu^{\frac{1}{2}}h \ge h^{1-\delta}$ as $\rho \ge \bar\rho$.

Therefore

(15.3.26) Contribution of $(\ell(\rho),\rho)$-element to the Tauberian remainder with $T = \mu h^{1-\delta'}\rho^{-1}$ does not exceed

$$C\mu^{-1}h^{-1}\ell(\rho)^2\bigl(\frac{\mu^2 h}{\rho} + 1\bigr) \asymp C + C\mu^{-2}h^{-1}\rho.$$

Then summation with respect to $\rho \in [\bar\rho, \epsilon\mu]$ results in

$$C|\log h| + C\mu^{-1}h^{-1}. \tag{15.3.27}$$

Thus we conclude that

(15.3.28) Contribution of zone $\{\rho \ge \bar\rho,\ \mu\ell^2 \le \rho\}$ to the Tauberian remainder with $T = \mu h^{1-\delta'}\rho^{-1}$ does not exceed (15.3.27).

Remark 15.3.3. We need to keep $\ell(\rho) \ge C\mu^{-1}$ but this is definitely case as $\mu \ge h^{-\frac{2}{5}}$ and we are ensured in this by condition (15.3.19).

Similarly, consider case $\mu\ell^2 \ge \rho \ge \bar\rho$. Then automatically $\ell \ge \ell(\bar\rho)$ and $\ell\rho \ge h^{1-\delta}$ and

(15.3.29) In this zone we can reset $\rho = \rho(\ell) = \mu\ell^2$.

Really, to avoid collision with $\mathcal{X}_{\mathsf{trans}}$ we select time direction in which ρ (and thus ξ_2) increases. It is possible because as long as $|t| \le \epsilon\mu\ell$ we remain in the same $\epsilon\ell$-vicinity. Again $\ell \ge C\mu^{-1}$ in virtue of condition (15.3.19).

Therefore

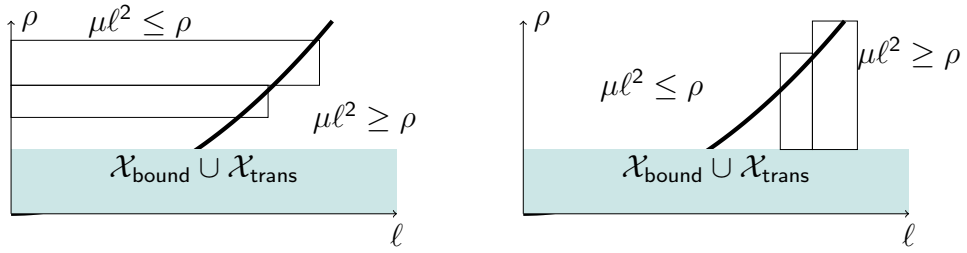
(15.3.30) Contribution of $(\ell, \rho(\ell))$-element to the Tauberian remainder with $T = h^{1-\delta'}\ell^{-2}$ does not exceed

$$C\mu^{-1}h^{-1}\rho(\ell)\bigl(\frac{\mu h}{\ell^2}+1\bigr) \asymp C + C\mu^{-1}h^{-1}\ell^2.$$

Then summation with respect to $\ell \in [\bar{\ell}, \epsilon]$ results in (15.3.27) where $\bar{\ell} = \mu^{-\frac{1}{2}}\bar{\rho}^{\frac{1}{2}}$.

Therefore we conclude that

(15.3.31) Contribution of zone $\{\rho \ge \bar{\rho},\ \mu\ell^2 \ge \rho\}$ to the Tauberian remainder with $T = \mu h^{1-\delta'}\rho^{-1}$ does not exceed (15.3.27).



(a) $\mu\ell^2 < \rho$ (b) $\mu\ell^2 > \rho$

Figure 15.8: Partition of two zones in $\mathcal{X}_{\mathsf{inn}}$

So we arrive to the following proposition:

Proposition 15.3.4. *Under assumptions* (15.2.41) *and* (15.2.7) *contribution of* $\mathcal{X}_{\mathsf{inn}}$ *to the Tauberian remainder with* $T = \mu h^{1-\delta'}\rho^{-1}$ *does not exceed* (15.3.27).

We are completely happy with this estimate unless $\mu \ge h^{-1}|\log h|^{-1}$ and we need to derive contribution of much more troublesome zone $\mathcal{X}_{\mathsf{trans}}$ before thinking if we should improve it.

15.3.3 Generic Case. Analysis in Transitional Zone

Let us consider zone $\mathcal{X}_{\mathsf{trans}}$. Recall that its contribution to the Tauberian remainder with $T = \epsilon\mu^{-1}$ (and thus with $T = h^{1-\delta}$) does not exceed $C\bar{\rho}h^{-1} = C(\mu h)^{\frac{2}{3}}h^{-1-\delta}$ in the general case and $C\bar{\rho}\bar{\ell}h^{-1} = C(\mu h)^{\frac{7}{6}}h^{-1-\delta}$ under condition (15.2.41) and therefore (as we assume (15.2.41)) we should consider only case

$$\mu \ge h^{\delta-\frac{7}{13}}. \tag{15.3.32}$$

Definitely $\mathcal{X}_{\mathsf{trans}}$ is leaner than $\mathcal{X}_{\mathsf{inn}}$ (albeit as $\mu \ge h^{\delta-1}$ it is thick enough to eliminate $\mathcal{X}_{\mathsf{bound}}$). However the main problem there is that in the current settings we are not aware of any lower bound of the propagation speed and the only bound we know is an upper bound $C_0\bar{\rho}^{\frac{1}{2}}$ in both directions.

Therefore there is no mechanism except drift with respect to ξ_2 to break periodicity and we must take

$$T_* = \epsilon h\ell^{-2} \tag{15.3.33}$$

and

$$T^* = \epsilon\bar{\rho}^{-\frac{1}{2}}\ell. \tag{15.3.34}$$

Sure we need to have $T_* \le T^*$, i.e. $\ell \ge \bar{\ell}_0$ with

$$\bar{\ell}_0 := \bar{\rho}^{\frac{1}{6}}h^{\frac{1}{3}} = (\mu h)^{\frac{1}{9}}h^{\frac{1}{3}-\delta}. \tag{15.3.35}$$

Obviously $\bar{\ell}_0 \le \bar{\ell} = (\mu h)^{\frac{1}{2}}h^{-\delta}$ unless $\mu \le h^{-\frac{1}{7}}$ the case we are not interested in.

Remark 15.3.5. Note that it it requires time $t^* \asymp \bar{\rho}\ell^{-1}$ to pierce through $\mathcal{X}_{\mathsf{trans}}$ and $t^* \le T^*$ as $\ell \le \bar{\ell}$ which is the case we are interested in. Otherwise we would be able to increase T^* further.

So, in the same manner as before contribution of $(\ell,\bar{\rho})$-element with $\ell \ge \bar{\ell}_0$ to the Tauberian remainder with any $T \ge T_*h^{-\delta}$ [7] by

$$C\mu^{-1}\bar{\rho}\ell h^{-1}\big(\mu T_* + 1\big) \times T^{*\,-1} = C\bar{\rho}^{\frac{3}{2}}\ell^{-2} + C\mu^{-1}h^{-1}\bar{\rho}^{\frac{3}{2}} \tag{15.3.36}$$

and summation over $\ell \ge \bar{\ell}_0$ results in

$$C\bar{\rho}^{\frac{3}{2}}\bar{\ell}_0^{-2} + C\mu^{-1}h^{-1}\bar{\rho}^{\frac{3}{2}}|\log h| \asymp C(\mu h)^{\frac{7}{9}}h^{-\frac{2}{3}-\delta}. \tag{15.3.37}$$

[7] As $T_*h^{-\delta} \ge T^*$ we reset $T = T^*$.

On the other hand, contribution of $(\bar{\ell}_0, \bar{\rho})$-element to the Tauberian remainder with $T \ge \epsilon\mu^{-1}$ does not exceed $C\bar{\rho}\bar{\ell}_0 h^{-1}$ which coincides with the (15.2.37).

Therefore we arrive to the following proposition:

Proposition 15.3.6. *Under condition* (15.2.41) *contribution of* $\mathcal{X}_{\mathsf{trans}}$ *with any* $T \ge T_* h^{-\delta} = h^{1-\delta}\ell^{-2}$ *(reset to* $T \ge h^{1-\delta}$ *as either* $\ell \le \bar{\ell}_0$ *or* $\ell \ge \bar{\ell}$*; see also* [7]*) does not exceed* (15.3.37)*.*

In particular, it does not exceed $C\mu^{-1}h^{-1}$ *as* $\mu \le h^{\delta-\frac{5}{8}}$*.*

Remark 15.3.7. (i) Surely we need to keep $\ell \ge C_0\mu^{-1}$ thus requiring $\bar{\ell}_0 \ge C_0\mu^{-1}$ i.e. $\mu \ge h^{-\frac{2}{5}}$ but this is a case.

(ii) We also need to keep an upper bound to speed greater than $C_0\mu^{-1}$, i.e. $\bar{\rho}^{\frac{1}{2}} \ge C_0\mu^{-1}$ i.e. $\mu \ge h^{-\frac{1}{4}}$ but this is also the case.

(iii) Further, we need to keep $\bar{\ell}_0\bar{\rho} \ge h^{1-\delta}$ i.e. $\mu \ge h^{-\frac{1}{7}}$, but this is again the case.

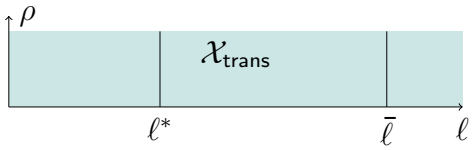


Figure 15.9: Zone $\mathcal{X}_{\mathsf{trans}}$

Can we increase T^* in these arguments? We need to do it only under condition (15.3.19). Then as we will prove later the following proposition:

Proposition 15.3.8. *Let Dirichlet boundary condition be given on* ∂X*. Then in the transitional zone* $\{|\rho| \le \bar{\rho}\}$ *as*

$$\mu \ge h^{-\frac{1}{4}-\delta} \tag{15.3.38}$$

average propagation speed in one direction (direction of hops when we have chosen time) is bounded by $C\bar{\rho}^{\frac{1}{2}}$ *and in the opposite direction it is bounded by* $C\mu^{-1}$*.*

Note, that condition (15.3.38) ensures that $\bar{\rho}^{\frac{1}{2}} \gg \mu^{-1}$.

So, assume that Dirichlet boundary condition be given on ∂X. Then selecting the time direction for given partition element so that ℓ *increases along hops* we can chose

$$T^* \asymp 1. \tag{15.3.39}$$

Really, we need to leave ℓ-element and come back. This comeback includes either going in the direction opposite of hops with the speed not exceeding $C_0\mu^{-1}$ and requires time $\asymp \mu\ell$ which is larger than (15.3.39) as $\ell \ge \mu^{-1}$ and $\mu \ge h^{-\frac{1}{4}}$ or leaving zone $\ell \le \epsilon$ which requires at least time $\asymp 1$ [8].

Actually $T^* \asymp 1$ is better (larger) than given by (15.3.34) we used before only as $\ell \le \bar{\rho}^{\frac{1}{2}}$, but this is only zone we need to care about.

So, contribution of the given element to the remainder does not exceed the left-hand expression of (15.3.36) which now becomes

$$C\mu^{-1}\bar{\rho}\ell h^{-1} \times \bigl(\mu h\ell^{-2}+1\bigr) \times \min\bigl(\bar{\rho}^{\frac{1}{2}}\ell^{-1},1\bigr) \asymp C\bar{\rho}\ell^{-1} + C\mu^{-1}h^{-1}\bar{\rho}\ell. \tag{15.3.40}$$

Summation of the second term in the right-hand expression over partition results in $O(\mu^{-1}h^{-1})$. Summation of the first term in the right-hand expression over partition with $\ell \ge \bar{\ell}_1$ results in the same term calculated as $\ell = \bar{\ell}_1$ and coincides with $C\bar{\rho}\bar{\ell}_1 h^{-1}$ which also estimates the contribution of zone $\{\ell \le \bar{\ell}_1\}$. Here again we chose $\bar{\ell}_j$ from condition $T_* \le T^*$ i.e. now we replace $\bar{\ell}_0$ defined by (15.3.35) by

$$\bar{\ell}_1 \coloneqq h^{\frac{1}{2}} \ll \bar{\ell}_0 \tag{15.3.41}$$

resulting in

$$C\bar{\rho}h^{-\frac{1}{2}} = C\mu^{\frac{2}{3}}h^{\frac{1}{6}-\delta}. \tag{15.3.42}$$

Therefore (with pending Proposition 15.3.8) we arrive to the following proposition:

Proposition 15.3.9. *Assume that Dirichlet boundary condition is given on ∂X. Then under assumption* (15.2.40) *the total contribution of $\mathcal{X}_{\mathsf{trans}}$ to the Tauberian remainder does not exceed* (15.3.42).

In particular, it does not exceed $C\mu^{-1}h^{-1}$ as $\mu \le h^{\delta-\frac{7}{10}}$ and it is always has an extra factor $\mu^{\frac{2}{3}}h^{\frac{7}{6}-\delta}$ in comparison with h^{-1}.

[8] As X is a bounded domain and $F \asymp 1$ in X this would mean run-around X along boundary which as we see later is impossible under certain assumptions.

Can we do better than this? Yes, as we can take

$$T^* \asymp \mu\ell. \tag{15.3.43}$$

provided ξ_2 and ℓ increase in the same time direction (on given ℓ-element) as $F_{12} < 0$.

Really, then selecting such time direction we remain in $\mathcal{X}_{\mathsf{trans}} \cap \mathcal{X}_{\mathsf{inn}}$ and as we reach $\ell \asymp 1$ dynamics is already in $\mathcal{X}_{\mathsf{inn}}$ where speed in both directions is $O(\mu^{-1})$.

Assuming that the critical point of $W := VF^{-1}|_{\partial X}$ is $x_2 = 0$ we note that hops are to the left (as $F_{12} < 0$). So, we select time direction of the sign of $(-x_2)$, so $\ell \asymp |x_2|$ increases.

Meanwhile those expressions have the same sign: $d\xi_2/dt$, $-\partial_{x_2} W$ and $(-\partial_{x_2}^2 W)x_2$ and our condition means that these expressions are positive, i.e. condition $(15.2.41)^-$ is fulfilled which matches to cases displayed at Figure 15.5(b), (d). So, under this condition we can select T^* according to (15.3.43), therefore replacing $\bar{\ell}_1$ defined by (15.3.41) by

$$\bar{\ell}_2 := \mu^{-\frac{1}{3}} h^{\frac{1}{3}-\delta} \ll \bar{\ell}_1 \tag{15.3.44}$$

(as $\mu \ge h^{-\frac{1}{2}-\delta}$) from condition $T_* \le T^*$ and arrive to remainder estimate

$$C\bar{\rho}\bar{\ell}_2 h^{-1} + C\mu^{-1}h^{-1} = C\mu^{\frac{1}{3}} h^{-\delta} + C\mu^{-1}h^{-1}. \tag{15.3.45}$$

So we have proven

Proposition 15.3.10. *Assume that Dirichlet boundary condition is given on ∂X. Then under assumption* $(15.2.41)^-$ *the total contribution of $\mathcal{X}_{\mathsf{trans}}$ to the Tauberian remainder does not exceed* (15.3.45).

In particular, it does not exceed $C\mu^{-1}h^{-1}$ as $\mu \le h^{\delta-\frac{3}{4}}$ and it has an extra factor $\mu^{\frac{1}{3}} h^{1-\delta}$ in comparison with h^{-1} (as $h^{\delta-\frac{3}{4}} \le \mu \le h^{-1}$).

Finally, assume that hops and magnetic drift have the same direction near point in question which is the case under condition

$$|\partial_{x_2} VF^{-1}| \le \epsilon \implies \partial_{x_1} VF^{-1} \ge \epsilon_0; \tag{15.2.7}^+$$

compare with original (15.2.7).

Then according to Proposition 15.3.8 there can be no roll-back, only run-around with the path of the length $\asymp 1$. This corresponds to Figure 15.5(a) and (b).

However if condition $(15.2.41)^-$ is also fulfilled (so we are in the framework of Figure 15.5(b)) fast run-around along boundary is not possible either, so the run around must contain the segment of the length $\asymp 1$ inside $\mathcal{X}_{\mathsf{inn}}$ (even inside $\{x, x_1 \ge \epsilon\}$) but the speed here is $O(\mu^{-1})$ and therefore $T^* \asymp \mu$. Then $T_* \le T^*$ defines

$$\bar{\ell}_3 := \mu^{-\frac{1}{2}} h^{\frac{1}{2}-\delta} \ll \bar{\ell}_2; \tag{15.3.46}$$

this leads to remainder estimate $C\bar{\rho}\bar{\ell}_3 h^{-1} = \mu^{\frac{1}{6}} h^{\frac{1}{6}-\delta}$ which is $O(\mu^{-1}h^{-1})$ as $\mu \le h^{\delta-1}$ and $O(h^{-\delta})$ otherwise. Thus we have proven

Proposition 15.3.11. *Assume that Dirichlet boundary condition is given on ∂X. Then under assumptions $(15.2.41)^-$ and $(15.2.7)^+$ the total contribution of $\mathcal{X}_{\mathsf{trans}}$ to the Tauberian remainder does not exceed $C\mu^{-1}h^{-1}$ as $\mu \le h^{\delta-1}$ and $Ch^{-\delta}$ otherwise.*

Conjecture 15.3.12. *The above Tauberian estimates hold with $\delta = 0$.*

To prove this conjecture we must reload $\mathcal{X}_{\mathsf{bound}}$ and $\mathcal{X}_{\mathsf{inn}}$ and to derive there the same remainder estimates as we proved in $\mathcal{X}_{\mathsf{inn}}$ but with $\bar{\rho} = (\mu h)^{\frac{2}{3}}$. It does not look difficult as we need just to rescale $x_1 \to x\gamma$ with $\gamma = h(\rho/\bar{\rho})^{-k}$ with arbitrarily large k. However then $h \mapsto \hbar = (\rho/\bar{\rho})^{-k}$ and it is not small enough to keep number of reflections below $\hbar^{-N}$. Really, we need to consider at least $\mu h\ell^{-2}$ number of reflections which means $\mu h\ell^{-2} \le \hbar^{-s}$ and as $\hbar$ goes to 1 we need to allow $\ell \ge (\mu h)^{\frac{1}{2}}$ only!

Another approach would be based on logarithmic uncertainty principle (albeit instead of $h^{-\delta}$ logarithmic factor $|\log h|^l$ would appear) but we do not have here theory similar to one of Section 2.3.

15.3.4 Propagation of Singularities in Transitional Zone

Let us study propagation of singularities in $\mathcal{X}_{\mathsf{trans}}$; later we assume that the Dirichlet boundary condition is given. We will use the technique developed Section 3.4. However there is a large difference: in Section 3.4 we treated basically trajectories with a single tangent point while here we need to treat μT such points in one shot.

Consider real function $\psi'(x, \xi, t, \tau)$ and by Weierstrass theorem replace it modulo $\big(\tau - a(x, \xi)\big)$ by a linear with respect to ξ_1 function

$$\psi'(x, \xi, t, \tau) = \psi_0(x, \xi_2, t, \tau) + \psi_1(x, \xi_2, t, \tau)\xi_1 = \psi(x, \xi, t, \tau) + \alpha(x, \xi, t, \tau)\big(\tau - a(x, \xi)\big). \tag{15.3.47}$$

Then as coefficient at ξ_1^2 in $a(x,\xi)$ is 1

$$\{\tau - a, \psi\} = \{\tau - a, \psi\} + \alpha_1\big(\tau - a(x,\xi)\big) \tag{15.3.48}$$

is at most quadratic with respect to ξ_1^2 and therefore the following identities hold:

$$\begin{aligned}((hD_t - A)u, \psi^{\mathsf{w}} u) = (u, (hD_t - A)\psi^{\mathsf{w}} u) + ih(hD_1 u, \psi^{\mathsf{w}} u)_{\partial X} - \\ ih(u, hD_1\psi^{\mathsf{w}} u)_{\partial X},\end{aligned}$$

$$(\psi^{\mathsf{w}} u, (hD_t - A)u) = (u, \psi^{\mathsf{w}}(hD_t - A)u) + ih(\psi_1^{\mathsf{w}} u, (hD_t - A)u)_{\partial X}$$

implying identity

$$\begin{aligned}2h^{-1}\operatorname{Re} i((hD_t - A)u, \psi^{\mathsf{w}} u) = -(ih^{-1}[(hD_t - A), \psi^{\mathsf{w}}]u, u) - \\ (hD_1 u, \psi^{\mathsf{w}} u)_{\partial X} + (u, hD_1\psi^{\mathsf{w}} u)_{\partial X} + (\psi_1^{\mathsf{w}} u, (hD_t - A)u)_{\partial X}.\end{aligned} \tag{15.3.49}$$

In this identity terms containing $(hD_t - A)u$ will be either negligible or under our control anyway; $[(hD_t - A), \psi^{\mathsf{w}}] \equiv -ih\{\tau - a, \psi\}^{\mathsf{w}}$ modulo smaller terms and only boundary terms

$$-(hD_1 u, \psi^{\mathsf{w}} u)_{\partial X} + (u, hD_1\psi^{\mathsf{w}} u)_{\partial X} \tag{15.3.50}$$

need special analysis. Under Dirichlet boundary condition we rewrite them as

$$-(hD_1 u, \psi_1^{\mathsf{w}} hD_1 u)_{\partial X} \tag{15.3.51}$$

and under Neumann boundary condition we rewrite them as

$$(u, h^2 D_1^2\psi_1^{\mathsf{w}} u)_{\partial X} = -(u, (hD_t - A)\psi_1^{\mathsf{w}} u)_{\partial X} + (u, (hD_t - A')\psi_1^{\mathsf{w}} u) \tag{15.3.52}$$

where $A' = A - h^2 D_1^2$ and the difference is that in the former case definiteness of this quadratic form is ensured by ψ_1 having definite sign but in the latter case not.

Remark 15.3.13. (i) The same difference would manifest itself as we would try to prove results of Section 3.4 for wave equation under Dirichlet and Neumann boundary condition. The truth is that the former satisfies Lopatinski condition while the latter does not: uniformity breaks in the tangent zone.

(ii) The same difference manifests itself through different behavior of eigenvalues $\lambda_{\mathsf{D},j}(\eta)$ and $\lambda_{\mathsf{N},j}(\eta)$ as $\eta \to +\infty$: while they both tend to $(2j+1)\mu h$ the former are tending to it from above and the latter from below. This implies that under Neumann boundary condition at least for a toy-model operator some singularities propagate in the direction opposite to hops and this propagation is faster than magnetic drift. Such difference will be the most transparent in the case of the intermediate and strong magnetic field in Section 15.4.

So, we assume that Dirichlet boundary condition is given on ∂X. We select

$$\psi_1 \ge 0 \tag{15.3.53}$$

and thus we should request

$$\{\tau - a, \psi'\} + \beta(\tau - a) \le 0 \qquad \text{with} \quad \beta = \beta(x, t, \xi_2, \tau). \tag{15.3.54}$$

We take originally

$$\psi' = \chi(\phi), \qquad \phi = \phi(x, \xi) \tag{15.3.55}$$

where $\mathscr{C}^\infty(\mathbb{R}) \ni \chi$ is a function of the same type as in Section 3.4: namely supported in $(-\infty, 0]$ and with $\chi' < 0$ on $(-\infty, 0)$.

Then as $\tau - a(x, \xi) = 0$ has two real roots $\pm\eta \ne 0$ we conclude that

$$\psi_1 = \frac{1}{2\eta}\Big(\chi(\phi(\xi_1 = \eta)) - \chi(\phi(\xi_1 = -\eta))\Big) \ge 0 \tag{15.3.56}$$

provided

$$-\partial_{\xi_1}\phi \ge \epsilon_0. \tag{15.3.57}$$

It does not satisfy (15.3.53) as $\tau - a' < 0$ but we will handle this in the same way as in Section 3.4).

Now, modulo terms of the type $\beta \cdot (\tau - a)$

$$\begin{aligned}\{\tau - a, \psi\} \equiv \{\tau - a, \psi'\} \equiv -\chi'(\phi) \cdot \{\tau - a, \phi\} \equiv \\ -\chi_1^2(\phi)\{\tau - a, \phi\} \equiv -\hat{\psi_1}^{\,2}\{\tau - a, \phi\}\end{aligned} \tag{15.3.58}$$

as $\chi_1 = \sqrt{-\chi'}$ is a smooth function (under correct choice of χ) and $\hat{\psi}$ is again linear with respect to ξ_1 function,

$$\hat{\psi} \equiv \chi_1(\phi)(\{\tau - a, \phi\})^{\frac{1}{2}} \tag{15.3.59}$$

and we used Weierstrass theorem again. We want

$$\{\tau - a, \phi\} \geq \epsilon_0. \tag{15.3.60}$$

Remark 15.3.14. Conditions (15.3.60) and (15.3.57) mean respectively that ϕ increases along trajectories as $x_1 > 0$ and at reflections.

Let us recall that instead of (15.3.53) we actually have a weaker inequality Namely instead of (15.3.53) we have

$$\psi_1 \geq -\chi_2(\phi)^2\chi_3(\tau - a')^2 \tag{15.3.53}'$$

with both χ_2 and χ_3 smooth functions supported in $(-\infty, 0]$ and notice that both operator $(hD_t - A)$ and Dirichlet boundary problem for it are elliptic as $\tau - a' < 0$ and we can apply elliptic arguments there.

So, repeating arguments of Section 3.4 we conclude that if

$$\mathsf{WF}^{s+1}\big((hD_t - A)u\big) \cap \big\{\phi^+ < \varepsilon\big\} \cap \{0 \leq t \leq T\} = \emptyset, \tag{15.3.61}$$

$$\mathsf{WF}^{s}\big(u|_{\partial X}\big) \cap \big\{\phi^+ < \varepsilon\big\} \cap \{0 \leq t \leq T\} = \emptyset, \tag{15.3.62}$$

$$\big(\mathsf{WF}^{s}(u) \cup \mathsf{WF}^{s}(hD_1 u)\big) \cap \big\{\phi^+ < \varepsilon\big\} \cap \{t = 0\} = \emptyset \tag{15.3.63}$$

and

$$\big(\mathsf{WF}^{s-\sigma}(u) \cup \mathsf{WF}^{s-\sigma}(hD_1 u)\big) \cap \{\phi^+ < \varepsilon\} \cap \{0 \leq t \leq T\} = \emptyset, \tag{15.3.64}$$

$$\mathsf{WF}^{s-\sigma}(hD_1 u|_{\partial X}) \cap \{\phi^+ < \varepsilon\} \cap \{0 \leq t \leq T\} = \emptyset \tag{15.3.65}$$

then

$$\big(\mathsf{WF}^{s}(u) \cup \mathsf{WF}^{s}(hD_1 u)\big) \cap \{\phi^+ < 0\} \cap \{0 \leq t \leq T\} = \emptyset, \tag{15.3.66}$$

$$\mathsf{WF}^{s}(hD_1 u|_{\partial X}) \cap \{\phi^+ < 0\} \cap \{0 \leq t \leq T\} = \emptyset. \tag{15.3.67}$$

Here $\sigma > 0$ is a sufficiently small exponents, $\epsilon > 0$ is an arbitrarily small constant and $T > 0$ is an arbitrary constant,

$$\phi^{\pm}(x_1, x_2, \xi_2, t, \tau) = \phi|_{\xi_1 = \pm\eta}, \qquad \eta = (\tau - a')^{\frac{1}{2}} \tag{15.3.68}$$

and WF^s are defined in terms of pseudodifferential operators $b(x, t, hD_2, hD_t)$.

As we can plug $(\phi - \varepsilon)$ instead of ϕ we by induction can get rid of assumptions (15.3.64), (15.3.65) (assuming that u is temperate).

We also can rescale $x \mapsto Tx$ (and $h \mapsto hT^{-1}$, $\mu \mapsto \mu T$) thus replacing ϵ by ϵT when we come back and we can also consider large T.

As $a = \xi_1^2 + (\xi_2 - \mu x_1^2) + V$ we select

$$\phi = \kappa t + (\mu x_2 - \xi_1). \tag{15.3.69}$$

In more general case we select

$$\phi = \kappa t + \mu x_2 + F_{12} p_1. \tag{15.3.70}$$

Note that (15.3.48) is fulfilled and also

$$\{\tau - a, \phi\} = \kappa - \partial_{x_1} V; \tag{15.3.71}$$

and (15.3.60) becomes $\kappa - \partial_{x_1} V \ge \epsilon$; so is fulfilled in two cases:

$$\kappa = C_0 \tag{15.3.72}$$

and

$$\partial_{x_1} V \ge \epsilon_0, \qquad |\kappa| \le \epsilon_1 \tag{15.3.73}$$

which allows us to prove respectively that "$(\mu x_2 - \xi_1)$" propagates with a speed bounded from below by $-C_0$ and, as $\partial_{x_1} V \ge \epsilon_0$ by ϵ_1. To get rid of ξ_1 we must to pass from ϕ^+ to ϕ but we need to notice that $\phi^+ \le \phi + c_0$.

Then we arrive to the following theorem:

Theorem 15.3.15. *Let Dirichlet boundary condition be given on ∂X and let $F_{12} < 0$ i.e. hops go to the left. Let $\chi \in \mathscr{C}_0^\infty([\frac{1}{2}, 1])$. Then*

(i) For $C_0 \le T \le T^*$

$$|F_{t \to h^{-1}\tau} \chi_T(t) \psi_2(x_2) U \psi_1(y_2)| \le C h^s \tag{15.3.74}$$

as

$$x_2 \le -C_0 \mu^{-1} T + y_2 \quad \forall x_2 \in \operatorname{supp} \psi_2, \forall y_2 \in \operatorname{supp} \psi_1; \tag{15.3.75}$$

(ii) Under condition (15.3.46) *for $C_0 \le T \le T^*$* (15.3.74) *holds as*

$$x_2 \le \epsilon_0 \mu^{-1} T + y_2 \quad \forall x_2 \in \operatorname{supp} \psi_2, \forall y_2 \in \operatorname{supp} \psi_1. \tag{15.3.76}$$

This theorem implies Proposition 15.3.8 and thus justifies the improved results of the previous subsection.

15.3.5 Calculations

Now we need to move from Tauberian to more explicit expressions for the principal part of asymptotics.

First of all recall that the Tauberian expression for $\Gamma\big(e(.,.,0)\,{}^tQ_y\big)$ is

$$h^{-1}\int_{-\infty}^{0} F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma\big(U\,{}^tQ_y\big)\,d\tau \tag{15.3.77}$$

and if we replace $\bar{\chi}_T(t)$ by $\chi_T(t)$ [9] we will get

$$T^{-1}\int_{-\infty}^{0} F_{t\to h^{-1}\tau}\tilde{\chi}_T(t)\Gamma\big(U\,{}^tQ_y\big)\,d\tau \tag{15.3.78}$$

where $\tilde{\chi}(t) = it^{-1}\chi(t)$.

Let us list different cases depending on our assumptions

General Discussion

As Q is supported in $\mathcal{X}_{\mathsf{trans}}$ an absolute value of this expression does not exceed

$$C\mu^{-1}\bar{\rho}\ell h^{-1}T^{-1}\times \mu T = C\bar{\rho}h^{-1} \tag{15.3.79}$$

as $T\ge \epsilon_0\mu^{-1}$ and therefore an approximation error should not exceed

$$C\bar{\rho}\ell h^{-1}\times Th^{-1}\Delta \tag{15.3.80}$$

where Δ is the size of perturbation and we know that *as we replace* $V(x_1,x_2)$ *by* $V(x_1,y_2)$

$$\Delta = C_0\ell(\delta x_2) + C_0(\delta x_2)^2 = C_0\ell(\mu^{-1}+\boldsymbol{v}T) + C_0(\mu^{-1}+\boldsymbol{v}T)^2 \tag{15.3.81}$$

as $\delta x_2 = (\mu^{-1}+\boldsymbol{v}T)$ and in the first term factor ℓ comes as bound for $|\partial_{x_2}V|$. Here $\boldsymbol{v}\asymp \bar{\rho}^{\frac12}$ is an upper bound for propagation speed in $\mathcal{X}_{\mathsf{trans}}$. However we will justify that under Dirichlet boundary condition we can take *in these calculations* $\boldsymbol{v}\asymp\mu^{-1}$. The difference between $\bar{\rho}^{\frac12} = (\mu h)^{\frac13-\delta}$ increases as μ increases.

[9] Where as usual $\bar{\chi},\chi\in \mathscr{C}_0^\infty([-1,1])$ and $\bar{\chi}=1$, $\chi=0$ on $[-\frac12,\frac12]$ and later we will decompose $\bar{\chi}_T$ in the sum of $\bar{\chi}_{\bar{T}}(t)$ and $\chi_{2^{-n}T}(t)$

Neglecting the second term in (15.3.81) (to be justified later), we acquire factor

$$Th^{-1}\Delta = C\ell(\mu^{-1}+\boldsymbol{v}T)Th^{-1} \tag{15.3.82}$$

which after plugging $T = h^{1-\sigma}\ell^{-2}$ becomes

$$C\ell\mu^{-1}Th^{-1} + C\boldsymbol{v}T^2h^{-1} = C\Big(\mu^{-1}\ell^{-1} + C\boldsymbol{v}h\ell^{-3}\Big)h^{-\sigma} \tag{15.3.83}$$

and it is $O(h^\sigma)$ as

$$\ell \ge \hat{\ell} := \max\big(\mu^{-1}, \boldsymbol{v}^{\frac{1}{3}}h^{\frac{1}{3}}\big)h^{-\sigma} \tag{15.3.84}$$

and the contribution of zone $\ell \le \hat{\ell}$ does not exceed

$$C\bar{\rho}\hat{\ell}h^{-1} = C\bar{\rho}\max\big(\mu^{-1}, \boldsymbol{v}^{\frac{1}{3}}h^{\frac{1}{3}}\big)h^{-1-\sigma} \tag{15.3.85}$$

and as $\mu \le h^{\delta-1}$ modulo $O(\mu^{-1}h^{-1})$ it is

$$\bar{\rho}\boldsymbol{v}^{\frac{1}{3}}h^{\frac{1}{3}}h^{-1-\sigma} \asymp \begin{cases} \mu^{\frac{7}{9}}h^{\frac{1}{9}-\sigma} & \text{as } \boldsymbol{v} \asymp \bar{\rho}^{\frac{1}{2}} \\ \mu^{\frac{1}{3}}h^{-\sigma} & \text{as } \boldsymbol{v} \asymp \mu^{-1} \end{cases} \tag{15.3.86}$$

which is $O(\mu^{-1}h^{-1})$ as <u>either</u> $\boldsymbol{v} \asymp \bar{\rho}^{\frac{1}{2}}$ and $\mu \le h^{\delta-\frac{9}{13}}$ <u>or</u> $\boldsymbol{v} \asymp \mu^{-1}$ and $\mu \le h^{\delta-\frac{3}{4}}$.

Meanwhile contribution of an element with $\ell \ge \hat{\ell}$ does not exceed (15.3.80) which is

$$C\bar{\rho}\ell h^{-1} \times \Big(\mu^{-1}\ell^{-1} + C\boldsymbol{v}h\ell^{-3}\Big)h^{-\sigma} \tag{15.3.87}$$

and summation over $\ell \ge \hat{\ell}$ results in the same expression with $\ell = \hat{\ell}$ and it is the same (15.3.86) as before.

On the other hand, (the ratio of) second term in (15.3.81) to the first one is $(\mu^{-1}+\boldsymbol{v}T)$ to ℓ and $\ell \ge \mu^{-1}$ so only comparison is $\boldsymbol{v}h\ell^{-2}$ vs ℓ and ℓ is larger by the choice.

Then we conclude that

(15.3.88) Under condition (15.2.41) contribution of $\mathcal{X}_{\mathsf{trans}}$ to approximation error is $O\big(\mu^{\frac{7}{9}}h^{\frac{1}{9}-\delta}\big)$ for Neumann boundary condition and $O\big(\mu^{\frac{1}{3}}h^{-\delta}\big)$ for Dirichlet boundary condition.

Case of Condition (15.2.7)$^+$. Assume now that Dirichlet boundary condition is given and condition (15.2.7)$^+$ is fulfilled. Then

$$T_* = h^{-\sigma}\min\bigl(\frac{(\mu h)}{\bar\rho}, \frac{h}{\ell^2}\bigr) \tag{15.3.89}$$

and therefore the above arguments should be applied only as $\ell \ge \hat\ell_1$ with

$$\hat\ell_1 = \mu^{-\frac{1}{6}} h^{\frac{1}{3}}; \tag{15.3.90}$$

otherwise we replace ℓ by $\hat\ell_1$ (single element) and $T = \mu^{\frac{1}{3}} h^{\frac{1}{3}-\sigma}$ and $\delta x_2 = \delta x_1 = O(\mu^{-1})$; so $\Delta \asymp \hat\ell_1 \mu^{-1}$. Here we assume that $\hat\ell_1 \ge C\mu^{-1}$ as otherwise we reset it to this value which would lead to the error less than $C\mu^{-1}h^{-1-\sigma}$.

Then obviously the total error under the same replacement $V(x_1,x_2)$ by $V(0,y_2) + (\partial_{x_1}V)(0,y_2)x_1$ does not exceed

$$C\bar\rho\hat\ell_1 h^{-1} \times (\mu h)^{\frac{1}{3}} h^{-1-\sigma}\hat\ell_1\mu^{-1} \asymp C\hat\ell_1^2 h^{-1-\sigma} = h^{-\sigma}\min\bigl(\frac{(\mu h)}{\bar\rho}, \frac{h}{\hat\ell_1^2}\bigr) \tag{15.3.91}$$

which in turn is $O\bigl(\mu^{-1}h^{-1} + h^{-\sigma}\bigr)$.

Non-Degenerate cCase. Finally, under condition (15.2.7) we take $T_* = h^{1-\sigma}$ and $\Delta = \mu^{-1}h^{-\sigma}$ so each next term acquires factor $C\mu^{-1}h^{-\sigma}$ i.e. the second term is $C\bar\rho h^{-1} \times \mu^{-1}h^{-\sigma}$ which is $O(\mu^{-1}h^{-1})$ unless $\mu \ge h^{\delta-1}$, in which case it is $O(h^{-\delta})$. However in the latter case the the third term is $O(1)$ so we will need to consider the second term in approximations as well.

Boundary Zone. Here we take $T_* = h^{1-\sigma}$ and $\Delta = \mu^{-1}h^{-\sigma}$ so each next term acquires factor $C\mu^{-1}h^{-\sigma}$ i.e. the second term is $Ch^{-1} \times \mu^{-1}h^{-\sigma}$ which is $O(\mu^{-1}h^{-1-\sigma})$. However the third term is $O(1)$ so we will need to consider the second term in approximations as well.

Inner Zone In inner zone we make usual transformation and apply the standard method of successive approximations at point y rather than $(0,y_2)$.

Calculations

Calculations. I. As a result in the zone $\{x_1 \ge C_0\mu^{-1}\}$ we get

$$h^{-2}\int \mathcal{N}^{\mathsf{MW}}(x,\tau,\mu h)\psi(x)\,dx$$

and in the zone $\{x_1 \le C_0\mu^{-1}\}$ we get something like this but with $\mathcal{N}^{\mathsf{MW}}$ replaced by $\mathcal{N}_*^{\mathsf{MW}}$ temporarily denoting eigenvalue counting function for operator with the boundary conditions. So this main part would come from parametrix $\bar{G}^{\pm}$ for operator $hD_t - A$ in the direction of $\pm t > 0$ (see previous and similar chapters) and this parametrix is equal to $\bar{G}_0^{\pm} + \bar{G}_1^{\pm}$ where "bar" refers to freezing coefficients in y.

Consider first term with $\bar{G}_0^{\pm}$. While $\bar{G}_0^{\pm}$ is what it was on $\mathbb{R}^2$, Γ is not as now old Γ is replaced by $\Gamma\theta(x_1) = \Gamma - \Gamma\theta(-x_1)$ and only the first term results in $h^{-2}\mathcal{N}^{\mathsf{MW}}(y, \tau, \mu h)$.

Note that contribution of terms with $\Gamma\theta(-x_1)\bar{G}_0^{\pm}$ and $\Gamma\theta(x_1)\bar{G}_1^{\pm}$ into main part of asymptotics are of magnitudes $\mu^{-1}h^{-2} \times \mu h = h^{-1}$ where μ^{-1} is the width of X_{bound} and μh is a semiclassical parameter after rescaling. Therefore replacing y_1 by 0 will bring really the estimate we had referred to and these terms together will result in $h^{-1}\mathcal{N}_{\mathsf{corr}}^{\mathsf{MW}}$.

Justification. To justify setting of the upper bound of propagation speed μ^{-1} rather than $\bar{\rho}^{\frac{1}{2}}$ under Dirichlet boundary conditions we just note that if $A' = A - A'$ then considering sandwich $\bar{G}^{\pm}A'\bar{G}^{\pm} \dots A'\bar{G}^{\pm}\delta$ (one of $\bar{G}^{\pm}$ may be replaced by $G^{\pm}$) we can decompose $A' = A'_0 + A'_+ + A'_-$ where A'_0, A'_+ and A'_- are copies of operator A' localized as $|x_2 - y_2| \le 2c_0\mu^{-1}(T+1)$, $(x_2 - y_2) \ge \frac{3}{2}c_0\mu^{-1}(T+1)$ and $(x_2 - y_2) \le -\frac{3}{2}c_0\mu^{-1}(T+1)$ respectively.

Note that due to the fact that in direction of $\pm t$ propagation speed does not exceed $c_0\mu^{-1}$ we conclude that as we consider time direction $\pm t > 0$ sandwiches containing at least one factor $A'_{\pm}$ are negligible, and sandwiches containing at least one factor $A'_{\mp}$ become negligible after we apply $x_2 = y_2$.

So, the first term in approximation results in what we claimed as an answer.

Calculations. II. Meanwhile *as we replace* $V(x_1, y_2)$ *by* $V(0, y_2) + (\partial_{x_1}V)(0, y_2)x_1$ we notice that $\Delta = C_0(\delta x_1)^2 \asymp C_0\mu^{-2}$ as $\delta x_1 = O(\mu^{-1})$ which is smaller than (15.3.81).

The same arguments show that removing $(\partial_{x_1}V)(0, y_2)x_1$ leads to the error not exceeding

$$C\mu^{-\frac{1}{2}}\bar{\rho}h^{-1} = C\mu^{\frac{1}{6}}h^{-\frac{1}{3}-\sigma} \tag{15.3.92}$$

as we need to use $\hat{\ell}_1 = \mu^{-\frac{1}{2}}$ instead of $\hat{\ell}$ defined by (15.3.85).

Unfortunately expression (15.3.92) may be larger than anything we got before. Fortunately there are alternatives: first, replacing x_1 by $W_0^{\frac{1}{2}}(x_2)$ leads to an error in operator $O(\mu^{-1}\bar\rho)$ and to the approximation error

$$C(\mu^{-1}\bar\rho)^{\frac{1}{2}}\bar\rho h^{-1} = C\mu^{\frac{1}{2}}h^{-\sigma}; \tag{15.3.93}$$

this would mean replacing $\mathcal{N}^{\mathsf{MW}}_{*,\mathsf{bound}}(W_0)$ by $\mathcal{N}^{\mathsf{MW}}_{*,\mathsf{bound}}(W_1)$ with $W_1 = W_0 + \mu^{-1}(\partial_{x_1}W)W_0^{\frac{1}{2}}$.

Second, we can replace x_1 by $\mu^{-1}\xi_2$ i.e. use $W(\mu^{-1}\xi_2, x_2)$ instead of $W_0(x_2)$ and we do not need to remove linear term $(\partial_{x_1}V)(0,y_2)(x_1-\mu^{-1}\xi_2)$ but just simply shift with respect to ξ_2. This would lead to $O(\mu^{-2})$ shift in $W(\mu^{-1}\xi_2, x_2)$ and this shift could be ignored.

Calculations. III. Consider the second term in approximations. There are two cases when we need to do this: in the boundary zone as $\mu \le h^{\delta-1}$ and under condition (15.2.8) as $\mu \ge h^{\delta-1}$.

In the former case however we need to consider only $T \le \epsilon\mu$ and then only one winding should be considered and perturbation does not exceed $C_0\mu^{-1}$, and we can take $T = h^{1-\sigma}$ so in fact we get an extra factor $\mu^{-1}h^{-\sigma}$. However scaling shows that in fact as we replace $\bar\chi_T$ by χ_T the error will be $C\mu^{-1}Th^{-2}(h/T)^s$ and summation results in $C\mu^{-1}h^{-1}$ error. Alternatively we could employ semiclassical approximation to show the same.

In the latter case there is no boundary zone anymore. However scaling shows that the same arguments work (without going to semiclassics).

15.3.6 Final Results

Let us compare Tauberian remainder and approximation error.

(i) Under assumptions (15.2.8) both Tauberian remainder and an approximation error are $O(\mu^{-1}h^{-1})$.

(ii) Under assumptions (15.2.7) and (15.2.41) Tauberian remainder is estimated by (15.3.37) + $O(\mu^{-1}h^{-1})$ and an approximation error is estimated by (15.3.86)+(15.3.92) and expressions (15.3.37) and (15.3.86) coincide, so the total error is (15.3.86) + (15.3.92) + $O(\mu^{-1}h^{-1})$.

(iii) Under assumptions (15.2.7) and (15.2.41) and Dirichlet boundary condition Tauberian remainder is estimated by (15.3.42) + $O(\mu^{-1}h^{-1})$ and approximation error is estimated by (15.3.86)+(15.3.92) and (15.3.42) is larger than (15.3.86) so the total error is (15.3.42) + (15.3.92) + $O(\mu^{-1}h^{-1})$.

(iv) Under assumptions(15.2.7) and $(15.2.41)^-$ and Dirichlet boundary condition Tauberian remainder is estimated by (15.3.45) and approximation error is (15.3.92)+$O(\mu^{-1}h^{-1}+h^{-\delta})$ and the total error is (15.3.45)+(15.3.92).

(v) Under assumptions $(15.2.7)^+$ and $(15.2.41)^-$ and Dirichlet boundary condition Tauberian remainder is $O(\mu^{-1}h^{-1}+h^{-\delta})$ and an approximation error is (15.3.92) + $O(\mu^{-1}h^{-1}+h^{-\delta})$ and this is a total error.

Thus we arrive to the following theorem:

Theorem 15.3.16. *Let $\psi \in \mathscr{C}^\infty(\bar{X})$ be a fixed function with a compact support contained in the small vicinity of ∂X and let conditions* (13.2.1)–(13.2.4), (13.3.1) *and and* (13.4.45) *be fulfilled on* $\mathsf{supp}\,\psi$.

Further, let condition (15.3.1) *be fulfilled. Then*

(i) Under non-degeneracy condition (15.2.8) R^{MW} *is* $O(\mu^{-1}h^{-1})$.

(ii) Under non-degeneracy conditions (15.2.7) *and* (15.2.41) R^{MW} *does not exceed* (15.3.86) + (15.3.92) + $O(\mu^{-1}h^{-1})$.

(iii) Under non-degeneracy conditions (15.2.7) *and* (15.2.41) $\mathsf{R}^{\mathsf{MW}}_{\mathsf{D}}$ *does not exceed* (15.3.42) + (15.3.92) + $O(\mu^{-1}h^{-1})$.

(iv) Under assumptions (15.2.7) *and* $(15.2.41)^-$ $\mathsf{R}^{\mathsf{MW}}_{\mathsf{D}}$ *does not exceed* (15.3.45)+(15.3.92).

(v) Under non-degeneracy conditions $(15.2.7)^+$ *and* $(15.2.41)^-$ $\mathsf{R}^{\mathsf{MW}}_{\mathsf{D}}$ *does not exceed* (15.3.92) + $O(\mu^{-1}h^{-1}+h^{-\delta})$.

We leave to the reader

Problem 15.3.17. Write down correction terms $\mathcal{N}^{\mathsf{MW}}_{*,\mathsf{bound},\mathsf{corr}}$ corresponding to the procedures described Paragraph *15.3.5.2.3 Calculations. II.*. Then $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}$ will be the same as in Theorem 15.3.16 albeit without (15.3.92).

One of the possible modifications of $\mathcal{N}^{\mathsf{MW}}_{*,\mathsf{bound}}$ is given by (15.4.81).

Conjecture 15.3.18. *All these estimates hold with $\delta = 0$.*

15.4 Strong Magnetic Field

15.4.1 Preliminary Analysis

In this section we consider cases of the *strong magnetic field* covering both *intermediate magnetic field*[10)]

$$\epsilon_0 h^{-1} \le \mu \le C_0 h^{-1} \tag{15.4.1}$$

and *really strong magnetic field*

$$C_0 h^{-1} \le \mu \tag{15.4.2}$$

In the latter case operator needs to be modified: we will replace V by $V - \mu h \mathfrak{z} F$ where constant $\mathfrak{z}$ will be specified later; so operator (15.2.1) is replaced by

$$A = \sum_{1\le j,k\le 2} P_j g^{jk} P_k - \mu h \mathfrak{z} F + V, \qquad \text{with } P_j = hD_j - \mu V_j. \tag{15.4.3}$$

Then

(15.4.4) We need only to consider eigenvalues $\lambda_n(\eta)$ of the toy-model operator $L(\eta)$ defined by (15.1.26) with $n \le N := N(\epsilon_0)$.

Really, as $n = 0, 1, 2, ...$

$$\inf_\eta \lambda_{\mathsf{D},n}(\eta) \ge (2n+1) \quad (n \ge 0) \tag{15.4.5}_{\mathsf{D}}$$

$$\inf_\eta \lambda_{\mathsf{N},n}(\eta) \ge (2n-1), \quad (n \ge 1). \tag{15.4.5}_{\mathsf{N}}$$

due to Proposition 15.A.1 and our operator is modelled by $(\mu h)^{-1} L(\eta)$.

However then

$$|\lambda_{*,j}(\eta) - \lambda_{*,n}(\eta)| \ge \epsilon(N) \qquad \forall m \ne n \le N \ \forall \eta \tag{15.4.6}$$

with subscript $*$ denoting either D or N and we can decompose into eigenfunctions $\upsilon_{*,n}(\mu x_1, \xi_2)$ of the toy-model operator. So basically we consider

[10)] Note that it is more narrow definition than (15.3.1) in the previous section.

1-dimensional diagonal operator perturbed and we can diagonalize it to $\mu hF(0,x_2)\mathcal{A}_n(x_2,hD_2)$ with

$$\mathcal{A}_n(x_2,hD_2) \coloneqq \bigl(\lambda_{*,n}(\hbar D_2) - \mathfrak{z} - (\mu h)^{-1}W(0,x_2)\bigr) + ... \tag{15.4.7}$$

where in *this* section

$$\hbar = \mu^{-\frac{1}{2}}h^{\frac{1}{2}}; \tag{15.4.8}$$

preliminary we scaled $x_1 \mapsto \hbar^{-1}x_1$. Let us ignore perturbation in (15.4.7); later in subsection (15.4.5) we will show that it could be made very small, albeit W will be modified.

Remark 15.4.1. In (15.4.7) we replaced x_1 by 0. Actually the better approximation would come by replacing x_1 by $\mu^{-1}hD_2$ but a need in it will not come out instantly.

Then we need to consider only scalar operators (15.4.7) and we are looking at fixed vicinity of some point $\bar{x}_2$. What we can say about ξ_2 apart of $\xi_2 \ge -C_0$ (as $\lambda_{*,n} \to +\infty$ as $\xi_2 \to -\infty$)?

There are two possibilities: either we would be inside the spectral gap if not a boundary, or not.

15.4.2 Almost Spectral Gaps

Assume first that we would be inside the spectral gap if not a boundary:

$$|(2j+1-\mathfrak{z})\mu hF + V - \tau| \ge \epsilon_0\mu h \qquad \forall j = 0,1,2,... \tag{15.4.9}$$

where $|\tau| \le C$ and usually $\tau = 0$.

We know (see Subsection 13.6.1) that inside domain (15.4.9) means a spectral gap. However it is not the case near the boundary: looking at behavior of $\lambda_{*,n}(\xi_2)\mu h$ interpreted as *Landau levels* we can conjecture that only lower spectral gap $(-\infty,.)$ survives and even it shrinks for Neumann boundary condition.

As $\lambda_{*,n}(\xi_2) \to (2n+1)\mu h$ as $\xi_2 \to +\infty$ due to Proposition 15.A.1 again and we conclude that

(15.4.10) Under condition (15.4.9) operators in question are elliptic as $\xi_2 \ge C_0$ and we need to consider only a compact interval $|\xi_2| \le C_0$.

Note that as

$$\lambda_{*,n}(\xi_2) - \mathfrak{z} - (\mu h)^{-1} W \ge \epsilon \mu h \qquad \forall \xi_2 \tag{15.4.11}_n$$

then operator $\mathcal{A}_n$ is elliptic for all ξ_2 and the contribution of it and interval $(-\infty, \tau)$ to the asymptotics is negligible. In the framework of (15.4.9) one can rewrite above condition as

$$(2n + 1 - \mathfrak{z} - \epsilon)\mu h F + V - \tau \ge 0, \tag{15.4.12}_{\mathsf{D},n}$$

$$(\lambda^*_{\mathsf{N},n} - \mathfrak{z} - \epsilon)\mu h F + V - \tau \ge 0, \tag{15.4.12}_{\mathsf{N},n}$$

for Dirichlet and Neumann boundary conditions respectively where

$$\lambda^*_{\mathsf{N},n} = \min_{\eta} \lambda_{\mathsf{N},n}(\eta) \in (2n - 1, 2n + 1). \tag{15.4.13}$$

One then concludes easily that

Theorem 15.4.2. *Let one of assumptions* (15.4.1), (15.4.2) *be fulfilled. Further, let* $(15.4.12)_{\mathsf{D},0}$ *or* $(15.4.12)_{\mathsf{N},0}$ *be fulfilled with* $\tau = 0$ *(matching to the boundary condition).*

Then

$$|e(x, x, 0)| \le C_s \mu^{-s} \tag{15.4.14}$$

with arbitrarily large exponent s*.*

Assume now that condition (15.4.9) is fulfilled for all j but $(15.4.12)_{*,0}$ fails. Then there is a large difference between Dirichlet and Neumann cases because only in the former case in virtue of Proposition 15.A.2

$$\lambda'_{*,n} \le -\epsilon(N, C_0) \qquad \forall n \le N \qquad \forall \xi_2 : |\xi_2| \le C_0 \tag{15.4.15}$$

where here and below $\lambda'_{*,n} \coloneqq \partial_{\xi_2} \lambda_{*,n}$ etc. For Neumann boundary condition we assume that

$$|\lambda'_{\mathsf{N},n}(\xi_2)| \le \epsilon_0 \implies |\big(\lambda_{\mathsf{N},n}(\xi_2) - \mathfrak{z}\big)\mu h F - W| \ge \epsilon_0 \mu h. \tag{15.4.16}$$

So, we arrive to the remainder estimate $O(1)$; transition from Tauberian expression to magnetic Weyl expression is trivial:

Theorem 15.4.3. *Let one of assumptions* (15.4.1), (15.4.2) *be fulfilled. Further, let* (15.4.9) *and in the case of the Neumann boundary condition also* (15.4.16) *be fulfilled. Then* $\mathsf{R}^{\mathsf{MW}}_{*} = O(1)$.

However there are other mechanisms to break degeneracy for Neumann problem: first of all we know (see Proposition 15.A.3) that

$$|\lambda'_{\mathsf{N},n}(\xi_2)| \le \epsilon_0 \implies \lambda''_{\mathsf{N},n}(\xi_2) \ge \epsilon_0 \qquad \forall n = 0,1,2,... \tag{15.4.17}$$

The first non-degeneracy assumption linked to derivatives with respect to x_2 is

$$|\bigl(\lambda_{\mathsf{N},n}(\xi_2) - \mathfrak{z}\bigr)\mu h - W| \le \epsilon_0 \mu h, \quad |\lambda'_{\mathsf{N},n}(\xi_2)| \le \epsilon_0 \implies |\partial_{x_2} W| \ge \epsilon_0 \qquad \forall m = 0,1,2,\dots, \tag{15.4.18}$$

where as usual $W = -VF^{-1}$; the next one is

$$|\bigl(\lambda_{\mathsf{N},n}(\xi_2) - \mathfrak{z}\bigr)\mu h - W| \le \epsilon_0 \mu h, \quad |\lambda'_{\mathsf{N},n}(\xi_2)| + |\partial_{x_2} W| \le \epsilon_0 \implies \pm\partial^2_{x_2} W \ge \epsilon_0 \qquad \forall n = 0,1,2,... \tag{15.4.19}$$

Theorem 15.4.4. *Let one of assumptions* (15.4.1), (15.4.2) *be fulfilled. Further, let conditions* (13.3.1) *and* (15.4.9) *be fulfilled in* $B(\bar{x},1)$ *and let* $\psi \in \mathscr{C}^{\infty}_0$ *be supported in* $B(\bar{x},\frac{1}{2}) \cap \{x : x_1 \le \epsilon'\}$. *Then*

(i) $\mathsf{R}^{\mathsf{MW}}_{\mathsf{N}} = O(1)$ *under one of assumptions* (15.4.18), $(15.4.19)^{+}$ *and*

(ii) $\mathsf{R}^{\mathsf{MW}}_{\mathsf{N}} = O\bigl(|\log h|\bigr)$ *under assumption* $(15.4.19)^{-}$.

Proof. As $\mu h \asymp 1$ we are in the framework of Subsection 5.2.3 and arrive immediately to conclusion of the theorem.

Consider case (15.4.2) of strong magnetic field and adapt arguments of Subsection 5.2.3. Then first we consider $\epsilon(1,\ell)$-admissible (with respect to (x_2,ξ_2)) with $\rho = \epsilon(|\lambda'_{\mathsf{N},n}|$ partition of zone $\{|\lambda'_{\mathsf{N},n}| \ge C(\mu h)^{-\frac{1}{2}}\}$. Then for operator (15.4.7)

$$T_* \asymp \hbar\rho^{-2}, \qquad T^* \asymp \rho^{-1}, \tag{15.4.20}$$

as propagation velocities with respect to x_2,ξ_2 are $\asymp \rho$ and $O\bigl((\mu h)^{-1}\bigr)$ respectively. Then contribution of an element to the Tauberian remainder

does not exceed $C\hbar^{-1}\rho \times T_* \times T^{*-1} \asymp C$ and the total contribution of zone in question does not exceed $C\int \rho^{-1}\,d\rho \asymp C$ as operator is elliptic unless ρ has a specific magnitude.

In the remaining zone $\{|\lambda'_{\mathsf{N},n}| \le C(\mu h)^{-\frac{1}{2}}$ let us introduce $\epsilon(\rho,\ell)$-admissible partition with

$$\rho = \bigl(|\lambda'_{\mathsf{N},n}|^2 + (\mu h)^{-1}|\partial_{x_2} W|^2\bigr)^{\frac{1}{2}} + \bar{\rho}, \quad \ell = (\mu h)^{\frac{1}{2}}\rho, \quad \bar{\rho} = \mu^{-\frac{1}{2}}. \tag{15.4.21}$$

Then for operator (15.4.7) divided by μh

$$T_* \asymp \hbar\rho^{-2}, \qquad T^* \asymp (\mu h)^{\frac{1}{2}}, \tag{15.4.22}$$

as propagation velocity with respect to $(x_2, (\mu h)^{\frac{1}{2}}\xi_2)$ is $\asymp \rho$. Therefore contribution of (ρ,ℓ) element to the Tauberian remainder does not exceed $C\hbar^{-1}\rho\ell \times T_* \times T^{*-1} \asymp C$ and the total contribution does not exceed $C\int \rho^{-1}\,d\rho$ which is $O(1)$ under one of assumptions (15.4.18), (15.4.19)$^+$ as in the former case $\rho \asymp \bar{\rho}_1 = (\mu h)^{-\frac{1}{2}}$ and in the latter case operator is elliptic unless ρ has a fixed magnitude.

Further, we get $O(|\log h|)$ under assumption (15.4.19)$^-$ as in this case the integral is taken from $\bar{\rho}$ to $\bar{\rho}_1$.

Transition from Tauberian expression to magnetic Weyl expression is trivial. □

Remark 15.4.5. Following Section 5.2.3 we would be able to establish remainder estimate $O(1)$ even under assumption (15.4.19)$^-$.

Remark 15.4.6. (i) The principal part of asymptotics in question is $O(\mu h^{-1})$ and as

$$(1+\mathfrak{z}) + F^{-1}(V-\tau) \le -\epsilon \tag{15.4.23}$$

it is $\asymp \mu h^{-1}$.

(ii) As Neumann boundary condition is considered and

$$\mu h(1+\mathfrak{z})F + V - \tau \ge \epsilon\mu h \tag{15.4.24}$$

the principal part of asymptotics in question is $O(h^{-1})$ and as

$$\mu hF(\lambda^*_{\mathsf{N},0} + \mathfrak{z}) + V - \tau \le -\epsilon\mu h \tag{15.4.25}$$

it is $\asymp \hbar^{-1} = \mu^{\frac{1}{2}}h^{-\frac{1}{2}}$.

(iii) As

$$\begin{aligned} &\mu h(2j-1-\mathfrak{z})F + V - \tau_1 \le -\epsilon\mu h, \\ &\mu h(2j+1-\mathfrak{z})F + V - \tau_2 \ge -\epsilon\mu h, \qquad \tau_1 < \tau_2, \end{aligned} \tag{15.4.26}$$

then the increment of principal part of asymptotics calculated as τ increases from τ_1 to τ_2 is $O(\hbar^{-1})$ and is $\asymp \hbar^{-1}$ as $|\tau_2 - \tau_1| \asymp \mu h$. Recall that without boundary it would be $O(\hbar^s)$.

We leave to the reader the following rather easy problem:

Problem 15.4.7. Investigate under weaker non-degeneracy conditions than (15.4.19)$^{\pm}$.

15.4.3 Landau Level

Preliminary Analysis

Now consider case when condition (15.4.9) is broken. Without any loss of the generality we can assume that

(15.4.27) Condition (15.4.9) breaks only for one value of $j = \bar{\jmath}$.

Really, if $\mu h \ge C$ this assumption would be fulfilled automatically; otherwise we achieve it by considering ϵ-admissible partition of unity.

Further, without any loss of the generality we can assume that

$$\mathfrak{z} = 2\bar{\jmath} + 1. \tag{15.4.28}$$

Really, let $\nu = |\mathfrak{z} - (2\bar{\jmath}+1)| \le 1$. If $\nu\mu h \le C$ then term $\pm\nu\mu hF$ could be absorbed into V; if $\nu\mu h \ge C$ then dividing operator by $\nu\mu h$ we find ourselves in the framework of the previous subsection with $\mu_{\mathsf{new}} = \mu^{\frac12} h^{-\frac12}\nu^{-\frac12}$ and $\mu_{\mathsf{new}} = \mu^{-\frac12} h^{\frac12}\nu^{-\frac12} \le 1$.

Then for all $n \ne \bar{\jmath}$ the previous arguments work and we need to consider $n = \bar{\jmath}$ only. Further, as $n = \bar{\jmath}$ these arguments work in zone $\{\xi_2 \le C_0\}$ and we need to consider only zone $\{C_0 \le \xi_2 \le \epsilon_0\mu\}$.

As a symbol of the operator $\mu h\mathcal{A}_n$ with $\mathcal{A}_n$ defined by (15.4.7) has a either first or second derivative with respect to x_2 disjoint from 0, but derivatives with respect to ξ_2 could be pretty small, we will see that we cannot skip without reservation $O(\hbar)$ terms like those with an extra factor x_1 even as

we derive Tauberian remainder estimate but we in our assumptions will be able in Tauberian arguments only to skip $O(\hbar^2)$ terms like those with an extra factor x_1^2 so we need to modify (15.4.7) a bit[11]. To do this we first drag out $F(x)$ out of $(A-\tau)$ leaving inside

$$(hD_1)^2+(hD_2-\mu x_1)^2-W(x_1,x_2)+\ldots= \tag{15.4.29}$$
$$(hD_1)^2+(hD_2-\mu x_1)^2-W(\mu^{-1}hD_2,x_2)-$$
$$(\partial_{x_1}W)(\mu^{-1}hD_2,x_2)(x_1-\mu^{-1}hD_2)+\ldots=$$
$$(hD_1)^2+\bigl(hD_2-\mu x_1+\frac{1}{2}\mu^{-1}(\partial_{x_1}W)(\mu^{-1}hD_2,x_2)\bigr)^2-W(\mu^{-1}hD_2,x_2)+\ldots$$

with $W=-(V-\tau)F^{-1}$ resulting after shift

$$hD_2\mapsto hD_2-\mu x_1-\frac{1}{2}\mu^{-1}(\partial_{x_1}W)(\mu^{-1}hD_2,x_2) \tag{15.4.30}$$

in

$$\mu h\mathcal{A}_n(x_2,hD_2)=\mu h\lambda_{*,n}(\hbar D_2)-W(\hbar^2D_2,x_2)+\ldots. \tag{15.4.31}$$

There could also be $O(x_1)$ terms[12]

$$\alpha(x_2)x_1\Bigl(h^2D_1^2-(hD_2-\mu x_1)^2\Bigr); \tag{15.4.32}$$

then replacing $x_1\mapsto x_1+\beta(x_2)x_1^2$ we also need to replace

$$hD_1\mapsto \bigl(1-2\beta x_1+O(x_1^2)\bigr)hD_1 \quad\text{and}\quad hD_2\mapsto \bigl(hD_2+O(x_1^2)hD_1\bigr)$$

so modulo $O(x_1^2)$

$$(hD_1)^2+(hD_2-\mu x_1)^2\mapsto (1-4\beta x_1)(hD_1)^2+(1+4\beta x_1)(hD_2-\mu x_1)^2+$$
$$2(hD_2-\mu x_1)\beta\mu^{-1}h^2D_2^2$$

and choosing $\beta=-\frac{1}{4}\alpha$ we arrive to

$$(hD_1)^2+(hD_2-\mu x_1+2\beta\mu^{-1}h^2D_2^2)^2$$

---

[11] Actually effectively x_1 could be marginally more than $\hbar$.

[12] After division by F intensity of magnetic field is 1 so there could be no terms (15.4.32) with "$-$" replaced by "$+$".

and thus we need to replace (15.4.31) by

$$\mu h\big(\mathcal{A}_n(x_2, hD_2) - 2n - 1\big) = \mu h\big(\lambda_{*,n}(\hbar D_2 + 2\beta\hbar^3 D_2^2) - 2n - 1\big) - W(\hbar^2 D_2, x_2(1+) + \dots . \tag{15.4.33}$$

Remark 15.4.8. In the case of the strong magnetic field we need to consider only zone

$$\big\{\xi_2,\ |\lambda_{*,n} - 2n - 1| \le C_0(\mu h)^{-1}\big\} \tag{15.4.34}$$

otherwise operator is elliptic.

Non-Degenerate Case

Consider first case

$$|(2n + 1 - \mathfrak{z})\mu hF + V| \le \epsilon_0 \implies |\nabla_{\partial X} VF^{-1}| \ge \epsilon_0 \qquad \forall n = 0, 1, 2, ... \tag{15.4.35}$$

Then obviously one can take $T_* \asymp \hbar$. Really, the propagation speed with respect to ξ_2 is $\asymp |\partial_{x_2} W| \asymp 1$.

Meanwhile consider time direction in which ξ_2 increases; we can select it due to assumption (15.4.35). To reach $\xi_2 = \epsilon\hbar^{-1}$ we need time $\asymp \hbar^{-1}\mu h$ as for $\xi_2 \asymp \hbar^{-1}$ speed with respect to ξ_2 is $(\mu h)^{-1}$. Consider evolution with respect to x_2; speed with respect to x_2 does not exceed $C\mu h|\lambda'_{*,n}| + C\hbar$. Then we can take for sure take

$$T^* = T^*(\xi_2) \asymp \min\big((\mu h|\lambda'_{*,n}|)^{-1}, \hbar^{-1}\big) \tag{15.4.36}$$

Then contribution of $B(\bar{x}, 1) \cap \{x : x_1 \le \epsilon'\}$ to the remainder does not exceed

$$C \int \hbar^{-1} \frac{T_*(\xi_2)}{T^*(\xi_2)}\, d\xi_2 \le \int \big(\hbar + C\mu h|\lambda'_{*,n}|\big)\, d\xi_2 \tag{15.4.37}$$

as the integrand in the left-hand expression does not exceed $C\hbar$ as $|\lambda'_{*,n}| \le \hbar(\mu h)^{-1}$ and it does not exceed $C\mu h|\lambda'_{*,n}|$ otherwise. As $\lambda_{*,n}(\xi_2)$ is monotone (at least for $\xi_2 \ge C$) and integral is taken over $\xi_2 \le \epsilon\hbar^{-1}$ also satisfying (15.4.34) the right-hand expression does not exceed C.

Then we conclude that

(15.4.38) Under conditions (15.4.1) or (15.4.2), (15.4.28) and (15.4.35) fulfilled in $B(\bar{x}, 1)$ contribution of $B(\bar{x}, \frac{1}{2}) \cap \{x : x_1 \le \epsilon'\} \cap \{\xi_2 \ge C\}$ to the Tauberian remainder does not exceed C.

Generic Case

Now look what happens as (15.4.35) is also broken and replaced by conditions

$$(15.4.39)^{\pm} \quad |(2n+1-\mathfrak{z})\mu hF+V|+|\nabla_{\partial X}VF^{-1}|\le \epsilon_0 \implies \mp\nabla^2_{\partial X}VF^{-1}\ge \epsilon_0 \qquad \forall n=0,1,2,...$$

and

$$(15.4.40)^{\pm} \quad |(2n+1-\mathfrak{z})\mu hF+V|+|\nabla_{\partial X}VF^{-1}|\le \epsilon_0 \implies \mp\nabla VF^{-1}\ge \epsilon_0 \qquad \forall n=0,1,2,...$$

Inner Zone. We define this zone preliminary by

$$\mathcal{X}_{\mathsf{inn}}=\{\xi_2\ge c_0(\log\mu)^{\frac12}\}$$

which means not only that $\mu h(\lambda_{*,n}-2n-1)$ and $\mu h\lambda'_{*,n}$ are negligible but that they remain so even if we replace ξ_2 by $(1-\epsilon)\xi_2$. Then under assumption (15.4.40) we can take

$$T_*\asymp \min\bigl(\xi_2^{-1},\hbar\ell^{-2}\bigr), \tag{15.4.41}$$

$$\ell\asymp |W_{x_2}| \tag{15.4.42}$$

Really, the propagation speed with respect to x_2 is $\asymp\hbar$ but as scale with respect to ξ_2 is $\asymp\xi_2$ there, uncertainty principle means that $\hbar T\cdot\xi_2\ge\hbar$. Similarly, propagation speed with respect to ξ_2 is $\asymp\ell$, scale with respect to x_2 is $\asymp\ell$ and uncertainty principle means that $\ell T\cdot\ell\ge\hbar$.

Consider propagation in the time direction $\mathsf{sgn}(W_{x_2})t>0$ in which ξ_2 increases. Then

$$\mathsf{sgn}(W_{x_2})\frac{d}{dt}\log|W_{x_2}|=-\hbar W_{x_1}\,\mathsf{sgn}\,|W_{x_2}|^{-1}W_{x_2x_2}$$

so $\log|W_{x_2}|$ also increases provided W_{x_1} and $W_{x_2x_2}$ have opposite signs. Then we can take $T^*=\hbar^{-1}$. This would lead to contribution of $\mathcal{X}_{\mathsf{inn}}\cap\{\xi_2\ell\ge\hbar\}$ to the Tauberian remainder not exceeding

$$C\iint \hbar^{-1}\frac{T_*(\xi_2)}{T^*(\xi_2)}\,d\xi_2dx_2=C\iint \min(\xi_2^{-1},\hbar\ell^{-2})\,d\ell d\xi_2\le C.$$

On the other hand, in $\mathcal{X}_{\mathsf{inn}} \cap \{\xi_2 \ell \le \hbar\}$ we can take $T_* \asymp \xi_2^{-1}$ and $T^* \asymp \xi_2$ and contribution of this subzone to the Tauberian remainder not exceeding

$$C \iint \hbar^{-1} \frac{T_*(\xi_2)}{T^*(\xi_2)}\, d\xi_2 dx_2 = C \int \xi_2^{-2}\, d\xi_2 \le C.$$

On the other hand, as $(15.4.39)^{\pm}$, $(15.4.40)^{\pm}$ with the same signs are fulfilled we should chose between $T^* \asymp \xi_2$ and $T^* \asymp \hbar^{-1}\ell$. So, as $\ell \ge (\hbar\xi_2)^{\frac{1}{2}}$ we can take $T_* = \hbar\ell^{-2}$, $T^* = \hbar\ell^{-1}$ and contribution of this subzone to the Tauberian remainder does not exceed

$$C \iint \hbar^{-1} \frac{T_*(\xi_2)}{T^*(\xi_2)}\, d\xi_2 dx_2 = C \int \ell^{-1}\, d\ell \le C$$

as in the case of the same signs $\ell^2 + \xi_2\hbar$ must have the same magnitude or operator is elliptic; so in this subzone ℓ must have the same magnitude.

As $\ell \le (\hbar\xi_2)^{\frac{1}{2}}$ we can take $T_* = \xi_2^{-1}$, but the same ellipticity argument means that actually we can upgrade $T^* \asymp \hbar$ to $T^* \asymp \hbar^{-\frac{1}{2}}\xi_2^{\frac{1}{2}}$ contribution of this subzone to the Tauberian remainder does not exceed

$$C \iint \hbar^{-1} \frac{T_*(\xi_2)}{T^*(\xi_2)}\, d\xi_2 dx_2 = C \int \xi_2^{-1}\, d\xi_2 \le C$$

as in this subzone ℓ must have the same magnitude.

We need to extend $\mathcal{X}_{\mathsf{inn}}$ up to

$$\mathcal{X}_{\mathsf{inn}} := \{\xi_2 \ge \bar{\xi}_2^{+}\} \tag{15.4.43}$$

with

$$\bar{\xi}_2^{\pm} = \Big(\log(\mu^{\frac{3}{2}}h^{\frac{1}{2}}) - (2n+2)\log\log(\mu^{\frac{3}{2}}h^{\frac{1}{2}}) \pm C\Big)^{\frac{1}{2}} \tag{15.4.44}$$

describing a zone where $\mu h|\lambda'_{*,n}(\xi_2)| \le \epsilon\hbar$ is dominated by $|\partial_{\xi_2} W(\xi_1\hbar, x_2)|$. Then the same arguments work there albeit scale with respect to ξ_2 is now $\xi_2 - \bar{\xi}_2^{+\prime}$ $\bar{\xi}_2^{\pm\prime}$ are defined by the same formula with C replaced by $C/2$.

Further, these arguments remain valid for $\xi_2 \ge \xi_2^{-}$ provided $\lambda'_{*,n}$ and $-W_{x_1}$ have the same sign.

Thus we arrive to the following proposition:

Proposition 15.4.9. *(i) Under conditions* (15.4.1) *or* (15.4.2), (15.4.28) *and* $(15.4.39)^{\pm}$, $(15.4.40)^{\pm}$ *(with the same or opposite signs) fulfilled in* $B(\bar{x}, 1)$ *contribution of* $B(\bar{x}, \frac{1}{2}) \cap \mathcal{X}_{\mathsf{inn}} \cap \{\xi_2 \ge C\}$ *to the Tauberian remainder does not exceed* C.

(ii) Further as $\lambda'_{*,n}$ *and* $-W_{x_1}$ *have the same signs*[13] *Statement (i) remains true for* $\mathcal{X}_{\mathsf{inn}} = \{\xi_2 \ge \bar{\xi}_2^-\}$.

Boundary Zone. Consider now boundary zone temporarily introduced as

$$\mathcal{X}_{\mathsf{bound}} \coloneqq \{\xi_2 \le \frac{1}{2}\bar{\xi}_2\}$$

which actually should be intersected with zone (15.4.34).

Then we can take

$$T_* \asymp \min \hbar\Big(\xi_2\big(\mu h|\lambda'_{*,n}|\big)^{-1}, \ell^{-2}\Big). \tag{15.4.45}$$

Really, in this zone the propagation speed with respect to x_2 is $\asymp \mu h|\lambda'_{*,n}|$ and scale with respect to ξ_2 is $\asymp \xi_2^{-1}$ to keep $\lambda'_{*,n}$ of the fixed magnitude.

As $\ell \ge \big(\xi_2^{-1}\mu h|\lambda'_{*,n}|\big)^{\frac{1}{2}}$ consider propagation in the time direction of ξ_2 increasing. Then

$$T^* \asymp \min\big((\mu h|\lambda'_{*,n}|)^{-1}, \hbar^{-1}\ell\big) \tag{15.4.46}$$

provided $|W_{x_2}|$ also increases in the same direction because speed with respect to x_2 is $\asymp (\mu h|\lambda'_{*,n}|)$ as long as we remain in $\mathcal{X}_{\mathsf{bound}}$ but then drops to $O(\hbar)$ outside of $\mathcal{X}_{\mathsf{bound}}$ but x_2 can go in the opposite direction there. Our extra assumption means exactly that

(15.4.47) $W_{x_2x_2}$ and $\lambda'_{*,n}$ have the same signs[14].

Then contribution of the subzone in question to the Tauberian remainder does not exceed

$$\begin{aligned} C\iint \hbar^{-1}\frac{T_*(\xi_2)}{T^*(\xi_2)}\,d\xi_2 dx_2 &\asymp C\iint \ell^{-2}\big(\mu h|\lambda'_{*,n}| + \hbar\ell^{-1}\big)\,d\ell d\xi_2 \asymp \\ &\asymp C\int \big((\mu h|\lambda'_{*,n}|\xi_2)^{\frac{1}{2}} + \hbar\xi_2(\mu h|\lambda'_{*,n}|)^{-1}\big)\,d\xi_2 \asymp \\ &C(\mu h|\lambda'_{*,n}|\xi_2)^{\frac{1}{2}}\xi_2^{-1}\big|_{\xi_2=\hat{\xi}_2} + C\hbar(\mu h|\lambda'_{*,n}|)^{-1}\big|_{\xi_2=\bar{\xi}_2} \asymp C \end{aligned}$$

[13] Which means that under Dirichlet or Neumann boundary condition $(15.4.40)^+$ and $(15.4.40)^-$ respectively must be fulfilled.

[14] Or equivalently that under Dirichlet or Neumann boundary condition $(15.2.39)^-$ and $(15.2.39)^+$ respectively must be fulfilled.

where $\hat{\xi}_2$ is defined from (15.4.34) as solution to $\mu h|\lambda_{*,n} - (2n+1)| = C$.

Consider now subzone

$$\{\ell \le (\xi_2^{-1}\mu h|\lambda'_{*,n}|)^{\frac{1}{2}},\ \ell \ge \xi_2\hbar\}$$

where the second condition is due to uncertainty principle and scaling with respect to ξ_2.

Then T^* is defined by (15.4.45) with ℓ replaced by $(\xi_2^{-1}\mu h|\lambda'_{*,n}|)^{\frac{1}{2}}$ as we can go into direction of increasing ξ_2. Then contribution of this subzone to the Tauberian remainder does not exceed

$$C\iint \hbar^{-1}\frac{T_*(\xi_2)}{T^*(\xi_2)}\,d\xi_2 dx_2 \asymp C\hbar\int d\xi_2 \ll C$$

again. Further, in subzone $\{\ell \le \hbar\xi_2\}$ one can take $T^* \asymp \xi_2^{-1}$ and its contribution is $\ll 1$ as well.

Consider now case when

(15.4.48) $W_{x_2x_2}$ and $\lambda'_{*,n}$ have the opposite signs[15)].

Then as $\ell \ge \ell^*$ we can do nothing better than

$$T^* = \ell(\mu h|\lambda'_{*,n}|)^{-1}. \tag{15.4.49}$$

Then contribution of the subzone in question to the Tauberian remainder does not exceed

$$C\iint \hbar^{-1}\frac{T_*(\xi_2)}{T^*(\xi_2)}\,d\xi_2 dx_2 \asymp C\iint \ell^{-3}(\mu h|\lambda'_{*,n}|)\,d\ell d\xi_2 \asymp$$
$$\asymp C\int d\xi_2 \asymp C\log\mu.$$

On the other hand, for $\ell \le \ell^*$ we can improve (15.4.49) to

$$T^* = \ell^{1-\delta}(\mu h|\lambda'_{*,n}|)^{-1-\frac{1}{2}\delta}. \tag{15.4.50}$$

[15)] Or equivalently that under Dirichlet or Neumann boundary condition $(15.2.39)^+$ and $(15.2.39)^-$ respectively must be fulfilled.

Then contribution of the subzone in question to the Tauberian remainder does not exceed

$$C \iint \hbar^{-1} \frac{T_*(\xi_2)}{T^*(\xi_2)}\, d\xi_2 dx_2 \asymp C \iint \ell^{-1+\delta} \ell^{*\,-\delta}\, d\ell d\xi_2 \asymp \\ \asymp C \int d\xi_2 \asymp C \log \mu.$$

Finally, extending $\mathcal{X}_{\mathsf{bound}}$ to

$$\mathcal{X}_{\mathsf{bound}} = \{\xi_2 \le \bar{\xi}_2^{-}\} \tag{15.4.51}$$

comes with no cost at all as scale with respect to ξ_2 is $\xi_2^{-1} \asymp (\bar{\xi}_2^{-\prime} - \xi_2)$ anyway.

So we arrive to the following proposition:

Proposition 15.4.10. *(i) Under conditions* (15.4.1) *or* (15.4.2), (15.4.28) *and* $(15.4.39)^{\pm}$, $(15.4.40)^{\pm}$ *(with the same or opposite signs) fulfilled in* $B(\bar{x}, 1)$ *contribution of* $B(\bar{x}, \frac{1}{2}) \cap \mathcal{X}_{\mathsf{bound}} \cap \{\xi_2 \ge C\}$ *to the Tauberian remainder does not exceed* $C \log \mu$.

(ii) Further as $\lambda'_{*,n}$ *and* $W_{x_2x_2}$ *have the same signs*[14] *this contribution does not exceed* C.

As $\lambda'_{*,n}$ and $-W_{x_1}$ have the same signs[13] we are done. Otherwise we need to consider

Transitional Zone. $\mathcal{X}_{\mathsf{trans}}$ is defined by

$$\bar{\xi}_2^{-} \le \xi_2 \le \bar{\xi}_2^{+} \tag{15.4.52}$$

with $\bar{\xi}_2^{\pm}$ defined by (15.4.44) and in this zone $\mu h \lambda'_{*,n}$ has magnitude $\hbar$ and $\mu h \lambda''_{*,n}$ has magnitude $\hbar \xi_2 \gg \hbar^2$ and we conclude that

Proposition 15.4.11. *(i) Under conditions* (15.4.1) *or* (15.4.2), (15.4.28) *and* $(15.4.39)^{\pm}$, $(15.4.40)^{\pm}$ *(with the same or opposite signs) fulfilled in* $B(\bar{x}, 1)$ *contribution of* $B(\bar{x}, \frac{1}{2}) \cap \mathcal{X}_{\mathsf{trans}} \cap \{\xi_2 \ge C\}$ *to the Tauberian remainder does not exceed* $C \log \mu$.

(ii) Further as $\lambda''_{*,n}$ *and* $W_{x_2x_2}$ *have the same signs*[14] *this contribution does not exceed* C.

Synthesis. So, we proved that the total Tauberian remainder does not exceed $C\log\mu$. To upgrade it to C we must assume that $W_{x_2x_2}$, $\lambda'_{*,n}$ and $-W_{x_1}$ have the same signs as $\lambda'_{*,n}$ and $\lambda''_{*,n}$ have opposite signs. So we proved

Proposition 15.4.12. *(i) Under conditions* (15.4.1) *or* (15.4.2), (15.4.28) *and* (15.4.39)$^{\pm}$, (15.4.40)$^{\pm}$ *(with the same or opposite signs) fulfilled in* $B(\bar{x},1)$ *contribution of* $B(\bar{x},\frac{1}{2})\cap\{\xi_2\ge C\}$ *to the Tauberian remainder does not exceed* $C\log\mu$.

(ii) Further, as $W_{x_2x_2}$, $\lambda'_{*,n}$ *and* $-W_{x_1}$ *have the same signs*[14)] *this contribution does not exceed* C.

15.4.4 From Tauberian to Magnetic Weyl Formula

In this subsection $T=T_*\mu^{\delta}$ but we will calculate with $\delta=0$ because if we consider time interval $[T,2T]$ with $T\ge T_*$ in the estimate of its contribution there will be an extra factor $T^{-s}T_*^{s}$ and the summation with respect to T will result in the same answer albeit with $T=T_*$.

We apply the successive approximation method to operator (15.4.7)

Inner Zone

Replacing x_2 by y_2 leads to a contribution of (ℓ,ρ) element to an error does not exceeding

$$C\hbar^{-1}\rho\ell\times\frac{\hbar T_*^2}{\hbar}=C\hbar^{-1}\rho\ell T_*^2 \tag{15.4.53}$$

as the propagation speed with respect to x_2 is $O(\hbar)$ in the time scale used here and we use time scale compatible with the choice of T_* in the previous subsection.

Under non-degeneracy condition (15.2.8) we can take $T_*=\hbar$ resulting in the contribution of $\mathcal{X}_{\mathsf{inn}}$ equal to $O(1)$ as $\rho\asymp\hbar^{-1}$.

In more general generic case we estimate a contribution of (ℓ,ρ) element to an error by

$$C\hbar^{-1}\rho\ell\times\frac{\hbar\ell T^2}{\hbar}=C\xi_2\ell^2T_*^2\asymp C\hbar^{-1}\xi_2\ell^2\min\bigl(\xi_2^{-2},\hbar^2\ell^{-4}\bigr) \tag{15.4.54}$$

where we plugged T_* defined by (15.4.41); an extra factor ℓ appears as we replace x_2 by y_2 because $\partial_{x_2}W=O(\ell)$ and at this moment we take original "narrow" $\mathcal{X}_{\mathsf{inn}}$ with scale $\rho\asymp\xi_2$ in ξ_2.

Therefore the contribution of $\mathcal{X}_{\mathsf{inn}}$ to an error does not exceed

$$C\hbar^{-1}\iint \min\bigl(\xi_2^{-2},\hbar^2\ell^{-4}\bigr)\ell\,d\ell d\xi_2 \asymp C\int \xi_2^{-1}\,d\xi_2 \asymp C\log\mu. \tag{15.4.55}$$

In the remaining part of $\mathcal{X}_{\mathsf{inn}}$ defined by (15.4.44) we must replace ξ_2 by $\rho \asymp \xi_2 - \bar{\xi}_2$ as the scaling is concerned and it redefines correspondingly T_* and instead of right-hand expression in (15.4.55) we get $C\int(\xi_2-\bar{\xi}_2)^{-1}\,d\xi_2$ which is also $\asymp C\log\mu$.

Boundary Zone

Due to the standard arguments contribution of subzone $\{\xi_2 \le C_0\}$ to the error is either $O(\log\mu)$ or $O(1)$ depending if there is a saddle point or no; one can see easily that this saddle points cannot come from $\lambda_{\mathsf{N},j}$ but only from $\lambda_{\mathsf{N},n}$ with $n\ne j$ and it is possible only as $\lambda h \asymp 1$.

Consider subzone $\{\xi_2 \ge C_0\}$. Here we need to replace (15.4.53) by

$$C\hbar^{-1}\rho\ell \times \frac{\mu h|\lambda'_{*,m}|\,T_*^2}{\hbar} = C\hbar^{-2}\mu h|\lambda'_{*,m}|\rho\ell\,T_*^2 \tag{15.4.56}$$

as the propagation speed with respect to x_2 is $\asymp \mu h|\lambda'_{*,m}|$. Plugging under condition (15.2.8) $\ell = 1$, $\rho = \xi_2^{-1}$

$$T_* = \hbar\min\bigl(\xi_2(\mu h|\lambda'_{*,j}|)^{-1},1\bigr)$$

we arrive to

$$C\mu h|\lambda'_{*,j}|\rho\min\bigl(\xi_2^2(\mu h|\lambda'_{*,j}|)^{-2},1\bigr)$$

and contributions of both subzones $\{\xi_2 \ge C,\ \mu h|\lambda'_{*,j}| \ge \xi_2\}$ and $\{C\hbar \le \mu h|\lambda'_{*,j}| \le \xi_2\hbar\}$ are $O(1)$.

In the same manner (15.4.54) is replaced by

$$Ch^{-1}\rho\ell^2 \times \frac{\mu h|\lambda'_{*,j}|\,T^2}{\hbar} \asymp C\hbar^{-2}\mu h|\lambda'_{*,j}|\rho\ell^2\,T_*^2 \tag{15.4.57}$$

and plugging

$$T_* = \hbar\min\bigl(\xi_2(\mu h|\lambda'_{*,m}|)^{-1},\ell^{-2}\bigr)$$

we arrive to

$$C\mu h|\lambda'_{*,m}|\rho\ell^2\min\bigl(\xi_2^2(\mu h|\lambda'_{*,j}|)^{-2},\ell^{-4}\bigr)$$

and summation with respect to ℓ results in

$$C\mu h|\lambda'_{*,m}|\rho\ell^{*,-2} = C\rho\xi_2$$

as $\ell^* = \xi_2^{-1}\mu h|\lambda'_{*,j}|$ and summation with respect to ξ_2 results in $C\bar{\xi}_2^{-2} \asymp C\log\mu$.

Transitional Zone

In this zone we can return to (15.4.53) and plug $T_* = \hbar$ under condition (15.2.8) resulting in $O(1)$.

Further we can return to (15.4.54) and plug in the generic case either $T_* = \min\bigl(\hbar\ell^{-2}, \xi_2^{-1}\bigr)$ or $T_* = \min(\hbar\ell^{-2}, |\xi_2 - \bar{\xi}_2|^{-1})$ resulting after summation with respect to ℓ in $C\xi_2^{-1}\rho$ or $C|\xi_2 - \bar{\xi}_2|^{-1}\rho$ respectively. In the former case we instantly get $O(1)$ while in the latter after summation with respect to subzone $\{|\xi_2 - \bar{\xi}_2| \ge \hbar\}$ we arrive to $O(\log\mu)$ while contribution of subzone $\{|\xi_2 - \bar{\xi}_2| \le \hbar, \ell \le \hbar\}$ to asymptotics is $O(1)$.

15.4.5 Justification: Reduction to Toy-Model Operator

Reduction: Step 1

The problem however is that the "kinetic" part of our operator is not exactly $h^2D_1^2 + (hD_2 - \mu x_1)^2$ but is different. We need to improve construction of Subsection 15.2.1). First we can assume that $V_1 = 0$ and that $g^{jk} = \delta_{jk}$, $V_2 = 0$, $\partial_1 V_2 = 1$ as $x_1 = 0$. Further, assume that

$$g^{12} = O(x_1^k), \qquad V_2 = x_1 + O(x_1^{k+1}) \tag{15.4.58}$$

with $k \ge 1$. Then changing $x_1 \mapsto x_1$, $x_2 \mapsto x_2 + \alpha(x_2)x_1^{k+1}$ we can achieve $g^{12} = O(x_1^{k+1})$, simultaneously preserving the second relation. Then redefining $x_1 := V_2(x)$ preserves the first relation (with $k+1$ instead of k). Continuing we can make k as large as we wish and thus we can take $k = \infty$.

One needs to remember that g^{11} and g^{22} differ from 1 by $O(x_1)$. However as we consider operator $F^{-1}A$ with the intensity 1 of magnetic field we conclude that $g^{11}g^{22} = 1 + O(x_1^\infty)$.

Our goal is to get rid of this perturbation which we rewrite as

$$\mathcal{P}^{\mathsf{w}} := \mu^2\Bigl(\hbar^2 D_1\sigma D_1 - \sigma(\hbar D_2 - x_1)\sigma(\hbar D_2 - x_1)\Bigr). \tag{15.4.59}$$

Consider Poisson brackets

$$\frac{1}{2}\bigl\{\bigl(\xi_1^2+(\xi_2-x_1)^2\bigr),\alpha+\beta\xi_1\bigr\} \tag{15.4.60}$$

and note that for coefficients at ξ_1 equal 0 i.e.

$$\alpha_{x_1}-(x_1-\xi_2)\beta_{x_2}=0 \tag{15.4.61}$$

it is equal to

$$\beta_{x_1}\xi_1^2-(\beta+\alpha_{x_2})(x_1-\xi_2). \tag{15.4.62}$$

To make it equal to (15.4.59) modulo $\xi_1^2+(x_1-\xi_2)^2$ one needs to satisfy

$$\alpha_{x_2}=2\sigma(x_1-\xi_2)-\beta_{x_1}(x_1-\xi_2)-\beta. \tag{15.4.63}$$

Compatibility with (15.4.61) requires

$$(\partial_{x_1}^2+\partial_{x_2}^2)\bigl((x_1-\xi_2)\beta\bigr)=2\partial_{x_1}\bigl((x_1-\xi_2)\sigma\bigr). \tag{15.4.64}$$

Consider instead it without ∂_{x_2}, with an extra condition "$\beta=0$ as $x_1=0$". We can solve this one-dimensional equation

$$\partial_{x_1}^2\bigl((x_1-\xi_2)\beta\bigr)=2\partial_{x_1}\bigl((x_1-\xi_2)\sigma\bigr), \tag{15.4.65}$$

then

$$\beta(x_1,.)=2(x_1-\xi_2)^{-1}\int_{\xi_2}^{x_1}(x_1'-\xi_2)\sigma(x_1',.)\,dx_1'+\rho \tag{15.4.66}$$

with ρ which does not depend on x_1; extra condition "$\beta=0$ as $x_1=0$" means that

$$\rho=-\xi_2^{-1}\int_0^{\xi_2}(x_1'-\xi_2)\sigma(x_1',.)\,dx_1' \tag{15.4.67}$$

which gives us smooth function β and α.

Remark 15.4.13. As $\alpha_{x_2\xi_2}=0$ as $x_1=\xi_2=0$ due to (15.4.63) and properties of σ,β we can select α such that $\alpha_{\xi_2}=0$ there as well. This would enable further calculations.

Then

$$\frac{1}{2}\{\big(\xi_1^2+(\xi_2-x_1)^2\big),\alpha+\beta\xi_1\}= \tag{15.4.68}$$
$$(\beta_{x_1}-\sigma)\big(\xi_1^2+(x_1-\xi_2)^2\big)+\sigma\big(\xi_1^2-(x_1-\xi_2)^2\big).$$

Let us pass to operators; to do this we consider corresponding $\hbar$-quantizations, divide by $-i\hbar$ and multiply by μ^2; then

$$-\frac{1}{i\hbar}[A,\mathcal{L}^{\mathsf{w}}]=\mathcal{P}^{\mathsf{w}}-\mathcal{P}_1^{\mathsf{w}}, \tag{15.4.69}$$

with

$$A:=\hbar^2D_1^2+(\hbar D_2-x_1)^2, \tag{15.4.70}$$

$$\mathcal{L}:=\frac{1}{2}(\alpha+\beta\xi_1), \tag{15.4.71}$$

$$\mathcal{P}_1=(\sigma-\beta_{x_1}\sigma)\big(\xi_1^2+(x_1-\xi_2)^2\big) \tag{15.4.72}$$

where as we consider Weyl quantizations and symmetrized products after multiplication by μ^2 the error is $O(\mu^2\hbar^2)=O(h^2)$.

One can see easily that in our settings $\sigma=\beta=0$ as $x_1=0$ and β, α_{x_2} has 0 of the second order as $x_1=\xi_1=\xi_2=0$ while α_{x_1} has 0 of the third order as $x_1=\xi_1=\xi_2=0$. Also note that due to (15.4.64) $\beta_{x_1}-\sigma=0$ as $x_1=\xi_2$ so

(15.4.73) symbol $(\sigma-\beta_{x_1})$ is divisible by $(x_1-\xi_2)$.

Consider transformation of $A+\mathcal{P}^{\mathsf{w}}$ by $\exp(i\hbar^{-1}\mathcal{L}^{\mathsf{w}})$:

$$\exp(i\hbar^{-1}\mathcal{L}^{\mathsf{w}})\,(A+\mathcal{P}^{\mathsf{w}})\exp(-i\hbar^{-1}\mathcal{L}^{\mathsf{w}})= \tag{15.4.74}$$
$$A+\mathcal{P}^{\mathsf{w}}+\frac{1}{i\hbar}[A+\mathcal{P}^{\mathsf{w}},\mathcal{L}^{\mathsf{w}}]+\frac{1}{2}\big(\frac{1}{i\hbar}\big)^2[[A+\mathcal{P}^{\mathsf{w}},\mathcal{L}^{\mathsf{w}}],\mathcal{L}^{\mathsf{w}}]+\dots$$

where as one can see easily we left out terms which are $o(h^2)$ and due to (15.4.69)

$$\equiv A+\mathcal{P}_1^{\mathsf{w}}+\frac{1}{2}\big(\frac{1}{i\hbar}\big)[\mathcal{P}^{\mathsf{w}},\mathcal{L}^{\mathsf{w}}]+\frac{1}{2}\big(\frac{1}{i\hbar}\big)[\mathcal{P}_1^{\mathsf{w}},\mathcal{L}^{\mathsf{w}}] \tag{15.4.75}$$

Consider first the symbol of the third term

$$-\frac{1}{4}\{\mathcal{P},\alpha+\beta\xi_1\}=$$
$$-\frac{1}{4}\{\sigma,\alpha+\beta\xi_1\}\big(\xi_1^2-(x_1-\xi_2)^2\big)-\frac{1}{4}\sigma\{\xi_1^2-(x_1-\xi_2)^2,\alpha+\beta\xi_1\};$$

as

$$\frac{1}{2}\{\xi_1^2-(x_1-\xi_2)^2, \alpha+\beta\xi_1\} = \big(\alpha_{x_1}+\beta_{x_2}(x_1-\xi_2)\big)\xi_1+\beta_{x_1}\xi_1^2+(x_1-\xi_2)(\alpha_{x_2}+\beta) = 2\beta_{x_2}(x_1-\xi_2)\xi_1 + \sigma\big(\xi_1^2+(x_1-\xi_2)^2\big) - (\sigma-\beta_{x_1})\big(\xi_1^2-(x_1-\xi_2)^2\big)$$

we conclude that

$$-\frac{1}{4}\{\mathcal{P}, \alpha+\beta\xi_1\} = \Big(-\frac{1}{4}\{\sigma, \alpha+\beta\xi_1\} + \frac{1}{2}\sigma(\sigma-\beta_{x_1})\Big)\big(\xi_1^2-(x_1-\xi_2)^2\big) - \sigma\beta_{x_2}(x_1-\xi_2)\xi_1 - \frac{1}{2}\sigma^2\big(\xi_1^2+(x_1-\xi_2)^2\big).$$

Consider now the symbol of the fourth term in (15.4.75):

$$-\frac{1}{4}\{\mathcal{P}_1, \alpha+\beta\xi_1\} = -\frac{1}{4}\{(\sigma-\beta_{x_1}), \alpha+\beta\xi_1\}\big(\xi_1^2+(x_1-\xi_2)^2\big) - \frac{1}{4}(\sigma-\beta_{x_1})\{\xi_1^2+(x_1-\xi_2)^2, \alpha+\beta\xi_1\};$$

as we know the last Poisson bracket we conclude that

$$-\frac{1}{4}\{\mathcal{P}_1, \alpha+\beta\xi_1\} = \Big(-\frac{1}{4}\{(\sigma-\beta_{x_1}), \alpha+\beta\xi_1\} + \frac{1}{2}(\sigma-\beta_{x_1})^2\Big)\big(\xi_1^2+(x_1-\xi_2)^2\big) - \frac{1}{2}(\sigma-\beta_{x_1})\sigma\big(\xi_1^2-(x_1-\xi_2)^2\big).$$

Then the sum of symbols of the third and the fourth terms in (15.4.75) is equal to

$$-\frac{1}{4}\{\sigma, \alpha+\beta\xi_1\}\big(\xi_1^2-(x_1-\xi_2)^2\big) - \sigma\beta_{x_2}(x_1-\xi_2)\xi_1 + \Big(-\frac{1}{4}\{(\sigma-\beta_{x_1}), \alpha+\beta\xi_1\} + \frac{1}{2}\big(-\sigma^2+(\sigma-\beta_{x_1})^2\big)\Big)\big(\xi_1^2+(x_1-\xi_2)^2\big). \tag{15.4.76}$$

Remark 15.4.14. However correction (15.4.59) is not a correct one as we need $g^{11}g^{22}=1$ and a more precisely is $\mathcal{P}_1^{\mathsf{w}}+\frac{1}{2}\sigma^2 A$ as $g^{11}=1+\sigma+\frac{1}{2}\sigma^2+\dots$, $g^{22}=1-\sigma+\frac{1}{2}\sigma^2+\dots$.

We can skip terms here which has 0 of degree 5 at $x_1=\xi_1=\xi_2=0$; this takes care of the middle term.

By the same reason we can skip in

$$-\frac{1}{4}\{\sigma, \alpha+\beta\xi_1\} = \frac{1}{4}\sigma_{x_2}\alpha_{\xi_2} + \frac{1}{4}\sigma_{x_2}\beta_{\xi_2}\xi_1 + \frac{1}{4}\sigma_{x_1}\beta$$

the middle term where we used that $\sigma = \sigma(x)$.

Consider two remaining terms (which have 0 of the second order at $x_1 = \xi_2 = 0$ and also equal 0 as $x_1 = 0$)

$$\sigma_1 := \frac{1}{4}\sigma_{x_2}\alpha_{\xi_2} + \frac{1}{4}\sigma_{x_1}\beta; \tag{15.4.77}$$

then we can repeat the same procedure as before thus replacing the whole first term in (15.4.76) by

$$(\sigma_1 - \beta_{1\,x_1})\big(\xi_1^2 + (x_1-\xi_2)^2\big)$$

with β_1 matching σ_1. One can see easily that then neither $\mathcal{P}_1$ nor the last term in (15.4.76) would change modulo $O(h^2)$ and therefore we arrive to operator $A + \mathcal{P}_2^{\mathsf{w}}$ with

$$\mathcal{P}_2 = \tag{15.4.78}$$
$$\Big((\sigma-\beta_{x_1})+\sigma_1-\beta_{1\,x_1}-\frac{1}{4}\{(\sigma-\beta_{x_1}),\alpha+\beta\xi_1\}+\frac{1}{2}(\sigma-\beta_{x_1})^2\Big)\big(\xi_1^2+(x_1-\xi_2)^2\big).$$

In this factor the first term has 0 of the first order, other terms have 0 of the second order or higher and we can skip those which are higher. We can also skip terms divisible by $(x_1-\xi_2)^2$ as those terms are $O(\hbar)$ on energy levels in question leading to $O(h^2)$ error in the final answer. Due to (15.4.73) we arrive to

$$(\sigma-\beta_{x_1}) + (\sigma_1-\beta_{1\,x_1}) - \frac{1}{4}\{(\sigma-\beta_{x_1}),\alpha+\beta\xi_1\}$$

where the last term is

$$-\frac{1}{4}\{(\sigma-\beta_{x_1}),\alpha\} - \frac{1}{4}\{(\sigma-\beta_{x_1}),\beta\}\xi_1 + (\sigma-\beta_{x_1})_{x_1}\beta.$$

One can prove easily that due to (15.4.63), (15.4.73) (which is valid for σ_1, β_1 as well) the remaining terms vanish as $x_1 = \xi_2$.

Therefore as we skip terms containing factor $(x_1-\xi_2)^2$, what is left of perturbation is $\mathcal{P}_3^{\mathsf{w}}$ with

$$\mathcal{P}_3 := \rho(x_2,\xi_2)(x_1-\xi_2)\big(\xi_1^2+(x_1-\xi_2)^2\big), \tag{15.4.79}$$

so, $\mathcal{P}_3^{\mathsf{w}}$ is a symmetric product of ρ^{w}, $(x_1-\mu^{-1}hD_2)$ and A (as factor μ^2 should be remembered).

Reduction: Step 2

This perturbation is of the same magnitude as the original perturbation $\mathcal{P}^{\mathsf{w}}$ (albeit $\hbar^{-\delta}$ in estimates does not pop-up) but it is much easier to handle. If we had no boundary[16)] we would use $\mathcal{L} = \omega(x_2,\xi_2)\xi_1\bigl(\xi_1^2 + (x_1-\xi_2)^2\bigr)$ to eliminate it generating lower order terms but closer to factor ξ_1 forbids it unless accompanied by factor x_1. Still replacing $\rho(x_2,\xi_2)$ in (15.4.79) by

$$\rho(x_2,\xi_2) = \rho(x_2,0) - \rho_{\xi_2}x_1 + \rho_{\xi_2}(x_1-\xi_2) + ...$$

we can eliminate the second term (producing exactly such component in $\mathcal{P}_3$) and skip the third and all terms as producing $O(h^2)$ contribution. Therefore

Remark 15.4.15. Without any loss of the generality one can assume that $\rho = \rho(x_2)$ in (15.4.79).

Still it does not help us much and we cannot use obvious an transformation by $\exp(i\hbar^{-1}\mathcal{L}^{\mathsf{w}})$ with $\mathcal{L} = \omega(x_2)\bigl(\xi_1^2 + (x_1-\xi_2)^2\bigr)$ due to trouble in the next commutators. Instead we recall that basically our operator (15.4.70) multiplied by μ^2 should be equal $W + \mu h\mathfrak{z}$ with constant $\mathfrak{z}$. Actually one needs to insert there also $-hD_t$ but at energy level 0 it is really small and we can actually by multiplication by $(I+Q)$ make our evolution equation linear with respect to hD_t. We leave absolutely standard detailed arguments to the reader.

So now perturbation becomes

$$\mathcal{P}_4^{\mathsf{w}} = \mu h\rho(x_2)(x_1-\hbar D_2)\mathfrak{z} + \rho(x_2)(x_1-\hbar D_2)\mathfrak{z} \tag{15.4.80}$$

and it is completely different game as we can get rid of the first term using transformation by $\exp(i\hbar^{-1}\mathcal{L}^{\mathsf{w}})$ with $\mathcal{L} = \mu h\omega(x_2)\mathfrak{z}$ as one can see easily that there will be no problems with the next commutators. Remaining perturbation is of the type $\bigl(W_1(x,\xi)(x_1-\xi_2)\bigr)^{\mathsf{w}}$ and is of magnitude $\hbar$. So we gained factor $(\mu h)^{-1}$ which is important only in the case of the strong magnetic field so we can skip this step in the case of the intermediate magnetic field.

Reduction: Step 3

Note that the only part of new perturbation which is not necessarily $O(h^2)$ comes from $\bigl(W_1(x_2,\xi_2)(x_1-\xi_2)\bigr)^{\mathsf{w}}$ and we can use the same approach as before to eliminate it.

[16)] Or were on the distance at least $\hbar^{\frac{1}{2}-\delta}$ from it.

Actually we can continue further. We can keep power of $\hbar D_1$ below 2 replacing $\hbar^2 D_1^2$ by $-(x_1 - \hbar^2 D_2)^2 + \hbar\mathfrak{z} - \mu^{-2}W$. We can eliminate $\hbar D_1$ transforming by $\mathsf{exp}(i\hbar^{-1}\mathcal{L}^{\mathsf{w}})$ with $\mathcal{L} = \rho(x_1 - \xi_2)$. Similarly, we can eliminate $x_1^p(x_1-\xi_2)^q$ with $p \ge q$ transforming by similar operator with each ξ_1 having cofactor x_1. Therefore having x_1^n with $n \ge 2$ we can decompose it into sum of terms of the type $x_1^p(x_1 - \hbar D_2)^q(\hbar D_2)^{n-p-q}$ where either $p \ge q$ or $p = 0, q = 1$ or $p = 0, q = 0$ and eliminate the first type by the method we just discussed and the second one transforming by $\mathsf{exp}(i\hbar^{-1}\mathcal{L}^{\mathsf{w}})$ with $\mathcal{L} = \mathcal{L}(x_2, \xi_2)$. It eliminates all powers – including those we skipped before as lesser than $O(h^2)$.

Remark 15.4.16. (i) So in the end we arrive to the toy-model operator. Sure W is going to be perturbed by $O(h^2)$ albeit it may be because of smallness of ξ_2; so actually we get in the end $W_{\mathsf{eff}}(\xi_2, x_2) = -(V/F)(\xi_2, x_2) + O(\xi_2^3 + h^2)$ albeit this does not change it basic properties.

(ii) Recall that in this subsection ξ_2 in contrast to the rest of the section corresponds to $\hbar D_2$ rather than $\hbar D_2$ and $\hbar = \hbar^2$.

Conjecture 15.4.17. $W_{\mathsf{eff}}(\xi_2, x_2) = -(V/F)(\xi_2, x_2) + O(\xi_2^\infty + h^2)$ *as* $\xi_2 < \hbar^\delta$.

15.4.6 Final Results

Thus we arrive to the final contribution of zone $\{\xi_2 \le \hbar^{-\delta}\}$ (or equivalently $\{x_1 \le \hbar^{1-\delta}\}$) to the asymptotics:

$$\sum_{n\ge 0}(2\pi\hbar)^{-1}\mu \iint \theta\big(\mu h\big(\lambda_{*,n}(\xi_2) - \mathfrak{z}\big) - W_{\mathsf{eff}}(\hbar\xi_2, x_2)\big)\psi(\xi_2\hbar^{-1}, x_2)\zeta(\xi_2\hbar^{\delta})\, dx_2 d\xi_2$$

where transformation of ψ is obvious. After change of variables $x_1 = \hbar\xi_2$

$$\sum_{n\ge 0}(2\pi h)^{-1} \iint \theta\big(\mu h\big(\lambda_{*,n}(x_1\hbar^{-1}) - \mathfrak{z}\big) - W_{\mathsf{eff}}(x_1, x_2)\big)\psi(x)\zeta(x_1\varepsilon^{-1})\, dx_1 dx_2$$

with $\varepsilon = \hbar^{1-\delta}$ and $\hbar = (\mu^{-1}h)^{\frac{1}{2}}$. Here $\zeta \in \mathscr{C}_0^\infty([-1,1])$ equals 1 on $[-\frac{1}{2}, \frac{1}{2}]$.

On the other hand, the final contribution of zone $\{\xi_2 \ge \hbar^{-\delta}\}$ (or equivalently $\{x_1 \ge \hbar^{1-\delta}\}$) to the asymptotics is

$$\sum_{n\ge 0}(2\pi h)^{-1}\mu \iint \theta\big(\mu h\big(2n+1\big) - \mathfrak{z}\big) - W_{\mathsf{eff}}(x_1, x_2)\big)\psi(x)\big(1 - \zeta(x_1\varepsilon^{-1})\big)\, dx_1 dx_2$$

and the final answer is

$$\sum_{n\ge 0}(2\pi h)^{-1}\mu\iint \theta\big(\mu h\big(2n+1\big)-\mathfrak{z}\big)-W_{\mathsf{eff}}(x_1,x_2)\big)\psi(x)dx_1dx_2+$$
$$\sum_{n\ge 0}(2\pi h)^{-1}\mu\iint \Big(\theta\big(\mu h\big(\lambda_{*,n}(x_1\hbar^{-1})-\mathfrak{z}\big)-W_{\mathsf{eff}}(x_1,x_2)\big)-$$
$$\theta\big(\mu h\big(2n+1-\mathfrak{z}\big)-W_{\mathsf{eff}}(x_1,x_2)\big)\Big)\psi(x)\zeta(x_1\varepsilon^{-1})\,dx_1dx_2.$$

where now obviously we can take $\varepsilon = C\hbar|\log\hbar|$ and even $\varepsilon = C\hbar$ as we are in the spectral gap situation.

This matches (15.3.18) with

$$\mathcal{N}^{\mathsf{MW}}_{*,\mathsf{bound}} :=$$
$$\sum_{n}(2\pi)^{-1}\mu\iint \Big(\theta\big(\mu h\big(\lambda_{*,n}(x_1\hbar^{-1})-\mathfrak{z}\big)-W_{\mathsf{eff}}(x_1,x_2)\big)-$$
$$\theta\big(\mu h\big(2n+1-\mathfrak{z}\big)-W_{\mathsf{eff}}(x_1,x_2)\big)\Big)\psi(x)\zeta(x_1\varepsilon^{-1})\,dx_1dx_2. \tag{15.4.81}$$

Therefore we arrive to our final theorem

Theorem 15.4.18. *Let $\psi\in\mathscr{C}^\infty(\bar{X})$ be a fixed function with a compact support contained in the small vicinity of ∂X and let conditions* (13.3.1), (15.2.7) *and* (15.2.41) *be fulfilled there. Then*

(i) R^{MW} *with* $\mathcal{N}^{\mathsf{MW}}_{\mathsf{bound}}$ *defined by* (15.4.81) *does not exceed* $C\log\mu$.

(ii) This estimate could be improved to $O(1)$ *in each of the following cases:*

- *(a) Assumption* (15.2.8) *is fulfilled.*
- *(b) Assumption* (15.4.9) *and in case of Neumann boundary problem one of conditions* (15.4.16), $(15.4.19)^+$ *is fulfilled.*
- *(c) Assumption* (15.4.9) *fails but there is Dirichlet boundary problem and both conditions* $(15.2.41)^-$ *and* $(15.2.7)^+$ *are fulfilled.*
- *(d) Assumption* (15.4.9) *fails but there is Neumann boundary problem and both conditions* $(15.2.41)^+$ *and* $(15.2.7)^-$ *are fulfilled.*

Remark 15.4.19. Here we redefined $\mathcal{N}^{\mathsf{MW}}_{*,\mathsf{bound}}$ including in it in contrast to what we used before (see f.e. Theorem 15.2.6) $W(x_1,x_2)$ rather than $W(0,x_2)$.

We leave to the reader to prove that replacing $W(x_1,x_2)$ by $W(0,x_2)$ leads to an error $O(1)$ under condition (15.4.8) and $O(\hbar^{-\frac{1}{2}}|\log\hbar|^L)$ under condition (15.2.41).

Remark 15.4.20. Recall that $\mathcal{N}_{*,\mathsf{bound}} = O(\mu^{\frac{1}{2}}h^{-\frac{1}{2}})$ and it is sharp.

15.5 Generalizations

15.5.1 Robin Boundary Value Problem

Consider boundary condition

$$\big(i\sum_{j,k}\nu_j g^{jk}P_k - \alpha\big)u\big|_{\partial X} = 0 \tag{15.5.1}$$

with real-valued $\alpha = \alpha(x) \ge 0$ where $\nu = (\nu_1,\dots,\nu_d)$ is an inner normal.

As $\alpha = 0$ we get Neumann boundary condition and in a formal limit $\alpha \to +\infty$ we get Dirichlet boundary condition.

Weak Magnetic Field. Obviously arguments of Section 15.2 remain valid. Recall that in that section we have not distinguished between Dirichlet and Neumann boundary conditions.

Moderate Magnetic Field. Obviously arguments of Section 15.3 remain valid. Further, under assumption $\alpha \ge \epsilon_0$ propagation Theorem 15.3.15 remains valid; really then boundary problem satisfies Lopatinski condition. Therefore under this assumption one can derive the same remainder estimates as under Dirichlet boundary condition.

Intermediate and Strong Magnetic Field. On the contrary, in this case we are practically in the framework of Neumann boundary condition: see Remark 15.A.8.

15.5.2 Boundary Meets Degeneration of V

The following problem seems to be rather easy and straightforward:

Problem 15.5.1. Get rid of condition (13.4.45) $V \le -\epsilon$ as $\mu h \le 1$ (we have not imposed this condition as $\mu h \ge 1$).

One should apply rescaling method and follow arguments of Subsection 13.8.1.

15.5.3 Domains with Corners

The following problem is far more interesting and challenging and I strongly believe that the detailed analysis merits a publication:

Problem 15.5.2. Consider domains with corners $\ne 0, \pi, 2\pi$ assuming that coefficients belong to $\mathscr{C}^\infty(X)$, all other assumptions are fulfilled and in the corners non-degeneracy condition $|\nabla_{\partial X} W| \asymp 1$ holds for each of two directions along ∂X.

Without any loss of the generality one can assume that locally either $X = \{x_1 > 0, x_2 > 0\}$ or $X = \mathbb{R}^2 \setminus \{x_1 < 0, x_2 < 0\}$. Then non-degeneracy condition means that

$$|W_{x_1}| \asymp 1, \qquad |W_{x_2}| \asymp 1. \tag{15.5.2}$$

We also assume that (13.4.45) is fulfilled.

Weak Magnetic Field Case

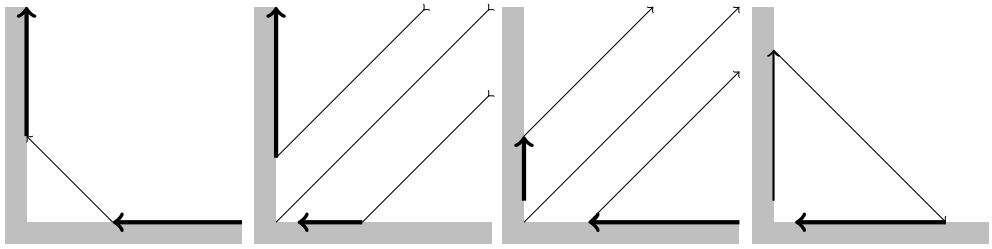
One can use rescaling method with $\ell = \epsilon|x|$ as long as $h_{\mathsf{eff}} = h\ell^{-1} \le 1$, $\mu_{\mathsf{eff}} = \mu\ell \ge 1$. In the inner zone $\mathcal{X}_{\mathsf{inn}}$ (defined exactly as before) magnetic drift is the only way to break periodicity and one can see easily that (15.3.22) should be replaced by

$$T_* := h^{1-\delta}\ell^{-1} \tag{15.5.3}$$

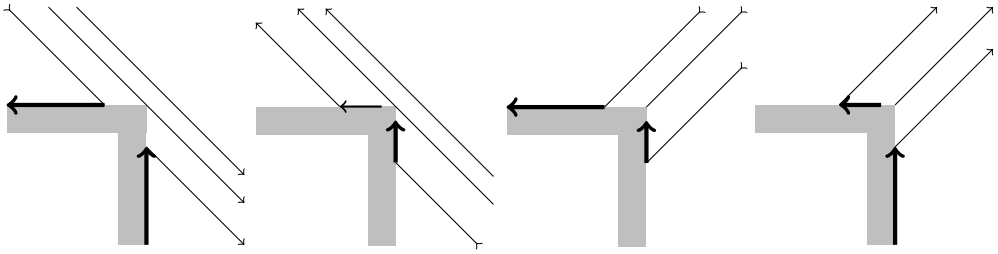
Then we are in the framework of the weak magnetic field approach as long as

$$\ell \ge \bar{\ell} := \mathsf{max}(C\mu^{-1}, \mu h^{1-\delta}). \tag{15.5.4}$$

Obviously we can take $T^* = \epsilon\mu\gamma$ with $\gamma := \mathsf{min}(|x_1|, |x_2|)$. To increase it consider dynamics on Figure 15.10.



(a) $W_{x_1} > 0, W_{x_2} > 0$ (b) $W_{x_1} > 0, W_{x_2} < 0$ (c) $W_{x_1} < 0, W_{x_2} > 0$ (d) $W_{x_1} < 0, W_{x_2} < 0$



(e) $W_{x_1} < 0, W_{x_2} < 0$ (f) $W_{x_1} > 0, W_{x_2} > 0$ (g) $W_{x_1} < 0, W_{x_2} > 0$ (h) $W_{x_1} > 0, W_{x_2} < 0$

Figure 15.10: Dynamics in the corner with angle $< \pi$ (a)–(d) and $> \pi$ (e)–(h). Bold lines show hop movement and thin lines show drift movement which is along level lines of W. Consistent with Figure 15.4.

Selecting a proper direction (away from the nearest boundary) we can take there $T^* = \epsilon\mu\ell$ and the contribution of $\mathcal{X}_{\mathsf{inn}}$ intersected with ℓ-element to the remainder does not exceed $Ch^{-1}\ell^2 \times (\mu\ell)^{-1} \asymp C\mu^{-1}h^{-1}\ell$ and summation over all inner zone returns $O(\mu^{-1}h^{-1})$.

Meanwhile contribution of ℓ-element intersected with the boundary zone $\mathcal{X}_{\mathsf{bound}}$ and transitional zone $\mathcal{X}_{\mathsf{trans}}$ does not exceed $C\mu^{-1}h^{-1}\ell \times \ell^{-1}$ as $T^* = \epsilon\ell$ here; then summation with respect to these zones returns $O(\mu^{-1}h^{-1}|\log h|)$.

One can get rid of logarithm noting that one can upgrade T^* to $T^* = \mathsf{min}(v^{-1}\ell, 1)$ as we are trying to move away from the corner; recall that in the transitional zone $\mathcal{X}_{\mathsf{trans}}$ speed in the opposite direction may be at most $v := C(\mu h)^{\frac{1}{3}}h^{-\delta} + C\mu^{-1}$ due to analysis of Section 5.4 and $\mu\ell \ge 1$. Further, in the transitional zone we have an extra factor $C(\mu h)^{\frac{2}{3}}h^{-\delta}$ due to analysis of Section 5.3.

We are left with the *corner zone* $\mathcal{X}_{\mathsf{corner}} := \{x : \ell(x) \le \bar{\ell}\}$ and in virtue of simple rescaling $x \mapsto \mu x$, $\mu \mapsto 1$ and $h \mapsto \mu h \le 1$ we can estimate its

contribution to the remainder by

$$C\mu h^{-1}\bar{\ell}^2 = C\mu^{-1}h^{-1}\big(1+\mu^4 h^{2-\delta}\big). \tag{15.5.5}$$

So, the total remainder estimate is also given by (15.5.5) which is $O(\mu^{-1}h^{-1})$ for

$$\mu \le h^{\delta-\frac{1}{2}}. \tag{15.5.6}$$

Moderate Magnetic Field Case

To improve estimate (15.5.5) (our ultimate goal is $O(\mu^{-1}h^{-1})$ as usual) as (15.5.6) breaks one needs to invoke arguments of Sections 15.2 and 15.3 and take into account dynamics near the corner (see Figure 15.10) in the same way we took into account dynamics near critical point of $W|_{\partial X}$ (see Figure 15.5) before.

Tauberian Remainder Estimate. As we mentioned in $\mathcal{X}_{\mathsf{inn}}$ one can take $T^* \asymp \mu\ell$ and $T_* = h^{1-\delta}\ell^{-1}$. Therefore contribution of ℓ-element intersected with $\mathcal{X}_{\mathsf{inn}}$ to the Tauberian remainder does not exceed

$$Ch^{-1}\ell^2\big(1+\mu T_*\big)T^{*\,-1} \asymp C\mu^{-1}h^{-1}\big(\ell+\mu h\big) \tag{15.5.7}$$

where due to arguments of Section 15.2 here and below in the estimates we can take $\delta = 0$ in the definition of T_*. Then summation over $\mathcal{X}_{\mathsf{inn}}\cap\{x, \ell(x)\ge \bar{\ell}\}$ results in $O\big(\mu^{-1}h^{-1}+|\log h|\big)$. Moreover, in cases (b), (c), (e)–(h) one can take $T^* \asymp \mu$ and the result will be $O(\mu^{-1}h^{-1})$.

The weak magnetic field arguments remain valid in $\mathcal{X}_{\mathsf{bound}}$ and its contribution to the Tauberian remainder is $O(\mu^{-1}h^{-1})$.

Meanwhile exactly as in Section 15.2 contribution of ℓ-element intersected with $\mathcal{X}_{\mathsf{trans}}$ to the Tauberian remainder does not exceed

$$C\mu^{-1}(\mu h)^{\frac{2}{3}}h^{-1-\delta}\ell \times \big(1+\frac{\mu h}{\ell}\big)\times\big(1+\frac{v}{\ell}\big)$$

as $T^* \asymp \min(1, v^{-1}\ell)$ where v is the propagation speed in the direction opposite to the hops and $(\mu h)^{\frac{2}{3}}h^{-\delta}$ is the width of $\mathcal{X}_{\mathsf{trans}}$; then the total contribution of $\mathcal{X}_{\mathsf{trans}}\cap\{x,\ell(x)\ge\bar{\ell}\}$ does not exceed

$$C\mu^{-1}(\mu h)^{\frac{2}{3}}h^{-1-\delta}\big(1+\mu h+v+\mu h v\bar{\ell}^{-1}\big);$$

meanwhile the contribution of $\mathcal{X}_{\mathsf{trans}} \cap \{x, C\hbar \le \ell \le \bar{\ell}\}$ does not exceed $C(\mu h)^{\frac{2}{3}} h^{-1-\delta}\bar{\ell}$ as we take $T^* \asymp \mu^{-1}$ here where here and below

$$\hbar := \max(\mu^{-1}, h^{1-\delta}). \tag{15.5.8}$$

Let us find $\bar{\ell}$ from $T^* = T_*$ but keep it greater than $\hbar$ i.e.

$$\bar{\ell} = C \max\bigl(\hbar, (vh)^{\frac{1}{2}}\bigr); \tag{15.5.9}$$

so the total contribution of $\mathcal{X}_{\mathsf{trans}} \cap \{x, \ell(x) \ge \hbar\}$ does not exceed

$$C\mu^{-1}h^{-1} + Ch^{-\delta} + Ch^{-1}(\mu h)^{\frac{2}{3}}(vh)^{\frac{1}{2}}h^{-\delta}.$$

Plugging $v = (\mu h)^{\frac{1}{3}}h^{-\delta}$ we arrive to

$$C\mu^{-1}h^{-1} + Ch^{-\delta} + Ch^{-\frac{1}{2}-\delta}(\mu h)^{\frac{5}{6}} \tag{15.5.10}$$

where the last term is not greater than the first two as $\mu \le h^{-\frac{8}{11}}$.

On the other hand, under Dirichlet boundary condition[17)] we can take $v = \mu^{-1}$ but then $\mu\ell \ge 1$ and we can take $T^* = 1$ and we arrive to $O(\mu^{-1}h^{-1} + h^{-\delta})$.

Remark 15.5.3. As it takes time $(\mu h)^{\frac{2}{3}}h^{-\delta}$ to punch through $\mathcal{X}_{\mathsf{trans}}$ and shift with respect to x_2 will be $v(\mu h)^{\frac{2}{3}}h^{-\delta}$ we can improve T^* as $v(\mu h)^{\frac{2}{3}}h^{-\delta} \le \ell$ which is not needed in the case of the Dirichlet[17)] problem and does not help in the case of the Neumann problem.

We are left with $\mathcal{X}_{\mathsf{corner}}$ but its contribution does not exceed $C\mu h^{-1}\bar{\ell}^2 = Ch^{-\delta}$.

Therefore remainder estimate is given by (15.5.10); if on the both sides of the corner Dirichlet[17) 18)] is given remainder estimate is $O(\mu^{-1}h^{-1} + h^{-\delta})$.

Magnetic Weyl Remainder Estimate. Transition from Tauberian to magnetic Weyl formula is obvious in $\mathcal{X}_{\mathsf{inn}}$. To run successive approximations one needs to assume that

$$(\mu^{-1} + vT_*)\,T_* \le h^{1+\delta}. \tag{15.5.11}$$

[17)] And Robin boundary condition with $\alpha \ge \epsilon$.

[18)] We can obviously mix conditions.

This is definitely the case in $\mathcal{X}_{\mathsf{bound}}$ and we need to consider $\mathcal{X}_{\mathsf{trans}}$ only. Plugging $T_* = h\ell^{-1}$ we arrive to $\ell \ge \bar{\ell}_1$ with $\bar{\ell}_1$ defined by

$$\bar{\ell}_1 = C\,\mathsf{max}\big(\mu^{-1}h^{-\delta}, (vh)^{\frac{1}{2}}\big) \tag{15.5.12}$$

which brings no extra terms in comparison with the Tauberian remainder.

So in contrast to Section 15.3 transition from Tauberian to magnetic Weyl formula does not increase an error.

Intermediate Magnetic Field Case

We need also to consider case of very strong magnetic field case $h^{\delta-1} \le \mu \le \epsilon h^{-1}$ under Dirichlet and improve remainder estimate $O(h^{-\delta})$ we got here. In this case analysis follows basically from the same arguments as in Subsection 15.3.1. Then in formula (15.5.7) one should replace $(\mu h)^{\frac{2}{3}}h^{-\delta}$ by 1 resulting in the remainder estimate $O(\mu^{-1}h^{-1} + |\log \mu|^K)$. In this approach actually corner zone becomes $\mathcal{X}_{\mathsf{corner}} = \{x, \ell(x) \le \bar{\ell}\}$ with $\bar{\ell} = C\mu^{-1} + Ch|\log h|^{K/2})$ or even $\bar{\ell} = C\mu^{-1}$.

Problem 15.5.4. (i) Investigate intermediate magnetic field case under Dirichlet[17)] and recover remainder estimate $O(\mu^{-1}h^{-1} + |\log \mu|^K)$.

(ii) Improve it under Dirichlet boundary condition to $O(\mu^{-1}h^{-1})$; is it possible?

(iii) Could be it done under Robin boundary condition?

Strong Magnetic Field Case

The similar arguments should work in the strong magnetic field case $\mu \ge \epsilon h^{-1}$ albeit we need to take $\bar{\ell} = \mu^{-\frac{1}{2}+\delta}h^{\frac{1}{2}}$, $\bar{\ell} = \mu^{-\frac{1}{2}}h^{\frac{1}{2}}|\log \mu|^{K/2}$ and $\bar{\ell} = C\mu^{-\frac{1}{2}}h^{\frac{1}{2}}$ to recover remainder estimates $O(\mu^{\delta})$, $O(|\log \mu|^K)$ and $O(1)$ respectively and we arrive to

Problem 15.5.5. (i) Investigate very strong magnetic field case under Dirichlet[17)] and recover remainder estimate $O(\mu^{\delta})$.

(ii) Improve it under Dirichlet boundary condition to $O(\mu^{-1}h^{-1} + |\log \mu|^K)$ and even to $O(\mu^{-1}h^{-1})$; is it possible?

(iii) Could be it done under Robin boundary condition?

15.5.4 Boundary Meets Degeneration of F

Set-up

Even more challenging (and in my opinion worth both publication and degree) seems to be a problem

Problem 15.5.6. Get rid of condition $|F| \ge \epsilon$.

Our goal (not necessary achievable in all cases) is to derive asymptotics as sharp as if there was no boundary. We will assume that $F \asymp \gamma(x)^{\nu-1}$ where $\gamma \in \mathscr{C}^\infty$, $\gamma(x) = 0 \implies |\nabla\gamma(x)| \asymp 1$, $x \in \partial X, \gamma(x) = 0 \implies |\nabla_{\partial X}\gamma(x)| \asymp 1$, $V < 0$.

Then we will have $X_{\mathsf{reg}} = \{x : |\gamma(x)| \ge \bar\gamma := C\mu^{-1/\nu}\}$ in the case of $\mu \le \epsilon h^{-\nu}$ and degeneration zone $X_{\mathsf{deg}} = \{|\gamma(x)| \le \bar\gamma\}$ and then we will have also four subzones:

$$X_{\mathsf{reg,inn}} = \{x : \gamma(x) \ge \bar\gamma,\ \ell(x) := \mathsf{dist}(x, \partial X) \ge C\mu^{-1}\gamma(x)^{1-\nu}\}, \tag{15.5.13}$$

$$X_{\mathsf{reg,bound}} = \{x : \gamma(x) \ge \bar\gamma,\ \ell(x) \le C\mu^{-1}\gamma(x)^{1-\nu}\}, \tag{15.5.14}$$

$$X_{\mathsf{deg,inn}} = \{x : \gamma(x) \le \bar\gamma,\ \ell(x) \ge C\mu^{-1}\bar\gamma^{1-\nu} = C\bar\gamma\}, \tag{15.5.15}$$

$$X_{\mathsf{corner}} = \{x : \gamma(x) \le \bar\gamma,\ \ell(x) \le C\bar\gamma\} \tag{15.5.16}$$

as on Figure 15.11(a) and each zone should be analyzed separately.

Remark 15.5.7. (i) Actually we need slightly increase $\bar\gamma$ as μ is close to $h^{-\nu}$ and we need increase "slightly" the threshold for $\ell(x)$ as $\gamma(x)$ is close to $(\mu h)^{-1/(\nu-1)}$ for $\mu \ge h^{\delta-1}$.

(ii) When we consider modified Schrödinger operators with $\mu h \ge 1$ so that domain $\{x,\ \gamma(x) \ge C(\mu h)^{-1/(\nu-1)}\}$ is no more forbidden, according to Section 15.4 we need to change threshold for $\ell(x)$ to "slightly more" than $\mu^{-1/2}h^{1/2}\gamma(x)^{(1-\nu)/2}$.

(iii) When we consider modified Schrödinger operators with $\mu h^\nu \ge 1$ according to Section 14.8 we need to change threshold for $\bar\gamma$ to "slightly more" than $\mu^{-1/2\nu}h^{1/2}$.

Without any loss of the generality one can assume that $X = \{x_1 > 0\}$ and $\Sigma = \{x_2 = 0\}$ and $F_{12} < 0$ as $x_2 > 0$. Then hops here are to the left (x_2 decays) and drift down (x_1 decays) because $\partial_{x_2}(-V)/F_{12} > 0$.

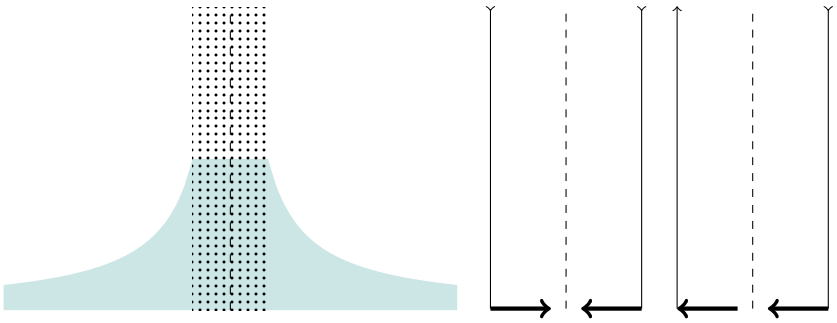


Figure 15.11: Zones (boundary zone is colored, degeneration zone is dotted) and dynamics near degeneration line of F (dashed). As $V<0$ we consider $F\asymp \gamma^{\nu-1}$ with smooth γ such that $|\nabla\gamma|\asymp 1$. Consistent with Figure 15.4.

Consider $x_2<0$; note that $F_{12}0$ changes sign and $\partial_{x_2}(-V)/F_{12}$ does not if ν is even and then hops are to the right and drift is down there (Figure 15.11(b)). On the other hand, $F_{12}0$ does not change sign and $\partial_{x_2}(-V)/F_{12}$ changes if ν is odd and then hops are to the left and drift is up there (Figure 15.11(c)).

Weak Magnetic Field Case

Using methods of the previous chapter and considering time direction in which drift up ne can prove easily that contribution of $X_{\mathsf{reg,inn}}$ to the remainder is $O(\mu^{-1/\nu}h^{-1})$ at least as $\mu\le h^{\delta-\nu}$.

Furthermore, using the methods of this chapter and and considering time direction in which hops away from Σ (and drift up) one can prove easily that contribution of $X_{\mathsf{reg,bound}}$ to the remainder is $O(\mu^{-1/\nu}h^{-1}$ at least as $\mu\le h^{\delta-\nu}$.

Consider zone $X_{\mathsf{deg,inn}}$. Then analysis of the previous chapter gives us remainder estimate $O\big(\mu^{-1/\nu}h^{-1}+(\mu^{1/\nu}h)^{1/2}h^{-1}\big)$. Recall that the last term is responsible for the correction which would come from near-periodic trajectories if $W=\mathsf{const}$. Now it is time to use the last non-degeneracy condition

$$|\nabla_\Sigma W|\asymp 1. \tag{15.5.17}$$

where V^* and $W=-V^*|_\Sigma$ are introduced by formulae (14.2.28) and (14.2.29). Then using a canonical form of the previous chapter near Σ (and thus

ignoring presence of the boundary) and considering evolution of ξ_2 one can take $T_* = h^{1-\delta}\ell^{-1}$ with $\ell \asymp |x_1|$ (and this takes in account the presence of ∂X).

As $T_* \le \epsilon\mu^{-1/\nu}$ one can then ignore all periodicity, which proves that the contribution of zone $X_{\mathsf{deg}} \cap \{x_1 \ge \bar{\ell}\}$ to the remainder is $O(\mu^{-1/\nu}h^{-1})$; meanwhile, the contribution of zone $X_{\mathsf{deg,inn}} \cap \{x_1 \le \bar{\ell}\}$ to the remainder is $O\big(\mu^{-1/\nu}h^{-1} + (\mu^{1/\nu}h)^{1/2}h^{-1}\bar{\ell}\big)$ with $\bar{\ell} = C\,\mathsf{max}(\mu^{1/\nu}h^{1-\delta}, \mu^{-1/\nu})$. By our standard technique we can actually make $\delta = 0$ in the estimate and then we arrive to the remainder estimate $O\big(\mu^{-1/\nu}h^{-1} + (\mu^{1/\nu}h)^{3/2}h^{-1}\big)$. In particular, remainder estimate is $O\big(\mu^{-1/\nu}h^{-1}\big)$ as $\mu \le h^{-3\nu/5}$.

Moderate Magnetic Field Case

Here comes the difficult part:

Problem 15.5.8. (i) Under condition (15.5.3) prove that if

$$|\xi_2 - \eta^*_\hbar W^{1/2}| \le \rho \le \ell^{1/2}, \qquad \text{with} \quad \rho\ell \ge h \tag{15.5.18}$$

then one can take $T_* \asymp h\ell^{-1}$ and $T^* \asymp 1$ and then contribution of such element to the Tauberian remainder does not exceed

$$C\mu^{-1/\nu}\rho\ell h^{-1}\big(1+\mu^{1/\nu}T_*\big)\,T^{*,-1} \asymp C\mu^{-1/\nu}h^{-1}\rho\ell\big(1+\mu^{1/\nu}h\ell^{-1}\big) \tag{15.5.19}$$

and therefore the total contribution to the Tauberian remainder of elements of this type with $\ell \ge \mathsf{max}(h^{2/3}, \mu^{-1/\nu})$ is $O(\mu^{-1/\nu}h^{-1})$.

(ii) Prove that if

$$|\xi_2 - \eta^*_\hbar W^{1/2}| \asymp \rho \ge \ell^{1/2}, \qquad \text{with} \quad \rho\ell \ge h \tag{15.5.20}$$

then one can take $T_* \asymp h\rho^{-2}$ and $T^* \asymp \rho$ and then contribution of such element to the Tauberian remainder does not exceed

$$C\mu^{-1/\nu}\rho\ell h^{-1}\big(1+\mu^{1/\nu}T_*\big)\,T^{*,-1} \asymp C\mu^{-1/\nu}h^{-1}\rho\ell\big(1+\mu^{1/\nu}h\rho^{-2}\big)\rho^{-1} \tag{15.5.21}$$

and therefore the total contribution to the Tauberian remainder of elements of this type with $\rho \ge \mathsf{max}(h^{1/3}, \mu^{-1/2\nu})$ is $O(\mu^{-1/\nu}h^{-1} + |\log\mu|)$.

(iii) Provide the same estimate for contribution of the remaining small zone $\{\rho \le \mathsf{max}(h^{1/3}, \mu^{-1/2\nu}),\ C\mu^{-1/\nu} \le \ell \le h^{2/3}, \rho\ell \le h\}$; combining with estimate for X_{corner} we arrive to the same estimate $O(\mu^{-1/\nu}h^{-1} + h^{-\delta})$.

(iv) Improve this estimate to $O(\mu^{-1/\nu}h^{-1}+1)$.

Also looks challenging the next step:

Problem 15.5.9. Pass from asymptotics with the Tauberian principal part to a more explicit ones.

There should be usual magnetic Weyl term, a term associated with the boundary and a term associated with degeneration albeit we do not expect any mixed boundary-degeneration term.

Strong and Superstrong Magnetic Field cCases

In the same way one needs to consider modified Schrödinger operator (with $\mathfrak{z}\mu hF^{\#}$ subtracted where $F^{\#}$ is "F but without absolute value") and taking into account Remark 15.5.7 follow the same path:

Problem 15.5.10. Derive spectral asymptotics for the modified Schrödinger operator:

(i) Consider the case of the strong magnetic field: $\epsilon h^{-1} \le \mu \le \epsilon h^{-\nu}$.

(ii) Consider the case of the superstrong magnetic field: $\mu \ge \epsilon h^{-\nu}$.

15.5.5 Case $h \ge 1$

The following problem does not seem to be too challenging but rather important not only on its own but due to its possible applications to the global asymptotics as in Chapter 23.

Problem 15.5.11. Extend our results to $\mu h \ge 1$, $h \ge 1$, $\mu \ge h$ (as it was the case without boundary).

15.A Appendices: Eigenvalues of $L(\eta)$

15.A.1 Basic Properties

Recall that $\lambda_{\mathsf{D},n}(\eta)$ and $\lambda_{\mathsf{N},n}(\eta)$ $n=0,1,...$ are eigenvalues of the operator[19)]

$$L(\eta) = -\partial_x^2 + (x+\eta)^2 \tag{15.A.1}$$

[19)] Obviously, transformation $x \mapsto -x$ reduces (15.1.26) to this one.

on $\mathbb{R}^-$ with Dirichlet and Neumann boundary condition respectively at 0, $u_n(x) = v_n(x+\eta, \eta)$ are (real-valued) eigenfunctions such that $\|v_n\| = 1$.

Proposition 15.A.1. *$\lambda_{\mathsf{D},n}(\eta)$ and $\lambda_{\mathsf{N},n}(\eta)$, $n = 0, 1, 2, \dots$ are real analytic functions with the following properties:*

(i) $\lambda_{\mathsf{D},n}(\eta)$ are monotone decreasing for $\eta \in \mathbb{R}$; $\lambda_{\mathsf{D},n}(\eta) \nearrow +\infty$ as $\eta \to -\infty$; $\lambda_{\mathsf{D},n}(\eta) \searrow (2n+1)$ as $\eta \to +\infty$; $\lambda_{\mathsf{D},n}(0) = (4n+3)$.

(ii) $\lambda_{\mathsf{N},n}(\eta)$ are monotone decreasing for $\eta \in \mathbb{R}^-$; $\lambda_{\mathsf{N},n}(\eta) \nearrow +\infty$ as $\eta \to -\infty$; $\lambda_{\mathsf{N},n}(\eta) < (2n+1)$ as $\eta \ge (2n+1)^{\frac{1}{2}}$; $\lambda_{\mathsf{N},n}(0) = (4n+1)$.

(iii) $\lambda_{\mathsf{N},n}(\eta) < \lambda_{\mathsf{D},n}(\eta) < \lambda_{\mathsf{N},(n+1)}(\eta)$; $\lambda_{\mathsf{D},n}(\eta) > (2n+1)$, $\lambda_{\mathsf{N},n}(\eta) > (2n-1)_+$.

Proof. Note that $\lambda_{\mathsf{D},n}(\eta)$ and $\lambda_{\mathsf{N},n}(\eta)$ are obtained by variational procedure from quadratic forms

$$Q^-(u) = \int_{x<0} \Big(|\partial_x u|^2 + (x+\eta)^2 |u|^2 \Big)\, dx, \quad Q_0^-(u) = \int_{x<0} |u|^2\, dx \tag{15.A.2}$$

respectively with Dirichlet or no boundary conditions at $x = 0$. Alternatively we can consider

$$Q^-(u) = \int_{x<\eta} \Big(|\partial_x u|^2 + x^2 |u|^2 \Big)\, dx, \quad Q_0^-(u) = \int_{x<\eta} |u|^2\, dx \tag{15.A.3}$$

respectively with Dirichlet or no boundary conditions at $x = \eta$ and we will shift notations between these two.

We know eigenvalues $\bar{\lambda}_n = (2n+1)$ and eigenfunctions of this operator on $\mathbb{R}$ (i.e. as $\eta = -\infty$). Further, we know that eigenvalues of $L_{\mathsf{D}}(0)$ and $L_{\mathsf{N}}(0)$ are $(2m+1)$ respectively with odd and even m we arrive to all assertions in Statement (i), and to all assertions in Statement (ii) except the third one.

Consider $\bar{v}_k(x)$ which are eigenfunctions of harmonic oscillator on $\mathbb{R}$, $k = 0, \dots, n$. As u belongs to their span

$$Q^-(u) = Q(u) - Q^+(u) \le (2n+1) Q_0(u) - Q^+(u) = \\ (2n+1) Q_0^-(u) + (2n+1) Q_0^+(u) - Q^+(u)$$

where Q, Q_0 denote the same forms on $\mathbb{R}$ and Q^+, Q_0^+ on $\{x > \eta\}$.

Let $\eta \ge (2n+1)^{\frac12}$. Obviously $(2n+1)Q_0^+(u) < Q^+(u)$ and therefore $Q^-(u) < (2n+1)Q_0^-(u)$ for u belonging to some $(n+1)$-dimensional space; then $\lambda_{\mathsf{N},n}(\eta) < (2n+1)$.

In Statement (iii) one needs only to prove that $\lambda_{\mathsf{D},n}(\eta) < \lambda_{\mathsf{N},(n+1)}(\eta)$. However in $(n+2)$ dimensional span of $\upsilon_{\mathsf{N},0}(x), \dots, \upsilon_{\mathsf{N},(n+1)}(x)$, functions vanishing at $x=\eta$ form $(n+1)$ dimensional subspace on which $Q^-(u) < \lambda_{\mathsf{N},(n+1)}Q_0^-(u)$ (strict inequality is due to the fact that only on $v = \upsilon_{\mathsf{N},(n+1)}(x)$ equality can be reached and this function does not belong to the subspace in question (as then $v(\eta) = \partial v(\eta) = 0$ which is impossible). □

So far we have not proved that $\lambda_{\mathsf{N},n}(\eta) \to (2n+1)$ as $\eta \to +\infty$; it will be done later.

15.A.2 More Properties

We follow M. Dauge–B. Helffer [1] here.

Consider equation on $x<0$

$$\bigl(L(\eta) - \lambda_n(\eta)\bigr)u_n = 0. \tag{15.A.4}$$

Differentiating by η and multiplying by u_n we arrive to

$$\begin{aligned}\partial_\eta\lambda_n(\eta) &= ((\partial_\eta L(\eta))u_n, u_n) = 2((x+\eta)u_n, u_n) = \\ &([\partial_x, (x+\eta)^2]u_n, u_n) = 2((x+\eta)^2u_n, \partial_x u_n) + \eta^2|u_n(0)|^2 = \\ &2(L(\eta)u_n, \partial_x u_n) + (\partial_x^2 u_n, \partial_x u_n) + \eta^2|u_n(0)|^2 = \\ &\bigl(\eta^2 - \lambda_n(\eta)\bigr)|u_n(0)|^2 - |\partial_x u_n(0)|^2.\end{aligned} \tag{15.A.5}$$

In particular,

$$\partial_\eta\lambda_{\mathsf{D},n}(\eta) = -|\partial_x\upsilon_{\mathsf{D},n}(\eta)|^2 \tag{15.A.6}_{\mathsf{D}}$$

and

$$\partial_\eta\lambda_{\mathsf{N},n}(\eta) = \bigl(\eta^2 - \lambda_{\mathsf{N},n}(\eta)\bigr)|\upsilon_{\mathsf{N},n}(\eta)|^2. \tag{15.A.6}_{\mathsf{N}}$$

We refer to them as *Dauge-Helffer formulae* [1].

We conclude immediately that

Proposition 15.A.2. *(i)* $\partial_\eta\lambda_{\mathsf{D},n}(\eta) < 0$.

(ii) $\partial_\eta \lambda_{\mathsf{N},n}(\eta) \gtreqless 0$ *if and only if*

$$\lambda_{\mathsf{N},n}(\eta) \lesseqgtr \eta^2. \tag{15.A.7}$$

Differentiating $(15.A.6)_{\mathsf{N}}$ we arrive to

$$\partial_\eta^2 \lambda_{\mathsf{N},n}(\eta) = \bigl(2\eta - \partial_\eta \lambda_{\mathsf{N},n}(\eta)\bigr)|\upsilon_{\mathsf{N},n}(\eta)|^2 + 2\bigl(\eta^2 - \lambda_{\mathsf{N},n}(\eta)\bigr)\partial_\eta \upsilon_{\mathsf{N},n}(\eta)\cdot \upsilon_{\mathsf{N},n}(\eta). \tag{15.A.8}$$

Therefore as $\partial_\eta \lambda_{\mathsf{N},n}(\eta) = 0$ in virtue of Proposition 15.A.1

$$\partial_\eta^2 \lambda_{\mathsf{N},n}(\eta) = 2\eta |\upsilon_{\mathsf{N},n}(\eta)|^2 \tag{15.A.9}$$

and therefore it is non-degenerate minimum. Thus we arrive to

Proposition 15.A.3. *(i)* $\lambda_{\mathsf{N},n}(\eta)$ *has a single stationary point*[20] η_n*, it is non-degenerate minimum, and in this point* (15.A.7), (15.A.9) *hold.*

(ii) In particular, $(\lambda_{\mathsf{N},n}(\eta) - \eta^2)$ *has the same sign as* $(\eta_n - \eta)$*.*

Therefore $\lambda_{\mathsf{N},n+1}(\eta_n) - \eta_n^2 > \lambda_{\mathsf{N},n}(\eta_n) - \eta_n^2 = 0$ implies that

$$\eta_{n+1} > \eta_n. \tag{15.A.10}$$

15.A.3 Estimates of $\lambda_{*,n}(\eta)$ as $\eta \to +\infty$

So we must investigate properties of $\lambda_n(\eta)$ as $\eta \to +\infty$. We know that then $\lambda_{\mathsf{D},n}(\eta) \ge (2n+1) \ge \lambda_{\mathsf{N},n}(\eta)$ and $\lambda_{\mathsf{D},n}(\eta)$ tends to $(2n+1)$.

Consider $\bar{\upsilon}_j(x)$. Note that

$$\int_\eta^{+\infty} \bar{\upsilon}_j(x)\bar{\upsilon}_k(x)\,dx = O\bigl(\eta^{j+k-1}e^{-\eta^2}\bigr) \tag{15.A.11}$$

(non-uniformly with respect to j,k) and as $j=k$ this is an exact magnitude.

Therefore

$$(\bar{\upsilon}_j, \bar{\upsilon}_k) = \delta_{jk} + O\bigl(\eta^{j+k-1}e^{-\eta^2}\bigr) \tag{15.A.12}$$

as our space is $\mathscr{L}^2((-\infty,\eta))$.

[20] And it must have one due to Proposition 15.A.1.

Consider bilinear form

$$Q^-(u,v)=\int_{-\infty}^{\eta}\bigl(u'v'+x^2uv\bigr)\,dx$$

(where $'$ means ∂_x). One can rewrite it as

$$Q^-(u,v)=\int_{-\infty}^{\eta}\bigl(-u''+x^2u\bigr)v\,dx+u'(\eta)v(\eta).$$

Then

$$Q^-(\bar{v}_j,\bar{v}_k)=(2j+1)(\bar{v}_j,\bar{v}_k)+\bar{v}_j'(\eta)\bar{v}_k(\eta).$$

Thus if we consider $u=\sum_{k\le n}\alpha_k\bar{v}_k$ with $\alpha_k\in\mathbb{R}$ such that $\sum_{k\le n}\alpha_k^2\asymp 1$ then

$$Q^-(u)-(2n+1)Q_0^-(u)=\sum_{k\le n}2(k-n)\alpha_k^2+O\bigl(e^{-\eta^2}\eta^{n+k+1}\bigr)$$

and this expression is negative provided $\alpha_k^2\ge e^{-\eta^2}\eta^{n+k+1}$ for some $k<n$; otherwise we can assume that $\alpha_n=1$ and

$$\begin{aligned}Q^-(u)-(2n+1)Q_0^-(u)\equiv\sum_{k\le n}\bigl(Q^-(\bar{v}_k)-(2n+1)Q_0^-(\bar{v}_k)\bigr)\alpha_k^2\le\\ \bigl(Q^-(\bar{v}_n)-(2n+1)Q_0^-(\bar{v}_n)\bigr)\equiv\bar{v}_n'(\eta)\bar{v}_n(\eta)\mod O\bigl(e^{-\frac{5}{4}\eta^2}\bigr)\end{aligned}$$

and we know that $\bar{v}_n'(\eta)\bar{v}_n(\eta)$ is negative and of magnitude $\eta^{2n+1}e^{-\eta^2}$ as $\eta\ge c_n$ and therefore due to variational principle

$$\lambda_{\mathsf{N},n}(\eta)\le(2n+1)-\epsilon\eta^{2n+1}e^{-\eta^2}\qquad\text{as }\ \eta\ge c_n.$$

Meanwhile let us replace $\bar{v}_k(x)$ by $\tilde{v}_k(x)=\bar{v}_n(x)+\phi_k$, $\phi_k=-\bar{v}_k(\eta)e^{\eta x-\eta^2}$ satisfying Dirichlet conditions as $x=\eta$. All the above conclusions except calculations of $Q(\bar{v}_n)$ remain true for these functions as well and for them

$$\begin{aligned}&Q^-(\tilde{v}_n)-(2n+1)Q_0^-(\tilde{v}_n)=\\ &\qquad\bar{v}_n'(\eta)\bar{v}_n(\eta)+2\bar{v}_n'(\eta)\phi_n(\eta)+Q^-(\phi_n)-(2n+1)Q_0^-(\phi_n)=\\ &\qquad\qquad-v_n'(\eta)v_n(\eta)+v_n^2(\eta)\bigl(\eta+O(\eta^{-1})\bigr)\asymp\eta^{2n+1}e^{-\eta^2}\end{aligned}$$

and therefore

$$\lambda_{\mathsf{D},n}(\eta) \le (2n+1) + C_n \eta^{2n+1} e^{-\eta^2} \qquad \text{as} \quad \eta \ge c_n.$$

On the other hand, consider $\mathscr{C} \ni u$ supported in $(-\infty, \eta]$ and orthogonal to $\bar{v}_0, \dots, \bar{v}_{n-1}$. Then (modulo constant factor) $u = \bar{v}_n(x) + v$ where v is orthogonal to $\bar{v}_n$ in $\mathbb{R}$ as well.

Note that

$$Q^-(u) = Q(u) = Q(\bar{v}_n) + Q(v) = (2n+1) + Q(v),$$

and $Q_0^-(u) = 1 + \|v\|^2_{\mathbb{R}}$, and

$$Q^-(u) - (2n+1)Q_0^-(u) = Q(v) - (2n+1)Q_0(v) \ge 2Q_0(v) \ge 2Q_0^+(v) = 2Q_0^+(\bar{v}_n) \ge \epsilon \eta^{2n+1} e^{-\eta^2}.$$

So, any $(n+1)$-dimensional subspace of $\mathscr{L}^2((-\infty, \eta))$ contains an element with $Q^-(u) \ge \big((2n+1) + \epsilon\eta^{2n+1}e^{-\eta^2}\big) Q_0^-(u)$ and we arrive to the left inequality of

$$\epsilon_n \eta^{2n+1} e^{-\eta^2} \le \lambda_{\mathsf{D},n}(\eta) - (2n+1) \le C_n \eta^{2k+1} e^{-\eta^2} \qquad \text{as} \quad \eta \ge c_n \tag{15.A.13}$$

as the right one has been already proven.

Finally, estimate $\lambda_{\mathsf{N},n}$ from below. Consider $\upsilon_n(x)$ and extend it as $\upsilon_n(\eta) e^{\frac{1}{2}(\eta^2 - x^2)}$ to (η, ∞). Let u be a resulting function. Note

$$Q^+(u) - (2n+1)Q_0^+(u) = |\upsilon_n(\eta)|^2 \big(\eta - 2n\Phi(\eta)\big),$$

with $\Phi(\eta) = \int_\eta^\infty e^{\eta^2 - x^2}\, dx$ and then as $n = 0$

$$\begin{aligned} \lambda_{\mathsf{N},n}(\eta) - (2n+1) = Q^-(\upsilon_n) - (2n+1)Q_0^-(\upsilon_n) = \\ Q(u) - (2n+1)Q_0(u) - Q^+(u) + (2n+1)Q_0^+(u) \ge \\ - |\upsilon_n(\eta)|^2 \big(\eta - 2n\Phi(\eta)\big) = -\frac{1}{2}\big(\eta - 2n\Phi(\eta)\big)(\eta^2 - \lambda_{\mathsf{N},n})^{-1} \partial_\eta \lambda_{\mathsf{N},n}(\eta) \end{aligned}$$

due to $(15.A.6)_{\mathsf{N}}$.Therefore

$$\big(2n + 1 - \lambda_{\mathsf{N},n}(\eta)\big)^{-1} \partial_\eta \lambda_{\mathsf{N},n}(\eta) \ge 2\big(\eta^2 - \lambda_{\mathsf{N},n}(\eta)\big)\big(\eta - \Phi(\eta)\big)^{-1}.$$

Note that $\Phi(\eta) = \frac{1}{2}\eta^{-1} + O(\eta^{-3})$ and $\lambda_n < 2n+1$. Then the right-hand side of the last inequality is greater than $2(\eta - (n+1)\eta^{-1}) - O(\eta^{-2})$ and integrating we conclude that

$$\log\bigl(2n+1-\lambda_{\mathsf{N},n}(\eta)\bigr)^{-1} \geq \eta^2 - 2(n+1)\log\eta - \log C_n$$

and

$$2n+1-\lambda_{\mathsf{N},n}(\eta) \leq \sigma_n(\eta) \coloneqq C_n e^{-\eta^2}\eta^{2n+2}$$

which is just one factor η greater than we got as an lower estimate for $(1-\lambda_n)$.

Finally, as $\lambda_{\mathsf{N},n+1}(\eta) > \lambda_{\mathsf{N},n+1}(\eta_{n+1}) > \lambda_{\mathsf{D},n}(\eta_{n+1}) > (2n+1)$ we conclude that $\lambda_{\mathsf{N},n+1}(\eta)$ and $\lambda_{\mathsf{N},n}(\eta)$ are disjoint as $\eta > C_n$ and therefore as $Q^-(\bar{v}_n) \leq Q^-(v_n) + C_n\sigma_n$ we conclude that $Q^-(v_n - \bar{v}_n) \leq C_n\sigma_n(\eta)$:

$$\|\partial_x(v_n - \bar{v}_n)\| + \|x(v_n - \bar{v}_n)\| \leq C_n\sigma_n(\eta)^{\frac{1}{2}} \tag{15.A.14}$$

where norms are calculated in $\mathscr{L}^2((-\infty,\eta))$ and then

$$|(v_n - \bar{v}_n)| \leq C_n\sigma_n(\eta)^{\frac{1}{2}}; \tag{15.A.15}$$

In particular $|v_n(\eta)| \leq C_n\sigma_n(\eta)^{\frac{1}{2}}$.

So far it was proven only for $n=0$. However we apply induction: as it was proven for $k<n$ with fast decaying $\sigma_k(\eta)$. Then $(v_n,\bar{v}_k) = O\bigl(\sigma_k(\eta)^{\frac{1}{2}}\bigr)$ for $k<n$ and extending v_n as before we arrive to $(u,\bar{v}_k) = O\bigl(\sigma_{n-1}(\eta)^{\frac{1}{2}}\bigr)$ for all $k<n$.

Then $Q(u) - (2n+1)Q_0(u) \geq -C\sigma_{n-1}$ and we arrive to

$$\lambda_{\mathsf{N},n} - (2n+1) \geq -\frac{1}{2}\bigl(\eta - 2n\Phi(\eta)\bigr)\bigl(\eta^2 - \lambda_{\mathsf{N},n}(\eta)\bigr)^{-1}\partial_\eta\lambda_{\mathsf{N},n}(\eta) - C\eta^{2n}e^{-\eta^2}$$

which implies the same conclusions.

We also conclude that (15.A.14) and (15.A.15) hold for $\bar{v}_n$ replaced by $v_{\mathsf{D},n}$ and that $|\partial_x v_{\mathsf{D},n}(\eta)| \leq C_n\eta\sigma_n(\eta)^{\frac{1}{2}}$.

15.A.4 Asymptotics of $\lambda_{*,n}(\eta)$ as $\eta \to +\infty$

Here will reproduce results of section 5 of C. Bolley–B. Helffer [1] and generalize them to arbitrary j and to Dirichlet problem (definitely their methods could do the same).

In contrast to this paper we will not build quasimodes but use Dauge-Helffer formulae. Let

$$\varepsilon_{*,n}(\eta) = \pm\big(\lambda_{*,n}(\eta) - 2n - 1\big) \tag{15.A.16}$$

with $* = \mathsf{D}, \mathsf{N}$ respectively. Then from

$$L(\eta)\upsilon_{*,n}(\eta) - \big(2n + 1 \pm \varepsilon_{*,n}(\eta)\big)\upsilon_{*,n} = 0, \qquad L(\eta)\bar{\upsilon}_n - (2n+1)\bar{\upsilon}_n = 0$$

we conclude that

$$\mp\varepsilon_{*,n}(\eta)(\upsilon_{*,n}, \bar{\upsilon}_n) = \upsilon_n'(\eta)\bar{\upsilon}_n(\eta) - \upsilon_n(\eta)\bar{\upsilon}_n'(\eta)$$

and as modulo $O\big(e^{-\frac{1}{2}\eta^2}\eta^K\big)$ $(\upsilon_{*,n}, \bar{\upsilon}_n) \equiv 1$ then

$$\mp\varepsilon_{*,n}(\eta) \equiv \upsilon_{*,n}'(\eta)\bar{\upsilon}_n(\eta) - \upsilon_{*,n}(\eta)\bar{\upsilon}_n'(\eta)$$

modulo $O\big(e^{-\frac{3}{2}\eta^2}\eta^K\big)$. Then modulo sign

$$-\varepsilon_{\mathsf{D},n}(\eta) \equiv \upsilon_{\mathsf{D},n}'(\eta)\bar{\upsilon}_n(\eta) = \big(-\partial_\eta\varepsilon_{\mathsf{D},n}\big)^{\frac{1}{2}}\bar{\upsilon}_n(\eta),$$

$$\varepsilon_{\mathsf{N},n}(\eta) \equiv \upsilon_{\mathsf{N},n}(\eta)\bar{\upsilon}_n'(\eta) = \big(-(\eta^2 - 2n - 1)^{-1}\partial_\eta\varepsilon_{\mathsf{N},n}\big)^{\frac{1}{2}}\bar{\upsilon}_n'(\eta)$$

and modulo $O\big(e^{\frac{1}{2}\eta^2}\eta^K\big)$

$$\partial_\eta\varepsilon_{\mathsf{D},n}^{-1}(\eta) \equiv \big(\bar{\upsilon}_n(\eta)\big)^{-2},$$

$$\partial_\eta\varepsilon_{\mathsf{N},n}^{-1}(\eta) \equiv (\eta^2 - 2n - 1)\big(\bar{\upsilon}_n'(\eta)\big)^{-2}$$

and

$$\varepsilon_{\mathsf{D},n}^{-1}(\eta) \equiv \int^\eta \big(\bar{\upsilon}_n(\eta)\big)^{-2}\, d\eta, \tag{15.A.17}_{\mathsf{D}}$$

$$\varepsilon_{\mathsf{N},n}^{-1}(\eta) \equiv \int^\eta (\eta^2 - 2n - 1)\big(\bar{\upsilon}_n'(\eta)\big)^{-2}\, d\eta \tag{15.A.17}_{\mathsf{N}}$$

and finally we arrive to

Theorem 15.A.4. *As* $\eta \to +\infty$

$$\lambda_{\mathsf{D},n}(\eta) \equiv 2n + 1 + \Big(\int^\eta \big(\bar{\upsilon}_n(\eta)\big)^{-2}\, d\eta\Big)^{-1}, \tag{15.A.18}_{\mathsf{D}}$$

$$\lambda_{\mathsf{N},n}(\eta) \equiv 2n + 1 - \Big(\int^\eta (\eta^2 - 2n - 1)\big(\bar{\upsilon}_n'(\eta)\big)^{-2}\, d\eta\Big)^{-1} \tag{15.A.18}_{\mathsf{N}}$$

modulo $O\big(e^{-\frac{3}{2}\eta^2}\eta^K\big)$ *with* $K = K(n)$.

Remark 15.A.5. One can see easily from $(15.A.18)_{\mathsf{D}}$, $(15.A.18)_{\mathsf{N}}$ that

$$e^{-\eta^2}\varepsilon_{*,n}^{-1}(\eta) \sim \sum_{k\ge 0} c'_{*,n,k}\eta^{-2n-1-2k}$$

with the leading coefficient $c'_{*,n,0} = \frac{1}{2}\kappa_n^{-2}$ where $\kappa_n = (n!)^{-\frac{1}{2}}\pi^{-\frac{1}{4}}2^{n/2}$ coefficient at x^n at $\bar{v}_n e^{x^2/2}$ and thus

$$e^{-\eta^2}\varepsilon_{*,n}(\eta) =\sim \sum_{k\ge 0} c_{*,n,k}\eta^{2n+1-2k} \tag{15.A.19}$$

where

$$c_{*,n,0} = 2\kappa_n^2 = \frac{2^{n+1}}{n!\sqrt{\pi}}. \tag{15.A.20}$$

Now we can estimate recurrently higher-order derivatives of $\varepsilon_{*,n}(\eta)$, $v_{*,n}(x,\eta)$ as $x = \eta$ arriving to

Corollary 15.A.6.

$$|\partial^j \lambda_{*,n}(\eta)| \asymp \eta^{2n+1+j}e^{-\eta^2} \tag{15.A.21}$$

as $\eta > c_{n,j}$.

15.A.5 Remarks

Remark 15.A.7. While we do not need it one can prove easily that

$$\lambda_{*,n}(\eta) \sim \eta^2 + \sum_{k\ge 0} c_{*,n,k}\eta^{\frac{2}{3}(1-2k)} \tag{15.A.22}$$

as $\eta \to -\infty$ with $c_{*,n,0}$ eigenvalues of operator $M(\eta) := \big(D^2 + 2x|\eta|\big)$ on $\mathbb{R}^+$ with the corresponding boundary condition (D or N) as $x = 0$ and thus are defined through zeros of Airy function or its derivative.

Remark 15.A.8. (i) For operator $D^2 + x^2$ on $\{x \le \eta\}$ albeit with the boundary condition $(u' + \alpha u)(\eta) = 0$ with $0 \le \alpha < \infty$ one can see easily that $(15.A.6)_{\mathsf{N}}$, $(15.A.17)_{\mathsf{N}}$ are replaced by

$$\partial_\eta \lambda_n(\eta,\alpha) = \big(\eta^2 - \alpha^2 - \lambda_n(\eta,\alpha)\big)|v_n(\eta,\alpha)|^2, \tag{15.A.6}_\alpha$$

$$\varepsilon_n^{-1}(\eta,\alpha) \equiv \int^\eta (\eta^2 - 2n - 1 - \alpha^2)(\bar{v}'_n(\eta,\alpha))^{-2}\,d\eta \tag{15.A.17}_\alpha$$

respectively.

(ii) One can prove easily that $\lambda_n(\eta;\alpha)$ monotonically depends on α and $\lambda_n(\eta;0) = \lambda_{\mathsf{N},n}(\eta)$, $\lambda_n(\eta;\alpha) \nearrow \lambda_{\mathsf{D},n}(\eta)$ as $\alpha \to +\infty$.

However, in virtue of (i) for all $\alpha > 0$ there is a single extremum which is a non-degenerate minimum at $\eta_n = \eta_n(\alpha) > 0$ such that $\eta_n^2 = \lambda_n(\eta_n) + \alpha^2$ and as $\alpha \to +\infty$ therefore $\eta_n(\alpha) \approx \alpha^2 + (2n+1)$ with a small negative error (see Remark 15.A.8).

Therefore as $\alpha \to +\infty$ these minima (see blue line on Figure 15.13 for $\alpha = 0$) become more shallow and move to the right and blue-like lines here can approximate red-like lines exactly as on the graphs of $e^{-\eta} - (\eta^2+\alpha^2)^{-1}$.).

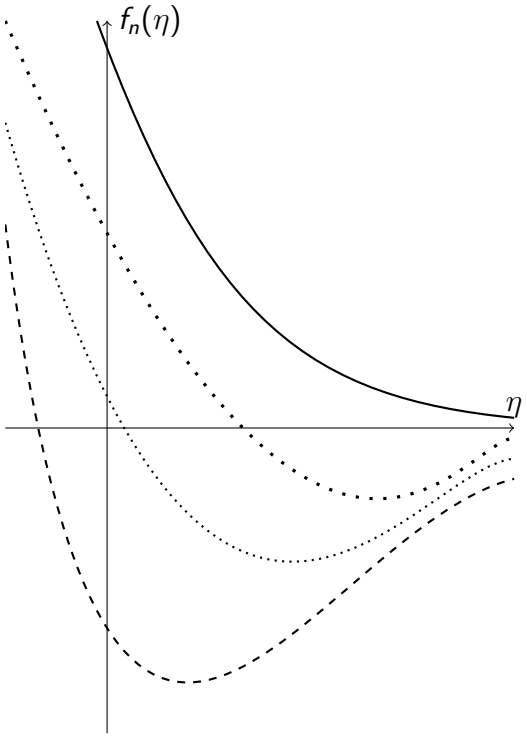


Figure 15.12: Schematics of $f_n(\eta) := \lambda_n(\eta;\alpha) - (2n+1)$.

Conjecture 15.A.9. *For each n function $\lambda_{\mathsf{D},n}(\eta)$ is convex and $\partial_\eta^2 \lambda_{\mathsf{D},n}(\eta) > 0$ while $\lambda_{\mathsf{N},n}(\eta)$ has a single non-degenerate inflection point η_n' and $\partial_\eta^2 \lambda_{\mathsf{D},n}(\eta) \gtrless 0$ as $\eta \lessgtr \eta_n'$.*

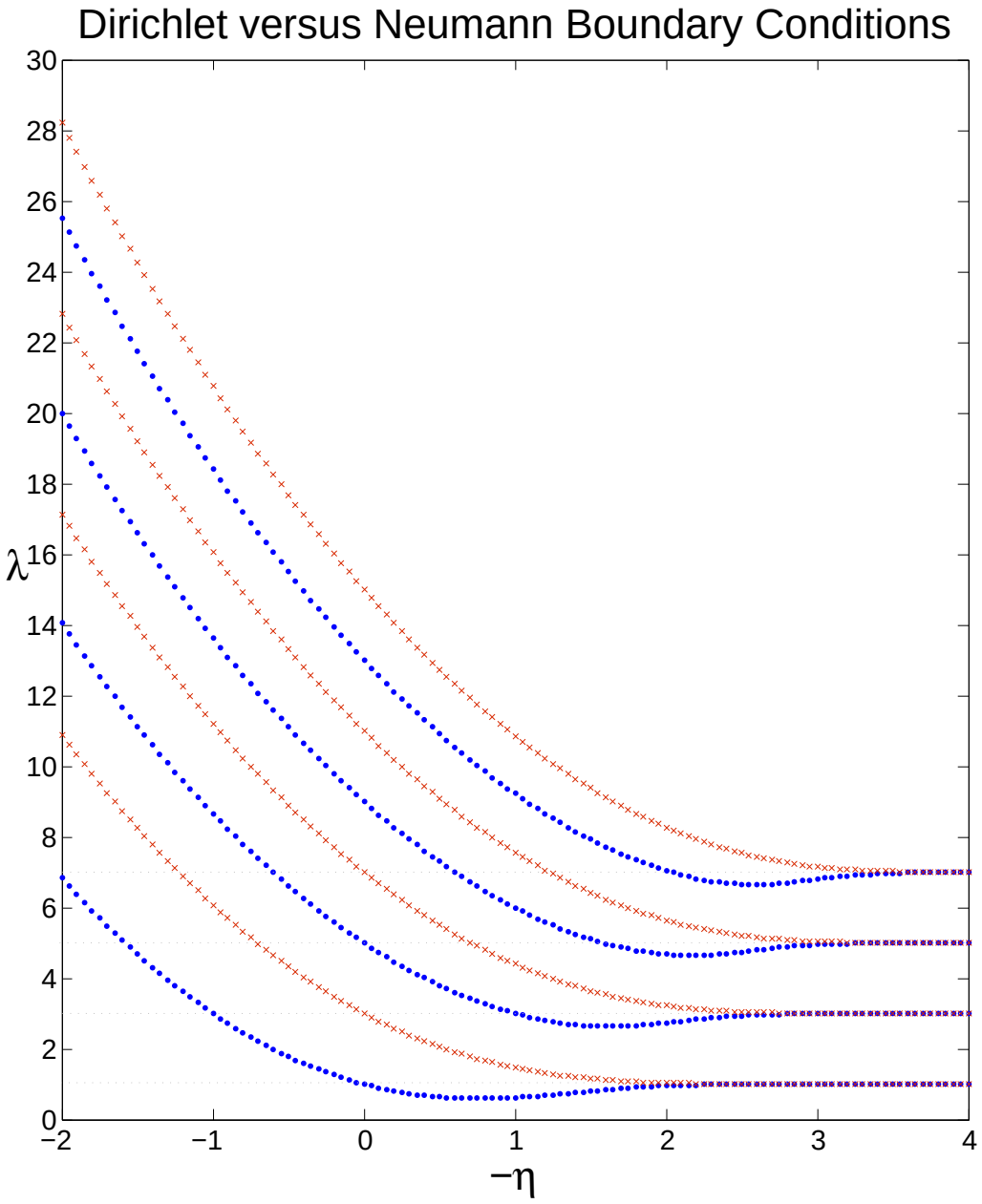


Figure 15.13: Plots of $\lambda_{\mathsf{N},n}(\eta)$ and $\lambda_{\mathsf{D},n}(\eta)$. Calculated by Matlab: Courtesy of Marina Chugunova.

Part VII

Smooth Theory in Dimensions 2 and 3 (Continued)

Chapter 16

Magnetic Schrödinger Operator: Short Loops, Pointwise Spectral Asymptotics and Asymptotics of Dirac Energy

16.1 Introduction

In this chapter we consider 2D- and 3D-magnetic Schrödinger operator (13.2.1) satisfying assumptions (13.2.2)–(13.2.5) and consider pointwise asymptotics of $e(x,x,0)$ and also asymptotics of expression (6.4.4):

$$\mathsf{I} := \iint e(x,y,0)e(y,x,0)\omega(x,y)\psi_2(x)\psi_1(y)\,dx\,dy \tag{16.1.1}$$

as well as of

$$\mathsf{J} := \iint e(x,x,0)e(y,y,0)\omega(x,y)\psi_2(x)\psi_1(y)\,dxdy \tag{16.1.2}$$

with a function $\omega(x,y)$ satisfying assumption (6.4.5):

(16.1.3) $\omega(x,y) := \Omega(x,y;x-y)$ where function Ω is smooth in $B(0,1)\times B(0,1)\times B(\mathbb{R}^2\setminus 0)$ and homogeneous of degree $-\kappa$ ($0<\kappa<2$) with respect to its third argument,

and with smooth cut-off functions ψ_1,ψ_2.

These two expressions play a role in the applications to the multiparticle quantum theory of Chapter 27. Actually, instead of asymptotics of (16.1.2) we consider related estimates of

$$\mathsf{K} := \iint \bigl(e(x,x,\tau) - h^{-d}\mathcal{N}_x(\tau)\bigr)\bigl(e(y,y,\tau) - h^{-d}\mathcal{N}_x(\tau)\bigr)\omega(x,y)\,dxdy \tag{16.1.4}$$

where $\mathcal{N}_x(\tau)$ denotes some approximation to $e(x,x,\tau)$.

We assume that V/F satisfies non-degeneracy conditions (13.4.45) and (13.4.54) i.e.

$$V \le -\epsilon_0 \qquad \text{in } B(0,1), \tag{16.1.5}$$

$$|F| \ge \epsilon \qquad \text{in } B(0,1), \tag{16.1.6}$$

$$|\nabla \frac{V}{F}| \asymp 1 \qquad \text{in } B(0,1) \tag{16.1.7}$$

in the asymptotics of expressions (16.1.1)–(16.1.2) we will be able to drop (16.1.5) and replace (16.1.7) by

$$|\nabla \frac{V}{F}| \le \epsilon_0 \implies |\det \mathsf{Hess}\, \frac{V}{F}| \ge \epsilon_0 \tag{16.1.8}$$

or

$$|\nabla \frac{V}{F}| \le \epsilon_0 \implies \det \mathsf{Hess}\, \frac{V}{F} \ge \epsilon_0. \tag{16.1.8}^+$$

As we consider Schrödinger-Pauli operator (13.6.3) and $\mu h \gtrsim 1$ conditions (16.1.7)–(16.1.8)$^+$ will be modified to

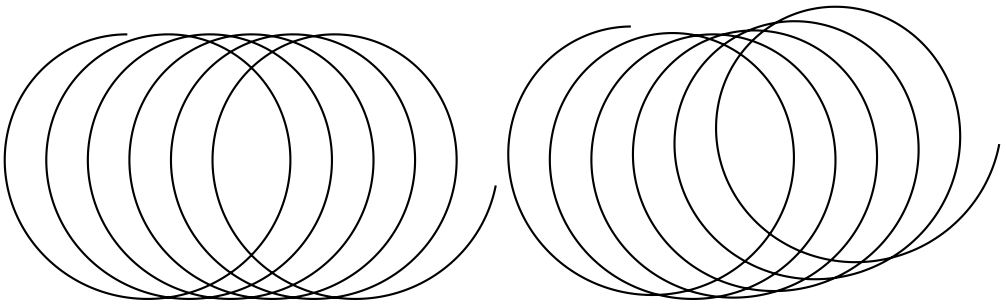
$$|\frac{V}{F} + (2m+1-\varsigma)| + |\nabla \frac{V}{F}| \asymp 1 \qquad \text{in } B(0,1)\ \forall m \in \mathbb{Z}^+; \tag{16.1.7}'$$

$$|\frac{V}{F} + (2m+1-\varsigma)| + |\nabla \frac{V}{F}| \le \epsilon_0 \implies |\det \mathsf{Hess}\, \frac{V}{F}| \ge \epsilon_0, \tag{16.1.8}'$$

$$|\frac{V}{F} + (2m+1-\varsigma)| + |\nabla \frac{V}{F}| \le \epsilon_0 \implies \det \mathsf{Hess}\, \frac{V}{F} \ge \epsilon_0 \tag{16.1.8}^{+\prime}$$

respectively.

Non-degeneracy assumptions (16.1.7)–(16.1.8)$^+$ eliminate periodicity at least as $\mu h \ll 1$ but there are plenty of loops as on figures 16.1(a) and 16.1(b) and we know that loops play the same important role in the



(a) prolate cycloid (b) perturbed prolate cycloid

Figure 16.1: Classical trajectories

pointwise asymptotics (of $e(x,x,0)$) as periodic trajectories play in the local asymptotics (of $\int e(x,x,0)\psi(x)\,dx$). There also will be many paths from x to y and back and these paths play important role in the asymptotics of expression (16.1.1).

Fortunately, even if there are plenty of looping points they are rather exceptional on the classical trajectory. Surely, one needs to remember that if we study $e(x,x,0)$ point x is fixed but direction ξ varies and thus trajectory[1)] moves so it passes through x and remains on the energy level 0 (see figures 16.2(a)–16.2(b) below). However for the geometrical simplicity we will sometimes talk about x moving along trajectory. As we study expression (16.1.1) x,y both move.

In fact, situation is more complicated than this due to the uncertainty principle: as μ increases not cyclotrons becomes smaller and tighter but the lines retain their width and become wider in comparison with the cyclotron, and symbolically we can consider figures 16.3(a)–16.3(c) below.

We also generalize Section 6.4 and we consider expression (16.1.1). It follows from Section 6.4 that $\mathsf{I} = \mathcal{I}^{\mathsf{W}} + O(h^{1-d-\kappa})$ as $\mu = O(1)$ with $\mathcal{I}^{\mathsf{W}}$ defined by the same formula but with $e(x,y,\tau)$ replaced by

$$e_y^{\mathsf{W}}(x,y,\tau) \coloneqq (2\pi h)^{-d} \int_{g(y,\xi)\le V(y)+\tau} e^{ih^{-1}\langle x-y,\xi\rangle}\,d\xi. \tag{16.1.9}$$

Then the standard rescaling technique implies the same asymptotics but with the remainder estimate $O(\mu h^{1-d-\kappa})$ provided $1 \le \mu = o(h^{-1})$.

[1)] Even if one calls them trajectories they are only *x-projections* of actual trajectories which live in (x,ξ)-space.

Let $d = 2$. Then *in the general case* it is the best remainder estimate possible while $O(\mu h^{-1})$ is the best possible remainder estimate for

$$\int e(x,x,\tau)\psi(x)\,dx; \tag{16.1.10}$$

recall that this estimate cannot be improved in the case of constant magnetic field F and $g^{jk} = \mathsf{const}$, $V = \mathsf{const}$.

On the other hand, *in the generic case* the remainder estimate for (16.1.10) is $o(\mu^{-1}h^{-1})$ and the principal part is $\int e^{\mathsf{MW}}(x,\tau)\psi(x)\,dx$ if F does not vanish and $\mu \le h^{-1}$.

Meanwhile for $d = 3$ the remainder estimate for I' is $O\big(h^{-2} + \mu h^{-1-\delta}\big)$ and under very mild non-degeneracy assumptions it is $O\big(h^{-2} + \mu h^{-1}\big)$.

Therefore our purpose is to get the sharper remainder estimate for (16.1.1) under the same non-degeneracy assumptions. This is a very daunting task since for (16.1.10) *periodic trajectories* were the main source of trouble and they were broken in the generic case; for (16.1.1) *loops* are also the source of trouble, and in the generic case former periodic trajectories generate a lot of loops.

It is rather easy to derive asymptotics with the sharp remainder estimate $O(\mu^{-1}h^{-1-\kappa})$ as $d = 2$ and $O(h^{-2})$ as $d = 3$ but with the principal part given by very implicit Tauberian formula when I^{T} is defined by (16.1.1) but with $e(x,y,\tau)$ replaced by its Tauberian approximation

$$e^{\mathsf{T}}(x,y,\tau) \coloneqq h^{-1}\int_{-\infty}^{\tau} F_{t\to h^{-1}\tau}\big(\bar{\chi}_T(t)u(x,y,t)\big)\,d\tau; \tag{16.1.11}$$

such remainder estimate is a rather easy corollary of the results of Chapter 13. However deriving of an asymptotics with the sharp remainder estimate and rather explicit principal part is much more difficult.

Plan of the chapter

In Sections 16.2 and 16.6 we consider respectively 2D- and 3D toy-models with the constant g^{jk} and F and linear V and derive representations of solutions and pointwise asymptotics as magnetic field is weak enough. We also derive asymptotics of expression (16.1.10) with $\psi(x)$ replaced by $\psi_\gamma(x)$ which is $\psi(x)$ scaled with a spatial scaling parameter γ; these asymptotics are instrumental in the asymptotics of (16.1.2) and estimates of (16.1.4).

In Sections 16.3 and 16.7 we prove the same asymptotics respectively for the general 2D- and 3D-operators; as magnetic field is strong we use explicit expressions for the toy-model operators as approximations.

In Sections 16.4 and 16.8 we prove important preliminary results and derive sharp asymptotics for expression (16.1.1) but with the Tauberian principal part.

In Sections 16.5 and 16.9 we pass from the Tauberian approximations to Weyl, magnetic Weyl or toy-model approximations.

Finally, in Section 16.10 we consider estimates of (16.1.4) with different approximations, leaving to the reader very similar asymptotics of expression (16.1.2).

The results of this Chapter are not always sharp or very explicit, but could be made either sharp or completely explicit.

16.2 Pointwise Asymptotics: 2D Toy-Model

16.2.1 Toy-Model in 2D: Propagator

Consider the *toy-model operator*

$$A = \bar{A} := h^2 D_1^2 + (hD_2 - \mu x_1)^2 + 2\alpha x_1. \tag{16.2.1}$$

We are interested in the Schwartz kernel $U(x,y,t)$ of the propagator $e^{ih^{-1}tA}$. Making h-Fourier transform with respect to $x_2 \mapsto \xi_2$ and rescaling $x_1 \mapsto \mu x_1$, $t \mapsto \mu t$ we arrive to

$$U(x,y,t) = (2\pi h)^{-1}\mu \int \boldsymbol{u}(x_1,y_1;\eta,t) e^{ih^{-1}(x_2-y_2)\eta}\, d\eta \tag{16.2.2}$$

with $\boldsymbol{u}(x_1,y_1;\eta,t)$ the Schwartz kernel of $e^{i\hbar^{-1}t\boldsymbol{a}}$ with 1D-Harmonic oscillator

$$\boldsymbol{a} = \hbar^2 D_1^2 + (x_1-\eta)^2 + 2\alpha\mu^{-1}x_1 = \underbrace{\hbar^2 D_1^2 + (x_1 - \eta + \alpha\mu^{-1})^2}_{\bar{\boldsymbol{a}}} + \mu^{-1}\alpha \underbrace{(2\eta - \alpha\mu^{-1})}_{\zeta(\eta)}. \tag{16.2.3}$$

Recall that for the Harmonic oscillator $\boldsymbol{b} = D^2 + x^2$ the Schwartz kernel of $e^{it\boldsymbol{b}}$ [2)] is

$$(2\pi)^{-\frac{1}{2}} e^{\frac{i\pi}{4}\sigma(t)} |\sin(2t)|^{-\frac{1}{2}} \exp\Big(-\frac{i}{2}\big(\cot(2t)(x^2+y^2) - 2xy\csc(2t)\big)\Big) \tag{16.2.4}$$

[2)] This is a metaplectic operator, see Section 1.2.5 or H. ter Morsche and P. J. Oonincx [1].

where $\sigma(t)$ is 2π-periodic equal to $1,2,3,4$ at $(0,\frac{\pi}{2}),(\frac{\pi}{2},\pi),(\pi,\frac{3\pi}{2}),(\frac{3\pi}{2},2\pi)$ respectively. Therefore for Harmonic oscillator $\boldsymbol{b}'=\hbar^2D^2+x^2$ the Schwartz kernel of $e^{it\hbar^{-1}\boldsymbol{b}'}$ is

$$(2\pi\hbar)^{-\frac{1}{2}}e^{\frac{i\pi}{4}\sigma(t)}|\sin(2t)|^{-\frac{1}{2}}\times \exp\Big(-\frac{i}{2}\hbar^{-1}\big(\cot(2t)(x^2+y^2)-2xy\csc(2t)\big)\Big); \tag{16.2.5}$$

one can prove it easily by rescaling $x\mapsto x\hbar^{\frac{1}{2}}$.

For operator (16.2.3) the Schwartz kernel of $e^{it\hbar^{-1}\boldsymbol{a}}$ is obtained by plugging $x:=x_1-\eta+\alpha\mu^{-1}$, $y:=y_1-\eta+\alpha\mu^{-1}$ and multiplication by $\exp\big(i\hbar^{-1}\mu^{-1}\alpha\zeta(\eta)\big)$. Therefore, multiplying (16.2.5) by

$$(2\pi\hbar)^{-1}\mu^2\exp\Big(i\hbar^{-1}\big(\mu^{-1}t\alpha(2\eta-\alpha\mu^{-1})+(x_2-y_2)\eta\big)\Big) \tag{16.2.6}$$

and integrating by η we arrive to[3)]

$$U(x,y,t)=(2\pi\hbar)^{-\frac{3}{2}}\mu^2\int e^{\frac{i\pi}{4}\sigma(t)}|\sin(2t)|^{-\frac{1}{2}}e^{i\hbar^{-1}\bar{\varphi}(x,y,\eta,t)}\,d\eta \tag{16.2.7}$$

with

$$\begin{aligned}\bar{\varphi}:=-\frac{1}{2}\cot(2t)\big((x_1-\eta+\alpha\mu^{-1})^2+(y_1-\eta+\alpha\mu^{-1})^2\big)+\\ \csc(2t)(x_1-\eta+\alpha\mu^{-1})(y_1-\eta+\alpha\mu^{-1})+(x_2-y_2)\eta+\\ t\mu^{-1}\alpha(2\eta-\alpha\mu^{-1}).\end{aligned} \tag{16.2.8}$$

We can rewrite (16.2.7)–(16.2.8) after integration by η as

$$U(x,y,t)=i(4\pi\hbar)^{-1}\mu^2\csc(t)\,e^{i\hbar^{-1}\bar{\phi}(x,y,t)} \tag{16.2.9}$$

with

$$\begin{aligned}\bar{\phi}:=-\frac{1}{4}\cot(t)(x_1-y_1)^2\\ +\frac{1}{2}(x_1+y_1+2\alpha\mu^{-1})(x_2-y_2+2t\alpha\mu^{-1})-\frac{1}{4}\cot(t)(x_2-y_2+2t\alpha\mu^{-1})^2-t\alpha^2\mu^{-2};\end{aligned} \tag{16.2.10}$$

[3)] After rescaling $x\mapsto\mu x$, $y\mapsto\mu y$, $t\mapsto\mu t$, $\mu\mapsto 1$, $h\mapsto\hbar=\mu h$; however we treat $U(x,y,t)$ and $e(x,y,\tau)$ as functions, not as densities with respect to y and it leads to the factor μ^2.

here the critical point with respect to η is

$$\eta = \frac{1}{2}(x_1 + y_1 + 2\alpha\mu^{-1}) - \frac{1}{2}\cot(t)(x_2 - y_2 + 2t\alpha\mu^{-1}). \tag{16.2.11}$$

So, we have proven

Proposition 16.2.1. *For the toy-model operator* (16.2.1) [3] *in* $\mathbb{R}^2$ *the Schwartz kernel* $\bar{U}(x,y,t)$ *of* $e^{ih^{-1}t\bar{A}}$ *is given by* (16.2.9)–(16.2.11) *and the Schwartz kernel* $e(x,y,\tau)$ *of the spectral projector is given by*

$$\partial_\tau e(x,y,\tau) = (2\pi\hbar)^{-1} F_{t\to\hbar^{-1}\tau} U(x,y,.) = (2\pi\hbar)^{-1}\int e^{-i\hbar^{-1}t\tau'} U(x,y,t)\,dt \tag{16.2.12}$$

and

$$e(x,y,\tau) = (2\pi\hbar)^{-1}\int^\tau \Big(\int e^{-i\hbar^{-1}t\tau'} U(x,y,t)\,dt\Big)\,d\tau = (2\pi)^{-1}\int (-it)^{-1} e^{-i\hbar^{-1}t\tau} U(x,y,t)\,dt \tag{16.2.13}$$

with the last integral taken in the sense of the principal (vrai) value at 0.

In the case of the stronger magnetic field we will need an alternative representation. Starting from formula for 1D-harmonic oscillator

$$e(x_1,y_1,\tau) = \sum_{m\in\mathbb{Z}^+} \upsilon_m(x_1)\upsilon_m(y_1)\theta\big(\tau - (2m+1)\big) \tag{16.2.14}$$

with Hermite functions υ_m we after the same rescaling and transition to the toy-model as before arrive to

Proposition 16.2.2. *For the toy-model operator* (16.2.1) [3] *in* $\mathbb{R}^2$ *the Schwartz kernel of the spectral projector is defined by*

$$e(x,y,\tau) = (2\pi)^{-1}\mu^2\hbar^{-1}\sum_{m\in\mathbb{Z}^+}\int \upsilon_m\big(\eta + \hbar^{-\frac12}(x_1-y_1)\big)\upsilon_m\big(\eta - \hbar^{-\frac12}(x_1-y_1)\big)\times \theta\Big(\tau - \alpha\mu^{-1}(x_1+y_1) - 2\alpha\mu^{-1}\hbar^{\frac12}\eta - \alpha^2\mu^{-2} - (2m+1)\hbar\Big) e^{i\hbar^{-\frac12}(x_2-y_2)\eta}\,d\eta \tag{16.2.15}$$

and thus

$$\partial_\tau e(x,y,\tau) = (4\pi\alpha)^{-1}\mu^3\hbar^{-\frac{3}{2}}\sum_{m\in\mathbb{Z}^+}\int \upsilon_m\bigl(\eta+\hbar^{-\frac{1}{2}}(x_1-y_1)\bigr)\upsilon_m\bigl(\eta-\hbar^{-\frac{1}{2}}(x_1-y_1)\bigr)\times e^{i\hbar^{-\frac{1}{2}}(x_2-y_2)\eta}\Big|_{\eta=\eta_m} \tag{16.2.16}$$

with

$$\eta_m := (2\alpha)^{-1}\mu\hbar^{-\frac{1}{2}}\bigl(\tau-\alpha\mu^{-1}(x_1+y_1)-\alpha^2\mu^{-2}-(2m+1)\hbar\bigr). \tag{16.2.17}$$

16.2.2 Tauberian Estimate

Consider now toy-model (16.2.1) in $B(0,1)\subset X\subset\mathbb{R}^2$. As dynamics (classical or microlocal) starts in $B(0,\epsilon)$ it is confined to $B(0,1)$ for $|t|\le T^*=\epsilon\mu$ and thus in this time interval we can use formulae of the previous subsection (modulo negligible term).

Preparatory Estimate

Let us rescale[3)] and set $x=y=0$ in (16.2.9)–(16.2.11):

$$U(0,0,t)\equiv i(4\pi\hbar)^{-1}\mu^2\csc(t)\,e^{i\hbar^{-1}\bar\phi(t)} \tag{16.2.18}$$

with

$$\bar\phi(t) := -t^2\alpha^2\mu^{-2}\cot(t)-\alpha^2\mu^{-2}t; \tag{16.2.19}$$

$$\eta = -t\alpha\mu^{-1}\cot(t)+\alpha\mu^{-1}. \tag{16.2.20}$$

We are interested in

$$F_{t\to\hbar^{-1}\tau}\bar\chi_T(t)U(0,0,t) \tag{16.2.21}$$

with $T=T^*$. Plugging (16.2.18) we arrive to

$$F_{t\to\hbar^{-1}\tau}\bar\chi_T(t)U(0,0,t)\equiv i(4\pi\hbar)^{-1}\mu^2\int \bar\chi_T(t)\csc(t)e^{i\hbar^{-1}(\bar\phi(t)-t\tau)}\,dt = \tag{16.2.22}$$

$$(4\pi\hbar)^{-1}\mu^2\int \bar\chi_T(t)\csc(t)\exp\Bigl(i\hbar^{-1}\bigl(-t^2\alpha^2\mu^{-2}\cot(t)-t\mu^{-2}\alpha^2-t\tau\bigr)\Bigr)\,dt.$$

Applying stationary phase to

$$\varphi(t) \coloneqq -t^2\mu^{-2}\alpha^2\cot(t) - t\mu^{-2}\alpha^2 - t\tau \tag{16.2.23}$$

we get

$$\mu^{-2}\alpha^2(t^2 - t\sin(2t)) = (\tau + \mu^{-2}\alpha^2)\sin^2(t). \tag{16.2.24}$$

Remark 16.2.3. Consider corresponding classical trajectory:

$$x_1 = \rho\cos(2s) + \eta - \alpha\mu^{-1}, \qquad \xi_1 = \rho\sin(2s) + \alpha\mu^{-1}s \tag{16.2.25}$$

and there is a self-intersection iff

$$\begin{cases} \cos(2s_1) = \cos(2s_2), \\ -\rho\sin(2s_1) + 2\alpha\mu^{-1}s_1 = -\rho\sin(2s_2) + 2\alpha\mu^{-1}s_2; \end{cases} \tag{16.2.26}$$

then $s_2 + s_1 = \pi k$ with $k \in \mathbb{Z}$ (because we cannot fulfill the second equation as $s_1 \neq s_2$ and $s_2 - s_1 = \pi k$); then $2s_1 = \pi k - t$, $2s_2 = \pi k + t$ and the second equation is $\rho\sin(\pi k - t) = -\alpha\mu^{-1}t$ or, equivalently, $\rho = -\alpha\mu^{-1}t\csc(\pi k - t)$.

Further, we need to satisfy $\rho\cos(\pi k - t) + \eta - \alpha\mu^{-1} = 0$ (as $x_1 = 0$ is a level of intersection) and $(\eta - \alpha\mu^{-1})^2 + 2\alpha\mu^{-1}(\eta - \alpha\mu^{-1}) + \alpha^2\mu^{-2} = \tau$ (to be on the energy level τ) and then $\rho^2\cos^2 t - 2\alpha\mu^{-1}\rho\cos(\pi k - t) = \tau - \alpha^2\mu^{-2}$.

Plugging $\rho = -\alpha\mu^{-1}t\csc(\pi k - t)$ we conclude that this equation becomes (16.2.24). We can find η to satisfy condition $\rho\cos(2s_1) + \eta - \alpha\mu^{-1} = 0$ (as $(0, .)$ is a point of intersection, or equivalently, $\eta = -\alpha\mu^{-1}t\cot(t) + \alpha\mu^{-1}$.

We need to justify the stationary phase method. Note first that the spacing between successive stationary points is $\sim \pi$. Therefore

$$t_k = -t_{-k}, \quad t_k \sim \pi k, \quad \sin(t_k) \sim \alpha\mu^{-1}\tau^{-\frac{1}{2}}\pi k \tag{16.2.27}$$

and

(16.2.28) The number of stationary points is $\sim 2\pi^{-1}|\alpha|^{-1}\mu\tau^{-\frac{1}{2}}$.

Recall that

$$\varphi'(t) = \alpha^2\mu^{-2}\big(t^2\csc^2(t) - 2t\cot(t) + 1\big) - \tau \tag{16.2.29}$$

and

$$\varphi''(t) = -2\alpha^2\mu^{-2}\csc(t)\Big(t^2\csc^2(t)\cos(t) - 2t\csc(t) + \cos(t)\Big); \tag{16.2.30}$$

therefore

(16.2.31) Let $k \neq 0$; then $\varphi''(t_k) \asymp -2\tau \cot(t_k)$ as $|\cos(t_k)| \geq C_0\varepsilon$ and moreover $\varphi''(t_k) \sim -2\tau \cot(t_k)$ as $|\cos(t_k)| \gg \varepsilon$.

Therefore as $|\sin(t_k)| \geq \epsilon$, $|\cos(t_k)| \geq \epsilon$ stationary points are non-degenerate and we can use the stationary phase method.

In the *near-pole zone* $\{|\sin(t)| \leq \epsilon\}$ we need to remember about singularity as $\sin(t) = 0$; so we need to introduce a scale $\ell \asymp |\sin(t)|$ and the stationary phase method is expected to work only as $|\varphi''|\ell^2 \geq C_0\hbar$ or equivalently $|\sin(t_k)| \geq C_0\hbar$ or, finally

$$|k| \geq \bar{k} := C_0\varepsilon^{-1}\hbar \tag{16.2.32}$$

with

$$\varepsilon := \alpha\mu^{-1}(\tau + \alpha^2\mu^{-2})^{-\frac{1}{2}}. \tag{16.2.33}$$

Summarizing, we arrive to

(16.2.34) Condition (16.2.32) is fulfilled for all $k \neq 0$ as $\varepsilon \geq C_0\hbar$; otherwise stationary phase method fails for $|k| \leq \bar{k}$. In particular, $\bar{k} = C_0\mu^2 h$ as $|\alpha| \asymp 1$, $\tau \asymp 1$ and condition (16.2.32) is fulfilled for all $k \neq 0$ as $\mu \leq \epsilon_0 h^{-\frac{1}{2}}$.

Meanwhile, there is no singularity in the *near-equator zone* $\{|\cos(t)| \leq \epsilon\}$ but there is a degeneration as $\cos(t) = 0$ and therefore we need to introduce a a scale $\ell \asymp |\cos(t)|$ and the stationary phase method is expected to work only as $|\varphi''|\ell^2 \geq C_0\hbar$ or equivalently $|\cos(t)| \geq C_0 \max(\varepsilon, \hbar^{\frac{1}{3}})$.

Definition 16.2.4. (i) $\{|\sin(t)| \leq \max(\epsilon_0\varepsilon, C_0\hbar)\}$ is a *near-pole singular zone*.

(ii) $\{|\cos(t)| \leq C_0 \max(\varepsilon, \hbar^{\frac{1}{3}})\}$ is a *near-equator singular zone*.

(iii) $\{|\sin(t)| \leq \max(\epsilon_0\varepsilon, C_0\hbar), |\cos(t)| \geq C_0 \max(\varepsilon, \hbar^{\frac{1}{3}})\}$ is a *regular zone*.

We apply the stationary phase method in the regular zone and justify it and we estimate contributions of the singular zones.

Note that the simple integration by parts brings factors $\hbar/(\varphi' \sin(t))$ and $\hbar\varphi''/(\varphi')^2$ and contribution of zones where these expressions are less than h^δ are negligible. However at this moment we want just to estimate.

Remark 16.2.5. As long as $\mu h \lesssim 1$ we will assume that $\tau \asymp 1$; while we can always achieve $\alpha \asymp 1$ by rescaling $x \mapsto |\alpha| x$, $\mu \mapsto \mu|\alpha|^{-1}$, $\hbar \mapsto \hbar$ we will need many intermediate results and it is simpler to deal with the general case $|\alpha| \le 1$ from the very beginning. Without any loss of the generality one can assume that

$$0 < \alpha \le 1 \qquad (\implies \varepsilon \le \mu^{-1}). \tag{16.2.35}$$

First, consider zone

$$\bigl\{t : \ |\sin t| \le (1-\epsilon_0)\varepsilon|t|, \ |t| \ge \epsilon_0\bigr\}. \tag{16.2.36}$$

Then $|\varphi'| \asymp \varepsilon^2 t^2 |\sin(t)|^{-2}$ and $|\varphi''| \lesssim \varepsilon^2 t^2 |\sin(t)|^{-3}$ and both factors $|\hbar/\varphi' \sin(t)|$ and $|\hbar\varphi''/(\varphi')^2|$ do not exceed $\hbar|\sin(t)|/\varepsilon^2 t^2 \le \hbar/\varepsilon|t|$ and are less than 1 as $\hbar|\sin(t)| \le \varepsilon^2 t^2$. This always holds in zone (16.2.36) if $\hbar \le \varepsilon$.

Then multiple integration by parts brings factor $C(\hbar|\sin(t)|/\varepsilon^2 t^2)^l$ with an arbitrarily large exponent l and contribution of k-th "tick"[4] to (16.2.22) does not exceed

$$C\mu^2\hbar^{-1}(\hbar|\sin(t)|/\varepsilon^2 t^2)^l |\sin(t)|^{-1}.$$

Therefore summation with respect to t along zone (16.2.36) intersected with k-th tick returns $C\mu h^{-1}(\hbar/\varepsilon|k|)^l$ as $|k| \ge \bar{k}$ with $\bar{k}$ defined by (16.2.32); then summation by k results in $C\mu h^{-1}(\hbar/\varepsilon)^l$ as $\hbar \le \varepsilon$ ($1 \le |k|$) and $C\mu h^{-1}\bar{k} \asymp C\mu^2\varepsilon^{-1}$ as $\hbar \ge \varepsilon$ ($\bar{k} \le |k|$).

In the latter case as $1 \le |k| \le \bar{k}$ we integrate by parts only as $|\sin(t)| \le \hbar^{-1}\varepsilon^2 t^2$ and contribution of k-th tick to (16.2.22) does not exceed $C\mu h^{-1}\log(\bar{k}/|k|)$ where logarithmic factor is an integral of $|\csc(t)|$ along k-th tick intersected with $\{t\colon \hbar^{-1}\varepsilon^2 t^2 \le |\sin(t)| \le \varepsilon|t|\}$. Then summation with respect to $1 \le |k| \le \bar{k}$ returns $O(\mu h^{-1}\bar{k})$ again.

Therefore we arrive to

(16.2.37) Contribution of zone (16.2.36) to expression (16.2.22) does not exceed $C\mu h^{-1}(\hbar/\varepsilon)^l$ as $\hbar \le \varepsilon$ and $C\mu^2\varepsilon^{-1}$ as $\hbar \ge \varepsilon$.

In the same time in the zone

$$\bigl\{t : \ |\sin(t)| \ge (1+\epsilon_0)\varepsilon|t|, \ |t| \ge \epsilon_0\bigr\} \tag{16.2.38}$$

[4] I.e. interval $\{t\colon |t - t_k| \le \pi - \epsilon_0\}$.

$|\varphi'| \asymp 1$ and $|\varphi''| \lesssim |\sin(t)|^{-1}$ and both factors do not exceed $\hbar/|\sin(t)|$. Again summation with respect to t along k-th tick intersected with zone (16.2.38) returns $C\mu h^{-1}(\hbar/\varepsilon|k|)^{\prime}$ as $|k| \ge \bar{k}$ and summation by k results in $C\mu h^{-1}(\hbar/\varepsilon)^{\prime}$ as $\hbar \le \varepsilon$ ($1 \le |k|$) and $\mu^2\varepsilon^{-1}$ as $\hbar \ge \varepsilon$ ($\bar{k} \le |k|$).

In the latter case as $|k| \le \bar{k}$ we do not integrate by parts and contribution of k-th tick to (16.2.22) does not exceed $C\mu h^{-1}$; summation with respect to $1 \le |k| \le \bar{k}$ returns $O(\mu^2\varepsilon^{-1})$. Therefore

(16.2.39) Contribution of zone (16.2.38) to expression (16.2.22) does not exceed $C\mu h^{-1}(\hbar/\varepsilon)^{\prime}$ as $\hbar \le \varepsilon$ and $C\mu^2\varepsilon^{-1}$ as $\hbar \ge \varepsilon$.

Note that as $\hbar \le \epsilon\varepsilon$ these considered zones (16.2.36) and (16.2.38) cover completely the singular near-pole zone. On the other hand, as $\hbar \ge \epsilon\varepsilon$ we did not integrate by parts in the second zone as $|k| \le \bar{k}$ and therefore it can be extended to $\{t\colon |\sin(t)| \ge \epsilon_0\varepsilon|t|\}$ and we arrive to

(16.2.40) Contribution of singular near-pole zone to expression (16.2.22) does not exceed $C\mu h^{-1}(\hbar/\varepsilon)^{\prime}$ as $\hbar \le \varepsilon$ and $C\mu^2\varepsilon^{-1}$ as $\hbar \ge \varepsilon$.

Claims (16.2.37), (16.2.39), (16.2.40) imply that as $|\cos(t_k)| \ge \epsilon$ we can apply the stationary phase method.

Then contribution of k-th stationary point to (16.2.22) does not exceed $C\mu h^{-1}(\hbar/\varepsilon|k|)^{\frac{1}{2}}$ and summation with respect to $k\colon \max(1,\bar{k}) \le |k| \le C_0\varepsilon^{-1}$ returns $C\mu h^{-1}(\hbar/\varepsilon|k|)^{\frac{1}{2}} \times |k|$ calculated as $|k| = \varepsilon^{-1}$ i.e. $C\mu^{\frac{3}{2}}\varepsilon^{-1}h^{-\frac{1}{2}}$ which is larger than what we got in (16.2.37)–(16.2.40) thus resulting in

(16.2.41) Contribution of zone $\{t\colon |\cos(t)| \ge \epsilon_0,\ |t| \ge \epsilon_0\}$ to expression (16.2.22) does not exceed $C\mu^{\frac{3}{2}}\varepsilon^{-1}h^{-\frac{1}{2}}$ as $1 \le \mu \le h^{-1}$.

Consider now the part of the near-equator zone $\{|\cos(t)| \le \epsilon_0,\ |t| \ge \epsilon_0\}$ not covered by (16.2.37), (16.2.39); there $(1-\epsilon)t^* \le |t| \le (1+\epsilon)t^*$ where

(16.2.42) $t^* = (k^* + \frac{1}{2})\pi$ is the closest to ε^{-1} number of the form $(k+\frac{1}{2})\pi$.

Plugging $t = t^* - s$ with $|s| \le \epsilon t^*$, $|\sin(s)| \le \epsilon$ we find that

$$\tau^{-1}\varphi'(t) = -\varepsilon^2 s(2t^* - s) + \sin^2(s) + O(\varepsilon^2) + O(\varepsilon|\sin(s)|). \tag{16.2.43}$$

Consider first zone

$$\bigl\{s\colon C_0 \le |s| \le \epsilon\varepsilon^{-1},\ |\sin(s)|^2 \le 2(1-\epsilon_1)\varepsilon|s|\bigr\}$$

where $|\varphi'| \gtrsim \varepsilon|s|$. Integrating by parts we acquire factor $(\hbar/\ell|\varphi'|) \asymp (\hbar/(\varepsilon|s|)^{\frac{3}{2}})$ because $\ell = (\varepsilon|s|)^{\frac{1}{2}}$ is the scale; thus we integrate multiple times by parts as $|s| \gtrsim \varepsilon^{-1}\hbar^{2/3}$; so contribution of k-th tick to (16.2.22) does not exceed

$$C\mu\varepsilon^{\frac{1}{2}}h^{-1}(\hbar^{2/3}/\varepsilon|k-k^*|)^{\frac{3}{2}l}|k-k^*|^{\frac{1}{2}}.$$

Then, as $\varepsilon \ge \hbar^{\frac{2}{3}}$ summation with respect to $k : |k-k^*| \ge C_0$ returns $C\mu\varepsilon^{\frac{1}{2}}h^{-1}(\hbar^{\frac{2}{3}}/\varepsilon)^l$. On the other hand, as $\varepsilon \le \hbar^{\frac{2}{3}}$ summation with respect to $k : |k-k^*| \ge \varepsilon^{-1}\hbar^{\frac{2}{3}}$ returns $C\mu\varepsilon^{\frac{1}{2}}h^{-1} \times \hbar^{\frac{2}{3}}\varepsilon^{-1} = C\mu^{\frac{5}{3}}\varepsilon^{-\frac{1}{2}}h^{-\frac{1}{3}}$.

Further, the contribution of the zone

$$\{s\colon C_0 \le |s| \le \epsilon\varepsilon^{-1},\ |\sin(s)|^2 \ge 2(1+\epsilon_1)\varepsilon|s|\}$$

is estimated in the same way. Finally, as $s < 0$ both terms in the right-hand expression of (16.2.43) have the same sign and we can extend zones in question to cover $\{s\colon \sin^2(s) \asymp \varepsilon|s|\}$ as well and we arrive to

(16.2.44) Contribution to (16.2.22) of the zone $\{s\colon |\sin(s)| \le \epsilon,\ |s| \le \epsilon\varepsilon^{-1}\}$ without subzone

$$\{s\colon s \ge C_0\max(1, \varepsilon^{-1}\hbar^{\frac{2}{3}}),\ (2-\epsilon)\varepsilon s \le \sin^2(s) \le (2+\epsilon)\varepsilon s\} \tag{16.2.45}$$

and the near-equator singular zone

$$\{s\colon |s| \le C_0\max(1, \varepsilon^{-1}\hbar^{\frac{2}{3}}),\ |\sin(s)| \le C_0\max(\varepsilon, \hbar^{\frac{1}{3}})\} \tag{16.2.46}$$

does not exceed $C\mu\varepsilon^{\frac{1}{2}}h^{-1}(\hbar^{\frac{2}{3}}/\varepsilon)^l$ as $\varepsilon \ge \hbar^{\frac{2}{3}}$ and $C\mu^{\frac{5}{3}}\varepsilon^{-\frac{1}{2}}h^{-\frac{1}{3}}$ as $\varepsilon \le \hbar^{\frac{2}{3}}$.

Obviously,

(16.2.47) Contribution to expression (16.2.22) of zone (16.2.46) does not exceed $C\mu h^{-1} \times \max(1, \varepsilon^{-1}\hbar^{\frac{2}{3}}) \times \max(\varepsilon, \hbar^{\frac{1}{3}}) = C\mu h^{-1} \times \max(\varepsilon, \varepsilon^{-1}\hbar, \hbar^{\frac{1}{3}})$.

Finally, in zone (16.2.45) we can use the stationary phase method. Recall that $\varphi'' \asymp \sin(s) \asymp (\varepsilon|k-k^*|)^{\frac{1}{2}}$; therefore the contribution of k-th tick does not exceed $C\mu h^{-1}(\hbar/|\sin(s)|)^{\frac{1}{2}} \asymp C\mu h^{-1}(\hbar/(\varepsilon|k-k^*|)^{\frac{1}{2}})^{\frac{1}{2}}$ and summation with respect to $k :\ |k-k^*| \le \varepsilon^{-1}$ returns $C\mu^{\frac{3}{2}}\varepsilon^{-1}h^{-\frac{1}{2}}$.

Combining all these estimates we arrive to estimate (16.2.49) below; adding

$$|F_{t\to\hbar^{-1}\tau}(\bar\chi\bar{\bar\tau}(t)U(0,0,t))| \le C\mu h^{-1} \tag{16.2.48}$$

which is the standard semiclassical estimate $O(\hbar^{-1})$ with an extra factor μ^2 due to rescaling we arrive to estimate (16.2.50):

Proposition 16.2.6. *For the toy-model operator* (16.2.1) *with* $\tau \asymp 1$, $\varepsilon \le \mu^{-1}$, $\mu \le h^{-1}$ *in* $\mathbb{R}^2$

$$|F_{t\to\hbar^{-1}\tau}\big((\bar\chi_T(t) - \bar\chi_{\bar T})U(0,0,t)\big)| \le C\mu^{\frac{3}{2}}\varepsilon^{-1}h^{-\frac{1}{2}} \tag{16.2.49}$$

as $\bar T = \epsilon_0 \le T \le \infty$ *and*

$$|F_{t\to\hbar^{-1}\tau}\big(\bar\chi_T(t)U(0,0,t)\big)| \le C\mu h^{-1} + C\mu^{\frac{3}{2}}\varepsilon^{-1}h^{-\frac{1}{2}}. \tag{16.2.50}$$

Tauberian Estimate

Remark 16.2.7. (i) Recall that for the toy-model operator (16.2.1) in domain X, $B(0,\ell) \subset X \subset \mathbb{R}^2$ with $\ell \ge C_0\mu^{-1}$, fundamental solution U coincides modulo negligible with one in $\mathbb{R}^2$ as $T \le T^* = \epsilon_0\mu\varepsilon^{-1}\ell$.

(ii) We have $T^* = \epsilon_0\mu\varepsilon^{-1}\ell$ rather than $T^* = \epsilon_0\varepsilon^{-1}\ell$ as in Chapter 13 only because we rescaled $t \mapsto \mu t$; without this rescaling we would have $F_{t\to h^{-1}\tau}$ rather than $F_{t\to\hbar^{-1}\tau}$ and the right-hand expressions (16.2.49), (16.2.50) should be multiplied by μ^{-1}.

Due to above we arrive to the Statement (i) of Proposition 16.2.8 below; then the standard Tauberian arguments yield that the Tauberian error does not exceed the right-hand expression of (16.2.50) divided by T^* and we derive Statement (ii):

Proposition 16.2.8. *For a self-adjoint operator in* $\mathscr{L}^2(X)$ *in the domain* X, $B(0,\ell) \subset X \subset \mathbb{R}^2$, $\ell \ge C_0\mu^{-1}$, *coinciding in* $B(0,\ell)$ *with the toy-model operator* (16.2.1) *with* $\tau \asymp 1$, $\varepsilon \le \mu^{-1}$, $\mu \le h^{-1}$,

(i) Estimates (16.2.49), (16.2.50) *hold as* $T \le T^* = \epsilon_0\mu\varepsilon^{-1}\ell$.

(ii) Tauberian estimate

$$|e(0,0,\tau) - e^{\mathsf{T}}(0,0,\tau)| \le C\big(\mu h^{-1} + \mu^{\frac{3}{2}}\varepsilon^{-1}h^{-\frac{1}{2}}\big)\mu^{-1}\varepsilon\ell^{-1} = C\big(\varepsilon h^{-1} + \mu^{\frac{1}{2}}h^{-\frac{1}{2}}\big)\ell^{-1} \tag{16.2.51}$$

holds with the Tauberian expression

$$e^{\mathsf{T}}(x,y,\tau) \coloneqq \hbar^{-1}\int_{-\infty}^{\tau}\Big(F_{t\to\hbar^{-1}\tau}\bar\chi_T(t)U(x,y,t)\Big)\,d\tau \tag{16.2.52}$$

with any $T \in (C_0\varepsilon^{-1}, \epsilon_0\mu\varepsilon^{-1}\ell)$.

Micro-Averaging

Note that the second terms in the right-hand expressions of (16.2.50), (16.2.51) are due to the loops and according to Chapter 13 these expressions would not appear if instead of $\Gamma_x e(.,.,\tau)$ we looked at $\Gamma\big(e(.,.,\tau)\psi\big)$ with the uniformly smooth function ψ. Now we want to consider $\Gamma\big(e(.,.,\tau)\psi_\gamma\big)$ where ψ_γ is γ-admissible function and $\gamma \le \ell$.

So, let us consider $\Gamma\big(e(.,.,\tau)\psi_\gamma\big)$. First of all note that all the above estimates will acquire factor γ^2 due to the integration, but there is more: integrating by parts with respect to x_1 and remembering that due to rescaling $x \mapsto \mu x$ we also rescale $\gamma \mapsto \mu\gamma$ we conclude that the contribution of k-th tick acquires factor $(\hbar/\varepsilon|k|\mu\gamma)^l = (h/\varepsilon\gamma|k|)^l$ as

$$|k| \ge \hat{k}(\gamma) \coloneqq h/\varepsilon\gamma. \tag{16.2.53}$$

This factor "forces" $|k| = \max(\hat{k}(\gamma), 1)$ in the estimate of the Fourier transform

$$F_{t\to\hbar^{-1}}\Gamma(U\psi_\gamma) \tag{16.2.54}$$

and there are four cases:

(a) $\gamma \ge h/\varepsilon$; then $\hat{k}(\gamma) = 1$ and expression (16.2.54) does not exceed $C\mu h^{-1}\min\big((\hbar/\varepsilon)^{\frac12}, 1\big) \times (h/\varepsilon\gamma)^l\gamma^2$.

(b) $h/\varepsilon \ge \gamma \ge \mu^{-1}$. Then $1 \le \hat{k}(\gamma) \le \bar{k} = \hbar/\varepsilon$ and (16.2.54) does not exceed $C\mu h^{-1}\hat{k}(\gamma)\gamma^2 = C\mu\varepsilon^{-1}\gamma$.

(c) $\min(h/\varepsilon, \mu^{-1}) \ge \gamma \ge h$. Then $\bar{k} \le \hat{k}(\gamma) \le \varepsilon^{-1}$ and (16.2.54) does not exceed $C\mu h^{-1}(\hbar/\varepsilon)^{\frac12}|\hat{k}(\gamma)|^{\frac12}\gamma^2 = C\mu^{\frac32}\varepsilon^{-1}\gamma^{\frac32}$.

(d) $\gamma \le h$. Then $\hat{k}(\gamma) \ge \varepsilon^{-1}$ and micro-averaging brings no improvement and estimates (16.2.55)–(16.2.58) below are due to estimates (16.2.49)–(16.2.51) without micro-averaging.

Therefore we arrive to

Proposition 16.2.9. *For the toy-model operator* (16.2.1) *with* $\tau \asymp 1$, $\varepsilon \le \mu^{-1}$, $\mu \le h^{-1}$ *in* $\mathbb{R}^2$ *estimates*

$$\gamma^{-2}|F_{t\to\hbar^{-1}\tau}\big(\big(\bar{\chi}_\tau(t) - \bar{\chi}_{\bar{\tau}}\big)\Gamma(U\psi_\gamma)\big)| \le C\mu\varepsilon^{-1}R^{\mathsf{T}}(\gamma) \tag{16.2.55}$$

and

$$\gamma^{-2}|F_{t\to\hbar^{-1}\tau}\bigl(\bar\chi_T(t)\Gamma(U\psi_\gamma)\bigr)| \le C\mu h^{-1} + C\mu\varepsilon^{-1}R^{\mathsf T}(\gamma) \tag{16.2.56}$$

hold as $\bar T = \epsilon_0 \le T \le \infty$ *where*

$$R^{\mathsf T}(\gamma) = \begin{cases} \mu^{\frac12}h^{-\frac12} & \text{as } \gamma \le h,\\ \mu^{\frac12}\gamma^{-\frac12} & \text{as } \min(h/\varepsilon, \mu^{-1}) \ge \gamma \ge h,\\ \gamma^{-1} & \text{as } h\varepsilon^{-1} \ge \gamma \ge \mu^{-1},\\ \varepsilon h^{-1}\min\bigl((\hbar/\varepsilon)^{\frac12},1\bigr)\times (h/\varepsilon\gamma)^{l} & \text{as } \gamma \ge h\varepsilon^{-1}. \end{cases} \tag{16.2.57}$$

Then the standard Tauberian arguments imply

Corollary 16.2.10. *In the framework of Proposition 16.2.8*

$$\gamma^{-2}|\Gamma\Bigl(\bigl(e(.,.,\tau) - e^{\mathsf T}(.,.,\tau)\bigr)\psi_\gamma\Bigr)| \le C\bigl(\varepsilon h^{-1} + R^{\mathsf T}(\gamma)\bigr)\ell^{-1}. \tag{16.2.58}$$

16.2.3 Calculations

Correction Term

We want however more explicit expression for (16.2.52) which we rewrite as

$$\hbar^{-1}\int_{-\infty}^{\tau}\Bigl(F_{t\to\hbar^{-1}\tau}\bar\chi_{\bar T}(t)U(x,x,t)\Bigr)\,d\tau + \int_{-\infty}^{\tau}\Bigl(F_{t\to\hbar^{-1}\tau}(-it)^{-1}\bigl(\bar\chi_T(t) - \bar\chi_{\bar T}(t)\bigr)U(x,x,t)\Bigr)\,d\tau \tag{16.2.59}$$

and as $\mu \le h^{\delta-1}$ the first term modulo negligible is a Weyl expression $h^{-2}\mathcal{N}_x^{\mathsf W} := (4\pi)^{-1}h^{-2}(\tau - V(x))$ (provided $V(x) \le \tau - \epsilon$) and the second expression is a *correction term* $h^{-2}\mathcal{N}_{x,\mathsf{corr}}$ and our purpose is either to estimate or calculate it.

Let us apply the stationary phase method whenever in makes sense; otherwise we just temporarily skip the corresponding zone. Then the main part of the answer is

$$\sum_{k\in\mathfrak{Z}}\frac{1}{4\pi\sqrt{\pi}}\mu^2\hbar^{-\frac12}(t_k\sin(t_k))^{-1}|\varphi''(t_k)|^{-\frac12}\times \exp\Bigl(\frac{i\pi}{4}\operatorname{sgn}\varphi''(t_k) + i\hbar^{-1}\varphi(t_k)\Bigr) \tag{16.2.60}$$

with

$$\varphi''(t_k) \sim 2\tau \cot(t_k) \qquad \text{as } |\cos(t_k)| \ge C\varepsilon^{\frac12} \tag{16.2.61}$$

and

$$\mathfrak{Z} := \{k \ne 0 : \ |\sin(t_k)| \ge \hbar,\ |\cos(t_k)| \ge C \max(\varepsilon^{\frac12}, \hbar^{\frac13})\} \tag{16.2.62}$$

where dimmed restriction is just temporary.

Estimating Correction Term

First, let us estimate the correction term. As $|\sin(t_k)| \ge \hbar$, $|\cos(t_k)| \ge \epsilon$ we have $t_k \asymp k$, $\sin(t_k) \asymp \varepsilon k$, $\varphi'' \asymp \varepsilon^{-1}k^{-1}$ and the contribution of k-th tick does not exceed $C\mu^{\frac32}\varepsilon^{-\frac12}h^{-\frac12}|k|^{-\frac32}$.

In comparison with the previous subsection we acquired factor $|k|^{-1}$ due to the factor $(-it)^{-1}$ in (16.2.59) and it is a game-changer: summation with respect to k returns $C\mu^{\frac32}\varepsilon^{-\frac12}h^{-\frac12}|k|^{-\frac12}$ with the *smallest* possible k: as $\hbar \le \varepsilon$ we sum from $|k| = 1$ and get $C\mu^{\frac32}\varepsilon^{-\frac12}h^{-\frac12}$; as $\hbar \ge \varepsilon$ we sum from $|k| = \bar{k} := \hbar\varepsilon^{-1}$ and get $C\mu h^{-1}$ which is the trivial estimate.

Further, contribution of k-th tick to the error is $C\mu h^{-1} \times (\hbar/\varepsilon|k|)^{r+\frac12}|k|^{-1}$ if we use r-term stationary phase approximation. Then summation with respect to $k : |k| \ge 1$ returns $C\mu h^{-1}(\hbar/\varepsilon)^{r+\frac12}$ as $\hbar \le \varepsilon$.

On the other hand, as $\hbar \ge \varepsilon$ summation with respect to $k : |k| \ge \bar{k}$ returns $C\mu h^{-1}$ and thus this approximation does not make any sense. Therefore

(16.2.63) We assume until the end of this subsection that $\hbar \le \varepsilon$.

This assumption takes care of many problems: it eliminates the near-pole singular zone and away from the stationary points we can many times integrate by parts; one can see easily that contribution of zones (16.2.37) and (16.2.39) does not exceed $C\mu h^{-1}(\hbar/\varepsilon)^l$ with arbitrarily large l.

Therefore we need an additional analysis only in the the near-equator zone $\{t : \ |\cos(t)| \le \epsilon,\ |t - t^*| \le \epsilon\varepsilon^{-1}\}$. It is easy to estimate its contribution to the correction term: in comparison with the previous subsection we gain factor ε (from factor $(-it)^{-1}$) and we get $C\mu^2\varepsilon^{-\frac12}h^{-\frac12} \times \varepsilon = C\mu^2\varepsilon^{\frac12}h^{-\frac12}$ which is less than the contribution of the near-polar zone. Thus we arrive to

Proposition 16.2.11. *For a toy-model operator* (16.2.1) *with* $\tau \asymp 1$, $\varepsilon \le \mu^{-1}$, $\hbar \le \varepsilon$

(i) Correction term $h^{-2}\mathcal{N}_{x,\mathsf{corr}}$ *does not exceed* $C\mu^{\frac{3}{2}}\varepsilon^{-\frac{1}{2}}h^{-\frac{1}{2}}$ *and therefore in the framework of Proposition 16.2.8*

$$\mathsf{R}_x^{\mathsf{W}} := |e^{\mathsf{T}}(x,x,0) - h^{-2}\mathcal{N}_x^{\mathsf{W}}| \le C\mu^{-1}h^{-1} + C\mu^{\frac{3}{2}}\varepsilon^{-\frac{1}{2}}h^{-\frac{1}{2}}. \tag{16.2.64}$$

(ii) In particular, $\mathsf{R}_x^{\mathsf{W}} \le C\mu^{-1}h^{-1}$ *as* $\varepsilon \ge \mu^5 h$.

Calculating Correction Term

Situation in the near-equator zone becomes more delicate if we want to estimate an error in r-term stationary phase approximation. Then the contribution of k-th tick does not exceed

$$C\mu\varepsilon h^{-1}(\hbar/|\cos(t_k)|^3)^{r+\frac{1}{2}} \asymp C\mu\varepsilon h^{-1}\bigl(\hbar/(\varepsilon|k-k^*|)^{\frac{3}{2}}\bigr)^{r+\frac{1}{2}}$$

which as $r \ge 1$ sums to $C\mu\varepsilon h^{-1}\bigl(\hbar/(\varepsilon|k-k^*|)^{\frac{3}{2}}\bigr)^{r+\frac{1}{2}}|k-k^*|$ calculated at the least possible value of $|k-k^*|$; so we get $C\mu\varepsilon h^{-1}\bigl(\hbar/\varepsilon^{\frac{3}{2}}\bigr)^{r+\frac{1}{2}}$ as we sum for $|k-k^*| \ge 1$, i.e. as $\varepsilon \ge \hbar^{\frac{2}{3}}$.

On the other hand, we get $C\mu\varepsilon h^{-1} \times \hbar^{\frac{2}{3}}\varepsilon^{-1} = C\mu^{\frac{5}{3}}h^{-\frac{1}{3}}$ as we sum for $|k-k^*| \ge \hbar^{\frac{2}{3}}\varepsilon^{-1}$, i.e. as $\varepsilon \le \hbar^{\frac{2}{3}}$. This takes care of zone (16.2.45); analysis of zones covered by (16.2.44) is easy and its contribution to the error is lesser.

Finally, the contribution to the error of zone (16.2.46) does not exceed $C\mu\varepsilon h^{-1}\max(\varepsilon, \varepsilon^{-1}\hbar, \hbar^{\frac{1}{3}})$ where in comparison with the analysis of the previous subsection we acquired factor ε.

Therefore, contribution of the near-equator zone to the error does not exceed

$$C\mu^{-1}h^{-1} + \begin{cases} C\mu\varepsilon h^{-1}\bigl(\hbar/\varepsilon^{\frac{3}{2}}\bigr)^{r+\frac{1}{2}} + C\mu^{\frac{4}{3}}\varepsilon h^{-\frac{2}{3}} & \text{as } \varepsilon \ge \hbar^{\frac{2}{3}}, \\ C\mu^{\frac{5}{3}}h^{-\frac{1}{3}} & \text{as } \varepsilon \le \hbar^{\frac{2}{3}} \end{cases} \tag{16.2.65}$$

and we arrive to

Proposition 16.2.12. *For a toy-model operator* (16.2.1) *with* $\tau \asymp 1$, $\varepsilon \le \mu^{-1}$, $\hbar \le \varepsilon$ *correction term* $h^{-2}\mathcal{N}_{x,\mathsf{corr}}$ *is delivered with* r*-term stationary phase approximation* $h^{-2}\mathcal{N}_{x,\mathsf{corr}(r)}$ *with an error not exceeding*

$$C\mu h^{-1}\bigl(\hbar/\varepsilon\bigr)^{r+\frac{1}{2}} + C\mu\varepsilon h^{-1}\bigl(\hbar/\varepsilon^{\frac{3}{2}}\bigr)^{r+\frac{1}{2}} + C\mu^{\frac{4}{3}}\varepsilon h^{-\frac{2}{3}} + C\mu^{-1}h^{-1} \quad \text{as } \varepsilon \ge \hbar^{\frac{2}{3}} \tag{16.2.66}$$

and

$$C\mu h^{-1}\big(\hbar/\varepsilon\big)^{r+\frac12} + \mu^{\frac53}h^{-\frac13} + C\mu^{-1}h^{-1} \qquad \text{as } \varepsilon \le \hbar^{\frac23}. \tag{16.2.67}$$

Micro-Averaging

In the current framework (including assumption $\hbar \le \varepsilon$) micro-averaging does not improve estimate of the correction term or the error as $|\mathsf{cos}(t)| \ge \epsilon_0$ in it unless $\gamma \ge h\varepsilon^{-1}$ (in comparison with the remainder estimate which was improved as $\gamma \ge h$). Then, as $\gamma \ge h\varepsilon^{-1}$ in estimates (16.2.66), (16.2.67) the first term gains factor $(h/\varepsilon\gamma)^l$, the last term is preserved and all other terms gain factor $(h/\gamma)^l \le \varepsilon^l$ which effectively kills them and we arrive to estimates (16.2.68), (16.2.71) below.

As $h \le \gamma \le h\varepsilon^{-1}$ the first and the last terms do not improve but all further terms gain factor $(h/\gamma)^l$ and we arrive to estimates (16.2.69) and (16.2.70) below. Finally, for $\gamma \le \mu h$ micro-averaging brings no improvement.

Proposition 16.2.13. *(i) For a toy-model operator* (16.2.1) *with* $\tau \asymp 1$, $\varepsilon \le \mu^{-1}$, $\varepsilon \ge \hbar$,

$$\mathsf{R}^{\mathsf{W}}_{x(r),\gamma} := \gamma^{-2}\Big|\int \Big(e^{\mathsf{T}}(x,x,0) - h^{-2}\big(\mathcal{N}^{\mathsf{W}}_x + \mathcal{N}_{x,\mathsf{corr}(r)}\big)\Big)\psi_\gamma\, dx\Big| \le C\mu h^{-1}\big(\hbar/\varepsilon\big)^{r+\frac12}\big(h/\varepsilon\gamma\big)^l + C\mu^{-1}h^{-1} \qquad \text{as } \gamma \ge h/\varepsilon, \tag{16.2.68}$$

$$\mathsf{R}^{\mathsf{W}}_{x(r),\gamma} \le C\mu h^{-1}\big(\hbar/\varepsilon\big)^{r+\frac12} + C\mu\varepsilon\Big(h^{-1}\big(\hbar/\varepsilon^{\frac32}\big)^{r+\frac12} + \mu^{\frac13}h^{-\frac23}\Big)\big(h/\gamma\big)^l + C\mu^{-1}h^{-1} \qquad \text{as } \varepsilon \ge \hbar^{\frac23},\ h \le \gamma \le h/\varepsilon \tag{16.2.69}$$

and

$$\mathsf{R}^{\mathsf{W}}_{x(r),\gamma} \le C\mu h^{-1}\big(\hbar/\varepsilon\big)^{r+\frac12} + C\mu^{\frac53}h^{-\frac13}\big(h/\gamma\big)^l + C\mu^{-1}h^{-1} \qquad \text{as } \varepsilon \le \hbar^{\frac23},\ h \le \gamma \le h/\varepsilon, \tag{16.2.70}$$

(ii) For a toy-model operator (16.2.1) *with* $\tau \asymp 1$, $\varepsilon \le \mu^{-1}$, $\varepsilon \le \hbar$

$$\mathsf{R}^{\mathsf{W}}_{x,\gamma} := \gamma^{-2}\Big|\int \Big(e^{\mathsf{T}}(x,x,0) - h^{-2}\mathcal{N}^{\mathsf{W}}_x\Big)\psi_\gamma\, dx\Big| \le \mu h^{-1}\big(h/\varepsilon\gamma\big)^l + C\mu^{-1}h^{-1} \qquad \text{as } \gamma \ge h/\varepsilon. \tag{16.2.71}$$

16.2.4 Strong and Superstrong Magnetic Field

Tauberian Estimate

Consider strong $\mu h \asymp 1$ and superstrong $\mu h \gtrsim 1$ magnetic field for a Schrödinger-Pauli toy-model operator

$$A := h^2 D_1^2 + (hD_2 - \mu x_1)^2 + 2\alpha x_1 - \mathfrak{z}\mu h \tag{16.2.72}$$

which alternatively means that we assume that $|\tau - \mathfrak{z}\mu h| \le C$ in the framework of the standard toy-model (16.2.1). Formulae (16.2.15)–(16.2.16) imply that

Proposition 16.2.14. *For a toy-model operator* (16.2.1) *with* $\mu h \gtrsim 1$ *in* $\mathbb{R}^2$ *as* $|x| \le C_0$, $|y| \le C_0$

(i) $e(x,y,\tau) \equiv 0 \mod O(\mu^{-\infty})$ *as* $\tau \le \mu h - \epsilon_0$ *(the lower spectral gap).*

(ii) $e(x,y,\tau) \equiv e(x,y,\tau')$ *and* $\partial_\tau e(x,y,\tau) \equiv 0 \mod O(\mu^{-\infty})$ *as* $(2n-1) + \epsilon_0 \le \tau' < \tau \le (2n+1)\mu h - \epsilon_0$ *(other spectral gaps).*

(iii) $e(x,x,\tau) - e(x,x,\tau') \asymp \mu h^{-1}$ *as* $\tau' < (2n-1)\mu h - \epsilon_0$ *and* $\tau > (2n-1)\mu h + \epsilon_0$.

(iv) As $\varepsilon > 0$ *the following estimates hold*

$$|\partial_\tau e(x,y,\tau)| \le C\mu^{\frac12}\varepsilon^{-1}h^{-\frac32} \tag{16.2.73}$$

and

$$|F_{t\to\hbar^{-1}\tau}U(x,y,t)| \le C\mu^{\frac32}\varepsilon^{-1}h^{-\frac12} \tag{16.2.74}$$

and these estimate are sharp as $x = y$ *and* τ *is close to Landau levels* $(2n+1)\mu h$.

Further, as before $T^* = \epsilon_0\mu\varepsilon^{-1}\ell$. Then we immediately arrive to

Corollary 16.2.15. *For a self-adjoint operator in domain* X, $B(0,\ell) \subset X \subset \mathbb{R}^2$, $\ell \ge C_0\mu^{-\frac12}h^{\frac12}$, *coinciding in* $B(0,\ell)$ *with the toy-model operator* (16.2.72) *with* $\tau \asymp 1$, $\varepsilon < 0$, $\mu \ge h^{-1}$

(i) Statements (i)–(iv) of Proposition 16.2.14 remain true.

(ii) Formula (16.2.15) *holds modulo* $O(\mu^{\frac12}h^{-\frac12}\ell^{-1})$.

(iii) Moreover, formula (16.2.15) *holds modulo* $O(\mu^{-\infty})$ *as* $(2n-1) + \epsilon_0 \le \tau \le (2n+1)\mu h - \epsilon_0$.

Micro-Averaging

Let us consider micro-averaging. First let us estimate Fourier transform where as before we rescaled $t \mapsto \mu t$; formulae (16.2.15)–(16.2.16) imply immediately

Proposition 16.2.16. *For a toy-model operator* (16.2.72) *in* $\mathbb{R}^2$ *with* $\mu h \ge 1$ *and* $\mathfrak{z} = (2n+1)$

$$\gamma^{-2}|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(U\psi_\gamma)| \le C\mu^2 + C\mu\varepsilon^{-1}R^{\mathsf{T}}(\gamma) \tag{16.2.75}$$

with

$$R^{\mathsf{T}}(\gamma) = \begin{cases} \mu^{\frac12}h^{-\frac12} & \text{as } \gamma \le \mu^{-\frac12}h^{\frac12}, \\ \gamma^{-1} & \text{as } \gamma \ge \mu^{-\frac12}h^{\frac12}. \end{cases} \tag{16.2.76}$$

Corollary 16.2.17. *In the framework of Corollary 16.2.15*

$$\gamma^{-2}|\Gamma\big(e(.,.,\tau) - e^{\mathsf{T}}(.,.,\tau)\big)\psi_\gamma| \le C\big(\mu\varepsilon + R^{\mathsf{T}}(\gamma)\big)\ell^{-1}. \tag{16.2.77}$$

16.2.5 Weyl and Magnetic Weyl Approximations

We can try to use a host of the different approximations but restrict ourselves now to Weyl approximation, which makes sense only as $\mu h \le 1$. Recall that

$$|e^{\mathsf{T}}(x,x,\tau) - h^{-2}\mathcal{N}_x^{\mathsf{W}}(\tau)| \le CR^{\mathsf{W}}, \tag{16.2.78}$$

$$\gamma^{-2}\Big|\int \big(e^{\mathsf{T}}(x,x,\tau) - h^{-2}\mathcal{N}_x^{\mathsf{W}}(\tau)\big)\psi_\gamma\, dx\Big| \le CR^{\mathsf{W}}(\gamma) \tag{16.2.79}$$

with

$$R^{\mathsf{W}} = \begin{cases} \mu^{\frac32}\varepsilon^{-\frac12}h^{-\frac12} & \text{as } \varepsilon \ge \hbar, \\ \mu h^{-1} & \text{as } \varepsilon \le \hbar, \end{cases} \tag{16.2.80}$$

$$R^{\mathsf{W}}(\gamma) = \begin{cases} R^{\mathsf{W}} & \text{as } \gamma \le h/\varepsilon, \\ R^{\mathsf{W}}(h/\varepsilon\gamma)^I & \text{as } \gamma \ge h/\varepsilon. \end{cases} \tag{16.2.81}$$

On the other hand, let us apply instead the magnetic Weyl approximation. Obviously, without micro-averaging $\mathcal{N}_x^{\mathsf{MW}}(\tau)$ cannot produce uniform with respect to τ remainder estimate better than μh^{-1} due to its own jumps. Meanwhile with an averaging (i.e. a micro-averaging with $\gamma = 1$) $\mathcal{N}_x^{\mathsf{MW}}(\tau)$

provides a better remainder estimate $O(\mu^{-1}h^{-1})$ (see Chapter 13; for a toy-model we need only assume that $\varepsilon \ge \mu^{-l}$). So, as $\mu h \le 1$ the magnetic Weyl approximation may provide a better remainder estimate as γ is not too small enough but for small γ Weyl approximation is better. Let us investigate this.

Proposition 16.2.18. *(i) For a toy-model operator* (16.2.1) *as* $\mu h \lesssim 1$

$$\mathsf{R}^{\mathsf{MW}}(\gamma) := \gamma^{-2}\Big|\int \big(e^{\mathsf{T}}(x,x,0) - h^{-2}\int \mathcal{N}^{\mathsf{MW}}\big)\psi_\gamma\,dx\Big| \le C\mu^{-1}h^{-1} + CR^{\mathsf{MW}}(\gamma) \tag{16.2.82}$$

with

$$R^{\mathsf{MW}}(\gamma) = \begin{cases} \mu^{-1}h^{-1}\gamma^{-2}(h/\varepsilon\gamma)^l & \text{as } \gamma \ge \max(\mu^{-1}, h/\varepsilon),\\ \mu^{-1}h^{-1}\gamma^{-2} & \text{as } \mu^{-1} \le \gamma \le h/\varepsilon,\\ \mu h^{-1}(h/\varepsilon\gamma)^l & \text{as } h/\varepsilon \le \gamma \le \mu^{-1},\\ \mu h^{-1} & \text{as } \gamma \le \min(\mu^{-1}, h/\varepsilon); \end{cases} \tag{16.2.83}$$

(ii) For a toy-model operator (16.2.72) *as* $\mu h \gtrsim 1$ *and* $\mathfrak{z} = 2n+1$ *estimate*

$$\mathsf{R}^{\mathsf{MW}}(\gamma) \le C + CR^{\mathsf{MW}}(\gamma) \tag{16.2.84}$$

holds with

$$R^{\mathsf{MW}}(\gamma) = \begin{cases} \gamma^{-2} & \text{as } \gamma \ge \mu^{-\frac{1}{2}}h^{\frac{1}{2}},\\ \mu h^{-1} & \text{as } \gamma \le \mu^{-\frac{1}{2}}h^{\frac{1}{2}}. \end{cases} \tag{16.2.85}$$

Proof. We need to investigate only case $\mu^{-1} \le \gamma \le h\varepsilon^{-1}$ as in all other cases we can easily pass from results for $\mathsf{R}^{\mathsf{W}}(\gamma)$.

Note that $h^{-2}\mathcal{N}_y^{\mathsf{MW}}(\tau)$ delivers $e(y,y,\tau)$ for a toy-model operator $\bar{A} := \bar{A}_y = \bar{A}_y(x, D_x)$ defined by (16.2.1) or (16.2.72) with αx_1 frozen at the point y (with respect to which we integrate later); so we consider our toy-model operator as a perturbation of this one.

Let us plug into Tauberian formula the Schwartz kernel of

$$e^{i\hbar^{-1}tA} = e^{i\hbar^{-1}t\bar{A}} + i\hbar^{-1}\int_0^t e^{i\hbar^{-1}t'\bar{A}}\big(A-\bar{A}\big)e^{i\hbar^{-1}(t-t')A}\,dt' \tag{16.2.86}$$

or its iteration for $\pm t > 0$

$$(16.2.86)_m \quad e^{i\hbar^{-1}tA} = e^{i\hbar^{-1}t\bar{A}_x} + \sum_{1\le j\le m-1} \times$$

$$(i\hbar^{-1})^j \int\limits_{\substack{\{\pm t_1>0,\pm t_2>0,\dots,\pm t_j>0,\\ t_1+t_2+\dots+t_j=t\}}} e^{i\hbar^{-1}t_1\bar{A}}\big(A-\bar{A}\big)e^{i\hbar^{-1}t_2\bar{A}}\big(A-\bar{A}\big)\cdots\big(A-\bar{A}\big)e^{i\hbar^{-1}t_j\bar{A}}\,dt_1dt_2\cdots dt_j+$$

$$(i\hbar^{-1})^m \int\limits_{\substack{\{\pm t_1>0,\pm t_2>0,\dots,\pm t_m>0,\\ t_1+t_2+\dots+t_j=t\}}} e^{i\hbar^{-1}t_1\bar{A}}\big(A-\bar{A}\big)e^{i\hbar^{-1}t_2\bar{A}}\big(A-\bar{A}\big)\cdots\big(A-\bar{A}\big)e^{i\hbar^{-1}t_mA}\,dt_1dt_2\cdots dt_m.$$

So let us plug $(16.2.86)_m$ into our expression with $\chi_T(t)$ instead of $\bar{\chi}_T(t)$. Then our standard methods imply that j-th term ($j = 0, \dots, m$) of what we got does not exceed

$$C\mu h^{-1}\big(T\|A-\bar{A}_x\|\hbar^{-1}\big)^j \times \big(1+\frac{T}{\bar{T}_2}\big)^{-l}\big(1+\frac{T}{\bar{T}_1}\big)^{-l} \tag{16.2.87}$$

with $\bar{T}_1 = C_0\mu\varepsilon^{-1}\bar{\gamma}$ where $\bar{\gamma} := \max(\mu^{-1}, \mu^{-\frac12}h^{\frac12})$ and $\bar{T}_2 = h/\varepsilon\gamma$; we leave easy details to the reader.

Then we can replace the norm by the "effective norm" which is a norm restricted to $\mu\bar{\gamma}$-vicinity of x (in rescaled coordinates) and equal to $C\alpha\bar{\gamma}$.

Note that $\bar{T}_2 \le \bar{T}_1$ as $\gamma \ge \bar{\gamma}$; then summation with respect to T returns

$$C\mu h^{-1}\big(\bar{T}_2\|A-\bar{A}_x\|\hbar^{-1}\big)^j \asymp C\mu h^{-1}\big(\bar{\gamma}/\gamma\big)^j$$

which provides required estimate (16.2.83) or (16.2.84) for all terms with $j \ge 2$.

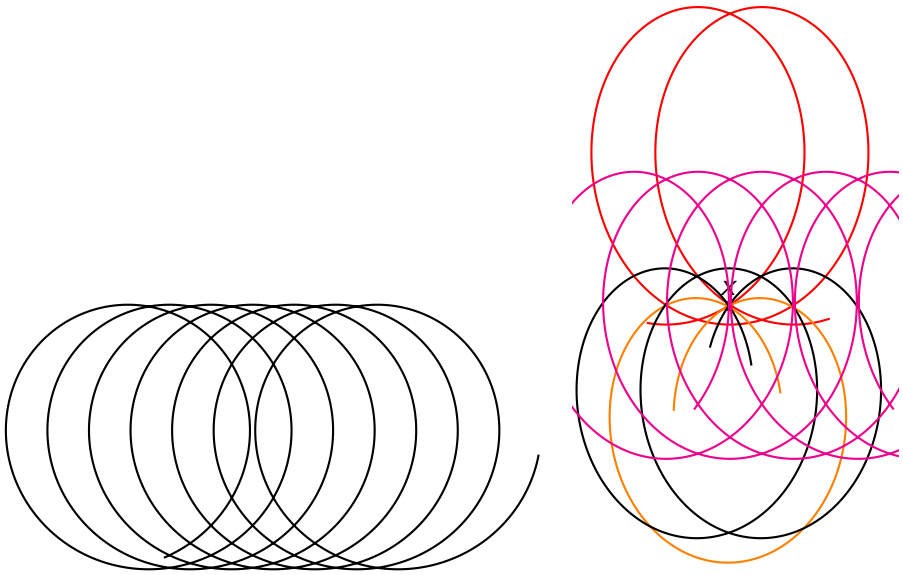
Finally note that while $\bar{A}_y(x-y, D_x)$ is even with respect to $(x-y, D_x)$ (we can always achieve it by the gauge transformation), perturbation $(A-\bar{A}_y)(x-y, D_x)$ is odd and therefore only terms with even j survive as we plug $x = y$. This takes care of term with $j = 1$. □

Remark 16.2.19. (i) For $\mu h \le 1$ magnetic Weyl formula provides better approximation as $\mu^{-1} \le \gamma \le h/\varepsilon$; otherwise Weyl formula provides either better or equally bad (μh^{-1}) approximation.

(ii) For $\mu h \ge 1$ magnetic Weyl formula provides approximation rather than just main term estimate as $\gamma \gg \mu^{-\frac12}h^{\frac12}$.

16.2.6 Geometric Interpretation

Remark 16.2.20. (i) As we mentioned t_k are length of the classical loops at point 0 (or x if we consider $e(x,x,\tau)$; then $t_k = t_k(x)$). While picture of the fixed trajectory and a point x on it (Figure 16.2(a)) is more geometrically appealing, the correct picture is of the fixed point and different trajectories passing through it (Figure 16.2(b)); due to assumption $\alpha \neq 0$ "radii" of these trajectories differ by $O(\mu^{-1})$.



(a) Intuitive (b) Correct picture

Figure 16.2: On the left more geometrically appealing picture, when x moves along given trajectory; on the right the correct one when we consider different trajectories passing through a fixed point.

Therefore, when we talk here about *loop* we mean a trajectory looping at the fixed point x rather the self-intersections of the fixed trajectory.

Similarly, when we talk about *pole* or *equator* we mean not x near a pole or an equator of the fixed trajectory but rather trajectory such that x is near its pole or equator.

So, all further remarks should be interpreted correctly.

(ii) How to interpret our results from heuristic uncertainty principle? Obviously these results mean that as $\varepsilon \gg \hbar$ then the majority of trajectories on Figure 16.2(b) do not return to the original point $x = 0$ after $k = 1$ tick[5] while otherwise this happens only after k ticks with $k\varepsilon \gg \hbar$. As the spatial shift is εk (after rescaling) this means that the thickness of the trajectory is $\asymp \hbar$ after rescaling (and therefore $\asymp h$ before rescaling).

It also means that if $|\sin(\theta)| \asymp 1$ where θ is the polar angle (between direction of the trajectory at 0 and the drift direction $(0,1)$) then we can take an interval in θ of the magnitude 1, so θ is a dual variable to x_2. Conversely, with the exception of the interval of the length $\hbar/\varepsilon|k|$ around θ_k (where θ_k corresponds to the classical trajectory returning to 0 after k ticks) trajectories do not return at 0; we can rewrite this condition as

$$|\sin(\theta - \theta_k)| \gg \hbar/(|k|\varepsilon). \tag{16.2.88}$$

This characteristic length matches to the stationary phase estimates we derived rigorously.

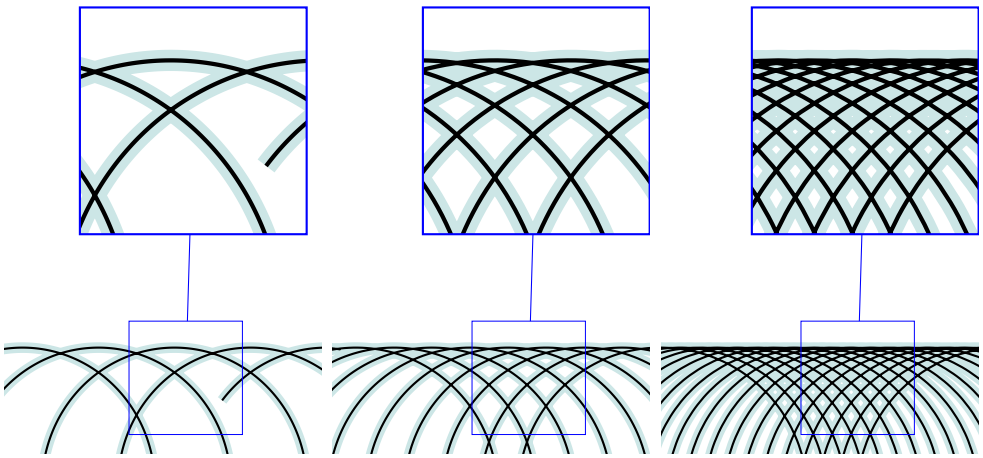


Figure 16.3: Looping near pole from microlocal point of view

(iii) Near equator situation changes drastically: spacing between self-intersections is $\asymp \varepsilon|\cos(\theta)|^{-1}$ as $|\cos(\theta)| \ge \epsilon_0\varepsilon^{\frac{1}{2}}$ and there are exceptional $0, 1, 2$

[5] In the sense that the difference between two points is observable.

self-intersections with $|\cos(\theta)| \le \epsilon_0 \varepsilon^{\frac{1}{2}}$. Uncertainty principle may prevent us to know how many of them are no matter how "not large" μ is.

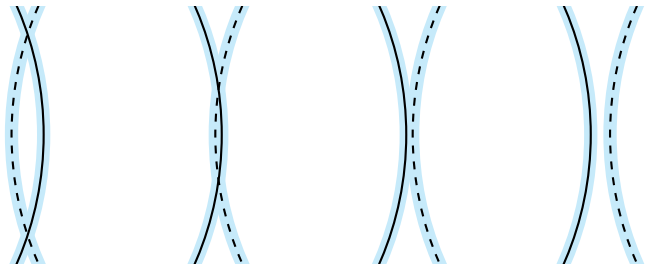


Figure 16.4: Self-intersections near equator. The first winding shown by a solid line, the closest to it near equator by a dashed one.

Consequent self-intersections are distinguishable as $|\cos(\theta)| \ge \hbar^{\frac{1}{3}}$ according to our calculations.

Problem 16.2.21. We know that estimates (16.2.73), (16.2.74) are sharp as $\mu h \gtrsim 1$. Are these estimates sharp as $\mu h \ll 1$?

16.3 Pointwise Asymptotics: General 2D-Operators

Let us recreate the approximation similar to the one of the previous section for general operators assuming that $\hbar \le \epsilon$ i.e.

$$\mu \le \epsilon h^{-1}. \tag{16.3.1}$$

We are interested in $|t| \le C_0 \mu^{-1} \varepsilon^{-1}$ for the original (not rescaled) operator because for $C_0 \mu^{-1} \varepsilon^{-1} \le |t| \le \mu^{-1} T^* := \epsilon_0 \varepsilon^{-1} \ell$ dynamics leaves $B(0, c_0 \mu^{-1})$ but remains in $B(0, \ell)$. Here

(16.3.2) $\varepsilon = \alpha^{-1} \mu^{-1}$ as we assume that $|\nabla V / F| \asymp \alpha$ in $B(0, \ell)$ with

$$C_1 \mu^{-1} \le \alpha \le 1 \qquad (\implies C_0 \mu^{-2} \le \varepsilon \le \mu^{-1}) \tag{16.3.3}$$

and $\ell \ge c_0 \mu^{-1}$ with sufficiently large constants C_1, c_0.

16.3.1 Classical Dynamics

First we consider classical dynamics starting from point $x = 0$. So far we use not-rescaled x, t. Let us freeze g^{jk} and F there and replace V_j and V by their linear germs:

$$\bar{g}^{jk} = g^{jk}(0), \quad \bar{V}(x) = V(0) + \langle \nabla V(0), x\rangle, \tag{16.3.4}$$
$$\bar{V}_j(x) = V_j(0) + \langle \nabla V_j(0), x\rangle \quad (\implies \bar{F} = F(0)),$$

denote corresponding Hamiltonian by $\bar{a}(x,\xi)$ and all the object associated with it will have bar. We consider our original dynamic system as a perturbation.

Without any loss of the generality we can assume that

$$g^{jk} = \omega^2 \delta_{jk} \qquad (\implies a(x,\xi) = \omega^2(p_1^2 + p_2^2) + V), \tag{16.3.5}$$

$$\omega(0) = F(0) = 1, \quad (\nabla\omega)(0) = 0, \tag{16.3.6}$$

$$V(0) = 0, \quad (\nabla V/F)(0) = (-\alpha, 0) \tag{16.3.7}$$

as we can achieve it by an appropriate change of variables (see footnote 9)) and

$$V_1(0) = 0, \quad V_2 = 0 \tag{16.3.8}$$

since we can achieve it by the gauge transformation.

Proposition 16.3.1. *Under conditions* (16.3.1)–(16.3.8) *as* $|t| \le c_0\mu^{-1}\varepsilon^{-1}$

(i) For drift flows $z_t = \Phi_t(0)$, $\bar{z}_t = \bar{\Phi}_t(0)$

$$z_t = \bar{z}_t + O\big(\mu^{-1}\varepsilon t^2\big); \tag{16.3.9}$$

(ii) If $F = 1$ *then for Hamiltonian flows* $(x_t, \xi_t) = \Psi_t(0,\eta)$, $(\bar{x}_t, \bar{\xi}_t) = \bar{\Psi}_t(0,\eta)$ *with magnetic parameters* μ *and* $\bar{\mu} = \mu(1 + O(\mu^{-2}))$ *defined by* (16.3.18) *on the energy level* $\tau \le c$

$$(x_t, \mu^{-1}\xi_t) = (\bar{x}_t, \mu^{-1}\bar{\xi}_t) + O\big(\mu^{-2}t\big). \tag{16.3.10}$$

Remark 16.3.2. Assumption (16.3.1) means that in $B(0,\mu^{-1})$ the constant part of ∇V dominates over its variable part and it implies that as

$$\mu^{-1}\varepsilon t^2 \le c_0\mu^{-2}|t| \le \kappa\varepsilon|t| \qquad \text{as} \quad |t| \le c_0\mu^{-1}\varepsilon^{-1} \tag{16.3.11}$$

with

$$\kappa = c_0\mu^{-2}\varepsilon^{-1} \le C_0^{-1}c_0 \tag{16.3.12}$$

and thus the perturbation of the drift is respectively small. In particular drift line deviates from the straight one by no more than $c_0\kappa$.

Also *equator* is defined as a point where x_t intersect its its first winding the last time deviates from $(\mu^{-1}, 0)$ by no more than $c_0\mu^{-1}\kappa$.

Proof of Proposition 16.3.1. (i) Proof of Statement (i) is trivial and left to the reader. Just recall that the drift flow is defined by (13.7.5) with an extra factor μ^{-1} in the right-hand expression and in our assumptions $\bar{\Phi}_t : (x_1, x_2) \mapsto (x_1 + \varepsilon t, x_2)$ after rescaling.

(ii) Recall that $p_j = \xi_j - \mu V_j(x)$, $\{p_j, x_k\} = \delta_{jk}$ and in virtue of the assumption $F = 1$

$$\{p_1, p_2\} = \mu\omega^{-2}. \tag{16.3.13}$$

According to Subsection 13.3.1 one can correct

$$\begin{aligned} y_1 &\coloneqq x_1 - \{p_1, p_2\}^{-1}p_2 = x_1 - \mu^{-1}\omega^2 p_2, \\ y_2 &\coloneqq x_2 + \{p_1, p_2\}^{-1}p_1 = x_2 + \mu^{-1}\omega^2 p_1 \end{aligned} \tag{16.3.14}$$

modulo $O(\mu^{-2})$ so that corrected expressions satisfy drift equation modulo $O(\mu^{-2})$. More precisely, in the current setup

$$\{a, y_1\} = \mu^{-1}\omega^2\Big(\{p_2, \omega^2\}(p_1^2 - p_2^2) - 2\{p_1, \omega^2\}p_1p_2 + \{p_2, V\}\Big)$$

and for $y_1' = y_1 + \mu^{-2}\beta_1 p_1 p_2 + \mu^{-2}\beta_2(p_1^2 - p_2^2)$ with

$$\beta_1 = -\frac{1}{2}\omega^2\{p_2, \omega^2\}, \qquad \beta_2 = \frac{1}{2}\omega^2\{p_1, \omega^2\}$$

we have

$$\{a, y_1'\} \equiv -\mu^{-1}\omega^2\{p_2, V\}$$

modulo terms which are $O(\mu^{-2})$ and also homogeneous polynomials of degrees 1 or 3 with respect to (p_1, p_2) and thus these terms could be corrected by adding to y_1' terms which are $O(\mu^{-3})$ and also homogeneous polynomials of degrees 2 or 4; then we arrive to

$$\{a, y_k''\} \equiv (-1)^k\mu^{-1}\omega^2\{p_{3-k}, V\} \mod O(\mu^{-3}) \qquad k = 1, 2$$

($k=2$ is considered in the same way as $k=1$). Note that the right-hand expressions are calculated at point $x=x_t$ rather than $y''=y_t''$; so we rewrite them as

$$\{a, y_k''\} \equiv (-1)^k \mu^{-1}\big(\omega^2\{p_{3-k}, V\}\big)(y'') + \sum_{j=1,2} \mu^{-2}(-1)^{j+k-1}(\partial_{3-j}\omega^2\partial_{3-k}V)(x)p_j \mod O(\mu^{-3}) \qquad k=1,2$$

and we can correct y_k'' by adding to y_k'' terms which are $O(\mu^{-3})$ and also homogeneous polynomials of degrees 2 with respect to (p_1,p_2).

Therefore

$$y_t''' \equiv z_t \mod O(\mu^{-3}t) \tag{16.3.15}$$

where $z_t=\Phi_t(0)$.

Meanwhile (16.3.13) implies

$$\frac{d}{dt}p_j = 2(-1)^j\mu p_{3-j} - (\omega^2)_{x_j}(p_1^2+p_2^2) - V_{x_j} = 2(-1)^j\mu p_{3-j} - 2\omega_{x_j}\omega^{-1}W + W_{x_j}, \qquad j=1,2 \tag{16.3.16}$$

where we replaced $p_1^2+p_2^2$ by $\omega^{-2}W$, $W=\tau-V$ (no error as τ is an energy level). Let us plug $x_1(t)\equiv z_1(t)+\mu^{-1}\omega^2(z_t)p_2$, $x_2(t)\equiv z_2(t)-\mu^{-1}\omega^2(z_t)p_1$ modulo $O(\mu^{-2})$ in the right-hand expressions; we get

$$\frac{d}{dt}\begin{pmatrix}p_1\\p_2\end{pmatrix} = J\begin{pmatrix}p_1\\p_2\end{pmatrix} + K \tag{16.3.17}$$

with

$$J \coloneqq 2\mu\begin{pmatrix} -\mu^{-2}\omega^2(\omega_{x_1}\omega^{-1}W+W_{x_1})_{x_2} & -\big(1-\mu^{-2}\omega^2(\omega_{x_1}\omega^{-1}W+W_{x_1})_{x_1}\big)\\ \big(1-\mu^{-2}\omega^2(\omega_{x_2}\omega^{-1}W+W_{x_2})_{x_2}\big) & \mu^{-2}\omega^2(\omega_{x_2}\omega^{-1}W+W_{x_2})_{x_1}\end{pmatrix},$$

and

$$K \coloneqq \begin{pmatrix}-2(\omega_{x_1}\omega^{-1}W+W_{x_1})\\ -2\omega_{x_2}\omega^{-1}W+W_{x_2}\end{pmatrix}.$$

and here we can calculate elements of J at any point of $B(0,\mu^{-1})$ we choose (and we choose 0) while elements of K are calculated at z_t. Then J becomes a constant coefficient matrix and since $\nabla\omega(0)=0$, $\omega(0)=1$

$$J=2\mu\begin{pmatrix} -\mu^{-2}(\omega_{x_1x_2}W+W_{x_1x_2}) & -\bigl(1-\mu^{-2}(\omega_{x_1x_1}W+W_{x_1x_1})\bigr)\\ \bigl(1-\mu^{-2}(\omega_{x_2x_2}W+W_{x_2x_2})\bigr) & \mu^{-2}(\omega_{x_2x_1}W+W_{x_2x_1})\end{pmatrix}$$

and its eigenvalues are $\pm 2\bar{\mu}i$ with

(16.3.18) $\bar{\mu}=$

$$\mu\Bigl(\bigl(1-\mu^{-2}(\omega_{x_1x_1}W+W_{x_1x_1})\bigr)\bigl(1-\mu^{-2}(\omega_{x_2x_2}W+W_{x_2x_2})-\mu^{-4}(\omega_{x_1x_2}W+W_{x_1x_2})^2\Bigr)^{\frac{1}{2}}$$
$$=\mu+\mu^{-1}(\Delta(\omega W))+O(\mu^{-3}).$$

Then $J=Q^{-1}\bar{J}Q$ with $\bar{J}=\bar{\mu}\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ and $Q=I+O(\mu^{-2})$.

Due to (16.3.17)

(16.3.19) $$p(t)\equiv e^{tJ}p(0)+\int_0^t e^{(t-t')J}K(z_{t'})\,dt'=$$
$$e^{tJ}p(0)+\int_0^t e^{(t-t')J}J^{-1}\frac{d}{dt'}K(z_{t'})\,dt'-J^{-1}K(z_t)+e^{tJ}J^{-1}K(z_0).$$

The second term in the right-hand expression is $O(\mu^{-1}\varepsilon t)$ as $J^{-1}=O(\mu^{-1})$ and $\frac{d}{dt'}K(z_{t'})=O(\varepsilon)$. Note that with $O(\mu^{-2}t)$ error one can replace in the two last terms J by $\bar{J}$ calculated for a toy-model. Finally, with an error $O(\mu^{-1}\varepsilon t)$ one can replace $K(z_t)$ by $K(z_0)$; however $K(z_0)$ and $\bar{K}(z_0)$ coincide as $\nabla\omega(0)=0$.

Replacing J by $\bar{J}$ we reduced evolution of p to those of the toy-model albeit with $\bar{\mu}$ instead of μ. Therefore $\mu^{-1}p_t\equiv\mu^{-1}\bar{p}_t \mod O(\mu^{-2}t)$. Replacing μ by $\bar{\mu}$ does not affect drift (in the framework of the indicated precision).

But then x_t could be found from z_t and p_t and their drift also is described by a toy-model in the framework of the same error and then it is true for $\mu^{-1}\xi_t$ as well. This concludes the proof of Statement (ii), □

Consider now the general case, i.e. F different from 1 and variable. Then differential equations describing $(x(t),\xi(t))$ for $a(x,\xi)$ coincide with

equations for $F^{-1}a(x,\xi)$ with $F \mapsto 1$ and $\tau - V \mapsto F^{-1}(\tau - V)$ but with the "time" θ satisfying

$$\frac{d\theta}{dt} = F(x_t) \equiv F(z_t) + (\omega^2 F^{-1})(z_t)\big(F_{x_1}(z_t)p_2 - F_{x_2}(z_t)p_1\big) \quad \mathsf{mod}\ O(\mu^{-2})$$

where $z_t = \Phi_t(0)$. We can correct θ by $O(\mu^{-1})$ eliminating linear terms in the right hand expression and therefore

$$\theta \equiv \int F(z_t)\,dt \equiv F(0)t + \frac{1}{2}\big(\frac{d}{dt}F(z_t)\big)(0)t^2 \quad \mathsf{mod}\ O(\mu^{-1} + \mu^{-2}|t|) \tag{16.3.20}$$

with the second term in the right-hand expression $O(\varepsilon\mu^{-1}t^2)$.

From now on we use rescaling $x \mapsto \mu x$, $t \mapsto \bar{\mu}t$, $\theta \mapsto \mu\theta$. Our analysis implies immediately

Proposition 16.3.3. *Let conditions* (16.3.1)–(16.3.8) *be fulfilled*[6)]. *Then (after rescaling* $x \to \bar{\mu}x$, $t \mapsto \bar{\mu}t$*) as* $|t| \le c_0\varepsilon^{-1}$

$$|D^\beta \Psi_t| \le C_\beta \qquad \forall \beta. \tag{16.3.21}$$

16.3.2 Semiclassical Approximation to $U(x,y,t)$

Proposition 16.3.4. *Let conditions* (16.3.1)–(16.3.8) *be fulfilled*[6)]. *Then*

(i) Uniformly with respect to $|t| \le c\varepsilon^{-1}$ $e^{i\hbar^{-1}tA}$ *is an* $\hbar$*-Fourier integral operator corresponding to Hamiltonian flow* Ψ_t.

(ii) As $|\sin(2\theta)| \ge \epsilon$ *with* θ *defined by* (16.3.20),

$$U(x,y,t) \equiv (4\pi\hbar)^{-1} i(\sin(\theta))^{-1} e^{i\hbar^{-1}\phi(x,y,t)} \sum_m b_m(x,y,t)\hbar^m \tag{16.3.22}$$

with ϕ *defined by* (16.3.25)–(16.3.29) *below and satisfying (with all derivatives)*

$$\phi = \bar{\phi}(\theta) + O(\mu^{-2}\varepsilon^{-1}), \tag{16.3.23}$$

$$b_m = \delta_{0m} + O(\mu^{-2}\varepsilon^{-1}) \tag{16.3.24}$$

with $\bar{\phi}$ *defined by* (16.2.10).

[6)] Before rescaling.

Proof. Both Statements (i) and (ii) of proposition are standard as $|t| \le T = \mathsf{const}$ but we need to extend them for larger t. Further, ϕ is an *action*[7)]

$$\phi = -\int_0^t L\big(x(t'), \dot{x}(t')\big)\, dt' \tag{16.3.25}$$

with the Lagrangian

$$L(x,\dot{x}) = \big(\sum_k \dot{x}_k\xi_k - a(x,\xi)\big)|_{\xi_k=(16.3.29)} = \frac{1}{4}\sum_{j,k} g_{jk}(x)\dot{x}_j\dot{x}_k + \frac{1}{2}\sum_j \dot{x}_j V_j(x) - V(x). \tag{16.3.26}$$

which is the Legendre transformation of the Hamiltonian

$$a(x,\xi) = \sum_{j,k} g^{jk}(x)\big(\xi_j - V_j(x)\big)\big(\xi_k - V_k(x)\big) + V(x) \tag{16.3.27}$$

$$\dot{x}_j = 2\sum_k g^{jk}(x)\big(\xi_k - V_k(x)\big), \tag{16.3.28}$$

$$\xi_k = \frac{1}{2}\sum\sum_j g_{jk}\dot{x}_j + V_k(x), \tag{16.3.29}$$

$\dot{f} := df/dt'$ and we are talking about trajectories from y as $t' = 0$ to x as $t' = t$.

To extend Statements (i) and (ii) to t: $|t| \le T = c\varepsilon^{-1}$ note that

$$e^{\pi t\hbar^{-1}A} = -e^{i\mu^{-1}\hbar^{-1}B} \tag{16.3.30}$$

with $\hbar$-pseudodifferential operator B; $B = 0$ if A is a toy-model operator with $\alpha = 0$. Therefore as usual

$$e^{\pi t\hbar^{-1}A} = e^{ik\hbar^{-1}A}e^{i(t-\pi k)\hbar^{-1}A} = (-1)^k e^{ik\mu^{-1}\hbar^{-1}B}e^{i(t-\pi k)\hbar^{-1}A} \tag{16.3.31}$$

with $k = \lfloor t/\pi \rfloor$ and the right-hand expression is $\hbar$-Fourier integral operator as $|k| \le c\varepsilon^{-1}$. Statement (i) is proven.

Note that the corresponding canonical manifold is well-projected to (x,y) space ($\mathbb{R}^4$) as $|\sin(2\theta)| \ge \epsilon$ and compare it with the canonical manifold

[7)] Sign "–" is due to considering of propagator $e^{i\hbar^{-1}tA}$ rather than $e^{-i\hbar^{-1}tA}$.

for the toy-model operator; this proves Statement (ii) as either $F = 1$ or $|t| \le \epsilon_1\varepsilon$ with sufficiently small $\epsilon_1 = \epsilon_1(\epsilon)$.

But then one can represent $e^{i\hbar^{-1}tA}$ as $e^{i\hbar^{-1}t_1A}e^{i\hbar^{-1}t_2A}\cdots e^{i\hbar^{-1}t_nA}$ with $n \le c_1$ and $|t_j| \le \epsilon_1\mu$, $|\sin(2t_j)| \ge \frac{1}{2}\epsilon$ which implies Statement (ii) in the general case. □

So far we exclude both vicinities of $\sin(\theta) = 0$ which matches $x = y$ (poles) and $\cos(\theta) = 0$ which matches x and y being antipodal points of the trajectory (equator) but we need to approach both of them. Actually exclusion of the latter was no more than a precaution but poles require a modification:

Proposition 16.3.5. *Let conditions* (16.3.1)–(16.3.8) *be fulfilled*[6)]. *Then*

(i) Decomposition (16.3.22) *remains valid as* $|\cos(\theta)| \le \epsilon$.

(ii) Decomposition (16.3.22) *remains valid as*

$$C\max\bigl(\hbar, \mu^{-1}\varepsilon|t|\bigr) \le |\sin(\theta)| \le \epsilon \tag{16.3.32}$$

albeit with an error not exceeding

$$C\hbar^{-1}|\sin(\theta)|^{-1}(\hbar/|\sin(\theta)|)^{l} \tag{16.3.33}$$

and with b_m *such that*

$$|D^{\beta}(\phi - \bar{\phi})| \le C_{\beta}\varepsilon|\sin(\theta)|^{-|\beta|} \qquad \forall \beta, \tag{16.3.34}$$

$$|D^{\beta}(b_m - \delta_{m0})| \le C_{m\beta}\varepsilon|\sin(\theta)|^{-m-|\beta|} \qquad \forall \beta, m. \tag{16.3.35}$$

Proof. As

$$U(x,y,t) = \mu^{-2}\int U(x,z,t')U(z,y,-t'')\,dz$$

for $t = t' - t''$ (where factor μ^{-d} is due to rescaling U as a function) we need to consider this oscillatory integral. If we consider oscillatory integral with the propagator for the toy-model operator we note that the standard stationary phase method applies with an effective semiclassical parameter $\hbar$ as $(\cot(t') - \cot(t'')) = -\sin(t)\sec(t')\sec(t'')$ disjoint from 0 i.e. also as $|\cos(t)| \le \epsilon$.

On the other hand, an effective semiclassical parameter is $\hbar/|\sin(t)|$ near poles.

In virtue of Proposition 16.3.4 both these claims remain true for a general operator as well albeit with t replaced by θ. □

Remark 16.3.6. So far we need only $|\nabla V/F| \lesssim \mu\varepsilon$ rather than $|\nabla V/F| \asymp \mu\varepsilon$ [6)].

16.3.3 Semiclassical Approximation to $e(x,x,\tau)$

Therefore in zone $\{|\sin(\theta)| \ge C\hbar\}$ all arguments of the toy-model theory work (with obvious modifications) under non-degeneracy assumptions

$$V - \tau \le -\epsilon_0, \tag{16.3.36}$$

$$|\nabla(\tau - V)F^{-1}| \asymp \varepsilon \tag{16.3.37}$$

we conclude that

(16.3.38) Contribution of zone $\{|\sin(\theta)| \ge C\hbar\}$ to $F_{t\to h^{-1}\tau}(1-\bar\chi_1(t))\Gamma_x U$ does not exceed $C\mu h^{-1} + C\mu^{\frac32}\varepsilon^{-1}h^{-\frac12}$.

Consider now zone $\{|\sin(\theta)| \le \epsilon|\}$. Both operators A and B (defined by (16.3.30) are ξ-microhyperbolic and unless factors they are coming with (namely $(\theta - \pi k)$ (with the closest πk) and εt respectively are of the same magnitude we can use ξ-microhyperbolicity to prove

(16.3.39) Contribution of zone $\{|t| \asymp T \ge \varepsilon^{-1}\max(\mu\hbar, C|\sin(\theta)|)\}$ to $F_{t\to h^{-1}\tau}\Gamma_x U$ does not exceed

$$C\mu^2\hbar^{-2}\times T\times \varepsilon T\times (\hbar/\varepsilon T)^{l+1} = C\mu^2 h^{-1}T(\hbar/\varepsilon T)^l$$

(with arbitrarily large l) while its contribution to expression (16.2.59) does not exceed the same expression albeit without factor T i.e.

$$C\mu^2 h^{-1}(\hbar/\varepsilon T)^l,$$

where factor μ^2 is due to rescaling, $T\times\varepsilon T$ is the measure of the zone $\{\theta : |\theta| \asymp T,\ |\sin(\theta)| \le \varepsilon T\}$ and other factors are standard; recall that B comes with the factor ε and therefore effective semiclassical parameter is $\hbar/\varepsilon$. We leave easy details to the reader.

After summation with respect to T we conclude that

(16.3.40) As $\varepsilon \ge C\hbar$ contributions of zone $\{|t| \ge 1,\ |\sin(\theta)| \le c\hbar\}$ to both $F_{t\to h^{-1}\tau}\Gamma_x U$ and expression (16.2.59) do not exceed $C\mu h^{-1}(\hbar/\varepsilon)^l$

and

(16.3.41) As $\varepsilon \le C_0\hbar \le 1$ contribution of zone $\{|t| \ge C_0\hbar/\varepsilon,\ |\sin(\theta)| \le c\hbar\}$ to $F_{t\to h^{-1}\tau}\Gamma_x U$ does not exceed $C\mu^2\varepsilon^{-1}$ while its contribution to expression (16.2.59) does not exceed $C\mu h^{-1}$.

So, as $\varepsilon \ge C\hbar$ we are done but as $\varepsilon \le C\hbar$ we need to consider zone $\{1 \le |t| \le c\hbar/\varepsilon,\ |\sin(\theta)| \le c\hbar\}$ and the same arguments imply

(16.3.42) As $\varepsilon \le C_0\hbar \le 1$ contribution of zone $\{1 \le |t| \le c\hbar/\varepsilon,\ |\sin(\theta)| \le c\hbar\}$ to $F_{t\to h^{-1}\tau}\Gamma_x U$ does not exceed

$$C\mu^2\hbar^{-2} \times \varepsilon^{-1}\hbar \times \hbar = C\mu^2\varepsilon^{-1}$$

where factor μ^2 is due to rescaling, $\varepsilon^{-1}\hbar \times \hbar$ is the measure of the zone $\{\theta : |\theta| \le \varepsilon^{-1}\hbar,\ |\sin(\theta)| \le \hbar\}$ and factor $\hbar^{-2}$ is the standard one.

Combining with the results for zone $\{|\sin(\theta)| \ge c\hbar\}$ we arrive to

Proposition 16.3.7. *For magnetic Schrödinger operator in domain X, $B(0,\ell) \subset X \subset \mathbb{R}^2$ under standard smoothness assumptions and non-degeneracy assumptions* (16.1.6) *and* (16.3.36)–(16.3.37), (16.3.3) *after rescaling $x \mapsto \mu x$, $t \mapsto \mu t$ (as $\mu \le h^{-1}$) estimates* (16.2.49)–(16.2.50) *of Proposition 16.2.6 and estimates* (16.2.55)–(16.2.57) *of Proposition 16.2.9 hold as $T \le c\varepsilon^{-1}$.*

Further, estimates (16.2.78)–(16.2.81) *hold as well.*

Now we apply the standard Tauberian arguments. We are interested mainly in the case of $\alpha \asymp 1$; then automatically $\ell \asymp 1$ and $T^* \asymp \mu^2$ inside domain. However in more general case $\ell \asymp \alpha \asymp \mu\varepsilon$ automatically and $T^* \asymp \ell/\varepsilon\mu\mu^2 \asymp \mu^2$ again and we arrive to

Proposition 16.3.8. *For magnetic Schrödinger operator in domain X, $B(0,\ell) \subset X \subset \mathbb{R}^2$ under standard smoothness assumptions and non-degeneracy assumptions* (16.1.6) *and* (16.3.36)–(16.3.37), (16.3.3) *Tauberian estimates* (16.2.51) *of Pproposition 16.2.8 and estimates* (16.2.57)–(16.2.58) *of Corollary 16.2.10 hold with the Tauberian expression* (16.2.52) *and $\ell \asymp \alpha$.*

16.3.4 Stationary Phase Calculations

Let apply the stationary phase method to the Tauberian expression. To do so we need to remember that assumptions $|t| \ll \varepsilon^{-1}$, $\varphi' = 0$ yield $\varphi'' \asymp |\sin(\theta)|^{-1}$ and all other estimates hold; therefore the stationary phase construction is available there. Also this construction works as long as $|t| \le c\varepsilon^{-1}$ and $|\varphi''| \ge \epsilon$ i.e. $|t - t^*| \ge \epsilon\varepsilon^{-1}$ where $T^* = \varepsilon^{-1}$.

Moreover, $|\varphi''| \ge \epsilon_0$ as $|t - t^*| \le \epsilon\varepsilon^{-1}$ and $\varphi' = 0$ and therefore all our estimates work here, in the near equator zone as well where

Definition 16.3.9. Equator just moves to the point where $|\varphi''|$ is minimal for $\varphi' = 0$. Let us in the near equator zone redefine θ in such way that equator is $\cos(\theta) = 0$.

Let us introduce

Definition 16.3.10. Consider r-term representation (16.3.22) (i.e. with summation over $m < r$) and plug it into Tauberian expression, excepting $|t| \le \epsilon_0$ and calculate it by the stationary phase method with r terms again. Let us call the result *r-term semiclassical approximation* and denote it by $h^{-2}\mathcal{N}_{x,\mathsf{corr}(r)}$.

Then its main term is delivered by modified (16.2.60)–(16.2.62)

$$\sum_{k\in\mathfrak{Z}} \frac{1}{4\pi\sqrt{\pi}}\mu\hbar^{-\frac{1}{2}}(t_k \sin(\theta_k))^{-1}|\varphi''(t_k)|^{-\frac{1}{2}} b_0(t_k)\times \exp\Big(\frac{i\pi}{4}\operatorname{sgn}\varphi''(t_k) + i\hbar^{-1}\varphi(t_k)\Big) \tag{16.3.43}$$

with

$$\varphi''(t_k) \sim 2\tau\cot(\theta_k) \qquad \text{as } |\cos(\theta_k)| \ge C\varepsilon^{\frac{1}{2}} \tag{16.3.44}$$

and

$$\mathfrak{Z} := \big\{k \ne 0 : |\cos(\theta_k)| \ge C\max\big(\varepsilon^{\frac{1}{2}}, \hbar^{\frac{1}{3}}\big)\big\} \tag{16.3.45}$$

where t_k is a time of k-th return to x (along k-th loop), $\theta_k = \theta(t_k)$ and $\varphi(t_k)$ is the corresponding action.

Then repeating arguments of the previous section we arrive to

Proposition 16.3.11. *For magnetic Schrödinger operator in domain X, $B(0,1) \subset X \subset \mathbb{R}^2$ under standard smoothness assumptions and non-degeneracy assumptions* (16.1.6) *and* (16.3.36)–(16.3.37), (16.3.3)

(i) As $\varepsilon \geq \hbar$ estimate (16.2.64) *of Proposition 16.2.11 holds.*

(ii) Estimates (16.2.66)–(16.2.67) *of Proposition 16.2.12, estimates* (16.2.68)–(16.2.71) *of Proposition 16.2.13 and estimates* (16.2.82)–(16.2.85) *of Proposition 16.2.18 hold.*

16.3.5 Approximation by a Toy-Model Operator

Weak Magnetic Field Case

We cannot do better in this framework; however using the toy-model operator as an approximation we could improve these results. Without any loss of the generality we can assume that (16.3.5)–(16.3.7) are fulfilled.

Assume temporarily that $F = 1$. Then one can prove easily that

$$t_k = \theta(t_k) \equiv \bar{t}_k, \qquad \phi(t_k) \equiv \bar{\phi}(\bar{t}_k) \mod \mu^{-2}|k| \tag{16.3.46}$$

where bar denotes objects related to the toy-model operator. Therefore $\hbar^{-1}\bigl(\phi(t_k) - \bar{\phi}(\bar{t}_k)\bigr) = O(\mu^{-2}\hbar^{-1}|k|)$ and it is less than 1 as

$$|k| \leq \tilde{k} := \mu^3 h. \tag{16.3.47}$$

Therefore it makes sense to apply this approach only as $\tilde{k} \geq 1$ i.e.

$$\mu \geq h^{-\frac{1}{3}}. \tag{16.3.48}$$

Under this assumption let us compare r-th terms for our operator and the toy model as $|k| \leq \tilde{k}$.

One can see easily that their difference does not exceed

$$C \sum_k \mu h^{-1} (\hbar/\varepsilon|k|)^{r+\frac{1}{2}} |k|^{-1} \times \mu^{-3} h^{-1} |k| \tag{16.3.49}$$

which as $r \geq 1$ does not exceed $C\mu^{-2}h^{-2}(\hbar/\varepsilon|k|)^{r+\frac{1}{2}}|k|$ calculated for the smallest possible $|k|$ which is 1 provided $\hbar \leq \varepsilon$. Therefore we get $C\mu^{-2}h^{-2}(\hbar/\varepsilon)^{r+\frac{1}{2}}$. On the other hand in (16.3.49) one should replace the last factor by 1 as $|k| \geq \tilde{k}$.

Therefore, if we start from l-term approximation and apply estimate (16.2.66), (16.2.67) as $\varepsilon \ge \hbar^{\frac{2}{3}}$, $\hbar \le \varepsilon \le \hbar^{\frac{2}{3}}$ respectively and pass from l-term to r-term approximation we get estimate

$$
\begin{aligned}
(16.3.50)\quad & \mathsf{R}^{\mathsf{W}\prime\prime}_{x(r)\gamma} := \\
& |e^{\mathsf{T}}(x,x,\tau) - h^{-2}\mathcal{N}_{x,\mathsf{corr}(r)} - \bar{e}^{\mathsf{T}}(x,x,\tau) + h^{-2}\bar{\mathcal{N}}_{x,\mathsf{corr}(r)}| \le \\
& \qquad C\mu^{-1}h^{-1} + C\mu^{-2}h^{-2}(\hbar/\varepsilon)^{r+\frac{1}{2}} + C\mu h^{-1}(\hbar/\varepsilon)^{l+\frac{1}{2}} + \\
& \qquad\qquad \begin{cases} C\mu\varepsilon h^{-1}\big(\hbar/\varepsilon^{\frac{3}{2}}\big)^{r+\frac{1}{2}} + C\mu^{\frac{4}{3}}\varepsilon h^{-\frac{2}{3}} & \text{as } \ \varepsilon \ge \hbar^{\frac{2}{3}} \\ C\mu^{\frac{5}{3}}h^{-\frac{1}{3}} & \text{as } \ \hbar \le \varepsilon \le \hbar^{\frac{2}{3}} \end{cases}
\end{aligned}
$$

where terms $-h^{-2}\mathcal{N}^{\mathsf{W}}_x$ and $h^{-2}\bar{\mathcal{N}}^{\mathsf{W}}_x$ obviously cancel one another.

Successive Approximations

Now we need to get rid of the third term in the right-hand expression of (16.3.50) without condition $\varepsilon \gg \hbar$ which allows to eliminate it. To do this we need more sophisticated arguments. Namely, let us apply the successive approximations.

Assuming that the original operator is perturbed by $O(\mu^{-2})$ we arrive to the heuristic conclusion that

(16.3.51) Contribution of $|t| \asymp T$ to $R^{\mathsf{W}\prime\prime}$ does not exceed the previous estimate of the contribution to $\mathsf{R}^{\mathsf{W}\prime}$ multiplied by $\mu^{-2}T\hbar^{-1}$;

this statement needs justification. We can achieve a better estimate by taking more terms in the successive approximation but it leads to a rather overcomplicated formula; so we take just one term.

Then, in the framework of (16.3.51) we have three cases:

(a) $\varepsilon \ge \hbar$; then all what successive approximation does is to eliminate the third term in the right-hand expression of (16.3.50).

(b) $\mu^{-3}h^{-1} \le \varepsilon \le \hbar$; then $1 \ge \bar{k} \le \varepsilon^{-1}$ and we need to adjust our arguments treating separately $|k| \le \bar{k}$ and $|k| \ge \bar{k}$ and recalculate the second term in the right-hand expression of (16.3.50); however equator zone is not affected.

(c) $\varepsilon \le \mathsf{min}(\hbar, \mu^{-3}h^{-1})$. In this case in addition we need to recalculate the last term in the right-hand expression of (16.3.50).

Let us reconsider the second term in the right-hand expression of (16.3.50). First, expression (16.3.49) with summation over $k : |k| \ge \bar{k}$ returns $C\mu^{-2}h^{-2}(\hbar/\varepsilon|k|)^{r+\frac{1}{2}}|k|$ calculated as $k = \bar{k}$; so we get $C\mu^{-2}h^{-2}\bar{k} = C\mu^{-1}h^{-1}\varepsilon^{-1}$.

Further, as $|k| \le \bar{k}$ we need to replace $(\hbar/\varepsilon|k|)^{r+\frac{1}{2}}$ by 1; so we sum $C\mu h^{-1}|k|^{-1} \times \mu^{-3}h^{-1}|k|$ which returns $C\mu^{-2}h^{-2}\bar{k}$ again. Therefore[8)] we conclude that

(16.3.52) As $\hbar \ge \varepsilon$, $r \ge 1$ one should replace the second term in (16.3.50) by $C\mu^{-1}h^{-1}\varepsilon^{-1}$.

Furthermore, as $\varepsilon \le \mathsf{min}(\hbar, \mu^{-3}h^{-1})$ one should multiply the last term in (16.3.50) i.e. $C\mu^{\frac{5}{3}}h^{-\frac{1}{3}}$ by $\mu^{-3}\varepsilon^{-1}h^{-1}$ resulting in $C\mu^{-\frac{4}{3}}\varepsilon^{-1}h^{-\frac{4}{3}}$. Therefore[8)] we conclude that

(16.3.53) As $\varepsilon \le \mathsf{min}(\hbar, \mu^{-3}h^{-1})$, $r \ge 1$ one should replace the second term in (16.3.50) by $C\mu^{-\frac{1}{3}}\varepsilon^{-1}h^{-\frac{4}{3}}$.

Therefore we (heuristically, the proof follows) arrive to

Proposition 16.3.12. *Consider two magnetic Schrödinger operators A and $\bar{A}$ satisfying the standard smoothness assumptions, non-degeneracy assumptions* (16.1.6) *and* (16.3.36)–(16.3.37), (16.3.3) *and such that*

$$(16.3.54)\quad g^{jk} = \bar{g}^{jk} + O(\mu^{-2}), \qquad V = \bar{V} + O(\mu^{-2}), \qquad V_j = \bar{V}_j + O(\mu^{-3}).$$

Then for $r \ge 1$

$$(16.3.55)\quad \mathsf{R}^{\mathsf{W}\prime\prime}_{x(r)} \le C\mu^{-1}h^{-1} + C\begin{cases} \mu^{-2}h^{-2}(\hbar/\varepsilon)^{r+\frac{1}{2}} & \text{as } \varepsilon \ge \hbar \\ C\mu^{-1}h^{-1}\varepsilon^{-1} & \text{as } \varepsilon \le \hbar \end{cases} + \begin{cases} C\mu\varepsilon h^{-1}\big(\hbar/\varepsilon^{\frac{3}{2}}\big)^{r+\frac{1}{2}} + C\mu^{\frac{4}{3}}\varepsilon h^{-\frac{2}{3}} & \text{as } \varepsilon \ge \hbar^{\frac{2}{3}} \\ C\mu^{\frac{5}{3}}h^{-\frac{1}{3}} & \text{as } \mathsf{min}(\hbar, \mu^{-3}h^{-1}) \le \varepsilon \le \hbar^{\frac{2}{3}} \\ C\mu^{-\frac{4}{3}}\varepsilon^{-1}h^{-\frac{4}{3}} & \text{as } \varepsilon \le \mathsf{min}(\hbar, \mu^{-3}h^{-1}) \end{cases}$$

[8)] Pending (16.3.51).

and

$$\mathsf{R}^{\mathsf{W}\prime\prime}_{x(r)\gamma} \coloneqq \tag{16.3.56}$$
$$\gamma^{-2}\Big|\int \Big(e^{\mathsf{T}}(x,x,\tau) - h^{-2}\mathcal{N}_{x,\mathsf{corr}(r)} - \bar{e}^{\mathsf{T}}(x,x,\tau) + h^{-2}\bar{\mathcal{N}}_{x,\mathsf{corr}(r)}\Big)\psi_\gamma\, dx\Big| \le$$
$$C\mu^{-1}h^{-1} + C\begin{cases}\mu^{-2}h^{-2}(\hbar/\varepsilon)^{r+\frac12}(\hbar/\varepsilon\gamma)^{l} & \text{as } \gamma\varepsilon \ge \hbar\\ \mu^{-2}h^{-2}(\hbar/\varepsilon)^{r+\frac12} & \text{as } \varepsilon \ge \hbar\\ \mu^{-1}h^{-1}\varepsilon^{-1} & \text{as } \varepsilon \le \hbar\end{cases} +$$
$$C\min\big(1,(h/\gamma)^{l}\big)\times\begin{cases}\mu\varepsilon h^{-1}\big(\hbar/\varepsilon^{\frac32}\big)^{r+\frac12} + C\mu^{\frac43}\varepsilon h^{-\frac23} & \text{as } \varepsilon \ge \hbar^{\frac23}\\ \mu^{\frac53}h^{-\frac13} & \text{as } \min(\hbar,\mu^{-3}h^{-1}) \le \varepsilon \le \hbar^{\frac23}\\ \mu^{-\frac13}\varepsilon^{-1}h^{-\frac43} & \text{as } \varepsilon \le \min(\hbar,\mu^{-3}h^{-1})\end{cases}$$

while for $r = 0$

$$\mathsf{R}^{\mathsf{W}\prime\prime}_{x(0)} \coloneqq |e^{\mathsf{T}}(x,x,\tau) - \bar{e}^{\mathsf{T}}(x,x,\tau)| \le C\mu^{-1}h^{-1} + \tag{16.3.57}$$
$$C\begin{cases}h^{-1}\varepsilon^{-\frac12} & \text{as } \mu^3 h \le \varepsilon^{-1},\\ \mu^{-\frac32}h^{-\frac32}\varepsilon^{-1} & \text{as } \mu^3 h \ge \varepsilon^{-1}\end{cases}$$

and

$$\mathsf{R}^{\mathsf{W}\prime\prime}_{x(0)\gamma} \coloneqq \gamma^{-2}\Big|\int \Big(e^{\mathsf{T}}(x,x,\tau) - \bar{e}^{\mathsf{T}}(x,x,\tau)\Big)\psi_\gamma\, dx\Big| \le C\mu^{-1}h^{-1} + \tag{16.3.58}$$
$$C\begin{cases}\mu^{-\frac32}h^{-1}\varepsilon^{-1}\gamma^{-\frac12} & \text{as } \mu^{-3}\varepsilon^{-1} \le \gamma \le \min(h\varepsilon^{-1},\mu^{-1}),\\ \mu^{-2}h^{-1}\varepsilon^{-1}\gamma^{-1} & \text{as } \mu^{-1} \le \gamma \le h\varepsilon^{-1},\\ \mu h^{-1}\min\big(1,(\mu h/\varepsilon)^{\frac12}\big)(h/\varepsilon\gamma)^{l} & \text{as } \gamma \ge h\varepsilon^{-1}.\end{cases}$$

Justification

To justify (16.3.51) and thus to prove Proposition 16.3.12 we need

Proposition 16.3.13. *Consider two magnetic Schrödinger operators* A *and* $\bar{A}$ *satisfying* (16.3.54). *Then as* $|t| \le \epsilon\mu^3 h$ *and* $|\sin(t)| \ge \max\big(\epsilon\varepsilon|t|, C\hbar\big)$

$$U(x,y,t)-\bar{U}(x,y,t)\equiv (4\pi\hbar)^{-1}i(\sin(\bar{\theta}(t)))^{-1}e^{i\hbar^{-1}\bar{\phi}(x,y,t)}\sum_m b'_m(x,y,t)\hbar^m \mod O\Big(\hbar^{-2}\mu^{-3}|t||\sin(\theta(t))|^{-1}\times(\hbar/|\sin(t)|)'\Big) \tag{16.3.59}$$

with

$$|D^\beta b'_m|\le C_{m\beta}\mu^{-3}|t|\hbar^{-1}|\sin(\theta(t))|^{-m-|\beta|}. \tag{16.3.60}$$

Proof. Obviously in our assumptions

$$U(x,y,t)\equiv (4\pi\hbar)^{-1}i(\sin(\bar{\theta}(t)))^{-1}e^{i\hbar^{-1}\bar{\phi}(x,y,t)}\sum_m b''_m(x,y,t)\hbar^m \mod O\Big(\hbar^{-1}|\sin(\bar{\theta}(t))|^{-1}\times(\hbar/|\sin(\bar{\theta}(t))|)'\Big)$$

with b''_m satisfying

$$|D^\alpha b''_m|\le C_{m\alpha}|\sin(\bar{\theta}(t))|^{-m-|\alpha|}$$

and plugging it into the right-hand expression of

$$e^{i\hbar^{-1}tA}-e^{i\hbar^{-1}t\bar{A}}=i\hbar^{-1}\int_0^t e^{i\hbar^{-1}t'A}(A-\bar{A})e^{i\hbar^{-1}(t-t')\bar{A}}\,dt' \tag{16.3.61}$$

we arrive to (16.3.59). Here to cover zones where either $|\sin(t')|\le\epsilon$ or $|\sin(t-t')|\le\epsilon$ we just pass to the standard representation through non-singular oscillatory integrals (with one or two extra variable) and then apply the stationary phase method. We leave easy but tedious details to the reader. □

Corollary 16.3.14. *In the framework of Proposition 16.3.13 equality* (16.3.59) *holds with* $U(x,y,t)$ *and* $\bar{U}(x,y,t)$ *replaced respectively by*

$$(4\pi\hbar)^{-1}i(\sin(\theta(t)))^{-1}e^{i\hbar^{-1}\phi(x,y,t)}\sum_m b_m(x,y,t)\hbar^m \tag{16.3.62}$$

and

$$(4\pi\hbar)^{-1}i(\sin(\bar{\theta}(t)))^{-1}e^{i\hbar^{-1}\bar{\phi}(x,y,t)}\sum_m \bar{b}_m(x,y,t)\hbar^m. \tag{16.3.63}$$

Reduction to the Toy-Model Operator

From now we are interested *only* in the estimates without micro-averaging.

The problem however is that $\bar{A}$ at this moment is a more general operator than the toy-model: it contains linear with respect to x terms in the original (before rescaling) g^{jk} and V, and also it contains quadratic terms in $V_2 = x_1 + \frac{1}{2}\beta_{11}x_1^2 + \beta_{12}x_1x_2 + \frac{1}{2}\beta_{22}x_2^2 + ...$ all of which generate $O(\mu^{-1})$ perturbation.

However one select a conformal coordinate system in such way that $\nabla\omega(0) = 0$ [9]. Then if we know that $F = F(0) + O(|x|^2)$ we conclude that $\beta_{11} = \beta_{12} = 0$ and by a gauge transformation we can make $\beta_{22} = 0$.

Further, we can assume that $\tau = 0$ and then make $V(0) = -1$ by division $V(0) \mapsto -1$, $h \mapsto h(-V(0))^{-\frac{1}{2}}$, $\mu \mapsto \mu(-V(0))^{-\frac{1}{2}}$ before rescaling. Rescaling μ (and changing coordinate orientation if needed we can make $F(0) = 1$.

Furthermore, we can assume that $F = 1$. Really, as we can consider energy level $\tau = 0$ we starting from equation $-\hbar D_t u = Au$, go to

$$F^{-1}Au = -\hbar F^{-1}D_t u = -hD_t + \hbar F^{-1}(F-1)D_t u \tag{16.3.64}$$

and on an interval $|\tau| \le \hbar h^{-\delta}T^{-1}$ (which is completely microlocally admissible as $|t| \asymp T$ the last term is $O(\mu^{-1}\hbar h^{-\delta}T^{-1})$ and multiplying by $T\hbar^{-1}$ we get $O(\mu^{-1}h^{-\delta})$ which allows us to use the successive approximation method.

We leave to the reader to use our standard methods and solve a rather easy

Problem 16.3.15. Prove that effectively in estimates one can make $\delta = 0$.

Note that $\mu^{-1} \le \mu^{-3}h^{-1}$ as $\mu \le h^{-\frac{1}{2}}$ and therefore this new error (F is replaced by $F(0)$) does not exceed the right-hand expressions of (16.3.55) and (16.3.57) as $r \ge 1$ and $r = 0$ respectively.

Meanwhile our new error does not exceed $C\mu h^{-1} \times \mu^{-1} = Ch^{-1}$ anyway and as $\mu \ge h^{-\frac{1}{2}}$ ($\implies \varepsilon \le \hbar$) it is less than $\mu^{\frac{5}{3}}h^{-\frac{1}{3}}$ leave alone $\varepsilon^{-\frac{1}{2}}h^{-1}$ which are parts of right-hand expressions of (16.3.55) and (16.3.57) respectively. Thus we arrive to

(16.3.65) Estimates (16.3.55) and (16.3.57) hold with $\bar{A}$ which is our original toy-model operator.

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[9] Really, changing conformal system to another conformal system is done with $y_1 = f$, $\nabla y_2 = (-f_{x_2}, f_{x_1})$ which can be satisfied iff $\Delta f = 0$; then ω is replaced by $\omega|\nabla f|$ and if $f = ax_1 + bx_1x_2 + \frac{1}{2}c(x_1^2 - x_2^2)$ then $|\nabla f| = a + bx_2 + cx_1 + O(|x|^2)$ and selecting a, b, c we can make $\omega(0) = 1$, $\nabla\omega(0) = 0$.

16.3.6 Strong Magnetic Field

Finally, let $\mu \ge h^{-1}$. Then as only matter is what is inside the circle of the radius $C_0\mu^{-\frac12}h^{\frac12}$ (we can get rid of $h^{-\delta}$ factor in estimates) we get in successive approximations $C\mu h^{-1} \times \mu^{-1}h \times \mu^{-\frac12}h^{\frac12}\varepsilon^{-1} \times h^{-1}$ where $\mu^{-1}h$ is the size of the perturbation, $\mu^{-\frac12}h^{\frac12}\varepsilon^{-1}$ is T (without rescaling); so we get $C\mu^{-\frac12}h^{-\frac12}\varepsilon^{-1}$.

We leave to the reader a rather easy

Problem 16.3.16. Prove that effectively in estimates one can make $\delta = 0$ as well in $\mu^{-\frac12+\delta}h^{\frac12}$ radius and $\mu^{-\frac12+\delta}h^{\frac12}\varepsilon^{-1}$ time.

So we arrive to

Proposition 16.3.17. *Under assumptions* (16.3.36), (16.3.37), $F = 1$ *and* (16.3.54) *estimate*

$$|e^{\mathsf{T}}(0,0,\tau) - \bar{e}^{\mathsf{T}}(0,0,\tau)| \le C\mu^{-\frac12}h^{-\frac12}\varepsilon^{-1} \qquad \text{as} \quad \mu \ge h^{-1}. \tag{16.3.66}$$

We leave to the reader yet another easy

Problem 16.3.18. Write correction to τ and may be α as $F \ne \mathsf{const}$ and $g^{jk} \ne \mathsf{const}$.

16.3.7 Main Theorem

Now we can write our main theorem of these two sections:

Theorem 16.3.19. *For magnetic Schrödinger operator in domain $X \subset \mathbb{R}^2$ such that $B(0,1) \subset X$ and satisfying conditions* (16.1.5)–(16.1.7) *the following estimates hold for $x \in B(0,\frac12)$*

(i) As $1 \le \mu \le h^{-\frac12}$

$$|e(x,x,0) - h^{-2}\mathcal{N}^{\mathsf{W}}_x(0)| \le C\mu^{-1}h^{-1} + C\mu^{\frac12}h^{-\frac12} + C\mu^2h^{-\frac12} \tag{16.3.67}$$

and

$$\begin{aligned} \mathsf{R}^{\mathsf{W}}_{x(r)} \coloneqq |e(x,x,0) - h^{-2}\big(\mathcal{N}^{\mathsf{W}}_x(0) + \mathcal{N}^{\mathsf{W}}_{x,\mathsf{corr}(r)}(0)\big)| \le \\ C\mu^{-1}h^{-1} + C\mu^{\frac12}h^{-\frac12} + C\mu h^{-1}\big(\mu^2h\big)^{r+\frac12} + \\ C\begin{cases} \Big(h^{-1}\big(h\mu^{\frac52}\big)^{r+\frac12} + \mu^{\frac13}h^{-\frac23}\Big) & \text{as} \quad \mu \le h^{-\frac25}, \\ \mu^{\frac53}h^{-\frac13} & \text{as} \quad \mu \ge h^{-\frac25}. \end{cases} \end{aligned} \tag{16.3.68}$$

(ii) As $h^{-\frac{1}{3}} \le \mu \le h^{-1}$

$$\mathsf{R}^{\mathsf{W}\prime\prime}_{x(r)} := \tag{16.3.69}$$
$$|\Big(e(x,x,\tau) - h^{-2}\mathcal{N}_{x,\mathsf{corr}(r)} - \bar{e}_x(x,x,\tau) + h^{-2}\bar{\mathcal{N}}_{x,\mathsf{corr}(r)}\Big)| \le$$
$$C\mu^{\frac{1}{2}}h^{-\frac{1}{2}} + C\begin{cases}\mu^{-2}h^{-2}(\mu^2 h)^{r+\frac{1}{2}} & \text{as } \mu \le h^{-\frac{1}{2}}\\ h^{-1} & \text{as } \mu \ge h^{-\frac{1}{2}},\end{cases} +$$
$$C\begin{cases}h^{-1}\big(\mu^{\frac{5}{2}}h\big)^{r+\frac{1}{2}} + \mu^{\frac{1}{3}}h^{-\frac{2}{3}} & \text{as } \mu \le h^{-\frac{2}{5}}\\ \mu^{\frac{5}{3}}h^{-\frac{1}{3}} & \text{as } h^{-\frac{2}{5}} \le \mu \le h^{-\frac{1}{2}},\\ \mu^{-\frac{1}{3}}h^{-\frac{4}{3}} & \text{as } \mu \ge h^{-\frac{1}{2}}\end{cases}$$

while for $r = 0$

$$\mathsf{R}^{\mathsf{W}\prime\prime}_{x(0)} := |e(x,x,\tau) - \bar{e}_x(x,x,\tau)| \le C\mu^{\frac{1}{2}}h^{-\frac{1}{2}} + \tag{16.3.70}$$
$$C\begin{cases}h^{-1}\mu^{\frac{1}{2}} & \text{as } \mu \le h^{-\frac{1}{2}},\\ \mu^{-\frac{1}{2}}h^{-\frac{3}{2}} & \text{as } \mu \ge h^{-\frac{1}{2}};\end{cases}$$

where here and in (iii) $\bar{e}_x$ *is constructed for a toy-model in* x*.*

(iii) For magnetic Schrödinger-Pauli operator with $\mu \ge h^{-1}$

$$|e(x,x,\tau) - \bar{e}_x(x,x,\tau)| \le C\mu^{\frac{1}{2}}h^{-\frac{1}{2}}. \tag{16.3.71}$$

16.3.8 Problems

Finally, let us formulate a series of the problems with probably simple parts (i) and really difficult parts (ii):

Problem 16.3.20. As $\mu \le h^{-1}$ drop condition (16.1.5) in the pointwise asymptotics:

(i) Use the simple rescaling technique $\mu \mapsto \mu\rho^{\frac{1}{2}}\mu$, $h \mapsto h\rho^{-\frac{3}{2}}$ with the scaling function $\rho(x) = \max(\epsilon_0|V(x)|, \mu h, \mu^{-1})$ reducing operator to the similar one either satisfying condition (16.1.5) or with $\mu h \asymp 1$ or with $\mu = 1$ and considered in Subsection 5.3.1; in two latter cases condition (16.1.5) is not needed.

(ii) Using propagation, improve what follows from the simple rescaling technique.

Problem 16.3.21. Derive asymptotics without condition (16.1.6) replaced by the non-degeneracy assumption (14.1.1) i.e. $|F| + |\nabla F| \asymp 1$:

(i) Use the simple rescaling technique $\mu \mapsto \mu\rho^2$, $h \mapsto h\rho^{-1}$ with the scaling function $\rho(x) = \max(\epsilon_0|F|, \mu^{-\frac{1}{2}})$.

(ii) Improve what follows from the simple rescaling technique.

Problem 16.3.22. Derive asymptotics under non-degeneracy assumption (16.1.8) rather than (16.1.7). The results of this section provide them at point x where $|\nabla V/F| \gg \bar{\gamma} := \max(\mu^{-1}, \mu^{-\frac{1}{2}}h^{\frac{1}{2}})$.

(i) Can one improve those results using that under condition $(16.1.8)^+$ and $(16.1.8)^-$ (when (16.1.8) holds but $(16.1.8)^+$ does not) the drift dynamic is elliptic elliptic and hyperbolic respectively?

(ii) Can one improve a trivial estimate $O(\mu h^{-1})$ at point x where $|\nabla V/F| \lesssim \bar{\gamma}$? Will be results affected by the difference between cases of $(16.1.8)^+$ and $(16.1.8)^-$?

16.4 Dirac Energy: 2D-Estimates

In this and the next sections we consider asymptotics of expression I defined by (16.1.2).

16.4.1 Tauberian Formula

Let us consider first contribution of the zone $\{(x,y)\colon |x-y| \ge C\gamma\}$.

Proposition 16.4.1. *Under conditions* (16.1.5)–(16.1.7) *the contribution of the zone* $\{(x,y)\colon |x-y| \ge C\gamma\}$ *to the remainder is* $O(\mu^{-1}h^{-1}\gamma^{-\kappa})$ *while the main part is given by the Tauberian approximation* I^{T}, *i.e. by the same expression* (16.1.1) *with* $e(x,y,0)$ *replaced by its standard implicit Tauberian approximation with* $T \asymp \epsilon\mu$ (16.1.11).

Proof. Recall that $e^{ih^{-1}tA}$ is the propagator of A and $U(x,y,t)$ is its Schwartz' kernel.

Consider expression (16.1.1) with $\omega(x,y)$ replaced by $\omega_\gamma(x,y)$ which is a cut-off of $\omega(x,y)$ in the zone $\{(x,y)\colon |x-y| \asymp \gamma\}$ and with the original functions ψ_1,ψ_2 replaced by 1. Let us replace *one* copy of $e(x,y,\tau)$ by $e(x,y,\tau,\tau') = \big(e(x,y,\tau) - e(x,y,\tau')\big)$ with $\tau' \le \tau$ and the second copy by $e(x,y,\tau'')$ and denote the resulting expression by $\mathsf{I}_\gamma(\tau,\tau',\tau'')$.

Now let us use decomposition

$$\omega_\gamma(x,y) = \gamma^{-d-\kappa}\int \psi_{1,\gamma}(x,z)\psi_{2,\gamma}(y,z)\,dz \tag{16.4.1}$$

with some $\psi_1,\psi_2 \in \mathscr{C}_0^\infty$; as before subscript γ means rescaling.

Then $\mathsf{I}_\gamma(\tau,\tau',\tau'')$ does not exceed

$$\sum_j C\gamma^{-\kappa}\|\varphi_j E(\tau,\tau')\varphi_j\|_1 \tag{16.4.2}$$

where $E(\tau,\tau') = E(\tau) - E(\tau')$, φ_j are real-valued γ-admissible functions supported in $C_0\gamma$-vicinities of z_j and balls $B(z_j, 2C_0\gamma)$ cover domain X with the multiplicity not exceeding C_0. Here we used that $\|E(\tau'')\| = 1$. Since $E(\tau,\tau')$ is a positive operator and $\varphi_j = \varphi_j^*$, one can replace the trace norm by the trace itself and get

$$\sum_j C\gamma^{-\kappa}\operatorname{Tr}\varphi_j E(\tau,\tau')\varphi_j = C\gamma^{-\kappa}\operatorname{Tr} E(\tau,\tau')\bar\psi \tag{16.4.3}$$

with $\bar\psi = \sum_j \varphi_j^2$.

Further, we know from the standard theory of Chapter 13 that under conditions (16.1.5)–(16.1.7)

$$\|E(\tau,\tau')\bar\psi\|_1 \le Ch^{-2}\big(|\tau-\tau'| + CT^{-1}h\big) \qquad \forall \tau,\tau' \in [-\epsilon,\epsilon],\ T = \epsilon\mu \tag{16.4.4}$$

and therefore

$$|\mathsf{I}_\gamma(\tau,\tau',\tau'')| \le C\gamma^{-\kappa}h^{-2}\big(|\tau-\tau'| + CT^{-1}h\big) \tag{16.4.5}$$

in the same framework, and therefore due to the standard Tauberian arguments we conclude that the contribution of zone $\{(x,y)\colon |x-y| \asymp \gamma\}$ to the Tauberian remainder estimate does not exceed $C\mu^{-1}h^{-1}\gamma^{-\kappa}$ which implies the statement immediately. □

However we need to consider also zone $\{(x,y)\colon |x-y|\le C\gamma\}$, complementary to one above. Assume that

$$\Omega_\kappa(z)=\sum_j D_{z_j}\Omega_{\kappa-1,j}+\Omega_{\kappa-1,0} \tag{16.4.6}$$

with the first subscript at Ω showing the degree of the singularity. Then

$$\omega_\kappa(x,x-y)\psi_\gamma(x-y)=\sum_j D_{x_j}\bigl(\omega_{\kappa-1,j}\psi_\gamma\bigr)+\omega_\kappa\psi'_\gamma+\omega_{\kappa-1}\psi''_\gamma \tag{16.4.7}$$

where $\psi_\gamma=\psi((x-y)\gamma^{-1})$ with ψ supported in $B(0,1)$ and equal 1 in $B(0,\frac12)$ while ψ'_γ is defined similarly with ψ' supported in $B(0,1)\setminus B(0,\frac12)$ and the last term gains 1 in the regularity.

After integration by parts expression $\mathsf{I}_{\kappa,\gamma}$, defined by (16.1.10) with Ω replaced by $\Omega\psi_\gamma$, becomes

$$-h^{-1}\sum_j\iint \omega_{\kappa-1,j}(x,y)(hD_{x_j})\bigl(e(x,y,\tau)\cdot e(y,x,\tau)\bigr)\,dxdy \tag{16.4.8}$$

plus two other terms: the term defined by (16.1.1) with the kernel $\Omega'_{\kappa,j}$ of the same singularity κ, albeit without factor h^{-1} and supported in the zone $\{(x,y)\colon |x-y|\ge\frac12\gamma\}$, and the term defined by (16.1.1) with the kernel $\Omega'_{\kappa-1,j}$, also without factor h^{-1} and of singularity $(\kappa-1)$.

The former term could be considered as before yielding to the same remainder estimate $O(\mu^{-1}h^{1-d}\gamma^{-\kappa})$. To the latter term we can apply the same trick again and again raising power (and these terms are treated in the same manner (but simpler) as we deal below with (16.4.8).

So, one needs to consider (16.4.8) and thus, denoting the second copy of $e(y,x,\tau)$ by $f(y,x,\tau)$ and without using that they are equal

$$\begin{aligned}&(hD_{x_j})\bigl(e(x,y,\tau)\cdot f(y,x,\tau)\bigr)=\\ &\bigl(hD_{x_j}e(x,y,\tau)\bigr)f(y,x,\tau)\ -\ e(x,y,\tau)\bigl(f(y,x,\tau)\,{}^t(hD_{x_j})\bigr)=\\ &\bigl(P_{j,x}e(x,y,\tau)\bigr)f(y,x,\tau)\ -\ e(x,y,\tau)\bigl(f(y,x,\tau)\,{}^tP_{j,x}\bigr).\end{aligned} \tag{16.4.9}$$

Recall that $P_j=hD_j-\mu V_j(x)$ and ${}^tP_j=-hD_j-\mu V_j(x)$ is the dual operator. Also recall that if $e(x,y,\tau)$ and $f(y,x,\tau)$ are Schwartz kernels of $E(\tau)$ and $F(\tau)$, then $P_{j,x}e(x,y,\tau)$ and $f(y,x,\tau)\,{}^tP_{j,x}$ are those of $P_jE(\tau)$ and $F(\tau)P_j$.

Therefore we are interested in the expressions of the type

$$h^{-1}\iint \omega_{\kappa-1}(x,y)e(x,y,\tau)f(x,y,\tau)\psi_\gamma\,dxdy. \tag{16.4.10}$$

If $\kappa\le 1$ then replacing $e(x,y,\tau)$ and $f(y,x,\tau)$ by their standard Tauberian expressions one gets an error not exceeding $Ch^{-1}\times\mu^{-1}h^{-2}\gamma^{1-\kappa}$ because $\|P_jE(\tau)\|\le C_0$, $\|P_jF(\tau)\|\le C_0$ where $F(\tau)$ is an operator with the Schwartz kernel $f(x,y,\tau)$ and also because

$$\begin{aligned}\sum_j \|\varphi_jP_jE(\tau,\tau')\varphi_j\|_1 \le \sum_j \|\varphi_jP_jE(\tau,\tau')\|_2\cdot\|E(\tau,\tau')\varphi_j\|_2 \le\\ \sum_j \|\varphi_jP_jE(\tau,\tau')\|_2^2+\sum_j\|E(\tau,\tau')\varphi_j\|_2^2=\\ \sum_j \mathsf{Tr}\,\varphi_jP_jE(\tau,\tau')P_j^*\varphi_j+\sum_j\mathsf{Tr}\,\varphi_jE(\tau,\tau')\varphi_j\\ \le Ch^{-2}\bigl(|\tau-\tau'|+C\mu^{-1}h\bigr)\qquad \forall\tau,\tau'\in[-\epsilon,\epsilon]\end{aligned} \tag{16.4.11}$$

which also easily follows from Chapter 13.

So, in this case one gets remainder estimate $O\bigl(\mu^{-1}h^{-1}\gamma^{-\kappa}+\mu^{-1}h^{-2}\gamma^{1-\kappa}\bigr)$ which is optimized to $O(\mu^{-1}h^{-1-\kappa})$ as $\gamma\asymp h$.

On the other hand, as $1<\kappa<2$ one can apply the same trick again since we did not use the fact that $e(.,.,.)$ and $f(.,.,.)$ coincide; then we arrive to the same estimates with P_j replaced by P_jP_k or even by $P^J:=P_{j_1}P_{j_2}\cdots P_{j_l}$:

$$\mathsf{Tr}\,P^JE(\tau,\tau')(P^J)^*\le Ch^{-2}(|\tau-\tau'|+\mu^{-1}h)\qquad\forall\tau,\tau'\in[-\epsilon,\epsilon]. \tag{16.4.12}$$

Finally,

Remark 16.4.2. note that as $\kappa\ne 1$, decomposition (16.4.6) is always possible. Further, as $\kappa=1$ this decomposition is possible as well provided one adds term $\varkappa(x)|x-y|^{-1}$ with an appropriate coefficient. On the other hand, if $\kappa=1$ and $\omega(x,y)=\varkappa(x)|x-y|^{-1}$ then this decomposition is also possible but with $\omega_{0,j}(x,y)=\varkappa(x)(x_j-y_j)|x-y|^{-1}\log|x-y|$.

So we arrive to

Proposition 16.4.3. *Let conditions* (16.1.5)–(16.1.7) *be fulfilled. Then*

(i) As $0 < \kappa < 2$ *and* <u>*either*</u> $\kappa \neq 1$ <u>*or*</u> $\kappa = 1$ *and* $\omega(x,y)$ *is replaced by* $\omega(x,y) - \varkappa(\frac{1}{2}(x+y))|x-y|^{-1}$ *with an appropriate smooth coefficient* $\varkappa(x)$*, with the error* $O(\mu^{-1}h^{-1-\kappa})$ *one can replace* $e(x,y,\tau)$ *by its standard Tauberian expression* (16.1.11) *in the formula* (16.1.11) *for* I.

(ii) As $\kappa = 1$ *and* $\omega = \varkappa(\frac{1}{2}(x+y))|x-y|^{-1}$*, with the error* $O(\mu^{-1}h^{-1-\kappa}|\log h|)$ *one can replace* $e(x,y,\tau)$ *by its standard Tauberian expression* (16.1.11) *in the formula* (16.1.11) *for* I.

Remark 16.4.4. (i) The arguments above show that in an appropriate sense one can consider arbitrary $\kappa \in \mathbb{R}$ and even in $\mathbb{C}$.

(ii) One needs only (16.4.12) rather than (16.1.5)–(16.1.7), and (16.4.12) holds as (16.1.7) is replaced by a weaker non-degeneracy condition $(16.1.8)^{+}$.

(iii) Furthermore (16.4.12) with an extra factor $(1+\mu h|\log h|)$ in the right-hand expression holds under condition (16.1.8).

The following problem seems to be difficult even without magnetiv field:

Problem 16.4.5. Can one prove the similar result for I_m with $m \geq 3$:

$$I_m \coloneqq \iint \omega(x^1,\dots,x^m)e(x^1,x^2,\tau)\psi_2(x^1)e(x^2,x^3,\tau)\cdots e(x^m,x^1,\tau)\psi_{m+1}(x^0)\times dx^1\cdots dx^m$$

with $x^{m+1} = x^1$, $\psi_{m+1} \coloneqq \psi_1$ etc and

$\omega(x^1,\dots,x^m) \coloneqq \Omega(x^1,\dots,x^m;\{x^j - x^{j+1}\}_{1\leq j\leq m})$ where function Ω is smooth in $B(0,1)^m \times B(\mathbb{R}^d\setminus 0)^{m-1}$ and homogeneous of degree $-(m-1)\kappa$ with respect to $\{x^j - x^k\}_{1\leq j<k\leq m}$. Moreover,

$$|D_z^\beta D_x^\alpha \Omega| \leq C_{\beta,\alpha}|z^1|^{-\kappa-|\beta^1|}\cdots|z^m|^{-\kappa-|\beta^m|} \qquad \text{as } \sum_k |z^k|^2 = 1,\ \sum_k z^k = 0$$

where $x = (x^1,\dots,x^m)$, $z = (z^1,\dots,z^m)$, etc.

Thus we arrive to

Proposition 16.4.6. *Let conditions* (16.1.5) *and* (16.1.6) *be fulfilled.*

(i) Further, let either condition (16.1.8)$^+$ *be fulfilled or condition* (16.1.8) *be fulfilled and* $\mu \le h^{-1}|\log h|^{-1}$. *Then* (16.1.11) *and Statements (i), (ii) of Proposition 16.4.3 hold.*

(ii) On the other hand, let condition (16.1.8) *be fulfilled and* $h^{-1}|\log h|^{-1} \le \mu \le h^{-1}$. *Then* (16.1.11) *and Statements (i), (ii) of Proposition 16.4.3 hold with an extra factor* $(1+\mu h|\log h|)$ *in the right-hand expressions.*

Remark 16.4.7. Under certain assumptions (see Chapter 19) this result could be generalized for $d \ge 4$. However in calculations we use that $d = 2$.

16.4.2 Strong Magnetic Field

Consider Schrödinger-Pauli operator as $\mu h \ge \epsilon_0$. Then we arrive immediately

Proposition 16.4.8. *Let* $\mu h \ge \epsilon_0$. *Then*

(i) Contribution of $\{|x-y| \ge \gamma\}$ *to* I *does not exceed* $C\mu h^{-1}\gamma^{-\kappa}$.

(ii) Further,

$$|\mathsf{I}| \le C\mu^{1+\frac{1}{2}\kappa}h^{-1-\frac{1}{2}\kappa}. \tag{16.4.13}$$

(iii) Furthermore, $\mathsf{I} = O(\mu^{-\infty})$ *provided*

$$V + \mu hF \ge \epsilon \qquad \forall x \in B(0,1); \tag{16.4.14}$$

in particular it is the case as $\mathfrak{z} < 1$ *and* $\mu h \ge C_0$.

Proof. Statement (i) trivially follows from the fact that $\mathscr{L}^2$ norm of $e(.,,,\tau)$ does not exceed $C\mu^{\frac{1}{2}}h^{-\frac{1}{2}}$.

Meanwhile contribution of the zone $\{(x,y) : x \in B(0,1), y \in B(0,1), |x-y| \le \gamma\}$ does not exceed $C\mu^2h^{-2}\gamma^{2-\kappa}$. Adding to (i) we conclude that the sum reaches its minimal $C\mu^{1+\frac{1}{2}\kappa}h^{-1-\frac{1}{2}\kappa}$ value as $\gamma = \mu^{-\frac{1}{2}}h^{\frac{1}{2}}$ which implies Statement (ii).

Finally, Statement (iii) is due to the fact that $e(x,y,0) = O(\mu^{-\infty})$ as long as condition (16.4.14) is fulfilled in one of points x, y; see Subsection 13.6.1.□

Proposition 16.4.9. *Let conditions* (16.1.1)–(16.1.5) *be fulfilled. Let* $\mu h \ge \epsilon$ *and one of the nondegeneracy conditions*

$$|\frac{V}{F} + (2m+1)\mu h| + |\nabla \frac{V}{F}| \ge \epsilon \tag{16.4.15}$$

or

$$|\frac{V}{F} + (2m+1)\mu h| + |\nabla \frac{V}{F}| \le \epsilon \implies \det \mathsf{Hess}\, \frac{V}{F} \ge \epsilon \tag{16.4.17}^{+}$$

be fulfilled.

(i) As $\gamma \ge C_0 \mu^{-\frac{1}{2}} h^{-\frac{1}{2}}$ *contribution of* $\{|x-y| \ge \gamma\}$ *to* I *does not exceed* $C\gamma^{-2-\kappa}$.

(ii) Estimate

$$|\mathsf{I} - \mathsf{I}^{\mathsf{T}}| \le C\mu^{\frac{1}{2}\kappa} h^{-\frac{1}{2}\kappa} \tag{16.4.16}$$

holds as $\kappa \ne 1$*; as* $\kappa = 1$ *this estimate holds with an extra factor* $|\log \mu|$ *in its right-hand expression.*

(iii) If instead of (16.4.15) *or* $(16.4.17)^{+}$ *we assume that*

$$|\frac{V}{F} + (2m+1)\mu h| + |\nabla \frac{V}{F}| \le \epsilon \implies |\det \mathsf{Hess}\, \frac{V}{F} \ge \epsilon| \tag{16.4.17}$$

then Statements (i), (ii) hold with an extra factor $|\log \mu|$ *in the right-hand expressions.*

(iv) $\mathsf{I} - \mathsf{I}^{\mathsf{T}} = O(\mu^{-\infty})$ *under ellipticity condition* $|V + (2m+1)\mu h F| \ge \epsilon$. *In particular, it happens as* $1 < \mathfrak{z} \notin 2\mathbb{Z}+1$ *and* $\mu h \ge C_0$.

Proof. (a) To prove Statement (i) recall that the drift speed does not exceed $C\mu^{-1}$ and therefore Hilbert-Schmidt norm of $\psi E(\tau)\psi'$ does not exceed C as ψ, ψ' are $\mathscr{C}_0^\infty$-functions with $\mathsf{dist}(\mathsf{supp}\,\psi, \mathsf{supp}\,\psi') \asymp 1$.

Really, it is true for a Hilbert-Schmidt norm of $\big(E(\tau) - E(\tau')\big)\psi$ with $|\tau - \tau'| \le \mu^{-1}h$ and then by Tauberian theorem it is true for a Hilbert-Schmidt norm of $\big(E(\tau) - E^{\mathsf{T}}(\tau)\big)\psi'$ with E^{T} operator with the Schwartz kernel e^{T} with time $T \asymp \mu$. However $\psi E^{\mathsf{T}} \psi'$ is negligible as $T \le \epsilon\mu$ due to propagation results.

Then rescaling $x \mapsto x/\gamma$, $y \mapsto y/\gamma$, $h \mapsto h/\gamma$, $\mu \mapsto \mu\gamma$ we conclude that for γ-admissible ψ, ψ' with $\mathsf{dist}(\mathsf{supp}\,\psi, \mathsf{supp}\,\psi') \asymp 1$ Hilbert-Schmidt norm of $\psi E(\tau)\psi'$ does not exceed C and therefore the contribution of

$$K(z) = \{(x,y) : |x-z| \le 2\gamma, |y-z| \le 2\gamma, |x-y| \ge \gamma\}$$

does not exceed $\gamma^{-\kappa}$; therefore taking partition we conclude that the contribution of zone $\{(x,y) \in B(0,1) \times B(0,1), |x-y| \ge \gamma\}$ does not exceed $C\gamma^{-2-\kappa}$.

(b) Statement (iv) follows from the fact $e - e^{\mathsf{T}} = O(\mu^{-\infty})$ under ellipticity condition.

(c) To prove Statement (ii) we apply the same arguments as in Proposition 16.4.3. While (16.4.1)–(16.4.3), (16.4.6)–(16.4.10) remain true, (16.4.4), (16.4.5) should be modified: factor h^{-2} must be replaced by μh^{-1}.

We claim that

(16.4.18) As $0 < \kappa < 1$ (16.4.11) holds with factor h^{-2} replaced by $\mu^{\frac{3}{2}} h^{-\frac{1}{2}}$.

Really, the same arguments are applied as before albeit now $P_j E(\tau)$ is bounded by $C_0(\mu h)^{\frac{1}{2}}$ rather than by C_0.

Then we arrive to the remainder estimate $O\big(\gamma^{-\kappa} + \mu^{\frac{1}{2}} h^{-\frac{3}{2}} \gamma^{1-\kappa}\big)$ and optimizing by $\gamma = \bar\gamma := C\mu^{-\frac{1}{2}} h^{\frac{1}{2}}$ we arrive to the remainder estimate $O(\bar\gamma^{-\kappa})$.

As $\kappa = 1$ we get the same estimate albeit with a logarithmic factor.

To tackle the case $1 < \kappa < 2$ one should replace (16.4.12) by

$$(16.4.12)' \quad \mathsf{Tr}\, P^J E(\tau,\tau')(P^J)^* \le C\mu h^{-1}(|\tau-\tau'| + \mu^{-1}h)(\mu h)^{|J|} \qquad \forall \tau, \tau' \in [-\epsilon, \epsilon]$$

as $\mu h \ge 1$; one can prove it easily as $\|P^J E(\tau)\| \le C(1+\mu h)^{\frac{1}{2}|J|}$.

Then as $1 < \kappa < 2$ we arrive to the remainder estimate $O(\gamma^{-\kappa} + \mu h^{-1}\gamma^{2-\kappa})$ and optimizing by $\gamma = \bar\gamma$ we again arrive to the remainder estimate $O(\bar\gamma^{-\kappa})$.

Finally, the above arguments imply Statement (iii). □

16.5 Dirac Energy: 2D-Calculations

16.5.1 Toy Model

Transformations

Consider toy-model operator (16.2.1). Let us rescale as before[10]. Then as $U(x,y,t)$ is defined by (16.2.9)–(16.2.10), and $e(x,y,\tau)$ is given by (16.2.13), and † means a complex conjugation we get

$$\begin{aligned}(16.5.1)\quad \mathsf{I} \coloneqq (2\pi)^{-2}\mu^{-4+\kappa} &\iint\iint \omega(\mu^{-1}x,\mu^{-1}y)(t't'')^{-1}\times\\ &e^{-i\hbar^{-1}(t'-t'')\tau}U(x,y,t')U^\dagger(x,y,t'')\,dt'dt''\,dxdy =\\ \frac{1}{4}(2\pi)^{-4}\hbar^{-2}\mu^{\kappa} &\iint\iint \omega(x,y)\csc(t')\csc(t'')(t't'')^{-1}\times\\ &\exp\Big(i\hbar^{-1}\big(\bar\phi(x,y,t')-\bar\phi(x,y,t'')-(t'-t'')\tau\big)\Big)\,dt'dt''\,dxdy\end{aligned}$$

with

$$\begin{aligned}(16.5.2)\quad \bar\phi(x,y,t')-\bar\phi(x,y,t'')-(t'-t'')\tau =&\\ -\frac{1}{4}\big(\cot(t')-\cot(t'')\big)(x_1-y_1)^2&\\ -\frac{1}{4}\cot(t')(x_2-y_2+2t'\varepsilon)^2+\frac{1}{4}\cot(t'')(x_2-y_2+2t''\varepsilon)^2&\\ +(x_1+y_1+2\varepsilon)(t'-t'')\varepsilon-(t'-t'')(\tau+\varepsilon^2).&\end{aligned}$$

Recall that in the original coordinates $\omega(x,y)=\Omega\big(\frac{1}{2}(x+y);x-y\big)$ where $\Omega(.,.)$ is uniformly smooth with respect to the first variable and positively homogeneous of degree $-\kappa$ with respect to the second one as $|x-y|\le 1$. Therefore without any loss of the generality one can assume that

$$(16.5.3)\qquad \omega(x,y)=\Omega\big(\frac{1}{2}(x+y);x-y\big)\psi(x-y)$$

where now Ω is positively homogeneous of degree $-\kappa$ and $\psi\in\mathscr{C}_0^\infty\big(B(0,1)\big)$ equal 1 in $B(0,\frac{1}{2})$; the difference would be of the same nature but with κ replaced by 0.

[10] We need to add factor $\mu^{-4+\kappa}=\mu^{-2}\times\mu^{-2}\times\mu^{-\kappa}$ where two factors μ^{-2} are coming from dx and dy and μ^{κ} from ω.

We can replace variables x, y with new variables $\mathsf{x} := \frac{1}{2}(x+y)$ and $z := (x-y)$ and rescale. Note that (16.5.2) depends on x in a very specific way: it does not depend on x_2 at all and it is linear with respect to x_1; thus (16.5.1) becomes[11)]

$$\frac{1}{4}(2\pi)^{-3}\hbar^{-2}\mu^{\kappa+2}\iint\iint \hat{\Omega}\big(2h^{-1}\varepsilon(t'-t''),z\big)\csc(t')\csc(t'')(t't'')^{-1}\times$$
$$\exp\Big(i\hbar^{-1}\Big(-\frac{1}{4}\big(\cot(t')-\cot(t'')\big)z_1^2-\frac{1}{4}\cot(t')(z_2+2t'\alpha\mu^{-1})^2+$$
$$\frac{1}{4}\cot(t'')(z_2+2t''\varepsilon)^2-(t'-t'')(\tau-\varepsilon^2)\Big)\Big)\,dt'dt''\,\psi(\mu^{-1}z)\,dz_1dz_2$$

where $\hat{\Omega}(\lambda;z) = F_{\mathsf{x}_1\to\lambda}\Omega$ is a partial Fourier transform. Rewriting the third line in (16.5.2) as

$$-\frac{1}{4}\big(\cot(t)-\cot(t')\big)(x_2-y_2+(t+t')\varepsilon)^2$$
$$-\frac{1}{4}\big(\cot(t)-\cot(t')\big)(t-t')^2\varepsilon^2-$$
$$\frac{1}{2}\big(\cot(t)+\cot(t')\big)\big(x_2-y_2+(t+t')\varepsilon\big)(t-t')\varepsilon$$

and replacing $t' = t+s$, $t'' = t-s$ we arrive to

(16.5.4) $\mathsf{I} =$
$$\frac{1}{2}(2\pi)^{-3}\hbar^{-2}\mu^{\kappa+2}\iint\iint \hat{\Omega}(4h^{-1}\varepsilon s,z)\csc(t+s)\csc(t-s)(t+s)^{-1}(t-s)^{-1}\times$$
$$\exp\Big(i\hbar^{-1}\Big[-\frac{1}{4}\big(\cot(t+s)-\cot(t-s)\big)\big(z_1^2+(z_2+2t\varepsilon)^2+4s^2\varepsilon^2\big)$$
$$-\big(\cot(t+s)+\cot(t-s)\big)s\varepsilon-2s(\tau-\varepsilon^2)\Big]\Big)\,dtds\,\psi(\mu^{-1}z)dz_1dz_2.$$

Remark 16.5.1. (i) For a sake of simplicity we assume (without any loss of the generalization) that $\alpha > 0$.

(ii) If $\alpha \ll 1$ in the above analysis applied to the general operators we need to take α-admissible with respect to x function Ω_α instead of Ω; thus

[11)] In these calculations we skip x_2 as an argument of Ω and integration by $d\mu^{-1}\mathsf{x}_2$

up to the shift we replace $\Omega(x,\cdot)$ by $\Omega(x\alpha^{-1},\cdot)$ and thus $\hat\Omega(2h^{-1}\varepsilon s,\cdot)$ by $\alpha^2\hat\Omega(2h^{-1}\varepsilon\alpha s,\cdot)$. In this case we will use notation I_α instead of I.

In the same time we need to consider separately $|z|\le\mu\alpha$ and $|z|\ge\mu\alpha$. For a while we will not assume that $\varepsilon\asymp\mu^{-1}\alpha$.

Case $\alpha^2\gg\mu h$

Assume first that

$$\alpha^2\ge\mu h^{1-\delta} \tag{16.5.5}$$

where for a sake of simplicity we assume that $\alpha>0$ (and therefore $\varepsilon>0$).

Remark 16.5.2. (i) In the virtue of the factor $\hat\Omega(2\hbar^{-1}\alpha^2 s,\cdot)$ under assumption (16.5.5) we need to consider only $|s|\le h^\delta$ and therefore we can consider separately $|t'|\le\epsilon_0, |t''|\le\epsilon_0$ and $|t'|\ge\epsilon_0, |t''|\ge\epsilon_0$.

(ii) Note that due to Section 6.4 contribution of zone $\{|t'|\le\epsilon_0, |t''|\le\epsilon_0\}$ defined by integral expressions (16.5.1) or (16.5.4) with an extra factor $\bar\chi_{\epsilon_0}(t')$ or $\bar\chi_{\epsilon_0}(t'')$ or $\bar\chi_{\epsilon_0}(t)$ differs from the same expression for non-magnetic Schrödinger operator by $O(\mu h^{-1-\kappa}\times\hbar^\kappa)=O(\mu^{\kappa+1}h^{-1})$ as $\kappa\ne1$ and by $O(\mu^2h^{-1}|\log\mu|)$ as $\kappa=1$.

(iii) Furthermore, if we remove from this expression for a non-magnetic Schrödinger operator cut-off $\{|t'|\ge\epsilon_0\}$ then the error would not exceed the same expression as well.

Let us consider contribution of the zone $\{|t'|\ge\epsilon_0, |t''|\ge\epsilon_0\}$ defined by an integral expressions (16.5.1) or (16.5.4) with an extra factor $\big(1-\bar\chi_{\epsilon_0}(t')\big)$. Due to Remark 16.5.2(i) we need to consider only t', t'' belonging to the same tick.

Let us consider first zone

$$\{|s|\ge\hbar/\alpha^2, |\sin(t)|\ge C|s|\}. \tag{16.5.6}$$

Then $\cot(t')-\cot(t'')\asymp-\sin^{-2}(t)s$ and integration by parts with respect to z delivers one of the factors

$$|s|^{-1}\hbar\big(|z_1|^2+|z_2|\cdot|z_2+\varepsilon t|\big)^{-1}|\sin(t)|^2, \tag{16.5.7}$$

$$|s|^{-1}\hbar\big(|z_1|^2+|z_2+\varepsilon t|^2\big)^{-1}|\sin(t)|^2. \tag{16.5.8}$$

Thus integrating by parts many times in the zone where both of these factors are less than 1 we acquire factors

$$\Big(1+|s|\hbar^{-1}\big(|z_1|^2+|z_2|\cdot|z_2+\varepsilon t|\big)|\sin(t)|^{-2}\Big)^{-l}, \tag{16.5.9}$$

$$\Big(1+|s|\hbar^{-1}\big(|z_1|^2+|z_2+\varepsilon t|^2\big)|\sin(t)|^{-2}\Big)^{-l} \tag{16.5.10}$$

respectively. Multiplying by $|z|^{-\kappa}$ and integrating we get after multiplication by $|\sin(t)|^{-2}$

$$\asymp |\sin(t)|^{-\kappa}|s|^{-1+\frac{1}{2}\kappa}\hbar^{1-\frac{1}{2}\kappa} \tag{16.5.11}$$

and integrating by t over one tick intersected with $\{t:\ |\sin(t)|\ge |s|\}$ we get

$$C\hbar^{1-\frac{1}{2}\kappa}\begin{cases} |s|^{-1+\frac{1}{2}\kappa} & \kappa<1,\\ |s|^{-\frac{1}{2}\kappa} & \kappa>1,\\ |s|^{-\frac{1}{2}}\big(1+|\log|s||\big) & \kappa=1.\end{cases} \tag{16.5.12}$$

This expression (16.5.12) must be <u>either</u> integrated by with respect to s: $|s|\le \hbar/\alpha^2$ <u>or</u> multiplied by $(\hbar/\alpha^2)^l|s|^{-l}$ due to factor $\hat\Omega$ and integrated over $|s|\ge \hbar/\alpha^2$, resulting in both cases in the same answer which is the value of (16.5.12) $\times |s|$ calculated as $s=\hbar/\alpha^2$ i.e.

$$C\hbar^{1-\frac{1}{2}\kappa}\begin{cases} (\hbar/\alpha^2)^{\frac{1}{2}\kappa} & \kappa<1,\\ (\hbar/\alpha^2)^{1-\frac{1}{2}\kappa} & \kappa>1,\\ (\hbar/\alpha^2)^{\frac{1}{2}}\big(1+|\log(\hbar/\alpha^2)|\big) & \kappa=1.\end{cases} \tag{16.5.13}$$

In addition to zone (16.5.6) we need to consider zone

$$\{|\sin(t')|\asymp |s|,\ |\sin(t'')|\le |s|\}; \tag{16.5.14}$$

its tween $\{|\sin(t')|\asymp |s|,\ |\sin(t'')|\le |s|\}$ is considered in the same way.

In zone (16.5.14) $|\cot(t')-\cot(t'')|\asymp |\sin(t'')|^{-1}$ and in this case (16.5.7), (16.5.8) are replaced by

$$\hbar\big(|z_1|^2+|z_2|\cdot|z_2+\varepsilon t|\big)^{-1}|\sin(t'')|, \tag{16.5.7$'$}$$

$$\hbar\big(|z_1|^2+|z_2+\varepsilon t|^2\big)^{-1}|\sin(t'')| \tag{16.5.8$'$}$$

and (16.5.9), (16.5.10) by

$$\Big(1+\hbar^{-1}\big(|z_1|^2+|z_2|\cdot|z_2+\varepsilon t|\big)|\sin(t'')|^{-1}\Big)^{-l}, \tag{16.5.9$'$}$$

$$\Big(1+\hbar^{-1}\big(|z_1|^2+|z_2+\varepsilon t|^2\big)|\sin(t'')|^{-1}\Big)^{-l}. \tag{16.5.10$'$}$$

Then, multiplying by $|z|^{-\kappa}$ and integrating we get after multiplication by $|\sin(t'')|^{-1}|\sin(t')|^{-1}$

$$\hbar^{1-\frac{1}{2}\kappa}|\sin(t'')|^{-\frac{1}{2}\kappa}|s|^{-1}; \tag{16.5.11$'$}$$

then integrating by $|t''|$ over one tick but intersected with $\{|\sin(t'')|\le |s|\}$ we get

$$\hbar^{1-\frac{1}{2}\kappa}|s|^{-\frac{1}{2}\kappa}. \tag{16.5.12$'$}$$

Finally, either integrating over $|s|\le \hbar/\alpha^2$ or multiplying by $|s|^{-l}(\hbar/\alpha^2)^l$ and integrating over $|s|\ge \hbar/\alpha^2$ we get in both cases the same answer $\hbar^{1-\frac{1}{2}\kappa}(\hbar\alpha^2)^{1-\frac{1}{2}\kappa}$ not exceeding (16.5.13).

Therefore the total contribution of zones (16.5.6) and (16.5.14) is given by expression (16.5.13). Then multiplying by $|k|^{-2}\mu^\kappa h^{-2}\alpha^2$ we get after summation with respect to $k: |k|\ge 1$ the value as $k=1$ i.e. (16.5.13) $\times$ $\mu^\kappa h^{-2}\alpha^2$.

Therefore we arrive to

Proposition 16.5.3. *For the toy-model operator with* $1\le \mu\le h^{-1}$ *and* $\alpha\ge \epsilon_0\mu^{-1}$ *under additional assumption* (16.5.5)

$$|\mathsf{I}^{\mathsf{T}}_\alpha-\mathsf{I}^{\mathsf{T}\prime}_\alpha|\le CR^{\mathsf{W}}(\alpha):= C\begin{cases}\mu^{1+\kappa}h^{-1}\alpha^{2-\kappa} & 0<\kappa<1,\\ \mu^2h^{-\kappa}\alpha^\kappa & 1<\kappa<2,\\ \mu^2h^{-1}\alpha(1+|\log(\mu h/\alpha^2)|) & \kappa=1,\end{cases} \tag{16.5.15}$$

where $\mathsf{I}^{\mathsf{T}\prime}_\alpha$ *is a Tauberian expression for* I *albeit with* $T=\epsilon_0\mu^{-1}$.

Now we need to explore the difference between $\mathsf{I}^{\mathsf{T}\prime}_\alpha$ and $\mathcal{I}^{\mathsf{W}}_\alpha$ defined by (16.1.9).

Proposition 16.5.4. *For the toy-model operator with* $1 \le \mu \le h^{-1}$ *and* $\alpha \ge \epsilon_0 \mu^{-1}$

$$|\mathsf{I}^{\mathsf{T}\,\prime}_{\alpha} - \mathcal{I}^{\mathsf{W}}_{\alpha}| \le C\mu^2 h^{-\kappa}\alpha^2. \tag{16.5.16}$$

Proof. Repeating arguments of Subsection 6.4.4 one can prove easily that

$$|\mathsf{I}^{\mathsf{T}\,\prime}_{\alpha} - \sum_{l:0\le l\le L-1, j=0,1} \mathcal{I}^{\mathsf{W}}_{(l)j,\alpha}\mu^l| \le C(\mu h)^L h^{-2-\kappa}\alpha^2 + C\mu^{\kappa+1}h^{-1}\alpha^2 \tag{16.5.17}$$

where $\mathcal{I}^{\mathsf{W}}_{(l)j,\mu^{-1}}$ is defined by (16.1.9) albeit with $\Omega(\frac{1}{2}(x+y), x-y)$ multiplied by a homogeneous polynomial of degree $(l+j)$ with respect to $(x-y)$ and for $l=j=0$ this polynomial is 1 and with integrals taken only over zone $\{(x,): |x-y| \lesssim \mu^{-1}\}$.

Since without any loss of the generality we can assume that Ω is even with respect to the second argument, all terms with odd $(l+j)$ vanish and since replacing μ by $-\mu$ we should arrive to the same result, all terms with odd l vanish and therefore

(16.5.18) In (16.5.17) all terms vanish except those with $j=0$ and even l.

Picking $L=2$ we arrive then to (16.5.16). □

Combining Propositions 16.5.3 and 16.5.4 and noting that the right-hand expression of (16.5.16) does not exceed $CR^{\mathsf{W}}(\alpha)$ we arrive to

Corollary 16.5.5. *For the toy-model operator with* $1 \le \mu \le h^{-1}$ *and* $\alpha \ge \epsilon_0\mu^{-1}$ *under additional assumption* (16.5.5)

$$|\mathsf{I}^{\mathsf{T}}_{\alpha} - \mathcal{I}^{\mathsf{W}}_{\alpha}| \le CR^{\mathsf{W}}(\alpha). \tag{16.5.19}$$

Improvement

In a certain case (under assumption (16.5.22) below) we can improve the results of the previous Subsubsection 16.5.1. To do this we note that (an easy proof is left to the reader)

Remark 16.5.6. Let $\zeta \le \varepsilon|t|$ with

$$\zeta := (\hbar/|s|)^{\frac{1}{2}}|\sin(t)|. \tag{16.5.20}$$

Then the sum of (16.5.9) and (16.5.10), multiplied by $|z|^{-\kappa}\,dz$ and integrated, does not exceed

$$C\begin{cases}\zeta^2(\varepsilon|t|)^{-\kappa} & 0<\kappa<1,\\ \zeta^{4-2\kappa}(\varepsilon|t|)^{\kappa-2} & 1<\kappa<2,\\ \zeta^2(\varepsilon|t|)^{-1}(1+|\log(\zeta/\varepsilon|t|)) & \kappa=1.\end{cases} \tag{16.5.21}$$

We need to multiply by $|\sin(t)|^{-2}\,dt$ and integrate. Then as $0<\kappa<1$ main contribution came from $\sin(t)\asymp 1$ and improvement is possible iff then $\zeta\le\varepsilon$ (as $|k|=1$ provided the large part of contribution). But as $s\asymp\hbar/\alpha^2$, $\sin(t)\asymp 1$ (which was the main contributor in the previous subsubsection) $\zeta\asymp\alpha$ while $\varepsilon=\mu^{-1}\alpha$.

So, improvement is possible only for $1<\kappa<2$. In this case main contribution came from $\sin(t)\asymp s$ and improvement is possible if then $\zeta\asymp(\hbar|s|)^{\frac12}\le\mu^{-1}\alpha$. Setting $|s|=\hbar/\alpha^2$ we have $\zeta\asymp\hbar/\alpha$ and therefore improvement is possible if

$$\alpha^2\ge\mu^2 h. \tag{16.5.22}$$

Consider zone (16.5.6) first. Then multiplication by $|\sin(t)|^{-2}\,dt$ and integration results instead of (16.5.12) in

$$C\begin{cases}\hbar^{\frac12}|s|^{-\frac12}\varepsilon^{1-\kappa} & 1<\kappa<\dfrac32,\\ \hbar^{2-\kappa}|s|^{1-\kappa}\varepsilon^{\kappa-2} & \dfrac32<\kappa<2,\\ \hbar^{\frac12}|s|^{-\frac12}\varepsilon^{-\frac12}(1+|\log(\hbar|s|/\varepsilon^2)) & \kappa=\dfrac32\end{cases} \tag{16.5.23}$$

where we set $|k|=1$. Finally, either integrating over $|s|\le\hbar/\alpha^2$ or multiplying by $|s|^{-l}(\hbar/\alpha^2)^l$ and integrating over $|s|\ge\hbar/\alpha^2$ we get instead of (16.5.13)

$$C\begin{cases}\mu^\kappa h\alpha^{-\kappa} & 1<\kappa<\dfrac32,\\ \mu^{6-3\kappa}h^{4-2\kappa}\alpha^{3\kappa-6} & \dfrac32<\kappa<2,\\ \mu^{\frac32}h\alpha^{-\frac32}(1+|\log(\mu^2h/\alpha^2)|) & \kappa=\dfrac32.\end{cases} \tag{16.5.24}$$

Consider zone (16.5.14); we apply Remark 16.5.6 albeit with

$$\zeta := (\hbar|\sin(t'')|)^{\frac12}. \tag{16.5.20}'$$

Again multiplying by $|\sin(t')|^{-1}|\sin(t'')|^{-1}$ and integrating by $|t''|$ over one tick but intersected with $\{|\sin(t'')| \le |s|\}$ and setting $|\sin(t')| = |s|$, and finally either integrating over $|s| \le \hbar/\alpha^2$ or multiplying by $|s|^{-l}(\hbar/\alpha^2)^l$ and integrating over $|s| \ge \hbar/\alpha^2$, we arrive to the same answer, not exceeding (16.5.24).

Finally, multiplying by $|k|^{-2}\mu^\kappa h^{-2}\alpha^2$ we get after summation with respect to $k : |k| \ge 1$ the following improvement to Proposition 16.5.3

Proposition 16.5.7. *For the toy-model operator with* $1 \le \mu \le h^{-1}$ *and* $\alpha \ge \max(\mu^{-1}, \mu h^{\frac12}, \mu^{\frac12} h^{\frac12-\delta})$

$$|\mathsf{I}^{\mathsf{T}}_\alpha - \mathsf{I}^{\mathsf{T}\prime}_\alpha| \le CR^{\mathsf{W}}_1(\alpha) := \tag{16.5.25}$$
$$C\begin{cases} \mu^{2\kappa}h^{-1}\alpha^{2-\kappa} & 1 < \kappa < \frac32,\\ \mu^{6-2\kappa}h^{2-2\kappa}\alpha^{3\kappa-4} & \frac32 < \kappa < 2,\\ \mu^3 h^{-1}\alpha^{\frac12}(1+|\log(\mu^2 h/\alpha^2)|) & \kappa = \frac32. \end{cases}$$

Remark 16.5.8. Right-hand expression of (16.5.25) in comparison with (16.5.15) has factor

$$C\begin{cases} (\mu^2 h/\alpha^2)^{\kappa-1} & 1 < \kappa < \frac32,\\ (\mu^2 h/\alpha^2)^{2-\kappa} & \frac32 < \kappa < 2,\\ (\mu^2 h/\alpha^2)^{\frac12}(1+|\log(\mu^2 h)|) & \kappa = \frac32. \end{cases} \tag{16.5.26}$$

While we cannot directly apply estimate (16.5.16) now as $\mu^2 h^{-\kappa}$ may be greater than $R^{\mathsf{W}}_1(\alpha)$, we can apply (16.5.17)–(16.5.18) with $L = 4$ resulting in

Proposition 16.5.9. *For the toy-model operator with* $1 \le \mu \le h^{-1}$ *and* $\alpha \ge \max(\mu^{-1}, \mu h^{\frac12}, \mu^{\frac12} h^{\frac12-\delta})$

$$|\mathsf{I}^{\mathsf{T}}_\alpha - \mathcal{I}^{\mathsf{W}}_\alpha - \mathcal{I}^{\mathsf{W}}_{\mathsf{corr}}| \le CR^{\mathsf{W}}_1(\alpha) \tag{16.5.27}$$

with $\mathcal{I}^{\mathsf{W}}_{\mathsf{corr}} := \mathcal{I}^{\mathsf{W}}_{(2)0,\alpha}$ *but taken only over zone* $\{|x-y| \le \mu^{-1}\}$.

Case $\alpha^2 \not\gg \mu h$

Assume now that $\alpha \ge \mu^{-1}$ but (16.5.5) fails. Then in contrast to the previous we will need to compute contributions of pair of ticks (k', k'') with $k' \ne k''$. Then if t', t'' belong to k'-th and k''-th ticks respectively we denote $r = k' - k''$ and $s = t' - t'' - 2\pi r$.

Contribution of $k' = k'' \ne 0$. Consider case $k' - k'' = r = 0$ first. As $\alpha^2 \ge \hbar$ applying the same arguments as before we get the same answer (16.5.13) as before. Meanwhile, as $\alpha^2 \le \hbar$ we need to integrate (16.5.12) over $\{s : |s| \le 1\}$ and we arrive to $\hbar^{1-\frac{1}{2}\kappa}$ for all $\kappa : 0 < \kappa < 2$.

Multiplication by $\mu^\kappa h^{-2}|k|^{-2}\alpha^2$ and summation with respect to k results in the right hand expression of (16.5.15) as $\alpha^2 \ge \hbar$ and

$$C\mu^\kappa h^{-2}\hbar^{1-\frac{1}{2}\kappa}\alpha^2 = C\mu^{\frac{1}{2}\kappa+1}h^{-1-\frac{1}{2}\kappa}\alpha^2 \tag{16.5.28}$$

as $\alpha^2 \le \hbar$.

Contribution of $0 \ne k' \ne k'' \ne 0$. Consider now the case $k' - k'' = r \ne 0$. As $\alpha^2 \ge \hbar$ expression (16.5.13) acquires factor $\big(|k' - k''|\alpha^2/\hbar\big)^{-l}$ and multiplying by $\mu^\kappa h^{-2}\alpha^2|k'|^{-1}|k''|^{-1}$ we get after summation with respect to $k' \ne 0$, $k'' \ne 0$ the same right-hand expression of (16.5.15) [12].

Meanwhile, as $\alpha^2 \le \hbar$ instead of (16.5.13) we get

$$C\hbar^{1-\frac{1}{2}\kappa}\big(1 + |k' - k''|\alpha^2/\hbar\big)^{-l}.$$

Multiplication by $\mu^\kappa h^{-2}\alpha^2|k'|^{-1}|k''|^{-1}$ and summation with respect to k', k'' returns

$$C\mu^{1+\frac{1}{2}\kappa}h^{-1-\frac{1}{2}\kappa}\alpha^2\big(1 + \big(\log(\hbar/\alpha^2)\big)_+\big)^2. \tag{16.5.29}$$

Contribution of $0 = k' \ne k''$. Consider now $k' = 0, k'' \ne 0$; its tween case $k' \ne 0, k'' = 0$ is addressed in the same way. To do this we need to modify our expression fo I: namely, we do not divide by t' and then take Fourier transform $F_{t' \to \hbar^{-1}\tau}$; instead we take Fourier transform $F_{t' \to \hbar^{-1}\tau'}$, then

[12] Actually, with an extra factor $(\hbar/\alpha^2)^l$ but we do not need it here.

integrate by τ' to τ and divide by $\hbar$; modifying this way (16.5.1) we arrive instead of (16.5.4) to

$$\mathsf{I}_\alpha = \frac{1}{2}(2\pi)^{-3}\hbar^{-3}\mu^{\kappa+2}\alpha^2 \int\iint\iint \hat{\Omega}(4\hbar^{-1}\alpha^2 s, z)\times \csc(t+s)\csc(t-s)(t+s)^{-1}(t-s)^{-1}\times \exp\Big(i\hbar^{-1}\Big[-\frac{1}{4}\big(\cot(t+s)-\cot(t-s)\big)\big(z_1^2+(z_2+2t\varepsilon)^2+4s^2\varepsilon^2\big) -\big(\cot(t+s)+\cot(t-s)\big)s\varepsilon-2s(\tau-\varepsilon^2)+(t+s)(\tau-\tau')\Big]\Big)\, dtds\, \psi(\mu^{-1}z)dz_1dz_2\, d\tau'. \tag{16.5.30}$$

Then repeating our above arguments we conclude that

(16.5.31) As $\alpha^2 \ge \hbar$ the resulting term (instead of (16.5.13)) does not exceed (16.5.13) $\times\, (\hbar/\alpha^2|k''|)^l \times \hbar^{-1}$ and as $\alpha^2 \le \hbar$ the resulting term does not exceed $C\hbar^{1-\frac{1}{2}\kappa} \times \big(1+\alpha^2|k''|\hbar^{-1}\big)^{-l} \times \hbar^{-1}$

and we can rewrite the result in both cases as

$$C\hbar^{-\frac{1}{2}\kappa}\big(1+\alpha^2|k''|\hbar^{-1}\big)^{-l}.$$

Multiplying by $\mu^\kappa h^{-2}|k''|^{-1}\alpha^2$ we get after summation with respect to $k'' \ne 0$

$$C\mu^{\frac{1}{2}\kappa}h^{-2-\frac{1}{2}\kappa}\alpha^2(\hbar/\alpha^2)^l \qquad \text{as } \alpha^2 \ge h, \tag{16.5.32}$$

$$C\mu^{\frac{1}{2}\kappa}h^{-2-\frac{1}{2}\kappa}\alpha^2\big(1+(\log(\hbar/\alpha^2))_+\big) \qquad \text{as } \alpha^2 \le \hbar. \tag{16.5.33}$$

So we have proven

Proposition 16.5.10. *For the toy-model operator with* $1 \le \mu \le h^{-1}$, $\alpha \ge \mu^{-1}$

$$|\mathsf{I}^{\mathsf{T}}_{(\alpha)} - \mathcal{I}^{\mathsf{W}}_{(\alpha)}| \le (16.5.15) + (16.5.32) \qquad \textit{as } \alpha^2 \ge \hbar, \tag{16.5.34}$$

$$|\mathsf{I}^{\mathsf{T}}_{(\alpha)} - \mathcal{I}^{\mathsf{W}}_{(\alpha)}| \le (16.5.28) + (16.5.29) + (16.5.33) \qquad \textit{as } \alpha^2 \le \hbar. \tag{16.5.35}$$

16.5.2 General Operators

Consider now general operators satisfying (16.1.5)–(16.1.6) and either (16.1.7) or (16.1.8) or $(16.1.8)^+$, $\tau = 0$.

First, using Proposition 16.4.1, Remark 16.4.4(i),(ii) and propagation results we conclude that

Proposition 16.5.11. *Let conditions* (16.1.5)–(16.1.6) *be fulfilled. Then*

(i) Contribution of zone $\{|x-y| \ge C_0\mu^{-1}\}$ *to* I^{T} *does not exceed* $C\mu^{\kappa+1}h^{-1}$ *and*

(ii) Under condition (16.1.7) *contribution of time* $\{|t| \ge C_0\}$ *(before rescaling) to* I^{T} *does not exceed* $C\mu^{\kappa+1}h^{-1}$.

Note that $O(\mu^{\kappa+1}h^{-1})$ is smaller than the remainder estimate $R^{\mathsf{W}}(1)$ or $R_1^{\mathsf{W}}(1)$ and as long as we are interested in Weyl approximation $\mathcal{I}^{\mathsf{W}}$ (even with correction term) we should be completely happy with it.

Under condition (16.1.7) we are done, immediately arriving to Statement (i) and related part of Statement (iii) of Theorem 16.5.13 below.

Under condition (16.1.8) let us introduce

$$\alpha(x) = \epsilon_0|\nabla VF^{-1}| + \frac{1}{2}\bar{\alpha}, \qquad \bar{\alpha} = C_0 \max\bigl(\mu^{-1}, (\mu h)^{\frac{1}{2}-\delta}\bigr) \tag{16.5.36}$$

and consider covering of $B(0,1)$ by α-admissible elements. In virtue of Proposition 16.5.11(i) we need to consider only pairs (x,y) belonging to the same element. Due to rescaling, contribution of one such element to $|I^{\mathsf{T}} - \mathcal{I}^{\mathsf{W}}|$ does not exceed $C\mu h^{-1-\kappa}\alpha^2$ and therefore

(16.5.37) Under condition (16.1.8) contribution of zone

$$\{(x,y) : |x-y| \le C_0\mu^{-1},\ |\nabla VF^{-1}| \le \alpha\}$$

to $|I^{\mathsf{T}} - \mathcal{I}^{\mathsf{W}}|$ does not exceed $C\mu h^{-1-\kappa}\alpha^2$.

Therefore we need to consider only zone where $|\nabla VF^{-1}(x)| \ge \alpha$ and $|x-y| \le C_0\mu^{-1}$. We are going to prove that

Proposition 16.5.12. *Let conditions* (16.1.5)–(16.1.6) *be fulfilled. Further, let* $\mu \le h^{-1}$. *Then*

(i) For each z *with* $\alpha(z) \ge \bar{\alpha}$ *contribution of zone*

$$\{(x,y) \in B(z,\alpha(z)),\ |x-y| \le C_0\mu^{-1}|\}$$

to $|\mathsf{I}^{\mathsf{T}} - \mathcal{I}^{\mathsf{W}}|$ *does not exceed* $R^{\mathsf{W}}(\alpha)$ *which is defined as the right-hand expression of* (16.5.15) *or* (16.5.25).

(ii) Further, as $\alpha^2 \ge \mu^2 h$, $1 < \kappa < 2$ *this contribution does not exceed* $R_1^{\mathsf{W}}(\alpha)$ *which is defined as the right-hand expression of* (16.5.25).

Note that both $R^{\mathsf{W}}(\alpha)$ and $R_1^{\mathsf{W}}(\alpha)$ contain α in the positive powers and thus after integration with respect $\alpha^{-1}\,d\alpha$ we get their values as $\alpha = 1$.

Therefore we arrive to Statement (ii) and related part of Statement (iii) of Theorem 16.5.13:

Theorem 16.5.13. *Let conditions* (16.1.5)–(16.1.6) *be fulfilled. Then as* $\mu \le h^{-1}$

(i) Under condition (16.1.7)

$$|\mathsf{I} - \mathcal{I}^{\mathsf{W}}| \le C\mu^{-1}h^{-1-\kappa} + R^{\mathsf{W}} \tag{16.5.38}$$

with

$$R^{\mathsf{W}} := C\begin{cases} \mu^{\kappa+1}h^{-1} & \text{as } \kappa < 1,\\ \mu^2 h^{-\kappa} & \text{as } \kappa > 1,\\ \mu^2 h^{-1}|\log(\mu h)| & \text{as } \kappa = 1.\end{cases} \tag{16.5.39}$$

(ii) Under condition (16.1.8)[13)]

$$|\mathsf{I} - \mathcal{I}^{\mathsf{W}}| \le C\mu^{-1}h^{-1-\kappa} + R^{\mathsf{W}} + C\mu^2 h^{-\kappa}(\mu h)^{-\delta}. \tag{16.5.40}$$

(iii) As $\mu^2 h \le 1$ *and* $1 < \kappa < 2$ *in these estimates one can replace* $\mathcal{I}^{\mathsf{W}}$ *by* $\mathcal{I}^{\mathsf{W}} + \mathcal{I}^{\mathsf{W}}_{\mathsf{corr}}$ *and* R^{W} *by*

$$R_1^{\mathsf{W}} := \begin{cases} \mu^{2\kappa}h^{-1} & 1 < \kappa < \dfrac{3}{2},\\ \mu^{6-2\kappa}h^{2-2\kappa} & \dfrac{3}{2} < \kappa < 2,\\ \mu^3 h^{-1}(1+|\log(\mu^2 h)|) & \kappa = \dfrac{3}{2}.\end{cases} \tag{16.5.41}$$

[13)] If $(16.1.8)^+$ fails the first term should acquire factor $(1+\mu h|\log h|)$ but it is important only as μ is close to h^{-1} but then the other terms dominate.

Remark 16.5.14. (i) Actually as $\mu^2 h \le 1$, $\kappa \ge \frac{3}{2}$ this correction term does not exceed $C\mu^{-1}h^{-1-\kappa}$.

(ii) Consider other terms in the estimates. Then (16.5.28), (16.5.29), (16.5.33) contain α in the positive powers and (16.5.32) contains α in the negative power and thus after integration with respect $\alpha^{-1}\,d\alpha$ we get their values as $\alpha = (\mu h)^{\frac{1}{2}}$ i.e. (16.5.28), (16.5.29) become $C\mu^{\frac{1}{2}\kappa+2}h^{-\frac{1}{2}\kappa}$ which is less than (16.5.39) and (16.5.41) while (16.5.32), (16.5.33) become $C\mu^{\frac{1}{2}\kappa+1}h^{-1-\frac{1}{2}\kappa}$ which is larger than (16.5.39). Therefore selecting $\alpha \ge (\mu h)^{\frac{1}{2}-\delta}$ saves us from all these problems for the price of $\delta \ne 0$.

Even as $h^{-1+\delta} \le \mu \le h^{-1}$ it works as $C\mu^{\frac{1}{2}\kappa+1}h^{-1-\frac{1}{2}\kappa}$ acquires factor $(\mu h)^l$ which makes it subordinate to (16.5.39).

The following problem seems to be challnging:

Problem 16.5.15. Prove above results with $\delta = 0$.

Proof of Proposition 16.5.12. (a) First of all applying Proposition 16.3.5 and the same arguments as in the analysis of Section 16.3 we arrive to

(16.5.42) Contribution of zone $\{|\sin(\theta(t'))| \ge C_0\hbar,\ |\sin(\theta(t''))| \ge C_0\hbar\}$ to the error $|\mathsf{I}^{\mathsf{T}} - \mathcal{I}^{\mathsf{W}}|$ does not exceed expression (16.5.15) or (16.5.41) (in its framework).

Therefore we need to explore contributions of three remaining zones

$$\{|\sin(\theta(t'))| \le 2C_0\hbar,\ |\sin(\theta(t''))| \le 2C_0\hbar\} \tag{16.5.43}$$

and

$$\{|\sin(\theta(t'))| \ge 2C_0\hbar,\ |\sin(\theta(t''))| \le C_0\hbar\} \tag{16.5.44}$$

(the same analysis would cover its tween zone $\{|\sin(\theta(t'))| \ge 2C_0\hbar,\ |\sin(\theta(t''))| \le C_0\hbar\}$).

Note first that the standard propagation arguments imply that we get a factor $C(|z_1| + |z_2|)^{-l}(|\sin(\theta(t'))| + \varepsilon|t'|)^l$ and the similar factor with t' replaced by t''.

(b) Consider zone (16.5.43) as $\varepsilon \ge C_1\hbar$. Then in the classical propagation shift $\varepsilon|k'|$ is always greater than $\hbar$.

Obviously contribution of zone (16.5.43), intersected with (k', k'') windings $\times\{(x,y) : |x-y| \ge \rho\}$ to the Tauberian expression does not exceed

$$C\mu^2 h^2 |k'|^{-1}|k''|^{-1}\rho^{-\kappa} \sup_t \iint |\tilde{U}(x,y,t)|^2\, dxdy \le C\mu^2 h^{2-d}\rho^{-\kappa}\alpha^d \tag{16.5.45}$$

where factor $\hbar^2$ is due to integration with respect to t', t'' and $\tilde{U}$ means cut-off of U to the energy level $|\tau| \le c$.

Plugging (as $d = 2$)

$$\rho = \varepsilon\mu^{-1}\max(|k'|, |k''|), \tag{16.5.46}$$

$\varepsilon = \mu^{-1}\alpha$ we get

$$C\mu^{2\kappa+2}|k'|^{-1}|k''|^{-1}\alpha^{2-\kappa}\big(\max(|k'|, |k''|)\big)^{-\kappa}. \tag{16.5.47}$$

Summation by k', k'' returns $C\mu^{2\kappa+2}\alpha^{2-\kappa}$ which is less than the right-hand expression of (16.5.15) or (as $\alpha^2 \ge \mu^2 h$, $1 < \kappa < 2$) the right-hand expression of (16.5.25).

Meanwhile, estimating $\tilde{U}(x,y,t)$ by h^{-d} and integrating by $|x-y| \le \rho$ and then integrating by t', t'' over $|\theta(t') - \pi k'| \le \hbar$, $|\theta(t'') - \pi k''| \le \hbar$ we get

$$Ch^{2-2d}\mu^2\rho^{d-\kappa}|k'|^{-1}|k''|^{-1}\big(h/\rho\big)^l\alpha^d \tag{16.5.48}$$

where the last factor is due to propagation results; plugging $d = 2$, (16.5.46) we get

$$C\mu^2 h^{-\kappa}|k'|^{-1}|k''|^{-1}\alpha^2\big(\hbar/\varepsilon\max(|k'|, |k''|)\big)^l; \tag{16.5.49}$$

after summation we get $C\mu^2 h^{-\kappa}\big(\hbar/\varepsilon\big)^l\alpha^2$ which is again less than the right-hand expression of (16.5.15) or (as $\alpha^2 \ge \mu^2 h$, $1 < \kappa < 2$) the right-hand expression of (16.5.25).

(c) Consider zone (16.5.43) as $\varepsilon \le C_1\hbar$. Taking sum of (16.5.47) over k', k'' with $\max(|k'|, |k''|) \ge \hbar/\varepsilon$ we get

$$C\mu^{2\kappa+2}\alpha^{2-\kappa}(1 + |\log \mu^2 h/\alpha|) \times (\mu^2 h/\alpha)^{-\kappa} = C\mu^2 h^{-\kappa}\alpha^2(1 + |\log \mu^2 h/\alpha|)$$

while taking sum of (16.5.49) we get the right-hand expression as well. Obviously, it is less than the right-hand expression of (16.5.15).

Thus we need to consider only $k', k'' : \mathsf{max}(|k'|, |k''|) \le \hbar/\varepsilon$.

Using the same arguments as before, we conclude that the contribution of zone (16.5.43), intersected with (k', k'') windings $\times\{(x, y) : |x - y| \ge \hbar\}$ to the Tauberian expression does not exceed $C\mu^2 h^{-\kappa}|k'|^{-1}|k''|^{-1}$; similarly contribution of zone (16.5.43), intersected with (k', k'') windings $\times\{(x, y) : |x - y| \le \hbar\}$ to the Tauberian expression does not exceed $C\mu^2 h^{-\kappa}|k'|^{-1}|k''|^{-1}\alpha^2$ as well.

After summation we get $C\mu^2 h^{-\kappa}(1 + |\log \mu^2 h/\alpha|)^2$ which again is less than the right-hand expression of (16.5.15).

(d) Furthermore, consider zone (16.5.44) as $\varepsilon \ge C_1\hbar$. Now shift with respect to $(x - y)$ is $\mu^{-1}\mathsf{max}\big(\varepsilon|t|, |\varepsilon t' \mp \mathsf{sin}(t')|\big)$; then repeating analysis in zone (16.5.43) we arrive to (16.5.47) for $1 < \kappa < 2$ and to (16.5.49) for $0 < \kappa < 2$; as $0 < \kappa \le 1$ one should replace (16.5.47) by

$$C\mu^{2\kappa+2}\hbar|k'|^{-1}|k''|^{-1} \tag{16.5.50}$$

with a logarithmic factor as $\kappa = 1$ and summation with respect to k', k'' results in $C\mu^{3\kappa+2}h(1 + |\log h|)^2$ which again is less than $R^{\mathsf{W}}(\alpha)$.

(e) Finally, consider zone (16.5.44) as $\varepsilon \le C_1\hbar$. With the above adjustment arguments of zone (16.5.43) work here as well. □

16.5.3 Perturbations

General Scheme

Contribution of $k' = k'' \ne 0$. Let us start from the case case $k' = k'' = k$. Assume first that $\alpha^2 \ge \hbar$.

Recall that k-th tick's contribution to I^{T} was not exceeding expression (16.5.13) multiplied by $\mu^\kappa h^{-2}\alpha^2|k|^{-2}$ i.e.

$$C\mu^{\kappa+2}\alpha^2|k|^{-2}\hbar^{-1-\frac{1}{2}\kappa}\begin{cases}(\hbar/\alpha^2)^{\frac{1}{2}\kappa} & \kappa < 1,\\ (\hbar/\alpha^2)^{1-\frac{1}{2}\kappa} & \kappa > 1,\\ (\hbar/\alpha^2)^{\frac{1}{2}}(1 + |\log(\hbar/\alpha^2)|) & \kappa = 1\end{cases} \tag{16.5.51}$$

(as $|\mathsf{sin}(t')| \le \hbar$ or $|\mathsf{sin}(t'')| \le \hbar$ one should use Proposition 16.5.12). Then we can suspect that the successive approximations bringing factor $\sigma T/\hbar$ as

U is replaced by U^0 simultaneously brings the same factor when we replace U by $\bar{U}$ in I; so contribution of k-th tick to the error is estimated by the previous expression multiplied by $C\sigma|k|/\hbar$ i.e.

$$C\sigma\mu^{\kappa+2}\alpha^2|k|^{-1}\hbar^{-2-\frac{1}{2}\kappa}\begin{cases}(\hbar/\alpha^2)^{\frac{1}{2}\kappa} & \kappa<1,\\ (\hbar/\alpha^2)^{1-\frac{1}{2}\kappa} & \kappa>1,\\ (\hbar/\alpha^2)^{\frac{1}{2}}(1+|\log(\hbar/\alpha^2)|) & \kappa=1.\end{cases} \tag{16.5.52}$$

and summation over $k:|k|\le \bar{k}:=\hbar/\sigma$ results in

$$C\sigma\mu^{\kappa+2}\alpha^2|\log\hbar/\sigma|\hbar^{-2-\frac{1}{2}\kappa}\begin{cases}(\hbar/\alpha^2)^{\frac{1}{2}\kappa} & \kappa<1,\\ (\hbar/\alpha^2)^{1-\frac{1}{2}\kappa} & \kappa>1,\\ (\hbar/\alpha^2)^{\frac{1}{2}}(1+|\log(\hbar/\alpha^2)|) & \kappa=1.\end{cases} \tag{16.5.53}$$

Remark 16.5.16. (i) Here σ is the magnitude of perturbation and as we approximate by toy-model it is $O(\mu^{-2})$ but if we are trying to use the magnetic Weyl approximation it is $O(\varepsilon)$.

(ii) Expression (16.5.53) provides us with the answer for $\alpha^2\ge\hbar$. On the other hand, if $\alpha^2\le\hbar$ we need to recall that as long as we are interested only in the zone $\{(x,y)\colon |x-y|\le C_0\mu^{-1}\}$ before rescaling then $|z|\le C_0$ and we need therefore there reset $\hbar/\alpha^2$ to 1 and also $|s|$ to 1 resulting in

$$C\sigma\mu^{\kappa+2}\alpha^2|\log\hbar/\sigma|\hbar^{-2-\frac{1}{2}\kappa} \tag{16.5.54}$$

(iii) In these settings we need to assume that $T\le C_0\varepsilon^{-1}$ i.e. $\bar{k}\le C_0\varepsilon^{-1}$ i.e.

$$\mu h\le \sigma/\varepsilon. \tag{16.5.55}$$

This inequality always holds as $\sigma=\varepsilon$ but it may fail as $\sigma=\mu^{-2}$; in this case we either reset $\bar{k}$ to $C\varepsilon^{-1}$ or reevaluate σ and analyze separately in both cases (see in the next subsubsection).

(iv) Note, as $\mu h\le\sigma/\varepsilon$ summation (16.5.51) for $k:|k|\ge\bar{k}$ brings the same expression albeit without logarithmic factor; otherwise if we reset $\bar{k}=C\varepsilon^{-1}$ this sum returns $C\mu^{\kappa}h^{-1}$.

(v) However, for the toy-model approximation we can reexamine the zone $\{(x,y)\colon |x-y|\ge C_0\mu^{-1}\}$ and improve the remainder estimate of (iv).

Contribution of $0 \neq k' \neq k'' \neq 0$. Next we need to consider $0 \neq k' \neq k'' \neq 0$. Recall that as $\alpha^2 \geq \hbar$ the contribution of such pair to I^{T} does not exceed

$$C\mu^{\kappa+2}\alpha^2|k'|^{-1}|k''|^{-1}(\hbar/\alpha^2|k'-k''|)^l\hbar^{-1-\frac{1}{2}\kappa}\times \begin{cases}(\hbar/\alpha^2)^{\frac{1}{2}\kappa} & \kappa<1,\\ (\hbar/\alpha^2)^{1-\frac{1}{2}\kappa} & \kappa>1,\\ (\hbar/\alpha^2)^{\frac{1}{2}}(1+|\log(\hbar/\alpha^2)|) & \kappa=1;\end{cases} \tag{16.5.56}$$

multiplying by $C\sigma|k'|/\hbar$ (as perturbation goes to the first "factor") we get

$$C\sigma\mu^{\kappa+2}\alpha^2|k''|^{-1}(\hbar/\alpha^2|k'-k''|)^l\hbar^{-2-\frac{1}{2}\kappa}\times \begin{cases}(\hbar/\alpha^2)^{\frac{1}{2}\kappa} & \kappa<1,\\ (\hbar/\alpha^2)^{1-\frac{1}{2}\kappa} & \kappa>1,\\ (\hbar/\alpha^2)^{\frac{1}{2}}(1+|\log(\hbar/\alpha^2)|) & \kappa=1.\end{cases} \tag{16.5.57}$$

Then summation with respect to k' of $(\hbar/\alpha^2|k'-k''|)^l$ returns $(\hbar/\alpha^2)^l$ as $\hbar \leq \alpha^2$ and summation with respect to k'' then returns (16.5.53) with an extra factor $(\hbar/\alpha^2)^l$ and therefore is less than (16.5.53).

On the other hand, as $\hbar \geq \alpha^2$ summation with respect to k' of 1 returns $\hbar/\alpha^2$ and as we recall that big left brace expression should be replaced by 1 we arrive after summation with respect to k'' to

$$C\sigma\mu^{\kappa+2}\hbar^{-1-\frac{1}{2}\kappa}(1+(\log\hbar/\sigma)) \tag{16.5.58}$$

which is (16.5.54) with an extra factor $(\hbar/\alpha^2)$ and therefore is greater than (16.5.53).

Contribution of $0 = k'' \neq k'$. If $k' \neq 0$, $k'' = 0$ we recall that $|k''|^{-1}$ should be replaced by $\hbar^{-1}$ in (16.5.56)–(16.5.57) and the perturbation brings a factor $\sigma|k'|/\hbar$, so we get instead of (16.5.57)

$$C\sigma\mu^{\kappa+2}\alpha^2(\hbar/\alpha^2|k'|)^l\hbar^{-3-\frac{1}{2}\kappa}\begin{cases}(\hbar/\alpha^2)^{\frac{1}{2}\kappa} & \kappa<1,\\ (\hbar/\alpha^2)^{1-\frac{1}{2}\kappa} & \kappa>1,\\ (\hbar/\alpha^2)^{\frac{1}{2}}(1+|\log(\hbar/\alpha^2)|) & \kappa=1.\end{cases}$$

Then summation with respect to k' returns

$$C\sigma\mu^{\kappa+2}\alpha^2(\hbar/\alpha^2)^l\hbar^{-3-\frac{1}{2}\kappa} \tag{16.5.59}$$

as $\hbar \le \alpha^2$ and

$$C\sigma\mu^{\kappa+2}\hbar^{-2-\frac{1}{2}\kappa} \tag{16.5.60}$$

as $\hbar \ge \alpha^2$; (16.5.60) is even greater than (16.5.58).

Contribution of $0 = k'$. With perturbation factors are no more of the same rights and we need to consider this case as well.

(i) If $k' = 0$, $k'' \ne 0$ we recall that $|k'|^{-1}$ should be replaced by $\hbar^{-1}$ in (16.5.56)–(16.5.57) but perturbation brings factor σ so together we get factor $\sigma\hbar^{-1}$ which falls into into case $k' = 1$, $k'' \ne 0$.

(ii) Similarly, if $k' = k'' = 0$ we fall into case $k' = 1$, $k'' = 0$.

In total we get (16.5.53)+(16.5.59) as $\alpha^2 \ge \hbar$ and (16.5.60) as $\alpha^2 \le \hbar$. So, we arrive to

Proposition 16.5.17. *Let conditions* (16.1.5)–(16.1.6) *be fulfilled in* $B(0,1)$ *and let* $\alpha := \epsilon_0|\nabla V/F(\bar{x})| \ge C_0\mu^{-1}$ *where* $\bar{x} \in B(0,\frac{1}{2})$.

Let $\bar{e}_z(x,y,\tau)$ *be a Schwartz kernel of the spectral projector of* <u>*either*</u> *the magnetic Weyl approximation (then* $\sigma = \alpha\mu^{-1}$*)* <u>*or*</u> *the toy-model approximation (then* $\sigma = \mu^{-2}$*) at point* z. *Let* $T \ge \min\bigl(\epsilon_0\hbar/\sigma, C_0\mu\alpha^{-1}\bigr) \ge 1$.

Let $\psi, \psi_1 \in \mathscr{C}_0^\infty(B(0,1))$.

(i) Then as $\alpha^2 \ge \hbar$, $\gamma = C_0\mu^{-1}$

$$\mathsf{R}_2 = \mathsf{R}_2(\alpha,\gamma) := \tag{16.5.61}$$
$$\Bigl|\int \Omega_{\alpha,\gamma}\bigl(\frac{1}{2}(x+y), x-y\bigr)\Bigl(|e^{\mathsf{T}}(x,y,0)|^2 - |\bar{e}^{\mathsf{T}}_{\frac{1}{2}(x+y)}(x,y,0)|^2\Bigr)\,dxdy\Bigr| \le$$
$$(16.5.53) + (16.5.59)$$

where

$$\Omega_{\alpha,\gamma}(x,z) := \psi\bigl(\alpha^{-1}(x-\bar{x})\bigr)\psi_1\bigl(\gamma^{-1}z\bigr).$$

(ii) As $\alpha^2 \le \hbar$, $\gamma = C_0\mu^{-1}$ *we get*

$$\mathsf{R}_2 \le (16.5.54) + (16.5.60). \tag{16.5.62}$$

(iii) Further, in this inequality one can replace $\bar{e}^{\mathsf{T}}$ *by* $\bar{e}$.

Proof. The easy albeit tedious proof following our standard technique we leave to the reader. □

Improvement

We want to apply arguments of the previous Subsubsection "Improvement" to improve results of the previous subsubsection. One can see easily that improvement makes sense only if we can improve $k' = k'' = 1$ and thus as $\alpha^2 \ge \mu^2 h$. This excludes magnetic Weyl approximation.

In this case $R_1^{\mathsf{W}}(\alpha)$ defined by (16.5.25) acquires factor $\mu^{-3}h^{-1}$:

$$C \begin{cases} \mu^{2\kappa-3}h^{-2}\alpha^{2-\kappa} & 1 < \kappa < \frac{3}{2}, \\ \mu^{3-2\kappa}h^{1-2\kappa}\alpha^{3\kappa-4} & \frac{3}{2} < \kappa < 2, \\ h^{-2}\alpha^{\frac{1}{2}}(1+|\log(\mu^2 h/\alpha^2)|) & \kappa = \frac{3}{2}. \end{cases} \tag{16.5.63}$$

This replaces (16.5.53) and (16.5.59) becomes negligible.

So, we arrive to

Proposition 16.5.18. *Let conditions of Proposition 16.5.17 be fulfilled, we consider the toy-model approximation and let* $\alpha^2 \ge \mu^2 h$. *Then*

(i) $\mathsf{R}_2(\alpha,\gamma)$ *does not exceed* (16.5.63).

(ii) Further, in this inequality one can replace $\bar{e}^{\mathsf{T}}$ *by* $\bar{e}$.

Proof. The easy albeit tedious proof following our standard technique we leave to the reader. □

Results. I

Let condition (16.1.7) be fulfilled.

Magnetic Weyl Approximation. If we are looking for the magnetic Weyl approximation, we should assume that $h^{-\frac{1}{2}} \le \mu \le h^{-1}$ and pick up $\sigma = \varepsilon = \mu^{-1}$, $\alpha = 1$ and look what happens with expressions (16.5.53) and (16.5.59). The former returns the last term in the estimate (16.5.64) below

and the latter returns the lesser expression $C\mu^{\kappa-1}h^{-2}\hbar'$. So we arrive to estimate (16.5.64) of Theorem 16.5.19 below albeit with an extra factor $\bigl(1+(\log \mu^2 h)_+\bigr)$ in front of the big left brace.

To get rid of it let us consider two-term successive approximation. Then to estimate a remainder one needs to sum the same expressions as before but with an extra factor which is either $\mu^{-2}h^{-1}|k'|$ or $\mu^{-2}h^{-1}|k''|$.

This really does not matter for $k'=k''\neq 0$ and we sum $\mu^{-2}h^{-1}$ instead of $|k|^{-1}$ from $|k|=1$ to $|k|=\mu^2 h$ resulting in 1 rather than logarithm. As $0\neq k'\neq k''\neq 0$ we get the same answer albeit with a factor $(\mu h)'$ as $\alpha\asymp 1$. If $k'=0$ and/or $k''=0$ we replace $|k'|^{-1}$ and/or $|k''|^{-1}$ by $\hbar^{-1}$ but this is more than compensated by $(\mu h)'$ but effectively $T'=\hbar$ and/or $T''=\hbar$ so in fact we reset these cases to $k'=1$ and/or $k''=1$.

On the other hand, the second term in the successive approximation refers to perturbation being strictly $\alpha(x_1-y_1)$ but this leads to 0 as Ω is even with respect to $(x-y)$. Thus we arrive to estimate (16.5.64) of Theorem 16.5.19 below.

Theorem 16.5.19. *Let conditions* (16.1.5)–(16.1.7) *be fulfilled. Then for* $h^{-\frac{1}{2}}\le \mu\le h^{-1}$

$$|\mathsf{I}-\mathcal{I}^{\mathsf{MW}}|\le C\mu^{-1}h^{-1-\kappa}+C\begin{cases}\mu^{\kappa-1}h^{-2} & 0<\kappa<1,\\ h^{-1-\kappa} & 1<\kappa<2,\\ h^{-2}(1+|\log(\mu h)|) & \kappa=1.\end{cases}\tag{16.5.64}$$

Toy-Model Approximation. If we are looking for the toy-model approximation, we should assume that $h^{-\frac{1}{3}}\le\mu\le h^{-1}$ and pick up $\sigma=\mu^{-2}$, $\varepsilon=\mu^{-1}$, $\alpha=1$. We need to consider variants:

(i) $h^{-\frac{1}{3}}\le\mu\le h^{-\frac{1}{2}}$. As $0<\kappa\le 1$ (16.5.53) becomes $C\mu^{\kappa-2}h^{-2}(1+|\log\mu^3 h|)$ (with another logarithmic factor as $\kappa=1$ and it is larger than $C\mu^{\kappa}h^{-1}$ (which is the contribution of $\{|x-y|\ge C_0\mu^{-1}\}$).

As $1<\kappa<2$ we can replace (16.5.53) by (16.5.63) which is then $O(\mu^{-1}h^{-1-\kappa})$ and $\mu^{\kappa}h^{-1}$ is also $O(\mu^{-1}h^{-1-\kappa})$.

Observing that (16.5.59) is negligible we arrive to statement (i) of Theorem 16.5.21 below.

(ii) $h^{-\frac{1}{2}}\le\mu\le h^{-1}$. In this case we note that (16.5.59) is still less than (16.5.53) but $\mu^{\kappa}h^{-1}$ may become leading term. To improve this term let us

consider contribution of zone $|x-y| \asymp \gamma \ge C_0\mu^{-1}$ into an error: we need to take $\sigma = \gamma^2$ and $T \asymp \mu^2\gamma$ so we get the estimate of perturbation

$$C\mu h^{-1}T^{-1}\gamma^{-\kappa} \times \sigma T/\hbar \asymp Ch^{-2}\gamma^{2-\kappa} \tag{16.5.65}$$

recall that unperturbed term does not exceed

$$C\mu h^{-1}T^{-1}\gamma^{-\kappa} \asymp C\mu^{-1}h^{-1}\gamma^{-\kappa-1} \tag{16.5.66}$$

because the drift propagation does not expand $B(x,\rho)$ with $\rho \ge C_0\mu^{-1}$ but merely shifts it. We sum (16.5.65) as $\gamma^2 T \le \hbar$ i.e. $\gamma^3 \le \mu^{-1}h$ i.e. $\gamma \le \bar{\gamma} = \mu^{-\frac{1}{3}}h^{\frac{1}{3}} = \mu^{-1}(\mu^2 h)^{\frac{1}{3}}$.

Then we get $Ch^{-\frac{4}{3}-\frac{1}{3}\kappa}\mu^{\frac{1}{3}\kappa-\frac{2}{3}}$ and we arrive to Statement (ii) of Theorem 16.5.21 below.

Remark 16.5.20. Obviously arguments of the magnetic Weyl approximation allowing us to get rid of logarithmic factor still work albeit the second term does not necessarily vanish as the perturbation is quadratic. The logarithmic factor in estimates (16.5.67) and (16.5.69) does not seem be worth of trouble to write this second term.

Theorem 16.5.21. *Let conditions* (16.1.5)–(16.1.7) *be fulfilled. Then*

(i) If $h^{-\frac{1}{3}} \le \mu \le h^{-\frac{1}{2}}$, *then*

$$|\mathsf{I}-\bar{\mathsf{I}}| \le C\mu^{-1}h^{-1-\kappa} + C|\log \mu^3 h| \begin{cases} \mu^{\kappa-2}h^{-2} & 0<\kappa<1,\\ 0 & 1<\kappa<2,\\ \mu^{-1}h^{-2}(1+|\log h|) & \kappa=1 \end{cases} \tag{16.5.67}$$

where

$$\bar{I} := \int \Omega_{1,1}\bigl(\frac{1}{2}(x+y), x-y\bigr)|\bar{e}_{\frac{1}{2}(x+y)}(x,y,0)|^2\,dxdy. \tag{16.5.68}$$

(ii) If $h^{-\frac{1}{2}} \le \mu \le h^{-1}$, *then*

$$|\mathsf{I}-\bar{\mathsf{I}}| \le C\mu^{-1}h^{-1-\kappa} + Ch^{-\frac{4}{3}-\frac{1}{3}\kappa}\mu^{\frac{1}{3}\kappa-\frac{2}{3}} + C|\log h| \begin{cases} \mu^{\kappa-2}h^{-2} & 0<\kappa<1,\\ \mu^{-1}h^{-1-\kappa} & 1<\kappa<2,\\ \mu^{-1}h^{-2}(1+|\log(\mu h)|) & \kappa=1. \end{cases} \tag{16.5.69}$$

Results. II

Assume now that condition (16.1.8) is fulfilled instead of (16.1.7) and estimate an approximation error again.

Magnetic Weyl Approximation Let us consider the magnetic Weyl approximation first. Then we should assume that $h^{-\frac{1}{2}} \le \mu \le h^{-1}$ and pick up $\sigma = \varepsilon = \alpha\mu^{-1}$. Recall that the contribution of $\{(x,y) : |x-y| \ge C_0\mu^{-1}\}$ does not exceed $C\mu^{\kappa}h^{-1}$ and therefore we need to consider zone

$$\{(x,y) : |x-y| \le C_0\mu^{-1},\ \alpha(x) \ge C_1\mu^{-1}\} \tag{16.5.70}$$

where also $(1-\epsilon) \le \alpha(y)/\alpha(x) \le (1+\epsilon)$ [14].

Therefore we can use α-admissible partition in zone (16.5.69) and we need just look what happens with expressions (16.5.53) and (16.5.59) after integration with respect to $\alpha^{-1}d\alpha$ as $\alpha \ge (\mu h)^{\frac{1}{2}}$ and (16.5.60) $\mu^{-1} \le \alpha \le (\mu h)^{\frac{1}{2}}$.

Obviously integral of (16.5.53) resets to its value as $\alpha = 1$ as under assumption (16.1.7); integrals of (16.5.59) and (16.5.60) reset to their values as $\alpha = (\mu h)^{\frac{1}{2}}$ which are equal and less than what we got already.

Therefore estimate (16.5.64) still holds[15] and we arrive to Statement (i) of Theorem 16.5.22 below after we apply the same arguments with two-term approximation as in the previous subsubsection to get rid of the logarithmic factor.

Toy-Model Approximation Let us consider the toy-model approximation; then again $\sigma = \mu^{-2}$. Then we need to consider two cases.

(a) $h^{-\frac{1}{3}} \le \mu \le h^{-\frac{1}{2}}$. In this case we are completely happy with the zone $\{(x,y) : |x-y| \ge C_0\mu^{-1}\}$.

As $0 < \kappa \le 1$ we need to integrate with respect to $\alpha^{-1}d\alpha$ (16.5.53) and (16.5.59) as $\alpha \ge (\mu h)^{\frac{1}{2}}$ and (16.5.60) as $\alpha \le (\mu h)^{\frac{1}{2}}$. The first integral resets to (16.5.53) as $\alpha = 1$; integrals of (16.5.59) and (16.5.60) reset to their values as $\alpha = (\mu h)^{\frac{1}{2}}$ which are equal (up to a logarithmic factor) and less than

[14] And add contribution of the zone $\{(x,y) : \alpha(x) \le \bar{\alpha} := C_1\mu^{-1},\ \alpha(y) \le \bar{\alpha}\}$ which would be $C\mu h^{-1-\kappa} \times \bar{\alpha}^2 = C\mu^{-1}h^{-1-\kappa}$ in virtue (16.1.8).

[15] Albeit the first term $C\mu^{-1}h^{-1-\kappa}$ acquires factor $|\log h|$ if assumption $(16.1.8)^+$ fails: see Tauberian estimate.

what we got already. Thus we arrive to Statement (ii) of Theorem 16.5.22 below.

As $1 < \kappa < 2$ we need to replace integral of (16.5.53) by its integral over zone $(\mu h)^{\frac12} \le \alpha \le \mu h^{\frac12}$ plus integral of (16.5.63) over zone $\mu h^{\frac12} \le \alpha \le 1$. These integrals are reset to their integrands values as $\alpha = (\mu h)^{\frac12}$ and $\alpha = 1$ respectively; the latter is larger than the former albeit not necessarily larger than integral of (16.5.60).

Thus we arrive to Statement (iii) of Theorem 16.5.22 below.

(b) $h^{-\frac12} \le \mu \le h^{-1}$. In this case the contribution of $\{(x,y) : |x-y| \le C_0\mu^{-1}\}$ does not exceed the right-hand expression of (16.5.69) but we need to reexamine zone $\{(x,y)\colon |x-y| \ge C_0\mu^{-1}\}$ which we split into two: $\{|x-y| \le \epsilon_0\alpha(x)\}$ and $\{|x-y| \ge \epsilon_0\alpha(x)\}$.

Repeating corresponding arguments of the analysis under assumption (16.1.7) we conclude that the contribution of the former zone does not exceed what we got then. In the latter zone $T \asymp \mu^2$ and its contribution does not exceed $C\mu^{-1}h^{-1-\kappa}$.

Thus we arrive to Statement (iv) of Theorem 16.5.22 below.

Theorem 16.5.22. *Let conditions* (16.1.5)–(16.1.6) *and* (16.1.8) *be fulfilled. Then*[15)]

(i) As $h^{-\frac12} \le \mu \le h^{-1}$ *estimate* (16.5.64) *holds.*

(ii) As $h^{-\frac13} \le \mu \le h^{-\frac12}$, $0 < \kappa \le 1$ *estimate* (16.5.67) *holds.*

(iii) As $h^{-\frac13} \le \mu \le h^{-\frac12}$, $1 < \kappa < 2$

$$|\mathsf{I} - \bar{I}| \le C\mu^{-1}h^{-1-\kappa} + C\mu^{\frac12\kappa-2}h^{-\frac12\kappa-2}(1 + |\log \mu^3 h|); \tag{16.5.71}$$

(iv) As $h^{-\frac12} \le \mu \le h^{-1}$ *estimate* (16.5.69) *holds.*

16.5.4 Strong Magnetic Field

Consider under assumption (16.1.8) the Schrödinger-Pauli operator with $\mu \ge h^{-1}$. Recall that according to Propositions 16.4.8(ii), 16.4.9(ii) respectively

$$|\mathsf{I}| \le C\mu^{1+\frac12\kappa}h^{-1-\frac12\kappa}$$

and

$$|\mathsf{I}-\mathsf{I}^{\mathsf{T}}| \le C\mu^{\frac{1}{2}\kappa}h^{-\frac{1}{2}\kappa}$$

(as $\kappa \ne 1$; as $\kappa = 1$ the latter estimate holds with an extra factor $|\log \mu|$ in its right-hand expression).

Let us assume that $F = 1$ to avoid some unpleasant correction terms as we reduce operator to one with $F = 1$. Recall that now $\hbar := \mu^{-\frac{1}{2}}h^{\frac{1}{2}}$ is the magnitude of the cyclotron radius.

Magnetic Weyl Approximation. Consider first the magnetic Weyl approximation. Then $\sigma = \alpha\hbar = \alpha\mu^{-\frac{1}{2}}h^{\frac{1}{2}}$ and $T = \sigma^{-1}\hbar = \alpha^{-1}\mu^{\frac{3}{2}}h^{\frac{1}{2}}$; then the the shift (unrescaled) due to magnetic drift does not exceed $C\alpha\mu^{-2}T = C\hbar$, which justifies the choice of σ.

Therefore as $\alpha \ge \bar{\alpha} = C_1\mu^{-\frac{1}{2}}h^{\frac{1}{2}}$ the contribution of α-ball intersected with $\{(x,y) : |x-y| \le \bar{\gamma} := C_0\mu^{-\frac{1}{2}}h^{\frac{1}{2}}\}$ to $|\mathsf{I}^{\mathsf{T}} - \mathcal{I}^{\mathsf{MW}}|$ does not exceed $C\mu h^{-1}\sigma \times \mu^{\frac{1}{2}\kappa}h^{-\frac{1}{2}\kappa} \asymp C\alpha\mu^{\frac{1}{2}+\frac{1}{2}\kappa}h^{-\frac{1}{2}-\frac{1}{2}\kappa}$. Then summation with respect to all such balls does not exceed $C\mu^{\frac{1}{2}+\frac{1}{2}\kappa}h^{-\frac{1}{2}-\frac{1}{2}\kappa}$, as in estimate (16.5.72) below.

Meanwhile the contributions to both I and $\bar{\mathsf{I}}$ of zone $\{(x,y) : |x-y| \ge \bar{\gamma}\}$ does not exceed $C\bar{\gamma}^{-1-\kappa}$ which is also of the same magnitude.

Finally, the contributions to both I and $\bar{\mathsf{I}}$ of zone $\{(x,y) : \alpha(x) \le \bar{\alpha},\ \alpha(y) \le \bar{\alpha}\}$ do not exceed $C\mu^{1+\frac{1}{2}\kappa}h^{-1-\frac{1}{2}\kappa} \times \bar{\alpha}^2 \asymp C\mu^{\frac{1}{2}\kappa}h^{-\frac{1}{2}\kappa}$ which is less than the right-hand expressions in both (16.5.72) and (16.5.73).

Thus we arrive to estimate (16.5.72) below.

Toy-Model Approximation. Meanwhile for the toy-model approximation $\sigma = \mu^{-1}h$ as long as $T \le T_* := \alpha^{-1}\mu^{\frac{3}{2}}h^{\frac{1}{2}}$; so we conclude that as $\alpha \ge \bar{\alpha}$, the contribution to $(\mathsf{I}-\bar{\mathsf{I}})$ of α-ball intersected with $\{(x,y) : |x-y| \le \bar{\alpha}\}$ does not exceed $C\alpha\mu^{\frac{1}{2}\kappa}h^{-\frac{1}{2}\kappa}$. After summation with respect to $\alpha^{-1}d\alpha$ we get $C\mu^{\frac{1}{2}\kappa}h^{-\frac{1}{2}\kappa}$ which is less than the right-hand expression in (16.5.73).

Now we need to consider larger T and γ; then $\sigma = \gamma^2$ and $T \asymp \alpha^{-1}\mu^2\gamma$. Then we should define $\bar{\gamma}$ so that $\sigma T \le \hbar$ iff $\gamma \le \bar{\gamma}$. Therefore we pick up $\bar{\gamma} = \mu^{-\frac{1}{3}}h^{\frac{1}{3}}\alpha^{\frac{1}{3}}$ and matching $\bar{T} = \mu^{\frac{4}{3}}h^{\frac{1}{3}}\alpha^{-\frac{2}{3}}$ and $\bar{\gamma} = \mu^{-\frac{2}{3}}h^{\frac{2}{3}}\alpha^{\frac{2}{3}}$.

Using the same arguments as above we conclude that the contribution to $(\mathsf{I}-\bar{\mathsf{I}})$ of the zone $\{(x,y) : \bar{\alpha} \le |x-y| \le \bar{\gamma}\}$ and contributions to both I and $\bar{I}$ of the zone $\{(x,y) : \bar{\alpha} \ge |x-y| \le \bar{\gamma}\}$ also do not exceed the right-hand expression in (16.5.73).

Thus we arrive to estimate (16.5.73) below.

Theorem 16.5.23. *For Schrödinger-Pauli operator under conditions* (16.1.5)–(16.1.6), (16.1.8), $\mu \ge h^{-1}$ *and* $F = 1$

$$|\mathsf{I} - \mathcal{I}^{\mathsf{MW}}| \le C\mu^{\frac{1}{2}+\frac{1}{2}\kappa} h^{-\frac{1}{2}-\frac{1}{2}\kappa} \tag{16.5.72}$$

and

$$|\mathsf{I} - \bar{\mathsf{I}}| \le C\mu^{\frac{1}{3}+\frac{1}{3}\kappa} h^{-\frac{1}{3}-\frac{1}{3}\kappa}. \tag{16.5.73}$$

16.5.5 Problems and Remarks

Remark 16.5.24. The main difference between cases $1 < \kappa < 2$ and $0 < \kappa < 1$ as $\mu h \ll 1$ is that in the former case the main contribution to the remainder is delivered by (x, y) close to one another ($|x - y| \ll \mu^{-1}$) while in the latter case by (x, y) with $|x - y| \asymp \mu^{-1}$.

We can calculate $\mathcal{I}^{\mathsf{MW}}$ and $\bar{\mathsf{I}}$ plugging corresponding expressions $e_z^{\mathsf{MW}}(x, y, 0)$ and $\bar{e}_z(x, y, 0)$ with $z = \frac{1}{2}(x + y)$ into I.

The following problems do not look too difficult:

Problem 16.5.25. Find nice expressions for $\mathcal{I}^{\mathsf{MW}}$ and $\bar{\mathsf{I}}$.

Problem 16.5.26. As $\mu h \le 1$ get rid of condition (16.1.5); according to Chapter 13 we do not need it for estimate $|\mathsf{I} - \mathsf{I}^{\mathsf{T}}|$ but we want to get rid of it in estimates for $|\mathsf{I} - \mathcal{I}^{\mathsf{W}}|$, $|\mathsf{I} - \mathcal{I}^{\mathsf{MW}}|$ and $|\mathsf{I} - \bar{\mathsf{I}}|$. To do this

(i) Assume first that condition

$$|V| + |\nabla V| \asymp 1 \tag{16.5.74}$$

is fulfilled and consider the scaling function $\alpha(x) = \mathsf{max}\big(\epsilon|V(x)|, \mu h, C\mu^{-1}\big)$.

(ii) Assume then that condition

$$|V| + |\nabla V| + |\det \mathsf{Hess}\, V| \asymp 1 \tag{16.5.75}$$

is fulfilled and consider the scaling function

$$\alpha(x) = \mathsf{max}\big(\epsilon(|V(x)| + |\nabla V(x)|^2)^{\frac{1}{2}}, (\mu h)^{\frac{1}{2}}, C\mu^{-1}\big).$$

Problem 16.5.27. Get rid of condition (16.1.6) assuming instead that $|F| + |\nabla F| \ge \epsilon$.

(i) Repeating arguments of Section 16.4 and using results of Chapter 14 one can prove easily that $|\mathsf{I}-\mathsf{I}^{\mathsf{T}}|\le C\mu^{-1/2}h^{-1-\kappa}$ as $\mu\le h^{-1}$. Derive similar estimates for Schrödinger and Schrödinger-Pauli operators as $h^{-1}\le\mu\le h^{-2}$ and for Schrödinger-Pauli operator as $\mu\ge h^{-2}$.

(ii) Introducing scaling function $\ell(x)=\epsilon|F(x)|$ in zone $\{x:\ \ell(x)\ge\mu^{-\frac12}\}$ one can rescale $\mu\mapsto\mu\ell^2$, $h\mapsto h\ell^{-1}$, $\mu h^{-1-\kappa}\mapsto\mu^{-1}h^{-1-\kappa}\ell^{-1+\kappa}$, after integration over $\ell^{-2}d\ell$ we get $C\mu^{-1}h^{-1-\kappa}\ell^{-2+\kappa}$ calculated as $\ell=\mu^{-\frac12}$ i.e. $\mu^{-\frac12\kappa}h^{-1-\kappa}$. This is an estimate for the contribution of the regular zone.

Prove the same estimate for the contribution of the singular zone as $\mu\le h^{-1}$. Derive similar estimates as $h^{-1}\le\mu\le h^{-2}$ and as $\mu\ge h^{-2}$.

(iii) Is it possible to upgrade the above estimate to $C\mu^{-\frac12}h^{-1-\kappa}$ as $\kappa<1$? Explore also cases $h^{-1}\le\mu\le h^{-2}$ and $\mu\ge h^{-2}$.

16.6 Pointwise Asymptotics: 3D Toy-Model

16.6.1 Toy-Model in 3D: Propagator

Consider the *pilot-model operator*

$$A=\bar A:=h^2D_1^2+(hD_2-\mu x_1)^2+h^2D_3^2+2\alpha x_1+2\beta x_3 \tag{16.6.1}$$

which is the sum of 2D toy-model (16.2.1) $\bar A_{(2)}$ and 1D Schrödinger operator

$$B:=h^2D_3^2+2\beta x_3 \tag{16.6.2}$$

and therefore propagator of $\bar A$ is

$$U(x,y,t)=U_{(2)}(x',y',t)U_{(1)}(x_3,y_3,t) \tag{16.6.3}$$

where $U_{(2)}(x',y',t)$ is the Schwartz kernel of the propagator $e^{ih^{-1}t\bar A_{(2)}}$ and therefore *after rescaling* it is defined by (16.2.1) while $U_{(1)}(x_3,y_3,t)$ is the Schwartz kernel of the propagator $e^{ih^{-1}tB}$ and then one can prove easily that *before rescaling*

$$\begin{aligned}&U_{(1)}(x_3,y_3,t)=\\ &(2\pi h)^{-1}\int\exp\Big(ih^{-1}\big((\zeta+\beta t)x_3-(\zeta-\beta t)y_3+\zeta^2t+\frac13\beta^2t^3\big)\Big)\,d\zeta=\\ &\frac12(2\pi h|t|)^{-\frac12}\exp\Big(ih^{-1}\big(\beta t(x_3+y_3)+\frac18t^{-1}(x_3-y_3)^2+\frac13\beta^2t^3\big)\Big);\end{aligned} \tag{16.6.4}$$

in particular

$$U_{(1)}(x_3,x_3,t)=\frac{1}{2}(2\pi h|t|)^{-\frac{1}{2}}\exp\Big(ih^{-1}\big(2\beta t x_3+\frac{1}{3}\beta^2t^3\big)\Big). \tag{16.6.5}$$

Therefore

(16.6.6) After standard rescaling $x\mapsto \mu x$, $t\mapsto \mu t$ $U(x,x,t)$ in comparison with $U_{(2)}(x',x',t)$ acquires factor

$$\frac{1}{2}\mu(2\pi\hbar|t|)^{-\frac{1}{2}}\exp\Big(i\hbar^{-1}\big(2\mu^{-1}\beta t x_3+\frac{1}{3}\mu^{-2}\beta^2t^3\big)\Big) \tag{16.6.7}$$

and therefore (16.2.23) and (16.2.24) are replaced respectively by

$$\varphi(t)\coloneqq -t^2\mu^{-2}\alpha^2\cot(t)-t\mu^{-2}\alpha^2+\frac{1}{3}\mu^{-2}\beta^2t^3-t\tau \tag{16.6.8}$$

and

$$\mu^{-2}\alpha^2\big(t^2-t\sin(2t)\big)+\mu^{-2}\beta^2t^2\sin^2(t)=\big(\tau+\mu^{-2}\alpha^2\big)\sin^2(t); \tag{16.6.9}$$

the latter equation could be rewritten as (16.2.24) with τ replaced by τ'

$$\mu^{-2}\alpha^2\big(t^2-t\sin(2t)\big)=\big(\tau'+\mu^{-2}\alpha^2\big)\sin^2(t) \tag{16.6.10}$$

and

$$\mu^{-2}\beta^2t^2=(\tau-\tau'); \tag{16.6.11}$$

the former equation shows the return times of 2D-movement on the energy level τ' and the former the looping time of 1D-movement associated with B (after rescaling) on the energy level $(\tau-\tau')$.

Remark 16.6.1. We should set $T=\epsilon_0\mu$ (in contrast to 2D-case). Really, unless we want to make assumptions outside of the ball $B(0,\ell)$ with a small constant ℓ we need to take $T\le\epsilon_0$ (before rescaling) and then $T\le\epsilon_0\mu$ (after rescaling). This absolves us from the analysis of the equatorial zone.

Let us compare solutions t_k^* of (16.6.9) and solutions t_k to (16.2.24). One can see easily that

(16.6.12) If $|k|\le\epsilon|\beta|^{-1}\mu$, then

$$t_k^*=t_k\big(1+O(\mu^{-2}\beta^2k^2)\big) \tag{16.6.13}$$

and

$$\varphi''(t_k^*)=\varphi''_{(2)}(t_k)\big(1+O(\mu^{-2}\beta^2k^2)\big) \tag{16.6.14}$$

with $\varphi_{(2)}$ defined by (16.2.23). Furthermore other properties of $\varphi_{(2)}$ are fulfilled for φ as well.

16.6.2 Tauberian Estimates

Then as $|k| \le \epsilon\mu$ in virtue of (16.6.6) and calculations of Section 16.2 contribution of k-th tick to $F_{t\to\hbar^{-1}\tau}\Gamma_x U$ does not exceed

$$C\begin{cases}\mu h^{-1}\times \big(\mu/h|k|\big)^{\frac12} & \text{as } 1\le |k|\le \bar{k},\\ \mu h^{-1}(\mu^2 h/\alpha|k|)^{\frac12}\times \big(\mu/h|k|\big)^{\frac12} & \text{as } \bar{k}\le |k|\le \epsilon\mu\ell,\end{cases} \tag{16.6.15}$$

where as before $\bar{k}=\epsilon\mu^2 h\alpha^{-1}$. Recall that without an extra factor $\big(\mu/h|k|\big)^{\frac12}$ we would have estimate for $F_{t\to\hbar^{-1}\tau}\Gamma_{x'}U_{(2)}$.

However, this extra factor is a game changer. As before there are three cases:

(a) $\alpha\ge \mu^2 h$; then we set $\bar{k}=1$ and the summation with respect to $k: 1\le |k|\le \epsilon\mu\ell$ returns

$$C\mu h^{-1}(\mu^2 h/\alpha)^{\frac12}\cdot(\mu/h)^{\frac12}\log(\mu\ell)\asymp C\mu^{\frac52}h^{-1}\alpha^{-\frac12}\log(\mu\ell).$$

(b) $\mu h\ell^{-1}\le\alpha\le\mu^2 h$; then $1\le\bar{k}\le\epsilon\mu\ell$ and summation with respect to $k:\bar{k}\le|k|\le\epsilon\mu\ell$ returns

$$C\mu^{\frac52}h^{-1}\alpha^{-\frac12}\log(\mu\ell)/\bar{k}\asymp C\mu^{\frac52}h^{-1}\alpha^{-\frac12}\big(1+(\log(\alpha\ell/\mu h))_+\big)$$

while summation with respect to $k: 1\le|k|\le\bar{k}$ returns the same expression albeit without logarithmic factor.

(c) $\alpha\le\mu h\ell^{-1}$; then $\bar{k}\ge\epsilon\mu\ell$ and we reset set $\bar{k}=\epsilon\mu\ell$ and summation with respect to k returns

$$C\mu h^{-1}\times\big(\mu/h\big)^{\frac12}\bar{k}^{\frac12}\asymp C\mu^2 h^{-\frac32}\ell^{\frac12}.$$

So we have proved

Proposition 16.6.2 [16]**.** *(i) After rescaling as* $T=\epsilon\mu\ell$, $\ell\ge C_0\mu^{-1}$

$$|F_{t\to\hbar^{-1}\tau}\bar{\chi}_T(t)\Gamma_x U|\le C\mu h^{-2}+ C\begin{cases}\mu^{\frac52}h^{-1}\alpha^{-\frac12}\big(1+(\log(\alpha\ell/\mu h))_+\big) & \text{as } \alpha\ell\ge\mu h,\\ \mu^2 h^{-\frac32}\ell^{\frac12} & \text{as } \alpha\ell\le\mu h.\end{cases} \tag{16.6.16}$$

[16] Cf. Proposition 16.2.6.

(ii) Therefore as operator A coincides with the toy-model operator (16.6.1) *in $B(0,\ell)$*

$$|e(0,0,\tau)-e^{\mathsf{T}}(0,0,\tau)|\le Ch^{-2}\ell^{-1}+ C\begin{cases}\mu^{\frac32}h^{-1}\alpha^{-\frac12}\bigl(1+(\log(\alpha\ell/\mu h))_+\bigr)\ell^{-1} & \text{as } \alpha\ell\ge\mu h,\\ \mu h^{-\frac32}\ell^{-\frac12} & \text{as } \alpha\ell\le\mu h.\end{cases} \tag{16.6.17}$$

(iii) In particular, as $\alpha\asymp\ell\asymp 1$ the right-hand expression in (16.6.17) *does not exceed $Ch^{-2}+C\mu^{\frac32}h^{-1}\bigl(1+|\log\mu h|\bigr)$ which is $O(h^{-2})$ as $\mu\le(h|\log h|)^{-\frac23}$.*

16.6.3 Weyl Estimates

Even simpler are Weyl estimates: in comparison with 2D-case they acquire factor $\mu^{\frac12}h^{-\frac12}$ and (16.2.64) becomes

$$\mathsf{R}^{\mathsf{W}}_x:=|e^{\mathsf{T}}(x,x,0)-h^{-3}\mathcal{N}^{\mathsf{W}}_x|\le C\mu^{-\frac12}h^{-\frac32}+C\mu^{\frac52}\alpha^{-\frac12}h^{-1} \tag{16.6.18}$$

and more generally we arrive to Statement (i) below

Proposition 16.6.3 [17]**.** *For a toy-model operator* (16.6.1) *with $\tau\asymp 1$,*

(i) As $\mu^2h\le\alpha\le 1$ estimate (16.6.18) *holds; moreover if we replace $\mathcal{N}^{\mathsf{W}}_x$ by $\mathcal{N}^{\mathsf{W}}_x+\mathcal{N}_{x,\mathsf{corr}(r)}$ with the correction term $h^{-3}\mathcal{N}_{x,\mathsf{corr}(r)}$ delivered by r-term stationary phase approximation then an error does not exceed*

$$C\mu^{-\frac12}h^{-\frac32}+C\mu^{\frac32}h^{-\frac32}\bigl(\mu^2h/\alpha\bigr)^{r+\frac12}. \tag{16.6.19}$$

(ii) As $\alpha\le\mu^2h$ estimate holds:

$$\mathsf{R}^{\mathsf{W}}_x:=|e^{\mathsf{T}}(x,x,0)-h^{-3}\mathcal{N}^{\mathsf{W}}_x|\le C\mu^{\frac32}h^{-\frac32}; \tag{16.6.20}$$

(iii) In particular, as $\mu\le h^{-\frac13}$ we have $\mathsf{R}^{\mathsf{W}}=O(h^{-2})$ without any assumptions to α,β: $|\alpha|\le 1$, $|\beta|\le 1$.

Recall that here we do not need to analyze the equatorial zone and therefore the difference between cases $\alpha\ge\mu\hbar^{\frac23}$ and $\alpha\le\mu\hbar^{\frac23}$ disappears together with corresponding terms in (16.2.66) or (16.2.67).

[17] Cf. Proposition 16.2.12.

16.6.4 Micro-Averaging

Micro-averaging adds yet another layer of complexity. Let us consider an *isotropic micro-averaging* with function $\psi_\gamma(x) = \psi(x/\gamma)$ and an *anisotropic micro-averaging* with function $\psi_\gamma(x) = \psi(x'/\gamma, x_3/\gamma_3)$ where $\boldsymbol{\gamma} = (\gamma, \gamma_3)$ is a scale with respect to (x', x_3).

Then one needs to integrate by parts with respect to x_3 taking in account factor $\exp\bigl(2ih^{-1}\beta t x_3\bigr)$ in (16.6.5) and rescaling with respect to t which brings factor $(\mu h/|\beta|\gamma_3|k|)^l$ as $(\mu h/|\beta|\gamma_3|k|) \le 1$ in addition to factor $(\mu h/|\alpha|\gamma|k|)^l$ as $(\mu h/|\alpha|\gamma|k|) \le 1$. Thus

(16.6.21) Micro-averaging brings factor $(\mu h/\nu(\boldsymbol{\gamma})|k|)^l$ as $(\mu h/\nu(\boldsymbol{\gamma})|k|) \le 1$ with $\nu(\boldsymbol{\gamma}) \coloneqq |\alpha|\gamma + |\beta|\gamma_3$.

Then we need to modify (16.2.53) replacing $|\alpha|\gamma$ by $\nu(\boldsymbol{\gamma})$:

$$|k| \ge \hat{k}(\boldsymbol{\gamma}) \coloneqq \mu h/\nu(\boldsymbol{\gamma}). \tag{16.6.22}$$

To understand how it affects a Tauberian estimate we need to consider the following cases:

(a) Micro-averaging has no effect as $\hat{k}(\boldsymbol{\gamma}) \ge \epsilon\mu\ell$ i.e. as $\nu(\boldsymbol{\gamma}) \le h\ell^{-1}$.

(b) Micro-averaging has a minimal effect (only decreasing the logarithmic factor) as $\epsilon\mu\ell \ge \hat{k}(\boldsymbol{\gamma}) \ge \bar{k}$. Obviously in this case $\bar{k} = \epsilon\,\mathsf{max}\bigl(\mu^2 h/\alpha, 1\bigr)$. So this case holds iff $h\ell^{-1} \le \nu(\boldsymbol{\gamma}) \le \mathsf{min}(\mu h, \alpha\mu^{-1})$. Then the logarithmic factor becomes $\mathsf{log}\bigl(\hat{k}(\boldsymbol{\gamma})/\bar{k}\bigr)$.

(c) Micro-averaging kills logarithmic factor and further leads to $\bar{k}^{\frac12}$ replaced by $\hat{k}(\boldsymbol{\gamma})^{\frac12}$ as $1 \le \hat{k}(\boldsymbol{\gamma}) \le \bar{k}$. Obviously in this case $\bar{k} = \epsilon\,\mathsf{min}\bigl(\mu^2 h/\alpha, \mu\ell\bigr)$. So this case holds iff $\mathsf{max}(h\ell^{-1}, \alpha\mu^{-1}) \le \nu(\boldsymbol{\gamma}) \le \mu h$. Then the right-hand expression in (16.6.24) becomes $C\mu^2 h^{-1}\nu^{-\frac12}\gamma^{-\frac12}$.

(d) Micro-averaging resets $\bar{k}$ to 1 and brings factor $(\mu h/\nu(\boldsymbol{\gamma}))^l$ as $\hat{k}(\boldsymbol{\gamma}) \le 1$ i.e. as $\nu(\boldsymbol{\gamma}) \ge \mu h$. Then the right-hand expression in (16.6.24) becomes $C\mu^{\frac32} h^{-\frac32}(\mu h/\nu(\boldsymbol{\gamma}))^l$.

Thus we arrive to

Proposition 16.6.4. *For the toy-model operator* (16.6.1) *with* $\tau \asymp 1$, $0 \le \alpha \le 1$, $|\beta| \le 1$, $\mu \le h^{-1}$ *in* $\mathbb{R}^3$ *estimates*

$$\gamma^{-2}\gamma_3^{-1}|F_{t\to h^{-1}\tau}\big(\big(\bar\chi_T(t)-\bar\chi_{\bar T}\big)\Gamma(U\psi_\gamma)\big)| \le C\mu R^{\mathsf T}(\boldsymbol\gamma) \tag{16.6.23}$$

and

$$\gamma^{-2}\gamma_3^{-1}|F_{t\to h^{-1}\tau}\big(\bar\chi_T(t)\Gamma(U\psi_\gamma)\big)| \le C\mu h^{-2} + C\mu R^{\mathsf T}(\boldsymbol\gamma) \tag{16.6.24}$$

hold as $T = \epsilon\mu\ell$ *where*

$$R^{\mathsf T}(\boldsymbol\gamma) = \tag{16.6.25}$$

$$\begin{cases} \mu^{\frac32}h^{-1}\alpha^{-\frac12}\big(1+(\log(\alpha\ell/\mu h))_+\big) & \text{as } \alpha\ell \ge \mu h,\ \nu(\boldsymbol\gamma)\le h\ell^{-1},\\ \mu h^{-\frac32}\ell^{\frac12} & \text{as } \alpha\ell \le \mu h,\ \nu(\boldsymbol\gamma)\le h\ell^{-1},\\ \mu^{\frac32}h^{-1}\alpha^{-\frac12}\big(1+(\log(\alpha/\mu\nu(\boldsymbol\gamma)))_+\big) & \text{as } h\ell^{-1}\le \nu(\boldsymbol\gamma)\le \min(\mu h,\alpha\mu^{-1}),\\ \mu h^{-1}\nu(\boldsymbol\gamma)^{-\frac12} & \text{as } \max(h\ell^{-1},\alpha\mu^{-1})\le \nu(\boldsymbol\gamma)\le \mu h,\\ \mu^{\frac12}h^{-\frac32}(\mu h/\nu(\boldsymbol\gamma))^l & \text{as } \nu(\boldsymbol\gamma)\ge \mu h. \end{cases}$$

In the right-hand expression of (16.6.25) two first lines correspond to the case (a).

Corollary 16.6.5. *In the framework of Proposition 16.6.4 the following estimate holds*

$$\gamma^{-2}\gamma_3^{-1}|\Gamma\big(e(.,.,\tau)-e^{\mathsf T}(.,.,\tau)\big)\psi_\gamma| \le C\big(h^{-2}+R^{\mathsf T}(\boldsymbol\gamma)\big)\ell^{-1} \tag{16.6.26}$$

Consider now how micro-averaging affects Weyl estimate (with the right-hand expression (16.6.19)). There is no effect as $\nu(\boldsymbol\gamma) \le \mu h$ and factor $(\mu h/\nu(\boldsymbol\gamma))^l$ comes out as $\nu(\boldsymbol\gamma)\ge \mu h$. Thus we arrive to

Proposition 16.6.6. *For the toy-model operator* (16.6.1) *with* $\tau \asymp 1$, $0 \le \alpha \le 1$, $\mu \le h^{-1}$ *in* $\mathbb{R}^3$

(i) As $\mu^2 h \le \alpha \le 1$ *estimate holds:*

$$\mathsf R^{\mathsf W}_{x(r)}(\boldsymbol\gamma) := \tag{16.6.27}$$

$$\gamma^{-2}\gamma_3^{-1}|\int \Big(e^{\mathsf T}(x,x,0)-h^{-3}\big(\mathcal N^{\mathsf W}_x+\mathcal N_{x,\mathsf{corr}(r)}\big)\Big)\psi_\gamma\,dx| \le$$

$$Ch^{-2}+C\mu^{\frac32}h^{-\frac32}\big(\mu^2h/\alpha\big)^{r+\frac12}\begin{cases}(\mu h/\nu(\boldsymbol\gamma))^l & \text{as } \nu(\boldsymbol\gamma)\ge\mu h,\\ 1 & \text{as } \nu(\boldsymbol\gamma)\le\mu h.\end{cases}$$

(ii) As $\alpha \le \mu^2 h$ estimate holds:

$$\mathsf{R}^{\mathsf{W}}_x(\gamma) := \gamma^{-2}\gamma_3^{-1}\Big|\int \Big(e^{\mathsf{T}}(x,x,0) - h^{-3}\mathcal{N}^{\mathsf{W}}_x\Big)\psi_\gamma\, dx\Big| \le \tag{16.6.28}$$

$$Ch^{-2} + C\mu^{\frac{3}{2}}h^{-\frac{3}{2}}\begin{cases} \big(\mu h/\nu(\gamma)\big)^l & \text{as } \nu(\gamma)\ge \mu h,\\ 1 & \text{as } \nu(\gamma)\le \mu h.\end{cases}$$

16.6.5 Transitional and Strong Magnetic Field

Tauberian Estimate

Consider transitional $\mu h \asymp 1$ and strong $\mu h \gtrsim 1$ magnetic field for the Schrödinger-Pauli toy-model operator

$$A = \bar{A} := h^2D_1^2 + (hD_2 - \mu x_1)^2 + h^2D_3^2 + 2\alpha x_1 + 2\beta x_3 - \mu h \tag{16.6.29}$$

which alternatively means that we assume that $|\tau - \mathfrak{z}\mu h| \le C$ in the framework of the standard toy-model (16.6.1), $\mathfrak{z} = 1$.

Proposition 16.6.7. *For a toy-model operator* (16.6.1) *with $\mu h \gtrsim 1$ in $\mathbb{R}^3$ as $|x| \le C_0$, $|y| \le C_0$, $\tau \le c\mu h$*

(i) $e(x,y,\tau) \equiv 0 \mod O(\mu h^\infty)$ as $\tau \le \mu h - \epsilon_0$ (lower spectral gap).

(ii) As $\alpha > 0$ the following estimates hold

$$|e(x,x,\tau+h) - e(x,y,\tau)| \le \tag{16.6.30}$$

$$C\mu h^{-1}\max_{j\in\mathbb{Z}^+}\big(\max(|V(x) + 2j\mu h - \tau|, h)\big)^{-\frac{1}{2}}$$

and

$$|F_{t\to\hbar^{-1}\tau}U(x,y,t)| \le C\mu^2 h^{-1}\max_{j\in\mathbb{Z}^+}\big(\max(|V(x) + 2j\mu h - \tau|, h)\big)^{-\frac{1}{2}} \tag{16.6.31}$$

and these estimate are sharp as $x = y$ and τ is close to Landau level μh. Recall that in (16.6.31) *t is rescaled.*

Proof. Statement (i) is obvious. To prove Statement (ii) note that

$$e(x,y,\tau) = \int e_B(x_3,y_3,\tau-\tau')\, d_{\tau'}e_{(2)}(x',y',\tau') = \tag{16.6.32}$$

$$\int e_{(2)}(x',y',\tau-\tau')\, d_{\tau'}e_B(x_3,y_3,\tau')$$

and therefore

$$e(x,y,\tau+h)-e(x,y,\tau)= \int \big(e_B(x_3,y_3,\tau-\tau'+h)-e_B(x_3,y_3,\tau-\tau')\big)\,\partial_{\tau'}e_{(2)}(x',y',\tau')\,d\tau'. \tag{16.6.33}$$

Recall that due to our analysis in Section 16.2 $\partial_{\tau'}e_{(2)}(x',y',\tau')$ is essentially supported in $C\mu^{-\frac12}h^{\frac12}\alpha$-vicinity of Landau level and fast decays out of it.

Finally recall that due to the analysis of Subsubsections 5.3.1–5.3.1

$$|e_B(x_3,y_3,\tau+h)-e_B(x_3,y_3,\tau)|\le C(\tau')^{-\frac12} \qquad \text{as } \tau'\ge C_0h. \tag{16.6.34}$$

Combining it with estimate (16.2.73) for $|\partial_{\tau'}e_{(2)}(x',y',\tau')|$ we conclude that the left-hand expression of (16.6.30) does not exceed

$$C\mu^{\frac32}\alpha^{-1}h^{-\frac32}\times \alpha\mu^{-\frac12}h^{\frac12}\times \max_{j\in\mathbb{Z}^+}\big(\max(|V(x)+2j\mu h-\tau|,h)\big)^{-\frac12}.$$

Estimate (16.6.31) is proven; estimate (16.6.31) follows from it. □

Further, as before $T^*=\epsilon_0\mu\ell$. Then we immediately arrive to

Corollary 16.6.8. *For a self-adjoint operator in domain X, $B(0,\ell)\subset X\subset \mathbb{R}^2$, $\ell\ge C_0h$, coinciding in $B(0,\ell)$ with the toy-model operator* (16.6.29) *with $\tau\asymp 1$, $0<\alpha\le 1$, $|\beta|\le 1$, $\mu\ge h^{-1}$*

(i) Statements (i) and (ii) of Proposition 16.6.7 remain true.

(ii) Formula (16.6.32) *holds modulo $O(\mu h^{-\frac32}\ell^{-1})$.*

Micro-Averaging

Let us consider micro-averaging. First let us estimate Fourier transform where as before we rescaled $t\mapsto \mu t$; formulae (16.6.16)–(16.6.17) imply immediately

Proposition 16.6.9. *For a toy-model operator* (16.6.29) *in $\mathbb{R}^3$ with $\mu h\ge 1$, $\gamma_3\ge h$*

$$\gamma^{-2}\gamma_3^{-1}|F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma(U\psi_\gamma)|\le C\mu^2h^{-1}\gamma_3^{-\frac12} \tag{16.6.35}$$

Corollary 16.6.10. *In the framework of Corollary 16.6.8 as $\ell\ge\gamma_3\ge h$*

$$\gamma^{-2}\gamma_3^{-1}|\Gamma\big(e(.,.,\tau)-e^{\mathsf T}(.,.,\tau)\big)\psi_\gamma|\le C\mu h^{-1}\gamma_3^{-\frac12}\ell^{-1}. \tag{16.6.36}$$

16.6.6 Magnetic Weyl Approximation

We can try to use a host of the different approximations but restrict ourselves now to the magnetic Weyl approximation. Recall (16.6.32) and note that

$$e^{\mathsf{MW}}(x,y,\tau) = \int e^{\mathsf{W}}_B(x_3,y_3,\tau-\tau')\,d_{\tau'}e^{\mathsf{MW}}_{(2)}(x',y',\tau') = \int e^{\mathsf{MW}}_{(2)}(x',y',\tau-\tau')\,d_{\tau'}e^{\mathsf{W}}_B(x_3,y_3,\tau'). \tag{16.6.37}$$

Pointwise Asymptotics

Proposition 16.6.11. *For the toy-model operator* (16.6.1) *with* $\tau \le c$, $\mu h \lesssim 1$ *and for the toy-model operator* (16.6.29) *with* $\tau \le c$, $\mu h \gtrsim 1$

$$|e^{\mathsf{MW}}(x,y,\tau) - e^{\mathsf{MW}}(x,y,\tau')| \le C \begin{cases} \mu^{\frac{3}{2}}h^{-1} & \text{as } \ \mu h \le 1,\\ \mu h^{-\frac{3}{2}} & \text{as } \ \mu h \ge 1 \end{cases} \tag{16.6.38}$$

as $|\tau-\tau'| \le h$.

Proof. Obviously the left-hand expression does not exceed

$$C\mu h^{-\frac{3}{2}} \sum_{j\le J} j^{-\frac{1}{2}} \asymp C\mu h^{-\frac{3}{2}} J^{\frac{1}{2}}$$

with $J = C_0 \max((\mu h)^{-1},1)$ which implies (16.6.38). □

Therefore one can hardly expect that the magnetic Weyl approximation provides a better error than the right hand expression of (16.6.38).

Proposition 16.6.12. *Let* $|\tau| \le \epsilon$. *Then for a toy-model operator* (16.6.1) *as* $\mu h \lesssim 1$ *and for a toy-model operator* (16.6.1) *as* $\mu h \gtrsim 1$

$$\mathsf{R}^{\mathsf{MW}} := |e^{\mathsf{T}}(x,x,0) - h^{-3}\mathcal{N}^{\mathsf{MW}}| \le Ch^{-2} + C\mu h^{-1} + C\mu h^{-\frac{5}{3}}|\beta|^{\frac{1}{3}}. \tag{16.6.39}$$

Proof. Without any loss of the generality we can assume that $x=0$, $\beta>0$. As contribution of k-th tick to the Tauberian expression is $O(\mu^{\frac{3}{2}}h^{-\frac{3}{2}}|k|^{-\frac{3}{2}})$, its contribution to the error when we replace β by 0 does not exceed $C\mu^{\frac{3}{2}}h^{-\frac{3}{2}}|k|^{-\frac{3}{2}} \times \mu^{-3}h^{-1}\beta^2|k|^3$ and summation with respect to

$$k : |k| \le \tilde{k}_1 := \mu\beta^{-\frac{2}{3}}h^{\frac{1}{3}} \tag{16.6.40}$$

returns

$$C\mu^{-\frac{3}{2}}h^{-\frac{5}{2}}\beta^2|k|^{\frac{5}{2}} = C\mu h^{-\frac{5}{3}}\beta^{\frac{1}{3}} \tag{16.6.41}$$

On the other hand, summation of $C\mu^{\frac{3}{2}}h^{-\frac{3}{2}}|k|^{-\frac{3}{2}}$ as $|k| \ge \tilde{k}_1$ returns $C\mu^{\frac{3}{2}}h^{-\frac{3}{2}}\tilde{k}_1^{-\frac{1}{2}}$ which is the same expression because $\tilde{k}_1$ was defined from $\mu^{-3}h^{-1}\beta^2k^3 = 1$.

Meanwhile, contribution of k-th tick to the error when we replace α by 0 does not exceed $C\mu^{\frac{3}{2}}h^{-\frac{3}{2}}|k|^{-\frac{3}{2}} \times \mu^{-2}h^{-1}\alpha|k|$ and summation with respect to

$$k : |k| \le \tilde{k}_2 \coloneqq \mu^2 h\alpha^{-1} \tag{16.6.42}$$

returns

$$C\mu^{-\frac{1}{2}}h^{-\frac{5}{2}}\alpha\tilde{k}_2^{\frac{1}{2}} = C\mu^{-\frac{1}{2}}h^{-2}\alpha^{\frac{1}{2}} \tag{16.6.43}$$

which is less than Ch^{-2}. On the other hand, summation of $C\mu^{\frac{3}{2}}h^{-\frac{3}{2}}|k|^{-\frac{3}{2}}$ as $|k| \ge \tilde{k}_2$ returns $C\mu^{\frac{3}{2}}h^{-\frac{3}{2}}\tilde{k}_2^{-\frac{1}{2}}$ which is the same expression because $\tilde{k}_2$ was defined from equation $\mu^{-2}h^{-1}\alpha\tilde{k}_2 = 1$. □

Remark 16.6.13. (i) We summed with respect to all k while it would be enough only with respect to $k : |k| \le \epsilon\mu$. However it provides us by no improvement.

(ii) So far we have not used factor $C(\mu^2h/\alpha|k|)^{\frac{1}{2}}$; using it we acquire in (16.6.41) factor $(\mu^2h/\alpha\tilde{k}_1)^{\frac{1}{2}} = \mu^{\frac{1}{2}}h^{\frac{1}{3}}\beta^{\frac{1}{3}}\alpha^{-\frac{1}{2}}$ resulting in

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{-2} + C\mu h^{-1} + C\mu^{\frac{3}{2}}h^{-\frac{4}{3}}\beta^{\frac{2}{3}}\alpha^{-\frac{1}{2}} \tag{16.6.44}$$

provided

$$\alpha \ge \mu h^{\frac{2}{3}}\beta^{\frac{2}{3}}. \tag{16.6.45}$$

(iii) In contrast to 2D the magnetic Weyl approximation now is better than Weyl approximation as $\mu \ge h^{-\frac{1}{3}}$ (when it matters) but Weyl approximation with the correction terms may be better still.

Micro-Averaging

Consider now micro-averaging. We need to redo only the first step of the proof of Proposition 16.6.12 and only in the case $\mu \ge h^{-\frac{1}{3}}\beta^{-2}$ and it is useful only if $\tilde{k}_1 \ge \hat{k}(\boldsymbol{\gamma}) = \mu h/\nu(\boldsymbol{\gamma})$ or equivalently

$$\nu(\boldsymbol{\gamma}) = \alpha\gamma + \beta\gamma_3 \ge \beta^{\frac{2}{3}}h^{\frac{2}{3}}. \tag{16.6.46}$$

In this case in the left-hand expression of (16.6.41) should be reset to $\tilde{k}_1$ replaced by $\hat{k}(\gamma)$ as $\hat{k}(\gamma) \ge 1$ or by 1 with an extra factor $(\mu h/\nu(\gamma))^l$ as $\hat{k}(\gamma) \le 1$ resulting in $CR_1^{\mathsf{MW}}(\gamma)$ with

$$R_1^{\mathsf{MW}}(\gamma) := C\mu \begin{cases} \beta^2 \nu(\gamma)^{-\frac{5}{2}} \big(\mu h/\nu(\gamma)\big)^l & \text{as } \nu(\gamma) \ge \mu h,\ \alpha \le \mu^2 h, \\ \beta^2 \nu(\gamma)^{-\frac{5}{2}} & \text{as } \nu(\gamma) \le \mu h,\ \mu\nu(\gamma) \ge \alpha, \\ \beta^2 \nu(\gamma)^{-\frac{5}{2}} (\mu^2 h/\alpha)^{\frac{1}{2}} \big(\mu h/\nu(\gamma)\big)^l & \text{as } \nu(\gamma) \ge \mu h,\ \alpha \ge \mu^2 h, \\ \beta^2 \nu(\gamma)^{-\frac{5}{2}} (\mu\nu(\gamma)/\alpha)^{\frac{1}{2}} & \text{as } \nu(\gamma) \le \mu h,\ \mu\nu(\gamma) \le \alpha \end{cases} \tag{16.6.47}$$

and therefore we arrive to

Proposition 16.6.14. *Let $|\tau| \le \epsilon$ and* (16.6.46) *be fulfilled. Then for a toy-model operator* (16.6.1) *as $\mu h \lesssim 1$ and for a toy-model operator* (16.6.1) *as $\mu h \gtrsim 1$ a*

$$\mathsf{R}^{\mathsf{MW}}(\gamma) := \gamma^{-2}\gamma_3^{-1} \Big| \int \big(e^{\mathsf{T}}(x,x,0) - h^{-3} \int \mathcal{N}^{\mathsf{MW}}\big)\psi_\gamma \, dx\Big| \le Ch^{-2} + C\mu h^{-1} + CR_1^{\mathsf{MW}}(\gamma) \tag{16.6.48}$$

with $R_1^{\mathsf{MW}}(\gamma)$ defined by (16.6.47).

Remark 16.6.15. If we replace only $e_{(2)}(.,.,.)$ by $e^{\mathsf{MW}}(.,.,.)$ we can skip the last term.

16.6.7 Geometric Interpretation

Note that there are two classical dynamics: in x' and in x_1. The former has return times $t'_k := t'_k(\tau')$ where τ' is a corresponding part of energy and t'_k are defined in Section 16.2 as $t_k(\tau)$ while the latter is

$$x_3(t) = x_3(0) + 2\xi_3(0)t - \beta t^2, \qquad \xi_3(t) = \xi_3(0) - \beta t. \tag{16.6.49}$$

and has return times (to 0) $t''_k(\tau - \tau') = 2\beta^{-1}(\tau - \tau')$ and therefore the total system has return times[18] t_k and return energy partitions $(\tau'_k, \tau - \tau'_k)$ defined from the pair of equations

$$t_k = t'_k(\tau'_k) = t''_k(\tau - \tau') \tag{16.6.50}$$

[18] To point 0.

which is equivalent to (16.6.9)–(16.6.10).

Therefore relatively sets of return directions ξ_k is thinner in 3D than in 2D which explains errors acquiring lesser factor (in comparison with the principal part) in 3D than in 2D.

In particular, as $\beta = 0$ 1D-dynamics does not return at all unless it stays at 0, therefore in this case (16.6.49) mean exactly that $\tau'_k = \tau$ and $t'_k(\tau) = t_k(\tau)$.

16.7 Pointwise Asymptotics: General 3D-Operators

16.7.1 Set-up

We assume that

$$F := \Big(\sum_{jk} g_{jk} F^j F^k\Big)^{\frac{1}{2}} \tag{16.7.1}$$

is disjoint from 0, where $\boldsymbol{F} = (F^1, F^2, F^3)$ and F are vector and scalar intensities of the magnetic field respectively. Temporarily we assume that $F = 1$ (as we can reduce the general case to this one by dividing operator by F and using our standard arguments).

Further, without any loss of the generality we will assume locally that

$$F^1 = F^2 = 0. \tag{16.7.2}$$

Remark 16.7.1. (i) Due to Frobenius theorem we can make locally $g^{3j} = 0$ for $j = 1, 2$ and $F^1 = F^2 = 0$ simultaneously if and only if $f_* \wedge df_* = 0$ where $f_* = \sum_{j,k} F^j g_{jk} dx^k$ is the 1-form[19)]

(ii) Since

$$F = \Big(\frac{1}{2} \sum_{i,j,k,l} g^{ij} g^{kl} F_{ik} F_{jl}\Big)^{\frac{1}{2}}, \qquad F_{jk} = \partial_k A_j - \partial_j A_k, \tag{16.7.3}$$

under assumption (16.7.2)

$$F = \big(g^{11} g^{22} - g^{12} g^{21}\big)^{\frac{1}{2}} |F_{12}|. \tag{16.7.4}$$

[19)] Similar statement holds in the multidimensional case provided $\mathsf{rank}(F_{jk})(x) = d - 1$ at each point x; then $\boldsymbol{F}$ is defined up to a scalar factor by $\boldsymbol{F}(x) \in \mathsf{Ker}(\mathsf{F}_{jk})(x)$.

However

(16.7.5) We can assume that $F^1 = F^2 = 0$ locally and simultaneously that $g^{jk} = \delta_{jk}$ along a single magnetic line $\{x' = 0\}$[19)].

Really, we can satisfy condition $f_* \wedge df_* = 0$ changing g^{jk} but preserving them at the chosen magnetic line. Note that $\boldsymbol{F}$ then acquires some scalar factor which does not affect this condition. So, we can assume $g^{3j} = 0$ for $j = 1, 2$ and $F^1 = F^2 = 0$ at the chosen magnetic line which (after shift) becomes then $\{x' = 0\}$. Changing $x_3 \mapsto \phi(x_3)$ makes $g^{33} = 1$ along this magnetic line. Then $F^3 = \pm F = 1$ (for an appropriate orientation). Changing $x' = B(x_3)x'$ with an appropriate matrix $B(x')$ makes $g^{jk} = \delta_{jk}$ along this line.

Further

(16.7.6) In the framework of assumption (16.7.5) we can assume that $g^{31} = 0$.

Really, one can achieve it by $x' \mapsto x'$, $x_3 \mapsto \phi(x)$ with $\phi(x) = x_3 + O(|x'|^2)$.

Furthermore, one can make

$$g^{jk} = \delta_{jk} + O(|x'|^2) \qquad \text{as} \quad x_3 = 0 \quad j, k = 1, 2 \tag{16.7.7}$$

by $x' \mapsto x' + Q(x')$ with Q quadratic respect to x'.

By a gauge transform one can make $A_3 = 0$. Then $F^1 = F^2 = 0$ imply that A_1, A_2 do not depend on x_3 and therefore we can assume that $A_1 = 0$ as well:

$$A_1 = A_3 = 0, \qquad A_2 = A_2(x') = x_2 + O(|x'|^3) \tag{16.7.8}$$

where the last equality is due to $F = 1$ and assumption (16.7.5).

16.7.2 Classical Dynamics

Let us consider a classical dynamics. We know that for time $|t| \le T = \epsilon_0$ Hamiltonian trajectory starting from $\{|x| \le c_0\mu^{-1}\}$ remains confined in $C_0\mu^{-1}$-tube $\{|x'| \le C_0\mu^{-1}\}$. Then in virtue of our assumptions

$$\frac{dx_3}{dt} = 2\xi_3 + O(\mu^{-1}) \qquad \frac{d\xi_3}{dt} = -V_{x_3}(0, 0, x_3) + O(\mu^{-1}) \tag{16.7.9}$$

and therefore

$$(x_3,\xi_3)(t) = (x_3^0,\xi_3^0)(t) + O(\mu^{-1}|t|), \tag{16.7.10}$$

where here $(x_3^0,\xi_3^0)(t)$ denotes dynamics for a 1D-Schrödinger $\xi_3^2 + V(0,0,x_3)$ with the same initial data (as $t=0$).

Then one can prove easily that

$$|\xi_3(t)| \le C\big(|\beta|\cdot|t| + |\xi_3(0)|\big), \qquad |x_3| \le C\big(|\beta|t^2 + |\xi_3(0)|\cdot|t|\big) \quad \text{if } x_3(0)=0 \text{ and } |t|\le\epsilon \tag{16.7.11}$$

with

$$\beta := -V_{x_3}(0). \tag{16.7.12}$$

Then

$$\xi_3 = \xi_3(0)\big(1+O(t^2)\big) + 2\beta t\big(1+O(t^2)\big), \tag{16.7.13}$$

$$x_3 = \xi_3(0)t\big(1+O(t^2)\big) + \beta t^2\big(1+O(t^2)\big). \tag{16.7.14}$$

Let us analyze evolution of $(x',\xi')(t)$ but we start from $(p_1,p_2)(t)$. Recall that

$$\mu^{-1}\{p_1,p_2\} \equiv 1, \qquad g^{11}g^{22} - g^{12}g^{21} \equiv 1 \mod O(\mu^{-2}) \tag{16.7.15}$$

in the tube in question.

Let $p_1' := p_1 + \gamma p_3$; then $p_2' := (g^{12}p_1 + g^{22}p_2 + g^{23}p_3) \equiv -\frac{1}{2}\mu^{-1}\{a,p_1'\} \mod O(\mu^{-1})$. Then

$$\frac{1}{2}\mu^{-1}\{a,p_2'\} \equiv (g^{11}p_1 + g^{12}p_2 + g^{13}p_3)g^{22} - (g^{12}p_1 + g^{22}p_2 + g^{23}p_3)g^{12} \equiv p_1 + (g^{13}g^{22} - g^{23}g^{12})p_3.$$

So we pick up $\gamma = g^{13}g^{22} - g^{23}g^{12}$ and then

$$p_1' := (g^{22})^{-\frac{1}{2}}(p_1+\gamma p_3), \qquad \gamma = g^{13}g^{22} - g^{23}g^{12}, \tag{16.7.16}$$

$$p_2' := (g^{22})^{-\frac{1}{2}}(g^{12}p_1 + g^{22}p_2 + g^{23}p_3), \tag{16.7.17}$$

satisfy

$$\frac{1}{2}\mu^{-1}\{a,p_1'\} \equiv -p_2', \qquad \frac{1}{2}\mu^{-1}\{a,p_2'\} \equiv p_1', \qquad \mu^{-1}\{p_1',p_2'\} \equiv 1 \tag{16.7.18}$$

modulo $O(\mu^{-1})$. We set $p'_3 = p_3$.

We do not assume that $g^{13} = 0$ identically; instead its choice will be different and it would be only $O(\mu^{-1})$ in the tube in question. Note that

$$\gamma \equiv g^{13}g^{22} \mod O(\mu^{-2}) \equiv g^{13} \mod O(\mu^{-2} + \mu^{-1}|x_3|), \tag{16.7.19}$$

$$\gamma_{x_j} \equiv g^{13} \mod O(\mu^{-1}). \tag{16.7.20}$$

Really, $g^{jk} = \delta_{jk} + O(|x'|^2 + |x'| \cdot |x_3|)$ $(j = 1, 2)$, $g^{3j} = O(|x'|)$.

Let us consider again $\{a, p'_1\}$, this time more precisely. Note that

$$\{a, p'_1\} = -2\mu p'_2 + 2\sum_j \gamma_{x_j} p_j p_3 - V_{x_1} + O(\mu^{-1}) \qquad \text{as } x_3 = 0$$

as $|x'| = O(\mu^{-1})$ due to our assumptions.

Therefore we arrive to the first equation below and the second is proven in the same way

$$\{a, p'_1\} \equiv -2\mu p'_2 + 2\sum_j g^{13}_{x_j} p_j p_3 - V_{x_1}, \tag{16.7.21}$$

$$\{a, p'_2\} \equiv 2\mu p'_1 + 2\sum_j g^{23}_{,x_j} p_j p_3 - V_{x_2} \mod O(\mu^{-1}). \tag{16.7.22}$$

We want

$$g^{23}_{x_2} = g^{13}_{x_1} = 0, \quad -g^{23}_{x_1} = g^{13}_{x_2} = \kappa(x_3) \qquad \text{as } x' = 0 \tag{16.7.23}$$

which is possible to arrange as the only value at $x' = 0$ which is fixed is

$$\kappa(x_3) := \frac{1}{2}\big(g^{13}_{x_2} - g^{23}_{x_1}\big)\big|_{x'=0} = -\frac{1}{2}\big(g_{13,x_2} - g_{23,x_1}\big)\big|_{x'=0}; \tag{16.7.24}$$

one can prove easily this by transformation $x_3 \mapsto x_3 + Q(x_3; x')$ with $Q(x_3; x')$ quadratic with respect to x'. Note that

$$f_* \wedge df_* = 2\kappa(x_3) dx_1 \wedge dx_2 \wedge dx_3 \qquad \text{as } x' = 0. \tag{16.7.25}$$

Then (16.7.21)–(16.7.22) become

$$\{a, p'_1\} \equiv -2(\mu + \kappa p_3)p'_2 - V_{x_1}, \tag{16.7.26}$$

$$\{a, p'_2\} \equiv 2(\mu + \kappa p_3)p'_1 - V_{x_2} \mod O(\mu^{-1}). \tag{16.7.27}$$

Obviously

(16.7.28) We can replace in equations (16.7.26)–(16.7.27) V_{x_j} by $V_{x_j}(x_3) := V_{x_j}|_{x'=0}$ and p_3 by $p_3^0(t)$.

If $\kappa = 0$ and V_{x_j} did not depend on x_3 we would get the toy-model operator. Unfortunately, it is not the case and (16.7.26), (16.7.27) coincide with those for the toy-model only modulo $O(\mu^{-1} + |p_3| + |x_3|)$. However note that $\kappa(x_3)p_3 = \frac{d}{dt}K(x_3)$ where

$$K = \frac{1}{2}\int \kappa \, dx_3. \tag{16.7.29}$$

Therefore

$$p'(t) \equiv \Omega(t)\Omega^{-1}(0)p'(0) + \int_0^t \Omega(t)\Omega^{-1}(t')V' \, dt \qquad \mathsf{mod}\ O(\mu^{-1}|t|) \tag{16.7.30}$$

with

$$\Omega(t) = \begin{pmatrix} \cos(\mu\theta) & -\sin(\mu\theta) \\ \sin(\mu\theta) & \cos(\mu\theta) \end{pmatrix}, \quad \theta(t) = t + \mu^{-1}K(x_3(t)) \tag{16.7.31}$$

$p = (p_1, p_2)^t$, $V' = (\nabla' V)^t$, $\Omega(t)$ fundamental matrix of system (16.7.26)–(16.7.27). Expressing $\Omega^{-1}(t')$ via its derivative and integrating by parts we conclude that

$$p'(t) \equiv \Omega(t)p'(0) + \mu^{-1}\Omega(t)V'(x_3(0)) - \mu^{-1}V'(x_3(t)). \tag{16.7.32}$$

Meanwhile, as $j = 1, 2$

$$\{a, (-1)^j \mu x_{3-j} + p_j\} \equiv -\sum_{k,l} g^{kl}_{x_j} p_k p_l - V_{x_j} \quad \mathsf{mod}\ O(\mu^{-1}) \tag{16.7.33}$$

and we can replace in the right-hand expression x' by 0, p_k by p'_k and as $k = 1, 2$ we can replace p'_k by the corresponding component of $\Omega(t)p'(0)$. Then integrating by parts we arrive to

$$\big((-1)^j \mu x_{3-j} + p_j\big)\big|_{t=0}^{t=t} \equiv \\ -\int_0^t \Big(\frac{1}{2}\sum_{k,l}\big(g^{11} + g^{22}\big)_{x_j}(p_1^2 + p_2^2) + V_{x_j}(x_3(t))\Big)\, dt \equiv -\int_0^t V_{x_j}(x_3(t))\, dt \tag{16.7.34}$$

modulo $O(\mu^{-1})$ as $\big(g^{11} + g^{22}\big)_{x_j} = O(\mu^{-1})$ in virtue of our assumptions including (16.7.15). Further, considering out-of integral terms we conclude that

(16.7.35) Equality (16.7.34) holds modulo $O\big(\mu^{-1}(|t| + |\sin(2\mu\theta) + |p_3|\big)$ (where $p_3(t) \equiv p_3(0) \mod |t|$).

16.7.3 Semiclassical Approximation

So, after rescaling, the classical dynamics of the general operator is close to one of the *generalized toy-model*

$$A^0 := h^2 D_1^2 + (hD_2 - \mu x_1)^2 + h^2 D_3^2 + V^0(x), \qquad V^0(x) := V(x_3) + \sum_{j=1,2} \alpha_j(x_3) x_j \tag{16.7.36}$$

with

$$V(x_3) = V(0,0,x_3), \qquad \alpha_j(x_3) = V_{x_j}(0,0,x_3), \quad j = 1,2 \tag{16.7.37}$$

and the latter one is close albeit with a larger error to the dynamics of the toy-model (16.6.1) with $V(x_3)$, $\alpha_j(x_3)$ replaced by $V(0) + \beta x_3$ ($\beta = V_{x_3}(0)$), $\alpha_j = \alpha_j(0)$ respectively.

Proposition 16.7.2 [20]**.** *Let conditions* (16.3.1)–(16.3.3), (16.7.7)–(16.7.8) *be fulfilled*[6]*. Then after rescaling*

(i) Uniformly with respect to $|t| \le \epsilon\mu$ the propagator $e^{i\hbar^{-1}tA}$ is an $\hbar$-Fourier integral operator corresponding to the Hamiltonian flow Ψ_t.

(ii) As $|\sin(2\theta_1)| \ge \epsilon$, $|t| \ge \epsilon_0$

$$U(x,y,t) \equiv \mu(2\pi\hbar)^{-\frac{3}{2}} |t|^{-\frac{1}{2}} i(\sin(\theta_1))^{-1} e^{i\hbar^{-1}\phi(x,y,t)} \Big(\sum_m b_m(x,y,t)\Big)\hbar^m \tag{16.7.38}$$

with θ_1 defined by

$$\theta_1(t) = \theta(t) + K\big(x_3(t)\big) - K\big(x_3(0)\big) \tag{16.7.39}$$

with $\theta(t)$, $K(x_3)$ defined by (16.3.20), (16.7.29) *respectively and with ϕ defined by* (16.3.25)–(16.3.29) *and satisfying (with all derivatives)*

$$\phi = \bar{\phi}(\theta) + O(\mu^{-1}t), \tag{16.7.40}$$

$$b_m = \delta_{0m} + O(\mu^{-1}t) \tag{16.7.41}$$

[20] Cf. Proposition 16.3.4.

with

$$\bar{\phi} = \beta\mu^{-1}t(x_3+y_3) + \frac{1}{8}(x_3-y_3)^2t^{-1} - \frac{2}{3}\beta^2\mu^{-2}t^3 + \bar{\phi}_2 \tag{16.7.42}$$

with $\bar{\phi}_2$ *defined by* (16.2.10).

Proof. Easy but tedious details we leave to the reader: comparing with the toy-model or generalized toy model we see that the structure of the canonical graph is close to one of the toy-model. □

Proposition 16.7.3 [21]**.** *Let conditions* (16.3.1)–(16.3.3), (16.7.7)–(16.7.8) *be fulfilled*[6]*. Then after rescaling*

(i) Decomposition (16.7.39) *remains valid as* $|t| \ge \epsilon_0$ *and* $|\cos(\theta_1)| \le \epsilon$.

(ii) Further, after rescaling decomposition (16.7.39) *remains valid as* $|t| \ge \epsilon_0$ *and*

$$C\max\bigl(\hbar, \mu^{-2}|t|\bigr) \le |\sin(\theta_1)| \le \epsilon \tag{16.7.43}$$

albeit with an error not exceeding

$$C\hbar^{-\frac{3}{2}}|t|^{-\frac{1}{2}}|\sin(\theta_1)|^{-1}\bigl(\hbar/|\sin(\theta_1)|\bigr)^{l} \tag{16.7.44}$$

and with ϕ, b_m *such that*

$$|D^\beta(\phi-\bar{\phi})| \le C_\beta\mu^{-1}|t|\alpha^{-1}|\sin(\theta_1)|^{-|\beta|} \qquad \forall\beta, \tag{16.7.45}$$

$$|D^\beta(b_m-\delta_{m0})| \le C_{m\beta}\mu^{-1}|t||\sin(\theta_1)|^{-m-|\beta|} \qquad \forall\beta, m. \tag{16.7.46}$$

Proof. Again a tedious but an easy proof we leave to the reader. □

16.7.4 Tauberian Estimates

Remark 16.7.4. (i) Let us set $x = y$. Then we get expression (16.7.38) for $U(y,y,t)$ albeit now due to definition $\theta_1 = \theta$.

[21] Cf. Proposition 16.3.5.

(ii) Now we can continue in the same manner as before but to get a better estimate one needs to observe that the shift in x' (after rescaling) is now $\mu^{-1}\int \nabla_{x'}V\cdot F^{-1}\,dt$ and its difference with $\mu^{-1}\alpha t$ does not exceed $C\mu^{-1}|t|\cdot|\operatorname{osc}(\nabla' V)|$ where $\operatorname{osc}(W)$ means an oscillation of W along trajectory (until time t).

However this trajectory must return to $x_3=y_3$ and therefore one can see easily that along it $p_3=O(\beta\mu^{-1}t)$ and $x_3-y_3=O(\beta\mu^{-2}t^2)$ and we need to assume that this is less than $\epsilon\alpha$ i.e.

$$|t|\le T_*=\epsilon\mu\min\bigl(1,\alpha^{\frac12}\beta^{-\frac12}\bigr),\quad \alpha:=|\nabla' V(y)|,\ \beta:=|\partial_{y_3}V(y)|. \tag{16.7.47}$$

Therefore only as $|t|\le T_*$ one can use nondegeneracy condition but it makes sense to consider $t:\ T_*\le |t|\le T^*=\epsilon\mu$. Then we have the following counterpart of Proposition 16.6.2 where we need to sum (16.6.15) to $k=T_*$ and compare $\bar k=\mu^2h\alpha^{-1}$ with T: namely, $\bar k\le T_*$ iff

$$\alpha\ge \max\bigl(\mu h,\beta^{\frac13}(\mu h)^{\frac23},\mu^{-1}\bigr). \tag{16.7.48}$$

Setting here $\beta=1$ (as without spatial averaging or micro-averaging $\partial_{\|F}V/F$ is our foe and we need only estimate it from above) and thus $T_*=\epsilon\mu\alpha^{\frac12}$ we arrive to estimate (16.7.50) below.

On the other hand, if we do not use non-degeneracy condition we need to sum

$$C\mu^{\frac32}h^{-\frac32}|k|^{-\frac12} \tag{16.7.49}$$

which results in $C\mu^{\frac32}h^{-\frac32}T^{\frac12}$ and we arrive to estimate (16.7.51) below.

Proposition 16.7.5 [22]**.** *Let conditions* (16.3.1)–(16.3.3), (16.7.7)–(16.7.8), (16.1.5)–(16.1.6) *be fulfilled*[6]*. Then*

(i) Assume that conditions (16.7.47)–(16.7.48) *be fulfilled as well. Then, after rescaling as* T *is given by* (16.7.47) *with* $\beta=1$,

$$|F_{t\to\hbar^{-1}\tau}\bar\chi_T(t)\Gamma_x U|\le C\mu h^{-2}+ C\begin{cases}\mu^{\frac52}h^{-1}\alpha^{-\frac12}\bigl(1+(\log(\alpha/(\mu h)^{\frac23}))_+\bigr) & \text{as } \alpha\ge(\mu h)^{\frac23},\\ \mu^{\frac32}h^{-\frac32}T^{\frac12} & \text{as } \alpha\le(\mu h)^{\frac23}.\end{cases} \tag{16.7.50}$$

[22] Cf. Proposition 16.6.2.

(ii) In the general case as $|t| \le T^*$

$$|F_{t\to\hbar^{-1}\tau}\bar{\chi}_T(t)\Gamma_x U| \le C\mu h^{-2} + C\mu^{\frac{3}{2}}h^{-\frac{3}{2}}T^{\frac{1}{2}}. \tag{16.7.51}$$

Then picking-up $T = T_*$ and dividing the left-hand expression of (16.7.50) by T_* or picking-up $T = T^*$ and dividing the left-hand expression of (16.7.51) by T^* we arrive to estimates (16.7.52), (16.7.53) below.

Corollary 16.7.6. *(i) In the framework of Proposition 16.7.5(i) with* $\alpha \ge (\mu h)^{\frac{3}{2}}$ *(which implies* $\mu \le h^{-1}$*)*

$$\mathsf{R}^{\mathsf{T}} := |e(0,0,\tau) - e^{\mathsf{T}}(0,0,\tau)| \le Ch^{-2}\alpha^{-\frac{1}{2}} + C\mu^{\frac{3}{2}}h^{-1}\alpha^{-1}\bigl(1 + (\log(\alpha/(\mu h)^{\frac{2}{3}}))_+\bigr). \tag{16.7.52}$$

(ii) In the general case

$$\mathsf{R}^{\mathsf{T}} \le Ch^{-2} + C\mu h^{-\frac{3}{2}}; \tag{16.7.53}$$

(iii) In particular, according to (16.7.52)*, as* $\mu \le h^{-1}$*,* $\alpha \asymp 1$

$$\mathsf{R}^{\mathsf{T}} \le Ch^{-2} + C\mu^{\frac{3}{2}}h^{-1}\bigl(1 + |\log \mu h|\bigr) \tag{16.7.54}$$

which is $O(h^{-2})$ *as* $\mu \le (h|\log h|)^{-\frac{2}{3}}$.

(iv) In particular, according to (16.7.53)*,* $\mathsf{R}^{\mathsf{T}} = O(h^{-2})$ *as* $\mu \le h^{-\frac{1}{2}}$.

(v) Estimate (16.7.52) *is better iff* $h^{-\frac{1}{2}} \le \mu \le h^{-1}$ *and*

$$\alpha \ge \mu^{-2}h^{-1} + (\mu h)^{\frac{1}{2}}|\log(\mu h)|. \tag{16.7.55}$$

16.7.5 Weyl Estimates

The proof of the following statement is rather obvious:

Proposition 16.7.7 [23]**.** *Let conditions* (16.3.1)–(16.3.3), (16.7.7)–(16.7.8), (16.7.47)–(16.7.48), (16.1.5)–(16.1.6) *be fulfilled*[6]*. Then*

[23] Cf. Proposition 16.6.3.

(i) As $\mu^2 h \le \alpha \le 1$ *estimate* (16.6.18) *holds and correction term* $h^{-3}\mathcal{N}_{x,\mathsf{corr}}$ *is delivered by* r*-term stationary phase approximation* $h^{-3}\mathcal{N}_{x,\mathsf{corr}(r)}$ *with an error not exceeding* (16.6.19)*.*

(ii) As $\alpha \le \mu^2 h$ *estimate* (16.6.19) *holds.*

(iii) In particular without any nondegeneracy condition $\mathsf{R}^{\mathsf{W}}_x = O(\mu^{\frac{3}{2}} h^{-\frac{3}{2}})$ *and it is* $O(h^{-2})$ *as* $\mu \le h^{-\frac{1}{3}}$*.*

(iv) On the other hand, $\mathsf{R}^{\mathsf{W}}_x = O(\mu^{-\frac{1}{2}} h^{-\frac{3}{2}} + \mu^{\frac{5}{2}} h^{-1})$ *as* $\alpha \asymp 1$ *and* $\mu \le h^{-\frac{1}{2}}$ *and therefore* $\mathsf{R}^{\mathsf{W}}_x = O(h^{-2})$ *as* $\alpha \asymp 1$ *and* $\mu \le h^{-\frac{2}{5}}$*.*

Combining with Corollary 16.7.6 we conclude that

Theorem 16.7.8. *Let conditions* (16.3.1)–(16.3.3), (16.7.7)–(16.7.8), (16.7.48), (16.1.5)–(16.1.6) *be fulfilled*[6)]*. Then,*

(i) In the general case

$$|e(x,x,0) - h^{-3}\mathcal{N}^{\mathsf{W}}_x(x,x,0)| \le Ch^{-2} + C\mu^{\frac{3}{2}} h^{-\frac{3}{2}}; \tag{16.7.56}$$

(ii) Under non-degeneracy condition

$$|\nabla_{\perp \boldsymbol{F}} V/F| \asymp 1 \tag{16.7.57}$$

as $\mu \le h^{-\frac{1}{2}}$ *estimates*

$$|e(x,x,0) - h^{-3}\mathcal{N}^{\mathsf{W}}_x(x,x,0)| \le Ch^{-2} + C\mu^{\frac{5}{2}} h^{-1} \tag{16.7.58}$$

and

$$|e(x,x,0) - h^{-3}\mathcal{N}^{\mathsf{W}}_x(x,x,0) - h^{-3}\mathcal{N}_{x,\mathsf{corr}(r)}| \le Ch^{-2} + C\mu^{\frac{3}{2}} h^{-\frac{3}{2}} (\mu^2 h)^{r+\frac{1}{2}} \tag{16.7.59}$$

hold.

16.7.6 Successive Approximations

Let us try successive approximation method. As an approximation we successively try the generalized toy-model operator A^0, the toy-model operator $\bar{A}$ and the magnetic Weyl approximation. As for $\mu \le h^{-\frac{1}{3}}$ Weyl approximation delivers $O(h^{-2})$ error we assume that

$$\mu \ge h^{-\frac{1}{3}}. \tag{16.7.60}$$

Generalized Toy-Model Approximation, $\mu h \le 1$

We claim that

(16.7.61) Effectively if $x = y (= 0)$ we can estimate $\|A - A^0\|$ by

$$\zeta(k) \coloneqq C\big(\mu^{-2}|k| + (\mu h)^{\frac12}\mu^{-1}|k|^{-\frac12}\big). \tag{16.7.62}$$

Really, due to our assumptions $g^{jl} = \delta_{jl} + O(|x'|^2) + O(|x'| \cdot |x_3|)$ with $|x'| = O(\mu^{-1})$, $x_3 = O(\mu^{-1}|k|)$ $(j, l = 1, 2)$, $g^{33} = 1 + O(\mu^{-2})$ (before rescaling) the only term which does not allow such estimate is $g^{j3}D_jD_3$ with $j = 1, 2$.

However for a classical trajectory to return back to time $t \asymp k$ we need to have $p_3 = O(\mu^{-1}k)$; in our analysis we will need also to satisfy uncertainty principle: $p_3^2|k| \ge \mu h$; so we estimate p_3 by $C\mu^{-1}|k| + C(\mu h)^{\frac12}|k|^{-\frac12}$ and as $g^{j3} = O(|x'|)$ this leads to (16.7.62).

Then the error in k-th winding does not exceed

$$C\mu^{\frac32}h^{-\frac32}|k|^{-\frac32} \times \big(\mu^{-2}|k| + (\mu h)^{\frac12}\mu^{-1}|k|^{-\frac12}\big) \times \mu^{-1}h^{-1}|k| \times \begin{cases} \big(\dfrac{\mu^2 h}{\alpha|k|}\big)^{r+\frac12} & \text{as } \alpha^{-1}\mu^2 h \le |k| \le (\mu^3 h)^{\frac12}, \\ 1 & \text{as } |k| \le \min\big(\alpha^{-1}\mu^2 h, (\mu^3 h)^{\frac12}\big). \end{cases} \tag{16.7.63}$$

Really, the successive approximation makes sense only as $\zeta(k)|k| \le \mu h$ which in view of (16.7.60) is equivalent to $|k| \le (\mu^3 h)^{\frac12}$.

Here as $|k| \ge \alpha^{-1}\mu^2 h$ we apply r-term stationary phase approximation as well. Also recall that this interval originally was $\alpha^{-1}\mu^2 h \le |k| \le \epsilon\mu\alpha^{\frac12}$ [24] but it is non-empty iff $\alpha \ge (\mu h)^{\frac23}$.

Then $(\mu^3 h)^{\frac12} \le \mu\alpha^{\frac12}$ and further $(\mu^3 h)^{\frac12} \ge \alpha^{-1}\mu^2 h$ iff $\alpha \ge (\mu h)^{\frac12}$. Only under this assumption we need to consider the first case in (16.7.63) but it is the most important case as it happens for $\alpha \asymp 1$.

Let us break (16.7.63) into two expressions: one with the second factor $\mu^{-2}|k|$ and another with the second factor $(\mu h)^{\frac12}\mu^{-1}|k|^{-\frac12}$.

[24] Here the upper limit meant that $|x_3| \le \alpha$ along trajectory before k-th winding. Since we look at the trajectory returning at k-th winding, $|x_3| \le (\mu^{-1}|k|)^2$ before it and we need to satisfy $(\mu^{-1}|k|)^2 \le \alpha$.

In the general case (when we do not know if $\alpha \ge (\mu h)^{\frac{1}{2}}$) summation of the first expression returns $Ch^{-2}(\mu^3 h)^{\frac{1}{4}}$ and of the second expression returns a lesser $Ch^{-2}(1+\log(\mu^3 h))$. Contribution of $k : |k| \ge (\mu^3 h)^{\frac{1}{2}}$ to the Tauberian expression is a sum of $C\mu^{\frac{3}{2}} h^{-\frac{3}{2}} |k|^{-\frac{3}{2}}$, which gives us $C\mu^{\frac{3}{2}} h^{-\frac{3}{2}} (\mu^3 h)^{-\frac{1}{4}}$ i.e. exactly the same answer. Therefore we arrive to

(16.7.64) The Tauberian expressions for the original operator and its generalized toy-model approximation differ by no more than $Ch^{-2}(\mu^3 h)^{\frac{1}{4}}$.

Assume now that $\alpha \ge (\mu h)^{\frac{1}{2}}$. Again break (16.7.63) into two expressions the same way as before. Obviously summation with respect to k of the second expression returns $Ch^{-2}\big(1+(\log(\mu^2 h/\alpha))_+\big)$ no matter what r is.

As $r=0$ summation of the first expression with respect to k returns $C\mu h^{-\frac{3}{2}} \alpha^{-\frac{1}{2}}$. One can see easily that the contribution of $k : |k| \ge (\mu^3 h)^{\frac{1}{2}}$ to the Tauberian expression does not exceed $C\mu h^{-\frac{3}{2}} \alpha^{-\frac{1}{2}}$.

Therefore we arrive to

(16.7.65) As $\alpha \ge (\mu h)^{\frac{1}{2}}$ the Tauberian expressions for the original operator and its generalized toy-model approximation differ by no more than $Ch^{-2} + C\mu h^{-\frac{3}{2}} \alpha^{-\frac{1}{2}}$.

As $r \ge 2$ summation of the first expression with respect to k returns its value ($\times |k|$) as $k = \max(1, \mu^2 h/\alpha)$:

$$C\mu^{\frac{3}{2}} h^{-1} \alpha^{-\frac{3}{2}} \times \begin{cases} \big(\frac{\mu^2 h}{\alpha}\big)^{r-1} & \text{as } 1 \ge \mu^2 h/\alpha, \\ 1 & \text{as } 1 \le \mu^2 h/\alpha \end{cases} \tag{16.7.66}$$

and as $r=1$ we get $C\mu^{\frac{3}{2}} h^{-1} \alpha^{-\frac{3}{2}} (1+\log(\alpha(\mu h)^{-\frac{1}{2}}))$. One can see easily that the contribution of $k : |k| \ge (\mu^3 h)^{\frac{1}{2}}$ to the Tauberian expression does not exceed these expressions as $r \ge 1$.

Therefore we arrive to

(16.7.67) As $\alpha \ge (\mu h)^{\frac{1}{2}}$ the Tauberian expressions with the subtracted $h^{-3}\mathcal{N}_{x,\mathsf{corr}(r)}$ i.e.

$$e^{\mathsf{T}}(x,x,\tau) - h^{-3}\mathcal{N}_{x,\mathsf{corr}(r)} \tag{16.7.68}$$

for the original operator and its generalized toy-model approximation differ by no more than

$$Ch^{-2} + Ch^{-2}(\log(\mu^2 h/\alpha))_+ + C\mu^{\frac{3}{2}}h^{-1}\alpha^{-\frac{3}{2}}\begin{cases} \bigl(1+\log(\alpha(\mu h)^{-\frac{1}{2}})\bigr) & \text{as } r=1, \alpha \ge \mu^2 h,\\ \bigl(1+\log(\mu^3 h)\bigr) & \text{as } r=1, \alpha \le \mu^2 h,\\ (\mu^2 h/\alpha)^{r-1} & \text{as } r\ge 2, \alpha \ge \mu^2 h,\\ 1 & \text{as } r\ge 2, \alpha \le \mu^2 h. \end{cases} \tag{16.7.69}$$

Combining with Corollary 16.7.6 we estimate expressions (16.7.70)–(16.7.72) below for different α; in particular, we arrive to:

Theorem 16.7.9. *Let conditions* (16.3.1)–(16.3.3), (16.7.7)–(16.7.8), (16.7.48), (16.1.5)–(16.1.6) *be fulfilled*[6)]. *Let* $e^0(.,.,.)$ *be a Schwartz kernel of the spectral projector for operator* A^0 *defined by* (16.7.36)–(16.7.37). *Then, as long as* $h^{-\frac{1}{3}} \le \mu \le h^{-1}$,

(i) In the general case

$$|e(x,x,0) - e^0(x,x,0)| \le C\mu^{\frac{3}{4}}h^{-\frac{7}{4}}. \tag{16.7.70}$$

(ii) Under non-degeneracy condition (16.7.57) *estimates*

$$|e(x,x,0) - e^0(x,x,0)| \le Ch^{-2} + C\mu h^{-\frac{3}{2}} \tag{16.7.71}$$

and

$$\begin{aligned} |e(x,x,0) - e^0(x,x,0) - h^{-3}\bigl(\mathcal{N}_{x,\mathsf{corr}(r)} - \mathcal{N}^0_{x,\mathsf{corr}(r)}\bigr)| \le{}& \\ Ch^{-2}\bigl(1+(\log \mu^2 h)_+\bigr) + C\mu^{\frac{3}{2}}h^{-1}\bigl(1+|\log \mu h|\bigr)+{}& \\ C\mu^{\frac{3}{2}}h^{-1}\begin{cases} \bigl(1+\log(\mu^3 h)\bigr) & \textit{as } r=1,\\ (\mu^2 h)^{r-1} & \textit{as } r\ge 2, \mu \le h^{\frac{1}{2}},\\ 1 & \textit{as } r\ge 2, \mu \ge h^{-\frac{1}{2}} \end{cases}& \end{aligned} \tag{16.7.72}$$

hold.

Remark 16.7.10. Replacing $\alpha(x_3)x_j$ by $\alpha(0)x_j$ we estimate the norm of the perturbation by $\zeta_1(k) = \mu^{-1}\cdot(\mu^{-2}|k|^2+\mu^{-1})$ which is less than expression (16.7.62) and thus leads to the lesser error. On the other hand, this new approximation is already a direct sum $A = A_{(2)} + B$ with the toy model operator $A_{(2)}$ and

$$B = h^2D_3^2 + V^0(x_3). \tag{16.7.73}$$

Therefore all above estimate remain valid with A^0 defined by (16.7.36)–(16.7.37) albeit with $\alpha_j = V_{x_j}(0,0,0)$.

Toy-Model Approximation, $\mu h \le 1$

Consider now the toy-model approximation $\bar{A}$ which differ from the approximation A^0 of Remark 16.7.10 by $O(x_3^2)$. Then

(16.7.74) Effectively if $x = y$ we can estimate $\|A^0 - \bar{A}\|$ by

$$\zeta(k) \coloneqq C\big(\mu^{-4}k^4 + \mu^{-1}h|k|\big). \tag{16.7.75}$$

Really, we need to estimate $O(x_3^2)$ and according to above arguments we estimate $|x_3|$ by $C\big(\mu^{-2}k^2 + (\mu h)^{\frac12}|k|^{-\frac12}\times\mu^{-1}|k|\big)$ which leads to (16.7.74).

Then we have estimate of the error in k-th winding not exceeding

$$C\mu^{\frac32}h^{-\frac32}|k|^{-\frac32}\times\big(\mu^{-4}k^4+\mu^{-1}h|k|\big)\times\mu^{-1}h^{-1}|k|\times \begin{cases} \big(\frac{\mu^2h}{\alpha|k|}\big)^{r+\frac12} & \text{as } |k|\ge\alpha^{-1}\mu^2h,\\ 1 & \text{as } |k|\le\alpha^{-1}\mu^2h.\end{cases} \tag{16.7.76}$$

The perturbation factor is $O(1)$ iff $|k|\lesssim \mu h^{\frac15}$ and we need to compare it with $\alpha^{-1}\mu^2h$: we have cases $\alpha\ge\mu h^{\frac45}$ when $\alpha^{-1}\mu^2h$ is less and $\alpha\le\mu h^{\frac45}$ when $\mu h^{\frac45}$ is less.

Case $\alpha\ge\mu h^{\frac45}$. Then we need to sum

(a) Expression (16.7.76) with the last factor 1 from $|k|=1$ to $|k|=\alpha^{-1}\mu^2h$ which returns $C\mu^{\frac12}h^{-\frac52}|k|^{\frac12}\times\big(\mu^{-4}k^4+\mu^{-1}h|k|\big)$ calculated as $|k| = \alpha^{-1}\mu^2h$:

$$C\mu^{\frac{11}{2}}h^2\alpha^{-\frac92} + C\mu^{\frac52}\alpha^{-\frac32};$$

as $\alpha \asymp 1$ the first term does not exceed $C\mu^{\frac{3}{2}}h^{-1}$ iff $\mu \le h^{-\frac{3}{4}}$ while for the second term it is always true.

(b) Expression (16.7.76) with the last factor $\bigl(\mu^2 h/\alpha|k|\bigr)^{r+\frac{1}{2}}$ from $|k| = \alpha^{-1}\mu^2 h$ to $k = \mu h^{\frac{1}{5}}$ which returns what we got earlier plus value of $C\mu^{\frac{1}{2}}h^{-\frac{5}{2}}|k|^{\frac{1}{2}} \times \bigl(\mu^{-4}k^4 + \mu^{-1}h|k|\bigr) \times (\mu^2 h/\alpha|k|)^{r+\frac{1}{2}}$ calculated at $k = \mu h^{\frac{1}{5}}$ (may be with the logarithmic term for exceptional values of $r = 4, 1$) i.e.

$$C\mu h^{-\frac{8}{5}}(\mu h^{\frac{4}{5}}/\alpha)^{r+\frac{1}{2}} + C\mu h^{-\frac{6}{5}}(\mu h^{\frac{4}{5}}/\alpha)^{r+\frac{1}{2}};$$

as $r = 1$ we get $C\mu^{\frac{5}{2}}h^{-\frac{2}{5}}\alpha^{-\frac{3}{2}}$; as $\alpha \asymp 1$ this is less than Ch^{-2} as $\mu \le h^{-\frac{16}{25}}$; note that the second term always is $O(\mu^{\frac{3}{2}}h^{-1} + h^{-2})$.

(c) $C\mu^{\frac{3}{2}}h^{\frac{3}{2}}|k|^{-\frac{3}{2}}(\mu^2 h/\alpha)^{r+\frac{1}{2}}$ as $|k| \ge C\mu h^{\frac{4}{5}}$ which returns the expression above.

In total, we get modulo $O\bigl(h^{-2} + \mu^{\frac{3}{2}}h^{-1}\bigr)$

$$C\mu^{\frac{11}{2}}h^2\alpha^{-\frac{9}{2}} + C\mu^{\frac{5}{2}}\alpha^{-\frac{3}{2}} + C\mu h^{-\frac{8}{5}}(\mu h^{\frac{4}{5}}/\alpha)^{r+\frac{1}{2}}. \tag{16.7.77}$$

Case $\alpha \le \mu h^{\frac{4}{5}}$**.** Then we need to sum

(a) Expression (16.7.76) with the last factor 1 from $|k| = 1$ to $|k| = \mu h^{\frac{1}{5}}$ which returns $C\mu^{\frac{1}{2}}h^{-\frac{5}{2}}|k|^{\frac{1}{2}} \times \bigl(\mu^{-4}k^4 + \mu^{-1}h|k|\bigr)$ calculated at $k = \mu h^{\frac{1}{5}}$ i.e.

$$C\mu h^{-\frac{8}{5}} + C\mu h^{-\frac{6}{5}};$$

here the first term is larger than $C\mu^{\frac{3}{2}}h^{\frac{3}{2}} + Ch^{-2}$ as $\mu \ge h^{-\frac{2}{5}}$ but the second term is always smaller than this.

(b) $C\mu^{\frac{3}{2}}h^{\frac{3}{2}}|k|^{-\frac{3}{2}}$ as $|k| \ge C\mu h^{\frac{4}{5}}$ which returns an above expression.

So, we get in this case

$$C\mu h^{-\frac{8}{5}}. \tag{16.7.78}$$

Thus we arrive to:

Theorem 16.7.11. *Let conditions* (16.3.1)–(16.3.3), (16.7.7)–(16.7.8), (16.7.48), (16.1.5)–(16.1.6) *be fulfilled*[6)]. *Let* $\bar{e}(.,.,.)$ *be a Schwartz kernel of the spectral projector for the toy-model operator* $\bar{A}$. *Then, as* $h^{-\frac{1}{3}} \le \mu \le h^{-1}$,

(i) In the general case

$$|e(x,x,0)-\bar{e}(x,x,0)| \le C\mu^{\frac{3}{4}}h^{-\frac{7}{4}} + C\mu h^{-\frac{8}{5}}. \tag{16.7.79}$$

(ii) Under non-degeneracy condition (16.7.57) *as* $\mu \le h^{-\frac{4}{5}}$ *estimates*

$$|e(x,x,0)-\bar{e}(x,x,0)| \le Ch^{-2} + C\mu h^{-\frac{3}{2}} + C\mu^{\frac{11}{2}}h^2 + C\mu^{\frac{5}{2}} + C\mu^{\frac{3}{2}}h^{-\frac{6}{5}} \tag{16.7.80}$$

and

$$\begin{aligned} |e(x,x,0)-\bar{e}(x,x,0)-h^{-3}\bigl(\mathcal{N}_{x,\mathsf{corr}(r)}-\mathcal{N}^0_{x,\mathsf{corr}(r)}\bigr)| \le \\ Ch^{-2}\bigl(1+(\log \mu^2 h)_+\bigr) + C\mu^{\frac{3}{2}}h^{-1}\bigl(1+|\log \mu h|\bigr)+ \\ C\mu^{\frac{3}{2}}h^{-1}\begin{cases} \bigl(1+\log(\mu^3 h)\bigr) & \text{as } r=1, \\ (\mu^2 h)^{r-1} & \text{as } r\ge 2, \mu \le h^{\frac{1}{2}}, \\ 1 & \text{as } r\ge 2, \mu \ge h^{-\frac{1}{2}} \end{cases} + \\ C\mu^{\frac{11}{2}}h^2 + C\mu^{\frac{5}{2}}C\mu h^{-\frac{8}{5}}(\mu h^{\frac{4}{5}})^{r+\frac{1}{2}}. \end{aligned} \tag{16.7.81}$$

hold.

Let us improve the above results. Note that we need only consider components of the estimate which are (in unrescaled x_3, p_3) due to $p_3 = O(\mu^{-1}k)$, $x_3 = O(\mu^{-2}k^2)$ rather than those which are due to the uncertainty principle $p_3 = O((\mu h)^{\frac{1}{2}}|k|^{-\frac{1}{2}})$, $x_3 = O(\mu^{-1}(\mu h)^{\frac{1}{2}}|k|^{\frac{1}{2}})$ as the latter brought a proper estimate in the above analysis.

Consider only case $\alpha \asymp 1$, $\beta \lesssim 1$. Then we need to consider only $\mu \ge h^{-\frac{16}{25}}$. Note that actually instead of $p_3 = O(t)$, $x_3 = O(t^2)$ we can use estimates $p_3 = O(\beta t)$, $x_3 = O(\beta t^2)$ and then our estimates acquire factor β^2 (recall that we are not discussing terms which are due to the uncertainty principle).

Case $h^{-16/25} \le \mu \le h^{-4/5}$. Consider (16.7.77) with $r = 1$; then the last term is the largest and the estimate with above improvement is $O(\beta^2\mu^{5/2}h^{-2/5})$ and it is $O(\mu^{3/2}h^{-1})$ as $\beta \le (\mu h^{\frac{3}{5}})^{-\frac{1}{2}}$. So we need to consider case

$$\beta \ge (\mu h^{\frac{3}{5}})^{-\frac{1}{2}} \qquad (\ge h^{1/10}). \tag{16.7.82}$$

As $r = 0$ we get $\mu^{\frac{3}{2}}h^{-\frac{6}{5}}$ which is less than $\mu h^{-\frac{3}{2}}$ in (16.7.71).

Case $h^{-4/5} \le \mu \le h^{-1}$. Then estimate is (modulo $O(\mu^{-3/2}h^{-1})$) $O(\beta^2\mu h^{-8/5})$ and it is $O(\mu^{3/2}h^{-1})$ as $\beta \le (\mu h^{\frac{6}{5}})^{\frac{1}{4}}$. So we need to consider case

$$\beta \ge (\mu h^{\frac{6}{5}})^{\frac{1}{4}} \qquad (\ge h^{1/10}). \tag{16.7.83}$$

Therefore we conclude that

Theorem 16.7.12. *As $\beta \coloneqq |\nabla_{\|\boldsymbol{F}} V/F| \le h^{-1/10}$ estimates* (16.7.70)–(16.7.72) *of Theorem 16.7.9 remain true for A^0 replaced by $\bar{A}$.*

So we need to consider the case when 1D-Schrödinger operator is almost non-degenerate and the corresponding dynamics is confined to zone $x_3 \le \beta$; rescaling $x_3 \mapsto x_{3,\mathsf{new}} = x_3/\beta$ we arrive to operator

$$\beta^2\Big(h_3^2D_3^2 + V_3(x_3)\Big), \qquad h_3 = h/\beta^2 \ll 1, \quad V_3(x_3) = \beta^{-2}V(x_3) \tag{16.7.84}$$

satisfies $\partial_3 V_3(x_3) \asymp 1$. Then we can construct its solution by WKB method. Note that before any rescaling

$$U_{(1)}(x_3, y_3, t) = (2\pi h)^{-1}\int \exp\Big(ih^{-1}\big(S(x_3, t, \zeta) - y_3\zeta\big)\Big)B(x_3, t, \zeta)\, d\zeta \tag{16.7.85}$$

where S solves

$$S_t = S_{x_3}^2 + V(x), \qquad S|_{t=0} = x_3\zeta \tag{16.7.86}$$

and $B \sim \sum_k B_k h^k$ in the standard way.

Then one can prove easily that

$$S = \bar{S} + O\Big(x_3^2|t| + |x_3|(|\zeta| + |t|)|t| + \zeta^2|t|^3 + |t|^5\Big) \tag{16.7.87}$$

where

$$\bar{S} = x_3\zeta + t(\zeta^2 - \beta x_3) + \zeta\beta t^2 + \frac{2}{3}\beta^2 t^3 \tag{16.7.88}$$

solves the same problem for toy-model $V = -\beta x_3$. Rescaling $x_3 \to x_3/\beta$, $h \mapsto h/\beta^2$ shows one can calculate S and $\bar{S}$ for $\beta = 0$, then plug $x_3 = x_3/\beta$, $\zeta = \zeta/\beta$ and multiply the result by β^2; (16.7.87) obviously survives but (16.7.88) improves to

$$S = \bar{S} + O\Big(x_3^2|t| + |x_3|(|\zeta| + \beta|t|)|t| + \zeta^2|t|^3 + |t|^5\Big). \tag{16.7.87*}$$

Obviously phase function

$$\bar{\phi}(x_3, y_3, t, \zeta) \coloneqq \bar{S}(x_3, t, \zeta) - y_3\zeta \tag{16.7.89}$$

is equivalent to phase function

$$\bar{\phi} \coloneqq (\zeta + \beta t)x_3 - (\zeta - \beta t)y_3 - 2\zeta^2 t - \frac{2}{3}\beta^2 t^3 \tag{16.7.90}$$

we used earlier.

Furthermore, obviously

(16.7.91) Phase function

$$\phi(x_3, y_3, t, \zeta) \coloneqq S(x_3, t, \zeta) - y_3\zeta \tag{16.7.92}$$

is 1D-action and therefore the total exponent is also 3D-action associated with the generalized toy-model.

Note that for a phase function (16.7.89) $\bar{\phi}_\zeta = 0$ iff $x_3 - y_3 + 2t\zeta + \beta t^2 = 0$ and then $\bar{\phi}_{\zeta\zeta} = -4t$ and therefore for $|t| \gg h$ we can apply a stationary phase method for both S and $\bar{S}$ and while the principal term would be of magnitude $(h|t|)^{-\frac{1}{2}}$ the error in l-term approximation will be of magnitude $(h|t|)^{-\frac{1}{2}}(h/|t|)^l$ or after all the substitution to $\boldsymbol{U}$ and rescalings it will be

$$C\mu^{\frac{3}{2}}h^{-\frac{3}{2}}|k|^{-\frac{3}{2}} \times (\mu h/|k|)^l \tag{16.7.93}$$

in the Tauberian expression. While almost useless for μh close to 1 and $k = 1$ the last factor is very important for larger k.

Then the contribution of zone $\{k : |k| \ge m\}$ to such error does not exceed (as $l = 1$)

$$C\mu^{\frac{5}{2}}h^{-\frac{1}{2}}m^{-\frac{3}{2}} \tag{16.7.94}$$

and it is $O(\mu^{-\frac{3}{2}}h^{-1})$ as $m \ge \mu^{\frac{2}{3}}h^{\frac{1}{3}}$. Obviously $m \ll \mu h^{\frac{1}{5}}$ and $m \ll \mu^2 h$ which means that in zone $k : |k| \le m$ we could perfectly deal with successive approximations.

On the other hand, replacing B by 1 and $\phi_{\zeta\zeta}$ by $-4t$ leads to an error

$$C\mu^{\frac{3}{2}}h^{-\frac{3}{2}}|k|^{-\frac{3}{2}} \times \mu^{-2}k^2 \times \min\bigl((\mu^2 h/|k|)^{\frac{1}{2}}, 1\bigr) \tag{16.7.95}$$

which after summation becomes $\mu^{\frac{1}{2}}h^{-1} \times \mu$ which is not as good as the Tauberian estimate. This spares us from more complicated formula. We leave to the reader

Problem 16.7.13. (i) Write the approximation due to described combination of successive approximations and WKB method.

(ii) Consider $\alpha \le 1$.

Magnetic Weyl Approximation

In virtue of the above results the result of Subsubsection 16.6.6 hold for general operators.

16.7.7 Transitional and Strong Magnetic Field

The standard method of successive approximations shows that replacing operator by the generalized toy-model brings an error with the contribution of $\{t : |t| \asymp T\}$ (before rescaling) not exceeding

$$C\mu h^{-2} \times \bigl(\mu^{-1}h + C\mu^{-\frac{1}{2}}h^{\frac{1}{2}}(T^{\frac{1}{2}} + h^{\delta})\bigr) \tag{16.7.96}$$

and summation with respect to T running from h to 1 returns

$$C\mu h^{-2} \times \bigl(\mu^{-1}h + C\mu^{-\frac{1}{2}}h^{\frac{1}{2}}(1 + h^{\delta}|\log h|)\bigr) \asymp C\mu^{\frac{1}{2}}h^{-\frac{3}{2}} + C\mu^{\frac{1}{2}}h^{\delta-\frac{3}{2}}|\log \mu|$$

which is less than the Tauberian estimate. Therefore

Theorem 16.7.14. *Let conditions* (16.3.1)–(16.3.3), (16.7.7)–(16.7.8), (16.7.48), (16.1.5)–(16.1.6) *be fulfilled*[6)]. *Then as* $\mu h \gtrsim 1$

$$|e(x,x,0) - e^{0}(x,x,0)| \le C\mu h^{-\frac{3}{2}}. \tag{16.7.97}$$

Similarly, transition to the toy model brings an error not exceeding

$$C\mu h^{-2} \times \bigl(\beta^2 T^4 + h^{\frac{4}{3}}\bigr) \tag{16.7.98}$$

and summation with respect from h to T_* returns $C\mu h^{-\frac{3}{2}} + C\mu h^{-2}\beta^2 T_*^4$. On the other hand, Tauberian approach shows that the contribution of $|t| \ge T_*$ should not exceed $C\mu h^{-\frac{3}{2}} T_*^{-1}$ and minimizing it sum we arrive to $T_* = \beta^{-\frac{2}{5}} h^{\frac{1}{10}}$ and it is

$$C\mu h^{-\frac{3}{2}}\Bigl(1 + \beta^{\frac{2}{5}} h^{-\frac{1}{10}}\Bigr). \tag{16.7.99}$$

Therefore

Theorem 16.7.15. *Let conditions* (16.3.1)–(16.3.3), (16.7.7)–(16.7.8), (16.7.48), (16.1.5)–(16.1.6) *be fulfilled*[6)]. *Then as* $\mu h \gtrsim 1$

$$|e(x,x,0) - \bar{e}(x,x,0)| \le C\mu h^{-\frac{8}{5}}. \tag{16.7.100}$$

Further, if we pick up $T_* = \min(\beta^{-\frac{1}{2}} h^{\frac{1}{8}}, 1)$ we conclude that the contribution of zone $\{t : T_* \le |t| \le \epsilon\}$ to an error does not exceed $C\mu h^{-\frac{3}{2}}$. So we need to consider zone $\{t : |t| \ge h^{\frac{1}{8}}\}$. However in this zone we can apply stationary phase expression for $U_{(31)}$ which gives an error

$$C \int_T^1 \mu h^{-2} \times \big(\frac{h}{t}\big)^{\frac{3}{2}}\, dt \asymp C\mu h^{-2} \big(\frac{h}{T}\big)^{\frac{3}{2}}$$

which is $O(\mu h^{-\frac{3}{2}})$ as $T \ge h^{\frac{2}{3}}$.

Problem 16.7.16. Construct this stationary phase approximation.

Again for the magnetic Weyl approximation error we refer to Subsection 16.6.2.

16.7.8 Micro-Averaging

We are interested only in Tauberian, Weyl and magnetic Weyl estimates as we consider toy-model and generalized toy-model approximations a bit too complicated. We consider here only isotropic micro-averaging (i.e. with $\gamma_3 = \gamma$).

Tauberian Estimates

Basically everything remains as in Subsection 16.6.4 but we need to answer a question: "what is ℓ?" Or rather "what T^* we need to take?" Clearly $T^* = \epsilon \min(\alpha, \beta)$ fits but can we do better than this?

(i) Let $\beta \le \alpha$. Then as we take $\gamma_3 = \gamma$ we have $\nu(\boldsymbol{\gamma}) \asymp \alpha\gamma$ so averaging with respect to x_3 does not matter and therefore we do not need to assume that $\gamma_3 \le \epsilon\beta$ (which in this case would mean $\gamma \le \epsilon\beta$). In this case our assumption is $\gamma \le \epsilon\alpha$.

Then as $\nabla_{\perp \boldsymbol{F}} V/F$ has a less oscillation then $\epsilon\alpha$ with $\alpha = |\nabla_{\perp \boldsymbol{F}} V/F|\big|_{t=0}$ for time $T^* = \epsilon_1\alpha$ we can take $T^* = \epsilon_1\alpha$.

(ii) Let $\beta \ge \alpha$. Then as we take $\gamma_3 = \gamma$ we have $\nu(\boldsymbol{\gamma}) \asymp \beta\gamma$ so averaging with respect to x' does not matter as well as the shift and therefore we do not need to assume that $\gamma \le \epsilon\alpha$ (which in this case would mean $\gamma_3 \le \epsilon\beta$).

Then as $\nabla_{\parallel \boldsymbol{F}} V/F$ has a less oscillation then $\epsilon\beta$ with $\beta = |\nabla_{\parallel \boldsymbol{F}} V/F|\big|_{t=0}$ for time $T^* = \epsilon_1\beta$ we can take $T^* = \epsilon_1\beta$. However in this case we should not include factor $(\mu h/\alpha|k|)^{\frac{1}{2}}$ which is a loss as $\mu h/\alpha < h/\beta\gamma$ i.e. as $\mu\gamma \le \alpha\beta$.

So, in both cases we can take $\gamma_3 = \gamma$ and $T^* \asymp |\nabla V/F|$ (but we obviously need $\gamma \le |\nabla V/F|$).

Weyl and Magnetic Weyl Estimates

As we do not care about T^* anymore we simply take $\gamma_3 = \gamma \le |\nabla V/F|$.

Problem 16.7.17. As β is rather an obstacle in the toy-model approximation, it would be interesting investigate this approximation with micro-averaging.

16.8 Dirac Energy: 3D-Estimates

In this and the next sections we consider asymptotics of the quantity I defined by (16.1.2).

16.8.1 Tauberian Formula

Let us consider first contribution of the zone $\{(x,y) : |x-y| \ge C\gamma\}$.

Proposition 16.8.1 [25]**.** *Let $\mu \le h^{-1}$. Then under conditions* (16.1.5)–(16.1.7) *the contribution of zone $\{(x,y) : |x-y| \ge C\gamma\}$, to the remainder is $O(h^{-2}\gamma^{-\kappa})$ while the main part is given by the same expression* (16.1.1) *with $e(x,y,0)$ replaced by its standard implicit Tauberian approximation* (16.1.11) *with $T \asymp \epsilon$.*

Proof. Proof repeats one of Proposition 16.4.1 albeit according to Chapter 13 estimate (16.4.4) is replaced by

$$\|E(\tau,\tau')\bar{\psi}\|_1 \le Ch^{-3}\big(|\tau-\tau'| + CT^{-1}h\big) \qquad \forall \tau,\tau' \in [-\epsilon,\epsilon],\ T = \epsilon\mu \tag{16.8.1}$$

[25] Cf. Proposition 16.4.1.

with $T = \epsilon$. □

Then we immediately arrive to:

Proposition 16.8.2 [26]**.** *Let conditions* (16.1.5)–(16.1.7) *be fulfilled. Then*

(i) As $0 < \kappa < 3$ *and* <u>*either*</u> $\kappa \neq 1, 2$ <u>*or*</u> $\kappa = 1, 2$ *and* $\omega(x, y)$ *is replaced by* $\omega(x, y) - \varkappa(\frac{1}{2}(x + y))|x - y|^{-\kappa}$ *with an appropriate smooth coefficient* $\varkappa(x)$*, with the error* $O(h^{-2-\kappa})$ *one can replace* $e(x, y, \tau)$ *by its standard Tauberian expression* (16.1.11) *in the formula* (16.1.11) *for* I*.*

(ii) As $\kappa = 1, 2$ *and* $\omega = \varkappa(\frac{1}{2}(x+y))|x-y|^{-\kappa}$*, with the error* $O(h^{-2-\kappa}|\log h|)$ *one can replace* $e(x, y, \tau)$ *by its standard Tauberian expression* (16.1.11) *in the formula* (16.1.11) *for* I*.*

Proposition 16.8.3 [27]**.** *Let conditions* (16.1.5) *and* (16.1.6) *be fulfilled.*

(i) Further, let <u>*either*</u> *condition* (16.1.8) [28] *be fulfilled* <u>*or*</u> $\mu \le h^{\delta-1}|\log h|^{-1}$ *with an arbitrarily small exponent. Then* (16.1.11) *and Statements (i), (ii) of Proposition 16.8.2 hold.*

(ii) Furthermore, let $h^{-1-\delta}\mu \le h^{-1}$*. Then* (16.1.11) *and Statements (i), (ii) of Proposition 16.8.2 hold with an extra factor* $(1 + \mu h^{1-\delta})$ *in the right-hand expressions.*

16.8.2 Strong Magnetic Field

Consider the Schrödinger-Pauli operator as $\mu h \ge \epsilon_0$ which is a bit more tricky:

Proposition 16.8.4 [29]**.** *Let* $\mu h \ge \epsilon_0$*. Then*

(i) Contribution of the zone $\{(x, y) : |x - y| \ge \gamma\}$ *to* I *does not exceed* $C\mu h^{-2}\gamma^{-\kappa}$*.*

---

[26] Cf. Proposition 16.4.3.

[27] Cf. Proposition 16.4.6.

[28] Which actually could be replaced by much weaker non-degeneracy condition of Chapter 13 for $d = 3$.

[29] Cf. Proposition 16.4.8.

(ii) Further,

$$|\mathsf{I}| \le C\mu h^{-2}\begin{cases} h^{-\kappa} & \text{as } \ 0<\kappa<1,\\ h^{-1}\bigl(1+(\log \mu h)_+\bigr) & \text{as } \ \kappa=1,\\ \mu^{\frac{1}{2}(\kappa-1)}h^{-\frac{1}{2}(\kappa+1)} & \text{as } \ 1<\kappa<3. \end{cases} \tag{16.8.2}$$

(iii) Furthermore, $\mathsf{I} = O(\mu h^{\infty})$ *under condition* (16.4.14)*; in particular it is the case as* $\mathfrak{z}<1$ *and* $\mu h \ge C_0$.

Proof. Statement (i) trivially follows from the fact that $\mathscr{L}^2$-norm of $e(.,.,\tau)$ does not exceed $C\mu^{\frac{1}{2}}h^{-1}$.

Meanwhile, an estimate of the contribution of the zone

$$\{(x,y) : x\in B(0,1), y\in B(0,1), |x-y|\le \gamma\}$$

is more subtle: if $0<\kappa<1$ $\omega = \partial_{x_3}\omega'$ with $\omega' = O(\gamma^{1-\kappa})$ as $|x-y|\le\gamma$. Then

$$\begin{aligned} I_\gamma = \int |e(x,y,\tau)|^2\omega(x,y)\psi'_\gamma(x-y)\,dxdy = \\ -2h^{-1}\operatorname{Re} i\int hD_{x_3}e(x,y,\tau)\cdot e(y,x,\tau)\omega'(x,y)\psi'_\gamma(x-y)\,dxdy - \\ \int |e(x,y,\tau)|^2\omega(x,y,\tau)\cdot\partial_{x_3}\psi'_\gamma(x-y)\,dxdy \end{aligned}$$

with the first and second terms not exceeding $C\gamma^{1-\kappa}h^{-1}\times\mu h^{-2}$ and $C\gamma^{-\kappa}\times\mu h^{-2}$ respectively where we used the fact that $\mathscr{L}^2$-norm of $P_j e(.,.,\tau)$ does not exceed $C\mu^{\frac{1}{2}}h^{-1}$. Setting $\gamma = h$ finishes the proof of Statement (ii) in this case.

As $1\le\kappa<3$ we consider a partition of $\{(x,y) : |x-y|\le h\}$ to subzones $\{(x,y) : |x'-y'|\gtrsim\sigma\}$ and $\{(x,y) : |x'-y'|\lesssim\sigma\}$ and in the former zone we repeat the same arguments as before albeit with $\omega' = O(\log|x'-y'|/h)$, $\omega' = O(|x'-y'|^{1-\kappa})$ for $\kappa=1$ and $1<\kappa<3$ respectively; then contribution of this zone does not exceed $Ch^{-1}|\log\sigma/h|\times\mu h^{-2}$ and $Ch^{-1}\sigma^{1-\kappa}\times\mu h^{-2}$ respectively; as $\sigma = \mu^{-\frac{1}{2}}h^{\frac{1}{2}}$ we arrive to estimate (16.8.2) for contribution of this zone as well which proves Statement (ii).

Finally, a contribution of the zone $\{(x,y) : |x'-y'|\le\sigma, |x_3-y_3\le\gamma\}$ does not exceed $C\mu^2h^{-4}\times\int|z|^{-\kappa}\,dz$ with the integral taken over the cylinder $\{|z'|\le\sigma, |z_3\le\gamma\}$ and not exceeding $C(1+|\log\sigma/\gamma|)\sigma^2$ and $C\sigma^{2-\kappa}$

respectively as $|e(x,y,\tau) = O(\mu h^{-2})$. Again for selected γ, σ we get estimate (16.8.2) for contribution of this zone as well.

Finally, Statement (iii) is due to the fact that $e(x,y,0) = O(\mu h^{\infty})$ since condition (16.4.14) is fulfilled in one of points x, y; see Subsection 13.6.4.□

Proposition 16.8.5 [30]**.** *Let conditions* (16.1.1)–(16.1.5) *be fulfilled. Let* $\mu h \ge \epsilon$ *and one of the nondegeneracy conditions* (16.4.15) *or* (16.4.17) *be fulfilled.*

(i) As $\sigma \ge C_0 \mu^{-\frac{1}{2}} h^{\frac{1}{2}}$ *contribution of the zone*

$$\{(x,y) : \ |x'-y'| \ge C_0 \mu^{-\frac{1}{2}} h^{\frac{1}{2}}, |x-y| \ge \gamma\}$$

to I *does not exceed* $C\mu h^{-1}\gamma^{-\kappa}$.

(ii) As $\gamma \ge C_0 h$ *contribution of the zone*

$$\{(x,y) : \ |x'-y'| \le C_0 \mu^{-\frac{1}{2}} h^{\frac{1}{2}}, |x_3-y_3| \ge \gamma\}$$

to I *does not exceed* $C\mu h^{-1}\gamma^{-1-\kappa}$.

(iii) Estimate

$$|\mathsf{I} - \mathsf{I}^{\mathsf{T}}| \le C\mu h^{-1} \begin{cases} h^{-\kappa} & \text{as } \ 0<\kappa<1, \\ h^{-1}\big(1+(\log \mu h)_+\big) & \text{as } \ \kappa = 1, \\ \mu^{\frac{1}{2}(\kappa-1)} h^{-\frac{1}{2}(\kappa+1)} & \text{as } \ 1<\kappa<3, \ \kappa \ne 2, \\ \mu^{\frac{1}{2}} h^{-\frac{3}{2}}\big(1+(\log \mu h)_+\big) & \text{as } \ \kappa = 2 \end{cases} \tag{16.8.3}$$

holds.

Proof. To prove Statement (i) recall that drift speed does not exceed $C\mu^{-1}$ and therefore Hilbert-Schmidt norm of $\psi E(\tau)\psi'$ does not exceed $C\mu^{\frac{1}{2}} h^{-\frac{1}{2}}$ as ψ, ψ' are $\mathscr{L}_0^{\infty}$-functions with $\mathsf{dist}(\mathsf{supp}\,\psi, \mathsf{supp}\,\psi') \ge C_0 \mu^{-\frac{1}{2}} h^{\frac{1}{2}}$.

Really, it is true for a Hilbert-Schmidt norm of $\big(E(\tau) - E(\tau')\big)\psi$ with $|\tau - \tau'| \le h$ and then by Tauberian theorem it is true for a Hilbert-Schmidt

[30] Cf. Proposition 16.4.9.

norm of $\big(E(\tau)-E^{\mathsf{T}}(\tau)\big)\psi'$ with E^{T} operator with the Schwartz kernel e^{T} with time $T\asymp 1$:

$$\|\big(E(\tau)-E^{\mathsf{T}}(\tau)\big)\psi'\|_{\mathsf{HS}}\le C\mu^{\frac12}h^{-\frac12}. \tag{16.8.4}$$

However $\psi E^{\mathsf{T}}\psi'$ is negligible as $T\le\epsilon$ due to propagation results. This implies Statement (i) obviously.

Statement (ii) is proven in the same way but now $T\asymp\gamma$.

To prove Statement (iii) we need to estimate contribution to $\mathsf{I}-\mathsf{I}^{\mathsf{T}}$ of the zones $\{(x,y):\ |x'-y'|\le h, h\le|x_3-y_3|\le\epsilon\}$ and $\{(x,y):\ |x-y|\le h\}$. However estimate of the contribution of the former is trivial due to estimate (16.8.4). So, consider zone $\{(x,y):\ |x-y|\le h\}$.

(a) If $0<\kappa<1$ we can replace in the same way as in the proof of Proposition 16.8.4 $\omega(x,y)$ by $\omega'(x,y)=O(h^{1-\kappa})$ in this zone and one copy of $\big(e(x,y,\tau)-e^{\mathsf{T}}(x,y,\tau)\big)$ to $D_3\big(e(x,y,\tau)-e^{\mathsf{T}}(x,y,\tau)\big)$; however Subsection 13.6.4 implies the following generalization of (16.8.4):

$$\|P_3^k P'^j\big(E(\tau)-E^{\mathsf{T}}(\tau)\big)\psi'\|_{\mathsf{HS}}\le C\mu^{\frac12}h^{-\frac12}(\mu h)^j \tag{16.8.5}$$

with $P'=(P_1,P_2)$ and therefore contribution of this zone to $(\mathsf{I}-\mathsf{I}^{\mathsf{T}})$ does not exceed $C\mu h^{-1-\kappa}$. In this case estimate (16.8.3) is proven.

(b) Let $1\le\kappa<3$. Then the above arguments imply

(16.8.6) If $1<\kappa<3$ the contribution of the zone $\{(x,y):\ |x'-y'|\ge\sigma\}$ with $\sigma\ge C_0\mu^{-\frac12}h^{\frac12}$ to I does not exceed $C\mu h^{-2}\sigma^{1-\kappa}$; if $\kappa=1$ it does not exceed $C\mu h^{-2}(1+|\log\mu h|)$.

Observe that with $\sigma=C_0\mu^{-\frac12}h^{\frac12}$ we get exactly the right-hand expression in (16.8.3). Therefore we need to estimate a contribution of the zone $\{(x,y):\ |x'-y'\le\sigma, |x_3-y_3|\le h\}$ with $\sigma=C_0\mu^{-\frac12}h^{\frac12}$.

Consider case $1\le\kappa<2$ first. Note that $\omega=(3-\kappa)^{-1}\sum_j\partial_{z_j}z_j\omega$ [31] and therefore we can replace ω either by $(x_j-y_j)\omega=O(|x'-y'|\cdot|x-y|^{-\kappa})$ with $j=1,2$ or by $(x_3-y_3)\omega=O(|x-y|^{1-\kappa})$, simultaneously applying $h^{-1}P_j$ to one of the factors $\big(e(x,y,\tau)-e^{\mathsf{T}}(x,y,\tau)\big)$. Then we can replace

[31] Recall that we replace (x,y) by $(\mathsf{x},z)=(\frac12(x+y),x-y)$.

$(x_j - y_j)\omega$ by $(x_j - y_j)\omega'$ simultaneously applying $h^{-1}P_j$ to one of the factors $\big(e(x,y,\tau) - e^{\mathsf{T}}(x,y,\tau)\big)$ (the same as before or another one).

So for $j = 1, 2$ we estimate the term by $C\mu h^{-1} \times h^{-2}(\mu h)^{\frac{1}{2}} \times \sigma \times \sigma^{1-\kappa}$ with factors $(\mu h)^{\frac{1}{2}}$ and σ coming from P_j and $(x_j - y_j)$; as $\kappa = 1$ the last factor is replaced by $|\log \sigma/\gamma|$.

For $j = 3$ these factors are replaced by 1 and h respectively with the same product. Note that when we drag the last ∂_{x_3} through x_3 they may cancel one another but the power becomes $(1-\kappa)$ which we treated already (and while the gain in this power is exactly compensated by an extra factor h^{-1}.)

Plugging in σ we get a proper estimate (16.8.3).

(c) Further, the same arguments work for $2 \le \kappa < 3$ as well albeit now $\omega = (3-\kappa)^{-2}\sum_{j,k} \partial_{z_j}\partial_{z_k} z_j z_k \omega$. Note that when we drag the last ∂_{x_3} through x_3 they may cancel one another but the power becomes $(1-\kappa)$ which we treated already (and while the gain in this power is exactly compensated by an extra factor h^{-1}.) However for $\kappa = 2$ we get power -1 and then the logarithmic factor comes in. □

16.9 Dirac Energy: 3D-Calculations

16.9.1 Toy Model

Transformations

Consider first the toy-model operator (16.6.1).

Let us rescale as before[32]. Then $U(x,y,t)$ in comparison with $U_{(2)}(x',y',t)$ has factor $U_{(1)}(x_3, y_3, t)$ defined by (16.6.4). Let us consider effect of all these changes ignoring other variables; again without any loss of the generality we assume that $\omega(x,y)$ is given by (16.5.3).

We can replace variables x, y with new variables $\mathsf{x} := \frac{1}{2}(x+y)$ and $z := (x-y)$ and rescale. Note that the phase of $U_{(1)}$ is linear with respect to x_3 and we can get rid of $\mu^{-1}d\mathsf{x}_3$ integration in the same manner we got rid of $\mu^{-1}\,d\mathsf{x}_1$: we replace $\Omega(\dots, \mathsf{x}_3, .)$ by its partial Fourier transform

[32] We need to add factor $\mu^{-6+\kappa} = \mu^{-3} \times \mu^{-3} \times \mu^{-\kappa}$ where two factors μ^{-3} are coming from dx and dy and μ^{κ} from ω.

$x_3 \to -\mu^{-1}h^{-1}(t'-t'')$ with an extra factor 2π; thus in comparison with 2D case we have

(a) $\hat\Omega\big(2\mu^{-1}h^{-1}\alpha(t'-t''), -\mu^{-1}h^{-1}\beta(t'-t''), z\big)$ where $\hat\Omega(.,.,z)$ is a partial Fourier transform with respect to (x_1, x_3) (instead of $\hat\Omega\big(2\mu^{-1}h^{-1}\alpha(t'-t''), z\big)$);

(b) An extra factor which after easy reductions becomes

$$\frac{1}{4}h^{-1}(t't'')^{-\frac{1}{2}}\exp\Big(i\hbar^{-1}(t'-t'')\big(-\frac{1}{8}(t't'')^{-1}z_3^2-\frac{2}{3}\beta^2\mu^{-2}(t'^2+t't''+t''^2)\big)\Big);$$

(c) Integration with respect to dz_3 which enters both this factor and $\hat\Omega(.,.,z)$.

Taking in account all these modifications we need to replace (16.5.4) by

$$\begin{aligned}(16.9.1)\quad \mathsf{I} = {}& \frac{1}{8}(2\pi)^{-3}\hbar^{-3}\mu^{\kappa+3}\iint\iiint \hat\Omega(4\hbar^{-1}\alpha s, -\hbar^{-1}\beta s, z)\times\\ & \csc(t+s)\csc(t-s)(t+s)^{-\frac{3}{2}}(t-s)^{-\frac{3}{2}}\times\\ \exp\Big(i\hbar^{-1}\Big[&-\frac{1}{4}\big(\cot(t+s)-\cot(t-s)\big)\big(z_1^2+(z_2+2t\mu^{-1}\alpha)^2+4s^2\mu^{-2}\alpha^2\big)\\ &-\big(\cot(t+s)+\cot(t-s)\big)s\mu^{-1}\alpha-2s(\tau-\mu^{-2}\alpha^2)+\\ 2s\big(&-\frac{1}{8}(t+s)^{-1}(t-s)^{-1}z_3^2-\frac{2}{3}\beta^2\mu^{-2}(3t^2+s^2)\big]\Big)\,dtds\,\psi(\mu^{-1}z)dz_1dz_2dz_3.\end{aligned}$$

Case $\lambda^2 \gg \mu h$

Assume first that

$$(16.9.2)\qquad \lambda^2 := \alpha^2+\beta^2 \ge h^{1-\delta}.$$

Remark 16.9.1[33]. (i) We take λ-admissible with respect to x function $\Omega_\lambda = \Omega(x/\lambda, .)$.

Then the virtue of the factor $\hat\Omega(4\hbar^{-1}\alpha\gamma s, -\hbar^{-1}\beta\gamma s, \cdot)$ under assumption (16.9.2) we need to consider only $|s| \le h^\delta$ and therefore we can consider separately $|t'| \le \epsilon_0, |t''| \le \epsilon_0$ and $|t'| \ge \epsilon_0, |t''| \ge \epsilon_0$.

[33] Cf. Remark 16.5.2.

(ii) Note that due to Section 5.4 contribution of the zone $\{|t'| \le \epsilon_0, |t''| \le \epsilon_0\}$ defined by integral expressions (16.9.1) etc with an extra factor $\bar{\chi}_{\epsilon_0}(t')$, or $\bar{\chi}_{\epsilon_0}(t'')$, or $\bar{\chi}_{\epsilon_0}(t)$ differs from the same expression for non-magnetic Schrödinger operator by $O(\mu h^{-2-\kappa} \times \hbar^{\kappa}) = O(\mu^{\kappa+1} h^{-2})$ as $\kappa \ne 1$ and by $O(\mu^2 h^{-2} |\log \mu|)$ as $\kappa = 1$.

(iii) Furthermore, if we remove from this expression for a non-magnetic Schrödinger operator cut-off $\{|t'| \ge \epsilon_0\}$ then the error would not exceed the same expression as well.

Let us consider contribution of the zone $\{|t'| \ge \epsilon_0, |t''| \ge \epsilon_0\}$ defined by an integral expressions (16.9.1) etc with an extra factor $\big(1 - \bar{\chi}_{\epsilon_0}(t')\big)$. Due to Remark 16.9.1(i) we need to consider only t', t'' belonging to the same tick.

Let us consider first zone

$$\{|s| \ge \hbar/\lambda^2, |\sin(t)| \ge C|s|\}. \tag{16.9.3}$$

Then integration by parts with respect to z' delivers one of the factors (16.5.7), (16.5.8). Thus integrating by parts many times in the zone where both of these factors are less than 1 we acquire factors (16.5.9), (16.5.10) respectively.

On the other hand, integration by parts with respect to z_3 delivers factor $\hbar |s|^{-1} t^2 |z_3|^2$; therefore integrating by parts many times in the zone $\{|z_3| \ge \hbar^{\frac{1}{2}} |s|^{-\frac{1}{2}} |t|\}$ delivers factor

$$\big(1 + |s| \hbar^{-1} t^{-2} |z_3|^2\big)^{-l}. \tag{16.9.4}$$

Case $1 < \kappa < 3$**.** In this case, multiplying by $|z|^{-\kappa}$ and integrating we get after multiplication by $|\sin(t)|^{-2}$ expression (16.5.11) albeit with κ replaced by $(\kappa - 1)$ i.e.

$$\asymp |\sin(t)|^{1-\kappa} |s|^{-\frac{3}{2}+\frac{1}{2}\kappa} \hbar^{\frac{3}{2}-\frac{1}{2}\kappa}. \tag{16.9.5}$$

Integrating by t over k-th tick intersected with $\{t : |\sin(t)| \ge |s|\}$ we get (16.5.12) modified the same way i.e.

$$C\hbar^{\frac{3}{2}-\frac{1}{2}\kappa} \begin{cases} |s|^{-\frac{3}{2}+\frac{1}{2}\kappa} & 1 < \kappa < 2, \\ |s|^{-\frac{1}{2}} (1 + |\log |s||) & \kappa = 2, \\ |s|^{\frac{1}{2}-\frac{1}{2}\kappa} & 2 < \kappa < 3. \end{cases} \tag{16.9.6}$$

This expression (16.9.6) must be either integrated with respect to s over $\{|s| \le \hbar/\lambda^2\}$ or multiplied by $(\hbar/\lambda^2)^l |s|^{-l}$ due to factor $\hat{\Omega}$ and integrated over $\{|s| \ge \hbar/\lambda^2\}$, resulting in both cases in the same answer which is the value of (16.9.6) $\times |s|$ calculated as $s = \hbar/\lambda^2$ which is similarly modified expression (16.5.13) i.e.

$$C\hbar \begin{cases} \lambda^{1-\kappa} & 1 < \kappa < 2, \\ (1 + |\log(\hbar/\lambda^2)|)\lambda^{-1} & \kappa = 2, \\ \hbar^{2-\kappa}\lambda^{-3+\kappa} & 2 < \kappa < 3. \end{cases} \tag{16.9.7}$$

In addition to zone (16.9.3) we need to consider zone (16.5.14) defined as $\{|\sin(t')| \asymp |s|,\ |\sin(t'')| \le |s|\}$; its tween $\{|\sin(t')| \asymp |s|,\ |\sin(t'')| \le |s|\}$ is considered in the same way.

In zone (16.5.14) $|\cot(t') - \cot(t'')| \asymp |\sin(t'')|^{-1}$ and in this case factors (16.5.7), (16.5.8) are replaced by (16.5.7)′, (16.5.8)′ and (16.5.9), (16.5.10) by 16.5.9, (16.5.10)′ while factor (16.9.4) is preserved.

Then, multiplying by $|z|^{-\kappa}$ and integrating we get after multiplication by $|\sin(t'')|^{-1}|\sin(t')|^{-1}$ similarly modified expression (16.5.11)′

$$\hbar^{\frac{3}{2}-\frac{1}{2}\kappa}|\sin(t'')|^{\frac{1}{2}-\frac{1}{2}\kappa}|s|^{-1}; \tag{16.9.5}'$$

then integrating by $|t''|$ over one tick but with $|\sin(t'')| \le |s|$ we get modified expression (16.5.12)′ i.e.

$$\hbar^{\frac{3}{2}-\frac{1}{2}\kappa}|s|^{\frac{1}{2}-\frac{1}{2}\kappa}. \tag{16.9.6}'$$

Finally, either integrating over $\{|s| \le h/\varepsilon\lambda\}$ or multiplying by $|s|^{-l}(h/\varepsilon\lambda)^l$ and integrating over $\{|s| \ge h/\varepsilon\lambda\}$ we get in both cases the same answer (16.9.6)′ $\times |s|$, calculated as $s = \hbar/\lambda^2$ not exceeding (16.9.7).

Therefore the total contribution of zones (16.9.3) and (16.5.14) is given by expression (16.9.7). Then multiplying by $|k|^{-3}\mu^\kappa h^{-3}\lambda^3$ we get after summation with respect to $k : |k| \ge 1$ the value as $k = 1$ i.e. (16.9.7) $\times$ $\mu^\kappa h^{-3}\lambda^3$.

Therefore we arrive to analogue of Proposition 16.5.3

$$|\mathsf{I}^{\mathsf{T}}_\lambda - \mathsf{I}^{\mathsf{T}\prime}_\lambda| \le R^{\mathsf{W}}(\lambda) := C\mu^{\kappa+1}h^{-2}\lambda^3 \begin{cases} \lambda^{1-\kappa} & 1 < \kappa < 2, \\ (1 + |\log(\hbar/\lambda^2)|)\lambda^{-1} & \kappa = 2, \\ \mu^{2-\kappa}h^{2-\kappa}\lambda^{-3+\kappa} & 2 < \kappa < 3 \end{cases} \tag{16.9.8}$$

where $\mathsf{I}^{\mathsf{T}\prime}_\alpha$ is a Tauberian expression for I albeit with $T = \epsilon_0\mu^{-1}$.

Case $0 < \kappa \le 1$. In this case integration over zone $\{z : |z| \le C_0\}$ [34] and multiplication by $|\sin(t)|^{-2}$ results in

$$C|\sin(t)|^{-2}\rho^2 \begin{cases} \zeta^{1-\kappa} & 0 < \kappa < 1, \\ (1 + (\log \zeta/\rho))_+) & \kappa = 1 \end{cases} \tag{16.9.9}$$

with

$$\zeta = \zeta(s,k) = \min\bigl((\hbar/|s|)^{\frac12}|k|, 1\bigr), \tag{16.9.10}$$

$$\rho = \min\bigl((\hbar/|s|)^{\frac12}, 1\bigr) \tag{16.9.11}$$

(cf. expression (16.9.5)).

One can see easily that

(16.9.12) Integral by t over one tick intersected with $\{t : |\sin(t)| \ge |s|\}$ does not exceed $C\hbar|s|^{-1}\zeta^{1-\kappa}$ as $|s| \ge \hbar$ and $C(\hbar|s|^{-1})^{\frac12}\zeta^{1-\kappa}$ as $|s| \le \hbar$. As $\kappa = 1$ one needs to include an extra factor $|\log \zeta/\rho|$, ρ calculated as $|\sin(t)| = 1$ for $|s| \ge \hbar$ and $\sin(t)| = (|s|/\hbar)^{\frac12}$ for $|s| \le \hbar$.

Meanwhile, in the zone (16.5.14) integration over $\{z : |z| \le C_0\}$ and multiplication by $s^{-1}|\sin(t'')|^{-1}$ results in expression (16.9.11) and integration by $\{t'' : |\sin(t'')| \le |s|\}$ results in much lesser term $C\hbar\zeta^{1-\kappa}$, with some logarithmic factor as $\kappa = 1$.

Now integral by $\{s : |s| \lesssim 1\}$ does not exceed $C\hbar|k|^{1-\kappa}$ as $0 < \kappa < 1$ and $C\hbar(1 + |\log\hbar| + |\log|k|)$ as $\kappa = 1$. Finally, multiplication by $\mu^{\kappa+3}\hbar^{-3}|k|^{-2}$ and summation with respect to $k : |k| \ge 1$ results in $C\mu^{\kappa+1}h^{-2}$ as $0 < \kappa < 1$ and $C\mu^2h^{-2}|\log\mu h|$ if $\kappa = 1$.

Therefore we arrive to the following analogue of Proposition 16.5.3:

$$|\mathsf{I}^{\mathsf{T}}_\lambda - \mathsf{I}^{\mathsf{T}\prime}_\lambda| \le R^{\mathsf{W}}(\lambda) := C\mu^{\kappa+1}h^{-2}\lambda^3 \begin{cases} 1 & 0 < \kappa < 1, \\ (1 + |\log(\mu h)|) & \kappa = 1. \end{cases} \tag{16.9.13}$$

Combining with (16.9.8) we arrive to:

Proposition 16.9.2 [35]**.** *(i) For the toy-model operator with $1 \le \mu \le h^{-1}$ under assumption* (16.9.2) *estimates* (16.9.8), (16.9.13) *hold for $1 < \kappa < 3$, $0 < \kappa \le 1$ respectively.*

[34] Corresponding to $|x - y| \le C_0\mu^{-1}$.

[35] Cf. cf. Corollary 16.5.3.

(ii) In particular, as $\lambda = 1$

$$|\mathsf{I}^{\mathsf{T}} - \mathsf{I}^{\mathsf{T}\prime}| \le R^{\mathsf{W}} := \tag{16.9.14}$$
$$C\mu^{\kappa+1}h^{-2}\begin{cases} 1 & 0<\kappa<2, \kappa\ne 1,\\ (1+|\log(\mu h)|) & \kappa = 1,2,\\ \mu^{2-\kappa}h^{2-\kappa} & 2<\kappa<3.\end{cases}$$

Following the same proof as in Proposition 16.5.4 one can prove easily that

$$|\mathsf{I}^{\mathsf{T}\prime} - \mathcal{I}^{\mathsf{W}}| \le C\mu^2 h^{-1-\kappa} \tag{16.9.15}$$

which does not exceed $R^{\mathsf{W}}(\lambda)$ implying the following:

Corollary 16.9.3 [36]**.** *In the framework of Proposition 16.9.2*

$$|\mathsf{I}^{\mathsf{T}}_{\lambda} - \mathcal{I}^{\mathsf{W}}_{\lambda}| \le CR^{\mathsf{W}}(\lambda). \tag{16.9.16}$$

Remark 16.9.4. After summation with respect to partition we restore estimate (16.9.14) under some non-degeneracy assumption. Comparing with the Tauberian remainder estimate $O(h^{-2-\kappa})$ we conclude that R^{W} is lesser as $\mu \le h^{-\kappa/(\kappa+1)}$ $(0<\kappa<2)$ and $\mu \le h^{-\frac{2}{3}}$ $(2\le\kappa<3)$ (with some correction for $\kappa = 1, 2$).

In particular, $\mu \le h^{-\frac{1}{2}}$ would suffice for $\kappa > 1$ and therefore improvement as in Subsubsection *16.5.1.3 Improvement* is not needed.

Case $\lambda^2 \not\gg \mu h$

Assume now that $\mu^{-1} \le \lambda$ and $\lambda^2 \le \mu h^{1+\delta}$. Then in contrast to the previous we will need to compute contributions of pair of ticks (k', k'') with $k' \ne k''$. Then if t', t'' belong to k'-th and k''-th ticks respectively we denote $r = k' - k''$ and $s = t' - t'' - 2\pi r$.

However, we still need to distinguish cases $1 < \kappa < 3$ and $0 < \kappa \le 1$.

[36] Cf. Corollary 16.5.5.

Case $1 < \kappa < 3$. In this case we according to the previous subsection need just to take results for $d = 2$, plug $\kappa := (\kappa - 1)$, $\alpha = \lambda$ and multiply by $h^{-1}\lambda$ resulting in

Proposition 16.9.5 [37]**.** *For the toy-model operator with* $1 \le \mu \le h^{-1}$, $1 < \kappa < 3$

$$|\mathsf{I}^{\mathsf{T}}_{(\lambda)} - \mathcal{I}^{\mathsf{W}}_{(\lambda)}| \le (16.9.8) + (16.9.18) \qquad \text{as } \lambda^2 \ge \mu h, \tag{16.9.17}$$

with

$$C\mu^{\frac{1}{2}\kappa-\frac{1}{2}} h^{-\frac{5}{2}-\frac{1}{2}\kappa}\lambda^3 (\mu h/\lambda^2)^l \tag{16.9.18}$$

and

$$|\mathsf{I}^{\mathsf{T}}_{(\lambda)} - \mathcal{I}^{\mathsf{W}}_{(\lambda)}| \le (16.9.20) + (16.9.21) + (16.9.22) \qquad \text{as } \lambda^2 \le \mu h \tag{16.9.19}$$

with

$$C\mu^{\frac{1}{2}\kappa+\frac{3}{2}} h^{-\frac{3}{2}-\frac{1}{2}\kappa}\lambda^3, \tag{16.9.20}$$

$$C\mu^{\frac{1}{2}+\frac{1}{2}\kappa} h^{-\frac{3}{2}-\frac{1}{2}\kappa}\lambda^3 \bigl(1 + (\log(\mu h/\lambda^2))_+\bigr)^2, \tag{16.9.21}$$

$$C\mu^{\frac{1}{2}\kappa-\frac{1}{2}} h^{-\frac{5}{2}-\frac{1}{2}\kappa}\lambda^3 \bigl(1 + (\log(\mu h/\lambda^2))_+\bigr) \tag{16.9.22}$$

Proof. Easy details are left to the reader. □

Case $0 < \kappa \le 1$. In this case we need to modify arguments of the previous subsubsection.

(i) Consider case $k' - k'' = r = 0$ first. As we did not use $\lambda^2 \gg \mu h$ estimate (16.9.13) for contributions of such terms remains valid.

(ii) Consider now case $k' - k'' = r \ne 0$. Note that then according to (16.9.4) one should use $\zeta = |k' + k''| \cdot (\hbar|k - k'|)^{\frac{1}{2}}$

If $\lambda^2 \ge \mu h$ the last expression before summation with respect to k $(= k')$ acquires factor $(|k' - k''|\lambda^2/\mu h)^{-l}$ and we get after summation with respect to $k' \ne 0$, $k'' \ne 0$ the same right-hand expression of (16.9.13).

Meanwhile, if $\lambda^2 \le \mu h$ we get before summation (as $0 < \kappa < 1$)

$$C\mu^{\kappa+1} h^{-2}\lambda^3 |k'|^{-1}|k''|^{-1} \bigl(|k' + k''|/|k' - k''|^{\frac{1}{2}}\bigr)\bigr)^{1-\kappa} \times \bigl(1 + |k' - k''|\lambda^2/\mu h\bigr)^{-l} \tag{16.9.23}$$

[37] Cf. Proposition 16.5.10.

(with a logarithmic factor as $\kappa = 1$).

It is enough to sum with as $1 \le k''$, $k' = k'' + r$, $r \ge 1$; then the sum is

$$\asymp C\mu^{\kappa+1}h^{-2}\lambda^3 \sum_{r\ge 1} r^{-\frac{1}{2}-\frac{1}{2}\kappa}\bigl(1 + r\lambda^2/\mu h\bigr)^{-l} \asymp C\mu^{\kappa+1}h^{-2}\lambda^3(\mu h/\lambda^2)^{\frac{1}{2}-\frac{1}{2}\kappa};$$

so we get

$$C\begin{cases} \mu^{\frac{3}{2}+\frac{1}{2}\kappa}h^{-\frac{3}{2}-\frac{1}{2}\kappa}\lambda^{2+\kappa} & 0 < \kappa < 1,\\ \mu^2 h^{-2}\lambda^3\bigl(1+\log(\mu h)\bigr)^2 & \kappa = 1 \end{cases} \tag{16.9.24}$$

where case $\kappa = 1$ we left to the reader.

(iii) Consider now $k' = 0$, $k'' \ne 0$; its tween case $k' \ne 0$, $k'' = 0$ is addressed in the same way. To do this we need to modify our expression fo I: namely, we do not divide by t' and then take Fourier transform $F_{t'\to\hbar^{-1}\tau}$; instead we take Fourier transform $F_{t'\to\hbar^{-1}\tau'}$, then integrate by τ' to τ and divide by $\hbar$; modifying this way (16.9.1) we arrive instead of (16.9.1) to the expression similar to (16.5.30).

Then repeating the similar arguments of Section 16.5 with the above modification, we arrive to

$$C\mu^{\kappa}h^{-3}\lambda^3|k''|^{-\frac{1}{2}-\frac{1}{2}\kappa}\bigl(1 + |k''|\lambda^2/\mu h\bigr)^{-l}$$

instead of (16.9.23), and summation with respect to $k'' : |k''| \ge 1$ returns $C\mu^{\kappa}h^{-3}\lambda^3(\mu h/\lambda^2)^l$ as $\lambda^2 \ge \mu h$ and $C\mu^{\kappa}h^{-3}\lambda^3(\mu h/\lambda^2)^{\frac{1}{2}-\frac{1}{2}\kappa}$ if $\lambda^2 \le \mu h$.

So we arrive to

$$C(\mu h/\lambda^2)^l\mu^{\kappa}h^{-3}\lambda^3\begin{cases} 1 & 0 < \kappa < 1,\\ \bigl(1+\log(\mu h)\bigr)^2 & \kappa = 1 \end{cases} \tag{16.9.25}$$

as $\lambda^2 \ge \mu h$ and

$$C\mu^{\frac{1}{2}+\frac{1}{2}\kappa}h^{-\frac{5}{2}-\frac{1}{2}\kappa}\lambda^{2+\kappa}\begin{cases} 1 & 0 < \kappa < 1,\\ \bigl(1+\log(\mu h)\bigr)^2 & \kappa = 1 \end{cases} \tag{16.9.26}$$

as $\lambda^2 \le \mu h$.

Thus we proved:

Proposition 16.9.6. *For the toy-model operator with* $1 \le \mu \le h^{-1}$, $0 < \kappa \le 1$

$$|\mathsf{I}^{\mathsf{T}}_{(\lambda)} - \mathcal{I}^{\mathsf{W}}_{(\lambda)}| \le (16.9.13) + (16.9.25) \qquad \text{as } \lambda^2 \ge \mu h, \tag{16.9.27}$$

and

$$|\mathsf{I}^{\mathsf{T}}_{(\lambda)} - \mathcal{I}^{\mathsf{W}}_{(\lambda)}| \le (16.9.13) + (16.9.24) + (16.9.26). \qquad \text{as } \lambda^2 \le \mu h. \tag{16.9.28}$$

16.9.2 General Operators

Consider now general operators satisfying conditions (16.1.5)–(16.1.6) and either (16.1.7) or (16.1.8) as $\tau = 0$.

First, using Proposition 16.8.1 and propagation results we conclude that

Proposition 16.9.7 [38]**.** *Let conditions* (16.1.5)–(16.1.6) *be fulfilled. Then*

(i) Contribution of the zone $\{(x,y) : |x-y| \ge \epsilon_0 \mu^{-1}\}$ *(before rescaling) to* I^{T} *does not exceed* $C\mu^{\kappa+1}h^{-2}$ *and*

(ii) Under condition (16.1.7) *contribution of time* $\{|t| \ge \epsilon_0\}$ *(before rescaling) to* I^{T} *does not exceed* $C\mu^{\kappa+1}h^{-2}$.

Note that $O(\mu^{\kappa+1}h^{-1})$ either coincides or is smaller than the remainder estimate (16.9.14) and as long as we are interested in Weyl approximation $\mathcal{I}^{\mathsf{W}}$ we should be completely happy with it.

Under condition (16.1.7) we are done arriving to Statement 16.9.9 of Theorem (i) below.

Under condition (16.1.8) let us introduce the scaling function

$$\delta(x) = \epsilon_0 |\nabla V F^{-1}| + \frac{1}{2}\bar{\delta}, \qquad \bar{\delta} = C_0 \mu^{-1} \tag{16.9.29}$$

and consider covering of $B(0,1)$ by δ-admissible elements. In virtue of Proposition 16.9.7(i) we need to consider only pairs (x,y) belonging to the same element. Due to the rescaling arguments, the contribution of one such element to $|I^{\mathsf{T}} - \mathcal{I}^{\mathsf{W}}|$ does not exceed $C\mu h^{-1-\kappa}\delta^3$ and therefore

[38] Cf. Proposition 16.5.11.

(16.9.30) Under condition (16.1.8) contribution of the zone

$$\{(x,y) : |x-y| \le C_0\mu^{-1},\ |\nabla VF^{-1}| \le \delta\}$$

to $|I^{\mathsf{T}} - \mathcal{I}^{\mathsf{W}}|$ does not exceed $C\mu h^{-2-\kappa}\delta^3$; in particular, if $\delta = \bar{\delta}$ we get $O(\mu^{-2}h^{-2-\kappa})$.

Therefore we need to consider only the zone

$$\{(x,y) : |x-y| \le C_0\mu^{-1},\ |\nabla VF^{-1}| \ge \bar{\lambda}\}.$$

We are going to prove that

Proposition 16.9.8 [39]**.** *Let conditions* (16.1.5)–(16.1.6) *be fulfilled. Further, let* $\mu \le h^{-1}$. *Then for each* z *contribution of zone* $\{(x,y) \in B(z,\delta(z)),\ |x-y| \le C_0\mu^{-1}|\}$ *with* $\delta(z) \ge C_0\mu^{-1}$ *to* $|\mathsf{I}^{\mathsf{T}} - \mathcal{I}^{\mathsf{W}}|$ *does not exceed*

(i) $R^{\mathsf{W}}(\lambda)$ *which is defined as the right-hand expression of* (16.9.8) *or* (16.9.13) *for* $1 < \kappa < 3$, $0 < \kappa \le 1$ *provided* $\lambda \ge (\mu h)^{\frac{1}{2}-\delta}$.

(ii) the right-hand expression of (16.9.17) *or* (16.9.27) *for* $1 < \kappa < 3$, $0 < \kappa \le 1$ *provided* $\lambda \ge (\mu h)^{\frac{1}{2}}$.

(iii) the right-hand expression of (16.9.19) *or* (16.9.28) *for* $1 < \kappa < 3$, $0 < \kappa \le 1$ *provided* $\lambda \le (\mu h)^{\frac{1}{2}}$.

Consider components of $R^{\mathsf{W}}(\lambda)$. Note that expressions (16.9.8), (16.9.13) and (16.9.20)–(16.9.22), (16.9.24), (16.9.26) contain λ in the positive powers while expressions (16.9.18), (16.9.25) contain λ in the negative powers. Therefore after integration with respect $\lambda^{-1}\,d\lambda$ we get expressions (16.9.8) and (16.9.13) as $\lambda = 1$ (i.e. we get (16.9.14) and we get all other expressions as $\lambda = (\mu h)^{\frac{1}{2}}$; one can check easily that these latter expressions are less than (16.9.14) + $Ch^{-2-\kappa}$.

Therefore we arrive to:

Theorem 16.9.9 [40]**.** *Let conditions* (16.1.5)–(16.1.6). *Let* $\mu \le h^{-1}$. *Then,*

[39] Cf. Proposition 16.5.12.

[40] Cf. Theorem 16.5.13.

(i) Under condition (16.1.7) *estimate*

$$|\mathsf{I}-\mathsf{I}^{\mathsf{W}}| \le CR^{\mathsf{W}} + Ch^{-2-\kappa}; \tag{16.9.31}$$

(ii) Under condition (16.1.8)

$$|\mathsf{I}-\mathsf{I}^{\mathsf{W}}| \le CR^{\mathsf{W}} + C\mu^{\frac{5}{2}}h^{-\frac{1}{2}-\kappa-\delta} + Ch^{-2-\kappa}. \tag{16.9.32}$$

Proof of Proposition 16.9.8. First of all applying Proposition 16.3.5 and the same arguments as in the analysis of Section 16.3 we arrive to

(16.9.33) Contribution of the zone $\{|\sin(\theta(t'))| \ge C_0\hbar,\ |\sin(\theta(t''))| \ge C_0\hbar\}$ to the error $|\mathsf{I}^{\mathsf{T}} - \mathcal{I}^{\mathsf{W}}|$ does not exceed $R^{\mathsf{W}}(\lambda)$.

Therefore we need to explore contributions of three remaining zones

$$\{|\sin(\theta(t'))| \le 2C_0\hbar,\ |\sin(\theta(t''))| \le 2C_0\hbar\} \tag{16.5.43}$$

and

$$\{|\sin(\theta(t'))| \ge 2C_0\hbar,\ |\sin(\theta(t''))| \le C_0\hbar\} \tag{16.5.44}$$

(the same analysis would cover its tween zone $\{|\sin(\theta(t'))| \ge 2C_0\hbar,\ |\sin(\theta(t''))| \le C_0\hbar\}$).

We would like to follow the arguments of the proof of Proposition 16.5.12 but there is a problem: the magnetic drift is controlled by α and it may be much less than λ. However let us apply few general arguments.

(a) Let us consider the zone (16.5.43) as $\lambda^2 \ge \mu h$. First, if we consider the zone $\{(x,y):\ |x-y| \ge \ell\}$ (in the non-rescaled coordinates) we estimate the contribution of the corresponding pair (k', k'') by $C\hbar^2\ell^{-\kappa}|k'|^{-1}|k''|^{-1} \times h^{-d}\lambda^d$ and summation with respect to $|k'| \le \mu\lambda$, $|k''| \le \mu\lambda$ returns

$$C\mu^2 h^{2-d}\ell^{-\kappa}(1+|\log \mu\lambda|)^2. \tag{16.9.34}$$

Meanwhile, if we consider zone $\{(x,y):\ |x-y| \le \ell\}$ (in the non-rescaled coordinates as well) then estimating $U(x,y,t)$ by h^{-d} (no rescaling, $d = 2,3$) we we estimate the contribution of the corresponding pair (k', k'') by $C\hbar^2 \times h^{-2d} \times \ell^{d-\kappa}\lambda^d$ and again summation with respect to k', k'' returns

$$C\mu^2 h^{2-2d}\ell^{d-\kappa}\lambda^d(1+|\log \mu\lambda|)^2. \tag{16.9.35}$$

Comparing (16.9.34) and (16.9.35) we see that their sum is minimal

$$C\mu^2 h^{2-d-\kappa}\lambda^d(1+|\log \mu\lambda|)^2 \tag{16.9.36}$$

for $\ell = h$.

Comparing with (16.9.8) and (16.9.13) we conclude that for $d = 3$ this is less than R^{W} as $0 < \kappa < 1$, $\lambda \ge \mu h$. To deal with $1 \le \kappa < 3$ let us note first that there must be a factor

$$C\big(1+|k'-k''|\lambda^2/\mu h\big)^{-l} \tag{16.9.37}$$

before summation. Really, as $|\xi_3+\eta_3| \ge \epsilon_0\lambda$ this factor is due to the shift in (x_3+y_3) and as $|\xi_3+\eta_3| \ge \epsilon_0\lambda$ but $|\nabla' V/F| \asymp \lambda$ or $|\nabla_{x_3} V/F| \asymp \lambda$ this factor is due to the shift in $(\xi_1+\eta_1)$ or $(\xi_3+\eta_3)$ respectively.

Therefore

(16.9.38) As $\lambda^2 \ge \mu h^{1-\delta}$ contribution of $k' \ne k''$ is negligible.

This allows us also eliminate logarithmic factors in (16.9.36) and (16.9.37).

Let us apply cut-off Q by hD_{x_3} and Q' by hD_{xy_3}. Assume first that <u>either</u> $|\xi_3| \le C\rho$ on the support of (the symbol of) Q <u>or</u> $|\eta_3| \le C\rho$ on the support of (the symbol of) Q', in our estimate we acquire factor ρ and we get an expression below $R^{\mathsf{W}}(\lambda)$.

Therefore we can assume that

(16.9.39) $|\xi_3| \ge \rho := \mathsf{max}\big((\mu h)^{\frac12}, \mu^{-1}\big)$ on the support of (the symbol of) Q

(or $|\eta_3| \ge \mathsf{max}\big((\mu h)^{\frac12}, \mu^{-1}\big)$ on the support of (the symbol of) Q'.

Let us consider corresponding classical dynamics. Assumption (16.9.39) implies that in k winding it travels the "observable" distance $k\xi_3$ and if there was no "turn back" the contribution of k-th winding to estimate would not exceed

$$C\mu^2 h^{-1}|k|^{-2-\kappa}\int_{|\xi_3|\ge\rho} |\xi_3|^{-\kappa}\, d\xi_3\lambda^3$$

where integral is $\asymp 1$ if $0 < \kappa < 1$, $\asymp \log\rho$ if $\kappa = 1$ and $\asymp \rho^{1-\kappa}$ as $1 < \kappa < 3$. Obviously multiplied by $\mu^2 h^{-1}$ it does not exceed $R^{\mathsf{W}}(\lambda)$ and therefore summation with respect to k returns $O(R^{\mathsf{W}}(\lambda))$.

Now we need to consider effect of the "turn back". Let $\beta = |\nabla_{x_3} V/F|$. Note first that as $|\xi_3| \ge \beta\lambda$ there is no turn back for time $|t| \le \epsilon\lambda$. Therefore we need to consider only $|\xi_3| \le \beta\lambda$.

There is come back on k-th winding as ξ_3 is close to $\zeta^*(k) \sim \mathsf{const}\beta\mu^{-1}k$ and the distance $|x-y| \asymp \mu^{-1}|k||\xi_3 - \zeta(k)|$ and then contribution of k-th winding (as $|x-y| \ge Ch$) does not exceed

$$C\mu^{2+\kappa}h^{-1}|k|^{-2-\kappa} \int_{|\xi_3-\zeta(k)|\ge\mu h|k|^{-1}} |\xi_3 - \zeta(k)|^{-\kappa}\, d\xi_3\lambda^3 \le C\lambda^3 \begin{cases} \mu^{2+\kappa}h^{-1}|k|^{-2-\kappa} & 0<\kappa<2, \\ \mu^4 h^{1-\kappa}|k|^{-4} & 2<\kappa<3, \end{cases}$$

with a logarithmic factor $(1+\log|k| + |\log\mu h|)$ as $\kappa = 2$. The sum of the expression above is the same as this expression for $k = 1$ (we denote it temporarily by $R\lambda^3$) which obviously less than $R^{\mathsf{W}}(\lambda)$.

Finally, we need to consider the contribution of k-th winding as $|x-y| \le h$. However it is done by the standard way as $U(x,y,t)$ after cut by ξ_3 and η_3 acquires factor Ch^2.

Therefore, under assumption $\lambda^2 \ge \mu h^{1-\delta}$ we are done.

On the other hand, if $\lambda^2 \ge \mu h$ (but $\lambda^2 \not\gg \mu h$) terms with $k' \ne k''$ are no more negligible but in the estimate we can ignore due to the factor (16.9.37) them except when $k' = 0$ or $k'' = 0$.

Applying the same arguments as in 2D-case i.e. replacing in term with $k' = 0$ factor $|k'|^{-1}$ by factor $\hbar^{-1}$ we arrive to $R\lambda^3 \times 1/(\mu h) \times (\mu h/\lambda^2)^l$ which is less than term (16.9.18) or (16.9.25) as $1 < \kappa < 3$ or $0 < \kappa \le 1$ respectively.

(b) Next, let us analyze the zone (16.5.43) if $\lambda^2 \le \mu h$. In this case we apply the same arguments but now effectively we sum over k', k'' with $|k'-k''| \le \mu h/\lambda^2$ which effectively boils out to the summation over $k' = 1,\ k'' = 1, \dots, \mu h/\lambda^2$ or $k' = 0,\ k'' = 1, \dots, \mu h/\lambda^2$; the former results in $R\lambda^3 \times \mu h/\lambda^2$ and the latter results $R\lambda^3 \times \mu h/\lambda \times 1/(\mu h)$ and one can see easily that the result is less than what we claimed.

(c) Further, consider the zone (16.5.44) if $\lambda^2 \ge \mu h$. Now there is always shift with respect to $(x-y)$ which is at least $\asymp |\sin(t')|$ and therefore as

$1 < \kappa < 3$ contribution of this zone does not exceed contribution of zone (16.5.43). As $0 < \kappa < 1$ we get $C\mu^{\kappa+1}h^{-2}\lambda^3|k|^{-2}$ and summation returns $C\mu^{\kappa+1}h^{-2}\lambda^3$ which is the answer as $\lambda^2 \gg \mu h$; let us denote it by $R\lambda^3$; for $\kappa = 1$ it includes logarithmic factor $(1 + |\log(\mu h)|)$.

The same answer estimates contribution of $k' \neq 0, k'' \neq 0$ as $\lambda^2 \geq \mu h$; contributions of $k' = 0, k'' \neq 0$ and $k'' \neq 0, k' = 0$ do not exceed $R\lambda^3 \times 1/(\mu h) \times (\mu h/\lambda^2)^l$ and one can see easily that the result is less than what we claimed.

(d) Finally, consider the zone (16.5.44) as $\lambda^2 \leq \mu h$. Again as $1 < \kappa < 3$ contribution of (16.5.44) does not exceed contribution of (16.5.43) and as $0 < \kappa \leq 1$ we estimate contribution of $k' \neq 0, k'' \neq 0$ by $R\lambda^3 \times \mu h/\lambda^2$, and contributions of $k' \neq 0, k'' = 0$ and $k' = 0, k'' \neq 0$ by $R\lambda^3 \times \mu h/\lambda^2 \times 1/(\mu h)$ and one can see easily that the result is less than what we claimed. □

16.9.3 Perturbations

Consider perturbations. However due to the presence of x_3, hD_3 there are rather different components here than in Subsection 16.5.3.

Perturbations. I

Let us start from the terms $O(|x' - y'|)$. During evolution these terms are $O(\mu^{-1})$ and therefore we need to multiply by $O(\mu^{-2}h^{-1}|k|)$.

Case $1 < \kappa < 3$**.** Consider first case $k' = k'' = k$. Assume that $\lambda^2 \geq \mu h$. Recall that k-th tick's contribution to I^{T} was not exceeding expression (16.9.7) multiplied by $\mu^{\kappa}h^{-3}\lambda^3|k|^{-3}$ i.e.

$$C\lambda^3|k|^{-3}\begin{cases}\mu^{\kappa+1}h^{-2}\lambda^{1-\kappa} & 1 < \kappa < 2,\\ \mu^3 h^{-2}(1 + |\log(\hbar/\lambda^2)|)\lambda^{-1} & \kappa = 2,\\ \mu^3 h^{-\kappa}\lambda^{-3+\kappa} & 2 < \kappa < 3\end{cases} \tag{16.9.40}$$

(as $|\sin(t')| \leq \hbar$ or $|\sin(t'')| \leq \hbar$ one should use Proposition 16.9.8) and multiplying by $C\mu^{-2}h^{-1}|k|^{-1}$ we get after summation with respect to k :

$|k| \ge 1$

$$C\lambda^3 \begin{cases} \mu^{\kappa-1}h^{-3}\lambda^{1-\kappa} & 1<\kappa<2,\\ \mu h^{-3}(1+|\log(\hbar/\lambda^2)|)\lambda^{-1} & \kappa=2,\\ \mu h^{-1-\kappa}\lambda^{-3+\kappa} & 2<\kappa<3 \end{cases} \tag{16.9.41}$$

and summation with respect to λ returns the same expression as $\lambda = 1$ which does not exceed $Ch^{-2-\kappa}$.

Next we need to consider $0 \ne k' \ne k'' \ne 0$. Recall that the contribution of such pair to I^{T} does not exceed

$$C\lambda^3|k'|^{-\frac{3}{2}}|k''|^{-\frac{3}{2}}(\mu h/\lambda^2|k'-k''|)^l\times \begin{cases} \mu^{\kappa+1}h^{-2}\lambda^{1-\kappa} & 1<\kappa<2,\\ \mu^3 h^{-2}(1+|\log(\mu h/\lambda^2)|)\lambda^{-1} & \kappa=2,\\ \mu^3 h^{-\kappa}\lambda^{-3+\kappa} & 2<\kappa<3. \end{cases} \tag{16.9.42}$$

Multiplying by $C\mu^{-2}h^{-1}|k'|$ (as the perturbation goes to the first "factor") we get

$$C\lambda^3|k'|^{-\frac{1}{2}}|k''|^{-\frac{3}{2}}(\mu h/\lambda^2|k'-k''|)^l\times \begin{cases} \mu^{\kappa-1}h^{-3}\lambda^{1-\kappa} & 1<\kappa<2,\\ \mu h^{-3}(1+|\log(\mu h/\lambda^2)|)\lambda^{-1} & \kappa=2,\\ \mu h^{-1-\kappa}\lambda^{-3+\kappa} & 2<\kappa<3. \end{cases} \tag{16.9.43}$$

Then summation with respect to k' of $|k'|^{-\frac{1}{2}}(\mu h/\lambda^2|k'-k''|)^l$ returns $|k''|^{-\frac{1}{2}}(\mu h/\lambda^2)^l$ as $\lambda^2 \ge \mu h$ and summation with respect to k'' then returns the same expression as we got considering $k' = k'' \ne 0$ with an extra factor $(\mu h/\lambda^2)^l$.

On the other hand, as $\lambda^2 \le \mu h$ one should replace (16.9.7) by $C(\mu h)^{\frac{3}{2}-\frac{1}{2}\kappa}$; multiplying by $\mu^{\kappa}h^{-3}\lambda^3|k'|^{-\frac{3}{2}}|k''|^{-\frac{3}{2}}(1+\lambda^2|k'-k''|/\mu h)^{-l} \times \mu^{-2}h^{-1}|k'|$ we get

$$C\mu^{-\frac{1}{2}+\frac{1}{2}\kappa}h^{-\frac{5}{2}-\frac{1}{2}\kappa}\lambda^3|k'|^{-\frac{1}{2}}|k''|^{-\frac{3}{2}}(1+\lambda^2|k'-k''|/\mu h)^{-l} \tag{16.9.44}$$

and one can see easily that summation with respect to k', k'' returns

$$C\mu^{-\frac{1}{2}+\frac{1}{2}\kappa}h^{-\frac{5}{2}-\frac{1}{2}\kappa}\lambda^3(\mu h/\lambda^2)^{\frac{1}{2}} \asymp C\mu^{\frac{1}{2}\kappa}h^{-2-\frac{1}{2}\kappa}\lambda^2$$

and summation with respect to λ returns $C\mu^{\frac{3}{2}\kappa}h^{-1-\frac{1}{2}\kappa}$ which again is less than $Ch^{-2-\kappa}$.

If $k' \neq 0$, $k'' = 0$ we recall that $|k''|^{-\frac{3}{2}}$ should be replaced by $\hbar^{-\frac{3}{2}}$ in (16.9.42)–(16.9.43) and perturbation brings factor $\mu^{-2}h^{-1}|k'|$ so we get instead of (16.9.43)

$$C\lambda^3|k'|^{-\frac{1}{2}}\mu^{-\frac{3}{2}}h^{-\frac{3}{2}}(\mu h/\lambda^2|k'|)^l\times \begin{cases} \mu^{\kappa-1}h^{-3}\lambda^{1-\kappa} & 1<\kappa<2,\\ \mu h^{-3}(1+|\log(\mu h/\lambda^2)|)\lambda^{-1} & \kappa=2,\\ \mu h^{-1-\kappa}\lambda^{-3+\kappa} & 2<\kappa<3 \end{cases}$$

and summation with respect to $|k'|$ and then $\lambda \geq (\mu h)^{\frac{1}{2}}$ returns $C\mu^{\frac{1}{2}\kappa-\frac{1}{2}}h^{-\frac{5}{2}-\frac{1}{2}\kappa}$.

Similarly, as $\lambda^2 \leq \mu h$ we start from (16.9.44) which is replaced by

$$C\mu^{-2+\frac{1}{2}\kappa}h^{-4-\frac{1}{2}\kappa}\lambda^3|k'|^{-\frac{1}{2}}(1+\lambda^2|k'|/\mu h)^{-l}$$

and summation with respect to k' and returns $C\mu^{-\frac{3}{2}+\frac{1}{2}\kappa}h^{-\frac{7}{2}-\frac{1}{2}\kappa}\lambda^2$ and then summation with respect to $\lambda \leq (\mu h)^{\frac{1}{2}}$ returns the same answer $C\mu^{\frac{1}{2}\kappa-\frac{1}{2}}h^{-\frac{5}{2}-\frac{1}{2}\kappa}$.

As $|k'| < |k''|$ the estimate of the contribution of the pair (k', k'') to an error is obviously less than the same estimate for the pair (k'', k'); this takes care of $k' = 0$, $k'' \neq 0$. Finally for $k' = k'' = 0$ we can use the standard arguments to estimate the same way.

As the result we conclude that the error will not exceed $Ch^{-2-\kappa}$.

Case $0 < \kappa \leq 1$. The similar arguments work for $0 < \kappa < 1$. However one must reconsider zone $\{(x, y) : |x - y| \geq C_0\mu^{-1}\}$ where contribution of the pair (k', k'') to Weyl asymptotics does not exceed

$$C\mu^2h^{-2}\zeta^{1-\kappa}|k|^{-3}\lambda^3 \asymp C\mu^{\kappa+1}h^{-2}|k|^{-2-\kappa}\lambda^3 \tag{16.9.45}$$

with $\zeta \asymp \mu^{-1}|k|$ as $k' = k'' = k \neq 0$. Summation with respect to $k : |k| \geq 1$ returns then $C\mu^{\kappa+1}h^{-2}\lambda^3$ as it should.

As $k' = k'' = k \neq 0$ it results in the approximation error not exceeding

$$C\mu^2h^{-2}\zeta^{1-\kappa}|k|^{-3}\lambda^3 \times \mu^{-2}h^{-1}|k| \asymp C\mu^{\kappa-1}h^{-3}|k|^{-1-\kappa}\lambda^3 \tag{16.9.46}$$

and summation with respect to k, λ returns $C\mu^{\kappa-1}h^{-3}$.

As $0 \neq k' \neq k' \neq 0$, $|k'| \geq |k''|$ it results in the approximation error not exceeding

$$C\mu^2 h^{-2}\bigl(\mu^{-1}|k'|\bigr)\bigr)^{1-\kappa}|k'|^{-\frac{3}{2}}|k''|^{-\frac{3}{2}}\times (1+\lambda^2|k'-k''|/\mu h)^{-l}\lambda^3 \times \mu^{-2}h^{-1}|k'| \asymp C\mu^{\kappa-1}h^{-3}|k'|^{\frac{1}{2}-\kappa}|k''|^{-\frac{3}{2}}(1+\lambda^2|k'-k''|/\mu h)^{-l}\lambda^3 \tag{16.9.47}$$

and the sum with respect to (k', k'') and $\lambda : \lambda^2 \geq \mu h$ does not exceed $C\mu^{\kappa-1}h^{-2}$.

Meanwhile as $\lambda^2 \leq \mu h$ the sum with respect to (k', k'') does not exceed $C\mu^{\kappa-1}h^{-3}(\mu h/\lambda^2)^{\frac{3}{2}-\kappa}\lambda^3$ and the sum with respect to λ is less than $C\mu^{\kappa-1}h^{-3}$.

As $k' \neq 0$, $k'' = 0$ (16.9.47) becomes

$$C\mu^{\kappa-1}h^{-3}|k'|^{\frac{1}{2}-\kappa}(\mu h)^{-\frac{3}{2}}(1+\lambda^2|k'|/\mu h)^{-l}\lambda^3 \tag{16.9.48}$$

and then summation with respect to k' returns $C\mu^{\kappa-1}h^{-3}(\mu h)^{-\frac{3}{2}}(\mu h/\lambda^2)^l\lambda^3$ as $\lambda^2 \geq \mu h$ and then the sum with respect to λ is less than $C\mu^{\kappa-1}h^{-3}$.

Meanwhile as $\lambda^2 \leq \mu h$ the sum with respect to k' does not exceed $C\mu^{\kappa-1}h^{-3}(\mu h/\lambda^2)^{\frac{3}{2}-\kappa}(\mu h)^{-\frac{3}{2}}\lambda^3$ and then the sum with respect to λ is less than $C\mu^{\kappa-1}h^{-3}$.

As $|k'| < |k''|$ the estimate of the contribution of the pair (k', k'') to an error is obviously less than the same estimate for the pair (k'', k'); this takes care of $k' = 0$, $k'' \neq 0$. Finally, for $k' = k'' = 0$ we can use the standard arguments to estimate contribution by $Ch^{-2-\kappa}$.

For $\kappa = 1$ we apply the same arguments with the obvious modifications, concluding that the error does not exceed $Ch^{-3}(1+|\log \mu h|)$.

Note that in contrast to the case $1 < \kappa < 3$ now the error is larger than $Ch^{-2-\kappa}$.

To improve this remainder estimate let us notice that the second term in the approximation is 0 and the third term also acquires factor $\mu^{-2}h^{-1}|k|$ (as $k' = k'' = k \neq 0$); so we get

$$C\mu^{\kappa-1}h^{-3}|k|^{-1-\kappa} \times \mu^{-2}h^{-1}|k|\lambda^3 \asymp C\mu^{\kappa-3}h^{-4}|k|^{-\kappa}\lambda^3$$

which we should sum as $|k| \leq \mu^2 h$; we also should sum

$$C\mu^2 h^{-2}\zeta^{1-\kappa}|k|^{-3}\lambda^3 \asymp C\mu^{\kappa+1}h^{-2}|k|^{-2-\kappa}\lambda^3$$

as $|k| \ge \mu^2 h$ and both sums return $C\mu^{\kappa+1}h^{-2}(\mu^2 h)^{-1-\kappa}\lambda^3 \asymp C\mu^{-1-\kappa}h^{-3-\kappa}\lambda^3$. Summation with respect to λ returns $C\mu^{-1-\kappa}h^{-3-\kappa}$. One can estimate contribution of all all other pairs (k', k'') in the same way. As $\kappa = 1$ we get $C\mu^{-2}h^{-4}(1 + |\log \mu h|)$.

So we arrive to

Theorem 16.9.10 [41]**.** *Let conditions* (16.1.5)–(16.1.6) *and either* (16.1.7) *or* (16.1.8) *be fulfilled. Let* $h^{-\frac12}\mu \le h^{-1}$. *Then*

(i) As $1 < \kappa < 3$

$$|\mathsf{I} - \mathsf{I}^0| \le Ch^{-2-\kappa} \tag{16.9.49}$$

where

$$\mathsf{I}^0 := \int \Omega_{1,1}\bigl(\frac12(x+y), x-y\bigr)\cdot |e^0_{\frac12(x+y)}(x,y,0)|^2\,dxdy \tag{16.9.50}$$

and $e^0_z(x,y,\tau)$ *is the Schwartz kernel of the spectral projector of the generalized toy-model operator*

$$A^0_z := h^2D_1^2 + (hD_2 - \mu x_1)^2 + h^2D_3^2 + V^0(x), \quad V^0(x) := V(z_1, z_2, x_3) \tag{16.9.51}$$

which in contrast to (16.7.36) V^0 *does not contain linear terms with respect to* (x_1, x_2).

(ii) As $0 < \kappa < 1$

$$|\mathsf{I} - \mathsf{I}^0| \le Ch^{-2-\kappa} + C\mu^{-1-\kappa}h^{-3-\kappa} \tag{16.9.52}$$

and as $\kappa = 1$

$$|\mathsf{I} - \mathsf{I}^0| \le Ch^{-3} + C\mu^{-2}h^{-4}|\log \mu h|. \tag{16.9.53}$$

Remark 16.9.11. Note that the remainder estimate does not exceed $Ch^{-2-\kappa}$ for $\mu \ge h^{-1/(1+\kappa)}$ as $0 < \kappa < 1$ and for $\mu \ge h^{-\frac12}(1 + |\log \mu h|)^{\frac12}$ as $\kappa = 1$.

Perturbations. II

If we go from the generalized toy-model approximation to the magnetic Weyl approximation then we need to counter $(x_3 - y_3)\lambda$ terms and this is $O(\mu^{-1}|k'|\lambda)$ due to the finite speed of propagation. Then dealing as before we acquire an extra factor $|k'|\lambda$.

[41] Cf. Theorem 16.5.21.

Case $1 < \kappa < 3$**.** Then as $k' = k'' = k \neq 0$ we have $|k|^{-1}$ rather than $|k|^{-2}$ and the sum with respect to k contains an extra logarithmic factor $(1+\log(\mu^2 h))\lambda$ in comparison to what we got in Subsubsection *16.9.3.1 Perturbations. I.*

(a) Consider case $0 \neq k' \neq k'' \neq 0$, $|k'| \geq |k''|$. Then if $\lambda^2 \geq \mu h$ we arrive to the same expression (but with an extra factor $(\mu h/\lambda^2)^l$). However if $\lambda^2 \leq \mu h$ we need to consider $|k'|^{-\frac{3}{2}}|k''|^{-\frac{3}{2}} \times |k'|^2\lambda$ with $k', k'' : |k' - k''| \leq \mu h/\lambda^2$ which in comparison to what we got before after summation returns an extra factor $\mu h/\lambda^2 \times \lambda$ and the sum with respect to λ does not exceed what we got above.

(b) Consider $0 \neq k'$, $k'' = 0$. Again no change if $\lambda^2 \geq \mu h$ and an extra factor $(\mu h/\lambda^2) \times \lambda$ if $\lambda^2 \leq \mu h$ and the sum with respect to λ does not exceed what we got above.

(c) Therefore, as a result we get $C\mu^{\kappa-1}h^{-3}(1 + |\log \mu^2 h|)$ if $1 < \kappa < 2$ and $C\mu h^{-1-\kappa}(1 + |\log \mu^2 h|)$ if $2 < \kappa < 3$ with an extra factor $(1 + |\log \mu h|)$ if $\kappa = 2$. In both cases it is less than $Ch^{-2-\kappa}$ unless $\mu \geq h^{-1}|\log h|^{-r}$.

To improve our result in this latter case we apply three term approximation and note that the second term is 0 while the third term contains summation of $R^{\mathsf{W}}\lambda^3|k|^{-3} \times (\mu^{-2}h|k|^2\lambda)^2$ for $k : |k| \leq \mu h^{\frac{1}{2}}$ and summation of $R^{\mathsf{W}}\lambda^3|k|^{-3}$ for $k : |k| \geq \mu h^{\frac{1}{2}}$ as $k' = k'' = k \neq 0$ where $R^{\mathsf{W}} = \mu^3 h^{-\kappa}$ as $2 < \kappa < 3$, $R^{\mathsf{W}} = \mu^{\kappa+1}h^{-2}$ as $1 < \kappa < 2$ with an extra factor $(1 + |\log \mu h|)$ as $\kappa = 2$.

Other pairs (k', k'') should be modified in the same way and we leave easy but tedious details to the reader. As a result we get $R^{\mathsf{W}}\mu^{-2}h^{-1}$ without any logarithmic factor thus arriving to the same estimate as before but without logarithmic factor; as $1 < \kappa < 3$ it does not exceed $Ch^{-2-\kappa}$. So we arrive to Statement (i) of Theorem 16.9.12 below.

Case $0 < \kappa \leq 1$**.** Now expression (16.9.46) acquires factor $|k|\lambda$ and becomes $C\mu^{\kappa-1}h^{-3}|k|^{-\kappa}\lambda^4$ and summation with respect to $k : |k| \leq \mu h^{\frac{1}{2}}\lambda^{-\frac{1}{2}}$ returns

$$C\mu^{\kappa-1}h^{-3}(\mu h^{\frac{1}{2}}\lambda^{-\frac{1}{2}})^{1-\kappa}\lambda^4 \asymp Ch^{-\frac{5}{2}-\frac{1}{2}\kappa}\lambda^{-\frac{7}{2}+\frac{1}{2}\kappa}.$$

We also should sum $C\mu^{\kappa+1}h^{-2}|k|^{-2-\kappa}\lambda^3$ for $|k| \geq \mu h^{\frac{1}{2}}\lambda^{-\frac{1}{2}}$ with the same result.

Finally summation with respect to λ returns $Ch^{-\frac{5}{2}-\frac{1}{2}\kappa}$.

Other pairs (k', k'') should be modified in the same way and we leave easy but tedious details to the reader. Three term approximation does not improve results. As $\kappa = 1$ we acquire factor $(1 + |\log h|)$. So we arrive to Statement (ii) of Theorem 16.9.12 below.

Theorem 16.9.12 [42]**.** *Let conditions* (16.1.5)–(16.1.6) *and either* (16.1.7) *or* (16.1.8) *be fulfilled. Let* $h^{-\frac{1}{2}} \le \mu \le h^{-1}$. *Then*

(i) If $1 < \kappa < 3$

$$|\mathsf{I} - \mathcal{I}^{\mathsf{MW}}| \le Ch^{-2-\kappa}; \tag{16.9.54}$$

(ii) If $0 < \kappa < 1$

$$|\mathsf{I} - \mathcal{I}^{\mathsf{MW}}| \le Ch^{-\frac{5}{2}-\frac{1}{2}\kappa} \tag{16.9.55}$$

and if $\kappa = 1$

$$|\mathsf{I} - \mathcal{I}^{\mathsf{MW}}| \le Ch^{-3}(1 + |\log \mu h|). \tag{16.9.56}$$

Perturbations. III

Now in the case $0 < \kappa \le 1$ we consider the toy-model approximation. Since (16.9.55) is larger than (16.9.53) we consider improvement of the previous subsubsection only. Now as perturbation is $(x_3 - y_3)^2$ which translates into $\mu^{-2}|k'|^2$ rather than μ^{-1} expression (16.9.46) acquires factor $\mu^{-1}|k'|^2$ and becomes $C\mu^{\kappa-2}h^{-3}|k|^{1-\kappa}\lambda^3$ and we need to sum this as $|k| \le \mu h^{\frac{1}{3}}$ resulting in

$$C\mu^{\kappa-2}h^{-3}(\mu h^{\frac{1}{3}})^{2-\kappa}\lambda^3 \asymp Ch^{-\frac{7}{3}-\frac{1}{3}\kappa}\lambda^3.$$

We should sum $C\mu^{\kappa+1}h^{-2}|k|^{-2-\kappa}\lambda^3$ for $|k| \ge \mu h^{\frac{1}{3}}$ with the same result.

Finally, summation with respect to λ returns $Ch^{-\frac{7}{3}-\frac{1}{3}\kappa}$.

Other pairs (k', k'') should be modified in the same way and we leave easy but tedious details to the reader. Three term approximation does not improve results. As $\kappa = 1$ we acquire factor $(1 + |\log h|)$. So we arrive to

[42] Cf. Theorem 16.5.19.

Theorem 16.9.13. *Let conditions* (16.1.5)–(16.1.6) *and either* (16.1.7) *or* (16.1.8) *be fulfilled. Let* $h^{-\frac{1}{2}} \le \mu \le h^{-1}$. *Then as* $0 < \kappa < 1$

$$|\mathsf{I} - \bar{\mathsf{I}}| \le Ch^{-2-\kappa} + Ch^{-\frac{7}{3}-\frac{1}{3}\kappa} + C\mu^{-1-\kappa}h^{-3-\kappa} \tag{16.9.57}$$

and as $\kappa = 1$

$$|\mathsf{I} - \bar{\mathsf{I}}| \le Ch^{-3} + C\mu^{-2}h^{-4}(1 + |\log \mu h|) \tag{16.9.58}$$

where $\bar{\mathsf{I}}$ *is defined for a simplified toy-model operator*

$$\bar{A}_z := h^2D_1^2 + (hD_2 - \mu x_1)^2 + h^2D_3^2 + \bar{V}(x), \qquad \bar{V}(x) := V(z) + \partial_{z_3}V(z)(x_3 - z_3). \tag{16.9.59}$$

16.9.4 Strong Magnetic Field

Consider the toy-model Schrödinger-Pauli operator (16.2.72). Recall that according to Propositions 16.8.4(ii), 16.8.5(ii) respectively

$$|\mathsf{I}| \le C\mu h^{-2}\begin{cases} h^{-\kappa} & \text{as } 0 < \kappa < 1,\\ h^{-1}\big(1 + (\log \mu h)_+\big) & \text{as } \kappa = 1,\\ \mu^{\frac{1}{2}(\kappa-1)}h^{-\frac{1}{2}(\kappa+1)} & \text{as } 1 < \kappa < 3 \end{cases} \tag{16.8.2}$$

and

$$|\mathsf{I} - \mathsf{I}^{\mathsf{T}}| \le C\mu h^{-1}\begin{cases} h^{-\kappa} & \text{as } 0 < \kappa < 1,\\ h^{-1}\big(1 + (\log \mu h)_+\big) & \text{as } \kappa = 1,\\ \mu^{\frac{1}{2}(\kappa-1)}h^{-\frac{1}{2}(\kappa+1)} & \text{as } 1 < \kappa < 3,\ \kappa \ne 2,\\ \mu^{\frac{1}{2}}h^{-\frac{3}{2}}\big(1 + (\log \mu h)_+\big) & \text{as } \kappa = 2. \end{cases} \tag{16.8.3}$$

Let us assume that $F = 1$ to avoid some unpleasant correction terms as reduced to $F = 1$.

(a) Consider the generalized toy-model approximation (16.9.51). Then $\sigma \asymp h$ and under condition

$$|V + 2m\mu h| \ge \epsilon_0 \qquad \forall m \in \mathbb{Z}^+ \tag{16.9.60}$$

the shift with respect to x_3 is $\asymp |t|$ which implies that we need to consider only $t' \asymp t''$ as the total contribution of other intervals will be less, and that contribution of intervals $|t'| \le C_0 h$, $|t''| \le C_0 h$ does not exceed the right-hand of (16.9.2) multiplied by Ch, and that contribution of intervals $t' \asymp T$, $t'' \asymp T$ ($T \ge C_0 h$) does not exceed $C\mu h^{-1} \times T^{-\kappa}$ so summation with respect to $T^{-1}dT$ returns $C\mu h^{-1-\kappa}$.

Then the error does not exceed the right-hand expression of (16.9.3).

(b) Consider the magnetic Weyl approximation. In this case $\sigma \asymp T$ and again under assumption (16.9.60) we can consider only $t' \asymp t''$, and we estimate contribution of intervals $|t'| \le C_0 h$, $|t''| \le C_0 h$ as above, but contribution of intervals $t' \asymp T$, $t'' \asymp T$ ($T \ge C_0 h$) does not exceed $C\mu h^{-2} \times T^{1-\kappa}$ and summation with respect to $T^{-1}dT$ returns $C\mu h^{-1-\kappa}$ as $1 < \kappa < 3$. As $0 < \kappa < 1$ we sum this way only for $T \le h^{\frac{1}{2}}$ resulting in $C\mu h^{-\frac{3}{2}-\frac{1}{2}\kappa}$ and we sum $C\mu h^{-1}T^{-\kappa}$ for $T \ge h^{\frac{1}{2}}$ with the same result. As $\kappa = 1$ we get $C\mu h^{-2}(1+|\log h|)$.

(c) Consider the toy-model approximation as $0 < \kappa < 1$. In this case $\sigma \asymp T^2$ (as we pass from the generalized toy-model approximation) and again under assumption (16.9.60) we can consider only $t' \asymp t''$, and we estimate contribution of intervals $|t'| \le C_0 h$, $|t''| \le C_0 h$ as above, but contribution of intervals $t' \asymp T$, $t'' \asymp T$ ($T \ge C_0 h$) does not exceed $C\mu h^{-2} \times T^{2-\kappa}$ summation with respect to $T^{-1}dT$ returns $C\mu h^{-\frac{4}{3}-\frac{1}{3}\kappa}$ as we sum with respect to $T^{-1}dT$ for $T \le h^{\frac{1}{3}}$; for $T \ge h^{\frac{1}{3}}$ we sum $C\mu h^{-1}T^{-1-\kappa}$ with the same result.

Then we arrive to

Theorem 16.9.14. *(i) For the Schrödinger-Pauli operator under conditions* (16.1.5)–(16.1.6) *and* (16.9.60)

$$|\mathsf{I} - \mathcal{I}^{\mathsf{MW}}| \le C \begin{cases} \mu h^{-\frac{3}{2}-\frac{1}{2}\kappa} & 0 < \kappa < 1, \\ \mu h^{-2}(1+|\log h|) & \kappa = 1, \\ \mu^{\frac{1}{2}(\kappa+1)} h^{-\frac{1}{2}(\kappa+3)} & 1 < \kappa < 3,\ \kappa \ne 2, \\ \mu^{\frac{3}{2}} h^{-\frac{5}{2}}(1+(\log \mu h)_+) & \kappa = 2. \end{cases} \tag{16.9.61}$$

(ii) Further, if $0 < \kappa \le 1$

$$|\mathsf{I} - \mathsf{I}^0| \le C \begin{cases} \mu h^{-1-\kappa} & 0 < \kappa < 1, \\ \mu h^{-2}(1 + (\log \mu h)_+) & \kappa = 1 \end{cases} \tag{16.9.62}$$

and

$$|\mathsf{I} - \bar{\mathsf{I}}| \le C \begin{cases} \mu h^{-1-\kappa} + \mu h^{-\frac{4}{3}-\frac{1}{3}\kappa} & 0 < \kappa < 1, \\ \mu h^{-2}(1 + |\log h|) & \kappa = 1 \end{cases} \tag{16.9.63}$$

Let us get rid of condition (16.9.60). First, we still can assume without any loss of the generality that (16.9.60) holds for all $m \in \mathbb{Z}^+$ save $m = \bar{m}$. Then making ℓ-admissible partitions in x and y with

$$\ell = \epsilon |V + 2\bar{m}\mu h| + h^{\frac{2}{3}} \tag{16.9.64}$$

one can replace (16.9.60) by (16.1.7) and then making ℓ-admissible partitions in x and y with

$$\ell = \epsilon \big(|V + 2\bar{m}\mu h| + |\nabla V|^2\big)^{\frac{1}{2}} + h^{\frac{1}{2}} \tag{16.9.65}$$

one can replace (16.1.7) by (16.1.8) arriving to

Theorem 16.9.15. *Theorem 16.9.14 remains valid with assumption* (16.9.60) *replaced by* (16.1.8)*.*

Proof. Easy but tedious details are left to the reader.

16.9.5 Problems and Remarks

Remark 16.9.16. The main difference between cases $2 < \kappa < 3$, $1 < \kappa < 2$ and $0 < \kappa < 1$ that in the first case the main contribution to the remainder is delivered by (x, y) close to one another ($|x-y| \ll \mu^{-1}$), in the second case by (x, y) with $|x - y| \asymp \mu^{-1}$ and in the third case by (x, y) with $|x - y| \asymp 1$.

We can calculate $\mathcal{I}^{\mathsf{MW}}$ and $\bar{\mathsf{I}}$ plugging corresponding expressions $e_x^{\mathsf{MW}}(x, x, 0)$ and $\bar{e}_x(x, x, 0)$ into I.

Problem 16.9.17. Find nice expressions for $\mathcal{I}^{\mathsf{MW}}$ and $\bar{\mathsf{I}}$.

Problem 16.9.18. As $\mu h \le 1$ get rid of condition (16.1.5); according to Chapter 13 we do not need it for estimate $|\mathsf{I}-\mathsf{I}^{\mathsf{T}}|$ but we want to get rid of it in estimates for $|\mathsf{I}-\mathcal{I}^{\mathsf{W}}|$, $|\mathsf{I}-\mathcal{I}^{\mathsf{MW}}|$ and $|\mathsf{I}-\bar{\mathsf{I}}|$. To do this

(i) Assume first that condition (16.1.7) is fulfilled and consider the scaling function $\ell(x) = \mathsf{max}\big(\epsilon|V(x)|, \mu h, C\mu^{-1}\big)$.

(ii) Assuming then that condition (16.1.8) is fulfilled and consider the scaling function $\ell(x) = \mathsf{max}\big(\epsilon(|V(x)| + |\nabla V(x)|^2)^{\frac{1}{2}}, (\mu h)^{\frac{1}{2}}, C\mu^{-1}\big)$.

Problem 16.9.19. Weaken non-degeneracy condition (16.1.8) which seems to be excessive here (in contrast to 2D case).

Problem 16.9.20. Get rid of condition (16.1.6) assuming instead that $|\boldsymbol{F}| + |\nabla \otimes \boldsymbol{F}| \ge \epsilon$.

16.10 Estimates of Expression (16.1.4)

16.10.1 Basics

In the multiparticle quantum theory we need to calculate expression (16.1.2) approximating energy of the electron-electron interaction or, more often, to estimate expression

$$\mathsf{K} := \iint \big(e(x,x,\tau) - h^{-d}\mathcal{N}_x(\tau)\big)\big(e(y,y,\tau) - h^{-d}\mathcal{N}_x(\tau)\big)\omega(x,y)\,dxdy \tag{16.1.4}$$

where $\omega(x,y) = \omega_\kappa(x,y)$ satisfies assumption (16.1.3) and $\mathcal{N}$ is *some* approximation. We concentrate on (16.1.4); asymptotics of (16.1.2) could be proven by the same arguments and we leave them to the reader.

Applying partition of unity we note that both of them depend on asymptotics of expression

$$\Gamma(e\psi) = \int e(x,x,\tau)\psi_\gamma(x)\,dx \tag{16.10.1}$$

with γ-admissible function ψ_γ.

Let us denote by $\gamma^d R(\alpha,\gamma)$ an upper estimate of an error occurring when we replace in (16.10.1) $e(x,x,0)$ by $h^{-d}\mathcal{N}$ and let

$$M = Ch^{-d} + C\mu h^{1-d} \tag{16.10.2}$$

be an upper estimate for $e(x,x,0)$.

Proposition 16.10.1. *Under condition* (16.1.7)

$$|\mathsf{K}| \le C\int \bigl(R(1,\gamma)\bigr)^2 \gamma^{d-1-\kappa}\,d\gamma \tag{16.10.3}$$

and

$$|\mathsf{J}-\mathcal{J}| \le CM\int R(1,\gamma)\gamma^{d-1-\kappa}\,d\gamma \tag{16.10.4}$$

and under condition (16.1.8)

$$\begin{aligned}|\mathsf{K}| \le C\iint_{\{\gamma\le\alpha\}} \bigl(R(\alpha,\gamma)\bigr)^2 \gamma^{d-1-\kappa}\alpha^{d-1}\,d\gamma d\alpha + \\ C\iint_{\{\alpha\le\beta\}} R(\alpha,\alpha)R(\beta,\beta)\alpha^{d-1}\beta^{d-1-\kappa}\,d\alpha d\beta\end{aligned} \tag{16.10.5}$$

and

$$\begin{aligned}|\mathsf{J}-\mathcal{J}| \le CM\iint_{\{\gamma\le\alpha\}} R(\alpha,\gamma)\gamma^{d-1-\kappa}\alpha^{d-1}\,d\gamma d\alpha + \\ CM\iint_{\{\alpha\le\beta\}} R(\alpha,\alpha)\alpha^{d-1}\beta^{d-1-\kappa}\,d\alpha d\beta\end{aligned} \tag{16.10.6}$$

where for $\alpha \le C_0\mu^{-1}$ we need to modify $\alpha^{-1}R(\alpha,\gamma)$ (see below)[43)].

Proof. Due to representation (16.4.1) and definition of $R(\alpha,\gamma)$ and M contribution of the zone $\{(x,y):\ |x-z|\asymp\gamma, |y-z|\asymp\gamma\}$ (note that $R(\alpha,\gamma)\le M$) to K does not exceed $C(R(\alpha,\gamma))^2\gamma^{2d-\kappa}$ for fixed z and γ where $\alpha=\alpha(z)$ provided $\gamma \le 3\epsilon\alpha$.

Under assumption (16.1.7) summation over z results in $C(R(1,\gamma))^2\gamma^{d-\kappa}$ and summation over γ results in estimate (16.10.3); estimate (16.10.4) is proven in the same way.

[43)] As $d=3$ we denote by α what we used to denote by γ in Sections 16.6–16.9 (i.e. $(\alpha^2+\beta^2)^{\frac12}$ in these sections notations.

On the other hand, under assumption (16.1.8) summation over $z : \alpha(z) \asymp \alpha$ results in $C(R(\alpha,\gamma))^2\alpha^d\gamma^{d-\kappa}$ and summation over γ, and then over α results the first term in the right-hand expression of (16.10.5); it estimates contribution of zone $\{(x,y) : |x-y| \le 3\epsilon\alpha\alpha(x)\}$ (in which $\alpha(y) \asymp \alpha(x)$).

Further, under assumption (16.1.8) if $|x-y| \ge 2\epsilon\alpha$, $\alpha := \alpha(x)$, $\beta := \alpha(y)$ then $|x-y| \asymp \max(\alpha,\beta)$. Therefore as $\beta \ge \alpha$ the contribution of zone

$$\{(x,y) : |x-y| \asymp \beta, \alpha(y) \asymp \alpha, \alpha(y) \asymp \beta\}$$

does not exceed $CR(\alpha,\alpha)\alpha^d \times R(\beta,\beta)\beta^{d-\kappa}$ and summation over α, β returns the second term in the in the right-hand expression of (16.10.5); it estimates contribution of zone $\{(x,y) : |x-y| \ge \epsilon\big(\alpha(x)+\alpha(y)\big)\}$. □

16.10.2 Case $d = 2$

Tauberian Asymptotics

(i) Consider first case $\mu \le h^{-1}$. Now according to (16.2.58) we need just to plug in

$$R(\alpha,\gamma) = \mu^{-1}h^{-1} + \alpha^{-1}R^{\mathsf{T}}(\alpha,\gamma) \tag{16.10.7}$$

as $h^{-2}\mathcal{N}_x = e^{\mathsf{T}}(x,x,0)$. Obviously the effect of $C\mu^{-1}h^{-1}$ term to both (16.10.3) and (16.10.5) will be term $C\mu^{-2}h^{-2}$.

Plugging into (16.10.3) or (16.10.5) $R(\alpha,\gamma) = \alpha^{-1}R^{\mathsf{T}}(\alpha,\gamma)$ with $R^{\mathsf{T}}(\alpha,\gamma)$ defined by (16.2.57) we obviously eliminate from integration the last two zones (defined in (16.2.57)) as a power of γ is negative [44)] and the first one as a power of γ is positive. Thus γ snaps to its minimal value in the last two zones and to its maximal value in the first one, and only the second zone matters.

Further, in virtue of (16.2.57) $R^{\mathsf{T}}(\alpha,\gamma) = C\mu^{\frac12}\gamma^{-\frac12}$ in the remaining second zone $\{h \le \gamma \le \min(\mu h/\alpha, \mu^{-1})\}$.

Then the power of γ is positive, 0 or negative as $0 < \kappa < 1$, $\kappa = 1$ or $1 < \kappa < 2$ respectively. Thus γ snaps to $\min(\mu h/\alpha, \mu^{-1})$ as $0 < \kappa < 1$ and γ snaps to h as $1 < \kappa < 2$.

[44)] Here and below talking about powers of α or β or γ we mean in the integral with respect to $\gamma^{-1}d\gamma$ or $\alpha^{-1}d\alpha$ or $\beta^{-1}d\beta$.

Then (16.10.3) becomes

$$|\mathsf{K}^{\mathsf{T}}| \le C\mu^{-2}h^{-2} + C\begin{cases}\mu h^{1-\kappa} & 1<\kappa<2,\\ \mu\min(|\log\mu h|, |\log\mu|) & \kappa=1,\\ \mu\min\big((\mu h)^{1-\kappa}, \mu^{-1+\kappa}\big) & 0<\kappa<1.\end{cases} \tag{16.10.8}$$

Consider the first term in the right-hand expression of estimate (16.10.5). If $1<\kappa<2$ as we mentioned γ snaps to h thus resulting in the integrand $C\mu h^{1-\kappa}$ and in $C\mu h^{1-\kappa}|\log h|$ after integration with respect to $\alpha^{-1}d\alpha$.

Let $0<\kappa\le 1$. Then γ snaps to $\gamma(\alpha) := \min(\mu h/\alpha, \mu^{-1})$, thus resulting in the integrand $C\mu\min\big((\mu h/\alpha)^{1-\kappa}, \mu^{\kappa-1}\big)$ as $0<\kappa<1$ and in $C\mu\log\min\big(\mu/\alpha, (\mu h)^{-1}\big)$ as $\kappa=1$.

Then after integration with respect to $\alpha^{-1}d\alpha$ we get $O(\mu^{\kappa}|\log h|)$ as $0<\kappa<1$ or $O(\mu^{\kappa}|\log h|\cdot|\log(\mu h)|)$ as $\kappa=1$ and this is less than the estimate of the contribution of zone (16.10.9) below unless $\mu\ge h^{-1}|\log h|^{-1}$.

However in (16.10.5) we need also calculate the contribution of the zone

$$\{(x,y):\ \alpha(x)\le C_0\mu^{-1},\ \alpha(y)\le C_0\mu^{-1}\} \tag{16.10.9}$$

which is not covered by our arguments and also the second term.

The former obviously does not exceed $C\mu^2h^{-2}\times\mu^{-4+\kappa} = C\mu^{-2+\kappa}h^{-2}$ where the second factor is $\int |x-y|^{-\kappa}dxdy$ over zone (16.10.9) and this also covers zone $\{\beta\le C_0\mu^{-1}\}$ in the second term of (16.10.5). Obviously $C\mu^{-2+\kappa}h^{-2}$ is larger than anything we got before.

Consider the second term in the right-hand expression of (16.10.5). Here we have $R^{\mathsf{T}}(\alpha,\alpha)=\alpha^{-1}$ as $(\mu h)^{\frac12}\ge\alpha\ge\mu^{-1}$ and $\mu^{-1}\alpha^{-1}h^{-1}\min\big((\mu^2h/\alpha)^{\frac12},1\big)\times(\mu h/\alpha)'$ as $(\mu h)^{\frac12}\le\alpha\le 1$ and therefore β is in the negative power and snaps to α; so we arrive to

$$\int \big(R^{\mathsf{T}}(\alpha,\alpha)\big)^2\alpha^{1-\kappa}\,d\alpha.$$

In this integral α is in the negative power and snaps to μ^{-1} resulting in $\big(R^{\mathsf{T}}(\mu^{-1},\mu^{-1})\big)^2\mu^{\kappa-2} = O(\mu^{\kappa})$ which is less than we have already.

Finally, we need to consider contribution of zone

$$\{(x,y):\ \alpha(x)\le C_0\mu^{-1}, \alpha(y)\ge C_0\mu^{-1}\}. \tag{16.10.10}$$

So, we get $C\mu^{-2} \times \mu h^{-1} \int R^{\mathsf{T}}(\beta,\beta)\beta^{1-\kappa}\,d\beta$; in this integral β is in the negative power for sure as $1<\kappa<2$ and it snaps to $\beta=\mu^{-1}$ resulting in $C\mu^{-1}h^{-1}R^{\mathsf{T}}(\mu^{-1},\mu^{-1})\mu^{\kappa-2}$ which is less than we got already.

As $0<\kappa<1$, β is also in the negative power except zone $\{\mu^{-1}\le\beta\le(\mu h)^{\frac12}\}$ which is possible as $\mu\ge h^{-\frac13}$; then as β snaps to $(\mu h)^{\frac12}$ resulting in $C(\mu h)^{-\frac12-\frac12\kappa}$ and one can see easily that it is less than what we got already. The same is true for $\kappa=1$ as well.

So we arrive to estimate

$$|\mathsf{K}^{\mathsf{T}}|\le C\mu^{-2+\kappa}h^{-2}+C|\log h|\mu\bigl(\mu^{\kappa-1}+h^{1-\kappa}\bigr). \tag{16.10.11}$$

(ii) Let $\mu\ge h^{-1}$. Now we need just to plug $R(\alpha,\gamma)=\alpha^{-1}R^{\mathsf{T}}(\alpha,\gamma)\ge 1$ with $R^{\mathsf{T}}(\alpha,\gamma)$ delivered by (16.2.76). Then in (16.10.3) we need to snap γ to $\mu^{-\frac12}h^{\frac12}$ and set $R(\gamma)=\mu^{\frac12}h^{-\frac12}$. So (16.10.3) becomes

$$|\mathsf{K}^{\mathsf{T}}|\le C\mu^{\frac12\kappa}h^{-\frac12\kappa}. \tag{16.10.12}$$

Meanwhile, under assumption (16.1.8)$'$ in the first term of the right-hand expression of (16.10.5) α is in power 0 and logarithmic factor appears; we get $C\mu^{\frac12\kappa}h^{-\frac12\kappa}|\log\mu|$.

Further, in (16.10.5) we need to estimate also contribution of zones

$$\{(x,y):\ \alpha(x)\le C_0\mu^{-\frac12}h^{\frac12},\alpha(y)\le C_0\mu^{-\frac12}h^{\frac12}\}, \tag{16.10.13}$$

$$\{(x,y):\ \alpha(x)\le C_0\mu^{-\frac12}h^{\frac12},\alpha(y)\ge C_0\mu^{-\frac12}h^{\frac12}\} \tag{16.10.14}$$

and of the second term in the right-hand expression and one can see easily that these contributions do not exceed $C\mu^{\frac12\kappa}h^{-\frac12\kappa}|\log\mu|$; and so we get

$$|\mathsf{K}^{\mathsf{T}}|\le C\mu^{\frac12\kappa}h^{-\frac12\kappa}|\log\mu|. \tag{16.10.15}$$

Thus we proved

Proposition 16.10.2. *(i) For $\mu\le h^{-1}$ under condition* (16.1.7) *estimate* (16.10.8) *holds and under condition* (16.1.8) *estimate* (16.10.11) *holds.*

(ii) Let $\mu\ge h^{-1}$. Then under condition (16.1.7)$'$ *estimate* (16.10.12) *holds and under condition* (16.1.8)$'$ *estimate* (16.10.15) *holds.*

Problem 16.10.3. Prove that estimates (16.10.8) and (16.10.12) hold under conditions (16.1.8)$^+$ and (16.1.8)$^{+\prime}$ respectively.

Weyl Approximation

Now we should add to what we got for K^{T} corresponding expressions with $R(\alpha,\gamma)=R^{\mathsf{W}}(\alpha,\gamma)$ calculated according to (16.2.80)–(16.2.81) and therefore γ must be snapped to $\mathsf{min}(\alpha,\mu h\alpha^{-1})$. Then (16.10.3) becomes

$$|\mathsf{K}^{\mathsf{W}}| \le C\mu^{-2}h^{-2} + C(R^{\mathsf{W}})^2(\mu h)^{2-\kappa} = C\mu^{-2}h^{-2} + C\begin{cases}\mu^4 h^{-1}(\mu h)^{2-\kappa} & \text{if } \mu \le h^{-\frac12},\\ \mu^2 h^{-2}(\mu h)^{2-\kappa} & \text{if } h^{-\frac12}\le \mu \le h^{-1}\end{cases} \tag{16.10.16}$$

which obviously is larger than the right-hand expressions of (16.10.8) and so we can neglect the Tauberian error.

Consider the first term in the right-hand expression of (16.10.5). Note that for $\alpha \le (\mu h)^{\frac12}$ γ snaps to α and then α is in the positive power. Also note that as $\alpha \le \mu^2 h$ (16.2.80) defines $R^{\mathsf{W}}=\mu h^{-1}$ and α is in the positive power as well. Therefore as $h^{-\frac12}\le \mu \le h^{-1}$ we snap α to 1 resulting in $C\mu^2h^{-2}(\mu h)^{2-\kappa}$ prescribed by (16.10.16) and $\mu \le h^{-\frac12}$ we need to consider interval $\mathsf{max}(\mu^{-1},\mu^2h)\le \alpha \le 1$; then automatically $\alpha \ge (\mu h)^{\frac12}$ and we arrive to

$$C\int \mu^4 h^{-1}(\mu h/\alpha)^{2-\kappa} d\alpha. \tag{16.10.17}$$

Here as $1<\kappa<2$, α is in the positive power and snaps to 1 resulting in $C\mu^4h^{-1}(\mu h)^{2-\kappa}$ prescribed by (16.10.16).

On the other hand, as $0<\kappa<1$, α is in the negative degree and snaps to $\mathsf{max}(\mu^{-1},\mu^2h)$ resulting in $C\mu^4h^{-1}(\mu h)^{2-\kappa}\mu^{1-\kappa}$ for $\mu \le h^{-\frac13}$ and $C\mu^4h^{-1}(\mu h)^{2-\kappa}(\mu^2h)^{\kappa-1}$ for $h^{-\frac13}\le\mu\le h^{-\frac12}$ and while the former expression is less than $C\mu^{-2+\kappa}h^{-2}$ the latter is not (at their respective intervals).

As $\kappa=1$ we get $C\mu^5(1+|\log \mu^2h|)$.

Consider the second term in the right-hand expression of (16.10.5). Then both $\alpha\le\beta$ effectively are bounded from above by $(\mu h)^{\frac12}$ which leaves interval empty for $\mu\le h^{-\frac13}$. On the other hand then $\beta\le\mu^2h$ for $h^{-\frac13}\le\mu\le h^{-1}$ and we arrive to

$$C\mu^2h^{-2}\iint_{\{\alpha\le\beta\le(\mu h)^{\frac12}\}} \alpha\beta^{1-\kappa}\,d\alpha d\beta \asymp C\mu^2h^{-2}(\mu h)^{2-\frac12\kappa}$$

which is less than the right-hand expression of (16.10.16) unless $h^{-\frac{1}{3}} \le \mu \le h^{-\frac{1}{2}}$ and $0 < \kappa < 1$ in which case it is less than $C\mu^4 h^{-1}(\mu h)^{2-\kappa}(\mu^2 h)^{\kappa-1}$.

Therefore we arrive to

$$|\mathsf{K}^{\mathsf{W}}| \le C\mu^{-2+\kappa}h^{-2} + C\begin{cases} \mu^4 h^{-1}(\mu h)^{2-\kappa} & \text{as } \mu \le h^{-\frac{1}{2}}, \\ \mu^2 h^{-2}(\mu h)^{2-\kappa} & \text{as } h^{-\frac{1}{2}} \le \mu \le h^{-1} \end{cases} + \tag{16.10.18}$$

$$C\begin{cases} \mu^4 h^{-1}(\mu h)^{2-\kappa}(\mu^2 h)^{\kappa-1} & 0 < \kappa < 1, \quad \mu \le h^{-\frac{1}{2}}, \\ \mu^5(1+|\log \mu^2 h|) & \kappa = 1, \quad \mu \le h^{-\frac{1}{2}}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we proved

Proposition 16.10.4. *For $\mu \le h^{-1}$ under condition* (16.1.7) *estimate* (16.10.16) *holds and under condition* (16.1.8) *estimate* (16.10.18) *holds.*

Magnetic Weyl Approximation

Finally, let us deal with magnetic Weyl approximation. Now we should add to what we got for K^{T} corresponding expressions with $R(\alpha,\gamma) = R^{\mathsf{MW}}(\alpha,\gamma)$ calculated according to (16.2.83) for $\mu \le h^{-1}$ and (16.2.85) for $\mu \ge h^{-1}$.

(i) Consider case $\mu \le h^{-1}$ first. Then according to (16.2.83) γ snaps to $\min(\mu^{-1}, \mu h\alpha^{-1})$ and $R(\alpha,\gamma) = \mu h^{-1}$ and (16.10.3) becomes

$$|\mathsf{K}^{\mathsf{MW}}| \le C\mu^{-2}h^{-2} + C\mu^2 h^{-2}\begin{cases} (\mu h)^{2-\kappa} & \mu \le h^{-\frac{1}{2}}, \\ \mu^{\kappa-2} & h^{-\frac{1}{2}} \le \mu \le h^{-1}. \end{cases} \tag{16.10.19}$$

Meanwhile in the first term of (16.10.5) α snaps to its largest possible value i.e 1. Further, in the second term α snaps to its largest value which is β and then β snaps to its largest value $(\mu h)^{\frac{1}{2}}$ as $h^{-\frac{1}{3}} \le \mu \le h^{-1}$ and it becomes $C\mu^2 h^{-2}(\mu h)^{2-\frac{1}{2}\kappa}$.

Furthermore, contributions of zones (16.10.9) and (16.10.10) are estimated as before; we arrive in the end to

$$|\mathsf{K}^{\mathsf{MW}}| \le C\mu^{-2+\kappa}h^{-2} + C\mu^2h^{-2}\begin{cases}(\mu h)^{2-\kappa} & \mu \le h^{-\frac12},\\ \mu^{\kappa-2} & h^{-\frac12} \le \mu \le h^{-1}\end{cases} + C\begin{cases}\mu^2h^{-2}(\mu h)^{2-\frac12\kappa} & h^{-\frac13} \le \mu \le h^{-1},\\ 0 & \text{otherwise.}\end{cases} \tag{16.10.20}$$

(ii) If $\mu h \ge 1$ γ snaps to $\mu^{-\frac12}h^{\frac12}$ and (16.10.3) becomes

$$|\mathsf{K}^{\mathsf{MW}}| \le C\mu^{1+\frac12\kappa}h^{-1+\frac12\kappa} \tag{16.10.21}$$

Consider the first term in expression (16.4.5); integrand contains α in the positive power and therefore α snaps to 1. Meanwhile in the second term both $\alpha \ge C_0\mu^{-\frac12}h^{\frac12}$ and $\beta \ge C_0\mu^{-\frac12}h^{\frac12}$ are in the negative powers and snap to $\mu^{-\frac12}h^{\frac12}$. Estimating as before contributions of zones (16.10.13) and (16.10.14) we arrive to estimate (16.10.21) again.

All these above estimates are worse than for K^{T}; thus we arrive to

Proposition 16.10.5. *(i) For $\mu \le h^{-1}$ under condition* (16.1.7) *estimate* (16.10.19) *holds and under condition* (16.1.8) *estimate* (16.10.20) *holds.*

(ii) For $\mu \ge h^{-1}$ under either condition (16.1.7)$'$, (16.1.8)$'$ *estimate* (16.10.21) *holds.*

16.10.3 Case $d = 3$

We consider case $d = 3$. Note that

Remark 16.10.6. (i) We use notations in (16.10.5), (16.10.6): λ instead of α and λ' instead of β (in the second term only).

(ii) We cannot expect estimate better than $O(h^{-4})$ for $\mu h \le 1$. Meanwhile under condition (16.1.8) contribution of zone

$$\{(x,y):\ \lambda(x) \le \bar\lambda,\ \lambda(y) \le \bar\lambda\} \tag{16.10.22}$$

with $\bar\lambda = C_0\mu^{-1}$ to any estimate does not exceed $C\mu^2h^{-4} \times \mu^{-6+\kappa} = O(h^{-4})$. We can even take $\bar\lambda = \mu^{-2/(6-\kappa)}$ here.

Tauberian Estimates

We will use results of Subsection 16.6.4 combined with arguments of Subsubsection 16.7.8 always setting $\gamma_3 = \gamma$.

Case $\mu h \le 1$. If $\gamma_3 = \gamma$ we have $\nu(\boldsymbol{\gamma}) \asymp \lambda\gamma$, $\ell \asymp \lambda$ and (16.6.25) becomes

$$R^{\mathsf{T}}(\gamma) = \begin{cases} \mu h^{-\frac{3}{2}}\lambda^{\frac{1}{2}} & \text{as } \lambda^2\gamma \le h, \\ \mu h^{-1}\lambda^{-\frac{1}{2}}\gamma^{-\frac{1}{2}} & \text{as } \lambda^2\gamma \ge h, \lambda\gamma \le \mu h, \\ \mu^{\frac{1}{2}} h^{-\frac{3}{2}}(\mu h/\lambda\gamma)^{l} & \text{as } \lambda\gamma \ge \mu h. \end{cases} \tag{16.10.23}$$

To estimate $|\mathsf{K}^{\mathsf{T}}|$ we need to calculate

$$K(\lambda) \coloneqq \int \bigl(R^{\mathsf{T}}(\gamma)\bigr)^2 \gamma^{2-\kappa}\, d\gamma \tag{16.10.24}$$

and as $\lambda \ge \mu^{-1}$ it is equal to $C\mu^2 h^{-2}\lambda^{-1}\gamma^{2-\kappa}$ calculated for $\gamma = \min(\mu h/\lambda, \lambda)$ as $0 < \kappa < 2$ and for $\gamma = h/\lambda^2$ as $2 < \kappa < 3$ resulting in

$$K(\lambda) \le Ch^{-4}\lambda^{3-\kappa} + C \begin{cases} C\mu^2 h^{-2}\lambda^{-1}(h/\lambda^2)^{2-\kappa} & 2 < \kappa < 3, \\ C\mu^2 h^{-2}\lambda^{-1}(1 + |\log \lambda\mu|) & \kappa = 2, \\ \mu^2 h^{-2}\lambda^{-1}\min(\mu h/\lambda, \lambda)^{2-\kappa} & 0 < \kappa < 2. \end{cases} \tag{16.10.25}$$

In particular, we conclude that

(16.10.26) As $\mu h \le 1$, $|\nabla V/F| \asymp 1$

$$|\mathsf{K}^{\mathsf{T}}| \le Ch^{-4} + C \begin{cases} \mu^2 h^{-\kappa} & 2 < \kappa < 3, \\ \mu^2 h^{-2}|\log \mu| & \kappa = 2, \\ 0 & 0 < \kappa < 2. \end{cases} \tag{16.10.27}$$

On the other hand, under assumption (16.1.8) we need to calculate the first term in the right-hand expression of (16.10.5) which is equal to $\int K(\lambda)\lambda^2\, d\lambda$ and one can see easily that it is (16.10.25) calculated as $\lambda = 1$.

Finally, one needs to calculate the second term in the right-hand expression of (16.10.5) and we leave to the reader to prove that it allows the same estimate. Thus we arrive to Statement (i) of Proposition 16.10.7 below.

On the other hand, assume that assumption (16.7.57) is fulfilled i.e. $|\nabla_{\perp \boldsymbol{F}} V/F| \asymp 1$. Then (16.6.25) becomes

$$R^{\mathsf{T}}(\gamma) = \begin{cases} \mu^{\frac{3}{2}} h^{-1} \left(1 + |\log(\mu h)|\right) & \text{as } \gamma \le h, \\ \mu^{\frac{3}{2}} h^{-1} \big(1 + (\log(1/\mu\gamma))_+\big) & \text{as } h \le \gamma \le \min(\mu h, \mu^{-1}), \\ \mu h^{-1} \gamma^{-\frac{1}{2}} & \text{as } \mu^{-1} \le \gamma \le \mu h, \\ \mu^{\frac{1}{2}} h^{-\frac{3}{2}} (\mu h/\gamma))' & \text{as } \gamma \ge \mu h. \end{cases} \tag{16.10.28}$$

and in this case instead of $\gamma = h$ the lower pick is $\gamma = \mu^{-1}$ in $C\mu^2 h^{-2} \gamma^{2-\kappa}$ which becomes $Ch^{-2}\mu^{\kappa}$ and we arrive to Statement (ii) of Proposition 16.10.7:

Proposition 16.10.7. *Let $\mu h \le 1$. Then*

(i) Under either non-degeneracy condition either (16.1.7) *or* (16.1.8) *estimate* (16.10.27) *holds.*

In particular, $|\mathsf{K}^{\mathsf{T}}| = O(h^{-4})$ as <u>either</u> $0 < \kappa < 2$ <u>or</u> $\kappa = 2$ and $\mu \le (h|\log h|)^{-1}$ <u>or</u> $2 < \kappa < 3$ and $\mu \le h^{-2+\frac{1}{2}\kappa}$ (and this is true for sure as $\mu \le h^{-\frac{1}{2}}$).

(ii) Under non-degeneracy condition (16.7.57)*, $2 < \kappa < 3$*

$$|\mathsf{K}^{\mathsf{T}}| \le Ch^{-4} + Ch^{-2}\mu^{\kappa}. \tag{16.10.29}$$

In particular, $|\mathsf{K}^{\mathsf{T}}| = O(h^{-4})$ as $\mu \le h^{-2/\kappa}$ (and this is true for sure as $\mu \le h^{-\frac{2}{3}}$).

Case $\mu h \ge 1$.

Proposition 16.10.8. *Let $\mu h \ge 1$. Then under either non-degeneracy condition* (16.1.7)$'$ *or* (16.1.8)$'$ *estimate*

$$|\mathsf{K}^{\mathsf{T}}| \le C\mu^2 h^{-2} + C \begin{cases} \mu^2 h^{-\kappa} & 2 < \kappa < 3, \\ \mu^2 h^{-2} |\log \mu| & \kappa = 2, \\ 0 & 0 < \kappa < 2. \end{cases} \tag{16.10.30}$$

holds.

Proof. An easy proof based on Proposition 16.6.9 is left to the reader. □

Weyl Asymptotics

Proposition 16.10.9. *Let $\mu h \le 1$. Then*

(i) Under non-degeneracy condition (16.1.7)

$$|\mathsf{K}^{\mathsf{W}}| \le Ch^{-4} + C\mu^3 h^{-3}(\mu h)^{3-\kappa}; \tag{16.10.31}$$

(ii) Under non-degeneracy condition (16.1.8)

$$|\mathsf{K}^{\mathsf{W}}| \le Ch^{-4} + C\mu^3 h^{-3}(\mu h)^{\frac{1}{2}(3-\kappa)}; \tag{16.10.32}$$

(iii) As $\mu \le h^{-\frac{1}{2}}$ under non-degeneracy condition (16.7.57)

$$|\mathsf{K}^{\mathsf{W}}| \le Ch^{-4} + C\mu^5 h^{-2}(\mu h)^{3-\kappa}. \tag{16.10.33}$$

Proof. An easy proof based on Proposition 16.6.6 is left to the reader. □

Magnetic Weyl Asymptotics

We will apply Proposition 16.6.14. In the isotropic settings $R^{\mathsf{MW}}(\boldsymbol{\gamma})$ given by (16.6.47) becomes

$$R^{\mathsf{MW}}(\gamma) \coloneqq C\mu \begin{cases} \lambda^{-\frac{1}{2}}\gamma^{-\frac{5}{2}}\bigl(\mu h/\lambda\gamma\bigr)^{l} & \text{as } \gamma \ge \mu h/\lambda, \\ \lambda^{-\frac{1}{2}}\gamma^{-\frac{5}{2}} & \text{as } \lambda^{-\frac{1}{3}}h^{\frac{2}{3}} \le \gamma \le \mu h/\lambda, \\ \lambda^{\frac{1}{3}}h^{-\frac{5}{3}} & \text{as } \gamma \le \lambda^{-\frac{1}{3}}h^{\frac{2}{3}}, \end{cases} \tag{16.10.34}$$

which immediately implies Statements (i) and (ii) of Proposition 16.10.10 below.

Meanwhile as $\alpha \asymp 1$, $\mu h \le 1$ (16.6.47) becomes

$$R_1^{\mathsf{MW}}(\gamma) \coloneqq C\mu \begin{cases} \gamma^{-\frac{5}{2}}(\mu h/\gamma)^{l} & \text{as } \gamma \ge \mu h,\ 1 \le \mu^2 h, \\ \gamma^{-\frac{5}{2}} & \text{as } \gamma \le \mu h,\ \mu\gamma \ge 1, \\ \gamma^{-\frac{5}{2}}(\mu^2 h)^{\frac{1}{2}}\bigl(\mu h/\gamma\bigr)^{l} & \text{as } \gamma \ge \mu h,\ 1 \ge \mu^2 h, \\ \gamma^{-\frac{5}{2}}(\mu\gamma)^{\frac{1}{2}} & \text{as } \gamma \le \mu h,\ \mu\gamma \le 1 \end{cases} \tag{16.10.35}$$

which immediately imply Statement (iii) of Proposition 16.10.10 below.

Proposition 16.10.10. *(i) Let $\mu h \le 1$. Then under either non-degeneracy condition* (16.1.7) *or* (16.1.8)

$$|\mathsf{K}^{\mathsf{MW}}| \le Ch^{-4} + C\mu^2 h^{-\frac{2}{3}(2+\kappa)}; \tag{16.10.36}$$

(ii) Let $\mu h \ge 1$. Then under either non-degeneracy condition$(16.1.7)'$ *or* $(16.1.8)'$

$$|\mathsf{K}^{\mathsf{MW}}| \le C\mu^2 h^{-2} + C\mu^2 h^{-\frac{2}{3}(2+\kappa)}; \tag{16.10.37}$$

(iii) Let $\mu \le h^{-\frac{2}{3}}$. Then under non-degeneracy condition (16.7.57)

$$|\mathsf{K}^{\mathsf{MW}}| \le Ch^{-4} + C\mu^3 h^{-\frac{2}{3}(1+\kappa)}. \tag{16.10.38}$$

16.10.4 Problems

Problem 16.10.11. For $d = 2, 3$ estimate $|\mathsf{J} - \mathsf{J}^{\mathsf{T}}|$, $|\mathsf{J} - \mathscr{J}^{\mathsf{W}}|$ and $|\mathsf{J} - \mathscr{J}^{\mathsf{MW}}|$.

Problem 16.10.12. For $d = 2, 3$ consider toy-model approximation.

Problem 16.10.13. For $d = 2, 3$ compare estimates for $|\mathsf{K}^{\mathsf{W}}|$ and $|\mathsf{J} - \mathscr{J}^{\mathsf{W}}|$ to those for $|\mathsf{K}^{\mathsf{MW}}|$ and $|\mathsf{J} - \mathscr{J}^{\mathsf{MW}}|$.

Problem 16.10.14. For $d = 2, 3$ as $\mu \le h^{-1}$ get rid of condition (16.1.5) replacing it by (16.1.7), or even $(16.1.8)^{+}$, or even (16.1.8).

Problem 16.10.15. Get rid of condition (16.1.6) assuming instead that $|F| + |\nabla F| \ge \epsilon$ for $d = 2$ (which will affect estimates for sure) and $|\boldsymbol{F}| + |\nabla \otimes \boldsymbol{F}| \ge \epsilon$ for $d = 3$ (which most likely would not affect estimates).

Problem 16.10.16. For $d = 3$ investigate K^{T}, K^{W}, K^{MW} under non-degeneracy conditions (16.1.7) and $|\nabla_{\perp \boldsymbol{F}} V/F| \le \epsilon \implies |\det \mathsf{Hess}_{\perp \boldsymbol{F}} V/F| \asymp 1$.

Chapter 17

Dirac Operator with the Strong Magnetic Field

In this chapter we analyze two- and three-dimensional (generalized) Dirac operators with the strong magnetic field. In the process we look at (generalized) Schrödinger-Pauli operators as well. The results and structure of this chapter are similar to those of Chapter 13: Sections 17.1–17.4 and Sections 17.6,17.7 correspond to Sections 13.2–13.5 and Sections 13.6 and 13.8 respectively.

However, we introduce a new Section 17.6 specially devoted to the spectral gaps and we collapse Dirac equivalent of Section 13.7 into just Subsection 17.7.1 containing also problems.

17.1 Preliminary Analysis

17.1.1 Toy-Model Operator

In this chapter we consider the *generalized Dirac operator*

$$A = \frac{1}{2}\sum_{j,l} \sigma_l\big(\omega^{jl}P_j + P_j\omega^{jl}\big) + \sigma_0 m + I\cdot V, \quad P_j = hD_j - \mu V_j \tag{17.1.1}$$

where ω^{jl}, V_j, V are real-valued functions, $m \ge 0$ is a constant parameter and the following conditions are fulfilled in $B(0,1)\subset X$:

$$(17.1.2)_{1-3} \quad |D^\alpha \omega^{jk}| \le c, \quad |D^\alpha V_j| \le c, \forall \alpha : |\alpha| \le K$$

$$|D^\alpha V| \le c \min\bigl(1, \frac{1}{m}\bigr) \quad \forall \alpha : 1 \le |\alpha| \le K,$$

$$(17.1.2)_4^- \qquad (\tau_1 - m - V)_- \le c \min\bigl(1, \frac{1}{m}\bigr),$$

$$(17.1.2)_4^+ \qquad (\tau_2 + m - V)_- \le c \min\bigl(1, \frac{1}{m}\bigr),$$

$$(17.1.3) \qquad \sum_{j,k} g^{jk}\xi_j\xi_k \ge \epsilon_0 |\xi|^2 \qquad \forall \xi \in \mathbb{R}^d, \qquad \det(\omega^{jk}) > 0$$

where

$$(17.1.4) \qquad g^{jk} = \sum_{l,r} \omega^{jl}\omega^{kr}\delta_{lr}$$

is an associated metric (we have rewritten conditions $(5.3.77)^*_*$ and (5.3.7)).

Furthermore, we assume that the matrices $\sigma_0, \dots, \sigma_d$ are $\mathsf{D} \times \mathsf{D}$ matrices satisfying conditions (5.A.21) and (5.A.22) (recall that we call them *Pauli matrices*).

Remark 17.1.1. (i) We also consider the *generalized massless Dirac operator*

$$(17.1.5) \qquad A = \frac{1}{2}\sum_{j,l} \sigma_l\bigl(\omega^{jl}P_j + P_j\omega^{jl}\bigr) + I \cdot V, \quad P_j = hD_j - \mu V_j$$

(here we do not need matrix σ_0 to exist) and the *generalized Schrödinger-Pauli operator*

$$(17.1.6) \qquad A = \Bigl(\frac{1}{2}\sum_{j,l} \sigma_l\bigl(\omega^{jl}P_j + P_j\omega^{jl}\bigr)\Bigr)^2 + V(x).$$

(ii) If $\omega^{jl} = \delta_{jl}$ we have standard Dirac operator

$$(17.1.7) \qquad A = \boldsymbol{\sigma} \cdot (-ih\nabla - \boldsymbol{V}(x)) + \sigma_0 m + V(x),$$

massless Dirac operator

$$(17.1.8) \qquad A = \boldsymbol{\sigma} \cdot (-ih\nabla - \boldsymbol{V}(x)) + V$$

and Schrödinger-Pauli operator

$$(17.1.9) \qquad A = \bigl(\boldsymbol{\sigma} \cdot (-ih\nabla - \boldsymbol{V}(x))\bigr)^2 + V(x).$$

Let us observe that

(17.1.10) $e(x,x,\tau_1,\tau_2)$ does not change after *gauge transformation* $V_j \mapsto V_j + \partial_j\phi$ as equivalently we replace A by the operator $e^{-i\mu h^{-1}\phi}Ae^{i\mu h^{-1}\phi}$.

Therefore without any loss of the generality one can replace condition $(17.1.2)_2$ by condition $(17.1.2)'_2 = (13.2.2)'_2$ where the tensor intensity of the magnetic field (F_{jk}) is defined by (13.2.7).

Let us first treat the case when $X = \mathbb{R}^d$ and ω^{jl}, V and F_{jk} are constant.

Theorem 17.1.2. *Let $X = \mathbb{R}^d$, let ω^{jk}, V, F_{jk} be constant and let condition* (17.1.3) *be fulfilled. Then for $\tau_1 < \tau_2$*

$$\operatorname{tr} e(x,x,\tau_1,\tau_2) = h^{-d}\mathcal{N}^{\mathsf{MW}}(\tau_1,\tau_2,\mu h), \tag{17.1.11}$$

where

(i) If $r := \frac{1}{2}\operatorname{rank}(F_{jk}) < d/2$ then

$$\begin{aligned}(17.1.12)_1\quad \mathcal{N}^{\mathsf{MW}}(\tau_1,\tau_2,\mu h) := \operatorname{sgn}(\tau - V)\times \\ \sum_{\alpha\in(\mathbb{Z}^+)^r}\Big((\tau-V)^2 - \sum_{1\le j\le r} 2\alpha_j f_j \mu h - m^2\Big)_+^{\frac{1}{2}(d-r)}\times \\ \frac{\mathsf{D}}{2}(2\pi)^{r-d}\varpi_{d-2r}2^{-\#\{j,\alpha_j=0\}}f_1\cdots f_r(\mu h)^r\sqrt{g}\Big|_{\tau=\tau_1}^{\tau=\tau_2},\end{aligned}$$

where $f_j > 0$, $\pm if_j$ with $j = 1,\dots,r$ are all the eigenvalues of the tensor $(F_k^n) = (\sum_l F_{kl}g^{ln})_{k,n}$ which are disjoint from 0 and multiplicities are taken into account.

(ii) If $r = d/2$ then for $\tau_1 < \tau_2$

$$(17.1.12)_2\quad \mathcal{N}^{\mathsf{MW}}(\tau_1,\tau_2,\mu h) := \sum_{(\varkappa,\alpha)\in\mathfrak{Z}:\ \tau_1\le H_{\varkappa\alpha}<\tau_2} c_{\varkappa\alpha}\frac{\mathsf{D}}{2}(2\pi)^{-r}f_1\dots f_r(\mu h)^r\sqrt{g},$$

where

$$H_{\varkappa\alpha} := V + \varkappa\Big(m^2 + \sum_k 2\alpha_k f_k\mu h\Big)^{\frac{1}{2}}, \tag{17.1.13}$$

$c_{\varkappa\alpha} = 2^{-\#\{j,\alpha_j=0\}}$ *for* $\alpha \neq 0$ *and if*

$$\sigma_0\sigma_1...\sigma_d = \varsigma i^{d/2} \tag{17.1.14}$$

with $\varsigma = \pm 1$ *then*

$$\mathfrak{Z} := \{-1,1\} \times (\mathbb{Z}^+)^r \setminus (\bar{\varkappa}, 0) \tag{17.1.15}$$

and $c_{-\bar{\varkappa},0} = 2^{1-r}$ *where* $\bar{\varkappa} = \pm\varsigma$ *if by a coordinate transformation with a positive (negative respectively) determinant one can reduce the matrix* (F_k^n) *to the canonical form*

$$F_k^n = \begin{cases} +f_j & \text{for } k = 2j-1, n = 2j \quad \text{with } j = 1,\dots,r, \\ -f_j & \text{for } k = 2j, n = 2j-1 \quad \text{with } j = 1,\dots,r, \\ 0 & \text{otherwise.} \end{cases} \tag{17.1.16}$$

Corollary 17.1.3. *In the framework of Theorem 17.1.2*

(i) For $r < d/2$ *the spectrum of* A *is absolutely continuous and*

$$\operatorname{Spec}(A) = (-\infty, V-m] \cup [V+m, \infty); \tag{17.1.17}_1$$

(ii) For $r = d/2$ *the spectrum of* A *is pure point of infinite multiplicity and*

$$\operatorname{Spec}(A) = \overline{\{H_{\varkappa\alpha}, (\varkappa,\alpha) \in \mathfrak{Z}\}} \tag{17.1.17}_2$$

where $\mathfrak{Z}$ *has been introduced under condition* (17.1.14) *and* $\mathfrak{Z} = \{-1,1\} \times (\mathbb{Z}^+)^r$ *if* (17.1.14) *holds with neither* $\varsigma = 1$ *nor* $\varsigma = -1$.

Remark 17.1.4. Appendix 5.A.2 yields that without any loss of the generality one can assume condition (17.1.14) provided d is even.

Proof of Theorem 17.1.2. Proposition 5.A.5 yields that without any loss of the generality one can assume that $g^{kn} = \delta_{kn}$, that F_{kn} has structure (17.1.16), and that $\mu = h = 1$, and therefore without any loss of the generality one can assume that $P_{2j} = D_{2j} + f_j x_{2j-1}$ for $j = 1,\dots,r$ and $P_k = D_k$ for $k \neq 2, 4, \dots, 2r$. Then the arguments of the proof of Theorem 13.2.1 yield that for $\tau_1 < \tau_2$

$$\operatorname{tr} e(x,x,\tau_1,\tau_2) = (2\pi)^{r-d} f_1 \dots f_r \int \boldsymbol{n}(\tau_1,\tau_2,\xi'')\, d\xi'' \tag{17.1.18}$$

where $\xi''=(\xi_{2r+1},\dots,\xi_d)\in\mathbb{R}^{d-2r}$, $\boldsymbol{n}(\tau_1,\tau_2,\xi'')$ is the number of eigenvalues of the operator

$$\mathcal{L}(\xi'')=\sum_{1\le k\le r}\sqrt{f_j}\big(\sigma_{2k-1}D_k+\sigma_{2k}x_k\big)+\sum_{2r+1\le k\le d}\sigma_k\xi_k+\sigma_0 m+V\cdot I \tag{17.1.19}$$

in $\mathscr{L}^2(\mathbb{R}^r,\mathbb{C}^{\mathsf{D}})$, residing in $[\tau_1,\tau_2)$. Let us rewrite the parenthesis in (17.1.19) in the form

$$\frac{1}{2}(\sigma_{2k}-i\sigma_{2k-1})Z_k+\frac{1}{2}(\sigma_{2k}+i\sigma_{2k-1})Z_k^*$$

with $Z_k=x_k+\partial_k$, $Z_k^*=x_k-\partial_k$.

Let us construct a unitary operator $\mathcal{T}:\mathscr{L}^2(\mathbb{R}^r,\mathbb{C}^{\mathsf{D}})\to\oplus\sum_\alpha\mathcal{H}_\alpha$ with $\alpha\in(\mathbb{Z}^+)^r$ in the following way:

(a) Let us decompose $u(x')\in\mathscr{L}^2(\mathbb{R}^r,\mathbb{C}^{\mathsf{D}})$ into the series $u(x')=\sum_\alpha\Upsilon_\alpha(x')w_\alpha$ where $\Upsilon_\alpha(x')=\upsilon_{\alpha_1}(x_1)\dots\upsilon_{\alpha_r}(x_r)$, $\upsilon.(.)$ are real-valued orthonormal Hermite functions and $w_\alpha\in\mathbb{C}^{\mathsf{D}}$;

(b) Then using Proposition 5.A.4 let us set $w_\alpha=\oplus\sum_\varepsilon v_{\alpha,\varepsilon}$ with $\varepsilon\in\{-1,1\}^r$, $w_{\alpha,\varepsilon}\in\cap_{1\le j\le r}\mathsf{Ker}(\sigma_{2j}-i\varepsilon_j\sigma_{2j-1})$;

(c) finally let us set $u_\alpha=\oplus\sum_\varepsilon w_{\beta(\alpha,\varepsilon),\varepsilon}$ where $\beta_j(\alpha,\varepsilon)=\alpha_j+\frac{1}{2}(1-\varepsilon_j)$.

Then $\mathcal{T}u=(u_\alpha)_{\alpha\in(\mathbb{Z}^+)^r}$ defines the operator in question.

Then Proposition 5.A.4 yields that

$$\mathcal{H}_0=\bigcap_{1\le j\le r}\mathsf{Ker}(\sigma_{2j}-i\sigma_{2j-1}), \tag{17.1.20}_0$$

$$\mathsf{dim}\ \mathcal{H}_\alpha=2^{-\#\{j,\alpha_j=0\}}\mathsf{D}\qquad\text{for}\quad\alpha\ne 0; \tag{17.1.20}_\alpha$$

moreover, for $d>2r$ $(17.1.20)_\alpha$ is true for $\alpha=0$.

Furthermore, $(13.5.32)_{1-2}$ yield that

$$\mathcal{T}\mathcal{L}\mathcal{T}^*|_{\mathcal{H}_\alpha}=\sum_{1\le k\le r}\sqrt{2\alpha_k f_k}\sigma'_{k\alpha}+\sum_{2r+1\le k\le d}\sigma_{k\alpha}\xi_k+\sigma_{0\alpha}+V\cdot I$$

where matrices $\sigma_{0\alpha},\sigma'_{1\alpha},\dots,\sigma'_{r\alpha},\sigma_{(2r+1)\alpha},\dots,\sigma_{d\alpha}$ transform $\mathcal{H}_\alpha$ into itself and form a system of Pauli matrices there. Then the eigenvalue counting function of the restriction of the operator $\mathcal{T}\mathcal{L}(\xi'')\mathcal{T}^*$ to $\mathcal{H}_\alpha$ is equal to

$$\frac{1}{2}\,\mathsf{dim}(\mathcal{H}_\alpha)\ \mathsf{sgn}(\tau-V)\theta\Big((\tau-V)^2-\sum_{1\le j\le r}2\alpha_j f_j-m^2-|\xi''|^2\Big)\Big|_{\tau=\tau_1}^{\tau=\tau_2}$$

provided either $d > 2r$ or $d = 2r$ and $\alpha \neq 0$; moreover, if $d = 2r$, $\alpha = 0$ and condition (17.1.14) is fulfilled then this function is equal to $\mathsf{dim}(\mathcal{H}_0)$ for $\tau_1 \le H_{-\zeta,0} < \tau_2$ and vanishes otherwise.

This immediately yields $(17.1.12)_{1,2}$. □

Remark 17.1.5. Let us note that under the assumptions of Theorem 17.1.2 for $V = 0$

$$A^2 = \sum_{j,k} g^{jk} P_j P_k + m^2 + \frac{1}{4i}\mu h \sum_{j,k,l,s} \omega^{jl}\omega^{ks} F_{jk}[\sigma_k, \sigma_s]. \tag{17.1.21}$$

(i) Then Corollary 5.A.7 yields that multiplying by $q^\dagger$ and q on the left and right respectively with an admissible unitary matrix q we transform A^2 into a direct sum of the Schrödinger operators B_ε (where ε runs over $\{-1,1\}^r$) with potentials $V_\varepsilon = m^2 + \sum_j \varepsilon_j f_j \mu h$ and with multiplicity $2^{-r}D$ and therefore Theorem 13.2.1 is applicable to the operator A^2.

(ii) Furthermore, let $2r < d$. Then one can assume without any loss of the generality that $g^{jk} = \delta_{jk}$, $F_{jd} = 0$. Then multiplying by σ_d on the left and on the right and replacing x_d by $x'_d = -x_d$ we transform for $V = 0$ operator A into operator $-A$. Therefore in this case

$$\mathsf{tr}\, e(x,x,0,\tau) = \mathsf{tr}\, e(x,x,-\tau,0) = \frac{1}{2}\,\mathsf{tr}\, e_{\mathsf{S}}(x,x,-\infty,\tau^2) \quad \text{for } \tau > 0, \tag{17.1.22}$$

where $e_{\mathsf{S}}(.,.,.,.)$ is the Schwartz kernel for the Schrödinger operator. These arguments easily yield $(17.1.12)_1$.

(iii) Similar arguments yield $(17.1.12)_1$ for τ_1 and τ_2 not in the spectrum of A provided $d = 2r$ and there exists a unitary matrix q such that $q^\dagger \sigma_j q = -\sigma_j$ for $j = 0, \dots, d$.

Remark 17.1.6. (i) For odd d one can treat the *principally massless* Dirac operator (17.1.5), where $\sigma_1, \dots, \sigma_d$ are Pauli matrices and there exists no admissible matrix σ_0. Then the proof of Theorem 17.1.2 yields that formula $(17.1.12)_1$ also remains true in this case.

(ii) Let us note that the connection between formulae obtained here and Weyl formulae is the same as for the Schrödinger operator (see Section 13.2)

but the approximate formula $(13.A.2)_1$ is replaced by $(13.A.2)_2$ and for $d = 2r$ the term

$$(17.1.23)\quad \frac{1}{2}\varsigma D\Big(\theta(\tau - V + m) - \theta(\tau - V - m)\Big)\Big|_{\tau=\tau_1}^{\tau=\tau_2} \times (4\pi)^{-r}(\mu h)^r f_1 \dots f_r\sqrt{g}$$

should be subtracted from $\mathcal{N}^{\mathsf{MW}}(\tau_1, \tau_2, \mu h)$.

(iii) Moreover, one can replace ςD by $i^{-r}\,\mathsf{tr}(\sigma_0 \dots \sigma_d)$ and then one can get rid of condition (17.1.14). Then for $d \geq 4$ term (17.1.23) is $O(h^2)$ for $\mu = 1$ while for $d = 2$ that term is connected with formula (5.3.96) for the coefficient $\mathsf{tr}\,\kappa_1$ in the spectral asymptotic decomposition.

Remark 17.1.7. In this chapter we assume that $d = 2, 3$ and $F \neq 0$. Then Appendix 5.A yields that we need to treat the following three cases:

(a) $d = 2$, $\mathsf{D} = 2$;

(b) $d = 3$, $\mathsf{D} = 4$;

(c) $d = 3$, $\mathsf{D} = 2$ and the operator is of the form 17.1.5.

Then the spectra are respectively

(A) $\{\eta\sqrt{m^2 + 2j\mu hF}, j \geq 0, \eta = \pm 1, j \geq 0, (\eta, j) \neq (\bar{\varkappa}, 0)\}$ is a pure point spectrum, as $m = 0$ it is symmetric,

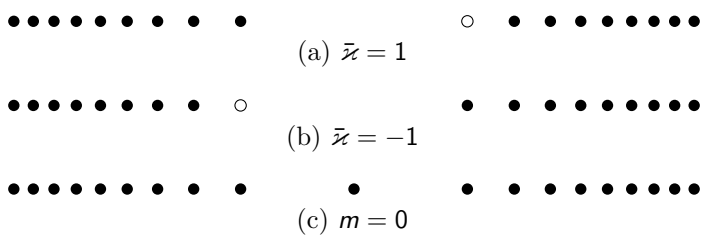
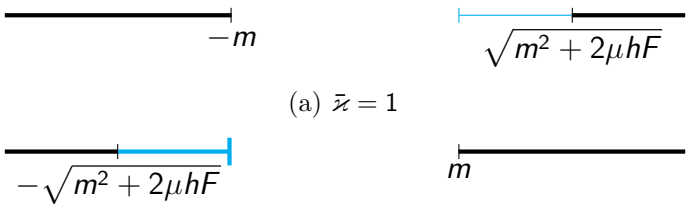


Figure 17.1: The "disappeared" point $(\bar{\varkappa}, 0)$ is not "filled"

(B) $(-\infty, -m) \cup (m, \infty)$ is a continuous spectrum,

Figure 17.2: The spectrum is the same as without magnetic field

(C) $(-\infty, -m) \cup (\sqrt{m^2 + 2\mu hF}, \infty)$ (as $\bar{\varkappa} = 1$) or $(-\infty, -\sqrt{m^2 + 2\mu hF}) \cup (m, \infty)$ (as $\bar{\varkappa} = -1$) is a continuous spectrum. and if $m = 0$ there is a spectral gap between 0 and $\bar{\varkappa}\sqrt{2\mu hF}$.



(a) $\bar{\varkappa}=1$

(b) $\bar{\varkappa}=-1$

Figure 17.3: The spectrum is pure continuous and the spectrum which "disappeared" is cyan

17.1.2 Rough estimates

Let us pass from the calculations to the analytical arguments as we did in Section 13.2. Let us first treat the main spectral gap $(-m+\max V, m+\min V)$ which is empty if $\max V - \min V \ge 2m$; Theorem 17.1.2 yields that it can be extended in only one direction and only in the case $d=2r$.

Theorem 17.1.8. *Let conditions* (17.1.1)–(17.1.3) *and* (13.2.5) *be fulfilled and let*

$$\tau_1 = -m + \max_{x\in B(0,1)} V, \qquad \tau_2 = m + \min_{x\in B(0,1)} V. \tag{17.1.24}$$

Let us assume that

$$\tau_2 - \tau_1 \ge 2\epsilon(1+\mu h)(m+1)^{-1} \tag{17.1.25}$$

with $\epsilon > 0$. *Then the following estimates hold:*

$$|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)U(x,y,t)| \le C'Th^s\mu^{-s}\bigl(\min(\tau_2-\tau,\tau-\tau_1)(m+1)+1\bigr)^{-s}, \tag{17.1.26}$$

$$|e(x,y,\tau_1',\tau_2')| \le C'h^s\mu^{-s}\bigl(\min(\tau_2-\tau_2',\tau_1'-\tau_1)(m+1)+1\bigr)^{-s} \tag{17.1.27}$$

$\forall x,y\in B(0,1-\epsilon) \quad \forall \tau,\tau_1',\tau_2' \in [\tau_1+\epsilon(1+\mu h)(m+1)^{-1}, \tau_2-\epsilon(1+\mu h)(m+1)^{-1}]$,

where $T \ge h^{1-\delta}$.

Proof. The proof of the first estimate is similar to the proof of Proposition 5.3.22, and the second estimate follows from the first one and the standard Tauberian arguments. □

One should replace Theorems 13.2.7, 13.2.9 by the exact equality

$$U(x,y,t)=0 \qquad \forall x,y\in B(0,1-C_0T): |x-y|\ge C_0T \tag{17.1.28}$$

which is due to the standard uniqueness theorem for symmetric hyperbolic systems.

Finally, one should replace Theorems 13.2.11, 13.2.13 by

Theorem 17.1.9. *Let conditions* (17.1.1)–(17.1.3) *and* (13.2.5) *be fulfilled. Let* $q\in \tilde{\mathsf{S}}^0_{h,\rho,1,K}$ *with* $\rho\ge \mathsf{max}(1,\mu h,\mu^{\frac12}h^{\frac12-\delta})$ *be supported in* $\{|\xi|\ge C_0\rho\}$, *let* $Q=\mathsf{Op}\bigl(q(x,h\xi_1-\mu V_1(x),\dots,h\xi_d-\mu V_d(x))\bigr)$ *and let* $\psi\in \mathscr{C}_0^K(B(0,1-\epsilon))$ *with* $\epsilon>0$. *Then the estimate*

$$|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)Q\psi U|\le C'Th^s\mu^{-s} \qquad \forall x,y\in B(0,1-\epsilon)\quad \forall \tau: |\tau|\le m+c(m+1)^{-1} \tag{17.1.29}$$

holds with the same $C_0, C', K, \delta, s, T\ge h^{1-\delta}$ *as in theorems 13.2.11, 13.2.13.*

Proof. This proof repeats the proof of Theorems 13.2.11, 13.2.13, and for $m\ge 1$ we apply the reduction made in the proof of Proposition 5.3.22. □

17.2 Microlocal Canonical Form

In this section we reduce operators (17.1.1) and 17.1.5 to their microlocal canonical forms assuming that $d=2,3$ and conditions (17.1.1)–(17.1.3) and

$$F\ge \epsilon_0 \qquad \text{in}\quad B(0,1) \tag{17.2.1}$$

are fulfilled. Let us recall that $B(0,1)\subset X$. Then Appendix 5.A yields that we need to treat the following three cases listed in Remark 17.1.7.

17.2.1 Principal Symbol Reduction

Then one can assume that in the cases (A) and (C) $\sigma_1,\sigma_2,\sigma_0$ and $\sigma_1,\sigma_2,\sigma_3=\sigma_0$ respectively are Pauli matrices (5.A.29) while in case (B) the matrices $\sigma_0,\dots,\sigma_3$ are given by (5.A.31) with α_j denoted as σ'_j now.

We will construct a reducing operator commuting with σ_0 in the cases (A) and (B). Moreover, in the case (B) this operator will commute with σ_4

where in the framework of (B), (5.A.31) the following auxiliary matrices are used:

$$\sigma_4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \sigma_5 = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}, \tag{17.2.2}$$

$$\hat{\sigma}_j = \sigma_4\sigma_j = \begin{pmatrix} \sigma'_j & 0 \\ 0 & \sigma'_j \end{pmatrix} \qquad (j = 1,2,3).$$

But then the case (B) is reduced to to the case (C) at *this* step of the analysis. So, $\mathsf{D} = 2$ and we treat operators (17.1.1) with $d = 2$ and 17.1.5 with $d = 3$. Let us start with the case $m = V = 0$, i.e., let us treat the operator

$$A^0 = \frac{1}{2}\sum_{j,l} \sigma_l(\omega^{jl}P_j + P_j\omega^{jl}). \tag{17.2.3}$$

Then the $\mu^{-1}h$-pseudodifferential operator $\mu^{-1}A^0$ has the principal symbol

$$a^0(x,\xi) = \sum_{j,l} \sigma_l\omega^{jl}(\xi_j - V_j) \tag{17.2.4}$$

in which case

$$\bigl(a^0(x,\xi)\bigr)^2 = g(x,\xi)I, \qquad g(x,\xi) = \sum_{j,k} g^{jk}(\xi_j - V_j)(\xi_k - V_k). \tag{17.2.5}$$

We use the notation of Section 13.3 in this section.

Proposition 17.2.1. *Let conditions* (13.2.5), (17.1.1)–(17.1.3) *and* (17.2.1) *be fulfilled with* $K = K(N)$. *Let* Ψ *be the symplectomorphism constructed in either Proposition 13.3.1(i) or (iii) for* $d = 2$ *and in Proposition 13.3.1(ii) and Remark 13.3.2(ii) for* $d = 3$ *with* N *replaced by* $N+2$. *Then there exists a smooth unitary matrix* $J = J(x,\xi)$ *in the neighborhood of* Λ^0 *such that*

(i) For $d = 2$

$$J = \exp(iw\sigma_0), \tag{17.2.6}$$

$$J^\dagger(a^0\circ\Psi)J = \bigl(\sigma_1\xi_2 + \sigma_2 x_2\bigr)\Bigl(\sum_{0\le n\le N} B_n(x_1,\xi_1)(x_2^2+\xi_2^2)^n\Bigr) + \sum_{1\le j\le 2}\sigma_j R_j, \tag{17.2.7}$$

$$R_j \in O_{2N+3}(\Lambda^0) \quad (j = 1,2)$$

where here and below w, w_k, B_n, R_j *are real-valued scalar symbols.*

(ii) For $d = 3$

$$J = \exp(i \sum_{1\le j\le 3} w^j \sigma_j), \tag{17.2.8}$$

$$J^\dagger (a^0 \circ \Psi) J = \tag{17.2.9}$$
$$\big(\sigma_1\xi_3 + \sigma_2 x_3 + \sigma_3\xi_2\big)\Big(\sum_{0\le n\le N} B_n(x',\xi_1)(x_2^2+\xi_2^2)^n\Big) + \sum_{1\le j\le 3}\sigma_j R_j,$$
$$R_j \in O_{2N+3}(\Lambda^0) \quad (j = 1,2,3).$$

(iii) Moreover, in the both cases

$$B_0 = F^{\frac12}\circ\bar\Psi \tag{17.2.10}$$

and under the assumptions of Proposition 13.3.1(iii)

$$B_1 = \frac18 F^{\frac12}\kappa\circ\bar\Psi. \tag{17.2.11}$$

Proof. Let Ψ be the indicated symplectomorphism. Then

$$a^0\circ\Psi = \sum_j \sigma_j U_j \qquad (j = 1,\dots,d) \tag{17.2.12}$$

where $U_j \in O_1(\Lambda^0)$, the linear hulls of the sets $\{dU_j,\ j = 1,\dots,d\}$ and $\{du_j,\ j = 1,\dots,d\}$ at points of Λ^0 coincide,

$$\sum_j U_j^2 \equiv \sum_j u_j^2\varphi^2 \qquad \mathrm{mod}\ O_{2N+3}(\Lambda^0), \tag{17.2.13}$$

$$\varphi = \Big(\sum_{1\le n\le N+1} b_n(x_d^2+\xi_d^2)^{n-1}\Big)^{\frac12}, \tag{17.2.14}$$

$u_1 = \xi_2$, $u_2 = x_2$ for $d = 2$ and $u_1 = \xi_3$, $u_2 = x_3$, $u_3 = \pm\xi_2$ for $d = 3$ and the sign of ξ_2 here will be fixed later.

Let us prove that

(17.2.15) in a neighborhood of Λ^0 there exists a smooth matrix $\rho = (\rho^{jk}) \in \mathrm{SO}(d)$ [1] such that

$$u_j \equiv \sum_k \rho^{jk} U_k \varphi^{-1} \qquad \mathrm{mod}\ O_n(\Lambda^0) \tag{17.2.16}_n$$

with $n = 2N+2$.

[1] Here and in what follows $\mathrm{O}(d)$ is the group of real orthogonal $d\times d$ matrices and $\mathrm{SO}(d)$ is its subgroup of matrices with determinant equal 1.

To prove (17.2.15) let us apply induction on n. Equality (17.2.13) yields that for $n = 2$, $(17.2.16)_n$ obviously holds with appropriate $\rho \in \mathsf{O}(d)$.

Moreover, if $d = 2$ then $\rho \in \mathsf{SO}(2)$ necessarily; this follows from the positiveness of $F \cdot \det(\omega^{jk})$ and the analysis of the Poisson brackets $\{U_1, U_2\}$ and $\{u_1, u_2\}$. For $d = 3$ one can obtain $\rho \in \mathsf{SO}(3)$ by choosing the sign at ξ_2.

Let us assume now that $(17.2.16)_n$ holds, $2 \le n \le 2N + 1$. Then

$$u_j \equiv \sum_k \rho^{jk} U_k \varphi^{-1} + v_j \qquad \mathsf{mod}\ O_{n+1}(\Lambda^0)$$

where $v_j \in O_n(\Lambda^0)$ and (17.2.13) yields that

$$\sum_j u_j v_j \equiv 0 \qquad \mathsf{mod}\ O_{n+2}(\Lambda^0). \tag{17.2.17}$$

Let $\phi \in O_{n-1}(\Lambda^0)$ be a real-valued skew-symmetric matrix. Then $\mathsf{SO}(d) \ni \exp\phi \equiv I + \phi \mod O_{n+1}(\Lambda^0)$ and $(17.2.16)_{n+1}$ holds with ρ replaced by $\exp(\phi)\rho$ provided

$$\sum_k \phi^{jk} u_k \equiv -v_j \qquad \mathsf{mod}\ O_{n+1}(\Lambda^0). \tag{17.2.18}$$

Therefore the possibility of making the induction step is due to the following:

Lemma 17.2.2. *Let $d = 2, 3$ and $v_j \in O_n(\Lambda^0)$ satisfy* (17.2.17) *with $n \ge 2$. Then there exists a real-valued skew-symmetric matrix $(\phi^{jk}) \in O_{n-1}(\Lambda^0)$ such that* (17.2.18) *holds.*

This lemma will be proven later; let us first complete the proof of Proposition 17.2.1. Thus, $U_k \equiv \sum_j \varphi\rho^{jk} u_j \mod O_{2N+2}(\Lambda^0)$. Plugging into (17.2.12) we obtain that

$$a^0 \circ \Psi \equiv \sum_{j,k} \varphi\rho^{jk} \sigma_k u_j. \tag{17.2.19}$$

Let us set $\bar\sigma_j = \sum_k \rho^{jk}\sigma_k$ $(j = 1, \dots, d)$. Moreover, for $d = 2$ let us introduce $\bar\sigma_3 = \sigma_3$ $(= \sigma_0)$.

Then $\bar\sigma_1, \bar\sigma_2, \bar\sigma_3$ are Pauli matrices and $\bar\sigma_1\bar\sigma_2\bar\sigma_3 = \sigma_1\sigma_2\sigma_3 = \varsigma i I$ and hence there exists an unitary unimodular matrix $J = J(\rho)$ such that $J^\dagger \bar\sigma_j J = \sigma_j$

$(j = 1, 2, 3)$. This matrix depends analytically on ρ, is defined uniquely modulo a factor ± 1 [2)] and it can be chosen as a one-valued smooth function of (x, ξ) in a Ω because Ω is simply connected.

Every unitary unimodular 2×2 matrix is of the form (17.2.8), and one can take smooth real-valued functions w^j. For $d = 2$, the matrix J commutes with σ_3 (because $\bar{\sigma}_3 = \sigma_3$) and hence it is of the form (17.2.6). Equalities (17.2.7) and (17.2.9) follow from (17.2.19) and the definition of J if B_n are chosen such that

$$\sum_{0 \le n \le N} B_n (x_d^2 + \xi_d^2)^n \equiv \varphi^{\frac{1}{2}} \qquad \mathsf{mod}\ O_{2N+1}(\Lambda^0).$$

Then $B_0 = \sqrt{b_1}$, $B_1 = b_2/2\sqrt{b_1}$ and (17.2.10), (17.2.11) follow from (13.3.12), (13.3.13). □

Proof of Lemma 17.2.2. Instead of $O_n(\Lambda^0)$ one can use $\mathcal{P}_n$, spaces of n-th degree homogeneous polynomials of $u = (u_1, \dots, u_d)$ with coefficients depending smoothly on the remaining coordinates in which case one should replace modulo equalities by exact equalities.

(i) If $d = 2$ then $\phi = \begin{pmatrix} 0 & \psi \\ -\psi & 0 \end{pmatrix}$ and (17.2.17) means exactly that $v_1 = \psi u_2$, $v_2 = -\psi u_1$; one can fulfill these equalities with an appropriate $\psi \in \mathcal{P}_{n-1}$ because (17.2.17) yields that $u_1 v_1 = -u_2 v_2$.

(ii) Let $d = 3$; then every real skew-symmetric matrix is given by a vector product: $\phi v = \psi \times v \ \forall v$ with $\psi \in \mathcal{P}_{n-1}^3 = (\mathcal{P}_{n-1})^3$. In this case (17.2.17) and (17.2.17) mean that $\langle u, v \rangle = 0$ and $\psi \times u = -v$ respectively.

Now let $\mathcal{P}_n$ be the spaces of n-th degree homogeneous polynomials of $u = (u_1, \dots, u_d)$ with constant coefficients. Let us consider the maps

$$\Phi_s : \mathcal{P}_s^3 \ni v \to u \times v \in \mathcal{P}_{s+1}^3,$$
$$\mathcal{L}_s : \mathcal{P}_s^3 \ni v \to \langle u, v \rangle \in \mathcal{P}_{s+1},$$
$$\mathcal{L}'_s : \mathcal{P}_s \ni w \to wu \in \mathcal{P}_{s+1}^3.$$

Obviously, $\mathsf{Ran}\, \Phi_s \subset \mathsf{Ker}\, \mathcal{L}_{s+1}$ and the lemma will be proven if we prove that $\mathsf{Ran}\, \Phi_s = \mathsf{Ker}\, \mathcal{L}_{s+1}$. In order to prove this equality it is sufficient to

[2)] The map $\mathsf{SO}(3) \ni \rho \to T(\rho)$ is called a spinor representation.

prove that the dimensions of these spaces are equal. It is an easy exercise if one notes that $\dim \mathcal{P}_s = \frac{1}{2}(s+1)(s+2)$, $\operatorname{Ker} \Phi_s = \operatorname{Ran} \mathcal{L}'_{s-1}$ and hence $\dim \operatorname{Ker} \Phi_s = \dim \mathcal{P}_{s-1}$ (because $\operatorname{Ker} \mathcal{L}'_{s-1} = 0$) and $\operatorname{Ran} \mathcal{L}_{s+1} = \mathcal{P}_{s+1}$. The calculation of the dimensions in question we leave to the reader; they are equal to $(s+1)(s+3)$. □

Remark 17.2.3. For $d = 2$, the function w is uniquely defined modulo $O_{2N}(\Lambda^0)$; for $d = 3$ the functions w^j are defined modulo the term λu_j (with scalar real-valued function λ) and modulo $O_{2N}(\Lambda^0)$.

17.2.2 Final Reduction

Let $\mathcal{F}$ be a scalar $\mu^{-1}h$-Fourier integral operator corresponding to the symplectomorphism Ψ and satisfying (13.3.61) and (13.3.63). Let J be a $\mu^{-1}h$-pseudodifferential operator with principal symbol $J(x,\xi)$ such that in a neighborhood of Λ^0

$$J^*J \equiv JJ^* \equiv I. \tag{17.2.20}$$

Since $J(x,\xi)$ commutes with σ_0 for $d = 2$ one can also take J commuting with σ_0 (for $d = 2$). Then in a neighborhood of Λ^0

$$J^*\mathcal{F}^*A^0\mathcal{F}J \equiv \mathcal{A}^0 \tag{17.2.21}$$

and in a neighborhood $\mathcal{W}$ of the point $\rho \in \Lambda$ for $d = 2$ (and in a neighborhood $\mathcal{W}$ of the magnetic line Γ for $d = 3$)

$$\mathcal{F}J\mathcal{A}^0J^*\mathcal{F}^* \equiv A^0 \tag{17.2.22}$$

where for $d = 2$

$$\mathcal{A}^0 = \sum_{0\le n\le N} \mu^{-2n} B_n(x_1, \hbar D_1)\Big((\sigma_1 hD_2 + \sigma_2 \mu x_2)(\mu^2 x_2^2 + h^2 D_2^2)^n\Big)^{\mathrm{w}} + \sum_{1\le j\le 2} \sigma_j R_j \tag{17.2.23}$$

and for $d = 3$

$$\mathcal{A}^0 = \sum_{0\le n\le N} \mu^{-2n} B_n(x', \hbar D_1)\Big((\sigma_1 h D_3 + \sigma_2 \mu x_3 + \sigma_3 h D_2)(\mu^2 x_3^2 + h^2 D_3^2)^n\Big)^{\mathsf{w}} + \sum_{1\le j\le 3} \sigma_j R_j + R_4 \cdot I \tag{17.2.24}$$

where here and in what follows $\hbar = \mu^{-1}h$, $R_j = R_{0j} + \hbar R_{1j}$, $R_{\dots}$ are scalar Hermitian $\hbar$-pseudodifferential operators uniformly with respect to all the parameters[3)]. Moreover,

(17.2.25) The symbols of R_{0j} belong to $O_{2N+3}(\Lambda^0)$, $R_{04} = 0$.

Here we quantize the total expression in the parentheses with the exterior superscript w.

Furthermore, for $d = 3$ we have written lower order terms of the general form and for $d = 2$ we noted that these terms are combinations of σ_1 and σ_2 because σ_0 skew-commutes with $\mathcal{A}^0$. Really, σ_0 skew-commutes with A^0 and commutes with $\mathcal{F}$ and J.

Calculation shows that for $d = 3$ the term R_{14} does not necessarily vanish.

For $d = 3$ one can apply the additional construction motivated by Remark 17.2.3. Let J' be an $\hbar$-pseudodifferential operator satisfying (17.2.20) with Weyl symbol $\exp(i\sum_j \sigma_j u_j w)$ (modulo $O(\hbar^2)$); this expression commutes with the principal symbol of $\mu^{-1}\mathcal{A}^0$; this fact and (5.A.30) yield that

$$J'^{*}\mu^{-1}\mathcal{A}^0 J' = \mu^{-1}\mathcal{A}^0 + \mu^{-1}h\Big(\sum_j \sigma_j R'_j + R'_4\Big)$$

where the principal symbol of R'_4 is equal to $\frac{1}{2}\{b_0(u_1^2+u_2^2+u_3^2), w\}$. Therefore replacing J by JJ', choosing an appropriate function w and repeating the proof of Proposition 13.3.1 we can obtain

$$R_4 = \hbar \sum_{0\le n\le N} B'_n(x', \hbar D_1)(x_3^2 + \hbar^2 D_3^2)^n + R''_4 + \hbar^2 R_{24}$$

[3)] Including m, μ, h.

in (17.2.24), where R_4'' is an $\hbar$-pseudodifferential operator with the principal symbol belonging to $O_{2N+1}(\Lambda^0)$; N is decreased if necessary but remains arbitrarily large.

Now by induction with respect to L one can construct operators $J = J_L$ satisfying (17.2.20) such that for $d = 2$ they commute with σ_0 and (17.2.23) holds with $B_n = B_n(x_1, \hbar D_1, \hbar)$ and

$$R_j = \sum_{0\le l\le L} R_{jl}\hbar^l. \tag{17.2.26}$$

On the other hand, for $d = 3$ we obtain that

$$\mathcal{A}^0 = \sum_{0\le n\le N} \Bigl(B_n(x', \hbar D_1, \hbar)(\sigma_1 hD_3 + \sigma_2\mu x_3 + \sigma_3 hD_2)(x_3^2 + \hbar^2 D_3^2)^n\Bigr)^{\mathsf{w}} + \sum_j \sigma_j R_j + R_4\cdot I. \tag{17.2.27}$$

Here R_j $(j = 1, \dots, 4)$ satisfy (17.2.26) and in both cases R_{jl} are operators with the symbols belonging to $O_{N_l}(\Lambda^0)$ and $N_l \to \infty$ as $N \to \infty$ for every fixed l.

Namely, in this construction $J_L = J_{L-1}J_L'J_L''J_L'''$ where J_L', J_L'' and J_L''' satisfy (17.2.20), J_L' is a scalar operator, J_L'' and J_L''' are matrix operators and $(J_L' - I)\hbar^{1-L}$ and $(J_L'' - I)\hbar^{-L}$ are $\hbar$-pseudodifferential operators uniformly with respect to all the parameters[3)], for $d = 2$ the matrix J_L'' commutes with σ_0 and $J_L''' \equiv I$ and for $d = 3$

$$\bigl(J_L''' - I - i\hbar^L \sum_j \alpha^j\sigma_j\bigr)\hbar^{-L-1}$$

is an $\hbar$-pseudodifferential operator uniformly with respect to all the parameters and α^j are scalar $\hbar$-pseudodifferential operators with real-valued symbols.

Checking this construction is left to the reader. So we have proven

Proposition 17.2.4. *Let conditions* (17.1.1)–(17.1.3) *and* (17.2.1) *be fulfilled with* $K = K(N)$ *and let* Ψ *be the symplectomorphism constructed in Proposition 13.3.1(i) or (iii) for* $d = 2$ *and in Proposition 13.3.1(ii) and Remark 13.3.2(ii) for* $d = 3$.

Then there exists a $\mu^{-1}h$*-pseudodifferential operator* J *satisfying* (17.2.20) *in a neighborhood of* Λ^0 *such that*

(i) For $d = 2$, J commutes with σ_0 and (17.2.21), (17.2.22), (17.2.23) *hold with $B_n = B_n(x_1, \hbar D_1, \hbar)$ and with R_j with symbols belonging to $O_{2N+1}(\Lambda^0)$.*

(ii) For $d = 3$ and $\mathsf{D} = 2$, equalities (17.2.21), (17.2.22), (17.2.27) *hold with $B_n = B_n(x', \hbar D_1, \hbar)$ and with R_j with symbols belonging to $O_{2N+1}(\Lambda^0)$; moreover the principal symbol of R_4 vanishes.*

Now we can also treat the case $d = 3$, $\mathsf{D} = 4$:

Corollary 17.2.5. *Let $d = 3$, $\mathsf{D} = 4$ and $\sigma_0, \dots, \sigma_3$ be defined by* (5.A.31). *Let us construct $J' (= J)$ as in Proposition 17.2.4(ii) for the operator $A^{0\prime}$ defined by* (17.2.3) *with σ'_j instead of σ_j and let us now set $J = \begin{pmatrix} J' & 0 \\ 0 & J' \end{pmatrix}$.*

Then for A^0 formulae (17.2.20)–(17.2.22) *and* (17.2.27)$^{\#}$ *hold where the superscript $^{\#}$ means that for R_4 the matrix I should be replaced by σ_4 with σ_4 defined by* (17.2.2).

Remark 17.2.6. For constant F_{jk} and ω^{jk} one can choose a constant matrix $J(x, \xi)$. Therefore the stability of the reduction procedure yields that if

$$(17.2.28)_{1-3} \quad |D^\alpha \omega^{jk}| \le c\bar\nu, \qquad |D^\alpha F_{jk}| \le c\bar\nu \qquad |D^\alpha V| \le c\bar\nu (m+1)^{-1} \quad \forall \alpha : 1 \le |\alpha| \le K,$$

then

$$(17.2.29) \quad B_0 = \bar F + \bar\nu \tilde B_0, \qquad B_n = \bar\nu \tilde B_n \quad (n \ge 1), \qquad B'_n = \bar\nu \tilde B'_n, \qquad R_j = \bar\nu \tilde R_j$$

where $\bar F = \mathsf{const}$, $\tilde B_n$, $\tilde B'_n$ and $\tilde R_j$ are $\mu^{-1}h$-pseudodifferential operators uniformly with respect to all parameters[3)] and the symbols of $\tilde R_j$ belong to $O_{2N+1}(\Lambda^0)$ uniformly with respect to all parameters. Surely condition $(17.2.28)_3$ is not used *here*. Condition $(17.2.28)_4$ will be introduced in the next section.

Let us consider operator

$$(17.2.30) \qquad \hat V = J^* \mathcal{F}^* (V \cdot I) \mathcal{F} J.$$

For $d = 2$ this operator commutes with σ_0 and for $d = 3$, $\mathsf{D} = 4$ it is of the form $\hat V = \hat V' \otimes I_2$. Therefore we obtain

Proposition 17.2.7. *Under the assumptions of Proposition 17.2.4*

$$\hat{V} = W\cdot I + \mu^{-1}hW_0\sigma_0 \qquad (d=2), \tag{17.2.31}_1$$

(17.2.32)

$$\hat{V} = W\cdot I + \sum_j \mu^{-1}hW_j\sigma_j \qquad (d=3,\ \mathsf{D}=2), \tag{17.2.31}_2$$

$$\hat{V} = W\cdot I + \sum_j \mu^{-1}hW_j\hat{\sigma}_j \qquad (d=3,\ \mathsf{D}=4) \tag{17.2.31}_1$$

where $(m+1)W - \mathsf{const}$ *and* $(m+1)W_j$ *(*$j=0,\dots,3$*) are scalar* $\mu^{-1}h$*-pseudodifferential operators uniformly with respect to all parameters and the principal symbol of* W *is equal to* $V\circ\Psi$*.*

Moreover, if conditions $(17.2.28)_{1-3}$ *are fulfilled then*

$$W_j = \bar{\nu}\tilde{W}_j \tag{17.2.33}$$

for all j *where* $\tilde{W}_j$ *are* $\mu^{-1}h$*-pseudodifferential operators uniformly with respect to all parameters.*

Let us note that we can make a more advanced (but incomplete, unfortunately) reduction of the operator

$$\mathcal{A} = \mathcal{A}^0 + \sigma_0 m + \hat{V} \tag{17.2.34}$$

but in the forthcoming sections we will use another method.

17.2.3 Reduced Propagator

Let us consider what we obtained in the treatment of U, etc. Let $\mu\ge 1$, $h\in(0,1]$ and $\mu h\le 1$ (we will get rid of this condition in Section 17.5) and let $\mathcal{F}$ and T be constructed above. Moreover, let us assume that Remark 13.3.12 is fulfilled.

Let us set $\mathcal{U} = J_x^*\mathcal{F}_x^* U\, {}^t\mathcal{F}_y^*\, {}^tJ_y^*$ where U is the Schwartz kernel of operator $\exp(ith^{-1}(m+1)A)$. Then

$$\bigl(hD_t - (m+1)\mathcal{A}\bigr)\mathcal{U}\equiv 0, \tag{17.2.35}$$

$$\mathcal{U}|_{t=0}\equiv \delta(x-y)\cdot I \tag{17.2.36}$$

modulo functions f and f' satisfying (13.3.81) and (13.3.82) respectively where the operator $\mathcal{A}$ is given by (17.2.34) (for $d = 3$, $D = 2$ term $\sigma_0 m$ is absent).

Moreover, $\mathcal{U}$ satisfies (13.3.84)–(13.3.86) and (13.3.87) is replaced by

$$|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)\big(U - \mathcal{F}_x J_x \mathcal{U}\, {}^t J_y\, {}^t\mathcal{F}_y\big)| \le C' T h^s \qquad \forall x, y \in \mathcal{W}_x \quad \forall \tau : |\tau| \le 4c \tag{17.2.37}$$

where $\mathcal{W}_x$ is the x-projection of $\mathcal{W}$.

17.3 Weak Magnetic Field

In this section we consider the case when condition (13.4.1) [4] is fulfilled and we obtain some results similar to the basic results of Section 13.4. Surely some of the auxiliary results of Section 13.4 do not remain valid here because the reduction of the operator $\mathcal{A}$ is now incomplete. For the sake of simplicity here and in Sections 17.4–17.6 below we assume that

$$m \le h^{-\bar{I}}\mu^{\bar{I}} \tag{17.3.1}$$

with some (arbitrarily large) exponent $\bar{I}$. Really, as $m \ge C_0$ we can apply reduction of Subsubsection *5.3.2.4 Asymptotics for $m \le c_0$* and arrive to the Schrödinger operator perturbed and as $m \ge h^{-\bar{I}}\mu^{\bar{I}}$ this perturbation will be negligible.

17.3.1 Analysis of Singularities. I

We start with from the analysis of singularities. First, let us prove

Proposition 17.3.1 [5]**.** *Let $d = 2, 3$ and suppose that condition* (13.4.1) *is fulfilled. Let $Q = \mathsf{Op}(q)$, $q \in \mathsf{S}_{\mu^{-1}h,\mu^{-1},\mu^{-1},K}(T^*\mathbb{R}^d)$ and let us assume that for both signs*

$$\mathsf{supp}(q) \cap \big\{\tau_1 \le \bar{W} \pm \big(m^2 + \mu^2 B_0^2(x_2^2 + \xi_2^2)\big)^{\frac{1}{2}} \le \tau_2\big\} = \emptyset, \tag{17.3.2}_{(2)}$$

$$\mathsf{supp}(q) \cap \big\{\tau_1 \le \bar{W} \pm \big(m^2 + \mu^2 B_0^2(x_3^2 + \xi_3^2 + \xi_2^2)\big)^{\frac{1}{2}} \le \tau_2\big\} = \emptyset, \tag{17.3.2}_{(3)}$$

[4] I.e., $\mu \in [\mu_0, h^{\delta-1}]$ with $\delta > 0$.

[5] Cf. Proposition 13.4.1.

for $d = 2, 3$ respectively where W is taken from $(17.2.31)_{1-3}$ *and the bar means that in this symbol we set $\hbar = 0$ and $x_2 = \xi_2 = 0$, $x_3 = \xi_3 = x_2 = 0$ for $d = 2, 3$ respectively.*

Then estimate (13.4.3) *remains true. Moreover, this estimate remains true if we replace Q_x by ${}^tQ_y$ (placed as usual to the right from U).*

Proof. The proof is by the standard elliptic methods; it is direct for $m \le C_0$ while for $m \ge C_0$ the reduction to the of Subsubsection *5.3.2.4 Asymptotics for $m \le c_0$* is used. We leave the details to the reader. □

It will be convenient for us to assume in what follows that for $m \ge 1$ one of the following conditions is fulfilled:

$$(17.3.3)_{\pm} \qquad |V \pm m| \le \frac{c}{(m+1)}.$$

In this case we obtain

Corollary 17.3.2. *Under condition* $(17.3.3)_{\pm}$ *Remark 13.4.2 remains true for $\tau_{1,2} : |\tau_j| \le 4c$.*

Theorem 17.3.3 [6]**.** *Let $d = 2, 3$ and let condition* (13.4.1) *be fulfilled with $\mu_0 = \mu_0(c, \delta)$.*

Let $Q' = \mathsf{Op}(q')$, $Q'' = \mathsf{Op}(q'')$, $q', q'' \in \mathsf{S}_{\boldsymbol{h},\rho,\gamma,K}(T^*\mathbb{R}^d)$ and condition (13.4.6) *be fulfilled.*

Furthermore, let <u>either</u> $d = 3$ and condition $(13.4.8)_{(3)}$ *and one of the conditions* $(13.4.7)_{1-4}$ *be fulfilled <u>or</u> $d = 2$ and condition* $(13.4.8)_{(3)}$ *and one of the conditions* $(13.4.7)_{1-2}$ *be fulfilled.*

Then estimate (13.4.9) *holds for $\tau : |\tau| \le 2c$.*

Proof. The proof is the standard energy (and commutator) proof (see the proof of Theorem 2.1.2) and for $m \ge C_0$ the reduction to the of Subsubsection *5.3.2.4 Asymptotics for $m \le c_0$* is used. □

Remark 17.3.4. (i) Theorem 13.4.3 and the standard elliptic arguments yield a statement similar to Statement (ii) of Theorem 13.4.3 but with an essentially stronger condition than (13.4.10) [7].

[6] Cf. Theorem 13.4.3.

[7] The left-hand expression of the inequality is the same as in (13.4.10) but the right-hand expression is essentially larger. Finding it is left to the reader.

(ii) Similarly, let us use the standard energy (and commutator) arguments with the reduction of subsubsection 5.3.2.4 for $m \ge C_0$ and with the equality

$$i[\,\sigma_1 hD_d + \sigma_2 \mu x_d\,,\, h^2 D_d^2 + \mu^2 x_d^2\,] = 2\mu h\omega(\sigma_1 hD_d + \sigma_2 \mu x_d) \tag{17.3.4}$$

for $m \le C_0$. In the latter case we write the terms in the parenthesis in the right-hand expression using $(m+1)hD_t - \hat{V}$, $m\sigma_0$ and other terms of $\mathcal{A}^0$ and we use the fact that the matrices $\omega = -\sigma_2\sigma_1$, $\omega\sigma_0$ and $\omega\sigma_3$ are skew-Hermitian. Then we can prove a similar assertion under a restriction stronger than (13.4.10), coinciding with (13.4.10) provided

$$|D^\alpha V| \le \frac{C}{(m+1)}\mu^{|\alpha|-l+d-3} \qquad \forall \alpha : 1 \le |\alpha| \le l+2-d. \tag{17.3.5}$$

Moreover, under some conditions one can prove a weakened version of Theorem 13.4.4. However we have no need for it; we will treat the case $d = 2$, $\nu < 1$ later by another method.

Theorem 17.3.3 immediately yields that Statements (ii) and (iii) of Theorem 13.4.5 remain true with $\tau : |\tau| \le c$ in estimate (13.4.20).

One can prove some statements similar to Propositions 13.4.7–13.4.9 with $\nu = 1$. However, we will now prove only a special statement; the general case will be treated in the following subsection by another method under certain additional condition (which we will get rid of later).

Proposition 17.3.5 [8)]**.** *(i) Let one of the following three conditions be fulfilled:*

$$|\nabla\big(\frac{V}{\sqrt{F}}\big)| \ge \epsilon_0 \quad \textit{in} \quad B(0,1), \qquad m \le \epsilon = \epsilon(c,\epsilon_0), \tag{17.3.6}$$

$$|\nabla\big(\frac{V^2-m^2}{F}\big)| \ge \epsilon_0 \qquad \textit{in} \quad B(0,1), \tag{17.3.7}$$

$$d = 3, \quad |\xi_2| \ge \epsilon_0 \qquad \textit{on} \quad \mathsf{supp}(q') \tag{17.3.8}$$

where $q', q'' \in \mathsf{S}_{h,\rho,\rho,K}(T^*\mathbb{R}^d)$ *with* $\rho \ge \mu^{\frac12} h^{\frac12(1-\delta)}$. *Let* $\chi, \bar\chi \in \mathscr{C}_0^K([-1,1])$, $\bar\chi = 1$ *on* $[-\frac12, \frac12]$ *and*

$$h^{1-\delta} \le T' \le T \le T^* := \mu\epsilon_0 \qquad \textit{for} \quad d = 2, \tag{17.3.9}_{(2)}$$

$$h^{1-\delta} \le T' \le T \le T^* := \epsilon_0 \qquad \textit{for} \quad d = 3, \tag{17.3.9}_{(3)}$$

[8)] Cf. Proposition 13.4.9.

with a sufficiently small constant $\epsilon_0 = \epsilon_0(c) > 0$. *Then estimate* (13.4.33) *holds for* $\tau \in [-\epsilon, \epsilon]$.

(ii) Therefore this estimate remains true if we replace $\mathcal{U}$, Q'_x *and* ${}^tQ''_y$ *by* U, $Q'_x\psi'(x)$ *and* $\psi''(y)\,{}^tQ''_y$ *respectively with* $\psi', \psi'' \in \mathscr{C}_0^K(B(0, \frac{1}{2}))$. *Then conditions* (17.3.6), (17.3.7) *do not change and condition* (17.3.8) *should be replaced by*

$$|\sum_j F^j(\xi_j - V_j)| \ge \epsilon_0 \qquad \text{on} \quad \operatorname{supp}(q'\psi'). \tag{17.3.8}'$$

Proof. The proof is similar to the proof of Corollary 2.1.17 but one should take into account the remarks made in the proof of Theorem 3.3.3. Here under condition (17.3.6) the reduced Dirac operator is treated directly while under conditions (17.3.7) and (17.3.8) the reduction of Subsubsection *5.3.2.4 Asymptotics for* $m \le c_0$ is used.

Let us note that assumption (17.3.7) yields that $m + |V| \ge \epsilon_1$ in $B(0,1)$ and that assumption (17.3.8) yields that the operator $hD_t - (m+1)\mathcal{A}$ is elliptic provided $|\tau| \le \epsilon$, $m + |V| \le \epsilon_0$. □

17.3.2 Analysis of Singularities. II

Let us now assume that

$$m + |V| \ge \epsilon_0 \qquad \text{in} \quad B(0,1). \tag{17.3.10}$$

This condition provides a reduction and we can assume that one of the conditions $(17.3.3)_\pm$ is fulfilled. Let us assume for definiteness that condition $(17.3.3)_+$ is fulfilled; then we can assume that the condition

$$(m - V) \ge \epsilon_0(m+1) \tag{17.3.11}$$

is fulfilled as well. For $m \ge C_0$ this condition is fulfilled automatically while for $m \le C_0$ one can reach it by an ϵ-admissible partition of unity in $B(0,1)$ with a small enough constant $\epsilon > 0$; in the latter case we replace A by $-A$ if necessary. Under this assumption one can apply the reduction to the equation

$$\big(-hD_t - (m+1)A\big)\,U^\pm = \pm ih\delta(x-y)\delta(t)I \tag{17.3.12}$$

directly. Recall that $U^{\pm} = \theta(\pm t)U$. Dividing U into four blocks U_{jk} and assuming that for $d = 2, 3$ the matrices σ_j are given by formulae (5.A.29), (5.A.31) respectively we obtain the following system in the zone $\{|\tau| \le \epsilon\}$:

$$\Big(-hD_t + (m+1)(V+m) - \mathcal{L}^* E \mathcal{L}\Big) U_{1r}^{\pm} \equiv \pm ihf_{1r}, \tag{17.3.13}$$

$$U_{2r}^{\pm} \equiv -E\mathcal{L}U_{1r}^{\pm} \pm ihf_{2r}. \tag{17.3.14}$$

Here E is a parametrix of the operator

$$-h(m+1)^{-1}D_t + (V - m) \tag{17.3.15}$$

which is elliptic in the indicated zone,

$$\mathcal{L} = \frac{1}{2}\sum_{j,l} \sigma_l'(\omega^{jl}P_j + P_j\omega^{jl}) \qquad \text{as} \quad \sigma_l = \begin{pmatrix} 0 & \sigma_l'^{\dagger} \\ \sigma_l' & 0 \end{pmatrix} \tag{17.3.16}$$

and

$$(f_{jk}) = \begin{pmatrix} I & -\mathcal{L}^* E \\ 0 & E \end{pmatrix} \delta(x-y)\delta(t). \tag{17.3.17}$$

In the case $d = 3$, $D = 2$ we can formally "double" our system setting $U' = U'' = U$:

$$(-hD_t + V)\begin{pmatrix} U'^{\pm} \\ U''^{\pm} \end{pmatrix} = \begin{pmatrix} 0 & -A_0 \\ -A_0 & 0 \end{pmatrix}\begin{pmatrix} U'^{\pm} \\ U''^{\pm} \end{pmatrix} \pm ih\delta(x-y)\delta(t)\begin{pmatrix} I \\ I \end{pmatrix};$$

then the above reduction can be applied to this system as well. Here and below $A_0 = A - V \cdot I$.

Let us note that without any loss of the generality one can assume that

$$|V| \ge \frac{\epsilon_0}{(m+1)} \qquad \text{in} \quad B(0,1). \tag{17.3.18}$$

In fact, for $m \ge C_0$ this condition follows from $(17.3.3)_{\pm}$ and for $m \le C_0$ one can easily see that under condition (17.3.10) the operator in the left-hand expression of (17.3.13) is elliptic for $|\tau| \le \epsilon$, $|V| \le \epsilon$ in which case all the statements below can be easily proven by means of simple elliptic arguments (instead of standard hyperbolic arguments).

Let E' be an operator with symbol equal to

$$F^{-\frac{1}{2}}\big((m - V) + (m+1)^{-1}\tau\big)^{\frac{1}{2}}$$

in the zone in question. Let us substitute $U_{1r} = E'u$ and note that

$$\mathcal{L}^* F^{-1} \mathcal{L} = \sum_{j,k} P_j F^{-1} g^{jk} P_k \cdot I + \mu h \mathcal{M}$$

where here and below $\mathcal{M}$ is a $(\mu^{-1}h, h)$-pseudodifferential operator with respect to (x,t) which we change if necessary. Then multiplying (17.3.13) by E' we obtain the equation

$$\Big(\sum_{j,k} P_j F^{-1} g^{jk} P_k + \mu h \mathcal{M} - F^{-1}\Big(\big(h(m+1)^{-1} d_t - V\big)2 - m^2\Big)\Big) u \equiv f$$

with $f = -E' f_{1r}$. One can rewrite this equation in the form

$$\Big(M + 2F^{-1}(m+1)^{-1} V h D_t - F^{-1}(m+1)^{-2}(hD_t)^2 + \mu h \mathcal{M}\Big) u \equiv f \tag{17.3.19}$$

with

$$M := \sum_{j,k} P_j (F^{-1} g^{jk}) P_k + v^{\#}, \qquad v^{\#} = \frac{1}{F}(m^2 - V^2). \tag{17.3.20}$$

Let us note that the operator in the left-hand expression in (17.3.19) is microhyperbolic in one of the directions $\pm\partial_\tau$ by condition (17.3.18); this statement remains true in what follows.

Let us consider the operator M as the Schrödinger operator of Chapter 13; then the scalar magnetic intensity equals 1. Let $\mathcal{F}$ be the corresponding reducing Fourier integral operator constructed in Section 13.3. Then (17.3.19) yields the equation

$$\Big(M' + M'' h D_t + M''' (hD_t)^2 + \mu h \mathcal{M}'\Big) u' \equiv f' \tag{17.3.21}$$

where $u' = \mathcal{F}_x^* u\, {}^t\mathcal{F}_y^*$, $f' = \mathcal{F}_x^*\, {}^t\mathcal{F}_y^* f$ and $M' = \mathcal{F}^* M \mathcal{F}$ is the reduced operator,

$$M'' = 2(m+1)^{-1} \mathcal{F}^* F^{-1} V \mathcal{F}, \qquad M''' = -(m+1)^{-2} \mathcal{F}^* F^{-1} \mathcal{F}$$

are scalar $\mu^{-1}h$-pseudodifferential operators and $\mathcal{M}'$ is an operator of the same type as $\mathcal{M}$.

We consider this equation in the zone $\{\epsilon\mu^{-1} \le |t| \le T^*,\ |\tau| \le \mu h^{1-\delta'}\}$ with $T^* = \epsilon_0$ for $d = 3$, $T^* = \mu\epsilon_0$ for $d = 2$ and with $T^* = \nu^{-1}\mu\epsilon_0$ for

$d = 2$ and some additional restriction where $\epsilon_0 = \epsilon_0(c) > 0$ is a small enough constant. The restriction $\epsilon\mu^{-1} \le T$ is due to the same reasons as in Section 13.4.

First of all the arguments used in the proof of Propositions 2.1.12 and 2.1.16 with the additional arguments used in the proof of Theorem 13.4.3 imply the following assertion:

Proposition 17.3.6 [9]**.** *(i) Let* $Q' = \mathsf{Op}(q')$, $Q'' = \mathsf{Op}(q'')$ *where* q', q'' *are symbols of type* (13.4.23) *and let*

$$|\nabla_{x',\xi'} M'(x,\xi)| \le c\gamma \tag{17.3.22}$$

on $\mathsf{supp}(q')$ *where*

$$\gamma \ge \mu^{-\frac{1}{2}} h^{\frac{1}{2}(1-\delta')} + C_0\mu h^{1-\delta'} + C_0\mu^{-l} \qquad \text{for } d = 2, \tag{17.3.23}_{(2)}$$

$$\gamma \ge h^{\frac{1}{2}(1-\delta')} + C_0\mu h^{1-\delta'} + C_0\mu^{-l} \qquad \text{for } d = 3. \tag{17.3.23}_{(3)}$$

Then estimate (13.4.9) *remains true for* u' *(instead of* $\mathcal{U}$*) for all* $x, y \in \mathbb{R}^d$ *and for all* $\tau : |\tau| \le \mu h^{1-\delta}$ *with arbitrary* T *such that*

$$T \in [\epsilon\mu^{-1}, \mu\epsilon_0] \qquad \text{for } d = 2, \tag{17.3.24}_{(2)}$$

$$T \in [\epsilon\mu^{-1}, \epsilon_0] \qquad \text{for } d = 3. \tag{17.3.24}_{(3)}$$

(ii) Moreover, let $d = 2$*, let conditions* $(17.2.28)_{1-3}$ *be fulfilled and suppose*

$$\nu \ge \bar{\nu}\mu h^{1-\delta'}, \tag{17.3.25}$$

$$\nu^{-1}(M'' - \tilde{M}) \in \mathbf{\Psi}_{\mu^{-1}h,K}(T^*\mathbb{R}) \tag{17.3.26}$$

where M'' *is obtained from* M' *by the substitution* $(\mu^2 x_2^2 + h^2 D_2^2) \mapsto \lambda \in [0, C_0]$ *and* $\tilde{M} = \mathsf{const}$*. Then under assumptions*

$$|\nabla_{x',\xi'} M'(x,\xi)| \le c\nu\gamma \tag{17.3.22}'$$

on $\mathsf{supp}(q')$ *and*

$$\gamma \ge \mu^{-\frac{1}{2}} h^{\frac{1}{2}(1-\delta')} + C_0\bar{\nu}\mu h^{1-\delta'} + C_0\mu^{-l} \qquad \text{for } d = 2 \tag{17.3.23}'_{(2)}$$

estimate (13.4.9) *for* u' *remains true for* $|\tau| \le \mu h^{1-\delta'}$ *and*

$$T \in [\epsilon\mu^{-1}, \nu^{-1}\mu\epsilon_0] \qquad \text{for } d = 2. \tag{17.3.24}'_{(2)}$$

[9] Cf. Proposition 13.4.8.

Remark 17.3.7. (i) Condition (17.3.26) means exactly that multiplying M' by an appropriate operator E'' we can obtain

$$E''M' = B(x_1, \hbar D_1) + \mu^2 x_2^2 + h^2 D_2^2$$

where B satisfies condition (13.4.11). One can use this reformulation even for $\nu = 1$ in the proof.

(ii) If $\nu \ge \epsilon_0 \mu^{-2}\bar{\nu}$ and (17.3.25) holds then assumption (17.3.26) is equivalent to

$$|D^\alpha v^\#| \le c\nu \qquad \forall \alpha : 1 \le |\alpha| \le K; \tag{17.2.28)$_4$}$$

recall that $v^\# = F^{-1}(m^2 - V^2)$.

Proposition 17.3.8 [10]**.** *(i) Let $Q' = \mathsf{Op}(q')$, $Q'' = \mathsf{Op}(q'')$ where q', q'' are symbols of type* (13.4.23) *and let*

$$\epsilon_0\gamma \le |\nabla_{x',\xi'} M'(x,\xi)| \le c\gamma \tag{17.3.27}$$

on $\mathsf{supp}(q')$ *where*

$$1 \ge \gamma \ge \mu^{\frac12} h^{\frac12(1-\delta')} + C_0\mu^{-1}. \tag{17.3.28}$$

Then estimate (13.4.33) *remains true for u' (instead of $\mathcal{U}$) for all $x, y \in \mathbb{R}^d$ and for all $\tau : |\tau| \le \mu h^{1-\delta}$ with arbitrary T satisfying* $(17.3.24)_{(d)}$.

(ii) Moreover, let $d = 2$, let conditions $(17.2.28)_{1-3}$, (17.3.25) *and* (17.3.26) *be fulfilled and let*

$$\epsilon_0\nu\gamma \le |\nabla_{x',\xi'} M'(x,\xi)| \le c\nu\gamma, \tag{17.3.27)$'$}$$

$$1 \ge \gamma \ge \mu^{\frac12} h^{\frac12(1-\delta')}\nu^{-\frac12} + C_0\mu^{-1}. \tag{17.3.28)$'$}$$

Then estimate (13.4.33) *remains true for $|\tau| \le \mu h^{1-\delta'}$ and T satisfying* $(17.3.24)'_{(2)}$.

Remark 17.3.9. (i) Both propositions remain true if we multiply Q' and Q'' by operators $\mathsf{Op}(q_1')$, $\mathsf{Op}(q_1'')$ with $q_1', q_1'' \in \mathsf{S}_{(\boldsymbol{h},h),K}(T^*(\mathbb{R}^d_x \times \mathbb{R}_t))$.

(ii) In what follows in conditions (17.3.27), (17.3.27)$'$ the parameter $\lambda = (\mu^2 x_d^2 + h^2\xi_d^2)$ is taken such that $|M'(x,\xi)| \le \mu h^{1-\delta'}$ in which case the concrete choice of λ is not essential; compare with Remark 17.3.7(i).

[10] Cf. Proposition 13.4.9.

17.3.3 Calculations

Let us pass from estimates to calculations. Here the basic result is the following statement (with arbitrary $d \ge 2$):

Proposition 17.3.10 [11]**.** *Let conditions* (17.1.1)–(17.1.3), (13.4.1), (13.4.58), $(17.3.3)_{\pm}$ *be fulfilled. Then*

(i) Statement (i) of Proposition 13.4.20 remains true albeit now

$$\operatorname{supp}(\varkappa'_{kn}) \subset \{\tau \le V(x) - m\} \cup \{\tau \ge V(x) + m\} \tag{17.3.29}$$

and we introduce only $\varkappa'_{kn}$ and $\varkappa_{kn}(x, \tau', \tau'')$ with finite τ', τ''.

(ii) Moreover, if the condition

$$V^2 - m^2 \ge \epsilon_0 \qquad \forall x \in B(0,1) \tag{17.3.30}$$

is fulfilled (instead of condition (13.4.45)*) then Statement (ii) of Proposition 13.4.20 also remains true.*

(iii) On the other hand, if the condition

$$|V^2 - m^2| + (m+1)|\nabla V| \ge \epsilon_0 \qquad \forall x \in B(0,1) \tag{17.3.31}$$

is fulfilled (instead of condition (13.4.46)*) then Statement (i) of Proposition 13.4.20 also remains true.*

(iv) Further, for $\tau_1 < \tau_2$

$$\operatorname{tr} \varkappa_{00}(x, \tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \operatorname{tr} \varkappa'_{00}(x, \tau) d\tau = (2\pi)^{-d} \frac{\mathsf{D}}{2} \sum_{\eta = \pm 1} \int_{\tau_1 \le H_\eta(x,\xi) \le \tau_2} q'_0(x,\xi) q''_0(x,\xi) \psi'(x) \psi''(x)\, d\xi \tag{17.3.32}$$

where

$$H_\eta(x,\xi) = V + \eta \Big(\sum_{j,k} g^{jk} (\xi_j - V_j)(\xi_k - V_k) + m^2 \Big)^{\frac{1}{2}} \tag{17.3.33}$$

and q'_0, q''_0 are the principal symbols of the operators Q', Q''.

[11] Cf. Proposition 13.4.20.

(v) Furthermore, for $q'_0 = q''_0 = 1$

$$\operatorname{tr} \varkappa_{00}(x,\tau_1,\tau_2) = (2\pi)^{-d}\varpi_d \frac{\mathsf{D}}{2}\sqrt{g}\times \operatorname{sgn}(\tau - V)\big((\tau - V)^2 - m^2\big)_+^{\frac{d}{2}}\big|_{\tau=\tau_1}^{\tau=\tau_2}\psi'\psi''. \tag{17.3.34}$$

(vi) Moreover, let $Q' = Q'' = I$ *and let the set of Pauli matrices contain* σ_0. *Then*

(a) $\operatorname{tr} \varkappa'_{kn} = 0$ *for odd* n.

(b) *Further, if* $d \not\equiv 2 \mod 4$ *then* $\operatorname{tr} \varkappa'_{kn} = 0$ *for even* k.

(c) *On the other hand, if* $d \equiv 2 \mod 4$ *then* $\operatorname{tr} \varkappa'_{kn}$ *does not depend on the choice of Pauli matrices of the given dimension for even* k, *and* $\operatorname{tr} \varkappa'_{kn}$ *depends only on the class of unitary equivalence of the set* $\{\sigma_0, \dots, \sigma_d\}$ *for odd* k: *namely, the replacement* $\sigma_0 \to -\sigma_0$ *yields the replacement* $\operatorname{tr} \varkappa'_{kn} \to -\operatorname{tr} \varkappa'_{kn}$.

(vii) Finally, let $Q' = Q'' = I$ *and suppose that the set of Pauli matrices does not contain* σ_0 *principally (principally massless particle; this is possible only for odd* d*). Then*

(a) *For* $d \equiv 3 \mod 4$ $\operatorname{tr} \varkappa'_{kn} = 0$ *for odd* n; *while for even* n $\operatorname{tr} \varkappa'_{kn}$ *does not depend on the choice of the Dirac matrices of the given dimension for odd* k, *and* $\operatorname{tr} \varkappa'_{kn}$ *depends only on the unitary equivalence class of the set* $\{\sigma_1, \dots, \sigma_d\}$ *for even* k: *namely, the replacement* $\sigma_1 \to -\sigma_1$ *yields the replacement* $\operatorname{tr} \varkappa'_{kn} \to -\operatorname{tr} \varkappa'_{kn}$.

(b) *On the other hand, if* $d \equiv 1 \mod 4$ *then* $\operatorname{tr} \varkappa'_{kn} = 0$ *for odd* n *and* $\operatorname{tr} \varkappa'_{kn}$ *does not depend on choice of Pauli matrices of the given dimension for even* k, *and* $\operatorname{tr} \varkappa'_{kn}$ *depends only on the class of the unitary equivalence of the set* $\{\sigma_1, \dots, \sigma_d\}$ *for odd* k: *namely, the replacement* $\sigma_1 \to -\sigma_1$ *yields the replacement* $\operatorname{tr} \varkappa'_{kn} \to -\operatorname{tr} \varkappa'_{kn}$.

Proof. (a) The proof of Statements (i)–(v) repeats the proof of Proposition 13.4.20 with the following modification: for $m \le C_0 h^{-l}$ we use direct calculation and for $m \ge C_0 h^{-l}$ we use reduction; we can choose $l \ge 0$ arbitrarily.

In order to prove Statements (vi) and (vii) let us note first that the replacement $\sigma_j \to q^\dagger \sigma_j q$ for all j with the unitary matrix q yields the replacement $U \to q^\dagger U q$ and does not change $\mathsf{tr}\, \varkappa'_{kn}$. Moreover, let us note that the complex conjugation of operator A also does not change $\mathsf{tr}\, \varkappa'_{kn}$ because A is a self-adjoint operator.

(b) Furthermore, in the case of Statement (vi) let us note that the replacement $(\mu, h) \to (-\mu, -h)$ is equivalent to the replacement $\sigma_j \to \sigma_0 \sigma_j \sigma_0$ for all j and therefore does not change $\mathsf{tr}\, \varkappa'_{kn}$; then $\mathsf{tr}\, \varkappa'_{kn} = 0$ for odd n (see also the arguments of the proof of Proposition 13.4.20 associated with the Jacobian and mollification).

Let us also note that the replacement $h \to -h$ is equivalent to the complex conjugation and the replacement $\{\sigma_0, \dots, \sigma_d\} \to \{\bar\sigma_0, \dots, \bar\sigma_d\}$ where the bar means a complex conjugation. For $d \not\equiv 2 \mod 4$ the latter set of Pauli matrices is unitarily equivalent to $\{\sigma_0, \dots, \sigma_d\}$ and for $d \not\equiv 0 \mod 4$ it is unitarily equivalent to the set $\{-\sigma_0, \dots, -\sigma_d\}$ (see formula (5.A.24)). This observation yields all the remaining assertions in (vi).

(c) On the other hand, in the case of Statement (vii) let us note that the replacement $\mu \to -\mu$ is equivalent to the complex conjugation of operator and the replacement $\{\sigma_1, \dots, \sigma_d\} \to \{\bar\sigma_1, \dots, \bar\sigma_d\}$; for $d \equiv 3 \mod 4$ these sets are equivalent and therefore $\mathsf{tr}\, \varkappa'_{kn} = 0$ for odd k. Moreover, the replacement $(\mu, h) \to (-\mu, -h)$ is equivalent to the replacement $\sigma_j \to -\sigma_j$ for all j and this observation concludes the proof of (vii) for $d \equiv 3$.

Finally, let us note that the replacement $h \to -h$ is equivalent to the complex conjugation of the operator A and the replacement $\{\sigma_1, \dots, \sigma_d\} \to \{\bar\sigma_1, \dots, \bar\sigma_d\}$. For $d \equiv 1$ these sets are equivalent and therefore $\mathsf{tr}\, \varkappa'_{kn} = 0$ for odd n. Using the replacement $\mu \to -\mu$ (see above) one can easily conclude the proof of (vii) for $d \equiv 1$. □

Repeating the corresponding arguments used in the proof of (13.4.69) one can prove the similar formula

$$\mathcal{N}^{\mathsf{MW}\prime}(\tau, \mu h) = \partial_\tau \mathcal{N}^{\mathsf{MW}}(x, \tau_1, \tau, \mu h) \sim \sum_{k \in \mathbb{Z}^+} \mathsf{tr}\, \varkappa'_{k0}(x, \tau)(\mu h)^k \tag{17.3.35}$$

in the weak sense.

In the calculation below the dependence on x is not essential and without any loss of the generality one can assume that $V = 0$.

Consider the case $d > 2r = \mathsf{rank}(F_{jk})$. Note that then $\mathcal{N}^{\mathsf{MW}\prime}(\tau, \mu h)$ is an even function of τ and therefore $\mathsf{tr}\, \varkappa'_{k0}(\tau)$ are also even functions of τ. Then one can introduce their odd (on τ) primitive functions $\mathcal{N}^{\mathsf{MW}}(\tau, \mu h)$ and $\mathsf{tr}\, \varkappa_{k0}(\tau)$.

The same is true for $d = 2r$ if we treat the even (with respect to τ) parts of $\mathcal{N}^{\mathsf{MW}\prime}(\tau, \mu h)$ and $\mathsf{tr}\, \varkappa'_{k0}(\tau)$ and the odd parts of $\mathcal{N}^{\mathsf{MW}}(\tau, \mu h)$ and $\mathsf{tr}\, \varkappa_{k0}(\tau)$. In any case one can treat $\mathcal{N}^{\mathsf{MW}}(\tau, \mu h)$ as Riemann sums given by (13.A.4) for the function

$$\mathcal{K}(\tau, z) = c_0 \big(\tau^2 - z_1 - \dots - z_r\big)_+^p \, \mathsf{sgn}\, \tau$$

with $p = \frac{1}{2}(d - 2r)$, $c_0 = (2\pi)^{r-d} 2^{-r-1} \varpi_{d-2r} \mathsf{D} \sqrt{g}$. We assert that

(17.3.36) The function $\mathcal{K}(\tau, z)$ mollified with respect to τ is smooth with respect to z.

In fact, any mollification with an even mollifying function $\phi(\tau)$ gives 0 while mollification with an odd mollifying function $\phi(\tau)$ (equal to $\phi_0(\tau^2)\tau$ with $\phi_0 \in \mathscr{C}_0^K(\mathbb{R})$) is equal to

$$\mathsf{const} \int_0^\infty \phi_0(\tau)(\tau - t)_+^p \, d\tau$$

with $t = z_1 + \dots + z_r + m^2$ where we change constants if necessary. Integrating by parts n times we obtain for $t \ge 0$

$$\mathsf{const} \int_0^\infty \phi_0^{(n)}(\tau)(\tau - t)_+^{p+n} \, d\tau$$

and the smoothness with respect to t is obvious. Therefore we have proven Statement (i) of the following

Proposition 17.3.11 [12]**.** *Let $Q' = Q'' = I$, $\psi' = \psi'' = 1$. Then*

(i) For $d > 2r$ the following equalities are true

$$\mathsf{tr}\, \varkappa_{k0} = \mathsf{D} \partial_t^k \, \mathsf{sgn}(\tau - V)\big((\tau - V)^2 - m^2 + t\big)_+^{\frac{d}{2}}\big|_{t=-0} \times \mathcal{P}_k(f_1^2, \dots, f_r^2)\sqrt{g} \tag{17.3.37}$$

[12] Cf. Proposition 13.4.22.

where $\mathcal{P}_k$ are homogeneous symmetric polynomials of degree $\frac{k}{2}$ also depending on d and r and vanishing for odd k. In particular,

$$\operatorname{tr} \varkappa_{20} = 2c\mathsf{D}\partial_t^2 \operatorname{sgn}(\tau - V)\big((\tau - V)^2 - m^2 + t\big)_+^{\frac{d}{2}}\big|_{t=-0} F^2\sqrt{g} \tag{17.3.38}$$

where c and F are introduced in Proposition 13.4.22.

(ii) For $d = 2r$ these equalities hold for the odd with respect to $(\tau - V)$ parts of $\varkappa_{k0}(\tau)$ and the even part of $\varkappa_{k0}(\tau)$ vanishes for $k \neq r$, while for $k = r$ under condition (17.1.15) *it is equal to*

$$\operatorname{tr} \varkappa_{r0\mathrm{even}} = -\bar{\varkappa}\frac{\mathsf{D}}{2}(4\pi)^{-r}\big(\theta(\tau - V + m) - \theta(\tau - V - m)\big) f_1 \dots f_r\sqrt{g} \tag{17.3.39}$$

with $\bar{\varkappa}$ introduced in Theorem 17.1.2. In particular, for $d = 2$ the coefficient in the right-hand expression is equal to $-\frac{\mathsf{D}}{8\pi}\varsigma F_{12}$.

Proof. In order to prove Statement (ii) it is sufficient to observe that outside of the discontinuity points the even part of $\mathcal{N}^{\mathsf{MW}}(\tau, \mu h)$ is equal to the right-hand expression of (17.3.39) multiplied by $(\mu h)^r$; this is due to formula (17.1.13).

Remark 17.3.12. In the manner of Remark 13.4.23 we conclude that if Q' is an operator with Weyl symbol q' even with respect to ξ and h and if $q'' = 1$, $\psi' = \psi'' = 1$ in a neighborhood of $\operatorname{supp}(q')$ then Statements (vi) and (vii) of Proposition 17.3.10 remain true.

17.3.4 Synthesis

Now, repeating the corresponding arguments of Section 13.4 we easily obtain the following statements:

Proposition 17.3.13 [13]**.** *Let conditions* (17.1.1)–(17.1.3), (17.2.1) *and* (13.4.1) *be fulfilled. Then*

(i) If one of the conditions (17.3.6), (17.3.7) *is fulfilled then estimate* (13.4.62) *holds for $|\tau| \le \epsilon$ with $T = \mu\epsilon_0$ for $d = 2$ and $T = \epsilon_0$ for $d = 3$.*

[13] Cf. Proposition 13.4.20.

(ii) Let $d = 2$ *and conditions* $(17.2.28)_{1-3}$, (17.3.25), (17.3.26), (17.3.30) *be fulfilled and let* $\tilde{Q}' = \mathsf{Op}(\tilde{q}')$, $\tilde{Q}'' = \mathsf{Op}(\tilde{q}'')$ *with* $\tilde{q}', \tilde{q}'' \in \mathsf{S}_{\hbar,\gamma,\gamma,K}(T^*\mathbb{R})$. *Moreover, let conditions* $(17.3.27)'$, $(17.3.28)'$ *be fulfilled on* $\mathsf{supp}(\tilde{q}')$ *and* $Q' = \mathcal{F}\tilde{Q}'\mathcal{F}^*$, $Q'' = \mathcal{F}\tilde{Q}''\mathcal{F}^*$. *Then estimate* (13.4.62) *holds for* $|\tau| \le \mu h^{1-\delta'}$ *with* $T = \nu^{-1}\mu\epsilon_0$.

(iii) Let $d = 3$, $\tilde{q}', \tilde{q}' \in \mathsf{S}_{\boldsymbol{h}',\gamma,\gamma,K}(T^*\mathbb{R}^2)$ *and let conditions* (17.3.30), (17.3.27), (17.3.28) *be fulfilled on* $\mathsf{supp}(\tilde{q}')$. *Let* $Q' = \mathcal{F}\tilde{Q}'\mathcal{F}^*$, $Q'' = \mathcal{F}\tilde{Q}''\mathcal{F}^*$. *Then estimate* (13.4.62) *holds for* $|\tau| \le \mu h^{1-\delta'}$ *with* $T = \epsilon_0$.

Theorem 17.3.14 [14]**.** *Let conditions* (17.1.1)–(17.1.3), (17.2.1), (13.4.1) *and one of conditions* (17.3.6), (17.3.7) *be fulfilled. Let us define*

$$\mathsf{R}^{\mathsf{W}\prime} := \Big| \iint \phi(\tau')\psi(x)\,\mathsf{tr}\big(e(x,x,\tau',\tau) - h^{-d}\sum_{0\le k\le M} \varkappa_{k0}(x,\tau',\tau)(\mu h)^k\big)\,dx d\tau \Big| \tag{17.3.40}$$

with $\phi \in \mathscr{C}_0^K([-c,c])$. *Then estimates* $(13.4.74)_{(2,3)}$ *hold for* $d = 2,3$ *respectively,* $|\tau| \le \epsilon(m+1)^{-1}$.

Proposition 17.3.15 [15]**.** *Let conditions* (17.1.1)–(17.1.3), (17.2.1), (13.4.1) *and* (17.3.30) *be fulfilled. Let* $\tilde{Q}' = \mathsf{Op}(\tilde{q}')$ *where* $\tilde{q}' \in \mathsf{S}_{\boldsymbol{h}',\gamma,\gamma,K}(T^*\mathbb{R}^{d-1})$, γ *satisfies* $(17.3.28)'$, $\mathsf{diam\ supp}\,\tilde{q}' \le c\gamma$, *and let* $Q' = \mathcal{F}\tilde{Q}'\mathcal{F}^*$. *Then*

(i) Let $d = 2$, *let conditions* $(17.2.28)_{1-3}$, (17.3.25), (17.3.26) *be fulfilled and let condition* $(17.3.27)'$, $(17.3.28)'$ *be fulfilled on* $\mathsf{supp}(\tilde{q}')$. *Let us define*

$$\mathsf{R}^{\mathsf{W}\prime\prime} := \Big| \iint \phi(\tau')\,\mathsf{tr}\big((Q'_x\psi(x)e)(x,x,\tau',\tau) - h^{-d}\sum_{k+n\le M} \varkappa_{Q'\psi kn}(x,\tau',\tau)(\mu h)^k h^n\big)\,dx d\tau' \Big|. \tag{17.3.41}$$

Then estimate $(13.4.76)_{(2)}$ *holds.*

(ii) Let $d = 3$, $\nu = 1$ *and let condition* (17.3.27) *be fulfilled on* $\mathsf{supp}(\tilde{q}')$. *Then estimate* $(13.4.76)_{(3)}$ *holds.*

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[14] Cf. Theorem 13.4.25.

[15] Cf. Proposition 13.4.27.

Proposition 17.3.16 [16]**.** *Let conditions* (17.1.1)–(17.1.3), (17.2.1), (13.4.1) *and* (17.3.30) *be fulfilled. Let* $\vartheta = 0$. *Then*

(i) Let $d = 2$ *and let conditions* $(17.2.28)_{1-3}$, (17.3.25), (17.3.26) *be fulfilled. Then estimate* $(13.4.78)_{(2)}$ *holds where* Ω_2 *is introduced for the reduced Schrödinger operator.*

(ii) For $d = 3$ *estimate* $(13.4.78)_{(3)}$ *holds where* Ω_2 *is introduced for the reduced Schrödinger operator.*

Theorem 17.3.17 [17]**.** *Let* $d = 3$ *and let conditions* (17.1.1)–(17.1.3), (17.2.1), (13.4.1) *and* (17.3.30) *be fulfilled. Then for* $\mu \le h^{-\frac{1}{3}+\delta}$ *estimate* $(13.4.74)_{(3)}$ *holds.*

Theorem 17.3.18 [18]**.** *(i) Under the conditions of Theorem 17.3.14 estimates* $(13.4.74)_{(2,3)}$ *(with* $\vartheta = 0$*) remain true for*

$$\mathsf{R}^{\mathsf{MW}} = \Big| \iint \phi(\tau)\psi(x)\,\mathsf{tr}\big(e(x,x,\tau,0) - h^{-d}\mathcal{N}^{\mathsf{MW}}(x,\tau,0,\mu h)\big)\,dx d\tau\Big|. \tag{17.3.42}$$

(ii) Under the assumptions of Proposition 17.3.16(i) estimate $(13.4.90)_{(2)}$ *(with* $\vartheta = 0$*) remains true with* V *replaced by*

$$V^{\#} = m^2 - V^2 \tag{17.3.43}$$

in the right-hand expression.

17.4 Temperate Magnetic Field

17.4.1 Separation of Variables and Decomposition

In this section we assume that

$$h^{-\delta} \le \mu \le \epsilon_1 h^{-1} \tag{17.4.1}$$

[16] Cf. Proposition 13.4.28.
[17] Cf. Theorem 13.4.25.
[18] Cf. Theorem 13.4.30.

where small enough ϵ_1 will be chosen later. The case $\mu \ge \epsilon_1 h^{-1}$ will be treated in the next section. Later conditions (17.3.30) and (13.4.1) will appear in some statements of this section. Let us make a decomposition similar to those of the proof of Theorem 17.1.2. Namely, let us note that every $\mathbb{H}$-valued function v on $\mathbb{R}$ (with $\mathbb{H} = \mathbb{C}^D$) can be written in the form

$$w = \sum_{j\in\mathbb{Z}^+} \bigl(w_j^+ \Upsilon_{j-1} + w_j^- \Upsilon_j\bigr), \qquad w_j^\pm \in \mathbb{H}^\pm, \quad w_0^+ = 0 \tag{17.4.2}$$

where $\mathbb{H}^\pm = \mathsf{Ker}(\sigma_1 \pm i\sigma_2)$, $\Upsilon_j = \Upsilon_j(x_d) = (\mu/h)^{\frac14} \upsilon_j\bigl(\sqrt{\mu/h}\, x_d\bigr)$, where υ_j are real-valued orthonormal Hermite functions satisfying $(13.5.32)_{1-2}$. Let us set $w_j = (w_j^+, w_j^-) \in \mathbb{H}$; so the isomorphism

$$\mathfrak{R} : \mathscr{L}^2(\mathbb{R}) \to \mathbb{H}^- \oplus \sum_{j\ge 1} \mathbb{H} \tag{17.4.3}$$

is introduced.

Formulae $(13.5.32)_{1-2}$ and the definition of $\mathbb{H}^\pm$ yield that for $w_j^\pm \in \mathbb{H}^\pm$

$$\bigl(hD_d\sigma_1 + \mu x_d \sigma_2\bigr)\bigl(w_j^+ \Upsilon_{j-1} + w_j^- \Upsilon_j\bigr) = \sigma_2 \sqrt{2j\mu h}\bigl(w_j^+ \Upsilon_{j-1} + w_j^- \Upsilon_j\bigr). \tag{17.4.4}$$

Similarly, every $\mathsf{Hom}(\mathbb{H},\mathbb{H})$-valued function $\mathcal{U}$ on $\mathbb{R}^2$ can be written in the form

$$\begin{aligned}\mathcal{U} = \sum_{j,k\in\mathbb{Z}^+} \bigl(&\mathcal{U}_{jk}^{++} \Upsilon_{j-1}(x_d)\Upsilon_{k-1}(y_d) + \mathcal{U}_{jk}^{+-} \Upsilon_{j-1}(x_d)\Upsilon_k(y_d) + \\ &\mathcal{U}_{jk}^{-+} \Upsilon_j(x_d)\Upsilon_{k-1}(y_d) + \mathcal{U}_{jk}^{--} \Upsilon_j(x_d)\Upsilon_k(y_d)\bigr)\end{aligned} \tag{17.4.5}$$

with $\mathcal{U}_{jk}^{\zeta\zeta'} \in \mathsf{Hom}(\mathbb{H}^\zeta, \mathbb{H}^{\zeta'})$ where for $j = 0$, $\zeta = +$ and for $k = 0$, $\zeta' = +$ the corresponding components vanish. Using these four blocks (with different ζ and ζ') one can introduce the blocks $\mathcal{U}_{jk}^{\zeta,} \in \mathsf{Hom}(\mathbb{H}^\zeta, \mathbb{H})$, $\mathcal{U}_{jk}^{,\zeta'} \in \mathsf{Hom}(\mathbb{H}, \mathbb{H}^{\zeta'})$ and $\mathcal{U}_{jk} \in \mathsf{Hom}(\mathbb{H},\mathbb{H})$.

Now let $\mathcal{U} = \mathcal{U}(x,y,t)$ be defined in Subsection 17.2.3. Then $\mathcal{U}_{jk} = \mathcal{U}_{jk}(x',y',t)$, etc., and the properties of $\mathcal{U}$ yield that

$$\mathcal{U}_{jk}(x',y',t) = \mathcal{U}_{kj}^\dagger(y',x',-t) \tag{17.4.6}$$

and

$$|F_{t\to h^{-1}\tau} \chi_{T,T'}(t)\mathcal{U}_{jk}| \le CT\mu h^{1-d}(1+|\tau|)^{d-1} \qquad \forall x',y' \in \mathbb{R}^{d-1} \quad \forall j,k \in \mathbb{Z}^+ \quad \forall \tau \in \mathbb{R}, \tag{17.4.7}$$

and Theorem 13.2.13 which obviously holds under condition (17.4.1) for the Dirac operator also yields that

$$(17.4.8)\quad |F_{t\to h^{-1}\tau}\chi_{T,T'}(t)\mathcal{U}_{jk}| \le CTh^s(1+j+k)^{-s}$$
$$\forall x', y' \in \mathbb{R}^{d-1} \quad \forall \tau \in \mathbb{R} : |\tau| \le c \quad \forall j,k \in \mathbb{Z}^+ : j+k \ge C_0\mu^{-1}h^{-1}+h^{-\delta}$$

where $T \ge h^{1-\delta}$ and T' are arbitrary.

Formula (17.4.4) yields that the operator $\mathcal{A} = J^*\mathcal{F}^*A\mathcal{F}J$ applied to $\mathcal{U}$ can be replaced[19] *in the indicated representation* by the operator

$$(17.4.9)\qquad \mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2$$

where we use the same notation, $\mathcal{A}_0$ and $\mathcal{A}_1$ are of the block-diagonal type[20] and

(i) For $d = \mathsf{D} = 2$ the blocks of $\mathcal{A}_0$ are

$$(17.4.10)_j\quad \mathcal{A}_{0(j)} = \sum_{0\le n\le N} B_n(x_1,\hbar D_1,\hbar)\mu^{-2n}(2j\mu h)^{n+\frac12}\sigma_2 + \sigma_0 m + \bar W(x_1,\hbar D_1,\hbar)I \qquad (j\ge 1),$$

$$(17.4.10)_0\qquad \mathcal{A}_{0(0)} = -\varsigma m + \bar W(x_1,\hbar D_1,\hbar)I,$$

and the blocks of $\mathcal{A}_1$ are

$$(17.4.11)_j\quad \mathcal{A}_{1(j)} = \sum_{0\le n\le N} B_n(x_1,\hbar D_1,\hbar)\mu^{1-2n}h\times \bigl(\mathcal{P}'_{n-1}(2j\mu h)\sigma_1 + \mathcal{P}''_{n-1}(2j\mu h)\sigma_2\bigr)\sqrt{2j\mu h} \qquad (j\ge 1),$$

$$(17.4.11)_0\qquad \mathcal{A}_{1(0)} = 0$$

respectively, $\mathcal{P}'_{n-1}$ and $\mathcal{P}''_{n-1}$ are absolute (i.e., independent of the operator) polynomials of degree $n-1$ and the operator

$$(17.4.12)\qquad \mathcal{A}_2 = T^*\mathcal{F}^*(V\cdot I)\mathcal{F}T - \bar W\cdot I$$

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[19] With an error $O(h^s)$ after this application.

[20] In what follows the blocks of an operator $\mathcal{B}$ of the block-diagonal type are denoted by $\mathcal{B}_{(j)}$.

has blocks

$$\mathcal{A}_{2,jk} = W_{jk}\cdot I + W^0_{jk}\sigma_0, \tag{17.4.13}$$

which for $j=0$ and (or) for $k=0$ have an additional factor Π_- and (or) $\Pi^\dagger_+$ from the left and (or) right respectively where $\Pi_\pm : \mathbb{H}\to\mathbb{H}^\pm$ are projectors and the adjoint operators $\Pi^\dagger_\pm : \mathbb{H}^\pm\to\mathbb{H}$ are embedding operators.

All these blocks can contain the following small factors: $\hbar$ arising from the commutation and surely contained in W^0_{jk} and (or) μ^{-1} (there are at least $|j-k|$ copies of this factor); moreover, there is surely a factor $(m+1)^{-1}$ and under conditions $(17.2.28)_{1-3}$ there is surely a factor $\bar\nu$. Therefore

$$W_{jk} = \bar\nu(m+1)^{-1}\mu^{-|j-k|}\big((1+j+k)\mu h\big)^{\frac12|j-k|}\tilde W_{jk}, \tag{17.4.14$_1$}$$

$$W^I_{jk} = \bar\nu(m+1)^{-1}h\mu^{-|j-k|-1}\big((1+j+k)\mu h\big)^{\frac12|j-k|}\tilde W^I_{jk} \tag{17.4.14$_2$}$$

where $W_{\dots}$ and $\tilde W_{\dots}$ here and below are $\boldsymbol{h}$-pseudodifferential operators uniformly with respect to all parameters[3)]. Moreover, for $j=k$ the operator W_{kk} surely contains either a factor $\hbar=\mu^{-1}h$ or μ^{-2} and therefore

$$W_{kk} = \bar\nu(m+1)^{-1}\mu^{-2}\big((1+j+k)\mu h\big)\tilde W_{kk}. \tag{17.4.14$_3$}$$

(ii) For $d=3$, $\mathsf{D}=4$ the blocks of $\mathcal{A}_0$ are

$$\begin{aligned}\mathcal{A}_{0(j)} = \sum_{0\le n\le N}&\big(B_n(x',\hbar D_1,\hbar)\mu^{-2n}(2j\mu h)^n\times\\ &(\sqrt{2j\mu h}\sigma_2 + hD_2\sigma_3)\big)^{\mathsf{w}} + \sigma_0 m + \bar W(x',\hbar D_1,\hbar)I \quad (j\ge1),\end{aligned} \tag{17.4.15$_j$}$$

$$\mathcal{A}_{0(0)} = \big(B_0(x',\hbar D_1,\hbar)\sigma'_3 hD_2\big)^{\mathsf{w}} + \sigma'_0 m, \tag{17.4.15$_0$}$$

and the blocks of $\mathcal{A}_1$ are

$$\begin{aligned}\mathcal{A}_{1(j)} = \sum_{0\le n\le N}&B_n(x',\hbar D_1,\hbar)\mu^{1-2n}h\times\\ &\big(\mathcal{P}'_{n-1}(2j\mu h)\sigma_1 + \mathcal{P}''_{n-1}(2j\mu h)\sigma_2\big)\sqrt{2j\mu h}+\\ &\sum_{0\le n\le N'} B'_n(x',\hbar D_1,\hbar)\mu^{-1-2n}h(2j\mu h)^n\sigma_4 \quad (j\ge1),\end{aligned} \tag{17.4.16$_j$}$$

$$(17.4.16)_0 \qquad \mathcal{A}_{1(0)} = B'_0(x', \hbar D_1, \hbar)\mu^{-1}h\sigma'_4$$

respectively, where σ'_0, σ'_3 and σ'_4 are the restrictions of σ_0, σ_3 and σ_4 to $\mathbb{H}^-$, σ_4 is defined in (17.2.2), $\mathcal{A}_2$ is defined by (17.4.12) and its blocks are

$$\mathcal{A}_{2,jk} = W_{jk} \cdot I + \sum_{l=1,2,3} W^l_{jk}\hat{\sigma}_l, \tag{17.4.17}$$

where W_{jk} and W^l_{jk} satisfy $(17.4.14)_{1-3}$ with $\bar{\nu} = 1$ and for $j = 0$ and (or) $k = 0$ the right-hand expression of (17.4.17) has an additional factor Π_- and (or) $\Pi^\dagger_+$ from the left and (or) right respectively.

(iii) Similar formulae are valid for $d = 3$, $\mathsf{D} = 2$ but in $(17.4.15)_j$, $(17.4.15)_0$ the terms $\sigma_0 m$ and $\sigma'_0 m$ are absent:

$$(17.4.18)_j \quad \mathcal{A}_{0(j)} = \sum_{0\le n\le N} \big(B_n(x', \hbar D_1, \hbar)\mu^{-2n}(2j\mu h)^n \times (\sqrt{2j\mu h}\sigma_2 + hD_2\sigma_3)\big)^{\mathsf{w}} + \bar{W}(x', \hbar D_1, \hbar)I \quad (j \ge 1),$$

$$(17.4.18)_0 \qquad \mathcal{A}_{0(0)} = \big(B_0(x', \hbar D_1, \hbar)\sigma'_3 hD_2\big)^{\mathsf{w}},$$

in $(17.4.16)_j$, $(17.4.16)_0$ the matrices σ_4, σ'_4 are replaced by I

$$(17.4.19)_j \quad \mathcal{A}_{1(j)} = \sum_{0\le n\le N} B_n(x', \hbar D_1, \hbar)\mu^{1-2n}h \times \big(\mathcal{P}'_{n-1}(2j\mu h)\sigma_1 + \mathcal{P}''_{n-1}(2j\mu h)\sigma_2\big)\sqrt{2j\mu h} + \sum_{0\le n\le N'} B'_n(x', \hbar D_1, \hbar)\mu^{-1-2n}h(2j\mu h)^n I \quad (j \ge 1),$$

$$(17.4.19)_0 \qquad \mathcal{A}_{1(0)} = B'_0(x', \hbar D_1, \hbar)\mu^{-1}hI$$

and in (17.4.17) the matrices $\hat{\sigma}_l$ are replaced by σ_l

$$\mathcal{A}_{2,jk} = W_{jk} \cdot I + \sum_{l=1,2,3} W^l_{jk}\sigma_l. \tag{17.4.20}$$

So far for $d = 3$, $W_{jk} = W_{jk}(x', \hbar D', \hbar)$ but later we will get rid of $\hbar D_2$.

17.4.2 Further Reduction

Thus we have the problem

$$\Big(hD_t - (m+1)(\mathcal{A}_{0(j)} + \mathcal{A}_{1(j)})\Big)\mathcal{U}_{jl} - (m+1)\sum_{|k-j|\le N} \mathcal{A}_{2,jk}\mathcal{U}_{kl} \equiv 0, \tag{17.4.21}$$

$$\mathcal{U}_{jl}|_{t=0} = \delta_{jl}\delta(x'-y')I \tag{17.4.22}$$

with $N = N(s,\delta)$.

Formulae (17.4.8), (17.4.7) and the uniform (with respect to all parameters[3)]) ellipticity of the operator $\big(hD_t - (m+1)(\mathcal{A}_{0(j)} + \mathcal{A}_{1(j)})\big)$ for the τ, j and ξ_2 (for $d=3$) indicated below and the fact that the operator $\mathcal{A}_{2,jk}$ surely contains the factor $(m+1)^{-1}\mu^{-1}$ yield the following

Proposition 17.4.1. *Let condition* (17.4.1) *be fulfilled with* $\epsilon_1 = \epsilon_1(c,\epsilon_0,s)$. *Then,*

(i) The estimate

$$|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)\mathcal{U}_{jk}| \le C'Th^s(1+j\mu h)^{-s} \tag{17.4.23}$$
$$\forall x',y' \in \mathbb{R}^{d-1} \quad \forall \tau : |\tau| \le \epsilon = \epsilon(c) \quad \forall j \ge C_0\mu^{-1}h^{-1}$$

holds with arbitrary $T \ge h^{1-\delta}$, T', s.

(ii) Moreover, if $d=2$ *and condition* (17.3.30) *is fulfilled then this estimate also holds for all* $j \le \epsilon\mu^{-1}h^{-1}$ *with a small enough constant* $\epsilon = \epsilon(c,\epsilon_0) > 0$.

(iii) If $d=3$, *condition* (17.3.30) *is fulfilled and* $Q = q(x', h'D')$ *with* q *is supported in the domain* $\{|\xi_2| \le \epsilon\}$ *with a small enough constant* $\epsilon = \epsilon(c,\epsilon_0) > 0$ *then the estimate*

$$|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)Q_x\mathcal{U}_{jk}| \le C'Th^s \tag{17.4.24}$$
$$\forall x',y' \in \mathbb{R}^{d-1} \quad \forall \tau : |\tau| \le \epsilon \quad \forall j \le \epsilon\mu^{-1}h^{-1}$$

holds with arbitrary $T \ge h^{1-\delta}$, T', s.

(iv) On the other hand, if $d=3$ *and* q *is supported in the domain* $\{|\xi_2| \ge \frac{1}{2}C_0\}$, *then estimate* (17.4.24) *holds for all* $j \in \mathbb{Z}^+$.

In what follows $\tilde{\Omega}$ denotes the domain Ω for $d=2$ and the domain $\Omega \times \{\xi_2, |\xi_2| \le C_0\}$ for $d=3$.

Proposition 17.4.2. *(i) There exist operators* $\mathcal{T}'_{(j)} = \mathcal{T}'_{(j)}(x', \boldsymbol{h}'D', \mu^{-1}, \boldsymbol{h}')$ *with* $j \in \mathcal{J} = \{\epsilon\mu^{-1}h^{-1} \le j \le C_0\mu^{-1}h^{-1}\}$ *such that in* $\tilde{\Omega}$ *the following equalities hold:*

$$\mathcal{T}'^{*}_{(j)}\mathcal{T}'_{(j)} \equiv \mathcal{T}'_{(j)}\mathcal{T}'^{*}_{(j)} \equiv I, \tag{17.4.25}$$

$$(m+1)\mathcal{T}'^{*}_{(j)}(\mathcal{A}_{0(j)} + \mathcal{A}_{1(j)})\mathcal{T}'_{(j)} \equiv \mathcal{B}_{(j)} = \begin{pmatrix} \mathcal{B}^{+}_{(j)} & 0 \\ 0 & \mathcal{B}^{-}_{(j)} \end{pmatrix} \tag{17.4.26}$$

where $\mathcal{B}^{\pm}_{(j)}$ *are operators with principal symbols* $(m+1)\big(\bar{W}(x', \xi') \pm L_{(j)}(x', \xi')\big)I$,

$$\bar{W} = V \circ \bar{\Psi}, \tag{17.4.27}$$

$$L_{(j)} = \big(m^2 + 2j\mu h\bar{B}^2\big)^{\frac{1}{2}} \qquad \text{for } \ d = 2, \tag{17.4.28}_{(2)}$$

$$L_{(j)} = \big(m^2 + (2j\mu h + \xi_2^2)\bar{B}^2\big)^{\frac{1}{2}} \qquad \text{for } \ d = 3, \tag{17.4.28}_{(3)}$$

$$\bar{B} = \sum_{0 \le n \le N} B_n(x', \xi', 0)(2j\mu h)^n; \tag{17.4.29}$$

(ii) Moreover, the inequalities

$$|\mathcal{D}^{\alpha}D_z^{\beta}(\mathcal{T}'_{(j)} - I)| \le c_{\alpha\beta}\bar{\nu}_{\alpha}(m+1)^{-1}, \tag{17.4.30}$$

$$\begin{aligned} |\mathcal{D}^{\alpha}D_z^{\beta}\big(\mathcal{B}_{(j)} - \sigma_0 m(m+1) - (m+1)\bar{W}\cdot I\big)| \le \\ c_{\alpha\beta}\bar{\nu}_{\alpha}(m+1)^{-1} \quad \forall \alpha, \beta : |\alpha| \le K', \beta \le K' \end{aligned} \tag{17.4.31}$$

hold where here and below $\mathcal{D} = \nabla_{(x,\xi,\boldsymbol{h}',\mu^{-1})}$, $z = 2j\mu h$, j *is now any real number in the interval* $(\epsilon\mu^{-1}h^{-1}, C_0\mu^{-1}h^{-1})$, $\bar{\nu}_{\alpha} = 1$ *for* $\alpha = 0$ *and* $\bar{\nu}_{\alpha} = \bar{\nu}$ *for* $|\alpha| \ge 1$ *and we assume that conditions* $(17.2.28)_{1-3}$ *are fulfilled for* $d = 2$ *and* $\bar{\nu} = 1$ *for* $d = 3$.

Remark 17.4.3. In all the inequalities below $j, k \in \mathbb{R}^+$ (i.e., all the functions of $j, k \in \mathbb{Z}^+$ are extended to functions of $j, k \in \mathbb{R}^+$ satisfying the indicated inequalities); here $\mathcal{C}_{jj}$ and $\mathcal{C}_{jk}$ are extended separately.

Proof. The proof is obvious because for $j \ge 1$ the principal symbol of $\mathcal{A}_{0(j)}$ is unitarily equivalent to the matrix $\big(\bar{W}\cdot I + L_{(j)}\sigma_0\big)$ [21)] and the operator realizing this unitary equivalence obviously also satisfies (17.4.30). □

Let us define $\mathcal{T}'_{(j)}$ for all $j \in \mathbb{R}^+$ so that inequalities (17.4.30)–(17.4.31) are fulfilled for all j and so that $\mathcal{T}'_{(j)} = I$ for $j \le \frac{1}{2}\epsilon(\mu h)^{-1}$ and for $j \ge 2C_0(\mu h)^{-1}$. Let $\mathcal{T}' = \big(\mathcal{T}'_{(j)}\big)_{j\in\mathbb{Z}^+}$ be an operator the of block-diagonal type. Let us set $\mathcal{A}' = \mathcal{T}'^{*}\mathcal{A}\mathcal{T}'$.

Proposition 17.4.4. *There exists an operator $\mathcal{K} = \big(\mathcal{K}_{jk}\big)_{j,k\in\mathbb{Z}^+}$ such that in $\tilde{\Omega}$*

$$\mathcal{K}^*\mathcal{K} \equiv \mathcal{K}\mathcal{K}^* \equiv I, \tag{17.4.32}$$

$$(m+1)\mathcal{K}^*\mathcal{A}'\mathcal{K} \equiv \mathcal{B} + \mathcal{B}' + \mathcal{R}, \tag{17.4.33}$$

$$\mathcal{R}_{jk} \equiv 0 \qquad \forall (j,k) \in \mathcal{J}\times\mathbb{Z}^+ \cup \mathbb{Z}^+\times\mathcal{J} \tag{17.4.34}$$

where $\mathcal{B}$ is an operator of the block-diagonal type with blocks $\mathcal{B}_{(j)}$,

$$\mathcal{B}'_{jk} = \begin{pmatrix} \mathcal{B}'^{+}_{jk} & 0 \\ 0 & \mathcal{B}'^{-}_{jk}\end{pmatrix} \tag{17.4.35}$$

and for $|\alpha| \le K'$, $\beta \le K'$

$$|\mathcal{D}^\alpha D^\beta_{(z)}\mathcal{K}_{jk}| \le c_{\alpha\beta}\mu^{-|j-k|}\bar{\nu}(m+1)^{-1} \qquad \text{for } j \ne k, \tag{17.4.36}$$

$$|\mathcal{D}^\alpha D^\beta_{z}(\mathcal{K}_{jj} - I)| \le c_{\alpha\beta}\mu^{-2}\bar{\nu}(m+1)^{-1} \qquad \text{for } j = k, \tag{17.4.37}$$

$$|\mathcal{D}^\alpha D^\beta_{(z)}\mathcal{B}'_{jk}| \le c_{\alpha\beta}\mu^{-|j-k|}\bar{\nu}, \tag{17.4.38}$$

$$|\mathcal{D}^\alpha D^\beta_{(z)}\mathcal{B}'_{jj}| \le c_{\alpha\beta}\mu^{-2}\bar{\nu} \tag{17.4.39}$$

where we set $z = 2j\mu h$, $z' = 2k\mu h$, $j,k \in \big(\epsilon(\mu h)^{-1}, C_0(\mu h)^{-1}\big)$ are not necessarily integers and $D_{(z)} = D_z + D_{z'}$.

Proof. By induction on M we construct an operator $\mathcal{K} = \mathcal{K}_M$ satisfying (17.4.32), (17.4.33), (17.4.35)–(17.4.39) and with $\mathcal{R}$ such that

$$|\mathcal{D}^\alpha D^\beta_{(z)}\mathcal{R}_{jk}| \le c_{\alpha\beta}\mu^{-\max(M,|j-k|)}\bar{\nu} \tag{17.4.40}_M$$
$$\forall (j,k) \in \mathcal{J}'\times\mathbb{R}^+ \cup \mathbb{R}^+\times\mathcal{J}', \quad \mathcal{J}' = \big(\epsilon(\mu h)^{-1}, C_0(\mu h)^{-1}\big).$$

[21)] Where here and below for $d = 3, D = 2$ the notation σ_0 is used instead of σ_4.

The latter inequality with large M provides (17.4.34). Obviously, one can set $\mathcal{K}_1 = I$. Let us assume that $(17.4.40)_M$ is fulfilled and let us set $\mathcal{K}_{M+1} = \mathcal{K}_M \exp i\mathcal{L}$ where the operator $\mathcal{L}$ is Hermitian, skew-commutes with σ_0 (i.e., $\mathcal{L} = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$) and the symbols of its blocks satisfy the inequalities

$$|\mathcal{D}^\alpha D^\beta_{(z)} \mathcal{L}_{jk}| \le c_{\alpha\beta}\mu^{-|j-k|}\bar\nu(m+1)^{-1}, \tag{17.4.41}$$

$$|\mathcal{D}^\alpha D^\beta_{z} \mathcal{L}_{jj}| \le c_{\alpha\beta}\mu^{-2}\bar\nu(m+1)^{-1} \tag{17.4.42}$$

and their principal symbols are defined by the equations

$$i\sigma_0(L_{(j)} + L_{(k)})(x',\xi')\mathcal{L}^0_{jk} = -(m+1)^{-1}\mathcal{R}^0_{jk} \qquad \forall (j,k) \in \mathcal{J}\times\mathbb{Z}^+ \cup \mathbb{Z}^+\times\mathcal{J} \tag{17.4.43}$$

where $\mathcal{R}^0_{jk}$ are the principal symbols of $\mathcal{R}_{jk}$. One can assume without any loss of the generality that they also skew-commute with σ_0 [22)] and the operator $\mathcal{R}$ is Hermitian.

It is easy to prove that $\mathcal{K}_{M+1}$ also satisfies (17.4.32), (17.4.33), (17.4.35), (7.4.31) and $(17.4.40)_{M+1}$. □

Remark 17.4.5. Here and in what follows the principal symbols of operators can contain the factors μ^{-1} and h separately.

Let us set $\mathcal{U}' = \mathcal{K}^*_x \mathcal{T}'^*_x \mathcal{U}\, {}^t\mathcal{T}'^*_y\, {}^t\mathcal{K}^*_y$. Then in $\tilde\Omega$

$$(hD_t - \mathcal{B}_{(j)})\mathcal{U}'_{jl} - \sum_{|j-k|\le N} \mathcal{B}'_{jk}\mathcal{U}'_{kl} \equiv 0, \tag{17.4.44}$$

$$\mathcal{U}'_{jl}|_{t=0} \equiv \delta_{jl}\delta(x'-y')I \tag{17.4.45}$$

for $j \in \mathcal{J}$. Let us decompose $\mathcal{U}'_{jl}$ into four blocks $\mathcal{U}'^{\zeta\zeta'}_{kl}$ with $\zeta,\zeta' = \pm$. Then (17.4.26), (17.4.35), (17.4.34) yields that system (17.4.44) is decomposed into four separate subsystems.

Further, if condition (17.3.30) is fulfilled then two of them are elliptic in the zone in question. Namely, subsystems with ζ such that

$$\zeta V > \epsilon_0 \qquad \text{in} \quad B(0,1) \tag{17.4.46}_\zeta$$

are elliptic. Taking into account that $\mathcal{U}'_{jk}$ also satisfies (17.4.6) and (17.4.7) we then obtain the following

[22)] Otherwise the part of $\mathcal{R}$ commuting with σ_0 can be included in $\mathcal{B}'$.

Proposition 17.4.6. *Under conditions* (17.3.30), $(17.4.46)_\zeta$ *the distribution* $\mathcal{U}'^{\zeta,}_{jk}$ *satisfies estimate* (17.4.24) *for all* $j \in \mathcal{J}$ *and* $\mathcal{U}'^{,\zeta}_{jk}$ *satisfies* (17.4.24) *for all* $k \in \mathcal{J}$.

Let us recall that the symbol of Q *is supported in* $\tilde{\Omega}$.

So we need to treat the problem

$$(hD_t - \mathcal{B}^+_{(j)})\mathcal{U}''_{jl} - \sum_{|k-j|\le N} \mathcal{B}'^+_{jk}\mathcal{U}''_{kl} \equiv 0, \tag{17.4.47}_+$$

$$\mathcal{U}''_{jk}|_{t=0} \equiv \delta_{jk}\delta(x'-y')I \tag{17.4.48}$$

where for definiteness we have assumed that condition $(17.4.46)_-$ is fulfilled and we set $\mathcal{U}''_{jk} = \mathcal{U}'^{++}_{jk}$.

Proposition 17.4.7. *There exists an operator* $\mathcal{K}' = (\mathcal{K}'_{jk})_{j,k\in\mathbb{Z}^+}$ *with scalar blocks satisfying* (17.4.32) *such that*

$$\mathcal{K}'^*(\mathcal{B}^+ + \mathcal{B}'^+ + \mathcal{R})\mathcal{K}' \equiv \mathcal{B}^+ + \mathcal{B}'' + \mathcal{R}' \tag{17.4.49}$$

where $\mathcal{R}'$ *satisfies* (17.4.34),

$$\mathcal{K}_{jk} \equiv 0 \qquad \text{for } |j-k| \ge N(1+\mu^{-2}h^{-1-\delta'}), \tag{17.4.50}$$

$$|\mathcal{D}^\alpha D^\beta_{(z)}\mathcal{B}''_{jk}| \le c_{\alpha\beta}\mu^{-|j-k|}\bar{\nu}\mu h \qquad \text{for } j \ne k \tag{17.4.51}$$

and $\mathcal{B}''_{jj}$ *satisfies* (17.4.39), $N = N(s,\delta,\delta')$ *and* $\delta' > 0$ *is arbitrary.*

Proof. We again apply induction on M fulfilling inequality $(17.4.40)_M$ at every step.

For $D = 2$ all the blocks are one-dimensional and for $D = 4$ they are two-dimensional but every block $\mathcal{B}'^+_{jk}$ is the sum of a scalar operator $\mathcal{B}'^+_{jkI}$ and an operator $\mathcal{B}'^+_{jkII}$ satisfying (17.4.51).

Let us again set $\mathcal{K}'_{M+1} = \mathcal{K}'_M \exp(i\mathcal{L}')$ where $\mathcal{L}'$ is a Hermitian operator with scalar blocks $\mathcal{L}'_{jk}$ such that

$$|\mathcal{D}^\alpha D^\beta_{(z)}\mathcal{L}'_{jk}| \le c_{\alpha\beta}\mu^{-\max(M,|j-k|)}\bar{\nu}(\mu h)^{-1}, \tag{17.4.52}$$

$\mathcal{L}'_{jj} = 0$ and the principal symbols of $\mathcal{L}'_{jk}$ are defined by the equations

$$i\big(L_{(j)} - L_{(k)}\big)\mathcal{L}'^0_{jk} + (m+1)^{-1}\mathcal{B}'^{+0}_{jkI} = 0 \qquad \text{for } j \ne k. \tag{17.4.53}$$

We see that $\mathcal{L}_{jk}$ satisfy (17.4.51).

On the other hand, even without equality (17.4.53) one can prove easily that if $\mathcal{L}'$ is a Hermitian operator with scalar blocks and $\mathcal{Q}_{(l)} = (\mathsf{Ad}\,\mathcal{L}')^l\mathcal{B}^+$, $\mathcal{Q}'_{(l)} = (\mathsf{Ad}\,\mathcal{L}')^l\mathcal{B}'^+_l$, $\mathcal{Q}'_{(l)} = (\mathsf{Ad}\,\mathcal{L}')^l\mathcal{B}''^+_{ll}$ then

$$|\mathcal{D}^\alpha D^\beta_{(z)}\mathcal{Q}_{(l),jk}| \le c_{\alpha\beta l}\mu^{-\max(M+l-1,|j-k|)}\bar\nu, \tag{17.4.54}$$

$$|\mathcal{D}^\alpha D^\beta_{(z)}\mathcal{Q}'_{(l),jk}| \le c_{\alpha\beta l}\mu^{-\max(M+l,|j-k|)}\bar\nu; \tag{17.4.55}$$

moreover, the second estimate with an additional factor μh in the right-hand expression holds for $\mathcal{Q}''_{(l),jk}$.

This yields that if $\mathcal{K}'_{M+1} = \exp(i\mathcal{L}')$ then

$$\mathcal{K}'^*_{(M+1)}(\mathcal{B}^+ + \mathcal{B}''^+ + \mathcal{R}_M)\mathcal{K}'_{(M+1)} = \mathcal{B}^+ + \mathcal{B}''^+ + i[\mathcal{B}^+, \mathcal{L}'] + \mathcal{R}'_{M+1} \tag{17.4.56}$$

where $\mathcal{R}'_{M+1}$ satisfies $(17.4.40)_{M+1}$.

Let us note that if we define $\mathcal{L}'$ by formula (17.4.53) then $(\exp(it\mathcal{L}'))_{jk} \equiv 0$ for $|j-k| > N$, $|t| \le \min(1, \mu^{1+M}h^{1+\delta'})$ and therefore $\mathcal{K}'_{(M+1)jk} \equiv 0$ for $|j-k| \ge N(1+\mu^{-1-M}h^{-1-\delta'})$.

This fact and (17.4.56) yield that $\mathcal{K}'_{M+1} = \mathcal{K}'_M\mathcal{K}'_{(M+1)}$ provides the induction step. □

Proposition 17.4.8. *There exists an operator $\mathcal{K}'' = (\mathcal{K}''_{jk})_{j,k\in\mathbb{Z}^+}$ which satisfies* (17.4.32)*, $\mathcal{K}''_{jk} = 0$ for $|j-k| \ge N$, and*

$$\mathcal{K}'^*(\mathcal{B}^+ + \mathcal{B}'' + \mathcal{R}')\mathcal{K}' \equiv \mathcal{B}^+ + \mathcal{B}''' + \mathcal{R}'' \tag{17.4.57}$$

where $\mathcal{R}''$ satisfies (17.4.34)*,*

$$|\mathcal{D}^\alpha D^\beta_{(z)}\mathcal{K}''_{jk}| \le c_{\alpha\beta}\mu^{-|j-k|}\bar\nu \qquad \text{for } j \ne k \tag{17.4.58}$$

$$|\mathcal{D}^\alpha D^\beta_z\mathcal{K}''_{jj}| \le c_{\alpha\beta}\mu^{-2}\bar\nu \qquad \text{for } j = k; \tag{17.4.59}$$

$\mathcal{B}'''$ is operator of the block-diagonal type and satisfies (17.4.39)*.*

Proof. The proof is similar to the proof of the previous proposition but now we have matrix operators $\mathcal{L}''_{jk}$ for $\mathsf{D} = 4$ which satisfy (17.4.52) with an additional factor μh in the right-hand expression; in particular, all of them are $O(\mu^{-1})$. □

Let us now set $\mathcal{U}''' = \mathcal{K}'''^{*}_{x}\mathcal{U}''\,{}^{t}\mathcal{K}'''^{*}_{y}$ with $\mathcal{K}''' = \mathcal{K}'\mathcal{K}''$. Then in $\tilde{\Omega}$

$$\big(hD_t - \mathcal{C}_{(j)}\big)\mathcal{U}'''_{jl} \equiv 0 \quad \forall j \in \mathcal{J}, \tag{17.4.60}$$

$$\mathcal{U}_{jl}|_{t=0} \equiv \delta_{jl}\delta(x'-y')I \tag{17.4.61}$$

where $\mathcal{C}_{(j)} = \mathcal{B}^{+}_{(j)} + \mathcal{B}'''_{(j)}$. The reduction is complete.

17.4.3 Tauberian Remainder Estimates

Let $H_{(j)} = H_{(j)}(x', \xi', \mu h, \mu^{-1}, \boldsymbol{h}')$ be the Weyl symbols of the operators $\mathcal{C}_{(j)}$. Our constructions yield that $H_{(j)}$ are smooth functions of $x', \xi', z = 2j\mu h, \mu^{-1}, \boldsymbol{h}'$ and

$$H_{(j)} = (m+1)\big(\bar{W} + L_{(j)}\big) + O\big(\mu^{-2}\bar{\nu}\big). \tag{17.4.62}$$

One can see easily that for $d = 2$ the operators $\mathcal{F}$ and $\mathcal{T}$ depend on μ and h only through $\mu^{-1}h$ and μ^{-2} and the operators are smooth functions of those arguments. Therefore the same is true for the operators W_{jk} and for $H_{(j)}$ as well. In particular,

$$H_{(j)} = (m+1)\big(\bar{W} + L_{(j)}\big) + \omega'_1\mu^{-2} + \omega'_2\mu^{-1}h + O\big(\bar{\nu}\mu^{-4} + \bar{\nu}\mu^{-2}h^2\big) \qquad \text{as } \ d = 2 \tag{17.4.63}$$

with yet unknown coefficients ω'_1 and ω'_2.

Repeating the corresponding arguments of Section 13.5 we see that the following statements similar to Theorem 13.5.1 and Propositions 13.5.2–13.5.3, 13.5.6 are true:

Theorem 17.4.9. *Let conditions* (17.3.30) *and* (17.4.1) *be fulfilled and let* q' *be supported in* $\{|\xi_2| \le 2\epsilon\}$ *for* $d = 3$. *Then*

(i) Let $|H_{(j)}| \ge \epsilon_0$ *for all* $(x, \xi') \in \tilde{\Omega}$. *Then for* $\mathcal{U}'''_{jk}$ *estimate* (17.4.24) *holds for* $|\tau| \le \epsilon$.

(ii) For $T \in [h^{1-\delta}, T^*]$ *with* $T^* = \mu\epsilon_0$ *for* $d = 2$ *and* $T^* = \epsilon_0$ *for* $= 3$, *estimate* (17.4.24) *holds for* $\mathcal{U}'''_{jk}$ *with* $j \ne k$; *therefore this estimate holds for* $\mathcal{U}_{jk}$ *with* $|j-k| \ge N(1 + \mu^{-2}h^{-1-\delta'})$ *as well; recall that* $T_0 = T_0(c, _0\,\epsilon) > 0$ *and* $\epsilon = \epsilon(c, \epsilon_0) > 0$ *are small enough constants.*

(iii) Let either $d = 2$ *and one of the conditions* $(13.4.7)_{1,2}$ *be fulfilled* or $d = 3$ *and one of the conditions* $(13.4.7)_{1-4}$ *be fulfilled.*

Then for $T \in [h^{1-\delta}, T^*]$ *the following estimate holds:*

$$|F_{t\to h^{-1}\tau}\chi_T(t)Q'_x\mathcal{U}'''_y\,{}^tQ''_{jk}| \le C'h^s \qquad \forall x', y' \in \mathbb{R}^{d-1} \quad \forall \tau : |\tau| \le \epsilon; \tag{17.4.64}$$

therefore this estimate holds for $\mathcal{U}_{jk}$ *as well.*

(iv) Let $d = 2$ *and let conditions* $(17.2.28)_{1-3}$, $(13.4.12)_{1,2}$ *and* (17.3.26) *be fulfilled. Then all the previous and the following statements remain true for* $T^* = \mu\nu^{-1}T_0$, $T \in [h^{1-\delta}\nu^{-1}, T^*]$ *and* $|\tau| \le \nu$.

(v) Under the assumptions of Proposition 13.5.6 (with conditions $(13.4.26)_{(2)}$ *or* $(13.4.26)_{(3)}$ *and* $(13.4.31)_{(2)}$ *or* $(13.4.31)_{(3)}$ *replaced by* (17.3.27)*) estimate* (13.5.36) *holds for* $\mathcal{U}'''$.

17.4.4 Calculations

Preliminary arguments

Let us pass from estimates to calculations. Let us note first of all that the operator $q = q(x, h'D', h')$ in the representation $\mathfrak{R}$ is transferred to the block-operator $Q = (Q_{jk})_{j,k\in\mathbb{Z}^+}$ where the symbols of the operators Q_{jk} satisfy the estimates

$$|\mathcal{D}^\alpha D^\beta_{(z)} Q_{jk}| \le c_{\alpha\beta}\mu^{-|j-k|} \qquad \forall j, k \in \mathcal{J}'. \tag{17.4.65}$$

Moreover, one can decompose these symbols into asymptotic series in μ^{-2} and h'. One can prove easily that if $\mathcal{M}$ is one of the operators $\mathcal{T}'$, $\mathcal{K}$, $\mathcal{K}'$, $\mathcal{K}'''$ constructed in Propositions 17.4.2, 17.4.4, 17.4.7 and 17.4.8 respectively (or if $\mathcal{M}$ is a product of these operators) then the operator $\mathcal{M}^*Q\mathcal{M}$ also has these properties.

Therefore using the arguments used in the proofs of Propositions 13.5.4, 13.5.5 and applying Theorem 17.4.9 we obtain the following

Proposition 17.4.10. *Let conditions* (17.3.30) *and* (17.4.1) *be fulfilled. Then*

(i) Let $d = 2$ *and* $T \in [h^{1-\delta}\nu^{-1}, T^*]$, $Q' = I$ *and let conditions* $(17.2.28)_{1-3}$, $(13.4.12)_{1,2}$ *and* (17.3.26) *be fulfilled. Then*

$$\sigma(t) := F_{t\to h^{-1}\tau}\chi_T(t)\operatorname{tr}\Gamma\big(Q'_x\psi(x)U\big) \equiv F_{t\to h^{-1}\tau}\chi_T(t)\sum_j \Gamma'(f_j\mathcal{U}'''_{jj}) \qquad \forall \tau \in [-\epsilon\nu, \epsilon\nu] \tag{17.4.66}$$

modulo $O(h^s)$ *with the summation over* $j \in \mathcal{J}$,

$$f_j = f(x', \boldsymbol{h}'D', 2j\mu h, \mu^{-2}, \boldsymbol{h}') \tag{17.4.67}$$

where f *is a smooth function of its arguments,*

$$f = f_0 \cdot I + O(\mu^{-2}\bar\nu), \qquad f_0 = \psi \circ \bar\Psi. \tag{17.4.68}_{(2)}$$

(ii) Let $d = 3$, *let* $Q = Q(x', \boldsymbol{h}'D', \boldsymbol{h}')$ *be a scalar operator with symbol supported in* $\{|\xi_2| \le \epsilon\}$, $Q' = \mathcal{F}TQT^*\mathcal{F}^*$ *and* $T \in [h^{1-\delta}, T^*]$. *Then equality* (17.4.66) *holds for* $\tau \in [-\epsilon, \epsilon]$.

Moreover, (17.4.67) *remains true and*

$$f = f_0 \cdot I + O(\mu^{-2} + h), \qquad f_0 = q_0(\psi \circ \bar\Psi) \tag{17.4.68}_{(3)}$$

where q *and* q_0 *are the symbol and the principal symbol of the operator* Q *respectively. Finally,*

$$\operatorname{supp}(f) \subset \operatorname{supp}(q). \tag{17.4.69}$$

From Tauberian to More Explicit Expression

Repeating the corresponding considerations of Section 13.5 we can easily prove that

Proposition 17.4.11. *(i) Under the assumptions of Proposition 13.5.7 and the additional condition* (17.3.30) [23] *for expression*

$$\mathsf{R}^{\mathsf{MW}}_{\infty} := \Big|\int \phi(\tau')\Big(\operatorname{tr}\Gamma\big(\tilde Q'_x\psi'(x)e\big)(\tau', \tau) - \sum_{j\in\mathcal{J}}\sum_{k,p,l\le N}\Theta_{j,kpl}(\tau', \tau)h^{1+l-d}\mu^{-1-2p-l}\Big)\,d\tau'\Big| \tag{17.4.70}$$

[23] And under the condition $\operatorname{supp}(q) \subset \{|\xi_2| \le 2\epsilon\}$ in the case $d = 3$.

estimate (13.5.43) *holds where*

$$\Theta_{j,kpl}(\tau',\tau) \coloneqq \Theta_{j,kpl}(\tau)\big|_{\tau=\tau'}^{\tau=\tau},$$

$\Theta_{j,kpl}$ *are given by formulae* (13.5.44) *with* $H_{(j)}$ *instead of* $\mathcal{A}_j$ *and with* f_{kpl} *satisfying* (13.5.45)–(13.5.46) *are fulfilled and, moreover, equality*

$$f_{000} = (2\pi)^{1-d}\frac{\mathsf{D}}{2}q'(\psi'\circ\bar{\Psi}_0) \tag{17.4.71}$$

holds instead of (13.5.47) *and we assume that* $\phi \in \mathscr{C}_0^K([-\epsilon,\epsilon])$.

(ii) Further, the right hand expression in estimate (13.5.43) $R_{1(0)}$ *is given by* (13.5.41).

(iii) Finally, under the indicated additional conditions in the assumptions of Proposition 13.5.8 estimate (13.5.51) *holds for* $\mathsf{R}^{\mathsf{MW}}_{\infty}$.

This proposition yields that in the framework of Propositions 13.5.10 and 13.5.9 as $d = 2, 3$, with $H_{(j)}$ instead of $\mathcal{A}_j$ and with $Q' = Q(I - Q'')$ instead of $Q' = I - Q''$ for $d = 3$ (provided the symbol of Q is supported in $\{|\xi_2| \le 2\epsilon\}$, $Q \sim I$ in $\{|\xi_2| \le \epsilon\}$) estimates $(13.5.60)_0$ and $(13.5.57)_0$ hold for $d = 2$ and for $d = 3$ respectively; we assume that condition (17.3.30) is fulfilled.

Moreover, applying the results of Section 5.3 to estimate contributions of zone in zone $\{\gamma \le \epsilon\}$ (as in Subsection 13.5.3) we obtain the following

Theorem 17.4.12 [24]**.** *Let conditions* (17.3.30) *and* (17.4.1) *be fulfilled and let* $\phi \in \mathscr{C}_0^K([-\epsilon,\epsilon])$, $\tau = 0$. *Then*

(i) Let $d = 2$, $Q' = I$ *and let conditions* $(17.2.28)_{1-3}$, (17.4.24) *and* $(13.4.12)_{1,2}$ *be fulfilled. Then estimate* (13.5.66) *holds under non-degeneracy condition* (13.5.62) *and estimate* (13.7.64) *holds under weak non-degeneracy condition* $(13.5.63)_m$.

(ii) For $d = 3$ *estimate* $\mathsf{R}^{\mathsf{MW}}_{\infty} \le Ch^{-2}$ *holds under condition* $(13.5.63)_m$ *and estimate* $\mathsf{R}^{\mathsf{MW}}_{\infty} \le Ch^{-2} + C\mu h^{-1-\delta}$ *holds in the general case.*

(iii) In particular, under non-degeneracy condition

$$|\nabla v^{\#}| \ge \epsilon_0 \qquad \text{in } B(0,1) \tag{17.4.72}$$

estimates $\mathsf{R}^{\mathsf{MW}}_{\infty} \le C\mu^{-1}h^{-1}$ *for* $d = 2$ *and* $\mathsf{R}^{\mathsf{MW}}_{\infty} \le Ch^{-2}$ *for* $d = 3$ *hold.*

[24] Cf. Theorem 13.5.16.

Effective asymptotics. I

Now our goal is to pass from estimates for $\mathsf{R}^{\mathsf{MW}}_{\infty}$ to estimates for

$$\mathsf{R}^{\mathsf{MW}} = \Bigl| \iint \phi(\tau')\psi(x)\bigl(\operatorname{tr} e(x,x,\tau',\tau) - h^{-d}\mathcal{N}^{\mathsf{MW}}(x,\tau',\tau,\mu h)\bigr)\,dx d\tau'\Bigr|. \tag{17.4.73}$$

To this end we apply the corresponding arguments of Section 13.5.

Remark 17.4.13. Surely, now the functions $f_{\dots}$ are not necessarily polynomials of $z = 2j\mu h$; however, these functions depend smoothly on z. Taking into account equalities $(17.4.68)_{(2,3)}$ and the fact that f_{000} does not depend on z at all, we easily conclude that the transformations of Section 13.5 and the estimate (13.5.82) for the error of using numerical integration formula also remain true with an additional term in the right-hand expression; this term is due to the non-polynomial character of $f_{\dots}$. This additional term is $O(1)$ for $d = 2$ and $O(h^{-2})$ for $d = 3$ and, moreover, it is $O(h^s)$ under condition (13.4.1).

Surely, for $d = 3$ we cannot set $Q' = I$; we can only take Q' with the symbol supported in $\{|\xi_2| \le 2\epsilon\}$. In order to avoid this difficulty we temporarily assume that condition (13.4.1) is fulfilled; the case $\mu \in [h^{1-\delta}, \epsilon h^{-1}]$ will be treated later. Then for the operator $(I - Q')$ with the symbol supported in $\{|\xi_2| \ge \epsilon\}$ the results of Section 17.3 apply and therefore we obtain the corresponding estimate for expression

$$\begin{aligned}\Bigl| \iint \phi(\tau)\phi_1(\tau')\Bigl(\int \psi(x)\bigl(\operatorname{tr} e(x,x,\tau',\tau) - h^{-d}\mathcal{N}^{\mathsf{MW}}(x,\tau',\tau)\bigr)\,dx \\ h^{-d}G_0(\tau,\mu^{-1}) - h^{1-d}G_1(\tau,\mu^{-1}) - h^{-d}G_2(\tau,\mu h) - h^{1-d}G_3(\tau,\mu h)\Bigr)\,d\tau d\tau'\Bigr|\end{aligned} \tag{17.4.74}$$

(i.e., for $Q' = I$) where G_0, G_1 are polynomials with respect to μ^{-1} and G_2, G_3 are polynomials with respect to μh.

Then applying arguments similar to those of Subsection 13.5.4 we easily prove that these polynomials vanish. In this proof we start with the results of Section 17.3 and the equality

$$\iint \phi_0(\tau)\phi(\tau')e(.,.,\tau',\tau)d\tau' d\tau = \int \Phi(\tau)d_\tau e(.,.,\tau_1,\tau)$$

where the choice of τ_1 does not matter and $\Phi \in \mathscr{C}_0^K([-\epsilon,\epsilon])$ is a primitive of the function $k_0\phi(\tau) - k\phi_0(\tau)$ with constants $k_\cdot = \int_{\mathbb{R}} \phi_\cdot(\tau)d\tau$.

As a result we obtain the following series of statements where *the following notational convention is used:* $^\#$ *means that* V *and* $v = V/F$ *are replaced by* $V^\# = m^2 - V^2$ *and* $v^\# = V^\#/F$ *respectively and* $(2j+1)\mu h$ *is replaced by* $2j\mu h$*:*

Theorem 17.4.14 [25]**.** *Let* $d = 3$, $\vartheta = 0$, $\tau = 0$ *and let conditions* (17.1.1)–(17.1.4), (13.3.1) *and* (17.3.30) *be fulfilled. Then estimate* $\mathsf{R}^{\mathsf{MW}} \le Ch^{-2} + C'h^s$ *holds under the additional assumption* (13.4.1)*.*

Theorem 17.4.15 [26]**.** *Let* $d = 2$, $\tau = 0$, $\vartheta = 0$ *and conditions* (17.1.1)–(17.1.4), (17.2.1), (17.3.30) *and* (17.4.1) *be fulfilled. Then*

(i) For $\mu^3 h \le C_0$ *estimate* $(13.5.107)^\#$ *holds. Moreover, under condition* $(13.5.102)^\#$ *estimate* (13.5.108) *holds and under condition* $(13.5.109)^\#$ *estimate* $(13.5.89)_{(2)}$ *holds.*

(ii) For $\mu^3 h \ge C_0$ *estimate* $(13.5.110)^\#$ *holds. Moreover, under condition* $(13.5.102)^\#$ *estimate* (13.5.111) *holds, under condition* $(13.5.109)_+{}^\#$ *estimate* $\mathsf{R}^{\mathsf{MW}} \le C\mu^{-1}h^{-1} + C'$ *holds and under condition* $(13.5.109)^\#$ *estimate* (13.5.111) *holds.*

(iii) Moreover, for $\mu^3 h \ge C_0$ *estimate* (13.5.119) *holds for*

$$\mathsf{R}^{\mathsf{MW}\prime} = \Big|\int \phi(\tau')\Big(\int \psi(x)\big(\operatorname{tr} e(x,x,\tau',0) - h^{-2}\mathcal{N}^{\mathsf{MW}}(x,\tau',0,\mu h)\big)\,dx - \sum_{x_l\in\mathfrak{M}} \frac{1}{\mu h}\Delta(x_l)\Big)\,d\tau'\Big| \tag{17.4.75}$$

where $\mathfrak{M}$ *is the set of saddle points of the function* $v^\#$ *and at every point* $\bar{x} \in \mathfrak{M}$, $\Delta(\bar{x})$ *is defined by formulae* $(13.5.116)^\#$ *with the additional factor* $\frac{\mathsf{D}}{2}$ *in the right-hand expression,* $(13.5.117)^\#$ *and*

$$\bar{\tau} = v^\#(\bar{x}) + 2j\mu h, \quad \bar{\tau}' = \bar{\tau} + \bar{\rho}_1\mu^{-2} + \bar{\rho}_2\mu^{-1}h, \quad \bar{\rho}_k = -2\bar{\omega}_k V^\# \tag{17.4.76}$$

where we do not yet know ω_1, ω_2*.*

[25] Cf. Theorem 13.5.23.

[26] Cf. Theorems 13.5.26, 13.5.28.

Theorem 17.4.16 [27]**.** *(i) Under the assumptions of Theorem 17.4.15 estimate* (13.5.60) *with* $\vartheta = 0$ *holds for expression* (17.4.74).

(ii) Under the additional condition (17.4.1) *both Statements (i) and (ii) of Theorem 13.5.31 remain true.*

Theorem 17.4.17. *Under additional condition* (17.4.1) *the statements of theorems 13.5.32, 13.5.33, 13.5.36 and Remark 13.5.34 remain true with obvious modifications. Without this condition they remain true only with an additional term* $O(\bar{\nu})$ *in the right-hand expression.*

Remark 17.4.18. One can see easily that for $\vartheta = 0$ all the above results hold without condition (13.4.1); the only exceptions are the effective asymptotics for $d = 3$ where we applied the results of the previous case in the zone $\{|\xi_2| \ge \epsilon_0\}$.

Problem 17.4.19. Problem 13.5.35, albeit now for the generalized Dirac and Schrödinger-Pauli operators.

Effective asymptotics. II

In order to finish our analysis in the case $d = 2$ we need to calculate the coefficients ω_1, ω_2 at a critical point of the function $v^{\#}$. To this end let us consider the estimate

$$(m+1)\|Au\| \ge \eta\mu^{-1}h\|u\| \qquad \forall u \in \mathscr{C}_0^K(B(\bar{x}, \zeta\mu^{-1}), \mathbb{C}^D) \tag{17.4.77}$$

with $\eta > 0$ and with small enough $\zeta = \zeta(\eta) > 0$ independent of μ and h provided $\mu h \le \epsilon = \epsilon(\eta)$. By reduction one can easily prove that this estimate is equivalent to the estimate

$$\|(A_{\mathsf{S}} I + \sigma_0 \mu h)u\| \ge \eta\mu^{-1}h\|u\| \qquad \forall u \in \mathscr{C}_0^K(B(\bar{x}, \zeta\mu^{-1}), \mathbb{C}^D) \tag{17.4.78}$$

for the Schrödinger operator

$$A_{\mathsf{S}} = \sum_{j,k} P_j (F^{-1} g^{jk}) P_k + v^{\#};$$

namely, estimate (17.4.77) yields estimate (17.4.78) and *vice versa* (in each implication we replace η by $C_0^{-1}\eta$).

[27] Cf. Theorems 13.5.30, 13.5.31.

On the other hand, the constructions of Section 13.5 and this section yield that estimates (17.4.77) and (17.4.78) are equivalent in the sense indicated above to the inequalities

$$(m+1)\bigl|(m^2+2j\mu hF)^{\frac12}+V+\omega_1\mu^{-2}+\omega_2\mu^{-1}h\bigr|\ge\eta\mu^{-1}h \tag{17.4.79}$$

and

$$\bigl|v^{\#}+2j\mu h+\omega_1'\mu^{-2}+\omega_2'\mu^{-1}h\bigr|\ge\eta\mu^{-1}h \tag{17.4.80}$$

respectively; these inequalities should be checked for all $j\in\mathbb{Z}^+$ and for $x=\bar x$. Let us take $j\in\mathbb{Z}^+$ such that

$$(m+1)\bigl|(m^2+2j\mu hF)^{\frac12}+V\bigr|\le C^{-1}\mu h \tag{17.4.81}$$

and

$$|v^{\#}+2j\mu h|\le C^{-1}\mu h; \tag{17.4.82}$$

one can easily see that for $\mu^3h\ge C$ these inequalities define the same number j.

Let us replace $v^{\#}$ by $v^{\#}(1+\lambda)$ and let us alter V in the corresponding manner. Then the variations of ω_k and ω_k' are $O(|\lambda|(m+1)^{-1})$ and $O(|\lambda|)$ respectively and for $|\lambda|\le\epsilon'''$ these variations are are not essential in the sense that they do not change estimates (17.4.79) and (17.4.80).

Thus estimates (17.4.79) and (17.4.80) remain equivalent after replacing $v^{\#}$ by $v^{\#}(1+\lambda)$ (preserving the critical points), and the corresponding replacement of V. Taking j satisfying (17.4.81) and taking λ such that $|\lambda|\le C\mu h$ and the left-hand expression of (17.4.79) vanishes we obtain (because $\eta>0$ is arbitrary) that $\omega_k=-\omega_k'F(2V)^{-1}$ at the point $\bar x$.

However, we have formula (13.5.113) for ω_1' at the point $\bar x$ and we have proven that ω_2' vanishes at this point. So we have proven

Proposition 17.4.20. *In formula* (17.4.76) *we can calculate* $\bar\rho_1=\bar\rho$ *according to* (13.5.118), (13.5.113) *with* $v^{\#}$ *instead of* v *and take* $\bar\rho_2=0$.

Effective asymptotics. III

Now our goal is to obtain similar results in the case when condition (17.3.30) is replaced by one of the conditions (17.3.6)–(17.3.8) and, in particular, to get rid of condition (13.4.1) in Theorem 17.4.14. Recall that we used condition (17.3.30) not only in the transition from $\mathsf{R}^{\mathsf{MW}}_\infty$ to R^{MW} but also in the reduction done in Subsection 17.4.2.

Let us first of all note that

(17.4.83) If at some point $(\bar{x}', \bar{\xi}')$ the inequality $|v^{\#} + 2j\mu h| \ge \epsilon > 0$ holds, if condition (17.4.1) is fulfilled with $\epsilon_1 = \epsilon_0(c, \epsilon_0, \epsilon, s) > 0$ and if the symbol of Q is supported in the ϵ_1-neighborhood of $(\bar{x}', \bar{\xi}')$ then estimate (17.4.24) is true for all τ with $|\tau| \le \epsilon_1$.

Therefore our standard arguments yield that if condition (17.4.1) and one of the conditions (17.3.6)–(17.3.8) are fulfilled then the following estimate holds:

$$|F_{t\to h^{-1}\tau}\chi_{T,\pm T}(t)\Gamma' Q'_x \mathcal{U}_{jk}| \le C' h^s \qquad \forall \tau : |\tau| \le \epsilon_1 \tag{17.4.84}$$

where $T \in [h^{1-\delta'}, T^*]$ is arbitrary, $T^* = \mu T_0$ for $d = 2$, $T^* = T_0$ for $d = 3$ and $T_0 = T_0(c, \epsilon_0, s) > 0$ as before.

On the other hand, let us apply the standard method of successive approximations to calculate

$$F_{t\to h^{-1}\tau}\chi_T(t)\,\mathrm{tr}\,\Gamma' Q'_x \mathcal{U}_{jk}\, {}^t Q''_y \tag{17.4.85}$$

and

$$F_{t\to h^{-1}\tau}\chi_T(t)\,\mathrm{tr}\,\Gamma \tilde{Q}'_x \psi'(x) U \psi''(y)\, {}^t\tilde{Q}''_y \tag{17.4.86}$$

for $T = h^{1-\delta'}$, $|\tau| \le \epsilon_0$ with the unperturbed operator

$$\bar{\mathcal{A}} = \sigma_1 \sqrt{2j\mu h} B_0 + \sigma_0 m + \bar{W} \cdot I \qquad \text{for} \quad d = 2, \tag{17.4.87}$$

$$\bar{\mathcal{A}} = (\sigma_1 \sqrt{2j\mu h} B_0 + \sigma_3 h D_2 B_0)^{\mathsf{w}} + \sigma_0 m + \bar{W} \cdot I \qquad \text{for} \quad d = 3. \tag{17.4.88}$$

Moreover, for $m \ge C_0$ one can either do the reduction beforehand or skip this procedure. Then applying the arguments of the proofs of Propositions 13.5.4, 13.5.5 we arrive to the following equality

$$F_{t\to h^{-1}\tau}\chi_T(t)\,\mathrm{tr}\,\Gamma Q'_x \mathcal{U}\, {}^t Q''_y \equiv \sum_{j,n} T\mu h^{1-d} \int \hat{\chi}\Big(\frac{(\tau-\tau')T}{h}\Big)\Theta_{j,n}^{(n+1)}(\tau')\,d\tau' \tag{17.4.89}$$

and a similar equality for $\tilde{Q}'_x \psi'(x) U \psi''(y)\, {}^t\tilde{Q}''_y$ modulo $O(h^s)$ where $\Theta_{j,n}(\tau)$ are given by (13.5.44) with $f_{j,n}(x', \xi', \mu h, \boldsymbol{h}', \mu^{-1})$ instead of f and satisfy

$$|\mathcal{D}^\alpha f_{j,n}| \le c' \mu^{-n} \qquad \forall \alpha : |\alpha| \le K. \tag{17.4.90}$$

Therefore, in particular,

(17.4.91) The absolute values of expressions (17.4.85) and (17.4.86) do not exceed Ch^{1-d}.

Let us calculate the second expression for $\psi''=1$, $Q''\equiv I$; this calculation will be done modulo $O(\mu^{-1})$ for $d=2$ and modulo $O(h^{-1+\delta'})$ with $\delta'>0$ for $d=3$; in this section calculations with $\delta'=0$ also apply. We assume that

$$\mu \ge h^{\delta-1} \tag{17.4.92}$$

with a small enough exponent $\delta>0$. Then we should treat terms of the following five types:

(i) Principal terms of the type

$$\sum_j \mu h^{2-d}\Theta^{(1)}_{j,0}(\tau) \tag{17.4.93}$$

with $\Theta_{j,0}(\tau)$ given by (13.5.44) with

$$f_{j,0}=(2\pi)^{1-d}q'(x',\xi')(\psi'\circ\bar\Psi)(x',\xi_1);$$

these terms originate from the terms

$$\pm ih\bar G^{\pm}\delta(x-y)\delta(t)I \tag{17.4.94}$$

of the successive approximation series for $\mathcal{U}$ where $\bar G^{\pm}$ are parametrices of the operator $\bar P = hD_t-(m+1)\bar A$ with coefficients frozen for $x=y$ and with μ^{-1}, h' replaced by 0.

The total contribution of these terms is of order h^{1-d}.

(ii) Terms with total contribution of order $\mu^{-2}h^{1-d}$. These terms are associated with $\mathcal{U}_{jj}$ and originate from the terms

$$\pm ih\bar G^{\pm}(W-\bar W\cdot I)\bar G^{\pm}(W-\bar W\cdot I)\bar G^{\pm}\delta(x-y)\delta(t)I \tag{17.4.95}$$

The total contribution of the other terms of the successive approximation series is $O(\mu^{-3}h^{1-d})$ and is less than required by remainder estimate in virtue of (17.4.92).

Note that the terms of the type

$$\pm ih\bar G^{\pm}(W-\bar W\cdot I)\bar G^{\pm}\delta(x-y)\delta(t)I \tag{17.4.96}$$

have 0 diagonal blocks because $W - \bar{W}\cdot I$ has the same property; so these terms make no contribution to the final answer.

One should consider terms of this type as well as terms of the following type only for $d = 2$ and one can skip all blocks of $(W - \bar{W}\cdot I)$ with $j \neq k \pm 1$ (we use assumption (17.4.92) again; recall that W_{jk} are of order $O\big((m+1)^{-1}\mu^{-|j-k|}\big)$).

Moreover, one can consider only terms with $j \geq 2$ because the total contribution of the terms with $j = 0, 1$ is $O(\mu^{-1})$. Let us take into account that

$$W_{j,j+1} \equiv \mu^{-1}\big(\sqrt{2j\mu h}\,\Pi^{+} + \sqrt{2(j-1)\mu h}\,\Pi^{-}\big)w$$

modulo $O(h(m+1)^{-1})$ where w is a scalar operator and $\Pi^{\pm}$ are orthogonal projectors to $\mathbb{H}^{\pm}$, and let us take into account that $W_{j,j-1} = W^{*}_{j-1,j}$ and that for $d = 2$

$$\big(\tau - (m+1)\bar{A}\big)^{-1} = \big(\tau_1^2 - (m+1)^2(2j\mu h B_0^2 + m^2)\big)^{-1}\times \\ \big(\tau_1 + (m+1)(\sigma_1\sqrt{2j\mu h}B_0 + \sigma_0 m)\big)$$

with $\tau_1 = \tau - (m+1)\bar{W}$. Then we obtain terms of type

$$D_\tau^3 \int \operatorname{sgn}\tau_1 \cdot \theta\big(\tau_1^2 - (m+1)^2(2j\mu h b_0 + m^2)\big)\times \\ \mathcal{P}(j\mu h, x_1, \xi_1, \mu h, \mu^{-1}, h)\mu^{-1}\,dx_1 d\xi_1$$

where $\mathcal{P}$ is a second order polynomial of $j\mu h$ with coefficients depending smoothly on all the other variables and $b_0 = B_0^2$.

Obviously, for $m + |\tau_1| \geq 3\epsilon_1$ the integrand is a smooth function of τ and $z = j\mu h$ and is a distribution with respect to (x_1, ξ_1); this follows from condition (17.3.7) and the inequality $|\tau| \leq \epsilon_1$. Using the arguments of the proof of Proposition 17.3.11 one can easily show that for fixed (x_1, ξ_1) and for $|\tau_1| \leq 3\epsilon_1$, $m \leq 3\epsilon_1$ the integrand is a smooth function of $z = j\mu h$ and a distribution with respect to τ.

Using condition (17.3.6) we then obtain that for $|\tau| \leq \epsilon_1$ the integrand is a smooth function of $(j\mu h, \tau)$ and a distribution with respect to (x_1, ξ_1). Therefore we can replace summation with respect to j by integration with respect to $z = j\mu h$ and we then obtain that the sum is equal to

$$\varkappa(\tau)\mu^{-2}h^{-1} + O(\mu^{-1}) \tag{17.4.97}$$

with $\varkappa(\tau)$ depending smoothly on τ.

(iii) Furthermore, we have terms with total contribution of the same magnitude; they are associated with $\mathcal{U}_{jj\pm1}$ and they originate from

$$\pm ih\bar{G}^{\pm}(W-\bar{W}\cdot I)\bar{G}^{\pm}\delta(x-y)\delta(t)I \tag{17.4.98}$$

of the successive approximation series for $\mathcal{U}$. In this case the diagonal blocks of the operator $\tilde{\psi}=\mathcal{T}^*\mathcal{F}^*\psi\mathcal{F}\mathcal{T}$ are replaced by blocks $\tilde{\psi}_{j\pm1,j}$ which are of the form

$$\mu^{-1}\psi'\bigl(\sqrt{2j\mu h}\,\Pi^{+}+\sqrt{2(j-1)\mu h}\,\Pi^{-}\bigr)$$

with the scalar operator ψ'. The same arguments as before again give expression of (17.4.97) type.

(iv) Moreover, we have terms associated with $\mathcal{U}_{jj}$ and originating from the lower order terms of $\tilde{\psi}_{jj}$; these terms originate from the same terms of the successive approximation series for $\mathcal{U}$ as terms of type (i).

For $d=2$ these terms are

$$\sum_j j\Theta^{(1)}_{(j,1)}(\tau)$$

where $\Theta_{j,1}(\tau)$ are defined by (13.5.44) with the function f independent of $j\mu h$, the total contribution of these terms have the same magnitude as in Part (ii) and the previous arguments give the final expression (17.4.97).

For $d=3$ the essential terms of this type are associated with the difference between the operator $\tilde{\psi}$ and the same operator with $\mu^{-1}hD_2$ replaced by 0; we obtain 0 if the symbol of the operator Q is even with respect to ξ_2.

(v) Finally, we have terms with total contribution of magnitude $O(\mu^{-1})$ for $d=2$ and $O(h^{-1})$ for $d=3$; therefore these terms should be treated only for $d=3$ and $\delta'>0$. These terms originate from the freezing of the coefficients and commutators. Simple calculations (see Section 5.3) give 0 if the symbol of the operator Q is even with respect to ξ_2.

Let us apply the standard Tauberian arguments. Moreover, let us prove that the term $\varkappa(\tau)\mu^{-2}$ in the final answer vanishes (we compare the final answer obtained here with the results of section 17.3; here one can use averaging with respect to τ). Consequently we obtain the following statements in which we can get rid of condition (17.4.84) because these

statements were proven under condition (13.4.1) (in Section 17.3 and this section above):

Theorem 17.4.21. *Let $d = 2, 3$, $\vartheta = 0$ and suppose conditions* (17.1.1)–(17.1.3)*,* (17.2.1)*,* (17.4.1) *and one of the conditions* (17.3.6)*,* (17.3.7) *are fulfilled.*

Then estimate (13.5.85) *holds.*

Theorem 17.4.22. *Let $d = 3$, $\vartheta = 0$. Then Theorem 17.4.21 remains true with condition* (13.4.1) *replaced by* (17.4.1)*.*

In the proof of this latter theorem we use the same arguments as in the proof of Theorem 17.4.21 but in the zone $\{|\xi_2| \ge \epsilon\}$ we use the arguments of this subsection instead of arguments of Section 17.3.

The same arguments are applicable in the case when conditions (17.3.6)–(17.3.8) are skipped but there is mollification on τ; under condition (13.4.1) we use the results of Section 17.3. So the following statement is proven:

Proposition 17.4.23. *If $d = 2, 3$ and conditions* (17.1.1)–(17.1.3)*,* (17.2.1) *and* (17.4.1) *are fulfilled then the following estimates hold:*

$$(17.4.99)_{(2)} \quad \mathsf{R}^{\mathsf{MW}\prime} := \Bigl| \iint \phi(\tau)\psi(x)\bigl(d_\tau e(x,x,\tau_1,\tau) - \partial_\tau \mathcal{N}^{\mathsf{MW}}(x,\tau_1,\tau,\mu h) h^{-d}\, d\tau\bigr)\, dx\Bigr| \le C \qquad \text{for } \ d = 2,$$

$$(17.4.99)_{(3)} \qquad \mathsf{R}^{\mathsf{MW}\prime} \le C h^{\delta'-2} \qquad \text{for } \ d = 3$$

with arbitrarily small exponent $\delta' > 0$.

This proposition yields the following

Corollary 17.4.24. *Theorems 17.4.14–17.4.17, Remark 17.4.18, Proposition 17.4.20 and theorems 17.4.21, 17.4.22 remain true when the condition $\mathsf{supp}(\phi) \subset [-\epsilon_0, \epsilon_0]$ is replaced by the condition $\mathsf{supp}(\phi) \subset [-c, c]$.*

Certain improvements and generalizations of Corollary 17.4.24 will appear in Section 17.7.

17.5 Strong Magnetic Field

In this section we consider the case when

$$\mu \ge 1, \quad \mu h \ge \epsilon_0 \tag{17.5.1}$$

forbidden for the Schrödinger operator A_{S} unless we replace it by $A_{\mathsf{S}} - \mathfrak{z} F \mu h$. This case is in fact rather simple, because we actually need to consider only one Landau level.

17.5.1 Additional Reduction

Let us note that the semiclassical parameter for $d = 2$ is $\mu^{-1}h$ instead of h. Therefore in this case we can weaken the condition $h \in (0, 1]$. Namely, we replace it by the condition

$$h \in (0, 1] \qquad \text{for} \quad d = 3; \tag{17.5.2}_{(3)}$$

$$h \in (0, \mu^{1-\delta}] \qquad \text{for} \quad d = 2,\ \delta > 0. \tag{17.5.2}_{(2)}$$

In Subsection 17.5.2 even this condition will be weakened for $d = 2$.

Under conditions (17.5.1) and $(17.5.2)_{(d)}$ equalities (17.2.35) and (17.2.36) are fulfilled modulo functions f and f' respectively such that

$$|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)Q_x f| \le C' T \mu^{1-s} h^s, \tag{17.5.3}$$

$$|Q_x f'| \le C' \mu^{1-s} h^s \tag{17.5.4}$$

with the same restrictions on Q, T as in (13.3.81), (13.3.82) for $d = 3$ and under the condition $T \ge h\mu^{\delta'}$ with $\delta' > 0$ for $d = 2$.

Moreover, (13.3.84), (13.3.86) hold and the left-hand expression in (13.3.87) does not exceed $C' T \mu^{1-s} h^s$. Finally, the norm of the left-hand expression of (13.3.85) does not exceed $C\mu^l$ with some exponent l; this rough estimate will be improved in what follows.

Let us introduce $\mathcal{U}_{jk}$ as we did it in Section 17.4; then estimate (17.4.7) and equality (17.4.6) remain true. Moreover, the construction of $\mathcal{U}$ and the fact that we multiplied $\mathcal{U}$ by two $\mu^{-1}h$-pseudodifferential operators with symbols vanishing outside of a neighborhood of Λ^0 (one pseudodifferential operator acts with respect to x and another acts with respect to y) yield that all our estimates remain true if we replace $\mathcal{U}$ by its cutoff

$$\sum_{j,k\le N} \mathcal{U}_{jk} \Upsilon_j(x_d) \Upsilon_k(y_d) \tag{17.5.5}$$

with $N = N(s,\delta)$. We also denote this cutoff by $\mathcal{U}$. The error will be $O(\mu^{-s})$.

We need the following statement which is obvious by the standard elliptic arguments:

Proposition 17.5.1. *Let us assume that $d = 3$ and that $Q = \mathsf{Op}(q)$ with $q \in \mathsf{S}_{h',\rho,1,K}(T^*\mathbb{R}^2)$ supported in $\{\rho \le |\xi_2| \le 2\rho\}$ with $\rho \ge C_0$. Then the estimate*

$$|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)Q_x\mathcal{U}_{jk}| \le C'T\mu h^s\rho^{-s} \tag{17.5.6}$$

holds for $|\tau| \le c$, $T \ge h^{1-\delta}\rho^\delta$, $j,k \le N$.

Proposition 17.5.2. *There exists an operator $\mathcal{K} = (\mathcal{K}_{j,k})_{j,k\le N}$ such that in the sense of h'-pseudodifferential operators in $\tilde\Omega$ (17.4.32) is fulfilled and*

$$(m+1)\mathcal{K}^*\mathcal{A}\mathcal{K} \equiv (m+1)\big(\mathcal{A}_0+\mathcal{A}_1\big) + \mathcal{B}' \tag{17.5.7}$$

where $\mathcal{B}'$ is of the block-diagonal type and

$$|\mathcal{D}^\alpha\mathcal{K}_{jk}| \le c_\alpha\bar\nu(\mu^{-1}h)^{\frac12|j-k|}(\mu h)^{-\frac12} \qquad (j\ne k), \tag{17.5.8}$$

$$|\mathcal{D}^\alpha(\mathcal{K}_{jj}-I)| \le c_\alpha\mu^{-2}\bar\nu^2, \tag{17.5.9}$$

$$|\mathcal{D}^\alpha\mathcal{B}'_{(j)}| \le c_\alpha\mu^{-2}\bar\nu^2. \tag{17.5.10}$$

Proof. Let us construct $\mathcal{K}$ by induction with respect to M assuming that at every step (17.5.7) holds with an additional term $\mathcal{R}$ in the right-hand expression such that

$$|\mathcal{D}^\alpha\mathcal{R}_{jk}| \le c_\alpha(\mu^{-1}h)^{\frac12\max(|j-k|,M)}. \tag{17.5.11}$$

Then we set $\mathcal{K}_{M+1} = \mathcal{K}_M \exp i\mathcal{L}$ with the Hermitian operator $\mathcal{L} = (\mathcal{L}_{jk})_{j,k\le N}$ where $\mathcal{L}_{jj} = 0$ and the principal symbols of the out-of-diagonal blocks of $\mathcal{L}$ are defined by the equations

$$i(m+1)\big(\mathcal{A}^0_{(j)}\mathcal{L}^0_{jk} - \mathcal{L}^0_{jk}\mathcal{A}^0_{(j)}\big) = \mathcal{R}^0_{jk}. \tag{17.5.12}$$

Note that the distances between the spectra of the Hermitian matrices $(m+1)\mathcal{A}^0_{(j)}$ and $(m+1)\mathcal{A}^0_{(k)}$ with $j\ne k$ are not less then $\epsilon_1\big((m+1)\sqrt{\mu h}+\mu h\big)$ with a constant $\epsilon_1 > 0$. Then Appendix 3.A.1 yields that $\mathcal{L}^0_{jk}$ are defined uniquely and satisfy the inequalities

$$|\mathcal{D}^\alpha\mathcal{L}^0_{jk}| \le c_\alpha\bar\nu(\mu^{-1}h)^{\frac12\max(|j-k|,M)}(\mu h)^{-1}. \tag{17.5.13}$$

All the assertions of our proposition follow from this construction. □

Let us now set $\mathcal{U}' = \mathcal{K}_x^* \mathcal{U}\, {}^t\mathcal{K}_y^*$; then in $\tilde{\Omega}$

$$(hD_t - \mathcal{B}_{(j)})\mathcal{U}'_{jk} \equiv 0, \tag{17.5.14}$$

$$\mathcal{U}'_{jk}|_{t=0} \equiv \delta_{jk}\delta(x'-y')I \tag{17.5.15}$$

where $\mathcal{B}_{(j)} = (m+1)(\mathcal{A}_{0(j)} + \mathcal{A}_{1(j)}) + \mathcal{B}'_{(j)}$.

Then the standard elliptic arguments yield the following

Proposition 17.5.3. *(i) If the support of the symbol of Q is small enough then there exists $j_0 \in \mathbb{Z}^+$ such that*

$$|F_{t\to h^{-1}\tau}\chi_{T,T'}(t)Q_x\mathcal{U}'_{jk}| \le C'T\mu^{-s} \tag{17.5.16}$$
$$\forall \tau : |\tau| \le \epsilon_0 \quad \forall T > \mu^{-\delta'}h \quad \forall j \ne j_0$$

where $\delta' > 0$.

(ii) In particular, if $\mu h \ge C_0$ then $j_0 = 0$ and this estimate remains true with $Q = I$.

Recall that the eigenvalues of the symbol $\mathcal{A}_{0(j)}(x',\xi')$ are

$$\lambda j,\pm := \bar{W} \pm (m^2 + 2j\mu h b_0)^{\frac12} \qquad \text{for} \quad d = 2 \tag{17.5.17}_{(2)}$$

and

$$\lambda j,\pm := \bar{W} \pm \bigl(m^2 + (2j\mu h + \xi_2^2)b_0\bigr)^{\frac12} \text{for} \quad d = 3, \tag{17.5.17}_{(3)}$$

where

(17.5.18) For $j \ge 1$ all these eigenvalues are disjoint while two eigenvalues with $j = 0$ are not necessarily disjoint.

Therefore the following statement holds:

Proposition 17.5.4. *For $\mu h \ge \epsilon_0$, $j \ge 1$ there exist operators $\mathcal{T}_{(j)}$ satisfying* (17.4.25) *and*

$$\mathcal{T}_{(j)}{}^*\mathcal{B}_{(j)}\mathcal{T}_{(j)} \equiv \tilde{\mathcal{B}}_{(j)} = \begin{pmatrix} \mathcal{B}^+_{(j)} & 0 \\ 0 & \mathcal{B}^-_{(j)} \end{pmatrix} \tag{17.5.19}$$

in $\tilde{\Omega}$ where the principal symbols of the $\mathbf{h}'$-pseudodifferential operators $\mathcal{B}^{\pm}_{(j)}$ are $(m+1)\lambda j,\pm I$ with $\lambda j,\pm$ defined by $(17.5.17)_{(d)}$ with $d = 2,3$ and the blocks are $\frac{\mathsf{D}}{2} \times \frac{\mathsf{D}}{2}$-dimensional.

Let us consider the case $j = 0$.

For $D = 2$ the operator $\mathcal{B}_{(0)}$ is 1×1-matrix from the beginning and its principal symbol is equal to

$$(17.5.20)_{(2)} \qquad (m+1)(\bar{W} + \varsigma m) \qquad \text{for } d = 2$$

and

$$(17.5.20)_{(3)} \qquad \bar{W} + \varsigma \xi_2 B_0 \qquad \text{for } d = 3,\ D = 2$$

where $m = 0$ automatically in the latter case.

In the case $d = 3$, $D = 4$ the operator $\mathcal{B}_{(0)}$ is 2×2-matrix and the eigenvalues of its principal symbol are equal to

$$(17.5.20)^*_{(3)} \qquad (m+1)\big(\bar{W} \pm (m^2 + \xi_2^2 b_0)^{\frac{1}{2}}\big);$$

these eigenvalues are disjoint if either $m \ge \epsilon_0$ or $|\xi_2| \ge \epsilon_0$ in $\tilde{\Omega}$ and in these two cases there exists an operator $\mathcal{T}_{(0)}$ satisfying (17.4.25), (17.5.19) with blocks which are now 1-dimensional.

So in all with the singular exception $d = 3$, $D = 4$, $j = 0$, $m \le \epsilon_0$, $|\xi_2| \le \epsilon_0$ on $\tilde{\Omega}$ our problem is reduced to the analysis of operators with scalar principal symbols and all the arguments of Section 5.3 are applicable with the following modification:

(17.5.21) We have $(d-1)$ instead of d and there is an additional factor μ in the right-hand expressions of the estimates.

Remark 17.5.5. (i) Since the real semiclassical parameter is $\mu^{-1}h$ in the case $d = 2$, all the negligibility estimates have right-hand expression $C'\mu^{-s}T$ in this case.

(ii) For $d = 3$ we have the real semiclassical parameter h and even $h/\sqrt{\nu}$ in the case when we treat the operator with the scalar principal symbol and $v^{\#}$ (defined by (17.3.20)) satisfies the condition

$$(17.5.22) \qquad |D^{\alpha}(v^{\#} + 2j\mu h)| \le c\nu \qquad \forall \alpha : |\alpha| \le K$$

with $\nu \in [h^2, 1]$.

In this case the negligibility estimates have $C'\mu h^{-1}\big(h/\sqrt{\nu}\big)^s T$ as their right-hand expression.

Finally, let us note that the standard elliptic arguments yield that

Remark 17.5.6. (i) We do not need to treat the case $j = 0$ for $|\tau| \le \epsilon_0$, $d = 2$ and for $|\tau| \le \epsilon_0$, $d = 3$ and symbol of Q supported in $\{|\xi_2| \le \epsilon_0\}$ provided condition (17.3.30) is fulfilled.

(ii) On the other hand, the opposite condition ($V^2 - m^2 \le \epsilon_1$) excludes the case $j \ge 1$ from the analysis.

17.5.2 Estimates

Estimates as $d = 3$

It is easy to see that Proposition 17.4.10 remains true with the following modification: we discuss smoothness with respect to $z = j\mu h$ neither here nor below and we do not replace summation by integration; the transition from $\mathsf{R}^{\mathsf{MW},\prime}$ to R^{MW} is now trivial. As a result we have the following statements (we leave the proof to the reader):

Theorem 17.5.7. *Let $d = 3$, $\tau = 0$, $\phi \in \mathscr{C}_0^K([-c, c])$ and let conditions* (17.1.1)–(17.1.3), (17.2.1), (17.5.1), $(17.5.2)_{(3)}$, $(13.4.85)^{\#}$ *be fulfilled with $\nu \in [h^2, 1]$. Recall that $^{\#}$ means modification described in Corollary 17.2.5. Then*

(i) Let either $m \ge 1$ or condition (17.3.30) *be fulfilled. Then the estimate*

$$\mathsf{R}^{\mathsf{MW}} \le C\mu h^{-1-\delta}\nu^{\frac{\delta}{2}} + C' \tag{17.5.23}$$

holds with arbitrarily small $\delta > 0$. Moreover, under the condition

$$\min_{j\in\mathbb{Z}^+} |v^{\#} + 2j\mu h| + \sum_{\alpha:1\le|\alpha|\le L} |D^\alpha v^{\#}| \ge \epsilon_0\nu \tag{17.5.24}$$

the estimate

$$\mathsf{R}^{\mathsf{MW}} \le C\mu h^{-1} \tag{17.5.25}$$

holds.

(ii) If one of the conditions (17.3.6), (17.3.7) *is fulfilled then estimate* (17.5.25) *holds.*

Let us note that we cannot apply this statement in the most general case (with $m \le \epsilon_0$) because there is a term with $j = 0$. Therefore let us treat the case $j = 0$ more carefully. Let us recall that for $j = 0$ we have an operator with the principal symbol $\big(W + \varsigma\xi_2\big)$ for $\mathsf{D} = 2$ and $\big(W \cdot I + \sigma_1\xi_2 + \sigma_0 m\big)$ for $\mathsf{D} = 4$. In the former case we have a scalar operator of principal type and therefore the contribution of this term to the remainder estimate does not exceed $C\mu h^{-1}$.

In the latter case we can assume without any loss of the generality that either $m = 1$ (a case which has been treated already) or $\nu = 1$; in Section 17.7 we will show that the general case can be reduced to one of these (maybe with a small "loss of information").

Applying the arguments of Subsection 5.3.2 (see theorems 5.3.36, 5.3.39[28]) one can easily prove the following

Theorem 17.5.8. *Let all the conditions of Theorem 17.5.8 be fulfilled and let $m \le 1$. Then estimate* (17.5.23) *holds with $\nu = 1$.*

Moreover, under condition (17.5.24) *(with $\nu = 1$) estimate* (17.5.25) *holds.*

Estimates as $d = 2$

Let us treat the case $d = 2$. In this case we obtain a scalar operator for $j = 0$ as well. The theory of Section 5.3 immediately yields the following two theorems (again, we leave their proofs to the reader):

Theorem 17.5.9. *Let $d = 2$, $\tau = 0$ and let conditions* (17.1.1)–(17.1.3), (17.2.1), (17.5.1) *and* $(17.5.2)_{(2)}$ *be fulfilled. Let $\phi \in \mathscr{C}_0^K([-c, c])$. Then*

(i) Under the conditions

$$(17.5.26)_+ \qquad |v^\# + 2j\mu h| + |\nabla v^\#| \le \epsilon_0 \implies \mathsf{Hess}\, v^\# \ge \epsilon_0 \quad \forall j \ge 1$$

$$(17.5.27)_+ \qquad |V_0| + |\nabla V_0| \le \epsilon_0 \implies \mathsf{Hess}\, V_0 \ge \epsilon_0$$

with

$$(17.5.28) \qquad V_0 \coloneqq (m+1)(V - \varsigma m)$$

the estimate

$$(17.5.29) \qquad \mathsf{R}^{\mathsf{MW}} \le C$$

holds.

[28] We apply the second theorem in order to get rid of the logarithmic factor.

(ii) Under the conditions

$$|v^{\#}+2j\mu h|+|\nabla v^{\#}|\le \epsilon_0 \implies |\operatorname{Hess} v^{\#}|\ge \epsilon_0 \quad \forall j\ge 1 \tag{17.5.26}$$

$$|V_0|+|\nabla V_0|\le \epsilon_0 \implies |\operatorname{Hess} V_0|\ge \epsilon_0 \tag{17.5.27)$_$}$$

the estimate

$$\mathsf{R}^{\mathsf{MW}} \le C|\log\hat\lambda| \le C\log\mu \tag{17.5.30}$$

holds where

$$\hat\lambda = \min_{j\in\mathbb{Z}^+, x\in\mathfrak{M}_j} \hat\lambda_{j,x}, \qquad \hat\lambda_{j,x}=|v^{\#}+2j\mu h| \quad (j\ge 1), \tag{17.5.31}$$

$$\hat\lambda_{j,x}=|V-\varsigma m| \quad (j=0)$$

and $\mathfrak{M}_j$ is the set of saddle points of $v_j^{\#}=v^{\#}$ for $j\ge 1$ and $v_0^{\#}=V_0$ for $j=0$.

(iii) Finally, under the conditions

$$\min_{j\in\mathbb{Z}^+, j\ge 1} |v^{\#}+2j\mu hF| + \sum_{\alpha:1\le|\alpha|\le L} |D^{\alpha}v^{\#}| \ge \epsilon_0\nu, \tag{17.5.24)$'$}$$

$$\sum_{\alpha:|\alpha|\le L} |D^{\alpha}V_0| \ge \epsilon_0 \tag{17.5.24)$''$}$$

the estimate

$$\mathsf{R}^{\mathsf{MW}} \le C\mu^{\delta} \tag{17.5.32}$$

holds with arbitrarily small $\delta>0$.

Theorem 17.5.10. *Let $d=2$, $\tau=0$ and let conditions* (17.1.1)–(17.1.3), (17.2.1), (17.5.1), $(17.5.2)_{(2)}$ *and* $(17.2.28)_{1,2}$ *be fulfilled. Moreover, let*

$$|D^{\alpha}v_j^{\#}|\le c\nu \qquad \forall \alpha : 1\le|\alpha|\le K \tag{17.5.33}$$

and

$$\bar\nu\ge\mu^{-l}, 1\ge\nu\ge C_0\bar\nu\mu^{-1}h \tag{17.5.34}$$

as j is defined by the inequality $|v_j^{\#}|\le\epsilon_1$ with arbitrarily small $\epsilon_1>0$. Then

(i) If

$$|v_j^{\#}| + |\nabla v_j^{\#}| \le \epsilon_0 \nu \implies \det \mathsf{Hess}\, v_j^{\#} \ge \epsilon_0 \nu^2 \tag{17.5.35}_+$$

then estimate (17.5.29) *holds for* $\mathsf{R}^{\mathsf{MW}\prime}$ *and*

$$\mathsf{R}^{\mathsf{MW}} \le C\frac{\bar\nu}{\nu}. \tag{17.5.36}$$

(ii) If

$$|v_j^{\#}| + |\nabla v_j^{\#}| \le \epsilon_0 \nu \implies |\det \mathsf{Hess}\, v_j^{\#}| \ge \epsilon_0 \nu^2 \tag{17.5.37}$$

then estimate (17.5.30) *holds for* $\mathsf{R}^{\mathsf{MW}\prime}$ *and*

$$\mathsf{R}^{\mathsf{MW}} \le C\frac{\nu}{\bar\nu}\bigl(\log \frac{\mu\nu}{\bar\nu h} + 1\bigr). \tag{17.5.38}$$

(iii) If

$$\sum_{\alpha:|\alpha|\le L} |D^\alpha v_j^{\#}| \ge \epsilon_0 \nu \tag{17.5.39}$$

then estimate (17.5.32) *holds for* $\mathsf{R}^{\mathsf{MW}\prime}$.

Estimates as $d = 2$ and h is Large

Let us weaken condition $(17.5.2)_{(2)}$ as $d = 2$. Namely, let us now assume that

$$\hbar = \mu^{-1} h \in (\mu^{-\delta}, 1] \tag{17.5.40}$$

with a small exponent $\delta > 0$ and let us treat $\hbar$ as a small parameter. Now the Schwartz kernels of operators with operator norm in $\mathscr{L}^2$ not exceeding $C'\hbar^s$ where t is treated as parameter are considered negligible (before the restriction to $x = y$ and integration).

Then the presence of a cut-off with respect to x_1 and $\hbar D_1$ yields that $\mathcal{U}$ is equal to decomposition (17.5.5) modulo a negligible function.

The first difficulty is that the remainder in the transformation of $\mathcal{A}^0$ is of order $\mu\bar\nu\hbar^N$ where we assume that conditions $(17.2.28)_{1,2}$ are fulfilled. In order to make this remainder negligible (i.e., with operator norm less than $C'\hbar^N$) we assume that

(17.5.41) Conditions $(17.2.28)_{1,2}$ are fulfilled and $\bar{\nu} \le c\mu^{-1}\hbar^{-l}$.

Here we do not divide the right-hand expression of this inequality by $(m+1)$ because the standard elliptic arguments yield that the components $\mathcal{U}_{jk}^{\zeta\zeta'}$ with $\zeta = \varsigma$ or (and) $\zeta' = \varsigma$ are Schwartz kernels of operators bounded even after being multiplied by $(m+1)$ ($(m+1)^2$ respectively) and cutting with respect to hD_t; i.e., after reduction is applied.

Then assuming that $\delta = \delta(s, l) > 0$ in (17.5.40) is small enough the standard elliptic arguments yield that the functions $T^{-1}F_{t\to h^{-1}\tau}\chi_{T,T'}(t)\mathcal{U}_{jk}$ are negligible for $T \ge \mu^{\delta'}h$, $j \ge 1$, $|\tau| \le c$ with $\delta' = 1/s$. Therefore the same is true for $k \ge 1$. So the only one essential component $\mathcal{U}_{00}$ remains and

$$\bigl(hD_t - (m+1)(\bar{W} - \varsigma m)\bigr)\mathcal{U}_{00} \equiv 0, \tag{17.5.42}$$

$$\mathcal{U}_{00}|_{t=0} \equiv \delta(x_1 - y_1) \tag{17.5.43}$$

where the first equality is fulfilled modulo functions negligible in the sense indicated above (in the same sense $\mathcal{U}_{jk}$ with $j + k \ge 1$ are negligible).

Therefore our routine arguments yield the following

Theorem 17.5.11. *Let $d = 2$ and let conditions* (17.1.1)–(17.1.3), (17.2.1), (17.5.1), (17.5.40) *and* (17.5.41) *be fulfilled where $\delta > 0$ in* (17.5.40) *is small enough.*

Let us assume that V_0 satisfies $(17.2.28)_3$ with $\bar{\nu}$ replaced by $\nu \in [\hbar^l, 1]$.

(i) Then all the statements of Theorem 17.5.10 remain true.

(ii) Moreover, under condition (17.5.39) *the estimate*

$$\mathsf{R}^{\mathsf{MW}\prime} \le C\hbar^{-\delta'} \tag{17.5.44}$$

holds with arbitrarily small $\delta' > 0$.

17.6 Semiclassical Spectral Gaps

17.6.1 Existence of Spectral Gaps

The majority of this section is devoted to the case $d = 2$ because for $d = 3$ there can exist only main spectral gap, namely $(-m + \sup V, m + \inf V)$ under the condition that the first number is less than the second. On the

other hand, for $d = 2$ other spectral gaps can appear. In this section we analyze the case $d = 2$ in two different situations: under condition (17.4.1) and under the condition

$$h^{\delta-1} \le \mu; \tag{17.6.1}$$

later we join these cases. Let us start with the statements that gaps exist. By the standard elliptic arguments one can prove easily the following two propositions:

Proposition 17.6.1. *Let $d = 2$ and let conditions* (17.1.1)–(17.1.3), (17.2.1), $(17.2.28)_{1-3}$, (17.4.1), (17.3.1) *and* $(13.4.12)_1$ *be fulfilled. Then*

(i) Let condition (17.3.30) *be fulfilled and for some $j \in \mathbb{Z}^+$ let*

$$|v^{\#} + 2j\mu h| \ge \zeta^2 \ge C_0\bar{\nu}\mu^{-2} \qquad \forall x \in B(0,1). \tag{17.6.2}$$

Then for $\mathcal{U}'''_{jk}$ constructed in Section 17.4 the following estimate holds:

$$|F_{t\to h^{-1}\tau}\chi_T(t)Q_x\mathcal{U}'''_{jk}| \le C'\mu^{-s}h^s \tag{17.6.3}$$
$$\forall x, y \in \mathbb{R}^2 \quad \forall T \ge h^{1-\delta}\zeta^{-2} \quad \forall k \in \mathbb{Z}^+ \quad \forall \tau : |\tau| \le \epsilon\zeta^2.$$

(ii) In particular, if condition (17.6.2) *is fulfilled for all $j \in \mathbb{Z}^+$ then estimate* (17.6.3) *holds for all j, k and hence this estimate holds for $\mathcal{U}$ instead of $\mathcal{U}'''$, and therefore the estimate*

$$|F_{t\to h^{-1}\tau}\chi_T(t)U| \le C'T\mu^{-s}h^s \tag{17.6.4}$$

holds for the same T and τ as in (17.6.3) *and for $x, y \in B(0, \frac{1}{2})$.*

(iii) Let

$$v^{\#} \ge \zeta^2 \ge C_0\bar{\nu}\mu^{-2} \qquad \forall x \in B(0,1). \tag{17.6.5}$$

Then estimates (17.6.3), (17.6.4) *hold for x, y, T, τ indicated above and for all j, k.*

Proposition 17.6.2. *Let $d = 2$ and let conditions* (17.1.1)–(17.1.3), (17.2.1), $(17.2.28)_{1-3}$, (17.5.1), $(17.5.2)_{(2)}$ *and* (17.3.1) *be fulfilled with $\bar{\nu} \ge \mu^{-l}$. Then*

(i) For some $j \in \mathbb{Z}^+$ *let*

$$|v^{\#} + 2j\mu h| \ge \zeta^2 \ge C_0\bar{\nu}\mu^{-1}h \qquad \forall x \in B(0,1). \tag{17.6.6}$$

Then for the $\mathcal{U}'_{jk}$ *constructed in Section 17.5 estimate* (17.6.3) *holds.*

(ii) particular, if condition (17.6.6) *is fulfilled for all* $j \in \mathbb{Z}^+$ *then estimate* (17.6.3) *holds for all* j, k*; hence the estimate holds for* $\mathcal{U}$ *instead of* $\mathcal{U}'$*, and therefore estimate* (17.6.4) *holds.*

(iii) For some $j \in \mathbb{Z}^+$ *let*

$$|H_{\varkappa j}(x)| \ge \zeta^2 \ge C_0(m+1)^{-1}\bar{\nu}\mu^{-1}h \qquad \forall x \in B(0,1) \quad \forall \varkappa : (\varkappa, j) \in \mathfrak{Z}, \tag{17.6.7}$$

where

$$H_{\varkappa j}(x) = V + \varkappa(2j\mu hF + m^2)^{\frac{1}{2}}, \tag{17.6.8}$$

$\varkappa = \pm$ *and* $\mathfrak{Z}$ *were introduced in Section 17.4. Then estimate* (17.6.3) *holds.*

(iv) In particular, if condition (17.6.7) *is fulfilled for all* j *then estimate* (17.6.3) *holds for all* j *and therefore this estimate holds for* $\mathcal{U}$ *instead of* $\mathcal{U}'$ *and hence estimate* (17.6.4) *holds.*

Let us discuss the case when conditions (17.5.1) and $(17.5.2)_{(2)}$ are fulfilled and $\mu h \ge C_0$ more carefully.

In this case one should check conditions of (17.6.2)-type only for $j = 0$, $\varkappa = -\varsigma$ and in order to prove estimates (17.6.3) and (17.6.4) it is sufficient to check the microlocal hypoellipticity of the block $\mathcal{B}_{(0)}$ of the operator $\mathcal{B} = (m+1)\mathcal{K}^*\mathcal{A}\mathcal{K}$; this operator is of the block-diagonal type.

Let us note that the distance between eigenvalues of the symbol $(m+1)\mathcal{A}^0$ corresponding to $j = 0$, $\varkappa = -\varsigma$ and eigenvalues corresponding to other $(j, \varkappa) \in \mathfrak{Z} \setminus (0, -\varsigma)$ surely is not less than $(m+1)\mu h(m + \sqrt{\mu h})^{-1}$ and that blocks $(m+1)\mathcal{A}_{0j}$ are uniformly $\mu^{-1}h$-pseudodifferential operators with an additional factor $\bar{\nu}(\mu^{-1}h)^{j/2}$ as $j \ge 1$.

Therefore operator $\mathcal{M} = \mathcal{B}_{(0)} - \mathcal{A}_{00}$ is equal to $\rho\tilde{\mathcal{M}}$ where

$$\rho = \bar{\nu}^2\mu^{-2}\bigl(1 + \sqrt{\mu h}(m+1)^{-1}\bigr) \tag{17.6.9}$$

and $\tilde{\mathcal{M}}$ is a $\mu^{-1}h$-pseudodifferential operator uniformly. Moreover,

$$\mathcal{A}_{00} = \mathcal{G}^*(V - \varsigma m)\mathcal{G}(m+1) \tag{17.6.10}$$

where $\mathcal{G} = \mathcal{T}\mathcal{F}L$, L is the operator: $v(x_1) \to Lv = v(x_1)\Upsilon_0(x_2)e$ and e is a fixed element of $\mathsf{Ker}(\sigma_1 - i\sigma_2)$ with $\|e\| = 1$.

So we are interested in the microlocal hypoellipticity of the operator

$$(m+1)\mathcal{G}^*(V - \varsigma m)\mathcal{G} + \mathcal{M}. \tag{17.6.11}$$

Let us consider the map $\bar{\Psi} : \Lambda^0 \ni (x_1, \xi_1) \to x \in B(0,1)$ introduced in Section 13.3.

One can see easily that if $\psi = 1$ in a neighborhood of $\bar{\Psi}(\mathsf{supp}(q))$ and $Q = \mathsf{Op}(q)$ then $Q\mathcal{G}^*\psi \equiv Q\mathcal{G}^*$ and $\psi\mathcal{G}Q \equiv \mathcal{G}Q$.

On the other hand, if $q = 1$ in a neighborhood of $\bar{\Psi}^{-1}(\mathsf{supp}(\psi))$ then $Q\mathcal{G}^*\psi \equiv \mathcal{G}^*\psi$ and $\psi\mathcal{G}Q \equiv \mathcal{G}Q$; here Q is a $\mu^{-1}h$-pseudodifferential operator with respect to x_1 and $\psi = \psi(x)$.

These observations easily yield that $\mathcal{B}_{(0)}$ is a microlocally hypoelliptic $\mu^{-1}h$-pseudodifferential operator provided

$$(m+1)|V - \varsigma m| \ge C_0\rho \tag{17.6.12}$$

(let us note that the left-hand expression equals $|H_{-\varsigma 0}|$). Moreover, one can see easily that

$$\mathcal{M} = \rho\Big(\sum_{p,r} f^{pr}\partial_p\tilde{V}\cdot\partial_r\tilde{V} + \sqrt{\mu^{-1}h}\sum_p f^p\partial_p\tilde{V} + \mu^{-1}hf\Big)\circ\bar{\Psi} \tag{17.6.13}$$

with uniformly smooth symbols f^{pr}, f^p, f and with $\tilde{V} = (m+1)\bar{\nu}^{-1}V$. Therefore condition (17.6.12) can be replaced by the condition

$$(m+1)|V - \varsigma m| \ge C_0\rho\mu^{-1}h. \tag{17.6.14}$$

So we have proven the following

Proposition 17.6.3. *Let $d = 2$ and let conditions* (17.1.1)–(17.1.3), (17.2.1), $(17.2.28)_{1-3}$, (17.5.1), $(17.5.2)_{(2)}$ *and* (17.3.1) *be fulfilled with $\bar{\nu} \ge \mu^{-l}$, $\mu h \ge C_0$. Moreover, let condition* (17.6.14) *be fulfilled.*

Then for all j, k estimate (17.6.3) *holds and estimate* (17.6.4) *holds as well.*

In a similar way one can prove

Proposition 17.6.4. *Let $d = 2$ and let conditions* (17.1.1)–(17.1.3), (17.2.1), $(17.2.28)_{1-3}$, (17.5.1), (17.5.42), (17.5.43), (17.5.33) *and* (17.3.1) *are fulfilled with*

$$\nu \ge \hbar^{l}, \qquad \bar{\nu} \le \nu\mu^{-1}\hbar^{-l} \tag{17.6.15}$$

where $\delta = \delta(l, s) > 0$ in condition (17.5.41) *is small enough. Moreover, let*

$$|H_{-\varsigma 0}| \ge \zeta^2 \ge C_0\nu\hbar \qquad \forall x \in B(0,1). \tag{17.6.16}$$

Then the estimate

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma'\mathcal{U}'_{jk}| \le C'\mu^{1-j-k}T\nu\zeta^{-2}\hbar^{s} \qquad \forall x, y \in \mathbb{R}^2 \tag{17.6.17}$$

holds for $T \ge h\hbar^{\delta'}$, $|\tau| \le \epsilon\zeta^2$. Therefore the estimate

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\psi U| \le C'\mu T\nu\zeta^{-2}\hbar^{s} \tag{17.6.18}$$

holds for the same T and τ.

Remark 17.6.5. Surely, one can weaken condition (17.6.16) in the spirit of (17.6.12) or (17.6.14).

As $d = 3$ by the standard elliptic arguments one can prove the following

Proposition 17.6.6. *Let $d = 3$ and let conditions* (17.1.1)–(17.1.3), (17.3.1), $(17.2.28)_{1-3}$, (17.5.33) *be fulfilled with $1 = \bar{\nu} \ge \nu \ge h^2$. Moreover, let*

$$V^2 - m^2 \le -\epsilon\nu \qquad \text{in } B(0,1). \tag{17.6.19}$$

Then the estimate

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\psi U| \le C'(\mu h^{-1} + h^{-2})\bigl(\frac{h}{\sqrt{\nu}}\bigr)^{s} \tag{17.6.20}$$

holds for the same $T \ge h^{1-\delta'}\nu^{\frac{1}{2}\delta'-1}$ and $|\tau| \le \nu$.

Starting from Propositions 17.6.1–17.6.4 and 17.6.6 by means of the standard Tauberian arguments we then obtain the following theorem:

Theorem 17.6.7. *Let conditions* (17.1.1)–(17.1.3), (17.2.1), (17.3.1) *and* $(17.2.28)_{1-3}$ *be fulfilled.*

(i) Let $d = 2$, *let conditions* (17.4.1), $(13.4.12)_1$, (17.3.30) *be fulfilled and let condition* (17.6.2) *be fulfilled for all* $j \in \mathbb{Z}^+$. *Then the estimate*

$$|e(x,y,\tau_1,\tau_2)| \le C'\mu^{-s}h^s \qquad \forall x,y \in B(0,\frac{1}{2}) \quad \forall \tau_1,\tau_2 \in (-\epsilon\zeta^2,\epsilon\zeta^2) \tag{17.6.21}$$

holds.

(ii) Let $d = 2$, *let conditions* (17.5.1), $(17.5.2)_{(2)}$ *be fulfilled and let* $\bar\nu \ge \mu^{-l}$.

Moreover, let <u>*either*</u> *condition* (17.6.6) *be fulfilled for all* $j \in \mathbb{Z}^+$ <u>*or*</u> *condition* (17.6.7) *be fulfilled for all* $(\varkappa, j) \in \mathfrak{Z}$. *Then estimate* (17.6.21) *also holds.*

(iii) Let $d = 2$, *let conditions* (17.5.1), $(17.5.2)_{(2)}$ *be fulfilled and let* $\mu h \ge C_0$. *Moreover, let* <u>*either*</u> *condition* (17.6.12) <u>*or*</u> *condition* (17.6.14) *be fulfilled. Then estimate* (17.6.21) *also holds.*

(iv) Let $d = 2$, *let conditions* (17.5.1), (17.5.42), (17.5.42), (17.5.33), (17.6.15), (17.6.16) *be fulfilled (and in 17.2.28 only the first two conditions should be checked) and let* $\delta = \delta(l,s) > 0$ *be small enough. Then the estimate*

$$|\Gamma\psi e(.,.,\tau_1,\tau_2)| \le C'\mu\nu\zeta^{-2}\hbar^s \qquad \forall \tau_1,\tau_2 \in (-\epsilon\zeta^2,\epsilon\zeta^2) \tag{17.6.22}$$

holds.

(v) Let $d = 3$, *let condition* (17.5.33) *hold with* $\nu \in [h^2,1]$, $\bar\nu = 1$ *in* $(17.2.28)_{1-3}$ *and let condition* (17.6.19) *be fulfilled. Then the estimate*

$$|\Gamma\psi e(.,.,\tau_1,\tau_2)| \le C'(\mu h^{-1} + h^{-2})\bigl(\frac{h}{\sqrt{\nu}}\bigr)^s \qquad \forall \tau_1,\tau_2 \in (-\epsilon\nu,\epsilon\nu) \tag{17.6.23}$$

holds.

17.6.2 Calculations in the Spectral Gaps

In what follows we assume that $d = 2$. In order to finish our analysis we need to treat only mollified asymptotics. Theorem 4.4.1 immediately yields

Proposition 17.6.8. *Let $d = 2$ and let conditions* (17.1.1)–(17.1.3), (17.2.1), (17.4.1), (17.3.1) *and* (17.3.30) *be fulfilled.*

Let $\phi \in \mathscr{C}_0^K((-\epsilon,\epsilon))$, $L \in [h^{1-\delta'},1]$. Then

$$|\int \phi\big(\frac{\tau}{L}\big)\Big(d_\tau(\Gamma\psi e)(.,\tau) - h^{-2}\int \psi d_\tau \mathcal{N}^{\mathsf{MW}}(x,.,\tau,\mu h)dx - \sum_{\varkappa,n,r:n\le 2r,1\le r\le N}\sum_j \Theta^{(n+1)}_{j\varkappa nr}(\tau)(\mu^{-1}h)^r dx\Big)| \le C'h^s \tag{17.6.24}$$

where

$$\Theta_{j\varkappa nr}(\tau) = \int \theta\big(\tau - H_{\varkappa j}\big) f_{nr\varkappa}(j\mu h,\mu^{-1},\mu h,x_1,\xi_1)\,dx_1 d\xi_1, \tag{17.6.25}$$

$H_{\varkappa j}$ are defined by (17.6.8) *and for $n = 0$*

$$f_{0r\varkappa} = \sum_{1\le|\alpha|\le K} F_{r\varkappa}(D^\alpha\psi)\circ\bar{\Psi}(j\mu h)^{\frac{|\alpha|}{2}} \tag{17.6.26}$$

and the coefficients $F_{r\varkappa}$ do not depend on ψ.

Moreover, for $j\mu h \ge \epsilon_0$ all the functions are uniformly smooth.

Let us now consider the method of successive approximations applied without the reduction construction and without condition (17.3.30) and with the non-perturbed operator $\mathcal{A}_0^0 + \sigma_0 m + \bar{W}\cdot I$ where in $\bar{W}$ we set $x = y$ and skip all the terms with the factor μ^{-2}.

Let us note first of all that this construction is provided by the assumption $T\mu^{-1}h^{-1} \le \mu^{-\delta'}$ with the arbitrary small exponent $\delta' > 0$ (under conditions (17.4.1) or (17.5.1)–(17.5.2)$_{(2)}$) and therefore we need only the assumption $L \ge \mu^{\delta''-1}$ on the mollification parameter where $\delta'' > 0$ is arbitrarily small.

Furthermore, let us note that for $j > k \ge 1$ the blocks W_{jk} of the perturbed operator are

$$W_{jk} = \alpha^-_{(j-k)}(\mu^{-1}h)^{\frac{1}{2}(j-k)}\big(\frac{j!}{k!}\big)^{\frac{1}{2}}\Pi_- + \alpha^+_{(j-k)}(\mu^{-1}h)^{\frac{1}{2}(j-k)}\big(\frac{(j-1)!}{(k-1)!}\big)^{\frac{1}{2}}\Pi_+ \tag{17.6.27}$$

where $\Pi_\pm = \frac{1}{2}(I \pm \sigma_0)$ are projectors and $\alpha^\pm_{(l)} = \alpha^\pm_{(l)}(x_1, \hbar D_1, \hbar, j\hbar)$ are polynomials with respect to $z = j\hbar$; the blocks with $k > j \geq 1$ can be obtained by Hermitian conjugation.

The blocks of the perturbing operator with $j = k \geq 1$ have similar form but in them $\alpha^\pm_{(l)}(.,.,0,0) = 0$. Moreover, the blocks of the operator

$$\tilde{\psi} = \mathcal{T}^*\mathcal{F}^*\psi\mathcal{F}\mathcal{T} - \psi'(x_1, \hbar D_1)I$$

where $\psi' = \psi \circ \bar{\Psi}$ have the same form and in this case $\alpha_{\dots}$ are expressed in terms of ψ in the following way: these coefficients (and therefore the coefficients $\tilde{\psi}_{jk}$ as well) are linear combinations of $(D^\beta\psi)\circ\bar{\Psi}$ with $1 \leq |\beta| \leq K$ and in $\tilde{\psi}_{jk}$ every derivative $D^\beta\psi$ surely has a factor $j^r\hbar^{|\beta|}$ with $r \leq |\beta|$.

Finally, one can see easily that in $\tilde{\psi}_{jk}$ as well as in W_{jk} the coefficients $\alpha^\pm_{(l)}$ satisfy the equality $\alpha^+_{(l)}(.,.,0,.) = \alpha^-_{(l)}(.,.,0,.)$.

On the other hand, let us note that

$$\bigl(\tau - \sigma_0 m - \sigma_1\sqrt{2j\mu\hbar f}\bigr)^{-1} = \bigl(\tau + \sigma_0 m + \sigma_1\sqrt{2j\mu\hbar f}\bigr)\bigl(\tau^2 - m^2 - 2j\mu\hbar f^2\bigr)^{-1},$$

that $\sigma_1\Pi_\pm = \Pi_\pm\sigma_1$ and that the trace of the product of any number of matrices σ_1 and σ_0 (taken in any order) vanishes unless there is an even number of matrices each kind.

These arguments yield that modulo terms generating $O(\mu^{-s})$ after mollification with the indicated mollification scale L, the expression

$$\mp i \operatorname{tr} F_{t\to h^{-1}\tau}\Gamma\psi U^\pm \tag{17.6.28}$$

is (for $\pm \operatorname{Im}\tau < 0$) a sum of terms of the following two types:

(i) Terms of the form

$$\prod_{1\leq\iota\leq n+1} (\tau_1^2 - m^2 - 2j_\iota\mu\hbar f^2)^{-1}\times \\ (j\mu^{-1}h)^p(\mu^{-1}h)^q(j\mu h)^r(\mu h)^l\tau_1^\gamma G(j_2 - j_1, j_3 - j_1, \dots, j_n + 1 - j_1) \tag{17.6.29}$$

with $\tau_1 = \tau - \bar{W}$, $j_1 = j$, $p, q, r, l \in \frac{1}{2}\mathbb{Z}$, $\gamma, n \in \mathbb{Z}$, such that $p + r, q + l \in \mathbb{Z}$, $2(p + q) \geq \max(n, |j_1 - j_2| + \dots + |j_n - j_{n+1}|)$, $n + 1 \geq 2(r + l) + \gamma$ and with bounded coefficients $G(\dots)$.

(ii) Terms of the same form but with $j \le C$ and with G depending arbitrarily on j (in a non-polynomial way; these terms arise because in all the matrices $j, k \ge 0$ and blocks with $j = 0$ or $k = 0$ are special).

However, after we take the difference of the traces from the upper and lower complex half-planes these terms of the type (ii) will produce 0 for $|\tau| \le \epsilon$ under conditions (17.3.30) and (17.4.1). Therefore taking the difference of the traces from the upper and lower complex half-planes we again obtain formula (17.6.24) but with a more explicit formula for $f_{nr\varkappa}$.

In particular, let us calculate $f_{0r\varkappa}$; namely, $f_{0r\varkappa}$ are polynomials of $j\mu^{-1}h$, $\mu^{-1}h$, $j\mu h$ and $(m^2 + 2j\mu h f^2)^{-\frac{1}{2}}$ and all the monomials in these polynomials satisfy the following conditions:

$$1 + 2(\deg_1 + \deg_2) \ge \deg_5 \ge 2(\deg_3 + \deg_4)$$

where $\deg_k$ means the degree with respect to the k-th argument, $k = 1, \dots, 5$. Since these and the original formulae for $f_{\dots}$ should agree for $h^{\delta'} \le \mu h \le \epsilon$ we arrive to the following proposition:

Proposition 17.6.9. *In formula* (17.6.25) *the functions $f_{0r\varkappa}$ are polynomials with respect to $j\mu^{-1}h$, $\mu^{-1}h$ $j\mu h$, μh and $(m^2 + 2j\mu h f^2)^{-\frac{1}{2}}$ with the properties indicated above.*

Let us now calculate all the "other" terms in the decomposition without conditions (17.3.30) and (17.4.1). Moreover, without any loss of the generality one can assume that (17.5.1) holds. Let us temporarily assume that condition $(17.5.2)_{(2)}$ also is fulfilled.

One can see easily that these additional terms are due to the non-standard form of the blocks $\mathcal{A}^0_0$, W_{0k}, W_{k0}, ψ_{00}, ψ_{0k} and ψ_{k0}. Further, one can see easily that these terms have the same form as above but with functions $f_{n\varkappa r}$ non-regularly depending on j. Namely, these functions are now $O\big((\mu^{-1}h)^{\max(j,1)}\big)$.

However, in all cases the following remark is true:

Remark 17.6.10. All terms with $n = 0$ which are different from terms contained in $\partial_\tau h^{-2}\mathcal{N}^{\mathrm{MW}}(x, \tau, \mu h)$ have functions $f_{\dots}$ depending linearly on $D^\beta\psi \circ \bar{\Psi}$ with $1 \le |\beta| \le K$.

17.6.3 Main Theorem

Now we can obtain the main statements of this section. We need the following two definitions:

Definition 17.6.11. Let $d = 2$ and let conditions (17.1.1)–(17.1.3), (17.2.1), (17.3.1) be fulfilled. As $\tau \neq 0$ let us replace the potential V by the potential $V_\tau = V - \tau$ in the operator A. Let us introduce different *lacunary intervals*, namely

(i) $\tau \in \mathcal{Z}_{L1}$ if conditions $(17.2.28)_{1-3}$, (17.4.1), $(13.4.12)_1$, (17.3.30) and (17.6.2) are fulfilled with

$$v_\tau^{\#} = -(V-\tau)^2 + m^2 \tag{17.6.30}$$

instead of $v^{\#}$.

(ii) $\tau \in \mathcal{Z}_{L2}$ if conditions $(17.2.28)_{1-3}$, (17.4.1), $(13.4.12)_1$ and (17.6.5) are fulfilled for all $j \in \mathbb{Z}^+$.

(iii) $\tau \in \mathcal{Z}_{L3}$ if conditions $(17.2.28)_{1-3}$, (17.5.1), $(17.5.2)_{(2)}$ and (17.6.6) are fulfilled for all $j \in \mathbb{Z}^+$ and $\bar{\nu} \geq \mu^{-l}$.

(iv) $\tau \in \mathcal{Z}_{L4}$ if conditions $(17.2.28)_{1-3}$, (17.5.1), $(17.5.2)_{(2)}$ and (17.6.7) are fulfilled for all $(\varkappa, j) \in \mathfrak{Z}$.

Definition 17.6.12. Let $d = 2$ and let conditions (17.1.1)–(17.1.3), $(17.5.2)_{(2)}$, (17.3.1), (17.5.1), (17.5.41), (17.5.33), (17.6.15) and (17.6.16) be fulfilled with small enough $\delta > 0$ in (17.5.41). Then $\tau \in \tilde{\mathcal{Z}}_{L5}$.

Now repeating arguments of theorems 13.6.6, 13.6.7 we easily obtain the following:

Proposition 17.6.13. *Let $d = 2$ and let conditions* (17.1.1)–(17.1.3), (17.2.1) *and* (17.3.1) *be fulfilled. Then*

(i) If $\tau_0 \in \mathcal{Z}_L := \mathcal{Z}_{L1} \cup \dots \cup \mathcal{Z}_{L4}$ then for the L indicated above the following estimate holds:

$$\Bigl|\int \phi\bigl(\frac{\tau-\tau_0}{L}\bigr)\Bigl(\Gamma\psi e(\tau,\tau_0) - h^{-2}\int \mathcal{N}^{\mathsf{MW}}(x,\tau,\tau_0,\mu h)\psi\,dx - \sum_{\varkappa,n,r:n\leq 2r,r\leq N}\sum_j \Theta^{(n)}_{j,\varkappa nr}(\tau,\tau_0)(\mu^{-1}h)^r\Bigr)\Bigr| \leq C'\mu^{-s}h^s \tag{17.6.31}$$

where $\Theta^{(n)}_{j,\dots}(\tau,\tau_0)=\Theta^{(n)}_{j,\dots}(\tau_0)-\Theta^{(n)}_{j,\dots}(\tau)$ *and* $\Theta^{(n)}_{j,\dots}(\tau_0)=0$ *here for* $r\ge 1$.

(ii) Moreover, $\Theta_{j,\dots}$ *are defined by* (17.6.25), (17.6.26) *and the functions* $f\dots$ *are uniformly bounded and smooth with respect to all their arguments for* $\tau\in\mathcal{Z}_{L1}$.

(iii) If $\tau_0\in\tilde{\mathcal{Z}}_{L5}$ *then the left-hand expression in* (17.6.31) *does not exceed* $C'\mu\nu\zeta^{-2}h_1^s$.

Definition 17.6.14. Let $d=2$ and let us assume that there exists parameter $\eta\in(0,1)$ such that after multiplication by η^{-1} the operator A satisfies (17.1.1)–(17.1.3) with $h'=h\eta^{-1}$, $\mu'=\mu\eta^{-1}$, $m'=m\eta^{-1}$.

Further, let us assume that after this procedure $\tau'=\tau\eta^{-1}\in\mathcal{Z}_{Lk}(A')$ where A' is a new operator. Then $\tau\in\mathcal{Z}'_{Lk}:=\mathcal{Z}'_{lk}(A)$ for the original operator. Furthermore, $\tilde{\mathcal{Z}}'_{L5}$ is defined in the same way.

Proposition 17.6.13 then obviously yields

Corollary 17.6.15. *Estimate* (17.6.31) *with* $L\eta$ *instead of* L *holds for* $\tau\in\mathcal{Z}'_L$.

The same arguments yield the following

Theorem 17.6.16. *Let* $d=2$ *and let conditions* (17.1.1)–(17.1.3), (17.2.1) *and* (17.3.1) *be fulfilled. Then for* $\tau_1,\tau_2\in\mathcal{Z}'_L$ *the following estimate holds:*

$$|\Gamma\psi e(\tau_1,\tau_2)-h^{-2}\int\mathcal{N}^{\mathsf{MW}}(x,\tau_1,\tau_2,\mu h)\psi\,dx- \sum_{\varkappa,r\le N}\sum_j\Theta_{j,\varkappa 0r}(\tau_1,\tau_2)(\mu^{-1}h)^r|\le C'\mu^{-s}h^s. \tag{17.6.32}$$

In this theorem the values of the parameter η *for* τ_1 *and* τ_2 *do not necessarily coincide.*

Remark 17.6.17. $\tau\in\tilde{\mathcal{Z}}'_{L5}$ can be defined similarly. The case when $\tau_1\in\tilde{\mathcal{Z}}'_{L5}$ and $\tau\in\mathcal{Z}'_L$ or when $\tau_1,\tau_2\in\tilde{\mathcal{Z}}'_{L5}$ can be treated using by Theorem 17.6.16 and Proposition 17.6.4.

17.7 Generalizations

17.7.1 Propagation of Singularities and Improved Asymptotics

We need to revise all the results of Sections 17.2–17.5 in the manner of Section 13.7. This is very routine task and we live it to the reader just writing down dynamics:

Case $d = 3$

In this case propagation is along magnetic lines on energy level 0 and equation of the fastest movement (as rotational number is 0) is

$$\frac{dx_j}{dt} = \frac{\sqrt{V^2 - m^2}}{FV} F^j \qquad (j = 1, 2, 3) \tag{17.7.1}$$

where F^j are components of the vector intensity of the magnetic field and we assume that $V^2 - m^2 \gtrsim 1$. The natural physical system (17.7.1) should be normalized as $m \gg 1$ by adding factor $(m+1)$ to the right-hand expression:

$$\frac{dx_j}{dt} = (m+1)\frac{\sqrt{V^2 - m^2}}{FV} F^j \qquad (j = 1, 2, 3) \tag{17.7.2}$$

and as $V = m + W/m$, $m \to \infty$ it becomes

$$\frac{dx_j}{dt} = (m+1)\frac{\sqrt{2W}}{F} F^j \qquad (j = 1, 2, 3) \tag{17.7.3}$$

while for $m = 0$ it becomes

$$\frac{dx_j}{dt} = \frac{1}{F} F^j \qquad (j = 1, 2, 3). \tag{17.7.4}$$

Case $d = 2$

In this case all the results for the Schrödinger operator remain true. Now under condition (17.3.30) the drift lines are defined by the equations

$$\frac{dx}{dt} = -\frac{1}{2\mu V}\bigl(\nabla\bigl(\frac{V^2 - m^2}{F}\bigr)\bigr)^{\perp} \tag{17.7.5}$$

as we assume that $V^2 - m^2 \gtrsim 1$. The natural physical system (17.7.5) should be normalized as $m \gg 1$ by adding factor $(m+1)$ to the right-hand expression:

$$\frac{dx}{dt} = -\frac{m+1}{2\mu V}\bigl(\nabla\bigl(\frac{V^2-m^2}{F}\bigr)\bigr)^{\perp} \tag{17.7.6}$$

and as $V = m + W/m$, $m \to \infty$ it becomes

$$\frac{dx}{dt} = -\mu^{-1}\bigl(\nabla\bigl(\frac{W}{F}\bigr)\bigr)^{\perp}. \tag{17.7.7}$$

17.7.2 Getting Rid of the Condition "$v^{\#} \le -\epsilon_0$"

Let us consider the violation of the condition $v^{\#} \le -\epsilon_0$. Recall that for $d = 3$ we need this condition only for $m \le h^{-1+\delta}$ and for $d = 2$ we need it only for $\mu h \le \epsilon_0$.

Case $d = 3$

Case $m \gtrsim 1$. Let us assume first that $m \gtrsim 1$. Then one can assume that

$$V = \pm m + \frac{1}{m}W$$

with

$$|D^{\alpha}W| \le c \qquad \forall \alpha : |\alpha| \le K.$$

Let us assume for the sake of definiteness that the sign is "+" here. Then the assumption $v^{\#} \le -\epsilon_0$ means exactly that

$$W \ge \epsilon_0 \tag{17.7.8}$$

(with another constant $\epsilon_0 > 0$). Let us treat the following cases:

(a) $v^{\#} \ge \epsilon_0 \nu$ in $B(0,1)$;

(b) $v^{\#} \le -\epsilon_0 \nu$ in $B(0,1)$;

(c) $|v^{\#}| \le c\nu$ in $B(0,1)$.

In the cases (a) and (b) let us divide the operator A by $r = |v^{\#}(0)|^{\frac{1}{2}}$ and let us introduce $h' = h/r$, $\mu' = \mu/r$, $m' = m/r$ and $\nu' = \nu/r^2$. One can easily see that

$$\frac{1}{r}V = m' + \frac{1}{m'}W' \qquad \text{with} \qquad |D^{\alpha}W'| \le c \quad \forall \alpha : |\alpha| \le K.$$

Therefore we are respectively in the frameworks of the cases

(a)′ $v^{\#\prime} \ge \epsilon_0$ in $B(0,1)$;

(b)′ $v^{\#\prime} \le -\epsilon_0$ in $B(0,1)$.

Applying the arguments of Subsection 13.8.1 we obtain that in the cases (a) and (b) the condition "$v^{\#} \le -\epsilon_0$" (or "$v^{\#} \ge \epsilon_0$") in our theorems is not necessary.

Similarly, in the case (c) we take $r = \sqrt{\nu}$ and apply the same arguments. In these all cases $h' \le 1$ for $d = 3$ (because we assume that $\nu \ge h^2$); for $d = 2$ the condition ${\mu'}^{1-\delta} \ge h'$ follows from the same condition for μ, h and the restriction $\nu \ge \mu^{-l}h^l$; the same assertion is also true for condition (17.5.40).

Repeating the arguments of Subsection 13.8.1 and referring to Theorem 17.5.7 for the case $\mu' h' \ge 1$ and with a partition of unity such that $\mu' h' \le 1$ we can easily prove

Proposition 17.7.1. *Theorems 17.4.14, 17.4.15, 17.4.16, 17.4.17, 17.4.21, 17.4.22 remain true for* $m \ge 1$ *without condition* (17.3.30).

Case $m \lesssim 1$. Now let $m \lesssim 1$. Then the following cases should be considered (if necessary we apply an ϵ-admissible partition of unity):

(a) $r = |v^{\#}(0)| + \nu + \mu h \le cm^2$;

(b) $|v^{\#}(0)| \ge \epsilon_0 r \ge cm^2 + \mu h$;

(c) $\mu h \ge \epsilon_0 r \ge m^2 + |v^{\#}(0)| + \nu$.

(d) $|v^{\#}(0)| \le \epsilon_0 \nu$, $m^2 + \mu h \le \epsilon_0 \nu$.

In the case (a) division by m reduces this case to the case $h' = h/m$, $\mu' = \mu/m$, $m' = 1$, $\nu'\nu/m^2$ which has been treated already.

Similarly, in the case (b) division by $\sqrt{r}$ reduces this case to the case $h' = h/\sqrt{r}$, $\mu' = \mu/\sqrt{r}$, $\nu' = \nu/r$, $m' = m/\sqrt{r}$, $|v^{\#}(0)| \ge \epsilon_0$.

In the case (c) we get $\mu' h' = 1$ and we do not need condition (17.3.30).

In the case (d) the reduction is not complete: we can only conclude that

$$|D^\alpha V| \le c\nu' \qquad \forall \alpha : |\alpha| \le K'$$

with $\nu' = \nu^{\frac{1}{2}-\delta'}$ with any exponent $\delta' > 0$ and K' (and then $K = K(K', \delta')$ in the original conditions). Therefore one can either divide by ν' or change variables $x' = x\nu^{-\delta'}$ and divide by $\sqrt{\nu}$. In the both cases some "loss of information" happens.

Let $\mu \le h^{-1+\delta}$ as before and let us now assume $m \le 1$. Let us introduce $\rho = \gamma$ by formula (13.8.4) and let us consider the zone $\{\rho \ge \bar\rho = m^{\frac{1}{2}} + (\mu h)^{\frac{1}{4}-\delta'}\}$. Let us consider a ρ-admissible partition of unity in this zone and let us divide the operator by ρ^2 on each element of the partition.

On this element let us introduce $m' = m\rho^{-2}$, $h' = h\rho^{-3}$, $\mu' = \mu\rho^{-1}$, $\nu' = \nu\rho^{-3}$. Then for $d = 3$ the contribution of every element of the partition to the remainder estimate does not exceed $Ch^{-2}\rho^4$ and the total contribution does not exceed Ch^{-2}. So we have proven the following:

Proposition 17.7.2. *Theorem 17.4.14 remains true without conditions* (17.3.30) *and* $m \gtrsim 1$.

The same arguments give us the following

Proposition 17.7.3. *Theorem similar to 13.7.18 remains true without conditions* (17.3.30) *and* $m \ge 1$.

Case $d = 2$

Let us treat now the case $d = 2$. The same arguments as before yield the following:

Proposition 17.7.4. *(i) Theorems 17.4.15, 17.4.16, 17.4.17, 17.4.21 and 17.4.21 remain true without condition* (17.3.30) *in all the cases excluding* $m \le 1$, $1 \ge \nu \ge \epsilon_0 m^2$.

(ii) In the latter case these theorems remain true under the additional restriction

$$|D^\alpha\big(\frac{V}{\sqrt{F}}\big)| \le c\sqrt{\nu'} \qquad \forall \alpha : 1 \le |\alpha| \le K \tag{17.7.9}$$

fulfilled with $\nu' = \nu$. *This condition is fulfilled automatically with* $\nu' = \nu^{1-\delta'}$ *with arbitrary* $\delta' > 0$.

17.7.3 Getting Rid of the Condition "$F \gtrsim \epsilon_0$"

Let us now assume that condition (13.3.1) is violated.

Case $d = 3$

We know that cases $d = 3$ and $d = 2$ are vastly different; let us consider case $d = 3$ first. Let condition $(13.8.19)_n$ be fulfilled[29]. Let us apply arguments of Subsection 13.8.2. If $\mu h \le 1$ then $\mu' h' \le 1$ also and therefore we have the following

Proposition 17.7.5. *Let $d = 3$, let conditions* (17.1.1)–(17.1.3) *be fulfilled and let $\mu h \le 1$. Then under condition* $(13.8.19)_n$ *estimate* (13.8.20) *holds and in the general case estimate* (13.8.23) *holds.*

Recall that the principal part is $O(h^{-3})$ and under an obvious condition it is $\asymp h^{-3}$.

On the other hand, if $\mu h > 1$ we need to consider the zone $\{\mu' h' \ge 1, h' \le 1\}$ since all other zones have been already considered in Subsection 13.8.2. Surely, we can always include h in γ and keep $\gamma \ge h$ but then in the zone $\{\gamma \le C_0 h\}$ condition (13.3.1) is not necessarily fulfilled after rescaling. One can treat this zone by the variational method but here we simply avoid the trouble assuming that $\mu \le h^{-2}$ (and then this zone does not appear). Therefore we have the following

Proposition 17.7.6. *Let $d = 3$, let conditions* (17.1.1)–(17.1.3) *be fulfilled and let $h^{-1} \le \mu \le h^{-2}$. Then the estimate* (13.8.23) *holds with arbitrarily small $\delta > 0$. Recall that the principal part is $O(\mu h^{-2})$ and under an obvious condition it is $\asymp \mu h^{-2}$.*

Remark 13.8.8 also remains true.

Case $d = 2$

Now let $d = 2$ and let condition (17.3.30) be fulfilled. We consider the following problem to be difficult and challenging enough (but not exceedingly for PhD thesis):

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[29] With n instead of m here and below.

Problem 17.7.7. Generalize results of Chapter 14 to magnetic Dirac operators. Here

(i) Analysis in the regular zone $X_{\mathsf{reg}} = \{|x_1| \ge C_0 \max(\mu^{-1\nu}, h)\}$ seems to be straightforward.

(ii) Analysis in the degenerate zone $X_{\mathsf{deg}} = \{|x_1| \le C_0\mu^{-1\nu}\}$ seems to be more subtle but not overly difficult as $\mu h^{\nu} \le \epsilon$.

(iii) However analysis in the degenerate zone $X_{\mathsf{deg}} = \{|x_1| \le C_1 h\}$ is definitely much more challenging as $\mu h^{\nu} \ge \epsilon$ because it requires analysis of the eigenvalues of 1-dimensional Dirac operator

$$\boldsymbol{a} = \sigma_1 D_1 + \sigma_2(\eta - x_1^{\nu}/\nu) + \sigma_0 m. \tag{17.7.10}$$

17.7.4 Other Generalizations

We consider the following problems to be difficult and challenging enough (but not exceedingly for PhD thesis):

Problem 17.7.8. Generalize results of Chapter 15 to

(i) Magnetic Schrödinger-Pauli operators. Here one needs to start from describing appropriate boundary conditions:

$$(u(x), \Lambda u(x)) \in E(x) \qquad \text{as } \ x \in \partial X, \qquad \Lambda := \sum_{j,l} \sigma_l \big(\omega^{jl} P_j + P_j \omega^{jl}\big) \tag{17.7.11}$$

where $E(x)$ is 2-dimensional space such that

$$(\sum_j \sigma_j n_j v, w) - (v, \sum_j \sigma_j n_j w) = 0 \qquad \forall v, w \in E(x), \tag{17.7.12}$$

(ii) Magnetic Dirac operators. Here one needs to start from describing appropriate boundary conditions:

$$u(x) \in E(x) \qquad \text{as } \ x \in \partial X, \tag{17.7.13}$$

where $E(x)$ is 1-dimensional space such that

$$(\sum_j \sigma_j n_j v, w) = 0 \qquad \forall v, w \in E(x), \tag{17.7.14}$$

$n(x)$ is a normal to ∂X at the point x.

Problem 17.7.9. Generalize results of Chapter 16 to

(i) Magnetic Schrödinger-Pauli operators. Again we need to assume that $V \le -\epsilon_0$.

(ii) Magnetic Dirac operators. Again we need to assume that $v^{\#} \le -\epsilon_0$.

Problem 17.7.10. Consider operator with m replaced by $J(x)$ satisfying

$$(17.7.15)_{1,2} \qquad |D^{\alpha} J(x)| \le cm \quad \forall \alpha : |\alpha| \le K, \qquad c^{-1} m \le J(x) \le cm.$$

It could be useful if we generalize the results of Chapter 23 for the global asymptotics from magnetic Schrödinger operators to the magnetic Dirac operators.

Microlocal Analysis, Sharp Spectral Asymptotics and Applications. I

Semiclassical Microlocal Analysis and Local and Microlocal Semiclassical Asymptotics

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Part VIII

Non-Smooth Theory and Higher Dimensions

Chapter 18

2D- And 3D-Magnetic SchrÖdinger Operator with Irregular Coefficients

18.1 Introduction

18.1.1 Preliminary

This Chapter is a continuation of Chapter 13 and Sections 4.6 and 5.4. Here we deal with the Schrödinger operator with the strong magnetic field in dimensions 2 and 3 under weak smoothness assumptions.

Keeping in mind the crucial role played by the microlocal canonical form in Chapter 13, devoted to such operators in the smooth case, the lack of the ultimate smoothness here seems to be the catastrophic because complete microlocal canonical form requires such smoothness. Instead, in this Chapter we will make a *partial reduction* with a remainder which is just small enough for our needs but not excessively small, and such reduction requires much lesser smoothness.

This reduction serves two purposes.

First, if the magnetic field is not very strong $\mu \le h^{-\frac{1}{3}}$ [1)], we will use it only to prove that the singularities leave the diagonal and propagate away from it and thus the time interval for which we have the standard decomposition

[1)] We skip logarithmic factors $|\log h|^{\beta}$ in this preliminary part.

of the propagator trace could be extended from $\{|t| \le T_* = \bar{T} := \epsilon\mu^{-1}\}$ [2)] to a larger one thus improving remainder estimate $O(\mu h^{1-d})$ which is due to this rescaling as well.

Namely, for $d = 2$ we use the non-degeneracy assumption $|\nabla(V/F)| \ne 0$ and extend time to $\{|t| \le T^* = \epsilon\mu\}$ thus improving estimate to $O(\mu^{-1}h^{-1})$. We also consider a weaker non-degeneracy assumption

$$|\nabla(V/F)| + |\det \mathsf{Hess}(V/F)| \ne 0$$

but it will require at the end more smoothness.

For $d = 3$ no non-degeneracy assumption at this stage is needed: we consider zone $\{|\xi_3| \ge C\mu^{-1}\}$ where ξ_3 is a velocity along magnetic field and we use this movement to prove that the singularities leave the diagonal, increasing time interval to $\{|t| \le T^* = T^*(\xi_3) = \epsilon|\xi_3|\}$ [3)] and another zone $\{|\xi_3| \le C\mu^{-1}\}$ which is small enough to estimate it contribution to the remainder by $O(\mu^{\frac{3}{2}}h^{-\frac{3}{2}}) = O(h^{-2})$.

On the other hand, if the magnetic field is strong enough, we use this canonical form to derive spectral asymptotics as well. For $d = 3$ and $\mu h \le 1$ we do this only in the zone $\{|\xi_3| \le C_0(\mu h)^{\frac{1}{2}}\}$, treating zone $\{|\xi_3| \ge C_0(\mu h)^{\frac{1}{2}}\}$ in the same way as before. Only at this stage the qualified estimate of the remainder in canonical form is needed.

Combining the idea of this "poor man canonical form" with the ideas of Chapter 13, and Sections 4.6 and 5.4 we manage to derive sharp spectral asymptotics under very moderate smoothness conditions: namely,

(a) For $d = 2$ assuming $\mathscr{C}^{2,1}$-regularity under non-degeneracy assumption $\nabla(V/F) \ne 0$ we recover remainder estimate $O\big(\mu^{-1}h^{-1} + 1\big)$ (and the same or almost the same remainder estimate under weaker non-degeneracy assumption) – exactly as in the smooth case; under weaker smoothness assumption remainder estimate would be larger.

(b) For $d = 3$, assuming $\mathscr{C}^{1,2}$-regularity we recover remainder estimates $O\big(h^{-2} + \mu h^{-1}\big)$ (and a the same remainder estimate under weaker non-degeneracy assumption) – exactly as in the smooth case.

(c) For $d = 3$ without any non-degeneracy assumption we recover remainder estimate $O\big(h^{-2} + \mu h^{-1-2/(l+2)}\big)$ in $\mathscr{C}^{l,\sigma}$-case, with appropriate σ.

[2)] This result is just a rescaling of the standard theory (i.e. the theory with $\mu = 1$).

[3)] Actually, a bit better but we skip logarithmic factors here.

Remark 18.1.1. (i) Actually in the case of stronger magnetic field we always assume that the metrics and the magnetic field should belong to $\mathscr{C}^{2,1}$.

(ii) To recover remainder estimate $O(1)$ for $d=2$ and $\mu \gg h^{-1}$ we need to assume that the metrics and the magnetic field should belong to $\mathscr{C}^{4}$.

Recall that here and below we use

Definition 18.1.2. $\mathscr{C}^{l,\sigma}$ denotes a space of functions which $\lfloor l \rfloor$-derivatives are continuous with the continuity modulus $|x-y|^{l-\lfloor l\rfloor}\big|\log|x-y|\big|^{-\sigma}$ if $l\notin\mathbb{N}$ or $l\in\mathbb{N}$ and $\sigma\ge 0$; for $l\in\mathbb{N}$ and $\sigma<0$ one should take $l-1$ instead of $\lfloor l\rfloor$. As $l\in\mathbb{N}$ and $\sigma=0$ $\mathscr{C}^{l,0}:=\mathscr{C}^{l}$.

Also σ will be positive and large enough in the most of the statements[4)].

In this Chapter we consider $d=2$ and $d=3$ only. For $d\ge 4$ the sharp remainder estimate was not derived so far in this book even in the smooth case due to lack of the canonical form[5)]. However, in the next forthcoming Chapters 19[6)] and 20[7)] we will to cover multidimensional case as well, using the idea of the partial canonical form introduced here. However, a higher smoothness than here may be required.

So, let us consider Schrödinger operator in the strong magnetic field (13.2.1) i.e.

$$A = A_0 + V(x), \qquad A_0 = \sum_{j,k\le d} P_j g^{jk}(x) P_k, \tag{18.1.1}$$

$$P_j = hD_j - \mu V_j(x), \quad \mu \ge 1$$

with real symmetric positive metric tensor (g^{jk}), real-valued V_j, V and small parameter h and large parameter μ.

We assume that the corresponding *magnetic field intensity*

$$F^{l}(x) = \frac{1}{2}\sum_{jk} \varepsilon^{jkl}\big(\partial_{x_j} V_k - \partial_{x_k} V_j\big) \tag{18.1.2}$$

[4)] In contrast to non-magnetic case we are not obsessed here to making σ as small as possible but we are still pushing hard l down.

[5)] Because of variable multiplicities of the eigenvalues of the matrix magnetic (F_{jk}) intensity and their "high-order resonances" when one of their linear combination with integer coefficients vanishes.

[6)] Provided $\mathsf{rank}(F_{jk})(x)=d$ at each point x.

[7)] Provided $\mathsf{rank}(F_{jk})=2r<d$ at each point x.

with the standard absolutely skew-symmetric tensor ε^{jkl} [8)] does not vanish in domain in question. Note that for $d=2$ automatically $F^1=F^2=0$ and thus magnetic field is reduced to F^3.

Remark 18.1.3. For $d=3$ we can straighten vector field $\boldsymbol{F}=(F^1,F^2,F^3)$ and direct it along x_3 by an appropriate choice of variables x_1,x_2 and make $V_3=0$ by an appropriate *gauge transformation* $A\mapsto e^{-i\mu h^{-1}\Phi(x)}Ae^{i\mu h^{-1}\Phi}$.

We assume that this is has been done and impose smoothness conditions to the reduced operator.

Thus we assume that for this reduced operator

$$V_3=0,\qquad \partial_{x_3}V_k=0\quad k=1,2 \tag{18.1.3}$$

where the second condition is due to the first one and assumption $F^1=F^2=0$; then one can reduce it further to $V_2=0$ and then

$$\boldsymbol{F}=(0,0,F^3),\quad F^3=g^{-\frac{1}{2}}(\partial_{x_1}V_2-\partial_{x_2}V_1),\qquad g=\det(g^{jk})^{-1} \tag{18.1.4}$$

and the scalar intensity of magnetic field is

$$F=g_{33}^{\frac{1}{2}}|F^3|=(g^{11}g^{22}-g^{12}g^{21})^{\frac{1}{2}}|(\partial_{x_1}V_2-\partial_{x_2}V_1)|, \qquad (g_{jk})=(g^{jk})^{-1}. \tag{18.1.5}$$

In particular, for $d=2$ one can assume that $g^{33}=1$, $g^{3j}=0$ for $j=1,2$; without taking absolute value we get *pseudo-scalar intensity* then.

Further, we assume that

$$F\ge c^{-1}. \tag{18.1.6}$$

Multiplying operator by $F^{-\frac{1}{2}}$ from both sides one can further reduce to the case

$$F=1; \tag{18.1.6)*}$$

then the operator in question will be

$$\sum_{j,k} P_jF^{-1}g^{jk}P_k+F^{-1}(V-h^2V') \tag{18.1.7}$$

[8)] With non-zero components equal $\pm g^{\frac{1}{2}}$ where here and in what follows $g=\det(g^{jk})$ and $(g_{jk})=(g^{jk})^{-1}$.

with

$$V' = F^{\frac{1}{2}} \sum_{j,k} \partial_j \bigl(g^{jk} \partial_k F^{-\frac{1}{2}}\bigr) \tag{18.1.8}$$

and this perturbation will not affect our results unless $d = 2$, $\mu \ge h^{-1}$ (in which case V should be corrected).

18.1.2 Smooth Case: Survey

From Chapter 13 we know that for $d = 2$ in the smooth case one can reduce operator further to the microlocal canonical form

$$\mu^2 \sum_{m+n+p\ge 1} \omega_{m,n,p}(x_2, \mu^{-1}hD_2)\bigl(x_1^2 + \mu^{-2}h^2D_1^2\bigr)^m \mu^{-2n-p}h^p \tag{18.1.9}$$

with the principal part

$$F(x_2, \mu^{-1}hD_2)\bigl(\mu^2 x_1^2 + h^2 D_1^2\bigr) + V(x_2, \mu^{-1}hD_2) \tag{18.1.9$_0$}$$

and for $d = 3$ in the smooth case one can reduce operator to the microlocal canonical form

$$\mu^2 \sum_{m+n+p+q\ge 1} \omega_{m,q,k,p}(x_2, x_3, \mu^{-1}hD_2)\times \bigl(x_1^2 + \mu^{-2}h^2D_1^2\bigr)^m (hD_3)^q \mu^{-2m-2q-2k-p}h^p \tag{18.1.10}$$

with the principal part

$$F(x_2, \mu^{-1}hD_2)\bigl(\mu^2 x_1^2 + h^2 D_1^2\bigr) + h^2D_3^2 + V(x_2, x_3, \mu^{-1}hD_2). \tag{18.1.10$_0$}$$

Then decomposing into Hermitian functions of $\mu^{\frac{1}{2}}h^{-\frac{1}{2}}x_1$ one can replace harmonic oscillator $(h^2D_1^2 + \mu^2 x_1^2)$ by its eigenvalues $(2j+1)\mu h$ with $j \in \mathbb{Z}^+$. So, basically for $d = 2$ our operator is a kind of the mixture of the usual second order operator and a family of $\mu^{-1}h$-pseudodifferential operators while for $d = 3$ it is a kind of the mixture of the usual second order operator and a family of h-differential 1-dimensional Schrödinger operators with respect to x_3 which are also $\mu^{-1}h$-pseudodifferential operators with respect to x_2. This explains the major differences between $d = 2$ and $d = 3$:

For $d = 2$ there are sharp Landau levels $(2j+1)\mu h$ (the eigenvalues of the infinite multiplicity as g^{ik} and V are constant and domain is $\mathbb{R}^2$); without

some non-degeneracy assumption one cannot expect the remainder estimate better than $O(\mu h^{-1})$ and with an appropriate non-degeneracy assumption one can improve it up to $O(\mu^{-1}h^{-1}+1)$ because only $\asymp (\mu h)^{-1}$ of Landau levels are not classically forbidden.

Meanwhile, for $d=3$ Landau levels are rather bottoms of the branches of the continuous spectrum and the non-degeneracy assumption is not that crucial: the best possible remainder estimate is $O\big(h^{-2}+\mu h^{-1}\big)$ while the worst possible remainder estimate depends on the smoothness of the reduced operator (presuming nothing wrong happened during reduction) and is $O\big(h^{-2}+\mu h^{-1-2/(l+2)}\big)$ in the heuristic accordance with Section 5.4.

Finally, for $d\ge 4$ Landau levels are $\sum_{1\le k\le r}(2j_k+1)f_k\mu h$ where $f_k>0$, $\pm i f_k$ are non-zero eigenvalues of the matrix magnetic intensity F, $2r=\mathsf{rank}\, F$ and $j\in \mathbb{Z}^{+\,r}$, and one can think operator to be similar to 2-dimensional as $\mathsf{rank}\, F=d$ and to 3-dimensional as $\mathsf{rank}\, F<d$ but there is no complete canonical form even if $\mathsf{rank}\, F$ and multiplicities of f_k are constant because $\sum_k j_k f_k$ can vanish for some $j\in\mathbb{Z}^r$ at some points (and we call this *resonances*). This leads to the difficulties in our analysis which we tackle in Chapters 19[6] and 20[7].

18.1.3 Non-Smooth Case: Heuristics as $d=2$

In the non-smooth case we do not have the complete canonical form but actually we do not need it. Further, if magnetic field is not very strong we do not need even logarithmic uncertainty principle[9] (18.1.12) below to justify quantization of $V(x_2,\xi_2)$ because we do not quantize it in this case.

Consider ξ_2-shift for time T in the propagation. Note that it does not exceed CT and under non-degeneracy assumption $|\partial_{x_2}V|\ge\epsilon$ the shift is exactly of magnitude $\asymp T$ as $T\le \epsilon\mu$. This shift is "observable" on the "quantum level" and the propagator trace vanishes provided logarithmic uncertainty principle $T\times\varepsilon\ge Ch|\log h|$ holds; here x_2- scale is ε.

In order to be able to take T as small as $\bar{T}=\epsilon\mu^{-1}$ we need to pick up

$$\varepsilon = C\mu h|\log h| \tag{18.1.11}$$

which in addition to the semiclassical remainder estimate $O(\mu^{-1}h^{-1})$ brings an approximation error $O(h^{-2}\varepsilon^{l}|\log h|^{-\sigma})$ for coefficients belonging to $\mathscr{C}^{l,\sigma}$.

[9] Recall that in the non-smooth case we approximate operator in question by operator with ε-smooth coefficients.

This error will be small for $\mu \le h^{1-\delta}$ and $\delta = 2/(l+2)$ which would take care of $\mu \le h^{1-\delta}$ in the smooth case but we do not have such luxury now.

As μ grows larger we need to treat our operator as a family of $\mu^{-1}h$-pseudodifferential operators and then we need the logarithmic uncertainty principle $\varepsilon \times \varepsilon \ge C\mu^{-1}h|\log \mu|$

$$\varepsilon = C(\mu^{-1}h|\log \mu|)^{\frac{1}{2}} \tag{18.1.12}$$

which in addition to the semiclassical remainder estimate $O(\mu^{-1}h^{-1})$ brings an approximation error $O\big(h^{-2}\varepsilon^{l}|\log h|^{-\sigma}\big)$ as $\mu \lesssim h^{-1}$.

On the other hand, for $\mu \gtrsim h^{-1}$ and we consider the Schrödinger-Pauli operator, the semiclassical remainder estimate is $O(1)$ and an approximation error is $O\big(\mu h^{-1}\varepsilon^{l}|\log h|^{-\sigma}\big)$ from approximation of V but also $O\big(\mu^{2}\varepsilon^{\bar{l}}|\log h|^{-\bar{\sigma}}\big)$ from approximation of g^{jk}, V_j where we assume that $g^{jk}, F \in \mathscr{C}^{\bar{l},\bar{\sigma}}$.

An actual reduction and calculations are the very significant part of our analysis.

For the best results we need to pick ε to be the minimum of (18.1.11) and (18.1.12) i.e.

$$\varepsilon \coloneqq C\begin{cases} \mu h|\log h| & \text{as } \ \mu \le (h|\log h|)^{-\frac{1}{3}} \\ (\mu^{-1}h|\log \mu|)^{\frac{1}{2}} & \text{as } \ \mu \ge (h|\log h|)^{-\frac{1}{3}} \end{cases} \tag{18.1.13}$$

and the threshold is $\mu_1^* \coloneqq (h|\log h|)^{-\frac{1}{3}}$. Below it magnetic field is considered to be *weak* and above it *temperate* (and we treat these two cases differently). The second threshold is $\mu_2^* \coloneqq h^{-1}$; for μ above it we modify our conditions and only finite number of Landau levels matter; we call magnetic field *strong* then.

18.1.4 Non-Smooth Case: Heuristics as $d = 3$

In contrast to 2-dimensional case, there is a movement along x_3 with the speed $|\xi_3|$ and for time T shift with respect to x_3 is $\asymp T\cdot|\xi_3|$ (provided $|\xi_3| \ge C\mu^{-1}$ and $T \le |\xi_3|$). This shift is "observable" on the "quantum level" and the propagator trace vanishes provided the logarithmic uncertainty principle $T|\xi_3|\times|\xi_3| \ge Ch|\log h|$ holds; also we need to assume that $\varepsilon\times|\xi_3| \ge Ch|\log h|$. Here scale with respect to ξ_3 is $\epsilon|\xi_3|$.

Plugging $T = \bar{T} = \epsilon\mu^{-1}$ we arrive to the inequality

$$|\xi_3| \ge \bar{\rho} \coloneqq C\max\big(\mu^{-1}, (\mu h|\log h|)^{\frac{1}{2}}\big) \tag{18.1.14}$$

and also we need to take ε depending on ξ_3: $\varepsilon = Ch|\log h|/|\xi_3|$. Then the contribution of the *outer zone* $\{|\xi_3| \ge \bar\rho\}$ to the approximation error will be $O(h^{-2})$ under $\mathscr{C}^{1,2}$ regularity of the coefficients.

Further, in this case one can replace $T^*(\xi_3) = \epsilon|\xi_3|$ by a bit larger $T^*(\xi_3) = \epsilon|\xi_3|\big|\log|\xi_3|\big|^2$ and then the contribution of this zone to the remainder does not exceed $Ch^{-2}\int T^{*\,-1}(\xi_3)\,d\xi_3$ which does not exceed Ch^{-2} for this increased $T^*(\xi_3)$.

On the other hand, the contribution of the *inner zone* $\{|\xi_3| \le \bar\rho\}$ to the remainder does not exceed $C\mu\bar\rho h^{-2}$ which in turn does not exceed Ch^{-2} as $\mu \le \mu_1^* := (h|\log h|)^{-\frac{1}{3}}$. In this case we take $\varepsilon = Ch\log h\bar\rho^{-1}$ in the inner zone and then its contribution to the approximation error does not exceed Ch^{-2} as well. In fact we will increase this threshold to $\mu_1^* := h^{-\frac{1}{3}}$.

For $\mu \ge h^{-\frac{1}{3}}$ we will consider the outer zone in the same way as above but in the inner zone $\{|\xi_3| \le \bar\rho\}$ we will treat our operator as a family of 1-dimensional Schrödinger operators. Then one needs to take $\varepsilon = Ch|\log h|/|\xi_3|$ in this zone as well. Now theory breaks in two cases.

(a) Without the non-degeneracy assumption. Here we apply in rather straightforward manner arguments of Subsection 5.4.2.

(b) Under the non-degeneracy assumption. It is still not very useful to use shift with respect to x_3, x_2, ξ_2 because it lands us with $\varepsilon = C\mu h|\log h|$ which leads to the total approximation error $O(h^{-2})$ only under excessive smoothness assumptions. The role of this condition is more subtle: after $|\xi_3|^2$-partition with respect to x_2, x_3, ξ_2 this condition assures that the ellipticity condition fails only for elements with the total relative measure $\asymp \min\big(|\xi_3|^2/\mu h, 1\big)$ and this leads us in the end to both remainder and approximation error not exceeding Ch^{-2}.

The logarithmic uncertainty principle requires condition $|\xi_3| \ge Ch^{\frac{1}{3}}|\log h|^{\frac{1}{3}}$ which can be weaken to $|\xi_3| \ge Ch^{\frac{1}{3}}$. Zone $\{|\xi_3| \le Ch^{\frac{1}{3}}\}$ is small enough and its total contribution does not exceed Ch^{-2} as well.

Finally, for $\mu \ge h^{-1}|\log h|^{-1}$ the outer zone disappears.

18.1.5 Plan of the Chapter

Let us discuss the plan of this chapter. Sections 18.2–18.5 are devoted to 2-dimensional case and Sections 18.5–18.8 are devoted to 3-dimensional case.

In Section 18.2 we consider case $d = 2$ when magnetic field is relatively weak. Using "precanonical form" we prove our first main Theorem 18.2.9.

In Section 18.3 we consider case $d = 2$ when magnetic field is relatively strong. We derive "poor man canonical form". The true (complete) canonical form does not exist because we lack the infinite smoothness. However precanonical form exists with a rather small error but we need to plug corrected potential W instead of V and their difference is the source of correction term. This corrected potential W is an original potential V averaged along magnetrons and as $(I,\sigma) \succeq (2,1)$ their difference is sufficiently small and therefore the correction term is sufficiently small as well.

In Section 18.4 we prove major Tauberian estimates. Then in Section 18.4.5 we make calculations, prove statements our main Theorem 18.5.4 and finish 2-dimensional case.

In Section 18.6 we consider case $d = 3$ when magnetic field is relatively weak. We analyze the outer zone defined by (18.1.14) (if (x_3,ξ_3) are "free"("non-magnetic") variables) and prove that the contribution of this zone to the remainder is $O(h^{-2})$; then we prove instantly Theorem 18.6.4 and we improve it to Theorem 18.6.12 to cover the case $\mu \le h^{-\frac{1}{3}}$ rather than $\mu \le (h|\log h|)^{-\frac{1}{3}}$.

In Section 18.7 we consider case $d = 3$ when magnetic field is relatively strong. We derive "poor man canonical form".

In Section 18.8 we analyze the inner zone defined by inequality opposite to (18.1.14) as $h^{-\frac{1}{3}} \le \mu \le Ch^{-1}$ and and prove major Tauberian estimates.

Then in Section 18.4.9 we make calculations, prove statements our main Theorem 18.9.4 and 18.9.6 and finish 2-dimensional case.

In what follows we consider operator (13.2.1) assumingly that conditions (13.2.1), (13.2.4), (13.2.5) and (13.4.45).

18.1.6 Standard Results Rescaled

By the *standard results* we mean results for $\mu = 1$. While rescaling final spectral asymptotics by $x \mapsto \mu x$, $h \mapsto \mu h$ we would get the standard remainder estimate $O(h^{1-d})$ converted into $O\big((\mu h)^{1-d}\cdot \mu^d\big) = O(\mu h^{1-d})$, something which we want to improve, rescaling of the intermediate results will be much more fruitful.

Let us consider some (γ,ρ)-admissible partition in (x,ξ) and ε-mollification with respect to x with ε depending on (x,ξ) such that[10)]

$$C_s\rho^{-1}h|\log h| \le \varepsilon \le \gamma. \tag{18.1.15}$$

At first we assume that ε is a numerical small parameter.

Let $\tilde{A}$ be such approximate operator and $\tilde{e}(x,y,\tau)$ be its spectral projector. Let $U(x,y,t)$ be the Schwartz kernel of the corresponding propagator $e^{-ih^{-1}t\tilde{A}}$. We do not write $\tilde{U}$ as we will never consider propagator for the non-mollified (i.e. original) operator.

Proposition 18.1.4. *Let ε be a small numerical parameter such that*

$$\varepsilon \ge C_s h|\log h|, \tag{18.1.16}$$

and

$$\mu \le C_s^{-1}h^{-1}|\log h|^{-1}. \tag{18.1.17}$$

(i) Then

$$|\int \bar{\phi}_\zeta(\tau)\Big(d_\tau \tilde{e}(x,x,\tau) - F_{t\to h^{-1}\tau}\big(\bar{\chi}_T(t)U(x,x,t)\big)\,d\tau\Big)| \le Ch^s \tag{18.1.18}$$

provided $\bar{\phi}(\tau)=1$ for $\tau<-1$, $\bar{\phi}(\tau)=0$ for $\tau>-\frac12$, $\bar{\chi}(t)=1$ for $|t|\le\frac12$, $\bar{\chi}(t)=0$ for $|t|\ge 1$[11)]*, $T=\bar{T}:=\epsilon_0\mu^{-1}$ with small enough constant $\epsilon_0>0$, $\zeta\ge C_s\mu h|\log h|$*[12)] *and $\bar{\chi}_T(t)=\bar{\chi}(t/T)$, $\bar{\phi}_\zeta(\tau)=\bar{\phi}(\tau/\zeta)$.*

(ii) Moreover as $|\tau|\le\epsilon$

$$|F_{t\to h^{-1}\tau}\big(\bar{\chi}_T(t)-\bar{\chi}_{T_*}(t)\big)U(x,x,t)| \le Ch^s \tag{18.1.20}$$

with $T=\bar{T}$ and $T_*:=C_sh|\log h|$,

$$|F_{t\to h^{-1}\tau}\bar{\chi}_{T_*}(t)\big(U(x,x,t)-U^0(x,x,t)\big)| \le Ch^{3-d}+Ch^{1-d}\vartheta(h) \tag{18.1.21}$$

[10)] In fact both γ and especially ρ could be vectors and then assumption (18.1.15) should hold for their components.

[11)] We pick up all auxiliary functions ϕ, χ etc satisfying assumptions of Section 2.3:

$$|D^\alpha f| \le (CN)^{(|\alpha|-m)_+} \qquad \forall \alpha : |\alpha| \le N = \epsilon_s|\log h| \tag{18.1.19}$$

with large enough exponent m.

[12)] So ζ and T also satisfy logarithmic uncertainty principle $\zeta\cdot T\ge C_sh|\log h|$.

with

$$U^0(x,x,t) \coloneqq h^{-d}\int e^{ih^{-1}t\tau}\, d_\tau \tilde{\mathcal{N}}_x^{\mathsf{MW}}(\tau) \tag{18.1.22}$$

and

$$|F_{t\to h^{-1}\tau}\bar{\chi}_{T_*}(t)\big(U(x,x,t)-U^0_{\mathsf{st}}(x,x,t)\big)| \le C\mu^2 h^{3-d} + Ch^{1-d}\vartheta(h) \tag{18.1.23}$$

with

$$U^0_{\mathsf{st}}(x,x,t) \coloneqq h^{-d}\int e^{ih^{-1}t\tau}\, d_\tau \tilde{\mathcal{N}}_x^{\mathsf{W}}(\tau) \tag{18.1.24}$$

where both $\tilde{\mathcal{N}}_x^{\mathsf{W}}(\tau)$ *and* $h^{-d}\tilde{\mathcal{N}}_x^{\mathsf{MW}}(\tau)$ *are constructed for the mollified operator* $\tilde{A}$ *and* $\vartheta(t) = t^l|\log t|^{-\sigma}$ *as* $g^{jk}, F_{jk}, V \in \mathscr{C}^{l,\sigma}$.

Proof. (a) Proof of estimate (18.1.18) is standard because in the scale $x\mapsto \mu x$, $t\mapsto \mu t$, $\tau\mapsto \mu^{-1}\tau$, $\mu\mapsto 1$, $h\mapsto \mu h$ we have the standard case (with $\mu = 1$). Note that

$$\int \bar{\phi}_\zeta(\tau) F_{t\to h^{-1}\tau}\Big(\bar{\chi}_T(t)U(x,x,t)\Big)\, d\tau = \bar{\phi}_\zeta(hD_t)\Big(\bar{\chi}_T(t)U(x,x,t)\Big)\Big|_{t=0}. \tag{18.1.25}$$

Inequality (18.1.20) also follows from the same rescaling arguments applied to the standard results of Section 4.6 as we take in account the ξ-microhyperbolicity which is equivalent to (18.1.26):

$$V \le -\epsilon_0. \tag{18.1.26}$$

(b) To prove (18.1.21)–(18.1.22) and (18.1.23)–(18.1.24) we also apply rescaling arguments and recall construction of $U(x,y,t)$ by the method of successive approximations as $|t| \le C_s h|\log h|$.

However, as an unperturbed operator one can also take

$$\bar{A}(x,y,hD_x) \coloneqq \sum_{j,k} g^{jk}(y)\bar{P}_j\bar{P}_k + V(y) \tag{18.1.27}$$

with

$$\bar{P}_j = hD_j - \mu V_j(y;x), \qquad V_j(y;x) = V_j(y) + \langle \nabla V_j(y), x-y\rangle \tag{18.1.28}$$

rather than more standard $V_j(y)$ (where the former includes magnetic field and the latter does not) and here we must plug mollified $\tilde{g}^{jk}$, $\tilde{V}_j$, and $\tilde{V}$ instead of the original (non-mollified) coefficients.

One can see easily that for such operator the restriction of the spectral kernel to the diagonal $\bar{e}(y,y,\tau)$ is exactly $h^{1-d}\tilde{\mathcal{N}}_y^{\mathsf{MW}}(\tau)$ and that all other terms of the successive approximation contribute no more than $O(h^{3-d})$.

Really, the contribution of the k-th term ($k = 1,2,3,...$) does not exceed $Ch^{-d-(k-1)}T^{2k-1}$ with $T = h|\log h|$ as long as $g^{jk}, V \in \mathscr{C}^{1,1}$ and $V_j \in \mathscr{C}^{2,1}$; then each next term acquires factor T^2h^{-1}.

Thus contribution of any term with $k \ge 4$ is clearly $O(h^{3-d})$ while contribution of the third term is $O(h^{3-d}|\log h|^5)$. Further, analysis of Subsection 4.6.4 (based on the rescaling technique) shows that one can take $T = h$ in this estimate and then the contribution of the second term does not exceed Ch^{3-d} as well.

Consider the second term

$$\frac{1}{2\pi i}\Big(\bar{G}^{+}(\tilde{A}-\bar{A})\bar{G}^{+} - \bar{G}^{-}(\tilde{A}-\bar{A})\bar{G}^{-}\Big)\delta(x-y)\delta(t) \tag{18.1.29}$$

where $\bar{G}^{\pm}$ are respectively forward and backward parametrices of the evolution operator $hD_t - \bar{A}$.

Consider $\tilde{A}-\bar{A} = R_1 + R_2$ where R_1, R_2 are second order operators with the coefficients $O\big(\mu|x-y|\vartheta(|x-y|)\big)$ and $O\big(\vartheta(|x-y|)\big)$ provided $\vartheta(z) \ge Cz^2$. The same rescaling procedure implies that if we replace $\tilde{A}-\bar{A}$ in (18.1.29) by R_1, R_2, the final error will be $O\big(h^{1-d}\mu h\vartheta(h)\big)$ and $O\big(h^{1-d}\vartheta(h)\big)$ respectively. On the other hand, for $\vartheta(z) = O(z^2)$ we get $O(h^{3-d})$.

It implies estimates (18.1.21)–(18.1.22).

(c) To prove estimates (18.1.23)–(18.1.24) we need to use the same construction but with the standard unperturbed operator $\bar{A}(y, hD_x)$ instead; it is given by (18.1.27) with $\bar{P}_j = hD_j - V_j(y)$. Then one can estimate the contribution of the k-th term in the successive approximation by $Ch^{-d+k}\mu^{k-1}$ and thus all the terms but the first two contribute less than $C\mu^2h^{2-d}$. The

contribution of the first term is exactly $h^{-d}\tilde{\mathcal{N}}_y^{\mathsf{W}}(\tau)$ and the only term we need to consider is the second one.

The same argument as above work but now

$$\tilde{A}-\bar{A}=R_0+R_1+R_2,\qquad R_0=\mu\langle x-y,\nabla_y\tilde{A}(y,hD_x)\rangle.$$

Then the contributions of the term given by (18.1.29) with $\tilde{A}-\bar{A}$ replaced by R_0 will be identically 0 we get a term odd with respect to $(x-y,D_x)$ and therefore its restriction to $\{x=y\}$ vanishes.

It implies estimates (18.1.23)–(18.1.24). □

Corollary 18.1.5. *In the framework of Proposition 18.1.4(ii)*

$$h^{-1}\Big|\int^{\tau} F_{t\to h^{-1}\tau}\bar{\chi}_{T_*}(t)\big(U(x,x,t)-U^0(x,x,t)\big)\,d\tau\Big|\le Ch^{2-d}+Ch^{-d}\vartheta(h) \tag{18.1.30}$$

and

$$h^{-1}\int^{\tau} \big|F_{t\to h^{-1}\tau}\bar{\chi}_{T_*}(t)\big(U(x,x,t)-U^0_{\mathsf{st}}(x,x,t)\big)\,d\tau\big|\le C\mu^2h^{2-d}+Ch^{-d}\vartheta(h). \tag{18.1.31}$$

18.2 $d=2$: Weak Magnetic Field

We start from 2-dimensional case which is more transparent and clean.

18.2.1 Heuristics

Toy-Model

The toy-model of such operator is $h^2D_1^2(hD_2-\mu x_1)^2+V(x)$ and metaplectic map $hD_1\mapsto hD_1$, $hD_2-\mu x_1\mapsto -\mu x_1$, $x_1\mapsto x_1+\mu^{-1}hD_2$, $x_2\mapsto x_2+\mu^{-1}hD_1$ transforms this operator into

$$h^2D_1^2+\mu^2x_1^2+V(x_1+\mu^{-1}hD_2,x_2+\mu^{-1}hD_1). \tag{18.2.1}$$

In some sense this operator is modelled by

$$h^2 D_1^2 + \mu^2 x_1^2 + V(\mu^{-1} h D_2, x_2), \tag{18.2.2}$$

Then along coordinates x_2 and $\xi_2 \sim \mu^{-1} h D_2$ singularities propagate with the speed $O(\mu^{-1})$ and under assumption $|\partial_{\xi_2} V| \asymp \nu$ for time T the shift with respect to x_2 is $\asymp \mu^{-1} T \nu$. To satisfy logarithmic uncertainty principle we should assume that

$$\mu^{-1} T \nu \cdot \varepsilon \geq C \mu^{-1} h |\log(\mu^{-1} h)| \tag{18.2.3}$$

because $V(\mu^{-1} h D_2, x_2)$ is $\mu^{-1} h$-pseudodifferential operator with respect to x_2 and ε is a characteristic size in ξ_2-variable. Therefore, $\Gamma U := \int U(x, x, t) dx$ is negligible for $T_* \leq |t| \leq T^*$ with

$$T_* := C \varepsilon^{-1} \nu^{-1} h |\log(\mu^{-1} h)| \tag{18.2.4}$$

and $T^* = \epsilon \mu$ (as we will show).

On the other hand, from the standard results rescaled we know ΓU for $|t| \leq \bar{T} := \epsilon \mu^{-1}$. In order these intervals to overlap we need

$$T_* = C \varepsilon^{-1} \nu^{-1} h |\log(\mu^{-1} h)| \leq \bar{T} = \epsilon \mu^{-1}$$

i.e. $\varepsilon \geq C \mu h |\log h| \nu^{-1}$ and therefore $\mu \leq h^{-1} |\log h|^{-1}$. So, let us pick up the minimal possible value

$$\varepsilon = C \mu h \nu^{-1} |\log h| \tag{18.2.5}$$

and then the approximation error will be $C h^{-2} \vartheta(\mu h \nu^{-1} |\log h|)$.

This construction works for pretty large μ; however, these arguments are optimal only for

$$\mu \leq \bar{\mu}_1 := (h |\log h|)^{-\frac{1}{3}}. \tag{18.2.6}$$

Remark 18.2.1. In the next section we will pick up $\varepsilon = C\big(\mu^{-1} h |\log(\mu^{-1} h)|\big)^{\frac{1}{2}}$ which (for $\nu \asymp 1$) is less than $C \mu h |\log h|$ as $\mu \geq \bar{\mu}_1$.

Note that we need inequality $\varepsilon \geq C\big(\mu^{-1} h |\log(\mu^{-1} h)|\big)^{\frac{1}{2}}$ even to consider $V(\mu^{-1} h D_2, x_2)$ as a legitimate $\mu^{-1} h$-pseudodifferential operator. So, in the framework of this section $V(\mu^{-1} h D_2, x_2)$ is not even defined properly. The short answer to this obstacle is that we should work directly with operator $V(x_1 + \mu^{-1} h D_2, x_2 + \mu^{-1} h D_1)$ or just not to make any reduction but still to be able to reproduce the above arguments.

Propagation of Singularities

The goal of the this subsubsection is to prove that on the energy level 0 and below singularities propagate with respect to x_1, x_2 with the finite speed, and that with respect to "variables" $Q_j := x_j + \sum_k \beta_{jk}\mu^{-1}P_k$ (with appropriate coefficients β_{jk}) singularities propagate with the speed not exceeding $C_0\mu^{-1}$; further, the same is true for $Q'_j := Q_j + \sum_{ki} \mu^{-2}\beta'_{jki}P_kP_l$ with appropriate coefficients β'_{jki}; finally, in the latter case singularities propagate along trajectories of the "Liouvillian" field of $(V-\tau)/F$ [13].

We define β_{jk} to have Poisson brackets vanish: $\{Q_j, P_k\} \equiv 0 \mod O(\mu^{-1})$, or simply

$$Q_1 = x_1 - F_{12}^{-1}P_2, \quad Q_2 = x_2 + F_{12}^{-1}P_1, \qquad \{P_1, P_2\} = -\mu F_{12} \tag{18.2.8}$$

as we would do in the smooth case; then

$$\{A_0, Q_j\} = \mu^{-1}\sum_{ki} \alpha_{jki}P_kP_i; \tag{18.2.9}$$

where $A_0 = A - V$. In the smooth case we would be able to eliminate α_{jkl} modulo $\omega_j A_0$ and $O(\mu^{-2})$ by an appropriate choice of coefficients β'_{jki} above but now α_{jki} have smoothness $(l-1, \sigma)$ and then β'_{jki} will have the same smoothness which is not enough to plug them into Poisson brackets unless $(l, \sigma) \succeq (2, 0)$.

(18.2.10) As $(1,1) \preceq (l,\sigma) \preceq (2,0)$ let us replace coefficients β'_{jki} by their μ^{-1}-mollifications[14] which provides $\mu\vartheta(\mu^{-1})$-approximation to them.

Then for these new coefficients we have $\{A_0, Q'_j\} \equiv \mu^{-1}\omega_j A^0$ and therefore

$$\{A, Q'_j\} \equiv \mu^{-1}\big(\omega_j(\tau+V) + F_{12}^{-1}\mathcal{L}_j V - \omega_j(\tau - A)\big) \mod O\big(\vartheta(\mu^{-1}) + \mu^{-2}\big) \tag{18.2.11}$$

with $\mathcal{L}_j := (-1)^{j-1}\partial_{x_{2-j}}$ (as $g^{jk} = \delta_{jk}$ at the given point).

---

[13] Recall that

$$\frac{dx}{dt} = \mu^{-1}(\nabla(V-\tau)/F))^{\perp}. \tag{18.2.7}$$

[14] Which makes sense only as $\mu^{-1} \ge \varepsilon$ i.e. $\mu \ge (h\nu^{-1}|\log h|)^{\frac{1}{2}}$.

In the smooth case we know from Chapter 13 that $\omega_j = \mathcal{L}_j F^{-1}$ and so it should be in our case as well:

$$\{A, Q_j'\} \equiv \mu^{-1}\Big(\mathcal{L}_j\big(\frac{V-\tau}{F}\big) + \omega_j(\tau - A)\Big) \quad \mathsf{mod}\ O\big(\vartheta(\mu^{-1}) + \mu^{-2}\big) \tag{18.2.12}$$

and this equality will be sufficient to prove that singularities propagate along trajectories of (18.2.7).

Remark 18.2.2. (i) The propagation speed of (18.2.7) is $\asymp \nu\mu^{-1}$. The right-hand side of (18.2.12) is larger than the error as

$$\nu \ge C_1\mu\vartheta(\mu^{-1}) + C_1\mu^{-1}. \tag{18.2.13}$$

In particular, as $(I,\sigma) \succeq (2,0)$ we need no μ^{-1}-mollification and $\nu \ge C\mu^{-1}$.

(ii) As there is a mollification with respect to ε we also need to assume that

$$\nu \ge C_1\varepsilon^{-1}\vartheta(\varepsilon) + C_1\varepsilon \tag{18.2.14}$$

but it is more important only as $\varepsilon \ge \mu^{-1}$ i.e. when $\mu \ge (h|\log h|)^{-\frac{1}{2}}$.

(iii) In what follows we assume that <u>either</u> $(I,\sigma) \succeq (1,1)$ and non-degeneration condition

$$|\nabla V/F| \ge \epsilon_0 \tag{18.2.15}$$

is fulfilled a <u>or</u> $(I,\sigma) \succeq (2,0)$ and non-degeneracy assumption

$$|\nabla V/F| + |\det \mathsf{Hess}\, V/F| \ge \epsilon_0 \tag{18.2.16}$$

is fulfilled leaving to the reader the following problem:

Problem 18.2.3. Investigate the case when $(1,1) \prec (I,\sigma) \prec (2,0)$ and assumption (18.2.16) is replaced by the estimate to $\mathsf{mes}(\{x : |\nabla V/F| \le \nu\})$ as $\nu \to 0$; in this case $\epsilon\eta$-vicinity should be replaced by γ-vicinity with $\gamma^{-1}\vartheta(\gamma) = \epsilon\nu$. Here $\gamma \ge C_2\mu^{-1}$ due to (18.2.13).

18.2.2 Propagation of Singularities: Rigorous Results

First let us prove the finite speed of the propagation:

Proposition 18.2.4. *Let* $d \ge 2$, $\mu \le h^{-1+\delta}$ *and let*

$$M \ge \sup_{x\in\Sigma}\Big(2\sum_k g^{jk}\xi_k\Big) + \epsilon \qquad \forall j \tag{18.2.17}$$

with arbitrarily small constant $\epsilon > 0$, *where* $\Sigma := \{x\colon : \sum_{jk} g^{jk}\xi_j\xi_k + V = 0\}$.

Let ϕ_1 *be supported in* $B(0,1)$, $\phi_2 = 1$ *in* $B(0,2)$, χ *be supported in* $[-1,1]$ [15]. *Let* $\bar{T}_* = Ch|\log h| \le T \le T^{*\prime} = \epsilon_0$.

Then

$$|F_{t\to h^{-1}\tau}\chi_T(t)\big(1-\phi_{2,MT}(x-\bar{x})\big)\phi_{1,MT}(y-\bar{x})U(x,y,t)| \le Ch^s \qquad \forall \tau \le \epsilon_1 \tag{18.2.18}$$

where here and below $\epsilon_1 > 0$ *is a small enough constant.*

Remark 18.2.5. Surely results will be much more precise in the distance according to $g_{jk}V^{-1}$ metrics.

Proof. Proof follows the proof of Theorem 2.3.2. Namely, let χ be the same function as there; recall that it was supported in $(-\infty, 0]$ and satisfying certain "regularity" conditions. Then as a main auxiliary symbol we pick up

$$\chi\Big(f(x,t)-\epsilon_3\Big) \qquad \text{with} \quad f(x,t) = \frac{1}{T}\big(|x-\bar{x}|^2+\epsilon_2\big)^{\frac{1}{2}} \mp M\frac{t}{T}, \tag{18.2.19}$$

We pick signs "$\mp$" analyzing cases $\pm t > 0$. □

Now let us investigate the magnetic drift.

Proposition 18.2.6. *Let* $d = 2$ *and condition* (18.1.6) *be fulfilled. Let* $\mu_0 \le \mu \le h^{-1+\delta}$ *where here and below* $\mu_0 > 0$, C_1, *and* $1/\epsilon_1$ *are large enough constants depending on all the other constants, exponents and the choice of axillary functions below. Let* <u>*either*</u> $(I,\sigma) \succeq (1,1)$ *and* $\nu = 1$ <u>*or*</u> $(I,\sigma) \succeq (2,0)$ *and*

$$\nu \ge C_0 \max(\mu^{-1}, \varepsilon). \tag{18.2.20}$$

Consider ball $B(\bar{x}, \epsilon\nu)$.

[15] Recall that all such auxiliary functions are appropriate in the sense of Section 2.3.

(i) Assume that $|\mathcal{L}_{V/F}(\bar{y})| \le \nu$. *Let*

$$T_* := C\varepsilon^{-1}\nu^{-1}h|\log h| + C\nu^{-1}(\mu h|\log h|)^{\frac{1}{2}} \le T \le T^* = \epsilon_1\mu. \tag{18.2.21}$$

Then[16)]

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\big(1-\phi_{2,\mu^{-1}\nu T}(Q_1-\bar{x}_1, Q_2-\bar{x}_2)\big)^{\mathsf{w}}U(x,y,t)\times {}^t\big(\phi_{1,\mu^{-1}\nu T}(Q_1-\bar{x}_1, Q_2-\bar{x}_2)\big)^{\mathsf{w}}| \le Ch^s \qquad \forall \tau, |\tau| \le \epsilon_1\nu; \tag{18.2.22}$$

(ii) Assume now that $|\mathcal{L}_{V/F}(\bar{y})| \asymp \nu$. *Let* χ *be supported in* $[1-\epsilon_0, 1+\epsilon_0]$.

Further, let $\bar{x} = \bar{y} + \mu^{-1}\mathcal{L}_{V/F}(\bar{y})T$ *with* T *satisfying* (18.2.21) *and* $T \le \epsilon_1\mu$.

Then

$$|F_{t\to h^{-1}\tau}\chi_T(t)\big(1-\phi_{2,M\epsilon\mu^{-1}\nu T}(Q_1-\bar{x}_1, Q_2-\bar{x}_2)\big)^{\mathsf{w}}U(x,y,t)\times {}^t\big(\phi_{1,\epsilon\mu^{-1}T}(Q_1-\bar{y}_1, Q_2-\bar{y}_2)\big)^{\mathsf{w}}| \le Ch^s \qquad \forall \tau, |\tau| \le \epsilon_1\nu. \tag{18.2.23}$$

Proof. Proofs of both statements follow the proof of Theorem 2.3.2. Namely, let χ be the same function as there; recall that it was supported in $(-\infty, 0]$ and satisfying certain "regularity" conditions. Then as a main auxiliary symbol we pick up

$$\chi\Big(f\big(Q_1(x,\xi), Q_2(x,\xi), t\big) - \epsilon_2\Big) \qquad \text{with} \tag{18.2.24}$$

$$f(Q,t) = \frac{1}{T}\big(\mu^2\nu^{-2}|Q-\bar{Q}|^2 + \epsilon_3\big)^{\frac{1}{2}} \mp M\frac{t}{T}$$

and

$$\chi\Big(f\big(Q_1'(x,\xi), Q_2'(x,\xi), t\big) - \epsilon_2\Big) \qquad \text{with} \tag{18.2.25}$$

$$f(Q',t) = \frac{1}{T}\big(\mu^2\nu^{-2}|Q'-\bar{Q}'-\mu^{-1}\ell t|^2 + \epsilon_3\big)^{\frac{1}{2}} \pm \epsilon_4\frac{t}{T}$$

in the proofs of Statements (i) and (ii) respectively where $\epsilon_k > 0$ are small constants. We pick signs "$\mp$" analyzing cases $\pm t > 0$. Furthermore, here $\ell = \mathcal{L}_{V/F}(\bar{y})$ in the proof of Statement (ii) . □

[16)] Recall that b^{w} means Weyl quantization of symbol b and due to condition (18.2.21) logarithmic uncertainty principle holds and this quantization of the symbols involved is justified, ${}^tB$ means the dual operator.

The following corollary follows immediately from Proposition 18.2.6(ii):

Corollary 18.2.7. *(i) Statement (ii) of Proposition 18.2.6 remains true for all* T *satisfying* (18.2.21) *if we redefine* $\bar{x}$ *as* $\bar{x} = \Psi_T(\bar{y})$ *where* Ψ_t *is Liouvillian flow defined by* (18.2.7).

(ii) As $C_0\nu^{-1} \le T \le \epsilon_1\mu$

$$|F_{t\to h^{-1}\tau}\Big(\chi_T(t)\Gamma_x\big(\psi(x)U\big)\Big)| \le Ch^s \qquad \forall \tau : |\tau| \le \epsilon_1\nu. \tag{18.2.26}$$

Recall that χ *is supported in* $[-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$.

Recall that under condition (13.4.45) i.e.

$$V \le -\epsilon_0 \tag{18.2.27}$$

estimate

$$|F_{t\to h^{-1}\tau}\Big(\chi_T(t)\Gamma\big(\psi(x)U\big)\Big)| \le Ch^s \qquad \forall \tau : |\tau| \le \epsilon_1. \tag{18.2.28}$$

holds for $Ch|\log h| \le T \le \bar{T} = \epsilon_1\mu^{-1}$. Note that $T_* \le \bar{T}$ iff

$$\nu \ge C(\mu^3 h|\log h|)^{\frac{1}{2}} \tag{18.2.29}$$

and ε is defined by (18.2.5) where we picked the smallest possible ε.

Therefore

(18.2.30) Under assumptions (18.2.27), (18.2.29) [17] and (18.2.5) estimate (18.2.28) holds for $T_* := Ch|\log h| \le T \le T^* := \epsilon_1\mu$.

Out of two conditions (18.2.29) and (18.2.5) the first is the nastiest. To weaken it we need a special analysis; however note that $\bar{T} \le \epsilon\nu^{-1}$ so we can restrict ourselves to $T \le \epsilon\nu^{-1}$ when evolution is restricted to $C_0\mu^{-1}$ vicinity of the original point.

Proposition 18.2.8. *Let assumptions of Proposition 18.2.6 be fulfilled.*

(i) Let $g^{jk} = \delta_{jk}$, $F = 1$. *Then* (18.2.28) *holds for* $T_* \le T \le T^*$ *with*

$$T_* := C\varepsilon^{-1}\nu^{-1}h|\log h|. \tag{18.2.31}$$

[17] Which we replace by (18.2.29)′.

(ii) In the general case (18.2.28) *holds for* $T_* \le T \le T^*$ *with*

$$T_* := C\varepsilon^{-1}\nu^{-1}h|\log h| + C\nu^{-1}\bigl(\bar\vartheta(\mu^{-1})h|\log h|\bigr)^{\frac12} \tag{18.2.32}$$

where $\bar\vartheta$ *is a modulus of continuity of* g^{jk} *and* F.

Proof. (i) In this case we can assume without any loss of the generality that $A_0 = h^2D_1^2 + (hD_2 - \mu x_1)^2$. Let us consider propagation with respect to ξ_2. We can use then function $f = T^{-1}\mu^{-1}\nu^{-1}\xi_2$ and as for time T shift with respect to ξ_2 is $\asymp \nu T$ and logarithmic uncertainty principle means exactly that $T \ge T_*$ with T_* defined by (18.2.31).

(ii) Consider the general case. Then without any loss of the generality we can assume that $F = 1$ (otherwise we can divide by it) and $g^{jk}(\bar x) = \delta_{jk}$, $\nabla g^{jk}(\bar x) = 0$, $V_1(\bar x) = V_2(\bar x) = 0$, $\nabla V_1(\bar x) = 0$. In this case $\mu Q_1 = \xi_2 + \mu\phi(x)$ with $\nabla\phi = O\bigl(\mu\bar\vartheta(\mu^{-1})\bigr)$ and quantization conditions is $T \ge T_*$ with T_* defined by (18.2.32). □

Now condition $T_* \le \bar T$ becomes (18.2.5) plus

$$\nu \ge C\bigl(\mu^3\bar\vartheta(\mu^{-1})h|\log h|\bigr)^{\frac12}. \tag{18.2.29}'$$

18.2.3 Main Theorem

Now we can prove main results of this section. Under assumption (18.2.15) we pick up $\varepsilon = C\mu h|\log h|$ provided $\nu = 1$ satisfies (18.2.29)′ i.e.

$$C\mu^3\bar\vartheta(\mu^{-1})h|\log h| \le 1; \tag{18.2.33}$$

as $(\bar l, \bar\sigma) = (1,1)$ it is equivalent to $\mu \le \epsilon_1 h^{-\frac12}$ and as $(\bar l, \bar\sigma) = (2,0)$ it is no restriction at all.

Under assumptions (18.2.16) (and thus $(l,\sigma) \succeq (2,0)$), (18.2.33) we introduce variable

$$\nu = \nu(x) = \epsilon|\nabla V/F| + \bar\nu, \qquad \bar\nu := C_0(\mu h|\log h|)^{\frac12} + C_0\mu^{-1} \tag{18.2.34}$$

and variable $\varepsilon = \varepsilon(x)$ by (18.2.5) where $\bar\nu$ ensures that (18.2.20) and (18.2.29)′ are fulfilled.

Let $\tilde{A}$ be ε-approximation[18)] of A.

Theorem 18.2.9. *Let $d = 2$ and A be a self-adjoint in $\mathscr{L}^2(X)$ operator defined by* (18.1.1). *Let conditions* (18.1.6) *and* (18.2.27) *be fulfilled in $B(0,1) \subset X \subset \mathbb{R}^2$.*

(i) Further, let $(l,\sigma) \succeq (1,1)$, condition (18.2.33) *be fulfilled and condition* (18.2.15) *be fulfilled in $B(0,1)$. Then estimate*

$$\mathsf{R}^{\mathsf{MW}} := \Big| \int \Big(\tilde{e}(x,x,0) - h^{-2}\mathcal{N}_2^{\mathsf{MW}}(x,0)\Big)\psi(x)dx\Big| \le C\mu^{-1}h^{-1} + Ch^{-2}\vartheta(\mu h|\log h|) \tag{18.2.35}$$

holds; this estimate also holds with the standard Weyl expression albeit with infinite[19)] *number of terms generated by the principal part:*

$$\mathsf{R}^{\mathsf{W}}_{\infty} := \Big| \int \Big(\tilde{e}(x,x,0) - h^{-2}\mathcal{N}^{\mathsf{W}}_{2,\infty}(x,0)\Big)\psi(x)dx\Big| \le C\mu^{-1}h^{-1} + Ch^{-2}\vartheta(\mu h|\log h|). \tag{18.2.36}$$

(ii) On the other hand, let $(l,\sigma) \succeq (2,1)$, condition (18.2.33) *be fulfilled and condition* (18.2.16) *be fulfilled in $B(0,1)$. Then estimate*

$$\mathsf{R}^{\mathsf{MW}} \le C\mu^{-1}h^{-1} + C\mu^2|\log h| \tag{18.2.37}$$

holds; this estimate also holds with the standard Weyl expression R^{W}.

Proof. (i) Due to Proposition 18.2.8 estimate (18.1.20) holds with $T = T^* = \epsilon\mu$ and $T_* = Ch|\log h|$ and therefore due to theory of Section 4.6

$$|F_{t\to h^{-1}\tau}\Big(\bar{\chi}_T(t)\Gamma\big(\psi(x)U\big)\Big)| \le Ch^{-1}. \tag{18.2.38}$$

[18)] One can take two copies $\tilde{A}^{\pm}$ of approximation such that the corresponding estimate (18.2.35) or (18.2.37) holds for both and $\tilde{A}^- \le A \le \tilde{A}^+$ in the operator sense. We call them framing approximations. This remark holds for all our Theorems 18.2.9, 18.6.10, 18.9.4, 18.9.6 in full.

[19)] Actually finite, due to finite smoothness.

Therefore due to the standard Tauberian procedure

$$|\Gamma\bigl(\tilde{e}(.,.,0)\psi\bigr) - h^{-1}\int_{-\infty}^{0}\Bigl(F_{t\to h^{-1}\tau}\bigl(\bar{\chi}_T(t)\Gamma\bigl(U\psi\bigr)\bigr)\Bigr)\,d\tau| \le \frac{C}{T^*}h^{-1} \asymp C\mu^{-1}h^{-1} \tag{18.2.39}$$

with $T = T^*$ and due to (18.1.20) with $T = Ch|\log h|$; but then

$$|\int\Bigl(\tilde{e}(x,x,0) \;-\; h^{-2}\tilde{\mathcal{N}}_2^{\mathsf{MW}}(x,0)\Bigr)\psi(x)dx| \qquad \le \qquad C\mu^{-1}h^{-1}. \tag{18.2.40}$$

Finally, one can see easily that replacing $\tilde{\mathcal{N}}_{2,\infty}^{\mathsf{W}}$ by $\mathcal{N}_2^{\mathsf{MW}}$ we make an error not exceeding $Ch^{-2}\vartheta(\varepsilon)$.

In these arguments one can replace $\mathcal{N}_2^{\mathsf{MW}}$ by $\mathcal{N}_{2,\infty}^{\mathsf{W}}$.

(ii) The same arguments work in the *exterior zone*

$$\mathcal{X}_{\mathsf{ext}} = \{x : |(\nabla V/F)(x)| \ge C\bar{\nu}\}; \tag{18.2.41}$$

therefore contribution of this zone to both the Tauberian remainder and to the left-hand expression (18.2.40) do not exceed $C\mu^{-1}h^{-1}$.

Further, contributions of the *inner zone*

$$\mathcal{X}_{\mathsf{inn}} = \{x : |(\nabla V/F)(x)| \le C\bar{\nu} \tag{18.2.42}$$

to both the Tauberian remainder and to the left-hand expression (18.2.40) do not exceed $C\mu h^{-1}\,\mathsf{mes}(\mathcal{X}') \asymp C\mu h^{-1}\bar{\nu}^2 \asymp C\mu^{-1}h^{-1} + C\mu^2|\log h|$.

Finally, one can see easily that replacing $\tilde{\mathcal{N}}_2^{\mathsf{MW}}$ by $\mathcal{N}_2^{\mathsf{MW}}$ we make an error not exceeding

$$Ch^{-2}\int_{\mathcal{X}_{\mathsf{ext}}}\vartheta\bigl(\mu h|\log h|/\nu^2\bigr)\,\nu d\nu + Ch^{-2}\vartheta(\bar{\varepsilon})\,\mathsf{mes}(\mathcal{X}')$$

with $\bar{\varepsilon} = C_0\mu h\bar{\nu}^{-1}|\log h| \asymp C_0\min\bigl((\mu h|\log h|)^{\frac{1}{2}}, \mu^2 h|\log h|\bigr)$; as $(l,\sigma) \succeq (2,1)$ we arrive to estimate (18.2.37). □

In these arguments one can replace $\mathcal{N}_2^{\mathsf{MW}}$ by $\mathcal{N}_{2,\infty}^{\mathsf{W}}$ but all extra terms do not exceed $C\mu^2$.

Corollary 18.2.10. *In the framework of Theorem 18.2.9 let* $(l,\sigma)\succeq(2,1)$ *and one of conditions* (18.2.15), (18.2.16) *be fulfilled. Then estimate*

$$\mathsf{R}^{\mathsf{MW}}=O(\mu^{-1}h^{-1}) \tag{18.2.43}$$

holds under assumption (18.2.6) *(i.e.* $\mu\le\bar\mu_1=(h|\log h|)^{\frac13}$*). The same estimate holds for* R^{W}*.*

Remark 18.2.11. Under stronger smoothness assumptions this estimate (18.2.43) holds for larger μ under assumption (18.2.15) but it does not hold for R^{W} under assumption (18.2.16).

18.3 $d=2$: Canonical Form

Now we analyze the case

$$\bar\mu_1:=(h|\log h|)^{-\frac13}\le\mu\le\bar\mu_2:=(h|\log h|)^{-1}; \tag{18.3.1}$$

the second restriction will be removed in Subsection 18.3.3.

Our arguments will not change much for larger μ as well but some of them become much simpler while other become a bit more complicated. We will take

$$\mu^{-1}\ge\varepsilon\ge C(\mu^{-1}h|\log h|)^{\frac12} \tag{18.3.2}$$

which makes it possible to use $\mu^{-1}h$-pseudodifferential operators.

18.3.1 Reduction

Main pPart of the Operator. I

Now our goal is to reduce our operator to a some kind of the canonical form. It would be easier for constant or at least smooth g^{jk} and F. However, we can overcome technical difficulties in a more general case as well. Instead of normal Fourier integral operators we will use "poor man Fourier integral operators" i.e. propagators of "rough" pseudodifferential operators.

Problem 18.3.1. In this section for a sake of simplicity of some arguments we assume that $\sigma\ge0$ but one can get rid of this assumption easily. We leave it to the reader.

In this subsection we will reduce A_0 to pre-canonical form which is really easy for constant metrics g^{jk} and F and is not needed at all for Euclidean metrics and constant F.

First of all, we can assume without any loss of the generality that

$$F^3 = F = 1. \tag{18.3.3}$$

Really, changing orientation, if necessary, we can make $F^3 > 0$. Then multiplying mollified operator by $F^{-\frac{1}{2}}$ both from the right and from the left and commuting it with P_j we will get operator of the same form with g^{jk}, V replaced by $F^{-1}g^{jk}$, $F^{-1}V$ respectively, modulo operator $h^2V'(x)$ where $V'(x)$ is a function, linear with respect to second derivatives of g^{jk} and F and thus V' does not exceed $C\bigl(1+\vartheta(\varepsilon)\varepsilon^{-2}\bigr)$; one can see easily that after multiplication by h^2 it does not exceed $C\vartheta(\varepsilon)$ which is an approximation error anyway (plus Ch^2 which is less than $C\mu^{-1}h$). Surely, such transformation would affect $e(x,x,0)$ but we will return to the original operator later.

In Chapter 13 on this step of reduction we applied $\mu^{-1}h$-Fourier integral operator and let us try the same now. Namely, let us transform our operator by operator $T(1)$, where

$$T(t) := e^{-i\mu^{-1}h^{-1}tL^{\mathsf{w}}}, \qquad L^{\mathsf{w}} = \frac{1}{2}\sum_{jk} P_j L^{jk}(x) P_k \tag{18.3.4}$$

with $L^{jk} \in \mathscr{F}^{\bar{l},\bar{\sigma}}$. Obviously L^{w} is a Weyl $\mu^{-1}h$-quantization of the symbol

$$L(x,\xi) := \mu^2 \sum_{j,k} L^{jk}(x) p_j(x,\xi) p_k(x,\xi), \qquad p_j := \xi_j - V_j(x). \tag{18.3.5}$$

Remark 18.3.2. In this subsection L, T, t do not denote the same things as everywhere else and in this and next chapters $\mathscr{F}^{l,\sigma}$ denotes space of functions functions satisfying

$$|\partial^{\nu}_{x,\xi} f| \le C_\nu \bigl(\varepsilon^{l-|\nu|}|\log\varepsilon|^{-\sigma}+1\bigr)$$

with the same constants C_ν as in Section 2.3.

Let us consider "Heisenberg evolution" $Q(t) = T(-t)\bar{Q}T(t)$ for $\mu^{-1}h$-pseudodifferential operator $\bar{Q}$; then

$$\partial_t Q(t) = i\mu^{-1}h^{-1}[L^{\mathsf{w}}, Q(t)], \qquad Q(0) = \bar{Q}. \tag{18.3.6}$$

Let us define first symbol $q(t)$ as a solution to

$$\partial_t q(t) = \{\mu^{-2} L, q\}, \qquad q(0) = \bar{q} \implies q(t) = \bar{q} \circ \Phi_t \tag{18.3.7}$$

with the standard Poison brackets where ϕ_t is a corresponding Hamiltonian flow. Let us consider differential equations defining Hamiltonian flow ϕ_t in terms of p_j and x_k:

$$\frac{d}{dt} x_j = \sum_k L^{jk} p_k, \qquad \frac{d}{dt} p_j = \sum_k \Lambda^{jk} p_k + \sum_{k,m} \beta_j^{km} p_k p_m \tag{18.3.8}$$

with $\Lambda := f(x)\ell J$, where $\Lambda = \bigl(\Lambda^{jk}\bigr)$, $\ell = \bigl(L^{jk}\bigr)$ and $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ are 2×2-matrices, here and below $\beta_* \in \mathscr{F}^{\bar{l}-1,\bar{\sigma}}$, $f(x) = \partial_{x_1} V_2 - \partial_{x_2} V_1$.

One can see easily that for $|t| \le c$

$$|p| \le c \implies |p \circ \phi_t| \le C|p|, \tag{18.3.9}$$

$$\phi_t \in \mathscr{F}^{\bar{l},\bar{\sigma}}, \tag{18.3.10}$$

$$p_j \circ \phi_t = e^{\Lambda(x)t} p + \sum_{j,k} \beta'_{jk} p_j p_k \qquad \text{on} \quad \{|p| \le c\mu^{-1}\}, \tag{18.3.11}$$

$$x_j \circ \phi_t - x_j = \sum_j \beta''_{jk} p_k \tag{18.3.12}$$

where $p = (p_1, p_2)$, $x = (x_1, x_2)$.

Further, one can see easily that

$$a_0 \circ \phi_t = \sum g_t^{jk} p_j p_k + M_t, \quad (g_t) = e^{t\,{}^t\Lambda(x)} (g) e^{t\Lambda(x)} \quad \text{with} \quad M_t = \sum_{i,j,k} \beta_{ijk;\,t} p_i p_j p_k \tag{18.3.13}$$

where $a_0 = \sum_{j,k} g^{jk} p_j p_k$ and (g), (g_t) denote corresponding matrices.

Now, let us consider in this region $\{|p| \le c\mu^{-1}\}$ the third and higher derivatives of the symbol $\alpha_{jk} p_j p_k$ with $\alpha_* \in \mathscr{F}^{\bar{l},\bar{\sigma}}$; then

$$|\partial^\nu_{x,\xi} (\alpha_{jk} p_j p_k)| \le C_\nu \varepsilon^{-|\nu|+l} |\log \mu|^{-\bar{\sigma}} \mu^{-2} + C \tag{18.3.14}$$

for $3 \le |\nu| \le c|\log \mu|$ where C_ν are standard constants of Section 3.2 even if $V_j \in \mathscr{F}^{\bar{l},\bar{\sigma}}$ only.

Similarly for $|\nu| \ge 3$ we have

$$|\partial^{\nu}_{x,\xi}(\beta_{ijk}p_ip_jp_k)| \le C_{\nu}\varepsilon^{-|\nu|+\bar{l}-1}|\log\mu|^{-\bar{\sigma}}\mu^{-3} + C; \tag{18.3.15}$$

and therefore $\mu^2 M_t \in \mathscr{F}^{\bar{l},\bar{\sigma}} \cap \mu^{-1}\mathscr{F}^{\bar{l}-1,\bar{\sigma}}$. Now let us pass from symbols to their quantizations. Let us recall that $(\bar{l},\bar{\sigma}) \succeq (2,0)$ and also that in the standard smooth situations and for the standard quantizations $[a^{\mathsf{w}}, b^{\mathsf{w}}] = -ih\{a,b\}^{\mathsf{w}} + O(h^3)$; then for the standard quantization of $\mu^2 a_0 \circ \phi_t$ equation (18.3.6) will be fulfilled modulo operators with A-bound[20)] not exceeding $C\mu^2(\mu^{-1}h)^2 \times \varepsilon^{-1}\mu^{-1} \times (\varepsilon^{-1}\mu^{-1} + \varepsilon^{-2}\mu^{-3})$ which in turn does not exceed $C\mu^{-1}h$.

Therefore,

(18.3.17) Transformed operator $T(-t)A_0T(t)$ differs from the quantization of the transformed symbol $\mu^2 a_0 \circ \phi_t$ by an operator with A-bound not exceeding $C\mu^{-1}h$.

Recall that A_0 is a Weyl $\mu^{-1}h$-quantization of a_0. This is as good as we need.

Now we can pick up $L^{jk} \in \mathscr{F}^{\bar{l},\bar{\sigma}}$ such that $e^{t\Lambda}(g)e^{\Lambda} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & g^{-1}\end{smallmatrix}\right)$ with $g = \det(g)^{-1}$ because e^{Λ} could be any matrix with determinant equal to 1. Therefore we get that modulo operator with A-bound not exceeding $\mu^{-1}h$

$$T(-1)A_0T(1) \equiv P_1^2 + P_2(g^{-1})^{\mathsf{w}}P_2 + \mu^2 M^{\mathsf{w}}, \tag{18.3.18}$$

$$M \coloneqq a_0 \circ \phi_1 - p_1^2 - g^{-1}p_2^2, \qquad g' \coloneqq g \circ \phi_1. \tag{18.3.19}$$

Recall that according to (18.3.13) $M = \sum_{i,j,k}\beta_{ijk}p_ip_jp_k$ with $\beta_{ijk} \in \mathscr{F}^{\bar{l}-1,\bar{\sigma}}$.

Consider now how the above transformation affects V. Note that

$$|\partial^{\nu}_{x,\xi}(V \circ \phi_t)| \le C_{\nu}\varepsilon^{l-|\nu|}|\log h|^{-\sigma}$$

in the same region $\{|p| \le c\mu^{-1}\}$. Then one can see easily that

$$T(-1)VT(1) \equiv (V \circ \phi_1)^{\mathsf{w}} \tag{18.3.20}$$

[20)] Here in contrast to the standard definition we use everywhere we say that A bound of B is R if

$$\|Bu\| \le R\big(\|u\| + \|A^m u\|\big) \tag{18.3.16}$$

with some exponent m which is fixed in advance.

modulo operator with A-bound not exceeding $C(\mu^{-1}h)^2\times\varepsilon^{-1}\mu^{-1}\times\varepsilon^{l-3}|\log h|^{-\sigma}$ (which is obviously less than $C\vartheta(\varepsilon)$).

Also note that we used only that $V_1, V_2 \in \mathscr{F}^{\bar l,\bar\sigma}$. Therefore without any loss of the generality one can assume that

$$V_1 = 0, \quad P_1 = hD_1, \quad V_2, \partial_{x_1} V_2 = g^{-\frac{1}{2}} \in \mathscr{F}^{\bar l,\bar\sigma} \tag{18.3.21}$$

where the third assertion follows from the first and the second ones and

$$p_2 = \alpha(x,\xi_2)\big(x_1 - \lambda(x_2,\xi_2)\big), \qquad \alpha,\lambda \in \mathscr{F}^{\bar l,\bar\sigma}. \tag{18.3.22}$$

Main Part of the Operator. II

In this subsection we will reduce A_0 to canonical form which is really easy for constant g^{jk} and F.

Our next transformation is $T'(1)$ with

$$T'(t) = e^{-it\lambda^{\mathsf w}D_1} \qquad \text{where } \ \lambda^{\mathsf w} = \lambda^{\mathsf w}(x_2, \mu^{-1}hD_2); \tag{18.3.23}$$

then

$$T'(-t)D_1T'(t) = D_1, \quad T'(-t)\lambda^{\mathsf w}T'(t) = \lambda^{\mathsf w}, \quad T'(-t)x_1T'(t) = x_1 + t\lambda^{\mathsf w} \tag{18.3.24}$$

precisely.

To calculate transformations of other operators we need to introduce the corresponding Hamiltonian flow ϕ'_t:

$$\xi_1 = \mathsf{const},\ \lambda(x_2,\xi_2) = \mathsf{const},\ x_1 = x_1(0) + t\lambda(x_2,\xi_2), \tag{18.3.25}$$

$$\frac{d}{dt}x_2 = \xi_1\big(\partial_{\xi_2}\lambda(x_2,\xi_2)\big), \qquad \frac{d}{dt}\xi_2 = -\xi_1\big(\partial_{x_2}\lambda(x_2,\xi_2)\big). \tag{18.3.26}$$

This flow is less regular than ϕ_t:

$$|\partial^\nu_{x,\xi}\phi'_t| \le C_\nu\mu^{-1}\big(1+\varepsilon^{\bar l-1-|\nu|}|\log\mu|^{-\bar\sigma}\big). \tag{18.3.27}$$

However, using again above arguments we can estimate the difference between operators $T'(-1)\alpha^{\mathsf w}T'(1)$ and $(\alpha\circ\phi'_1)^{\mathsf w}$ with $\alpha\in\mathscr{F}^{\bar l,\bar\sigma}$: its A-bound does not exceed $(\mu^{-1}h)^2\times\varepsilon^{-1}\mu^{-1}\times\varepsilon^{-2}\mu^{-1}$ which in turn does not exceed $C\mu^{-1}h$. The same statement is true for $\mu^{-1}\beta^{\mathsf w}$ with $\beta\in\mathscr{F}^{\bar l-1,\bar\sigma}$.

Therefore, *modulo operator with* $\bar{A}_0$*-bound not exceeding* $C\mu^{-1}h$

$$T'(-1)T(-1)A_0T(1)T'(1) \equiv \bar{A}_0 + \mu^2(M')^{\mathrm{w}}, \tag{18.3.28}$$

where

$$\bar{A}_0 := \bar{P}_1^2 + \bar{P}_2^2, \quad \bar{P}_1 = hD_1, \bar{P}_2 = -\mu x_1, \tag{18.3.29}$$

$$M' = a_0 \circ \Phi - \bar{a}_0 = \sum_{i,j,k} \beta'_{ijk}\bar{p}_i\bar{p}_j\bar{p}_k \tag{18.3.30}$$

with $\bar{p}_1 := \xi_1$, $\bar{p}_2 := -x_1$, $\Phi := \phi_1 \circ \phi_1'$.

Similarly, *modulo operator with* $\bar{A}_0$*-bound not exceeding* $C\vartheta(\varepsilon)$

$$T'(-1)T(-1)VT(1)T'(1) \equiv (V \circ \Phi)^{\mathrm{w}}. \tag{18.3.31}$$

Potential

Now we have $\mu^{-1}h$-pseudodifferential operator $\bar{A}$, with the symbol $\mu^2\bar{a}_0 = \mu^2(x_1^2 + \xi_1^2)$, perturbed by $\mu^2(M')^{\mathrm{w}}$ and $(V')^{\mathrm{w}}$ with $V' = (V \circ \Phi)^{\mathrm{w}}$. The first perturbing operator would vanish for constant g^{jk} and F but the second one would still cause trouble even in this case because it's symbol depends on x_1, ξ_1 as well[21)].

Our goal is to make a perturbation a $\mu^{-1}h$-pseudodifferential operator with the symbol depending on x_2, ξ_2, and $r^2 := x_1^2 + \xi_1^2$ only.

Let us consider the smooth case first as a toy-model; then we can decompose $V \circ \Phi$ into asymptotic series with respect to x_1, ξ_1. To get rid of linear terms we need to make a shift in x_1, ξ_1 of a magnitude μ^{-2}; then the increment of $\mu^2\bar{a}_0$ will absorb these terms.

We can continue further[22)] but what we need really is an error in magnetic Weyl expression not exceeding $C\bigl(\mu^{-1}h + \vartheta(\varepsilon)\bigr)h^{-2}$. In the 3-dimensional case we will need much less than this (an error in operator $O(\mu^{-2})$ will be almost sufficiently small) but now we should request $O(\mu^{-1}h)$ error which can be as large as $O(\mu^{-4})$ now and to run things in a "smooth" manner we would need $l = 4$ while our objective is to get the best estimate as $l = 2$ [23)].

[21)] In this case Φ is a linear symplectomorphism.

[22)] We cannot eliminate terms of even degrees $2m$ completely, but we can reduce them to $\mu^2\omega_m(x_2, \xi_2)(x_1^2 + \xi_1^2)^m$; see Chapter 13.

[23)] Actually we need $l = 3$ because the "weak magnetic field" approach leads to $O\bigl(h^{-2}(\mu h|\log h|)^l\bigr)$ error and the "strong magnetic field" approach in the discussed above smooth version produces $O\bigl(h^{-2}\mu^{-l}\bigr)$ error; threshold is as $\mu = h^{-\frac{1}{2}}|\log h|^{-\frac{1}{2}}$ and to keep

Let us return to the non-smooth case and try to repeat the above arguments. To make the shift described above we need to use transformation $T''(-t) = e^{-i\mu h^{-1}tS^{\mathsf{w}}}$ with $\mu^{-1}h$-pseudodifferential operator S^{w} with the symbol belonging to $\mu^{-2}\mathscr{F}^{l,\sigma}$; on the other hand, to accommodate M' we will need to add $\mu^{-1}h$-pseudodifferential operator with the symbol satisfying improved estimate (18.3.15). Namely, one can see easily that

$$|\partial^{\nu}_{x,\xi}M'| \le C_{\nu}\varepsilon^{-|\nu|+1}|\log h|^{-1}\mu^{-3} + C\mu^{(-3+\nu_1)_-} \tag{18.3.32}$$

with ν_1 counting derivatives with respect to (x_1,ξ_1) only.

So in fact

$$S = S' + S'', \qquad S' \in \mu^{-2}\mathscr{F}^{l,\sigma}, \qquad S'' \in \mu^{-2}\mathscr{F}^{1,0} \tag{18.3.33}$$

and S'' satisfies (18.3.32); some adjustments to be done later.

Remark 18.3.3. One can check easily that such operator $T''(t)$ would be $\mu^{-1}h$-pseudodifferential operator iff

$$\mu \ge \bar{\mu}_1^* := Ch^{-\frac{1}{3}}|\log h|^{\frac{1}{3}}. \tag{18.3.34}$$

It is much simpler to analyze transformation by $\mu^{-1}h$-pseudodifferential operators than by $\mu^{-1}h$-Fourier integral operators with not very regular symbols and one can avoid some hassle if (18.3.34) holds. This restriction is not smoothness related: one just need to take away from $\mu^2 M' + V'$ a term of magnitude μ^{-1}.

One can cover case $\mu \le \bar{\mu}_1^*$ by results of the previous section; this would require $(l,\sigma) = (\bar{l},\bar{\sigma}) = (2,3)$ and assumption (18.2.15) only to achieve the best possible estimate $O(\mu^{-1}h)$. The second restriction would be the most unfortunate.

However, we will use more sophisticated arguments to cover $\mu \in [\bar{\mu}_1, \bar{\mu}_1^*]$ here and prove the best possible estimate for $\sigma = \bar{\sigma} = 1$ instead.

Let us consider corresponding Hamiltonian flow ψ_t. One can see easily that

$$|\partial^{\nu}_{x,\xi}(\psi_t - I)| \le C_{\nu}\mu^{-2}\varepsilon^{l-1-|\nu|}|\log h|^{-\sigma} + C_{\nu}\mu^{-3}\varepsilon^{-|\nu|} \tag{18.3.35}$$

error below $\mu^{-1}h$ we need $(l,\sigma) = (3,\frac{3}{2})$ and we will proceed in this assumption in our next Chapters 19 and 20 devoted to the higher dimensions.

where I is an identity map.

Using arguments of the previous subsection one can prove easily that $T''(-t)\bar{P}_j T''(t) \equiv \mu(\bar{p}_j \circ \psi_t)^{\mathsf{w}}$ modulo operator with the upper $\bar{A}_0$-bound

$$\mu(\mu^{-1}h)^2 \times \bigl(1+\mu^{-2}\varepsilon^{I-3}|\log\mu|^{-\sigma}+\mu^{-3}\varepsilon^{-2}|\log\mu|^{-1}\bigr)\times \bigl(\mu^{-2}\varepsilon^{I-4}|\log h|^{-\sigma}+\mu^{-3}\varepsilon^{-3}|\log\mu|^{-1}\bigr); \tag{18.3.36}$$

then the same is true for $T''(-t)\bar{A}_0 T''(t) \equiv \mu^2(\bar{a}_0 \circ \psi_t)^{\mathsf{w}}$.

The bad news is that this expression is not necessarily less than $C\mu^{-1}h$ even if $(I,\sigma)=(2,1)$; the trouble appears as μ is close to $h^{-\frac{1}{3}}$ in which case expression (18.3.36) is close to h. Also, expression (18.3.36) is not less than $C\vartheta(\varepsilon)$ for $I<\frac{3}{2}$ and μ close to $h^{-\frac{1}{3}}$ again.

We could avoid these problems by slightly increasing $\bar{I}$ and assuming that $I\ge\frac{3}{2}$ but there is a better way. Note that (18.3.35) with ε replaced by $\bar{\epsilon}:=\mu^{-1-\delta}$ with small enough $\delta>0$ is less than $C\bigl(\mu^{-1}h+\vartheta(\varepsilon)\bigr)$ (with the original parameter ε in the last expression). Then let us assume that

(18.3.37) $S'\in \mu^{-2}\bar{\mathscr{F}}^{I,\sigma}$ and S'' satisfies (18.3.32) with ε replaced by $\bar{\varepsilon}:=\mu^{-1-\delta}$; here and below $\bar{\mathscr{F}}^*$ means that in the definition of the class ε replaced by $\bar{\varepsilon}$ while $\mathscr{F}^*$ denotes the original class.

Then the same arguments show that

$$T''(-t)(\mu^2M'+V')^{\mathsf{w}}T''(t)\equiv\bigl((\mu^2M'+V')\circ\psi_t\bigr)^{\mathsf{w}}$$

with the same error. So, modulo operator with A-bound not exceeding $C\bigl(\mu^{-1}h+\vartheta(\varepsilon)\bigr)$ we need to consider the quantization of

$$\bigl(\mu^2\bar{a}_0+\mu^2M'+V'\bigr)\circ\psi_1. \tag{18.3.38}$$

One can see easily that $\bar{p}_j\circ\psi_t-\bar{p}_j-t\{\bar{p}_j,S'+S''\}\in\mu^{-4}\bar{\mathscr{F}}^{I-2,\sigma}$ and therefore

$$\mu^2\bigl(\bar{a}_0\circ\psi_t-t\bar{a}_0-\{\bar{a}_0,S'+S''\}\bigr)\in\mu^{-2}\bar{\mathscr{F}}^{I-1,\sigma};$$

we conclude that if $V'\in\bar{\mathscr{F}}^{I,\sigma}$ and M' satisfied (18.3.32) with ε replaced by $\bar{\varepsilon}$ then $(\mu^2M'+V')\circ\psi_t-(\mu^2M'+V')$ would belong to the same class. In this case "the main part" of (18.3.38) would be $\mu^2\bar{a}_0+\mu^2\{\bar{a}_0,S'+S''\}+\mu^2M'+V'$ and we would define S',S'' from equations

$$\{\bar{a}_0,S'\}=-\mu^{-2}(V'-W'), \tag{18.3.39}$$

$$\{\bar{a}_0,S''\}=-(M'-\mu^{-2}W''). \tag{18.3.40}$$

Since $\{\bar{a}_0, q\} = (\xi_1\partial_{x_1} - x_1\partial_{\xi_1})q$ these equations would be solvable and solutions would satisfy (18.3.37) if and only if

$$W'(x_2,\xi_2,r) = \frac{1}{2\pi}\int V'(r\cos t, r\sin t; x_2,\xi_2)\,dt, \tag{18.3.41}$$

$$W''(x_2,\xi_2,r) = \frac{1}{2\pi}\mu^2\int M'(r\cos t, r\sin t; x_2,\xi_2)\,dt, \tag{18.3.42}$$

where we recall that $r^2 = x_1^2 + \xi_1^2$.

One can calculate easily the difference between $\bigl(\mu^2\bar{a}_0 + \mu^2 M' + V'\bigr)\circ\psi_1$ and $\mu^2\bar{a}_0+\mu^2\{\bar{a}_0, S'+S''\}+\mu^2M'+V'$; it will be $O(\mu^{-2})$ which is small enough to be taken care of by $\mu^{-1}h$-pseudodifferential operator transformations in what follows. More precisely

$$N := \Bigl(\mu^2\bar{a}_0 + \mu^2 M' + V'\Bigr)\circ\psi_1 - \Bigl(\mu^2\bar{a}_0 + \mu^2\{\bar{a}_0, S'+S''\} + \mu^2 M' + V'\Bigr) \in \mu^{-2}\bar{\mathscr{F}}^{l-1,\sigma}. \tag{18.3.43}$$

Since our symbols belong to $\mathscr{F}^*$ rather than $\bar{\mathscr{F}}^*$, we just replace V', M' by their $\bar{\varepsilon}$-mollifications $\bar{V}', \bar{M}'$ and define $\bar{S}', \bar{S}'', \bar{W}', \bar{W}''$ by (18.3.39)–(18.3.42) with $\bar{V}', \bar{M}'$ instead of V', M'.

Then we have unaccounted term K^{w} in the operator *before* the last transformation with $K = (\mu^2\tilde{M}' + \tilde{V}')$, where $\tilde{M}' = M' - \bar{M}'$, $\tilde{V}' = V' - \bar{V}'$ and then $K \in \vartheta(\bar{\varepsilon})\mathscr{F}^{0,0}$ and it also belongs to the same class as $V' + \mu^2 M'$ did.

To finish this part we need to understand how operator K^{w} is transformed by $T''(1)$. One can see easily that $K = K_0 + K_1 + ... + K_m$ where K_j is the difference between ε_j and ε_{j+1}-mollifications, $\varepsilon_j = 2^j\varepsilon$ and $\varepsilon_m = \bar{\varepsilon}$. Obviously $K_j \in \vartheta(\varepsilon_j)\mathscr{F}_j^{0,0}$ where

(18.3.44) $\mathscr{F}_j^*$ denotes corresponding class with ε replaced by ε_j.

Then one can see easily that $T''(-1)K_j^{\mathrm{w}}T''(1) \equiv K_j^{\mathrm{w}}$ modulo operator with the norm not exceeding $\mu^{-2}\varepsilon_j^{l-2}|\log h|^{-\sigma}$ and then $T''(-1)K^{\mathrm{w}}T''(1) \equiv K^{\mathrm{w}}$ modulo operator with the norm not exceeding $\mu^{-2}\varepsilon^{l-2}|\log h|^{-\sigma}$ as $l<2$ or $\mu^{-2}|\log h|^{1-\sigma}$ as $l=2$ and in both cases we got $O\bigl(\vartheta(\varepsilon)\bigr)$ estimate unless $l=2, \mu \le \bar{\mu}_1^*$.

The similar calculations imply that $T''(-1)K^{\mathrm{w}}T''(1) \equiv (K')^{\mathrm{w}}$ modulo operator with the norm not exceeding $C\vartheta(\varepsilon)$ with $K' = K + \{K, \bar{S}' + \bar{S}''\}$ as $l=2$ and $\mu \in [\bar{\mu}_1, \bar{\mu}_1^*]$.

Final Reduction

Thus we arrived to operator $\bar{A}_0 + \bar{W}^{\mathsf{w}} + (L + K')^{\mathsf{w}}$ where $\bar{W} = W' + W''$, symbol L is defined by (18.3.43) and

$$K' = \sum_{0\le j\le m} K_j, \qquad K_j \in \vartheta(\varepsilon_j)\mathscr{F}_j^{0,0} \subset \mathscr{F}_j^{l,\sigma}. \tag{18.3.45}$$

Fortunately, both operators L^{w} and K'^{w} are small enough to be reduced by $\mu^{-1}h$-pseudodifferential operator.

Namely, let us consider transformation by $U = \bigl(e^{-i\mu^{-1}h^{-1}(S'''+S^{IV})}\bigr)^{\mathsf{w}}$ where S''' and S^{IV} are symbols of the same class as L and K respectively; in particular, S^{IV} admits decomposition of type (18.3.45).

One can check easily that

$$e^{-i\mu^{-1}h^{-1}(S'''+S^{IV})} \in \mathscr{F}^{0,0}$$

as long as $\mu^{-l}h^{-1}|\log h|^{-\sigma} \le C\varepsilon^{-1}$ which is always the case because $\mu \ge C(h|\log h|)^{-\frac{1}{3}}$ and we assume that

$$(l,\sigma) \succeq (1,2). \tag{18.3.46}$$

Also one can see easily that modulo operator with $\bar{A}_0$-bound not exceeding $C\bigl(\mu^{-1}h + \vartheta(\varepsilon)\bigr)$

$$U^{-1}\Bigl(\bar{A}_0 + \bigl(\bar{W} + K' + L\bigr)^{\mathsf{w}}\Bigr)U \equiv \bar{A}_0 + \bigl(\{\bar{a}_0, S''' + S^{IV}\} + \bar{W} + K + L + R\bigr)^{\mathsf{w}} \tag{18.3.47}$$

where *one needs to include* $R \in \mu^{-4}\mathscr{F}^{l-2,\sigma}$ *only for* $(l,\sigma) \succeq (2,0)$.

Now we need just to define $S''', S^{IV}, W''', W^{IV}$ from equations similar to (18.3.39)–(18.3.42):

$$\{\bar{a}_0, S'''\} = -(L - W'''), \tag{18.3.48}$$

$$\{\bar{a}_0, S^{IV}\} = -(K' - W^{IV}), \tag{18.3.49}$$

$$W'''(x_2,\xi_2,r) = \frac{1}{2\pi}\int L(r\cos t, r\sin t; x_2,\xi_2)\,dt, \tag{18.3.50}$$

$$W^{IV}(x_2,\xi_2,r) = \frac{1}{2\pi}\int K'(r\cos t, r\sin t; x_2,\xi_2)\,dt. \tag{18.3.51}$$

So, *modulo operator with* $\bar{A}_0$*-bound not exceeding* $C\bigl(\mu^{-1}h + \vartheta(\varepsilon) + \mu^{-4}\bigr)$ *we reduced* A *to* $\bar{A}_0 + W^{\mathsf{w}}$ *where* $W = \bar{W} + W'' + W'''$ *depends on* x_2, ξ_2, r *only*.

This is almost the end of the story: for $(l,\sigma) \succeq (2,0)$ and $\mu \in [\bar{\mu}_1, h^{-\frac{1}{3}}]$ one needs to make one step more: namely, there is operator R^{w} in the right-hand expression of (18.3.47) and we should make transformation $U' = \bigl(e^{-i\mu^{-1}h^{-1}S^V}\bigr)^{\mathsf{w}}$ with $S^V \in \mu^{-4}\mathscr{F}^{l-2,\sigma}$ defined by

$$\{\bar{a}_0, S'''\} = -(R - W^V) \tag{18.3.52}$$

with

$$W^V(x_2,\xi_2,r) = \frac{1}{2\pi}\int R(r\cos t, r\sin t; x_2,\xi_2)\,dt. \tag{18.3.53}$$

So, *modulo operator with* $\bar{A}_0$*-bound not exceeding* $C\bigl(\mu^{-1}h + \vartheta(\varepsilon)\bigr)$ *we reduced* A *to* $\bar{A}_0 + W^{\mathsf{w}}$ *where for* $(l,\sigma) \succeq (2,0)$ *one needs to include* W^V *into* W *as well*.

18.3.2 Calculations

Calculations. I

Now we will need to calculate W more explicitly. To do this we repeat our construction, paying attention to explicit expressions we get rather than their smoothness. Also we will consider Weyl symbols rather than operators themselves. Let us start from the beginning of Subsubsection *18.3.1.3 Potential*. There we started from symbol $\bigl(\mu^2\bar{a}_0 + b\bigr)$ with $\bigl(b = \mu^2 M' + V'\bigr)$. Then we applied ψ_1 transformation and arrived to the symbol

$$\begin{aligned}\bigl(\mu^2\bar{a}_0 + b\bigr)\circ\psi_1 = \mu^2\bar{a}_0 \,+\,& \\ \mu^2\{\bar{S},\bar{a}_0\} + \frac{1}{2}\mu^2\{\bar{S},\{\bar{S},\bar{a}_0\}\} \,+\, &\frac{1}{6}\mu^2\{\bar{S},\{\bar{S},\{\bar{S},\bar{a}_0\}\}\}+\\ b \quad + \quad \{\bar{S},b\} \quad + \quad &\frac{1}{2}\{\bar{S},\{\bar{S},b\}\} \quad + \ \ldots\end{aligned} \tag{18.3.54}$$

where $\bar{S} = \bar{S}' + \bar{S}''$ and we skipped terms not exceeding $C\bigl(\mu^{-1}h + \vartheta(\varepsilon)\bigr)$.

Plugging $\mu^2\{\bar{S},\bar{a}_0\} = -\bar{b} + \bar{W}_0$ where $\bar{b}$ is a $\bar{\varepsilon}$-mollification of b and

$$\bar{W}_0 = \mathcal{M}_r(\bar{b}) \coloneqq \frac{1}{2\pi}\int_0^{2\pi} b(r\cos t, r\sin t; x_2,\xi_2)\,dt, \tag{18.3.55}$$

we get $\mu^2\bar{a}_0 + \bar{W}_0 + b_1$ with

$$b_1 = \frac{1}{2}\{\bar{S}, -\bar{b} + \bar{W}_0\} + (b - \bar{b}) + \{\bar{S}, b\} + \frac{1}{6}\{\bar{S}, \{\bar{S}, -\bar{b} + \bar{W}_0\}\} + \frac{1}{2}\{\bar{S}, \{\bar{S}, b\}\} \tag{18.3.56}$$

as a result of Subsubsectio *18.3.1.3 Potential*.

To calculate W as a result Subsubsection *18.3.1.4 Final Reduction* (without the last step) we need to apply $\mathcal{M}_r$ to this expression:

$$W = \bar{W}_0 + (\mathcal{M}_r b_1) = \mathcal{M}_r(b) + \mathcal{M}_r\big(\{\bar{S}, -\frac{1}{2}\bar{b} + \frac{1}{2}\bar{W}_0 + b\}\big) + \frac{1}{2}\mathcal{M}_r\big(\{\bar{S}, \{\bar{S}, -\frac{1}{3}\bar{b} + \frac{1}{3}\bar{W}_0 + b\}\}\big). \tag{18.3.57}$$

Finally, if $l = 2$, $\sigma > 0$ and we want to eliminate $O(\mu^{-4})$ from an error, we need to make a last step and define S_1 from the equation

$$\mu^2\{S_1, \bar{a}_0\} = -b_1 + (W - \bar{W}_0), \tag{18.3.58}$$

calculate

$$b_2 = \frac{1}{2}\mu^2\{S_1, \{S_1, \bar{a}_0\}\} + \{S_1, \bar{W}_0\} + b_1 - (W - \bar{W}_0) = \{S_1, -\frac{1}{2}b_1 + \frac{1}{2}W + \frac{1}{2}\bar{W}_0\} + b_1 - (W - \bar{W}_0) \tag{18.3.59}$$

and add to W term $\mathcal{M}_r(b_2)$.

Let $W_0 \coloneqq \mathcal{M}_r(b)$ and $b' = \frac{1}{2}\bar{W}_0 - \frac{1}{2}\bar{b} + b$. Then

$$\partial_\phi \bar{S} = \frac{1}{2}\mu^{-2}\bar{\beta}, \qquad \text{with} \quad \beta = b - W_0, \quad \bar{\beta} = \bar{b} - \bar{W}_0. \tag{18.3.60}$$

One can see easily that $W_0 \in \mathscr{F}^{l,\sigma}$ and that $W_0 \equiv b \mod O\big(\vartheta(\bar{\varepsilon}) + \mu^{-2}\big)$.

Note that we can take $\bar{S} = 0$ as $x_1 = \xi_1 = 0$ and in this case

$$|\partial_{x_2,\xi_2}\bar{S}| \le C\mu^{-2}\big(\mu^{-1} + \bar{\varepsilon}^{l-1}|\log\mu|^{-\sigma}\big). \tag{18.3.61}$$

One can check easily that with the final error $O\big(\mu^{-1} + \vartheta(\varepsilon)\big)$ in all the above calculations one can replace the standard Poisson brackets $\{.,.\}$ by the "short" ones $\{.,.\}'$, involving only x_1, ξ_1

$$\{f, g\}' = \partial_{\xi_1} f \cdot \partial_{x_1} g - \partial_{\xi_1} g \cdot \partial_{x_1} f \tag{18.3.62}$$

which in polar coordinates (r,ϕ) at (x_1,ξ_1)-plane are

$$\{f,g\}' = r^{-1}\big((\partial_\phi f)\cdot(\partial_r g) - (\partial_\phi g)\cdot(\partial_r f)\big). \tag{18.3.63}$$

Then integrating by parts if b' is differentiated by ϕ we get easily that

$$\mathcal{M}_r\big(\{\bar S, b'\}'\big) = r^{-1}\partial_r\mathcal{M}_r(b'\partial_\phi\bar S) = \frac{1}{2}\mu^{-2}r^{-1}\partial_r\mathcal{M}_r\big(b'\bar\beta\big) \equiv W_1 \coloneqq \frac{1}{4}\mu^{-2}r^{-1}\partial_r\mathcal{M}_r\big((W_0-b)^2\big). \tag{18.3.64}$$

Further, for $(l,\sigma)\succeq(2,0)$ let us consider the last terms in (18.3.55) and $\mathcal{M}_r(b_2)$ [24]. One can see easily that only "short" Poisson brackets (18.3.62) should be considered.

Then using the same arguments as above one can prove easily that modulo $O\big(\mu^{-1}h+\vartheta(\varepsilon)\big)$ these terms are equal $\frac{1}{8}\mu^{-4}r^{-1}\partial_r\mathcal{M}_r\big(r^{-1}(b-W_0)^2W\big)$ and $\frac{1}{4}\mu^{-2}r^{-1}\partial_r\mathcal{M}_r\big((b_1-W_1)^2\big)$ respectively with b_1 *redefined as*

$$b_1 \coloneqq \frac{1}{2}\{S, b+W_0\},\quad W_1=\mathcal{M}_r(b_1). \tag{18.3.56}$$*

Calculations. II

Now we want more explicit expressions for W, W_1 via g^{jk}, V.

First of all, note that for a constant metrics g^{jk} we have $b=V'=V\circ\Phi$, where $\Phi : T^*\mathbb{R}^2\to T^*\mathbb{R}^2$ is a linear map. Then $W_0=\mathcal{M}_rV'$ and since under map Φ a circle

$$\mathcal{C}_r=\{(x_2,\xi_2)=\mathsf{const}, x_1=r\cos t, \xi_1=r\sin t\} \tag{18.3.65}$$

translates into ellipse with x-projection

$$\mathcal{E}_r(y)=\{\sum_{j,k} g_{jk}(x_j-y_j)(x_k-y_k)=r^2\} \tag{18.3.66}$$

where $y=\Phi(\bar x_2,\bar\xi_2)$ is its center and g_{jk} are elements of the inverse matrix $(g^{jk})^{-1}$, we get that

$$W_0=\mathcal{M}_{\mathcal{E}_r}(\Phi(\bar x_2,\bar\xi_2)) \tag{18.3.67}$$

[24] One can prove easily that they are not essential as $(l,\sigma)\preceq(2,0)$ and we will see that they are $\asymp\mu^{-4}$ as $(l,\sigma)\succeq(2,0)$.

which is a corresponding average of V along $\mathcal{E}_r(y)$.

Similarly, we see that

$$W_1 \equiv -\frac{1}{4}\mu^{-2}|\nabla V'|^2 \equiv -\frac{1}{4}\mu^{-2}\sum g^{jk}(\partial_{x_j}V)(\partial_{x_k}V)\circ\Phi \mod O(\mu^{-4}). \tag{18.3.68}$$

Problem 18.3.4. In the general case, the deviation of $\mathcal{C}_r\circ\Phi$ from the ellipse $\mathcal{E}_r$ (with g_{jk} calculated in the center) is $O(\mu^{-2})$ and thus

$$W \equiv \mathcal{M}_{\mathcal{E}_r}(\Phi(\bar{x}_2,\bar{\xi}_2)) \mod O(\mu^{-2}). \tag{18.3.69}$$

This does not require $l>1$. It would be nice to get more precise answer in the general case as well.

18.3.3 Strong Magnetic Field $\mu \ge \epsilon(h|\log h|)^{-1}$

In this case we cannot claim anymore that $|p_j|\le C\mu^{-1}$ in the microlocal sense even if $\mu\le ch^{-1}$ because $f\big(\mu p_1(x,\xi),\mu p_2 x,\xi),x\big)$ is no more quantizable symbol. However, now $\mu^{-1}\le\varepsilon$ and $f\big(\varepsilon^{-1}p_1(x,\xi),\varepsilon^{-1}p_2(x,\xi),x\big)$ is quantizable symbol and we can claim that $|p_j|\le C\varepsilon$ in the microlocal sense in the zone of $\tau\le C_0(1+\mu h)$ due to the standard ellipticity arguments. Basically, this makes our construction much simpler; however, we need to remember that h^2 is no more non-essential term.

Construction of Subsubsections *18.3.1.1 Main pPart of the Operator. I* and *18.3.1.2 Main Part of the Operator. II* remains the same but we need to take in account correction (18.1.8) appearing when multiplying A by $F^{-\frac12}$ from the left and right and moving both copies of $F^{-\frac12}$ inside to g^{jk}.

Further, since the principal part of operator A is of magnitude μh rather than of magnitude 1 as it was before, the error due to approximation of the main part of operator is $O\big(\mu^2\varepsilon^{\bar l}|\log\mu|^{-\bar\sigma}\big)$ and it is non-essential (i.e. $O(\mu^{-1}h+1)$) for $(\bar l,\bar\sigma)=(4,2)$ only.

In the very input of Subsubsection *18.3.1.2 Main Part of the Operator. II* we can take a Taylor decomposition with respect to (x_1,ξ_1) (as we did in the smooth case) and then modulo "non-essential symbol" (which belongs to $\big(\mu^{-1}h+\vartheta(\varepsilon)\big)\mathscr{F}^{0,0}$) we get

$$\mu^2(x_1^2+\xi_1^2)+V(x_2,\xi_2)+ \mu^2\sum_{j=0}^{3}\beta_j(x_2,\xi_2)x_1^j\xi_1^{3-j}+\sum_{j=0}^{1}\alpha_j(x_2,\xi_2)x_1^j\xi_1^{1-j}+ \mu^2\sum_{j=0}^{4}\beta'_j(x_2,\xi_2)x_1^j\xi_1^{4-j}+\sum_{j=0}^{2}\alpha'_j(x_2,\xi_2)x_1^j\xi_1^{2-j} \tag{18.3.70}$$

with $\beta_j\in \mathscr{F}^{1,1}$, $\beta_j\in \mathscr{F}^{0,1}$, $\alpha_j\in \mathscr{F}^{l-1,\sigma}$, $\beta_j\in \mathscr{F}^{l-2,\sigma}$ and we need the last term only if $l\ge 2$.

However, we need to remember the terms in decomposition of the main part; we can eliminate the third order term but we will reduce the fourth-order term. More precisely, to get rid of the third and the forth terms in (18.3.70) we use again the construction of Subsubsection *18.3.1.3 Potential* but we look explicitly for

$$S'=\sum_{j=0}^{1}\hat\alpha_j(x_2,\xi_2)x_1^j\xi_1^{1-j},\qquad S''=\mu^2\sum_{j=0}^{3}\hat\beta_j(x_2,\xi_2)x_1^j\xi_1^{3-j} \tag{18.3.71}$$

with $\hat\alpha_j,\hat\beta_j$ of the same regularity as α_j,β_j.

Now we have expression (18.3.70) without the third and the fourth terms; the fifth and the sixth terms are modified but retain $\mathscr{F}^{l-2,\sigma}$-regularity provided $(l,\sigma)\succ(2,0)$; for $(l,\sigma)\preceq(2,0)$ we get non-essential terms. For $\mu\ge c^{-1}h^{-1}$ these terms are non-essential as well. So, *in these two cases our construction is complete*.

To complete reduction for $\mu\in[(h|\log h|)^{-1},ch^{-1}]$, $(l,\sigma)\succ(2,0)$ we apply again "poor man Fourier integral operator", this time with

$$S'=\sum_{j=0}^{2}\hat\alpha'_j(x_2,\xi_2)x_1^j\xi_1^{2-j},\qquad S''=\mu^2\sum_{j=0}^{4}\hat\beta'_j(x_2,\xi_2)x_1^j\xi_1^{4-j} \tag{18.3.72}$$

with $\hat\alpha'_j,\hat\beta'_j\in\mathscr{F}^{l-2,\sigma}$. However, we cannot eliminate these terms completely, we can only to reduce them to

$$\bar\alpha(x_2,\xi_2)\big(x_1^2+\xi_1^2\big),\qquad \bar\beta(x_2,\xi_2)\big(x_1^2+\xi_1^2\big)^2 \tag{18.3.73}_{1,2}$$

respectively with $\bar\alpha,\bar\beta\in\mathscr{F}^{l-2,\sigma}$.

One can calculate easily that the final expression is resembling one from Subsubsections *18.3.2.1 Calculations. I* and *18.3.2.2 Calculations. II* because in our situation $\mathcal{M}_C f = f + \frac{1}{2}\mu^{-2}\Delta_{x_1,\xi_1} f$ modulo non-essential term and one can recalculate this way W_1 too. However, construction here reminds one of the smooth theory.

So, we have proven

Proposition 18.3.5. *Let* $d = 2$, $\mu \ge (h|\log h|)^{-\frac{1}{3}}$ *and* $(\bar{l}, \bar{\sigma}) \succeq (2, 0)$, $(l, \sigma) \succeq (1, 2)$.

Then modulo operator with $\bar{A}_0$*-bound not exceeding* $C\big(\mu^{-1}h + \vartheta(\varepsilon)\big)$ *with* $\varepsilon = C(\mu^{-1}h|\log h|)^{\frac{1}{2}}$ *one can reduce operator* A *to operator* $\bar{A}_0 + W^{\mathsf{w}}$ *where* $W = W_0 + W_1 + W_2$ *depends on* x_2, ξ_2 *and* $\mu^{-1}A_0^{\frac{1}{2}}$ *(or* $\mu^{-1}\bar{A}_0^{\frac{1}{2}}$*) only.*

18.4 $d = 2$: Tauberian Theory

18.4.1 Decomposition

We again consider more difficult case $\mu \le \epsilon(h|\log h|)^{-1}$ first; the opposite case can be considered in the same way with rather obvious modifications and simplifications.

There is not much what is left: removing non-essential operator and compensating this by adding $\mp C_0\vartheta(\varepsilon)$ to the lower/upper approximation (which in turn will not violate our estimates; see below), we get a 1-dimensional operator-valued $\mu^{-1}h$-pseudodifferential operator

$$\bar{A}_0 + W(x_2, \mu^{-1}hD_2, \mu^{-1}\bar{A}_0) \tag{18.4.1}$$

with an operator-valued symbol, acting in auxiliary space $\mathscr{L}^2(\mathbb{R})$.

Taking as in Section 13.5 decomposition into

$$\Upsilon_n(x_1) \coloneqq (\mu^{-1}h)^{-\frac{1}{4}} v_n\big((\mu^{-1}h)^{-\frac{1}{2}}x_1\big) \tag{18.4.2}$$

with Hermitian functions v_n and coefficients which are functions of x_2 we get a family of 1-dimensional $\mu^{-1}h$-pseudodifferential operators

$$\mathcal{A}_n \coloneqq r_n^2 + W(x_2, \mu^{-1}hD_2, \mu^{-1}r_n), \qquad r_n \coloneqq \big((2n+1)\mu h\big)^{\frac{1}{2}} \tag{18.4.3}$$

provided $\mu hn \le C_0$.

Even if we cannot apply directly spectral projectors associated with $\mathcal{A}_n$, we can consider propagator $e^{ih^{-1}t\tilde{A}}$ and its Schwartz kernel $u(x,y,t)$ and prove easily that for $t, |t| \le T_1 := \epsilon\mu$

$$U(x,y,t) \equiv \mathcal{T}\Big(\sum_{n\ge 0} U_n(x_2,y_2,t)\Upsilon_n(x_1)\Upsilon_n(y_1)\Big)\mathcal{T}^{-1} \tag{18.4.4}$$

modulo Schwartz kernel of an operator which becomes negligible multiplied by $\big(\psi(x)q(\mu p_1,\mu p_2)\big)^{\mathsf{w}}$ where ψ, q are smooth and supported in $B(0,\frac{1}{2})$ and $B(0,C_0)$ respectively. Recall that we call operator *negligible* if its norm does not exceed $C(\mu^{-1}h)^s$ with large enough exponent s.

Here U is the Schwartz kernel of the propagator $e^{ih^{-1}t\tilde{A}}$ of the transformed back reduced operator[25),26)]

$$\tilde{A} := \mathcal{T}\Big(\bar{A}_0 + W(x_2,\mu^{-1}hD_2,\mu^{-1}\bar{A}_0)\Big)\mathcal{T}^{-1} \tag{18.4.5}$$

which is 1-dimensional $\mu^{-1}h$-pseudodifferential operator with operator-valued symbol, $\mathcal{T}$ is a full transformation constructed in Subsection 18.3.1, and

(18.4.6) U_n are Schwartz kernels of the propagators $e^{ih^{-1}t\mathcal{A}_n}$ for $\mathcal{A}_n$.

Further, by definition $W = W_0 + W_1$ because W_2 is not essential in our sense.

18.4.2 Estimate under Assumption (18.2.15)

So, for $T = T_1$, $\tau \le c$

$$F_{t\to h^{-1}\tau}\Big(\chi_T(t)\Gamma\big(\psi(x)u(x,y,t)\big)\Big) \equiv \tag{18.4.7}$$
$$F_{t\to h^{-1}\tau}\Big(\chi_T(t)\sum_{n\ge 0}\Gamma\Big(\tilde{\psi}\big(U_n(x_2,y_2,t)\Upsilon_n(x_1)\Upsilon_n(y_1)\big)\Big)\Big)$$

modulo $O(\mu^{-s})$ where as before Γ is a trace for Schwartz kernel (restriction to the diagonal and integration) and

$$\tilde{\psi} = \mathcal{T}^{-1}\psi\mathcal{T} \tag{18.4.8}$$

[25)] Apart of just reduction we removed some "non-essential" terms.

[26)] Actually we have two of them: $\tilde{A}^{\pm}$ such that $\tilde{A}^{-} \le A \le \tilde{A}^{+}$ and A-bounds of operators $A - \tilde{A}^{\pm}$ do not exceed $C\big(\mu^{-1}h + \vartheta(\varepsilon)\big)$ (so they are operators described in footnote [18)]).

is $\mu^{-1}h$-pseudodifferential operator rather than a function; its symbol is not very regular (see below).

In this decomposition we can consider only $n \le C_1(\mu h)^{-1}$ because for any other n interval $(-\infty, C_0)$ is a classically forbidden zone for operator $\mathcal{A}_n$ and one can prove easily that their contributions do not exceed $C\mu^{-s}(\mu hn+1)^{-s}$ with the arbitrarily large s and thus their total contribution is negligible.

Note that all operators $\mathcal{A}_n$ are non-degenerate at level 0 in the sense that

$$|\mathcal{A}_n| \le \epsilon_0 \implies |\nabla_{x_2,\xi_2}\mathcal{A}_n| \asymp 1 \tag{18.4.9}$$

if and only if the original operator satisfied the non-degeneracy assumption (18.2.15) because $|\nabla(W - V/F)| \le C\mu^{-1}$.

Let us assume first that this non-degeneracy assumption holds. Then in the standard way we can prove easily that if $Q_x = q(x_2, \mu^{-1}hD_{x_2})$ and $Q'_y = q'(y_2, \mu^{-1}hD_{y_2})$ are $\mu^{-1}h$-pseudodifferential operators with symbols supported in ε-vicinity of $(\bar{x}_2, \bar{\xi}_2)$ in which condition (18.4.9) is fulfilled, then

$$F_{t\to h^{-1}\tau}\chi_T(t)\big(Q_x U_n(x_2, y_2, t)\, {}^tQ'_y\big) \tag{18.4.10}$$

is negligible provided $T \le T_1$ and $\mu^{-1}T \times \varepsilon \ge C\mu^{-1}h|\log h|$ or equivalently

$$T_* := C\varepsilon^{-1}h|\log h| \le T \le T^* := \epsilon\mu. \tag{18.4.11}$$

Recall that χ is an admissible function supported in $[-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$. Then

$$F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\big(\psi(x)U(x, y, t)\big) \tag{18.4.12}$$

is negligible as well.

This statement does not look strong enough because T_* seems to be much larger than we need due to the factor ε^{-1}: the direct assault shows that $|\Gamma\psi KU|$ does not exceed Ch^{-2} (where $K = k\big(x, \mu p(x, \xi)\big)^{\mathsf{w}}$) is $\mu^{-1}h$-pseudodifferential operator cut-off; then

$$|F_{t\to h^{-1}\tau}\big(\bar{\chi}_T(t)\Gamma(\psi KU)\big)| \le Ch^{-2}T_* \tag{18.4.13}$$

for $\bar{\chi}$ supported in $[-1, 1]$ and equal 1 at $[-\frac{1}{2}, \frac{1}{2}]$ and $T \in [T_*, T^*]$; here we can take $K = I$ because $\tau \le c$ and we can apply ellipticity arguments in

zone $\{|p| \ge C'\mu^{-1}\}$; then (18.4.13) and the standard Tauberian arguments imply Tauberian spectral remainder estimate

$$\mathsf{R}^{\mathsf{T}} := |\Gamma(\psi Q_x e)(\tau) - h^{-1}\int_{-\infty}^{\tau}\Big(F_{t\to h^{-1}\tau'}\big(\bar\chi_T(t)\Gamma(\psi QU)\big)\Big)\,d\tau'| \le Ch^{-2}\frac{T_*}{T^*} \asymp Ch^{-1}\mu^{-1}\varepsilon^{-1}|\log h| \tag{18.4.14}$$

and it is close to what we are looking for (but still is not that good) only for $\varepsilon = 1$. Sure by rescaling arguments we can make $T_* = h\varepsilon^{-1}$ in the right-hand expressions of the estimates only and thus get rid of logarithmic factor (and we will do it) but it alone does not solve our problem.

The source of this trouble is that *we have ε-scale in both x_2 and ξ_2*, while in the non-magnetic case treated in Section 4.6 we had scale 1 in ξ_2.

However, there is a way to improve our analysis: what we need is an estimate

$$|F_{t\to h^{-1}\tau}\big(\bar\chi_T(t)\Gamma(\psi U)\big)| \le Ch^{-1} + C\mu h^{-2}\vartheta(\varepsilon) \tag{18.4.15}$$

which we are going to prove; then dividing the right-hand expression of (18.4.15) by T^* we get the required remainder estimate $O\big(\mu^{-1}h^{-1}+h^{-2}\vartheta(\varepsilon)\big)$.

(i) First of all, we want to deal with U_n directly. Note that

$$\tilde\psi = \psi_0(x_2,\mu^{-1}hD_2)+ \psi_1(x_2,\mu^{-1}hD_2)x_1 + \psi_2(x_2,\mu^{-1}hD_2)\mu^{-1}hD_1 + \tilde\psi'(x,\mu^{-1}hD)$$

with symbol of $\tilde\psi'$ belonging to $\mu^{-l}|\log h|^{-\sigma}\mathscr{F}^{0,0}$; for $l=2,\sigma>0$ one needs to include also the second order terms

$$\psi_{11}(x_2,\mu^{-1}hD_2)\,x_1^2 + \psi_{12}(x_2,\mu^{-1}hD_2)\,(x_1\mu^{-1}hD_1+\mu^{-1}hD_1x_1)+ \psi_{11}(x_2,\mu^{-1}hD_2)\,(\mu^{-1}hD_1)^2.$$

Let us consider $|F_{t\mapsto h^{-1}\tau}\big(\bar\chi_T(t)\tilde\psi' U\big)|$ first; due to the analysis above it does not exceed $Ch^{-2}T_*\times\mu^{-l}|\log h|^{-\sigma}$; one can see easily that it does not exceed the right-hand expression of (18.4.15).

Further, note that plugging terms linear with respect to $(x_1,\mu^{-1}hD_1)$ kills the trace while plugging quadratic terms is equivalent to plugging $(n+\frac12)\mu^{-1}h\psi'(x_2,\mu^{-1}hD_2)$ with $\psi'=\psi_{11}+\psi_{22}$.

Now we have pure $\mu^{-1}h$-pseudodifferential operators, acting on x_2 alone; we consider u_n and replace Γ by Γ' acting with respect to x_2 alone.

(ii) Second, we need to improve T_* slightly, removing $|\log h|$ factor and to get better estimates for Fourier transform. Let us consider ε-admissible partition in (x_2, ξ_2): $1 = \sum_\nu q_{(\iota)}$ and consider element $q_{(\iota)}$ of this partition.

One can easily prove by the standard rescaling technique the following estimate

$$|\Gamma'\bigl(\chi_T(t) q_{(\iota)} u_n\bigr)| \le C\varepsilon^2 \mu h^{-1} \bigl(\frac{T}{\bar{T}_*}\bigr)^s \tag{18.4.16}$$

for arbitrarily large s and $T \in [\bar{T}_*, T^*]$ where

$$\bar{T}_* := C_0 \varepsilon^{-1} h = T_*/|\log h|. \tag{18.4.17}$$

Then

$$|F_{t\to h^{-1}\tau}\bigl(\chi_T(t)\Gamma'(q_{(\iota)}U_n)\bigr)| \le C\varepsilon^2 \mu h^{-1} \bigl(\frac{T}{\bar{T}_*}\bigr)^s T \tag{18.4.18}$$

and thus

$$|F_{t\to h^{-1}\tau}\bigl(\bar{\chi}_T(t)\Gamma'(q_{(\iota)}U_n)\bigr)| \le C\varepsilon^2 \mu h^{-1} \bar{T}_* \tag{18.4.19}$$

for any $T \in [\bar{T}_*, T^*]$.

Further, note that due to the standard ellipticity arguments

$$F_{t\to h^{-1}\tau}\bigl(\bar{\chi}_T(t)\Gamma'(q_{(\iota)}U_n)\bigr)$$

is negligible for $|\tau| \le Ch|\log h| T^{-1}$ unless

$$|\mathcal{A}_n(x_2, \xi_2)| \le C\bigl(\varepsilon + Ch|\log h| T^{-1}\bigr) \qquad \text{on} \quad \operatorname{supp}(q_{(\iota)}), \tag{18.4.20}$$

and that

(18.4.21) Under assumption (18.2.15) the set

$$\Omega_{\tau,n} := \{(x_2, \xi_2)\colon |\mathcal{A}_n(x_2, \xi_2) - \tau| \le C\varepsilon\} \tag{18.4.22}$$

has measure $O(\varepsilon)$ for each τ, n.

Therefore as $T \ge T_*$ (and therefore $Ch|\log h|\,T^{-1} \lesssim \varepsilon$ we arrive to the estimate

$$|F_{t\to\mu^{-1}h}\bar\chi_T(t)\Gamma'(QU_n)| \le C\mu h^{-1}\,\bar T_* \times \varepsilon = C\mu \tag{18.4.23}$$

which implies (18.4.15) since we need to multiply by the number of the admissible indices (which is $\asymp (\mu h)^{-1}$) resulting in Ch^{-1} and take in account contribution of the neglected terms. Then in virtue of the standard Tauberian arguments (with $T^* \asymp \mu$) we arrive to the estimate

$$\mathsf{R}^{\mathsf{T}} \le C\mu^{-1}h^{-1} + Ch^{-2}\vartheta(\varepsilon) \tag{18.4.24}$$

with R^{T} defined by (18.4.14) $\bar\chi = 1$ on $[-\frac12, \frac12]$, $T \in [T_*, T^*]$ which is exactly estimate we want.

18.4.3 Estimate under Assumption (18.2.16)

Now let $(I,\sigma) \succeq (2,0)$ and at some point $\bar z := (\bar x_2, \bar\xi_2)$ the following generalization of (18.4.9) holds:

$$|\nabla \mathcal{A}_n| \le \epsilon_0\nu \implies |\nabla \mathcal{A}_n(x_2,\xi_2)| \asymp \nu \tag{18.4.9}'$$

where

$$\nu \ge C_0\mu^{-1} \tag{18.4.25}$$

which in turn is larger than $C_0\varepsilon$ if $\mu \le \epsilon(h|\log h|)^{-1}$. Then (18.4.9)$'$ holds in $B(\bar z, \epsilon_0\nu)$.

Then we can take

$$T_* := C\varepsilon^{-1}\nu^{-1}h|\log h| \le T^* = \epsilon\mu \tag{18.4.17}'$$

and estimates (18.4.18) and (18.4.19) hold. On the other hand, one needs to replace (18.4.20) by

$$|\mathcal{A}_n(x_2,\xi_2)| \le C\bigl(\varepsilon\nu + Ch|\log h|\,T^{-1}\bigr) \qquad \text{on} \quad \mathsf{supp}(q_{(\iota)}), \tag{18.4.20}'$$

and instead of (18.4.21) observe that

(18.4.26) Under assumption (18.4.9)$'$

$$\Omega_{\tau,n,\nu} := \{(x_2,\xi_2) \in B(\bar z, \epsilon_0\nu)\colon |\mathcal{A}_n(x_2,\xi_2) - \tau| \le C\varepsilon\nu\} \tag{18.4.22}'$$

has measure $O(\varepsilon\nu)$ for each τ, n.

Then, as Q is supported in $B(\bar{z}, \epsilon_0\nu)$, (18.4.23) is replaced by

$$|F_{t\to\mu^{-1}h}\bar{\chi}_T(t)\Gamma'(Qu_n)| \le C\mu h^{-1}\bar{T}_* \times \nu\varepsilon = C\mu \tag{18.4.23)'}$$

and summation over n returns $C\mu(\nu^2/(\mu h)+1)$ while summation over z returns

$$C\int \mu(\nu(z)^2/(\mu h)+1)\nu^{-2}(z)\,dz \asymp Ch^{-1} + C\mu|\log\mu| \tag{18.4.27}$$

due to assumption (18.2.16). As $\mu \lesssim (h|\log h|)^{-1}$ the second term in the right-hand expression is less than the first one.

Therefore if Q is supported in the ball of radius $\asymp 1$ and is negligible on $\mathcal{X}_{\mathsf{inn}} := \{z : |\nabla W| \le C_0\mu^{-1}\}$ inequality (18.4.23) remains true and the contribution of Q to Tauberian remainder does not exceed $C\mu^{-1}h^{-1}$.

Meanwhile contribution of $\mathcal{X}_{\mathsf{inn}}$ (intersected with the ball of radius $\asymp 1$) to the Tauberian remainder does not exceed $C\mu h^{-1} \times \mathsf{mes}\,\mathcal{X}_{\mathsf{inn}} = O(\mu^{-1}h^{-1})$ in virtue of (18.2.16) again. Thus we arrive to the Tauberian estimate (18.4.24) again. Therefore we arrive to

Proposition 18.4.1. *Let $d = 2$ and $(\bar{l}, \bar{\sigma}) \succeq (2, 0)$. Further, assume that $h^{-\frac{1}{3}} \lesssim (h|\log h|)^{-\frac{1}{3}} \le \mu \le \epsilon(h|\log h|)^{-1}$ and <u>either</u>*

(i) $(l, \sigma) \succeq (1, 2)$ and assumption (18.2.15) *be fulfilled <u>or</u>*

(ii) $(l, \sigma) \succeq (2, 0)$ and assumption (18.2.16) *be fulfilled.*

Then the Tauberian estimate (18.4.24) *holds where we need to calculate the Tauberian expression*

$$h^{-2}\mathcal{N}^{\mathsf{T}} := h^{-1}\int_{-\infty}^{\tau}\Big(F_{t\to h^{-1}\tau'}\big(\bar{\chi}_T(t)\Gamma(\psi u)\big)\Big)\,d\tau' \tag{18.4.28}$$

with $T \ge T_* = C\varepsilon^{-1}h|\log h|$ in the framework of (i); in the framework of (ii) we need to take the discussed above partition and take $T \ge T_{*\iota} = C\varepsilon^{-1}\nu_\iota^{-1}h|\log h|$ on each element of this partition.

18.4.4 Case $\mu \ge \epsilon(h|\log h|)^{-1}$

As $\mu \ge \epsilon(h|\log h|)^{-1}$ we need to remember that now $\varepsilon \ge \mu^{-1}$; therefore estimates

$$|x_1| \le C_0\mu^{-1}, \qquad |\mu^{-1}hD_1| \le C_0\mu^{-1} \tag{18.4.29)$_{1,2}}$$

should be replaced by

$$|x_1| \le C_0\varepsilon, \qquad |\mu^{-1}hD_1| \le C_0\varepsilon \tag{18.4.30}_{1,2}$$

in the "domain of interest"; however, as $\mu \lesssim h^{-1}$, estimate (18.4.28) still holds in the operator sense.

Further, as $\mu \ge ch^{-1}$ we need to consider the Schrödinger-Pauli operator

$$A = \sum_{j,k} P_j g^{jk} P_k + V - \mathfrak{z}\mu hF, \qquad \text{with } P_j = hD_j - \mu V_j \tag{18.4.31}$$

with $\mathfrak{z} \in \mathbb{R}$ and to avoid hitting the spectral gap we need to consider only $\mathfrak{z} = 1, 3, 5, \dots$. Then (18.4.30) holds in microlocal sense and also in operator sense (but in the latter case one can take $\varepsilon = C(\mu^{-1}h)^{\frac{1}{2}}$).

Furthermore, one need to remember that $n \le C_0/\mu h$ should be replaced by $n \le C_0$ as $\mu \ge ch^{-1}$. This takes care of the case of assumption (18.2.15) and we arrive to the estimate (18.4.24) as $\mu \le h^{-1}$ and estimate

$$\mathsf{R}^{\mathsf{T}} \le C + C\mu h\vartheta(\varepsilon) \tag{18.4.32}$$

as $\mu \ge h^{-1}$.

Under assumption (18.2.16) the additional analysis is required since for $\{\nu \le \varepsilon\}$ the uncertainty principle breaks but the contribution of such zone to the remainder is $Ch^{-2}(\mu h + 1)\varepsilon^2$ (i.e. estimates (18.4.24) and (18.4.32) acquire additional terms $C\mu^{-1}h^{-1}|\log\mu|$ and $C|\log\mu|$ respectively.

Thus we arrive to the following:

Proposition 18.4.2. *Let* $d = 2$ *and* $(\bar{l}, \bar{\sigma}) \succeq (2, 0)$.

(i) Let $(l, \sigma) \succeq (1, 2)$ *and the non-degeneracy assumption* (18.2.15) *be fulfilled. Then for* $(h|\log h|)^{-1} \le \mu \le h^{-1}$ *the Tauberian estimate* (18.4.24) *holds for the Schrödinger operator and for* $\mu \ge h^{-1}$ *the Tauberian estimate* (18.4.32) *holds for the Schrödinger-Pauli operator* (18.4.31) *where we need to calculate the Tauberian expression* (18.4.32) *with* $T \ge T_* = C\varepsilon^{-1}h|\log h|$.

(ii) Let $(l, \sigma) \succeq (2, 0)$ *and assumption* (18.2.16) *be fulfilled. Then for* $(h|\log h|)^{-1} \le \mu \le h^{-1}$ *the Tauberian estimate*

$$\mathsf{R}^{\mathsf{T}} \le C\mu^{-1}h^{-1} + C|\log\mu| + Ch^{-2}\vartheta(\varepsilon) \tag{18.4.33}$$

holds for the Schrödinger operator and for $\mu \ge h^{-1}$ *Tauberian estimate*

$$\mathsf{R}^{\mathsf{T}} \le C|\log \mu| + C\mu h^{-1}\vartheta(\varepsilon) \tag{18.4.34}$$

holds for the Schrödinger-Pauli operator(18.4.31) *where we need to take partition and to calculate Tauberian expression* (18.4.32) *with* $T \ge T_{*\iota} = C\varepsilon^{-1}\nu_\iota^{-1}h|\log h|$.

Repeating arguments of the smooth case (in Section 13.6) one can prove easily the following:

Remark 18.4.3. (i) For $\mu \gtrsim h^{-1}$ as usual the non-degeneracy assumptions (18.2.15) and (18.2.16) could be weaken to

$$|(2n+1-\mathfrak{z})\mu h + V| + |\nabla V/F| \ge \epsilon_0 \qquad \forall n \in \mathbb{Z}^+ \tag{18.2.15)*}$$

and

$$|(2n+1-\mathfrak{z})\mu h + V| + |\nabla V/F| + |\det \mathsf{Hess}\, V/F| \ge \epsilon_0 \qquad \forall n \in \mathbb{Z}^+ \tag{18.2.16)*}$$

respectively.

(ii) For $\mu \lesssim h^{-1}$ under condition

$$|\nabla V/F| \le \epsilon_0 \implies \det \mathsf{Hess}\, V/F \ge \epsilon_0 \tag{18.2.16)$^+$}$$

estimate (18.4.24) holds and as $\mu \gtrsim h^{-1}$ under condition

$$|(2n+1-\mathfrak{z})\mu h + V| + |\nabla V/F| \le \epsilon_0 \implies \det \mathsf{Hess}\, V/F \ge \epsilon_0 \qquad \forall n \in \mathbb{Z}^+ \tag{18.2.16)$^{*+}$}$$

estimate (18.4.32) holds.

(iii) As $\mu \gtrsim h^{-1}$ in the case of the spectral gap

$$|(2n+1-\mathfrak{z})\mu h + V| \ge \epsilon_0 \qquad \forall n \in \mathbb{Z}^+ \tag{18.4.35}$$

$\mathsf{R}^{\mathsf{T}} = O(\mu^{-s})$ as in the smooth case.

Here in Statement (i) we use the fact that under condition (18.2.16)$^+$ or (18.2.16)$^{*+}$ (in the corresponding settings) $\mathcal{A}_n$ is is either elliptic unless $\nu \asymp \bar\nu$ for some $\bar\nu \ge C_0 \max\bigl(\mu^{-1}, (\mu^{-1}h)^{\frac12}\bigr)$ or unless $\nu \le \bar\nu = C_0 \max\bigl(\mu^{-1}, (\mu^{-1}h)^{\frac12}\bigr)$.

For $\bar\nu \in [(\mu^{-1}h)^{\frac12}, (\mu^{-1}h|\log h|)^{\frac12}]$ we use the standard rescaling technique.

18.5 $d = 2$: Calculations and Main Theorems

Now our purpose is to prove main results as $\mu \ge \bar{\mu}_1$, i.e. Theorems 18.5.4 and 18.5.7 below. In the calculations we can assume that $(18.1.6)^*$ holds, i.e. $F = 1$.

18.5.1 Calculations

Step I under Assumption (18.2.15)

To calculate the Tauberian expression (18.4.28) we rewrite it modulo negligible terms as the sum with respect to n, ι of

$$\mathcal{S}_{n(\iota)} := h^{-1} \int_{-\infty}^{\tau} \Big(F_{t\to h^{-1}\tau'} \big(\bar{\chi}_T(t) \Gamma \big(\tilde{\psi}(x, \mu^{-1} hD) q^{\mathsf{w}}_{(\iota)} \big(u_n \Upsilon_n(x_1) \Upsilon_n(x_1) \big) \big) \big) \Big) \, d\tau' \tag{18.5.1}$$

with $q_{(\iota)} = q^{\mathsf{w}}_{(\iota)}(x_2, \mu^{-1} hD_2)$.

One can see easily that on the *elliptic elements* (with failed (18.4.20) or $(18.4.20)'$)

$$\mathcal{S}_{n(\iota)} \equiv \frac{1}{2\pi} \mu h^{-1} \iint \theta\big(\tau - \mathcal{A}_n(x_2, \xi_2)\big) \tilde{\psi}(x, \xi) q_{(\iota)}(x_2, \xi_2) \, dx d\xi \tag{18.5.2}$$

modulo negligible expression; here $\theta\big(\tau - \mathcal{A}_n(x_2, \xi_2)\big) = \mathsf{const}$ on $\mathsf{supp}(q_{(\iota)})$ and we need to calculate the difference between these two expressions on the *non-elliptic elements* (i.e. satisfying (18.4.20) or $(18.4.20)'$).

The total contribution of such terms is not very large: as $\mu h \lesssim 1$ and $\nu = 1$ it does not exceed $Ch^{-1} \times \mu \times \varepsilon \times 1/(\mu h)$ where the first factor is just present in (18.5.1), the second factor is the estimate (18.4.23) of the Fourier transform $(F_{t\to \mu^{-1}h} \bar{\chi}_T(t) \Gamma'(Qu_n))$, the third factor is here because this Fourier transform is negligible for $|\tau' - \tau| \ge C\varepsilon$ and the last factor is the number of n for which (18.4.20) holds on $\mathsf{supp}(q_{(\iota)})$. Still resulting expression $C\varepsilon h^{-2}$ is larger than what we need.

We apply the method of the successive approximations to calculate these terms; as unperturbed operator we take

$$\mathcal{B}_n = \mathcal{A}_n(y, \mu^{-1} hD_x). \tag{18.5.3}$$

Then

$$\mathcal{R}_n \coloneqq \mathcal{A}_n - \mathcal{B}_n = (x_2 - y_2)\mathcal{Q}_n(x, y, \mu^{-1}hD_x) \tag{18.5.4}$$

and then

$$U_n^{\pm}(x_2, y_2, t) \coloneqq \theta(\pm t)U(x_2, y_2, t) = \mp ih\mathcal{G}_n^{\pm}\delta(x_2 - y_2)\delta(t) = \mp ih\bar{\mathcal{G}}_n^{\pm}\delta(x_2 - y_2)\delta(t) \mp ih\bar{\mathcal{G}}_n^{\pm}\mathcal{R}_n\mathcal{G}_n^{\pm}\delta(x_2 - y_2)\delta(t) \tag{18.5.5}$$

where $\bar{\mathcal{G}}^{\pm}$ and $\mathcal{G}^{\pm}$ are forward/backward parametrices for the Cauchy problem for operators $hD_t - \mathcal{B}_n$ and $hD_t - \mathcal{A}_n$ respectively. Similarly,

$$U_n^{\pm}(x_2, y_2, t)\,{}^t\!Q_y = \mp ih\mathcal{G}_n^{\pm}\delta(x_2 - y_2)\,{}^t\!Q_y\delta(t) = \mp ih\bar{\mathcal{G}}_n^{\pm}\delta(x_2 - y_2)\,{}^t\!Q_y\delta(t) \mp ih\bar{\mathcal{G}}_n^{\pm}\mathcal{R}_n\mathcal{G}_n^{\pm}\delta(x_2 - y_2)Q_y^t\delta(t) \tag{18.5.6}$$

for $Q_y = Q(y_2, \mu^{-1}hD_{y_2})$ where as usual we write pseudodifferential operator acting on y on the right in accordance to matrix theory and ${}^t\!Q$ denotes the transposed operator.

So, $U_n = U_n^0 + U_n^1$ where U^0 is a Schwartz kernel of the propagator for $\mathcal{B}_n$ and

$$U_n^{1\pm} = \bar{\mathcal{G}}_n^{\pm}\mathcal{R}_n U_n^{\pm}. \tag{18.5.7}$$

As usual we take $(x_2 - y_2)$ and carry it forward to the left using commuting relation

$$[\bar{\mathcal{G}}_n^{\pm}, (x_2 - y_2)] = \bar{\mathcal{G}}_n^{\pm}[\mathcal{B}_n, x_2 - y_2]\bar{\mathcal{G}}_n^{\pm} \tag{18.5.8}$$

and in the very front this factor will be killed by Γ'. Then for our purpose we can replace $U_n^{1\pm}$ by

$$\bar{\mathcal{G}}_n^{\pm}[\mathcal{B}_n, x_2 - y_2]\bar{\mathcal{G}}_n^{\pm}\mathcal{Q}_n U_n^{0\pm}. \tag{18.5.9}$$

Let us dump it in (18.4.28). Consider first

$$F_{t\to h^{-1}\tau}\Big(\varphi_T(t)\Gamma'\Big(\big(U_n^1(\tilde{\psi}\,{}^t\!q_{(\iota)}^{\mathsf{w}})_y\big)\Big)\Big) \tag{18.5.10}$$

with <u>either</u> $\varphi_T = \bar{\chi}_T$, $T = \bar{T}_*$ <u>or</u> $\varphi_T = \chi_T$, $T \ge \bar{T}_0$; recall that $\bar{T}_* = Ch/\varepsilon$ as $\nu = 1$.

In the first case ($\varphi_T = \bar{\chi}_T$, $T = \bar{T}_*$) our estimate is rather straightforward. Note that the norm of $\bar{\mathcal{G}}_n^{\pm}$ in the strip $\{t\colon |t| \le T\}$ does not exceed $h^{-1}T$ and the norm of $[\mathcal{B}_n, x_2 - y_2]$ does not exceed $C\mu^{-1}h$. Then the factor $(x_2 - y_2)$ translates into an extra factor $C\mu^{-1}h^{-1}T^2$ in the norm estimate. For $T = \bar{T}_*$ this factor is $C\mu^{-1}h\varepsilon^{-2} = C|\log h|^{-1}$ and in the general case it will be $C|\log h|^{-1}T^2/\bar{T}_*^2$.

Further, we can assume that

(18.5.11) The operator norm of $\mathcal{R}_n$ does not exceed $C\bigl(\vartheta(\varepsilon) + \varepsilon^2\bigr)$.

Indeed, we always can make $\partial_{x_2} W$ vanish at some point of $\mathsf{supp}(q_{(\iota)})$ by an appropriate Fourier integral operator corresponding to some linear symplectomorphism at (x_2, ξ_2)-plane.

Then in this case expression (18.5.10) does not exceed $C\mu\varepsilon^{-1}\bigl(\vartheta(\varepsilon) + \varepsilon^2\bigr)$; recall that for U^0 it was just $C\mu$.

In the second case ($\varphi_T = \chi_T$, $T \ge \bar{T}_*$) we, using the standard rescaling arguments, gain an extra factor $(\bar{T}_*/T)^s$ with arbitrarily large exponent s which takes care of all $(T/\bar{T}_*)^k$ factors and summation with respect to partition of unity with respect to t. Now we can conclude that expression (18.5.10) does not exceed $C\mu\varepsilon^{-1}\bigl(\vartheta(\varepsilon) + \varepsilon^2\bigr)$ for $\varphi = \bar{\chi}$, $T = T_*$.

Then applying arguments we used to estimate the total contribution of our "special" elements $q_{(\iota)}$ to (18.4.28), we see that

(18.5.12) The total contribution of these "special" elements $q_{(\iota)}$ to (18.4.28) with U replaced by U^1 does not exceed $Ch^{-2}\vartheta(\varepsilon) + Ch^{-2}\varepsilon^2$

what is exactly what we expect mollification error would be as $(I, \sigma) \preceq (2, 0)$.

To deal with the case $(I, \sigma) \succ (2, 0)$ we note that due to the propagation speed $O(\mu^{-1})$ we can estimate $(x_2 - y_2)$ by $C\mu^{-1}T$ and as effectively T is estimated by $\bar{T}_*$ we conclude that we can assume that

(18.5.13) As $(I, \sigma) \succ (2, 0)$ the operator norm of $\mathcal{R}_n$ does not exceed $C\varepsilon'^2$ with $\varepsilon' := C\mu^{-1}h/\varepsilon$

and replace (18.5.12) by

(18.5.14) As $(I, \sigma) \succeq (2, 0)$ the total contribution of these "special" elements $q_{(\iota)}$ to (18.4.28) with U replaced by U^1 does not exceed $Ch^{-2}\varepsilon'^2 = O(\mu^{-1}h^{-1})$.

Now let us consider contribution of U^0. One can see easily that it is exactly equal to the left-hand expression of

$$\Gamma'\Big(\theta\big(\tau-\mathcal{A}_n(y_2,\mu^{-1}hD_{x_2})\big)K_{(\iota)}\Big)= (2\pi)^{-1}\mu h^{-1}\iint \theta\big(\tau-\mathcal{A}_n(x_2,\xi_2)\big)k_{(\iota)}(x_2,\xi_2)\,dx_2d\xi_2 \tag{18.5.15}$$

where $K_{(\iota)}=\tilde{\psi}_n q^{\mathsf{w}}_{(\iota)}$, $k_{(\iota)}$ is its pq-symbol[27] and

$$\tilde{\psi}_n=\langle\tilde{\psi}\Upsilon_n,\Upsilon_n\rangle \tag{18.5.16}$$

where $\Upsilon_n=\Upsilon_n(x_1)$ and $\langle .,.\rangle$ is an inner product in $\mathscr{L}^2(\mathbb{R}_{x_1})$.

The problem is that pq-symbols are not invariant with respect to Fourier integral operators and the difference between pq- and Weyl symbols is not small because of $q_{(\iota)}$. However, there is a simple walk-around. Cover our total (x_2,ξ_2)-domain by two, with $\partial_{\xi_2}W$ and $\partial_{x_2}W$ disjoint from 0 respectively. We consider the first domain; the second case will be reduced to the first one easily, by means of $\mu^{-1}h$-Fourier transform.

We can consider partition of unity (depending on n) $q'_{(\iota_1)}(x_2)q'_{(\iota_2)}(\xi_2)$ such that for each ι_1 (and each n) there is no more than one number ι_2 such that condition (18.4.20) holds on $\mathsf{supp}(q'_{(\iota_1)})\times\mathsf{supp}(q''_{(\iota_2)})$. Then all the contributions of all other elements do not depend on this Fourier integral operator and summing with respect to ι_2 we get just $q'_{(\iota_1)}(x_2)$.

But we can take the required linear symplectomorphism $(x_2,\xi_2)\mapsto(x_2,\xi_2-r_{(\iota_1)}x_2)$ and the corresponding Fourier integral operator will be just multiplication by $e^{-i\mu h^{-1}r_{(\iota)}x_2^2/2}$ which does not affect $q'_{(\iota_1)}(x_2)$. On the other hand, for $\tilde{\psi}$ the difference between different symbols does not exceed $C\mu^{-1}h$ and therefore without increasing our remainder estimate we can rewrite our final answer as

$$(2\pi)^{-1}\mu h^{-1}\sum_{n\ge 0}\iint \theta\big(\tau-\mathcal{A}_n(x_2,\xi_2)\big)\tilde{\psi}_n(x_2,\xi_2)\,dx_2d\xi_2. \tag{18.5.17}$$

Recall that so far $\mu\le(h|\log h|)^{-1}$ and assumption (18.2.15) was supposed to be fulfilled.

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[27] Which means that $K=k(\overset{1}{x},\mu^{-1}\overset{2}{h}D_x)$.

Step I under Assumption (18.2.16)

If assumption (18.2.15) is not fulfilled we must observe that instead of claim (18.5.12) we now have that

(18.5.18) The total contribution of these "special" ν-elements $q_{(\iota)}$ to (18.4.28) with U replaced by U^1 does not exceed $Ch^{-2}\big(\vartheta(\varepsilon)+\varepsilon^2\big)\times\big(\nu^2+\mu h\big)$.

Really, in comparison with the case $\nu=1$, factor T_* (or $\bar T_*$) brings a factor ν^{-1}, and volume brings factor ν but the number of indices "n" is now $\asymp(\nu^2/(\mu h)+1)$ rather than $\asymp 1/(\mu h)$.

Further, claim (18.5.13) is preserved as factor ν in the propagation speed is compensated by factor ν^{-1} in $\bar T_*$ and therefore (18.5.14) should be replaced by

(18.5.19) As $(l,\sigma)\succeq(2,0)$ the total contribution of these "special" ν-elements $q_{(\iota)}$ to (18.4.28) with U replaced by U^1 does not exceed $Ch^{-2}\varepsilon'^2\times\big(\nu^2+\mu h\big)$.

Then we have an extra term

$$C\mu h^{-1}\big(\vartheta(\varepsilon)+\varepsilon'^2\big) \tag{18.5.20}$$

which after summation with respect to ν under assumption (18.2.16) simply acquires factor $|\log h|$; further, under assumption $(18.2.16)^+$ only ν of the same magnitude should be taken in account near each critical point and no logarithmic factor is acquired. Thus we arrive to

(18.5.21) Under assumption (18.2.16) the total contribution of these "special" elements $q_{(\iota)}$ to (18.4.28) with U replaced by U^1 does not exceed

$$C\big(h^{-2}\vartheta(\varepsilon)+\mu^{-1}h^{-1}\big)\big(1+\mu h|\log h|\big)\asymp C\big(h^{-2}\vartheta(\varepsilon)+\mu^{-1}h^{-1}\big)+C\big(\mu h^{-1}\vartheta(\varepsilon)+1\big)|\log\mu|$$

and

(18.5.22) Under assumption $(18.2.16)^+$ the total contribution of these "special" elements $q_{(\iota)}$ to (18.4.28) with U replaced by U^1 does not exceed $C\big(h^{-2}\vartheta(\varepsilon)+\mu^{-1}h^{-1}\big)$.

Sure, in our case $\mu\lesssim(h|\log h|)^{-1}$ these extra terms with logarithmic factor do not matter but they will be important otherwise.

Step II under Assumption (18.2.15)

To finish the proof of the main theorems we need to bring expression (18.5.17) with $\tau = 0$ to the more explicit form.

First of all, we need to use original (x_1, x_2)-coordinates rather than (x_2, ξ_2)-coordinates after reduction. The natural expression of the area form in (x_1, x_2)-coordinates is $\omega = \sqrt{g}\,dx_1 \wedge dx_2$ which can be considered as the area form on $\Sigma = \{p_1 = p_2 = 0\}$ such that $\omega \wedge dp_1 \wedge dp_2 = dx \wedge d\xi$.

Then after construction of Subsubsections 18.3.1.1 and 18.3.1.2 we have a symplectic map Φ which transforms Σ into $\Sigma_0 = \{x_1 = \xi_1 = 0\}$; let us denote $\Phi|_\Sigma$ by ϕ. Further, Φ transforms $g^{-\frac{1}{2}} dp_1 \wedge dp_2$ at points of Σ exactly into $-dx_1 \wedge d\xi_1$ at points of Σ_0. Therefore

$$\phi_* \omega = -dx_2 \wedge d\xi_2, \qquad \phi = \Phi|_\Sigma. \tag{18.5.23}$$

Consider next transformations of Subsection 18.3.1. If $V = 0$ then these transformations will preserve Σ_0 and be identical on it. In the general case one can check easily that these transformations transform point $(0, 0; x_2, \xi_2) \in \Sigma_0$ into

$$\begin{aligned}&\big(\zeta_1(x_2, \xi_2), \zeta_2(x_2, \xi_2); x_1 + \eta_1(x_2, \xi_2), x_2 + \eta_2(x_2, \xi_2)\big) \in \Sigma_0'\\ &\qquad \text{with } \ |\zeta_j| + |\nabla \zeta_j| \le C\mu^{-2},\ |\eta_j| + |\nabla \eta_j| \le C\mu^{-4} + C\varepsilon^l |\log \varepsilon|^{-\sigma}\end{aligned}$$

and therefore after projecting Σ_0' onto Σ_0: $(x_1, \xi_1; x_2, \xi_2) \mapsto (0, 0; x_2, \xi_2)$ we have $(1 + k) dx_2 \wedge d\xi_2$ with $k = 0$ unless $(l, \sigma) \succeq (2, 0)$ and μ is close to $h^{-\frac{1}{3}}$ and $k = \mu^{-4} k_1(x)$ in the latter case.

Therefore, instead of $-dx_2 \wedge d\xi_2$ in our formula we can use $(1+k)\sqrt{g}\,dx_1 dx_2$ and instead of $W(x_2, \xi_2, \rho_n)$ we can use $W_n := W(\phi(x_1, x_2), r_n)$.

Further, obviously one can find $w(x_1, x_2)$ of the same regularity as W such that $(2n+1)\mu h + W_n < 0$ if and only if $(2n+1)\mu h + w < 0$ and replace $W_n(x)$ by $w(x)$. One should not be very concerned with the difference between ψ and ψ_n because we are interested in the calculations when $\psi = 1$ and then $\psi_n = 1$ as well.

Anyway, one can prove easily that

$$\tilde{\psi}_n(x_2, \xi_2) = \psi \circ \phi^{-1} + \mu^{-2} \psi_n' + \mu^{-4} \psi_n''$$

where ψ_n', ψ_n'' are defined by (18.5.16) with $\tilde{\psi}$ replaced by ψ', ψ'' respectively, and with symbols ψ', ψ'' belonging to $\mathscr{F}^{l-1,\sigma}$, $\mathscr{F}^{l-2,\sigma}$ and not depending on

μ, h and one can take ψ_n'', ψ'' equal to 0 unless $(l,\sigma) \succeq (2,0)$ and μ is close to $h^{-\frac{1}{3}}$.

Then we get the final formula with the correction term

$$h^{-2}\mathcal{N}^{\mathsf{MW}}_{2,\mathsf{corr}} = \tag{18.5.24}$$
$$\frac{1}{2\pi}h^{-2}\int \sum_n \Big(\theta\big(-w-(2n+1)\mu h\big)-\theta\big(-\frac{V}{F}-(2n+1)\mu h\big)\Big)\mu h\cdot \psi(x)\,dx+$$
$$\mu^{-1}h^{-1}\int \sum_n \theta\big(-w-(2n+1)\mu h\big)(\psi_n'\circ\phi)\,dx+$$
$$\mu^{-3}h^{-1}\int \sum_n \theta\big(-w-(2n+1)\mu h\big)(\psi_n''\circ\phi + k)\,dx.$$

Consider the second term in (18.5.24). It is equal to

$$\mu^{-1}h^{-1}\int \Big(\mathsf{Tr}_1\big(\Pi_{-w}\psi'(x_1,x_2,\mu^{-1}hD_1,\xi_2)\big)\circ\phi\Big)\,dx \tag{18.5.25}$$

where Π_τ is the spectral projector in $\mathscr{L}^2(\mathbb{R})$ ($\mathbb{R}=\mathbb{R}_{x_1}$) for harmonic oscillator $\mu^2x_1^2+h^2D_1^2$ and Tr_1 is an operator trace in $\mathscr{L}^2(\mathbb{R})$, $\tau=-w$ is the spectral parameter.

One can apply Weyl formula to calculate this expression (18.5.25). The error will not exceed $C\mu^{-1}h^{-1}\times\big(1+\varepsilon^{l-2}|\log h|^{-\sigma}\big)$ where the last factor is the remainder estimate for Weyl formula for $\mathsf{Tr}_1\big(\Pi_{-w}\psi'(x_1,hD_1)\big)$ [28]. Therefore this error does not exceed desired $Ch^{-2}\big(\mu^{-1}h+\vartheta(\varepsilon)\big)=C\mu^{-1}h^{-1}+Ch^{-2}\vartheta(\varepsilon)$.

On the other hand, the Weyl answer (for the same term) is

$$\mu h^{-1}\iint \theta\big(-w-r^2\big)\psi'(x,\mu^{-1}r\cos\alpha,\mu^{-1}r\sin\alpha)\,rdrd\alpha; \tag{18.5.26}$$

plugging into (18.5.25) we get

$$h^{-2}\iiint \theta\big(-w-r^2\big)\psi'(x,\mu^{-1}r\cos\alpha,\mu^{-1}r\sin\alpha)\,rdrd\alpha dx. \tag{18.5.27}$$

Recall that

$$w-V=O\big(\vartheta(\mu^{-1})\big) \tag{18.5.28}$$

[28] Actually, C is the remainder estimate and $C\varepsilon^{l-2}|\log h|^{-\sigma}$ is an estimate for error in the calculations.

and then due to assumption (18.2.15) replacing w by V we make an error not exceeding $C\mu^{-2-l}|\log h|^{-\sigma}h^{-2}$; this expression does not exceed $Ch^{-2}\vartheta(\varepsilon)$ as $l \le 2$ and $C\mu^{-1}h^{-1}$ as $l > 2$; recall that $\mu \ge (h|\log h|)^{-\frac{1}{3}}$.

We can treat the third term in (18.5.24) in the same way. Therefore

(18.5.29) Without deterioration of the remainder estimate, the second and third terms in (18.5.24) can be rewritten as $K_1\mu^{-2}h^{-2}$ and $K_2\mu^{-4}h^{-2}$ respectively with some constants K_1, K_2.

We also claim that

(18.5.30) With an error not exceeding $C\mu^{-1}h^{-1} + Ch^{-2}\vartheta(\mu h)$ one can replace in the first term in (18.5.24) the Riemann sum by the corresponding integral.

Really, without any loss of the generality one can assume that $|\partial_{x_j}V| \ge \epsilon_0$ with either $j = 1$ or $j = 2$ and the same is true for w. Then the first part of the first term in (18.5.24) is equal to

$$h^{-2}\int \sum_n \mu h\theta\big(\tau - w - (2n+1)\mu h\big)(\partial_{x_j}w)(\partial_{x_j}w')^{-1}\psi(x)\,dx + \tag{18.5.31}$$

$$h^{-2}\mu h\int \sum_n \theta\big(\tau - w - (2n+1)\mu h\big)(\partial_{x_j}w)\big((\partial_{x_j}w)^{-1} - (\partial_{x_j}w')^{-1}\big)\psi(x)\,dx$$

where w' is μh-mollification of w. Note that $\big((\partial_{x_j}W_n)^{-1} - (\partial_{x_j}W')^{-1}\big)$ does not exceed $C(\mu h)^{l-1}|\log \mu h|^{-\sigma}$ and therefore passing in the corresponding term from the Riemann sum to the corresponding integral brings an error not exceeding $Ch^{-2}\times \mu h \times (\mu h)^{l-1}|\log \mu h|^{-\sigma}$. On the other hand, the first term in (18.5.31) is equal to

$$h^{-2}\int \sum_n \big(\tau - w - (2n+1)\mu h\big)_+\big(\partial_{x_j}(\partial_{x_j}w')^{-1}\psi(x)\big)\mu h\,dx$$

with $\big(\partial_{x_j}(\partial_{x_j}w')^{-1}\psi(x)\big)$ not exceeding $(\mu h)^{l-2}|\log \mu h|^{-\sigma}$ and passing from the Riemann sum to the corresponding integral brings an error not exceeding $Ch^{-2}\times(\mu h)^2\times(\mu h)^{l-2}|\log \mu h|^{-\sigma}$ for $(l,\sigma) \preceq (2,0)$; for $(l,\sigma) \succeq (2,0)$ we need to repeat this procedure one more time to recover the same estimate.

Thus, claim (18.5.30) has been proven and with an error not exceeding $C\bigl(\mu^{-1}h^{-1}+h^{-2}\vartheta(\mu h)\bigr)$ the first term in (18.5.24) is equal to

$$\frac{1}{4\pi}h^{-2}\int \bigl(w_- - V_-\bigr)\psi(x)\,dx = \frac{1}{4\pi}h^{-2}\int (V-w)\psi(x)\,dx$$

due to condition (18.2.27).

So we get that with the indicated error the correction term is

$$\frac{1}{4\pi}h^{-2}\int (V-w)\psi(x)\,dx + h^{-2}\bigl(K_1\mu^{-2}+K_2\mu^{-4}\bigr) \tag{18.5.32}$$

with V, K_1, K_2 not depending on μ^{-1} and w depending on it.

Comparing with the results for the weak magnetic field (see Section 18.2) when correction term is 0 [29] we see that expression (18.5.32) vanishes modulo $O\bigl(\mu^{-1}h^{-1}+\vartheta(\varepsilon)h^{-2}\bigr)$, with $\varepsilon = (\mu^{-1}h|\log h|)^{\frac12}$ and then (18.5.24) transforms into

$$h^{-2}\mathcal{N}_{2,\mathsf{corr}} \coloneqq \tag{18.5.33}$$
$$h^{-2}\int \sum_n \Bigl(\theta\bigl(\tau - W - (2n+1)\mu h\bigr) - \theta\bigl(\tau - V - (2n+1)\mu h\bigr)\Bigr)\mu h\cdot\psi(x)\,dx - \frac12 h^{-2}\int (V-W)\psi(x)\,dx$$

where we replaced notation w by W. Thus we arrive under assumption (18.2.15) to Proposition 18.5.1 below.

Step II under Assumption (18.2.16)

Note that up to (18.5.31) we have not used any non-degeneracy assumption. Next step is done on each ν-ball with $\nu \ge C_0\max\bigl((\mu h)^{\frac12}, \mu^{-1}\bigr)$ and then the factor ν^{-2} appears (in comparison with the calculations of the previous subsubsection) and it is compensated by factor ν^2 from the ball volume; summation over all balls brings either factor $\log\mu$ under assumption (18.2.16).

[29] Surely for $\mu \asymp (h|\log h|)^{-\frac13}$ we can take the same mollification but we also removed operator with A_0-bound not exceeding $\vartheta(\varepsilon)$ and one can see easily that it brings error not exceeding $Ch^{-2}\vartheta(\varepsilon)$.

However, the same factor $\log \mu$ or 1 appears under corresponding assumption in the remainder estimate as well and therefore without deterioration of the remainder we arrive to the same expression (18.5.33) for the correction term, arriving under assumption (18.2.16) to the following:

Proposition 18.5.1. *Let* $h^{-\frac{1}{3}} \lesssim \mu \lesssim (h|\log h|)^{-1}$ *and condition* (18.2.27) *be fulfilled. Then*

(i) Under non-degeneracy assumption (18.2.15) *Tauberian expression* (18.4.28) *equals*

$$h^{-2} \int \bigl(\mathcal{N}^{\mathsf{MW}}_{x}(0) + \mathcal{N}^{\mathsf{MW}}_{x,\mathsf{corr}}\bigr)\psi(x)\,dx \tag{18.5.34}$$

modulo $O\bigl(\mu^{-1}h^{-1} + h^{-2}\vartheta(\varepsilon)\bigr)$ *with* $h^{-2}\mathcal{N}^{\mathsf{MW}}_{x,\mathsf{corr}}$ *defined according to* (18.5.33).

(ii) Under non-degeneracy assumption (18.2.16) *Tauberian expression* (18.4.28) *equals* (18.2.34) *modulo* $O\bigl(\mu^{-1}h^{-1} + h^{-2}\vartheta(\varepsilon)|\log h|\bigr)$.

Calculations: $\mu \gtrsim h^{-1}|\log h|^{-1}$

In this case calculations are easy. We just should remember that as $\mu \gtrsim h^{-1}$ the correction $h^2 W'$ with W' defined by (18.5.42)–(18.5.43) below to potential.

Comparisons

We need to compare $\int \tilde{\mathcal{N}}^{\mathsf{MW}}(x,\tau)\psi(x)\,dx$ with $\int \mathcal{N}^{\mathsf{MW}}(x,\tau)\psi(x)\,dx$:

Proposition 18.5.2. *(i) Under non-degeneracy assumptions* (18.2.15) *or* $(18.2.16)^{+}$ *as* $\mu \lesssim h^{-1}$ *and* $(18.2.15)^{*}$ *or* $(18.2.16)^{*+}$ *as* $\mu \gtrsim h^{-1}$

$$h^{-2}\Bigl|\int \Bigl(\tilde{\mathcal{N}}^{\mathsf{MW}}(x,\tau) - \mathcal{N}^{\mathsf{MW}}(x,\tau)\Bigr)\psi(x)\,dx\Bigr| \le C(1+\mu h)h^{-2}\vartheta(\varepsilon) \tag{18.5.35}$$

for $|\tau| \le \epsilon$.

(ii) Under non-degeneracy assumptions (18.2.16) *as* $\mu \lesssim h^{-1}$ *and* $(18.2.16)^{*}$ *as* $\mu \gtrsim h^{-1}$

$$h^{-2}\Bigl|\int \Bigl(\tilde{\mathcal{N}}^{\mathsf{MW}}(x,\tau) - \mathcal{N}^{\mathsf{MW}}(x,\tau)\Bigr)\psi(x)\,dx\Bigr| \le C(1+\mu h)h^{-2}\vartheta(\varepsilon)|\log \varepsilon| \tag{18.5.36}$$

for $|\tau| \le \epsilon$.

Proof. Without any loss of the generality we can assume that $F = 1$.

(a) Assume first that $\vartheta(\varepsilon) \le \epsilon\mu h$ and either assumption (18.2.15) or assumption $(18.2.16)^+$ is fulfilled.

Let us introduce γ-admissible partition of unity with $\gamma = \epsilon\vartheta(\varepsilon)$. If $|V + (2j+1)\mu hF - \tau| \ge C\gamma$ for all $j \in \mathbb{Z}^+$ on some element of the partition, then on this element $\tilde{\mathcal{N}}^{\mathsf{MW}}(x,\tau)$ and $\mathcal{N}^{\mathsf{MW}}(x,\tau)$ coincide while on all other elements their difference does not exceed $C\mu h^{-1}$. However, due to (18.2.15) or $(18.2.15)^*$ the total area of the latter elements does not exceed $C\gamma(1+\mu^{-1}h^{-1})$ which yields the necessary estimate.

(b) Assume now that $\vartheta(\varepsilon) \ge \epsilon\mu h$ and assumption (18.2.15) is fulfilled. Then $\theta\big(\tau - V - (2n+1)\mu h\big)$ and $\theta\big(\tau - \tilde{V} - (2n+1)\mu h\big)$ differ for each x for no more than $C_0\vartheta(\varepsilon)/\mu h + 1$ number of indices "n" which yields the necessary estimate.

The similar analysis under one of the assumptions (18.2.16)–$(18.2.16)^{*+}$ we leave to the reader. □

We also need to compare W and V/F as $\mu \lesssim (h|\log h|)^{-1}$ (i.e. as $\mu^{-1} \gtrsim \varepsilon$). One can see easily that it is $O\big(\vartheta(\mu^{-1}) + \mu^{-2}\big)$ and we arrive to the following:

Proposition 18.5.3. *Let* $\mu \lesssim (h|\log h|)^{-1}$. *Then*

(i) Under assumption (18.2.15) *or* $(18.2.16)^+$

$$|\mathcal{N}^{\mathsf{MW}}_{2,\mathsf{corr}}| \le C \begin{cases} \vartheta(\mu h) & \text{as } \ \mu \le h^{-\frac12}, \\ \vartheta(\mu^{-1}) & \text{as } \ \mu \ge h^{-\frac12},\ (l,\sigma) \preceq (2,0), \\ \mu^{-4}h^{-2}\vartheta(\mu h) & \text{as } \ \mu \ge h^{-\frac12},\ (l,\sigma) \succ (2,0). \end{cases} \tag{18.5.37}$$

In particular, $h^{-2}\mathcal{N}^{\mathsf{MW}}_{2,\mathsf{corr}} = O(\mu^{-1}h^{-1})$ *for sure as* $(l,\sigma) \succeq (3,0)$.

(ii) Under assumption (18.2.16)

$$|\mathcal{N}^{\mathsf{MW}}_{2,\mathsf{corr}}| \le C|\log h| \begin{cases} \vartheta(\mu h) & \text{as } \ \mu \le h^{-\frac12}, \\ \vartheta(\mu^{-1}) & \text{as } \ \mu \ge h^{-\frac12},\ (l,\sigma) \preceq (2,0), \\ \mu^{-4}h^{-2}\vartheta(\mu h) & \text{as } \ \mu \ge h^{-\frac12},\ (l,\sigma) \succ (2,0). \end{cases} \tag{18.5.38}$$

In particular, $h^{-2}\mathcal{N}^{\mathsf{MW}}_{2,\mathsf{corr}} = O(\mu^{-1}h^{-1})$ *for sure as* $(I,\sigma) \succeq (3,1)$.

Proof. An easy proof is left to the reader; as $(I,\sigma) \succ (2,0)$ we need to take into account some extra correction terms of the type $K\mu^{-2}h^{-2}$ because then $W - V/F = k\mu^{-2} + O\big(\vartheta\mu^{-1})\big)$. □

18.5.2 Main Theorems

Now, assembling results of this section we arrive to our main

Theorem 18.5.4. *Let* $d = 2$ *and* A *be a self-adjoint in* $\mathscr{L}^2(X)$ *Schrödinger operator defined by* (18.1.1).

Let $g^{jk}, F, \psi \in \mathscr{C}^{\bar{I},\bar{\sigma}}$, $V \in \mathscr{C}^{I,\sigma}$ *with* $(\bar{I},\bar{\sigma}) \succeq (2,0)$, $(\bar{I},\bar{\sigma}) \succeq (I,\sigma) \succeq (1,2)$ *and let conditions* (18.1.6) *and* (18.2.27) *be fulfilled in* $B(0,1) \subset X \subset \mathbb{R}^2$.

Let $(h|\log h|)^{-\frac{1}{3}} \le \mu \le Ch^{-1}$. *Then there are two framing approximations* $\tilde{A} = \tilde{A}^{\pm}$ *as in footnote*[18)] *such that for each of them*

(i) Under assumption (18.2.15) *estimate*

$$\mathsf{R}^{\mathsf{MW}}_{*} := \Big|\int \Big(\tilde{e}(x,x,0) - h^{-2}\mathcal{N}^{\mathsf{MW}}_{2,x}(0) - h^{-2}\mathcal{N}^{\mathsf{MW}}_{2,x\,\mathsf{corr}}(0)\Big)\psi(x)\,dx\Big| \le C\mu^{-1}h^{-1} + C\vartheta(\varepsilon)h^{-2} \tag{18.5.39}$$

holds with the standard magnetic Weyl expression $h^{-2}\mathcal{N}^{\mathsf{MW}}_{2,x}(\tau)$ *and with the correction term* $h^{-2}\mathcal{N}^{\mathsf{MW}}_{2,x\,\mathsf{corr}}(\tau)$ *defined by* (18.5.31) *with the appropriate corrected potential* $W = V/F + O\big(\vartheta(\mu^{-1}) + \mu^{-2}\big)$; *here and below* $\varepsilon = (\mu^{-1}h|\log h|)^{\frac{1}{2}}$.

(ii) Let $(I,\sigma) \succeq (2,0)$; *then under assumption* $(18.2.16)^{+}$ *the same estimate* (18.5.39) *holds.*

(iii) Let $(I,\sigma) \succeq (2,0)$; *then under assumption* (18.2.16)

$$\mathsf{R}^{\mathsf{MW}}_{*} \le C\mu^{-1}h^{-1} + C\vartheta(\varepsilon)h^{-2}|\log h| + C|\log h|. \tag{18.5.40}$$

Remark 18.5.5. Recall that the correction term has been estimated in Proposition 18.5.3. In particular, it implies estimates for R^{MW} without the correction term.

Problem 18.5.6. Either improve estimates (18.5.37), (18.5.38) of Proposition 18.5.3 or construct counter-examples showing that the improvement is impossible.

Theorem 18.5.7. *Let* $d = 2$ *and* A *be a self-adjoint in* $\mathscr{L}^2(X)$ *Schrödinger-Pauli operator defined by* (18.1.29).

Let $g^{jk}, F, \psi \in \mathscr{C}^{\bar{l},\bar{\sigma}}$, $V \in \mathscr{C}^{l,\sigma}$ *with* $(\bar{l},\bar{\sigma}) \succeq (2,0)$, $(\bar{l},\bar{\sigma}) \succeq (l,\sigma) \succeq (1,1)$ *and condition* (18.1.6) *be fulfilled in* $B(0,1) \subset X \subset \mathbb{R}^2$.

Then there are two framing approximations $\tilde{A} = \tilde{A}^{\pm}$ *as in footnote*[18] *such that for each of them*

(i) Under assumption $(18.2.15)^*$ *estimate*

$$\mathsf{R}_*^{\mathsf{MW}} \le C + C\vartheta(\varepsilon)\mu h^{-1} + C\bar{\vartheta}(\varepsilon)\mu^2 \tag{18.5.41}$$

holds with the correction term $h^{-2}\mathcal{N}^{\mathsf{MW}}_{2,x\,\mathsf{corr}}$ *defined again by* (18.5.24)[30].

(ii) Let $(l,\sigma) \succeq (2,0)$*; then under assumption* $(18.2.16)^{*+}$ *the same estimate* (18.5.40) *holds.*

(iii) Let $(l,\sigma) \succeq (2,0)$*; then under assumption* $(18.2.16)^*$

$$\mathsf{R}_*^{\mathsf{MW}} \le C\bigl(1 + \vartheta(\varepsilon)\mu h^{-1} + \bar{\vartheta}(\varepsilon)\mu^2\bigr)\,|\log \mu|. \tag{18.5.44}$$

(iv) Finally, under assumption (18.4.35) *estimate*

$$|\tilde{e}(x,x,0) - h^{-2}\mathcal{N}^{\mathsf{MW}}_{2,x}(x,0)| \le C_s\mu^{-s} \qquad \text{for } x \in B(0,\frac{1}{2}) \tag{18.5.45}$$

holds.

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[30] However, now $W = V + W'h^2$ with

$$W' = -\frac{1}{12}\kappa(\mathfrak{z}^2 + 2) + F^{\frac12}\sum_{j,k}\partial_j\bigl(g^{jk}\partial_k F^{-\frac12}\bigr) \tag{18.5.42}$$

and κ a curvature associated with metrics $g^{jk}F^{-1}$ (and this correction is not due to the lack of the smoothness);

$$g^{jk}F^{-1} = \alpha^2\delta_{jk} \implies \kappa = 2\Delta\log\alpha. \tag{18.5.43}$$

Further, the sum in the definition contains only a finite number of terms.

We leave to the reader an easy

Problem 18.5.8. Investigate conditions to the regularity (i.e. to (l,σ) and $(\bar{l},\bar{\sigma})$) depending on μ, h and reversely conditions to μ, h depending on the regularity required to have the same results as in the smooth case.

18.5.3 Generalizations

Vanishing V

We want to get rid of assumption (18.2.27) $V \le -\epsilon_0$. We need to deal only with the case $\mu \le \epsilon_1 h^{-1}$. We should use the same rescaling arguments as in Subsection 13.8.1 but we need to take care of ε.

Let us introduce admissible scaling functions $\gamma(x) = \epsilon|V(x)|$ and $\rho(x) = \gamma(x)^{\frac12}$. Then rescaling ball $B\big(x,\gamma(x)\big)$ to $B(0,1)$ and dividing operator by ρ^2 we find ourselves again in our settings but with h, μ replaced by $h_{\mathsf{new}} = h/\rho\gamma$, $\mu_{\mathsf{new}} = \mu\gamma/\rho$ respectively. However, we need a cut-off: in fact $\gamma(x) = \epsilon|V(x)| + \bar{\gamma}_0$ with an appropriate parameter $\bar{\gamma}_0$. Modifications in some cases are needed.

Consider first the case of Theorem 18.5.4 when $\bar{\mu}_1 = \big(h|\log h|\big)^{-\frac13} \le \mu \le h^{-1}$. Then as $\mu_{\mathsf{new}} h_{\mathsf{new}} = \mu h/\rho^2 = \mu h/\gamma$ and since we do not need assumption (18.2.27) as $\mu_{\mathsf{new}} h_{\mathsf{new}} \gtrsim 1$ we will cut of γ by picking up $\bar{\gamma}_0 = \mu h$.

Plugging μ_{new}, γ_{new} and corresponding mollification parameter

$$\varepsilon_{\mathsf{new}} = \big(\mu_{\mathsf{new}}^{-1} h_{\mathsf{new}}|\log h_{\mathsf{new}}|\big)^{\frac12} = (\mu^{-1}h)^{\frac12}\gamma^{-1}|\log(h/\gamma^{\frac32})|$$

to the remainder estimate (18.5.39) we see that the contribution of each partition element to the remainder does not exceed

$$C\mu^{-1}h^{-1}\gamma + C(\mu^{-1}h)^{\frac{l}{2}}|\log(h/\gamma^{\frac32})|^{\frac{l}{2}-\sigma}h^{-2}\gamma^{3-l}$$

and the total remainder estimate will be then due to assumption (18.2.15)

$$C\int \mu^{-1}h\gamma^{-1}\,d\gamma + C(\mu^{-1}h)^{\frac{l}{2}}\int |\log(h/\gamma^{\frac32})|^{\frac{l}{2}-\sigma}h^{-2}\gamma^{3-l}\,d\gamma.$$

The second integral is equal to the second term in the right-hand expression of (18.5.39) for $l < 2$; to handle the case $l = 2$ we need to notice that the second term in (18.5.39) is due to mollification error only but in the rescaled coordinates mollification parameter is $\varepsilon_{\mathsf{new}}$ which in the original

coordinates becomes $\varepsilon_{\mathsf{new}}\gamma = (\mu^{-1}h)^{\frac{1}{2}}|\log(h/\gamma^{\frac{3}{2}})| \le (\mu^{-1}h|\log h|)^{\frac{1}{2}}$ and thus mollification error does not exceed the second term in (18.5.39) anyway. Therefore

(18.5.46) Theorem 18.5.4(i) without condition (18.2.27) remains true with the remainder estimate $C\mu^{-1}h^{-1}|\log(\mu h)| + R_2$ where R_2 is the second term in the right-hand expression in (18.5.39).

The analysis of the case of Theorem 18.2.9 is even simpler: one need to pick $\bar{\gamma}_0 = \mu^{-2}$; then on each element after rescaling either condition (18.2.27) is fulfilled or $\gamma \asymp \bar{\gamma}_0$ and $\mu_{\mathsf{new}} \asymp 1$ and condition (18.2.27) is not needed while $h_{\mathsf{new}} \asymp \mu^3 h \le 1$.

Note that in this case $\varepsilon_{\mathsf{new}} = \mu_{\mathsf{new}}h_{\mathsf{new}}|\log h| = \mu h\gamma^{-1}|\log(h/\gamma^{\frac{3}{2}})|$ and after rescaling to original coordinates it is $\varepsilon_{\mathsf{new}}\gamma \le \mu h|\log h|$.

Then one can prove easily that

(18.5.47) Theorem 18.2.9(i) without condition (18.2.27) remains true with the remainder estimate $C\mu^{-1}h^{-1}|\log(\mu h)| + R_1$ where R_1 is the second term in the right-hand expression in (18.2.35).

Actually one can get rid of this logarithmic factor considering propagation of singularities which is going in direction orthogonal to ∇V with the speed $\asymp \mu^{-1}$. Details in the smooth case can be found in Subsection 13.8.1.

Thus we can replace $T^* = \epsilon\mu$ by $T^* = \epsilon\mu \min\bigl(|\log\gamma|, |\log(\gamma/\bar{\gamma}_0|\bigr)^{\alpha}$ in the Tauberian theorem provided $(1,\alpha) \preceq (I,\sigma)$. Also, one can calculate "the second term" and to prove that it vanishes.

After summation we get an extra term in the remainder estimate

$$C\mu^{-1}h^{-1}\int \gamma^{-1}\Bigl(|\log\gamma|^{-\sigma} + |\log(\gamma/\bar{\gamma}_0|^{-\sigma}\Bigr)\,dx \asymp C\mu^{-1}h^{-1}.$$

Therefore

(18.5.48) Theorems 18.5.4(i) and 18.2.9(i) remain true without assumption (18.2.27).

Combining with the other statements of Theorems 18.5.4 and 18.2.9 we conclude that these theorems remain true under assumption (18.2.27) replaced by

$$|V| + |\nabla V| \ge \epsilon_0. \tag{18.5.49}$$

This conclusion enables us to run rescaling again but now with the scaling function $\gamma = \epsilon\big(|V| + |\nabla V|^2\big)^{\frac{1}{2}}$ and $\rho \asymp \gamma$; then $\mu^{-1}_{\mathsf{new}} h^{-1}_{\mathsf{new}} \asymp \mu^{-1}h^{-1}\gamma^2$. Then in the framework of Theorem 18.5.4 $\varepsilon_{\mathsf{new}} = \big(\mu^{-1}h|\log(h/\gamma^2)|\big)^{\frac{1}{2}}\gamma^{-1}$, which in the original coordinates is $\varepsilon_{\mathsf{new}}\gamma \lesssim \big(\mu^{-1}h|\log h|\big)^{\frac{1}{2}}$ again.

Meanwhile in the framework of Theorem 18.2.9 $\varepsilon_{\mathsf{new}} = \mu h|\log(h/\gamma^2)|\gamma^{-2}$ and a mollification error term in the virtue of assumption (18.2.16) is

$$Ch^{-2}\int \gamma^4\big(\mu h|\log(h/\gamma^2)|\big)^l\gamma^{-2l}|\log\big(\mu h|\log(h/\gamma^2)|\big)|^{-\sigma}\,\gamma^{-1}d\gamma$$

which as one can see easily resets to the value of such term in Theorem 18.2.9. Therefore

Theorem 18.5.9. *Theorems 18.5.4 and 18.2.9 hold without assumption* (18.2.27).

Other Generalizations

We leave to the reader an easy parts (i) of the Problems 18.5.10 –18.5.12 below; on the contrary, parts (ii) look rather challenging.

Problem 18.5.10. To get rid of condition (18.1.6) (while (18.2.27) holds) under assumption

$$|\nabla(F/V)| \ge \epsilon. \tag{18.5.50}$$

(i) Use the rescaling method and recover remainder estimate $O(h^{-1})$ as $\mu \lesssim h^{-2}$.

(ii) Generalize results of *Chapter 14.* 2D-*Schrödinger Operator with the Strong Degenerating Magnetic Field* to the non-smooth case.

Problem 18.5.11. To get rid of condition $B(0,1) \subset X$.

(i) Use the rescaling method and recover remainder estimate $O(h^{-1})$ as $\mu \lesssim h^{-2}$.

(ii) Generalize results of *Chapter 15.* 2D-*Schrödinger Operator with the Strong Magnetic Field near the Boundary* to the non-smooth case.

Problem 18.5.12. (i) Generalize our results to the generalized Schrödinger-Pauli operator (17.1.6).

(ii) Generalize our results to the generalized Dirac operator (17.1.1).

Remark 18.5.13. (i) One could think about generalizing *Chapter 16. agnetic Schrödinger Operator: Short Loops, Pointwise Spectral Asymptotics and Asymptotics of Dirac Energy* to the non-smooth case but as we do not have monotonicity of $e(x,x,\tau)$ with respect to operator it does not make much sense as we need to deal with the framing approximations.

(ii) Sure, we do not have such monotonicity of $\int e(x,x,\tau)\psi(x)\,dx$ either but this kind of expressions are building blocks for $\mathsf{N}(\tau) = \int e(x,x,\tau)\,dx$.

18.6 $d = 3$: Weak Magnetic Field

In this section we consider the case

$$\mu \le \bar{\mu}_1 := C(h|\log h|)^{-\frac{1}{3}}. \tag{18.6.1}$$

In the smooth case then one needs to reduce operator to canonical form only to analyze relatively long (comparing with $\epsilon_0\mu^{-1}$) propagation of singularities but not to calculate asymptotics. The same situation preserves in non-smooth case and to recover sharp asymptotics one needs neither non-degeneracy nor extra smoothness. This is the same threshold (18.2.6) as in Section 18.2 for $d = 2$ despite completely different mechanism; in some sense this is just coincidence.

18.6.1 Preliminary Remarks

We consider the microlocally mollified Schrödinger operator in the strong magnetic field given by expression (18.1.1) with the symmetric positive matrix (g^{jk}), real-valued V_j, V and small parameters h, ε and large parameter μ. We assume that the corresponding magnetic field $\boldsymbol{F} = (F^1, F^2, F^3)$ with

$$F^l(x) := \frac{1}{2}\sum_{jk} \epsilon^{lkj} F_{kj}, \qquad F_{kj} := (\partial_{x_j} V_k - \partial_{x_k} V_j)$$

does not vanish in $B(0,1)$; then we can straighten it and direct along x_3 by an appropriate change of variables and make $V_3 = 0$ by an appropriate gauge

transformation. We assume that this has been done and impose smoothness conditions to the reduced operator.

Namely we assume that for this reduced operator conditions (18.1.3), (18.1.6) hold and

$$g^{jk}, V, F^3 \in \mathscr{F}^{1,\sigma}(B(0,1)), \qquad F^1 = F^2 = 0. \tag{18.6.2}$$

We start from the case (further restrictions to follow)

$$\varepsilon \geq Ch|\log h|, \quad \mu \leq C^{-1}h|\log h|^{-1}. \tag{18.6.3}$$

Then

$$h^{-1}[A, x_j] = \sum_k \bigl(g^{jk}P_k + P_k g^{jk}\bigr) \tag{18.6.4$_1$}$$

$$h^{-1}[A, P_3] = h^{-1}\sum_{j,k} P_j[g^{jk}, P_3]P_k + h^{-1}[V, P_3] \tag{18.6.4$_2$}$$

$$\mu^{-1}h^{-1}[A, P_j] = i(-1)^{j+1}F_{12}\sum_k (g^{53-j,k}P_k + P_k g^{3-j,k}) + \mu^{-1}h^{-1}\sum_{jk} P_j[g^{jk}, P_i]P_k + \mu^{-1}h^{-1}[V, P_i] \quad \text{for} \quad i = 1, 2 \tag{18.6.4$_3$}$$

have symbols of class $\mathscr{F}^{1,\sigma}$ in any domain $\{(x,\xi) : \ a(x,\xi) \leq c\mu^{-2}\}$ of bounded energy where

$$a(x,\xi) = a_0(x,\xi) + \mu^{-2}V, \quad a_0(x,\xi) = \sum_{j,k} g^{jk}p_j(x,\xi)p_k(x,\xi), \qquad p_j(x,\xi) \coloneqq (\xi_j - V_j). \tag{18.6.5}$$

To finish the preliminary remarks let us notice that the following statement holds

Proposition 18.6.1. *Let $f \in \mathscr{C}_0^\infty(\mathbb{R}^3)$, $f = 1$ in $B(0, C_0)$ with large enough constant C_0. Let $T \geq h^{1-\delta}$. Then under condition* (18.6.2) *with large enough constant $C = C_s$*

$$|F_{t\to h^{-1}\tau}\Bigl(1 - f\bigl(p_1(x,\xi), p_2(x,\xi), p_3(x,\xi)\bigr)\Bigr)^{\mathsf{w}}\chi_T(t)U| \leq Ch^s \qquad x, y \in B(0, \frac{1}{2}), \tau \leq c. \tag{18.6.6}$$

Proof. One can see easily that symbol $f\bigl(p_1(x,\xi), p_2(x,\xi), p_3(x,\xi)\bigr)$ is quantizable under condition (18.6.3) and the proof is rather obvious. We leave it to the reader. □

18.6.2 Heuristics

Let us consider operator (18.1.1) with the Euclidean metrics. Then for time $T(\xi_3) := \epsilon_0|\xi_3|$ the shift with respect to ξ_3 will be less than $\frac{1}{2}|\xi_3|$ and so ξ_3 will remain of the same magnitude and sign. Then the shift with respect to x_3 will be of magnitude $|\xi_3|T$ because ξ_3 is a speed of the propagation along x_3 and in order it to be observable we need logarithmic uncertainty principle

$$|\xi_3|T \cdot |\xi_3| \geq Ch|\log h| \tag{18.6.7}$$

because we need to use $|\xi_3|$-sized scale along ξ_3. This gives us the value T_* for which we can claim that $(\Gamma_x U)(t)$ is negligible for $|t| \in [T_*, T^*]$:

$$T_* = T_*(\xi_3) := Ch|\log h| \cdot |\xi_3|^{-2}. \tag{18.6.8}$$

Since we know $(\Gamma_x U)(t)$ for $|t| \leq \bar{T} := \epsilon\mu^{-1}$ due to condition (18.2.27) and rescaling, we would like to have these intervals overlap: $\bar{T} \geq T_*$ or, equivalently,

$$|\xi_3| \geq \bar{\rho}_1 := C(\mu h|\log h|)^{\frac{1}{2}} \tag{18.6.9}$$

exactly as in the smooth case in Section 13.4.

Then the contribution of the zone $\mathcal{Z}_1^c := \{(x,\xi) : \ |\xi_3| \geq \bar{\rho}_1\}$ to the remainder does not exceed

$$Ch^{-2} \int T^{*\,-1}(\xi_3)\, d\xi_3. \tag{18.6.10}$$

This integral diverges logarithmically at $\xi_3 = 0$ and is equal to $Ch^{-2}|\log \bar{\rho}_1|$; to improve remainder estimate to Ch^{-2} we will need to increase $T^*(\xi_3)$ to $T^*(\xi_3) := \epsilon|\xi_3| \cdot \big|\log|\xi_3|\big|^{\sigma'}$ with $\sigma' > 1$.

On the other hand, the contribution of the zone $\mathcal{Z}_1 := \{(x,\xi) : \ |\xi_3| \leq \bar{\rho}_1\}$ to the remainder does not exceed $Ch^{-2}\bar{T}^{-1}\bar{\rho}_1 = C\mu\bar{\rho}_1 h^{-2}$ and in order to keep it below Ch^{-2} we need to have $\mu\bar{\rho}_1 \leq C$ which is exactly condition (18.6.1).

But in this case zone $\mathcal{Z}_2 := \{(x,\xi) : \ |\xi_3| \leq \bar{\rho}_2 = C\mu^{-1}\}$ is covered by these latter arguments; therefore we can take $C\mu^{-1}$ as a critical value and replace (18.6.9) by a stronger restriction

$$|\xi_3| \geq \bar{\rho}_2 := C\mu^{-1}. \tag{18.6.11}$$

Condition (18.6.9) will be restored in Sections 18.7–18.9 where we analyze the case $\mu \geq C(h|\log h|)^{-\frac{1}{3}}$ and restriction (18.6.9) is stronger.

This analysis is basically "smoothness-independent". It allows us to apply weak magnetic field approach without any non-degeneracy assumptions.

18.6.3 Variable Parameter ε

So far we did not discuss the choice of ε; however, since the scale with respect to ξ_3 is $C|\xi_3|$ in the zone $\mathcal{Z}_2^c := \{(x,\xi) : |\xi_3| \ge \bar\rho_2\}$ the logarithmic uncertainty principle requires $\varepsilon \ge Ch|\log h| \cdot |\xi_3|^{-1}$. To reduce approximation error we pick up the smallest allowed ε there:

$$\varepsilon = Ch|\log h| \cdot |\xi_3|^{-1}. \tag{18.6.12}$$

Then

(18.6.13) The contribution of the zone $\mathcal{Z}_2^c$ to the approximation error does not exceed $Ch^{-3}\int \vartheta\big(\varepsilon(\xi_3)\big)\,d\xi_3$ which is $O(h^{-2})$ as $(l,\sigma) \succeq (1,2)$.

On the other hand, in the zone $\mathcal{Z}_2 := \{(x,\xi) : |\xi_3| \le \bar\rho_2\}$ the scale with respect to ξ_3 is $\asymp \bar\rho_2$ and the logarithmic uncertainty principle requires there $\varepsilon = \bar\varepsilon_2$ with

$$\bar\varepsilon_2 := Ch|\log h| \cdot \bar\rho_2^{-1} = C\mu h|\log h|. \tag{18.6.14}$$

Then the contribution of this zone to the approximation error will be $Ch^{-3}\bar\rho_2\vartheta(\bar\varepsilon_2)$ which is $O(h^{-2})$ as $(l,\sigma) \succeq (1,1)$.

However, if we tried to use $\varepsilon = \bar\varepsilon_2$ everywhere we would get an approximation error $Ch^{-3}\vartheta(\bar\varepsilon_2)$ which is not $O(h^{-2})$ unless stronger smoothness condition is satisfied: namely, to cover $\mu \asymp (h|\log h|)^{\frac13}$ we would need $(l,\sigma) \succeq (\frac32, 1)$.

Thus, for $d = 3$ we will need ε to be a function of ξ_3 and may be of x. We already met such situation in Section 4.6. We will pick ε to be a temperate function:

$$\rho|\nabla_\xi \varepsilon| + \gamma|\nabla_x \varepsilon| \le c\varepsilon \tag{18.6.15}$$

where (γ,ρ) is (x,ξ)-scale. In this case we introduce

$$\tilde A = \sum \psi_\alpha^* \tilde A_\alpha \psi_\alpha \tag{18.6.16}$$

where ψ_α are (γ,ρ)-admissible functions, $\psi_\alpha^*\psi_\alpha$ is a partition of unity and $\tilde A_\alpha$ are mollifications of A with $\varepsilon = \varepsilon_\alpha$ calculated at some points of the corresponding elements.

18.6.4 Rigorous Analysis

As we already mentioned we consider now the case of the weak magnetic field: $\mu \le \bar{\mu}_1 := (h|\log h|)^{-\frac{1}{3}}$ assuming that

$$l = 1, \quad \sigma > 1 \tag{18.6.17}$$

and we will not press very hard to reduce σ. Let us consider point $(\bar{x}, \bar{\xi})$ with

$$\rho := |\bar{\xi}_3| \ge \bar{\rho}_2 := C\mu^{-1}, \qquad \varepsilon = Ch\rho^{-1}|\log h|. \tag{18.6.18}$$

Recall that the second condition (to ε) means the logarithmic uncertainty principle; thus quantization is possible; we will take ε variable but depending on ξ_3 only.

Let us consider first a classical propagation starting from this point. Obviously $|d\xi_3/dt| \le C_0$ due to condition $(18.6.4)_3$ which implies that ξ_3 keeps it magnitude during time $|t| \le \epsilon_0\rho$ even in the general (not Euclidean) case but we need a bit better. Let us consider

$$\frac{d}{dt}\xi_3 = \{\mu^2 a(x,\xi), \xi_3\} = \beta(x,\xi)\xi_3 + \mu \sum_{j,k=1}^{2} \beta_{jk} p_j p_k - \partial_{x_3} V$$

with $\beta_* \in \mathscr{F}^{0,\sigma}$. In the smooth case we would be able to reduce the second term in the right-hand expression to the form $\beta'' a_0(x,\xi)$ by replacing ξ_3 by

$$\ell = \xi_3 + \sum_{j,k=1}^{2} \beta'_{jk} p_j p_k \tag{18.6.19}$$

with appropriate coefficients β'_* found from the linear algebraic system which we would get from equality

$$\{\mu^2 a(x,\xi), \ell(x,\xi)\} \equiv \beta'(x,\xi)\xi_3 + \mu\beta'' a_0(x,\xi) - \partial_{x_3} V \mod O\bigl(\mu^{-1}\bigr) \tag{18.6.20}$$

(since $|p_j| \le C\mu^{-1}$). However, in on our non-smooth case we replace β'_{jk} by their $h^{-\delta}$-mollifications with the small enough exponent $\delta > 0$; then we

would get the above equations (18.6.20) modulo $O(|\log\rho|^{-\sigma})$. Finally, one can find that $\mu^2\beta''=\partial_3 F$ and therefore

$$\{\mu^2 a(x,\xi),\ell(x,\xi)\}\equiv \beta'(x,\xi)\ell(x,\xi)+\mu^2 Fa(x,\xi)-F\partial_{x_3}(V/F) \mod O(|\log\varrho|^{-\sigma}). \tag{18.6.21}$$

Then in an appropriate time direction[31)] inequality $\pm\ell\ge\frac12\rho$ holds for $|t|\in[0,T^*]$ with

$$T^*=T^*(\rho)\coloneqq \epsilon\rho|\log\rho|^{\sigma}. \tag{18.6.22}$$

This additional logarithmic factor is a small but a crucial progress in comparison with $T^*(\rho)=\epsilon_0\varrho$ we had before.

Then using the following function with an appropriate sign ς in the standard (see Section 2.3) propagation arguments

$$\chi\Big(\pm\frac{1}{\rho}\big((\ell(x,\xi)-\ell(\bar x,\bar\xi)\big)+c\varsigma\frac{t}{T}\Big) \tag{18.6.23}$$

one can prove the similar statement on microlocal level (due to conditions (18.6.2), $(18.6.4)_{1-3}$, (18.6.17)–(18.6.18) all symbols will be quantizable):

Proposition 18.6.2. *Let conditions* (18.6.2), (18.6.3), (18.6.17)–(18.6.18) *be fulfilled.*

Let Q be h-pseudodifferential operator with the symbol supported in ρ-vicinity of $(\bar x,\bar\xi)$ with $|\bar\xi_3|=\rho\ge C\mu^{-1}$ and let Q' be h-pseudodifferential operator with the symbol equal to 1 in ρ'-vicinity of $(\bar x,\bar\xi)$ (with $\rho'=C\rho|\log\rho|^{\sigma}$), intersected with $\{|\xi_3|\ge\frac12\rho\}$.

Then $(I-Q')e^{-ih^{-1}At}Q$ is negligible for $\varsigma t\in[0,T^*]$ with $T^*=T^*(\rho)$ defined by (18.6.22) *and appropriate $\varsigma=\pm1$.*

Proof. The standard proof is left to the reader. □

This statement shows that for $\varsigma t\in[0,T^*]$ singularity of $e^{-ih^{-1}At}Q$ is confined to ρ'-vicinity of $(\bar x,\bar\xi)$ intersected with $\{|\xi_3|\ge\frac12\rho\}$.

Now let us consider classical propagation along x_3; then

$$dx_3/dt=\{\mu^2 a,x_3\}=2g^{33}\xi_3+\mu\sum_{j=1}^{2}2g^{3j}p_j.$$

[31)] That means for t of an appropriate sign; namely the sign of $-\xi_3\partial_{x_3}(V/F)$.

Replacing x_3 by

$$\phi = x_3 + \sum_{j=1}^{2} \alpha_j p_j \tag{18.6.24}$$

with appropriate coefficients α_j found from an appropriate linear algebraic system we would able in the smooth case get

$$\{\mu^2 a, \phi\} \equiv 2g^{33}\xi_3 \qquad \mathsf{mod}\ O(\mu^{-1}). \tag{18.6.25}$$

In the non-smooth case we do the same and no mollification is needed since $\alpha_j \in \mathscr{F}^{1,\sigma}$. So, the classical shift with respect to ϕ for time t with $\varsigma t \in [0, T^*]$ will be at least $\epsilon_0\rho t$ which is "observable" if $\epsilon_0\rho|t| \times \rho \geq Ch|\log h|$.

On the microlocal level using the same arguments of Section 2.3 with function

$$\chi\Big(\varsigma_1 \frac{1}{\rho T}\big(\phi(x,\xi) - \phi(y,\xi)\big) \pm c\frac{t}{T}\Big) \tag{18.6.26}$$

(which again is quantizable due to our conditions) with an appropriate sign $\varsigma_1 = \pm 1$ one can easily prove that for $\varsigma t \in [T_*, T^*]$,

$$T_* = C\rho^{-2}h|\log h| \tag{18.6.27}$$

and $\psi(x_3)$, supported in ϵ-vicinity of $\bar{x}_3$, $\psi(x_3)e^{-ih^{-1}At}Q\psi(x_3) \equiv 0$.

Then $\mathsf{Tr}\big(e^{-ih^{-1}At}Q\big) \equiv 0$. Furthermore this is true for $|t| \in [T_*, T^*]$ *with no restriction to the sign because of the trace*. Therefore we arrive

Proposition 18.6.3. *Let the above assumptions be fulfilled. Then for any* $T \in [T_*, T^*]$

$$F_{t\to h^{-1}\tau}\Big(\big(\bar{\chi}_{T^*}(t) - \bar{\chi}_T(t)\big)\Gamma(U\,{}^tQ_y)\Big) \equiv 0. \tag{18.6.28}$$

Proof. Again, the standard details are left to the reader. □

Recall that χ, $\bar{\chi}$ are admissible functions, supported in $[-1,1]$ and equal to $0,1$ respectively on $[-\frac{1}{2},\frac{1}{2}]$.

Then the same statement holds for $Q = \psi(x)\chi_\rho(hD_3)$ (we can sum with respect to the partition of unity in (x,ξ_1,ξ_2)). *From now on operator* Q *is given by this expression*.

On the other hand, from the standard theory rescaled we already know $F_{t\to h^{-1}\tau}\bigl(\bar\chi_{\bar T}(t)\Gamma(u\,{}^tQ_y)\bigr)$ with $\bar T = \epsilon_0\mu^{-1}$ and in order to combine these two results we need to have $T_* \ge \bar T$ which is equivalent to (18.6.9); this condition is wider than our current framework; right now we have $(\mu h|\log h|)^{\frac12} \lesssim \mu^{-1}$ which exactly means that $\mu \lesssim \bar\mu_1 = (h|\log h|)^{-\frac13}$.

Then from the standard theory rescaled we conclude that

$$|F_{t\to h^{-1}\tau}\bigl(\bar\chi_T(t)\Gamma(U\,{}^tQ_y)\bigr)| \le C\rho h^{-2} \tag{18.6.29}$$

for any $T \in [T_*, T^*]$. With this estimate the standard Tauberian arguments imply that

$$\mathsf{R}_Q^{\mathsf{T}} := |\Gamma(\tilde e\,{}^tQ_y)(\tau) - h^{-1}\int_{-\infty}^{\tau}\Bigl(F_{t\to h^{-1}\tau'}\bigl(\bar\chi_T(t)\Gamma(U\,{}^tQ_y)\bigr)\Bigr)\,d\tau'| \le C\rho h^{-2}T^{*-1} = Ch^{-2}|\log\rho|^{-\sigma}. \tag{18.6.30}$$

Let us sum with respect to partition on ξ_3 in the zone $\mathcal{Z}_2^{\mathsf{c}} = \{(x,\xi) : |\xi_3| \ge \bar\rho_2\}$. Then the right-hand expression of this estimate transforms into

$$Ch^{-2}\int_{\rho_2^*}^1 \rho^{-1}|\log\rho|^{-\sigma}\,d\rho$$

which does not exceed Ch^{-2} as long as $\sigma > 1$.

On the other hand, it follows from the standard theory rescaled that estimate (18.6.29) holds for $Q_\rho = \psi(x)\bar\chi_\rho(hD_3)$ and $T = \bar T = \epsilon\mu^{-1}$. Therefore for $\rho = \bar\rho_2$ and $\bar Q = Q_{\bar\rho_2}$ we have

$$\mathsf{R}_{\bar Q}^{\mathsf{T}} \le C\bar\rho_2 h^{-2}/\bar T = Ch^{-2}.$$

Combining this estimate with the result of the previous analysis we get the final inequality

$$\mathsf{R}_{\psi}^{\mathsf{T}} \le Ch^{-2}. \tag{18.6.31}$$

Further, from the standard theory rescaled we know $F_{t\to h^{-1}\tau}\bigl(\bar\chi_T(t)\Gamma(U\,{}^tQ_y)\bigr)$ for $T = \bar T$ and we get the final estimate[32)]

$$|\int\Bigl(\tilde e(x,x,\tau) - h^{-3}\tilde{\mathcal{N}}_{3,x}^{\mathsf{W}}(\tau)\Bigr)\psi(x)\,dx| \le Ch^{-2} \tag{18.6.32}$$

[32)] There should be also a term $C\mu^2h^{-1}$ in the right-hand expression but for $\mu \le h^{-\frac12}$ it is less than Ch^{-2}. We will discuss this expression with more details later.

where

$$h^{-3}\tilde{\mathcal{N}}^{\mathsf{W}}_{3,x}(\tau) = (2\pi)^{-3}\int \theta\big(\tau - \tilde{A}(x,\xi)\big)\,d\xi \tag{18.6.33}$$

and $\tilde{A}$ is a symbol of the mollified operator and this is h-pseudodifferential operator because we take ε depending on ξ_3 [33].

Now let us consider an approximation error in $h^{-3}\tilde{\mathcal{N}}^{\mathsf{W}}$. It does not exceed

$$Ch^{-3}\int \varepsilon(\rho)^{l}|\log\varepsilon(\rho)|^{-\sigma}\,d\rho \asymp Ch^{-2}|\log h|^{1-\sigma}\log\mu \tag{18.6.34}$$

which is $O(h^{-2})$ as $\sigma = 2$ [34].

This completes the proof of the following theorem:

Theorem 18.6.4. *Let $d = 3$ and A be a self-adjoint in $\mathscr{L}^2(X)$ Schrödinger operator* (18.1.1). *Let conditions* (18.1.3), (18.1.6), (18.2.27) *fulfilled in $B(0,1) \subset X \subset \mathbb{R}^3$. Let $(l,\sigma) \succeq (1,1)$.*

Let $\mu \le \bar{\mu}_1 = (h|\log h|)^{-\frac{1}{3}}$. Then for two framing approximations as in footnote [18] *estimate*

$$\mathsf{R}^{\mathsf{W}} := \Big|\int \Big(\tilde{e}(x,x,0) - h^{-3}\mathcal{N}^{\mathsf{W}}_{3,x}(0)\Big)\psi(x)\,dx\Big| \le Ch^{-2} + Ch^{-3}\vartheta(\mu h|\log h|) \tag{18.6.35}$$

holds with the standard Weyl expression

$$h^{-3}\mathcal{N}^{\mathsf{W}}_{3,x}(\tau) = \frac{1}{6\pi^2}h^{-3}\big(\tau - V(x)\big)^{\frac{3}{2}}_{+}\sqrt{g}.$$

In particular, $\mathsf{R}^{\mathsf{W}} = O(h^{-2})$ as $(l,\sigma) \succeq (1,2)$.

Recall that this theorem does not require any non-degeneracy assumption and as it is it cannot be improved significantly in the framework of the

[33] Which means that we take appropriate partition of unity, on each element take its own ε, mollify and then add. One can check easily that perturbation is small enough to preserve all our arguments.

[34] Exactly in this place we need to have variable $\varepsilon = \varepsilon(\rho)$ rather than to take its largest value $\varepsilon(\bar{\rho}_2) = C\mu h|\log h|$ which would require too restrictive smoothness conditions to get approximation error $O(h^{-2})$.

weak magnetic field approach without non-degeneracy assumptions even under improved smoothness assumptions. Later in Theorem 18.6.10 we will slightly improve this theorem with no additional assumptions (removing logarithmic factor)

In the rest of this chapter we are going to improve it under non-degeneracy assumption (18.2.15) or (18.2.16); it appears that extra smoothness will be required.

18.6.5 Outer Zone

First of all, for $\mu \le c^{-1}(h|\log h|)^{-1}$ one can apply exactly the same arguments as before[35] with

$$\rho := |\bar{\xi}_3| \ge \bar{\rho}_1 := C\big(\mu h|\log h|\big)^{\frac{1}{2}} \quad \text{and} \quad \varepsilon = Ch\rho^{-1}|\log h| \tag{18.6.36}$$

and get an estimate (18.6.30) for Q with symbol supported in $\mathcal{Z}_1^c$. Then to derive asymptotics of $\Gamma(\tilde{e}\,{}^tQ_y)$ (at this stage we include ψ in Q) we need to calculate the Tauberian expression

$$h^{-1}\int_{-\infty}^{\tau}\Big(F_{t\to h^{-1}\tau'}\big(\bar{\chi}_T(t)\Gamma(u\,{}^tQ_y)\big)\Big)\,d\tau' \tag{18.6.37}$$

with $T = \bar{T} = \epsilon_0\mu^{-1}$. Note that

(18.6.38) Under assumption (18.2.27) expression (18.6.37) with $T = \bar{T}$ equals modulo $O(h^s)$ to the same expression with $T = T_0 := Ch|\log h|$.

Really, let us break Q into few operators $Q_{(\iota)}$, with the symbols supported in ϵ-vicinity of points $(\bar{x}_{(\iota)}, \bar{\xi}_{(\iota)})$. Without any loss of the generality one can assume that $g^{jk} = \delta_{jk}$ at $\bar{x}_{(\iota)}$ [36]. Then if $|\bar{\xi}_{(\iota)j}| \ge \epsilon_0$ and $\epsilon > 0$ is small enough then both the propagation speed and scale with respect to x_j are disjoint from 0, $j = 1,2,3$ and we can apply arguments of Section 2.3.

Now, to calculate expression (18.6.37) with $T = T_0$ we can apply the method of successive approximations. However, it is not very convenient to take unperturbed operator in our standard way $\bar{A} = A(y, hD_x)$ because

[35] And we need $(\bar{l},\bar{\sigma}) \succeq (l,\sigma) \succeq (1,2)$ only.

[36] We can achieve it by a linear change of coordinates preserving direction of the magnetic field $\boldsymbol{F}$.

then one can estimate $(m+1)$-th term of (18.6.37) with the approximation plugged only by

$$h^{-3}|\log h|\big(\mu T_0^2 h^{-1}\big)^m \asymp h^{-3}|\log h|\big(\mu h|\log h|^2\big)^m$$

which is not good as μ close to $(h|\log h|)^{-1}$ even if we reduce powers of logarithms (which is not very difficult).

Instead we take unperturbed operator (18.1.27)–(18.1.28) exactly as in 2-dimensional case. Recall that without any loss of generality we assume $V_3 = 0$ and $V_j = V_j(x_1, x_2) \in \mathscr{F}^{2,2}$.

With this choice of an unperturbed operator one can estimate $(m+1)$-th term of (18.6.37) with the approximation plugged by

$$h^{-4} T_0\big(\mu T_0^3 h^{-1} + T_0^2 h^{-1}\big)^m \asymp h^{-3}|\log h| \cdot \big(h|\log h|^2\big)^m;$$

recall that $\mu \le C(h|\log h|)^{-1}$ and $T_0 \asymp Ch|\log h|$.

Using rescaling arguments one can prove rather easily that we can take effectively $T_0 \asymp h$ and thus to reduce the power of the logarithmic factors but it is not worth of our efforts now; we just notice that modulo $O(h^{-2})$ expression (18.6.37) is equal to the same expression with U replaced by $U^0 + u^1$ where U^0 and U^1 are first two terms of approximation:

$$U^0 = -ih \sum_{\varsigma=\pm} \varsigma \bar{\mathcal{G}}^{\varsigma} \delta(x-y)\delta(t), \tag{18.6.39}$$

$$U^1 = -ih \sum_{\varsigma=\pm} \varsigma \bar{\mathcal{G}}^{\varsigma} \mathcal{R} \bar{\mathcal{G}}^{\varsigma} \delta(x-y)\delta(t) \tag{18.6.40}$$

where $\bar{\mathcal{G}}^{\pm}$ are forward and backward parametrices respectively for $(hD_t - \bar{A})$ and $\mathcal{R} = A - \bar{A}$.

Now we want to get rid of the contribution of U^1 and only extra logarithmic factors in its estimate prevent us. Then we need just notice that due to the rescaling method the corresponding term in (18.6.37) with $\bar{\chi}_T$ replaced by χ_T with $T \ge T_0 := Ch^{-1}$ does not exceed $CTh^{-4} \times T^2 h^{-1} \times \big(T_0/T\big)^s$ with an arbitrarily large exponent s. Then the sum with respect to t-partition does not exceed Ch^{-2}.

On the other hand, this term calculated with $\bar{\chi}_T$ and $T = T_0$ does not exceed Ch^{-2} as well. Therefore the total contribution of U^1 to the final answer does not exceed Ch^{-2}.

Finally, let us calculate expression (18.6.37) with $U = U^0$. Without any loss of the generality one can assume that $g^{jk}(y) = \delta_{jk}$; we need later to get the answer in the invariant form. Then

$$U^0(x,y,t) = h^{-1}\iint e^{-ih^{-1}\left(-t(\xi_3^2+\tau'+V(y))+(x_3-y_3)\xi_3\right)}\, d\xi_3\, d_{\tau''}\bar{e}_y(x_1,x_2;y_1,y_2;\tau'') \tag{18.6.41}$$

where $\bar{e}_y(x_1,x_2;z_1,z_2;\tau)$ is the Schwartz kernel of the spectral projector of $\bar{P}_1^2+\bar{P}_2^2$ and "y" indicates the same point as in (18.1.27). Direct calculations of Chapter 13 show that

$$\bar{e}_y(x_1,x_2;y_1,y_2;\tau'') = \sum_{n\ge 0}\theta\big(\tau''-(2n+1)\mu hF(y)\big)F(y)\mu h^{-1} \tag{18.6.42}$$

and plugging into (18.6.41) we arrive to

$$\Gamma_y\big(U^0\,{}^tQ_y\big) = \int\sum_{n\ge 0} e^{ih^{-1}t\left(\xi_3^2+(2n+1)\mu hF(y)+V(y)\right)}\, q(\xi_3)\, d\xi_3 \times F(y)\mu h^{-2} \tag{18.6.43}$$

and consequently to

$$F_{t\to h^{-1}\tau'}\Big(\bar{\chi}_T(t)\Gamma(u\,{}^tQ_y)\Big) = \sum_{n\ge 0} T\int \widehat{\bar{\chi}}\Big(Th^{-1}\big(\tau'-(2n+1)F(y)\mu h - V(y)-\xi_3^2\big)\Big)\times q(\xi_3)\, d\xi_3 \times F(y)\mu h^{-2} \tag{18.6.44}$$

where $\widehat{\bar{\chi}}$ is a Fourier transform of $\bar{\chi}$ and modulo $O(h^{-2})$ expression (18.6.37) is finally equal to

$$h^{-3}\int \tilde{\mathcal{N}}_Q^{\mathsf{MW}}(x,\tau)\psi(x)\, dx \tag{18.6.45}$$

with

$$\tilde{\mathcal{N}}_Q^{\mathsf{MW}} := \sum_{n\ge 0}\int \theta\Big(\tau-(2n+1)F\mu h-\tilde{V}-\xi_3^2\Big)q(\xi_3)\, d\xi_3\times F\mu h. \tag{18.6.46}$$

One can check easily that $\varepsilon|\log h|^{-2}\le \rho^2$ for $\rho\ge\bar{\rho}_1$ and therefore the approximation error of $h^{-3}\tilde{\mathcal{N}}_Q^{\mathsf{MW}}$ does not exceed $C\int h^{-3}\varepsilon|\log h|^{-2}d\rho \le Ch^{-2}$ (recall that $(I,\sigma)\succeq(1,2)$). So we arrive to

Proposition 18.6.5. *Let* $(l,\sigma)\succeq(1,2)$ *and* $\bar\mu_1\le\mu\le c^{-1}(h|\log h|)^{-1}$. *Let* $Q=q(hD_3)$ *with* $q=1-\bar\chi_{\bar\rho_1}(\xi_3)$.

Then

$$|\int\Big(\Gamma_x(\tilde e\,{}^tQ_y)-h^{-3}\tilde{\mathcal N}_Q^{\mathsf{MW}}(x,\tau)\Big)\psi(x)\,dx|\le Ch^{-2} \tag{18.6.47}$$

with $\tilde{\mathcal N}_Q^{\mathsf{MW}}(x,\tau)$ *defined by* (18.6.46).

18.6.6 Analysis under Assumption (18.2.15)

Let us prove that under assumption (18.2.15) $\Gamma(U\psi\,{}^tQ_y)$ is negligible for $\bar T\le|t|\le\epsilon$ as

$$\varepsilon=C\mu h|\log h|\quad\text{in}\quad\mathcal Z_1 \tag{18.6.48}$$

and $Q=q(hD_3)$ with symbol supported in $\mathcal Z_1$. Recall that $\mathcal Z_1=\{(x,\xi):|\xi_3|\le\bar\rho_1\}$.

Proposition 18.6.6. *Let condition* (18.2.15) *be fulfilled,* $Q=q(hD_3)$ *with appropriate symbol* q *supported in* $\mathcal Z_1$. *Then*

$$|F_{t\to h^{-1}\tau}\big(\chi_T\Gamma u\psi\,{}^tQ_y\big)|\le Ch^s\qquad\forall\tau:|\tau|\le\epsilon \tag{18.6.49}$$

as $T\in[T_*,T^*]$ *where* χ *an admissible function supported in* $[-1,-\frac12]\cup[\frac12,1]$ *and here*

$$T_*=C\varepsilon^{-1}h|\log h|,\qquad T^*=\epsilon. \tag{18.6.50}$$

In particular, (18.6.49) *holds as* $T\in[\bar T,\epsilon]$ *with* $\bar T=\epsilon_0\mu^{-1}$ *for* ε *given by* (18.6.48).

We prove this proposition below, it implies

Corollary 18.6.7. *Under conditions* (18.2.15), (18.2.27) *and* (18.6.48)

$$|\int\Big(\tilde e(x,x,\tau)-\int_{-\infty}^{\tau}F_{t\to h^{-1}\tau'}\big(\bar\chi_{\bar T}u(x,x,t)\big)\Big)\psi(x)\,dx|\le Ch^{-2} \tag{18.6.51}$$

and

$$|\int\Big(\tilde e(x,x,\tau)-h^{-3}\tilde{\mathcal N}^{\mathsf{MW}}(x,\tau)\Big)\psi(x)\,dx|\le Ch^{-2}. \tag{18.6.52}$$

Remark 18.6.8. (i) Note that under assumption (18.2.15) and ε given by (18.6.48) the contribution of zone $\mathcal{Z}_1$ to the approximation error does not exceed

$$C(\mu h|\log h|)^{l+\frac12}|\log h|^{-\sigma}h^{-3} \tag{18.6.53}$$

which is in turn does not exceed Ch^{-2} as long as

$$\mu \le \mu^*_{(l,\sigma)} := Ch^{-(2l-1)/(2l+1)}|\log h|^{-1+2\sigma/(2l+1)} \tag{18.6.54}$$

while remainder estimate does not exceed Ch^{-2} anyway.

(ii) Note that $\mu^*_{(1,2)} = h^{-\frac13}|\log h|^{\frac13}$ and $\mu^*_{(l,\sigma)} \sim h^{-1+\delta}$ in the smooth case when l is large.

Proof of Proposition 18.6.6. We will use the canonical form reduction of the next Section 18.7.

Making ϵ_1-partition we can assume that on it $|\nabla V - \ell| \le \epsilon$ at each point with some fixed vector ℓ, $|\ell| \ge 4\epsilon$.

Let us consider first the case when $|\ell_\perp| \ge \frac12\epsilon$, $\ell_\perp = (\ell_1, \ell_2)$ and examine propagation of singularities with respect to (x_2, ξ_2) for the transformed operator.

To time T the shift with respect to x_2 will be $\asymp \epsilon|\ell_2|\mu^{-1}$ and the logarithmic uncertainty principle means that $\epsilon_2|\ell_2|\mu^{-1}T \times \varepsilon \ge C\mu^{-1}h|\log h|$ which for $|\ell_2| \ge \epsilon$ is equivalent to $T \ge T_*$ with T_* defined by (18.6.50).

Similarly, to time T the shift with respect to ξ_2 will be $\asymp \epsilon|\ell_1|\mu^{-1}$ and the logarithmic uncertainty principle means that $\epsilon_2|\ell_1|\mu^{-1}T \times \varepsilon \ge C\mu^{-1}h|\log h|$ which for $|\ell_1| \ge \varepsilon$ is equivalent to $T \ge T_*$ again.

We can justify this analysis by our standard propagation arguments with functions

$$\chi\bigl(\mu\frac{x_2 - y_2}{\ell_2 T} \pm \epsilon\frac{t}{T}\bigr), \qquad \chi\bigl(\mu\frac{\xi_2 - \eta_2}{\ell_1 T} \pm \epsilon\frac{t}{T}\bigr) \tag{18.6.55}$$

respectively.

On the other hand, let us consider the case $|\ell_\parallel| \ge \frac12\epsilon$ with $\ell_\parallel = \ell_3$ and examine propagation of singularities with respect to ξ_3. To time T the shift will be $\asymp \epsilon|\ell_3|$ and and the logarithmic uncertainly principle means that $\epsilon|\ell_3|\mu^{-1} \times \varepsilon \ge Ch|\log h|$ which is equivalent to $T \ge T_*$ again.

We can justify this analysis by standard means with a function

$$\chi\bigl(\frac{\xi_3-\eta_3}{\ell_3 T}\pm\epsilon\frac{t}{T}\bigr). \tag{18.6.55}_3$$

Also, in all upper bound for T is a small constant because for this time (x_2,x_3,ξ_2,ξ_3) keeps in ϵ_1-vicinity of the point of origin[37] which can be justified by standard means with functions

$$\begin{aligned}&\chi\bigl(\frac{t}{T}\pm\epsilon\mu\frac{\xi_2-\eta_2}{T}\bigr), && \chi\bigl(\frac{t}{T}\pm\epsilon\mu\frac{x_2-y_2}{T}\bigr),\\ &\chi\bigl(\frac{t}{T}\pm\epsilon\frac{\xi_3-\eta_3}{T}\bigr), && \chi\bigl(\frac{t}{T}\pm\epsilon\frac{x_3-y_3}{T}\bigr).\end{aligned} \tag{18.6.56}_{1-4}$$

Remark 18.6.9. The above proof uses canonical form (see the next Section 18.7) and therefore requires $(\bar{l},\bar{\sigma})=(2,1)$. However, one can provide an alternative proof, similar to ones in Subsection 18.2.3, which uses "precanonical" form (i.e. variables Q_1, Q_2); now it is enough $(\bar{l},\bar{\sigma})=(l,\sigma)\succeq(1,2)$. We leave easy details to the reader.

Now Statement (i) of Theorem 18.6.12 below follows from Propositions 18.6.5, 18.6.6, Corollary 18.6.7 and Remark 18.6.8.

18.6.7 Analysis under Assumption (18.2.16)

Let $(l,\sigma)\succeq(2,0)$. Then we can introduce

$$\nu=\epsilon|\nabla(V/F)|+\bar{\nu},\qquad \bar{\nu}=C_1(\mu h|\log h|)^{\frac12}\ge C_0\mu^{-1} \tag{18.6.57}$$

where the last inequality is due to $\mu\ge\epsilon(h|\log h|)^{-\frac13}$ and as $|\xi_3|\le\bar{\rho}_1$ and $\nu\ge 2\bar{\nu}$ we can apply arguments of the previous subsection with

$$\varepsilon=C_0\mu h|\nu^{-1}\log h| \tag{18.6.58}$$

and take $T_*=Ch|\log h|\nu^{-1}\varepsilon^{-1}\le\epsilon\mu^{-1}$ and $T^*\asymp 1$.

Then the contribution of the zone $\mathcal{Z}_1\cap\{|\nabla(V/F)|\ge 2\bar{\nu}\}$ to the Tauberian remainder is $O(h^{-2})$ and in virtue of assumption (18.2.16) its contribution to the approximation error does not exceed expression (18.6.53) as $l<3$.

[37] Actually it can be longer for small ℓ_3 but there is no benefit from this observation.

On the other hand, the contribution of the zone $\mathcal{Z}_1 \cap \{|\nabla(V/F)| \le 2\bar\nu\}$ to the Weyl remainder does not exceed $C\mu h^{-2}(\mu h|\log h|)^2$ where an extra factor (in comparison with the general case) $(\log \mu h|\log h)^{\frac12}$ is due to assumption (18.2.16). Finally, contribution of this zone to the approximation error also does not exceed (18.6.53).

These arguments imply Statement (ii) of Theorem 18.6.12 below.

18.6.8 Weakly-Degenerate Case

We need this subsection to improve Theorem 18.6.4 to Theorem 18.6.10 below. Let us consider point $\bar x$ in which

$$|\nabla(V/F)| \asymp \zeta \ge C'|\log h|^{-\sigma'}, \qquad (l,\sigma) \succeq (1,\sigma'). \tag{18.6.59}$$

Then the same is true in its γ-vicinity with $\gamma = h^\delta$ with some small $\delta > 0$ provided $C' = C'(\delta)$ is large enough if $(l,\sigma) = (1,\sigma')$; otherwise no restriction on C' is needed. Thus all the arguments of the previous subsection remain valid but now for time $\bar T = \epsilon\mu^{-1}$ the shift with respect to ξ_3 will be $\asymp \zeta\bar T$ provided $|\nabla_\parallel(V/F)| \asymp \zeta$ and the shift with respect to (x_2,ξ_2) will be $\asymp \mu^{-1}\zeta\bar T$ provided $|\nabla_\perp(V/F)| \asymp \zeta$ and therefore logarithmic uncertainty principle is fulfilled if

$$\varepsilon \ge C\zeta^{-1}\mu h|\log h|. \tag{18.6.60}$$

We need to assume also that $\gamma \ge C\varepsilon$ which is always the case provided $\mu \le h^{\delta_0-1}$ with $\delta_0 > \delta$.

Then we can take

$$T^* = \begin{cases} \gamma & \text{for } l = 1,\\ \epsilon\zeta & \text{for } l > 1 \end{cases} \tag{18.6.61}$$

and the contribution of this particular element $\psi_\gamma(x)q_\rho(hD_3)$ to the remainder estimate will be $O\big(\rho\gamma^3h^{-2}T^{*\,-1}\big)$ (we assume that q is compactly supported).

We can take always $\rho = C\bar\rho_1$. Recall that we estimated contribution of the partition element $\psi_\gamma(x)\big(1-q_\rho(hD_3)\big)$ to the remainder as $O\big(\gamma^3h^{-2}\big)$. Therefore the contribution of $\psi_\gamma(x)$ to the remainder is $O(\gamma^3h^{-2})$ anyway.

Now let us consider its contribution to the approximation error. In our assumptions it does not exceed

$$Ch^{-2}\gamma^3 + C\rho\gamma^3h^{-3}\vartheta(\varepsilon) \asymp Ch^{-2}\gamma^3 + C\gamma^3h^{-3}(\mu h|\log h|)^{l+\frac12}\zeta^{-l}|\log h|^{-\sigma} \tag{18.6.62}$$

and to keep it below $C\gamma^3 h^{-2}$ we need to assume that

$$\zeta \ge \mu^{(2l+1)/(2l)} h^{(2l-1)/(2l)} |\log h|^{(2l-2\sigma-1)/(2l)}; \tag{18.6.63}$$

in particular, for $(l,\sigma)=(1,2)$

$$\zeta \ge \mu^{\frac{3}{2}} h^{\frac{1}{2}} |\log h|^{-\frac{1}{2}} \tag{18.6.64}$$

which is possible only for $\mu \le h^{-\frac{1}{3}} |\log h|^{\frac{1}{3}}$.

18.6.9 Main Theorems

Now we prove

Theorem 18.6.10. *As $(\bar{l},\bar{\sigma})=(2,1)$ the conclusion of Theorem 18.6.4 remains true for $\mu \le h^{-\frac{1}{3}}$.*

Proof. To prove this theorem we need to analyze zone

$$\{\mathsf{max}\big(\mu^{-1}, (\mu h)^{\frac{1}{2}}\big) \le |\xi_3| \le (\mu h|\log h|)^{\frac{1}{2}}\}. \tag{18.6.65}$$

Let us make γ-admissible partition with $\gamma = h^{\delta}$. Then on the elements with $|\nabla(V/F)| \ge \bar{\zeta} := C_0 |\log h|^{-\sigma'}$ we can apply arguments of the previous subsection and conclude that their contribution to the Weyl remainder does not exceed $Ch^{-2}\gamma^3$.

On the other hand, consider elements with $|\nabla(V/F)| \le \bar{\zeta}$. Consider ξ_3-partition in zone (18.6.65). Using rescaling arguments one can prove easily that the contribution of such elements to the remainder does not exceed $Ch^{-2}\gamma^3\big(\rho + C\mu\rho(\mu h/\rho^2)^s\big)$; we leave the standard arguments to the reader.

Then summation with respect to ρ returns $Ch^{-2}\gamma^3\big(C\mu(\mu h)^{\frac{1}{2}}\big)$ and summation with respect to γ returns $Ch^{-2}\big(C\mu(\mu h)^{\frac{1}{2}}\big)$. □

Corollary 18.6.11. *In the framework of Theorem 18.6.4 let $(l,\sigma) \succeq (1,2)$ and $\mu \le \mu_1^* := h^{-\frac{1}{3}}$. Then $\mathsf{R}^{\mathsf{MW}} = O(h^{-2})$.*

Consider now the non-degenerate case. We have proven already

Theorem 18.6.12. *Let $d=3$ and A be a self-adjoint in $\mathscr{L}^2(X)$ Schrödinger operator* (18.1.1). *Let conditions* (18.1.3), (18.1.6), (18.2.27) *fulfilled in $B(0,1) \subset X \subset \mathbb{R}^3$. Let $(l,\sigma) \succeq (1,1)$.*

Let $h^{-\frac{1}{3}} \le \mu \le C(h|\log h|)^{-1}$, $(\bar{l},\bar{\sigma})=(2,1)$. Then for two framing approximations as in footnote[18)]

(i) Under assumption (18.2.15) *asymptotics*

$$|\int \Big(\tilde{e}(x,x,0) - h^{-2}\mathcal{N}_3^{\mathsf{MW}}(x,0)\Big)\psi(x)\,dx| \le Ch^{-2} + Ch^{-3}\vartheta(h|\log h|) + Ch^{-3}(\mu h|\log h|)^{\frac{1}{2}}\vartheta(\mu h|\log h|) \tag{18.6.66}$$

holds.

(ii) Let $(l,\sigma) \succeq (2,0)$ *and* $l < 3$. *Then under assumption* (18.2.16) *asymptotics* (18.6.66) *holds.*

We leave to the reader the following easy problem:

Problem 18.6.13. Prove that

(i) Theorems 18.6.10 and 18.6.12 remain true for $\mathsf{R}^{\mathsf{W}}_{\infty}$ and, if $\mu \le h^{-\frac{1}{2}}$, also for R^{W}.

(ii) Corollary 18.6.11 remains true for R^{W}.

18.7 $d = 3$: Canonical Form

From now on we will consider more difficult case

$$\mu \ge \bar{\mu}_1 := C^{-1}(h|\log h|)^{-\frac{1}{3}}; \tag{18.7.1}$$

further restrictions will be added later. Actually due to Theorem 18.6.10 we can consider $\mu \ge \mu_1^* := h^{-\frac{1}{3}}$.

We already know from Subsection 18.5 that the contribution of the zone $\mathcal{Z}_1^c := \{|\xi_3| \ge \bar{\rho}_1\}$ with $\bar{\rho}_1 := C(\mu h|\log h|)^{\frac{1}{2}}$ to the remainder does not exceed Ch^{-2} and therefore we can limit ourselves by zone $\mathcal{Z}_1 := \{|\xi_3| \le \bar{\rho}_1\}$.

18.7.1 Canonical Form. Preliminary Remarks

To treat this zone $\mathcal{Z}_1$ properly one needs to reduce operator to the canonical form as in Section 13.3. However, as for $d = 2$, to follow this procedure one must assume that $\varepsilon \ge C\mu^{-1}|\log h|$ at least and then the approximation error will be too large unless the regularity condition holds with $l \ge 2$.

On the other hand, these arguments still work, but for much larger μ. One of the reasons is that even if we neglect an approximation error, the remainder estimate would be at least $\mu h^{-1} \times \mathcal{R}_1$ where $\mathcal{R}_1$ is the remainder estimate for 1-dimensional Schrödinger operator.

Unless we assume that V/F is non-degenerate (i.e. either $\nabla(V/F) \neq 0$ or $\nabla^2(V/F) \neq 0$) we can derive nothing better than $\mathcal{R}_1 = O\big(h^{-2/(2+l)}\big)$ according to Section 4.6 (we neglect logarithmic factors here). Therefore we should not expect anything better than estimate $O\big(\mu h^{-1-2/(2+l)}\big)$ for our magnetic 3-dimensional Schrödinger operator anyway. For $l = 1$ this estimate coincides with $O(h^{-2})$ for $\mu \asymp h^{-\frac{1}{3}}$ exactly.

However, for μ close to $h^{-\frac{1}{3}}$ we need to modify reduction arguments properly and the rest of the section is devoted to this.

We pay mainly attention to the case

$$C_0(h|\log h|)^{-\frac{1}{3}} \le \mu \le C_0^{-1}(h|\log h|)^{-1} \tag{18.7.2}$$

and analyze zone $\mathcal{Z}_1$.

First of all we need to decide what does *non-essential* mean.

Non-Degenerate Case. Under non-degeneracy assumption (18.2.15) the contribution of zone $\mathcal{Z}_1$ to the approximation error in $h^{-3}\mathcal{N}^{\mathsf{MW}}$ does not exceed $Ch^{-3}\bar{\rho}_1 R$ where R is an upper bound for a perturbation. To prove this one needs to notice that for $\rho \le \rho_1^* := (\mu h)^{\frac{1}{2}}$ due to this assumption the measure of the zone

$$\mathcal{Y}_\rho := \big\{\rho^2 \le \min_{n\ge 0} |\tau - V - (2n+1)\mu h| \le 2\rho^2\big\} \tag{18.7.3}$$

does not exceed $C\rho^2/(\mu h)$ and its contribution to the approximation error in $h^{-3}\mathcal{N}^{\mathsf{MW}}$ does not exceed $C\big(\bar{\rho}_1 \mu h^{-2}\rho^{-1}\big)R \times \rho^2/\mu h$. Then the total contribution of $\mathcal{Z}_1$ to the approximation error does not exceed this expression integrated by $\rho^{-1}d\rho$ which gives exactly $Ch^{-3}\bar{\rho}_1 R$.

On the other hand, one can prove easily that under assumption (18.2.15) the remainder estimate is $O\big(h^{-2}\big)$ and therefore to keep approximation error below it we need to make calculations with the precision

$$R \le R^* := C\mu^{-\frac{1}{2}} h^{\frac{1}{2}} |\log h|^{-\frac{1}{2}}. \tag{18.7.4}$$

Therefore

(18.7.5) In the non-degenerate case as $(l,\sigma)=(1,2)$ in $\mathcal{Z}_1$ we can increase ε up to

$$\varepsilon^* = C\mu^{-\frac{1}{2}}h^{\frac{1}{2}}|\log h|^{\frac{3}{2}} \tag{18.7.6}$$

Case of the non-degeneracy assumption (18.2.16) does not warrant its own analysis since one expects to have the remainder estimate $O(h^{-2})$ and the approximation error as above but with an extra term $C\mu h^{-2}R^2$ which would be smaller.

General Case. In the general case (i.e. without non-degeneracy assumption) the contribution of the zone $\mathcal{Z}_1$ to the approximation error in $h^{-3}\mathcal{N}^{\mathsf{MW}}$ does not exceed $Ch^{-3}\bar{\rho}_1R+C\mu h^{-2}R^{\frac{1}{2}}$ while we expect the remainder estimate to be $O\big(h^{-2}+\mu h^{-1-2/(l+2)}|\log h|^{-\sigma/(l+2)}\big)$; then to keep approximation error below it we need to make calculations with precision

$$R_* = C\min\big(\mu^{-\frac{1}{2}}h^{\frac{1}{2}}|\log h|^{-\frac{1}{2}},\ \mu^{-2}+h^{2l/(2+l)}|\log h|^{-2\sigma/(l+2)}\big) \tag{18.7.7}$$

where $(l,\sigma)\preceq(2,1)$. Note that R given by this formula is larger than $\varepsilon^l|\log h|^{-\sigma}$ with ε defined by (18.7.1). Therefore

(18.7.8) In the general case we can increase ε up to

$$\varepsilon_* = C(R_*)^{1/l}|\log h|^{\sigma/l}. \tag{18.7.9}$$

Later further increase will be made.

18.7.2 Canonical Form. I

Let us assume that $g^{jk}\in\mathscr{F}^{2,1}$ even if one can weaken this assumption. We call term *non-essential* if its A-bound does not exceed R_* given by (18.7.7); under assumption (18.2.15) or (18.2.16) we can increase it to R^*.

Therefore, in the zone $\mathcal{Z}_1$ we have

$$\varepsilon = C(\mu^{-1}h|\log h|)^{\frac{1}{2}} \tag{18.7.10}$$

or larger. Note that for ε given by (18.7.10) the mollification error $\varepsilon^l|\log h|^{-\sigma}$ does not exceed R_*.

To transform operator A to the canonical form we would like first to transform its principal part to the "diagonal form" as it was done for $d = 2$. This can be achieved by the same transformation (18.3.4)

$$T(t) \coloneqq e^{-i\mu^{-1}h^{-1}tL^{\mathrm{w}}} \qquad \text{with} \quad L^{\mathrm{w}} = \frac{1}{2}\sum_{j,k} P_j L^{jk} P_k. \tag{18.3.4}$$

Then (18.3.5)–(18.3.8) are preserved but now we have 3×3 commutator matrix

$$J = \begin{pmatrix} J' & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{with} \quad J' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$\Lambda = f(x)\ell J = \begin{pmatrix} \Lambda' & 0 \\ \Lambda'' & 0 \end{pmatrix} \qquad \text{with} \quad \Lambda' = \begin{pmatrix} \ell^{12} & -\ell^{21} \\ \ell^{22} & -\ell^{21} \end{pmatrix}, \quad \Lambda'' = (\ell^{32} \ -\ell^{31}).$$

Then

$$e^{t\Lambda} = \begin{pmatrix} e^{t\Lambda'} & 0 \\ \Lambda'' K(t) & 1 \end{pmatrix}$$

with matrix $K(t) = (\Lambda')^{-1}(e^{t\Lambda'} - I)$ which is well defined even if Λ' is degenerate.

Therefore we can achieve $e^{t\Lambda} G e^{\Lambda}$ to be diagonal with the top-left element equal to 1. However, we cannot change g^{33} by this method unless we use "much larger" operator L (something like $\omega(x)\mu h D_3$) which is not a good idea at this moment in the non-smooth case.

The rest of the analysis of Subsubsection *18.3.1.1 Main pPart of the Operator. I* does not change and and therefore we conclude that

(18.7.11) Modulo operator with A-bound not exceeding $\mu^{-1}h$

$$T(-1)A_0 T(1) \equiv P_1^2 + P_2 (g')^{\mathrm{w}} P_2 + P_3 (g'')^{\mathrm{w}} P_3 + \mu^2 M^{\mathrm{w}}, \tag{18.7.12}$$

and

$$M \coloneqq a_0 \circ \phi_1 - p_1^2 - g'^{22} p_2^2 - g'^{33} p_2^2 \tag{18.7.13}$$

(compare with (18.3.18),(18.3.19)).

Recall that according to (18.3.13) $M = \sum_{i,j,k} \beta_{ijk}\, p_i p_j p_k$ with $\beta_{ijk} \in \mathscr{F}^{1,1}$. Finally, modulo a non-essential operator

$$T(-1)VT(1) \equiv (V \circ \phi_1)^{\mathrm{w}}. \tag{18.7.14}$$

Now, we can assume that g', g'' are functions of x only because we can achieve it modifying M but not changing its type. Further, condition $F = 1$ means that $\sqrt{g'}f = 1$ and we can apply the same construction as in Subsubsection *18.3.1.2 Main Part of the Operator. II* and therefore we can assume without any loss of the generality that

$$V_1 = V_3 = 0, \quad P_1 = hD_1 \implies V_2 = V_2(x_2, x_3), \tag{18.7.15}$$

$$V_2, \partial_{x_1} V_2 \in \mathscr{F}^{2,1} \tag{18.7.16}$$

and

$$p_2 = \alpha(x, \xi_2)\big(x_1 - \lambda(x_2, x_3, \xi_2)\big), \qquad \alpha, \lambda \in \mathscr{F}^{2,1}. \tag{18.7.17}$$

Repeating then construction of Subsubsection *18.3.1.2 Main Part of the Operator. II*, we take the next transformation $T'(t) = e^{-i\lambda^{\mathrm{w}} D_1}$. As a result we will get the operator

$$T'(-1)T(-1)A_0T(1)T'(1) \equiv \bar{A}_0 + \big(\mu^2 M'\big)^{\mathrm{w}}, \tag{18.7.18}$$

$$\bar{A}_0 \coloneqq \bar{P}_1^2 + \bar{P}_2^2 + \bar{P}_3 \alpha_0^{\mathrm{w}} \bar{P}_3, \qquad \bar{P}_1 = hD_1,\ \bar{P}_2 = -\mu x_1,\ \bar{P}_3 = hD_3, \tag{18.7.19}$$

$$M' = a_0 \circ \Phi - \bar{a}_0 = \sum_{i,j,k} \beta'_{ijk} \bar{p}_i \bar{p}_j \bar{p}_k \tag{18.7.20}$$

with $\bar{p}_1 \coloneqq \xi_1$, $\bar{p}_2 \coloneqq -x_1$, $\bar{p}_3 \coloneqq \xi_3$, $\Phi \coloneqq \phi_1 \circ \phi'_1$, where ϕ_t, ϕ'_t are corresponding Hamiltonian flows, $\alpha_0 = \alpha_0(x_2, x_3, \xi_2) \in \mathscr{F}^{2,1}$ (compare with (18.3.28)–(18.3.30)).

Further, modulo an operator with $\bar{A}_0$-bound not exceeding $C\vartheta(\varepsilon)$

$$T'(-1)T(-1)A_0T(1)T'(1) \equiv (V \circ \Phi)^{\mathrm{w}}. \tag{18.7.21}$$

So far precision was better than we actually needed: it was $\varepsilon^l |\log h|^{-\sigma}$, exactly as in Subsubsection *18.3.1.2 Main Part of the Operator. II*. We could achieve $\alpha_0 = 1$ but it would be not very useful.

18.7.3 Canonical Form. II

Now we will follow Subsubsection *18.3.1.3 Potential* but with a twist. Namely, we define S', S'', W', W'' *exactly* by (18.3.39)–(18.3.42) and apply transformation $T''(-t) = e^{-it\mu h^{-1} S^{\mathrm{w}}}$ with the phase $S = S' + S''$. Then modulo a non-essential term

$$T''(-t)\Big(\bar{A}_0 + \big(\mu^2 M' + V'\big)^{\mathrm{w}}\Big)T''(t) \equiv \Big(\big(\mu^2 \bar{a}_0 + \mu^2 M' + V'\big) \circ \psi_t\Big)^{\mathrm{w}} \tag{18.7.22}$$

where ψ_t is a corresponding Hamiltonian flow. Let us calculate symbol

$$\bigl(\mu^2\bar{a}_0 + \mu^2 M' + V'\bigr)\circ\psi_t = \mu^2\bar{a}_0 + \bigl(\mu^2 M' + V'\bigr) - t\mu^2\{\bar{a}_0, S\} \\ - t\{\mu^2 M' + V', S\} - \frac{t^2}{2}\{\{\bar{a}_0, S\}, S\} + \dots,$$

and plug $t = 1$. Then

$$\mu^2\{\bar{a}_0, S\} = 2\mu^2(\xi_1\partial_{x_1} - x_1\partial_{\xi_1})S + 2\mu^2\alpha_0^2\xi_3(\partial_{x_3}S) \ + \ \{\alpha_0^2, S\}\xi_3^2 \\ = \mu^2 M' + V' - W_0 + 2\mu^2\alpha_0^2\xi_3(\partial_{x_3}S) \ + \ \{\alpha_0^2, S\}\xi_3^2$$

where the last equality is due to (18.3.39), (18.3.40) and $W_0 = W' + W''$.

Further, let us notice that $|\xi_3| \le \mu^{-1}\bar{\rho}_1 = C(\mu^{-1}h|\log h|)^{\frac{1}{2}}$ in our zone $\mathcal{Z}_1$ where an extra factor μ^{-1} appears because in the reduction part we consider $\mu^{-1}h$-pseudodifferential operator. Then one can see easily that modulo non-essential operator (in the general sense) our transformed operator equals to the quantization of

$$\mu^2\bar{a}_0 + W_0 - 2\mu^2\alpha_0^2\xi_3(\partial_{x_3}S) - \frac{1}{2}\{\mu^2 M' + V' - W_0, S\}' \tag{18.7.23}$$

where $\{.,.\}'$ again mean "short" Poisson brackets (with respect to x_1, ξ_1 only).

Obviously the last term in (18.7.23) is $O(\mu^{-2})$ and it is essential only for $\mu \le \hat{\mu}_1 := \epsilon_0 h^{-\frac{1}{3}}|\log h|^{\frac{1}{3}}$.

Furthermore, $\partial_{x_3}S = O\bigl(\mu^{-1-l}|\log h|^{-\sigma} + \mu^{-3}\bigr)$ and therefore the second term in (18.7.23) is $O\bigl((\mu^{1-l}|\log h|^{-\sigma} + \mu^{-2})\cdot(\mu^{-1}h|\log h|)^{\frac{1}{2}}\bigr)$ which does not exceed R^* but is not necessarily smaller than R_*.

Therefore, if one of the conditions (18.2.15), (18.2.16) is fulfilled and $\mu \ge \hat{\mu}_1$ then we have finished because only the first term in (18.7.23) is essential. In this case we have arrived to the operator

$$\bar{A}_0 + W_0(x_2, x_3, \mu^{-1}hD_2, \mu^{-1}hD_3, \mu^{-1}\bar{A}_0^{\frac{1}{2}})^{\mathrm{w}} \tag{18.7.24}$$

with

$$W_0(x_2, x_3, \xi_2, \xi_3, \rho) = \mathcal{M}_\rho(\mu^2 M' + V'). \tag{18.7.25}$$

Furthermore, under one of the assumptions (18.2.15), (18.2.16) we will see later that for $\mu \le \hat{\mu}_1$ we do not need the full canonical form at all.

However, in the general case the second and the third terms in (18.7.23) are essential and we need to continue in the same way as in Subsubsection *18.3.1.4 Final Reduction*. Let us define S_1, S''', W_1, W''' in the same way as before:

$$W''' = \alpha_0^2 \partial_{x_3} \mathcal{M}_\rho S, \qquad W_1 = -\frac{1}{2}\mathcal{M}_\rho \{\mu^2 M' + V' - W_0, S\}', \tag{18.7.26}$$

$$2\mu^2(\xi_1\partial_{x_1} - x_1\partial_{\xi_1})S''' = \alpha_0^2\partial_{x_3}(S - \mathcal{M}_\rho S), \tag{18.7.27}$$

$$2\mu^2(\xi_1\partial_{x_1} - x_1\partial_{\xi_1})S_1 = -\frac{1}{2}\{\mu^2 M' + V' - W_0, S\}' + W_1; \tag{18.7.28}$$

Recall that $\alpha_0 = \alpha_0(x_2, x_3, \xi_2)$. Let us apply $\mu^{-1}h$-pseudodifferential operator transformation

$$T'''(t) = \bigl(e^{-it\mu h^{-1}(S'''\xi_3 + S_1)}\bigr)^{\mathrm{w}};$$

then

$$T'''(-1)\Bigl(\mu^2\bar{a}_0 + W_0 - 2\mu^2\alpha_0^2\xi_3(\partial_{x_3}S) - \frac{1}{2}\{\mu^2 M' + V' - W_0, S\}'\Bigr)^{\mathrm{w}} T'''(1) \equiv$$
$$\Bigl(\mu^2\bar{a}_0 + W_0 - 2\mu^2\alpha_0^2\xi_3(\partial_{x_3}S) - \frac{1}{2}\{\mu^2 M' + V' - W_0, S\}' - \mu^2\{\bar{a}_0, S'''\xi_3 + S_1\} + \dots\Bigr)^{\mathrm{w}}.$$

Plugging in (18.7.26)–(18.7.27) we see easily that

$$T'''(-1)\Bigl(\mu^2\bar{a}_0 + W_0 - 2\mu^2\alpha_0^2\xi_3(\partial_{x_3}S) - \frac{1}{2}\{\mu^2 M' + V' - W_0, S\}'\Bigr)^{\mathrm{w}} T'''(1) \equiv \bar{A}_0 + Wx_2, x_3, \mu^{-1}hD_2, \mu^{-1}hD_3, B) \tag{18.7.29}$$

where $W = W_0 + W'''\xi_3 + W_1$.

The last step is not really necessary, but we'll do it for the sake of the simple canonical form.

First of all, transformation $T^V(1)$, with $T^V(t) = e^{-it\omega^{\mathrm{w}}D_3}$ and with $\omega = \omega(x_2, x_3, \xi_2) \in \mathscr{F}^{2,1}$, transforms D_3 into $(\partial_{x_3}\omega)^{\mathrm{w}}D_3 + \beta^{\mathrm{w}}\mu^{-1}hD_3^2$ with $\beta = \beta(x_2, x_3, \xi_2, \xi_3) \in \mathscr{F}^{1,1}$ and therefore one can transform $\alpha_0^{\mathrm{w}}D_3$ into $D_3 + \beta^{\mathrm{w}}\mu^{-1}hD_3^2$. Thus, modulo a non-essential operator

$$T^V(-1)\bar{A}_0 T^V(1) \equiv h^2D_1^2 + \mu^2 x_1^2 + D_3^2. \tag{18.7.30}$$

One can also see easily that $T^V(-1)W^{\mathsf{w}}T^V(1) \equiv W^{\mathsf{w}} + K^{\mathsf{w}}\mu^{-1}hD_3$ with $K \in \mathscr{F}^{l-1,\sigma}$.

Further, modulo a non-essential term we can rewrite our reduced operator in the form $\bar{A}_0 + \bar{W}^{\mathsf{w}} + \bar{K}^{\mathsf{w}}\mu^{-1}hD_3$ with $\alpha_0 = 1$ in the expression (18.7.19) for $\bar{A}_0$ and $\bar{W} = \bar{W}(x_2, x_3, \xi_2, \mu^{-1}\bar{A}_0^{\frac{1}{2}})$ where $\bar{W} := W|_{\xi_3=0} \in \mathscr{F}^{l,\sigma}$ and $\bar{K} = \bar{K}(x_2, x_3, \xi_2, \mu^{-1}\bar{A}_0^{\frac{1}{2}})$ with $\bar{K} := (\partial_{\xi_3}W)|_{\xi_3=0} \in \mathscr{F}^{l-1,\sigma}$.

Finally, applying transformation $T^{Vl}(1)$ with $T^{Vl}(t) = e^{-it\mu^{-1}h^{-1}\omega'^{\mathsf{w}}}$ and $\omega' = \omega'(x_2, x_3, \xi_2, \mu^{-1}\bar{A}_0^{\frac{1}{2}})$, $\omega' \in \mathscr{F}^{l-1,\sigma}$ we will get (for an appropriate symbol ω') modulo negligible operator $\bar{A}_0 + \bar{W}^{\mathsf{w}}$.

18.7.4 Canonical Form. III. $\mu \ge \epsilon h^{-1}|\log h|^{-1}$

In this case again an A-bound for $\mu^{-1}P_j$ does not exceed $C\varepsilon$ and construction is absolutely straightforward as in the smooth case; we just note that $R^* = Ch$ and

$$R_* = C\min\bigl(h, h^{2l/(2+l)}|\log h|^{-2\sigma/(l+2)}\bigr) = C\begin{cases} h & \text{for } (l,\sigma) \preceq (2,0),\\ h|\log h|^{-\frac{\sigma}{2}} & \text{for } l = 2, 0 < \sigma \le 1.\end{cases} \tag{18.7.31}$$

18.8 $d = 3$: Tauberian tTheory

In this section we derive Tauberian estimates. First of all, we have decomposition similar to one we had in the 2-dimensional case.

Using this decomposition we analyze the case $\mu \ge \bar{\mu}_1$ in two different settings: under one of non-degeneracy assumptions (18.2.15), (18.2.16) we prove that the Tauberian remainder estimate does not exceed $C(1+\mu h)h^{-2}$ and that in the general case we prove that it does not exceed expression $Ch^{-2} + C\mu h^{-1}\mathcal{R}_1$ where $\mathcal{R}_1$ is the remainder estimate for 1-dimensional Schrödinger operator.

18.8.1 Decomposition

First of all, as we already mentioned, we have decomposition, similar to (18.4.3)–(18.4.4): we reduce our operator to the family of 1-dimensional

Schrödinger operators with respect to x_3 which are also $\mu^{-1}h$-pseudodifferential operators with respect to x_2

$$\mathcal{A}_n = hD_3^2 + r_n^2 + W(x_2, x_3, \mu^{-1}hD_2, \mu^{-1}r_n), \qquad r_n = \big((2n+1)\mu h\big)^{\frac{1}{2}}, \tag{18.8.1}$$

with $x' = (x_1, x_2)$ and arrive to decomposition

$$U(x, y, t) \equiv \mathcal{T}\Big(\sum_{n\ge 0} U_n(x', y', y_3, t)\Upsilon_n(x_1)\Upsilon_n(y_1)\Big)\mathcal{T}^{-1} \tag{18.8.2}$$

where $U_n(x', y', t)$ are Schwartz kernels of the propagators for $\mathcal{A}_n$.

Here we assume that $\mu \le C_0 h^{-1}$; as $\mu \ge C_0 h^{-1}$ we consider Schrödinger-Pauli operator with $\mathfrak{z} = 1$ and $r_n = 0$.

18.8.2 Reduction in the Zone $\mathcal{Z}_1$

As we mentioned, in this case we have essentially 1-dimensional Schrödinger operator (or rather a family of them). Due to Section 18.6 we need to consider zone $\mathcal{Z}_1 = \{(x, \xi) : |\xi_3| \le \bar\rho_1 = C(\mu h|\log h|)^{\frac{1}{2}}\}$ only.

Assuming first that $\vartheta(\eta) \ge \eta^2$ et us introduce admissible functions $\ell = \ell_{(n)}$ and $\varrho = \varrho_{(n)}$ of (x_2, x_3, ξ_2) similar to those introduced in Section 5.2:

$$\ell = \epsilon \min\Big\{\eta : \ |W + (2n+1)\mu h| \le \vartheta(\eta), |\nabla W| \le \eta^{-1}\vartheta(\eta)\Big\} + \bar\ell, \tag{18.8.3}$$

$$\varrho = \vartheta(\ell)^{\frac{1}{2}}$$

with $\bar\ell$ defined from equation

$$\bar\ell\vartheta(\bar\ell)^{\frac{1}{2}} = C_1 h|\log h|. \tag{18.8.4}$$

Note that as $\vartheta(\eta) \le \eta|\log\eta|^{-2}$ we conclude that

$$\bar\ell \ge (h|\log h|)^{\frac{2}{3}} \ge \bar\varepsilon = (\mu^{-1}h)^{\frac{1}{2}}|\log h| \tag{18.8.5}$$

where the last inequality is due to $\mu \ge \bar\mu_1$. Recall that $\bar\varepsilon$ was the approximation parameter in the inner zone.

We need to have approximation parameter $\varepsilon_n = \varepsilon_{(n)}$ such that $\varepsilon\varrho_n \ge Ch|\log h|$ to satisfy uncertainty principle with respect to x_3 and therefore we replace $\varepsilon_{(n)}$ by

$$\varepsilon := \varepsilon_{(n)} + \varepsilon'_{(n)}, \qquad \varepsilon'_{(n)} := Ch\varrho_{(n)}^{-1}|\log h|; \tag{18.8.6}$$

then

$$\varepsilon_{(n)} \lesssim \ell_{(n)} \quad \text{and} \quad \varepsilon_{(n)} \asymp \ell_{(n)} \iff \ell_{(n)} \asymp \bar{\ell}. \tag{18.8.7}$$

Note first that

(18.8.8) If on the given partition element

$$|W + (2n+1)\mu h| \asymp \varrho^2 = \vartheta(\ell) \geq C\mu h|\log h| \tag{18.8.9}$$

then on this element operator $\mathcal{A}_n$ is elliptic in $\mathcal{Z}_1$ and its contribution to $F_{t\to h^{-1}\tau}\bar{\chi}_T(t)U_n$ is negligible on energy levels $\tau : |\tau| \leq \epsilon_1\varrho^2$.

On the other hand,

(18.8.10) If on the given partition element

$$|\nabla W| \asymp \varrho^2\ell^{-1} = \vartheta(\ell)\ell^{-1}, \qquad \varrho^2 \geq C\mu h|\log h| \tag{18.8.11}$$

then on this element operator $\mathcal{A}_n$ is microhyperbolic with respect to (x_2, x_3, ξ_2) in $\mathcal{Z}_1$ and its contribution to Tauberian estimate with the given index n does not exceed $C\mu h^{-1}\ell^2$.

Really, microhyperbolicity with respect to (x_2, ξ_2) and propagation with respect to (ξ_2, x_2) gives us

$$T_* = Ch|\log h|/(\varrho^2\ell^{-1}\varepsilon) \tag{18.8.12}$$

and microhyperbolicity with respect to x_3 and propagation with respect to ξ_3 gives us the same answer as in these case we consider $\mu^{-1}h$-pseudodifferential and h-pseudodifferential operators respectively. One can see easily that $\varrho^2\ell^{-1} \geq C(\mu h|\log h|)^{\frac{1}{2}}$ and therefore $T_* \leq \epsilon$.

Remark 18.8.1. However, the actual estimate is based not on the propagation but on fine subpartition and ellipticity arguments exactly like in Section 18.4. We leave details to the reader.

Meanwhile one can see easily that

(18.8.13) On elements with $\varrho \geq (\mu h)^{\frac{1}{2}}$ additional approximation is not needed because $Ch\varrho^{-1}|\log h| \leq C\bar{\varepsilon}$.

Since over such element with $\varrho^2 \ge C\mu h$ ellipticity is violated only for $\asymp \varrho^2/\mu h$ indices n then due to (18.8.10) its contribution to the Tauberian remainder does not exceed $C\mu h^{-1}\ell^{-1} \times \varrho^2/\mu h \asymp Ch^{-2}\varrho^2\ell^2 \le Ch^{-2}\ell^3$. Then summation over all such elements results in $O(h^{-2})$.

Therefore we arrive to

Proposition 18.8.2. *As $(l,\sigma) \preceq (1,2)$ contribution to the Tauberian remainder with $T = \bar{T} = \epsilon$ of the complement of the zone*

$$\mathcal{Z} := \{(x,\xi) : |\xi_3| + \varrho \le C\bar{\rho}_1\} \tag{18.8.14}$$

with $\varrho := \min_n \varrho_n$, is $O(h^{-2})$.

Therefore we need to consider zone $\mathcal{Z}$ only and we divide it into an *inner zone*

$$\mathcal{Z}_{\mathsf{inn}} := \{(x,\xi) : \ |\xi_3| + \varrho \le \bar{\rho}_0 = \epsilon_0(\mu h)^{\frac{1}{2}}\} \tag{18.8.15}$$

and a *transitional zone*

$$\mathcal{Z}_{\mathsf{trans}} := \{(x,\xi) : \ \bar{\rho}_0 \le |\xi_3| + \varrho \le \bar{\rho}_1\}. \tag{18.8.16}$$

Corollary 18.8.3. *As $(l,\sigma) \preceq (2,0)$ the Tauberian remainder is $O(h^{-2})$ as $\mu \le (h|\log h|)^{-1}$.*

We leave to the reader to prove using rescaling technique (with defined above ℓ, ϱ) a bit more stronger

Proposition 18.8.4. *As $(l,\sigma) \preceq (2,0)$ contribution to the Tauberian remainder with $T = \bar{T} = \epsilon$ of the complement of the zone $\mathcal{Z}_{\mathsf{inn}}$ with $\varrho := \min_n \varrho_n$ is $O(h^{-2})$.*

Corollary 18.8.5. *As $(l,\sigma) \preceq (2,0)$ the Tauberian remainder is $O(h^{-2})$ as $\mu \lesssim h^{-1}$.*

18.8.3 Analysis in the Zone $\mathcal{Z}_{\mathsf{inn}}$

Let us consider more difficult inner zone; as $\mu \ge h^{-1}$ and we consider the Schrödinger-Pauli operator with $\mathfrak{z} = 1$. Then we can apply the same arguments as above but over one partition element ellipticity assumption

is violated for no more than 1 index n. Further, as $\ell \asymp \bar{\ell}$ (and then $\bar{\varrho}\bar{\ell} \asymp \mu h|\log h|$ contribution of the element to the asymptotics for given index n does not exceed $C\mu h^{-2}\bar{\varrho}\bar{\ell}^3 \asymp C\mu h^{-2}|\log h|\bar{\ell}^2$.

Therefore we arrive to

Proposition 18.8.6. *As $(l,\sigma) \preceq (2,0)$ contribution to the Tauberian remainder with $T = \bar{T} = \epsilon$ of the zone $\mathcal{Z}_{\mathsf{inn}}$ does not exceed*

$$C\mu h^{-1}\Big(\int_{\{\ell\ge C_0\bar{\ell}\}} \ell^{-1}\,dx'd\xi_2 + \bar{\ell}^{-1}|\log h| \int_{\{\ell\le C_0\bar{\ell}\}} dx'd\xi_2\Big) \tag{18.8.17}$$

and an extra approximation error does not exceed

$$C\mu h^{-2}\int_{\{\ell\ge C_0\bar{\ell}\}} \vartheta\big(h|\log h|/\varrho\big)\,dx'd\xi_2 + C\mu h^{-1}|\log h|\bar{\ell}^{-1}\int_{\{\ell\le C_0\bar{\ell}\}} dx'd\xi_2 \tag{18.8.18}$$

with $\ell := \min_n \ell_n$, $\varrho = \big(\vartheta(\ell)\big)^{\frac{1}{2}}$ and $\bar{\ell}$ defined by (18.8.4).

Then, under non-degeneracy assumptions (18.2.15) (as $(l,\sigma) \succeq (1,2)$) or (18.2.16) (as $(l,\sigma) = (2,0)$) both expressions (18.8.17) and (18.8.17) do not exceed $C\mu h^{-1}$ while in the general case it does not exceed $C\mu h^{-1}|\log h|\bar{\ell}^{-1}$. Combining with Proposition 18.8.4 as as $\mu h \lesssim 1$ we arrive to

Proposition 18.8.7. *(i) Let $\mu \lesssim h^{-1}$ and either $(l,\sigma) \succeq (1,2)$ and condition* (18.2.15) *be fulfilled or $(l,\sigma) \succeq (2,0)$ and condition* (18.2.16) *be fulfilled.*

Then both the Tauberian remainder and an additional approximation error do not exceed Ch^{-2}.

(ii) Let $\mu \gtrsim h^{-1}$ and either $(l,\sigma) \succeq (1,2)$ and condition (18.2.15)* *be fulfilled or $(l,\sigma) \succeq (2,0)$ and condition* (18.2.16)* *be fulfilled.*

Then both the Tauberian remainder and an additional approximation error do not exceed $C\mu h^{-1}$.

(iii) In the general case as $(\ell,\sigma) \preceq (2,0)$ both the Tauberian remainder and an additional approximation error do not exceed

$$Ch^{-2} + C\mu h^{-1}|\log h|\bar{\ell}^{-1}. \tag{18.8.19}$$

We leave to the reader the following moderately difficult problem:

Problem 18.8.8. Using rescaling arguments prove that as $(l,\sigma) \preceq (2,0)$ both contribution to the Tauberian remainder with $T = \bar{T} = \epsilon$ of the zone $\mathcal{Z}_{\mathsf{inn}}$ and extra approximation error do not exceed

$$Ch^{-2} + C\mu h^{-1}\bar{\ell}^{-1} \tag{18.8.19)*}$$

with $\bar{\ell}$ redefined by

$$\bar{\ell}\vartheta(\bar{\ell})^{\frac{1}{2}} = C_1 h. \tag{18.8.4)*}$$

where ε is also redefined as

$$\varepsilon := \varepsilon_{(n)} + \varepsilon'_{(n)}, \qquad \varepsilon'_{(n)} := Ch\varrho_{(n)}^{-1}\big(h/(\varrho_{(n)}\ell_{(n)})\big)^{-\delta'}; \tag{18.8.6)*}$$

We can generalize estimate (18.8.19)* to $(l,\sigma) \succ (2,0)$. Namely, as $(m,0) \preceq (l,\sigma) \prec (m+1,0)$ we introduce scaling functions

$$\ell = \ell_m := \epsilon \min\Big\{\eta : \ |\nabla^\alpha\big(W + (2n+1-\mathfrak{z})\mu h\big)| \le \vartheta(\eta)\eta^{-|\alpha|} \ \forall \alpha : |\alpha| \le m\Big\}, \tag{18.8.20}$$

$$\varrho = \vartheta(\ell)^{\frac{1}{2}}$$

and define $\bar{\ell}$ from (18.8.4)*.

Let us consider ℓ-admissible partition. Then we can apply the same arguments as before to cover zone $\{(x_2,x_3,\xi_2) : \varrho^2 \gtrsim \mu h\}$ and to prove that its contribution to the Tauberian remainder does not exceed Ch^{-2}.

Meanwhile contribution of zone $\{(x_2,x_3,\xi_2) : \ \ell \lesssim \bar{\ell}\}$ does not exceed $C\mu h^{-2}\bar{\varrho} \asymp C\mu h^{-1}|\log h|\bar{\ell}^{-1}$ with $\bar{\varrho} = \vartheta(\bar{\ell})^{\frac{1}{2}}$.

So, we need to cover zone $\{(x_2,x_3,\xi_2), \ell \ge C_0\bar{\ell}\}$. Consider particular element, and on this element

$$\sum_{\alpha:|\alpha|\le m} |\nabla^\alpha\big(W + (2n+1-\mathfrak{z})\mu h\big)|\ell^{|\alpha|} \asymp \varrho^2. \tag{18.8.21}$$

We will need

Proposition 18.8.9. *The contribution of each such element both to the Tauberian remainder and to an approximation error for two appropriate framing approximations do not exceed*

$$Ch^{-2}\ell^3 + C\mu h^{-1}\ell^2(h/\varrho\ell)^{-\delta} \tag{18.8.22}$$

with arbitrarily small exponent $\delta > 0$.

Proof. The proof is due to arguments of Section 5.2 based on the series of the scaling functions ℓ_k, ϱ_k with $k = m-1, \dots, 1$ defined by (18.8.20) with $\varrho(\eta) = \eta^{k+1}$ and ε_k defined by (18.8.6)* with $\delta' = \delta'_k > 0$, with the following modifications:

(a) When $\varrho_k^2 \ge \mu h$ (as $\mu h \le C_0$ only) we have an extra factor $\varrho_k^2/(\mu h)$ and therefore the first term in (18.8.21) carries no extra factor (unlike the second term).

(b) Due to factor $(h/\varrho\ell)^{-\delta}$ in the second term we do not need sophisticated arguments of Subsection 5.2.2 - the only ones where we needed a large smoothness.

Note that $\ell(h/\varrho\ell)^{\delta} \ge \bar{\ell}$. Really as δ is small enough this expression is monotone increasing function of ℓ; recall that $\ell \ge \bar{\ell}$. □

Therefore expression (18.8.22) does not exceed $C\big(h^{-2} + C\mu h^{-1}\bar{\ell}^{-1}\big)\ell^3$ and summation over partition returns $C\big(h^{-2} + C\mu h^{-1}\bar{\ell}^{-1}\big)$.

Remark 18.8.10. It follows from the analysis of the Problem 18.8.8 that under an extra non-degeneracy assumption

$$\sum_{\alpha:\ 1\le|\alpha|\le m} |\nabla^\alpha W| \ge \epsilon_0 \tag{18.8.23}_m$$

(as $\mu h \lesssim 1$) and

$$\sum_{\alpha:\ |\alpha|\le m} |\nabla^\alpha \big(W + (2n+1) - \mathfrak{z}\big)| \ge \epsilon_0 \tag{18.8.23}^*_m$$

(as $\mu h \gtrsim 1$) and with $(m,0) \prec (l,\sigma)$ both both contribution to the Tauberian remainder with $T = \bar{T} = \epsilon$ of the zone $\mathcal{Z}_{\mathsf{inn}}$ and extra approximation error do not exceed

$$Ch^{-2} + C\mu h^{-1-\delta}. \tag{18.8.24}$$

We leave to the reader the following moderately difficult problem:

Problem 18.8.11. Improve (18.8.24) to

$$Ch^{-2} + C\mu h|\log h|^{m-1}. \tag{18.8.25}_m$$

To do this one needs to improve (18.8.22) in an obvious way.

18.9 $d = 3$: Calculations and Main Theorems

18.9.1 Assembling what We Got

After corresponding estimates were derived in the previous section, following Subsection 13.5.2, we can rewrite (without increasing an error) the Tauberian expression

$$h^{-1}\int_{-\infty}^{0} F_{t\to h^{-1}\tau}\Big(\bar{\chi}_T(t)\Gamma\big(\psi U\big)\Big)\,d\tau$$

as

$$\frac{1}{4\pi^2}\mu h^{-2}\sum_n \int \theta\big(-\xi_3^2-(2n+1)\mu h-W(x_2,x_3,\xi_2)\big)\alpha(x_2,x_3,\xi_2,\xi_3)\,dx_2dx_3d\xi_2d\xi_3$$

which in turn we can rewrite as

$$\frac{1}{2\pi^2}\mu h^{-2}\sum_n \int \big(-(2n+1)\mu h - W(x_2,x_3,\xi_2)\big)_+^{\frac12}\bar{\alpha}(x_2,x_3,\xi_2)\,dx_2dx_3d\xi_2$$

and then as

$$\frac{1}{2\pi^2}\mu h^{-2}\sum_n \int \big(-(2n+1)\mu h - W\big)_+^{\frac12}(1+\alpha')\psi F\sqrt{g}\,dx$$

where now $W = W(x,\mu^{-1})$, which in turn we rewrite as the sum of the main part $h^{-3}\mathcal{N}^{\mathsf{MW}}\psi dx$ and the correction term

$$\frac{1}{2\pi^2}\mu h^{-2}\sum_n \int \big(-(2n+1)\mu h - W\big)_+^{\frac12}(1+\alpha')\psi F\sqrt{g}\,dx- \frac{1}{2\pi^2}\mu h^{-2}\sum_n \int \big(-(2n+1)\mu h - V/F\big)_+^{\frac12}\psi F\sqrt{g}\,dx. \tag{18.9.1}$$

Note that

(18.9.2) As $\mu \gtrsim (h|\log h|)^{-1}$ the correction term does not exceed the an approximation error and thus we can skip it.

Indeed, in this case $\varepsilon \gtrsim \mu^{-1}$. Therefore in this case we are done and our main Theorems 18.9.4 and 18.9.6 below are proven.

18.9.2 Correction Term

However, in the case $h^{-\frac{1}{3}} \le \mu \lesssim C(h|\log h|)^{-1}$ we need to provide an alternative expression and the estimate for the correction term (18.9.1).

Proposition 18.9.1. *Let* $(1,2) \preceq (I,\sigma) \preceq (2,0)$. *Then modulo* $O(h^{-2})$ *one can rewrite the correction term* (18.9.1) *as*

$$h^{-3}\mathcal{N}^{\mathsf{MW}}_{\mathsf{3corr}}(x) = \frac{1}{4\pi^2}\sum_{n\ge 0}\Big(\big(\tau - W - (2n+1)\mu hF\big)_+^{\frac{1}{2}} - \big(\tau - V - (2n+1)\mu hF\big)_+^{\frac{1}{2}}\Big)\mu h^{-1}F - \frac{1}{6\pi^2}\big(W_-^{\frac{3}{2}} - V_-^{\frac{3}{2}}\big)h^{-3}. \tag{18.9.3}$$

Proof. Consider

$$\frac{1}{2\pi^2}\mu h^{-2}\sum_n \int \big(-(2n+1)\mu h - W\big)_+^{\frac{1}{2}}\alpha'\psi F\sqrt{g}\,dx$$

and replace the Riemannian sum by the corresponding integral; then with an error not exceeding

$$C\vartheta(\mu^{-1})(\mu h)^{\frac{3}{2}}h^{-3} \ll \mu^{\frac{1}{2}}h^{-\frac{3}{2}}$$

we can replace it by

$$\frac{1}{6\pi^2}\mu h^{-2}\int W_-^{\frac{3}{2}}\alpha\psi F\sqrt{g}\,dx \tag{18.9.4}$$

where with the same error we can replace W by V.

Consider (18.9.1) without α' and replace all Riemannian sums by integrals; then we get the second term in the right-hand expression of (18.9.1) with the opposite sign, making an error $O\big((\mu h)^{\frac{3}{2}}h^{-3}\big)$.

For $\mu \le h^{-\frac{1}{3}}$ the sum of this two expressions should be $O(h^{-2})$ and therefore (18.9.4) must be equal to the right-hand expression of (18.9.1). Then it is true for $\mu \ge h^{-\frac{1}{3}}$. □

Now let us estimate the correction term.

Proposition 18.9.2. *(i) Under non-degeneracy assumption* (18.2.15) *as* $(l,\sigma)\preceq(2,0)$

$$h^{-3}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}\le C(\mu h)^{\frac12}h^{-3}\begin{cases}\vartheta(\mu h) & \text{as } \mu\le h^{-\frac12},\\ \vartheta(\mu^{-1}) & \text{as } h^{-\frac12}\le\mu\le h^{-1};\end{cases}\tag{18.9.5}$$

in particular, it is $O(h^{-2})$ *provided* $(l,\sigma)\succeq(\frac32,0)$.

(ii) Under non-degeneracy assumption (18.2.16) *as* $(l,\sigma)=(2,0)$ *the correction term can be ignored.*

(iii) In the general case

$$h^{-3}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}\le C(\mu h)^{\frac12}\vartheta(\mu^{-1})h^{-3}+C\mu h^{-2}(\vartheta(\mu^{-1}))^{\frac12};\tag{18.9.6}$$

in particular, it is $O(h^{-2})$ *provided* $(l,\sigma)\succeq(2,0)$.

Proof. (a) To prove Statements (i) and (i) we first restore ξ_3 replacing $z_+^{\frac12}$ by $\frac12\int\theta(z-\xi_3^2)d\xi_3$. Then for fixed ξ_3 we apply arguments of the proof of Proposition 18.5.3, recovering estimate (18.9.5) albeit without factor $(\mu h)^{\frac12}$.

Then integration over zone $\{|\xi_3|\le(\mu h)^{\frac12}\}$ results in the right-hand expression in (18.9.5).

On the other hand, in the zone $\{|\xi_3|\ge(\mu h)^{\frac12}\}$ we can "save" one derivative by integrating by parts over ξ_3 rather x, so there will be an extra factor $(\mu h/\xi_3^2)$ which after integration over ξ_3 results in $(\mu h)^{\frac12}$ again resulting in the right-hand expression in (18.9.5).

(b) Proof of Statement (iii) is trivial. □

18.9.3 Approximation Error

Similarly to to Proposition 18.9.2 one can prove the following proposition (we leave easy details to the reader):

Proposition 18.9.3. *(i) Under non-degeneracy assumption* (18.2.15) *as* $(l,\sigma)\preceq(2,0)$ *an approximation error does not exceed*

$$C(\mu h)^{\frac12}h^{-3}\vartheta(\bar\varepsilon) \qquad \text{as } \mu\le h^{-1}\tag{18.9.7}$$

$$C\mu h^{-2}\vartheta(\bar\varepsilon) \qquad \text{as } \mu\ge h^{-1}\tag{18.9.8}$$

with

$$\bar{\varepsilon} = (\mu^{-1}h)^{\frac{1}{2}}|\log \mu|; \tag{18.9.9}$$

therefore this error is $O\big((1+\mu h)h^{-2}\big)$ *as* $(l,\sigma) \succeq (1,2)$.

(ii) As $(l,\sigma) = (2,0)$ *an approximation error is* $O\big((1+\mu h)h^{-2}\big)$.

(iii) In the general case an approximation error does not exceed (modulo remainder estimate)

$$C(\mu h)^{\frac{1}{2}}\vartheta(\bar{\varepsilon})h^{-3} + C\vartheta(\bar{\varepsilon})^{\frac{1}{2}}\mu h^{-2} \tag{18.9.10}$$

where the first term is $O\big((1+\mu h)h^{-2}\big)$ *and the second term does not exceed* $C\mu h^{-1-2/(l+2)}|\log h|^{-\sigma/(l+2)}$ *(which is the part of the remainder estimate) for sure.*

18.9.4 Main Theorems

Getting all our beans together and getting rid of assumption $F = 1$ we arrive to the following theorem:

Theorem 18.9.4. *Let* $d = 3$ *and* A *be a self-adjoint in* $\mathscr{L}^2(X)$ *operator given by* (18.1.1). *Let conditions* (18.1.3), (18.1.6) *be fulfilled in* $B(0,1) \subset X \subset \mathbb{R}^3$ *with* $(1,2) \preceq (l,\sigma) \preceq (\bar{l},\bar{\sigma}) \succeq (2,1)$. *Let* <u>*either*</u> *non-degeneracy assumption* (18.2.15) *be fulfilled* <u>*or*</u> $(l,\sigma) \succeq (2,0)$ *and non-degeneracy assumption* (18.2.16) *be fulfilled.*

(i) As $h^{-\frac{1}{3}} \le \mu \le Ch^{-1}$ *under additional condition* (18.2.27) *for two framing approximations as in footnote*[18)] *asymptotics*

$$|\int \Big(\tilde{e}(x,x,0) - h^{-3}\mathcal{N}_3^{\mathsf{MW}}(x,0) - h^{-3}\mathcal{N}_{3,\mathsf{corr}}^{\mathsf{MW}}(x,0)\Big)\psi(x)\,dx| \le Ch^{-2} \tag{18.9.11}$$

holds with magnetic Weyl expression given by $(13.2.9)_{(3)}$ *and with correction term* (18.9.3), *satisfying estimate* (18.9.5).

(ii) As $h^{-1} \le \mu$ *we consider Schrödinger-Pauli operator with* $\mathfrak{z} = 1$ *for two framing approximations as in footnote*[18)] *asymptotics*

$$\mathsf{R}^{\mathsf{MW}} := |\int \Big(\tilde{e}(x,x,0) - h^{-3}\mathcal{N}_3^{\mathsf{MW}}(x,0)\Big)\psi(x)\,dx| \le C\mu h^{-1} \tag{18.9.12}$$

holds with magnetic Weyl expression given by $(13.2.9)_{(3)}$.

Remark 18.9.5. Using rescaling technique we will be able in the next subsection to get rid of condition (18.2.27) in this and in the next theorems.

Now we are going to drop non-degeneracy assumption as $d = 3$.

Theorem 18.9.6. *Let $d = 3$ and A be a self-adjoint in $\mathscr{L}^2(X)$ operator given by* (18.1.1). *Let conditions* (18.1.3), (18.1.6) *be fulfilled in $B(0,1) \subset X \subset \mathbb{R}^3$ with $(1,2) \preceq (l,\sigma) \preceq (\bar{l},\bar{\sigma}) \succeq (2,1)$. Then*

(i) As $h^{-\frac{1}{3}} \le \mu \le Ch^{-1}$ under additional condition (18.2.27) *for two framing approximations as in footnote*[18] *asymptotics*

$$|\int \Big(\tilde{e}(x,x,0) - h^{-3}\mathcal{N}^{\mathsf{MW}}(x,0) - h^{-3}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x)\Big)\psi(x)\,dx| \le Ch^{-2} + C\mu h^{-1-2/(l+2)}|\log h|^{-\sigma/(l+2)} \tag{18.9.13}$$

holds with the magnetic Weyl expression given by $(13.2.9)_{(3)}$ *and with the correction term* (18.9.3), *satisfying estimate* (18.9.6).

(ii) As $h^{-1} \le \mu$, $F = 1$ we consider Schrödinger-Pauli operator with $\mathfrak{z} = 1$ for two framing approximations as in footnote[18] *asymptotics*

$$|\int \Big(\tilde{e}(x,x,0) - h^{-3}\mathcal{N}^{\mathsf{MW}}(x,0)\Big)\psi(x)dx| \le C\mu h^{-1-2/(l+2)}|\log h|^{-\sigma/(l+2)} \tag{18.9.14}$$

holds with magnetic Weyl expression given by $(13.2.9)_{(3)}$.

Remark 18.9.7. (i) Note that the second term in the right-hand expression (18.9.14) prevails as $\mu \ge h^{-l/(l+2)}|\log h|^{\sigma/(l+2)}$.

(ii) Using Remark 18.8.10 one can prove that under non-degeneracy assumption $(18.8.23)_m$ as $\mu h \lesssim 1$ and under non-degeneracy assumption $(18.8.23)^*_m$ as $\mu h \gtrsim 1$ with $W = V/F$, $2 \le m$, $(m,0) \prec (l,\sigma)$ remainder estimate 18.8.24 holds for R^{MW}:

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{-2} + C\mu h^{-1-\delta}. \tag{18.9.15}$$

(iii) Furthermore, using problem 18.8.11 one can prove that in the same framework remainder estimate $(18.8.25)_m$ holds for R^{MW}:

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{-2} + C\mu h|\log h|^{m-1}. \tag{18.9.16}_m$$

Problem 18.9.8. Either improve estimates (18.9.5), (18.9.6) of Proposition 18.9.2 or construct counter-examples, showing that the improvement is impossible.

18.9.5 Generalizations

Vanishing V

For $d = 3$ we need condition (18.2.27) only for $\mu h \le 1$.

(i) Consider first Theorem 18.6.10 with $\mu \le h^{-\frac{1}{3}}$. Then applying the standard scaling with $\varrho = \gamma^{\frac{1}{2}}$ and $\bar{\gamma}_0 = \mu^{-2}$ we will find ourselves either in the case of condition (18.2.27) fulfilled or in the case of $\mu_{\mathsf{new}} = 1$ when it is not needed. Then contribution of each partition element to the remainder is $O\big(h_{\mathsf{new}}^{-2}\big) = O\big(h^{-2}\gamma^3\big)$ and then the total remainder is $O(h^{-2})$. Therefore

(18.9.17) Theorem 18.6.10 holds without condition (18.2.27).

(ii) Consider now $h^{-\frac{1}{3}} \le \mu \le h^{-1}$. Then in the non-degenerate case (under assumption (18.2.15) or (18.2.16)) the remainder estimate was $O(h^{-2})$; plugging h_{new} we get the contribution of such ball to the remainder estimate does not exceed $Ch^{-2}\varrho^2\gamma^2 \asymp Ch^{-2}\gamma^3$ which results in the total contribution of all such balls $O(h^{-2})$. So,

(18.9.18) Theorem 18.9.4 holds without condition (18.2.27).

(iii) Consider the general case; recall that then estimate contains an extra term

$$R_3 := C\mu h^{-1-2/(l+2)} |\log h|^{-\sigma/(l+2)}. \tag{18.9.19}$$

However, this term does not translates well: after we plug μ_{new}, h_{new} it produces

$$C\mu h^{l/(l+2)} h^{-2} |\log(h/\varrho\gamma)|^{-\sigma/(l+2)} \times \gamma^{3-(l-1)/(2(l+2))}$$

and after multiplication by γ^{-3} we have γ in the negative degree.

On the other hand, we can introduce a different kind of scaling. Namely, introduce γ defined by

$$\gamma = \min\bigl\{\eta : \ |V| \le C\vartheta(\eta),\ |\nabla V| \le C\eta^{-1}\vartheta(\eta)\bigr\}, \tag{18.9.20}$$

$$\varrho = \gamma^{\frac{l}{2}} |\log\gamma|^{-\frac{\sigma}{2}}$$

where we assume so far that $(l,\sigma) \preceq (2,0)$.

(a) If $|V| \asymp \gamma^l |\log\gamma|^{-\sigma}$ we are in the degenerate situation with V_{new} disjoint from 0.

Plugging h_{new} into Ch^{-2} we get $Ch^{-2}\varrho^2\gamma^2$ which does not exceed $Ch^{-2}\gamma^3$. Plugging h_{new}, μ_{new} into R_3 we get

$$C\mu h^{-1-2/(l+2)} |\log(\frac{h}{\varrho\gamma})|^{-\sigma/(l+2)} \times \varrho^{2/(l+2)}\gamma^{-l/(l+2)} \times \gamma^3 \asymp \tag{18.9.21}$$

$$R_3\Bigl(|\log(\frac{h}{\rho\gamma})| \cdot |\log\gamma|/|\log h|\Bigr)^{-\sigma/(l+2)} \times \gamma^3$$

which does not exceed $R_3\gamma^3$ as $\sigma \ge 0$ and $R_3'\gamma^3$ as $\sigma < 0$ with

$$R_3' := C\mu h^{-1-2/(l+2)} |\log h|^{-\sigma/(l+2)} |\log(\mu h)|^{-\sigma/(l+2)}. \tag{18.9.22}$$

(b) If $|\nabla V| \asymp \gamma^{l-1}|\log\gamma|^{-\sigma}$ then after rescaling we find ourselves in the non-degenerate situation with V_{new} not necessarily disjoint from 0. However, we already examined this before and found that contribution of the ball to the remainder estimate would not exceed Ch_{new}^{-2} which in turn does not exceed $Ch^{-2}\gamma^3$ and then the total contribution of such balls does not exceed Ch^{-2}. So,

(18.9.23) Let $(l,\sigma) \preceq (2,0)$. Then Theorem 18.9.6 holds without condition (18.2.27) as $\sigma \ge 0$ and with the last term in the right-hand expression of (18.9.13) replace by (18.9.22) as $\sigma < 0$.

Problem 18.9.9. (i) Get rid of assumption (18.2.27) in Remark 18.9.7(ii), (iii). To do this use a scaling functions

$$\gamma = \min\bigl\{\eta : \sum_{\alpha:\,|\alpha|\le l-1} |\nabla^\alpha V/F| \le C\eta^{-|\alpha|}\vartheta(\eta)\bigr\}, \tag{18.9.24}$$

$$\varrho = \gamma^{\frac{l}{2}} |\log\gamma|^{-\frac{\sigma}{2}}.$$

(ii) Using this get rid of assumption $(I,\sigma)\preceq(2,0)$ in claim (18.9.23).

(iii) Improve our arguments and prove that one can replace R_3' by R_3 even as $\sigma<0$.

Vanishing F

We assume that g^{jk}, F^k have an extra smoothness 1 in comparison which is required in theorems we refer to. This assumption allows us to avoid problems straightening magnetic field. Let us introduce

$$\gamma=\epsilon|F|+\bar\gamma,\qquad \bar\gamma=\mu^{-\frac12}. \tag{18.9.25}$$

Then after recalling we have $\mu_{\mathsf{new}}=\mu\gamma^2$ and $h_{\mathsf{new}}=h/\gamma$.

(i) Consider first case $\mu\le h^{-\frac13}$. Then $\mu_{\mathsf{new}}\le h_{\mathsf{new}}^{-\frac13}$ as well and the contribution of $B\bigl(x,\gamma(x)\bigr)$ to the remainder is $O(h_{\mathsf{new}}^{-2})=O(h^{-2}\gamma^2)$ and the remainder does not exceed

$$Ch^{-2}\int\gamma(x)^{-1}\,dx \tag{18.9.26}$$

which does not exceed $Ch^{-2}\mu^{\frac12}$ in the general case, and $Ch^{-2}|\log\mu$, $Ch^{-2}|\log\mu$ under assumptions

$$|\nabla \boldsymbol{F}/V|\ge\epsilon_0, \tag{18.9.27}_1$$

$$\sum_{j,k}|\Delta_{jk}(\nabla\otimes\boldsymbol{F}/V)|\ge\epsilon_0 \tag{18.9.27}_2$$

respectively where $\Delta_{jk}(\nabla\otimes\boldsymbol{F})$ denote 2×2 minors of the matrix $(\nabla\otimes\boldsymbol{F})=(\partial_{x_j}F^k)$.

Note that assumption $(18.9.27)_1$ implies that after rescaling assumption (18.2.15) is fulfilled.

(ii) Consider now case $h^{-\frac13}\le\mu\le h^{-1}$. Under condition $(18.9.27)_1$ contribution of $B\bigl(x,\gamma(x)\bigr)$ to the remainder is $O(h_{\mathsf{new}}^{-2})=O(h^{-2}\gamma^2)$ and then the total remainder is $O(h^{-2}\log\mu)$ and it is $O(h^{-2})$ under assumption $(18.9.27)_2$.

We leave to the reader rather standard and tedious

Problem 18.9.10. Consider the case when assumption $(18.9.27)_1$ fails.

(iii) Finally, for $h^{-1} \lesssim \mu \lesssim h^{-2}$ magnitude of the remainder and the principal part strongly depend on the behavior of $\mathsf{mes}\{x, F(x) \le \eta\}$ as $\eta \to +0$. In particular, in the generic case

$$|\det(\nabla \otimes \boldsymbol{F})| \ge \epsilon_0 \tag{18.9.27}_3$$

the remainder is $O(\mu^{-2}h^{-4})$ while the principal part is $O(\mu^{-3}h^{-6})$ - if we consider Schrödinger operator (then for $\mu \ge C_0 h^{-2}$ principal part is 0 and remainder is $O(\mu^{-\infty})$).

(iii)* On the other hand, for Schrödinger-Pauli operator with $\mathfrak{z} = 1$ under assumption $(18.9.27)_2$ the principal part is $O(\mu h^{-2})$ and the remainder is $O(\mu h^{-1})$ as $\mu \ge h^{-1}$.

Other Generalizations

Problem 18.9.11. Get rid of condition $B(0,1) \subset X$. Use the rescaling method and recover remainder estimate $O(h^{-1})$ as $\mu \lesssim h^{-2}$.

Problem 18.9.12. (i) Generalize our results to the generalized Schrödinger-Pauli operator (17.1.6).

(ii) Generalize our results to the generalized Dirac operators (17.1.1) and (17.1.5).

Here Part (i) is easy but Part (ii) looks rather challenging.
Remark 18.5.13 stays.

Chapter 19

Multidimensional Magnetic Schrödinger Operator. Full-Rank Case

19.1 Introduction

19.1.1 Preface

In this and the next chapters we consider multidimensional Schrödinger operator

$$A = A_0 + V(x), \qquad A_0 = \sum_{j,k\le d} P_j g^{jk}(x) P_k, \tag{19.1.1}$$

$$P_j = hD_j - \mu V_j(x), \quad h \in (0,1],\ \mu \ge 1.$$

It is characterized by the *magnetic field intensity tensors* (F_{jk}) with

$$F_{jk} = \partial_k V_j - \partial_j V_k, \tag{19.1.2}$$

which is skew-symmetric $d \times d$-matrix, and $(F^j_p) = (g^{jk})(F_{kp})$ which is equivalent to the skew-symmetric matrix $(g^{jk})^{\frac12}(F_{jk})(g^{jk})^{\frac12}$.

Compare this with 2D and 3D cases when we could characterize intensity by a (pseudo)scalar F and (pseudo)vector $\boldsymbol{F}$ respectively.

Then

(19.1.3) All eigenvalues of (F^j_k) are $\pm i f_p$ ($f_l > 0$, $p = 1, \dots, r$) and 0 of multiplicity $d - 2r$ where $2r = \mathsf{rank}(F^j_k)$.

In this chapter we assume that the magnetic field intensity tensor has a full rank at each point x:

$$\mathsf{rank}(F_k^j)(x) = 2r = d \quad \text{and} \quad |(F_k^j)^{-1}(x)| \le c_0 \tag{19.1.4}$$

and (under certain conditions) we derive sharp spectral asymptotics (with the remainder estimate $O\bigl(\mu^{-1}h^{1-d}\bigr)$ as $\mu h \lesssim 1$ while the principal part has a magnitude h^{-d}. As $\mu h \gtrsim 1$ we consider the corresponding Schrödinger-Pauli operator and derive remainder estimate $O\bigl(\mu^{r-1}h^{1-r}\bigr)$ hile the principal part has a magnitude $\mu^r h^{-r}$. The typical (but different from the general case) example (already studied in Chapters 13 and 18) is 2D magnetic Schrödinger operator with $r = 1$.

As usual, we consider operator in some domain or on some manifold X with some boundary conditions, assuming that it is self-adjoint in $\mathscr{L}^2(X)$ and denote by $e(x, y, \tau)$ Schwartz' kernel of it spectral projector.

As usual we assume that conditions (13.2.4) and (13.2.5) are fulfilled i.e.

$$\epsilon_0 \le \sum_{j,k} g^{jk}\eta_j\eta_k \cdot |\eta|^{-2} \le c \quad \forall \eta \in \mathbb{R}^d \setminus 0 \quad \forall x \in B(0,1) \tag{19.1.5}$$

and

$$X \supset B(0,1). \tag{19.1.6}$$

19.1.2 Canonical Form

Recall (see e.g. Subsection 13.2.2) that if $X = \mathbb{R}^d$ and g^{jk}, F_{jk}, V are constant then operator (19.1.1) is unitary equivalent to

$$\sum_{1\le j\le r} f_j(h^2D_j^2 + \mu^2 x_j^2) + \sum_{r+1\le j\le d-2r} h^2D_j^2 + V \tag{19.1.7}$$

and $e(x, x, \tau) = h^{-d}\mathcal{N}_d^{\mathsf{MW}}(\tau)$ is defined by (13.2.9); in particular, under condition (19.1.4) decomposition (19.1.7) becomes

$$\sum_{1\le j\le r} f_j(h^2D_j^2 + \mu^2 x_j^2) + V \tag{19.1.8}$$

and

$$h^{-d}\mathcal{N}_d^{\mathsf{MW}}(\tau) := (2\pi)^{-r}\mu^r h^{-r} \sum_{\alpha\in\mathbb{Z}^{+r}} \theta\Bigl(\tau - \sum_j (2\alpha_j+1)f_j\mu h - V\Bigr) f_1\cdots f_r\sqrt{g} \tag{19.1.9}$$

with $g = \det(g_{jk})$, $(g_{jk}) = (g^{jk})^{-1}$ and Heaviside function θ.

In particular,

(19.1.10) For the toy-model operator under condition (19.1.4) (and only in this case) the spectrum is pure-point (of the infinite multiplicity) consisting of *Landau levels*

$$E_\alpha := \sum_j (2\alpha_j + 1) f_j, \qquad \alpha \in \mathbb{Z}^{+r}. \tag{19.1.11}$$

Now in the general case (i.e. for $X \neq \mathbb{R}^d$, and variable V and, may be, g^{jk}, F_{jk}) we are interested in asymptotics of the spatially mollified spectral function, i.e.

$$\mathsf{N}^{\mathsf{MW}}_\psi := \int e(x,x,0)\psi(x)\,dx \tag{19.1.12}$$

as $h \to +0, \mu \to +\infty$ where ψ is a fixed function, smooth and compactly supported in X. As $\mu h \ge C_0$ the result will be trivial and therefore instead of the original Schrödinger operator we will need to consider the generalized Schrödinger-Pauli operator

$$A - \sum_j \mathfrak{z}_j f_j \mu h \tag{19.1.13}$$

with constants $\mathfrak{z}_j \in \mathbb{R}$.

The principal part of such asymptotics under some reasonable and obvious conditions is of magnitude h^{-2r} as $\mu h \lesssim 1$ and $\mu^r h^{-r}$ as $\mu h \gtrsim 1$.

On the other hand, the remainder estimate is $O(\mu h^{1-2r} + \mu^r h^{-r})$ unless one imposes some non-degeneracy assumption while under the strongest possible assumption one can expect remainder estimate $O\big(\mu^{-1}h^{1-2r} + \mu^{r-1}h^{1-r}\big)$.

Remark 19.1.1. However, even if g^{jk}, F_{jk}, V are constant the remainder estimate is as bad as μh^{1-d} (for $\mu h \ll 1$) only if $f_1, \dots, f_r$ are commensurable. Otherwise as $\mu h \lesssim 1$ the remainder estimate is between $O\big(\mu^r h^{-r}\big)$ and $O\big(\mu h^{1-2r}\big)$ depending on the non-commensurability of $f_1, \dots, f_r$.

Probably it is the same remainder estimate as an error in the "equality"

$$h^{-d}\mathcal{N}^{\mathsf{MW}}(0) \approx h^{-d}\mathcal{N}^{\mathsf{W}}_\infty(0) \tag{19.1.14}$$

where $h^{-d}\mathcal{N}_\infty^{\mathsf{W}}$ denotes Weyl approximation with many terms. Considering the left-hand expression as an r-dimensional Riemannian sum, passing to the integral and correcting it so if instead of θ there was a smooth function we would get an error $O(h^{-d}(\mu h)^\infty)$, we get a right-hand expression.

Also obviously all Hamiltonian trajectories are periodic if and only if $f_1, \dots, f_r$ are commensurable.

Problem 19.1.2. Explore error in (19.1.14) when $f_1, \dots, f_r$ are non-commensurable. It definitely would depend on how these numbers are non-commensurable (something related to their diophantine properties - as $r = 2$ it is related to Liouville's exponent for f_1/f_2).

Recall from Section 13.3 that in the smooth case for $d = 2$ one can reduce operator in question to the canonical form

$$\sum_{m+k+j\ge 1} \mu^{2-2k-2m-j} h^j a_{m,k,j}(x_2, \mu^{-1}hD_2)\big(h^2D_1^2 + \mu^2 x_1^2\big)^m \tag{19.1.15}$$

with $a_{1,0} = F\circ\Psi_0$, $a_{0,1} = V\circ\Psi_0$ and a certain diffeomorphism $\Psi_0 : T\mathbb{R}^1 \to \mathbb{R}^2$. Then one can replace harmonic oscillator $h^2D_1^2+\mu^2x_1^2$ by one of its eigenvalues $(2\alpha+1)\mu h$ with $\alpha \in \mathbb{Z}^+$; we ignore terms with $j \ge 1$.

Thus, for $\mu h \le \epsilon_0$ our operator looked like a family (with $C_0(\mu h)^{-1}$ elements) of 1-dimensional $\mu^{-1}h$-pseudodifferential operators with the principal symbols $\big((2\alpha+1)\mu hF + V\big)\circ\Psi_0$.

Spectral asymptotics for such operators are very sensitive to the degeneracy of the symbol but under the non-degeneracy condition $|b| + |\nabla b| \ge \epsilon_1$ with $b = a_{0,1}\lambda + a_{1,0}$ with $\lambda \in \mathbb{R}^+$ this asymptotics has a principal part of the magnitude μh^{-1} and a remainder $O(1)$. This non-degeneracy condition is equivalent to $|\nabla V/F| \ge \epsilon_1$ and the final asymptotics has its principal part of magnitude h^{-2} and a remainder $O(\mu^{-1}h^{-1})$ [1)].

On the other hand, as $\mu h \ge \epsilon_0$ we had the family of no more than C_0 of $\mu^{-1}h$-pseudodifferential operators, then the non-degeneracy condition became $|(2\alpha+1)\mu h + (V/F)| + |\nabla(V/F)| \ge \epsilon_1$ and the final asymptotics has its principal part of magnitude μh^{-1} and a remainder $O(1)$ [1)].

Non-smoothness prevented us from the complete canonical form but we had a "poor man" canonical form which was sufficient; see Section 18.3.

[1)] We could also consider a weaker non-degeneracy condition invoking second derivatives.

Multidimensional case is much more tricky. Under assumption (19.1.4) one could expect a canonical form

$$\sum_{m\in\mathbb{Z}^{+r},k,j:|m|+k+j\ge 1} \mu^{2-2k-2|m|-j}h^j a_{m,k,j}(x'',\mu^{-1}hD'')H^m \tag{19.1.16}$$

with $H=(H_1,\dots,H_r)$, harmonic oscillators $H_i=h^2D_j^2+\mu^2x_i^2$ $a_{(m,0,0)}=f_j\circ\Psi_0$ as $|m|=0$ with $m_i=\delta_{ji}$ and $a_{(0,1,0)}=V\circ\Psi_0$ where now we consider diffeomorphism $\Psi_0: T\mathbb{R}^r=\mathbb{R}^{2r}\to\mathbb{R}^{2r}$ with $x''=(x_{r+1},\dots,x_{2r})$.

In this case for $\mu h\le\epsilon_0$ one would get a family (with $C_0(\mu h)^{-r}$ elements) of r-dimensional $\mu^{-1}h$-pseudodifferential operators, and *for each of them* the principal part of asymptotics would be of magnitude $\mu^r h^{-r}$, and under proper non-degeneracy assumption remainder $O(\mu^{r-1}h^{1-r})$ and the final asymptotics has its principal part of the magnitude $\mu^r h^{-r}\times(\mu h)^{-r}h^{-2r}=h^{-2r}$ and a remainder $O(\mu^{r-1}h^{1-r}\times(\mu h)^{-r}h^{-2r})=O(\mu^{-1}h^{1-2r})$.

If $\mu h\ge\epsilon_0$ and we consider operatore (19.1.13) one would have the family of no more than C_0 of $\mu^{-1}h$-pseudodifferential operators, under proper non-degeneracy condition $|b|+\nabla b|\ge\epsilon_1$, $b:=\sum_j(2\alpha_j+1)\mu h f_j+V$, the final asymptotics has its principal part of magnitude $\mu^r h^{-r}$ and a remainder $O(\mu^{-1}h^{1-r})$.

Remark 19.1.3. As $f_j=\mathsf{const}$ this non-degeneracy assumption becomes $|\nabla V|\ge\epsilon_0$ and we will be able to replace it by a weaker assumption $|\nabla V|\le\epsilon_0 \implies |\det \mathsf{Hess}\, V|\ge\epsilon_0$.

19.1.3 Resonances

However, the resonances prevent us from reducing our operator to the desired canonical form (19.1.6) even in the smooth case. In fact, *m-th order resonances*

$$\sum_j f_j\gamma_j=0 \qquad \text{as } \gamma\in\mathbb{Z}^r \text{ and } |\gamma|:=\sum_j|\gamma_j|=m\ge 2 \tag{19.1.17}$$

prevent us from reducing properly terms

$$\sum_{\alpha,\beta:|\alpha|+|\beta|=m;\ k,j} \mu^{2-2k-|\alpha|-|\beta|-j}h^j a_{\alpha,\beta;k,j}(x'',\mu^{-1}hD'')(hD')^\alpha(\mu x')^\beta. \tag{19.1.18}$$

In particular, due to the 2-*nd order resonances* $f_j \approx f_k$ ($k \neq j$) we can reduce the main part of operator only to

$$\sum_{\mathfrak{m}\in\mathfrak{M}} \sum_{j,k\in\mathfrak{m}} a_{jk}(x'', \mu^{-1}hD'')Z_j^* Z_k + a_0(x'', \mu^{-1}hD'') \tag{19.1.19}$$

with $Z_k = hD_k + i\mu x_k$ where $\mathfrak{m} \in \mathfrak{M}$ are disjoint subsets of $\{1, \dots, r\}$ and eigenvalues of each of matrices $(a_{jk})_{j,k\in\mathfrak{m}}$ are close to one another (and to f_j). This leads to the necessity of the matrix rather than the scalar non-degeneracy (microhyperbolicity) condition.

Further, the 3-*rd order resonances* $f_i \approx f_j + f_k$ (with possible $j = k$) prevent us from getting rid of the terms

$$\mu^{-1} a_{jkl}(x'', \mu^{-1}hD'')Z_j^* Z_k Z_l \tag{19.1.20}$$

and their adjoints

$$\mu^{-1} a_{jkl}^*(x'', \mu^{-1}hD'')Z_j Z_k^* Z_l^* \tag{19.1.20}^*$$

and these terms appear as the perturbations of the main part of the operator unless F_{jk} and g^{jk} are constant.

Furthermore, the fourth order resonances $f_i = f_j + f_k + f_l$ and $f_i + f_j = f_k + f_l$ where i, j, k, l may coincide but in the second case $(i, j) \neq (k, l)$ leave us with the whole bunch of terms instead of just $\mu^{-2} a_{jk} H_j H_k$ and but these terms are smaller than those produced by the third order resonances and rather harmless.

More precisely, the symbol of the reduced operator is

$$\mu^2 \sum_{\mathfrak{m}\in\mathfrak{M}} \sum_{j,k\in\mathfrak{m}} a_{jk}\zeta_j^\dagger \zeta_k + a_0 + \mu^2 \sum_{\mathfrak{n}\in\mathfrak{N}} \sum_{j,k,l\in\mathfrak{n}} \operatorname{Re} a_{jkl}\zeta_j^\dagger \zeta_k \zeta_l + \dots \tag{19.1.21}$$

where $\zeta_j = \xi_j + ix_j$ are symbols of $\mu^{-1}h$ differential operators $\mu^{-1}Z_j$ and precise definition of $\mathfrak{N}$ will appear later. Recall that † means a complex conjugation for scalars and a Hermitian conjugation for matrices.

Remark 19.1.4. (i) Resonances of m-th order become important only as smoothness is large enough (at least $(m, 0)$).

(ii) If we assume that g^{jk} and F_{jk} are constant then m-th order resonances affect only terms (19.1.18) without factor μ^2 i.e.

$$\sum_{\alpha,\beta:|\alpha|+|\beta|=m;\ k,j} \mu^{-2k-|\alpha|-|\beta|-j} h^j a_{\alpha,\beta;k,j}(x'', \mu^{-1}hD'')(hD')^\alpha (\mu x')^\beta. \tag{19.1.18}'$$

19.1.4 Dynamics and Microhyperbolicity

If operator in its canonical form is

$$\sum_{1\le j\le r} f_j(x'',\mu^{-1}hD'')(h^2D_j+\mu^2x_j^2)+V(x'',\mu^{-1}hD'') \tag{19.1.22}$$

then in the classical dynamics $(\xi_j^2+\mu^2x_j)^2=\rho_j=\mathsf{const}$ and dynamics in (x',ξ') is described by the Hamiltonian

$$b(x'',\xi'',\rho_1,\dots,\rho_r)\coloneqq \sum_{1\le j\le r} f_j(x'',\xi'')\rho_j+V(x'',\xi''); \tag{19.1.23}$$

where with respect to $x''=(x_{r+1},\dots,x_d)$ we use $\hbar$-quantization with $\hbar=\mu^{-1}h$ and therefore this dynamics is

$$\frac{dx''}{dt}=\mu^{-1}\nabla_{\xi''}b(x'',\xi''),\qquad \frac{d\xi''}{dt}=-\mu^{-1}\nabla_{x''}b(x'',\xi'') \tag{19.1.24}$$

$$b(x'',\xi'')=\tau$$

and therefore the microhyperbolicity assumption $|b-\tau|+|\nabla b|\ge\epsilon_0$ depends on $\boldsymbol{\rho}$; for each $\boldsymbol{\rho}$ with $|b(x'',\xi'',\boldsymbol{\rho})-\tau|\le\epsilon_0$ there exists direction $\ell=\ell(\boldsymbol{\rho})$ such that $\langle\ell,\nabla\rangle b(x'',\xi'',\boldsymbol{\rho})\ge\epsilon_0$ and we know that this is preserved until time $T^*=\epsilon\mu$.

However, if there are 3-rd order resonances situation becomes more complicated: ρ_j are evolving with the speed $O(1)$ and we can take only $T^*=\epsilon$ (which leads to the less sharp remainder estimate).

To remedy this situation we may assume that ℓ does not depend on $\boldsymbol{\rho}$ which is the case when $f_j=\mathsf{const}$ (then microhyperbolicity condition is $|\nabla V|\ge\epsilon_0$). More generally, we need to assume that $\{1,\dots,r\}$ is partitioned into subsets $\mathfrak{n}\in\mathfrak{N}$ such that 2-nd and 3-rd order resonances involve indices from the same subset $\mathfrak{n}$ only, and that ℓ depends on $\boldsymbol{\lambda}=(\lambda_{\mathfrak{n}})_{\mathfrak{n}\in\mathfrak{N}}$ with $\lambda_{\mathfrak{n}}=\sum_{j\in\mathfrak{n}}f_j\rho_j$ rather than on $\boldsymbol{\rho}$.

More general definitions of the microhyperbolicity and $\mathfrak{N}$-microhyperbolicity working also in the cases when even the main part is not in the form (19.1.22) will be given later (see Definitions 19.2.4 and 19.2.5).

19.1.5 Regularity Assumptions and Mollification

We assume that

$$g^{jk}\in \mathscr{C}^{\bar l,\bar\sigma},\qquad V\in\mathscr{C}^{l,\sigma}, \tag{19.1.25}$$

and

$$V_j = \partial_j \phi_j, \qquad \phi_j \in \mathscr{C}^{\bar{l}+2,\bar{\sigma}} \tag{19.1.25}_2$$

where the last assumption is a bit stronger than more natural $F_{jk} \in \mathscr{C}^{\bar{l},\bar{\sigma}}$ or $V_{\in} \mathscr{C}^{\bar{l}+1,\bar{\sigma}}$.

Also due to problems appearing when we reduce to a canonical form, mollification parameter in the intermediate magnetic field case is $\varepsilon = C\mu^{-1}$ rather than $\varepsilon = C(\mu^{-1}h|\log h|)^{\frac{1}{2}}$ as in the previous Chapter 18, therefore the threshold between weak magnetic field ($\varepsilon = C\mu h|\log h|$) and intermediate one is $\mu = (h|\log h)^{-\frac{1}{2}}$ rather than $\mu = (h|\log h)^{-\frac{1}{3}}$ and we also need to assume a larger regularity than there.

19.1.6 Plan of the Chapter

This Chapter consists of six more Sections.

As usual, we start from the Section 19.2 devoted to the weak magnetic field when asymptotics is defined by evolution to time $T = \epsilon\mu^{-1}$. We also study the classical dynamics here.

Again, following the standard scheme, Section 19.3 is devoted to the canonical form, in Section 19.4 we consider the case of the temperate magnetic field $\mu \le \epsilon(h|\log h|)^{-1}$ (overlapping with Section 19.2). Next, in Section 19.5 we consider the case of the intermediate magnetic field $\epsilon(h|\log h|)^{-1}\mu \le \epsilon h^{-1}$.

In the general case we assume some microhyperbolicity assumption (see Definition 19.2.4 but there is also stronger $\mathfrak{N}$-microhyperbolicity, see Definition 19.2.5). If there are no 2-nd order resonances, then this condition means that $\lambda_1\nabla(V/f_1) + ... + \lambda_r\nabla(V/f_r) \ne 0$ as long as $\lambda_1 \ge 0, ..., \lambda_r \ge 0$ and $\lambda_1 + ... + \lambda_r > 0$.

We also consider case of constant g^{jk}, F_{jk} and in this case instead of the microhyperbolicity assumption $\nabla V \ne 0$ we assume a weaker non-degeneracy assumption $\nabla V = 0 \implies \det \mathsf{Hess}\, V \ne 0$.

Finally, for the main course, in Section 19.6 we consider two cases of the strong magnetic field $\epsilon h^{-1} \le \mu \le Ch^{-1}$ and $\mu \ge Ch^{-1}$ respectively.

Furthermore, in Section 19.7 we consider the case when the rank of $\{\nabla(V/f_1), ..., \nabla(V/f_r)\}$ is less than r (usually $(r-1)$) at some points. There we assume that coefficients are regular.

19.2 Weak Magnetic Field

As we mentioned, in the weak magnetic field we consider original (x,ξ) coordinates. As we explained, we can take x-scale exactly as in Chapter 18

$$\varepsilon = C\mu h|\log h| \tag{19.2.1}$$

and therefore to satisfy logarithmic uncertainty principle we can take ξ-scale μ^{-1} [2)]. By no means ε is larger than $C(\mu^{-1}h|\log h|)^{\frac{1}{2}}$ needed to consider $\mu^{-1}h$-pseudodifferential operator with the symbol smooth in ε-scale in both x and ξ and therefore we cannot make a reduction here; only a some kind of the quasireduction as in Chapter 18 in the case the of the weak magnetic field is possible.

19.2.1 Heuristics. Classical dynamics

Smooth theory. Classical Dynamics

Here we are interested in the classical dynamics generated by symbol $a(x,\xi) \coloneqq \mu^2 a^0(x,\xi) + V(x)$ with

$$a^0(x,\xi) \coloneqq \sum_{j,k} g^{jk} p_j p_k, \qquad p_j = \mu^{-1}\xi_j - V_j(x) \tag{19.2.2}$$

on the energy levels close to 0 for time $t : |t| \le T^*$ with $T^* = \epsilon\mu$. On the energy levels below c_0 we have

$$|p_j| \le C\mu^{-1}. \tag{19.2.3}$$

At this heuristic stage we will not look at the smoothness (thus assuming that l is large enough).

First of all, exactly as in the previous Chapter 18, we can find $\phi_{jk} \in \mathscr{F}^{\bar{l},\bar{\sigma}}$ (see Remark 18.3.2) such that for $q_j \coloneqq x_j - \sum_k \phi_{jk}(x)p_k$

$$\{p_m, q_j\} = O(\mu^{-2}). \tag{19.2.4}$$

Really,

$$\{p_m, p_k\} = \mu^{-1}F_{km} \tag{19.2.5}$$

[2)] Recall that we consider $\mu^{-1}h$-differential operators.

and therefore $\{p_m, q_j\} = \mu^{-1}\delta_{mj} - \mu^{-1}\sum_{j,k}\phi_{jk}F_{km}$ and one needs to take

$$(\phi_{jk}) = (F_{jk})^{-1}. \tag{19.2.6}$$

Then one can see easily that

$$\{q_j, q_k\} \equiv \{x_j, q_k\} \equiv \mu^{-1}\phi_{jk} \qquad \mathsf{mod}\ O(\mu^{-2}) \tag{19.2.7}$$

and therefore

$$\mu^2\{a^0, q_j\} = O(\mu^{-1}), \qquad \{V, q_j\} = O(\mu^{-1}) \tag{19.2.8}$$

which implies that

(19.2.9) In the classical evolution at energy levels below c_0 for time $t : |t| \le \epsilon\mu$ increment of q_j does not exceed $C\mu^{-1}|t| \le \epsilon_1$;

(19.2.10) In the classical evolution at energy levels below c_0 for time $t : |t| \le \epsilon\mu$ increment of x_j does not exceed $C\mu^{-1}(|t|+1) \le \epsilon_1$.

One can see easily that in the case of constant g^{jk}, F_{jk}

$$\mu^2\{a^0, q_j\} = 0, \qquad \{V, q_j\} = \mu^{-1}\sum_k \phi_{jk}\partial_k V \tag{19.2.8}'$$

and therefore

(19.2.11) In the case of constant g^{jk}, F_{jk} the classical evolution for time $t : |t| \le \epsilon\mu$ is approximately described by equation[3)]

$$\frac{d}{dt}x_j = \mu^{-1}\sum_k \phi_{jk}\partial_k V \tag{19.2.12}$$

along which V is preserved.

To describe evolution more precisely in the general case let us consider a point $\bar{x}$. Consider $\mathfrak{M}(\bar{x})$ a partition of $\{1, \dots, r\}$ such that $|(f_i - f_j)(\bar{x})| \le \epsilon$ iff i, j belong to the same element $\mathfrak{m}$ of this partition (so 2-nd order resonances involve indices from the same subset $\mathfrak{m}$ only).

[3)] This equation describes approximately coordinates x_j but *not* velocities.

Then we can introduce

$$\zeta_j(x,\xi) = \sum_{1\le k\le 2r} \alpha_{jk}(x) p_k(x,\xi), \qquad j = 1, \dots, r \tag{19.2.13}$$

with complex coefficients $\alpha_{jk} \in \mathscr{F}^{\bar{I},\bar{\sigma}}$ such that

$$\{\zeta_j, \zeta_k\} = 0, \quad \{\zeta_j^\dagger, \zeta_k\} = 2i\mu^{-1}\delta_{jk} \quad \text{at} \ \ \Sigma^0 := \big\{(x,\xi) : \ p_1 = \dots = p_{2r} = 0\big\} \tag{19.2.14}$$

where as usual † means complex conjugation, and therefore

$$\{\zeta_j, \zeta_k\} \equiv 0, \quad \{\zeta_j^\dagger, \zeta_k\} \equiv 2i\mu^{-1}\delta_{jk} \qquad \mathsf{mod}\ O(\mu^{-2}) \quad \text{as} \ \ |p| \le c\mu^{-1} \tag{19.2.15}$$

and also

$$a^0 = \frac{1}{2}\mu^2 \sum_{\mathfrak{m}} \sum_{j,k\in\mathfrak{m}} a_{jk}(x)\zeta_j^\dagger(x,\xi)\zeta_k(x,\xi) \tag{19.2.16}$$

with Hermitian matrices $\big(a_{jk}\big)_{j,k\in\mathfrak{m}}$ close to scalar matrices $f_{\mathfrak{m}} I_{\#\mathfrak{m}}$ of the same dimensions.

Note that Σ^0 is a symplectic manifold and $\mathbb{R}^d \ni x$ inherits its symplectic form, a volume form, and a Hamiltonian field (we call it "Liouvillian field") respectively:

$$\frac{1}{2}\sum_{j,k} F_{jk}\, dx_j \wedge dx_k, \tag{19.2.17}$$

$$|\det F|^{\frac{1}{2}}\, dx_1 \wedge \dots \wedge dx_d = f_1 \cdots f_r \sqrt{g}\, dx, \tag{19.2.18}$$

$$\mathcal{L}_\psi = -\sum_{j,k} \phi_{jk}(\partial_j \psi)\partial_k. \tag{19.2.19}$$

Note also that

$$a^0_{\mathfrak{m}} := \frac{1}{2}\mu^2 \sum_{j,k\in\mathfrak{m}} a_{jk}(x)\zeta_j^\dagger(x,\xi)\zeta_k(x,\xi) \tag{19.2.20}$$

satisfy

$$\{a^0, a^0_{\mathfrak{m}}\} = O(1), \qquad \{V, a^0_{\mathfrak{m}}\} = O(1). \tag{19.2.21}$$

Therefore an increment of $a^0_{\mathfrak{m}}$ for time t does not exceed $C|t|$ and thus $a^0_{\mathfrak{m}}$ are *moderate* but not necessarily *slow variables* and their increment for $t = T^* \asymp \mu$ could be rather large.

Now we can replace ζ_j by

$$\zeta_{j,\mathsf{new}} = \zeta_j - \sum_{kl} \beta_{jkm}(x)\eta_k\eta_m \tag{19.2.22}$$

with $\beta_{jkm} \in \mathscr{F}^{\bar{l}-1,\bar{\sigma}}$ to make

$$\{\zeta_j, \zeta_k\} \equiv 0, \quad \{\zeta_j^\dagger, \zeta_k\} \equiv 2i\mu^{-1}\delta_{jk} \qquad \mathsf{mod}\ O(\mu^{-3}) \tag{19.2.23}$$

where for the sake of simplicity we use notations $\eta_{2k-1} \coloneqq \mathsf{Re}\,\zeta_k$, $\eta_{2k} \coloneqq \mathsf{Im}\,\zeta_k$, $k = 1, \dots, r$.

Then

$$a^0 = \frac{1}{2}\mu^2 \sum_{\mathfrak{m}} \sum_{j,k\in\mathfrak{m}} a_{jk}(q)\zeta_j^\dagger\zeta_k + \frac{1}{3}\mu^2 \sum_{j,k,m} a_{jkl}\,\eta_j\eta_k\eta_m + \dots, \tag{19.2.24}$$

$$V = V(q) + \sum_j a_j(q)p_j + \dots \tag{19.2.25}$$

where dots denote terms with $O(\mu^{-1})$ gradients.

These cubic terms in (19.2.24) and linear terms in (19.2.25) are the only sources of the trouble because Poisson brackets $\{\tilde{a}^0_{\mathfrak{m}}, \tilde{a}^0_{\mathfrak{m}'}\} = O(\mu^{-1})$ and $\{\tilde{a}^0_{\mathfrak{m}}, V(q)\} = O(\mu^{-1})$ where $\tilde{a}^0_{\mathfrak{m}}$ are defined by (19.2.20) with $a_{jk}(x)$ replaced by $a_{jk}(q)$.

To get rid of these terms we can redefine ζ_j again replacing them by $\zeta_j + \{S, \zeta_j\}$ with

$$S = \mu \sum_{k,m,n} \gamma_{kmn}\eta_k\eta_m\eta_n + \mu^{-1} \sum_m \gamma_m\eta_m. \tag{19.2.26}$$

This replacement preserves (19.2.14), (19.2.15) and (19.2.19) and, modulo terms with $O(\mu^{-1})$ gradients, it is equivalent to the replacement of $a^0 + V$ by

$$a^0 + V + \{S, a^0 + V\}. \tag{19.2.27}$$

We can always choose γ_j killing a_j and if there is no 3-rd order resonances we can choose γ_{jkm} killing a_{jkm}; in the general case we can reduce $a^0 + V$ to

$$\sum_{\mathfrak{m}} \tilde{a}^0_{\mathfrak{m}} + \mu^2\,\mathsf{Re} \sum_{j,k,m:|f_j - f_k - f_m|\le\epsilon} a_{jkm}(q)\,\zeta_j^\dagger\zeta_k\zeta_m + V(q). \tag{19.2.28}$$

Example 19.2.1. Consider $p_j = \xi_j$ as $j = 1, \dots, r$ and $p_{j+r} = \xi_{r+j} + x_j$; then $\zeta_j = \xi_j + i(\xi_{r+j} + x_j)$. Let consider a symbol which is the real part of the quadratic form of $(\zeta_1, \zeta_1^\dagger, \dots \zeta_r, \zeta_r^\dagger)$ with a linear coefficients coinciding at $x = 0$ with $\sum_j f_j \zeta_j^\dagger \zeta_j$ with constant $f_1, \dots, f_r$.

Let the only term with non-constant coefficients be

$$2\,\mathsf{Re}\big(\alpha x_1 \zeta_2 \zeta_3\big) = \mathsf{Re}\big(\alpha i(\zeta_1^\dagger - \zeta_1 + 2i\xi_{r+1})\zeta_2\zeta_3\big) = \mathsf{Re}\big(\alpha i \zeta_1^\dagger \zeta_2 \zeta_3\big) - \mathsf{Re}\big(\alpha i \zeta_1 \zeta_2 \zeta_3\big) - 2\,\mathsf{Re}\big(\alpha \xi_{r+j} \zeta_2 \zeta_3\big)$$

with $\alpha \in \mathbb{C}$. Here ξ_{1+r} is nothing but a constant parameter in the dynamics; let pick it equal to 0.

Consider two important cases:

(i) $r = 2$, $\zeta_3 := \zeta_2$; one can see easily that we arrive exactly to the case with non-zero cubic terms (the last term in the right-hand expression could be removed but not the first one as $f_1 = 2f_2$).

(ii) $r = 3$; one can see easily that we arrive exactly to the case with non-zero cubic terms (the last term in the right-hand expression could be removed but not the first one as $f_1 = f_2 + f_3$).

Example 19.2.2. (i) Consider

$$\mu^{-2} a = f_1 \zeta_1^\dagger \zeta_1^\dagger + f_2 \zeta_2^\dagger \zeta_2^\dagger + \mathsf{Re}\big(\alpha \zeta_1^\dagger \zeta_2^2\big)$$

with $f_1 = 2f_2$; let $\varrho_j = \mu^2 \zeta_j^\dagger \zeta_j$; then

$$\frac{d\varrho_2}{dt} = \{a, \varrho_2\} = \mu^4\, \mathsf{Re}\{\alpha \zeta_1^\dagger \zeta_2^2, \zeta_2^\dagger \zeta_2\} = -4\mu^3\, \mathsf{Re}\big(\alpha i \zeta_1^\dagger \zeta_2^2\big);$$

$$\frac{d}{dt}\mu^3\, \mathsf{Re}(\alpha \zeta_1^\dagger \zeta_2^2) = \{a, \mu^3\, \mathsf{Re}(\alpha \zeta_1^\dagger \zeta_2^2)\} = 0 \implies \mu^3\, \mathsf{Re}(\alpha \zeta_1^\dagger \zeta_2^2) = |\alpha|\sigma (= \mathsf{const})$$

where σ can accept any value in the interval $[-\varrho_1^{\frac{1}{2}} \varrho_2, \varrho_1^{\frac{1}{2}} \varrho_2]$ and then

$$\frac{d^2\varrho_2}{dt^2} = \{a, \{a, \varrho_2\}\} = -4\mu^5 \{\mathsf{Re}(\alpha \zeta_1^\dagger \zeta_2^2), \mathsf{Re}(\alpha i \zeta_1^\dagger \zeta_2^2)\} = -2|\alpha|^2 \varrho_2(\varrho_2 - 4\varrho_1);$$

therefore as $2\varrho_1 + \varrho_2 = \lambda = (\mathsf{const})$ we arrive to

$$\frac{d^2\varrho_2}{dt^2} = -2|\alpha|^2(3\varrho_2^2 - 2\lambda). \tag{19.2.29}$$

Equation (19.2.29) describes a kind of oscillations in the real time in the interval

$$J_\lambda = \{\varrho_2 \ge 0,\ \ \varrho_2^3 - \lambda\varrho_2^2 + 2\sigma^2 \le 0\}.$$

Thus $a_j^0 = \varrho_j$ are really moderate but not slow variables.

(ii) Consider

$$\mu^{-2}a = f_1\zeta_1^\dagger\zeta_1^\dagger + f_2\zeta_2^\dagger\zeta_2^\dagger + f_3\zeta_3^\dagger\zeta_3^\dagger + \mathsf{Re}\big(\alpha\zeta_1^\dagger\zeta_2\zeta_3\big)$$

with $f_1 = f_2 + f_3$; let $\varrho_j = \mu^2\zeta_j^\dagger\zeta_j$; then for $j = 2, 3$

$$\frac{d\varrho_j}{dt} = \{a, \varrho_j\} = \mu^4\,\mathsf{Re}\{\alpha\zeta_1^\dagger\zeta_2\zeta_3, \zeta_j^\dagger\zeta_j\} = -2\mu^3\,\mathsf{Re}\big(\alpha i\zeta_1^\dagger\zeta_2\zeta_3\big)$$

and again $\mu^3\,\mathsf{Re}(\alpha\zeta_1^\dagger\zeta_2\zeta_3) = |\alpha|\sigma$ $(= \mathsf{const})$ where σ can accept any value in the interval $[-(\varrho_1\varrho_2\varrho_3)^{\frac12}, (\varrho_1\varrho_2\varrho_3)^{\frac12}]$

$$\frac{d^2\varrho_2}{dt^2} = \{a, \{a, \varrho_j\}\} = -2\mu^5\{\mathsf{Re}(\alpha\zeta_1^\dagger\zeta_2\zeta_3), \mathsf{Re}(\alpha i\zeta_1^\dagger\zeta_2\zeta_3)\} = \\ - |\alpha|^2(\varrho_2\varrho_3 - \varrho_1\varrho_2 - \varrho_1\varrho_3).$$

Due to the first equation with $j = 2, 3$ $\varrho_2 - \varrho_3 = \mathsf{const}$ and $\varrho_{2,3} = \varrho \pm \eta$ with constant η and $\varrho = \frac12(\varrho_2 + \varrho_3)$; therefore as $f_1\varrho_1 + f_2\varrho_2 + f_3\varrho_3 = \mathsf{const}$ we conclude that $\varrho_1 = -\varrho + \lambda$ and arrive to equation

$$\frac{d^2\varrho}{dt^2} = -|\alpha|^2(3\varrho^2 - 2\lambda\varrho + \eta^2). \tag{19.2.30}$$

Again, equation (19.2.30) describes a kind of oscillations in the real time in the interval

$$J_{\lambda,\eta} = \{\varrho \ge 0,\ \ \varrho^3 - \lambda\varrho^2 + \eta^2\varrho - \lambda\eta^2 + \sigma^2 \le 0\}.$$

Thus $a_j^0 = \varrho_j$ again are really moderate but not slow variables.

To have a_j^0 as slow variables we need either to assume that there are no cubic terms (i.e. either F_{jk}, g^{jk} are constant or there are no third order resonances) or to group them together.

The important role will be played by partition $\mathfrak{N} = \mathfrak{N}_\epsilon(\bar{x})$ of $\{1, \dots, r\}$ such that

(19.2.31) $j \neq k$ belong to the same element $\mathfrak{n} \in \mathfrak{N}$ if there exists l such that either $|f_j - f_k - f_l| \le \epsilon$ or $|f_k - f_j - f_l| \le \epsilon$ or $|f_l - f_j - f_k| \le \epsilon$.

Remark 19.2.3. (i) Then, according to this definition l also belongs to the same element $\mathfrak{n}$. We do not exclude $l = j$ or $l = k$.

(ii) Obviously $\mathfrak{M}$ is a subpartition of $\mathfrak{N}$; recall that $\mathfrak{M}$ groups together only indices j, k such that $f_j \approx f_k$.

(iii) It may happen that indices j, k must belong to the same element $\mathfrak{n} \in \mathfrak{N}$ even if they are not part of the same resonance equation (f.e. $1, 2$ are swapped together by $f_1 \approx 2f_2$ and $2, 3$ are swapped together by $f_2 \approx f_3$). This was not the case with $\mathfrak{M}$.

(iv) We do not assume that $\mathfrak{N}$ is the finest partition satisfying (19.2.31) as we do not want to exclude possibility to take $\#\mathfrak{N} = 1$ (i.e. no partition at all).

Then, after we got rid of all non-resonant cubic term let us define

$$\tilde{a}^0_{\mathfrak{n}} := \sum_{\mathfrak{m}\subset\mathfrak{n}} \tilde{a}_{\mathfrak{m}} + \mu^2 \operatorname{Re} \sum_{j,k,m\in\mathfrak{n}} a_{jkm}(q)\zeta_j^\dagger \zeta_k \zeta_m. \tag{19.2.32}$$

Then

$$\{\tilde{a}^0_{\mathfrak{n}}, \tilde{a}^0_{\mathfrak{n}'}\} = O(\mu^{-1}), \qquad \{\tilde{a}^0_{\mathfrak{n}}, V(q)\} = O(\mu^{-1}) \tag{19.2.33}$$

and therefore

(19.2.34) In the classical evolution at energy levels below c_0 for time $t : |t| \le \epsilon\mu$ increment of $\tilde{a}^0_{\mathfrak{n}}$ does not exceed $C\mu^{-1}|t| \le \epsilon_1$ (so $\tilde{a}^0_{\mathfrak{n}}$ are slow variables);

and

(19.2.35) In the classical evolution at energy levels below c_0 for time $t : |t| \le \epsilon\mu$ increment of $a^0_{\mathfrak{n}} := \sum_{\mathfrak{m}\subset\mathfrak{n}} a^0_{\mathfrak{m}}$ does not exceed $C\mu^{-1}(|t| + 1) \le \epsilon_1$.

It follows from (19.2.7) that

$$\{q_j, q_k\} \equiv \mu^{-1} f_{jk}(q) + O(\mu^{-2}) \tag{19.2.7}'$$

and then one can reintroduce $q_j := x_j - \sum_k f_{jk}(x)\eta_k - \sum_{k,m} \rho_{jkm}(x)\eta_k\eta_m$ to make

$$\{\eta_k, q_j\} = O(\mu^{-3}). \tag{19.2.36}$$

Then

$$\{a^0 + V, q_j\} \equiv \mu^2 \sum_{k,m} \{a_{km}, q_j\}\zeta_k\zeta_m^\dagger + \{V(q), q_j\} \quad \mathsf{mod}\ O(\mu^{-3}). \tag{19.2.37}$$

Microhyperbolicity

Let us introduce microhyperbolicity condition provided a_0 does not vanish i.e.

$$V \le -\epsilon_0; \tag{19.2.38}$$

we are not interested in the classically forbidden case of $V \ge \epsilon_0$.

Definition 19.2.4. We call the magnetic Schrödinger operator *microhyperbolic* at point $\bar{x}$ if for each $\boldsymbol{\tau} = (\tau_{\mathfrak{m}})_{\mathfrak{m}\in\mathfrak{M}}$, such that $|\sum_{\mathfrak{m}\in\mathfrak{M}} \tau_{\mathfrak{m}} + V| \le \epsilon$, there exists vector $\ell = \ell(\bar{z}, \boldsymbol{\tau}) \in \mathbb{R}^{2r}$, such that

$$-\mu^2 \sum_{j,k} \bigl(\ell(a_{jk}a_0^{-1})\bigr)\zeta_j^\dagger\zeta_k \ge \epsilon_0 \tag{19.2.39}$$

as long as

$$\mu^2 \sum_{j,k\in\mathfrak{m}} a_{jk}\zeta_j^\dagger\zeta_k = \tau_{\mathfrak{m}} \qquad \forall \zeta \in \mathbb{C}^r \quad \forall \mathfrak{m} \in \mathfrak{M}. \tag{19.2.40}$$

Definition 19.2.5. Let $\mathfrak{N} = \mathfrak{N}(\bar{x})$ satisfy (19.2.31). We call the magnetic Schrödinger operator $\mathfrak{N}$-*microhyperbolic* at point $\bar{x}$ if for each $\boldsymbol{\tau} = (\tau_{\mathfrak{n}})_{\mathfrak{n}\in\mathfrak{N}}$, such that $|\sum_{\mathfrak{n}\in\mathfrak{N}} \tau_{\mathfrak{n}} + V| \le \epsilon$, there exists vector $\ell = \ell(\bar{z}, \boldsymbol{\tau}) \in \mathbb{R}^{2r}$ such that (19.2.39) holds as long as

$$\mu^2 \sum_{j,k\in\mathfrak{n}} a_{jk}\zeta_j^\dagger\zeta_k = \tau_{\mathfrak{n}} \qquad \forall \zeta \in \mathbb{C}^r \quad \forall \mathfrak{n} \in \mathfrak{N}. \tag{19.2.41}$$

Then we conclude that

(19.2.42) If magnetic Schrödinger operator is microhyperbolic at point $\bar{x}$ then $\sum_j \ell_j q_j$ will increase with the exact rate $\asymp \mu^{-1}$ for time $|t| \le T^* = \epsilon$

and

(19.2.43) If magnetic Schrödinger operator is either $\mathfrak{N}$-microhyperbolic at point $\bar{x}$ or contains no cubic terms[4] then in the frames of the smooth theory one can take $T^* = \epsilon\mu$.

Here both microhyperbolicity and $\mathfrak{N}$-microhyperbolicity obviously mean "on the energy level 0".

Example 19.2.6. (i) If F_{jk}, g^{jk} are constant, then the microhyperbolicity condition means exactly that $|\nabla V| \ge \epsilon$. In this case also $\mathfrak{N}$-microhyperbolicity condition is fulfilled with $\#\mathfrak{N} = 1$.

(ii) If there are no 2-nd order resonances[5] then the microhyperbolicity condition means exactly that

$$|\sum_k \lambda_k \nabla \log(-V/f_k)| \ge \epsilon_0 \qquad \forall \lambda_1 \ge 0, \dots, \lambda_r \ge 0 : \ \lambda_1 + \dots + \lambda_r = 1. \tag{19.2.44}$$

Then there exists vector ℓ such that

$$\sum_k \lambda_k \langle \ell, \nabla \rangle \log(-V/f_k) \ge \epsilon_0 \qquad \forall \lambda_1 \ge 0, \dots, \lambda_r \ge 0 : \ \lambda_1 + \dots + \lambda_r = 1$$

and then we have $\mathfrak{N}$-microhyperbolicity with $\#\mathfrak{N} = 1$. So in this case the notions of microhyperbolicity and $\mathfrak{N}$-microhyperbolicity also coincide.

This assumption (19.2.44) is fulfilled provided $\nabla(V/f_1), \dots, \nabla(V/f_r)$ are linearly independent.

Problem 19.2.7. Either prove that the the notions of microhyperbolicity and $\mathfrak{N}$-microhyperbolicity coincide in the general case or construct a counter-example.

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[4] Which happens for example if either there are no 3-rd order resonances or if F_{jk}, g^{jk} are constant.

[5] Or, more generally, if eigenvalues f_j of (F_k^j) have constant multiplicities.

Non-Smooth Theory

Consider what happens for not very large smoothness. First of all, all our analysis above obviously holds as $(l,\sigma)\succeq(2,0)$.

Furthermore, as long as we do not need to analyze $a_{\mathfrak{n}}^0$ (i.e. if we have $\#\mathfrak{N}=1$ in the $\mathfrak{N}$-microhyperbolicity condition) $\mathscr{C}^1$-smoothness is sufficient: to estimate by $O(\mu^{-1})$ the rate of change of q we need to have only $\{\zeta_j, q_k\}=O(\mu^{-2})$ and $\{x_j, q_k\}=O(\mu^{-1})$.

Microhyperbolicity arguments require $\{\zeta_j, q_k\}=o(\mu^{-2})$ and calculation of $\{x_j, q_k\}$ modulo $o(\mu^{-1})$. To do it we just need to mollify $\mathscr{C}$-coefficients ρ_{jkm} with the mollification parameter $o(1)$.

If $(l,\sigma)\prec(2,0)$ and $\#\mathfrak{N}>1$ in the $\mathfrak{N}$-microhyperbolicity condition, we need to analyze evolution of $a_{\mathfrak{n}}^0$ more carefully. One can see easily that in this case

$$\{a_{\mathfrak{n}}^0, a\}=O\big(\mu^{-l+1}|\log\mu|^{-\sigma}\big) \tag{19.2.45}$$

because one can consider 3-rd order and linear terms and replace them by their μ^{-1}-mollifications with respect to η and Q; then for mollified terms this estimate would hold and for mollification error it would hold as well since the gradient of the error would be $O(\mu^{1-l}|\log\mu|^{-\sigma})$.

Therefore under $\mathfrak{N}$-microhyperbolicity assumption with $\#\mathfrak{N}>1$ one can take

$$T^*=\epsilon\begin{cases}\mu & (l,\sigma)\succeq(2,0),\\ \mu^{l-1}|\log\mu|^{\sigma} & (l,\sigma)\prec(2,0)\end{cases} \tag{19.2.46}$$

and in the end of the day semiclassical error will be $O\big(h^{1-d}T^{*-1}\big)$ which is exactly the first term in estimate (19.2.87) below.

19.2.2 Semiclassical Propagation

Now moving from the classical dynamics to the rigorous analysis we prove few statements, assuming that

$$\mu_0\le\mu\le\mu_2^*=\epsilon_1(h|\log h|)^{-1} \tag{19.2.47}$$

and

$$\varepsilon\ge Ch|\log h| \tag{19.2.48}$$

where here and below μ_0, C are large enough constants.

(19.2.49) In the statements of this subsubsection ϕ_1 is supported in $B(0,1)$, $\phi_2 = 1$ in $B(0,2)$ and χ is supported in $[-1,1]$ [6].

General Theory

Proposition 19.2.8 [7]**.** *Let* $\mu \le \epsilon h^{-1}|\log h|^{-1}$ [8] *and let*

$$M \ge \sup_{\{\sum_{j,k} g^{jk}\xi_j\xi_k + V = 0\}} 2\sum_k g^{jk}\xi_k + \epsilon \tag{19.2.50}$$

with arbitrarily small constant $\epsilon > 0$.

Let

$$Ch|\log h| \le T \le T^{*\prime} := \epsilon_0. \tag{19.2.51}$$

Then

$$|F_{t\to h^{-1}\tau}\chi_T(t)\big(1-\phi_{2,MT}(x-\bar{x})\big)\phi_{1,MT}(y-\bar{x})U(x,y,t)| \le Ch^s \quad \forall \tau \le \epsilon_1 \tag{19.2.52}$$

where here and below $\epsilon_1 > 0$ *is a small enough constant.*

Proposition 19.2.9 [9]**.** *Let condition* (19.1.4) *be fulfilled and let*

$$T_* := C\varepsilon^{-1}h|\log h| + C(\mu h|\log h|)^{\frac{1}{2}} \le T \le T^* = \epsilon_0\mu. \tag{19.2.53}$$

(i) Then

$$|F_{t\to h^{-1}\tau}\chi_T(t)\big(1-\phi_{2,\mu^{-1}MT}(q_1-\bar{x}_1,\dots,q_d-\bar{x}_d)\big)^{\mathsf{w}}_x \times U(x,y,t)\,{}^t\big(\phi_{1,\mu^{-1}MT}(q_1-\bar{x}_1,\dots,q_d-\bar{x}_d)\big)^{\mathsf{w}}_y| \le Ch^s \qquad \forall \tau : |\tau| \le \epsilon_1 \tag{19.2.54}$$

where here and below $\mu_0 > 0$, C_1, M *are large enough constants. Here and below* b^{w} *means Weyl* $\mu^{-1}h$*-quantization of symbol* b *and due to condition* (19.2.53) *logarithmic uncertainty principle holds and this quantization of the symbols involved is justified.*

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[6] Recall that all such auxiliary functions are appropriate in the sense of Section 2.3.

[7] Finite speed of propagation with respect to x; cf. Proposition 18.2.4.

[8] Lower bound condition is not needed in this statement.

[9] Magnetic propagation; cf. Proposition 18.2.6.

(ii) In particular, for $T \ge C_1$ this inequality holds for $q_1, \dots, q_d$ replaced by $x_1, \dots, x_d$.

Proposition 19.2.10. *Let $\nu = C\mu h\varepsilon^{-1}|\log h| + C(\mu h|\log h|)^{\frac12}$.*

(i) Let $\#\mathfrak{M} \ge 2$. Then

$$|F_{t\to h^{-1}\tau}\chi_T(t)\bigl(1-\phi_{2,MT+\nu}(a_{\mathfrak m}-\tau_{\mathfrak m})\bigr)^{\mathsf w}_x\times U(x,y,t)\,{}^t\bigl(\phi_{1,MT+\nu}(a_{\mathfrak m}-\tau_{\mathfrak m})\bigr)^{\mathsf w}_y| \le Ch^s \qquad \forall \tau : |\tau|\le \epsilon_1. \tag{19.2.55}$$

(ii) Let $\#\mathfrak{N} \ge 2$ and $\nu = T^{*\,-1}$ with T^* defined by (19.2.46)*. Then*

$$|F_{t\to h^{-1}\tau}\chi_T(t)\bigl(1-\phi_{2,\nu T+\nu}(a_{\mathfrak n}-\tau_{\mathfrak n})\bigr)^{\mathsf w}_x\times U(x,y,t)\,{}^t\bigl(\phi_{1,\nu T+\nu}(a_{\mathfrak n}-\tau_{\mathfrak m})\bigr)^{\mathsf w}_y| \le Ch^s \qquad \forall \tau : |\tau|\le \epsilon_1. \tag{19.2.56}$$

Proofs of Propositions 19.2.8–19.2.10. All Propositions 19.2.8–19.2.10 are proven by the same scheme as in Section 2.3:

(a) To prove that the speed with respect to x does not exceed M one can use function

$$\chi\Bigl(\bigl(\frac{|x-\bar x|^2}{T^2}+\epsilon^2\bigr)^{\frac12} - C\varsigma\frac{t}{T}\Bigr) \tag{19.2.57}$$

with χ function of the same type as used in the proof of Theorem 2.3.1, $\varsigma = \pm 1$ depending on the time direction and arbitrarily small constant $\epsilon > 0$.

(b) To prove that the speed with respect to Q does not exceed $C\mu^{-1}$ one can use function

$$\chi\Bigl(\bigl(\frac{\mu^2|q-\bar q|^2}{T^2}+\epsilon^2\bigr)^{\frac12} - C\varsigma\frac{t}{T}\Bigr). \tag{19.2.58}$$

(c) To prove that the speed with respect to $a_{\mathfrak n}$ does not exceed $v = T^{*\,-1}$ with T^* defined by (19.2.46) one can use function

$$\chi\Bigl(\bigl(\frac{|a_{\mathfrak n}-\tau_{\mathfrak n}|^2}{v^2T^2}+\epsilon^2\bigr)^{\frac12} - C\varsigma\frac{t}{T}\Bigr) \tag{19.2.59}$$

and to prove that the speed with respect to $a_{\mathfrak m}$ does not exceed $v = C$ one can use the same function (19.2.59) with $\mathfrak n$ replaced by $\mathfrak m$. □

Corollary 19.2.11. *(i) Let* $\#\mathfrak{M} \ge 2$. *Then (in the microlocal sense)* $\{\mathsf{a}_{\mathfrak{m}}\}$ *stays in the* ϵ*-vicinity of value* $\boldsymbol{\tau}$ *for time* $T^* = \epsilon$.

(ii) Let $\#\mathfrak{N} \ge 2$. *Then (in the microlocal sense)* $\{\mathsf{a}_{\mathfrak{n}}\}$ *stays in the* ϵ*-vicinity of value* $\boldsymbol{\tau}$ *for time* T^* *defined by* (19.2.46).

Microhyperbolic Theory

In this subsubsection function χ is supported in $[-1,-\frac{1}{2}]\cup[\frac{1}{2},1]$ and $\bar\chi$ is supported in $[-1,1]$ and $\bar\chi = 1$ on $[-\frac{1}{2},\frac{1}{2}]$ [6)].

Proposition 19.2.12. [10)] *Let* $(I,\sigma) \succeq (1,2)$ *and*

$$\mu_0 \le \mu \le \mu_1^* := \epsilon_1(h|\log h|)^{-\frac{1}{2}}, \tag{19.2.60}$$

$$\varepsilon = C'\mu h|\log h|, \tag{19.2.61}$$

$$\bar T = \epsilon\mu^{-1} \tag{19.2.62}$$

where large enough constant C' *is chosen in the very last moment. Then*

(i) Under microhyperbolicity condition (see Definition 19.2.4) let us pick up $T^* = \epsilon$.

(ii) Under $\mathfrak{N}$*-microhyperbolicity condition (see Definition 19.2.5) let us define* T^* *by* (19.2.46) *as* $\#\mathfrak{N} \ge 2$ *and* $T^* = \epsilon\mu$ *as* $\#\mathfrak{N} = 1$ [11)].

Then for $T \in [\bar T, T^*]$, $\tau \in [-\epsilon_1,\epsilon_1]$

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\psi U| \le Ch^s. \tag{19.2.63}$$

Proof. (a) Let us first try function

$$\chi\Big(\frac{\mu}{T}\langle \ell, q-\bar q\rangle + \epsilon\varsigma\frac{t}{T}\Big) \tag{19.2.64}$$

with $\ell = \ell(y;\tau_1,\dots,\tau_{\#\mathfrak{N}})$.

One can check easily that it is an admissible symbol as long in scale $\rho = \gamma = \mu T$ with respect to x and ξ and the logarithmic uncertainty

[10)] Weak magnetic field case.

[11)] So under $\mathfrak{N}$-microhyperbolicity condition $T^* = \epsilon\mu$ unless $\#\mathfrak{N} \ge 2$ and $(I,\sigma) \prec (2,0)$.

principle $\rho\gamma \ge C\mu^{-1}h|\log h|$ is fulfilled[12)] as

$$T \ge T_* := C(\mu h|\log h|)^{\frac12}; \tag{19.2.65}$$

then $T_* \le \bar{T} = \epsilon_1\mu^{-1}$ as

$$\mu \le \epsilon_3(h|\log h|)^{-\frac13}. \tag{19.2.66}$$

Then our standard analysis like in Theorem 2.3.1, together with Propositions 19.2.8–19.2.9 and Corollary 19.2.11[13)] imply estimate (19.2.63).

(b) Now let us consider the case when

$$\epsilon_3(h|\log h|)^{-\frac13} \le \mu \le \epsilon_1(h|\log h|)^{-\frac12}, \tag{19.2.67}$$

$$\epsilon\mu^{-1} \le T \le C(\mu h|\log h|)^{\frac12}; \tag{19.2.68}$$

then

$$\varepsilon \ge C(\mu^{-1}h|\log h|)^{\frac12} \ge \mu^{-1}T, \tag{19.2.69}$$

function (19.2.64) is not necessarily admissible symbol and one needs a bit more subtle arguments.

Let us consider a small vicinity of y. Without any loss of the generality one can assume that $\phi_{jk}(y) = 0$ as $|j-k| \ne r$ (otherwise one can achieve it by a rotation). After this, without any loss of the generality one can assume that $\ell = (\ell', 0)$ where $x' = (x_1, \dots, x_r)$, $x'' = (x_{r+1}, \dots, x_{2r})$ etc. as we can achieve it by rotations in the planes (x_j, x_{j+r}), $j = 1, \dots, r$.

Recall that by definition

$$q_j = x_j - \sum_k \phi_{jk}p_k + \frac12\sum_{i,k}\beta_{jik}p_ip_k,$$

where β_{jik} are (mollified) coefficients chosen to eliminate linear with respect to $(p_1, \dots, p_d)$ terms in $\{p_k, q_j\}$. Consider function

$$\chi\Big(\frac{\mu}{T}\langle\ell, q-\bar{q}\rangle + \epsilon\varsigma\frac{t}{T}\Big) = \chi\Big(\frac{\mu}{T}\big(\langle\ell', x-\bar{x}'\rangle - \sum_{j,k}\ell_j\phi_{jk}p_k + \frac12\sum_{i,j,k}\ell_j\beta_{jik}p_ip_k\big)\Big) \tag{19.2.70}$$

[12)] To fulfill logarithmic uncertainty principle we also need $\rho\varepsilon \ge C\mu^{-1}h|\log h|$, but it will be fulfilled automatically as $T \ge \bar{T}$ and $\varepsilon \ge C'\mu h|\log h|$.

[13)] If we are in framework of Statement (ii) with $\#\mathfrak{N} \ge 2$.

and use a vector scale $(\boldsymbol{\gamma}, \boldsymbol{\rho})$ with respect to (x, ξ) with

$$\rho_j \gamma_j \ge C\mu^{-1}h|\log h|. \tag{19.2.71}$$

Consider the first derivatives of the argument of χ; we want them to be bounded *after rescaling*.

Let us take $\gamma_j = \gamma$, $\rho_j = \rho$ as $j = 1, \dots, r$ and $\gamma_j = \rho$, $\rho_j = \gamma$ as $j = r+1, \dots, 2r$ with $\rho \ge \gamma$; then $|\phi_{jk} - \phi_{jk}(y)| \le c\rho$.

Note that to have the first derivatives with respect to x' bounded by $C_0\gamma^{-1}$ one needs to take $\gamma = \mu^{-1}T$ [14] as $j = 1, \dots, r$.

Further, to have the first derivatives with respect to ξ' bounded by $C_0\rho^{-1}$ one needs to take $\rho = (\mu^{-1}T)^{\frac{1}{2}}$ as $j = 1, \dots, r$. Then the first derivatives with respect to ξ'' are also bounded by $C_0\gamma^{-1}$ and to have the first derivatives with respect to x'' bounded by $C_0\rho^{-1}$ one needs to have $|\partial_j p_k| \le c\rho$ as $j = 1, \dots, r$, $k = j + r$ or, equivalently, $\partial_j V_k(y) = 0$ for indicated j, k; this could be achieved by a gauge transformation.

To satisfy (19.2.71) one needs then

$$T \ge C\mu(\mu^{-1}h|\log h|)^{\frac{2}{3}}; \tag{19.2.72}$$

one can see easily that $T = \bar{T} = \epsilon\mu^{-1}$ fits if and only if μ satisfies (19.2.60). Then picking up $\gamma = C\min\bigl(\mu h|\log h|, \mu^{-1}T\bigr)$ and $\rho = \gamma^{\frac{1}{2}}$ we see that (19.2.71) holds and (19.2.70) is an admissible symbol.

Then our standard analysis like in Theorem 2.3.1, together with Propositions 19.2.8–19.2.9 and Corollary 19.2.11[13] imply estimate (19.2.63). □

Applying the same arguments as in Parts (a) and (b) of the proof, in the cases $T \ge C(\mu h|\log h|)^{\frac{1}{2}}$ and $T \le C(\mu h|\log h|)^{\frac{1}{2}}$ respectively one can prove easily

Proposition 19.2.13. [15] *Let $(l, \sigma) \succeq (1, 2)$, microhyperbolicity condition be fulfilled and*

$$\epsilon_1(h|\log h|)^{-\frac{1}{2}} \le \mu \le \epsilon(h|\log h|)^{-1}, \tag{19.2.73}$$

$$C'(\mu^{-1}h|\log h|)^{\frac{1}{2}} \le \varepsilon \le C'\mu h|\log h|, \tag{19.2.74}$$

$$T_* = C\max\Bigl(\varepsilon^{-1}h|\log h|,\ \mu(\mu^{-1}h|\log h|)^{\frac{2}{3}}\Bigr) \tag{19.2.75}$$

[14] Or less; we always take the maximal value.

[15] Weak magnetic field approach.

where large enough constant C' is chosen in the very last moment. Then

(i) For $T \in [T_*, T^*]$, $\tau \in [-\epsilon_1, \epsilon_1]$ estimate (19.2.63) *holds where $T^* = \epsilon$.*

(ii) Furthermore, under $\mathfrak{N}$-microhyperbolicity this estimate holds with T^* defined as in Proposition 19.2.12(ii).

Then we arrive immediately to

Corollary 19.2.14. *In the framework of Propositions 19.2.12, 19.2.13*

$$|F_{t\to h^{-1}\tau}\bigl(\bar\chi_T(t) - \bar\chi_{T'}(t)\bigr)\Gamma\psi U| \le Ch^s \tag{19.2.76}$$

with $T_* \le T' \le T \le T^*$ with T_* defined by (19.2.75) *and T^* defined in Proposition 19.2.12(i), (ii).*

Recall that $\bar\chi$ is an admissible function supported in $[-1,1]$ and equal 1 at $[-\frac{1}{2}, \frac{1}{2}]$.

19.2.3 Tauberian Theory

Corollary 19.2.15. *In the framework of Proposition 19.2.12*

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma(\psi U)| \le Ch^{1-d} \qquad \forall \tau : |\tau| \le \epsilon \tag{19.2.77}$$

and

$$\mathsf{R}^{\mathsf{T}} := |\Gamma(\psi\tilde e)(\tau) - h^{-1}\int_{-\infty}^{\tau}\Bigl(F_{t\to h^{-1}\tau'}\bar\chi_T(t)\Gamma(\psi U)\Bigr)\,d\tau'| \le CT^{*\,-1}h^{1-d} \qquad \forall \tau : |\tau| \le \epsilon \tag{19.2.78}$$

for any $T \in [\bar T, T^*]$.

Proof. Rescaling $x \mapsto \mu x$, $\varepsilon \mapsto \mu\varepsilon$, $t \mapsto \mu t$, $h \mapsto \mu h$, $T_0 \mapsto \epsilon$ we arrive to the standard Schrödinger operator (i.e. with $\mu = 1$), we can apply our standard methods. Then estimate (19.2.77) holds for $T = \bar T$. Combining with (19.2.76) we arrive to (19.2.77) with arbitrary $T \in [\bar T, T^*]$.

Applying Tauberian arguments we arrive then to the Tauberian estimate (19.2.78). □

19.2.4 Main Theorem

Now under assumption (19.2.60) we need to calculate the Tauberian expression

$$h^{-1}\int_{-\infty}^{\tau}\Big(F_{t\to h^{-1}\tau'}\bar{\chi}_T(t)\Gamma(\psi U)\Big)\,d\tau'. \tag{19.2.79}$$

with $T = Ch|\log h|$.

Using rescaling to the standard Schrödinger operator we get that it is equal to

$$\sum_{m,n\ge 0}\kappa_{nm}h^{-d+2m+2n}\mu^{2n} \tag{19.2.80}$$

as all other coefficients vanish under assumption (19.2.38).

Taking only term $\kappa_{00}h^{-d}$ we get the standard Weyl expression

$$h^{-d}\int \mathcal{N}^{\mathsf{W}}(\tau,x)\psi(x)\,dx \tag{19.2.81}$$

with an error $O(\mu^2h^{2-d})$ which is $O(\mu^{-1}h^{1-d})$ as $\mu\le h^{-\frac{1}{3}}$.

Further, taking only terms $\kappa_{00}h^{-d}+\kappa_{10}h^{2-d}\mu^2$ we commit an error $O(h^{2-d}+\mu^4h^{4-d})$ which is $O(\mu^{-1}h^{1-d})$ as $\mu\le h^{-\frac{3}{5}}$. One can calculate this correction term $\kappa_{10}h^{2-d}\mu^2$ easily. More generally, taking n-terms we make an error $O(\mu^{2n}h^{2n-d})$ arriving to

$$|\Gamma(\psi\tilde{e})\tau) - h^{-d}\int \tilde{\mathcal{N}}^{\mathsf{W}}_{(n)}(x,\tau)\psi(x)\,dx| \le CT^{*-1}h^{1-d}+C\mu^{2n}h^{2n-d} \qquad \forall \tau : |\tau|\le \epsilon \tag{19.2.82}$$

where $h^{-d}\mathcal{N}^{\mathsf{W}}_{(n)}$ denotes n-term Weyl expression i.e. cut the asymptotic sum

$$h^{-d}\mathcal{N}^{\mathsf{W}}_{(\infty)} := \sum_{n\ge 0}\varkappa_{n0}(x)h^{-d}(\mu h)^{2n}. \tag{19.2.83}$$

Instead let us notice that $\kappa_{n0}=\int \varkappa_{n0}(x,\tau)\psi(x)\,dx$ where $\varkappa_{n0}(x,\tau)$ depends only on g^{jk}, V and $\partial_k V_j$ calculated at point x, and thus with the indicated error to calculate (19.2.79) with Γ_y instead of Γ and with $\psi = 1$

and one can consider Schrödinger operator

$$\bar{A} = \sum_{j,k} \bar{g}^{jk} \bar{P}_j \bar{P}_k + \bar{V}, \qquad \bar{g}^{jk} = g^{jk}(y),\ \bar{V} = V(y), \tag{19.2.84}$$

$$\bar{P}_j = hD_j - V_j(y) - \sum_k (\partial_k V_j)(y)(x_k - y_k)$$

and for this operator expression in question is exactly $h^{-d}\tilde{\mathcal{N}}^{\mathsf{MW}}(y,\tau)$.

So, under assumption (19.2.60) we have proven

$$|\Gamma(\psi\tilde{e})\tau) - h^{-d}\int \tilde{\mathcal{N}}^{\mathsf{MW}}(x,\tau)\psi(x)\,dx| \le CT^{*-1}h^{1-d} \tag{19.2.85}$$

$$\forall \tau : |\tau| \le \epsilon$$

where for $\mu \le Ch^{-\frac{1}{3}}$ one can replace $\tilde{\mathcal{N}}^{\mathsf{MW}}$ by $\tilde{\mathcal{N}}^{\mathsf{W}}$.

Thus taking into account that the difference between either Weyl or magnetic Weyl expressions for the original and mollified operator does not exceed $C\varepsilon^{l}|\log\varepsilon|^{-\sigma}h^{-d}$ we arrive to the main theorem of this section:

Theorem 19.2.16. *Let assumptions* (19.1.4)–(19.1.6), $(19.1.25)_{1-3}$ *with* $(\bar{l},\bar{\sigma}) \succeq (l,\sigma) \succeq (1,2)$ *and* (19.2.38) *be fulfilled. Let* $\mu \le \mu_2^* = C(h|\log h|)^{-\frac{1}{2}}$. *Then there are two framing approximations*[18)] *(see f.e. Theorem 18.2.9) such that the following statements are true:*

(i) Let $\mathfrak{N}$*-microhyperbolicity condition (see Definition 19.2.5) be fulfilled. Then if* <u>*either*</u> $(l,\sigma) \succeq (2,0)$ <u>*or*</u> $\#\mathfrak{N} = 1$, *then*

$$\mathsf{R}^{\mathsf{MW}} := |\int \Big(\tilde{e}(x,x,0) - h^{-d}\mathcal{N}^{\mathsf{MW}}(x,0)\Big)\psi(x)\,dx| \le \tag{19.2.86}$$

$$C\mu^{-1}h^{1-d} + C(\mu h)^{l}|\log h|^{l-\sigma}h^{-d}$$

and if $(l,\sigma) \prec (2,0)$ <u>*and*</u> $\#\mathfrak{N} \ge 2$ *then*

$$\mathsf{R}^{\mathsf{MW}} \le C\mu^{l-l}h^{1-d}|\log\mu|^{-\sigma} + C(\mu h)^{l}|\log h|^{l-\sigma}h^{-d}. \tag{19.2.87}$$

(ii) Let microhyperbolicity condition (see Definition 19.2.4) be fulfilled. Then

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d} + C(\mu h)^{l}|\log h|^{l-\sigma}h^{-d}. \tag{19.2.88}$$

Corollary 19.2.17. *If* $(l,\sigma) \succeq (3,1)$ *then in the framework of Syatement (i) asymptotics*

$$\mathsf{R}^{\mathsf{MW}} \le C\mu^{-1}h^{1-d} \tag{19.2.89}$$

holds as $\mu \le C(h|\log h|)^{-\frac{1}{2}}$.

Remark 19.2.18. (i) The above estimates hold for $\mathsf{R}^{\mathsf{W}}_{(2)}$ as well.

(ii) Theorem 19.2.16 can be extended to the case $C(h|\log h|)^{-\frac{1}{2}} \le \mu \le h^{\delta-1}$ but it will be our theorem of choice only as $\mu \le h^{-\frac{1}{2}}$.

(iii) In the case $C(h|\log h|)^{-\frac{1}{2}} \le \mu \le h^{\delta-1}$ we need to take n such that $(2n,0) \succeq (l,\sigma)$.

(iv) This remark also applies to Theorem 19.2.19 below.

The general case of weak magnetic field is completed.

19.2.5 Special Case of Constant g^{jk}, F_{jk}

Assume now that g^{jk} and F_{jk} are constant. Then without any loss of the generality one can assume that

$$g^{jk} = \delta_{jk}, \qquad F_{jk} = f_j\delta_{k,j+r} - f_k\delta_{j,k-r} \tag{19.2.90}$$

and

$$V_j = 0 \quad \text{as} \quad j = 1,\dots,r \qquad V_j = f_{j-r}x_{j-r} \quad \text{as} \quad j = r+1,\dots,2r. \tag{19.2.91}$$

Then all the arguments of the previous Subsections could be easily simplified as we can assume without any loss of the generality that $\ell = (0,\ell'')$ and consider the shift with respect to ξ''. Then the logarithmic uncertainty principle would mean $T\varepsilon \ge Ch|\log h|$ and thus we can take $\varepsilon = C\mu h|\log h|$ to accommodate $T = \bar{T} \asymp \mu^{-1}$.

However, we can make better than this and allow critical points. Let us first consider a point $\bar{x}$ such that at this point

$$|\nabla V| \asymp \nu \ge \bar{\nu} := C_0 \max\bigl(\mu^{-1}, \mu h|\log h|\bigr) \tag{19.2.92}$$

where we pick $\bar{\nu}$ exactly as in the 2D-case.

Then $|\nabla V| \asymp \nu$ in $B(\bar{x}, 2\epsilon_1\nu)$ and it follows from our analysis that the evolution (including microlocal one) which starts at $B(\bar{x}, \epsilon_1\nu)$ does not leave $B(\bar{x}, 2\epsilon_1\nu)$ for time $T^* = \epsilon\mu$.

On the other hand, the shift in ξ'' is $\asymp T\nu$ and the logarithmic uncertainty principle requires $T\nu \cdot \varepsilon \ge Ch|\log h|$ and we can take

$$\varepsilon = C\begin{cases} \mu h\nu^{-1}|\log h| & \nu \ge \bar{\nu},\\ \mu h\bar{\nu}^{-1} & \nu \le \bar{\nu}\end{cases} \tag{19.2.93}$$

where $\nu \le \bar{\nu}$ means that $|\nabla V| \le C\bar{\nu}$ in $\bar{x}$ and its $2\bar{\nu}$-vicinity.

Then contribution of $B(x, \nu)$ to the Tauberian remainder does not exceed $C\mu^{-1}h^{1-d}\nu^d$ as $\nu \ge \bar{\nu}$ and $C\mu h^{1-d}\bar{\nu}^d$ as $\nu \le \bar{\nu}$ and in the latter case we take $T^* = \epsilon\mu^{-1}$. Summation over partition returns

$$\mathsf{R}^{\mathsf{T}} \le C\mu^{-1}h^{1-d} + C\mu h^{1-d}\,\mathsf{mes}\big(\{x:\ |\nabla V(x)| \le \bar{\nu}\}\big) \tag{19.2.94}$$

which in turn implies the same estimate for $\mathsf{R}^{\mathsf{W}}_{(\infty)}$ calculated for mollified operator and under assumption (19.2.38) an error when we pass from mollified to the original operator error becomes

$$Ch^{-d}\int \varepsilon^l|\log\varepsilon|^{-\sigma}\,dx \asymp Ch^{-d}\int (\mu h)^l|\log(\mu h/\nu)|^{-\sigma}\nu^{-l}\,dx. \tag{19.2.95}$$

Assuming that $(l, \sigma) \succeq (2, 0)$ and imposing generic non-degeneracy condition

$$|\nabla V| \le \epsilon \implies |\det \mathsf{Hess}\, V| \ge \epsilon \tag{19.2.96}$$

we conclude that expression (19.2.94) does not exceed

$$C\mu^{-1}h^{1-d} + C\mu(\mu h|\log h|)^{d/2}h^{1-d} \tag{19.2.97}$$

which even in the worst case $d = 4$ does not exceed $C\mu^{-1}h^{1-d}$ as $\mu \le (h|\log h|)^{-\frac{1}{2}}$.

Meanwhile as $l < d$ expression (19.2.95) does not exceed $C(\mu h)^l|\log h|^{-\sigma}h^{-d}$ and as $l \ge d$ this expression does not exceed (19.2.97) and therefore both expressions (19.2.94) and (19.2.95) do not exceed

$$C\mu^{-1}h^{1-d} + C\mu(\mu h|\log h|)^{d/2}h^{1-d} + C(\mu h)^l|\log h|^{-\sigma}h^{-d}. \tag{19.2.98}$$

Then we arrive to

Theorem 19.2.19. *Let assumptions* (19.1.4)–(19.1.6), $(19.1.6)_3$ *with* $(I,\sigma)\succeq(1,2)$ *and* (19.2.38) *be fulfilled. Let* g^{jk}, F_{jk} *be constant.*

Then there are two framing approximations[18] *(see Chapter 18) such that the following statements are true:*

(i) Under microhyperbolicity assumption $|\nabla V|\ge\epsilon$ *estimate* (19.2.86) *holds.*

(ii) Under assumptions $(I,\sigma)\succeq(2,0)$ *and* (19.2.96) R^{MW} *does not exceed* (19.2.98).

19.3 Canonical Form

19.3.1 Toy-Model

Assume now that g^{jk} and F_{jk} are constant. Then without any loss of the generality we can assume that $V_j(x)$ are linear functions.

Then A^0 is transformed into exactly

$$\sum_{1\le j\le r} f_j\Big(h^2D_j^2+\mu^2x_j^2\Big) \tag{19.3.1}$$

by $\mu^{-1}h$-metaplectic transformation which consists of the following steps:

(i) Change of variables $(x,\mu^{-1}hD)\mapsto(Qx,\ {}^tQ^{-1}\mu^{-1}hD)$ transforming g^{jk} into δ_{jk} and F into matrix with $F_{j,j+r}=f_j$, $F_{j+r,j}=-f_j$ and other elements 0. It transforms $V(x)$ into $V(Qx)$.

(ii) Gauge transformation (multiplication by $e^{i\mu h^{-1}S(x)}$ with quadratic form $S(x)$, transforming $V_j(x)$ into 0 and $V_{j+r}(x)$ into f_jx_j for $j=1,\dots,r$ [16]. Then A^0 is transformed into

$$h^2\sum_{1\le j\le r}\Big(D_j^2+\big(D_{j+r}-f_j\mu h^{-1}x_j\big)^2\Big) \tag{19.3.2}$$

and $V(x)$ is preserved.

(iii) Partial $\mu^{-1}h$-Fourier transform:

$$(x',x'';\mu^{-1}hD',\mu^{-1}hD')\mapsto(x',-\mu^{-1}hD'';\mu^{-1}hD',x'')$$

[16] Thus we achieve (19.2.90)–(19.2.91).

transforming A^0 into

$$h^2 \sum_{1\le j\le r} \Big(D_j^2 + \mu^2 h^{-2} \big(x_{j+r} - f_j x_j\big)^2 \Big) \tag{19.3.3}$$

and $V(x)$ into $V(x', -\mu^{-1}hD'')$.

(iv) Change of variables

$$(x', x''; \mu^{-1}hD', \mu^{-1}hD') \mapsto (x' - Kx'', x''; \mu^{-1}hD', \mu^{-1}hD'' + {}^tK\mu^{-1}hD')$$

with $K_{j,j+r} = f_j\delta_{j,k-r}$, transforming A^0 into

$$\sum_{1\le j\le r} \Big(h^2 D_j^2 + f_j^2\mu^2 x_j^2 \Big)$$

and V into $\tilde{V} \coloneqq V(x' - Kx'', \mu^{-1}hD'' - {}^tK\mu^{-1}hD')$ with Weyl quantization. Finally, $x_j \mapsto f_j^{\frac{1}{2}} x_j$, $D_j \mapsto f_j^{-\frac{1}{2}} D_j$ reduces operator A^0 into (19.3.1) and transforms $\tilde{V}$ accordingly.

This example already demonstrates why we need $\varepsilon \gtrsim \mu^{-1}$: we need to fulfill equation similar to one considered in Section 18.3

$$\sum_j f_j \partial_{\phi_j} L \approx \tilde{V} - W \tag{19.3.4}$$

where L is a symbol of the operator generating transformation, $\tilde{V}$ is defined above, W is its replacement in the canonical form and (ρ_j, ϕ_j) are polar coordinates in (x_j, ξ_j)-plane.

But then if $f_1, \dots, f_r$ are commensurable, for generic $\tilde{V}$ one cannot satisfy this equation with an error better than $O(\mu^{-l}|\log\mu|^{-\sigma})$ and with L which is 2π-periodic with respect to all arguments ϕ_j and $W = W(\rho_1, \dots, \rho_r)$ as W would need to depend on some linear combinations of ϕ_j.

Problem 19.3.1. Explore what is possible when $f_1, \dots, f_r$ are non-commensurable, and we try to satisfy (19.3.4) with a better error. It definitely would depend on how these numbers are non-commensurable (something related to their diophantine properties - as $r = 2$ it is related to the Liouville's exponent for f_1/f_2).

19.3.2 General Case: Framework

Now we need to reduce our operator to a canonical form assuming that $(\bar{l},\bar{\sigma}) \succeq (2,1)$ and

$$\mu_1^* := \epsilon_1(h|\log h|)^{-\frac{1}{2}} \le \mu \le \mu_2^* := \epsilon(h|\log h|)^{-1} \tag{19.3.5}$$

(case $\mu \ge \mu_2^*$ and especially case $\mu \ge \mu_3^* := \epsilon h^{-1}$ we consider later). As we mentioned we will need to assume that

$$\varepsilon \ge C_0\mu^{-1} \tag{19.3.6}$$

rather than

$$\varepsilon \ge C(\mu^{-1}h|\log h|)^{\frac{1}{2}} \tag{19.3.7}$$

as in Chapter 18. This larger ε makes certain parts of our construction much simpler but leads to a larger error. In this and following Sections we consider $\mu^{-1}h$-quantization.

19.3.3 Reducing the Main Part

First we need to reduce the main part. Consider point x in the vicinity of $\bar{x}$ and consider Hamiltonian fields H_{η_j} of η_j reduced to Σ_x^0 where we recall that η_j are linear combinations of $p_j = \xi_j - V_j$: $\eta_{2k-1} := \operatorname{Re}\zeta_k$, $\eta_{2k} := \operatorname{Im}\zeta_k$, $k = 1,\dots,r$ and ζ_k are defined by (19.2.13)–(19.2.14) (rather than redefined by (19.2.22)–(19.2.23)). Also recall that $\Sigma^0 = \{\eta_1 = \dots = \eta_{2r} = 0\}$.

Consider symplectic map e^{tH_q} where $q = q(x;\eta_1,\dots,\eta_d)$ is a quadratic form

$$q(x;\eta_1,\dots,\eta_d) = \frac{1}{2}\sum_{i,j} q_{ij}(x)\eta_i\eta_j \tag{19.3.8}$$

with $q_{ij} \in \mathscr{F}^{\bar{l},\bar{\sigma}}$ where $\mathscr{F}^{l,\sigma}$ are defined in Remark 18.3.2. Note that

$$\frac{d}{dt}(\eta_k \circ e^{tH_q}) = \{q,\eta_k\}\circ e^{tH_q} \equiv \sum_{i,j} q_{ij}\{\eta_i,\eta_k\}\eta_i \circ e^{tH_q} \tag{19.3.9}$$

modulo quadratic form. Then using arguments of Subsection 18.3.1 one can prove easily that

$$\eta_k \circ e^{H_q} = \sum_j \beta_{kj}(x)\eta_j + \sum_{i,j}\beta'_{kij}(x,\xi)\eta_i\eta_j \tag{19.3.10}$$

with $\beta_{jk} \in \mathscr{F}^{\bar{l},\bar{\sigma}}$, $\beta'_{kij} \in \mathscr{F}^{\bar{l}-1,\bar{\sigma}}$. Here $\mathcal{B} := (\beta_{jk}) = e^{\mathcal{Q}\Lambda}$ with $\mathcal{Q} = (q_{kj})$ and $\Lambda = (\Lambda_{jm})$ where $\Lambda_{jm} = 1$ as $j = 2i-1$, $m = 2i$, $\Lambda_{jm} = -1$ as $j = 2i$, $m = 2i-1$ and $\Lambda_{jm} = 0$ in all other cases. Recall that $\{\eta_j, \eta_m\}\big|_{\Sigma^0} = \Lambda_{jm}$.

Obviously,

(19.3.11) For a given matrix $\mathcal{B} \in \mathscr{F}^{\bar{l},\bar{\sigma}}$ one can find a symmetric matrix $\mathcal{Q} = (q_{ij})$ of the same regularity such that $\mathcal{B} = e^{\mathcal{Q}\Lambda}$ if and only if $\mathcal{B}$ is a *symplectic matrix* i.e.

$$\mathcal{B}^{\dagger}\Lambda\mathcal{B} = \Lambda. \tag{19.3.12}$$

In particular, one can transform (modulo quadratic forms): η_1 into p_1, η_2 into $\beta_{22}p_2$ with β_2 disjoint from 0 [17], and η_3 into $p_{33} + \beta_{31}p_1 + \beta_{32}p_2$, and η_4 into $\beta_{44}p_3 + \beta_{41}p_1 + \beta_{42}p_2$ etc:

$$\mathcal{B}\eta_m = \eta'_m := \qquad p_m + \sum_{j\le 2k-2} \beta_{mj}p_j \qquad \text{as} \quad m = 2k-1, \tag{19.3.13}$$

$$\mathcal{B}\eta_m = \eta'_m := \beta_{mm}p_m + \sum_{j\le 2k-2} \beta_{mj}p_j \qquad \text{as} \quad m = 2k \tag{19.3.14}$$

where coefficients $\beta_{jk} \in \mathscr{F}^{\bar{l},\bar{\sigma}}$ are chosen to satisfy

$$\{\eta'_j, \eta'_m\} = \Lambda_{jm}. \tag{19.3.15}$$

Now let us consider $\mu^{-1}h$-differential operator $Q = q^{\mathrm{w}}$ and transformation $T(t) = e^{it\mu h^{-1}Q}$ ("poor man's Fourier integral operator").

Using arguments of Subsection 18.3.1 one can prove easily that

$$T(1)\eta_k^{\mathrm{w}}T(-1) \equiv (\eta'_k)^{\mathrm{w}} + \Big(\sum_{i,j} \beta'_{kij}\eta'_i\eta'_j\Big)^{\mathrm{w}} \tag{19.3.16}$$

modulo operator with the norm not exceeding $C\mu^{-2}h$. Then for

$$A^0 = \mu^2\Big(\sum_{jk} a_{jk}(x)\,\eta_j\eta_k\Big)^{\mathrm{w}} \tag{19.3.17}$$

[17] Since $\{p_1, p_k\}$ is disjoint from 0 for some k and we just rename x_k into x_2 and v.v.

we have

$$T(1)A^0T(-1)\equiv \mu^2\Big(\sum_{j,k} a_{jk}(x)\,\eta'_j\eta'_k\Big)^{\mathsf{w}}+\mu^2\Big(\sum_{i,j,k} b'_{ijk}\,\eta'_i\eta'_j\eta'_k\Big)^{\mathsf{w}} \tag{19.3.18}$$

modulo operator with the norm not exceeding $C\mu^{-1}h$ where $b'_{ijk}\in \mathscr{F}^{\bar{l}-1,\bar{\sigma}}$; for simplicity of notations we skip $'$ in what follows.

Also, the same arguments show that

$$T(-1)V(x)T(1)\equiv V(x)+\sum_j (b''_j\,\eta_j)^{\mathsf{w}} \tag{19.3.19}$$

modulo operator with the norm not exceeding $C\mu^{-2}+C\mu^{-l}|\log h|^{-\sigma}$ where $b''_j\in \mathscr{F}^{l-1,\sigma}$.

Recall that while a_{jk}, V are functions depending on x only, b'_{ijk}, b''_j are complete symbols. In the end of the day we decompose them in the powers of $\eta_1,\dots,\eta_{2r}$ as far as their smoothness allows.

Due to construction $\eta_1=p_1=\xi_1-V_1(x)$ [18]. Note that we can get rid of V_1 using gauge transform, after which condition $(19.1.25)_2$ still holds.

Remark 19.3.2. Exactly here we need this condition $(19.1.25)_2$ rather than more natural condition $V_j\in \mathscr{C}^{\bar{l}+1,\bar{\sigma}}$. Otherwise we would be forced to confine ourselves to $C\varepsilon$-vicinity of $\bar{x}$ which would cause problems in Section 19.4.

Then according to commutation property $\partial_{x_1}\eta_2=1$ on Σ^0. Further, by its construction η_2 does not depend on $\xi_1,\xi_3,\dots,\xi_d$ and therefore

$$\eta_2=\alpha(x,\xi_2)\big(x_1-\lambda(x',\xi_2)\big),\qquad \alpha\big|_{\Sigma^0}=1 \tag{19.3.20}$$

with $\alpha\in \mathscr{F}^{\bar{l},\bar{\sigma}}$, $\lambda\in \mathscr{F}^{\bar{l}+1,\bar{\sigma}}$ and $x'=(x_2,\dots,x_d)$.

Let us redefine

$$\eta_2\coloneqq \big(x_1-\lambda(x',\xi_2)\big). \tag{19.3.21}$$

(19.3.22) Without any loss of the generality we assume that $\bar{x}=0$; otherwise in what follows one should replace x by $x-\bar{x}$.

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[18] Recall that η_k means η'_k defined by (19.3.13).

Now we want to transform η_2 into x_1. To achieve this goal let us consider $T'(t) = e^{i\mu h^{-1}(\xi_1\lambda)^w}$. Then we have the series of exact equalities

$$T'(-t)D_1T(t) = D_1, \quad T'(-t)\lambda^{\mathsf{w}}T'(t) = \lambda^{\mathsf{w}}, \quad T'(-t)x_1T'(t) = x_1 + t\lambda^{\mathsf{w}}. \tag{19.3.23}$$

Therefore $T'(1)$ will transform "new" η_2^{w} into x_1. Now we need to check how this transformation will affect η_j^{w} with $j \ge 3$ and also α^{w}, b_{jk}^{w}, b_{ijk}^{w}, and finally V and $(b_j''\eta_j)^{\mathsf{w}}$.

Note that Hamiltonian function $p_1\lambda$ belongs to $\mathscr{F}^{\bar l+2,\bar\sigma}$ [19]. Therefore Hamiltonian map $\Phi_t = e^{tH_{p_1\lambda}}$ belongs to $\mathscr{F}^{\bar l+1,\bar\sigma}$ as $t \in [0,1]$ and then

$$\|T'(-t)\eta_j^{\mathsf{w}}T(t) - (\eta_j\circ\Phi_t)^{\mathsf{w}}\| \le C(\mu^{-1}h)^2\Big(1+\varepsilon^{\bar l+2-3}|\log\varepsilon|^{-\bar\sigma}\Big)\times\Big(1+\varepsilon^{\bar l+1-3}|\log\varepsilon|^{-\bar\sigma}\Big) \le C\mu^{-2}h, \tag{19.3.24}$$

$$\|T'(-t)b_{jk}^{\mathsf{w}}T(t) - (b_{jk}\circ\Phi_t)^{\mathsf{w}}\| \le C(\mu^{-1}h)^2\Big(1+\varepsilon^{\bar l-3}|\log\varepsilon|^{-\bar\sigma}\Big)\times\Big(1+\varepsilon^{\bar l+1-3}|\log\varepsilon|^{-\bar\sigma}\Big) \le C\mu^{-1}h, \tag{19.3.25}$$

$$\|T'(-t)b_{ijk}^{\mathsf{w}}T(t) - (b_{ijk}\circ\Phi_t)^{\mathsf{w}}\| \le C(\mu^{-1}h)^2\Big(1+\varepsilon^{\bar l-4}|\log\varepsilon|^{-\bar\sigma}\Big)\times\Big(1+\varepsilon^{\bar l+1-3}|\log\varepsilon|^{-\bar\sigma}\Big) \le Ch, \tag{19.3.26}$$

Furthermore, (19.3.25) also holds for α (coefficient in (19.3.20)) instead of b_{jk}.

Therefore

(19.3.27) Transformation of A^0 given by (19.3.17) by $T'(1)$ leads to the same expression (19.3.17) but all the symbols η_j', b_{jk}' and b_{ijk}' are replaced by $\eta_j'\circ\Phi_1$, $b_{jk}'\circ\Phi_1$ and $b_{ijk}'\circ\Phi_1$ respectively and the total error does not exceed $C\mu^{-1}h$.

Remark 19.3.3. Now η_1, η_2 are redefined as $p_1\circ\Phi_1 = \xi_1$, $p_2\circ\Phi_1 = x_1$ respectively.

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[19] Really, $\lambda \in \mathscr{F}^{\bar l+1,\sigma}$ and multiplication by p_1 increases regularity by 1 due to inequalities $|p_j| \le c\mu^{-1} \le \varepsilon$ due to (19.2.2).

Let us consider η_j with $j \ge 3$. Note that $\{p_1, \eta_j\} = 0$ on Σ^0 and therefore

$$\eta_j \circ \Phi_t = \eta_j + k_j \eta_1 + \sum_{k,i} \gamma_{jik} p_i p_k. \tag{19.3.28}$$

The same arguments hold for x_j with $j \ge 2$. Therefore, moving errors arising in the quadratic part of A^0 into its cubic part, we can replace $\eta_j \circ \Phi_t$ by $\eta_j + k_j p_1$ for $j \ge 3$, and $x_j \circ \Phi_t$ by x_j for $j \ge 2$; moreover, in b_{jk} we can replace $x_1 \circ \Phi_1 = x_1 - \lambda(x', \xi_2)$ just by $-\lambda(x', \xi_2)$.

Further, from the beginning η_j with $j \ge 3$ could depend on ξ_1 (see (19.3.13), (19.3.14)). After above transformation $\eta_j \circ \Phi_1$ could acquire "more" ξ_1. Note however that originally $\{\eta_j, \eta_2\} = 0$ on Σ^0 and $\eta_2 \circ \Phi_1 = x_1$; therefore $\{\eta_j \circ \Phi_1, x_1\} = 0$ on Σ^0 and therefore in $\eta_j \circ \Phi_1$, which are linear combination of $p_k \circ \Phi_1$, we can replace $p_k \circ \Phi_1$ by $\xi_k - V_k(0, x'')$ as $k \ge 3$ (modifying again cubic terms) where temporarily $x'' = (x_2, \dots, x_d)$.

Thus we arrive to the same quadratic expression but with $p_1 = \xi_1$, $p_2 = x_1$, $p_k = \xi_k - V_k(x_2, \dots, x_d; \xi_2)$, $b_{jk} = b_{jk}(x_2, x_3, \dots, x_d; \xi_2)$.

Repeating this process, in the end of the day we arrive to the same quadratic expression with $\eta_{2k-1} = \xi_k$, $\eta_{2k} = x_k$ $(k = 1, \dots, r)$, $b_{jk} = b_{jk}(x'', \xi'')$ where now $x'' := (x_{r+1}, \dots, x_{2r})$, $\xi'' := (\xi_{r+1}, \dots, \xi_{2r})$:

$$\mu^2 \sum_{j,k} b_{jk}(x'', \mu^{-1} h D'')\, \eta_j^{\mathrm{w}} \eta_k^{\mathrm{w}} + \mu^2 \Big(\sum_{j,k,m} b_{jkm}\, \eta_j \eta_k \eta_m \Big)^{\mathrm{w}} \tag{19.3.29}$$

with $b_{jkm} = b_{jkm}(x, \xi) \in \mathscr{F}^{\bar{l}-1, \bar{\sigma}}$.

Also, considering transformation of $V(x)$ we see that at each step the semiclassical error does not exceed $C(\mu^{-1}h)^2 \varepsilon^{l-3} |\log h|^{-\sigma} \le C\mu^{-1}h$ and in the end of the day we arrive to

$$W(x'', \mu^{-1} h D'') + \Big(\sum_j b_j\, \eta_j \Big)^{\mathrm{w}} \tag{19.3.30}$$

with $W \in \mathscr{F}^{l,\sigma}$ and $b_j = b_j(x, \xi) \in \mathscr{F}^{l-1,\sigma}$.

Therefore we have proven

Proposition 19.3.4. *Let conditions* (19.1.4), (19.3.5) *and* (19.3.6) *be fulfilled,* $\bar{x}$ *be a fixed point and* $\varepsilon \le R \le \epsilon_2$ *with a small enough constant* ϵ_2. *Then there exists a bounded operator* $\mathcal{T}$ *such that*

(i) For operators $Q = q^{\mathrm{w}}$ *with symbol* q *supported in* $B(0,R) \subset \mathbb{R}^{2d}$ *and* $Q' = q_1'^{\mathrm{w}}$ *with* q_1' *supported in* $B((\bar{x},0), 2C_1R) \subset \mathbb{R}^{2d}$ *and equal* 1 *in* $B((\bar{x},0), C_1R) \subset \mathbb{R}^{2d}$ *the following equalities hold modulo negligible operators*

$$(I - Q_1')\mathcal{T}Q \equiv 0, \tag{19.3.31}$$

$$\mathcal{T}^*\mathcal{T}Q \equiv Q \tag{19.3.32}$$

and modulo operators with norm not exceeding $C\mu^{-1}h$

$$\mathcal{T}^*\psi\mathcal{T} \equiv \bar{\psi}^{\mathrm{w}}, \tag{19.3.33}$$

$$\mathcal{T}^*A\mathcal{T}Q \equiv \mathcal{A}Q \tag{19.3.34}$$

with

$$\mathcal{A} = \sum_{i,j} b_{ij}(x'', \mu^{-1}hD'')\bar{P}_i\bar{P}_j + \mu^2\Big(\sum_{i,j,k} b_{ijk}\bar{p}_i\bar{p}_j\bar{p}_k\Big)^{\mathrm{w}} + b_0(x'', \mu^{-1}hD'') + \Big(\sum_j b_i\bar{p}_i\Big)^{\mathrm{w}} \tag{19.3.35}$$

where $b_{ij} \in \mathscr{F}^{\bar{l},\bar{\sigma}}$, $b_{ijk} = b_{ijk}(x,\xi) \in \mathscr{F}^{\bar{l}-1,\bar{\sigma}}$, $b_0 \in \mathscr{F}^{l,\sigma}$, $b_i = b_i(x,\xi) \in \mathscr{F}^{l-1,\sigma}$ *are real-valued,*

$$\bar{p}_{2k-1} = \xi_k, \quad \bar{p}_{2k} = x_k, \qquad k = 1, \dots, r, \tag{19.3.36}$$

$\bar{P}_i = \mu\bar{p}_i^{\mathrm{w}}$, $x' = (x_1, \dots, x_r)$, $x'' = (x_{r+1}, \dots, x_{2r})$ *etc; recall that* $\psi = \psi(x)$ *is a smooth function.*

(ii) For operators $Q' = q'^{\mathrm{w}}$ *with symbol* q' *supported in* $B((\bar{x},0), R) \subset \mathbb{R}^{2d}$ *and* $Q_1 = q_1^{\mathrm{w}}$ *with symbol* q_1 *supported in* $B(0, 2C_1R) \subset \mathbb{R}^{2d}$ *and equal* 1 *in* $B(0, C_1R) \subset \mathbb{R}^{2d}$

$$\|Q'\mathcal{T}(1 - Q_1)\| \le Ch^s. \tag{19.3.37}$$

Remark 19.3.5. (i) Calculations show that

$$b_{ij} = b_{ij}' \circ \Psi_0, \qquad b_0 = V \circ \Psi_0, \quad \bar{\psi} = \psi \circ \Psi \tag{19.3.38}$$

where Ψ is a symplectic map $\mathbb{R}^{2d} \supset B(0,R) \to B((\bar{x},0), 2C_1R)$ and $\Psi_0 = \Pi_x\Psi\big|_{\bar{\Sigma}^0}$ with $\bar{\Sigma}^0 := \{x' = \xi' = 0\}$ and

$$|\det D\Psi_0| = f_1 \cdots f_r. \tag{19.3.39}$$

(ii) Recall that one can rewrite quadratic part as

$$\mu^2 \sum_{\mathfrak{m}\in\mathfrak{M}} \sum_{j,k\in\mathfrak{m}} a_{jk}(x'', \mu^{-1}hD'')\, Z_j^* Z_k - \mu h \sum_k a_{kk}(x'', \mu^{-1}hD'') \tag{19.3.40}$$

with $Z_k = \mu^{-1}hD_k - ix_k$, $Z_k^* = \mu^{-1}hD_k + ix_k$.

(iii) In particular, if $\#\mathfrak{m} = 1$ for all $\mathfrak{m} \in \mathfrak{M}$ (no 2-nd order resonances) one can rewrite quadratic form as

$$\sum_{1\le k\le r} f_j(x'', \mu^{-1}hD'') \left(h^2 D_k^2 + \mu^2 x_k^2\right). \tag{19.3.41}$$

19.3.4 Reducing Next Terms

There is very little what we can do else because of all the resonances. We already finished the transformation of the "main part"

$$\mathcal{A}_0 := \sum_{i,j} b_{ij}(x'', \mu^{-1}hD'')\bar{P}_i\bar{P}_j + b_0(x'', \mu^{-1}hD'') \tag{19.3.42}$$

where one can rewrite quadratic part as (19.3.40); in what follows the quadratic part and the potential of operator generate corrections which could be written in the form

$$\mu^2 \sum_{\alpha:\ 3\le|\alpha|\le\lceil(\bar{l},\bar{\sigma})\rceil+1} b''_\alpha(x'', \mu^{-1}hD'')\,(\bar{p}^\alpha)^{\mathsf{w}} \tag{19.3.43}$$

and

$$\sum_{\alpha:1\le|\alpha|\le\lceil(l,\sigma)\rceil-1} b'''_\alpha(x'', \mu^{-1}hD'')\,(\bar{p}^\alpha)^{\mathsf{w}} \tag{19.3.44}$$

modulo terms with norms not exceeding $C\mu^{-\bar{l}}|\log\mu|^{-\bar{\sigma}}$ and $C\mu^{-l}|\log\mu|^{-\sigma}$ respectively with $b''_\alpha \in \mathscr{F}^{\bar{l}+2-|\alpha|,\bar{\sigma}}$ and $b'''_\alpha \in \mathscr{F}^{l-|\alpha|,\sigma}$.

Here and below $\lceil(l,\sigma)\rceil = \lceil l\rceil$ unless $l \in \mathbb{Z}, \sigma > 0$ in which case $\lceil(l,\sigma)\rceil = l+1$ etc.

The major problem are resonances. The 2-nd order resonances ($f_i = f_j$ as $i \ne j$) prevent us from the diagonalization of the quadratic form as in (19.3.41). We can only rewrite quadratic form as (19.3.40).

Now it is easy to get rid of non-resonant terms of the 3-rd order in the main part and of 1-st order terms in V. Let us consider transformation

$$U = \Big(e^{i\mu^{-1}h^{-1}\left(\mu^2 S_3(x'',\xi'';x',\xi')+S_1(x'',\xi'';x',\xi')\right)}\Big)^{\mathsf{w}} \tag{19.3.45}$$

which is a pseudodifferential operator as $\mu \ge Ch^{-\frac{1}{3}}|\log h|^{\frac{1}{3}}$. As a result the principal part $\mathcal{A}_0$ does not change, and the next terms

$$\mathcal{A}_1 \coloneqq \mu^2 a_1^{\mathsf{w}} \qquad a_1 \coloneqq \sum_{\alpha:|\alpha|=3} b''_\alpha(x'',\xi'')\,\bar p^\alpha + \sum_{\alpha:|\alpha|=1} b'''_\alpha(x'',\xi'')\,\bar p^\alpha \tag{19.3.46}$$

(which have norms $O(\mu^{-1})$) are replaced by

$$\mathcal{A}_1 + \big\{\tilde a^0, \mu^2 S_3 + S_1\big\}'^{\mathsf{w}} + \dots \tag{19.3.47}$$

with $\tilde a^0 = \sum_{i,j} b_{ij}(x'',\xi'')\,p_i p_j$ and the "short" Poisson brackets $\{.,.\}'$ (i. e. with respect to (x',ξ') only).

Here dots denote terms of the same type (19.3.43) with $|\alpha| \ge 4$ (modulo $O(\mu^{-\bar l}|\log\mu|^{-\bar\sigma})$) and of the type (19.3.44) with $|\alpha| \ge 2$ (modulo $O(\mu^{-l}|\log\mu|^{-\sigma})$).

One can see easily that by an appropriate choice of S_3 and S_1 we can eliminate all terms in the "main part" $\mu^2 a_1 + \big\{\tilde a^0, \mu^2 S_3 + S_1\big\}'$ save corresponding to the 3-rd order resonances, namely

$$\mathsf{Re}\Big(\mu^2 \sum_{m,j,k} b''_{mjk}(x'',\xi'')\,\zeta_m^\dagger \zeta_j \zeta_k\Big) \tag{19.3.48}$$

where $\zeta_m = \xi_m + ix_m$, $m = 1,\dots,r$ as we can assume without any loss of the generality that

$$\{\tilde a^0, \zeta_m\}' = \sum_n i f_{mn}\zeta_n \tag{19.3.49}$$

where matrix (f_{mn}) has only positive eigenvalues f_k.

Note that sum (19.3.48) is restricted to indices m, j, k such that f_m is not disjoint from $f_j + f_k$; in particular $b''_{mjk} = 0$ unless $m, j, k \in \mathfrak{n}$ with $\mathfrak{n} \in \mathfrak{N}$.

Further, one can get rid of the non-resonant 4-th order terms in (19.3.43) and 2-nd order terms in (19.3.44) in the same way. However, while the 2-nd order resonant terms are only

(19.3.50) $\zeta_j^\dagger \zeta_k$ with j, k belonging to the same $\mathfrak{m}$-group,

the 4-th order resonant terms include

(19.3.51) $\zeta_j^\dagger \zeta_k \zeta_m \zeta_n$ and $\zeta_j \zeta_k^\dagger \zeta_m^\dagger \zeta_n^\dagger$ with f_j not disjoint from $f_k + f_m + f_n$

and also

(19.3.52) $\zeta_j^\dagger \zeta_k^\dagger \zeta_m \zeta_n$ with $f_j + f + k$ not disjoint from $f_m + f_n$

which we never intended to cover. The same is true for 5-th order resonant terms as well etc.

Still we need to have some concern about only the 3-rd, the 4-th and the 5-th order terms in (19.3.43) and the 2-nd and the 3-rd order terms in (19.3.44) because for $\mu \ge \epsilon(h|\log h|)^{-\frac{1}{2}}$ only these terms could be larger than $C\mu^{-1}h$.

Thus we arrive to

Proposition 19.3.6. *Let conditions* (19.1.4), (19.3.5) *and* (19.3.6) *be fulfilled, $\bar{x}$ be a fixed point. Then there exists a bounded operator $\mathcal{T}$ such that*

(i) For operators $Q = q^{\mathsf{w}}$ with the symbol q supported in $B(0,R) \subset \mathbb{R}^{2d}$ and $Q' = {q_1'}^{\mathsf{w}}$ with the symbol q_1' supported in $B((\bar{x},0),1) \subset \mathbb{R}^{2d}$ and equal 1 in $B((\bar{x},0),1) \subset \mathbb{R}^{2d}$ equalities (19.3.31) *and* (19.3.32) *hold modulo negligible operators and equalities* (19.3.33) *and* (19.3.34) *hold modulo operators with the norms not exceeding $C\bigl(\mu^{-1}h + \mu^{-l}|\log h|^{-\sigma}\bigr)$ with*

$$\mathcal{A} = \mathcal{A}_0 + \mu^2 \operatorname{Re} \sum_{\alpha,\beta:\ 3\le|\alpha|+|\beta|\le\lceil(l,\sigma)\rceil+1} b''_{\alpha\beta}(x'', \mu^{-1}hD'')\left(\zeta^\alpha \zeta^{\dagger\,\beta}\right)^{\mathsf{w}} + \operatorname{Re} \sum_{\alpha,\beta:\ 1\le|\alpha|+|\beta|\le\lceil(l,\sigma)\rceil-1} b'''_{\alpha\beta}(x'', \mu^{-1}hD'')\left(\zeta^\alpha \zeta^{\dagger\,\beta}\right)^{\mathsf{w}} \tag{19.3.53}$$

with $\mathcal{A}_0$ defined by (19.3.42) *(and equal to* (19.3.40)*) and with the symbols $b''_{\alpha\beta}, b'''_{\alpha\beta} \in \mathscr{F}^{l-|\alpha|-|\beta|,\sigma}$ and $b''_{\alpha\beta} = b'''_{\alpha\beta} = 0$ unless*

$$|\sum_j (\alpha_j - \beta_j) f_j| \le \epsilon \tag{19.3.54}$$

with arbitrarily small constant $\epsilon > 0$.

(ii) Further,

$$\bar{\psi} = \sum_{\alpha,\beta\colon 1\le|\alpha|+|\beta|\le\lceil(l,\sigma)\rceil-1} \bar{\psi}_{\alpha\beta}(x'',\xi'')\,\zeta^{\alpha}\zeta^{\dagger\beta}, \tag{19.3.55}$$

where $\bar{\psi}_{\alpha\beta}\in \mathscr{F}^{l-|\alpha|-|\beta|,\sigma}$ *and*

$$\bar{\psi}_{00} = \psi\circ \Psi_0, \tag{19.3.56}$$

$$\bar{\psi}_{\alpha\beta} = \sum_{\gamma\colon 1\le|\gamma|\le|\alpha|+|\beta|} c_{\alpha\beta\gamma}(\partial_x^{\gamma}\psi)\circ\Psi_0 \quad \textit{as}\ \ |\alpha|+|\beta|\ge 1; \tag{19.3.57}$$

(iii) Statement (ii) of Proposition 19.3.4 remains true.

Recall that only cubic terms are obstacles in the proof that $\tilde{a}^0_{\mathsf{m}}$ evolve slowly i.e. with the speed $O(\mu^{-1})$; we prove this statement only for $\tilde{a}^0_{\mathsf{n}}$.

19.3.5 Intermediate and Strong Magnetic Field Cases

Intermediate Magnetic Field Case

The same construction works in the intermediate magnetic field case

$$\mu_2^* := \epsilon(|\log h|)^{-1} \le \mu \le \mu_3^* := \epsilon h^{-1} \tag{19.3.58}$$

but in this case one needs to take parameter ε according to (19.3.7), i.e. $\varepsilon = C(\mu^{-1}h|\log h|)^{\frac12}$ or larger since it is larger now than $C\mu^{-1}$. Then restriction to p_j is now

$$|p_j|\le C\varepsilon. \tag{19.3.59}$$

However, in the operator rather than microlocal sense we have still

$$\|p_j^{\mathsf{w}}\|\le C\mu^{-1} \tag{19.3.60}$$

on the energy levels below c_0 and we should estimate perturbations based on this estimate rather than (19.3.59).

Note, that now we need to take in account only the 3-rd and the 4-th order terms in the first sum in the right-hand expression of (19.3.53) and only the 1-st and the 2-nd order terms in the second sum in the right-hand expression of (19.3.53). Similarly, in (19.3.55) we need to sum as $|\alpha|+|\beta|\le 2$.

Strong Magnetic Field Case

The same construction also works in the strong magnetic field case

$$\mu \ge \mu_3^* := \epsilon h^{-1} \tag{19.3.61}$$

we should take again ε by (19.3.7) and inequality (19.3.59) again holds in the microlocal sense but (19.3.60) is replaced by

$$\|p_j^{\mathsf{w}}\| \le \varsigma := C(\mu^{-1}h)^{\frac{1}{2}} \tag{19.3.62}$$

Really, then

$$\mathcal{A} \equiv \mu^2 \sum_{\mathfrak{m}\in\mathfrak{M}} \sum_{j,k\in\mathfrak{m}} a_{jk}(x', \mu^{-1}hD')\, Z_j^* Z_k + b_0'(x', \mu^{-1}hD') \tag{19.3.63}$$

with symbol $b_0' \in \mathscr{F}^{l,\sigma}$.

Then the principal part of operator $\bar{A}_0$ is of magnitude $C\mu h$ while norms of perturbations $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$, $\bar{A}_4$ do not exceed $C\mu h\varsigma$, $C\mu h\varsigma^2 = Ch^2$, $C\mu h\varsigma^3$, $C\mu h\varsigma^4 = C\mu^{-1}h^3$ respectively and only the last one could be ignored for really large μ. This is the reason beyond correction $h^{-d}\mathcal{N}^{\mathsf{MW}}_{\mathsf{2corr}}$ in Theorem 19.6.26 below. It is an also one of the reasons beyond the last term in the right-hand expression of estimate (19.6.65) and the corresponding extra-smoothness requirement there to get the best possible estimate; another reason is the mollification error.

Anyway, in both intermediate and strong magnetic field cases we arrive to

Proposition 19.3.7. *All statements of Proposition 19.3.6 remains true in both intermediate and strong magnetic field cases with* (19.3.34) *fulfilled modulo operator $\mathcal{R}$ such that*

$$\|\mathcal{R}v\| \le C\Big(\varepsilon^{l}|\log\mu|^{-\sigma} + \mu h\varepsilon^{\bar{l}}|\log\mu|^{-\bar{\sigma}}\Big)\|v\| + \varepsilon^{-\bar{l}}|\log\mu|^{-\bar{\sigma}}\|\mathcal{A}v\| \qquad \forall v \tag{19.3.64}$$

with ε given by (19.3.7).

19.4 Temperate Magnetic Field

Let us consider temperate magnetic field case (19.3.5). While generally we assume that microhyperbolicity condition is fulfilled, in Subsection 19.4.7 we consider the case of constant g^{jk}, F_{jk} when V has critical points.

19.4.1 Mid-Range Propagation

By mid-range propagation we assume propagation for operator $\mathcal{A}$ and thus for operator $\mathcal{T}^*\mathcal{A}\mathcal{T}$ as $T \in [T'_*, T^{*\prime}]$ with

$$T^{*\prime} = C(\mu h|\log h|)^{\frac{1}{2}}, \tag{19.4.1}$$

$$T'_* = C\varepsilon^{-1}h|\log h| \le C\mu h|\log h|; \tag{19.4.2}$$

we are interested almost exclusively in the extending Proposition 19.2.12. Here $T^{*\prime}$ is what used to be T_* [20] and T'_* is the same as T_* as if $\mathcal{A}$ was $\mu^{-1}h$-pseudodifferential operator $\mathcal{A}(x'', \mu^{-1}hD'')$ satisfying microhyperbolicity condition.

Remark 19.4.1. Recall that then $\varepsilon \gtrsim \mu^{-1}$ but we also assume that $\varepsilon \le h^{\delta}$ to avoid some complications as otherwise logarithmic uncertainty principle should be replaced by a microlocal uncertainty principle and below $T'_* = Ch^{1-\delta}$.

However, T'_* is still pretty large unless we increase ε and thus smoothness assumptions. Instead, in the next Section we consider (smoothness-dependent) rate of decay of the left-hand expression of (19.2.63) on interval $[T''_*, T^{*\prime\prime}]$ with $T^{*\prime\prime} = T'_*$ and $T''_* = Ch$.

Consider some point $(\bar{x}, \bar{\xi})$. During time $T^{*\prime}$ (as now we assume that $(\bar{\ell}, \bar{\sigma}) \succeq (2, 0)$) (x'', ξ'') will stay in $C(\mu^{-1}h|\log h|)^{\frac{1}{2}}$-vicinity of $(\bar{x}'', \bar{\xi}'')$ in both classical and microlocal senses (we can follow Proposition 19.2.9 for the proof in the microlocal sense).

According to Subsection 19.3.4

$$A_{\mathcal{T}} := \mathcal{T}^* A \mathcal{T} = \mathcal{A} + \mathcal{R}, \tag{19.4.3}$$

with

$$\mathcal{A} = \sum_{0\le m\le \lceil (l,\sigma)\rceil -1} \mathcal{A}_m, \tag{19.4.4}$$

$$\mathcal{R} = \rho(x, \mu^{-1}hD), \qquad \rho \in \mu^{-l}|\log \mu|^{-\sigma}\mathscr{F}^{0,0}, \tag{19.4.5}$$

[20] See (19.2.53) and (19.2.65) as, for a intermediate magnetic field we have already proven that trace is negligible as $T'_* \le T \le T^*$.

$\mathcal{A}_0$ given by (19.3.42) (where one can rewrite quadratic part as (19.3.40) in the general case and as (19.3.41) as $\#\mathfrak{m}=1$ for all $\mathfrak{m}$) and

$$\mathcal{A}_m = \mu^2 \operatorname{Re} \sum_{\alpha,\beta:\ 3\le|\alpha|+|\beta|=m+2} b''_{\alpha\beta}(x'', \mu^{-1}hD'')\left(\zeta^{\alpha}\zeta^{\dagger\,\beta}\right)^{\mathsf{w}} + \operatorname{Re} \sum_{\alpha,\beta:1\le|\alpha|+|\beta|=m} b'''_{\alpha\beta}(x'', \mu^{-1}hD'')\left(\zeta^{\alpha}\zeta^{\dagger\,\beta}\right)^{\mathsf{w}} \tag{19.4.6}$$

(see (19.3.53)) and $\zeta_j = \xi_j + ix_j$, $\zeta_j^{\dagger} = \xi_j - ix_j$, $j=1,\dots,r$.

Remark 19.4.2. Surely we need to remember that $\mathcal{A}$ and $A_{\mathcal{T}} \coloneqq \mathcal{T}^*A\mathcal{T}$ are close but not equal. In Chapter 18 we considered $\mathcal{A}$ and took $\mathcal{T}\mathcal{A}\mathcal{T}^*$ as an approximation[21)] but now we have the global reduction only in the special case of condition (19.1.25) while until the next Section we need only a weaker assumption $V_j \in \mathscr{C}^{\bar{l}+1,\bar{\sigma}}$.

The following statements hold both for propagators $\boldsymbol{U}' = \boldsymbol{U}_{\mathcal{T}} \coloneqq \mathcal{T}^* e^{-h^{-1}\boldsymbol{A}t}\mathcal{T}$ and its Schwartz kernel $U' \coloneqq U_{\mathcal{T}}$ and for $\boldsymbol{U}' = e^{ih^{-1}\mathcal{A}t}$ and its Schwartz kernel U'.

Proposition 19.4.3. *Let U' be defined in Remark 19.4.2. Let ϕ_1 be supported in $B(0,1)$, $\phi_2=1$ in $B(0,2)$, χ be supported in $[-1,1]$. Let $Q_k = \phi_{k,M\mu^{-1}\mathcal{T}}(x''-\bar{x}'',\xi''-\bar{\xi}'')^{\mathsf{w}}$.*

(i) Then for $T^{*\prime} \le T \le C$ and large enough constant M

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\bigl(1-Q_{2x}\bigr)U_{\alpha\beta}(x,y,t)\,{}^tQ_{1,y}| \le C\mu^{-s} \qquad \forall \tau \le c. \tag{19.4.7}$$

(ii) Further, the same estimate holds for $T'_* \le T \le C$ if we replace $\phi_*(x'',\xi'')$ by $\phi_*(x''_I, \mu^{-1}hD''_{II})$ with an arbitrary partition $x''=(x''_I;x''_{II})$.

Proposition 19.4.4. *Let U' be defined in Remark 19.4.2. Assume that $m \coloneqq \#\mathfrak{M} \ge 2$ and let ϕ_j be functions as before.*

Then for $C(\mu h|\log h|)^{\frac12} \le T \le c_0$

$$|F_{t\to h^{-1}\tau}\chi_T(t)\bigl(1-\phi_{2,T}(\mu^2 a^0_{\mathfrak{m}_1}-\tau_1,\dots,\mu^2 a^0_{\mathfrak{m}_m}-\tau_m)\bigr)^{\mathsf{w}}\times \times U'(x,y,t)\,{}^t\bigl(\phi_{1,T}(\mu^2 a^0_{\mathfrak{m}_1}-\tau_1,\dots,\mu^2 a^0_{\mathfrak{m}_m}-\tau_m)\bigr)^{\mathsf{w}}_y| \le C\mu^{-s} \quad \forall \tau:\ |\tau|\le\epsilon_1. \tag{19.4.8}$$

[21)] Recall that actually we consider A_ε instead of A, so we already have an approximation.

Proof of Propositions 19.4.3–19.4.4. Both Propositions 19.4.3–19.4.4 are proven by the same scheme as in Chapter 18:

To prove Proposition 19.4.3(i), (ii) one can use functions

$$\chi\Big(\big(\frac{\mu^2(|x''-\bar{x}''|^2+|\xi''-\bar{\xi}''|^2)}{T^2}+\epsilon^2\big)^{\frac{1}{2}}-C\varsigma\frac{t}{T}\Big) \tag{19.4.9}$$

and

$$\chi\Big(\big(\frac{\mu^2(|x_I''-\bar{x}_I''|^2+|\xi_{II}''-\bar{\xi}_{II}''|^2)}{T^2}+\epsilon^2\big)^{\frac{1}{2}}-C\varsigma\frac{t}{T}\Big) \tag{19.4.10}$$

respectively with χ function of the same type as used in Theorem 2.3.1 and $\varsigma=\pm 1$ depending on time direction and arbitrarily small constant $\epsilon>0$.

To prove Proposition 19.4.4 one can use function

$$\chi\Big(\big(\frac{\mu^4\sum_j|a_{\mathfrak{m}_j}-\tau_{\mathfrak{m}_j}|^2}{T^2}+\epsilon^2\big)^{\frac{1}{2}}-C\varsigma\frac{t}{T}\Big). \tag{19.4.11}$$

Because $T\le\epsilon_0$ and we picked up velocity with respect to $a_{\mathfrak{m}_j}$, the 3-rd order resonances do not pose any problem here. □

Even if at this stage the decomposition into Hermitian functions is not very useful, we can study propagation using only functions of (x'',ξ'',t).

Proposition 19.4.5. *Let U' be defined in Remark 19.4.2. Let microhyperbolicity condition (see Definition 19.2.4) be fulfilled. Then for $T\in[T'_*,c_0]$, $\tau\in[-\epsilon',\epsilon']$*

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma QU|\le C\mu^{-s} \tag{19.4.12}$$

where $Q=Q(x,\mu^{-1}hD)$.

Proof. Observe that without any loss of the generality one can assume that $\ell(\bar{x}'',\bar{\xi}'';\bar{\tau}_1,\dots,\bar{\tau}_m)$ is equal to $(\ell',0)$. Otherwise one can achieve it by a linear symplectic transformation in (x'',ξ'') and corresponding FIO (which will be a metaplectic operator in this case). Then using

$$\chi\Big(\mu T^{-1}\langle\ell',x''-y''\rangle\pm\epsilon tT^{-1}\Big) \tag{19.4.13}$$

which is an admissible pseudodifferential symbol one can prove in the standard way that

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma''Qu|\le C\mu^{-s} \tag{19.4.14}$$

where here and below Γ'' is a partial trace (with respect to x'' only). □

Propositions 19.2.13 and 19.4.5 imply immediately

Corollary 19.4.6. *(i) Under microhyperbolicity condition estimate* (19.2.63) *holds for* $T \in [T'_*, T^*]$ *with* $T^* = \epsilon$.

(ii) Under $\mathfrak{N}$*-microhyperbolicity condition one can take* T^* *defined by* (19.2.46).

(iii) Also estimate (19.2.76) *holds with* T_* *replaced by the lesser value* T'_*.

Corollary 19.4.7. *Estimates* (19.2.86)–(19.2.88) *hold under corresponding assumptions of Theorem 19.2.16 in the case of* $\mu \le h^{\delta-1}$ *(albeit they are not optimal).*

Proof. Picking $\varepsilon = C\mu h|\log h|$ we get approximation error $Ch^{-d}(\mu h)^l|\log h|^{l-\sigma}$ and $T'_* = \epsilon\mu^{-1}$. Then we again have estimates (19.2.77), (19.2.78) and need just calculate expression (19.2.79) which is equivalent (modulo negligible) to (19.2.80) where without spoiling remainder estimate we can take only terms with $m = 0$ and $n \le n(\delta)$ as $\mu \le h^{-1+\delta}$ with arbitrarily small exponent $\delta > 0$. The rest repeats arguments of Subsection 19.2.4. □

19.4.2 Short-Range Theory: Framework

Still, $T_* = C\varepsilon^{-1}h|\log h|$ is too large. Really, the Tauberian method implies that the remainder does not exceed

$$\frac{1}{T}\sup_{|\tau|\le\epsilon}|F_{t\to h^{-1}\tau}\bigl(\bar\chi_T(t)\Gamma U\psi\bigr)| \tag{19.4.15}$$

while the principal part is

$$\frac{1}{h}\int_{-\infty}^{0}\Bigl(F_{t\to h^{-1}\tau}\bigl(\bar\chi_T(t)\Gamma U\psi\bigr)\Bigr)\,d\tau. \tag{19.4.16}$$

We will reduce (19.4.16) to a more explicit form in Subsection 19.4.5 (where this Subsection approach will be crucial as well) but now let us consider expression (19.4.15). To decrease it one should increase T but generally supremum also grows proportionally T and there is no improvement; the following estimate

$$|F_{t\to h^{-1}\tau}\bigl(\bar\chi_T(t)\Gamma U\psi\bigr)| \le CT\Bigl(h^{-2r}+\mu^r h^{-r}\Bigr) \qquad \forall \tau \le c \tag{19.4.17}$$

rather easily follows from

$$|\phi(hD_t)(\bar{\chi}_T(t)\Gamma U\psi)| \le C\Big(h^{-2r}+\mu^r h^{-r}\Big). \tag{19.4.18}$$

However, results of the previous Subsection shows that while taking $T=T^*$ one can take $\bar{\chi}_{T_*}(t)$ and then (19.4.15) becomes $CT_*T^{*-1}\big(h^{-2r}+\mu^r h^{-r}\big)$. It equals to our dream remainder estimate as $T_* := C_0 h$. Recall that $T_* := C\varepsilon^{-1}h|\log h|$.

Note that even in the smooth case T_* is larger[22)].

We cannot just take $T_* = C_0 h$ because $F_{t\to h^{-1}\tau}\big(\chi_T(t)\Gamma U\psi\big)$ is not negligible for $T\in [C_0h,\, T_*]$; instead our goal is to prove under microhyperbolicity condition that for $T\in (C_0h,\, T_*)$

$$|F_{t\to h^{-1}\tau}\big(\chi_T(t)\Gamma U\psi\big)| \le CT\Big(h^{-2r}+\mu^r h^{-r}\Big)\times \big(\frac{h}{T}\big)^l |\log(\frac{h}{T})|^{-\sigma} \qquad \forall \tau : |\tau|\le \epsilon'. \tag{19.4.19}$$

As $(l,\sigma)\succ(1,1)$ it would imply that left-hand expression of (19.4.17) with $T=T_*$ does not exceeds its right-hand expression with $T=C_0h$ i.e. that in the estimates effectively $T_*=C_0h$.

To achieve this goal we apply η-mollification to $A_{\mathcal{T}}$; this will lead to approximation error in $F_{t\to h^{-1}\tau}\big(\bar{\chi}_T(t)\Gamma U\psi\big)$ which we are going to estimate and to (kind of) negligibility estimate of it after mollification. Minimizing the sum by η (which will depend on T) we will arrive to (19.4.19) for non-mollified operator (surely, the original mollification will be still here).

Alternatively we can try the partition approach like in the previous Chapter 18.

19.4.3 Toy-Model: Some Classes of Pseudodifferential Operators

We start from related simple results (we will need them anyway):

Proposition 19.4.8. *Let $A=A(x,hD)$ be a (matrix) self-adjoint h-pseudodifferential operator in $\mathbb{R}^r$ with symbol $a\in \mathscr{F}^{l,\sigma}$ and satisfying microhyperbolicity condition on level $\tau=0$:*

$$\langle(\ell a)v, v\rangle \ge \epsilon\|v\|^2 - C\|(a-\tau)v\|^2 \qquad \forall v \quad \forall (x,\xi). \tag{19.4.20}$$

[22)] Since "no mollifications needed" has its own pitfalls, we would need actually request $T_*=h^{1-\delta}$ with arbitrarily small exponent $\delta>0$ rather than $T_*=Ch|\log h|$.

Let U be a Schwartz kernel of $e^{-ih^{-1}A}$. Finally, let $\psi \in \mathscr{C}_0^\infty$ be supported in the small vicinity of 0.

(i) Let $I > 1$. Then for $T \in [Ch^{1-\delta}, T^*]$ with arbitrarily small exponent $\delta > 0$ and small enough constant T^*

$$|\phi(hD_t)\chi_T(t)(\Gamma U\psi_y)| \le Ch^{-r}\bigl(\frac{h}{T}\bigr)^{I-1}\bigl(1+\frac{T\varepsilon}{h}\bigr)^{-s}|\log\frac{h}{T}|^{-\sigma}, \tag{19.4.21}$$

$$|F_{t\to h^{-1}\tau}\chi_T(t)(\Gamma U\psi_y)| \le Ch^{1-r}\bigl(\frac{h}{T}\bigr)^{I-1}\bigl(1+\frac{T\varepsilon}{h}\bigr)^{-s}|\log\frac{h}{T}|^{-\sigma} \tag{19.4.22}$$

as $\tau : |\tau| \le \epsilon$ with arbitrarily large exponent s.

(ii) Let $I = 1, \sigma \ge 2$. Then for $T \in [Ch^{1-\delta}, T^*]$ with arbitrarily small exponent $\delta > 0$ and small enough constant T^*

$$|\phi(hD_t)\chi_T(t)(\Gamma U\psi_y)| \le Ch^{-r}|\log\frac{h}{T}|^{-\sigma}\bigl(1+\frac{T\varepsilon}{h}\bigr)^{-s} + Ch^{-r}|\log\frac{h}{T}|^{-s}, \tag{19.4.23}$$

$$|F_{t\to h^{-1}\tau}\chi_T(t)(\Gamma U\psi_y)| \le Ch^{1-r}|\log\frac{h}{T}|^{-\sigma}\bigl(1+\frac{T\varepsilon}{h}\bigr)^{-s} + Ch^{1-r}|\log\frac{h}{T}|^{-s} \tag{19.4.24}$$

as $\tau : |\tau| \le \epsilon$ with arbitrarily large exponent s.

Toy-model. Before an actual proof, let us start with a simple example (actually arising as $d = 2$[23]):

In the case of scalar operator $\Gamma(U\psi)$ is something like $h^{-d}I_1$ with

$$I_1 = \int e^{ih^{-1}\lambda(z)t}q(z)\,dz \tag{19.4.25}$$

and $F_{t\to h^{-1}\tau}\chi_T(t)\Gamma(U\psi)$ is $h^{1-d}I_2$ with

$$I_2 = h^{-1}\iint \chi(t/T)e^{ih^{-1}\lambda(z)t}q(z)\,dz\,dt. \tag{19.4.26}$$

[23] In this case one can apply successive approximations on this stage and to arrive to oscillatory integral expression.

Let us consider the case $T \le \varepsilon^{-1}h$, $l > 1$.

Replacing λ by its η-mollification we make an error $O\big(\vartheta(\eta)\big)$ in λ which leads to error $O\big(Th^{-1}\vartheta(\eta)\big)$ in I_1. In the same time a (multiple) integration by parts with respect to t shows that the error in I_2 is does not exceed

$$Ch^{-2}T^2 \int (Th^{-1}|\lambda - \tau| + 1)^{-2}\, dz \times \vartheta(\eta) \le Ch^{-1}T\vartheta(\eta). \tag{19.4.27}$$

Consider now I_1 and I_2 with mollified λ assuming that $|\partial_{z_1}\lambda| \ge \epsilon$. A multiple integration by parts with respect to z_1 will transform modified I_1 into $(h/T)^k \int e^{ih^{-1}\lambda(z)t} q_k(z)\, dz$ where $q_k = O\big(\eta^{-k-1}\vartheta(\eta)\big)$ as k is large enough and therefore modified expression I_1 is $O\big((h/T)^k\eta^{-k-1}\vartheta(\eta)\big)$.

We can treat expression I_2 in the similar way, but we add an integration by parts with respect to t as well and use of nondegeneracy condition as above. Then modified expression I_2 has the same upper estimate.

Therefore both original expressions I_1 and I_2 do not not exceed

$$CTh^{-1}\vartheta(\eta) + \big(\frac{h}{T}\big)^k \eta^{-k-1}\vartheta(\eta).$$

Picking (near) optimal $\eta = h/T$ we get estimates

$$|I_k| \le C\big(\frac{h}{T}\big)^{-1}\vartheta\big(\frac{h}{T}\big).$$

It proves (19.4.21), (19.4.22) in this special case. □

Proof of Proposition 19.4.8. (a) Note first that due to the standard propagation results for $T \ge Ch\varepsilon^{-1}|\log h|$ left-hand expressions in (19.4.21), (19.4.22) are negligible. Thus we need to consider only $T \le T'_* := Ch\varepsilon^{-1}|\log h|$. We can consider χ supported in $[\frac12, 1]$ and until the end of the proof t, T_k are positive.

Consider η-mollification A_η of operator A with respect to (x,ξ) [24]:

$$1 \gg \eta \ge \varepsilon. \tag{19.4.28}$$

Then the error in the operator (the difference between A_η and A) will not exceed $C\vartheta(\eta)$ in the sense that

$$\|(A_\eta - A)\| \le C\vartheta(\eta) \tag{19.4.29}$$

[24] It means that we mollify its Weyl symbol; recall that original symbol is regular in ε-scale.

and identity

$$e^{ih^{-1}tA} = e^{ih^{-1}tA_\eta} + ih^{-1}\int_0^t e^{ih^{-1}(t-t')A}(A-A_\eta)e^{ih^{-1}t'A_\eta}\,dt' \tag{19.4.30}$$

implies that

$$\|\big(e^{ih^{-1}tA} - e^{ih^{-1}tA_\eta}\big)\| \le CTh^{-1}\vartheta(\eta) \qquad \text{as } |t|\le T, \tag{19.4.31}$$

$$|\chi_T(t)\Gamma\big((U_\eta - U)\psi_y\big)| \le Ch^{-d-1}T\vartheta(\eta) \tag{19.4.32}$$

for the standard cut-off ψ in $B\big((\bar{x},\bar{\xi}),1\big)\subset\mathbb{R}^{2r}_{(x,\xi)}$; this is exactly what we got in the toy-model.

On the other hand, microhyperbolicity condition (19.4.20) and Theorem 2.3.1 imply that $\phi(hD_t)\big(\chi_T(t)\Gamma U_\eta\psi_y\big)$ is negligible provided

$$h^\delta \ge \eta \ge ChT^{-1}|\log h| \tag{19.4.33}$$

(and $\mathsf{supp}(\phi)\subset[-\epsilon,\epsilon]$) and picking minimal $\eta = ChT^{-1}|\log h|$ we arrive to inequality

$$|\phi(hD_t)\Gamma\chi_T(t)U\psi| \le Ch^{-d}\big(\frac{h}{T}\big)^{l-1}\cdot|\log h|^{l-\sigma} \tag{19.4.34}$$

due to $T\ge h^{1-\delta}$. This inequality is almost as good as (19.4.21): we have an extra factor $|\log h|^l$ in the right-hand expression.

(b) To get gid of this $|\log h|^l$ factor we need a more delicate analysis. First, let us iterate (19.4.30):

$$\begin{aligned} e^{ih^{-1}tA} = \sum_{0\le p\le m-1}\frac{1}{p!}i^p h^{-p}\int_{\Delta_p} e^{ih^{-1}(t-t_1-\dots-t_p)A_\eta}(A-A_\eta)\times \\ e^{ih^{-1}t_1A_\eta}(A-A_\eta)e^{ih^{-1}t_2A_\eta}\cdots e^{ih^{-1}t_{p-1}A_\eta}(A-A_\eta)e^{ih^{-1}t_pA_\eta}\,dt_1\cdots dt_p + R_m \end{aligned} \tag{19.4.35}$$

with the remainder

$$\begin{aligned} R_m = \frac{1}{m!}i^m h^{-m}\int_{\Delta_m} e^{ih^{-1}(t-t_1-\dots-t_m)A_\eta}(A-A_\eta)\times \\ e^{ih^{-1}t_1A_\eta}(A-A_\eta)e^{ih^{-1}t_2A_\eta}\cdots e^{ih^{-1}t_{m-1}A_\eta}(A-A_\eta)e^{ih^{-1}t_mA}\,dt_1\cdots dt_m. \end{aligned} \tag{19.4.36}$$

Then the trace norm of the remainder does not exceed $C\big(Th^{-1}\vartheta(\eta)\big)^m h^{-d}$ and it is less than the right-hand expression of (19.4.21) for large enough m (because we gain factor $(h/T)^{l-1}|\log(h/T)|^{l-\sigma}$ on each step).

Now as we take $p=1$ we need to consider

$$e^{ith^{-1}A_\eta}\big(ih^{-1}t(A-A_\eta)\big)\psi. \tag{19.4.37}$$

Without any loss of the generality one can assume that the direction of microhyperbolicity is $\ell=\partial_{\xi_1}$. Let us apply ϵT-admissible partition with respect to x_1: $\psi'=\psi'\sum_\nu \varphi_k(x_1)$. Here and below φ (with different indices) are ϵT-admissible functions.

Due to (19.4.33) the logarithmic uncertainty principle $T\times\eta\ge C'h|\log h|$ holds and therefore due to the microhyperbolicity $\varphi_\nu e^{ith^{-1}A}\equiv \varphi_\nu e^{ith^{-1}A}\tilde\varphi_\nu$ (modulo operator, negligible after cut-off $\phi(hD_t)$) as $\frac12 T\le t\le T$ where the distance between $\operatorname{supp}(\tilde\varphi_\nu)$ and $\operatorname{supp}(\varphi_\nu)$ is exactly of magnitude $\mu^{-1}T$. Then to prove (19.4.21) it is sufficient to estimate properly a trace norm of $\sum_\nu \tilde\varphi_\nu B\psi\varphi'_\nu\times(h/T)^{-1}$. To do this it is sufficient to prove that

(19.4.38) The operator norm of each "sandwich" $\tilde\varphi_\nu B\psi'\varphi_\nu$ does not exceed

$$M:=\big(\frac{h}{T}\big)^l\big(1+\frac{h}{T\varepsilon}\big)^{-s}|\log\big(\frac{h}{T}\big)|^{-\sigma}.$$

Really, then the trace norm of each "sandwich" will be $CTh^{-r}\times M_k$ where T is x_1-size of $\operatorname{supp}(\varphi_\nu)$. Then after summation with respect to ν factor T disappears and we estimate the trace norm of (19.4.37) (after $\phi(hD_t)$ cut-off) by $h^{-r}M\big(h/T\big)^{-1}$, which implies (19.4.21) and (19.4.23).

For $p\ge 2$ we replace the integral over Δ_p by the sum of the integrals over $\Delta_p\cap\{|t_j-\bar t_j|\le\epsilon' T,\ j=1,\dots,p\}$ with very small constants ϵ'.

We also place partitions elements around each copy of the operator exponent, so $e^{ih^{-1}t_jA_\eta}$ is now replaced by $\varphi_{\nu_{2j}}e^{ih^{-1}t_jA_\eta}\varphi_{\nu_{2j+1}}$, $t_0=t-t_1-\dots-t_p$.

Then <u>either</u> supports of $\varphi_{\nu_{2j+1}}$ and $\phi_{\nu_{2j+2}}$ are disjoint by a distance at least $\epsilon\mu^{-1}T$ for some j and we have a sandwich $\varphi_{\nu_{2j+1}}B\varphi_{\nu_{2j+2}}$ of the type described in (19.4.38), <u>or</u> supports of φ_{ν_0} and $\varphi_{\nu_{2p+1}}$ are disjoint by a distance at least $\epsilon\mu^{-1}T$ (and then after taking trace we get negligible term) <u>or</u> $\varphi_{\nu_{2j}}e^{ih^{-1}t_jA_\eta}\varphi_{\nu_{2j+1}}$ is a negligible operator.

Since any sandwich has a norm not exceeding $C\eta^l|\log\eta|^{-\sigma}$ and is accompanied by a factor T/h, it brings as a result a factor $(T/h)^{l-1}|\log(T/h)|^{l-\sigma}$.

To estimate the norm of a sandwich coming from (19.4.36)-term let us consider the sequence of mollifications A_ρ with $\rho = \rho_n = 2^n\rho_0$, $\rho_0 = \max\bigl(h/T, \varepsilon\bigr)$ and $n = 1, 2, \dots, \bar n$ such that $\rho_{\bar n+1} = \eta$.

Then operator $B = \sum_{0\le n\le \bar n} B_n$ with $B_n = A_{\rho_n} - A_{\rho_{n+1}}$ has a $\vartheta(\rho)\mathscr{F}^{0,0}_\rho$ symbol with $\rho := \rho_n$. Then the norm of the sandwich $\tilde\varphi_\nu B_n\varphi_\nu$ does not exceed $C\vartheta(\rho)\times(h/T\rho)^s$ with arbitrarily large s and summation with respect to n does not exceed the same expression with $\rho = \rho_0 = hT^{-1}$; so we get $\vartheta(h/T)\bigl(1 + T\varepsilon/h\bigr)^{-s}$ which is exactly M as defined in (19.4.38).

Thus (19.4.21) and (19.4.23) are proven.

(c) To prove (19.4.22) consider first the case $l > 1$. Then plugging (19.4.35)–(19.4.36) we see that for large enough m we can skip R_m. Also we can skip term with $p = 0$ because it involves only η-admissible operators. Applying the above arguments we need to estimate

$$|F_{t\to h^{-1}\tau}\chi_T(t)(\Gamma v_p\psi_y)| \tag{19.4.39}$$

with v_p a Schwartz kernel of the term in (19.4.35) with the same notations as above. Let us consider $p = 1$ first. One can see easily that

$$\begin{aligned}
&F_{t\to h^{-1}\tau}\Bigl(\chi_T(t)e^{ith^{-1}A_\eta}\Bigr) = \\
&(2\pi)^{-1}\int \chi_T(t)e^{ith^{-1}(A_\eta-\tau)}\,dt = \\
&(2\pi)^{-1}\int \chi_T(t)\bigl(-h^2\partial_t^2+\gamma^2\bigr)^q e^{ith^{-1}(A_\eta-\tau)}\bigl((A_\eta-\tau)^2+\gamma^2\bigr)^{-q}\,dt = \\
&(2\pi)^{-1}\bigl(\frac{h}{T}\bigr)^q\int \chi_{(q),T}(t)e^{ith^{-1}(A_\eta-\tau)}\bigl((A_\eta-\tau)^2+\gamma^2\bigr)^{-q}\,dt
\end{aligned} \tag{19.4.40}$$

with $\chi_{(q)}(t) = (-\partial_t + \gamma^2T^2h^{-2})^q\chi(t)$. Let us pick up $\gamma = \rho_0$.

Let us multiply (19.4.40) from the left by φ'_ν which is equal to 1 in ϵT-vicinity of $\operatorname{supp}(\varphi_\nu)$. Then modulo negligible we can multiply it from the right by $\tilde\varphi_\nu$ which is equal to 0 in ϵT-vicinity of $\operatorname{supp}(\varphi_\nu)$:

$$\begin{aligned}
&F_{t\to h^{-1}\tau}\bigl(\chi_T(t)\varphi'_\nu\, e^{ith^{-1}A_\eta}\bigr) \equiv \\
&\qquad (2\pi)^{-1}\bigl(\frac{h}{T}\bigr)^q\int \chi_{(q),T}(t)\varphi'_\nu\, e^{ith^{-1}(A_\eta-\tau)}\bigl((A_\eta-\tau)^2+\gamma^2\bigr)^{-q}\tilde\varphi_\nu\,dt
\end{aligned} \tag{19.4.41}$$

and therefore

$$(19.4.42)\quad F_{t\to h^{-1}\tau}\bigl(\chi_T(t)\varphi'_\nu\, e^{ith^{-1}A_\eta}\bigr)B\varphi_\nu \equiv$$
$$(2\pi)^{-1}\bigl(\frac{h}{T}\bigr)^q \int \chi_{(q),T}(t)\varphi'_\nu\, e^{ith^{-1}(A_\eta-\tau)}\tilde\varphi_\nu\bigl((A_\eta-\tau)^2+\gamma^2\bigr)^{-q}\tilde\varphi'_\nu\cdot \underbrace{\tilde\varphi_\nu B\varphi_\nu}_{\text{sandwich}}\,dt$$

where the origin of both factors $\tilde\varphi'_\nu$ is clear: the first one is again due to the propagation and the second one is just taken of $\tilde\varphi_\nu$.

Let us note that $\chi_{(q)}$ is a function bounded by $C\bigl(1+\varepsilon T/h\bigr)^{2q}$. Further, note that we already proved that the operator norm of the "sandwich" does not exceed $C\gamma^{lp}|\log h|^{-p\sigma}\bigl(1+\varepsilon T/h\bigr)^{-s}$ and therefore (19.4.22) for term with $p=1$ follows from estimate

$$(19.4.43)\qquad \sum_\nu \|\tilde\varphi'_\nu\bigl((A_\eta-\tau)^2+\gamma^2\bigr)^{-q}\tilde\varphi'_\nu\|_1 \le Ch^{-r}\gamma^{1-2q}$$

where $\|.\|_1$ means the trace norm.

(d) Let us prove (19.4.43). Because operator is positive, its left-hand expression does not exceed

$$(19.4.44)\qquad \mathsf{Tr}\bigl(\bigl((A_\eta-\tau)^2+\gamma^2\bigr)^{-q}\psi\bigr).$$

Let us consider γ-admissible partition; recall that $\gamma\ge\varepsilon$. Because in $B(z,\gamma)$ with $z=(x,\xi)$ variation of symbol of A_η does not exceed $C\gamma$, we conclude that (19.4.44) does not exceed

$$Ch^{-r}\int_{\mathcal{D}} \mathsf{Tr}'\bigl((a(z)-\tau)^2+\gamma^2\bigr)^{-s}\,dz = C\mu^{-r}h^r\int_{\mathcal{D}}\sum_J\bigl((\omega_J-\tau)^2+\gamma^2\bigr)^{-s}\,dz$$

where $a(z)$ is a (matrix) symbol of A_η and Tr' means a (matrix) trace, ω_J are eigenvalues of a (in order) and integrals are taken over bounded domain in $\mathcal{D}\subset\mathbb{R}^{2r}_{x'',\xi''}$.

Then (19.4.43) follows from

$$(19.4.45)\qquad \int\bigl(|\omega_j(z)-\tau|+\gamma\bigr)^{-q}\,dz \le C\gamma^{1-2q}$$

as $q>\frac12$ which, in turn, follows from the microhyperbolicity. Really, without any loss of the generality one can assume that $\ell=\partial_{z_1}$. Then uniformly with

respect to $z' = (z_2, \dots, z_d)$ $\epsilon \le \partial_{z_1}\omega_j \le C$ which instantly yields that even if we restrict ourselves by integral over z_1, (19.4.45) still holds.

Therefore *for a term with* $p = 1$ *estimate* (19.4.22) *is proven*. For $p \ge 2$ we need to modify our trace trick. Consider $(p+1)$-dimensional integrals representing corresponding terms

$$(19.4.46)\quad (2\pi)^{-1}h^{-p}\int_{\{t\ge t_1+\dots+t_p\}} \chi_T(t) e^{ih^{-1}\left((t-t_1-\dots-t_p)A_\eta - t\tau\right)} B\times e^{it_1h^{-1}A_\eta} B\cdots B e^{it_kh^{-1}A_\eta}\, dt_1\cdots dt_p\, dt.$$

Substituting

$$e^{ih^{-1}\left((t-t_1-\dots-t_p)A_\eta - t\tau\right)} = (-h^2\partial_t^2+\gamma^2)(A_\eta^2+\gamma^2)^{-1} e^{ih^{-1}\left((t-t_1-\dots-t_p)A_\eta - t\tau\right)}$$

and integrating one time by parts with respect to t we gain factor $(h/T)^2\big((A_\eta-\eta)^2+\gamma^2\big)^{-1}$ with χ, replaced by $\chi_{(1)} = (-\partial_t^2 + h^{-2}T^2\gamma^2)\chi$, but we also get new terms

$$(19.4.47)\quad ch^{1-p}\int \chi_T(t_1+\dots+t_p)\big((A_\eta-\eta)^2+\gamma^2\big)^{-1}\times e^{-ih^{-1}t\tau} B e^{it_1h^{-1}A_\eta} B\cdots B e^{it_ph^{-1}A_\eta}\, dt_1\cdots dt_p$$

and may be

$$(19.4.48)\quad ch^{2-p}\int \chi_T(t_1+\dots+t_{p-1})\big((A_\eta-\eta)^2+\gamma^2\big)^{-1}\times e^{-ih^{-1}t\tau} B e^{it_1h^{-1}A_\eta} B\cdots B e^{it_{p-1}h^{-1}A_\eta}\, dt_1\cdots dt_{p-1}$$

with some other functions χ. These new terms are just lesser-dimensional integrals of the same type. As before we frame operator exponents by φ_* functions before applying this trick.

(e) Now let $l = 1$, $\sigma > 1$. We need to consider R_m and generated by its expression which is $(m+1)$-dimensional integral

$$(19.4.49)\quad (2\pi)^{-1}h^{-m}\int_{\{t\ge t_1+\dots+t_m\}} \chi_T(t) e^{ih^{-1}\left((t-t_1-\dots-t_m)A_\eta - t\tau\right)} B\times e^{it_1h^{-1}A_\eta} B\cdots B e^{it_mh^{-1}A}\, dt_1\cdots dt_m\, dt.$$

We treat it in the same way as before, but in the end we apply estimate

$$\|(A_\eta + i\gamma)^{-1}\psi\|_1 \le Ch^{-r}\begin{cases} \gamma^{1-q} & \text{as } q>1,\\ |\log\gamma| & \text{as } q=1,\end{cases} \tag{19.4.50}$$

which is proven the same way (but simpler) as (19.4.43); we see that (19.4.49) does not exceed $Ch^{1-r}\bigl(\eta|\log h|^{-\sigma}Th^{-1}\bigr)^m|\log h| = Ch^{1-r}|\log h|^{m(1-\sigma)+1}$ as $T\ge h^{1-\delta}$; for large m it is less than the right-hand expression in (19.4.24).

Further, one can treat m-dimensional integral (19.4.48) in the same way. So, *estimates* (19.4.22) *and* (19.4.24) *are proven completely*. □

19.4.4 Short-Term Estimates

So, let us consider dynamics generated by our reduced operator as

$$T : h^{1-\delta}\le T\le T_* := C\mu h|\log h| \tag{19.4.51}$$

with arbitrarily small exponent $\delta>0$. Note that we need to consider only $T\ge \bar T := \epsilon_0\mu^{-1}$ because for $|t|\le \bar T$ standard theory takes place. Another restriction from below will appear later.

Recall that as $T\ll 1$ we do not need to distinguish microhyperbolicity and $\mathfrak{N}$-microhyperbolicity.

Proposition 19.4.9. *Let microhyperbolicity condition be fulfilled. Let ϕ be supported in the small vicinity of 0.*

(i) Let $l>1$. Then for $T\in[Ch^{1-\delta}, T_*]$ with arbitrarily small exponent $\delta>0$

$$|\phi(hD_t)\chi_T(t)(\Gamma U\psi)| \le Ch^{-d}\bigl(\frac{h}{T}\bigr)^{l-1}|\log\frac{h}{T}|^{-\sigma}\bigl(1+\frac{T\varepsilon}{h}\bigr)^{-s}, \tag{19.4.52}$$

$$|F_{t\to h^{-1}\tau}\chi_T(t)(\Gamma U\psi)| \le Ch^{1-d}\bigl(\frac{h}{T}\bigr)^{l-1}|\log\frac{h}{T}|^{-\sigma}\bigl(1+\frac{T\varepsilon}{h}\bigr)^{-s} \tag{19.4.53}$$

as $|\tau|\le\epsilon$ with arbitrarily large exponent s.

(ii) Let $l=1, \sigma\ge 2$. Then for $T\in[Ch^{1-\delta}, T_*]$ with arbitrarily small exponent $\delta>0$ estimates

$$|\phi(hD_t)\chi_T(t)(\Gamma U\psi)| \le Ch^{-d}\big(\frac{h}{T}\big)^{l-1}|\log\frac{h}{T}|^{-\sigma}\big(1+\frac{T\varepsilon}{h}\big)^{-s} + Ch^{-d}|\log\frac{h}{T}|^{-s}, \tag{19.4.54}$$

$$|F_{t\to h^{-1}\tau}\chi_T(t)(\Gamma U\psi)| \le Ch^{1-d}\big(\frac{h}{T}\big)^{l-1}|\log\frac{h}{T}|^{-\sigma}\big(1+\frac{T\varepsilon}{h}\big)^{-s} + Ch^{1-d}|\log\frac{h}{T}|^{-s} \tag{19.4.55}$$

as $|\tau|\le\epsilon$ *with arbitrarily large exponent* s.

Proof. As in the previous Subsection 19.4.3, we can consider χ supported in $[\frac12,1]$ and in what follows t, T_k are positive.

Consider η-mollification $\mathcal{A}_\eta$ of operator $A_\mathcal{T}$ with respect to x'',ξ'' with η satisfying (19.4.28). Then the error in the operator (i.e. the difference between A_η and A) will not exceed $C\eta^l|\log\eta|^{-s}$ in the sense that

$$\|(A_\eta - A)v\| \le C\eta^l|\log\eta|^{-s}\Big(\|Av\|+\|v\|\Big), \tag{19.4.56}$$

and then we just follow the proof of Proposition 19.4.8 with the following modifications:

(a) we consider $\mu^{-1}h$-pseudodifferential operators with respect to x'' with "matrix" symbols a; furthermore, factor $(\mu h)^{-r}$ comes from the "matrix" trace.

(b) Between different copies of $\mathcal{B}$ we place not only $\varphi_{(\lambda)\nu j}$ but also Q'_j where Q'_j are cut-offs with respect to (x',ξ') keeping all in zone $\{|x'|+|\xi'|\le C\mu^{-1}\}$ (and also $a_\mathfrak{m}$ in vicinity of $\tau_\mathfrak{m}$ for all $\mathfrak{m}\in\mathfrak{M}$).

(c) We remember that $\psi_\mathcal{T} = \psi(x',\mu^{-1}D';x'',\mu^{-1}hD'')$ with $\mathscr{F}^{l,\sigma}$-symbol and we need to mollify ψ as well. Then there will be either $\psi_{\mathcal{T}\eta}$ or $(\psi_{\mathcal{T}\eta}-\psi_\mathcal{T})$ factor; in the latter case it is possible that it is the only factor containing difference. In the former case we follow Subsection 19.4.2, in the latter case we just note that the norm of $(\psi_{\mathcal{T}\eta}-\psi_\mathcal{T})$ is $O(\eta^l|\log\eta|^{-\sigma})$ exactly as in $(A_{\mathcal{T}\eta}-A_\mathcal{T})$ but this time it is not accompanied by a rather large factor T/h and this makes things much more comfortable. □

Then summing over partition of $[-T_*,T_*]\setminus[-T'_*,T'_*]$ with $T'_* = Ch^{1-\delta}$ we get immediately inequality

$$|F_{t\to h^{-1}\tau}\big(\bar\chi_{T_*}(t)-\bar\chi_{T'_*}(t)\big)\Gamma(U\psi)| \le Ch^{1-d}. \tag{19.4.57}$$

Really, the left-hand expression here does not exceed the right-hand expression of (19.4.52) integrated over $\frac{dT}{T}$ from T'_* to T_* which, in turn, does not exceed Ch^{1-d}; contribution of extra term (19.4.54) as $l = 1$ does not exceed Ch^{1-d} as well. On the other hand, we know from rescaling of the standard results that (for $\mu \le \mu_2^*$)

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(U\psi)| \le Ch^{1-d} \tag{19.4.58}$$

as $T = T'_*$ and therefore it holds for $T = T_*$ and it follows from Section 19.2, that (19.4.58) holds for $T = T^*$. Recall that under microhyperbolicity assumption $T^* = \epsilon$ and under $\mathfrak{N}$-microhyperbolicity assumption $T^* = \epsilon\mu$ as either $\#\mathfrak{N} = 1$ or $(l,\sigma) \succeq (2,0)$ and $T^* = \epsilon\mu^{l-1}|\log\mu|^\sigma$ otherwise.

Then standard the Tauberian arguments imply immediately

Corollary 19.4.10. *Under microhyperbolicity condition as $\mu \le h^{\delta-1}$ estimate* (19.2.78) *holds.*

In the case $h^{\delta-1} \le \mu \le \mu_2^*$ we will need analysis of the next Subsection 19.4.5 to prove this statement. Anyway, we need it to get more explicit formula than (19.2.78) provides.

19.4.5 Calculations

In this subsection we still under assumption (19.3.5) will replace rather implicit Tauberian expression formula by a more explicit one and we finish the proof of the crucial inequality

$$|F_{t\to h^{-1}\tau}\Gamma\bar{\chi}_{T_*}(t)U\psi| \le Ch^{1-2r} + C\mu^r h^{1-r}. \tag{19.4.59}$$

Our tool will be method of successive approximations on a rather short interval $[-T_*, T_*]$.

Preliminary Remarks

Now we have a remainder estimate proven in some cases (see Corollary 19.4.10). However, its principal part is given by rather implicit Tauberian expression

$$h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\,\mathsf{Tr}\big(e^{ih^{-1}tA_T}\psi_T\big)\Big)\,d\tau \tag{19.4.60}$$

with arbitrary $T \ge T_* = C\varepsilon^{-1}h|\log h|$ and $\psi_{\mathcal{T}} = \mathcal{T}^*\psi\mathcal{T}$.

Surely, microhyperbolicity condition is fulfilled only at levels close to 0 but as usual we do not need it otherwise. Really, let us decompose expression (19.4.60) into sum of

$$\int_{-\infty}^{0} \bar{\phi}_L(\tau)\Big(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\,\mathsf{Tr}\big(e^{ih^{-1}t\mathcal{A}}\psi_{\mathcal{T}}\big)\Big)\,d\tau \tag{19.4.61}$$

and

$$\int_{-\infty}^{\infty} \phi_L(\tau)\Big(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\,\mathsf{Tr}\big(e^{ih^{-1}t\mathcal{A}}\psi_{\mathcal{T}}\big)\Big)\,d\tau \tag{19.4.62}$$

where $\bar{\phi}$, ϕ are admissible functions, $\bar{\phi}$ is supported in $[-1,1]$ and equal 1 in $[-\frac{1}{2},\frac{1}{2}]$, $\phi = \theta(-\tau)\big(1-\bar{\phi}(\tau)\big)$, L is a small constant. Now we can use microhyperbolicity in the first term and replace $T = T^*$ by any $T \in [T_*, T^*]$. On the other hand, one can rewrite the second expression as

$$\Big(\phi_L(hD_t)\big(\bar{\chi}_T(t)\,\mathsf{Tr}\big(e^{ih^{-1}t\mathcal{A}}\psi_{\mathcal{T}}\big)\big)\Big)\Big|_{t=0} \tag{19.4.63}$$

and then one can replace here $T = T^*$ by any $T \in [T', T^*]$ where $T' = CL^{-1}h|\log\mu| \ll T_*$. One can see easily that expression (19.4.63) is of magnitude $h^{-d} + \mu^r h^{-r}$ and therefore we expect the principal part of asymptotics of the same magnitude. We assume that microhyperbolicity is in the direction ∂_{ξ_1}.

Recall that

$$\int_{-\infty}^{0} h^{-1}\Big(F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\big(U\psi\big)\Big)\,d\tau = T^{-1}\Big(F_{t\to h^{-1}\tau}\check{\chi}_T(t)\Gamma\big(U\psi\big)\Big)\Big|_{\tau=0} \tag{19.4.64}$$

with $\check{\chi}(t) = it^{-1}\chi(t)$; this formula as usual plays very important role below.

Remark 19.4.11. It follows from Propositions 19.4.9, 19.5.10–19.6.20 that in their framework expression (19.4.64) does not exceed $Ch^{-d}\big(h/T\big)^{l}|\log\big(h/T\big)|^{-\sigma}$ and thus

$$\int_{-\infty}^{0} h^{-1}\Big|\Big(F_{t\to h^{-1}\tau}(\bar{\chi}_{T_*}(t)-\bar{\chi}_T(t))\Gamma\big(U\psi\big)\Big)\,d\tau\Big| \le C\big(h^{-d}+\mu^r h^{-r}\big)\big(\frac{h}{T}\big)^{l}|\log\big(\frac{h}{T}\big)|^{-\sigma}. \tag{19.4.65}$$

In particular for $T = \bar{T}_* := \varepsilon^{-1}h$ the right-hand expression does not exceed the remainder estimates we want to prove (in any of our cases). We can even reduce T if the smoothness allows.

In particular, as $T = \epsilon\mu^{-1}$ (and therefore $\varepsilon = \mu h$) we get a remainder estimate $Ch^{-d}(\mu h)^l|\log\mu|^{-\sigma}$ which is a bit better than in Theorem 19.2.16.

Successive Approximation Method

We apply method of successive approximations to calculate both terms (19.4.61) and (19.4.62). We consider our operator $A_{\mathcal{T}}(x', hD'; x'', hD_{x''})$ as a matrix operator and then we take $A_{\mathcal{T}}(x', hD', y'', hD_{x''})$ for unperturbed operator $\bar{A}_{\mathcal{T}}$ (simplifying it later).

Then one can see easily that each next term of successive approximations gets an extra factor $\bar{\mathcal{G}}^{\pm}R$ or $\mathcal{G}^{\pm}R$ where $\bar{\mathcal{G}}^{\pm}$ and $\mathcal{G}^{\pm}$ are forward and backward parametrices of $hD_t - \bar{A}_{\mathcal{T}}$ and $hD_t - A_{\mathcal{T}}$ respectively and by Duhamel principle their operator norms on interval $[-T, T]$ do not exceed Th^{-1}.

Mollified by τ Asymptotics

Each next term of successive approximations when plugged there adds a factor $\|R\|$ to its estimate from above.

Really, it follows from the fact that $t^{-1}\bar{\mathcal{G}}^{\pm}$ and $t^{-1}\mathcal{G}^{\pm}$ are operators with norms h^{-1} and factor t either annuls the corresponding term, or is replaced by its commutator with $\phi_L(hD_t)$ thus releasing factor h. So, with expression (19.4.63) any perturbation with norm $O(h^{\delta})$ would be good enough to have only a bounded number of successive approximations to be considered, but $(x_j - y_j)$ is even better: as usual

$$\bar{\mathcal{G}}^{\pm}(x_j - y_j) = (x_j - y_j)\bar{\mathcal{G}}^{\pm} - \bar{\mathcal{G}}^{\pm}[\bar{A}, x_j]\bar{\mathcal{G}}^{\pm} \tag{19.4.66}$$

and similarly for $\mathcal{G}^{\pm}$, and as explained, these parametrices plugged into expression (19.4.63) do not increase an upper estimate, while the norm of $[\bar{A}, x_j]$ is $O(\mu^{-1}h)$.

Further, each perturbation of the form $(A_{\mathcal{T}} - \mathcal{A})$ adds a factor $\mu^{-l}|\log h|^{-\sigma}$ to the corresponding term in (19.4.63); we take $\bar{\mathcal{A}}$ as unperturbed operator in this Subsubsection.

So any term of the successive approximation but the first one, leads to term in (19.4.63) not exceeding $\bigl(\mu^{-1}h + \mu^{-l}|\log h|^{-\sigma}\bigr) \times h^{-d}$ which, in turn, does not exceed the remainder estimate. Therefore

(19.4.67) Modulo remainder estimate one can replace in (19.4.63) operator $A_{\mathcal{T}}$ by $\bar{\mathcal{A}}$ leading to expression

$$(19.4.68)\quad (2\pi)^{-r}\mu^r h^{-r}\times \int\int \bar{\phi}_L(\tau)\, d_\tau\Big(\mathsf{Tr}'\Big(\theta\big(\tau-\mathcal{A}(x'',\xi'')\big)\psi_{\mathcal{T}}(x'',\xi'')\Big)\Big)\, dx''d\xi''$$

where both $\mathcal{A}(x'',\xi'')$ and $\psi_{\mathcal{T}}(x'',\xi'')$ are considered as "matrices" i.e. as operators in auxiliary space $\mathbb{H}=\mathscr{L}^2(\mathbb{R}^r)$, $\theta\big(\tau-\mathcal{A}(x'',\xi'')\big)$ is its spectral projector and $\mathsf{Tr}'=\mathsf{Tr}_{\mathbb{H}}$ is a "matrix" trace.

Unmollified Asymptotics. I

Unfortunately things are not that good for expression (19.4.61) because parametrices are "worth" of T/h and therefore in view of (19.4.66) factor (x_j-y_j) is "worth" of $T^2h^{-2}\times\mu^{-1}h \asymp \varepsilon^{-2}\mu^{-1}h|\log h|^2 \asymp \mu h|\log h|^2$ (as $T=T_*$) which is not that small, especially for large μ.

In 2D-case considered in the previous Chapter 18 we could always arrange (after an appropriate symplectic map[25)]) to have (x_j-y_j) accomplished by an extra factor $\big(\varepsilon+\varepsilon^{l-1}|\log\varepsilon|^{-\sigma}\big)+\varepsilon^2$ because we considered a scalar symbol.

Here we have essentially a matrix symbol and this trick does not work (however it will work for a strong magnetic field). A bit larger ε does not help much as μ is close to h^{-1}. Further, perturbation $(A_{\mathcal{T}}-\mathcal{A})$ is "worth" of $Th^{-1}\times\mu^{-l}|\log\mu|^{-\sigma}\asymp\varepsilon^{-1}\mu^{-l}|\log h|^{1-\sigma}$ (again as $T=T_*$). In 2D-case we just removed such term at the very beginning.

However, using methods of Subsection 19.4.4 we will be able to insert a factor $\big(h/T\big)^{l}|\log(h/T)|^{-\sigma}$ into estimates of successive the approximation terms which will take care of few copies of factor Th^{-1}.

So, our goal now is to evaluate properly

$$(19.4.69)\qquad F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\big(U\psi\big)=F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\big(U_{\mathcal{T}}\psi_{\mathcal{T}}\big)$$

as $|\tau|\le\epsilon$.

Recall the successive approximation method is the following procedure: rewriting down

$$(19.4.70)\qquad (hD_t-A_{\mathcal{T}})U_{\mathcal{T}}^{\pm}=\mp ih\delta(t)\delta(x-y)$$

[25)] Which after quantization gives us a metaplectic transformation.

with $U^{\pm} = U\theta(\pm t)$ and thus

$$U^{\pm}_{T} = \mp ih\mathcal{G}^{\pm}\delta(t)\delta(x-y) \tag{19.4.71}$$

as

$$(hD_t - \bar{A}_{T})U^{\pm}_{T} = \mp ih\delta(t)\delta(x-y) + (A_{T} - \bar{A}_{T})U^{\pm} \tag{19.4.72}$$

with $U^{\pm} = U\theta(\pm t)$ and thus

$$U^{\pm}_{T} = \mp ih\bar{\mathcal{G}}^{\pm}\delta(t)\delta(x-y) + \bar{\mathcal{G}}^{\pm}(A_{T} - \bar{A}_{T})U^{\pm} \tag{19.4.73}$$

we can iterate the last equation few times:

$$\begin{aligned} U^{\pm}_{T} = \mp ih \sum_{0\le k\le m-1} \Big(\bar{\mathcal{G}}^{\pm}(A_{T} - \bar{A}_{T})\Big)^{k} \bar{\mathcal{G}}^{\pm}\delta(t)\delta(x-y) \\ \mp ih\Big(\bar{\mathcal{G}}^{\pm}(A_{T} - \bar{A}_{T})\Big)^{m} \mathcal{G}^{\pm}\delta(t)\delta(x-y); \end{aligned} \tag{19.4.74}$$

however, our latest technique allows usually to take $m = 1$. Applying ψ_{T} from the right we get that

$$\begin{aligned} U^{\pm}_{T} = \mp ih \sum_{0\le k\le m-1} \Big(\bar{\mathcal{G}}^{\pm}(A_{T} - \bar{A}_{T})\Big)^{k} \bar{\mathcal{G}}^{\pm}\delta(t)\mathcal{K}_{\psi}(x,y) \\ \mp ih\Big(\bar{\mathcal{G}}^{\pm}(A_{T} - \bar{A}_{T})\Big)^{m} \mathcal{G}^{\pm}\delta(t)\mathcal{K}_{\psi}(x,y) \end{aligned} \tag{19.4.75}$$

where $\mathcal{K}_{\psi}(x,y)$ is the Schwartz kernel of ψ_{T}.

(19.4.76) From now on Γ means that we set both Schwartz kernel arguments equal to y' (the same we used in the successive approximations) and then integrate.

Proposition 19.4.12. *In the framework of Proposition 19.4.9 for* $T \in [T', T^{*}]$, $T' := h^{1-\delta}$,

$$\begin{aligned} T^{-1}|F_{t\to h^{-1}\tau}\chi_{T}(t)\Gamma\Big(\bar{\mathcal{G}}^{\pm}(A_{T} - \bar{A}_{T})\mathcal{G}^{\pm}\delta(t)\mathcal{K}_{\psi}\Big)| \le \\ C\mu^{-1}h^{1-d}\big(\frac{h}{T}\big)^{l-2}|\log h|^{-\sigma}. \end{aligned} \tag{19.4.77}$$

Therefore

$$|\int_{-\infty}^{0} F_{t\to h^{-1}\tau}\bigl(\bar{\chi}_{T^*}(t)-\bar{\chi}_{T'}(t)\bigr)\Gamma\Bigl(\bar{\mathcal{G}}^{\pm}(A_{\mathcal{T}}-\bar{A}_{\mathcal{T}})\mathcal{G}^{\pm}\delta(t)\mathcal{K}_{\psi}\Bigr)\,d\tau| \le C\begin{cases}\mu^{-1}h^{1-d} & \text{as } (l,\sigma)\succeq(2,1),\\ \mu^{1-l}h^{1-d}|\log h|^{-\sigma} & \text{as } (l,\sigma)\prec(2,1)\end{cases} \tag{19.4.78}$$

and thus does not exceed remainder estimate of Theorem 19.4.17 below.

Proof. Rewriting

$$A_{\mathcal{T}}-\bar{A}_{\mathcal{T}}=\sum_j (x_j-y_j)R_j,\qquad R_j=R_j(x',\mu^{-1}hD';x'',y'',\mu^{-1}hD'') \tag{19.4.79}$$

we see that

$$\Bigl(\bar{\mathcal{G}}^{\pm}(A_{\mathcal{T}}-\bar{A}_{\mathcal{T}})\mathcal{G}^{\pm}\delta(t)\mathcal{K}_{\psi}\Bigr)= \sum_j\Bigl(\bar{\mathcal{G}}^{\pm}R_j\mathcal{G}^{\pm}\delta(t)\mathcal{K}_{\psi,j}\Bigr)-i\mu^{-1}h\sum_j\Bigl(\bar{\mathcal{G}}^{\pm}R_j\mathcal{G}^{\pm}A_{\mathcal{T}}^{(j)}\mathcal{G}^{\pm}\delta(t)\mathcal{K}_{\psi}\Bigr) \tag{19.4.80}$$

where we used the standard commutation relation (19.4.66) and $\mathcal{K}_{\psi j}=(x_j-y_j)\mathcal{K}_{\psi}$ is the Schwartz kernel of $[x_j,\psi]$. Note that symbols of $\bar{A}_{\mathcal{T}}^{(j)}$, R_j belong to $\mathscr{F}^{l-1,\sigma}$ while symbol of $[x_j,\psi]$ belongs to $\mu^{-1}h\mathscr{F}^{l-1,\sigma}$.

Therefore due to the Duhamel formula the left-hand expression of (19.4.77) is an absolute value of the sum of

$$\begin{aligned}&-i\mu^{-1}hT^{-1}F_{t\to h^{-1}\tau}\chi_{\mathcal{T}}(t)\,\mathsf{Tr}\sum_j\Bigl(\bar{\mathcal{G}}^{\pm}R_j\mathcal{G}^{\pm}A_{\mathcal{T}}^{(j)}\mathcal{G}^{\pm}\psi_{\mathcal{T}}\Bigr)=\\ &-i\mu^{-1}h^{-1}\,T^{-1}F_{t\to h^{-1}\tau}\chi_{\mathcal{T}}(t)\times\\ &\sum_j\mathsf{Tr}\Bigl(\int_{\{t_1+t_2\le t\}}e^{ih^{-1}(t-t_1-t_2)\bar{A}_{\mathcal{T}}}R_j e^{ih^{-1}t_1A_{\mathcal{T}}}A_{\mathcal{T}}^{(j)}e^{ih^{-1}t_2A_{\mathcal{T}}}\psi_{\mathcal{T}}\,dt_1\,dt_2\Bigr)\end{aligned} \tag{19.4.81}$$

and

$$\begin{aligned}&T^{-1}F_{t\to h^{-1}\tau}\chi_{\mathcal{T}}(t)\,\mathsf{Tr}\sum_j\Bigl(\bar{\mathcal{G}}^{\pm}R_j\mathcal{G}^{\pm}[x_j,\psi_{\mathcal{T}}]\Bigr)=\\ &T^{-1}F_{t\to h^{-1}\tau}\chi_{\mathcal{T}}(t)\sum_j\mathsf{Tr}\Bigl(\int_{\{t_1\le t\}}e^{ih^{-1}(t-t_1)\bar{A}_{\mathcal{T}}}R_j e^{ih^{-1}t_1A_{\mathcal{T}}}[x_j,\psi_{\mathcal{T}}]\,dt_1\Bigr).\end{aligned} \tag{19.4.82}$$

For $\eta = Ch T^{-1}|\log h|$ let us make η-mollification $A_{\mathcal{T}\eta}$ of (symbol) of $A_{\mathcal{T}}$ and also of ψ; thus we will get also $\bar{A}_{\mathcal{T}\eta}$. Note that the left-hand expression of (19.4.77) with $A_{\mathcal{T}}$, $\bar{A}_{\mathcal{T}}$ replaced by $A_{\mathcal{T}\eta}$, $\bar{A}_{\mathcal{T}\eta}$ is negligible again due to Proposition 19.4.3. Really, both operators $A_{\mathcal{T}\eta}$, $\bar{A}_{\mathcal{T}\eta}$ are microhyperbolic in the same direction.

Consider the difference between expressions (19.4.80) for η-mollified and original operators.

To do this we apply formula (19.4.35)–(19.4.36) assuming first that $l > 1$ and skipping the negligible remainder. Then the difference in (19.4.81) will be operator

$$-i\mu^{-1}h^{-1-k}F_{t\to h^{-1}\tau}\chi_T(t)\sum_j \mathsf{Tr}$$

applied to the sum of the following terms (with the constant coefficients)

$$\mu^{-1}h^{-1-k-k_1-k_2}\int_{\Delta_k} e^{ih^{-1}(t-t_1-t_2-\ldots-t_k)\bar{A}_{\mathcal{T}\eta}}\mathcal{C}_1 e^{ih^{-1}t_1\bar{A}_{\mathcal{T}\eta}}\mathcal{C}_2\cdots\mathcal{C}_k e^{ih^{-1}t_k\bar{A}_{\mathcal{T}\eta}}\psi'$$

where operators $\mathcal{C}_k$ are in a some order:

(a) one copy of $\bar{A}^{(j)}_{\mathcal{T}\eta}$ or $(\bar{A}^{(j)}_{\mathcal{T}\eta} - A^{(j)}_{\mathcal{T}\eta})$,

(b) one copy of $R_{j\eta}$ or $(R_{j\eta} - R_j)$ and

(c) $(\bar{A}_{\mathcal{T}} - \bar{A}_{\mathcal{T}\eta})$ constitute the rest;

also $\psi' = \psi_{\mathcal{T}\eta}$ or $\psi' = (\psi_{\mathcal{T}} - \psi_{\mathcal{T}\eta})$ and among all the factors there must be at least one factor which is the difference between mollified and non-mollified operators.

Acting as in the proof of Proposition 19.4.8 we can also gain the factor $h^{2q}T^{-2q}\big((\bar{A}_{\mathcal{T}\eta}-\tau)^2+\gamma^2\big)^{-q}$ replacing χ by $(-\partial_t^2+\gamma^2T^2h^{-2})^q\chi$. Further, 2D-integral can also appear instead of 3D one (one extra integral comes from the Fourier transform).

Note that

(a) each factor $(\bar{A}_{\mathcal{T}} - \bar{A}_{\mathcal{T}\eta})$ or $(A_{\mathcal{T}} - A_{\mathcal{T}\eta})$ is worth of $\eta'|\log h|^{-\sigma}$ and is accompanied by a factor not exceeding Th^{-1} while

(b) each factor $(\bar{A}^{(j)}_{\mathcal{T}} - \bar{A}^{(j)}_{\mathcal{T}\rho})$ or $(R_{j\eta} - R_j)$ is worth of $\eta'^{-1}|\log h|^{-\sigma}$ and $(\psi_{\mathcal{T}} - \psi_{\mathcal{T}\eta})$ is worth of $\eta'|\log h|^{-\sigma}$.

Then using estimate (19.4.50) with $\gamma = h/T$ one can derive easily that the difference does not exceed $C\mu^{-1}h^{1-d}\big(h/T\big)^{l-2}|\log h|^{l-\sigma}$ and we arrive to the estimate different from the desired one by an extra factor $|\log h|^{l}$.

There are many methods to get rid of this factor; let us apply the cheapest one. First, let we apply m iterations in successive approximations and note that each extra iteration brings the factor $\mu^{-1}h \times T^2h^{-2} \le \mu h \le |\log h|^{-1}$ and use the same arguments as above; we conclude that instead of $\mathcal{G}^{\pm}$ we need to consider $\big(\bar{\mathcal{G}}^{\pm}(A_{\mathcal{T}} - \bar{A}_{\mathcal{T}})\big)^k\bar{\mathcal{G}}^{\pm}$; so we got rid of $\mathcal{G}^{\pm}$ but paid for this by a bit more complicated expression. This expression includes also $[x_j - x_k, R_j]$, $[x_j - x_k, A^{(j)}]$ etc but one can see easily that the corresponding terms would satisfy the desired estimate.

Also, because $l > 1$ we see from the above mollification arguments that all the terms with more than one factor of $(\bar{A}_{\mathcal{T}} - \bar{A}_{\mathcal{T}\eta})$, $(\bar{A}^{(j)}_{\mathcal{T}} - \bar{A}^{(j)}_{\mathcal{T}\eta})$ and $(R_j - R_{j\eta})$ satisfy the desired estimate and thus we need to consider only terms with exactly one such factor. Further, terms containing factor $(\psi_{\mathcal{T}} - \psi_{\mathcal{T}\eta})$ satisfy this estimate for sure.

Furthermore, if we replace in such factor η by $\rho_0 = h/T$ we will get terms which can be properly estimated and thus in the terms under consideration we can replace $(\bar{A}_{\mathcal{T}} - \bar{A}_{\mathcal{T}\eta})$ by $(\bar{A}_{\mathcal{T}\rho_0} - \bar{A}_{\mathcal{T}\eta}) = \sum_n \mathcal{B}_n$, also replace $(\bar{A}^{(j)}_{\mathcal{T}} - \bar{A}^{(j)}_{\mathcal{T}\eta})$ by $(\bar{A}^{(j)}_{\mathcal{T}\rho_0} - \bar{A}^{(j)}_{\mathcal{T}\eta}) = \sum_n \mathcal{B}'_n$, and replace $(R_j - R_{j\eta})$ by $(R_{j\rho_0} - R_{j\eta}) = \sum_n \mathcal{B}''_n$ where $\mathcal{B}_n := (\bar{A}_{\mathcal{T}\rho_n} - \bar{A}_{\mathcal{T}\rho_{n+1}})$ etc where again $\rho_n = 2^n\rho_0$.

Now we can just repeat arguments of the proof of Proposition 19.4.8, related to $\mu^{-1}T$-partition with respect to x_1 assuming that microhyperbolicity direction is ∂_{ξ_1}.

On the other hand, if $l = 1$, using (19.4.49) and related integration by part trick (but without $\mu^{-1}T$-partition), one can see easily that the far terms in the successive approximation satisfy estimate in question and thus we got rid of $\mathcal{G}^{\pm}$ leaving only $\bar{\mathcal{G}}^{\pm}$.

We can apply the mollification and formulae (19.4.35)–(19.4.36) skipping terms generated by (19.4.36); then we arrive to an estimate which contains in comparison with the desired one an extra factor $|\log h|$ and all the terms save the same as above surely satisfy the desired estimate. Now we just apply the same $\mu^{-1}T$-partition with respect to x_1 etc as above.

Analysis of the expression (19.4.82) uses the same technique but is simpler. □

Unmollified Asymptotics. II

Now we are in the shortest-range zone $|t| \le \bar{T} := h^{1-\delta}$. Then as $\mu \le \epsilon h^{\delta-1}$ we can apply rescaling arguments and arrive to estimate (19.4.59) which implies the corresponding Tauberian estimate.

However, as $\epsilon h^{\delta-1} \le \mu \le \epsilon(h|\log h|)^{-1}$ we still need to estimate $F_{t\to h^{-1}\tau}\bar{\chi}_T\Gamma(U_T\psi_T)$. Using a successive approximation method in its rough form we conclude that modulo term not exceeding $CTh^{-d}\times \mu^{-1}h(T/h)^2 \asymp \mu^{-1}h^{1-d-3\delta}$ we can replace here U by U^0_T which is a Schwartz kernel of $e^{ih^{-1}t\bar{A}}$. Then calculations show that for $T=\bar{T}$ and $\bar{\chi}=1$ on $[-\frac12,\frac12]$

$$F_{t\to h^{-1}\tau}\bar{\chi}_T\Gamma(u^0_T\psi_T) \equiv (2\pi)^{-r}\mu^r h^{1-r}\partial_\tau \int \mathsf{Tr}'\Big(\theta\big(\tau-\mathcal{A}(x'',\xi'')\big)\psi_T(x'',\xi'')\Big)\,dx''d\xi'' \tag{19.4.83}$$

modulo negligible term where $\theta\big(\tau-\mathcal{A}(x'',\xi'')\big)$ is the spectral projector of $\mathcal{A}$ considered as operator in $\mathbb{H}$, $\psi(x'',\xi'')$ is also operator in $\mathbb{H}$ and Tr' is a trace in $\mathbb{H}$.

One can see easily that due to the microhyperbolicity condition for small increment $d\tau$ eigenvalue $\omega_\jmath(x'',\xi'')$ of matrix $\mathcal{A}(x'',\xi'')$ belongs to $[\tau,\tau+d\tau]$ only for (x'',ξ'') belonging to the set of measure not exceeding $Cd\tau$. Since there are $\asymp(\mu h)^{-r}$ of such eigenvalues we conclude that expression (19.4.83) does not exceed Ch^{1-d}.

This and the previous arguments imply immediately estimate (19.4.57) with $T=\bar{T}$ and thus the same estimate (19.4.57) with $T=T_*$ and thus (19.4.57) with $T=T^*$. Then the standard Tauberian arguments imply estimate (19.2.78). So we have proven

Proposition 19.4.13. *Under either microhyperbolicity or $\mathfrak{N}$-microhyperbolicity condition for $\mu\le\epsilon(h|\log h|)^{-1}$ estimates* (19.4.57) *and* (19.2.78) *hold.*

Still we need to get more a explicit formula:

Proposition 19.4.14. *Under either microhyperbolicity or $\mathfrak{N}$-microhyperbolicity condition for $\mu\le\epsilon(h|\log h|)^{-1}$*

$$|\Gamma(\psi\tilde{e})(\tau) - (2\pi)^{-r}\mu^r h^{-r}\int \mathsf{Tr}'\Big(\theta\big(\tau-A_T(x'',\xi'')\big)\psi_T(x'',\xi'')\Big)\,dx''d\xi''| \le C\mu^{-1}h^{1-d}\Big(1+\varepsilon'^{-2}|\log h|^{-\sigma}\Big). \tag{19.4.84}$$

Proof. In view of what we have proven already we need to prove that

$$|h^{-1}\int_{-\infty}^{0} F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma\Big(\bar{\mathcal{G}}^{\pm}(A_{\mathcal{T}}-\bar{A}_{\mathcal{T}})\mathcal{G}^{\pm}\delta(t)\mathcal{K}_{\psi}\Big)\,d\tau| \tag{19.4.85}$$

with $T=\bar{T}$ does not exceed right-hand expression of (19.4.84).

Consider expression (19.4.85) first. Clearly it does not exceed $C\mu^{-1}h^{-d-2}T^3$ and for $T=\bar{T}$ it is just $C\mu^{-1}h^{1-d-3\delta}$ and therefore the required estimate holds as $l<2$; so we need to consider case $l\ge 2$ only.

Applying then the brute-force successive approximations we conclude that replacing in (19.4.85) $\mathcal{G}^{\pm}$ by $\bar{\mathcal{G}}^{\pm}$ leads to a much smaller error. Also, note that

$$A_{\mathcal{T}}-\bar{A}_{\mathcal{T}}=\sum_j (x_j-y_j)\bar{A}_{\mathcal{T}(j)}+\sum_{j,k}(x_j-y_j)(x_k-y_k)R'_{jk}$$

with $R'_{jk}\in \mathscr{F}^{l-2,\sigma}$. Then using the commutator equality for all (x_j-y_j) factors, we conclude that the contribution of these terms will be also much smaller. In the end we are left with the term

$$-i\mu^{-1}\sum_j F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\,\mathsf{Tr}\Big(\bar{\mathcal{G}}^{\pm}\bar{A}_{\mathcal{T}(j)}\bar{\mathcal{G}}^{\pm}\bar{A}^{(j)}_{\mathcal{T}}\bar{\mathcal{G}}^{\pm}\psi_{\mathcal{T}}\Big) \tag{19.4.86}$$

which is reduced to

$$\begin{aligned} i(2\pi)^{-r}\mu^{r-1}h^{-r}\sum_j F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\,\mathsf{Tr}'\int\Big(&\mathcal{G}^{\pm}(x'',\xi'')A_{\mathcal{T}(j)}\\ &(x'',\xi'')\mathcal{G}^{\pm}(x'',\xi'')A^{(j)}_{\mathcal{T}}(x'',\xi'')\mathcal{G}^{\pm}(x'',\xi'')\psi^0(x'',\xi'')\Big)\,dx''d\xi'' \end{aligned} \tag{19.4.87}$$

where we have just matrix-valued symbols $a(x'',\xi'')=A_{\mathcal{T}}(x'',\xi'')$ and $\mathcal{G}^{\pm}(x'',\xi'')$ and we replaced matrix-valued function $\psi_{\mathcal{T}}(x'',\xi'')$ by the scalar-valued one $\psi^0(x'',\xi'')$; this is legitimate since $\psi_{\mathcal{T}}(x'',\xi'')\equiv\psi^0(x'',\xi'')$ modulo $O(\mu^{-1})$.

Using that $\partial_{x_j}\mathcal{G}^{\pm}(x'',\xi'')=\mathcal{G}^{\pm}(x'',\xi'')A_{\mathcal{T}(j)}(x'',\xi'')\partial_{x_j}\mathcal{G}^{\pm}(x'',\xi'')$ and similar identity for ∂_{ξ_j} one can rewrite (19.4.87) due to the trace property and scalar nature of ψ^0 as

$$\begin{aligned} &(2\pi)^{-r}\mu^{r-1}h^{-r}F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\times\\ &\mathsf{Tr}'\int\Big(\mathcal{G}^{\pm}(x'',\xi'')B(x'',\xi'')\mathcal{G}^{\pm}(x'',\xi'')\Big)\psi^0(x'',\xi'')\,dx''d\xi''+\\ &\frac{i}{2}(2\pi)^{-r}\mu^{r-1}h^{-r}F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\,\mathsf{Tr}'\int\mathcal{G}^{\pm}(x'',\xi'')\psi'(x'',\xi'')\,dx''d\xi'' \end{aligned} \tag{19.4.88}$$

with $B = -\frac{i}{2}\sum_j A^{(j)}_{\mathcal{T}(j)}$ and $\psi' = \sum_j \psi^{0(j)}_{(j)}$.We can rewrite the first term here as the convolution with respect to τ of $L\cdot \widehat{\bar{\chi}}(L\tau)$ (with $L = Th^{-1}$) with

$$(19.4.89)\quad (2\pi)^{-r}\mu^{r-1}h^{-r}\,\mathsf{Tr}'\,\partial_\tau\times \int \theta\big(\tau - A_{\mathcal{T}}(x'',\xi'')\big)B(x'',\xi'')\psi^0(x''\xi'')\,dx''d\xi''.$$

Due to the microhyperbolicity and monotonicity of θ with respect to τ this convolution does not exceed $C\mu^{r-1}h^{1-r}\times(\mu h)^{-r}\times(1+\mu^{2-l}|\log\mu|^{-\sigma})$ where the last factor is just upper bound of $B(x'',\xi'')$ and $(\mu h)^{-r}$ comes from the trace Tr' and what we got does not exceed the right-hand expression of (19.4.84).

We can rewrite the second term in (19.4.89) as the convolution with respect to τ of $L\widehat{\bar{\chi}}(L\tau)$ (with $L = Th^{-1}$) with

$$(19.4.90)\qquad (2\pi)^{-r}\mu^{r-1}h^{-r}\,\mathsf{Tr}'\int \theta\big(\tau - A_{\mathcal{T}}(x'',\xi'')\big)\psi'(x''\xi'')\,dx''d\xi''$$

and this convolution does not exceed the same expression where factor $(1+\mu^{2-l}|\log\mu|^{-\sigma})$ this time is an upper bound for ψ'. □

Unmollified Asymptotics. III

Now we can decompose expression

$$(19.4.91)\qquad (2\pi)^{-r}\mu^{r}h^{-r}\int \mathsf{Tr}'\,\theta\big(\tau - A_{\mathcal{T}}(x'',\xi'')\big)\psi_{\mathcal{T}}(x'',\xi'')\,dx''d\xi''$$

as

$$(19.4.92)\quad (2\pi)^{-r}\mu^{r}h^{-r}\int \mathsf{Tr}'\,\theta\big(\tau - \mathcal{A}_0(x'',\xi'')\big)\psi^0(x'',\xi'')\,dx''d\xi'' + h^{-d}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$$

with

$$(19.4.93)\quad \mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}} := (2\pi)^{-r}(\mu h)^r\int \mathsf{Tr}'\Big(\theta\big(\tau - A_{\mathcal{T}}(x'',\xi'')\big) - \theta\big(\tau - \mathcal{A}_0(x'',\xi'')\big)\Big)\times \psi^0(x'',\xi'')\,dx''d\xi''+$$
$$(2\pi)^{-r}(\mu h)^r\int \mathsf{Tr}'\,\theta\big(\tau - A_{\mathcal{T}}(x'',\xi'')\big)\times \big(\psi_{\mathcal{T}}(x'',\xi'') - \psi^0(x'',\xi'')\big)\,dx''d\xi''$$

Then making change of variables Ψ_0 in the integration and using (19.2.18) and (19.3.39) for the Jacobian we conclude that

(19.4.94) The first term in (19.4.92) equals to

$$h^{-d}\int \mathcal{N}^{\mathsf{MW}}(x,\tau)\psi(x)\,dx.$$

Let us represent expression (19.4.93) as

$$\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}} = \int \mathcal{N}^{\mathsf{MW}}_{\mathsf{corr},x}(\tau)\psi(x)\,dx. \tag{19.4.95}$$

While it is clearly the case with the first term in (19.4.93), in the second term we can rewrite

$$\psi_{\mathcal{T}}(x'',\xi'') - \psi^0(x'',\xi'') = \sum_{1\le|\alpha|+|\beta|\le m} \psi_{\alpha\beta}(x'',\xi'')(x')^{\alpha}(\mu^{-1}hD')^{\beta} + O(\mu^{-l}|\log\mu|^{-\sigma}) \tag{19.4.96}$$

with $m = \lfloor (l,\sigma)\rfloor$; then we can just skip the last term (because the output of it will be $O(\mu^{-l}|\log\mu|^{-\sigma}h^{-d})$). Since

$$\psi_{\alpha\beta}\circ\Psi_0 = \sum_{\gamma:1\le\gamma\le|\alpha|+\beta|} \rho_{\alpha\beta\gamma}\partial^{\gamma}\psi(x)$$

we can always rewrite the second term in the required form as well. What we are lacking is the estimate for these two terms.

Proposition 19.4.15. *(i) In the case of the intermediate magnetic field under the microhyperbolicity condition*

$$\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}} \le C\begin{cases} \mu^{-l}|\log\mu|^{-\sigma} & \text{as } l<2;\\ \mu^{-2}(\mu h)^{l-2}|\log(\mu h)|^{-\sigma} + \mu^{-1}h & \text{as } l>2.\end{cases} \tag{19.4.97}$$

(ii) In particular, $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}} \le C\mu^{-1}h$ *for* $(l,\sigma)\succeq(3,0)$.

Therefore $(2,0)\prec(l,\sigma)\prec(3,0)$ is the only case when we cannot skip $h^{-d}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ in the final answer.

Proof. (i) Assume first that $(l,\sigma)\preceq(2,0)$.

First of all, let us replace $A_{\mathcal{T}}$ by $\mathcal{A}$. Due to the microhyperbolicity and the fact that $(A_{\mathcal{T}}-\mathcal{A})$ is bounded by $C\mu^{-l}|\log\mu|^{-\sigma}$, such operation causes an admissible error $O(\mu^{-l}|\log\mu|^{-\sigma}h^{-d})$.

Replacing in the second term in (19.4.93) $\mathcal{A}$ by $\mathcal{A}_0$ we get an error $O(\mu^{-2})$. Really, each factor $\bigl(\theta(\tau-\mathcal{A})-\theta(\tau-\mathcal{A}_0)\bigr)$ and $(\psi_{\mathcal{T}}-\psi^0)$ contributes the factor μ^{-1} into the remainder estimate. So we need to consider the second term in (19.4.93) with $A_{\mathcal{T}}$ replaced by $\mathcal{A}_0$ and with $(\psi_{\mathcal{T}}-\psi^0)$ replaced by ψ^1,

$$\psi^m=\sum_{|\alpha|+|\beta|=m}\psi_{\alpha\beta}(x'',\xi'')(x')^{\alpha}(\mu^{-1}hD')^{\beta} \tag{19.4.98}$$

but then after taking trace Tr' we get 0.

In the first term in (19.4.93) $\mathcal{A}=\mathcal{A}_0+\mathcal{A}_1$. Consider $\mathcal{A}_0$ as an unperturbed operator and $\mathcal{A}_0+\mathcal{A}_1$ as a perturbed one. Let us return back to representation (19.2.78).

We claim that

(19.4.99) As $m>l+1$, the contribution of m-th term of successive approximations does not exceed $Ch^{-d}\vartheta(\mu^{-1})$.

Really, replace $\mathcal{A}_0$ and $\mathcal{A}_1$ by their η-mollifications. Then, in virtue of our previous arguments, replacing η-mollification of $\mathcal{A}_0$ by its 2η-mollification we get an error not exceeding

$$Ch^{-d}\vartheta(\eta)\times\Bigl(\frac{\mu^{-1}T}{h}\Bigr)^{m-1}\times\Bigl(1+\frac{\eta T}{h}\Bigr)^{-s} \tag{19.4.100}$$

and similarly replacing η-mollification of $\mathcal{A}_1$ by its 2η-mollification we get an error not exceeding the same expression with an factor $\vartheta(\eta)$ replaced by $hT^{-1}\vartheta(\eta)\eta^{-1}$; thus we get the same expression (19.4.100).

Summation with respect to η returns $Ch^{-d}\vartheta\bigl(h/T\bigr)\times\bigl(T/\mu h\bigr)^{m-1}$ and then summation with respect to $T\le C\mu h$ returns this expression as $T=\mu h$ i.e. $Ch^{-d}\vartheta(\mu^{-1})$. On the other hand, as $T\ge C\mu h$ we just do not need to consider successive approximations but the whole difference.

Further, as we have η-mollified $\mathcal{A}_0$ and $\mathcal{A}_1$, the contribution of this term does not exceed

$$Ch^{1-d}T^{-1}\times \big(\frac{\mu^{-1}T}{h}\big)^{m-1}\times \big(1+\frac{\eta T}{h}\big)^{-s}$$

and taking $\eta=\epsilon$ we get $Ch^{-d}\mu^{1-m}$ after summation with respect to T. So (19.4.99) is proven.

Therefore for $l<2$ we can take $m=3$ i.e. take only two terms of the successive approximations.

Note that the first term of the successive approximations annihilates with the corresponding expression for an unperturbed operator.

Recall that

$$\mathcal{A}_0=\Big(\mu^2\sum_{j,k} a_{jk}\zeta_j^\dagger\zeta_k+V\Big)^{\mathsf{w}}$$

preserves spaces $\mathscr{H}_n$ and

$$\mathcal{A}_1=\Big(2\operatorname{Re}\mu^2\sum_{j,k,p} b_{jkp}\zeta_j^\dagger\zeta_k\zeta^p+\sum_k b_k\zeta_k\Big)^{\mathsf{w}}$$

maps $\mathscr{H}_n$ into $\mathscr{H}_{n+1}\oplus\mathscr{H}_{n-1}$ and therefore after taking trace the second term of successive approximations results in 0 for ψ_T replaced by ψ^0.

We need also to consider the similar expressions for ψ_T replaced by ψ^1 but in this case the first term of successive approximations vanishes and the second term has a desired estimate.

(ii) On the other hand, for $l>2$, $(l,\sigma)\preceq(3,0)$ we need to remember that $\mathcal{A}=\mathcal{A}_0+\mathcal{A}_1+\mathcal{A}_2$, $\psi_T=\psi^0+\psi^1+\psi^2$ and also we need to consider four extra terms (the rest definitely is estimated in a desired way) which are[26)]

$$ih^{-1}\int_{-\infty}^0 F_{t\to h^{-1}\tau}\bar\chi_T(t)\operatorname{Tr}\Big(\sum_{\pm}\mp\mathcal{G}_0^{\pm}\mathcal{A}_1\mathcal{G}_0^{\pm}\mathcal{A}_1\mathcal{G}_0^{\pm}\psi^0\Big)\,d\tau, \tag{19.4.101}$$

$$ih^{-1}\int_{-\infty}^0 F_{t\to h^{-1}\tau}\bar\chi_T(t)\operatorname{Tr}\Big(\sum_{\pm}\mp\mathcal{G}_0^{\pm}\mathcal{A}_2\mathcal{G}_0^{\pm}\psi^0\Big)\,d\tau, \tag{19.4.102}$$

$$ih^{-1}\int_{-\infty}^0 F_{t\to h^{-1}\tau}\bar\chi_T(t)\operatorname{Tr}\Big(\sum_{\pm}\mp\mathcal{G}_0^{\pm}\mathcal{A}_1\psi^1\Big)\,d\tau, \tag{19.4.103}$$

$$ih^{-1}\int_{-\infty}^0 F_{t\to h^{-1}\tau}\bar\chi_T(t)\operatorname{Tr}\Big(\sum_{\pm}\mp\mathcal{G}_0^{\pm}\psi_2\Big)\,d\tau \tag{19.4.104}$$

[26)] As we return back to parametrices, cut-off on interval $[-\hat T,\hat T]$ etc.

with $T = \hat{T} = \mu h$ and $\mathcal{G}_0^{\pm}$ parametrices of $hD_t - \mathcal{A}_0$. Here Tr means that we take $x = y$ and then integrate; recall that all operators here depend on y.

Really, even as $l = 3$ we need to consider to $m = 4$ but this term vanishes in virtue the above arguments related to $\mathscr{H}_n$-scale.

Replacing $\bar{\chi}_T(t)$ by $\chi_T(t)$ and applying the same technique as before we find that such modified expressions (19.4.101)–(19.4.104) do not exceed $C\mu^{-2}h^{-d}(h/T)^{l-2}|\log(h/T)|^{-\sigma}$.

Then after summation over t-partition we find that the contribution of segment $[\bar{T}, \hat{T}]$ is estimated now by $C\mu^{-2}h^{-d}(h/\bar{T})^{l-2}|\log(h/\bar{T})|^{-\sigma}$. We take $\bar{T} = \epsilon\mu^{-1}$. This is clearly possible provided

$$\mu \le h^{\delta-1}. \tag{19.4.105}$$

Assuming this, let us consider contribution of $[-\bar{T}, \bar{T}]$. Returning from $\mathcal{T}$ reduction to the initial settings we can rewrite (19.4.101)–(19.4.104) (modulo terms, estimated properly) as the similar expressions but with $\mathcal{G}_0^{\pm}$ replaced by parametrices $\bar{G}^{\pm}$ of operator $\bar{A}$, given by (19.2.84):

$$\bar{A} = \sum_{j,k} \bar{g}^{jk}\bar{P}_j\bar{P}_k + \bar{V}, \qquad \bar{g}^{jk} = g^{jk}(y), \bar{V} = V(y), \tag{19.4.106}$$

$$\bar{P}_j = hD_j - V_j(y) - \sum_k (\partial_k V_j)(y)(x_k - y_k),$$

and $\mathcal{A}_j$ and ψ^j are replaced by

$$\bar{A}_j = \sum_{\alpha:|\alpha|\ge 2+j} \mu^{2-|\alpha|} b_\alpha(y)\bar{P}^\alpha + \sum_{\alpha:|\alpha|\ge j} \mu^{-|\alpha|} b_\alpha(y)\bar{P}^\alpha, \tag{19.4.107}$$

and

$$\bar{\psi}^j = \sum_{\alpha:|\alpha|\ge j} \mu^{-|\alpha|}\psi_\alpha(y)\bar{P}^\alpha \tag{19.4.108}$$

respectively.

Note that the coefficients of all operators are smooth in ε-scale with $\varepsilon = \mu^{-1}$ which is larger than $Ch|\log h|$ required by the standard theory which is applicable then (after we scale $x \mapsto \mu x$, $h \mapsto \mu h$, $t \mapsto \mu t$ producing that

(a) before rescaling the contributions of intervals $[-\epsilon\mu^{-1}, -Ch|\log h|]$ and $[Ch|\log h|, \epsilon\mu^{-1}]$ are negligible and

(b) the contribution of interval $[-Ch|\log h|, Ch|\log h|]$ is equal to $\mu^{-2}\sum_{j\ge 0}\kappa_j(\mu h)^j$ with the coefficients κ_j which do not depend on μ, h.

We should not care about terms with $j \ge 1$ here because these terms do not exceed $C\mu^{-1}h^{1-d}$. Therefore,

$$h^{-d}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}} = \kappa_0\mu^{-2}h^{-d} + O\big(\mu^{-2}(\mu h)^{l-2}|\log(\mu h)|^{-\sigma} + \mu^{-1}h\big)$$

(we can alway replace it modulo properly estimated).

Now notice that the "main part" of asymptotics is $h^{-d}\int \mathcal{N}^{\mathsf{MW}}(x)\psi(x)\,dx$ which in comparison with Theorem 19.2.16 for $\mu \asymp (h|\log h|)^{-\frac{1}{2}}$ implies that $\kappa_0 = 0$.

Therefore, under assumption (19.4.105) statement (ii) of proposition is proven.

(iii) To get rid of assumption (19.4.105) let us return to terms (19.4.101)–(19.4.104) with $\bar{\chi}$ replaced by χ and with $T \ge \epsilon\mu^{-1}$.

Each of these expressions is equal to $\mu^{-2}h^{-d}f(T/h, \mu h)$ and therefore the above arguments imply estimate

$$\mu h|\log h| \le \epsilon, \quad T \ge h^{1-\delta}, \quad 1 \ge \varepsilon \ge C_0\mu^{-1} \implies |f(\frac{T}{h}, \mu h)| \le C(\frac{h}{T})^{l-2}|\log(\frac{h}{T})|^{-\sigma}. \tag{19.4.109}$$

This is a really strange inequality because its assumptions involve T and h separately but its conclusion contains them only as T/h. This gives us a certain flexibility. Namely, let us assume that

$$h \le T \le h^{1-\delta} \tag{19.4.110}$$

(otherwise everything is fine) and replace $h \mapsto h\lambda$, $T \mapsto T\lambda$, $\mu \mapsto \mu\lambda^{-1}$ which affects neither μh nor (h/T). Assumptions of (19.4.109) should be fulfilled now with these modified h, T, μ. To fulfil the second one we take $\lambda = h^{-1}(hT^{-1})^{1/\delta}$ and this is greater than 1 due to (19.4.110).

The first of the assumptions of (19.4.109) will survive[27] but the third one is now replaced by a more restrictive condition $\varepsilon \ge C_0\mu^{-1}\lambda$. To get rid

[27] In fact, it will be fulfilled as long as $\mu h|\log(h/T)| \le \epsilon'$.

of it let us apply $C\mu^{-1}\lambda$-mollification which leads to the error in $f(T/h, \mu h)$, not exceeding

$$C(\mu^{-1}\lambda)^{l-1}|\log \mu^{-1}\lambda|^{-\sigma}\bigl(\frac{T}{h}\bigr)^2 + C(\mu^{-1}\lambda)^{l}|\log \mu^{-1}\lambda|^{-\sigma}\bigl(\frac{T}{h}\bigr)^3.$$

This is clearly less than the right-hand expression in (19.4.109) provided $T \ge h(\mu h)^{-\delta'}$ with arbitrarily small exponent $\delta' > 0$ (δ depends on it). It also implies $1 \ge \varepsilon$.

Now inequality (19.4.109) is proven under this humble assumption. Then one can take $\bar{T} = \epsilon\mu^{-1}$ which clearly satisfies it.

The rest of arguments of Part (ii) hold without any modifications. □

Remark 19.4.16. Case $l = 2$ is obviously missing in the statement of Proposition 19.4.15.

(i) Using the same arguments as above we find that contribution of zone $\{\bar{T} \le |t| \le \hat{T}\}$ does not exceed $C\mu^{-2}h^{-d}|\log(\mu h)|^{1-\sigma}$ as $\sigma > 1$ and $C\mu^{-2}h^{-d}|\log h|^{1-\sigma}$ as $\sigma < 1$ and $C\mu^{-2}h^{-d}|\log\bigl(|\log(\mu h)|/|\log h|\bigr)|$ as $\sigma = 1$.

On the other hand, contribution of zone $\{|t| \le \bar{T}\}$ will be smaller as $\sigma \le 1$ and $\kappa_0\mu^{-2}h^{-d} + O\bigl(\mu^{-2}h^{-d}|\log(\mu h)|^{1-\sigma}\bigr)$ as $\sigma > 1$ and the same arguments as above show that $\kappa_0 = 0$.

Therefore in comparison with the case $l \ne 2$ an extra factor $|\log \mu h|$ appears as $\sigma \ne 1$; as $\sigma = 1$ it is $|\log \mu h| \cdot |\log\bigl(|\log(\mu h)|/|\log h|\bigr)|$.

(ii) Further, if <u>either</u> there are no third-order resonances <u>or</u> g^{jk} and F_{jk} are constant, then $\mathcal{A}_1 = 0$ after proper reduction. We can exploit it properly only for $\sigma \le 0$: one can see easily that then $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}} \le C\mu^{-2}|\log h|^{-\sigma}$.

This concludes the proof of Theorem 19.4.17 below.

19.4.6 Main Theorem

Now combining all the results of this Section we arrive to

Theorem 19.4.17. *Let assumptions* (19.1.4)–(19.1.6), $(19.1.25)_{1-3}$ *with* $(\bar{l}, \bar{\sigma}) \succeq (l, \sigma) \succeq (1, 2)$, $(\bar{l}, \bar{\sigma}) \succeq (2, 1)$ *and* (19.2.38) *be fulfilled. Let*

$$\mu_1^* := C(h|\log h|)^{-\frac{1}{2}} \le \mu \le \mu_2^* := \epsilon'(h|\log h|)^{-1} \tag{19.4.111}$$

with sufficiently small constant $\epsilon' > 0$.

Then there are two framing approximations[18] *(see Chapter 18) such that*

(i) Let $\mathfrak{N}$*-microhyperbolicity condition (see Definition 19.2.5) be fulfilled. Then if* <u>*either*</u> $(l,\sigma) \succeq (2,0)$ <u>*or*</u> $\#\mathfrak{N} = 1$*, then*

$$\mathsf{R}_1^{\mathsf{MW}} := \Bigl| \int \Bigl(\tilde{e}(x,x,0) - h^{-d}\mathcal{N}^{\mathsf{MW}}(x,0) - h^{-d}\mathcal{N}^{\mathsf{MW}}_{1\mathsf{corr}}(x,0)\Bigr)\psi(x)\,dx\Bigr| \le C\mu^{-1}h^{1-d} + C\mu^{-l}|\log h|^{-\sigma}h^{-d} \tag{19.4.112}$$

and if $(l,\sigma) \prec (2,0)$ <u>*and*</u> $\#\mathfrak{N} \ge 2$ *then*

$$\mathsf{R}_1^{\mathsf{MW}} \le C\mu^{l-l}h^{1-d}|\log\mu|^{-\sigma} + C\mu^{-l}|\log h|^{-\sigma}h^{-d}. \tag{19.4.113}$$

(ii) Let microhyperbolicity condition (see Definition 19.2.4) be fulfilled. Then

$$\mathsf{R}_1^{\mathsf{MW}} \le Ch^{1-d}| + C\mu^{-l}|\log h|^{-\sigma}h^{-d}. \tag{19.4.114}$$

Remark 19.4.18. (i) Condition (19.4.111) means exactly that estimate (19.4.112) is stronger than (19.2.86), (19.2.87).

(ii) If $(l,\sigma) \succeq (3,1)$ then the best remainder estimate (19.2.89) holds as $\mu \le \mu_2^*$.

19.4.7 Special Case of Constant g^{jk}, F_{jk}

Framework; Mid- and Long-Range Propagation

Consider now the case of constant g^{jk} and F_{jk}, $(l,\sigma) \succeq (2,0)$ and V having only non-degenerate critical points. Our goal is to extend results obtained for V which does not have any critical points.

We assume that $l < 4$ as $l > 3$ will be sufficient for the remainder estimate $O(\mu^{-1}h)$. Note that the contribution of $\{x:\ |\nabla V| \le \bar{\nu} := C_1\mu^{-\frac{1}{2}}\}$ to the remainder does not exceed $C\bar{\nu}^d\mu h^{1-d} = O(\mu^{-1}h^{-1})$ and therefore we need to consider domain $\{x:\ |\nabla V| \ge \bar{\nu}\}$ only.

Let us introduce in this domain

$$\varepsilon = C_0\mu^{-1}\nu^{-1} \le \nu, \qquad \nu := |\nabla V|. \tag{19.4.115}$$

One can see easily that

(19.4.116) Mollification error does not exceed $C\mu^{-l}|\log h|^{-\sigma}h^{-d}$

exactly as in the case when V does not have critical points.

Exactly as in Subsection 19.4.1 one can prove easily that

(19.4.117) At $B(y,\nu(y))$ with $\nu(y)\ge\bar\nu$ estimates (19.4.7) and (19.4.8) hold for $T\in[T_*,T^*]$ with $T^*=\epsilon\mu$ and

$$T_*=C\varepsilon^{-1}\nu^{-1}h|\log h|\asymp C\mu h|\log h|. \tag{19.4.118}$$

Short-Range Theory

Short-range theory as in Subsections 19.4.2–19.4.4 is more complicated:

Proposition 19.4.19. *Let $\psi\in\mathscr{C}_0^\infty(B(y,\nu(y))$ with $\nu(y)\ge\bar\nu$ and let ϕ be supported in L-vicinity of 0, $L\ge\nu^2+\mu h+h/T$.*

Then for $T\in[Ch^{1-\delta},T^*]$ with arbitrarily small exponent $\delta>0$ such that $T\ge\nu^{-2}h$

$$|\phi(hD_t)\chi_T(t)(\Gamma U\psi)|\le CL\nu^{d+1}h^{-d}\Big(\frac{h}{\nu T}\Big)^{l-1}|\log\frac{h}{\nu T}|^{-\sigma}\Big(1+\frac{h}{\nu T\varepsilon}\Big)^{-s}, \tag{19.4.119}$$

$$|F_{t\to h^{-1}\tau}\chi_T(t)(\Gamma U\psi)|\le CL\nu^{d-1}h^{1-d}\Big(\frac{h}{\nu T}\Big)^{l-1}|\log\frac{h}{\nu T}|^{-\sigma}\Big(1+\frac{h}{\nu T\varepsilon}\Big)^{-s} \tag{19.4.120}$$

as $\tau\le c$ with arbitrarily large exponent s.

Idea of the proof. The proof follows the same scheme as Proposition 19.4.9. Instead of giving the full proof, which we leave to the reader, let us consider the toy-model for the proof of Proposition 19.4.8.

Consider again I_1,I_2 defined by (19.4.25) and (19.4.26) respectively. Replace λ by its η-mollification with $\eta\le\epsilon\nu h^\delta$. Then an error in I_1 is $O\big(h^{-1}T\vartheta(\eta)\nu^d\big)$ (just acquires factor ν^d as a volume of $B(y,\nu(y))$ in comparison with the original toy-model) while an error in I_2 is given by the left-hand expression of (19.4.27) which does not exceed $Ch^{-1}T\vartheta(\eta)\nu^{d-2}$ now.

Obviously I_1 with mollified λ does not exceed $C(h/\nu T)^k\eta^{-k-1}\vartheta(\eta)\nu^d$ and I_2 with mollified λ does not exceed $C\nu^{-1}(h/\nu T)^k\eta^{-k-1}\vartheta(\eta)\nu^d$ and therefore

$$I_1+\nu^2I_2\le Ch^{-1}T\Big(1+\nu(h/\nu T)^{k+1}\eta^{-k-1}\Big)\vartheta(\eta)\nu^d.$$

Taking near optimal $\eta = h/T\nu^{1-\delta'}$ we conclude that

$$I_1 + \nu^2 I_2 \le Ch^{-1}T\vartheta(h/\nu T)\nu^d.$$

Note that $\eta \le h^\delta \nu$ as $T \ge \max(h^{1-\delta}, h/\nu^2)$.

This would imply for a single $\mu^{-1}h$-operator estimates as in the proof of Proposition 19.4.9 but with T replaced by $T\nu$ and with extra factors ν^{d+1} and ν^{d-1}. However, we also need to take into account that the ellipticity is broken for no more than $CL(\mu h)^{-r}$ multiindices $\alpha \in \mathbb{Z}^{+r}$ rather than $(\mu h)^{-r}$ as it was for $\nu \asymp 1$ and this leads to an extra factor CL.

Easy details are left to the reader. □

Then summation of the right-hand expression of (19.4.120) for $T \ge h/\nu^2$ returns $O(\nu^d h^{1-d})$.

On the other hand note that

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)(\Gamma U\psi)| \le CT\nu^d h^{-d}$$

and plugging $T = h/\nu^2$ into the right-hand expression we get $C\nu^{d-2}h^{1-d}$.

Corollary 19.4.20. *In the framework of Proposition 19.4.19 as either $\mu \le h^{\delta-1}$ or $\nu \le h^\delta$*

(i) Estimate holds

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)(\Gamma U\psi)| \le C\nu^{d-2}h^{1-d} \tag{19.4.121}$$

for $T \le T^* = \epsilon\mu$.

(ii) Therefore contribution of $B(y,\nu(y))$ to the Tauberian remainder estimate does not exceed $C\mu^{-1}\nu^{d-2}h^{1-d}$.

Calculations

Following Subsection 19.4.5 we arrive to expression (19.4.83) where now $\mathcal{A} = \mathcal{A}_0 + \mu^{-2}\mathcal{A}_2 + \mathcal{A}'$ with $\mathcal{A}' = O(\varepsilon^l|\log\varepsilon|^{-\sigma})$ and applying the method of the successive approximations we conclude first that modulo $o(h^{1-d})$ we can replace here $\mathcal{A}$ by $\mathcal{A}_0$ and then (19.4.121) holds without restriction "either $\mu \le h^{\delta-1}$ or $\nu \le h^\delta$".

Summation with respect to a partition of unity results in the remainder estimate $O(\mu^{-1}h^{1-d})$ under assumption (19.2.26). Therefore

(19.4.122) Under assumption (19.2.96) Tauberian remainder is $O(\mu^{-1}h^{1-d})$.

Further, let us consider expression (19.4.83) integrated with respect to τ. Let us skip $\mathcal{A}'$; one can see easily that the contribution of $B(y,\nu(y))$ to the error does not exceed $C\big(h^{-d}+\mu h^{1-d}\nu^{-2}\big)\nu^d\varepsilon^l|\log\varepsilon|^{-\sigma}$ and summation with respect to partition of unity results in $O(h^{-d}\varepsilon^l|\log\varepsilon|^{-\sigma})$.

Furthermore, skipping term $\mu^{-2}\mathcal{A}_2$ results in the correction term which after replacing a Riemann sum by an integral as $\nu^2\ge\mu h$ results in the expression $\mu^{-2}h^{-d}\int\kappa_0(y)\,dy$; contribution of $B(y,\nu(y))$ into an error does not exceed

$$C\mu^{-2}h^{-d}(\mu h/\nu^2)^{l-2}|\log(\mu h/\nu^2)|^{-\sigma}\nu^d$$

and summation with respect to partition of unity results in

$$C\mu^{-2}h^{-d}(\mu h)^{l-2}|\log(\mu h)|^{-\sigma}. \tag{19.4.123}$$

Meanwhile, skipping term $\mu^{-2}\mathcal{A}_2$ as $\nu^2\le\mu h$ results in an error; one can prove easily that the contribution of $B(y,\nu(y))$ into this error does not exceed $C\mu^{-1}h^{1-d}\nu^{d-2}$. Treating all other terms (arising when we replace ψ by ψ^0) in the similar way we arrive to a correction term not exceeding (19.4.123) as "final" κ_0 must be 0.

Main Theorem

Then we arrive to

Theorem 19.4.21. *Let assumptions* (19.1.4)–(19.1.6), $(19.1.25)_3$ *with* $(l,\sigma)\succeq(1,2)$ *and* (19.2.38) *be fulfilled. Let* g^{jk}, F_{jk} *be constant.*

Then there are two framing approximations[18)] *(see Chapter 18) such that the following statements are true:*

(i) Under microhyperbolicity assumption $|\nabla V|\ge\epsilon$ *estimate* (19.4.112) *holds.*

(ii) Under assumptions $(l,\sigma)\succeq(2,0)$ *and* (19.2.96) *estimate* (19.4.112) *holds.*

(iii) We can skip the correction term unless $(2,0)\prec(l,\sigma)\prec(3,0)$ *in which case it does not exceed* (19.4.123).

(iv) We have remainder estimate $O(\mu^{-1}h^{1-d})$ *as* $\mu\ge(h|\log h|)^{-\frac12}$ *provided* $(l,\sigma)\succeq(3,1)$.

19.5 Intermediate Magnetic Field

19.5.1 Framework and Special Cases

Let now

$$\mu_2^* := \epsilon(h|\log h|)^{-1} \le \mu \le \mu_3^* := \epsilon h^{-1}. \tag{19.5.1}$$

Then we need to modify our arguments in several ways. First of all now, as $(\mu^{-1}h|\log h|)^{\frac{1}{2}} \ge \mu^{-1}$ we must assume that

$$\varepsilon \ge C(\mu^{-1}h|\log h|)^{\frac{1}{2}}. \tag{19.5.2}$$

Then under $\mathfrak{N}$-microhyperbolicity condition with $\#\mathfrak{N} = 1$ we are done and we have immediately the first clause (either) of our final Theorem 19.5.2 below which differs from Theorem 19.4.17(i) only by a choice of ε and thus by a mollification error.

Furthermore, in the case of the constant g^{jk}, F_{jk} we are done as well and we have immediately our final Theorem 19.5.3 which differs from Theorem 19.4.21 in the same way.

However, under either microhyperbolicity condition with $\#\mathfrak{M} \ge 2$ or $\mathfrak{N}$-microhyperbolicity condition with $\#\mathfrak{N} \ge 2$ we were dealing with the partition of energies and then variables (x', ξ') are no more microlocal as logarithmic uncertainty principle for them fails and our approach here needs to be modified.

Still, we are also done if we either use microhyperbolicity assumption and there are no 2-nd order resonances or we use $\mathfrak{N}$-microhyperbolicity assumption and there are no 2-nd and 3-rd order resonances.

Really, in these cases we need to consider operators

$$a_j = f_j(x'', \mu^{-1}hD'')(h^2D_j^2 + \mu^2 x_j^2) \tag{19.5.3}$$

with $\mathfrak{m} = \{j\}$ and $\mathfrak{n} = \{j\}$ respectively and instead we can consider commuting operators $(h^2D_j^2 + \mu^2 x_j^2)$ and we do not need to have logarithmic uncertainty principle for these variables to establish the following proposition similar to Proposition 19.2.10(i),(ii) with small constant νT:

Proposition 19.5.1. *Let either there are no 2-nd order resonances and $T^* = \epsilon$ or there are no 2-nd and 3-rd order resonances and $T^* = \epsilon\mu$. Let $\mu \le \epsilon_0 h^{-1}$.*

Let $B_j = (h^2 D_j^2 + \mu^2 x_j^2)$ *and* $\boldsymbol{B} = (B_1, \dots, B_d)$. *Let* ϕ_1, ϕ_2 *be two functions with*

$$\operatorname{dist}(\operatorname{supp}(\phi_1), \operatorname{supp}(\phi_2)) \ge (T/T^* + \epsilon_1) \tag{19.5.4}$$

the following estimate holds:

$$\|\big(\phi_2(\boldsymbol{B})\big)_x U\, {}^t\big(\phi_1(\boldsymbol{B})\big)_y\| \le Ch^s. \tag{19.5.5}$$

Then we have immediately clause or of our final Theorem 19.5.2.

Thus we arrive to our first main theorem in the case of a strong magnetic field:

Theorem 19.5.2. *Let assumptions* (19.1.4)–(19.1.6), $(19.1.25)_{1-3}$ *with* $(\bar l, \bar\sigma) \succeq (l, \sigma) \succeq (1, 2)$, $(\bar l, \bar\sigma) \succeq (2, 1)$ *and* (19.2.38) *be fulfilled. Let* μ *satisfy* (19.5.1).

Then there are two framing approximations[18] *(see Chapter 18) such that*

(i) Let $\mathfrak{N}$*-microhyperbolicity condition (see Definition 19.2.5) be fulfilled and either* $\#\mathfrak{N} = 1$ *or there be no* 2*-nd and* 3*-rd order resonances. Then*

$$\mathsf{R}_1^{\mathsf{MW}} := \Big| \int \Big(\tilde e(x, x, 0) - h^{-d} \mathcal{N}^{\mathsf{MW}}(x, 0) - h^{-d} \mathcal{N}^{\mathsf{MW}}_{1\mathsf{corr}}(x, 0) \Big) \psi(x)\, dx \Big| \le C\mu^{-1} h^{1-d} + C\mu^{-\frac{l}{2}} |\log h|^{\frac{l}{2}-\sigma} h^{-d+\frac{l}{2}} \tag{19.5.6}$$

(ii) Let microhyperbolicity condition (see Definition 19.2.4) be fulfilled and there are be no 2*-nd order resonances. Then*

$$\mathsf{R}_1^{\mathsf{MW}} \le Ch^{1-d} + C\mu^{-\frac{l}{2}} |\log h|^{\frac{l}{2}-\sigma} h^{-d+\frac{l}{2}}. \tag{19.5.7}$$

Theorem 19.5.3. *Let assumptions* (19.1.4)–(19.1.6), $(19.1.25)_3$ *with* $(l, \sigma) \succeq (1, 2)$ *and* (19.2.38) *be fulfilled. Let* g^{jk}, F_{jk} *be constant.*

Then there are two framing approximations[18] *(see Chapter 18) such that the following statements are true:*

(i) Under microhyperbolicity assumption $|\nabla V| \ge \epsilon$ *estimate* (19.5.6) *holds.*

(ii) Under assumptions $(l, \sigma) \succeq (2, 0)$ *and* (19.2.96) *estimate* (19.5.6) *holds.*

19.5.2 Partition of the Energy Space

Consider the general case now. To prove a statement similar to Proposition 19.4.5 under either microhyperbolicity condition with $\#\mathfrak{M} \ge 2$ or $\#\mathfrak{N}$-microhyperbolicity condition with $\#\mathfrak{N} \ge 2$ one needs to consider evolution of $a_{\mathfrak{m}}$ or $a_{\mathfrak{n}}$ and the main trouble is that we can microlocalize them only in the box of size $\mu^2\varepsilon^2 \asymp \mu h|\log h|$ which is by no means smaller than ϵ. We need instead *operator localization*: something like Proposition 19.5.1 but with $\boldsymbol{B} = (B_1, \dots, B_m)$ with $B_{\mathfrak{m}}$ operators "close" to $a^0_{\mathfrak{m}} - \tau_{\mathfrak{m}}$, $m = \#\mathfrak{M}$ [28)].

Therefore we need to define function of operators and thus operators need to be self-adjoint and "almost commute".

Let B be an operator of the form

$$B = \sum_{i,k\in\mathfrak{m}} \bigl(\beta^{\mathrm{w}}_{j,k}(x'', \mu^{-1}hD'')\, Z_j Z_k^*, \qquad Z_j = hD_j + i\mu x_j \tag{19.5.8}$$

where $\beta_{jk} \in \mathscr{F}^{2,0}$, $\beta_{jk} = \beta^{\dagger}_{kj}$ and the complex sesquilinear form $\sum_{j,k} \beta_{jk}\zeta_j\zeta_k^{\dagger}$ is positive definite; here $\mathfrak{m}$ is just a subset in $\{1, \dots, d\}$.

This will be sufficient to study propagation if either $T^* = \epsilon$ or $T^* = \epsilon\mu$ and there are no 3-rd order resonances; however if $T^* = \epsilon\mu$ and there are 3-rd order resonances we consider

$$\begin{aligned} B = \sum_{j,k\in\mathfrak{n}} \bigl(\beta^{\mathrm{w}}_{jk}(x'', \mu^{-1}hD'')\, Z_j Z_k + \\ \mu^{-1} \sum_{j,k,m\in\mathfrak{n}} \Bigl(\beta^{\mathrm{w}}_{jkm}(x'', \mu^{-1}hD'')\, Z_j Z_k Z_m^* + \beta^{\dagger\,\mathrm{w}}_{jkm}(x'', \mu^{-1}hD'')\, Z_m Z_k^* Z_j^*\Bigr) + \\ C_0\mu^{-2}\Bigl(\sum_{j\in\mathfrak{n}} Z_j^* Z_j\Bigr)^2 \end{aligned} \tag{19.5.8$'$}$$

where $\beta_{jkm} \in \mathscr{F}^{1,0}$ and the last term is added to make the symbol positive.

Then this operator is self-adjoint and

$$\phi(B) = \int e^{itB}\hat{\phi}(t)\, dt \tag{19.5.9}$$

where $\hat{\phi}$ is a Fourier transform.

[28)] Or similar estimate with $\mathfrak{m}$ and $\mathfrak{M}$ replaced by $\mathfrak{n}$ and $\mathfrak{N}$.

Let temporarily $A = \alpha^{\mathrm{w}}(x'', \mu^{-1}hD'')$ or

$$A = \sum_j \Big(\alpha^{\mathrm{w}}(x'', \mu^{-1}hD'')Z_j + \alpha^{\dagger\mathrm{w}}(x'', \mu^{-1}hD'')Z_j^*\Big) \tag{19.5.10}$$

with $\alpha \in \mathscr{F}^{1,0}$. Let us consider $[\phi(B), A]$. Note first that

$$e^{itB}Ae^{-itB} = \sum_{0\le k\le m-1} \frac{1}{k!}(it)^k \operatorname{Ad}_B^k(A) + \frac{1}{(m-1)!}(it)^m \int_0^1 (1-z)^{m-1} e^{itzB} \operatorname{Ad}_B^m(A) e^{-itzB}\, dz$$

where

$$\operatorname{Ad}_B^0 A \coloneqq A, \quad \operatorname{Ad}_B^m A \coloneqq \big[B, \operatorname{Ad}_B^{m-1} A\big]. \tag{19.5.11}$$

and therefore

$$[e^{i\tau B}, A] = \sum_{1\le k\le m-1} \frac{1}{k!}(it)^k \operatorname{Ad}_B^k(A) e^{itB} + \frac{1}{(m-1)!}(it)^m \int_0^1 (1-z)^{m-1} e^{itzB} \operatorname{Ad}_B^m(A) e^{it(1-z)B}\, dz$$

and

$$[\phi(B), A] = \sum_{1\le k\le m-1} \frac{1}{k!} \operatorname{Ad}_B^k(A)\phi^{(k)}(B) + \frac{1}{m!}\int_{-\infty}^{\infty}\int_0^1 e^{itzB} \operatorname{Ad}_B^m(A) e^{it(1-z)B} \widehat{\phi^{(m)}}(t)\, dzdt. \tag{19.5.12}$$

Consider repeated commutators $\operatorname{Ad}_B^k(A)$. Note first that if B is of the form (19.5.8), A is defined by (19.5.10) and if in the commutators we consider only derivatives with respect to (x', ξ') then obviously these shortened commutators would be also of the form (19.5.10) but with norms $C_1(C_0\mu h)^k$.

On the other hand, in virtue of Proposition 1.A.1 we can find uniformly smooth function $\phi = \phi_n$, $\phi_n \ge \frac{1}{2}$ on the given interval and such that $|\phi^{(k)}| \le C_0^{k+1} n^k$ as $k \le m+2 \le n$. Then terms in the sum in the right-hand

expression in (19.5.12) would not exceed $C_1(C_0\mu h)^k n^k k^{-k}$ which is $O(h^s)$ as $\mu h \le \epsilon$ and

$$k \ge C_s|\log h|/|\log \mu h|, \qquad n \le \epsilon_0 k/(\mu h). \tag{19.5.13}$$

Let us now consider complete commutators. Then as $\beta_{jk} \in \mathscr{F}^{2,0}$, $\alpha_j \in \mathscr{F}^{1,0}$ we conclude that in comparison with $\mathsf{Ad}_B^{m-1}(A)$ each term in $\mathsf{Ad}_B^m(A)$ acquires either factor $C_0\mu h$ (if there are only differentiations with respect to (x',ξ') or $C_1\mu^{-1}h\varepsilon^{-1}N$ where $N = \mu h^{-1}\varepsilon^2$; it happens as long as $k \le N$, which is obviously the case; recall that $\varepsilon \ge C(\mu^{-1}h|\log h|)^{\frac12}$.

Further, similarly one can estimate an integral term in (19.5.12). Furthermore, similar arguments work for operator B defined by (19.5.8)$'$.

Therefore we arrive to

Proposition 19.5.4. *As ε is defined by* (19.5.2) *(or larger)*

$$[\phi(B), A] \equiv \sum_{1\le k\le M-1} \frac{1}{k!}\,\mathsf{Ad}_B^k(A)\phi^{(k)}(B), \tag{19.5.14}$$

$$\text{with} \quad M = C_1|\log h|/|\log(\mu h)|.$$

Similarly one can see easily that

$$e^{itB_1}e^{it'B_2}e^{-itB_1}e^{-it'B_2} \sim I + \sum_{p,q} \frac{1}{p!q!}(it)^p(it')^q L_{pq}$$

with

$$L_{pq} = \sum_{1\le j\le p}\sum_{1\le k\le q} \frac{p!q!}{j!k!(p-j)!(q-k)!} B_1^j B_2^k (-B_1)^{p-j}(-B_2)^{q-k} = \tag{19.5.15}$$

$$\begin{cases} \displaystyle\sum_{1\le k\le q} \frac{q!}{k!(q-k)!}\,\mathsf{Ad}_{B_1}^p(B_2^k)(-B_2)^{q-k} = \\ \displaystyle\sum_{1\le j\le p} \frac{p!}{j!(p-j)!} B_1^{p-j}\,\mathsf{Ad}_{B_2}^q((-B_1)^j); \end{cases}$$

we will use the first (the second) expression as $q \le p$ ($p \le q$) respectively.

Then one can prove easily

Proposition 19.5.5. *As ε is defined by* (19.5.2) *(or larger)*

$$[\phi_1(B_1),\phi_2(B_2)] \equiv \sum_{1\le q,p\le M-1} \frac{1}{p!q!} L_{pq}\phi_2^{(q)}(B_2)\phi_1^{(p)}(B_1), \tag{19.5.16}$$

$$\text{with}\quad M = C_1|\log h|/|\log(\mu h)|.$$

Proof. We leave an easy proof using the same arguments as the proof of Proposition 19.5.4 to the reader. □

Corollary 19.5.6. *If* $\mathsf{dist}(\mathsf{supp}(\phi_1),\mathsf{supp}(\phi_k)) \ge \epsilon_0$, $i_1 = i_k$ *and* $\mu h \le \epsilon$ *then*

$$\phi_1(B_{i_1})\phi_2(B_{i_2})\cdots\phi_k(B_{i_k}) \equiv 0. \tag{19.5.17}$$

Proof. If $k = 3$ multiplying (19.5.16) with B_1, B_2 replaced by B_{i_1}, B_{i_2} respectively by $\phi_k(B_{i_1})$ on its right and using $\phi_1^{(p)}(B_{i_1})\phi_{i_3}(B_{i_3}) = 0$ we arrive to the required equality.

In the general case let φ be functions such that $\varphi = 1$ in vicinity of $\mathsf{supp}(\phi_k)$ and $\varphi = 0$ in vicinity of $\mathsf{supp}(\phi_1)$. Then in virtue of (19.5.17) with $k = 3$ we can insert $\varphi(B_{i_1})$ between $\phi_{k-2}(B_{i_{k-2}})$ and $\phi_{k-1}(B_{i_{k-1}})$ and apply induction with respect to k. □

This is very important: we can localize with respect to $B_1,\dots,B_k$ *simultaneously* by $\phi_1(B_1)\phi_2(B_2)\cdots\phi_k(B_k)$.

19.5.3 Propagation

Now we can apply the same arguments as in Theorem 2.3.1. Consider operator

$$\chi\bigl(\epsilon^{-1}(B-s)+\frac{t}{T}\bigr) \tag{19.5.18}$$

and apply it to $U^{\pm} = \theta(t)U$. Then

$$(hD_t - A)U^{\pm} = \mp ih\delta(t)\delta(x-y) \tag{19.5.19}$$

and applying $\mu^{-1}h$-pseudodifferential operator cutoff we get

$$(hD_t - A)W^{\pm} = \mp ihf(x,y) \qquad \text{with} \tag{19.5.20}$$

$$W = U\,{}^t\bar{\varphi}_y(B)\,{}^tQ_y,\quad f = \delta(t)\delta(x-y)\,{}^t\bar{\varphi}_y(B)\,{}^tQ_y,$$

where $\bar{\varphi}$ is supported in vicinity of $\tau \in \mathbb{R}$ while χ is our standard function and then

$$|T\,\mathsf{Re}\, ih^{-1}\Big(\Big[(hD_t - A), \chi\big(\epsilon^{-1}(B-\tau)+\frac{t}{T}\big)\Big]W^{\pm}, W^{\pm}\Big)| \le Ch^{2s}. \tag{19.5.21}$$

Transforming this inequality we get

$$|\Big(\chi'\big(\epsilon^{-1}(B-\tau)+\frac{t}{T}\big)W^{\pm}, W^{\pm}\Big)| \le T\epsilon^{-1}|\Big(\Big[A, \chi\big(\epsilon^{-1}(B-\tau)+C_0\frac{t}{T}\big)\Big]W^{\pm}, W^{\pm}\Big)| + Ch^{2s} \tag{19.5.22}$$

where χ' is derivative of χ.

Approximating commutator by Poisson brackets (multiplied by $(-ih)$) we get

$$\epsilon^{-1}T\{a,b\}\chi'\big(\epsilon^{-1}(B-\tau)+\frac{t}{T}\big)$$

with the first factor being an operator with a norm not exceeding $\frac{1}{2}$ provided

$$T \le T^* := \begin{cases} \epsilon_1 & \text{as } b = a_{\mathfrak{m}}, \\ \epsilon_1\mu & \text{as } b = a_{\mathfrak{n}} + O(\mu^{-2}), \quad (l,\sigma) \succeq (2,0), \\ \epsilon_1\mu^{l-1}|\log\mu|^{-\sigma} & \text{as } b = a_{\mathfrak{n}} + O(\mu^{-2}), \quad (l,\sigma) \prec (2,0) \end{cases} \tag{19.5.23}$$

where ϵ_1 is small enough constant.

Therefore, taking χ to be a primitive of function φ^2, considering next terms of the commutator and assuming that

$$\|\varphi\big(\epsilon^{-1}(B-\tau)+\frac{t}{T}-\epsilon_2\big)W^{\pm}\| \le Ch^{s-\delta} \tag{19.5.24}$$

with a small exponent $\delta > 0$, we get

$$\|\varphi\big(\epsilon^{-1}(B-\tau)+\frac{t}{T}\big)W^{\pm}\|^2 \le C_1\Big(\sum_{1\le m\le M}\frac{1}{m!}\eta^m\|\varphi^{(m)}\big(\epsilon^{-1}(B-\tau)+\frac{t}{T}\big)W^{\pm}\|\Big)^2 + Ch^{2s} \tag{19.5.25}$$

with $\eta = C_0\mu h$.

Plugging $\frac{1}{m!}\eta^m\varphi^{(m)}$ instead of φ and $(M-m)$ instead of M into inequality (19.5.25) we conclude that

$$\Big(\frac{1}{m!}\eta^m\|\varphi^{(m)}\big(\epsilon^{-1}(B-\tau)+\frac{t}{T}\big)W^{\pm}\|\Big)^2 \tag{19.5.26}$$

also does not exceed the right-hand expression of (19.5.25).

Taking a sum with respect to m, $0\le m\le M$, and again increasing C_0, C_1 we get that expression

$$\sum_{0\le m\le M}\Big(\frac{1}{m!}\eta^m\|\varphi^{(m)}\big(\epsilon^{-1}(B-\tau)+\frac{t}{T}\big)W^{\pm}\|\Big)^2 \tag{19.5.27}$$

does not exceed itself multiplied by $C_1\eta$ plus Ch^{2s}. Taking η small enough we conclude that (19.5.24) holds with $\epsilon_2=0$ and $\delta=0$. Consequently increasing s by δ and doubling ϵ_2 we get rid of assumption (19.5.24). Thus

$$\|\varphi\big(\epsilon^{-1}(B-\tau)+C_0\mu^{-1}t\big)U\,{}^t\bar\varphi_y(B)\,{}^tQ_yg\|\le Ch^s. \tag{19.5.28}$$

Further, we can plug $-(B-\tau)$ instead of $(B-\tau)$ into (19.5.28). Therefore we have proven

Proposition 19.5.7. *Let $\bar\varphi$ be supported in ϵ-vicinity of τ and $\varphi=1$ in 2ϵ-vicinity of τ. Then for $T\le T^*$ defined by* (19.5.23)

$$\|\big(I-\varphi(B)\big)U\,{}^t\bar\varphi_y(B)\,{}^tQ_y\|\le Ch^s. \tag{19.5.29}$$

Using Corollary 19.5.6 which allows us to apply simultaneous cut-off by

$$\Phi(B_1,\dots,B_\nu):=\varphi_1(B_1)\cdot\varphi_2(B_2)\cdots\varphi_\nu(B_\nu) \tag{19.5.30}$$

we arrive to

Corollary 19.5.8. *Let functions $\bar\varphi_j$ be supported in ϵ-vicinities of τ_j and let $\varphi_j=1$ in 2ϵ-vicinities of τ_j. Then for $T\le T^*$ defined by* (19.5.23)

$$\|\big(1-\Phi(B_1,\dots,B_\nu)\big)U\,{}^t\bar\Phi_y(B_1,\dots,B_\nu)\,{}^tQ_y\|\le Ch^s. \tag{19.5.31}$$

Recall that we consider set of B_j coinciding either with $\{a_{\mathfrak m}\}_{\mathfrak m\in\mathfrak M}$ or with $\{a_{\mathfrak n}+O(\mu^{-2})\}_{\mathfrak n\in\mathfrak N}$.

19.5.4 Propagation and Microhyperbolicity

Consider now the proof that singularities leave diagonal – again as Theorem 2.3.1 with

$$\phi = \langle \ell, x-y\rangle \mp \epsilon t T^{-1}. \tag{19.5.32}$$

We plug-in

$$W = U\,{}^t\big(\Phi(B_1,\dots,B_\nu)\big)_y\,{}^tQ_y \tag{19.5.33}$$

localizing x near y and B_j near τ_j for all j, and take $\ell = \ell(y,\boldsymbol{\tau})$.

To recover the proof we need to estimate from below quadratic form

$$(\mu^2 \sum_{j,k} p_j^{\mathrm{w}} K_{jk} p_k^{\mathrm{w}} w, w)$$

with $w = \chi(\phi)^{\mathrm{w}} W$, $W = \bar{\Phi}(B_1,\dots,B_\nu)U$ and $K_{jk} = \ell(g^{jk})$; one can assume that $V = -1$.

Due to Proposition 19.5.8 one can replace W by $\Phi(B_1,\dots,B_\nu)W$ with a bit more wide supports of φ_j than those of $\bar{\varphi}_j$.

Let us introduce μ^{-1}-admissible partition $\psi_\nu(x)$ and on each element of it apply gauge transformation making $|V_j| \le C\mu^{-1}$. Rescaling $x \mapsto \mu(x-\bar{x}_\nu)$ we get (μh)-pseudodifferential operators μp_j^{w} and $\mu^2 \sum_{j,k} p_j^{\mathrm{w}} K_{jk} p_k^{\mathrm{w}}$ while $\Phi(B)$ is a legitimate (μh)-pseudodifferential operator as well.

Then for $\mu h \le \epsilon_1$

$$\Phi(B_1,\dots,B_\nu)^*\Big(\mu^2 \sum_{j,k} p_j^{\mathrm{w}} K_{jk} p_k^{\mathrm{w}} - \epsilon_0 + C\mu h\Big)\Phi(B_1,\dots,B_\nu)$$

is non-negative operator and

$$\Phi(B_1,\dots,B_\nu)^*\mu^2 \sum_{j,k} p_j^{\mathrm{w}} K_{jk} p_k^{\mathrm{w}} \Phi(B_1,\dots,B_\nu) \ge \frac{1}{2}\epsilon_0 \Phi(B_1,\dots,B_\nu)^*\Phi(B_1,\dots,B_\nu)$$

in the operator sense and therefore

$$\Big(\mu^2 \sum_{j,k} p_j^{\mathrm{w}} K_{jk} p_k^{\mathrm{w}} w, w\Big) \ge \frac{1}{2}\epsilon_0 \|w\|^2 - Ch^s; \tag{19.5.34}$$

the rest of the proof needs no modifications.

Now applying the same approach as in the proof of Proposition 19.2.13 we arrive to

Proposition 19.5.9. *Let* either $\mathfrak{M}$*-microhyperbolicity assumption* or $\mathfrak{N}$*-microhyperbolicity assumption be fulfilled.*

Then estimate (19.2.63) *holds for* $T \in [T_*, T^*]$ *with* T^* *defined by* (19.5.23) *and* $T_* = C\varepsilon^{-1}h|\log h| \le \epsilon_1(\mu h|\log h|)^{\frac12}$.

Proof. An easy proof we leave to the reader. □

19.5.5 Short-Range Estimates

In this strong magnetic field case we can apply the same arguments as in Subsection 19.4.4 combined with Corollary 19.5.8 and extend Proposition 19.4.9:

Proposition 19.5.10. *Both statements (i), (ii) of Proposition 19.4.9 remain true in the case of the strong magnetic field as well.*

Proof. Just repeating proof of Proposition 19.4.9 without any significant difference. We use that in the operator sense $|x'| + |\xi'| \le C\mu^{-1}$ still even if in the microlocal sense one should put $C(\mu^{-1}h|\log h|)^{\frac12}$ instead.

Easy details are left to the reader. □

19.5.6 Calculations

In the strong magnetic field case one can apply the same arguments in the successive approximation method remembering that now $\varepsilon \ge C(\mu^{-1}h|\log h|)^{\frac12}$ and $\bar{T}_0 \le h\varepsilon^{-1} \le (\mu h|\log h|^{-1})^{\frac12}$.

Then Propositions 19.4.12–19.4.14 and their proofs remain true with this minor modification:

Proposition 19.5.11. *Under microhyperbolicity condition for* $\mu \le \epsilon h^{-1}$

(i) Estimates (19.4.77) *and* (19.4.78) *hold.*

(ii) Estimates (19.4.57), (19.2.78) *and* (19.4.84) *hold.*

Furthermore, arguments of Proposition 19.4.15 related to $T \ge \bar{T} = \epsilon\mu^{-1}$ do not change either. However, we need to reconsider contribution of the segment $[-\bar{T}, \bar{T}]$ as $l > 2$. Again, we need to consider only the sum of terms (19.4.101)–(19.4.104).

We already know from the proof of Proposition 19.4.15(ii) that under assumptions $\mu h|\log h \le \epsilon$ and $\varepsilon \ge C\mu^{-1}$ this sum does not exceed $C\mu^{-2}h^{-d}(\mu h)^{l-2}|\log(\mu h|)^{-\sigma}$.

On the other hand, all terms (19.4.101)–(19.4.104) with $T = \bar{T}$ and their sum have the form $\mu^{-2}h^{-d}f(\mu h)$ and then repeating arguments of Part (iii) of the proof of Proposition 19.4.15 with $\lambda = \mu^K h^{K-1}$ we arrive to the same estimate $C\mu^{-2}h^{-d}(\mu h)^{l-2}|\log(\mu h|)^{-\sigma}$ for the sum as $\mu h \le \epsilon'$. Furthermore, Remark 19.4.16 remains true as well:

Proposition 19.5.12. *Statements of Proposition 19.4.15 and Remark 19.4.16 remain true as well.*

19.5.7 Main Theorems

Thus we arrive to the following

Theorem 19.5.13. *Let assumptions* (19.1.4)–(19.1.6), $(19.1.25)_{1-3}$ *with* $(\bar{l}, \bar{\sigma}) \succeq (l, \sigma) \succeq (1, 2)$, $(\bar{l}, \bar{\sigma}) \succeq (2, 1)$ *and* (19.2.38) *be fulfilled.*

Let

$$\mu_2^* := \epsilon(h|\log h|)^{-1} \le \mu \le \mu_3^* := \epsilon' h^{-1} \tag{19.5.35}$$

with sufficiently small constant $\epsilon' > 0$. *Then for two framing approximations*

(i) Under $\mathfrak{N}$*-microhyperbolicity condition (see Definition 19.2.5) estimate* (19.5.6) *holds.*

(ii) Under microhyperbolicity condition (see Definition 19.2.4) estimate (19.5.7) *holds.*

Remark 19.5.14. (i) Obviously Theorem 19.5.13 generalizes Theorem 19.5.2; recall that our second main theorem is Theorem 19.5.3.

(ii) Theorems 19.5.2 and 19.5.3 hold for very strong magnetic field (19.6.1) as well but then we can still relax definition of $\mathfrak{N}$-hyperbolicity (see condition (19.6.13) below).

19.6 Intermediate and Strong Magnetic Field

19.6.1 Framework

The remaining case $\mu \ge \mu_3^* = \epsilon_1 h^{-1}$ is split into two subcases:

$$\mu_3^* = \epsilon_1 h^{-1} \le \mu \le \mu_4^* = C_0 h^{-1}, \tag{19.6.1}$$

$$\mu \ge \mu_4^* = C_0 h^{-1} \tag{19.6.2}$$

which we refer as *very strong* and *superstrong* magnetic field cases respectively.

In the former subcase we know that the spatial speed of the propagation is $O(\mu^{-1})$ but we have not estimated the propagation speed with respect to the energy partition and thus we cannot use microhyperbolicity condition except $\mathfrak{N}$-microhyperbolicity condition with $\#\mathfrak{N} = 1$. Further, this condition should be modified to accommodate spectral gaps.

In the latter subcase (19.6.2) we have nothing about propagation at all. Recall that in this case we need to consider the generalized Schrödinger-Pauli operator (19.1.13).

19.6.2 Hermitian Decomposition

In both cases, however, we can apply a very usefully decomposition

$$U_{\mathcal{T}}(x,y,t) = \sum_{\alpha\in\mathbb{Z}^{+r}} U_{\alpha\beta}(x'',y'',t)\Upsilon_\alpha(x')\Upsilon_\beta(y') \tag{19.6.3}$$

with

$$\Upsilon_\alpha(x') \coloneqq \mu^{\frac12 r} h^{-\frac12 r} \upsilon_\alpha\big(\mu^{\frac12} h^{-\frac12} x'\big), \qquad \upsilon_\alpha(x) = \prod_{1\le j\le r} \upsilon_{\alpha_j}(x_j). \tag{19.6.4}$$

Recall that υ_{α_j} are 1-dimensional Hermite functions.

Note that

$$\mathcal{A}_0\Big(u_\alpha(x'')\Upsilon_\alpha(x')\Big) = \sum_{\beta:|\beta|=|\alpha|,\,|\beta-\alpha|=0,2} \Big(\mu h \mathcal{A}^0_{\beta\alpha}(x'',\mu^{-1}hD'') + \delta_{\alpha\beta}\, q_0(x'',\mu^{-1}hD'')\Big) u_\alpha\cdot \Upsilon_\beta(x') \tag{19.6.5}$$

where

$$\mathcal{A}^0_{\beta\alpha} = \begin{cases} \sum_j (2\alpha_j + 1) b_{jj} & \text{as } \beta = \alpha, \\ 2\sqrt{\alpha_j \beta_k}\, b_{jk} & \text{as } \beta_i - \alpha_i = \delta_{ik} - \delta_{ij}. \end{cases} \tag{19.6.6}$$

Further,

(19.6.7) $\mathbb{H}_n \coloneqq \{\sum_{\alpha:|\alpha|=n} v_\alpha(x'')\Upsilon_\alpha(x')\}$ are invariant subspaces of both $\mathcal{A}_0$ and $\mathcal{A}^0$

and

$$\mathsf{Spec}\big(\mathcal{A}^0_{\alpha\beta}(x'',\xi'')\big)_{\alpha,\beta}\Big|_{\mathbb{H}_n} = \Big\{\sum_j (2\alpha_j+1) f_j,\ |\alpha| = n\Big\}. \tag{19.6.8}$$

19.6.3 Propagation

Special Case of Constant g^{jk}, F_{jk}

This is far the easiest case. First, we can assume without any loss of the generality that

$$\mathcal{A} = \mathcal{A}_0. \tag{19.6.9}$$

Really, no cubic terms appear from the "kinetic" part of Hamiltonian $\sum_j g^{jk} P_j P_k$ and the first-order terms appearing from the potential are eliminated in the process of reduction.

Further, we can skip an error $O(\varepsilon^l |\log \varepsilon|^{-\sigma})$ and for $(l,\sigma) \succeq (2,0)$ we can skip quadratic terms appearing from the potential because in the operator sense they are less than $\mu^{-2}(\mathcal{A}^0 + 1)$ and since our reduction in this case is global, we can just take approximate operator as $\mathcal{T}\mathcal{A}_0\mathcal{T}^*$.

Furthermore, we can assume without any loss of the generality that $b_{jk} = f_j \delta_{jk}$ and then

$$\mathcal{A}\Big(u_\alpha(x'')\Upsilon_\alpha(x')\Big) = \Big(\sum_j (2\alpha_j+1) f_j \mu h + q_0\Big) u_\alpha(x'') \cdot \Upsilon_\alpha(x'). \tag{19.6.10}$$

Now we are dealing with r-dimensional scalar $\mu^{-1}h$-pseudodifferential operators and the proofs of the following two statements are obvious:

Proposition 19.6.1. *Let g^{jk}, F_{jk} be constant and one of conditions* (19.6.1), (19.6.2) *be fulfilled. Let $Q = q(x', \mu^{-1}hD')$. Assume that on the support of q an ellipticity condition*

$$|\sum_i (2\alpha_i + 1) f_i \mu h + V - \tau| \ge \epsilon_1 \qquad \forall \alpha \in \mathbb{Z}^{+r}. \tag{19.6.11}$$

is fulfilled.

Then for $T \ge T_* := Ch|\log \mu|$

$$|F_{t\to h^{-1}\tau}\Big(\bar{\chi}_T(t) U_T\, {}^tQ_y\Big)| \le CT\mu^{-s}. \tag{19.6.12}$$

Proof. An easy proof based on our standard elliptic arguments is left to the reader. □

Proposition 19.6.2. *Let g^{jk}, F_{jk} be constant and one of conditions* (19.6.1), (19.6.2) *be fulfilled. Let $Q = q(x', \mu^{-1}hD')$.*

(i) Assume that on the support of q a microhyperbolicity condition

$$|\sum_i (2\alpha_i + 1) f_i \mu h + V - \tau| + |\nabla V| \ge \epsilon_1 \qquad \forall \alpha \in \mathbb{Z}^{+r}. \tag{19.6.13}$$

is fulfilled. Then for $T \in [T_*, T^*]$

$$|F_{t\to h^{-1}\tau}\Big(\chi_T(t) \Gamma' U_T\, {}^tQ_y\Big)| \le C\mu^{-s}. \tag{19.6.14}$$

(ii) Assume that on the support of q condition

$$|\sum_i (2\alpha_i + 1) f_i \mu h + V - \tau| \le \epsilon \implies |\nabla V| \asymp \epsilon_1 \nu \qquad \forall \alpha \in \mathbb{Z}^{+r} \tag{19.6.15}$$

with $C_0(\mu^{-1}h|\log h|)^{\frac{1}{2}} \le \nu \le \epsilon$. Then for $T \in [T_*, T^*]$ (19.6.14) *holds where now $T_* = C_0\nu^{-1}\mu h|\log \mu|$.*

Proof. An easy proof based on our standard microhyperbolicity arguments is left to the reader. □

Intermediate Magnetic Field: General Operators

Assume that condition (19.6.1) holds.

Consider operator $A_{\mathcal{T}}$ as a $\mu^{-1}h$-pseudodifferential operator with respect to x' with a "matrix" symbol with values in $\mathcal{L}(\mathbb{H},\mathbb{H})$ where $\mathbb{H}=\mathscr{L}^2(\mathbb{R}^r_{x'})$ and $\mathcal{L}(\mathbb{H}_1,\mathbb{H}_2)$ is the space of linear operators from $\mathbb{H}_2$ to $\mathbb{H}_1$.

Obviously, operator $\mathcal{A}$ does not belong to this class, but we can also replace $A_{\mathcal{T}}$ by $\mathcal{A}_0+Q_1^*(A_{\mathcal{T}}-\mathcal{A}_0)Q_1$ where $Q_1=I-\phi(\varepsilon^{-1}x',\varepsilon^{-1}\mu^{-1}hD')$ and ϕ is supported in $2C_1$-vicinity and equal 1 in C_1-vicinity of $0\in\mathbb{R}^d_{x'',\xi''}$. Applied to U, Q_1 produces negligible output[29]. Therefore:

Proposition 19.6.3. *Let condition* (19.6.1) *be fulfilled. Assume that on the support of* q *the ellipticity condition* (19.6.11) *is fulfilled and let* $Q=q(x'',\mu^{-1}hD'')$. *Then for* $T\ge Ch|\log h|$, $\tau\in[-\epsilon',\epsilon']$ *(with small enough constant* $\varepsilon'>0$*) estimate* (19.6.12) *holds.*

Furthermore,

Proposition 19.6.4. *Let condition* (19.6.1) *be fulfilled. Assume that on the support of* q *for a symbol* $\mathcal{A}_0$ *the standard microhyperbolicity assumption*

$$\big\langle\big(\ell\mathcal{A}_0(x'',\xi'')\big)v,v\big\rangle\ge\epsilon_0\|v\|^2-C\|\mathcal{A}_0v\|^2\qquad\forall v\in\mathbb{H} \tag{19.6.16}$$

is fulfilled and let $Q=q(x'',\mu^{-1}hD'')$. *Then for* $T\in[T'_*,T^*]$ *estimate* (19.6.14) *holds.*

Proof. An easy proof based on the standard arguments of the proof of Theorem 2.3.1 is left to the reader. □

Remark 19.6.5. Note that we need the single direction ℓ. We just cannot localize in $(\mu^2\mathsf{a}^0_{\mathsf{n}_1},\dots,\mu^2\mathsf{a}^0_{\mathsf{n}_\nu})$ even after our reduction. This is really frustrating because Poisson brackets $\big\{\mu^2\mathsf{a}^0_{\mathsf{n}_j},\mu^2\mathsf{a}^0_{\mathsf{n}_k}\big\}$ are $O(\mu^3h|p|^4)$ which is $O(\mu h^3|\log h|^2)$ even in the microlocal sense.

Strong Magnetic Field Case. Reduction

Assume that condition (19.6.2) holds. Then we should take into account that unless F_{jk},g^{jk} are constant (the case we already considered), generally

[29] After applying $F_{t\to h^{-1}\tau}\bar{\chi}_T(t)$ with $\tau\le c$ and $Ch|\log h|\le T\le ch^{-s}$.

variations of $\mu h f_j$ are of magnitude $\mu h \gg 1$. To overcome this difficulty we need to modify our assumptions.

First, we need to replace the Schrödinger operator by the generalized Schrödinger-Pauli operator (19.1.13).

Ellipticity condition would mean that

$$|\sum_j \big(\mathfrak{z}_j - (2\alpha_j+1)\big) f_j| \ge \epsilon \qquad \forall \alpha \in \mathbb{Z}^{+r} \tag{19.6.17}$$

but we concentrate mainly on the opposite case

$$\mathfrak{z}_j = 2\alpha_j + 1 \qquad \text{as} \quad \alpha \in \mathfrak{A} \subset \mathbb{Z}^{+r}, \tag{19.6.18}$$

$$|\sum_j (\mathfrak{z}_j - 2\alpha_j - 1) f_j| \ge \epsilon \qquad \forall \alpha \in \mathbb{Z}^{+r} \setminus \mathfrak{A} \tag{19.6.19}$$

and $C_0 = C_0(\epsilon)$ in condition (19.6.2). Then obviously $\mathfrak{A}$ is a finite subset.

We still need to deal with the fact that we have only local reduction and thus $A_{\mathcal{T}}$ differs from $\mathcal{A}$ by operator $\mathcal{A}'$ with symbol belonging to $\omega \mathscr{F}^{0,0}$ with

$$\omega = \varepsilon^l |\log \mu|^{-\sigma} + \mu h \varepsilon^{\bar l} |\log \mu|^{-\bar\sigma}. \tag{19.6.20}$$

Proposition 19.6.6. *Let $b \in \mathscr{F}^{0,0}$ be supported in $\{|x'| + |\xi'| \le c\varepsilon\}$. Then operator*

$$b^{\mathsf{w}}_{\alpha\beta} : \mathbb{K} \ni v \to \langle b^{\mathsf{w}}(v \otimes \Upsilon_\beta), \Upsilon_\alpha\rangle \in \mathbb{K}, \qquad \mathbb{K} := \mathscr{L}^2(\mathbb{R}^r_{x''}) \tag{19.6.21}$$

has norm

$$\|b^{\mathsf{w}}_{\alpha\beta}\| \le C e^{-\epsilon|\alpha-\beta|}. \tag{19.6.22}$$

Proof. Obviously, it is sufficient to prove this proposition as $x' \in \mathbb{R}^1$ (with $x'' \in \mathbb{R}^r$). Then since

$$H_j \Upsilon_\beta = (2\beta_j + 1)\mu h \Upsilon_\beta, \qquad H_j := \mu h (|\zeta_j|^2)^{\mathsf{w}} = \mu^2 x_j^2 + h^2 D_j^2 \tag{19.6.23}$$

we conclude that

$$b^{\mathsf{w}}_{\alpha\beta} = (\alpha_j - \beta_j)^{-1} \big\{|\zeta_j|^2, b\big\}^{\mathsf{w}}_{\alpha\beta} \tag{19.6.24}$$

and continuing this process we conclude that $b^{\mathsf{w}}_{\alpha\beta} = (\alpha_j - \beta_j)^{-n} (b_{(n)})^{\mathsf{w}}_{\alpha\beta}$ with $b_{(0)} = b$ and $b_{(n)} = \big\{|\zeta_j|^2, b_{(n-1)}\big\}$.

One can see easily $b_{(n)} \in C^n n! \mathscr{F}^{0,0}$ uniformly with respect to n [30] operator norm of $b^{\mathsf{w}}_{(n)}$ in $\mathcal{L}(\mathbb{H}\otimes\mathbb{K}, \mathbb{H}\otimes\mathbb{K})$ does not exceed $C^n n!$ and then operator norm of $b^{\mathsf{w}}_{\alpha\beta}$ in $\mathcal{L}(\mathbb{K},\mathbb{K})$ does not exceed $|\alpha_j-\beta_j|^{-n} C^n n!$. It reaches minimum $e^{-\epsilon|\alpha_j-\beta_j|}$ as $n = C^{-1}|\alpha_j-\beta_j|$. □

No 2-nd Order Resonances Case. Assume first that all f_j are disjoint. Then

$$\mathcal{A}_0 = \sum_{1\le j\le r} b_j(x'', \mu^{-1}hD'')\big(\mu^2 x_j^2 + h^2 D_j^2\big) + b_0(x'', \mu^{-1}hD'') \tag{19.6.25}$$

and

$$\mathcal{A}_0\big(v(x'')\Upsilon_\alpha(x')\big) = (W_\alpha v)\Upsilon_\alpha(x') \tag{19.6.26}$$

where W_α is $(\mu^{-1}h)$-pseudodifferential operator with the principal symbol

$$W_\alpha = \big(V + \sum_j (2\alpha_j + 1 - \mathfrak{z}_j) f_j \mu h\big) \circ \Psi_0. \tag{19.6.27}$$

Proposition 19.6.7. *Let ellipticity condition* (19.6.17) *be fulfilled and let condition* (19.6.2) *be fulfilled at $\bar{x}$ with $C_0 = C_0(\epsilon)$. Then modulo $O(\mu^{-s})$*

$$F_{t\to h^{-1}\tau}\bar{\chi}_T(t)U_{\alpha\beta} \equiv 0 \qquad \forall \tau : |\tau| \le \epsilon'\mu h \qquad \text{in } \ \Omega\times\Omega \tag{19.6.28}$$

where $\Omega = B(\bar{z}'', C_0\varepsilon) \subset \mathbb{R}^{2r}_{x'',\xi''}$ is a domain in which reduction is done, $\bar{z}'' = (\bar{x}'', \bar{x}'') = \Psi_0^{-1}(\bar{x})$.

Proof. Proof due to the standard elliptic arguments is left to the reader. □

Proposition 19.6.8. *Let conditions* (19.6.18)–(19.6.19) *be fulfilled and let condition* (19.6.2) *be fulfilled at $\bar{x}$ with $C_0 = C_0(\epsilon)$. Assume that all f_j are disjoint. Let $T = \mu^m$ with large enough m. Then*

(i) As $\alpha \notin \mathfrak{A}$

$$F_{t\to h^{-1}\tau}\bar{\chi}_T(t)U_{\alpha\beta} \equiv \sum_{\alpha'\in\mathfrak{A}} E_{\alpha\alpha',x} F_{t\to h^{-1}\tau}\bar{\chi}_T(t)U_{\alpha'\beta} \tag{19.6.29}$$
$$\forall \tau : |\tau| \le \epsilon'\mu h \qquad \text{in } \ \Omega\times\Omega$$

[30] With simultaneous decay of the number of derivatives checked, but everything works as long as $n \le \epsilon|\log\varepsilon|$.

where $E_{\alpha\alpha'} = E_{\alpha\alpha'}(\tau, x'', \mu^{-1}hD'')$ *are pseudodifferential operators with the norms*

$$\|E_\alpha\| \le C(\mu h)^{-1}\Big((\mu^{-1}h)^{\frac{1}{2}\delta(\alpha)} + e^{-|\alpha-\bar\alpha|}\omega\Big) \tag{19.6.30}$$

and a similar dual equation holds; here

$$\delta(\alpha) := \min_{\alpha'\in\mathfrak{A}} |\alpha - \alpha'|. \tag{19.6.31}$$

(ii) As $\alpha \notin \mathfrak{A}$ *and* $\beta \neq\notin \mathfrak{A}$

$$F_{t\to h^{-1}\tau}\bar\chi_T(t)U_{\alpha\beta} \equiv \sum_{\alpha',\beta'\in\mathfrak{A}} E_{\alpha\alpha',x}F_{t\to h^{-1}\tau}\bar\chi_T(t)U_{\alpha'\beta'}\,{}^tE_{\beta\beta',y} \qquad \forall \tau : |\tau| \le \epsilon'\mu h \quad \text{in } \Omega\times\Omega. \tag{19.6.32}$$

(iii) Further, $\boldsymbol{U} := \{U_{\alpha\beta}\}_{\alpha,\beta\in\mathfrak{A}}$ *satisfies*

$$F_{t\to h^{-1}\tau}\bar\chi_T(t)\Big(\big(hD_t - W_\alpha(x'', \mu^{-1}hD'')\big)U_{\alpha\beta} - \sum_{\alpha'\in\mathfrak{A}} W'_{\alpha\alpha'}(x'', \mu^{-1}hD'')U_{\alpha'\beta}\Big) \equiv 0 \qquad \forall \tau : |\tau| \le \epsilon'\mu h \quad \text{in } \Omega\times\Omega \tag{19.6.33}$$

and adjoint equation with respect to y'' *where* W_α *is defined by* (19.6.27) *and* $W'_{\alpha\alpha'}$ *are* $\mu^{-1}h$*-pseudodifferential operator with symbol belonging to* $(\omega + h^2)\mathscr{F}^{0,0}$.

Proof. (a) Plugging representation (19.6.3) into equation $\big(hD_t - A_{\mathcal{T},x}\big) = 0$ and using (19.6.26)–(19.6.27), form of $\mathcal{A}$ and applying Proposition 19.6.6 to $A_{\mathcal{T}} - \mathcal{A}$ we arrive to the system

$$\big(hD_t - W_\alpha - W'_\alpha\big)U_{\alpha\beta} \equiv \sum_{\gamma\neq\alpha} B_{\alpha\gamma}\,U_{\gamma\beta} \qquad \text{in } \Omega\times\mathbb{R}^{2r} \tag{19.6.34}$$

with W'_α not including hD_t. Here and below all operators are acting with respect to x''. We also get an adjoint system

$$U_{\alpha\beta}\,{}^t\big(hD_t - W_\beta - W'_\beta\big) \equiv \sum_{\gamma\neq beta} U_{\alpha\gamma}\,{}^tB_{\gamma\beta} \qquad \text{in } \mathbb{R}^{2r}\times\Omega \tag{19.6.34}'$$

with operators acting with respect to y''.

Applying $F_{t\to h^{-1}\tau}\bar\chi_T(t)$ with T described above, we arrive to the system

$$\big(\tau - W_\alpha - W'_\alpha\big)v_{\alpha\beta} - \equiv \sum_{\gamma\neq\alpha} B_{\alpha\gamma}\, v_{\gamma\beta} \qquad \text{in } \ \Omega\times\mathbb{R}^{2r}, \tag{19.6.35}$$

with $v_{\alpha\beta} = F_{t\to h^{-1}\tau}\bar\chi_T(t)U_{\alpha\beta}$ and a similar dual system.

Due to assumptions (19.6.18)–(19.6.19) all operators $\big(\tau - W_\alpha - W'_\alpha\big)$ with $\alpha\notin\mathfrak{A}$ are elliptic with inverse operators norms of magnitude $\big(\mu h\delta(\alpha)\big)^{-1}$ as $|\tau|\le \epsilon\mu h$.

Note that even in the worst case $B_{\alpha\beta}$ are operators with the symbols belonging to $\mu h(\mu^{-1}h|\alpha|)^{\frac12}\mathscr{F}^{0,1}$. Then by the successive approximations

$$v_{\alpha\beta} = \sum_{\alpha'\in\mathfrak{A}} E_{\alpha\alpha'}v_{\alpha'\beta} \qquad \text{as } \ \alpha\notin\mathfrak{A} \tag{19.6.36}$$

where each $E_{\alpha\alpha'}$ is the sum of terms of the following type:

$$(\tau - W_\alpha)^{-1}B'_{\alpha\alpha_1}(\tau - W_{\alpha_1})^{-1}B'_{\alpha_1\alpha_2}\cdots(\tau - W_{\alpha_k})^{-1}B'_{\alpha_k\alpha'} \tag{19.6.37}$$

with $k\ge 0$ where $B'_{\gamma\gamma'}$ are operators with symbols belonging to $\mu h(\mu^{-1}h|\alpha|)^{\frac12}\mathscr{F}^{0,1}$. For the sake of simplicity we included W'_α into W_α.

Similarly,

$$v_{\alpha\beta} = \sum_{\beta'\in\mathfrak{A}} v_{\alpha\beta'}\,{}^tE_{\beta\beta'} \qquad \text{as } \ \alpha\notin\mathfrak{A} \tag{19.6.36}'$$

Then both Statements (i) and (ii) follow then from formulae (19.6.36), (19.6.36)′ and (19.6.37).

(b) Plugging (19.6.27), (19.6.3) into equation $(hD_t - A_{\mathcal{T}x})u = 0$ and a dual equation with respect to y, we arrive to an equation similar to (19.6.33) (as well as an adjoint equation with respect to y):

$$\big(\tau - W_\alpha\big)v_{\alpha\beta} \equiv \sum \mathcal{W}_{\alpha\alpha'}v_{\alpha'\beta} \qquad \text{in } \ \Omega\times\mathbb{R}^{2r} \tag{19.6.38}$$

where $\mathcal{W}_{\alpha\alpha'}$ is a sum of terms of of (19.6.37) type with an extra factor on their left:

$$(\mu^{-1}h)^{\frac p2}B'_{\alpha\alpha_1}(\tau - W_{\alpha_1})^{-1}B'_{\alpha_1\alpha_2}\cdots(\tau - W_{\alpha_k})^{-1}B'_{\alpha_k\alpha'} \tag{19.6.39}$$

with $B'_{\bar\alpha\alpha}$ of the same type as above; $p=0$ here. Note that

$$(\tau - W_\gamma)^{-1} = (W_{\alpha'} - W_\gamma)^{-1} - (\tau - W_\gamma)^{-1}(\tau - W_{\alpha'})(W_{\alpha'} - W_\gamma)^{-1} \tag{19.6.40}$$

in $\Omega \times \mathbb{R}^{2r}$.

We apply this formula to $(\tau - W_{\alpha_k})^{-1}$. Then we drag $(\tau - W_{\alpha'})$ to the right. If it perishes at commuting, we get the same expression with the same k but with p replaced by $p+1$ because in the commuting we gain $\varepsilon^{-1}\mu^{-1}h$ factor. Continuing this process with $(\tau - W_{\alpha_{k-1}})$ etc we arrive to

$$\bigl(\tau - W_\alpha\bigr)v_{\alpha\beta} \equiv \sum_{\alpha'\in\mathfrak{A}} \Bigl(W''_{\alpha\alpha'} + \mathcal{W}''_{\alpha\alpha'}\,(\tau - W_{\alpha'}) + \mathcal{W}'''_{\alpha\alpha'}\Bigr)v_{\alpha'\beta} \qquad \text{in } \ \Omega\times\mathbb{R}^{2r} \tag{19.6.41}$$

where $W''_{\alpha\alpha'}$ does not depend on τ, $\mathcal{W}'''_{\alpha\alpha'}$ is the sum of products of (19.6.39) type with p replaced by $p+1$, $\mathcal{W}''_{\alpha\alpha'}$ is the sum of products of (19.6.39) type with p replaced by $p+2$ and k replaced by $k-1$. Considering system (19.6.41) as a $(\#\mathfrak{A})\times(\#\mathfrak{A})$-matrix equation and multiplying by $(I+\mathcal{W}'')$ one can rewrite this system as (19.6.38) with W_α replaced by $W_\alpha + W''_{\alpha\alpha}$ and $\mathcal{W}'''_{\alpha\alpha'}$ of the same type with p replaced by $p+1$.

Continuing this process we arrive to negligible $\mathcal{W}'''_{\alpha\alpha'}$. Statement (iii) is proven. □

Remark 19.6.9. (i) It follows from our construction that an added correction modulo $(\omega+\mu^{-1}h)\mathscr{F}^{0,0}$ is equal to

$$W'_{\alpha\alpha'} = \sum_{\beta:|\alpha-\beta|=|\alpha-\beta|=1} B_{\alpha\beta}(W_\beta)^{-1}B_{\beta\alpha'} \tag{19.6.42}$$

with $B_{\alpha\beta}$ appearing exclusively from cubic terms in $\mathcal{A}$.

(ii) Further, note that these terms are $b_{m;jk}Z_m^*Z_jZ_k$ and $b^*_{m;jk}Z_mZ_j^*Z_k^*$ with f_m not disjoint from f_j+f_k. Conditions (19.6.18)–(19.6.19) and $\#\mathfrak{A}=1$, $\alpha\in\mathfrak{A}$ imply that $\alpha_m=0$ <u>and</u> either $\bar\alpha_j=0$ or $\bar\alpha_k=0$ if $j\ne k$ and $\alpha_j=0,1$ as $j=k$. Therefore cubic terms, applied to $v(x'')\Upsilon_{\bar\alpha}(x')$, produce 0, and therefore $W'_{\alpha\alpha'}\in(\omega+\mu^{-1}h)\mathscr{F}^{0,0}$.

However, correction $W'_{\alpha\alpha'}\equiv h^2\sum_j \kappa_j Z_j^*Z_j \mod O(\omega+\mu^{-1}h)$ has been already generated in the process of reduction in Section 19.3. Still we see that it 0 in the case of constant g^{jk}, F_{jk}.

2-nd Order Resonances Case. Consider now a more general case when some of f_j are not disjoint; assume that f_j is disjoint from the rest of eigenvalues as $j = 1, \dots, p$ and is not as $j = p+1, \dots, r$. Then

$$\mathcal{A}_0 = \sum_{1\le j\le p} f_j(x'', \mu^{-1}hD'')\big(\mu^2 x_j^2 + h^2 D_j^2\big) + \sum_{p+1\le j,k\le r} b_{jk}(x'', \mu^{-1}hD'')\big(Z_j^* Z_k + \mu h\delta_{jk}\big) + b_0(x'', \mu^{-1}hD''). \tag{19.6.43}$$

Assume that

$$\alpha_j = 0 \quad \text{as} \quad j = p+1, \dots, r \qquad \forall \alpha \in \mathfrak{A}. \tag{19.6.44}$$

Note that conditions (19.6.18)–(19.6.19) and $\#\mathfrak{A} = 1$ yield (19.6.44).

Due to (19.6.18)–(19.6.19) and (19.6.44) $\mathcal{A}_0$ restricted to $v(x'')\Upsilon_{\bar\alpha}(x')$ equals

$$\mathcal{A}_0 = \sum_{1\le j\le p} f_j(x'', \mu^{-1}hD'')\big(2\bar\alpha_j + 1\big)\mu h + b_0(x'', \mu^{-1}hD''); \tag{19.6.45}$$

then arguments of the proof of Proposition 19.6.7 still work but instead of the individual subspaces $\mathscr{H}_\alpha := \big\{v(x'')\Upsilon_\alpha(x')\big\}$ for $\alpha = (\alpha'; \alpha'') = (\alpha_1, \dots, \alpha_p; \alpha_{p+1}, \dots, \alpha_r)$ one should consider subspaces

$$\mathscr{H}_{\alpha', n} := \Big\{ \sum_{|\alpha''|=n} v_{(\alpha';\alpha'')}(x'')\, \Upsilon_{(\alpha';\alpha'')}(x') \Big\}.$$

Therefore we have

Proposition 19.6.10. *Propositions 19.6.7 and 19.6.8 holds even if f_i are not necessary disjoint but* (19.6.44) *is fulfilled.*

Strong Magnetic Field Case. Propagation

So under assumption $\#\mathfrak{A} = 1$ basically we reduced our operator to a single r-dimensional $\mu^{-1}h$-pseudodifferential operator $W_{\bar\alpha}$ with a principal symbol $V \circ \Psi_0$ while under assumption (19.6.44) we have a matrix-operator with the diagonal principal part instead. Then in the framework of (19.6.18)–(19.6.19) we can impose an ellipticity

$$|V| \ge \epsilon_0 \tag{19.6.46}$$

or a microhyperbolicity

$$|V| + |\nabla V| \ge \epsilon_0 \tag{19.6.47}$$

or as $(l,\sigma) \succeq (2,0)$ a non-degeneracy assumption

$$|V| + |\nabla V| + |\det \mathsf{Hess}\, V| \ge \epsilon_0. \tag{19.6.48}$$

Proposition 19.6.11. *Let conditions* (19.6.2), (19.6.18)–(19.6.19), *and* (19.6.44) *be fulfilled. Then under assumption* (19.6.46) *estimate* (19.6.28) *holds.*

Proof. The standard elliptic arguments applied to equation (19.6.33) yield that under condition (19.6.46) estimate (19.6.28) holds as $\tau \le \epsilon'$, $\alpha, \beta \in \mathfrak{A}$. Then in virtue of Propositions 19.6.8 and 19.6.10 this is true for all α, β. Easy details we leave to the reader. □

Proposition 19.6.12. *Let conditions* (19.6.2), (19.6.18)–(19.6.19), *and* (19.6.44) *be fulfilled.*

Then for $T_* \le T \le T^* = \epsilon_0 T$, $T \ge C\varepsilon^{-1}h|\log \mu|$ *and large enough constant* M

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t) Q'_x U_{\alpha\beta}(x,y,t)\, {}^tQ''_y| \le C\mu^{-s} \qquad \forall \tau \le c. \tag{19.6.49}$$

as $\tau \le \epsilon'$ *and distance between supports of (symbols of)* Q' *and* Q'' *is at least* $C\mu^{-1}T$.

Proof. The standard propagation arguments applied to equation (19.6.33) yield that (19.6.49) holds as $\alpha, \beta \in \mathfrak{A}$. Then in virtue of Propositions 19.6.8 and 19.6.10 this is true for all α, β. Easy details we leave to the reader. □

Proposition 19.6.13. *Let conditions* (19.6.2) *and* (19.6.18)–(19.6.19), (19.6.44) *be fulfilled. Further, let microhyperbolicity condition* (19.6.47) *be fulfilled.*

Then for $T_* \le T \le T^* = \epsilon_0 T$, $T \ge C\varepsilon^{-1}h|\log \mu|$ *and large enough constant* M

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t) Q'_x U_{\alpha\beta}(x,y,t)\, {}^tQ''_y| \le C\mu^{-s} \qquad \forall \tau \le c. \tag{19.6.50}$$

as $\tau \le \epsilon'$ *and distance between the most distant points of supports of (symbols of)* Q' *and* Q'' *is at most* $C\epsilon\mu^{-1}T$.

Proof. The standard propagation arguments applied to equation (19.6.33) yield that (19.6.50) holds as $\alpha, \beta \in \mathfrak{A}$. Then in virtue of Propositions 19.6.8 and 19.6.10 this is true for all α, β. Easy details we leave to the reader. □

Corollary 19.6.14. *(i) In the framework of Proposition 19.6.12*

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\psi'(x)U(x,y,t)\psi''(y)| \le C\mu^{-s} \qquad \forall \tau \le c \tag{19.6.51}$$

as $\tau \le \epsilon'$ and distance between supports of (symbols of) ψ' and ψ'' is at least $C\mu^{-1}T$ under additional assumption

$$T \ge C_0(\mu h|\log h|)^{\frac{1}{2}}. \tag{19.6.52}$$

(ii) In the framework of Proposition 19.6.13

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\psi'(x)U(x,y,t)\psi''(y)| \le C\mu^{-s} \qquad \forall \tau \le c \tag{19.6.53}$$

as $\tau \le \epsilon'$ and distance between the most distant points of supports of (symbols of) Q' and Q'' is at most $C\epsilon\mu^{-1}T$ under additional assumption (19.6.52).

Corollary 19.6.15. *In the framework of Proposition 19.6.13*

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma\psi'(x)U(x,y,t)\psi''(y)| \le C\mu^{-s} \qquad \forall \tau \le c. \tag{19.6.54}$$

Remark 19.6.16. (i) One can prove easily that under assumptions (19.6.2), (19.6.18)–(19.6.19) and (19.6.44) as $(I,\sigma) \succeq (2,0)$ singularities propagate along trajectories

$$\frac{dx_j}{dt} = \mu^{-1}\sum_k \Phi^{jk}\partial_{x_k}V \tag{19.6.55}$$

($\tau : |\tau| \le \epsilon'$) where $(\Phi^{jk}) = (F_{jk})^{-1}$. In particular, $V_{\bar{\alpha}}$ is an integral.

(ii) For $d = 2$ even condition $(I,\sigma) \succeq (2,0)$ is not necessary.

Remark 19.6.17. As $g^{jk} = \mathsf{const}$, $F_{jk} = \mathsf{const}$ all our results remain true even if there are 2-nd order resonances and (19.6.44) fails.

We leave to the reader the following

Problem 19.6.18. Formulate and prove similar results as $|\nabla V| \asymp \nu \ge C\varepsilon$ (instead of $|\nabla V| \asymp 1$).

19.6.4 Short-Range Estimates

Intermediate Magnetic Field

As before the case $\epsilon_1 h^{-1} \le \mu \le C_0 h^{-1}$ is just a variation of the strong magnetic field case but with the mandatory assumption $\#\mathfrak{N} = 1$ and the microhyperbolicity condition required only for $\tau = \sum_j (2\alpha_j + 1)\mu h f_j$ with $\alpha \in \mathbb{Z}^{+r}$. Using the same arguments as in the proof of Proposition 19.4.9 we arrive to

Proposition 19.6.19. *Let $\epsilon h^{-1} \le \mu \le C_0 h^{-1}$, $\varepsilon \ge (\mu^{-1} h |\log h|)^{\frac{1}{2}}$, and $T_* = C\varepsilon^{-1} h |\log h|$.*

Let microhyperbolicity condition (19.6.13) *be fulfilled and let $T \in [h^{1-\delta}, T_*]$ with an arbitrarily small exponent $\delta > 0$. Then both Statements (i) and (ii) of Proposition 19.4.9 remain true.*

Proof. Easy details are left to the reader. □

Strong Magnetic Field

Let us assume now that $\mu \ge \mu_4^* = C_0 h^{-1}$ and we consider generalized Schrödinger-Pauli operator (19.1.13).

We need to assume now that either g^{jk}, F_{jk} are constant or conditions (19.6.18)–(19.6.19), (19.6.44) are fulfilled. Exactly the same method as before leads us to similar results:

Proposition 19.6.20. *Let $\mu \ge C_0 h^{-1}$, $\varepsilon \ge (\mu^{-1} h |\log h|)^{\frac{1}{2}}$, and $T_* = C\varepsilon^{-1} h |\log h|$. Let conditions* (19.6.18)–(19.6.19), (19.6.44), *and microhyperbolicity condition* (19.6.47) *be fulfilled.*

(i) Let $l > 1$. Then for $T \in [C\mu^{\delta} h, T_*]$ with an arbitrarily small exponent $\delta > 0$

$$|\phi(hD_t)\chi_T(t)(\Gamma U \psi_y)| \le C\mu^r h^{-r} \Big(\frac{h}{T}\Big)^{l-1} \Big(1 + \frac{h}{T\varepsilon}\Big)^{-s} |\log \frac{h}{T}|^{-\sigma} \tag{19.6.56}$$

and

$$|F_{t\to h^{-1}\tau}\chi_T(t)(\Gamma U \psi_y)| \le C\mu^r h^{1-r} \Big(\frac{h}{T}\Big)^{l-1} \Big(1 + \frac{h}{T\varepsilon}\Big)^{-s} |\log \frac{h}{T}|^{-\sigma} \tag{19.6.57}$$

as $|\tau| \le \epsilon$ with arbitrarily large exponent s.

(ii) Let $l = 1, \sigma \ge 2$. *Then for* $T \in [Ch^{1-\delta}, T_*]$ *with an arbitrarily small exponent* $\delta > 0$ *and small enough constant* T_*

$$|\phi(hD_t)\chi_T(t)(\Gamma U\psi_y)| \le C\mu^r h^{-r}|\log\frac{h}{T}|^{-\sigma}\big(1+\frac{T\varepsilon}{h}\big)^{-s} + C\mu^r h^{-r}|\log\frac{h}{T}|^{-s}, \tag{19.6.58}$$

and

$$|F_{t\to h^{-1}\tau}\chi_T(t)(\Gamma U\psi_y)| \le C\mu^r h^{1-r}|\log\frac{h}{T}|^{-\sigma}\big(1+\frac{T\varepsilon}{h}\big)^{-s} + C\mu^r h^{1-r}|\log\frac{h}{T}|^{-s} \tag{19.6.59}$$

as $|\tau| \le \epsilon$ *with arbitrarily large exponent* s.

Remark 19.6.21. As $g^{jk} = \mathsf{const}, F_{jk} = \mathsf{const}$ all our results remain true even if there are 2-nd order resonances and (19.6.44) fails.

Problem 19.6.22. Formulate and prove similar results as $|\nabla V| \asymp \nu \ge C\varepsilon$ (instead of $|\nabla V| \asymp 1$).

19.6.5 Calculations

Now Propositions 19.6.19 and 19.6.20 imply immediately

Proposition 19.6.23. *In the frameworks of Propositions 19.6.19 and 19.6.20 the following estimates hold:*

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma(U\psi)| \le C\mu^r h^{1-r}, \tag{19.6.60}$$

$$T^{-1}|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\big(\bar{\mathcal{G}}^{\pm}(A_T-\bar A_T)\mathcal{G}^{\pm}\delta(t)\mathcal{K}_\psi\big)| \le C\mu^{r-1}h^{1-r}\big(\frac{h}{T}\big)^{l-2}|\log\mu|^{-\sigma} \tag{19.6.61}$$

and

$$|\Gamma(\psi\tilde e)(\tau) - h^{-1}\int_{-\infty}^{\tau}\big(F_{t\to h^{-1}\tau'}\bar\chi_T(t)\Gamma(\psi u)\big)\,d\tau'| \le C\mu^{r-1}h^{1-r} \quad \forall \tau : |\tau| \le \epsilon \tag{19.6.62}$$

with $T \in [T_*, T^*]$, $T_* = Ch\mu^\delta$, $T^* := \epsilon\mu$, $|\tau| \le \epsilon$.

In turn, Proposition 19.6.23 immediately implies

Proposition 19.6.24. *In the frameworks of Propositions 19.6.19 and 19.6.20*

$$|h^{-1}\int_{-\infty}^{\tau}\bigl(F_{t\to h^{-1}\tau'}\bar{\chi}_T(t)\Gamma(\psi u)\bigr)\,d\tau'- (2\pi)^{-r}\mu^{r-1}h^{-r}\,\mathsf{Tr}'\,\theta\bigl(\tau-\mathcal{A}(x'',\xi'')\bigr)\psi^0(x'',\xi'')\,dx''d\xi''| \le C\mu^{r-1}h^{1-r}+C\mu^r h^{-r}\varepsilon'|\log\mu|^{-\sigma} \qquad \forall \tau : |\tau|\le\epsilon. \tag{19.6.63}$$

19.6.6 Main Theorems

Finally we immediately arrive to two Theorems 19.6.25 and 19.6.26 below:

Theorem 19.6.25. *Let assumptions* (19.1.4)–(19.1.6), $(19.1.25)_{1-3}$ *with* $(\bar{l},\bar{\sigma})\succeq(l,\sigma)\succeq(1,2)$, $(\bar{l},\bar{\sigma})\succeq(2,1)$ *be fulfilled. Let*

$$\mu_3^*:=\epsilon h^{-1}\le\mu\le\mu_4^*:=C_0h^{-1} \tag{19.6.64}$$

with sufficiently small constant $\epsilon'>0$. *Assume that the microhyperbolicity condition* (19.6.16) *is fulfilled.*

Then there are two framing approximations[18] *(see Chapter 18) such that:*

$$\mathsf{R}^{\mathsf{MW}}:=|\int\Bigl(\tilde{e}(x,x,0)-h^{-d}\mathcal{N}^{\mathsf{MW}}(x,0)\Bigr)\psi(x)\,dx|\le C\mu^{r-1}h^{1-r}+C\mu^r h^{-r}(\mu^{-1}h)^{\frac{l}{2}}|\log\mu|^{\frac{l}{2}-\sigma}+C\mu^{r+1}h^{1-r}(\mu^{-1}h)^{\frac{\bar{l}}{2}}|\log\mu|^{\frac{\bar{l}}{2}-\bar{\sigma}}. \tag{19.6.65}$$

Theorem 19.6.26. *Let assumptions* (19.1.4)–(19.1.6), $(19.1.25)_{1-3}$ *with* $(\bar{l},\bar{\sigma})\succeq(l,\sigma)\succeq(1,2)$, $(\bar{l},\bar{\sigma})\succeq(2,1)$ *be fulfilled. Let*

$$\mu_4^*:=C_0h^{-1}\le\mu \tag{19.6.66}$$

with sufficiently small constant $\epsilon'>0$. *Consider generalized Schrödinger-Pauli operator. Assume that* (19.6.18)–(19.6.19) *and* <u>*either*</u> *microhyperbolicity condition* (19.6.47) <u>*or*</u> *nondegeneracy condition* (19.6.48) *are fulfilled.*

Then there are two framing approximations[18] *(see Chapter 18) such that:*

$$\mathsf{R}_2^{\mathsf{MW}}:=|\int\Bigl(\tilde{e}(x,x,0)- h^{-d}\mathcal{N}^{\mathsf{MW}}(x,0)-h^{-d}\mathcal{N}_{2\mathsf{corr}}^{\mathsf{MW}}(x,0)\Bigr)\psi(x)\,dx|\le C\mu^{r-1}h^{1-r}+C\mu^r h^{-r}(\mu^{-1}h)^{\frac{l}{2}}|\log\mu|^{\frac{l}{2}-\sigma}+C\mu^{r+1}h^{1-r}(\mu^{-1}h)^{\frac{\bar{l}}{2}}|\log\mu|^{\frac{\bar{l}}{2}-\bar{\sigma}} \tag{19.6.67}$$

with

$$h^{-d}\mathcal{N}^{\mathsf{MW}}_{2\mathsf{corr}}(x,\tau) := (2\pi)^{-r}\mu^r h^{2-r}\sum_{\alpha\in\mathfrak{A}\subset\mathbb{Z}^{+r}}\delta\Big(\tau-\sum_j(2\alpha_j+1)f_j\mu h - V\Big)W'_\alpha \tag{19.6.68}$$

with $W'_\alpha = O(1)$.

Remark 19.6.27. The exact formula for W'_α is rather complicated; however $W'_\alpha = 0$ provided g^{jk}, F_{jk} are constant.

Problem 19.6.28. Derive explicit formula for W'_α.

Finally, consider spectral gaps:

Theorem 19.6.29. *Let assumptions* (19.1.4)–(19.1.6), $(19.1.25)_{1-3}$ *with* $(\bar I,\bar\sigma)\succeq(I,\sigma)\succeq(1,2)$, $(\bar I,\bar\sigma)\succeq(2,1)$ *be fulfilled. Consider two cases:*

(i) Let assumptions (19.6.64) *and* (19.6.11) *be fulfilled.*

(ii) Let assumptions (19.6.66), (19.6.18)–(19.6.19) *and either* $\#\mathfrak{A}=0$ *or* (19.6.46) *be fulfilled.*

Then in both cases (i) and (ii) there are two framing approximations[18)] *(see Chapter 18) such that*

$$\mathsf{R}^{\mathsf{MW}}\le C\mu^{-s} \tag{19.6.69}$$

with arbitrarily large s.

19.7 Simple Degeneration

Now we want to consider the case of the degeneration but completely different from one when g^{jk}, F_{jk} constant and V having non-degenerate critical points.

19.7.1 Framework

Assume now that

(19.7.1) There are no 2-nd order resonances i.e. eigenvalues f_j of (F_k^j) are simple

albeit the microhyperbolicity condition (see Definition 19.2.4) is violated i.e. there exists $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r)$ with

$$\lambda_1 \ge 0, \dots, \lambda_r \ge 0, \quad \lambda_1 + \dots + \lambda_r = 1 \tag{19.7.2}$$

such that

$$\sum_k \lambda_k \nabla \log(-V/f_k)(\bar{x}) = 0; \tag{19.7.3}$$

let $\mathbb{K}$ be the set of $\boldsymbol{\lambda}$, satisfying (19.7.2).

Further, let us assume that

$$\operatorname{rank}\bigl\{\nabla \log(-V/f_1)(\bar{x}), \dots, \nabla \log(-V/f_r)(\bar{x})\bigr\} \ge m \tag{19.7.4}$$

with $1 \le m \le r-1$.

Note that if $m = r-1$ (case of the *simple degeneration* then $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r) \in \mathbb{K}$ in (19.7.3) is defined uniquely; more generally, $\boldsymbol{\lambda} \in \mathbb{K}$ is defined uniquely modulo elements of $(r-m-1)$-dimensional subspace $\mathbb{L}$ and

$$|\sum_k \lambda_k \nabla \log(-V/f_k)(\bar{x})| \asymp \nu := \operatorname{dist}(\boldsymbol{\lambda}, \mathbb{L}') \qquad \forall \boldsymbol{\lambda} \in \mathbb{K} \tag{19.7.5}$$

where affine subspace $\mathbb{L}'$ is defined by (19.7.3) and $\lambda_1 + \dots + \lambda_r = 1$.

We consider only the smooth case i.e. assume that

$$g^{jk}, V_j, V \in \mathscr{C}^K \tag{19.7.6}$$

with sufficiently large $K = K(d)$. Then (19.7.5) remains true in $\epsilon\nu$-vicinity of x.

19.7.2 Weak Magnetic Case

Using standard reduction we reach

$$\mathcal{A} = \mu h \sum_j f_j(x'', \mu^{-1}hD'')\big(x_j^2 + D_j^2\big) + O(\mu^{-l}) \tag{19.7.7}$$

with $l = 2, 1$ if there are no 3-rd order resonances and in the general case respectively.

We leave to the reader the following

Problem 19.7.1. (i) Prove by our standard method that one can localize with respect to $\mu h\big(x_j^2 + D_j^2\big)$ modulo $O(\mu^{1-l}T + C\mu h|\log h|)$.

(ii) Prove by the method of Section 19.5 that one can localize with respect to $\mu h\big(x_j^2 + D_j^2\big)$ modulo $O(\mu^{1-l}T + C\mu h)$.

Assume that the localization is done in vicinity of $\boldsymbol{\tau}$. Consider symbol

$$b(x'', \xi''; \boldsymbol{\tau}) = \sum_j f_j(x'', \xi'')\tau_j. \tag{19.7.8}$$

Consider point $(\bar{x''}, \bar{\xi''})$. Then if in its γ-vicinity

$$|\nabla b| \asymp \nu\gamma^{-1}, \quad |b| \lesssim \nu, \qquad \nu = \gamma^2, \tag{19.7.9}$$

$$\gamma \ge C_0 \max(\mu^{-1}, (\mu^{-1}h|\log h|)^{\frac{1}{2}}); \tag{19.7.10}$$

then the shift for time T is $\asymp \mu^{-1}\nu T$ and it is observable if $\mu^{-1}\nu T \times \gamma \ge C\mu^{-1}h|\log h|$ and one can take $T = \epsilon_0\mu^{-1}$ as long as

$$\gamma \ge \bar{\gamma} := C_0 \max\big((\mu h|\log h|)^{\frac{1}{2}}, \mu^{-l}\big); \tag{19.7.11}$$

here the last requirement ($\gamma \ge C_0\mu^{-l}$) is needed to counter the irreducible terms of order $l+2$ but it contributes only $C\mu h^{1-d} \times \mu^{-l} = C\mu^{1-l}h^{1-d}$ into the final remainder estimate. Then one can take

$$T^* \asymp \mu^{l-1}\gamma \tag{19.7.12}$$

and $T_* = \epsilon_0\mu^{-1}$ and then contribution of γ-vicinity to Tauberian remainder does not exceed $C\gamma^{2d}h^{1-d}T^{*-1} \asymp C\gamma^{2d-1}\mu^{1-l}h^{1-d}$ and then contribution of zone $\{\gamma \ge \bar{\gamma}\}$ to such remainder does not exceed

$$C\mu^{1-l}h^{1-d} \int_{\{\gamma \ge \bar{\gamma}\}} \gamma^{-1}\, dx''d\xi''d\boldsymbol{\tau}, \tag{19.7.13}$$

while contribution of zone $\{\gamma \le \bar{\gamma}\}$ does not exceed

$$C\mu h^{1-d}\,\mathsf{mes}\big(\{\gamma \le \bar{\gamma}\}\big) \tag{19.7.14}$$

and the total remainder does not exceed (19.7.13)+(19.7.14):

$$C\mu^{1-l}h^{1-d}\int_{\gamma\ge\bar{\gamma}} \gamma^{-1}\,dM(\gamma) + C\mu h^{1-d}M(\bar{\gamma}) \tag{19.7.15}$$

with

$$M(\gamma) \coloneqq \mathsf{mes}_{3r}\big(\{(x'',\xi'',\boldsymbol{\tau}) : |\nabla_{x'',\xi''}b(x'',\xi'',\boldsymbol{\tau})| \asymp \gamma\}\big) = \int M(x'',\xi'';\gamma)\,dx''d\xi'', \tag{19.7.16}$$

$$M(x'',\xi'';\gamma) = \mathsf{mes}_{r}\big(\{(\boldsymbol{\tau}) : |\nabla_{x'',\xi''}b(x'',\xi'',\boldsymbol{\tau})| \asymp \gamma\}\big). \tag{19.7.17}$$

Note that

(19.7.18) Under assumption $(19.7.4)_m$ $M(x'',\xi'';\gamma) = O(\gamma^m)$.

Problem 19.7.2. Justify the following statement: we have $(3r-1)$-dimensional variable $(x,\boldsymbol{\lambda})$ (due to restriction $\lambda_1+...+\lambda_r=1$) and (19.7.3) imposes $2r$ restrictions; thus it happens generically on a variety of dimension $(r-1)$. However, $(19.7.4)_m$ with $m=(r-2)$ imposes $(2r-1)$ further restrictions more and thus never happens generically.

So let us assume that assumption $(19.7.4)_{r-1}$ is fulfilled. Then there exists a single $\boldsymbol{\lambda}=\boldsymbol{\lambda}(x)$ minimizing

$$|\sum_k \lambda_k \nabla \log(-V/f_k)|^2. \tag{19.7.19}$$

Let us assume that

$$\nabla(\sum \lambda_i \log(-V/f_i)) = 0 \implies \mathsf{rank}\big\{\sum_i \partial_j(\lambda_i\partial_k \log(-V/f_i))\big\}_{j,k=1,\dots,2r} = n. \tag{19.7.20}_n$$

Note that

(19.7.21) Under assumptions $(19.7.4)_{r-1}$ and $(19.7.20)_n$ $M(\gamma) = O(\gamma^{m+n})$.

Problem 19.7.3. Prove that generically (19.7.9) is fulfilled with $n = r+1$ and therefore $M(\gamma) = \gamma^{2r}$.

Therefore under assumption $(19.7.4)_{r-1}$ we conclude that

$$\mathsf{R}^{\mathsf{T}} \le C\mu^{-1}h^{1-d} + C\mu(\mu h|\log h|)^{\frac{1}{2}(r-1)}h^{1-d} \tag{19.7.22}$$

and under assumptions $(19.7.4)_{r-1}$ and $(19.7.20)_{r+1}$

$$\mathsf{R}^{\mathsf{T}} \le C\mu^{1-l}h^{1-d} + C\mu(\mu h|\log h|)^{r}h^{1-d}. \tag{19.7.23}$$

Further, the same estimates hold for $\mathsf{R}^{\mathsf{W}}_{(\infty)}$.

Remark 19.7.4. Actually summation with respect to the partition as $r = 2$ and only the first assumption is fulfilled returns $\mu^{1-l}h^{1-d}\log\mu$ instead of the first term.

However, as $m = 1$ we can consider propagation in the direction of ν increasing and we would be able to take $T^* \asymp \mu\nu^{1-\delta}$ if there are no 3-rd order resonances and $T^* \asymp \nu^{1-\delta}$ otherwise (we used similar arguments multiple times, for example in the analysis of the exterior zone for 3D magnetic Schrödinger) and then we would arrive to (19.7.22) again.

Remark 19.7.5. Using rescaling technique one can get rid of logarithmic factors arriving to

$$\mathsf{R}^{\mathsf{T}} \le C\mu^{-1}h^{1-d} + C(\mu h)^{\frac{1}{2}(r+1)}h^{-d} \tag{19.7.22}^*$$

and

$$\mathsf{R}^{\mathsf{T}} \le C\mu^{1-l}h^{1-d} + C(\mu h)^{r+1}h^{-d}. \tag{19.7.23}^*$$

respectively.

Therefore we arrive to the same estimates for $\mathsf{R}^{\mathsf{W}}_{(\infty)}$.

Theorem 19.7.6. *Let assumptions* (19.1.4)–(19.1.6), (19.7.6) *and* (19.2.38) *be fulfilled.*

Then under assumption $(19.7.4)_{r-1}$

$$\mathsf{R}^{\mathsf{MW}} \le C\mu^{-1}h^{1-d} + C(\mu h)^{\frac{1}{2}(r+1)}h^{-d} \tag{19.7.24}$$

and under assumptions $(19.7.4)_{r-1}$ *and* $(19.7.20)_{r+1}$

$$\mathsf{R}^{\mathsf{MW}} \le C\mu^{1-l}h^{1-d} + C(\mu h)^{r+1}h^{-d}. \tag{19.7.25}$$

respectively.

Proof. We leave to the reader the transition from $\mathsf{R}^{\mathsf{W}}_{(\infty)}$ to R^{MW}. □

19.7.3 Temperate Magnetic Field

Let us assume now that either $(19.7.4)_{r-1}$ is fulfilled and $(\mu h)^{(r+1)/2} \ge C\mu^{-1}h$ which is equivalent to

$$C_0 h^{-(r-1)/(r+3)} \le \mu \le \epsilon h^{-1} \tag{19.7.26}$$

or both $(19.7.4)_{r-1}$ and $(19.7.20)_{r+1}$ are fulfilled and $(\mu h)^{(r+1)} \ge C\mu^{-1}h$ which is equivalent to

$$C_0 h^{-r/(r+2)} \le \mu \le \epsilon h^{-1}. \tag{19.7.27}$$

Tauberian Remainder: the Toy-Model

Assume temporarily that there are no resonances of order less than L with large enough L except the trivial ones i.e. with the irreducible terms

$$\mu^{2-L} b_\beta(x'', \mu^{-1}hD'')(h^2D_1^2 + \mu^2 x_1^2)^{\beta_1} \cdots (h^2D_r^2 + \mu^2 x_r^2)^{\beta_r}; \tag{19.7.28}$$

then in view of (19.7.26) or (19.7.27) we can skip all the terms corresponding to resonances of order $\ge L$ (with a very small error) and we get a family of r-dimensional $\mu^{-1}h$-pseudodifferential operators and we can apply our standard theory to them.

Under the non-degeneracy assumption $(19.7.20)_{r+1}$ all of these operators satisfy non-degeneracy assumption "no degenerate critical points" and therefore the individual remainder estimate would be $O\big((\mu^{-1}h)^{1-r}\big)$; then multiplying by the number of operators $\asymp (\mu h)^{-r}$ we get $O(\mu^{-1}h^{1-d})$.

Meanwhile without assumption $(19.7.20)_{r+1}$ these operators could be highly degenerate and the individual remainder estimate would be $O\big((\mu^{-1}h)^{-r}\big)$ and thus we do not expect a total remainder estimate better than this one.

Let us prove that this is also a total remainder estimate. According to Subsection 5.2.4 we can introduce an admissible function $\ell = \ell(z, \boldsymbol{\lambda})$ with $z = (x'', \xi'')$ such that in $B(z, \ell(z))$

$$\ell(z, \boldsymbol{\lambda}) \asymp \sum_{\alpha:|\alpha|\le m} |\nabla^\alpha b(z, \boldsymbol{\lambda})|^{\frac{1}{m+1-|\alpha|}} \tag{19.7.29}$$

and then we redefine

$$\ell(z, \boldsymbol{\lambda}) \coloneqq \max\bigl(\ell(z, \boldsymbol{\lambda}), \bar{\ell}\bigr), \qquad \bar{\ell} = C_0(\mu^{-1}h)^{\frac{1}{2}(1-\delta)} \tag{19.7.30}$$

with arbitrarily large m and arbitrarily small $\delta > 0$.

Then again according to the same Subsection 5.2.4 the Tauberian remainder for individual $\boldsymbol{\lambda}$ does not exceed

$$C\mu^{r-1}h^{1-r-\delta'} \int \ell(z, \boldsymbol{\lambda})^{-2}\, dz + C\mu^r h^{-r} \int_{\ell(z,\boldsymbol{\lambda})\le\bar{\ell}} dz \tag{19.7.31}$$

and therefore the total Tauberian remainder does not exceed

$$C\mu^{r-1}h^{1-r-\delta'} \sum_{\boldsymbol{\lambda}} \int \ell(z, \boldsymbol{\lambda})^{-2}\, dz + C\mu^r h^{-r} \sum_{\boldsymbol{\lambda}} \int_{\ell(z,\boldsymbol{\lambda})\le\bar{\ell}} dz \tag{19.7.32}$$

where $\boldsymbol{\lambda}$ runs through the lattice.

Note that $\gamma \coloneqq \ell_1 \le C\ell^m$ (where $\ell = \ell_m$). Therefore $\ell \asymp \bar{\ell}$ implies that $\gamma \lesssim \mu h$ provided

$$m(r+1) > 6. \tag{19.7.33}$$

But then for each z there exist no more than C_0 values of $\boldsymbol{\lambda}$ such that $\ell(z, \boldsymbol{\lambda}) \le \bar{\ell}$ and therefore the second term in (19.7.32) is $O(\mu^r h^{-r})$. This statement remains valid as we redefine $\bar{\ell} = (\mu h)^{1/(m-1)}$.

Meanwhile for any given z the number of $\boldsymbol{\lambda}$ such that $\ell(z, \boldsymbol{\lambda}) \le \ell$ does not exceed $C_0(\mu h)^{-r}\gamma^r + C_0 \lesssim C_0(\mu h)^{-r}\ell^{(m-1)r} + C_0$ and therefore the first term in (19.7.32) does not exceed

$$C\mu^{-1}h^{1-d-\delta'} \int \ell^{mr-2}\, dz + C\mu^r h^{-r}. \tag{19.7.34}$$

Since $mr > 2$ this term is $O(\mu^{-1}h^{1-d-\delta'})$ which is almost exactly what we need. And it is less than $C\mu^r h^{-r}$ unless $\mu \le Ch^{-(r-1)/(r+1)-\delta''}$ so we need to consider only this case.

To cover this case we need to make $\delta' = 0$. Then we need to consider only elements of ℓ_m-partition with $\ell = \ell_m \ge h^{\delta}$. However, if we consider elements of ℓ_{m-1}-partition with $\ell_{m-1} \le h^{\delta_1}$ (but $\ell_m \ge h^{\delta}$), their contribution to the remainder will be properly estimated as well, so we need to consider only elements of ℓ_{m-1}-partition with $\ell_{m-1} \ge h^{\delta_1}$; continuing this process we conclude that we need to consider only balls with $\ell_1 \ge h^{\delta_{m-1}}$. But this is exactly γ and we can apply weak magnetic field approach.

Tauberian Estimates: the General Case

As far as Tauberian estimate is concerned everything remains the same as long as $\ell^{m+1} \ge \mu^{-L}$ where μ^{-L} is the magnitude of the highest irreducible "extra" terms. Further, let

$$(19.7.29)' \quad \ell(z,\boldsymbol{\lambda}) \asymp \sum_{\alpha:1\le|\alpha|\le m} |\nabla^{\alpha} b(z,\boldsymbol{\lambda})|^{\frac{1}{m+1-|\alpha|}} + \min_{s:|s|\le C_0\mu^{-L}} |b(z,\boldsymbol{\lambda})+s|^{\frac{1}{m+1}}$$

(in contrast to (19.7.29) term with $\alpha = 0$ is modified).

However, this would be absorbed by the case $\ell^m \le \mu h$ provided $\mu^{-L} \le \mu h$.

(a) If $L = 1$ we are are looking at the remainder estimate $O(h^{1-d})$ i.e. we need to consider the case $\mu \ge h^{-(r-1)/(r+1)}$ and therefore $\mu \ge h^{-\frac12}$ as $r \ge 3$ and $\mu \ge h^{-\frac13}$ as $r = 2$. However, condition of absorption is $\mu^{-1} \le \mu h$; so as $r \ge 3$ everything is fine: we arrive to the remainder estimate $O\big(h^{1-2r} + \mu^r h^{-r}\big)$.

If $r = 2$ we also arrive to this remainder estimate unless $h^{-\frac13} \le \mu \le h^{-\frac12}$ in which case the remainder estimate is

$$(19.7.35) \quad Ch^{-3} + C\min\big((\mu h)^{\frac32}, \mu^{-2}\big)h^{-4} = Ch^{-3} + Ch^{-4}\begin{cases} (\mu h)^{\frac32} & \text{as } \mu \le h^{-\frac37},\\ \mu^{-2} & \text{as } \mu \ge h^{-\frac37}.\end{cases}$$

(b) If $L = 2$ we are are looking at remainder estimate $O(\mu^{-1}h^{1-d})$ and therefore we need to consider case of (19.7.26) i.e. $\mu \ge h^{-\frac13}$ as $r \ge 3$ and $\mu \ge h^{-\frac15}$ as $r = 2$. However, condition of absorption is $\mu^{-2} \le \mu h$; so as $r \ge 3$ everything is fine: we arrive to the remainder estimate $O\big(\mu^{-1}h^{1-2r} + \mu^r h^{-r}\big)$.

If $r=2$ the same arguments work for $L=4$. As $L=2$ we also get this remainder estimate unless $h^{-\frac{1}{5}} \le \mu \le h^{-\frac{1}{3}}$ when we get

$$C\mu^{-1}h^{-3} + C\min\big((\mu h)^{\frac{3}{2}}, \mu^{-4}\big)h^{-4} = C\mu^{-1}h^{-3} + Ch^{-4}\begin{cases}(\mu h)^{\frac{3}{2}} & \text{as } \mu \le h^{-\frac{3}{11}},\\ \mu^{-2} & \text{as } \mu \ge h^{-\frac{3}{11}}.\end{cases} \tag{19.7.36}$$

(c) If $r=2$, $L=3$ contribution of perturbation is $O(\mu^{-6}h^{-3}) = O(\mu^{-1}h^{-1})$ as $\mu \ge h^{-\frac{1}{5}}$ and therefore we get a proper remainder estimate.

Therefore we arrive to

Proposition 19.7.7. *Let $\mu \le \epsilon h^{-1}$ and assumption $(19.7.4)_{r-1}$ be fulfilled. Then*

(i) If $L=1$

$$\mathsf{R}^{\mathsf{T}} \le Ch^{1-2r} + C\mu^r h^{-r} \tag{19.7.37}$$

unless $r=2$, $h^{-\frac{1}{3}} \le \mu \le h^{-\frac{1}{2}}$ when R^{T} does not exceed expression (19.7.35).

(ii) If $L=2$

$$\mathsf{R}^{\mathsf{T}} \le C\mu^{-1}h^{1-2r} + C\mu^r h^{-r} \tag{19.7.38}$$

unless $r=2$, $h^{-\frac{1}{5}} \le \mu \le h^{-\frac{1}{3}}$ when R^{T} does not exceed expression (19.7.36); furthermore, as $r=2$, $L=3$ estimate (19.7.38) holds.

Assume now that extra assumption $(19.7.20)_{r+1}$ is fulfilled. Then

(d) If $L=1$ we are looking for the remainder $O(h^{1-d})$ i.e. we need to consider a case $(\mu h)^{r+1} \ge h$ i.e. $\mu \ge h^{-\frac{2}{3}}$ and then the contribution to the remainder of the degeneration will be $O\big(((\mu^{-1/2})^{2r}h^{-2r}\big) = O(h^{1-2r})$.

(e) If $L=2$ we are looking for the remainder $O(\mu^{-1}h^{1-d})$ i.e. we need to consider a case of (19.7.27) i.e. $\mu \ge h^{-\frac{1}{2}}$ and then contribution to the remainder of the degeneration will be $O\big(((\mu^{-2/2})^{2r}h^{-2r}\big) = O(\mu^{-1}h^{1-2r})$.

Proposition 19.7.8. *Let $\mu \le \epsilon h^{-1}$ and assumptions $(19.7.4)_{r-1}$ and $(19.7.20)_{r+1}$ be fulfilled. Then*

(i) If $L = 1$

$$\mathsf{R}^{\mathsf{T}} \le Ch^{1-d}. \tag{19.7.39}$$

(ii) If $L = 2$

$$\mathsf{R}^{\mathsf{T}} \le C\mu^{-1}h^{1-d}. \tag{19.7.40}$$

Calculations: the Toy-Model

In the toy-model case we get the Tauberian expression $h^{-d}\mathcal{N}^{\mathsf{T}}$ with

$$\mathcal{N}^{\mathsf{T}} := \sum_{\alpha} \mu^r h^r \theta\big(1 - b_0(x,\alpha) - \mu^{-2}b_1(x,\alpha) - ...\big)\times \big(\psi_0(x) + \mu^{-2}\psi_1(x,\alpha) + ...\big) f_1 \cdots f_r \sqrt{g}\, dx \tag{19.7.41}$$

and if we remove all powers of μ^{-2} then we get the final answer.

If $\mu \ge h^{-\frac{1}{3}}$ we can following our approach skip instantly $O(\mu^{-4})$; we can also remove $O(\mu^{-2})$ but add a correction term instead which can be written as $\kappa\mu^{-2}$ and prove that an error does not exceed $C\mu^{-1}h$ provided $r \ge 3$; then we conclude that $\kappa = 0$ and therefore as $r \ge 3$ we arrive modulo $O(\mu^{-1}h)$ to $\mathcal{N}^{\mathsf{MW}}$. Therefore for the toy-model operator with $r \ge 3$ we arrive to the desired formula.

For $r = 2$ situation is more complicated: we can assume only that $\mu \ge h^{-\frac{1}{5}}$ and then we can skip instantly only $O(\mu^{-6})$ (rather than $O(\mu^{-4})$); we can also remove $O(\mu^{-4})$ but add a correction term instead which will be given by (19.4.93).

For $r = 2$ and $\mu \ge h^{-\frac{1}{3}}$ we can remove $O(\mu^{-2})$ and add a correction term instead which will be 0 but only modulo $O(\mu^{-2}(\mu h)^{\frac{1}{2}}h^{-4}) = O(\mu^{-\frac{3}{2}}h^{-\frac{7}{2}})$.

On the other hand, under assumption $(19.7.20)_{r+1}$ we can deal with $O(\mu^{-2})$ terms with an error $O(\mu^{-2}(\mu h)^r h^{-2d})$ which is $O(\mu^{-1}h^{1-2d})$ even as $r = 2$. We leave all details to the reader.

Thus we arrive to the following theorem:

Theorem 19.7.9. *Let assumptions* (19.1.4)–(19.1.6), (19.7.6) *and* (19.2.38) *be fulfilled. Let there be no resonances of the* 2*-nd and* 3*-rd order, and no non-trivial resonances of order less than* L *(with large enough* L*) and let assumption* $(19.7.4)_{r-1}$ *be fulfilled. Then*

(i) As $r \ge 3$, $\mu \le \epsilon(h|\log h|)^{-1}$

$$\mathsf{R}^{\mathsf{MW}} \le C\mu^{-1}h^{1-2r} + C\mu^{r}h^{-r}. \tag{19.7.42}$$

(ii) If $r = 2$, $h^{-\frac{1}{5}} \le \mu \le \epsilon(h|\log h|)^{-1}$

$$\mathsf{R}^{\mathsf{MW}}_1 := \Big| \int \Big(\tilde{e}(x,x,0) - h^{-d}\mathcal{N}^{\mathsf{MW}}(x,0) - h^{-d}\mathcal{N}^{\mathsf{MW}}_{1\mathsf{corr}}(x,0)\Big)\psi(x)\,dx\Big| \le C\mu^{-1}h^{1-d} + C\mu^{2}h^{-2} \tag{19.7.43}$$

where $\mathcal{N}^{\mathsf{MW}}_{1\mathsf{corr}}(x,0)\Big)$ *is defined by* (19.4.93) *and* $\mathcal{A}_{\mathcal{T}} = \mathcal{A}_0 + \mu^{-2}\mathcal{A}_2$*; in particular, as* $r = 2$, $h^{-\frac{1}{3}} \le \mu \le \epsilon(h|\log h|)^{-1}$ *this correction term does not exceed* $C\mu^{-\frac{3}{2}}h^{-\frac{7}{2}}$ *and therefore one can skip it as* $\mu \ge h^{-\frac{3}{7}}$.

(iii) On the other hand, under assumption $(19.7.20)_{r+1}$

$$\mathsf{R}^{\mathsf{MW}} \le C\mu^{-1}h^{1-2r}. \tag{19.7.44}$$

Calculations: the General Case

Consider now the general case. Assume first that there are no 3-rd order resonances. Then both arguments of the proof and statements of Theorem 19.7.9 obviously remain true with the singular exception of Statement (ii) where now we need to take

$$\mathcal{A}_{\mathcal{T}} = \mathcal{A}_0 + \mu^{-2}\mathcal{A}_2 + \mu^{-3}\mathcal{A}_3 \tag{19.7.45}$$

and $\mathcal{A}_2$ may contain also a non-diagonal part (if there are non-trivial 4-th order resonances) and $\mathcal{A}_3$ is purely a non-diagonal term (if there are 5-th order resonances).

Now we would like to get rid of all non-diagonal terms in $\mu^{-2}\mathcal{A}_2$ and $\mu^{-3}\mathcal{A}_3$. It works as long as $\mu \ge h^{-\frac{1}{3}}$ and $\mu \ge h^{-\frac{1}{4}}$ respectively. Then we will have another correction term. However, it will be a bit smaller than $\mu^{-L}(\mu h)^{\frac{1}{2}}h^{-4}$. Namely if we take in account only the leading term in ψ, i.e. ψ_0, we get a non-diagonal term with the trace 0 in the second approximation term and the error will be $O(\mu^{-\frac{3}{2}L}h^{-4})$ which is a bit better than $O(\mu^{-L}(\mu h)^{\frac{1}{2}}h^{-4})$ if $\mu^{L+1}h \ge 1$.

Modulo this term there will be term coming from $\psi_1\mu^{-1}$ and our standard arguments yield that it will not exceed $O(\mu^{-L-1}h^{-4})$ which is exactly what we got before if $L=2$. However, if $L=3$ one can see easily that the first perturbation term would be out-of-diagonal by more than 1 and we need "interact" it with $\psi_2\mu^{-2}$ to get non-0 trace and this term will be $O(\mu^{-L-2}h^{-4})$.

Therefore we arrive to

Theorem 19.7.10. *Let assumptions* (19.1.4)–(19.1.6), (19.7.6) *and* (19.2.38) *be fulfilled. Let there be no resonances of the* 2*-nd and* 3*-rd order and let assumption* $(19.7.4)_{r-1}$ *be fulfilled. Then*

(i) Statements (i) and (iii) of Theorem 19.7.9 hold; Statement (ii) remains true without any modifications if there are no non-trivial 4*-th and no* 5*-th order resonances.*

(ii) In Statement (ii) of Theorem 19.7.9 one should take $\mathcal{A}_{\mathcal{T}}$ *in the form* (19.7.45) *but the conclusion about magnitude of the correction term remains true.*

(iii) If $r=2$ *and there are non-trivial* 4*-th order resonances and we take* $\mathcal{A}_{\mathcal{T}}=\mathcal{A}_0+\mu^{-2}\mathcal{A}_2$ *and remove from* $\mathcal{A}_2$ *non-diagonal terms, estimate*

$$\mathsf{R}_1^{\mathsf{MW}}\le C\mu^{-1}h^{-3}+C\mu^{2}h^{-2}+C\mu^{-3}h^{-4} \tag{19.7.46}$$

holds as $\mu\ge h^{-\frac{1}{3}}$*; in particular* (19.7.43) *holds as* $\mu\ge h^{-\frac{2}{5}}$.

(iv) If $r=2$ *and there are no non-trivial* 4*-th order resonances but there are* 5*-th order resonances and we take* $\mathcal{A}_{\mathcal{T}}=\mathcal{A}_0+\mu^{-2}\mathcal{A}_2$*, estimate*

$$\mathsf{R}_1^{\mathsf{MW}}\le C\mu^{-1}h^{-3}+C\mu^{2}h^{-2}+C\mu^{-\frac{9}{2}}h^{-4} \tag{19.7.47}$$

holds as $\mu\ge h^{-\frac{1}{4}}$*; in particular* (19.7.43) *holds as* $\mu\ge h^{-\frac{2}{7}}$.

Assume now that there are 3-rd order resonances. Then for $r\ge 3$ the weak magnetic field approach gives remainder estimate $O(h^{1-d})$ as $\mu\le h^{-\frac{1}{2}}$ and an error when we remove $O(\mu^{-1})$ non-diagonal term is $O(\mu^{-2}h^{-d})=O(h^{1-d})$ as $\mu\ge h^{-\frac{1}{2}}$. Therefore we are done.

If $r=2$ and both assumptions $(19.7.4)_1$ and $(19.7.20)_3$ are fulfilled, the weak magnetic field approach gives remainder estimate $O(h^{-3})$ as $(\mu h)^3\le h$

i.e. as $\mu \le h^{-\frac{2}{3}}$ and an error when we remove $O(\mu^{-1})$ non-diagonal term is $O(\mu^{-2}h^{-4}) = O(h^{-3})$ as $\mu \ge h^{-\frac{2}{3}}$. Therefore we are done as well.

Consider case $r = 2$ and only assumption $(19.7.4)_1$ is fulfilled. Then the weak magnetic field approach gives remainder estimate $O(h^{-3})$ as $\mu \le h^{-\frac{1}{4}}$. Then we have a proper remainder estimate with the correction term defined by

$$\mathcal{A}_{\mathcal{T}} = \mathcal{A}_0 + \mu^{-1}\mathcal{A}_1 + \mu^{-2}\mathcal{A}_2 + \mu^{-3}\mathcal{A}_3; \tag{19.7.48}$$

however in virtue of our previous arguments skipping $O(\mu^{-3})$ term would lead to an error $O(\mu^{-4}(\mu h)^{\frac{1}{2}}h^{-4}) = O(h^{-3})$ then. Therefore we arrive to

Theorem 19.7.11. *Let assumptions* (19.1.4)–(19.1.6), (19.7.6) *and* (19.2.38) *be fulfilled. Let there be no resonances of the* 2*-nd order and let assumption* $(19.7.4)_{r-1}$ *be fulfilled. Then*

(i) If $r \ge 3$, $\mu \le \epsilon(h|\log h|)^{-1}$

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{1-2r} + C\mu^r h^{-r}. \tag{19.7.49}$$

(ii) If $r = 2$, $h^{-\frac{1}{5}} \le \mu \le \epsilon(h|\log h|)^{-1}$

$$\mathsf{R}_1^{\mathsf{MW}} := \Big| \int \Big(\tilde{e}(x,x,0) - h^{-d}\mathcal{N}^{\mathsf{MW}}(x,0) - h^{-d}\mathcal{N}^{\mathsf{MW}}_{1\mathsf{corr}}(x,0)\Big)\psi(x)\,dx\Big| \le Ch^{-3} + C\mu^2 h^{-2} \tag{19.7.50}$$

where $\mathcal{N}^{\mathsf{MW}}_{1\mathsf{corr}}(x,0)$ *is defined by* (19.4.93) *and* $\mathcal{A}_{\mathcal{T}}$ *is defined by* (19.7.48) *without the last term; in particular, as* $r = 2$, $h^{-\frac{1}{2}} \le \mu \le \epsilon(h|\log h|)^{-1}$ *this correction term does not exceed* $C\mu^{-\frac{3}{2}}h^{-4}$ *and therefore one can skip it as* $\mu \ge h^{-\frac{4}{7}}$.

(iii) On the other hand, under assumption $(19.7.20)_{r+1}$

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{1-2r}. \tag{19.7.51}$$

Remark 19.7.12. Further, skipping $O(\mu^{-2})$ term in $\mathcal{A}_{\mathcal{T}}$ leads to an error $O(\mu^{-2}(\mu h)^{\frac{1}{2}}h^{-4})$ which is $O(h^{-3})$ as $\mu \ge h^{-\frac{1}{3}}$.

Furthermore, skipping $O(\mu^{-1})$ non-diagonal term leads to an error $O(\mu^{-\frac{3}{2}}h^{-4}) = O(h^{-3})$ as $\mu \ge h^{-\frac{2}{3}}$.

19.7.4 Intermediate Magnetic Field

The case of the intermediate magnetic field is easy:

Theorem 19.7.13. *Let assumptions* (19.1.4)–(19.1.6), (19.7.6) *and* (19.2.38) *be fulfilled. Let there be no resonances of the* 2*-nd order, and let assumption* $(19.7.4)_{r-1}$ *be fulfilled. Let* $\epsilon(h|\log h|)^{-1} \le \mu \le \epsilon h^{-1}$.

(i) Assume that there no resonances of the 3*-rd order as well. Then estimate* (19.7.42) *holds and under assumption* $(19.7.20)_{r+1}$ *estimate* (19.7.44) *holds.*

(ii) In the general case estimate (19.7.49) *holds and under assumption* $(19.7.20)_{r+1}$ *estimate* (19.7.50) *holds.*

Proof. An easy proof using arguments of Section 19.5 is left to the reader. □

19.7.5 Conclusion to Section

There is no theory of intermediate and strong magnetic field separate from theory of Section 19.6

Problem 19.7.14. Prove similar results under assumptions $(19.7.4)_m$ and $(19.7.20)_n$ with $0 \le m < r-1$ and $1 \le m+n \le 2r$.

19.7.6 Final Remarks

Vanishing V

Remark 19.7.15. (i) In Sections 19.2–19.5 we divided by V in the microhyperbolicity or $\mathfrak{N}$-microhyperbolicity assumptions. However, it is not a problem: we could assume in advance that

$$|V| + |\nabla V| \ge \epsilon_0 \tag{19.7.52}$$

and then the microhyperbolicity or $\mathfrak{N}$-microhyperbolicity conditions will be needed in zone $\{x : |V(x)| \ge \epsilon'\}$ only.

(ii) We can assume instead that

$$|V| + |\nabla V| \le \epsilon_0 \implies |\det \mathsf{Hess}\, V| \ge \epsilon_0. \tag{19.7.53}$$

Really then due to a rescaling technique we can ensure microhyperbolicity condition (except a small zone) after rescaling as $|V| \le \epsilon'$ and then the microhyperbolicity or $\mathfrak{N}$-microhyperbolicity conditions will be needed in zone $\{x : |V(x)| \ge \epsilon'\}$ only.

We leave to the reader the following problem:

Problem 19.7.16. (i) Using rescaling technique like in Subsection 18.9.5 get rid of assumption $|V| \ge \epsilon_0$ in the framework of Sections 19.2, 19.4, and 19.5.

(ii) Using rescaling technique get rid of assumption $|V| \ge \epsilon_0$ in the framework of Sections 19.2, 19.4, and 19.5.

Pointwise Asymptotics

We strongly believe that the following problem is both extremely challenging and interesting:

Problem 19.7.17. For operators of the type considered in this Chapter (namely, Sections 19.2–19.6) construct theory similar to theory of Chapter 16.

(i) Let us discuss the toy-model with constant g^{jk}, F_{jk}, and linear V. Then A is a sum of 2D-operators and the $U(x,y,t)$ is a product of 2D-propagators which after rescaling $x \mapsto \mu x$, $t \mapsto \mu t$ are given by (16.2.9)–(16.2.10) with t replaced by $f_j t$ and α_j replaced by $f_j^{-1}\alpha_j$. Then we come to the oscillatory integral with respect to t of (16.2.22) type (and we will need to multiply it later by $\mu^{-r}h^{-r}$) but with a phase function which is the sum of (16.2.23)-type expressions

$$\phi(t) = -t^2 \sum_j \alpha_j^2 \cot(f_j t) - t(\tau - \beta), \qquad \beta = -\mu^{-2} \sum_j \alpha_j^2 f_j^{-1} \tag{19.7.54}$$

and with factor $\mu^2 \csc(t)$ replaced by $\mu^{2r} \csc(f_1 t)\cdots\csc(f_r t)$. Now analysis of the stationary points becomes really difficult. The stationary phase equation

$$-2t \sum_j \alpha_j^2 \cot(f_j t) + t^2 \sum_j \alpha_j^2 f_j \csc^2(f_j t) - (\tau - \beta) = 0 \tag{19.7.55}$$

is equivalent to

$$-2t\cot(f_j t) + t^2\alpha_j^2 f_j \csc^2(f_j t) - \tau_j = 0, \tag{19.7.56}$$

$$\sum \tau_j = \tau - \beta. \tag{19.7.57}$$

This system is not easy to handle for large t if f_j are not commensurable.

(ii) However, in the case when $f_1 = ... = f_r = 1$ which is supposedly the worst case scenario we get almost (16.2.23) exactly

$$\phi(t) = -t^2\cot(t) + t\mu^{-2}\alpha^2 - t\tau \tag{19.7.58}$$

but factor $\mu^{2r}(\csc(t))^r$ with $r \geq 2$ rather than $r = 1$ will be a game-changer in all Tauberian estimates as $\mu \leq h^{-1}$.

Problem 19.7.18. (i) Prove that for the toy-model with $f_1 = ... = f_r$

$$\mathsf{R}^{\mathsf{T}} \leq C \begin{cases} \mu^{-1}h^{1-d} & \text{as } \mu_0 \leq \mu \leq h^{-\frac{1}{2}}, \\ \mu h^{2-d} & \text{as } h^{-\frac{1}{2}} \leq \mu \leq h^{-1}, \\ \mu^{r-\frac{1}{2}}h^{-r+\frac{1}{2}} & \text{as } \mu \geq h^{-1}; \end{cases} \tag{19.7.59}$$

(ii) Prove the same in the case of commensurable $f_1, ..., f_r$.

(iii) Investigate the case of non-commensurable $f_1, ..., f_r$.

Problem 19.7.19. (i) Calculate Tauberian expressions using stationary phase methods as $\mu \leq h^{\delta-\frac{1}{2}}$.

(ii) Investigate the matching cases when V is non-linear.

Chapter 20

Multidimensional Magnetic Schrödinger Operator. Non-Full-Rank Case

20.1 Introduction

20.1.1 Preface

In this chapter we consider multidimensional Schrödinger operator (19.1.1)

$$A = A_0 + V(x), \qquad A_0 = \sum_{j,k\le d} P_j g^{jk}(x) P_k, \tag{20.1.1}$$

$$P_j = hD_j - \mu V_j(x), \quad h \in (0,1],\ \mu \ge 1.$$

Recall that it is characterized by *magnetic field intensity tensors* (F_{jk}) with

$$F_{jk} = \partial_k V_j - \partial_j V_k, \tag{20.1.2}$$

which is skew-symmetric $d \times d$-matrix, and $(F^j_p) = (g^{jk})(F_{kp})$, which is unitarily equivalent to the skew-symmetric matrix $(g^{jk})^{\frac12}(F_{jk})(g^{jk})^{\frac12}$. Then all the eigenvalues of (F^j_k) (with multiplicities) are $\pm i f_p$ ($f_p > 0$, $p = 1, \dots, r$) and 0 of multiplicity $q = d - 2r$ where $2r = \mathsf{rank}(F^j_k)$.

In this chapter we assume first that the magnetic field intensity matrix has a constant but not full rank:

$$\mathsf{rank}(F_{jk}) = 2r = d - q, \quad q \ge 1, \tag{20.1.3}$$

and

$$|\big((F_{jk})\big|_{\mathcal{K}^\perp}\big)^{-1}| \le c_0 \tag{20.1.4}$$

where $\mathcal{K} = \mathsf{Ker}(F_{jk})$ and $\mathcal{K}^\perp = \mathsf{Ran}(F_{jk})$ and (under certain conditions) we derive sharp spectral asymptotics (in this case with the remainder estimate $O(h^{1-d} + \mu^r h^{1-r-q})$. The typical example (already studied in Chapters 13 and 18) is 3D magnetic Schrödinger operator.

Remark 20.1.1. Obviously, the most interesting case is $q = 1$ which is the generic case for skew-symmetric real $d \times d$-matrices. However it does not exclude thi type generic matrix-valued functions with extra degenerations at some points.

Condition (20.1.3)–(20.1.4) is equivalent to (20.1.3) and

$$|f_p| \ge \epsilon_0 \qquad p = 1, \dots, r; \tag{20.1.5}$$

however, later we replace it with a *multiscale assumption* according to which $f_1, \dots, f_r$ are broken into groups of different magnitudes which will allow us to analyze generalizations.

Recall that we consider operator in some domain or on some manifold X with some boundary conditions, assuming that it is self-adjoint in $\mathscr{L}^2$ and denote by $e(x, y, \tau)$ Schwartz' kernel of it's spectral projector.

There is a profound difference between the full-rank case considered in Chapter 19 and non-full-rank case considered here: in the full-rank case in its canonical form the magnetic Schrödinger operator is reduced to the family of r-dimensional $\mu^{-1}h$-pseudodifferential operators and therefore we can expect asymptotics with the remainder estimate (in the smooth case) as good as

$$C(\mu h^{-1})^{r-1} \times \big(1 + (\mu h)^{-r}\big) = C\big(\mu^{-1}h^{1-d} + \mu^{r-1}h^{1-r}\big),$$

where the first factor is the remainder estimate for a single scalar $\mu^{-1}h$-pseudodifferential operator and the second factor is the number of these operators to be taken into account; however, one needs some non-degeneracy (microhyperbolicity) assumption; otherwise the remainder estimate could be as bad as $C\big(\mu h^{1-d} + \mu^r h^{-r}\big)$ which is the magnitude of the principal part as $\mu \ge h^{-1}$.

On the other hand, in the non-full-rank case in its canonical form the magnetic Schrödinger operator is reduced to the family of q-dimensional

Schrödinger operators which are also r-dimensional $\mu^{-1}h$-pseudodifferential operators and in the best case we can expect asymptotics with the remainder estimate

$$Ch^{1-q}\times(\mu h^{-1})^r\times\bigl(1+(\mu h)^{-r}\bigr)\asymp C\bigl(h^{1-d}+\mu^r h^{1-r}\bigr),$$

with the principal part of magnitude

$$h^{-q}\times(\mu h^{-1})^r\times\bigl(1+(\mu h)^{-r}\bigr)\asymp h^{-d}+\mu^r h^{r-d}$$

however, the non-degeneracy condition is not needed at all (if smoothness allows) as $q\ge 2$; case $q=1$ is slightly more complicated.

We see that there is a more subtle distinction between $q=1$ and $q\ge 2$; actually $q=2$ might stay apart from $q\ge 3$ due to the lack of the smoothness. There is even more subtle distinction between $r=1$ and $r\ge 2$ but it makes the difference only as $q=0,1$ setting $d=2,3$ a bit apart (and in fact it plays role mainly in the non-smooth case).

20.1.2 Assumptions

Note first that $\mathcal{K}=\mathsf{Ker}\,F=\mathsf{Ker}\,\omega_F$ with $\omega_F=\sum_{j,k}F_{jk}dx_j\wedge dx_k=\frac{1}{2}d\sum_j V_j dx_j$ and therefore $d\omega_F=0$ and in virtue of Frobenius theorem

(20.1.6) If $\mathsf{rank}\,F=d-q$ at each point x, then $\mathcal{K}(x)=\mathsf{Ker}\,F(x)$ is an integrable foliation of q-dimensional spaces, which means that for any y there exists a q-dimensional manifold $K=K_y\ni y$ such that $T_xK=\mathcal{K}(x)$ at any point $x\in K$.

Then there exists a coordinate system $x=(x''';x^\perp)=(x_1,\dots,x_q;x_{q+1},\dots,x_d)$ with $x^\perp=\mathsf{const}$ along leafs K_y of the above foliation. This (almost) fixes $x^\perp$ and also implies that in this coordinate system

$$F_{jk}=0\qquad\text{if } j=1,\dots,q \text{ or } k=1,\dots,q. \tag{20.1.7}$$

Then there exists function ϕ, such that $V_j=\partial_j\phi$ for $j=1,\dots,q$ and one can eliminate V_j for all $j=1,\dots,q$ by a gauge transformation[1)]. So, we can assume that

$$V_j=0 \text{ if } j=1,\dots,q \quad(\implies V_j=V_j(x^\perp) \text{ if } j=q+1,\dots,d. \tag{20.1.8}$$

[1)] Really, we always can eliminate V_1; but then $V_{j,1}=V_{1,j}=0$ and $V_j=V_j(x_2,\dots,x_d)$. Then we can eliminate V_2 etc repeating previous arguments for $V_1,\dots,V_q$. Actually we can eliminate also V_{q+1}.

Now we assume that these reductions are already made and after these reductions

$$g^{jk} \in \mathscr{C}^{\bar{l},\bar{\sigma}}, \qquad V \in \mathscr{C}^{l,\sigma}, \tag{20.1.9}_{1,3}$$

and

$$V_j = \partial_j \phi_j, \qquad \phi_j \in \mathscr{C}^{\bar{l}+2,\bar{\sigma}} \tag{20.1.9}_{2}$$

where the last assumption is a bit stronger than a more natural assumption $F_{jk} \in \mathscr{C}^{\bar{l},\bar{\sigma}}$ or $V_{\in}\mathscr{C}^{\bar{l}+1,\bar{\sigma}}$; cf. $(19.1.25)_{1-3}$. Again, $(\bar{l},\bar{\sigma}) \succeq (l,\sigma) \succeq (1,1)$. Assume also that

$$q = 1 \implies g^{jd} = \delta_{jd}, . \tag{20.1.10}$$

We also assume that

$$|V| \geq \epsilon. \tag{20.1.11}$$

20.1.3 Canonical Form

Recall (see Section 13.2.2) that if $X = \mathbb{R}^d$ and g^{jk}, F_{jk}, V are constant then the magnetic Schrödinger operator is unitarily equivalent to operator

$$\sum_{1\leq j\leq r} f_j(h^2D_j^2 + \mu^2 x_j^2) + \sum_{r+1\leq j\leq d-2r} h^2D_j^2 + V \tag{20.1.12}$$

and that $e(x,x,\tau) = h^{-d}\mathcal{N}_d^{\mathsf{MW}}(\tau)$ where

$$\mathcal{N}_d^{\mathsf{MW}}(\tau) := \varpi_{d-2r}(2\pi)^{-d+r}\mu^r h^r \sum_{\alpha\in\mathbb{Z}^{+r}} \Big(\tau - \sum_j (2\alpha_j+1) f_j \mu h - V\Big)_+^{\frac{q}{2}} f_1\cdots f_r\sqrt{g} \tag{20.1.13}$$

and ϖ_k is a volume of unit ball, $g = \det(g^{jk})^{-1}$.

Now in the general case (for variable V and, may be, g^{jk}, F_{jk}) we are interested in the asymptotics of $\int e(x,x,0)\psi(x)\,dx$ as $h\to+0, \mu\to+\infty$ where ψ is a fixed function, smooth and compactly supported in X.

It is known from Section 13.3 that in the smooth case for $d=3$ one can reduce magnetic Schrödinger operator to its canonical form

$$\mu^2 \sum_{m+p+j+k\geq 1} a_{m,p,j,k}(x_2,x_3,\mu^{-1}hD_2)\times \\ \times \big(x_1^2 + \mu^{-2}h^2D_1^2\big)^m (hD_3)^{2p}\mu^{-2m-2j-2p-k}h^k \tag{20.1.14}$$

with the "main part"

$$(20.1.14)_0 \quad a_{1,0,0}(x_2,x_3,\mu^{-1}hD_2)\big(\mu^2x_1^2+h^2D_1^2\big)+ h^2D_3^2+a_{0,1,0}(x_2,x_3,\mu^{-1}hD_2).$$

with $a_{1,0,0}=F\circ\Psi_0$, $a_{0,1,0}=V\circ\Psi_0$ and certain diffeomorphism $\Psi_0: T\mathbb{R}^1\times\mathbb{R}\to\mathbb{R}^3$; we ignore terms with $k\ge 1$ or $m+p+j\ge 2$. Then one can replace harmonic oscillator $\big(h^2D_1^2+\mu^2x_1^2\big)$ by one of its eigenvalues $(2\alpha+1)\mu h$.

Multidimensional case is much more tricky. The main part of the canonical form will be

$$(20.1.15) \quad \sum_{j,k\le r} Z_k^*\,a_{jk}^{\mathrm{w}}(x'',x''',\mu^{-1}hD'')Z_j+ \sum_{2r+1\le j,k\le d} h^2D_j\,b_{jk}^{\mathrm{w}}(x'',x''',\mu^{-1}hD'')D_k+a_0^{\mathrm{w}}(x'',x''',\mu^{-1}hD'')$$

with $Z_j=(hD_j-i\mu x_j)$, $Z_j^*=(hD_j+i\mu x_j)$ $j=1,\dots,r$, $x''=(x_{r+1},\dots,x_{2r})$, $x'''=(2r+1,\dots,d)$. Here $(a_{jk})_{j,k=1,\dots,r}$ is a Hermitian positive definite matrix with eigenvalues $f_1,\dots,f_r$ and $(b_{jk})_{j,k=2r+1,\dots,d}$ is a real symmetric positive definite matrix. We would like to have (a_{jk}) in the diagonal form; then instead of (20.1.15) we would have

$$(20.1.16) \quad \sum_{j} f_j^{\mathrm{w}}(x'',x''',\mu^{-1}hD'')\big(H_j-\mu h\big)+ \sum_{2r+1\le j,k\le d} h^2D_j\,b_{jk}^{\mathrm{w}}(x'',x''',\mu^{-1}hD'')D_k+a_0(x'',x''',\mu^{-1}hD'')^{\mathrm{w}}$$

with 1D-harmonic oscillators $H_j=(h^2D_j^2+\mu^2x_j^2)$. However, second-order resonances $f_j=f_k$ with $j\ne k$ prevent us; in the general case we can assume only that locally

$$(20.1.17) \quad \sum_{\mathfrak{m}\in\mathfrak{M}}\sum_{j,k\in\mathfrak{m}} a_{jk}(x'',x''',\mu^{-1}hD'')Z_j^*Z_k+ \sum_{2r+1\le j,k\le d} h^2D_j\,b_{jk}^{\mathrm{w}}(x'',x''',\mu^{-1}hD'')D_k+a_0(x'',\mu^{-1}hD'',x'')$$

where $\mathfrak{m}\in\mathfrak{M}$ are disjoint subsets of $\{1,\dots,r\}$ and eigenvalues of each of matrices $(a_{jk})_{j,k\in\mathfrak{m}}$ are close to one another. This leads to the necessity

of the matrix rather than the scalar non-degeneracy (microhyperbolicity) condition.

The reduction of non-principal terms of operator are prevented by higher-order resonances (see Chapter 19); we discuss it later.

20.1.4 Microhyperbolicity

Thus we arrive to the final the *microhyperbolicity assumption* at point $\bar{z} = (\bar{x}'', \bar{x}''', \bar{\xi}'')$ and thus at point $\bar{x} = \Psi_0^{-1}(z)$:

Definition 20.1.2. Assume that

(20.1.18) Set $\{1, \dots, r\}$ is split into disjoint groups $\mathfrak{m} \in \mathfrak{M}(\bar{z})$ and matrix (a_{ij}) is block-diagonal and in a small vicinity of $\bar{z}$ its blocks are close to scalar matrices $f_{\mathfrak{m}} I_{\#\mathfrak{m}}$ where $f_{\mathfrak{m}}$ are disjoint.

Then we call operator *microhyperbolic* if for each real vector $\boldsymbol{\tau} = (\tau_{\mathfrak{m}})_{\mathfrak{m}\in\mathfrak{M}}$ such that $|\sum_{\mathfrak{m}\in\mathfrak{M}} \tau_{\mathfrak{m}} + V| \le \epsilon$ there exists vector $\ell = \ell(\bar{z}, \boldsymbol{\tau}) \in \mathbb{R}^{2r+q}$ such that

$$\mu^2 \sum_{j,k} \bigl(\ell\bigl(a_{jk} a_0^{-1}\bigr)\bigr)\zeta_j^\dagger \zeta_k \ge \epsilon_1 \qquad \forall \zeta \in \mathbb{C}^r \tag{20.1.19}$$

as long as

$$a_{\mathfrak{m}} = \sum_{j,k\in\mathfrak{m}} a_{jk}\zeta_j^\dagger \zeta_k = \mu^2 \tau_{\mathfrak{m}} \qquad \forall \mathfrak{m} \in \mathfrak{M}. \tag{20.1.20}$$

Remark 20.1.3. (i) To impose this condition we need to assume first that symbol a_0 does not vanish; thus we need assumption (20.1.11).

(ii) Exactly as in Chapters 13 and 18 we do not need microhyperbolicity assumption if magnetic field is weak enough.

(iii) In contrast to Chapters 13 and 18 we do not need the microhyperbolicity assumption if $q \ge 3$, and also in $q = 2$ if there are no 3-rd order resonances.

(iv) In contrast to Chapter 19 we no longer consider $\mathfrak{N}$ partition of $\{1, \dots, r\}$ joining in one group all possible 3-rd order resonances as well; the reason is that now we need to consider evolution with $|t| \le \epsilon_0$ instead of $|t| \le \epsilon_0 \mu$ as it was in Chapter 19 and therefore 3-rd order resonances are not a problem anymore here.

In the case of $\mu h \ge \epsilon$ we will need to reformulate microhyperbolicity assumption.

20.1.5 Plan of the Chapter

In Section 20.2 we develop weak magnetic field approach to cover both weak magnetic field case and the outer zone in the temperate magnetic field case. Definition of the "weak" magnetic field depends on q.

In Section 20.3 we reduce operator to its canonical form which basically is what we got in previous Chapter 19 albeit with the coefficient depending on all variable x plus a "free" Hamiltonian (i.e. without magnetic field) in q-dimensional space.

Sections 20.4 and 20.5 are devoted to the remainder estimate with the Tauberian main part of asymptotics in the cases of the temperate and a stronger[2)] magnetic field respectively. In all these two and the next two sections the presence of 3-rd order resonances and corresponding non-removable terms poses one of the main obstacles. As $q \ge 3$ ($q \ge 2$ provided there are no such terms) we recover the remainder estimate $O\bigl(h^{1-d}(1+\mu h)\bigr)$ but in the remaining cases the remainder estimate is not as good unless we impose microhyperbolicity or non-degenerateness condition.

Calculations and simplifications of the Tauberian main part are done in Sections 20.6 and 20.7 in the cases of the intermediate and a stronger[2)] magnetic field respectively.

Finally, in Section 20.8 we consider the case when one pair of the non-zero eigenvalues of F_K^j vanishes on the manifold of codimension 3. It is more complicated than in 3D-case because different eigenvalues have different magnitudes and a simple rescaling does not work.

20.2 Weak Magnetic Field Case

Analysis in this case follows one of Section 18.6 without serious modifications. We assume that $\mu \le h^{\delta-1}$. Let us assume that f operator conditions (20.1.3)–(20.1.4), (20.1.9) are fulfilled.

[2)] I.e. either intermediate, or strong magnetic field.

20.2.1 Preliminary Remarks

Let us start from the case (further restrictions to follow)

$$\varepsilon \ge Ch|\log h|, \quad \mu \le \epsilon(h|\log h|)^{-1}. \tag{20.2.1}$$

Obviously

$$[P_j, P_k] = i\mu h F_{jk}. \tag{20.2.2}$$

Recall that ε is a mollification parameter. Consider in the framework of (20.1.7) commutators

$$h^{-1}[A, x_m] = \sum_k \bigl(g^{mk}P_k + P_k g^{mk}\bigr), \qquad m = 1, \dots, d, \tag{20.2.3}$$

$$h^{-1}[A, P_m] = h^{-1}\sum_{j,k} P_j[g^{jk}, P_m]P_k + h^{-1}[V, P_m] \quad m = 1, \dots, q, \tag{20.2.4}$$

and

$$\mu^{-1}h^{-1}[A, P_m] = \sum_{j,k}\bigl(g^{jk}F_{jm}P_k + P_j g^{jk}F_{km}\bigr) + \mu^{-1}h^{-1}\sum_{jk} P_j[g^{jk}, P_m]P_k + \mu^{-1}h^{-1}[V, P_m] \qquad m = q+1, \dots, d. \tag{20.2.5}$$

These commutators have symbols of the class $\mathscr{F}^{l,\sigma}$ in any domain $\{(x,\xi) : a(x,\xi) \le c\}$ of the bounded energy where

$$a(x,\xi) = a_0(x,\xi) + V, \qquad a_0(x,\xi) = \mu^2 \sum_{j,k} g^{jk} p_j(x,\xi)p_k(x,\xi), \tag{20.2.6}$$

$$p_j(x,\xi) \coloneqq (\xi_j - V_j);$$

recall that classes $\mathscr{F}^{l,\sigma}$ are introduced in Chapter 18. To finish the preliminary remarks let us notice that the following statement holds

Proposition 20.2.1. [3] *Let $f \in \mathscr{C}_0^\infty(\mathbb{R}^d)$, $f = 1$ in $B(0, C_0)$ with large enough constant C_0. Let $T \ge h^{1-\delta}$. Then under condition* (20.2.1) *with large enough constant $C = C_s$*

$$|F_{t\to h^{-1}\tau}\Bigl(1 - f\bigl(p_1(x,\xi), p_2(x,\xi), \dots, p_d(x,\xi)\bigr)\Bigr)^{\mathrm{w}}\chi_T(t)u| \le Ch^s \tag{20.2.7}$$

$$\forall x, y \in B(0, \frac{1}{2}) \qquad \forall \tau \le c.$$

[3] Microlocal boundaries for P_j. Compare with Proposition 18.6.1.

Proof. One can see easily that symbol $f\big(p_1(x,\xi), p_2(x,\xi), \dots, p_d(x,\xi)\big)$ is quantizable under condition (20.2.1) and the proof of Proposition 20.2.1 is rather obvious. Further standard details we leave to the reader. □

20.2.2 Tauberian Estimate

Outer Zone: General Case

Let us prove several results about propagation:

Proposition 20.2.2 [4]**.** *Let* $\mu \le \epsilon(h|\log h|)^{-1}$ *and let*

$$M \ge \sup_{\sum_{jk} g^{jk}\xi_j\xi_k + V = 0} 2\sum_k g^{jk}\xi_k + \epsilon \tag{20.2.8}$$

with arbitrarily small constant $\epsilon > 0$.

Let ϕ_1 *be supported in* $B(0,1)$, $\phi_2 = 1$ *in* $B(0,2)$, χ *be supported in* $[-1,1]$ [5]. *Finally, let* $T_* = Ch|\log h| \le T \le T^* = \epsilon_0$.

Then

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\psi_{2x}U(x,y,t)\,\psi_{1y}| \le Ch^s \qquad \forall x,y\ \forall \tau \le \epsilon_1 \tag{20.2.9}$$

for

$$\psi_{2x} = \big(1 - \phi_{2,MT}(x-\bar{x})\big), \quad \psi_{1y} = \phi_{1,MT}(y-\bar{x})$$

where here and below $\epsilon_1 > 0$ *is a small enough constant.*

Proposition 20.2.3 [6]**.** *Let* $\mu \le \epsilon h^{\delta-1}$ *and let*

$$T_* = C\varepsilon^{-1}h|\log h| \le T \le T^* = \epsilon_0. \tag{20.2.10}$$

Then

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)Q_{2x}\psi_{2x}U(x,y,t)\psi_{1y}\,{}^tQ_{1y}| \le Ch^s \qquad \forall x,y\ \forall \tau \le \epsilon_1 \tag{20.2.11}$$

for $Q_{2x} = \big(1 - \phi_{2,MT}(hD_x''' - \bar{\xi}''')\big)$ *and* $Q_{1y} = \phi_{1,MT}(hD_y''' - \bar{\xi}''')$, *where here and below* $\psi_{1,2}$ *are admissible functions supported in* $B(0,\frac{1}{2})$ *and* $\psi_{1x} = \psi_1(x)$, $\psi_{2y} = \psi_2(y)$.

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[4] Finite speed of propagation with respect to x; cf. Propositions 18.2.4 and 19.2.8.

[5] Recall that we pick up all such auxiliary functions to be admissible in the sense of Section 2.3.

[6] Finite speed of propagation with respect to P'''; cf. Proposition 18.6.2.

Proposition 20.2.4 [7]**.** *Let* $\mu \le \epsilon h^{\delta-1}$ *and condition* (20.2.10) *be fulfilled, and let*

$$|\bar{\xi}'''| := \rho \ge C\varepsilon^{-1}h|\log h| + C\mu^{-1}. \tag{20.2.12}$$

Then for

$$T_* := C(\mu^{-1}h|\log h|)^{\frac12} + C\rho^{-2}h|\log h| \le T \le T^* := \epsilon\rho \tag{20.2.13}$$

estimate

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma U(x,y,t)\psi_y\, {}^tQ_y| \le Ch^s \qquad \forall \tau \le \epsilon_1 \tag{20.2.14}$$

holds for $Q_y = \phi_{1,\frac12\rho}(hD_y''' - \bar{\xi}''')$.

Proofs of Propositions 20.2.2–20.2.4. Proofs of these propositions repeat proofs of the corresponding propositions of Chapters 18 and 19 and are left to the reader. □

Now let us pick up

$$\varepsilon = C\rho^{-1}h|\log h|. \tag{20.2.15}$$

Then any $T \in [\epsilon_0\mu^{-1}, \epsilon\rho]$ satisfies condition (20.2.13) as long as

$$\rho \ge \bar{\rho}_1 := C\max\bigl(\mu^{-1}, (\mu h|\log h|)^{\frac12}\bigr). \tag{20.2.16}$$

This is a definition of the *outer zone* and ε is defined by (20.2.15) there.

Therefore the contribution of the zone $\{|\xi''| \asymp \rho\}$ with given $\rho \ge \bar{\rho}_1$ to the Tauberian remainder does not exceed $Ch^{1-d}T^{*\,-1}\rho^{q-1}$, where ρ^{q-1} is the measure of $\{\xi''' :\ a(x,\xi) = 0,\ |\xi'''| \asymp \rho\}$; then the total contribution of the outer zone to the Tauberian remainder does not exceed

$$Ch^{1-d}\int \frac{1}{T^*(\rho)}\rho^{q-1}\,d\rho \tag{20.2.17}$$

with all integrals here and until the end of the section taken from $\bar{\rho}_1$ to 1, and plugging $T^*(\rho) \asymp \rho$ we get $Ch^{1-d}\int \rho^{q-2}\,d\rho \asymp h^{1-d}$ provided $q \ge 2$.

So, we have proven

Proposition 20.2.5. *Let* $q \ge 2$, ε *and* $\bar{\rho}_1$ *be defined by* (20.2.15) *and* (20.2.16)*, and* $(l,\sigma) = (1,1)$*. Then the contribution of the outer zone* $\{|\xi'''| \ge \bar{\rho}_1\}$ *to the Tauberian remainder with* $T = \epsilon\mu^{-1}$ *is* $O(h^{1-d})$.

[7] Singularities leave diagonal; cf. Proposition 18.6.3.

Outer Zone: Special Case $q = 1$

As $q = 1$ expression (20.2.17) with $T^*(\rho) = \epsilon\rho$ results in $Ch^{1-d}|\log\mu|$ and to get rid of this factor $|\log\mu|$ we need to increase $T^*(\rho)$. We consider few cases in the increased generality and complexity.

(i) If g^{jk} and F_{jk} are constant it is easy: we can take

$$T^*(\rho) = C\rho|\log\rho|^2 \tag{20.2.18}$$

as in section Section 18.6 provided $(l,\sigma) \succeq (1,2)$ and we can consider propagation in an appropriate time direction (in which ξ_1 increases); see Section 18.6 for details. Namely, let us cover $B(0,1)$ by balls of radii $\gamma = \rho^{\delta_1}$ (with a small exponent $\delta_1 > 0$). In each element one of the following cases holds:

$$|\partial_{x_1} V(x,\xi)| \asymp \varrho \ge \bar{\varrho} := C_0|\log\rho|^{-2}, \tag{20.2.19}$$

$$|\partial_{x_1} V(x,\xi)| \le \bar{\varrho}. \tag{20.2.20}$$

During propagation the classical trajectory does not leave the initial ball. Furthermore, in the case (20.2.20) $|d\xi_1/dt| = |\partial_{x_1} V|$ does not exceed $\bar{\varrho}$ and $|\xi_1| \ge \frac{1}{2}\rho - \bar{\varrho}|t| \ge \frac{1}{3}\rho$ as $|t| \le T^*(\rho)$ defined by (20.2.18).

On the other hand, in the case (20.2.19) $|d\xi_1/dt|$ is larger than ϱ and does not change sign for $|t| \le T^*(\rho)$. Then $|\xi_1| \ge \frac{1}{2}\rho + \varrho|t|$ for $0 \le \pm t \le T(\rho)$ with an appropriate sign.

In both cases x_1 shifts away from its initial position and for time t (of the appropriate sign in the case (20.2.19)) this shift is of magnitude $\rho|t| + \varrho|t|^2$ as $\epsilon\mu^{-1} = T_* \le \pm t \le T^*(\rho)$ (with an appropriate sign) and we can take a prescribed $T^*(\rho) = \epsilon\rho|\log\rho|^2$ and recover $O(h^{1-d})$ in (20.2.17).

(ii) Similar arguments work if f_j have constant multiplicities. Really, we can rewrite then the symbol as

$$a(x,\xi) = \xi_1^2 + \mu^2 \sum_{1\le j\le r} f_j(\eta_{2j-1}^2 + \eta_{2j}^2) + V \tag{20.2.21}$$

with η_k being linear combinations of $p_1, \dots, p_{2r}$ (with $\mathscr{F}^{1,2}$ coefficients) such that

$$\mu\{\eta_k, \eta_m\} \equiv \Lambda_{km} \tag{20.2.22}$$

modulo linear combinations of $\eta_1, \dots, \eta_{2r}$ with $\mathscr{F}^{0,2}$ coefficients where $\Lambda_{km} = 1$ if $k = 2j-1, m = 2j$ for some j, $\Lambda_{km} = -1$ if $k = 2j, m = 2j-1$ for some j, $\Lambda_{km} = 0$ otherwise.

Then the derivative of $\mu^2(\eta_{2j-1} + i\eta_{2j})(\eta_{2k-1} - i\eta_{2k})$ along classical trajectories

$$|\mu^2\{a, (\eta_{2j-1} + i\eta_{2j})(\eta_{2k-1} - i\eta_{2k})\}| \le M(1 + \mu|f_j - f_k|) \qquad \forall j, k \tag{20.2.23}$$

is bounded as $f_j = f_k$ and during time T its variation does not exceed CT which is way less than $\epsilon\bar{\varrho}$. This could be justified on the quantum level as well because one can quantize symbol

$$f\big(\bar{\varrho}^{-1}\big((\eta_{2j-1} + i\eta_{2j})(\eta_{2k-1} - i\eta_{2k}) - \lambda\big)\big)$$

with function f supported in $[-c, c]$ and $|\lambda| \le c$.

Consider now $\{a, \xi_1\}$; modulo $O(\rho)$ it is equal to $-\partial_{x_1} V + \mu^2 \sum_{j,k} \alpha_{jk}\eta_j\eta_k$ with $\mathscr{F}^{0,2}$ coefficients. Then we can find α'_{jk} such that

$$\begin{aligned}\big\{a, \xi_1 - \mu \sum_{j,k} \alpha_{jk}\eta_j\eta_k\big\} \equiv \\ - \partial_{x_d} V + \mu^2 \sum_{j,k:f_j=f_k} \alpha''_{jk}(\eta_{2j-1} + i\eta_{2j})(\eta_{2k-1} - i\eta_{2k})\end{aligned} \tag{20.2.24}$$

modulo $O(\bar{\varrho})$: we remove all non-resonance terms but previously we make ρ^δ-mollification of α_{jk} thus making $O(\bar{\varrho})$ error. But then all the above arguments work since correction $\mu \sum_{j,k} \alpha_{jk}\eta_j\eta_k$ does not exceed $C\mu^{-1} \le \epsilon\rho$. The role of $-\partial_{x_1} V$ is played now by the right-hand expression of (20.2.24).

(iii) Consider now the general case. First of all let us make again $\gamma = \rho^{\delta_1}$-covering and consider one element. Then the symbol could be rewritten in the form

$$a(x, \xi) = \xi_1^2 + \sum_{\mathfrak{m}} a_{\mathfrak{m}} + V, \tag{20.2.25}$$

where

$$a_{\mathfrak{m}} = \mu^2 \sum_{j,k\in\mathfrak{m}} a_{jk}(\eta_{2j-1} - i\eta_{2j})(\eta_{2k-1} + i\eta_{2k}) \tag{20.2.26}$$

with Hermitian matrices $a_{\mathfrak{m}} = (a_{jk})_{j,k\in\mathfrak{m}}$ whose eigenvalues are $f_j(x)$, f_j and f_k differ by less than ϵ (more than 2ϵ) as j, k belong to the same group $\mathfrak{m}$ (different groups, respectively).

Consider traceless matrices $b_{\mathfrak{m}} = a_{\mathfrak{m}} - \frac{1}{(\#\mathfrak{m})}\operatorname{tr} a_{\mathfrak{m}}\cdot I_{\mathfrak{m}}$ with the corresponding unit matrices $I_{\mathfrak{m}}$ and matrices $c_{\mathfrak{m}} = \partial_{x_1} b_{\mathfrak{m}}$ calculated at some fixed point of this partition element. Without any loss of the generality one can assume that each $c_{\mathfrak{m}}$ is a diagonal matrix; let classify its eigenvalues into groups: λ_j and λ_k differ less than $\epsilon\bar{\varrho}$ (more than $2\epsilon\bar{\varrho}$) as j, k belong to the same subgroup $\mathfrak{n} \subset \mathfrak{m}$ (different subgroups, respectively).

Then one can cover each element of partition by $\gamma_1 = \rho^{\delta_1+\delta_2}$ subelements of two kinds:

(a) With f_j, f_k different by less than γ_1 for some j, k belonging to the different subgroups. One can prove easily that the total measure of this type elements (i.e. of all elements together) does not exceed $C\bar{\varrho}$ and thus their total contribution to the Tauberian remainder does not exceed $Ch^{1-d}|\log\rho|^{-2}$ which after integration over $d\rho/\rho$ results in $O(h^{1-d})$.

(b) With f_j, f_k different by more than γ_1 for all j, k belonging to the different subgroups. Then we can rewrite (20.2.25) with $\mathfrak{n}$ (which indicates the finer partition) instead of $\mathfrak{m}$ and to prove that $|\{a, a_{\mathfrak{n}}\}| \le C\gamma^{-\delta_3}$ where exponent δ_3 could be made arbitrarily small (by taking all the previous exponents small). Then $a_{\mathfrak{n}}$ are conserved modulo $O(\bar{\varrho})$ in the classical evolution.

Moreover, we can make even finer subpartition depending on subelement such that $|f_j - f_k| \le C\bar{\varrho}$ if j, k belong to the same element of it.

But then $f_{\mathfrak{n}}\mu^2\sum_{j\in\mathfrak{n}}(\eta_{2j-1}^2 + \eta_{2j}^2)$ are conserved modulo $O(\bar{\varrho})$ and then it is also true for $a_{\mathfrak{n}}$ and $a_{\mathfrak{m}}$ as well and also

$$\{a, \xi_1 - \mu\sum_{j,k}\alpha_{jk}\eta_j\eta_k\} \equiv \sum_j \lambda_j(\eta_{2j-1}^2 + \eta_{2j}^2) \mod O(\bar{\varrho})$$

for appropriate α_{jk}.

But then arguments of Parts (i) and (ii) work and the total contribution of elements of type (b) does not exceed $Ch^{1-d}|\log\rho|^{-2}$ as well and after integration over $d\rho/\rho$ we get $O(h^{1-d})$ again. So we have proven

Proposition 20.2.6. *Let $q = 1$, ε and $\bar{\rho}_1$ be defined by* (20.2.15) *and* (20.2.16)*, and $(I, \sigma) = (1, 2)$. Then the contribution of the outer zone $\{|\xi'''| \ge \bar{\rho}_1\}$ to the Tauberian remainder with $T = \epsilon\mu^{-1}$ is $O(h^{1-d})$.*

Remark 20.2.7. Similar arguments were applied in Section 7.5 to study propagation near the boundary.

Conclusion

We have not specified ε for $\rho = |\xi'''| \le \bar{\rho}_1$ but it is reasonable to take there $\varepsilon = C\bar{\rho}_1^{-1}h|\log h|$ and therefore

$$\varepsilon = C\begin{cases} \rho^{-1}h|\log h| & \rho \gtrsim \bar{\rho}_1, \\ \bar{\rho}_1^{-1}h|\log h| & \rho \lesssim \bar{\rho}_1. \end{cases} \tag{20.2.27}$$

Remark 20.2.8. Note that the threshold for $\bar{\rho}_1$ is $\mu = \bar{\mu} = C(h|\log h|)^{-\frac{1}{3}}$: $\bar{\rho}_1 = C\mu^{-1}$ as $\mu \le \bar{\mu}$, and $\bar{\rho}_1 = C(\mu h|\log h|)^{\frac{1}{2}}$ as $\mu \ge \bar{\mu}$.

On the other hand, the standard estimate in the complementary *inner zone* $\{|\xi'''| \le \bar{\rho}_1\}$ brings the Tauberian remainder estimate

$$Ch^{1-d} \times \mu\bar{\rho}_1^q \asymp Ch^{1-d} \times \mu(\mu h|\log h|)^{\frac{q}{2}} + Ch^{1-d}\mu^{1-q}. \tag{20.2.28}$$

Combining with Propositions 20.2.5 and 20.2.6 we arrive to

Proposition 20.2.9. *(i) In the framework of either Proposition 20.2.5 or 20.2.6*

$$\mathsf{R}^{\mathsf{T}} := |\Gamma(\tilde{e}\psi)(\tau) - h^{-1}\int_{-\infty}^{\tau} F_{t\to h^{-1}\tau'}\bar{\chi}_T(t)\Gamma(U\psi)| \le Ch^{1-d} + C\mu(\mu h|\log h|)^{\frac{q}{2}}h^{1-d} \qquad \forall \tau : |\tau| \le \epsilon_1 \tag{20.2.29}$$

where the left-hand expression is the total Tauberian remainder with $T = \epsilon\mu^{-1}$.

(ii) Furthermore, this estimate holds with any $T \ge \bar{T}$ *with* $\bar{T} = Ch|\log h|$.

20.2.3 Main Theorems

General Case

Now in virtue of the standard results rescaled we can replace the Tauberian expression by the multiterm Weyl expression $h^{-d}\tilde{\mathcal{N}}^{\mathsf{W}}_{(\infty)}$ defined by (19.2.83) for

approximate operators. Then we can replace R^{T} by $\tilde{\mathsf{R}}^{\mathsf{W}}_{(\infty)}$ also for approximate operators (where tilde means that we consider approximate operators).

Note that the contribution of the outer zone to the approximation error does not exceed

$$C\int \Big(\frac{h|\log h|}{\rho}\Big)^{l}\Big(1+\frac{\mu h|\log h|}{\rho}\Big)\,|\log(\rho^{-1}h|\log h|)|^{-\sigma}\rho^{q-1}\,d\rho\times h^{-d} \tag{20.2.30}$$

and this expression is $O(h^{1-d})$ provided

$$\text{Either}\quad q\ge 2,\ (l,\sigma)\succeq(1,1)\quad \underline{\text{or}}\quad q=1,\ (l,\sigma)\succeq(1,2). \tag{20.2.31}$$

Also one can prove easily that replacing $h^{-d}\mathcal{N}^{\mathsf{W}}_{(\infty)}$ by the magnetic Weyl expression $h^{-d}\mathcal{N}^{\mathsf{MW}}$ does not increase an error and therefore we arrive to the following

Theorem 20.2.10. *Let conditions* (20.1.1), (20.1.5), (20.1.7), (20.1.8)–(20.1.11), *and* (20.2.31) *be fulfilled, and* $\mu\le h^{\delta-1}$.

Then there exist two framing approximations (see footnote [18)] *of Chapter 18) such that for both of them*

$$\mathsf{R}^{\mathsf{MW}}=\Big|\int \Big(\tilde{e}(x,x,\tau)-h^{-d}\mathcal{N}^{\mathsf{MW}}(x,\tau)\Big)\psi(x)\,dx\Big|\le Ch^{1-d}+C\mu h^{1-d}(\mu h|\log h|)^{\frac{q}{2}}; \tag{20.2.32}$$

the same estimate holds for $\mathsf{R}^{\mathsf{W}}_{(\infty)}$.

Remark 20.2.11. (i) Theorem 20.2.10 uses no non-degeneracy condition and it does not benefit from the stronger smoothness assumptions.

(ii) For $\mu\le\bar{\mu}_q$,

$$\bar{\mu}_{(q)}\coloneqq (h|\log h|)^{-q/(q+2)} \tag{20.2.33}$$

estimate

$$\mathsf{R}^{\mathsf{MW}}\le Ch^{1-d} \tag{20.2.34}$$

holds. In particular, $\bar{\mu}_{(1)}=C(h|\log h|)^{-\frac{1}{3}}=\bar{\mu}_1$, $\bar{\mu}_{(2)}=C(h|\log h|)^{-\frac{1}{2}}$ and $\bar{\mu}_{(q)}\ge C(h|\log h|)^{-\frac{3}{5}}$ as $q\ge 3$.

Problem 20.2.12. Using rescaling technique get rid of logarithms in the definitions of $\bar{\rho}_1$ and thus in estimates (20.2.29), (20.2.32) and in (20.2.33):

$$\mathsf{R}^{\mathsf{T}} \le Ch^{1-d} + C(\mu h)^{1+\frac{q}{2}} h^{-d} \tag{20.2.29}^*$$

and

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d} + C(\mu h)^{1+\frac{q}{2}} h^{-d}. \tag{20.2.32}^*$$

Microhyperbolic Case

To push μ in Theorem 20.2.10 up[8)] we need the microhyperbolicity assumption and some extra smoothness.

First of all, let us improve our results about propagation of singularities:

Proposition 20.2.13. *Let $\bar{\mu}_{(q)} \le \mu \le \epsilon h^{-1}|\log h|^{-1}$ and the microhyperbolicity assumption (see Definition 20.1.2) be fulfilled. Then for $C\varepsilon^{-1}h|\log h| \le T \le T^* = \epsilon_1$ estimate* (20.2.14) *holds.*

Proof. An easy proof based on technique of subsections 18.6.6 and 19.2.2 we leave to the reader. □

In the inner zone $\{\rho \le \bar{\rho}_1\}$ we can select $\varepsilon = C\mu h|\log h|$ (if $\mu \ge (h|\log h|)^{-\frac{1}{3}}$) and then for Q with the symbol supported there estimate (20.2.14) holds for $T_* = \epsilon\mu^{-1} \le T \le T^* = \epsilon_1$.

Redefining in the outer zone ε, so finally

$$\varepsilon = C \begin{cases} \max\bigl(\mu h|\log h|\bigl(\frac{\bar{\rho}_1}{\rho}\bigr)^s, \rho^{-1}h|\log h|\bigr) & \rho \ge \bar{\rho}_1, \\ \bar{\varepsilon} := \mu h|\log h| & \rho \le \bar{\rho}_1 \end{cases} \tag{20.2.35}$$

and combining with Proposition 20.2.4 we get then that for Q with the symbol supported in the outer zone estimate also (20.2.14) holds for $T_* \le T \le T^*$.

Thus we arrive to

Proposition 20.2.14. *Let $(h|\log h|)^{-\frac{1}{3}} \lesssim \mu \le \epsilon(h|\log h|)^{-1}$ and the microhyperbolicity assumption (see Definition 20.1.2) be fulfilled. Let ε be defined by* (20.2.35).

Then estimate (20.2.14) *holds with $Q = I$ and any $T \in [T_*, T^*]$, $T_* = \epsilon\mu^{-1}$, $T^* = \epsilon_1$.*

[8)] Actually we need to do this for $q = 1$ only.

Combining with the standard results rescaled and applying the standard Tauberian technique we arrive to estimate (20.2.29) again.

Note that an approximation error in the operator does not exceed

$$C\varepsilon^{q}|\log\varepsilon|^{-\sigma}| + C\mu\varepsilon^{\bar{l}+1}|\log\varepsilon|^{-\bar{\sigma}} \tag{20.2.36}$$

where one can skip the second term as $\varepsilon \lesssim \mu^{-1}$ which is the case in the inner zone iff $\mu \lesssim (h|\log h|)^{-\frac{1}{3}}$; then it is true in the outer zone as well.

Then one can prove easily that an approximation error in both R^{MW} and $\mathsf{R}^{\mathsf{W}}_{(\infty)}$ does not exceed (20.2.35) calculated as $\varepsilon = \bar{\varepsilon}$, multiplied by $\bar{\rho}_1^{\frac{q}{2}} h^{-d}$ i.e.

$$C(\mu h|\log h|)^{l+\frac{q}{2}}|\log h|^{-\sigma}h^{-d} + C(\mu h|\log h|)^{2+\bar{l}+\frac{q}{2}}|\log h|^{-1-\bar{\sigma}}h^{-1-d} \tag{20.2.37}$$

where one can skip the second term if $\mu \lesssim (h|\log h|)^{-\frac{1}{2}}$. This proof uses the microhyperbolicity assumption for $q = 1$ and we consider R^{MW}.

However if $\ell_\perp = 0$ we can take mollification with $\varepsilon = Ch\rho^{-1}|\log h|$ with respect to all variables x and also an extra mollification with ε defined by (20.2.35) with respect to x'''; then an approximation error will be given by (20.2.37) but without the second term.

One can prove easily that the difference between $h^{-d}\mathcal{N}^{\mathsf{MW}}$ and $h^{-d}\mathcal{N}^{\mathsf{W}}_{(\infty)}$ does not exceed (20.2.37) without second term.

This implies

Theorem 20.2.15. *Let conditions* (20.1.1), (20.1.5), (20.1.7), (20.1.8)–(20.1.11), *and* (20.2.31) *be fulfilled, and* $\mu \le h^{\delta-1}$.

Let the microhyperbolicity assumption (see Definition 20.1.2) be fulfilled. Then there exist two framing approximations (see footnote [18)] *of Chapter 18) such that for both of them*

(i) The following estimate holds

$$\begin{aligned}\mathsf{R}^{\mathsf{MW}} \le{}& Ch^{1-d}+\\ & C(\mu h|\log h|)^{l+\frac{q}{2}}|\log h|^{-\sigma}h^{-d} + C(\mu h|\log h|)^{2+\bar{l}+\frac{q}{2}}|\log h|^{-1-\bar{\sigma}}h^{-1-d}.\end{aligned} \tag{20.2.38}$$

(ii) Furthermore, if <u>*either*</u> $\mu \le \bar{\mu}_2 = C(h|\log h|)^{-\frac{1}{2}}$ <u>*or*</u> $\ell_\perp = 0$ *in the microhyperbolicity assumption, then one can skip the last term in the right-hand expression of* (20.2.38)*:*

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d} + C(\mu h|\log h|)^{l+\frac{q}{2}}h^{-d}|\log h|^{-\sigma}. \tag{20.2.39}$$

(iii) In particular, $\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d}$ *as* $q=1$, $(l,\sigma)=(\frac{3}{2},1)$ *and* $\mu \le \bar{\mu}_2$.

Recall that all these estimates hold for $\mathsf{R}^{\mathsf{W}}_{(\infty)}$ as well.

Problem 20.2.16. Using rescaling technique replace $\mu h|\log h|$ by μh in the definitions of $\bar{\rho}_1$ and ε and thus in estimates (20.2.38) and (20.2.39):

$$(20.2.38)^* \quad \mathsf{R}^{\mathsf{MW}} \le Ch^{1-d} + C(\mu h)^{l+\frac{q}{2}}|\log h|^{-\sigma}h^{-d} + C(\mu h)^{2+\bar{l}+\frac{q}{2}}|\log h|^{-\bar{\sigma}}h^{-1-d}$$

and

$$(20.2.39)^* \qquad \mathsf{R}^{\mathsf{MW}} \le Ch^{1-d} + C(\mu h)^{l+\frac{q}{2}}h^{-d}|\log h|^{-\sigma}.$$

Special Case of Constant g^{jk}, F_{jk}

Consider now the case of constant g^{jk} and F_{jk}; then without any loss of the generality one can assume that

(20.2.40) $g^{jk}=\delta_{jk}$ and $F_{jk}=\pm f_i$ for $j=q+i, k=q+i+r$ and $j=q+i+r, k=q+i$ with $i=1,\dots,r$ and $F_{jk}=0$ otherwise.

Then the microhyperbolicity assumption is equivalent to

$$(20.2.41) \qquad |\nabla V| \ge \epsilon_0.$$

We replace it by the non-degeneracy condition

$$(20.2.42) \qquad |\nabla V| \le \epsilon_0 \implies |\det \mathsf{Hess}\, V| \ge \epsilon_0.$$

We leave to the reader the following

Problem 20.2.17. Let conditions (20.1.1), (20.1.5), (20.1.7), (20.1.8), $(20.1.9)_3$, (20.1.11), and (20.2.31) be fulfilled, and $\mu \le h^{\delta-1}$.

Assume that g^{jk}, F_{jk} are constant and (20.2.42) is fulfilled.

(i) Using arguments similar to those of Subsection 19.2.5 prove that for appropriate framing approximations

$$(20.2.43) \qquad \mathsf{R}^{\mathsf{T}} \le Ch^{1-d} + C\mu(\mu h|\log h|)^{q+r}h^{1-d}$$

where $q+r=\frac{1}{2}q+\frac{1}{2}d$ and

$$(20.2.44) \quad \mathsf{R}^{\mathsf{MW}} \le Ch^{1-d} + C\mu(\mu h|\log h|)^{q+r}h^{1-d} + C(\mu h|\log h|)^{l+\frac{q}{2}}|\log h|^{-\sigma}h^{-d}.$$

(ii) Adding rescaling arguments prove that for appropriate framing approximations

$$\mathsf{R}^{\mathsf{T}} \le Ch^{1-d} + C(\mu h)^{q+r+1}h^{-d} \tag{20.2.43}^*$$

and

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d} + C(\mu h)^{q+r+1}h^{-d} + C(\mu h)^{l+\frac{q}{2}}|\log h|^{-\sigma}h^{-d}. \tag{20.2.44}^*$$

20.3 Canonical Form

20.3.1 Pilot-Model

Assume temporarily that g^{jk} and F_{jk} are constant. Then without any loss of the generality we can assume $V_j(x)$ are linear functions.

Then A^0 is transformed into exactly

$$\sum_{1\le k\le q} h^2D_k^2 + \sum_{1\le j\le r} f_j\Big(h^2D_{r+j}^2 + \mu^2 x_{r+j}^2\Big) \tag{20.3.1}$$

by $\mu^{-1}h$-metaplectic transformation which consists of the following steps:

(i) Change of variables $(x, \mu^{-1}hD) \mapsto (Qx, {}^tQ^{-1}\mu^{-1}hD)$ transforming g^{jk} into δ_{jk} and F into matrix satisfying (20.2.40). It transforms $V(x)$ into $V(Qx)$.

(ii) Gauge transformation (multiplication by $e^{i\mu h^{-1}S(x)}$ with quadratic form $S(x)$), transforming $V_j(x)$ into 0 for $j = 1, \dots, q+r$ and $V_{j+r+q}(x)$ into $f_j x_{j+q}$ for $j = 1, \dots, r$. Then A^0 is transformed into

$$\sum_{1\le k\le q} h^2D_k^2 + h^2\sum_{1\le j\le r}\Big(D_{q+j}^2 + \big(D_{q+j+r} - f_j\mu h^{-1}x_{q+j}\big)^2\Big) \tag{20.3.2}$$

and $V(x)$ is preserved.

(iii) [(iii)–(iv)] Partial $\mu^{-1}h$-Fourier transform, change of variables like in Subsection 19.3.1 A^0 into

$$\sum_{1\le k\le q} h^2D_k^2 + \sum_{1\le j\le r}\Big(h^2D_j^2 + f_j^2\mu^2x_j^2\Big)$$

and V into $\tilde{V} := V(x''', x' - Kx'', \mu^{-1}hD'' - {}^tK\mu^{-1}hD')$ with Weyl quantization. Finally, $x_j \mapsto f_j^{\frac{1}{2}}x_j$, $D_j \mapsto f_j^{-\frac{1}{2}}D_j$ reduces operator A^0 into (20.3.1) and transforms $\tilde{V}$ accordingly.

20.3.2 General Case: Framework

We can assume now that

$$\bar{\mu}_{(q)} = C(h|\log h|)^{-\frac{q}{q+2}} \le \mu \le \epsilon(h|\log h|)^{-1} \tag{20.3.3}$$

(a case of larger μ we'll consider later; we also can redefine $\bar{\mu}_{(q)} = Ch^{-\frac{q}{q+2}}$).

Anyway we need to consider an inner zone

$$\mathcal{Z}_{\mathsf{inn}} := \{|\xi'''| \le \bar{\rho}_1 = C(\mu h|\log h|)^{\frac{1}{2}}\} \tag{20.3.4}$$

and in this zone in the general case we already set $\varepsilon = C(\mu^{-1}h|\log h|)^{\frac{1}{2}}$. However we reset it to a far larger value $\varepsilon = C\mu^{-1}$ thus defining

$$\varepsilon = \begin{cases} C\mu^{-1}\bigl(\dfrac{\bar{\rho}_1}{\rho}\bigr)^s + C\rho^{-1}h|\log h| & \text{as } \rho \ge \bar{\rho}_1, \\ C\mu^{-1} & \text{as } \rho \le \bar{\rho}_1 \end{cases} \tag{20.3.5}$$

with a possible increase later.

From this section point of view there are two different cases $q = 1$ and $q \ge 2$; if either the microhyperbolicity assumption is fulfilled or g^{jk}, F_{jk} are constant and the non-degeneracy condition (20.2.42) is fulfilled, then $\bar{\mu}_{(1)}$ is pushed up and $q = 1$ falls in the latter case as well.

Really, for $q \ge 2$, $\mu \ge \bar{\mu}_{(2)} = (h|\log h|)^{-\frac{1}{2}}$ (and thus $\mu^{-l}|\log h|^{-\sigma} \le \mu h$ even as $(l,\sigma) = (1,1)$) this will bring an approximation error not exceeding

$$C\mu^r h^{-r}(\mu h|\log h|)^{\frac{q}{2}-1}\mu^{-l}|\log h|^{-\sigma} \times (\mu h)^{1-r}|\log h|$$

where the first factor estimates an error in each term in the sum defining $h^{-d}\mathcal{N}^{\mathsf{MW}}$ and the second factor estimates the number of the terms effected; the result is $O\bigl(h^{1-d}\bigr)$ for sure.

As $q = 1$ we need to assume that[9)]

$$\bar{\rho}_2 := \bigl(\mu^{-l}|\log h|^{-\sigma}\bigr)^{\frac{1}{2}} \le \rho_1^* := (\mu h)^{\frac{1}{2}} \iff \mu \ge (h|\log h|^{\sigma})^{1/(l+1)}). \tag{20.3.6}$$

[9)] This is not necessarily true as $q = 1$; however without this assumption no improvement of the previous section results is possible.

Consider strip $\{\rho \le |\xi'''| \le 2\rho\}$ with $\bar\rho_2 \le \rho \le \bar\rho_1$ where an approximation error can be estimated similarly by

$$Ch^{-d}\rho\bar\rho_2^2 + C\mu h^{1-d}\rho^{-1}\bar\rho_2^2$$

and after summation over ρ we conclude that the contribution of the zone $\{\bar\rho_2 \le |\xi'''| \le \bar\rho_1\}$ to an approximation error does not exceed

$$Ch^{-d}(\mu h|\log h|)^{\frac12}\mu^{-l}|\log h|^{-\sigma} + Ch^{1-d}\mu^{1-\frac{l}{2}}|\log h|^{-\frac{\sigma}{2}}. \tag{20.3.7}$$

Meanwhile, in the zone $\{|\xi'''| \le C\bar\rho_2\}$ no more than $C(\mu h)^{1-r}$ terms in the sum defining $h^{-d}\mathcal{N}^{\mathsf{MW}}$ are affected and an error in each does not exceed $C\mu^r h^{-r}\bar\rho_2$; this brings an approximation error estimate $C\mu h^{1-d}\bar\rho_2$ which is the exactly the second term in (20.3.7).

Remark 20.3.1. Expression (20.3.7) is $O\big(h^{1-d}\big)$ for all $\mu \ge C(h|\log h|)^{-\frac13}$ iff $(l,\sigma) \succeq (2,1)$.

On the other hand, if $q = 1$ and the microhyperbolicity assumption (see Definition 20.1.2) is fulfilled, then an the approximation error does not exceed

$$Ch^{-d}(\mu h|\log h|)^{\frac12}\mu^{-l}|\log h|^{-\sigma} \tag{20.3.8}$$

and it should be weighted against Theorem 20.2.15; so in this case for $\mu \le \bar\mu_{(2)} = C(h|\log h|)^{-\frac12}$ we use Theorem 20.2.15 and for $\mu \ge \bar\mu_{(2)}$ we will use the intermediate magnetic field construction; an approximation error estimate in both cases is $O\big(h^{1-d}\big)$ for all corresponding μ iff $(l,\sigma) \succeq (\frac32, 1)$.

Similar arguments work if $q = 1$, g^{jk}, F_{jk} are constant and non-degeneracy condition (20.2.42) is fulfilled; we refer in this case to Problem 20.2.17 as $\mu \le \bar\mu_{(2)}$.

Thus we have proven

Proposition 20.3.2. *Let ε be defined by* (20.3.5). *Then*

(i) If $q \ge 2$, $(l,\sigma) \succeq (1,1)$ and $\mu \ge (h|\log h|)^{-\frac12}$, then a mollification error is $O(h^{1-d})$.

(ii) If $q = 1$, $(l,\sigma) \succeq (1,2)$, $\mu \ge (h|\log h|)^{-\frac13}$ and condition (20.3.6) *is fulfilled, then a mollification error does not exceed* (20.3.7).

(iii) If $q = 1$, $(l,\sigma) \succeq (1,2)$, $\mu \ge (h|\log h|)^{-\frac12}$ and <u>either</u> the microhyperbolicity assumption is fulfilled <u>or</u> g^{jk}, F_{jk} are constant and non-degeneracy condition (20.2.42) *is fulfilled, then a mollification error does not exceed* (20.3.8).

20.3.3 Reduction. Main Part

So we can start from the operator

$$A=\sum_{1\le j,k\le d} P_j^* g^{jk} P_k + V, \qquad P_j = hD_j - \mu V_j \tag{20.3.9}$$

with

$$[P_j, P_k] = i\mu h F_{jk} \tag{20.3.10}$$

where

$$F_{jk}=0 \quad \text{as} \quad j\le q \text{ or } \quad k\le q;\ \operatorname{rank}(F_{jk})_{j,k=1,\dots,2r}=2r, \tag{20.3.11}$$

and we can assume without any loss of the generality at some point $\bar{x}$ matrix (F_{jk}) is in the canonical form; namely

$$F_{jk}(\bar{x})=\begin{cases} f_i & \text{as } \ j=q+i, k=q+r+i,\ i=1,\dots,r,\\ -f_i & \text{as } \ k=q+i, j=q+r+i,\ i=1,\dots,r,\\ 0 & \text{otherwise.}\end{cases}$$

Then without any loss of the generality we can assume that in its vicinity

$$F_{jk}=0\neq 0 \implies j\ge q+1,\ k\ge q+1 \quad \text{and} \quad \exists \mathfrak{m}\in\mathfrak{M}: j,k\in\mathfrak{m}. \tag{20.3.12}$$

Proposition 20.3.3. *Let*

$$\mathcal{L}=\mu^2\Big(\sum_{j,k\le d}\beta^{jk}p_jp_k\Big)^{\mathsf{w}} \tag{20.3.13}$$

with $\beta^{jk}=\beta^{kj}\in \mathscr{F}^{\bar{l},\bar{\sigma}}$. *Then modulo operators with symbols belonging to* $\mu^{-2}\mathscr{F}^{\bar{l}-1,\bar{\sigma}}$

$$e^{\frac{i}{2}\mu^{-1}h^{-1}t\mathcal{L}}(p_j)^{\mathsf{w}}e^{-\frac{i}{2}\mu^{-1}h^{-1}t\mathcal{L}}\equiv\Big(\sum_k \big(e^{\Lambda t}\big)_j^k p_k\Big)^{\mathsf{w}} \tag{20.3.14}$$

with matrix $\Lambda=(\Lambda_{jk})_{j,k}=(F_{jp})(\beta^{pk})$.

Proof. An easy proof following arguments of Sections 18.7 and 19.3 is left to the reader. □

One can see easily that due to (20.1.7)–(20.1.8)

$$e^{\Lambda t} = \begin{pmatrix} I & 0 \\ \int_0^t e^{\Lambda' t'} \Lambda''\, dt' & e^{\Lambda' t} \end{pmatrix}$$

with $\Lambda' = (\Lambda_{jk})_{j,k\ge q+1}$, $\Lambda'' = (\Lambda_{jk})_{j\le q+1, k\le q}$. It provides us with the first step of the reduction:

Step 1 Applying transformation $\exp(\frac{i}{2}\mu^{-1}h^{-1}\mathcal{L})$ with operator $\mathcal{L}$ defined by (20.3.13) with appropriate coefficients β^{jk} one can transform operator A to the block form

$$\mu^2 \sum_{q+1\le j,k\le q+r} \zeta_j^{\dagger\,\mathrm{w}} a^{jk\,\mathrm{w}} \zeta_k^{\mathrm{w}} + \mu^2 \sum_{j,k\le q} p_j^{\mathrm{w}} a^{jk\,\mathrm{w}} p_k^{\mathrm{w}} + a_0^{\mathrm{w}} + \frac{1}{2}\big(B + B^*\big), \tag{20.3.15}$$

$$B = \Big(\mu^2 \sum_{j,k,m} b^{jkm} p_j p_k p_m + \sum_j b^j p_j\Big)^{\mathrm{w}} \tag{20.3.16}$$

$$\zeta_j = p_j + i p_{j+r} \text{ with } j = q+1, \dots, q+r, \tag{20.3.17}$$

$$a^{jk} \ne 0 \implies \exists \mathfrak{m} : j, k \in \mathfrak{m}; \tag{20.3.18}$$

let us recall that $\mathfrak{M}$ is the resonance partition at point $\bar{x}$.

(20.3.19) Here and below $a^{jk} \in \mathscr{F}^{\bar{l},\bar{\sigma}}$, $a_0 \in \mathscr{F}^{l,\sigma}$, $b^{jkl} \in \mathscr{F}^{\bar{l}-1,\bar{\sigma}}$, $b^j \in \mathscr{F}^{l-1,\sigma}$ and while $a^{jk} = a^{jk}(x)$ we have $b^{jkl} = b^{jkl}(x, \mu^{-1}\xi)$, $b^j = b^j(x,\mu^{-1}\xi)$ and c^{w} means h- rather than $\mu^{-1}h$-quantization of symbol c.

Step 2 Applying an appropriate gauge transformation we can achieve

$$V_k = 0 \text{ as } k \le q+r \text{ and } V_k = V_k(x_{q+1}, \dots, x_d) \text{ for } k = q+r+1, \dots, d \tag{20.3.20}$$

where condition $(20.1.9)_2$ provides $\mathscr{F}^{\bar{l}+1,\bar{\sigma}}$-regularity of new V_k. This is the only reason for this condition. Then

$$p_j = 0 \ \ \forall j = q+r+1, \dots, d \iff x_j = \lambda_j(x'', \xi') \ \ \forall j = q+1, \dots, q+r \tag{20.3.21}$$

with $x' = (x_{q+1}, \dots, x_{q+r})$, $\xi' = (\xi_{q+1}, \dots, \xi_{q+r})$, $x'' = (x_{q+r+1}, \dots, x_{q+2r})$, $\xi'' = (\xi_{q+r+1}, \dots, \xi_{q+2r})$, and $\lambda_j \in \mathscr{F}^{\bar{l}+1,\bar{\sigma}}$.

After this, following arguments of Sections 18.3, 18.7, and 19.3 and applying transformation $e^{i\mu^{-1}\mathcal{L}'}$ with $\mathcal{L}' = \sum_{q+1\le j\le q+r} \lambda_j^{\mathrm{w}} D_{j+r}$ we can transform operator (20.3.15) to the form (20.3.15)–(20.3.16) but now with

$$\zeta_j = \eta_j + i\eta_{j+r},\ \eta_j = \mu^{-1}\xi_j,\ \eta_{j+r} = x_j \qquad j = q+1, \dots, q+r, \tag{20.3.22}$$

$$\eta_j = \mu^{-1}\xi_j \qquad j = 1, \dots, q. \tag{20.3.23}$$

However in contrast to the previous step of the reduction now

$$a^{jk} = a^{jk}(x', \mu^{-1}\xi''; x'''), \quad a_0 = a_0(x', \mu^{-1}\xi''; x''')$$

rather than $a^{jk} = a^{jk}(x)$, $a_0 = a_0(x)$.

Remark 20.3.4. (i) One can see easily that if f_j have constant multiplicities, then the operator can be reduced on *Step 1* to the form

$$\mu^2 \sum_{q+1\le j\le q+r} \zeta_j^{\dagger\,\mathrm{w}} f_j^{\mathrm{w}} \zeta_j^{\mathrm{w}} + \mu^2 \sum_{j,k\le q} p_j^{\mathrm{w}} a^{jk\,\mathrm{w}} p_k^{\mathrm{w}} + a_0^{\mathrm{w}} + \frac{1}{2}\bigl(B + B^*\bigr) \tag{20.3.24}$$

and this form will be preserved after *Step 2*.

(ii) One can see easily that if $q = 1$, then after *Step 2* by means of transformation $e^{ih^{-1}\mathcal{L}''}$ with $\mathcal{L}'' = \frac{1}{2}(hD_1 k^{\mathrm{w}} + k^{\mathrm{w}} hD_1)$, $k = k(x', x_d, \mu^{-1}\xi'')$ one can reduce operator to the same form (20.3.15) but with $a^{11} = 1$.

(iii) In the general case we can achieve $a^{jk} = \delta_{jk}$ for $j, k = 1, \dots, q$ only assuming that g^{jk}, F_{jk} are constant.

20.3.4 Reduction. Junior Terms

Recall that junior terms are of the form $\frac{1}{2}(B+B^*)$ with B defined by (20.3.16) with $b^{jkm} = b^{jkm}(x, \mu^{-1}\xi)$ and $b^j = b^j(x, \mu^{-1}\xi)$.

Remark 20.3.5. As $(1,1) \preceq (l,\sigma) \preceq (2,0)$ one can rewrite B in the same form but with

$$b^{jkm} = b^{jkm}(x', \mu^{-1}\xi''; x'''), \qquad b^j = b^j(x', \mu^{-1}\xi''; x''') \tag{20.3.25}$$

modulo operator B' with the symbol belonging to $\mu^{-l}|\log\mu|^{-\sigma}\mathscr{F}^{0,0}$. Our reduction is done modulo operators of this type.

Then applying operator $e^{i\mu^{-2}h^{-1}\mathcal{L}_1}$ with

$$\mathcal{L}_1 = \Big(\sum_{\alpha:1\le|\alpha|\le 3} \beta^{(\alpha)}\mu^{|\alpha|}\eta^{\alpha}\Big)^{\mathsf{w}}, \tag{20.3.26}$$

where $\eta^{\alpha} = \eta_1^{\alpha_1}\cdots\eta_d^{\alpha_d}$, one can reduce operator to the same form (20.3.15) with

$$\begin{aligned} B \equiv \mu^2\Big(&\sum_{q+1\le j,k,m\le q+r} b^{jkm}\zeta_j^{\dagger}\zeta_k\zeta_m + \\ &\sum_{q+1\le j,k\le q+r,\ 1\le m\le q} b^{jkm}\eta_m\zeta_j^{\dagger}\zeta_k + \sum_{1\le j,k,m\le q} b^{jkm}\eta_m\eta_j\eta_k + \sum_{1\le j\le q} b^{j}\eta_j\Big)^{\mathsf{w}}. \end{aligned} \tag{20.3.27}$$

Here in the first term $b^{jkm} = 0$ unless $|f_j - f_k - f_m| \le \epsilon$ and in the second term $b^{jkm} = 0$ unless $|f_j - f_k| \le \epsilon$ (so there are the 3-rd and 2-nd order resonances respectively). Further, $b^{jkm} = 0$ always for constant g^{jk}, F_{jk}.

Also, in (20.3.26) $\beta^{(\alpha)}$ belongs to $\mathscr{F}^{\bar{l}-1,\bar{\sigma}}_{x'}$, $\mathscr{F}^{l-1,\sigma}_{x'}$ as $|\alpha|\ge 2$, $|\alpha| = 1$ respectively; also $b^{jkm} \in \mathscr{F}^{\bar{l}-1,\bar{\sigma}}_{x'}$, $b^{j} \in \mathscr{F}^{l-1,\sigma}_{x'}$.

Remark 20.3.6. (i) In the intermediate magnetic field case $|\eta_j| \le C\mu^{-1}$ in the microlocal sense (on the energy levels below c) and therefore junior terms are of magnitude μ^{-1} and skipping them we would arrive to an approximation error $Ch^{-d}(\mu h|\log h|)^{\frac{q}{2}}\mu^{-1}$ (as either $q\ge 2$, or $q = 1$ and the microhyperbolicity assumption is fulfilled, or $(l,\sigma)\succeq(2,0)$, and non-degeneracy assumption is fulfilled) which is $O(h^{1-d})$ as $q\ge 3$ and $O(h^{1-d}|\log h|)$ as $q = 2$.

Thus, in the case of an intermediate magnetic field only for $q = 1, 2$ we will need to take junior terms into account at all. This will be a minor annoyance for $q = 2$ but a major obstacle for $q = 1$, when without either microhyperbolicity or non-degeneracy assumptions this error would be $O\big(h^{-d}(\mu h|\log h|)^{\frac{1}{2}}\mu^{-1} + \mu^{\frac{1}{2}}h^{1-d}\big)$ but even under the microhyperbolicity or non-degeneracy assumptions these terms cannot be simply ignored.

(ii) Actually, not all of these terms are equally bad: terms containing factor(s) η_m with $1\le m\le q$ are of magnitude $O(\mu^{-1}|\xi'''|)$ and can be simply ignored if $q = 2$ but even if $q = 1$ they are minor annoyances. Only the first term in (20.3.27) are really important.

(iii) One can see easily that if $q = 1$ and f_j have constant multiplicities, then by means of transformation $e^{i\mu^{-1}h^{-1}\mathcal{L}'_1}$ with

$$\mathcal{L}'_1 = \mu^2\Big(\sum_{\mathfrak{m}}\sum_{j,k\in\mathfrak{m}} c^{jk}\zeta_j^\dagger\zeta_k + c_0\eta_1^2 + c_1\Big)^{\mathsf{w}}$$

with $c^{jk} = c^{jk}(x'', x_d, \mu^{-1}\xi'')$, $c_j = c_j(x'', x_d, \mu^{-1}\xi'')$ one can remove from B all terms containing factor η_1 thus arriving to

$$B \equiv \mu^2\Big(\sum_{q+1\le j,k,m\le q+r} b^{jkm}\zeta_j^\dagger\zeta_k\zeta_m\Big)^{\mathsf{w}}. \tag{20.3.28}$$

Therefore, we constructed operator $\mathcal{T}$ with the properties described below:

Proposition 20.3.7. *Let $(\bar{l},\bar{\sigma}) = (2,1)$, $(1,1) \preceq (l,\sigma) \preceq (2,0)$. Then there exists an operator $\mathcal{T}$ such that*

$$\mathcal{T}^*\mathcal{T}\bar{Q}\bar{\psi} \equiv \bar{Q}\bar{\psi}, \tag{20.3.29}$$

$$\mathcal{T}^*Q_1\mathcal{T}(I-\bar{Q})\bar{\psi} \equiv 0, \tag{20.3.30}$$

$$\mathcal{T}^*(I-Q_2)\mathcal{T}\bar{Q}\bar{\psi} \equiv 0 \qquad \mathsf{mod}\ \mu^{-\infty}(\mathscr{F}^{0,0})^{\mathsf{w}}, \tag{20.3.31}$$

$$\mathcal{T}^*\mathcal{A}\mathcal{T}\bar{Q}\bar{\psi} \simeq A\bar{Q}\bar{\psi}, \qquad \mathsf{mod}\ \mu^{-l}|\log h|^{-\sigma}(\mathscr{F}^{0,0})^{\mathsf{w}} \tag{20.3.32}$$

as $\bar{\psi} \in \mathscr{C}_0^\infty(B(0,1))$, $\bar{Q} = \bar{Q}(hD''')$ has a symbol which is supported in $\{|\xi'''| \le 2C\bar{\rho}_1\}$ and equal to 1 in $\{|\xi'''| \le 2C\bar{\rho}_1\}$, Q_1 and Q_2 are the same type operators but with symbols which are supported in $\{|\xi'''| \le 2\bar{\rho}_1\}$ and $\{|\xi'''| \le 2C^2\bar{\rho}_1\}$ and equal to 1 in $\{|\xi'''| \le \bar{\rho}_1\}$ and $\{|\xi'''| \le C^2\bar{\rho}_1\}$ respectively with large enough constant C, $\mathcal{A}$ is the reduced operator in the form (20.3.15) *with junior terms defined by* (20.3.27).

Further, if $q = 1$ and f_j have constant multiplicities, junior are terms defined by (20.3.28).

Remark 20.3.8. (i) Note that while (20.3.29)–(20.3.31) hold modulo negligible operator, (20.3.32) is satisfied modulo $\mu^{-1}h$-pseudodifferential operators with symbols of $\mu^{-l}|\log h|^{-\sigma}\mathscr{F}^{0,0}$, and to emphasize the difference we use "$\simeq$" instead of "$\equiv$".

(ii) In contrast to almost all this Chapter in Propositions 20.3.7, 20.3.9 b^{w} means $\mu^{-1}h$-quantization of symbol b.

We also need the following properties of operator constructed above:

Proposition 20.3.9. *Furthermore the constructed operator has the following properties:*

(i) For functions $\bar{\psi}=1$ *in* $B(0,\frac{3}{4})$ *and* $\psi\in \mathscr{C}_0^\infty(B(0,\frac{1}{2}))$

$$\psi\bar{Q}\bar{\psi}\equiv \mathcal{T}^*\tilde{\psi}^{\mathsf{w}}\mathcal{T}\bar{Q}\bar{\psi}, \tag{20.3.33}$$

$$\tilde{\psi}\simeq \psi_0(x'',x''',\xi'')+\sum_{1\le j\le d}\psi_j(x'',x''',\xi'')\eta_j \qquad \mathsf{mod}\ \mu^{-l}|\log h|^{-\sigma}\mathscr{F}^{0,0}. \tag{20.3.34}$$

(ii) For operator $Q=Q^{\mathsf{w}}(hD''')$ *as above*

$$\mathcal{T}^*Q\mathcal{T}\bar{\psi}\simeq \big(Q(f(x,\xi))\big)^{\mathsf{w}}\bar{\psi} \qquad \mathsf{mod}\ \mu^{-l}|\log h|^{-\sigma}(\mathscr{F}^{0,0})^{\mathsf{w}}, \tag{20.3.35}$$

$$f(x,\xi)\simeq \xi'''-\sum_{1\le j\le 2r}\kappa_j\eta_j-\mu^{-1}\kappa_0 \qquad \mathsf{mod}\ \mu^{-l}|\log h|^{-\sigma}\mathscr{F}^{0,0} \tag{20.3.36}$$

with coefficients $\kappa_j=\kappa_j(x'',x''',\mu^{-1}\xi'')$ *and all estimates holding for* $|\eta_j|\le C\mu^{-1}$, $\psi_j,\alpha_j\in \mathscr{F}^{l-1,\sigma}$.

Proof. We prove more difficult Statement (ii) leaving easier Statement (i) to the reader.

(i) On *Step 1* of the main part reduction we considered operator $e^{\frac{i}{2}\mu^{-1}h^{-1}t\mathcal{L}}$ with the corresponding symplectomorphism which is equivalent to the map $f_t:(x,\eta_1,\dots,\eta_d)\to(x,\eta_1,\dots,\eta_d)$ which is of $\mathscr{F}^{\bar{l},\bar{\sigma}}$ class. Let $Q_t=\big(Q(f_t)\big)^{\mathsf{w}}$; then the principal symbol of operator

$$\frac{\partial}{\partial t}Q_t+\frac{i}{2}\mu^{-1}h^{-1}\big[\mathcal{L},Q_t\big] \tag{20.3.37}$$

vanishes and then as long as $(\bar{l},\bar{\sigma})\succeq(2,1)$ and $\varepsilon\ge C(\mu^{-1}h|\log h|)^{\frac{1}{2}}$ the norm of this operator does not exceed

$$C\mu^{-1}h^{-1}\times\big(\frac{h}{\rho}\big)^2\le C\mu^{-1}h(\mu h|\log h)^{-1}=C\mu^{-2}|\log h|^{-1}$$

for $\rho\ge C(\mu h|\log h|)^{\frac{1}{2}}$. Then the norm of the operator

$$\frac{d}{dt}\Big(e^{-\frac{i}{2}\mu^{-1}h^{-1}t\mathcal{L}}Q_te^{\frac{i}{2}\mu^{-1}h^{-1}t\mathcal{L}}\Big) \tag{20.3.38}$$

does not exceed $C\mu^{-2}|\log h|^{-1}$ and then the same is true for operators

$$e^{-\frac{i}{2}\mu^{-1}h^{-1}t\mathcal{L}}Q_te^{\frac{i}{2}\mu^{-1}h^{-1}t\mathcal{L}}-Q \quad\text{and}\quad e^{\frac{i}{2}\mu^{-1}h^{-1}t\mathcal{L}}Qe^{-\frac{i}{2}\mu^{-1}h^{-1}t\mathcal{L}}-Q_t. \tag{20.3.39}$$

(ii) The same arguments work perfectly on *Step 2* of the main part reduction.

(iii) Now consider the junior terms reduction. First of all, there are removable terms $\sum_{q+1\le j\le q+2r}\beta_j\eta_j$ with $\beta_j=\beta_j(x'',\mu^{-1}\xi'')$, $\beta_j\in\mathscr{F}^{l-1,\sigma}$ which are removed by a transformation of the same type as above with $\mathcal{L}=\mu^{-1}\sum_{1\le j\le 2r}\beta'_j\eta_j$ and $\beta'_j=\beta'_j(x'',\mu^{1}\xi'')$, $\beta'_j\in\mathscr{F}^{l-1,\sigma}$ as well.

Then the corresponding Hamiltonian map is of class $\mathscr{F}^{l,\sigma}$ and then the principal symbol of operator (20.3.34) vanishes and the norm of this operator does not exceed

$$C\mu^{2-l}|\log h|^{-\sigma}\times\mu^{-1}h^{-1}\times\Big(\big(\frac{h}{\rho}\big)^2+\mu^{-1}h\big(\frac{h}{\rho}\big)\times\mu^{2-l}|\log h|^{-\sigma}\Big)\le C\mu^{-l}|\log h|^{-\sigma}\times\frac{\mu h}{\rho^2}$$

which is less than $C\mu^{-l}|\log h|^{-\sigma}$ and then the same estimate holds for operators (20.3.38)–(20.3.39).

With the removable third and second order terms the construction is the same but simpler since these terms are coming from the principal part which belongs to $\mathscr{F}^{2,1}$ for sure. □

Remark 20.3.10. In Chapter 18 in the case $d=3$ we took $\varepsilon=C(\mu^{-1}h|\log h|)^{\frac12}$ instead of $C\mu^{-1}$. The above construction would not work. In this case note first that the rescaling $x\mapsto\varepsilon^{-1}(x-\bar x)$ and taking $\hbar=\mu^{-1}h\varepsilon^{-2}=C^{-1}|\log h|^{-1}$ we will see that the original operator $\bar\chi(hD_1/\rho)$ becomes $\bar\chi(\hbar D_1)$ and $\mu^{-1}h^{-1}\mathcal{L}$ becomes $\nu\ell(x,\hbar D)$ with $\nu=\mu^{-2}\varepsilon^{-1}=C^{-1}(\mu^3h|\log h|)^{-\frac12}\le\epsilon_1$ and therefore $e^{i\mu^{-1}h^{-1}\mathcal{L}}$ becomes $\hbar$-pseudodifferential operator with "analytic" symbol (more precisely its symbol and those of Q satisfy assumptions of Subsection 1.1.3).

Then $e^{-i\mu^{-1}h^{-1}\mathcal{L}}Qe^{i\mu^{-1}h^{-1}\mathcal{L}}$ becomes $\hbar$-pseudodifferential operator which means that in the original scale it is $\mu^{-1}h$-pseudodifferential operator with $\mathscr{F}^{0,0}$ symbol supported in $\{|\xi_3|\le 2C\rho\}$ and equal 1 in $\{|\xi_3|\le C\rho\}$ with $\rho=\bar\rho_1=C(\mu h|\log h|)^{\frac12}$.

20.3.5 Temperateg and Stronger Magnetic Field

In this case we have

$$\mu\ge\epsilon(h|\log h|)^{-1}\tag{20.3.40}$$

and

$$\varepsilon = C(\mu^{-1}h|\log\mu|)^{\frac{1}{2}}. \tag{20.3.41}$$

Then construction of Subsections 3.3–3.4 works on every its step. Now in a microlocal sense $|\eta_j| \le C(\mu^{-1}h|\log h|)^{\frac{1}{2}}$ (on the energy levels below c), and therefore junior terms seem to be of magnitude $C\mu^2(\mu^{-1}h|\log\mu|)^{\frac{3}{2}}$ and cannot be skipped; even next junior terms are of magnitude $C\mu^2(\mu^{-1}h|\log\mu|)^2$ which is $O(h)$ as $h|\log\mu|^2 \le C$ only.

In fact, however, as we will see, junior terms will have no impact because the microlocal estimate will be replaced by an operator one $|\eta_j^{\mathsf{w}}| \le C(\mu^{-1}h)^{\frac{1}{2}}$ which is sufficient to estimate junior terms by Ch as long as $\mu \le C_0h^{-1}$ (strong and very strong magnetic field).

Further, for $\mu h \ge C_0h^{-1}$ (*strong* magnetic field, we consider Schrödinger-Pauli operator then) only the lowest Landau level is important where irreducible terms due to the 3-rd order resonances vanish and irreducible terms due to the 4-th order resonances are of magnitude $C\mu^2(\mu^{-1}h)^2 = O(h^2)$ in the operator sense.

20.4 Temperate Magnetic Field: Estimates

In this section we assume that the magnetic field is intermediate i.e. condition (20.3.3) is fulfilled. In this case we know that the contribution of the outer zone $\Omega_{\mathsf{out}} := \{|\xi'''| \ge \bar\rho_1 := C(\mu h|\log h|)^{\frac{1}{2}}\}$ to the Tauberian estimate with $T = \epsilon\mu^{-1}$ is $O(h^{-2})$ and therefore we need to consider a complementary zone $\Omega^{\mathsf{c}}_{\mathsf{out}} = \{|\xi'''| \le \bar\rho_1 := C(\mu h|\log h|)^{\frac{1}{2}}\} = \Omega_{\mathsf{intm}} \cup \Omega_{\mathsf{inn}}$ with an *intermediate zone*

$$\Omega_{\mathsf{intm}} := \bigl\{\bar\rho_1^* := C\max\bigl(\mu^{-1}, (\mu h)^{\frac{1}{2}}\bigr) \le |\xi'''| \le \bar\rho_1 = (\mu h|\log h|)^{\frac{1}{2}}\bigr\} \tag{20.4.1}$$

and an inner zone $\Omega_{\mathsf{inn}} = \{|\xi'''| \le \bar\rho_1^*\}$.

20.4.1 Intermediate Zone

General Arguments

Let us consider an intermediate zone (20.4.1). We split it into strips $\mathcal{Z}_\rho$ with ρ in the indicated framework:

$$\mathcal{Z}_\rho := \{(x,\xi) :\ \rho \le |\xi'''| \le 2\rho\}, \qquad \bar\rho_1^* \le \rho \le \bar\rho_1. \tag{20.4.2}$$

Proposition 20.4.1. *Let condition* (20.3.3) *be fulfilled and let* $Q = Q^{\mathrm{w}}$ *be an operator with the symbol supported in the strip* $\mathcal{Z}_\rho$*, and such that*

$$|\partial^\alpha_{x,\xi} Q| \le C_\alpha \rho^{-|\alpha|} \qquad \forall \alpha, \tag{20.4.3}$$

and let $\psi \in \mathscr{C}_0^\infty(B(0,1))$.

Let $\chi \in \mathscr{C}_0^\infty([-1,-\frac12] \cup [\frac12, 1])$. *Then estimate*

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma Q\psi U| \le Ch^{-d}T\rho^q\bigl(\rho^2 + \rho T + \mu^{-1}\bigr)\bigl(\frac{h}{\rho^2 T}\bigr)^s \tag{20.4.4}$$

holds as

$$T_* = \frac{h}{\rho^2} \le T \le T^* = \epsilon\rho \tag{20.4.5}$$

with an arbitrarily large exponent s.

Proof. Proof repeats those of analysis in Section 18.8. First of all, let us make γ-partition with respect to (x'', ξ'', x''') with

$$\gamma \ge C\mu^{-1} + C(\mu^{-1}h|\log h|)^{\frac12} \tag{20.4.6}$$

and consider some element $\psi'(x)$ of this partition supported in $B(\bar x, \gamma)$. Let us reduce operator to its canonical form in $B(\bar x, 2\gamma)$; without any loss of the generality one can assume that this canonical form is

$$\sum_{1\le j\le r} \bar f_j(h^2 D^2_{q+j} + \mu^2 x^2_{q+j}) + \sum_{1\le j\le q} h^2 \bar g^{jk} D_j D_k + \bar V + O(\gamma) \tag{20.4.7}$$

there where $\bar f_j$, $\bar g^{jk}$ and $\bar V$ are constant (and we can assume that $\bar g^{jk} = \delta_{jk}$, $O(\gamma)$ covers variations of f_j, V as well as unremovable terms and we are discussing only $B(\bar x, 2\gamma)$.

Let us decompose $\mathsf{U} := \mathcal{T}U$ into the sum

$$(\mathcal{T}U)(x,y,t) = \sum_{\alpha,\beta\in\mathbb{Z}^{+\,r}} \mathsf{U}_{\alpha,\beta}(x'', x''', y'', y''', t)\Upsilon_\alpha(x')\Upsilon_\beta(y') \tag{20.4.8}$$

with $\Upsilon_\alpha(x') = \hbar^{-\frac{r}{2}}\prod_j \upsilon_{\alpha_j}(x_{q+j}/\hbar)$, $\hbar = (\mu^{-1}h)^{\frac12}$ and Hermite functions υ_*.

Then modulo $O(\gamma)$ in $\mathscr{L}^2$-norm operator $\mathcal{A}$ applied to the terms containing $\Upsilon_\alpha(x')$ in this sum becomes

$$\bar{\mathcal{A}}_\alpha = \sum_{1\le j\le r} \bar f_j(2\alpha_j + 1)\mu h + \sum_{1\le j\le q} h^2 D_j^2 + \bar V \tag{20.4.9}$$

which is a temporary notation.

Let us strengthen condition (20.4.6) assuming that

$$\gamma \ge \rho^2 + \mu h + \mu^{-1}. \tag{20.4.10}$$

Then using our standard methods one can prove easily that for $Q' = Q\mathcal{T}\psi'\mathcal{T}^*$ the following inequality holds

$$\begin{aligned} |F_{t\to h^{-1}\tau}\chi_T(t)\Gamma' Q'\psi \mathsf{U}_{\alpha,\beta}| \le \\ C\mu^r h^{-r-q}\rho^q\gamma^d T \times \Big(\big(\frac{h}{\rho^2 T}\big)^s + \big(\frac{h}{\rho\gamma}\big)^s + \big(\frac{\mu^{-1}h}{\gamma^2}\big)^s\Big)\times \\ \Big(\frac{\gamma}{\gamma + |\mu h \sum_j \bar{f}_j(\alpha_j - \bar{\alpha}_j)|}\Big)^s \times \Big(\frac{\gamma}{\gamma + |\mu h \sum_j \bar{f}_j(\beta_j - \bar{\alpha}_j)|}\Big)^s \end{aligned} \tag{20.4.11}$$

with some $\bar{\alpha} = \bar{\alpha}(\tau) \in \mathbb{Z}^{+r}$ where

$$\Gamma' v := \int v|_{x''=y'',x'''=y'''}\, dy''\, dy'''. \tag{20.4.12}$$

Here factor $\mu h^{1-d}\rho^q\gamma^d$ in the right-hand expression comes as a trace norm of $Q'\mathsf{U}_{\alpha,\beta}$; factor T is a result of integration with respect to t in the Fourier transform, the second factor is the sum of three terms: the first one $\big(h/(\rho^2 T)\big)^s$ is due to the microhyperbolicity with respect to ξ''' and rescaling while terms $\big(h/(\rho\gamma)\big)^s$ and $\big(\mu^{-1}h/(\gamma^2)\big)^s$ are operator calculus errors.

Finally, two last factors in the right-hand expression of (20.4.11) are due to the ellipticity of operators $\big(hD_t - \bar{\mathcal{A}}_\alpha\big)$ and $\big(hD_t - \bar{\mathcal{A}}_\beta\big)$ respectively.

More precisely, let us note that operator $\big(\bar{\mathcal{A}}_\alpha + O(\gamma) - \tau\big)$ is elliptic in this vicinity as long as $\alpha \notin \mathfrak{A}$ where

$$\mathfrak{A} := \{\alpha \in \mathbb{Z}^{+r} : |\sum_j (2\alpha_j + 1)\bar{f}_j\mu h + \bar{V} - \tau| \le \gamma\}. \tag{20.4.13}$$

Really, we can express there $\mathsf{U}_{\alpha,*}$ with $\alpha \notin \mathfrak{A}$ via $\mathsf{U}_{\alpha',*}$ with $\alpha' \in \mathfrak{A}$. Namely, for $\alpha \notin \mathfrak{A}$ we have

$$\begin{aligned} |F_{t\mapsto h^{-1}\tau}\chi_T(t)Q'\mathsf{U}_{\alpha,\beta}| \le \\ \max_{\alpha'\in\mathfrak{A}} |F_{t\mapsto h^{-1}\tau}\chi_T(t)Q'\mathsf{U}_{\alpha',\beta}| \times \Big(\gamma\big(|\sum_j (2\alpha_j+1)\bar{f}_j\mu h + \bar{V}| + \gamma\big)^{-1}\Big)^s \end{aligned} \tag{20.4.14}$$

The similar inequality holds for $\mathsf{U}_{*,\beta}$ with $\beta \notin \mathfrak{A}$. Finally, for $\mathsf{U}_{\alpha,\beta}$ with arbitrary $\alpha, \beta \notin \mathfrak{A}$

$$|F_{t\mapsto h^{-1}\tau}\chi_T(t)Q'U\alpha,\beta| \le \max_{\alpha',\beta'\in\mathfrak{A}} |F_{t\mapsto h^{-1}\tau}\chi_T(t)Q'\mathsf{U}_{\alpha',\beta'}|\times \tag{20.4.15}$$
$$\Big(\gamma\big(|\sum_j (2\alpha_j+1)\bar{f}_j\mu h+\bar{V}|+\gamma\big)^{-1}\Big)^s \times \Big(\gamma\big(|\sum_j (2\beta_j+1)\bar{f}_j\mu h+\bar{V}|+\gamma\big)^{-1}\Big)^s$$

which in combination with the microhyperbolicity arguments justifies (20.4.11).

Note also that $\#\mathfrak{A} \le C\gamma(\mu h)^{-r}$. Then (20.4.11) implies

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma Q'\psi U| \le \tag{20.4.16}$$
$$C\mu^r h^{-r-q}\rho^q\gamma^d T \times \Big(\big(\frac{h}{\rho^2 T}\big)^s + \big(\frac{h}{\rho\gamma}\big)^s + \big(\frac{\mu^{-1}h}{\gamma^2}\big)^s\Big) \times \gamma(\mu h)^{-r} =$$
$$Ch^{-d}T\rho^q\gamma^{d+1}\Big(\big(\frac{h}{\rho^2 T}\big)^s + \big(\frac{h}{\rho\gamma}\big)^s + \big(\frac{\mu^{-1}h}{\gamma^2}\big)^s\Big).$$

Restoring to original Q, ψ (by summation over x-partition) we lose factor γ^d. As exponent s is large enough, the optimal value of γ honoring (20.4.10) is $\gamma = \rho T + \rho^2 + \mu^{-1}$ because for $\rho \ge \mu^{-1}$ (which we assume) $\gamma^2 \ge \mu^{-1}\rho^2 T$.

Then we arrive to estimate (20.4.4). □

Proposition 20.4.1 implies immediately estimate

$$|F_{t\to h^{-1}\tau}\big(\bar{\chi}_T(t) - \bar{\chi}_{\bar{T}}(t)\big)\Gamma Q\psi U| \le Ch^{-d}\mu^{-1}\rho^q(\rho^2+\mu^{-1}) \times \big(\frac{\mu h}{\rho^2}\big)^s \tag{20.4.17}$$

with $\bar{\chi} \in \mathscr{C}_0^\infty([-1,1])$ equal 1 on $[-\frac{1}{2},\frac{1}{2}]$, $T_* = \epsilon\mu^{-1}$ as T satisfies

$$\bar{T} = \epsilon\mu^{-1} \le T \le T^* = \epsilon\rho. \tag{20.4.18}$$

In turn (20.4.17) implies

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma Q\psi U| \le Ch^{-d}\mu^{-1}\rho^q(\rho^2+\mu^{-1}) \times \big(\frac{\mu h}{\rho^2}\big)^s + Ch^{1-d}\rho^q \tag{20.4.19}$$

where $C\rho^q h^{1-d}$ estimates $|F_{t\to h^{-1}\tau}\bar{\chi}_{T_*}(t)\Gamma Q\psi U|$ due to the standard results rescaled.

Therefore the standard Tauberian arguments immediately imply estimate

$$\mathsf{R}^{\mathsf{T}}_{Q} \coloneqq |\Gamma(Q\psi\tilde{e}) - h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\bar{\chi}_{T}(t)\Gamma Q\psi u\Big)\,d\tau| \le \tag{20.4.20}$$
$$CT^{-1}\Big(Ch^{-d}\mu^{-1}\rho^{q}(\rho^{2}+\mu^{-1})\times\big(\frac{\mu h}{\rho^{2}}\big)^{s} + Ch^{1-d}\rho^{q}\Big) =$$
$$Ch^{-d}\mu^{-1}\rho^{q-1}(\rho^{2}+\mu^{-1})\times\big(\frac{\mu h}{\rho^{2}}\big)^{s} + Ch^{1-d}\rho^{q-1}$$

with any

$$T_{*} \coloneqq Ch|\log h|/\rho^{2} \le T \le \epsilon\rho. \tag{20.4.21}$$

Case $q \ge 2$

Consider $q \ge 2$ (we analyze the case of $q = 1$ later). Integrating the first term in the right-hand expression of (20.4.20) by $d\rho/\rho$ from $\bar{\rho}_{1}^{*} = C(\mu h)^{\frac{1}{2}}$ to $\bar{\rho}_{1}$ we get its value as $\rho = \bar{\rho}_{1}^{*}$; one can see easily that the result will be then less than Ch^{1-d}. The same integration by $d\rho/\rho$ applied to the second term in the right-hand expression of (20.4.20) results in $O(h^{1-d})$ for sure.

Thus for $q \ge 2$ in the whole intermediate zone $\{\bar{\rho}_{1}^{*} \le |\xi'''| \le \bar{\rho}_{1}\}$ we get a proper remainder estimate

$$\mathsf{R}^{\mathsf{T}}_{1\,Q} \coloneqq |\Gamma(Q\psi\tilde{e}) - h^{-1}\int_{-\infty}^{0}\sum_{0\le n\le \bar{n}}\Big(F_{t\to h^{-1}\tau}\bar{\chi}_{T_{n}}(t)\Gamma Q_{n}\psi U\Big)\,d\tau| \le \tag{20.4.22}$$
$$Ch^{1-d}$$

as $Q = \sum_{0\le n\le \bar{n}} Q_{n}$, where Q_{n} are operators with the symbols supported in $\{\rho_{n} \le |\xi'''| \le \rho_{n+1}\}$, $\rho_{n} = 2^{n}\bar{\rho}_{1}^{*}$, $\rho_{\bar{n}} \asymp \bar{\rho}_{1}$, and satisfying (20.4.3), $Ch|\log h|/\rho_{n}^{2} \le T_{n} \le \epsilon\rho_{n}$. Therefore we arrive to

Proposition 20.4.2. *For $q \ge 2$ in the framework of Proposition 20.4.1 estimate* (20.4.22) *holds for operator $Q = Q^{\mathsf{w}}$ with the symbol supported in $\Omega_{\mathsf{intm}} = \{\bar{\rho}_{1}^{*} \le |\xi'''| \le \bar{\rho}_{1}\}$ and satisfying* (20.4.3).

Now let us consider the expression

$$h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\chi_{T}(t)\Gamma Q\psi U\Big)\,d\tau \tag{20.4.23}$$

where again Q is an operator with the symbol supported in $\mathcal{Z}_\rho$, $T \in [T_*, T^*]$, $T^* = \epsilon\rho$, $\rho \in [\bar\rho_1^*, \bar\rho_1]$. Recall that $\chi \in \mathscr{C}_0^\infty([-1,-\frac12]\cup[\frac12,1])$. Also recall that one can rewrite (20.4.23) as

$$T^{-1}\Bigl(F_{t\to h^{-1}\tau}\check\chi_T(t)\Gamma Q\psi U\Bigr)\Big|_{\tau=0} \tag{20.4.24}$$

with $\check\chi(t) = it^{-1}\chi(t)$ and due to Proposition 20.4.1 expression (20.4.24) does not exceed

$$Ch^{-d}\rho^{q+2}\times \bigl(\frac{h}{T\rho^2}\bigr)^s. \tag{20.4.25}$$

Making a summation with respect to $T \in [T_*, T^*]$ we get its value at $T = \bar T = \epsilon\mu^{-1}$ i.e. $Ch^{-d}\rho^{q+2}(\mu h/\rho^2)^s$. Then after summation with respect to $\rho \in [\bar\rho_1^*, \bar\rho_1]$ we get its value at $\rho = \bar\rho_1^*$ i.e. $Ch^{-d}(\mu h)^{\frac{q}{2}+1}$. Estimating roughly contribution of the inner zone $\Omega_{\mathsf{inn}} = \{|\xi'''| \le C\bar\rho_1^*\}$ by $C\mu h^{1-d}(\bar\rho_1^*)^q$ and estimating contribution of the outer zone by Ch^{1-d} due to the results of Section 20.2 we arrive to the estimate

$$\mathsf{R}^{\mathsf{T}} = |\Gamma(\psi\tilde e) - h^{-1}\int_{-\infty}^0 \Bigl(F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma\psi U\Bigr)\,d\tau| \le Ch^{1-d} + C(\mu h)^{\frac{q}{2}+1}h^{-d} \tag{20.4.26}$$

with $T = \bar T = \epsilon\mu^{-1}$.

On the other hand, due to the standard results rescaled for $q \ge 2$

$$|h^{-1}\int_{-\infty}^0 \Bigl(F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma\psi U\Bigr)\,d\tau - h^{-d}\int \mathcal{N}^{\mathsf{MW}}(x,\tau)\psi(x)\,dx| \le Ch^{1-d} + C(\mu h)^{\frac{q}{2}+1}h^{-d} \tag{20.4.27}$$

as $T = \bar T$ because with our choice (20.3.5) for ε, a mollification error does not exceed Ch^{1-d} as well.

Thus we arrive to

Theorem 20.4.3. *Let conditions* (20.1.1), (20.1.5), (20.1.7), (20.1.8)–(20.1.11), *and* (20.2.31) *be fulfilled,* $(\bar l, \bar\sigma) = (2,1)$, $(l,\sigma) = (1,1)$.

Let $q \ge 2$ *and condition* (20.3.3) *be fulfilled.*

(i) Then there exist two framing approximations (see footnote [18)] of Chapter 18) such that for both of them

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d} + Ch^{-d}(\mu h)^{\frac{q}{2}+1}; \tag{20.4.28}$$

in particular $\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d}$ *as*

$$\mu \le \bar{\mu}^*_{(q)} := Ch^{-\frac{q}{q+2}}. \tag{20.4.29}$$

(ii) Furthermore, the same estimates hold for $\mathsf{R}^{\mathsf{W}}_{(\infty)}$.

Now we can assume instead of (20.3.3) that

$$\bar{\mu}^*_{(q)} \le \mu \le \epsilon(h|\log h|)^{-1}. \tag{20.4.30}$$

In this case we will use the following corollary of (20.4.20):

Proposition 20.4.4. *Let* $q \ge 2$ *and* (20.4.30) *hold and* $Q = Q(hD''')$, $Q = Q(\xi''')$ *be operator with symbol supported in* $\mathcal{Z}_\rho$, $\bar{\rho}^*_1 \le \rho \le \bar{\rho}_1$, $\psi \in \mathscr{C}^\infty_0(B(0,1))$. *Then*

$$\mathsf{R}^{\mathsf{T}}_Q = |\Gamma(Q\tilde{e}) - h^{-1}\int_{-\infty}^0 \Big(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma Q\psi U\Big)\,d\tau| \le Ch^{1-d}\rho^{q-1} \tag{20.4.31}$$

as $Ch|\log h|/\rho^2 \le T \le T^* = \epsilon\rho$ *and*

$$\mathsf{R}^{\mathsf{T}}_{\bar{T},Q} = |\Gamma(Q\tilde{e}) - h^{-1}\int_{-\infty}^0 \Big(F_{t\to h^{-1}\tau}\bar{\chi}_{\bar{T}}(t)\Gamma Q\psi U\Big)\,d\tau| \le Ch^{1-d}\rho^{q-1} + Ch^{-d}\big(\frac{\mu h}{\rho^2}\big)^s\rho^q. \tag{20.4.32}$$

with $\bar{T} = \epsilon\mu^{-1}$.

Note that the right-hand expression of (20.4.31) integrated by $d\rho/\rho$ does not exceed Ch^{1-d}.

Case $q \ge 2$ (Some Calculations)

Now we can apply the technique of the previous subsubsection to calculate the Tauberian expression

$$h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\phi_T(t)\Gamma Q\psi U\Big)\,d\tau \tag{20.4.33}$$

with $\phi=\bar\chi$. Let us consider the difference

$$h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\phi_T(t)\Gamma Q\psi (U-U')\Big)\,d\tau \tag{20.4.34}$$

between (20.4.33) and the same expression for the perturbed operator $A'=A+O(\mu^{-1})$. Then expression (20.4.34) is equal to $I_0+I_1+...+I_{\bar n}$ where I_0 is defined by (20.4.34) with $\phi=\bar\chi$ and T replaced by $T_*=\epsilon\mu^{-1}$ and I_n are defined by (20.4.34) with $\phi=\chi$ and $T=2^n\bar T$, $n=1,...,\bar n$, $\bar n=\lceil T/\bar T\rceil$.

Applying arguments of the previous subsubsection and observing that

(20.4.35) An operator norm of $\Big(e^{ih^{-1}tA}-e^{ih^{-1}tA'}\Big)$ does not exceed $C\mu^{-1}h^{-1}T$ as $|t|\asymp T$,

we conclude that I_n does not exceed expression (20.4.25) multiplied by $C\mu^{-1}h^{-1}T$ as $n\ge 1$, i.e.

$$|h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\chi_T(t)\Gamma Q\psi (U-U')\Big)\,d\tau|\le Ch^{-d}\rho^{q+2}\times\big(\frac{h}{\rho^2T}\big)^s\times\mu^{-1}h^{-1}T. \tag{20.4.36}$$

Recall that with cut-off χ_T we can replace h^{-1} by T^{-1} and χ by $\check\chi$ like in (20.4.24), also $\rho^2\ge\mu h\ge\mu^{-1}$ due to $\mu\ge h^{-\frac12}$ as $q\ge 2$.

After summation with respect to n (i.e. T) we get the same expression I_0 with $T=\bar T$:

$$|h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\big(\bar\chi_T(t)-\bar\chi_{\bar T}(t)\big)\Gamma Q\psi (U-U')\Big)\,d\tau|\le C\mu^{-2}h^{-d-1}\rho^{q+2}\big(\frac{\mu h}{\rho^2}\big)^s. \tag{20.4.37}$$

Note that the right-hand expression in (20.4.37), integrated by $d\rho/\rho$ from $\bar{\rho}_1^*$ to 1, does not exceed its value as $\rho = \bar{\rho}_1^*$ i.e. $C\mu^{-2}h^{-d-1}(\mu h)^{\frac{1}{2}(q+2)} = Ch^{1-d}(\mu h)^{\frac{1}{2}(q-2)}$.

Therefore for $q \ge 2$ and Q supported in the intermediate zone expression (20.4.34) is equal to I_0 modulo $O(h^{1-d})$. However, for I_0 we can apply the standard theory rescaled and replace it by the difference of the Weyl expressions for A and A' (we will do it in Section 20.6). So far we proved

Proposition 20.4.5. *In the framework of Proposition 20.4.4*

$$|\Gamma(Q\psi\tilde{e}) - h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma Q\psi U^0 - F_{t\to h^{-1}\tau}\bar{\chi}_{\bar{T}}(t)\Gamma Q\psi(U-U^0)\Big)\,d\tau| \le Ch^{1-d} \tag{20.4.38}$$

where U^0 is a Schwartz kernel of $e^{ih^{-1}tA^0}$, $A^0 = \mathcal{T}^*\mathcal{A}^0\mathcal{T}$,

$$\mathcal{A}^0 := \mu^2 \sum_{q+1\le j,k\le q+r} \zeta_j^{\dagger\mathsf{w}} a^{jk\mathsf{w}}\zeta_k^{\mathsf{w}} + h^2\sum_{j,k\le q} D_j g^{jk\mathsf{w}} D_k + a_0^{\mathsf{w}} \tag{20.4.39}$$

is the main part of $\mathcal{A}$.

Case $q = 1$

Let us repeat arguments of Subsubsection 20.4.1.1 first as $q = 1$,

$$(h|\log h|)^{-\frac{1}{3}} \le \mu \le \epsilon(h|\log h|)^{-1}, \tag{20.4.40}$$

$$|\xi_1| \asymp \rho \ge \bar{\rho}_1^* = C(\mu h)^{\frac{1}{2}} + C\mu^{-1}. \tag{20.4.41}$$

The first problem is that the factor $(\rho^2 + \mu^{-1})$ in the right-hand expression of (20.4.17) is too large as $\mu \le Ch^{-\frac{1}{2}}$; further, even if there are no unremovable cubic terms, one should take factor $(\rho^2 + \rho^{-1}h|\log h|)$ instead because we need to take $C\rho^{-1}h|\log h|$ scale with respect to x_1. We handled this problem in Chapter 18 as $d = 3$ and the idea from there (may be combined with the idea used in Subsection 20.2.3) is applicable now.

(i) The factor problem is rather simple. Let us assume first that g^{jk}, F_{jk} are constant (and in particular there no unremovable cubic terms).

Then this problem arrises only as $\rho^3 \le h|\log h|$ which implies $\mu \le h^{-\frac{1}{3}}|\log h|^{\frac{2}{3}}$.

Let us consider a γ-covering with $\gamma = \rho^\delta$ and consider separately two types of elements:

(a) Elements with $|\nabla V| \ge \zeta = |\log h|^{-2}$. If $|\partial_{x_1} V| \ge \zeta$ we have that ξ_1-shift for time $T \in [T_*, \rho^{\delta_1}]$, $T_* = \epsilon\mu^{-1}$ is of magnitude no less than ζT and then since logarithmic uncertainty principle is fulfilled

$$\mu^{-1}\zeta \times \varepsilon \asymp \mu^{-2}\zeta \ge Ch|\log h|$$

we conclude that $\chi_T \Gamma Q\psi u$ is negligible; otherwise $|\nabla_\perp V| \ge \zeta$ and similar arguments work for (x'', ξ'')-shifts.

(b) Elements with $|\nabla V| \le \zeta$. We can take on them $\gamma = C(\mu^{-1}h)^{\frac{1}{2}}|\log h|$ which is larger than $C\rho^{-1}h|\log h|$ and replace the term γ by $C(\zeta\gamma + \gamma'|\log h|^{-\sigma})$ which does not exceed $C\mu h$. In this case one can take $T_1 = \epsilon\rho\zeta^{-1}$.

(ii) Similar analysis combined with ideas of Subsection 20.2.3 works in the general case as long as there are no unremovable cubic terms. We leave all the details to the reader.

(iii) If there are unremovable terms of this type one can replace decomposition (20.4.8) by a similar decomposition with Υ_α orthonormal eigenfunctions of an auxiliary operator

$$\boldsymbol{a} = \mathcal{A}_0 + \mu^{-1} \sum_{j,k,m \le 2r} \beta_{jkm} L_j L_k L_m + C_0 \mu^{-2} \mathcal{A}_0^2 \tag{20.4.42}$$

with

$$\mathcal{A}_0 = \sum_{1 \le j \le r} \bar{f}_j \bigl(\mu^2 h^2 D_{q+j}^2 + x_{q+j}^2\bigr), \tag{20.4.43}$$

where $L_{2j-1} = \mu h D_{q+j}$, $L_{2j} = x_{q+j}$ and Υ_β are replaced by their complex conjugates). We added the last term in (20.4.42) to make operator non-negative and self-adjoint; this term is $O(\mu^{-2})$ on U and thus it is included in the approximation error estimate $O(\mu^{-l}|\log\mu|^{-\sigma})$ anyway. We got operator (20.4.43) by rescaling $x' \mapsto \mu^{-1}x'$, $D_j \mapsto \mu D_j$.

For an eigenvalue counting function $\boldsymbol{n}(\tau, \hbar)$ of this operator $\boldsymbol{a}$ in $\mathscr{L}^2(\mathbb{R}^r)$ the standard semiclassical eigenvalue asymptotics holds

$$\boldsymbol{n}(\tau, \hbar) = \boldsymbol{n}^{\mathsf{W}}(\tau, \hbar) + O(\hbar^{1-r}), \qquad \boldsymbol{n}^{\mathsf{W}} \asymp \hbar^{-r} \qquad \text{as } \hbar \to +0 \tag{20.4.44}$$

and in particular

$$\boldsymbol{n}(\tau + \hbar, \hbar) - \boldsymbol{n}(\tau, \hbar) = O(\hbar^{1-r}) \tag{20.4.45}$$

where $\boldsymbol{n}^{\mathsf{W}}$ means Weyl expression for $\boldsymbol{a}$ and we plug $\hbar = \mu h$.

Then the same modifications hold for estimates (20.4.17) and (20.4.19) as well and then the standard Tauberian arguments improve estimate (20.4.20) to

$$\mathsf{R}^{\mathsf{T}}_{Q} \le C\Big(h^{-d}\mu^{-1}\rho^2 \times \big(\frac{\mu h}{\rho^2}\big)^{s} + Ch^{1-d}\Big)\,|\log \rho|^{-2} \tag{20.4.46}$$

as $\mu \le h^{\delta-1}$ because $T^* \ge \epsilon_1\rho|\log\rho|^2$ now.

Integrating the right-hand expression over $d\rho/\rho$ from $\bar{\rho}_1^*$ to $\bar{\rho}_1$ we get $O(h^{1-d})$ then as long as $\mu \le h^{\delta-1}$. However for $h^{\delta-1} \le \mu \le \epsilon(h\log h|)^{-1}$ we can take only $T^* = \epsilon_1\rho$ and recover estimate $O(h^{1-d}|\log h|)$.

On the other hand, under the microhyperbolicity assumption for $\mu \le \epsilon(h|\log h|)^{-1}$ we can take $T^* = \epsilon_1$ and recover estimate $O(h^{1-d})$ again.

Thus we have proven

Proposition 20.4.6. *Let $q = 1$, $(l,\sigma) = (1,2)$. Then*

(i) Estimate (20.4.28) *holds. In particular for $\mu \le Ch^{-\frac{1}{3}}$ sharp remainder estimate $\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d}$ holds.*

(ii) In the framework of Proposition 20.4.2 $\mathsf{R}^{\mathsf{T}}_{1\,Q}$ defined by (20.4.22) *does not exceed $Ch^{1-d}|\log h|$ where again $T_n = \epsilon\rho_n$.*

(iii) If <u>either</u> $\mu \le h^{\delta-1}$ <u>or</u> f_j have constant multiplicities <u>or</u> the microhyperbolicity condition is fulfilled at level 0, then $\mathsf{R}^{\mathsf{T}}_{1\,Q}$ defined by (20.4.22) *does not exceed Ch^{1-d} where $T_n = \rho_n|\log\rho_n|^2$.*

(iv) Furthermore, if Q is supported in $\{|\xi'''| \ge \rho\}$ and equal 1 in $\{2|\xi'''| \ge \rho\}$ then $\mathsf{R}^{\mathsf{T}}_{\bar{T}\,Q}$ defined by (20.4.32) *does not exceed $Ch^{1-d} + Ch^{-d}\rho\big(\mu h/\rho^2\big)^{s}$.*

Case $q = 1$ (Calculations)

Let us apply for $q = 1$ the same arguments as we did for $q \ge 2$ in Subsubsection 20.4.1.3. However as perturbation is $O(\mu^{-1})$ it makes sense only for $\mu \ge h^{-\frac{1}{2}}$ resulting in $Ch^{1-d}(\mu h)^{-\frac{1}{2}} = C\mu^{-\frac{1}{2}}h^{\frac{1}{2}-d}$ which is less than the weak magnetic field estimate only in this case.

On the other hand, as perturbation is $O(\mu^{-1})$ these arguments make sense as $\mu \ge h^{-\frac{1}{3}}$ resulting in $C\mu^{-\frac{3}{2}}h^{\frac{1}{2}-d} = O(h^{1-d}$.

Leaving easy details to the reader we arrive to the following

Proposition 20.4.7. [10] *In the framework of Proposition 20.4.6*

(i) Estimate (20.4.38) *holds as either there are no* 3*-rd order resonances or we include in unperturbed operator* $\mathcal{A}^0$ *non-removable* $O(\mu^{-1})$ *terms like in* (20.4.42)*.*

(ii) If there are non-removable $O(\mu^{-1})$ *terms and we do not include them into* $\mathcal{A}^0$ *then as* $\mu \ge h^{-\frac{1}{2}}$ *the left-hand expression of* (20.4.38) *does not exceed* $C\mu^{-\frac{1}{2}}h^{\frac{1}{2}-d}$*.*

We will return to the intermediate zone (for $q = 1$) later under microhyperbolicity or non-degeneracy assumptions.

20.4.2 Inner Zone

Now let us consider an *inner zone*

$$\Omega_{\mathsf{inn}} = \{|\xi'''| \le \bar{\rho}_1^* = (\mu h)^{\frac{1}{2}}\}. \tag{20.4.47}$$

Case $q \ge 2$

Assume that $q \ge 2$ and (20.4.30) holds. Then one can see easily that

(20.4.48) The perturbation of the magnitude $O(\mu^{-1})$ in $\Omega_{\mathsf{inn}} \cup \Omega_{\mathsf{intm}}$ leads to an error $O(\mu^{-1}\bar{\rho}_1^{*q}h^{-d}) = O(h^{1-d})$ in $h^{-d}\mathcal{N}_Q^{\mathsf{MW}}$.

Therefore for $q \ge 2$ instead of $\mathcal{A}$ one can consider in $\Omega_{\mathsf{inn}} \cup \Omega_{\mathsf{intm}}$ a reduced operator $\mathcal{A}^0$ with the coefficients

$$a^{jk\,\mathsf{w}} = a^{jk}(x'', x''', \mu^{-1}hD''), \qquad g^{jk\,\mathsf{w}} = g^{jk}(x'', x''', \mu^{-1}hD''),$$
$$a_0^{\mathsf{w}} = a_0(x'', x''', \mu^{-1}hD''), \qquad \zeta_j^{\mathsf{w}} = \mu^{-1}hD_j + ix_j$$

where $a^{jk} = 0$ unless $\exists \mathfrak{m} : j, k \in \mathfrak{m}$; this substitution leads to an error $O(h^{1-d})$ in $\mathsf{R}_Q^{\mathsf{T}}$. However we provided different arguments for inner zone Ω_{inn} and intermediate zone Ω_{intm} to justify this replacement.

Moreover, if f_j had constant multiplicities we would have even

$$\mathcal{A}^0 = \mu^2 \sum_{j\le r} f_j\big(h^2D_j^2 + \mu^2x_j^2 - \mu h\big) + h^2 \sum_{j,k\ge 2r+1} D_j^{\mathsf{w}} g^{jk\,\mathsf{w}} D_k^{\mathsf{w}} + a_0^{\mathsf{w}} \tag{20.4.49}$$

[10] Cf. Proposition 20.4.5

and decomposing $\mathcal{T}U$ into (20.4.8) series we would arrive to the family of operators

$$\mathcal{A}_\alpha := h^2 \sum_{j,k\ge 2r+1} D_j g^{jk\,\mathsf{w}} D_k + W_\alpha^{\mathsf{w}}, \qquad W_\alpha = a_0 + \sum_{j\le r} (2\alpha_j+1)\mu h f_j \tag{20.4.50}$$

with $\alpha \in \mathbb{Z}^{+r}$ and we would be able to apply without any significant modifications analysis of Sections 4.6 and 5.4.

However, we need to consider more general operator (20.4.39). Consider operator $Q = Q^{\mathsf{w}}$ with symbol supported in $\mathcal{Z}_\rho$ with $\rho \le \bar\rho_1^*$.

Repeating arguments of the proof of Proposition 20.4.1 we arrive to estimate (20.4.16) where $Q' = \mathcal{T} Q \psi' \mathcal{T}^*$, ψ' is an element of γ-admissible partition in (x'', x''', ξ''), $\gamma \ge \mu h$.

Restoring to Q we arrive to the estimate

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma Q\psi U| \le Ch^{-d} T\rho^q\gamma\Big(\big(\frac{h}{\rho^2 T}\big)^s + \big(\frac{h}{\rho\gamma}\big)^s + \big(\frac{\mu^{-1}h}{\gamma^2}\big)^s\Big). \tag{20.4.51}$$

As $q \ge 2$ and $\mu \ge \bar\mu^*_{(q)}$ we can take $\gamma \ge \mu h$ and $\rho \ge \bar\rho_0 \ge \mu^{-\frac12}$; remaining *inner core*

$$\Omega^0_{\mathsf{inn}} = \{|\xi'''| \le \bar\rho_0\} \tag{20.4.52}$$

will be considered separately. Then both $h/(\rho\gamma) = 1/(\mu\rho)$ and $\mu^{-1}h/\gamma^2 = 1/(\mu^3 h)$ do not exceed h^δ and we arrive to the estimate

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma Q\psi U| \le C\mu h^{1-d} T\rho^q\big(\frac{h}{\rho^2 T}\big)^s \tag{20.4.53}$$

which in turn implies

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma Q\psi U| \le C\mu h^{1-d}\rho^q \times \frac{h}{\rho^2} = C\mu h^{2-d}\rho^{q-2}, \tag{20.4.54}$$

$$\text{as} \quad T_* := \epsilon\frac{h}{\rho^2} \le T \le T^* = \epsilon\rho.$$

Therefore, the contribution of zone $\mathcal{Z}_\rho$ to the remainder is $O(\mu h^{2-d}\rho^{q-3})$; integrated over $d\rho/\rho$ it results in $O(\mu h^{2-d}) = O(h^{1-d})$ for $q \ge 4$, $O(\mu h^{2-d}|\log h|) = O(h^{1-d})$ for $q = 3$ and $O(\mu h^{2-d}\bar\rho_0^{-1})$ for $q = 2$.

On the other hand, contribution of the inner core Ω^0_{inn} to the remainder is $O(\mu h^{1-d}\bar\rho_0^q)$.

Therefore we arrive to the remainder estimate $O\big(h^{1-d}+\mu h^{1-d}\bar\rho_0^q\big)$ as $q\ge 3$ and $O\big(\mu h^{1-d}(\bar\rho_0^2+h\bar\rho_0^{-1})\big)$ as $q=2$ and we need to pick-up

$$\bar\rho_0=\max\big(h^{\frac13},\mu^{-\frac12}\big), \tag{20.4.55}$$

then finally arriving to the remainder estimate $O(h^{1-d})$ as $q\ge 3$ and $O\big(h^{1-d}+\mu h^{\frac53-d}\big)$ as $q=2$ [11].

Thus we have proven

Proposition 20.4.8. *Let $q\ge 2$, condition* (20.4.30) *be fulfilled and $Q=Q(hD''')$ be operator with the symbol $Q(\xi''')$ supported in $\{|\xi'''|\le C\bar\rho_1\}$, $\psi\in\mathscr{C}_0^\infty(B(0,1))$. Then*

(i) Estimate

$$\mathsf{R}_Q^{\mathsf{T}}\le Ch^{1-d} \tag{20.4.56}$$

holds provided <u>either</u> $q\ge 3$ <u>or</u> $q=2$ and $\mu\le h^{-\frac23}$.

(ii) Estimate

$$\mathsf{R}_Q^{\mathsf{T}}\le C\mu h^{\frac23-d} \tag{20.4.57}$$

holds as $q=2$ and $\mu\ge h^{-\frac23}$

where in both cases $\bar\chi\in\mathscr{C}_0^\infty([-1,1])$, $\bar\chi=1$ on $[-\frac12,\frac12]$, $Q=\sum_{0\le n\le\bar n}Q_n$, Q_n are operators with symbols supported in $\{\frac12\rho_n\le|\xi'''|\le 2\rho_n\}$ as $n=1,\dots,\bar n$, and in $\{|\xi'''|\le 2\rho_0\}$ as $j=0$, $\rho_n=2^n\rho_0^*$, $\bar n=\lfloor\log\bar\rho_1/\rho_0^*\rfloor+1$, $T_n=\epsilon\rho_n$ ($n\ge 1$), $T_0=\epsilon\mu^{-1}$.

Further, both statements remain true with U replaced by U^0 which is the Schwartz kernel of $e^{ih^{-1}tA_0}$.

Case $q=1$

In contrast to the previous case $q\ge 2$ skipping $O(\mu^{-1})$ terms is not now generally justified. Because of this we need to modify now arguments of Subsubsection 20.4.2.1, in the same manner as we modified arguments of Subsubsections 20.4.1.2–20.4.1.3 in Subsubsections 20.4.1.4–20.4.1.4.

[11] Similarly we would get $O\big(h^{1-d}+\mu h^{\frac43-d}\big)$ as $q=1$ but there is a lot of other things to consider.

So let us proceed without skipping such terms first, assuming that

$$Ch^{-\frac{1}{3}} \le \mu \le \epsilon(h|\log h|)^{-1}, \tag{20.4.58}$$

$$C\mu^{-1} \le |\xi_1| \asymp \rho \le \bar{\rho}_1^* = (\mu h)^{\frac{1}{2}}. \tag{20.4.59}$$

Starting from (20.4.51) with $\gamma = \mu h$ due to the above arguments we arrive to

$$|F_{t\to h^{-1}\tau}\bigl(\bar{\chi}_T(t) - \bar{\chi}_{T_*}(t)\bigr)\Gamma Q\psi U| \le C\mu h^{1-d}\rho\Bigl(T_*\bigl(\frac{h}{\rho^2 T_*}\bigr)^s + T\bigl(\frac{h}{\rho\gamma}\bigr)^s + T\bigl(\frac{\mu^{-1}h}{\gamma^2}\bigr)^s\Bigr) \tag{20.4.60}$$

if $T_* = h/\rho^2 \le T \le T^* = \epsilon\rho$ and thus

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma Q\psi U| \le C\mu h^{1-d}\rho\Bigl(\frac{h}{\rho^2} + T\bigl(\frac{h}{\rho\gamma}\bigr)^s + T\bigl(\frac{\mu^{-1}h}{\gamma^2}\bigr)^s\Bigr). \tag{20.4.61}$$

Therefore due to the standard Tauberian arguments

$$|\Gamma(Q\psi\tilde{e}) - h^{-1}\int_{-\infty}^{0}\Bigl(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma Q\psi U\Bigr)\,d\tau| \le C\mu T^{-1}h^{1-d}\rho\Bigl(\frac{h}{\rho^2} + T\bigl(\frac{h}{\rho\gamma}\bigr)^s + T\bigl(\frac{\mu^{-1}h}{\gamma^2}\bigr)^s\Bigr) \asymp C\mu h^{1-d}\rho\Bigl(h\rho^{-3} + \bigl(\frac{h}{\rho\gamma}\bigr)^s + \bigl(\frac{\mu^{-1}h}{\gamma^2}\bigr)^s\Bigr). \tag{20.4.62}$$

After integration the right-hand expression over $d\rho/\rho$ with ρ ranging from $\bar{\rho}_0$ to $\bar{\rho}_1^*$,

$$\bar{\rho}_0 = Ch^{\frac{1}{3}} \ge C\mu^{-1} \qquad \text{as} \quad \mu \ge h^{-\frac{1}{3}}, \tag{20.4.63}$$

we get

$$C\mu h^{1-d}\rho\Bigl(h\rho^{-3} + \bigl(\frac{h}{\rho\gamma}\bigr)^s\Bigr)\Big|_{\rho=\bar{\rho}_0} + C\mu h^{1-d}\rho\bigl(\frac{\mu^{-1}h}{\gamma^2}\bigr)^s\Big|_{\rho=\bar{\rho}_1^*} \tag{20.4.64}$$

where all terms except $C\mu h^{2-d}\bar{\rho}_0^{-2} = C\mu h^{\frac{4}{3}-d}$ do not exceed Ch^{1-d}.

Also, contribution of the inner core $\Omega^0_{\mathsf{inn}} = \{|\xi_1| \le \bar{\rho}_0\}$ does not exceed $C\mu h^{1-d}\bar{\rho}_0 = C\mu h^{\frac{4}{3}-d}$ as well.

Therefore, the final estimate is $Ch^{1-d} + C\mu h^{\frac{4}{3}-d}$. Thus we arrive to

Proposition 20.4.9. *Let $q = 1$, condition* (20.4.30) *be fulfilled and $Q = Q(hD''')$ be operator with the symbol $Q(\xi''')$ supported in $\{|\xi'''| \le C\bar\rho_1\}$, $\psi \in \mathscr{C}_0^\infty(B(0,1))$. Then*

$$\mathsf{R}^{\mathsf{T}}_{1\,Q} \le Ch^{1-d} + C\mu h^{\frac{4}{3}-d} \tag{20.4.65}$$

where $\mathsf{R}^{\mathsf{T}}_{1\,Q}$ is defined by (20.4.22) *with $Q = \sum_{0\le n\le \bar n} Q_n(hD''')$, Q_n are operators with symbols supported in $\{\frac12\rho_n \le |\xi'''| \le 2\rho_n\}$ as $n = 1, \dots, \bar n$, and in $\{|\xi'''| \le 2\rho_0\}$ as $n = 0$, $\rho_n = 2^n\bar\rho_0$, $\bar n = \lfloor \log \bar\rho_1/\bar\rho_0\rfloor + 1$, $T_n = \epsilon\rho_n$ ($n \ge 1$), $T_0 = \epsilon\mu^{-1}$.*

20.4.3 Cases $q = 1, 2$ Revised

We repeat arguments of Subsection 20.4.2 as $q = 1, 2$, but we will use certain additional assumptions to improve remainder estimates.

Microhyperbolicity Assumption

Let us assume first that the microhyperbolicity assumption (see Definition 20.1.2) is fulfilled. Note that (20.4.11) implies that

$$\begin{aligned}|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma Q'\psi U| \le C\mu^r h^{r-d} T\rho^q \times \\ \Big(\big(\frac{h}{\rho^2 T}\big)^s + \big(\frac{h}{\rho\gamma}\big)^s + \big(\frac{\mu^{-1}h}{\gamma^2}\big)^s\Big)\times \\ \gamma^s \int_{B(z,2\gamma)} \|\operatorname{tr}\big(\mathcal{A}(x', x''', \xi'') + i\gamma\big)^{-s}\|\, dx''\, dx'''\, d\xi''\end{aligned} \tag{20.4.66}$$

where $B(z,\gamma) \supset \operatorname{supp}(Q')$, $\mathcal{A}$ is an operator reduced and considered as an operator in $\mathscr{L}^2(\mathbb{R}^r, \mathbb{C})$ and $\|.\|$ is an operator norm there. Then the same estimate (with an integral over $B(0,1)$) holds as Q' is replaced by Q.

Further, one can prove easily that

(20.4.67) Due to the microhyperbolicity assumption the last integral in (20.4.66) is of magnitude $\gamma(\mu h)^{-r}$ as $\gamma \ge \rho^2$; even condition $\gamma \ge \mu^{-1}$ is not needed anymore if we consider operator (20.4.41) from the very beginning[12)].

[12)] If $q \ge 2$ we can skip all $O(\mu^{-1})$ terms. If $q = 1$ we preserve all the terms which do not contain factor hD_1, we skip all such terms containing $(hD_1)^2$ because this leads to the approximation error $\rho^2\mu^{-1}$ and then we remove as in Chapter 18 all such terms containing hD_1 exactly in power 1.

Then (20.4.66) becomes

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma Q\psi U| \le Ch^{-d}\gamma T\rho^q \times \Big(\big(\frac{h}{\rho^2 T}\big)^s + \big(\frac{h}{\rho\gamma}\big)^s + \big(\frac{\mu^{-1}h}{\gamma^2}\big)^s\Big) \tag{20.4.68}$$

as $T_* = h/\rho^2 \le T \le T^* = \epsilon\rho$ and picking up $\gamma = \rho^2$, $\rho \ge \bar\rho_0 = h^{\frac{1}{3}-\delta} \ge \mu^{-1}$ we rewrite the right-hand expression here as $Ch^{-d}T\rho^{q+2}\big(\frac{h}{\rho^2 T}\big)^s$.

Remark 20.4.10. Note that we changed here (under the microhyperbolicity assumption) definition of $\bar\rho_0$ and the corresponding notion of the inner core Ω^0_{inn}.

Then (20.4.68) implies that

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma Q\psi U| \le Ch^{1-d}\rho^q \tag{20.4.69}$$

as long as $T \le \epsilon\rho$, $\rho \ge \bar\rho_0$.

However, due to the microhyperbolicity assumption $F_{t\to h^{-1}\tau}\chi_T(t)\Gamma Q\psi U$ is negligible as long $Ch|\log h|\varepsilon^{-1} \le T \le \epsilon$ and $T = \epsilon\rho$ satisfies this inequality as long as

$$\rho \ge C\bar\rho_0, \quad \varepsilon \ge Ch\rho^{-1}|\log h|. \tag{20.4.70}$$

Therefore, (20.4.70) yields that (20.4.69) holds for $T \le \epsilon$.

Then we can apply the standard Tauberian arguments resulting in the estimate with the right-hand expression $Ch^{1-d}\rho^q$. This last estimate holds for $\rho \le \bar\rho_1$ as well. Integration of this expression over $d\rho/\rho$ implies the remainder estimate $O(h^{1-d})$.

Consider now the contribution of the inner core $\Omega^0_{\mathsf{inn}} = \{|\xi'''| \le \bar\rho_0\}$. In this zone we get estimate

$$|F_{t\to h^{-d}\tau}\bar\chi_T(t)\Gamma Q\psi U| \le Ch^{-d}\bar\rho_0^{q+3}, \tag{20.4.71}$$

as $T = \rho$ and $\rho := \bar\rho_0$ there; then the standard microhyperbolicity arguments push it to $T = \epsilon$ under condition (20.4.70) and then the standard Tauberian arguments imply that the contribution of this zone to the remainder does not exceed $Ch^{-d}\bar\rho_0^{q+3}$ which is $O(h^{1-d})$ as $q \ge 1$ and δ small enough.

Thus under microhyperbolicity assumption the Tauberian remainder estimate is $O(h^{1-d})$ even as $q = 1, 2$ but we need to look at an approximation error; one can prove easily that for smallest $\varepsilon := C\rho^{-1}h|\log h|$ satisfying (20.4.70) it is also $O(h^{1-d})$.

Thus we arrive to

Proposition 20.4.11. *Under the microhyperbolicity assumption* $\mathsf{R}^{\mathsf{T}}_Q$ *does not exceed* Ch^{1-d} *where* $\mathsf{R}^{\mathsf{T}}_Q$ *is given by* (20.4.20) *with* $T = \epsilon$.

Special Case of Constant g^{jk}, F_{jk}

Recall that if F_{jk}, g^{jk} are constant then there are no non-removable $O(\mu^{-1})$ terms and the microhyperbolicity assumption is (20.2.41).

Using arguments similar to those used in Subsection 19.4.7 one can prove easily that one can replace it by non-degeneracy condition (20.2.42):

Problem 20.4.12. Let conditions (20.1.1), (20.1.5), (20.1.7), (20.1.8), $(20.1.9)_3$, (20.1.11), and (20.2.31) be fulfilled, and $\mu \le \epsilon(h|\log h|)^{-1}$.

Further, let g^{jk}, F_{jk} be constant, $(I, \sigma) = (2, 0)$ and assumption (20.2.42) be fulfilled.

Prove that $\mathsf{R}^{\mathsf{T}} = O(h^{1-d})$.

Case of f_j Having Constant Multiplicities

Assume that f_j have constant multiplicities. Then in both inner and intermediate zones after reduction and skipping $O(\mu^{-1})$ terms which are due to the 3-rd order resonances and all smaller terms[13)] instead of "matrix" operator we get a family of "scalar" operators

$$A_\alpha(x'', x''', \mu^{-1}hD'', hD''') = \sum_{q+1\le j,k\le q+r} (hD_j)g^{jk}(x'', x''', \mu^{-1}hD''')(hD_k) + V_\alpha(x'', x''', \mu^{-1}hD''') \tag{20.4.72}$$

and we can study them separately. Let us apply the standard rescaling procedure: for each index α we introduce functions $\gamma = \gamma_\alpha$ and $\varrho = \varrho_\alpha$ by

$$\gamma = \inf\bigl\{t : \vartheta(t) \ge \rho^2 + |V_\alpha(x'', x''', \xi'')|, \vartheta(t)t^{-1} \ge |\nabla V_\alpha(x'', x''', \xi'')|\bigr\} + \bar\gamma, \tag{20.4.73}$$

[13)] The case of $q = 1$ where there are such terms will be considered in details in the next Subsection 20.4.3.4

$$\varrho = \vartheta(\gamma)^{\frac{1}{2}}, \tag{20.4.74}$$

with

$$\bar{\varrho} = \vartheta(\bar{\gamma})^{\frac{1}{2}}, \qquad \bar{\varrho}\bar{\gamma} = Ch \implies \bar{\varrho} = h^{\frac{l}{l+2}} |\log \mu|^{-\frac{l\sigma}{2(l+2)}}, \quad \bar{\gamma} = h^{\frac{2}{l+2}} |\log h|^{\frac{\sigma}{l+2}} \tag{20.4.75}$$

where we assume that

$$(1,1) \preceq (l,\sigma) \preceq (2,0). \tag{20.4.76}$$

Note that $\gamma^2 \ge C\mu^{-1}h|\log h|$.

(i) Consider first elements with $\gamma \ge C\bar{\gamma}$. Calculating contribution of each element of $(\gamma;\rho)$-partition with respect to $(x'',x''',\xi'';\xi''')$ we can apply elliptic arguments unless $\epsilon\varrho \le \rho \le C\varrho$; in the latter case we can apply microhyperbolic arguments; then the contribution of each element to the remainder estimate does not exceed $C\mu^r h^{r-d+1}\varrho^{q-1}\gamma^{d-1}$ in the latter case and

$$C\mu^r h^{r-d+1}\rho^{q-1}\gamma^{d-1} \times \Big(\frac{h}{(\rho+\varrho)\gamma}\Big)^s$$

in the former one.

Then the contribution of each such element $\mathcal{W}$ of γ-partition with respect to (x'',x''',ξ') [14] to the remainder also does not exceed $C\mu^r h^{r-d+1}\varrho^{q-1}\gamma^{d-1}$.

Then the total contribution of such elements to the Tauberian remainder for given α does not exceed

$$C\int \mu^r h^{r-d+1}\varrho_\alpha^{q-1}\gamma_\alpha^{-1}\,dz$$

with $z=(x'',x''',\xi'')$ and after summation over all indices we get

$$C\sum_{\alpha\in\mathbb{Z}^{+r}} \mu^r h^{r-d+1}\int \varrho_\alpha^{q-1}\gamma_\alpha^{-1}\,dz. \tag{20.4.77}$$

[14] More precisely, of $\mathcal{W}\times\{|\xi'''|\le C\rho_1^*\}$.

(ii) On the other hand, contribution of each element of γ-partition with $\gamma \le C\bar\gamma$ does not exceed $C\mu^r h^{r-d}\gamma^d \bar\varrho^q$ and their total contribution does not exceed (20.4.77) as well.

Since for given (z,ρ)

$$\#\{\alpha \in \mathbb{Z}^{+r} : \epsilon\rho^2 \le \varrho_\alpha^2 \le c\rho^2\} \le C(\mu h)^{-r}(\rho^2+\mu h), \tag{20.4.78}$$

expression (20.4.77) does not exceed

$$C\mu h^{2-d}\int \varrho_1^{q-1}\gamma_1^{-1}\,dz + Ch^{1-d}\int \rho^q \gamma(\rho)^{-1}\,d\rho \tag{20.4.79}$$

where $\gamma_1 = \min_\alpha \gamma_\alpha$ and ϱ_1 corresponds to it, $\gamma(\rho)$ is defined from equation $\vartheta(\gamma)=\rho$.

One can see easily that the second term in (20.4.79) does not exceed Ch^{1-d}, while the first term does not exceed

$$C\mu h^{2-d}\int \gamma^{\frac12(q-1)l-2}|\log\gamma|^{-\frac12(q-1)\sigma}\,d\gamma$$

which in turn does not exceed $C\mu h^{1-d}\bar\varrho^q$ unless $q=2$, $l=2$. In the latter case $\sigma\le 0$ and the first term does not exceed $C\mu h^{2-d}|\log h|^{-\sigma}$. In both cases we arrive to the remainder estimate

$$\mathsf{R}^{\mathsf{T}}_{1\,Q} \le Ch^{1-d} + C\mu h^{1-d+\frac{ql}{l+2}}|\log\mu|^{-\frac{q\sigma}{(l+2)}} \tag{20.4.80}$$

which is exactly remainder estimate (20.6.89) below. Recall that it is $C\mu h^{2-d}$ as $(l,\sigma)=(2,0)$. Here $T=\gamma/\varrho$.

Thus we have proven

Proposition 20.4.13. *Let $q=1,2$, $\mu \ge \bar\mu^*_{(q)}$, f_j have constant multiplicities and $\mathcal{A}$ contain no unremovable cubic terms*[15]*. Let $Q=Q(hD''')$, $Q=Q(\xi''')$ be operator with the symbol supported in $\{|\xi'''|\le C\bar\rho_1\}$, $\psi\in \mathscr{C}_0^\infty(B(0,1))$.*

Then estimate (20.4.80) *holds with $\mathsf{R}^{\mathsf{T}}_{1\,Q}$ defined by* (20.4.22) *with $Q=\sum_{0\le n\le \bar n} Q_n(hD''')$, where Q_n are operators with symbols supported in $\{\frac12\gamma_n\le\gamma\le 2\gamma_n\}$ as $n=1,\dots,\bar n$, and in $\{\gamma\le 2\bar\gamma\}$ as $n=0$, $\bar n = \lfloor \log\bar\gamma\rfloor+1$, $\gamma_n = 2^{-n}$, $T_n = C\gamma_n^{1-\frac{l}{2}}|\log\gamma_n|^{-\frac{\sigma}{2}}$ $(n\ge 1)$, $T_0=\epsilon\mu^{-1}$.*

[15] May be we just dropped them in the interior zone causing an error $O(h^{1-d})$ if either $q=2$ or $q=1$ and each of these terms contained factor $\mu^{-1}hD_1$ and an error $O(\mu^{\frac12}h^{1-d})$ in the general case as $q=1$.

On the other hand, if $q = 1$ skipping unremovable cubic terms in $\mathcal{A}$ comes with a price of $O(\mu^{\frac12} h^{1-d} + \mu^{-\frac12} h^{\frac12-d})$ error. If we would prefer not to remove them, we cannot treat separate equations.

Case of Constant f_j

However we can do as well as before assuming that

$$f_j = \mathsf{const}, \quad j = 1, \dots, r \tag{20.4.81}$$

which by no means excludes variable g^{jk}, F_{jk} or unremovable cubic terms.

Really, under assumption (20.4.81) we can introduce

$$\gamma = \mathsf{inf}\big\{ t : \vartheta(t) \ge \rho^2, \vartheta(t) t^{-1} \ge |\nabla V(x'', x''', \xi'')| \big\} + \bar{\gamma}, \tag{20.4.82}$$

and ϱ by (20.4.74) and $\bar{\varrho}$, $\bar{\gamma}$ by (20.4.75). Then we can apply our standard arguments as long as

$$\rho^2 + |\nabla V| \ge C\mu^{-1} \tag{20.4.83}$$

which will be fulfilled automatically for $\gamma \ge C\bar{\gamma}$ as long as

$$\bar{\rho} \ge C\mu^{-\frac12} \iff \mu \ge Ch^{-\frac{2l}{l+2}} |\log h|^{\frac{2\sigma}{l+2}} \tag{20.4.84}$$

with the right-hand expression exceeding $h^{-\frac12}$ for sure.

Therefore, under condition (20.4.84) estimate (20.4.80) holds. On the other hand, if condition (20.4.84) is violated, we need also consider elements with violated condition (20.4.83), in particular, with $|\nabla V| \le C\mu^{-1}$ and

$$\gamma \le \hat{\gamma} = C\mu^{-\frac1l} |\log h|^{\frac{\sigma}{l}}, \tag{20.4.85}$$

Instead of γ partition we consider $\mathsf{max}(\gamma, \hat{\gamma})$-partition and on elements with $\gamma = \hat{\gamma}$ let us introduce

$$\gamma' = \hat{\gamma}\mu\rho^2 + \bar{\gamma}', \qquad \varrho' = \rho + \bar{\rho}', \tag{20.4.86}$$

$$\bar{\gamma}' = \mu^{\frac13} \hat{\gamma}^{\frac13} h^{\frac23} = h^{\frac23} \mu^{\frac{l-1}{3l}} |\log h|^{\frac{\sigma}{3l}}, \qquad \bar{\rho}' = \mu^{-\frac13} \hat{\gamma}^{-\frac13} h^{\frac13} \tag{20.4.87}$$

Then we arrive to the remainder estimate $Ch^{1-d} + C\mu h^{2-d} \bar{\gamma}'^{-1}$:

Proposition 20.4.14. *Let $q = 1$, $\mu \ge \bar{\mu}^*_{(1)}$ and condition* (20.4.81) *be fulfilled. Then*

(i) Under condition (20.4.84) *estimate* (20.4.80) *holds.*

(ii) If condition (20.4.84) *is violated then estimate*

$$\mathsf{R}^{\mathsf{T}}_{1\,Q} \le Ch^{1-d} + C\mu h^{1-d+\frac{l}{l+2}}|\log\mu|^{-\frac{\sigma}{(l+2)}} + C\mu^{\frac{2l+1}{3l}}h^{\frac{4}{3}-d}|\log\mu|^{-\frac{\sigma}{3l}} \tag{20.4.88}$$

holds.

Remark 20.4.15. (i) Condition (20.4.84) means exactly that the third term in the right-hand expression of (20.4.88) does not exceed the second one.

(ii) If condition (20.4.81) is violated one needs to take in account that as the main part is of the form $\xi_1^2 + \sum_{1\le j\le r} f_j^{\mathsf{w}}(\mu^2 x_{1+j}^2 + h^2 D_{1+j}^2) + V^{\mathsf{w}}$ with $f_j = f_j(x'', x_1, \xi'')$, $V = V(x'', x_1, \xi'')$, its gradient with respect to x'', x_d, ξ'' depends on the localization with respect to $\bigl\{(\mu^2 x_{1+j}^2 + h^2\xi_{1+j}^2), j = 1, \dots, r\bigr\}$ and there is at least one obstacle to it: due to the logarithmic uncertainty principle these quantities are defined with $C\mu h|\log h|$ precision.

Therefore one must take $\hat\gamma^{l-1}|\log\hat\gamma|^{-\sigma}| \ge C\mu h|\log h|$ in the arguments leading to Proposition 20.4.14 which deteriorates estimate (20.4.80). It does not look worth needed efforts.

(iii) Another approach would be to drop unremovable cubic terms as $\rho \ge \hat\varrho$ paying the price of $O(h^{2-d}\hat\varrho^{-1})$ for this and apply the same arguments as above. Then the remainder estimate is $C\mu h^{\frac{4}{3}-d}\hat\varrho^{\frac{2}{3}}\hat\gamma^{\frac{1}{3}}$ where $\hat\gamma = \hat\varrho^{\frac{2}{l}}|\log h|^{\frac{\sigma}{l}}$ and one should optimize the answer with respect to $\hat\varrho$ which is not $\mu^{-\frac{1}{2}}$ anymore.

We leave to the reader the following generalization of Problem 20.4.12:

Problem 20.4.16. Under assumption (20.4.81) (and, in particular, in the case of constant g^{jk}, F_{jk}) prove remainder estimate $\mathsf{R}^{\mathsf{T}} = O(h^{1-d})$ under non-degeneracy assumption (20.2.42).

20.4.4 Improved Remainder Estimates

There are peculiar cases when without microhyperbolicity or non-degeneracy assumptions one can improve general remainder estimate $\mathsf{R}^{\mathsf{T}}_{1\,Q} \le Ch^{1-d} + C\mu h^{1+\frac{q}{3}-d}$ or estimates (20.4.80), (20.4.88). It happens only for $r \ge 2$.

Note that the factor estimating the number of indices one needs to take in account is actually $\bigl(\boldsymbol{n}(\tau,\mu h) - \boldsymbol{n}(\tau - C\rho^2, \mu h)\bigr)$ where $\boldsymbol{n}(\tau,\mu h)$ is an eigenvalue counting function for operator $\boldsymbol{a}$ in $\mathscr{L}^2(\mathbb{R}^r,\mathbb{C})$ calculated at any point (x'', x''', ξ'') at $\mathsf{supp}(Q)$ while so far we used estimate $C\bigl(\rho^2 + \mu h\bigr)(\mu h)^{-r}$ for this difference.

Therefore as $(l,\sigma) \preceq (2,0)$ one can replace term $C\mu h^{1-d}\bar\varrho^q$ which we have in the framework of Proposition 20.4.13 by

$$
\begin{aligned}
(20.4.89)\quad & C\mu^r h^{r-d+1}\times \\
& \int\int_{\bar\varrho}^{\rho_1^*} \bigl(\boldsymbol{n}(x,\tau,\mu h) - \boldsymbol{n}(x,\tau - C\rho^2,\mu h)\bigr)\rho^{q-1}\gamma^{-1}\,d\rho\,dx + \\
& C\mu^{r+1}h^{r-d}\int \bigl(\boldsymbol{n}(x,\tau,\mu h) - \boldsymbol{n}(x,\tau - C\bar\varrho^2,\mu h)\bigr)\bar\varrho^q\,dx
\end{aligned}
$$

with $\gamma = \varrho^{2/l}|\log\rho|^{\sigma/l}$, $\bar\varrho = h^{l/(l+2)}|\log h|^{-\sigma/(l+2)}$ where we already returned to coordinates x.

Also, in the general settings one can replace term $C\mu h^{1+\frac{q}{3}-d}$ by expression (20.4.89) with $\gamma = \rho^2$ and $\bar\varrho = h^{\frac13}$.

Consider two examples

Example 20.4.17. Consider eigenvalue counting function $\boldsymbol{n}_0 = \boldsymbol{n}_0(\tau,\hbar)$ of operator $\boldsymbol{a}_0 = \mathcal{A}_0$ in $\mathscr{L}^2(\mathbb{R}^r,\mathbb{C})$; since $\mathcal{A}_0$ depends on (x'',x'',ξ''), so does $\boldsymbol{n}_0$ and we can always return to variable x by map Ψ_0. Since $\mathcal{A}_0$ contains only quadratic terms one can calculate $\boldsymbol{n}_0$ explicitly

$$
(20.4.90)\qquad \boldsymbol{n}_0(\tau,\hbar) = \#\bigl\{\alpha\in\mathbb{Z}^{+\,r} : \sum_j (2\alpha_j+1)f_j\hbar + V < \tau\bigr\}.
$$

If all $f_1,\dots,f_r$ are commensurable then

$$
\boldsymbol{n}_0(\tau,\hbar) - \boldsymbol{n}_0(\tau-\lambda,\hbar) \le C\hbar^{1-r}(\lambda+\hbar)
$$

is the best possible estimate ($\lambda\ge 0$); otherwise from

$$
\begin{aligned}
(20.4.91)\quad \boldsymbol{n}_0(\tau,\hbar) - \boldsymbol{n}_0(\tau-\lambda,\hbar) = \sum_{n\ge 0} \#\bigl\{\alpha'\in\mathbb{Z}^{+\,r-1} : \sum_{j\ge 2}\frac{f_j}{f_1}(2\alpha_j+1)\hbar \in \\
[f_1^{-1}(\tau-V) - 2(n+1) - f_1^{-1}\lambda,\, f_1^{-1}(\tau-V) - 2(n+1))\bigr\}
\end{aligned}
$$

one can derive a better estimate $C\hbar^{-r}(\lambda+\nu(\hbar))$ with $\nu(\hbar) = C\hbar^{\kappa}$, $\kappa>1$ depending on Diophantine properties of $\{f_2/f_1,\dots,f_r/f_1\}$ (see G. Harman [1]).

Example 20.4.18. Let us assume that[16)]

$$
(20.4.92)\qquad \operatorname{rank}\{\nabla\frac{f_2}{f_1},\dots,\nabla\frac{f_r}{f_1}\} \ge \kappa - 1 \qquad \forall x.
$$

[16)] Actually we need a quantitative version of this condition and only at points where ∇V is a linear combination of $\nabla f_1,\dots,\nabla f_r$.

One can easily recover an integrated version of (20.4.91), namely

$$|\int_{B(0,1)} \big(\boldsymbol{n}_0(x,\hbar,\tau') - \boldsymbol{n}_0(x,\hbar,\tau')\big)\,dx| \le C\hbar^{-r}\big(|\tau-\tau'| + \nu(\hbar)\big) \tag{20.4.93}$$
$$\forall \hbar \in (0,1)\ \forall \tau,\tau' : |\tau| \le \epsilon, |\tau'| \le \epsilon$$

with $\nu(\hbar) = C\hbar^{\kappa}$. Really, one needs just to consider ζ-admissible partition in $(\alpha\hbar, x)$-space with the scaling function $\zeta = \epsilon|\nabla(\tau - V - \sum_j (2\alpha_j+1)f_j\hbar)|$.

Furthermore, these arguments show that for $\boldsymbol{n}$ the left-hand expression of (20.4.93) does not exceed $C\hbar^{1-r}\big(\lambda + \hbar^{\kappa} + \mu^{-\kappa}\big)$.

Then we arrive to

Proposition 20.4.19. *Let condition* (20.4.93) *hold for* $\boldsymbol{n}_0$ *which is an eigenvalue counting function for* $\mathcal{A}_0$. *Let* $\bar{\mu}^*_{(q)} \le \mu \le \epsilon(h\log h|)^{-1}$.

(i) Let <u>*either*</u> $q=2$ <u>*or*</u> $q=1$ *and* $\mathcal{A}$ *contain no unremovable cubic terms*[17]. *Then*

$$\mathsf{R}^{\mathsf{T}}_{1\,Q} \le Ch^{1-d} + C\nu(\mu h)h^{\frac{q}{3}-d}. \tag{20.4.94}$$

(ii) Assuming instead that $\boldsymbol{n}$ *which is an eigenvalue counting function for* $\boldsymbol{a}$, *satisfies* (20.4.93), *we need in Statement (i) neither assumption "*$\mathcal{A}$ *contains no unremovable cubic terms" nor* (20.4.93) *for* $\boldsymbol{n}_0$.

(iii) Let $q=1$ *and* $\mathcal{A}$ *contain unremovable cubic terms and* $\mu \ge (h|\log h|)^{-\frac{1}{2}}$. *Then*

$$\mathsf{R}^{\mathsf{T}}_{1\,Q} \le Ch^{1-d} + C\nu(\mu h)h^{\frac{1}{3}-d} + C\mu^{-1}h^{\frac{1}{3}-d}. \tag{20.4.95}$$

Proof. Statements (i) and (ii) follow directly from the above arguments. In Statement (iii) one needs to notice that (20.4.93) for $\boldsymbol{n}_0$ implies

$$\int \big(\boldsymbol{n}(x,\tau,\hbar) - \boldsymbol{n}(x,\tau-\lambda,\hbar)\big)\,dx \le C\big(\lambda + \mu^{-1} + \nu(\hbar)\big)\hbar^{-r}. \tag{20.4.96}$$

Easy details we leave to the reader. □

[17] Unremovable cubic terms containing hD_1 are allowed.

Remark 20.4.20. (i) If $q = 1$ and (20.4.93) holds then skipping unremovable cubic terms leads to an error not exceeding

$$C\nu(\mu h)\mu^{-\frac{1}{2}}h^{-d} + C\mu^{-\frac{3}{2}}h^{-d}. \tag{20.4.97}$$

(ii) If $\mu \ge h^{-\frac{2}{3}}$ last terms in the right-hand expression of (20.4.95) and in (20.4.97) do not exceed Ch^{1-d}.

Similarly one can prove

Proposition 20.4.21. *Let $q = 1, 2$ and f_j have constant multiplicities. Let condition* (20.4.93) *hold for $\boldsymbol{n}_0$. Let $\bar{\mu}^*_{(q)} \le \mu \le \epsilon(h|\log h|)^{-1}$.*

(i) Let <u>either</u> $q = 2$ <u>or</u> $q = 1$ and $\mathcal{A}$ contain no unremovable cubic terms. Then

$$\mathsf{R}^{\mathsf{T}}_{1\,Q} \le Ch^{1-d} + C\nu(\mu h)\cdot h^{-d+\frac{lq}{l+2}}|\log\mu|^{-\frac{lq\sigma}{2(l+2)}}. \tag{20.4.98}$$

(ii) Let $q = 1$, condition (20.4.81) *be fulfilled, $\mathcal{A}$ contain unremovable cubic terms, estimate* (20.4.93) *hold for $\boldsymbol{n}$ as well and $\mu \ge (h|\log h|)^{-\frac{1}{2}}$.*

Then estimate (20.4.95) *holds if* (20.4.84) *is fulfilled and estimate*

$$\mathsf{R}^{\mathsf{T}}_{1\,Q} \le Ch^{1-d} + C\nu(\mu h)\cdot\Big(h^{-d+\frac{l}{l+2}}|\log\mu|^{-\frac{l\sigma}{2(l+2)}} + C\mu^{-\frac{l-1}{3l}}h^{-\frac{2}{3}-d}|\log h|^{-\frac{\sigma}{3l}}\Big) \tag{20.4.99}$$

holds if (20.4.84) *fails.*

(iii) Let $q = 1$, condition (20.4.81) *be fulfilled, $\mathcal{A}$ contain unremovable cubic terms and $\mu \ge \epsilon(h|\log h|)^{-\frac{1}{2}}$. Then estimate*

$$\mathsf{R}^{\mathsf{T}}_{1\,Q} \le Ch^{1-d} + C\big(\nu(\mu h) + \mu^{-1}\big)\cdot h^{-d+\frac{lq}{l+2}}|\log\mu|^{-\frac{lq\sigma}{2(l+2)}}; \tag{20.4.100}$$

holds if (20.4.84) *is fulfilled and estimate*

$$\mathsf{R}^{\mathsf{T}}_{1\,Q} \le Ch^{1-d} + C\big(\nu(\mu h) + \mu^{-1}\big)\cdot\Big(h^{-d+\frac{l}{l+2}}|\log\mu|^{-\frac{l\sigma}{2(l+2)}} + C\mu^{-\frac{l-1}{3l}}h^{-\frac{2}{3}-d}|\log h|^{-\frac{\sigma}{3l}}\Big) \tag{20.4.101}$$

holds if (20.4.84) *fails.*

20.5 Stronger Magnetic Field: Estimates

In this section we derive Tauberian remainder estimates the remaining cases: the *intermediate magnetic field* $\epsilon(h|\log h|)^{-1} \le \mu \le \epsilon h^{-1}$, *intermediate-strong magnetic field* $\epsilon h^{-1} \le \mu \le C_0 h^{-1}$ and *strong magnetic field* $\mu \ge C_0 h^{-1}$.

20.5.1 Intermediate Magnetic Field

General Settings

Consider now the intermediate magnetic field case

$$\epsilon(h|\log h|^{-1}) \le \mu \le \epsilon h^{-1}. \tag{20.5.1}$$

Recall that the reduction to the canonical form was done with a greater

$$\varepsilon = C\bigl(\mu^{-1}h|\log h|\bigr)^{\frac{1}{2}} \tag{20.5.2}$$

rather than $C\mu^{-1}$; furthermore, an outer zone is empty now (but intermediate zone still present) and $\bar\rho_1 = C$ rather than $\bar\rho_1 = C\bigl(\mu h|\log h|\bigr)^{\frac{1}{2}}$.

As $q \ge 2$ throwing away $O(\mu^{-1})$ terms brings an $O(h^{1-d})$ error of $h^{-d}\mathcal{N}^{\mathsf{MW}}$. Then all the arguments of Subsections 20.4.1 and 20.4.2 remain true and the Tauberian remainder estimate will be $O(h^{1-d})$ if $q \ge 3$, $O(\mu h^{\frac{5}{3}-d})$ if $q = 2$ and $O(\mu h^{\frac{4}{3}-d})$ if $q = 1$; in the latter case contribution of the intermediate zone is estimated by $Ch^{1-d}|\log h|$ rather than by Ch^{1-d}. Therefore in the general settings previous results hold without any modifications:

Proposition 20.5.1. *Propositions 20.4.1, 20.4.2, 20.4.4–20.4.9 and theorem 20.4.3 remain valid for the strong magnetic field case* (20.5.1) *as well.*

Therefore in what follows we need to consider cases $q = 1, 2$ only.

Microhyperbolic Case

We must distinguish two cases:

(i) The microhyperbolicity direction ℓ does not depend on $\boldsymbol{\tau}$. In this case all the arguments of Subsubsection 20.4.3.1 remain true and the remainder estimate is $O(h^{1-d})$.

In particular, the case of f_j having constant multiplicities is covered. Really, in this case the microhyperbolicity condition means exactly that at each point x

$$\sum_{1\le j\le r_1} \tau_j \nabla\big(\frac{f_j}{V}\big) = 0, \quad \forall_j\ \tau_j \ge 0 \implies \forall_j\ \tau_j = 0; \tag{20.5.3}$$

then there exists a vector $\ell = \ell(x)$ such that

$$\langle \ell, \nabla\big(\frac{f_j}{V}\big)\rangle > 0 \qquad \forall j \tag{20.5.4}$$

which exactly means microhyperbolicity in direction ℓ.

(ii) Microhyperbolicity condition is fulfilled with ℓ depending on $\boldsymbol{\tau}$. Then as in Subsection 19.5.2 one needs to study partition of the phase space and select ϵ small enough in the upper bound for μ in (20.5.1); then all the previous results will still hold.

So, we have proven

Proposition 20.5.2. *In the intermediate magnetic field case Proposition 20.4.11 holds provided either microhyperbolicity direction does not depend on $\boldsymbol{\tau}$ or ϵ small enough in the upper bound for μ in* (20.5.1).

Special Case of Constant g^{jk}, F_{jk}

We leave to the reader the following

Problem 20.5.3. [18)] Prove that in the intermediate magnetic field case (20.5.1) $\mathsf{R}^{\mathsf{T}} = O(h^{-1})$ provided g^{jk} and F_{jk} are constant and non-degeneracy assumption (20.2.42) is fulfilled.

Case of f_j Having Constant Multiplicities

Assuming that there are no cubic terms we get a family of separate operators $\mathcal{A}_\alpha$ and all the previous arguments hold bringing us remainder estimate (20.4.80). On the other hand, as $q = 1$ dropping unremovable cubic terms produces an error $C\mu^{\frac12}h^{1-d}$ which is less than the right-hand expression of (20.4.80) unless $l = 2$, $\sigma \in (-4, 0]$. In this case assuming (20.4.81) one can prove easily (20.4.80) as well. So we arrive to

[18)] Cf. Problem 20.4.12.

Proposition 20.5.4. *(i) In the intermediate magnetic field case as f_j have constant multiplicities estimate* (20.4.80) *holds unless $q = 1$, $(I, \sigma) \succeq (2, -4)$ and there are unremovable cubic terms.*

(ii) In this exceptional case should include an extra term $C\mu^{\frac{1}{2}}h^{1-d}$ in the right-hand expression.

Case of Constant f_j

We leave to the reader the following

Proposition 20.5.5. *In the intermediate magnetic field case Proposition 20.4.14 holds.*

and

Problem 20.5.6. [19] Prove that in the intermediate magnetic field case (20.5.1) $\mathsf{R}^{\mathsf{T}} = O(h^{-1})$ provided f_j are constant and the non-degeneracy assumption (20.2.42) is fulfilled.

Improvement without Microhyperbolicity

Furthermore, under condition (20.4.93) for $\boldsymbol{n}_0$ we get remainder estimate (20.4.94) in the framework of Proposition 20.4.19(i) which is an extremely small improvement now.

Moreover, if f_j have constant multiplicities we can recover (20.4.98) in the framework of Proposition 20.4.21(i).

Proposition 20.5.7. *In the intermediate magnetic field Propositions 20.4.19 and 20.4.21 hold.*

20.5.2 Intermediate-Strong Magnetic Field

Consider now the intermediate-strong magnetic field case

$$\epsilon h^{-1} \le \mu \le C_0 h^{-1}. \tag{20.5.5}$$

Then the same arguments as before but in simplified form hold and one can prove easily

[19] Cf. Problem 20.4.16.

Proposition 20.5.8. *In the intermediate-strong magnetic field case*

(i) In the general settings remainder estimate

$$\mathsf{R}^{\mathsf{T}} \le Ch^{1-d}\begin{cases} 1 & \text{for } q \ge 3,\\ h^{-\frac{1}{3}} & \text{for } q = 2,\\ h^{-\frac{2}{3}} & \text{for } q = 1 \end{cases} \tag{20.5.6}$$

(ii) If $q = 1, 2$ and f_j have constant multiplicities then the remainder estimate

$$\mathsf{R}_1^{\mathsf{T}} \le Ch^{1-d} + Ch^{-d+\frac{lq}{l+2}}|\log h|^{-\frac{lq\sigma}{2(l+2)}} \tag{20.5.7}$$

holds.

Note that we can ignore cubic terms (which are $O(h)$ now). In the strong magnetic field case we need to modify the microhyperbolicity assumption:

Definition 20.5.9. In the case of the intermediate-strong magnetic field we call operator *microhyperbolic* (on energy level τ) if there exists vector $\ell = \ell(\bar{z}) \in \mathbb{R}^{2r+q}$ such that

$$\mu^2 \sum_{j,k\ge q+1} \bigl(\ell(a_{jk}a_0^{-1})\bigr)\zeta_j^{\dagger}\zeta_k \ge \epsilon_1 \qquad \forall \zeta \in \mathbb{C}^r \tag{20.1.19}$$

as long as

$$a = \mu^2\tau; \tag{20.5.8}$$

we also need to modify non-degeneracy assumption (20.2.42) (see (20.5.10) below).

Proposition 20.5.10. *In the case intermediate-strong magnetic field*

(i) If $q = 1, 2$ and the microhyperbolicity condition (see Definition 20.5.9) is fulfilled then the remainder estimate $\mathsf{R}^{\mathsf{T}} \le C\mu h^{2-d}$ holds.

(ii) If f_j are constant and either the microhyperbolicity condition

$$|V - \tau - \sum_j (2\alpha_j + 1)\mu h \mathsf{f}_j| + |\nabla V| \ge \epsilon_0 \tag{20.5.9}$$

or the non-degeneracy condition

$$|V - \tau - \sum_j (2\alpha_j + 1)\mu h \mathsf{f}_j| + |\nabla V| \le \epsilon_0 \implies |\det \mathsf{Hess}\, V| \ge \epsilon_0 \tag{20.5.10}$$

is fulfilled then the remainder estimate $\mathsf{R}^{\mathsf{T}} \le C\mu h^{2-d}$ holds (in the latter case we assume that $(l, \sigma) \succeq (2, 0)$).

20.5.3 Strong Magnetic Field

The last and the easiest case to consider is the strong magnetic field case $\mu \ge C_0 h^{-1}$. In this case we need to consider Schrödinger-Pauli operator. Then we arrive to a single operator

$$\mathcal{A}_0 = h^2 \sum_{2r+1\le j,k\le d} D_j g^{jk}(x'', x''', \mu^{-1}hD'')D_k + a_0(x'', x''', \mu^{-1}hD'') \tag{20.5.11}$$

where we recall that

$$a_0 = W \circ \Psi_0, \qquad W = V - \sum_j f_j \mu h \tag{20.5.12}$$

belongs to $\mathscr{F}^{l,\sigma}$ uniformly.

Surely we get a system, but all other components $\mathsf{U}_{\alpha,\beta}$ (with $(\alpha,\beta)\ne 0$) could be expressed via $\mathsf{U}_{0,0}$. Then the principal part of asymptotics (as $|W|\le c$) is of magnitude $\mu^r h^{r-d}$. For the remainder estimate we have

Proposition 20.5.11. *In the case of the strong magnetic field*

(i) For $q\ge 3$ remainder estimate $\mathsf{R}^{\mathsf{T}} \le C\mu^r h^{r+1-d}$ holds.

(ii) For $q=1,2$ and the microhyperbolicity condition (20.5.9) *fulfilled, the remainder estimate $\mathsf{R}^{\mathsf{T}} \le C\mu^r h^{r+1-d}$ holds.*

(iii) For $q=2$ and $(l,\sigma)\succeq(2,1)$ the remainder estimate $\mathsf{R}^{\mathsf{T}} \le C\mu^r h^{r+1-d}$ holds.

(iv) For $q=1$, $(l,\sigma)\succeq(2,1)$, and non-degeneracy condition (20.5.10) *fulfilled the remainder estimate $\mathsf{R}^{\mathsf{T}} \le C\mu^r h^{r+1-d}$ holds.*

(v) For $q=2$, $(l,\sigma)\prec(2,0)$ and for $q=1$ the remainder estimate

$$\mathsf{R}^{\mathsf{T}} \le C\mu^r h^{r-d} \begin{cases} h^{\frac{2l}{l+2}} |\log h|^{-\frac{2\sigma}{l+2}} & \text{for } q=2,\\ h^{\frac{l}{l+2}} |\log h|^{-\frac{\sigma}{l+2}} & \text{for } q=1 \end{cases} \tag{20.5.13}$$

holds.

On the other hand, both mollification and approximation errors in the operator are $O(h)$; then if either $q\ge 2$ or microhyperbolicity or non-degeneracy

assumptions are fulfilled then both mollification and approximation errors in the magnetic Weyl expression are $O(h)$ multiplied by $\mu^r h^{r-d}$ i.e. $\mu^r h^{r+1-d}$.

Furthermore, if $q = 1$ and we consider the general case, then both mollification and approximation errors are $O(h^{\frac{1}{2}})$ multiplied by $\mu^r h^{r-d}$ i.e. $\mu^r h^{r+\frac{1}{2}-d}$.

In all these cases both mollification and approximation errors do not exceed the remainder estimate.

20.6 Intermediate Magnetic Field: Calculations

Now we need to derive more explicit expressions rather than the Tauberian expression

$$h^{-1}\int_{-\infty}^{0} F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(U\psi_y\, {}^tQ_y)\,d\tau \tag{20.6.1}$$

or the sum of those with different T where $\psi_y = \psi(y)$, $Q_y = Q(y, hD_y)$.

20.6.1 Weak Magnetic Field Redone

Recall that the weak magnetic field case is when $\mu \le h^{\delta-1}$ and we derived asymptotics with the principal part given by (20.6.1) with $Q = I$, $T = \epsilon\mu^{-1}$ and with some remainder estimate derived without reduction to the canonical form.

Note that the mollified operator $\tilde{A}$ is microhyperbolic in the direction $\langle \xi, \partial_\xi\rangle$ due to condition (20.1.11). Further, in the zone $\{|\xi'''| \ge \epsilon'\}$ this operator is a differential operator since we take there $\varepsilon = Ch|\log h|$ which does not depend on $|\xi'''|$.

On the other hand, in the zone $\{|\xi'''| \le \frac{1}{2}\epsilon\}$ operator $\tilde{A}$ is microhyperbolic in direction $\langle \xi', \partial_{\xi'}\rangle + \langle \xi'', \partial_{\xi''}\rangle$ and it is a differential operator with respect to x', x'' since mollification parameter $\varepsilon = C\rho^{-1}h|\log h|$ does not depend on ξ', ξ''.

Therefore due to the standard results rescaled $F_{t\to h^{-1}\tau}\chi_T(t)\Gamma(U\psi_y)$ is negligible as $|\tau| \le \epsilon'$ and $T \in [T_*, \bar{T}]$, $T_* = Ch|\log h|$, $\bar{T}_0 = \epsilon\mu^{-1}$. Therefore, due to (20.2.29) we can take $T = T_*$ in (20.6.1) with only $O(h^s)$ difference. It happens in the proof of Theorems 20.2.10, 20.2.15, and 20.4.3.

Then we can launch the successive approximation method with the unperturbed operator

$$\bar{A} = \sum_{j,k} \bar{P}_j \bar{g}^{jk} \bar{P}_k + \bar{V}, \tag{20.6.2}$$

$$\bar{g}^{jk} = g^{jk}(y),\ \bar{V} = V(y),\ \bar{P}_k = hD_k - \mu V_k(y) - \mu \sum_j (\partial_k V)(y)(x_k - y_k)$$

and plug it into (20.6.1). In what we get then the first term does not exceed $CTh^{-1-d} = Ch^{-d}|\log h|$. Perturbation can be written as

$$\sum B_j(x_j - y_j) + \mu \sum_{j,k} B_{jk}(x_j - y_j)(x_k - y_k) + ...$$

with h-differential operators $B_{...}$ and since each factor $(x_j - y_j)$ according to our standard approach leads to an extra factor T in the estimate, one can see easily that each next term in the successive approximations gains factor $Ch^{-1}T\big(T^2 + \mu T^3\big) \le Ch|\log h|$ [20].

Therefore as we are looking for $O(h^{1-d})$ error, only first two terms should be considered. The first term results exactly (20.6.1) in with $U(x,y,t)$ replaced by $\bar{U}(x,y,t)$ and it is exactly

$$h^{-d} \int \tilde{\mathcal{N}}^{\mathsf{MW}}(y,\tau)\psi(y)\,dy \tag{20.6.3}$$

with $O(h^s)$ error[21].

The second term consists of two parts; one of them is generated by the perturbation

$$R_1 = \sum_{j,k,m} \bar{P}_j(\partial_m g^{jk})(y)(x_m - y_m)\bar{P}_k + \sum_m (\partial_m V)(y)(x_m - y_m) \tag{20.6.4}$$

and it is obviously 0, while the second part is generated by

$$R_2 = \sum_{j,k} \Big((P_j - \bar{P}_j)\bar{g}^{jk}\bar{P}_k + \bar{P}_j \bar{g}^{jk}(P_k - \bar{P}_k) + (P_j - \bar{P}_j)\bar{g}^{jk}(P_k - \bar{P}_k) \Big) \tag{20.6.5}$$

and is obviously $O(h^{-d-2}\mu T^3) = O(h^{1-d})$.

Therefore

---

[20] Recall that factor Th^{-1} comes from Duhamel principle.

[21] After we replaced U by $\bar{U}$ we can replace T by ∞ with $O(h^s)$ error since then (20.2.29) holds for any $T \ge \epsilon\rho$ with $O(h^s T^{-s})$ error and it will be exactly (20.6.3).

(20.6.6) In the case of the weak magnetic field approach we can replace the Tauberian expression (20.6.1) by (20.6.3) without deterioration of the remainder estimate[22)].

Remark 20.6.1. This takes care of Theorems 20.2.10 and 20.4.3–with R^{MW} rather than $R^{\mathsf{W}}_{(\infty)}$ remainder. Also under microhyperbolicity or non-degeneracy conditions it takes care of Theorem 20.2.15 and Problem 20.2.17 respectively.

20.6.2 Decomposition

Decomposition. Part I

Now let us consider temperate magnetic field $\bar{\mu}^*_{(q)} \le \mu \le \epsilon(h|\log h|)^{-1}$.

Then there is a *standard zone* where we use the already derived estimate for $\mathsf{R}^{\mathsf{T}}_{Q}$ with $T = \bar{T}$; this standard zone contains an outer zone $\Omega_{\mathsf{out}} = \{|\xi'''| \ge \bar{\rho}_1\}$ but could be wider due to the microhyperbolicity or non-degeneracy assumptions. Recall that due to assumption (20.1.11) in this standard zone we can replace $\bar{T}$ by $T_* = Ch|\log h|$.

The unexpectedly difficult (as $q = 1$) problem is to join asymptotics derived in the standard zone with a cut-off operator $I - \mathcal{T}\mathcal{Q}\mathcal{T}^*$ and in the remaining non-standard zone where we employ cut-off operator $\mathcal{T}\mathcal{Q}\mathcal{T}^*$. To tackle it better let us rewrite the formula we derived for the answer:

$$h^{-1}\sum_n \int_{-\infty}^{0} F_{t\to h^{-1}\tau}\bar{\chi}_{T_n}(t)\Gamma(U\psi_y\, {}^tQ_n y)\, d\tau \tag{20.6.7}$$

where Q_n are appropriate elements of the partition of I and T_n are already chosen. The asymptotics with this principal part and different remainder estimates were derived in Section 20.4. Actually one can replace here T_n by larger values without the affecting remainder estimate, but let us do a bit differently. Considering some term here with $Q = Q_n$ and $T = T_n$ one can rewrite it as

$$h^{-1}\int_{-\infty}^{0} \sum_{0\le m<\infty} \phi_{m,L_m}(\tau)\Big(F_{t\to h^{-1}\tau}\bar{\chi}_{T_n}(t)\Gamma\big(U\psi_y\, {}^tQ_y\big)\Big)\, d\tau \tag{20.6.8}$$

where $\mathsf{supp}(\phi_0) \subset [-2,2]$, $L_0 = C\bar{\rho}_1$, $\mathsf{supp}(\phi_m) \subset [-2,-1]$, $L_m = 2^m L_0$ as $m \ge 1$ and ϕ_{m,L_m} form partition of 1: $\sum_{0\le m<\infty}\phi_{m,L_m}(\tau) = 1$ as $\tau \le 0$.

[22)] We need also to replace $\tilde{\mathcal{N}}^{\mathsf{MW}}$ by $\mathcal{N}^{\mathsf{MW}}$ but estimate of the approximation error is already done; in what follows we will not distinguish between $\mathcal{N}^{\mathsf{MW}}$ and $\tilde{\mathcal{N}}^{\mathsf{MW}}$ keeping in mind that we always must include an approximation error estimate in our final statement.

Note that one can rewrite each term with $m>0$ in (20.6.8) as

$$\phi_{m,L_m}(hD_t)\Big(\varphi_T(t)\Gamma\big(U\psi_y\,{}^tQ_y\big)\Big)\Big|_{t=0}, \quad \text{with } \varphi=\bar{\chi},\ T=T_n. \tag{20.6.9}$$

Let us consider (20.6.9) with $\varphi=\chi$ and $T\ge hL_m^{-1}$. Using arguments of Subsection 20.4.1 one can estimate such term by

$$Ch^{-d-1}\rho^q L_m^2 T\times \big(\frac{h}{TL_m}\big)^s \tag{20.6.10}$$

where factors L_m, T are due to the integrations with respect to τ, t respectively and another factor L_m is due to the calculation of the number of contributing indices α; we assume that

$$L_m\ge C(\mu^{-1}+\rho^2). \tag{20.6.11}$$

Further, the same $Ch^{-d}\rho^q L_m$ estimate holds for (20.6.9) terms with $\varphi=\bar{\chi}$ and $TL_m=1$ as well. Then the sum with respect to t-partition results in $Ch^{-d}\rho^q L_m$, and then the sum with respect to τ partition results in $C\rho^q h^{-d}$ because $\max L_m\asymp 1$ here; terms with $L_m\ge C$ are less than $Ch^s L_m^{-s}$ due to the standard ellipticity arguments.

Let us employ a method of the successive approximations described in Subsection 20.6.1 and plug it into (20.6.9). Obviously, estimate $C\rho^q h^{-d}$ for the first term is fulfilled.

A perturbation with factor (x_j-y_j) will result either in the factor CT^2h^{-1} if commuting with A or $\bar{A}$, or in the factor $CT\rho^{-1}$ if commuting with Q, or, what is equivalent, in factors hL_m^{-2} and $hL_m^{-1}\rho^{-1}$ respectively thus resulting in the estimates

$$Ch^{1-d}\rho^q L_m^{-1}\times\big(\frac{h}{TL_m}\big)^s \quad\text{and}\quad Ch^{1-d}\rho^{q-1}\times\big(\frac{h}{TL_m}\big)^s \tag{20.6.12}$$

respectively.

Note that the commutator with Q_n (and thus the second of these expressions) appears in the sum only as $T\ge \epsilon T_n$; as for $T\le \epsilon T_n$ commutators from adjacent elements compensate one another.

Therefore taking the sum with respect to t-partition and then with respect to m we get

$$Ch^{1-d}\rho^q L_0^{-1}, \qquad Ch^{1-d}\rho^{q-1}\times\big(\frac{h}{T_nL_0}\big)^s \tag{20.6.13}$$

respectively. We can always take $L_0 = \bar{\rho}_1$ since $T_n \ge \mu^{-1}$ for sure and get $O(h^{1-d})$. As $q \ge 2$ we can do even better but it does not matter. Then both terms in (20.6.13) do not exceed Ch^{1-d} and remain this way after we sum over partition in zone $\{|\xi'''| \le \bar{\rho}_1\}$.

Therefore, in the terms with $m > 0$ we can completely ignore partition and consider only the first term of the successive approximations, Further, after this we can take T_n arbitrarily large (thus we can take $T_n = +\infty$); then after easy calculations we get

$$h^{-d} \int_{-\infty}^{0} \int \bigl(1 - \phi_{0,L_0}(\tau)\bigr)\, d_\tau \mathcal{N}^{\mathsf{MW}}(x,\tau)\, \psi(x)\, dx. \tag{20.6.14}$$

So we proved estimate

$$\begin{aligned} |h^{-1} \int_{-\infty}^{0} \bigl(1 - \phi_{0,L_0}(\tau)\bigr) \sum_n \Bigl(F_{t\to h^{-1}\tau}\bar{\chi}_{T_n}(t)(\Gamma U \psi_y\, {}^t Q_{ny})\Bigr)\, d\tau - \\ h^{-d} \int_{-\infty}^{0} \int \bigl(1 - \phi_{0,L_0}(\tau)\bigr)\, d_\tau \mathcal{N}^{\mathsf{MW}}(x,\tau)\, \psi(x)\, dx| \le Ch^{1-d}. \end{aligned} \tag{20.6.15}$$

Now we need to consider terms with $m = 0$ i.e.

$$h^{-1} \int_{-\infty}^{0} \phi_{0,L_0}(\tau) \Bigl(F_{t\to h^{-1}\tau}\bar{\chi}_{T}(t)\Gamma(U\psi_y\, {}^t Q_{ny})\Bigr)\, d\tau, \qquad T = T_n. \tag{20.6.16}$$

Instead we consider the same expression (20.6.16) but with $T = \bar{T} = \epsilon\mu^{-1}$ and consider a correction later. We remember that $T_n \ge \bar{T}$ in the intermediate and interior zones. Thus, let us consider expression (20.6.16) with $T = \bar{T}$.

Now we want to replace $\bar{T}$ by a lesser value. We could replace it by $T_* = Ch|\log h|$ with a negligible error before but it is not the case anymore since as a result of mollification and transformation the symbol of operator satisfies only an estimate

$$|\partial_\xi^\alpha a| \le C\bar{\rho}_1^{l-|\alpha|} |\log \bar{\rho}_1|^{-\sigma} + C \tag{20.6.17}$$

in the intermediate and interior zones. However then we can take there

$$T_* = C\bar{\rho}_1^{-1} h|\log h| \le \bar{T} \tag{20.6.18}$$

Now let us apply the same successive approximations to calculate expression (20.6.16) with $T = T_*$.

Then the perturbation with the factor $(x_j - y_j)$ results in the factor $T^2h^{-1} = \bar{\rho}_1^{-2}h|\log h|^2$ and extra factor $\bar{\rho}_1^{-2}|\log h|^2$ (in comparison with h) is well absorbed by ρ^{q+1} at least as $q \ge 2$, $\mu \le h^{-1}|\log h|^{-2}$.

One can consider both cases $q = 1$ and the exceptional case $\mu \ge h^{-1}|\log h|^{-2}$ either using rescaling arguments thus punishing $T \ge h/\bar{\rho}_1$ by the factor $\big(h/(\bar{\rho}_1 T)\big)^s$ or just by taking the two-terms approximation instead of the one-term and proving that the second term is identically 0.

Furthermore, a perturbation containing factor $\mu(x_j - y_j)(x_k - y_k)$ will get an extra factor $\mu T^3 h^{-1} = \mu h^2|\log h|^3\bar{\rho}_1^{-3}$ which should be treated in the same way. We leave the easy standard details to the reader.

Therefore we can again consider only the first term of the successive approximations but in contrast to the previous part we cannot tend $T = T_n$ to infinity in this approximation term unless in the outer zone. Then the answer will be similar to one given in (20.6.14), namely

$$h^{-1}\int_{-\infty}^{0} \phi_{0,L_0}(\tau)\Big(F_{t\to h^{-1}\tau}\bar{\chi}_{T_*}(t)\Gamma\big(\bar{U}\psi_y\big)\Big)\,d\tau = h^{-d}\int_{-\infty}^{0}\int \big(\phi(\tau)+\omega(\tau)\big)\,d_\tau \mathcal{N}^{\mathsf{MW}}(x,\tau)\,\psi(x)\,dx \tag{20.6.19}$$

with

$$\omega(\tau) = T_* h^{-1}\int_{-\infty}^{0} \phi(\tau')(\tau')\hat{\bar{\chi}}\big(\frac{(\tau'-\tau)T_*}{h}\big)\,d\tau' - \phi(\tau), \tag{20.6.20}$$

where $\phi = \phi_{0,L_0}$.

Here $\omega(\tau) \ne 0$ because we cannot replace T_* by $+\infty$. Instead we replaced back T_* by $\bar{T}$ since operator $\bar{A}$ has the same propagation with respect to x properties as A and therefore even with Γ_y instead of Γ the difference is negligible.

Let us define

$$\mathcal{N}_{\mathcal{Q}}^{\mathsf{MW}}(x,0) := (2\pi)^{-d+2r}h^r\mu^r \sum_{\alpha\in\mathbb{Z}^{+\,r}}\int_{\Omega_\alpha(x,\tau)} f_1\cdots f_r\,\frac{\sqrt{g}}{\sqrt{g}'''}\,\mathcal{Q}(\xi''')\,d\xi''', \tag{20.6.21}$$

with

$$\Omega_\alpha(x,\tau) = \big\{\xi : V + \sum_j (2\alpha_j+1)\mu h + \sum_{1\le j,k\le q} g^{jk}\xi_j\xi_k + V \le \tau\big\}, \tag{20.6.22}$$

where g^{jk} with $j,k=1,\dots,q$ denote coefficients after we reduce operator $\bar{A}$ to its canonical form

$$\sum_{1\le j\le r} f_j(h^2D_{q+j}^2+\mu^2x_{q+j}^2)+\sum_{1\le j,k\le q} g^{jk}h^2D_jD_k+V, \tag{20.6.23}$$

and $g'''=\det(g^{jk})^{-1}_{j,k=1,\dots,q}$.

Then $\mathcal{N}^{\mathsf{MW}}=\mathcal{N}^{\mathsf{MW}}_{\mathcal{Q}}+\mathcal{N}^{\mathsf{MW}}_{1-\mathcal{Q}}$. Plugging this into (20.6.14), (20.6.19) and adding we get in our calculation a candidate to the final answer

$$\begin{aligned} &h^{-d}\int \mathcal{N}^{\mathsf{MW}}(x,0)\psi(x)\,dx+\\ &\qquad h^{-d}\int_{-\infty}^{0}\int \omega(\tau)d_\tau\mathcal{N}^{\mathsf{MW}}(x,\tau)\,\psi(x)\,dx\equiv\\ &h^{-d}\int \mathcal{N}^{\mathsf{MW}}(x,0)\psi(x)\,dx+h^{-d}\int_{-\infty}^{0}\int \omega(\tau)d_\tau\mathcal{N}^{\mathsf{MW}}_{\mathcal{Q}}(x,\tau)\,\psi(x)\,dx \end{aligned} \tag{20.6.24}$$

modulo negligible error where $\mathcal{Q}$ is supported in $\{|\xi'''|\le 2\bar{\rho}_1\}$ and equal 1 in $\{|\xi'''|\le 2\bar{\rho}_1\}$.

Now it is a time to recall a correction

$$h^{-1}\sum_n\int_{-\infty}^{0}\phi_{0,L_0}(\tau)\Big(F_{t\to h^{-1}\tau}\big(\bar{\chi}_{T_n}(t)-\bar{\chi}_{\bar{T}}(t)\big)\Gamma\big(U\psi_y\,{}^t\mathcal{Q}_{ny}\big)\Big)\,d\tau \tag{20.6.25}$$

because we replaced T_n by $\bar{T}$ in the intermediate and inner zones. Thus the following statement is proven:

Proposition 20.6.2. *In the framework of Proposition 20.4.4 expression* (20.6.7) *with summation over all zones modulo* $O(h^{1-d})$ *is equal to*

$$\begin{aligned} &h^{-d}\int \mathcal{N}^{\mathsf{MW}}(x,0)\psi(x)\,dx+\\ &\qquad h^{-d}\int_{-\infty}^{0}\int \omega(\tau)d_\tau\mathcal{N}^{\mathsf{MW}}_{\mathcal{Q}}(x,\tau)\,\psi(x)\,dx+\\ &h^{-1}\sum_n\int_{-\infty}^{0}\phi_{0,L_0}(\tau)\Big(F_{t\to h^{-1}\tau}\big(\bar{\chi}_{T_n}(t)-\bar{\chi}_{\bar{T}}(t)\big)\Gamma\big(U\tilde{\psi}_y\,{}^t\mathcal{Q}_{ny}t\big)\Big)\,d\tau \end{aligned} \tag{20.6.26}$$

where $U=\mathcal{T}^*U\,{}^t\mathcal{T}_y^*$, $\tilde{\psi}=\mathcal{T}^*\psi\mathcal{T}$, $\mathcal{Q}_n=\mathcal{T}^*Q_n\mathcal{T}$ *and here we can take* $\mathcal{Q}_n=\mathcal{Q}_n(hD''')$ *elements of partition in zone* $\{|\xi'''|\le 2\bar{\rho}_1\}$, $\mathcal{Q}=\sum_n\mathcal{Q}_n$.

Now what we need is to estimate the sum of the second and the third terms in the right-hand expression of (20.6.26) (but not separately).

Decomposition. Part II

Let us consider the remaining inexplicit terms in the the the sum in expression (20.6.26)

$$h^{-1}\int_{-\infty}^{0}\phi(\tau)\Big(F_{t\to h^{-1}\tau}\big(\bar{\chi}_{T_n}(t)-\bar{\chi}_{\bar{T}}(t)\big)\Gamma\big(U\tilde{\psi}_y\,{}^t\mathcal{Q}_y\big)\Big)\,d\tau \tag{20.6.27}$$

where we renamed ϕ_{0,L_0} into $\phi(\tau)$ (so, in (20.6.20) ϕ is no more ϕ_{0,L_0}) and for the sake of simplicity we skip index n.

Let us apply the same decomposition technique to $\phi(\tau)$ as we did to 1: $\phi(\tau)=\sum_{m\ge 0}\phi_{m,L_m}(\tau)$ with lesser L_m than before and rewrite (20.6.27) as the sum of (20.6.9)-type terms

$$h^{-1}\int_{-\infty}^{0}\phi_{m,L_m}(\tau)\Big(F_{t\to h^{-1}\tau}\big(\bar{\chi}_{T_n}(t)-\bar{\chi}_{\bar{T}}(t)\big)\Gamma\big(U\tilde{\psi}_y\,{}^t\mathcal{Q}_y\big)\Big)\,d\tau. \tag{20.6.28}$$

Replacing $\big(\bar{\chi}_{T_n}(t)-\bar{\chi}_{\bar{T}}(t)\big)$ by $\chi_T(t)$ we find that each obtained term with $m>0$ does not exceed (20.6.10) again; however, there is a difference: after summation with respect to T ranging from $\bar{T}$ now to T_n we get that expression (20.6.28) does not exceed

$$C\rho^q h^{-d}L_m\times\big(\frac{\mu h}{L_m}\big)^s \tag{20.6.29}$$

as long $L_0\ge\mu h$ and $m>0$. After summation with respect to L_m it gives us $C\mu\rho^q h^{1-d}$.

These arguments would enable us to set $L_0=\mu h$ albeit condition $L\ge\rho^2+\mu^{-1}+\mu h$ prevents us.

Let us assume now that $\mu\ge h^{-\frac12}$; then we can take

$$L_0=C\rho^2+C\mu h; \tag{20.6.30}$$

analysis in the case $q=1$, $h^{-\frac13}\le\mu\le h^{-\frac12}$ will be done later with or without microhyperbolicity or non-degeneracy assumptions.

Let us employ the same method of the successive approximations with an unperturbed operator $\bar{\mathcal{A}}=\mathcal{A}_0(y'',y''',\mu^{-1}hD'',hD''')$ considered as an operator with operator valued symbol in $\mathscr{L}^2(\mathbb{R}^d)$. Then the perturbation is

$$\sum_{1\le j\le q+r}(x_j-y_j)a_j(x'',y'',x''',y''',\mu^{-1}hD'',hD''')+ \\ b(x'',y'',x''',y''',\mu^{-1}hD'',hD''') \tag{20.6.31}$$

with $b = O(\mu^{-1})$.

Consider the second term of the successive approximations plugged into (20.6.28) with $m > 0$ and transform it in our usual way. Then the factor $(x_j - y_j)$ in the first part of perturbation (20.6.28) should commute with either $\mathcal{G}^{\pm}$ bringing thus factor $\rho T^2 h^{-1}$ as $1 \le j \le q$ or the smaller factor $\mu^{-1} T^2 h^{-1}$ as $q+1 \le j \le q+r$ or with $\mathcal{Q}$ bringing factor $\rho^{-1} T$.

However again commuting with $\mathcal{Q}$ in the final run (after summation with respect to n) should be considered only as $T \ge T_n$ and since $T_n \ge \epsilon h \rho^{-2}$ due to the analysis of Section 20.4 the first factor is larger anyway.

Thus from the first part of perturbation we get terms estimated by (20.6.10) multiplied by one these factors: so we get

$$
\begin{aligned}
&(20.6.32)_1 \qquad && Ch^{-d}\rho^q L \times \bigl(\frac{h}{TL}\bigr)^s \times \rho T^2 h^{-1},\\
&(20.6.32)_2 \qquad && Ch^{-d}\rho^q L \times \bigl(\frac{h}{TL}\bigr)^s \times \mu^{-1} T^2 h^{-1},\\
&(20.6.32)_3 \qquad && Ch^{-d}\rho^q L \times \bigl(\frac{h}{TL}\bigr)^s \times T\rho^{-1}
\end{aligned}
$$

respectively because in our analysis $T \ge \epsilon\mu^{-1}$, $L \ge \mu h$.

On the other hand, the second part of the perturbation (20.6.31) brings factor $C\mu^{-1}Th^{-1}$ or equivalently $C\mu^{-1}L^{-1}$ thus giving us

$$
C\mu^{-1}\rho^q h^{-d} \times \bigl(\frac{h}{TL}\bigr)^s. \tag{20.6.33}
$$

Summation of $(20.6.32)_1$–$(20.6.32)_3$, (20.6.33) with respect to L, T results in the same expressions but with the minimal possible values, i.e. $T = \mu^{-1}$ and $L = \rho^2 + \mu h$; so the result of the summation for each of the expressions $(20.6.32)_{1-3}$ does not exceed

$$
C\mu^{-1}\rho^{q+1} h^{-d} \times \bigl(\frac{\mu h}{\rho^2 + \mu h}\bigr)^s, \tag{20.6.34}
$$

while the result for (20.6.33) does not exceed

$$
C\mu^{-1}\rho^{q} h^{-d} \times \bigl(\frac{\mu h}{\rho^2 + \mu h}\bigr)^s. \tag{20.6.35}
$$

Integration over $d\rho/\rho$ brings instead of (20.6.34), (20.6.35) their values as $\rho = \bar{\rho}_1^*$; the first one is always $O(h^{1-d})$ while the second one is $O(h^{1-d})$ as

$q \ge 2$ only and is $O(\mu^{-\frac12} h^{\frac12-d})$ as $q = 1$. We can do better than this but marginally. We will do it later.

Therefore, replacing U by its one-term approximation

$$\bar{\mathsf{U}} = -ih^{-1}\bar{\mathcal{G}}^{+}\delta(x-y)\delta(t) + ih^{-1}\bar{\mathcal{G}}^{-}\delta(x-y)\delta(t) \tag{20.6.36}$$

in each term of (20.6.28) with $m > 0$ brings the total error $O(h^{1-d})$ unless $q = 1$ and $\mathcal{A}$ contains (unremovable) cubic terms; in the latter case the total error is $O(\mu^{-\frac12} h^{\frac12-d})$.

Thus, let us plug $\bar{\mathsf{U}}$ into expression (20.6.28). Recall that the symbol of $\tilde{\psi}$ is given by (20.3.34). But then the remainder estimate $O(\mu^{-1}|\log h|^{-1})$ in this expression should be multiplied by $C\bar{\rho}_1\rho^q h^{-d}$ (recall that an extra factor $\bar{\rho}_1$ comes from the support of ϕ) and it results in $O(h^{1-d})$.

Furthermore, terms containing factors η_j^{w} with $j = q+1, \dots, d$ result in 0 because these terms applied to $v(x'', x''')\Upsilon_\alpha(x')$ result in

$$\sum_{\beta:|\alpha-\beta|=1} v_\beta(x'', x''')\Upsilon_\alpha(x')$$

while $\mathcal{A}_0$ "honors" $|\alpha|$.

Finally, terms containing η_j^{w} with $j = 1, \dots, q$ also result in 0 since we can take $\mathcal{Q}_n$ even with respect to ξ''' and $\mathcal{A}_0$ is even as well.

Therefore it suffices to plug only ψ_0 instead of $\tilde{\psi}$. We can also replace T_n by any larger value, say infinity. But then using the change of the coordinates Ψ_0 we get exactly the second term of (20.6.26) with the opposite sign and with ω defined by (20.6.20) for $\phi = \phi_{m,L_m}$. Then we arrive to (20.6.26) with $L_0 = \mu h + \rho^2$ (which now depends on n).

We also need to look at the case $m = 0$. So far we considered both intermediate and inner zones. Consider now intermediate zone only. Then all the above arguments remain true with the only difference that factor $\big(h/(\rho^2 T)\big)^s$ is now due to estimate (20.4.4) rather than τ-mollification.

Therefore we conclude that

(i) If <u>either</u> $q \ge 2$ <u>or</u> $q = 1$ and $\mathcal{A}$ contains no (unremovable) cubic terms then the expression

$$\begin{aligned} |\sum_n h^{-d} \int_{-\infty}^{0} \int \omega(\tau) d_\tau \mathcal{N}^{\mathsf{MW}}_{\mathcal{Q}_n}(x,\tau)\, \psi(x)\, dx + \\ h^{-1} \sum_n \int_{-\infty}^{0} \phi_{0,L_0}(\tau) \Big(F_{t\to h^{-1}\tau}\big(\bar{\chi}_{T_n}(t) - \bar{\chi}_{\bar{T}}(t)\big) \Gamma \mathsf{U} \tilde{\psi}_y\, {}^t\mathcal{Q}_{ny} \Big)\, d\tau| \end{aligned} \tag{20.6.37}$$

with summation over n such that $\mathcal{Q}_n$ belongs to the intermediate zone does not exceed Ch^{1-d}.

(ii) If $q=1$ and $\mathcal{A}$ contains (unremovable) cubic terms then this expression (20.6.37) does not exceed $C\mu^{-\frac{1}{2}}h^{\frac{1}{2}-d}$.

Combining Proposition 20.6.2 we arrive to

Proposition 20.6.3. *In the framework of Proposition 20.4.4 let us assume that* $h^{-\frac{1}{2}}\le \mu\le \epsilon(h|\log h|)^{-1}$.

Then one can rewrite expression (20.6.7) *as* (20.6.26) *with* $L_0=C\mu h$ *and* ϕ, ω *defined accordingly and one should take* $\mathcal{Q}_n=\mathcal{Q}_n(hD''')$ *elements of partition in the inner zone* $\{|\xi'''|\le \rho_1^*\}$, $\mathcal{Q}=\sum_n \mathcal{Q}_n$ *and the error is*

(i) $O(h^{1-d})$ *if* <u>*either*</u> $q\ge 2$ <u>*or*</u> $q=1$ *and* $\mathcal{A}$ *contains no (unremovable) cubic terms.*

(ii) $O(\mu^{-\frac{1}{2}}h^{\frac{1}{2}-d})$ *if* $q=1$ *and* $\mathcal{A}$ *contains (unremovable) cubic terms.*

Recall that the only cubic terms which matter are those which do not contain hD_j with $1\le j\le q$.

20.6.3 Inner Zone: General Settings

Now in virtue of Proposition 20.6.3 we need to consider only an inner zone $\Omega_{\mathsf{inn}}=\{|\xi'''|\le \bar{\rho}_1^*\}$. In the general setting there is no point to take $L_0\le \mu h$ because we cannot improve estimate $O\big((\mu h)^{-r}\big)$ for the number of indices making contributions as τ runs an interval of length L_0. Because of this (as $m=0$) one should replace $(20.6.32)_{1-3}$ by

$$(20.6.38)_1 \qquad C\mu\rho^q h^{1-d}\times \min\Big(\big(\frac{h}{T\rho^2}\big)^s,1\Big)\times \rho T^2h^{-1},$$

$$(20.6.38)_2 \qquad C\mu\rho^q h^{1-d}\times \min\Big(\big(\frac{h}{T\rho^2}\big)^s,1\Big)\times \mu^{-1} T^2h^{-1},$$

$$(20.6.38)_3 \qquad C\mu\rho^q h^{1-d}\times \min\Big(\big(\frac{h}{T\rho^2}\big)^s,1\Big)\times T\rho^{-1}$$

respectively where factor $(h/T\rho^2)^s$ as $T\ge h/\rho^2$ is due to the estimate (20.4.4). Furthermore, $(20.6.38)_3$ one should count with $T\ge \epsilon T_n\asymp \rho$ only.

After summation with respect to T we get instead of expressions $(20.6.38)_{1-2}$ their values as $T = h/\rho^2$ and instead of expression $(20.6.38)_3$ its value as $T = \rho$ i.e.

$$(20.6.39)_{1-3} \qquad C\mu\rho^{q-3}h^{2-d}, \quad C\rho^{q-4}h^{2-d}, \quad C\mu\rho^{q}h^{1-d}\bigl(\frac{h}{\rho^3}\bigr)^s$$

respectively.

Finally, after summation with respect to $\rho \ge h^{\frac{1}{3}}$ we get $O(h^{1-d})$ as $q \ge 3$, $O(\mu h^{\frac{5}{3}-d})$ as $q = 2$, and $O(\mu h^{\frac{4}{3}-d})$ as $q = 1$.

On the other hand, instead of (20.6.33) we get

$$(20.6.40) \qquad C\mu\rho^{q}h^{1-d} \times \min\Bigl(\bigl(\frac{h}{T\rho^2}\bigr)^s, 1\Bigr) \times \mu^{-1}Th^{-1}$$

and summation with respect to T results in $C\rho^{q-2}h^{1-d}$.

Summation of this expression with respect to ρ should be taken over $\rho \ge \max\bigl(\mu^{-\frac{1}{2}}, h^{\frac{1}{3}}\bigr)$ only; it results in $O(h^{1-d})$ as $q \ge 3$, $O(h^{1-d}|\log \mu^2 h|)$ as $q = 2$, and $O(\mu^{\frac{1}{2}}h^{1-d})$ as $q = 1$.

To get rid of the factor $|\log \mu^2 h|$ as $q = 2$ one should consider two-term approximation; then (20.6.40) will be replaced by

$$(20.6.40)^* \qquad C\mu\rho^{q}h^{1-d} \times \min\Bigl(\bigl(\frac{h}{T\rho^2}\bigr)^s, 1\Bigr) \times \mu^{-2}T^2h^{-2}$$

which results in $C\mu^{-1}\rho^{-2}h^{1-d}$ after summation with respect to T and in $O(h^{1-d})$ after summation with respect to ρ.

On the other hand, consider the second part of perturbation (20.6.31); one can rewrite it as $b(y'', y''', \mu^{-1}hD'', hD''')$. Really, since this part contains unremovable cubic terms, symbol b belongs to $\mathscr{F}^{1,1}$ and thus freezing $x'' = y''$, $x''' = y'''$ one makes an error which could be accommodated in the first part of (20.6.31) and treated correspondingly.

However, the second term in the approximation of U is

$$(20.6.41) \qquad \bar{\mathsf{U}}' = -ih\bar{\mathcal{G}}^{+}\bar{B}\bar{\mathcal{G}}^{+}\delta(x-y)\delta(t) + ih\bar{\mathcal{G}}^{-}\bar{B}\bar{\mathcal{G}}^{-}\delta(x-y)\delta(t).$$

and plugging ψ_0 instead of $\tilde{\psi}$ results in the trace equal to 0.

Really, operators $\mathcal{A}_0$ and thus $\bar{\mathcal{G}}^{\pm}$ transform $v(x'', x''')\Upsilon_\alpha(x')$ into the sum of terms $v_\beta(x'', x''')\Upsilon_\beta(x')$ with $|\beta - \alpha| = 0, 2$ while operator $\bar{B}'$ transforms

$v(x'',x''')\Upsilon_\alpha(x')$ into the sum of terms $v_\beta(x'',x''')\Upsilon_\beta(x')$ with $|\beta-\alpha|=1,3$ where

$$\bar{b}'=\sum_{(j,k,m)\in\mathfrak{J}_3\cup\mathfrak{J}_1} b_{jkm}\eta_j\eta_k\eta_m, \qquad \bar{b}''=\sum_{(j,k,m)\in\mathfrak{J}_2\cup\mathfrak{J}_0} b_{jkm}\eta_j\eta_k\eta_m$$

and $\mathfrak{J}_s=(j,k,m)$ with exactly s of j,k,m in $\{q+1,\dots,d\}$.

Meanwhile symbol $\bar{b}''$ is odd with respect to ξ''' while everything else is even and thus the corresponding part of the second term vanishes after integration with respect to ξ'''.

Furthermore, plugging $(\tilde{\psi}-\psi_0)$ instead of $\tilde{\psi}$ produces an extra factor μ^{-1} in the estimate which is more than enough to compensate the logarithmic factor.

Now we need to estimate the contribution of the inner core $\Omega^0_{\mathsf{inn}}=\{|\xi'''|\le\bar{\rho}_0\}$ with

(a) $\bar{\rho}_0=h^{\frac{1}{3}}$ if <u>either</u> there are no unremovable cubic terms <u>or</u> $\mu\ge h^{-\frac{2}{3}}$,

(b) $\bar{\rho}_0=\mu^{-\frac{1}{2}}$ if there are unremovable cubic terms <u>and</u> $\mu\le h^{-\frac{2}{3}}$.

However, it follows from Section 20.4 that we can consider this zone as a single partition element and set $T_n=\bar{T}=\epsilon\mu^{-1}$ here thus bringing an extra term $C\bar{\rho}_0^q\bar{T}^{-1}h^{1-d}=C\bar{\rho}_0^q\mu h^{1-d}$ into the remainder estimate; the latter expression is $O(h^{1-d})$ as $q\ge 3$, and $C\mu h^{\frac{5}{3}-d}$ as $q=2$.

As $q=1$ this expression is equal to $C\mu h^{\frac{4}{3}-d}$ if either $\mu\ge h^{-\frac{2}{3}}$ or there are no unremovable cubic terms in $\mathcal{A}$; otherwise (if $\mu\le h^{-\frac{2}{3}}$ and there are unremovable cubic terms in $\mathcal{A}$) this expression is equal to $C\mu^{\frac{1}{2}}h^{1-d}$.

Therefore with the error described one can replace U by $\bar{\mathsf{U}}$ in (20.6.27). Furthermore, we can replace $\tilde{\psi}$ by ψ_0 there with $O(h^{1-d})$ error.

Now we can preserve the remainder while increasing T_n to ∞. Really, consider

$$h^{-1}\int_{-\infty}^{0}\phi_{0,L_0}(\tau)\Big(F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\bar{\mathsf{U}}\psi_{0\,y}\,{}^t\mathcal{Q}_y\Big)\,d\tau \tag{20.6.42}$$

with $T\ge Ch\rho^{-2}$; it does not exceed

$$Ch^{-d}\times\mu h\times\rho^q\times\Big(\frac{h}{\rho^2 T}\Big)^s \tag{20.6.43}$$

exactly as would be for U but now we can take any $T \ge Ch\rho^{-2}$ because $\bar{\mathcal{A}}$ is a constant coefficient operator-valued operator. Summation with respect to $T \ge T_n$ results in the value of (20.6.43) with $T = T_n$ which is ρ in these settings i.e. we get

$$Ch^{1-d}\bigl(\rho^2+\mu h\bigr)\rho^q \times \Bigl(\frac{h}{\rho^3}\Bigr)^s \tag{20.6.44}$$

and after summation with respect to ρ we get its value as $\rho = h^{\frac{1}{3}}$ which is $C\mu h^{\frac{q}{3}+1-d}$.

But then after obvious calculations the second and the new third term in formula (20.6.26) just cancel one another and we are left with just the first term.

So we had proven

Proposition 20.6.4. *In the framework of Proposition 20.4.4 let us consider* $h^{-\frac{1}{2}} \le \mu \le \epsilon(h|\log h|)^{-1}$.

Then one can rewrite expression (20.6.7) *as* $h^{-d}\int \mathcal{N}^{\mathsf{MW}}(x,0)\,\psi(x)\,dx$ *with an error*

(i) $O(h^{1-d})$ *if* $q \ge 3$.

(ii) $O(h^{1-d}+\mu h^{\frac{5}{3}-d})$ *if* $q = 2$.

(iii) $O(\mu h^{\frac{4}{3}-d})$ *if* $q = 1$ *and either* $\mu \ge h^{-\frac{2}{3}}$ *or* $\mathcal{A}$ *contains no (unremovable) cubic terms.*

(iv) $O(\mu^{\frac{1}{2}}h^{1-d})$ *if* $q = 1$, $\mu \ge h^{-\frac{2}{3}}$ $q = 1$ *and* $\mathcal{A}$ *contains (unremovable) cubic terms.*

Remark 20.6.5. (i) When we say "unremovable cubic terms" we mean *only* terms $b_{jkm}\eta_j\eta_k\eta_m$ with $j \ge q+1$, $k \ge q+1$, $m \ge q+1$. Really, any other term bears an extra factor ρ and if $\mathcal{A}$ contains only these terms one can take $\bar{\rho}_0 = Ch^{\frac{1}{3}}$ and the above analysis of Subsection 20.6.2 and this Subsection produces an error $O(h^{1-d})$ (we get rid of factor $|\log \mu^2 h|$ in the same way as we did it for $q = 2$ in the proof of Proposition 20.6.4).

(ii) In the next Subsection 20.6.4 we show that under either microhyperbolicity or non-degeneracy assumptions these unremovable cubic terms do not cause any corrections even as $q = 1$. Further, in Subsection 20.6.4 we also derive asymptotics with a better remainder estimate.

(iii) In Subsection 20.6.5 we derive asymptotics with a correction term due to the unremovable cubic terms and with a better remainder estimate.

Proposition 20.6.4 together with the remainder estimate of Proposition 20.4.4 and with a mollification error estimate imply Theorem 20.6.17 below.

20.6.4 Improved Error Estimates

Microhyperbolicity Assumption

If either $q=2$ or $q=1$ and there are no unremovable cubic terms, everything is easy: under the microhyperbolicity condition one can replace in the estimates $(20.6.38)_{1-3}$ factor μh by factor ρ^2 as $\rho \ge \bar{\rho}_0 = Ch^{\frac{1}{3}}$; then instead of expressions $(20.6.39)_{1-3}$ we will get expressions

$$(20.6.45)_{1-3} \qquad C\rho^{q-1}h^{1-d}, \qquad C\mu^{-1}\rho^{q-2}h^{1-d}, \qquad C\rho^{q+2}h^{-d}\bigl(\frac{h}{\rho^3}\bigr)^s$$

respectively which will result after integration over $d\rho/\rho$ in $O(h^{1-d})$ in each of them[23].

Further, contribution of the inner core is estimated in Section 20.4 by $C\bar{T}^{-1}\bar{\rho}_0^{q+2}h^{1-d} = O(\mu h^{2-d})$ as we take $T_n = \bar{T}$ there. So we can again replace U by $\bar{\mathsf{U}}$ and then $\tilde{\psi}$ by ψ_0 and after this we replace T_n by ∞. Then after obvious calculations we estimate expression (20.6.28) by Ch^{1-d} and thus expression (20.6.26) will be reduced modulo $O(h^{1-d})$ to its first term.

On the other hand, factor μh is replaced by ρ^2 in the expression(20.6.40) as well thus resulting after summation with respect to T in $C\mu^{-1}\rho^q h^{-d}$; after summation with respect to ρ we get $C\mu^{-\frac{1}{2}}h^{\frac{1}{2}-d}$ as $q=1$. Even use of two-term approximation gives us $(20.6.40)^*$ multiplied by $\rho^2/(\mu h)$ and after summation with respect to T and then by ρ we get $C\mu^{-2}h^{-\frac{1}{3}-d}$ which is $O(h^{1-d})$ only for $\mu \ge h^{-\frac{2}{3}}$.

However, if $\mathcal{A}$ contains no unremovable cubic terms in the sense of Remark 20.6.5(i), one should replace there factor $\mu^{-1}T$ by $\mu^{-1}\rho T$; thus we get

$$C\mu^{-1}T\rho^4 h^{-d}\min\Bigl(\bigl(\frac{h}{T\rho^2}\bigr)^s, 1\Bigr)$$

[23] Sure in the first expression as $q=1$ we get $Ch^{1-d}|\log h|$ but we consider two-term approximation as above and also we replace $\tilde{\psi}$ by ψ_0; then the error term results in $O(h^{1-d})$ while the second term (20.6.41) resulting in 0.

in both cases instead of (20.6.40). Then after summation with respect to T we get $C\mu^{-1}\rho^2 h^{-d}$ and after integration over $d\rho/\rho$ we get $C\mu^{-1}\bar{\rho}_1^{*2}h^{-d} = Ch^{1-d}$.

Also, we need to cover also the case $(h|\log h|)^{-\frac{1}{2}} \le \mu \le h^{-\frac{1}{2}}$ (lesser μ are covered by the weak magnetic field case due to the microhyperbolicity) but one can easily weaken an assumption $\mu \ge h^{-\frac{1}{2}}$ if $\mathcal{A}$ contains no unremovable cubic terms.

So, we can plug $\bar{\mathsf{U}}$ instead of U (and ψ_0 instead of $\tilde{\psi}$) and thus we have proven

Proposition 20.6.6. *Let $(h|\log h|)^{-\frac{1}{2}} \le \mu \le \epsilon(h|\log h|)^{-1}$. Then in the framework of Proposition 20.4.4 under the microhyperbolicity assumption (see Definition 20.1.2) one can rewrite expression* (20.6.7) *as*

$$h^{-d}\int \mathcal{N}^{\mathsf{MW}}(x,0)\psi(x)\,dx \tag{20.6.46}$$

with $O(h^{1-d})$ error provided <u>either</u> $q = 2$ <u>or</u> $q = 1$ and $\mathcal{A}$ contains no unremovable cubic terms (in the sense of Remark 20.6.5(i)).

However, we want to get rid of this extra assumption and thus we want to consider case "$q = 1$, $\mathcal{A}$ contains unremovable cubic terms". To do this we must reexamine an intermediate zone as well.

Our main idea is to estimate decay of $F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\psi_0\mathcal{Q}U$ more accurately, using the microhyperbolicity conditions it was done in Subsections 19.4.1–19.4.4.

Proposition 20.6.7. *Let $(h|\log h|)^{-\frac{1}{2}} \le \mu \le \epsilon(h|\log h|)^{-1}$. Then in the framework of Proposition 20.4.4 under the microhyperbolicity assumption (see Definition 20.1.2)*

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\tilde{\psi}\mathcal{Q}U| \le CT\rho^{q+2}h^{-d}\bigl(\frac{h}{T}\bigr)^{l-1}|\log\bigl(\frac{h}{T}\bigr)|^{-\sigma}\times \min\Bigl(\bigl(\frac{h}{\rho^2 T}\bigr)^{s},1\Bigr) \tag{20.6.47}$$

as $\bar{\rho}_0 = h^{-\frac{1}{3}} \le \rho \le \bar{\rho}_1$, $|\tau| \le \epsilon$.

Proof. The proof repeats those of Propositions 19.4.3 and 19.4.4 and is based on η-approximation of $\mathcal{A}$ with $hT^{-1} \le \eta \le ChT^{-1}|\log h|$. Easy details are left to the reader. □

After we improved our estimate by an extra factor $(h/T)^{l-1}|\log(h/T)|^{-\sigma}$ we can do more precise estimates.

Recall that

$$h^{-1}\int_{-\infty}^{0}\phi_{0,L_0}(\tau)\Big(F_{t\to h^{-1}\tau}\chi_T(t)\tilde{\psi}\mathcal{Q}_n\mathsf{U}\Big)\,d\tau = iT^{-1}\Big(F_{t\to h^{-1}\tau}\check{\chi}_T(t)\tilde{\psi}\mathcal{Q}_n\mathsf{U}\Big)\Big|_{\tau=0} - iT^{-1}L_0^{-1}\int_{-\infty}^{0}\phi'_{0,L_0}(\tau)\Big(F_{t\to h^{-1}\tau}\check{\chi}_T(t)\tilde{\psi}\mathcal{Q}_n U\Big)\,d\tau \tag{20.6.48}$$

with $\check{\chi}(t)=t^{-1}\chi(t)$ and ϕ'_0 derivative of ϕ_0; recall that $\phi_0(0)=1$. Here L_0 could be larger than ρ^2 to accommodate a weaker assumption $\rho\le\mu^{-\frac12}$. Then Proposition 20.6.7 implies that for $q=1$ the right-hand expression of (20.6.48) does not exceed

$$C\rho^3h^{-d}\big(\frac{h}{T}\big)^{l-1}|\log\big(\frac{h}{T}\big)|^{-\sigma}\times\min\Big(\big(\frac{h}{\rho^2T}\big)^{s},1\Big). \tag{20.6.49}$$

Integrating this expression (20.6.49) over dT/T from $T=\bar{T}$ to $T=\infty$ we get this expression as $T=\bar{T}$ and then integrating over $d\rho/\rho$ from $\bar{\rho}_)=h^{\frac13}$ to $\rho=\bar{\rho}_1$ as $(h|\log h|)^{-\frac12}\le\mu\le h^{-\frac23}$ we get this expression as $\rho=\bar{\rho}_0$ i.e.

$$C(\mu h)^{l+\frac12}|\log h|^{-\sigma}h^{-d} \tag{20.6.50}$$

which does not exceed an approximation error

$$C\mu^{-l}\bar{\rho}_1|\log h|^{-\sigma}h^{-d}=C\mu^{\frac12-l}h^{\frac12-d}|\log h|^{\frac12-\sigma} \tag{20.6.51}$$

as $\mu\le h^{-\frac12}$. Therefore, in this case we can plug $\bar{\mathsf{U}}$ instead of U (and ψ_0 instead of $\tilde{\psi}$).

Thus it remains to consider the case $h^{-\frac12}\le\mu\le h^{-\frac23}$. Consider expression (20.6.49) with $T=\mu^{-1}$ and the last factor equal to 1. Integrating it with respect to $d\rho/\rho$ from $\bar{\rho}_0=h^{-\frac13}$ to $\rho=\mu^{-\frac12}$, we get its value as $\rho=\mu^{-\frac12}$ which is

$$C\mu^{-\frac32}(\mu h)^{l-1}|\log h|^{-\sigma}h^{-d} \tag{20.6.52}$$

and which does not exceed approximation error as $l\le\frac32$ and Ch^{1-d}) as $l>\frac32$. These arguments estimate contribution of the inner core as well. Therefore we estimated properly contribution of the inner zone $\{|\xi'''|\le C\mu^{-\frac12}\}$.

Consider now ρ ranging from $\mu^{-\frac{1}{2}}$ to $\bar{\rho}_1$. We also apply the above arguments as $T \ge \mu h \ge h/\rho^2$, resulting after integration over dT/T in

$$C\rho^3\mu^{1-l}|\log h|^{-\sigma}\bigl(\frac{1}{\mu\rho^2}\bigr)^s h^{-d}. \tag{20.6.53}$$

As $\bar{T} \le T \le \mu h$ we apply the two-term approximation method with the second term resulting in 0 in the final answer and with an error estimate

$$C\rho^3 h^{-d}\bigl(\frac{h}{T}\bigr)^{l-1}|\log\bigl(\frac{h}{T}\bigr)|^{-\sigma} \times \bigl(\frac{T}{\mu h}\bigr)^2 \times \min\Bigl(\bigl(\frac{h}{\rho^2 T}\bigr)^s, 1\Bigr) \tag{20.6.54}$$

which can be easily proven by arguments of Propositions 19.4.3, 19.4.4 and 19.4.15. After integration over dT/T we get

$$C\rho^{2l-3}\mu^{-2}|\log\rho|^{-\sigma}h^{-d} \times \min\Bigl(\bigl(\frac{\mu h}{\rho^2}\bigr)^s, 1\Bigr). \tag{20.6.55}$$

Finally, integration of (20.6.53) over $d\rho/\rho$ in the indicated limits results in its value at the lower limit $\rho = \mu^{-\frac{1}{2}}$; this result does not exceed $\mu^{-\frac{1}{2}-l}|\log h|^{-\sigma}h^{-d}$ which again does not exceed an approximation error (20.6.51). This is also true for expression (20.6.55) as $l < \frac{3}{2}$; as $l = \frac{3}{2}$ we get $C\mu^{-2}|\log h|^{-\sigma}h^{-d} \times |\log \mu^2 h|$ which also does not exceed approximation error.

So again we can replace U by $\bar{\mathsf{U}}$ and $\tilde{\psi}$ by ψ_0. Moreover, after this replacement we can replace $T_n = \epsilon$ by $T_n = \infty$ with a negligible error. Thus we have proven

Proposition 20.6.8. *Let $q = 1$ and $(h|\log h|)^{-\frac{1}{2}} \le \mu \le \epsilon(h|\log h|)^{-1}$. Then in the framework of Proposition 20.4.4 under microhyperbolicity assumption (see Definition 20.1.2) one can rewrite expression* (20.6.7) *as* (20.6.46) *with $O\bigl(h^{1-d} + \mu^{\frac{1}{2}-l}h^{\frac{1}{2}-d}|\log h|^{\frac{1}{2}-\sigma}\bigr)$ error.*

Proposition 20.6.8 together with remainder estimate of Proposition 20.4.11 and with mollification error estimate imply Theorem 20.6.15 below.

Case of f_j Having Constant Multiplicities

If we drop all junior terms we get a family of scalar operators and everything is easy. However while covering case $q = 2$ this would not cover case $q = 1$

even if the unremovable cubic terms contain hD_1. So we need more delicate arguments.

Note that instead of $\mathcal{Q}_n(hD''')$ we have now $\mathcal{Q}_n(x'', x''', \mu^{-1}hD'')\mathcal{Q}'_m(hD''')$ where $\mathcal{Q}'_m$ are elements of ρ-partition, $\mathcal{Q}_n$ are elements of γ-partition with γ, ϱ introduced by (20.4.73)–(20.4.74).

(i) Consider first elements with $\varrho \ge C\bar{\varrho}$ and $\rho \ge \epsilon\varrho$.

Then one should multiply (20.4.49) by $\gamma^d \times \varrho^2/\gamma$ where factor γ^d comes from $\mathcal{Q}_n$ and factor ϱ^2/γ (which is greater than μ^{-1}) comes from the estimate of $|\nabla_{x'',x''',\xi'''}A|$.

Then expressions $(20.6.39)_{1-3}$ become

$$(20.6.56)_{1-3} \quad C\mu\varrho^{q-1}h^{2-d}\gamma^{d-1}, \qquad C\varrho^{q-2}h^{2-d}\gamma^{d-1}, \qquad C\mu\varrho^{q+2}h^{1-d}\min\Big(\big(\frac{h}{\varrho^3}\big)^s, 1\Big)\gamma^{d-1}$$

respectively and the second and the third expressions do not exceed the first one.

(ii) On the other hand, as $\rho \le \epsilon\varrho$, $\varrho \ge C\bar{\varrho}$ we can rescale problem to the microhyperbolic one with $x \mapsto x\gamma$, $h \mapsto h/(\varrho\gamma)$, $\mu^{-1}h \mapsto \mu^{-1}h\gamma^{-2}$. Then the total contribution of elements with fixed n and different m does not exceed $(20.6.56)_1$ again. therefore, $(20.6.56)_1$ estimates the contribution of element $\mathcal{Q}_n$.

After summation over partition $(20.6.56)_1$ results in

$$\mu h^{2-d}\int \varrho^{q-1}\gamma^{-1}\,dx''dx'''d\xi'' \tag{20.6.57}$$

which does not exceed the right hand expression in (20.6.90) (or (20.4.80)).

(iii) Finally, the same estimate remains true as $\varrho \asymp \bar{\varrho}$ because now we estimate the contribution to each of the second and the third term of (20.6.7) rather than to their sum.

(iv) Similarly, if unremovable cubic terms contain factor hD_1 then we get that the contribution of $\mathcal{Q}_n$ instead of (20.6.51) is estimated by

$$C\mu h^{1-d}\varrho^q\gamma^d\min\big(\mu^{-1}\varrho^{-1}, 1\big) \le Ch^{1-d}\varrho^{q-1}\gamma^d \tag{20.6.58}$$

and the total contribution of all elements does not exceed Ch^{1-d} because then μ scales as $\mu\varrho$ [24)].

(v) On the other hand, if there are unremovable cubic terms without factor hD_1 then (20.6.58) is replaced by

$$C\mu h^{1-d}\varrho^q\gamma^d \min\bigl(\mu^{-1}\varrho^{-2},1\bigr) \le Ch^{1-d}\varrho^{q-1}\gamma^d \tag{20.6.59}$$

and summation results in $C\mu^{\frac12}h^{1-d}$ and Ch^{1-d} as $q=1,2$ respectively.

Thus we arrive to

Proposition 20.6.9. *Let $q=1,2$ and f_j have constant multiplicities.*

(i) Let <u>either</u> $q=2$ <u>or</u> $q=1$ and there are no unremovable cubic terms. Then with an error not exceeding the right-hand expression of (20.4.79) *one can rewrite* (20.6.7) *as* (20.6.46).

(ii) Let $q=1$ and there are unremovable cubic terms. Then with an error not exceeding the right-hand expression of (20.4.80) *plus $C\mu^{\frac12}h^{1-d}$ one can rewrite* (20.6.7) *as* (20.6.46).

Proposition 20.6.9 together with remainder estimate of Proposition 20.4.13 and with mollification error estimate imply Theorem 20.6.23 below.

We leave to the reader the following easy:

Problem 20.6.10. Let f_j be constant and non-degeneracy condition (20.2.42) be fulfilled. Prove that with $O(h^{1-d})$ error one can rewrite (20.6.7) as (20.6.46).

Number-Theoretical Arguments

Now it is easy to prove

Proposition 20.6.11. *In the framework of Proposition 20.4.19(i) one can rewrite expression* (20.6.7) *as* (20.6.46) *with an error not exceeding the right-hand expressions of* (20.6.82), (20.6.84) *below for $q=2,1$ respectively.*

[24)] Parameter $\mu^{-1}h$ scales independently from μ and h because we deal with the reduced form of operator.

Proposition 20.6.11 together with remainder estimate of Proposition 20.4.19 and with mollification error estimate imply Statements (i) and (ii) ofTheorem 20.6.21 below.

A bit more sophistication requires

Proposition 20.6.12. *In the framework of Proposition 20.4.21(i) one can rewrite expression* (20.6.7) *as* (20.6.46) *with an error not exceeding the right-hand expressions of* (20.6.90), (20.6.91) *below as* $q = 2, 1$ *respectively.*

Proof. One needs just to replace $\tilde{\psi}$ by ψ_0, then consider two term approximations (the second term will result in 0 in the end), and then replace T_n by ∞. In the intermediate zone estimates repeat those of the generic case.

In the inner zone we need to consider integral (20.6.57) and notice that contribution in it of all elements with $c^{-1}t \le \varrho \le ct$, $t \le (\mu h)^{\frac{1}{2}}$ does not exceed

$$Ch^{1-d}\varrho^{q-1}\gamma^{-1}\big(\varrho^2 + \nu(\mu h)\big)$$

with ϱ defined by given γ and thus this integral does not exceed

$$Ch^{1-d}\int \varrho^{q+1}\gamma^{-1}\,dx + C\nu(\mu h)h^{1-d}\int \varrho^{q-1}\gamma^{-1}\,dx$$

and plugging $\varrho = \gamma^{\frac{l}{2}}|\log\gamma|^{-\frac{\sigma}{2}}$ and replacing dx by $\gamma^{-1}d\gamma$ we find that the first term does not exceed Ch^{1-d} provided <u>either</u> $q = 2$ <u>or</u> $q = 1$, $(l,\sigma) \succ (1,1)$ while the second integral does not exceed

$$C\nu(\mu h)h^{1-d}\bar{\varrho}^{q-1}\bar{\gamma}^{-1} \asymp C\nu(\mu h)\bar{\varrho}^{q}h^{-d}$$

unless $q = 2$, $l = 2$ in which case the second term does not exceed $C\nu(\mu h)h^{1-d}|\log h|^{-\sigma}$. □

Proposition 20.6.12 together with remainder estimate of Proposition 20.4.21 and with mollification error estimate imply Theorem 20.6.25 below.

20.6.5 $q = 1$: Asymptotics with Correction

We want to improve remainder estimates in asymptotics as $q = 1$ and $\mathcal{A}$ has non-removable cubic terms. To do this we need to add some correction to $h^{-d}\mathcal{N}^{\mathsf{MW}}$.

First of all, note that if both $\bar{\chi}$ and ϕ are proper functions[25)] and $L_0 \ge C\mu h|\log h|$, we can rewrite Tauberian expression (20.6.7) in (20.6.26) form with $O(h^s)$ error because (20.6.9) with $\varphi_T = \chi_T$ and with $m > 0$ is negligible as $TL_m \ge Ch|\log h|$.

To calculate the last term in (20.6.26) we apply the same successive approximation method as before but with an unperturbed operator $\bar{\mathcal{A}} = \mathcal{A}_1(y'', y''', \mu^{-1}hD'', hD''')$ where in $\mathcal{A}_1$ we include unremovable cubic terms

$$\mathcal{B} = \mu^2 \sum_{j,k,m,\ge 2} \bigl(b_{jkm}\eta_j\eta_k\eta_m\bigr)^{\mathsf{w}} \tag{20.6.60}$$

but do not include the similar terms with at least one of the indices j, k, m equal 1[26)]; obviously such unremovable terms contain either one or three such indices.

Then the last term in the perturbation (20.6.31) will contain at least one factor η_1 and thus we get an extra factor ρ in the corresponding estimates. Then instead of an unwanted term $C\rho\mu h^{1-d} \times (\mu\rho^2)^{-2}$ calculated as $\rho = \max\bigl(\mu^{-\frac{1}{2}}, \bar{\rho}_0\bigr)$ in an estimate, we get $C\rho\mu h^{1-d} \times (\mu\rho)^{-2}$, also calculated as $\rho = \max\bigl(\mu^{-\frac{1}{2}}, \bar{\rho}_0\bigr)$, which results in the error estimate $O(h^{1-d})$; here we take the same two-term approximation because we still need to consider the second term to avoid extra $|\log h|$ factor.

However, we need to consider it only as b contains exactly one factor hD_1 because otherwise we gain one more extra factor ρ. We also can replace $\tilde{\psi}$ by ψ_0 with impunity because the we get an error not exceeding $CL_0\rho\mu^{-1}h^{-d} = C\rho|\log h|h^{1-d}$ which does not exceed Ch^{1-d} as $\mu \le h^{-1}|\log h|^{-3}$ and is much less than the remainder estimate otherwise (actually we can replace in the above estimate L_0 by μh but we do not need it). But then the second term in this approximation will be odd with respect to ξ_1 and result in 0 in the final answer.

So, all estimates of Propositions 20.6.4 , 20.6.9, and 20.6.11 related to the case "$q = 1$ and $\mathcal{A}$ contains no unremovable cubic terms" hold for an error term arising if we replace in the last term of (20.6.26) U by $\hat{\mathsf{U}}$ and also replace $\tilde{\psi}$ by ψ_0. Here and below $\hat{\mathsf{U}}$ is our "new" first term, i.e. the first term in our new approximations.

Therefore in comparison with our previous calculations we get an extra

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[25)] In Section 2.3 sense.

[26)] We also include $C\mu^{-2}\mathcal{A}_0^2$ to have operator semibounded from below

term

$$h^{-1}\sum_n \int_{-\infty}^{0} \phi_{0,L_0}(\tau)\times \Big(F_{t\to h^{-1}\tau}\big(\bar\chi_{T_n}(t)-\bar\chi_{\bar T}(t)\big)\Gamma\big((\hat{\mathsf{U}}-\bar{\mathsf{U}})\psi_{0y}\,{}^t\mathcal{Q}_{ny}\big)\Big)\,d\tau \tag{20.6.61}$$

where $\bar{\mathsf{U}}$ is the "old" first term in the successive approximations.

This is a correction term in question but we want to rewrite it in a more explicit form. First of all, we can replace here T_n by $+\infty$ (using the same arguments as before) and also replace $\bar T$ by $\bar T' = Ch|\log h|$ because $\big(F_{t\to h^{-1}\tau}\chi_T(t)\Gamma(\mathsf{U}'\psi_{0y}\,{}^t\mathcal{Q}_{ny})\big)$ is negligible as $T\in[\bar T', \bar T]$, $|\tau|\le CL_0$ for both $\mathsf{U}'=\bar{\mathsf{U}}$ and $\mathsf{U}'=\hat{\mathsf{U}}$.

However if we consider expression

$$h^{-1}\sum_n \int_{-\infty}^{0} \phi_{0,L_0}(\tau)\Big(F_{t\to h^{-1}\tau}\bar\chi_{\bar T'}(t)\Gamma\big((\hat{\mathsf{U}}-\bar{\mathsf{U}})\psi_{0y}\,{}^t\mathcal{Q}_{ny}\big)\Big)\,d\tau \tag{20.6.62}$$

and apply the method of successive approximation to calculate it (considering $\bar{\mathcal{A}}_0$ as an unperturbed operator and $\bar{\mathcal{A}}_1$ as a perturbed one, we can get easily an error estimate $C\rho|\log h|^2h^{1-d}$ for it which is again less than the remainder estimate we are looking for.

Thus modulo term not exceeding the remainder estimate we can rewrite expression (20.6.61) as the same expression but with $\bar\chi_T$ replaced by 1 and with $\bar\chi_{\bar T}$ replaced by 0 i.e. as

$$h^{-1}\sum_n \int_{-\infty}^{0} \phi_{0,L_0}(\tau)\Big(F_{t\to h^{-1}\tau}\Gamma(\hat{\mathsf{U}}-\bar{\mathsf{U}})\psi_{0y}\,{}^t\mathcal{Q}_{ny}\Big)\,d\tau \tag{20.6.63}$$

which in turn is exactly equal to $h^{-d}\int \mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x,0)\psi(x)\,dx$ with

$$\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}} := (2\pi)^{-d+r}(\mu h)^r \int \big(1-\phi_{0,L_0}(\tau)\big)\times d_\tau\Big(\int \big(\boldsymbol{n}(x,\tau-z^2)-\boldsymbol{n}_0(x,\tau-z^2)\big)\mathcal{Q}(z)\,dz\times f_1\cdots f_r\sqrt{g}\Big). \tag{20.6.64}$$

Thus we arrive to

Proposition 20.6.13. *Let* $q = 1$, $(h|\log h|)^{-\frac{1}{3}} \le \mu \le (h|\log h|)^{-1}$ *and* $\mathcal{A}$ *contain unremovable cubic terms. Let* $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}$ *be the difference between expression* (20.6.7) *and*

$$h^{-d}\int \bigl(\mathcal{N}^{\mathsf{MW}}(x,0)+\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x,0)\bigr)\,\psi(x)dx \tag{20.6.65}$$

where $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ *is defined by* (20.6.64) *with* $L_0 = C_0\mu h|\log h|$ *with large enough constant* C_0 *and a proper symbol* $\mathcal{Q}$ *equal* 1 *in* $\{|z| \le C\bar{\rho}_1\}$ *and* $\boldsymbol{n}_0$ *and* $\boldsymbol{n}$ *are eigenvalue counting functions for* $1\mathsf{D}$*-operators* $\boldsymbol{a}_0$ *and* $\boldsymbol{a}$ *respectively. Then*

(i) In the general settings $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}$ *does not exceed* $C\mu h^{\frac{4}{3}-d}$.

(ii) In the framework of Proposition 20.4.14 $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}$ *does not exceed the right-hand expression of* (20.4.88).

(iii) In the framework of Proposition 20.4.19(iii) $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}$ *does not exceed the right-hand expression of* (20.4.95).

(iv) In the framework of Proposition 20.4.21(ii) $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}$ *does not exceed the right-hand expression of* (20.4.99) *as condition* (20.4.85) *is violated.*

(v) In the framework of Proposition 20.4.21(iii) $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}$ *does not exceed the right-hand expression of* (20.4.101) *as condition* (20.4.85) *is violated.*

This proposition together with corresponding results of Section 20.4 implies all statement of theorems dealing with $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}$ rather than R^{MW} of Theorems 20.6.19(i) and 20.6.23(ii) below.

It follows from Proposition 20.6.13(iii) that in the framework of (20.4.93) with $\nu(\hbar) = \hbar^2$ we do not need any correction. On the other hand, the following example shows the need in the correction term.

Example 20.6.14. Consider the case $q = 1$, $r = 2, 3$ and $f_j = \mathsf{const}$ and commensurable and there are unremovable cubic terms in $\mathcal{A}$ so that the $\mathcal{A}_1 = \mathcal{A}_0 + \mu^{-1}\mathcal{E}$ with $\mathcal{E}$ commuting with $\mathcal{A}$ (at each point x). Furthermore, let us assume that one of the following conditions is fulfilled:

$$r = 2, \quad f_1 = 2f_2, \tag{20.6.66)$_2$}$$

$$r = 3, \quad f_1 = f_2 + f_3, \quad f_2 \ne f_3. \tag{20.6.66)$_3$}$$

Then

$$\mathcal{A}_0 = 2Z_1^*Z_1 + Z_2^*Z_2 + D_1^2, \qquad \mathcal{E} = (\omega Z_1^*Z_2^2) + (\omega Z_1^*Z_2^2)^*,$$
$$\mathcal{A}_0 = f_1Z_1^*Z_1 + f_2Z_2^*Z_2 + f_3Z_3^*Z_3 + D_1^2, \quad \mathcal{E} = (\omega Z_1^*Z_2Z_3) + (\omega Z_1^*Z_2Z_3)^*$$

respectively with $\omega = \omega(x'', x''', \mu^{-1}hD'')$ [27].

We can already apply Proposition 20.6.13, getting for $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}$ better estimate than estimate $C\mu^{\frac12}h^{1-d}$ for R^{MW}, but we can improve further the former one.

Let us assume that $|\omega| \ge \epsilon$. Then (assuming large enough smoothness of symbols) one can apply a transformation by operator $e^{i\hbar^{-1}\mathcal{L}}$ with $\mathcal{L} = \sum_j \beta_j Z_j^*Z_j$ with Hermitian $\beta_j = \beta_j(x'', x''', \mu^{-1}hD'')$ and transform operator (modulo terms containing factors $\mu^{-1}hD_1$ or μ^{-2}) to the same form but with Hermitian operator ω. So, $\mathcal{E} = \omega\mathcal{E}_0$ with

$$\mathcal{E}_0 = \begin{cases} Z_1^*Z_2^2 + Z_1Z_2^{*\,2} & r = 2 \\ Z_1^*Z_2Z_3 + Z_1Z_2^*Z_3^* & r = 3 \end{cases} \tag{20.6.67}$$

commuting with $\mathcal{E}_0$. Then one can decompose with respect to common eigenfunctions of $\mathcal{A}_0$ and $\mathcal{E}_0$ and to break the systems into separate equations and one can easily prove that

$$\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}} \le C\mu h^{\frac32-d} \tag{20.6.68}$$

without any microhyperbolicity or non-degeneracy condition; note that the right-hand expression is much less than $C\mu^{\frac12}h^{1-d}$ as $\mu h \ll 1$).

Let us select $V = -\sum_j (2\bar\alpha_j + 1)f_j\mu h$ and $\mu \ge h^{-\frac12}$.

Further, let us replace $\boldsymbol{n}(x, -z^2, \hbar)$ by

$$\#\bigl\{\alpha : \sum_j (2\alpha_j + 1)f_j\mu h + V < 0\bigr\} + \boldsymbol{m}(x, -\xi_d^2, \hbar)\theta(\rho^2 - z^2) \tag{20.6.69}$$

where $\boldsymbol{m}(x, \lambda, \hbar)$ is an eigenvalue counting function for operator $\mathcal{E}$ restricted to the linear span of Υ_α such that $\sum_j (2\alpha_j + 1)f_j\mu h + V = 0$. Then the error in $h^{-d}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ will not exceed $C\mu\rho h^{1-d} \times (\mu\rho^2)^{-2}$. Taking $\rho = \bar\rho_1^*$ we get this error not exceeding $C\mu^{-\frac52}h^{-\frac12-d} \ll \mu^{\frac12}h^{1-d}$ as long as $\mu \gg h^{-\frac12}$.

[27] One should not forget to add $C\mu^{-2}\mathcal{A}_0^2$ to avoid some non-semi-boundedness related problems.

But then

$$\int \mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x,0)\psi(x)\,dx = (\mu h)^r \iint \boldsymbol{m}(x,-\xi_d^2,\hbar)\theta(\rho^2-\xi_d^2)\psi(x)\,dx\,d\xi_1 + O(C\mu^{-\frac{5}{2}}h^{-\frac{1}{2}}) \tag{20.6.70}$$

which is exactly of magnitude $\mu^{-\frac{1}{2}}\times \mu h = \mu^{\frac{1}{2}}h^{1-d}$ and one cannot skip it without increasing the remainder estimate.

20.6.6 Main tTheorems

Microhyperbolicity Assumption

We assume first that the microhyperbolicity condition (see Definition 20.1.2) is fulfilled. Assume that

$$\bar{\mu}_{(2)} = C(h|\log h|)^{-\frac{1}{2}} \le \mu \le \epsilon(h|\log h|)^{-1}. \tag{20.6.71}$$

The low bound is justified as $q\ge 2$ since otherwise Theorem 20.2.10 provides $O(h^{1-d})$ remainder estimate. For $q=1$ justification is that we will need to take $\varepsilon = C\mu^{-1}$ and it is smaller than $\varepsilon = C\mu h|\log h|$ used in the previous results if and only if $\mu \ge \bar{\mu}_{(2)}$. The only exception (when we can take $\varepsilon = C(\mu^{-1}h|\log h|)^{\frac{1}{2}}$) is the case $r=1$ but then $q\ge 2$ since we assume that $d\ge 4$.

We also assume that

(20.6.72) Either $q\ge 2$ and $(l,\sigma)\succeq(1,1)$ or $q=1$ and $(l,\sigma)\succeq(1,2)$.

Theorem 20.6.15. *Let conditions* (20.1.1)–(20.1.11), (20.6.62) *and* (20.6.71) *be fulfilled. Further, let the microhyperbolicity condition (see Definition 20.1.2) be fulfilled. Then there exist two framing approximations (see footnote* [18)] *of Chapter 18) such that*

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d} + Ch^{-d}(\mu h|\log h|)^{\frac{q}{2}}\mu^{-l}|\log h|^{-\sigma}. \tag{20.6.73}$$

Corollary 20.6.16. *Under the microhyperbolicity condition for* $\mu \le \epsilon(h|\log h|)^{-1}$ *the sharp remainder estimate* $\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d}$ *holds as either* $q\ge 2$ *and* $(l,\sigma)\succeq(1,1)$ *or* $q=1$ *and* $(l,\sigma)\succeq(\frac{3}{2},1)$.

General Case

Now let us consider the temperate magnetic field without microhyperbolicity assumption.

Theorem 20.6.17. *Let conditions* (20.1.1)–(20.1.11), (20.6.72) *and* (20.6.71) *be fulfilled. Then there exist two framing approximations such that*

(i) For $q \ge 3$, $(l,\sigma) = (1,1)$, $(\bar{l},\bar{\sigma}) = (2,1)$ *estimate* $\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d}$ *holds.*

(ii) For $q = 2$, $(l,\sigma) = (1,1)$, $(\bar{l},\bar{\sigma}) = (2,1)$ *estimate*

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d} + C\mu h^{\frac{5}{3}-d} \tag{20.6.74}$$

holds; in particular, $\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d}$ *as* $\mu \le h^{-\frac{2}{3}}$.

(iii) For $q = 1$, $(l,\sigma) = (1,2)$ *estimate*

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d} + C\mu h^{\frac{4}{3}-d} + C\mu^{\frac{1}{2}}h^{1-d} \tag{20.6.75}$$

holds.

Remark 20.6.18. (i) Note that estimates (20.6.74), (20.6.75) are better than (20.2.32) with $q = 2, 1$ respectively iff $\mu \ge (h|\log h|)^{-\frac{1}{2}}$.

(ii) The second terms in (20.6.74), (20.6.75) are due to the remainder estimates for q-dimensional Schrödinger operator (it is not important that operator in question is also r-dimensional $\mu^{-1}h$-pseudodifferential operator). These estimates correspond to the $\mathscr{C}^{1,1}$-smoothness and cannot be improved in the general case even if we assume much the larger smoothness because we have a matrix, not a scalar h-pseudodifferential operator as for $d = 3$. However, in certain cases we can do better than this; we need to consider $q = 1, 2$ only.

(iii) In (20.6.75) the third term estimates an approximation error arising when we skip irreducible $O(\mu^{-1})$ terms due; we can prevent these terms under certain conditions.

Theorem 20.6.19. *Let* $q = 1$, $(1,2) \preceq (l,\sigma) \preceq (2,0)$ *and conditions* (20.1.1)–(20.1.11), *and* (20.6.71) *be fulfilled. Then there exist two framing approximations such that*

(i) Estimate

$$\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}} := \Bigl|\int \Bigl(e(x,x,\tau) - h^{-d}\mathcal{N}^{\mathsf{MW}}(x,\tau) - h^{-d}\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x,\tau)\Bigr)\psi(x)\,dx\Bigr| \le Ch^{1-d} + C\mu h^{\frac{4}{3}-d} + C\mu^{1-\frac{l}{2}}h^{1-d}|\log h|^{-\frac{\sigma}{2}} \tag{20.6.76}$$

holds where

$$|\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x,\tau)| \le C\mu^{\frac{1}{2}}h \tag{20.6.77}$$

is defined by formula (20.6.70) *and in the general case one cannot improve estimates* (20.6.75), (20.6.77).

(ii) If one of the following conditions

(20.6.78) *There are no third-order resonances;*

$$g^{jk} = \mathsf{const},\ F_{jk} = \mathsf{const} \tag{20.6.79}$$

is fulfilled then estimate (20.6.76) *holds for* R^{MW}.

As $r \ge 2$ and condition (20.6.71) is fulfilled let us consider

$$\boldsymbol{n}_0(x,\hbar,\tau) := \#\bigl\{\alpha \in \mathbb{Z}^{+r}, \sum_j (2\alpha_j+1)f_j(x)\hbar + V(x) < \tau\bigr\}. \tag{20.6.80}$$

Let us assume that estimate (20.4.93) holds with $\nu(\hbar) = o(\hbar)$.

Remark 20.6.20. Estimate (20.4.93) holds with $\nu(\hbar) = \hbar$ for sure and it holds with $\nu(\hbar) = \hbar^{\kappa}$, $\kappa > 1$ if <u>either</u> f_j have constant multiplicities and

$$\mathsf{rank}\{\nabla\frac{f_2}{f_1}, \dots, \nabla\frac{f_r}{f_1}\} \ge \kappa - 1 \qquad \forall x \tag{20.6.81}$$

<u>or</u> $\frac{f_j}{f_1} = \mathsf{const}$ and the system $\bigl\{\frac{f_2}{f_1}, \dots, \frac{f_r}{f_1}\bigr\}$ has some Diophantine properties[28)].

Under condition (20.4.93) we can improve Theorems 20.6.17 and 20.6.19 as $q = 1, 2$:

[28)] See Examples 20.4.17 and 20.4.18 for details.

Theorem 20.6.21. *Let* $q = 1, 2$ *and conditions* (20.1.1)–(20.1.11), (20.6.71), (20.4.93) *be fulfilled. Then there exist two framing approximations such that*

(i) For $q = 2$, $(l, \sigma) = (1, 1)$, $(\bar{l}, \bar{\sigma}) = (2, 1)$ *estimate*

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d} + C\nu(\mu h)h^{\frac{2}{3}-d} \tag{20.6.82}$$

holds.

(ii) For $q = 1$, $(1, 2) \preceq (l, \sigma) \preceq (2, 0)$ *estimates*

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d} + C\mu^{-\frac{1}{2}}h^{\frac{1}{2}-d}|\log h|^{\frac{1}{2}} + C\nu(\mu h)\bigl(h^{\frac{1}{3}-d} + \mu^{-\frac{1}{2}}h^{-d}\bigr), \tag{20.6.83}$$

$$\begin{aligned}\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}} \le{}& Ch^{1-d} + C\mu^{\frac{1}{2}-l}h^{\frac{1}{2}-d}|\log h|^{\frac{1}{2}-\sigma}+\\ & C\bigl(\nu(\mu h) + \mu^{-1}\bigr) \times \bigl(h^{\frac{1}{3}-d} + \mu^{-\frac{l}{2}}h^{-d}|\log h|^{-\frac{\sigma}{2}}\bigr)h^{-d}\end{aligned} \tag{20.6.84}$$

hold.

(iii) For $q = 1$, $(1, 2) \preceq (l, \sigma) \preceq (2, 0)$ *as one of conditions* (20.6.78), (20.6.79) *is fulfilled the following estimate holds:*

$$\begin{aligned}\mathsf{R}^{\mathsf{MW}} \le{}& Ch^{1-d} + C\mu^{\frac{3}{2}-l}h^{\frac{3}{2}-d}|\log h|^{\frac{1}{2}-\sigma}+\\ & C\nu(\mu h)\bigl(h^{\frac{1}{3}-d} + \mu^{-\frac{l}{2}}h^{-d}|\log h|^{-\frac{\sigma}{2}}\bigr).\end{aligned} \tag{20.6.85}$$

Recall that Proposition 20.6.13 estimates $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}$ defined as a difference between expressions (20.6.7) and (20.6.65). One can prove easily that

Theorem 20.6.22. *In the framework of Statements (i)–(v) of Proposition 20.6.13 the corresponding estimates hold also for* $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}$ *defined by* (20.6.76) *and* (20.6.64).

Case of f_j Having Constant Multiplicities

Assume now that

(20.6.86) Matrix (F^j_k) has eigenvalues $\pm i f_m$ of constant multiplicities.

Then the microhyperbolicity assumption is equivalent to

$$|V + \sum_{1\le j\le r} f_j\tau_j| + |\nabla(V + \sum_{1\le j\le r} f_j\tau_j)| \ge \epsilon \qquad \forall \tau_1, \dots, \tau_r \in \mathbb{R}^+. \tag{20.6.87}$$

Further, if

$$f_2/f_1 = \mathsf{const}, \dots, f_r/f_1 = \mathsf{const} \tag{20.6.88}$$

then (20.6.87) is equivalent to

$$|\nabla \frac{V}{f_1}| \ge \epsilon. \tag{20.6.89}$$

Theorem 20.6.23. *Let conditions* (20.1.1)–(20.1.11), (20.6.72), *and* (20.6.86) *be fulfilled. Then there exist two framing approximations such that*

(i) For $q = 2$, $(1,1) \preceq (l,\sigma) \preceq (2,0)$, $(\bar{l},\bar{\sigma}) = (2,1)$ *estimate*

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d} + C\mu h^{1-d+\frac{ql}{l+2}} |\log \mu|^{-\frac{q\sigma}{(l+2)}} \tag{20.6.90}$$

holds.

In particular, $R^{\mathsf{MW}} \le Ch^{1-d}$ *as* $(l,\sigma) = (2,0)$.

(ii) As $q = 1$, $(1,2) \preceq (l,\sigma) \preceq (2,0)$, $(\bar{l},\bar{\sigma}) = (2,1)$ *estimate*

$$\begin{aligned} \mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}} \le{}& Ch^{1-d} + C\mu h^{2-d-\frac{2}{l+2}} |\log h|^{-\frac{\sigma}{(l+2)}} + \\ & Ch^{1-d}(\mu h|\log h|)^{\frac{1}{2}} \mu^{1-l} |\log h|^{-\sigma} + C\mu^{1-\frac{l}{2}} h^{1-d} |\log h|^{-\frac{\sigma}{2}} + \\ & C\mu^{\frac{2l+1}{3l}} h^{\frac{4}{3}-d} |\log h|^{-\frac{\sigma}{3l}} \end{aligned} \tag{20.6.91}$$

holds.

(iii) As $q = 1$, $(1,2) \preceq (l,\sigma) \preceq (2,0)$, $(\bar{l},\bar{\sigma}) = (2,1)$ *and one of conditions* (20.6.78), (20.6.79) *is fulfilled estimate*

$$\begin{aligned} \mathsf{R}^{\mathsf{MW}} \le{}& Ch^{1-d} + C\mu h^{2-d-\frac{2}{l+2}} |\log h|^{-\frac{\sigma}{(l+2)}} + \\ & Ch^{1-d}(\mu h|\log h|)^{\frac{1}{2}} \mu^{1-l} |\log h|^{-\sigma} + C\mu^{1-\frac{l}{2}} h^{1-d} |\log h|^{-\frac{\sigma}{2}} \end{aligned} \tag{20.6.92}$$

holds.

Remark 20.6.24. (i) Note that the right-hand expression of (20.6.91) contains an extra term (in comparison with (20.6.92).

(ii) Also note that for $q = 1$ estimate (20.6.90) is better than (20.2.32) as

$$\mu^{l+1} h |\log h|^{\sigma+1} \ge 1; \tag{20.6.93}$$

in particular, as $(l, \sigma) = (2, 0)$ we get estimate $\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d} + C\mu h^{\frac{3}{2}-d}$ which always is better than (20.2.32) as $\mu \ge \bar{\mu}_{1(1)}$.

Under condition (20.4.93) one can improve Theorem 20.6.23:

Theorem 20.6.25. *Let conditions* (20.1.1)–(20.1.11), (20.6.72) *and* (20.6.86), *and* (20.4.93) *be fulfilled. Then there exist two framing approximations such that*

(i) As $q = 2$, $(1, 1) \preceq (l, \sigma) \preceq (2, 0)$, $(\bar{l}, \bar{\sigma}) = (2, 1)$ *estimate*

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d} + C\nu(\mu h) h^{1-d+\frac{l-2}{l+2}} |\log h|^{-\frac{2\sigma}{l+2}} \tag{20.6.94}$$

holds.

(ii) As $q = 1$, $(1, 2) \preceq (l, \sigma) \preceq (2, 0)$, $(\bar{l}, \bar{\sigma}) = (2, 1)$ *and one of conditions* (20.6.78), (20.6.79) *is fulfilled estimate*

$$\begin{aligned} \mathsf{R}^{\mathsf{MW}} \le {} & Ch^{1-d} + Ch^{-d}(\mu h |\log h|)^{\frac{1}{2}} \mu^{-l} |\log h|^{-\sigma} \\ & C\nu(\mu h)\Big(h^{1-d-\frac{2}{l+2}} |\log h|^{-\frac{\sigma}{(l+2)}} + C\mu^{-\frac{l}{2}} h^{-d} |\log h|^{-\frac{\sigma}{2}}\Big) \end{aligned} \tag{20.6.95}$$

holds.

(iii) If $f_1 = \mathsf{const}$, *...*, $f_r = \mathsf{const}$, $(l, \sigma) \succeq (2, 0)$ *and nondegeneracy assumption* (20.2.42) *is fulfilled then* $\mathsf{R}^{\mathsf{MW}} = O(h^{1-d})$.

20.7 Stronger Magnetic Field: Calculations and Main Results

In this section we consider the cases of the intermediate magnetic field as $\epsilon(h|\log h|)^{-1} \le \mu \le \epsilon h^{-1}$, or as the intermediate-strong magnetic field $\epsilon h^{-1} \le \mu \le C_0 h^{-1}$ and of strong magnetic field as $\mu \ge C_0 h^{-1}$.

20.7.1 Intermediate Magnetic Field

Calculations

Consider now case of the intermediate magnetic field $\epsilon(h|\log h|)^{-1} \le \mu \le \epsilon h^{-1}$. The same arguments work again, with no outer zone and necessity to join outer and intermediate zones. Because of this one should deal directly with (20.6.7) rather than (20.6.26).

Therefore (20.6.7) is preserved modulo $O(h^s)$ as T_n are replaced by some lesser values which depend on the assumptions described in the previous subsections; then we apply the method of the successive approximations in the same way as above replacing U by $\bar{\mathsf{U}}$ and $\tilde{\psi}$ by ψ_0 and estimating an error in (20.6.7) cause by this; finally we replace T'_n by T_n again with $O(h^s)$ error and then T_n by $+\infty$. We leave easy details to the reader.

Then one can easily prove several propositions below; we leave easy and standard details to the reader. The first proposition holds for both intermediate and intermediate-strong magnetic field:

Proposition 20.7.1. *Let* $\epsilon(h|\log h|)^{-1} \le \mu \le C_0 h^{-1}$, $(l,\sigma) = (1,1)$ *as* $q \ge 2$, $(l,\sigma) = (1,2)$ *as* $q = 1$. *Then one can rewrite* (20.6.7) *as* (20.6.46) *with an error not exceeding* Ch^{1-d} *as* $q \ge 2$, $C\mu^{\frac{1}{2}}h^{1-d}$ *as* $q = 1$ [29].

Proposition 20.7.2. *Let* $\epsilon h^{-1}|\log h|^{-1} \le \mu \le \epsilon h^{-1}$, $(l,\sigma) \succeq (1,1)$ *as* $q \ge 2$, $(l,\sigma) \succeq (1,2)$ *as* $q = 1$. *Then under the microhyperbolicity condition (see Definition 20.1.2) one can rewrite* (20.6.7) *as* (20.6.46) *with an error not exceeding* Ch^{1-d}.

Proposition 20.7.3. *Statements of Propositions 20.6.9–20.6.13, Theorem 20.6.22, Problem 20.6.10 and Example 20.6.14 remain true for* $\epsilon(h|\log h|)^{-1} \le \mu \le \epsilon h^{-1}$.

Proofs of Propositions 20.7.1–20.7.3. We leave all easy details to the reader.□

Main Results

Recall that in the case of the intermediate magnetic field we pick up

$$\varepsilon \asymp C(\mu^{-1}h|\log h|)^{\frac{1}{2}} + Ch|\log h| \asymp h|\log h| \tag{20.7.1}$$

[29] One can improve it as there are no unremovable cubic terms but there is no point as the general remainder estimate $O(\mu h^{\frac{4}{3}-d})$ is larger.

The first of our theorems hold for both intermediate and intermediate-strong magnetic field:

Theorem 20.7.4. *Let* $\epsilon(h|\log h|)^{-1} \le \mu \le C_0 h^{-1}$. *Let conditions* (20.1.1)–(20.1.11) *and* (20.6.72) *be fulfilled. Then there exist two framing approximations (see footnote* [18)] *of Chapter 18) such that*

(i) For $q \ge 3$, $(l,\sigma) = (1,1)$, $(\bar{l},\bar{\sigma}) = (2,1)$ *sharp remainder estimate* $\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d}$ *holds.*

(ii) For $q = 2$, $(l,\sigma) = (1,1)$, $(\bar{l},\bar{\sigma}) = (2,1)$ *estimate* (20.6.74) *holds:* $\mathsf{R}^{\mathsf{MW}} \le C\mu h^{\frac{5}{3}-d}$.

(iii) For $q = 1$, $(1,2) \preceq (l,\sigma) \preceq (\bar{l},\bar{\sigma}) = (2,1)$ *estimate* (20.6.75) *holds:* $\mathsf{R}^{\mathsf{MW}} \le C\mu h^{\frac{4}{3}-d}$.

In all other cases we need to restrict ourselves here to the intermediate magnetic field case. Here conclusion is very simple: all results of Subsection 20.6.6 remain valid:

Theorem 20.7.5. *Let* $\epsilon(h|\log h|)^{-1} \le \mu \le \epsilon h^{-1}$. *Let conditions* (20.1.1)–(20.1.11), *and* (20.6.72) *be fulfilled.*

Further, let <u>*either*</u> *the microhyperbolicity assumption (see Definition 20.1.2)* <u>*or*</u> $(l,\sigma) = (2,0)$, $f_1 = \mathsf{const}, \dots, f_r = \mathsf{const}$ *and the nondegeneracy assumption* (20.2.42) *be fulfilled.*

Then there exist two framing approximations such that $\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d}$.

Theorem 20.7.6. *In the case of the intermediate magnetic field* $\epsilon(h|\log h|)^{-1} \le \mu \le \epsilon h^{-1}$ *all statements of Theorems 20.6.23 and 20.6.25 remain true.*

Proofs of Theorems 20.7.4–20.7.6. We leave all easy details to the reader.□

20.7.2 Intermediate-Strong Magnetic Field

Calculations

Consider now case of the intermediate-strong magnetic field $\epsilon h^{-1} \le \mu \le C_0 h^{-1}$. The same arguments work again, with no intermediate zone anymore. We do not need consider the general case as it is covered by Theorem 20.7.4. Thus we should consider only $q = 1, 2$.

On the other hand, the microhyperbolicity and non-degeneracy assumptions should be modified to Definition 20.5.9 and (20.5.10) (again the non-degeneracy assumption is considered as $f_1, \dots, f_r/f_1$ are constant).

Finally, number theoretical arguments are no more applicable at all.

Again one can prove easily propositions below; we leave all easy details to the reader:

Proposition 20.7.7. *Let* $\epsilon h^{-1} \le \mu \le C_0 h^{-1}$, *and* $(l,\sigma) = (1,1)$ *as* $q \ge 2$, $(l,\sigma) = (1,2)$ *as* $q = 1$. *Then one can rewrite expression* (20.6.7) *as* (20.6.46) *with an error not exceeding* Ch^{1-d} *as* $q \ge 2$, $Ch^{\frac{1}{2}-d}$ *as* $q = 1$.

Proposition 20.7.8. *Let* $\epsilon h^{-1} \le \mu \le C_0 h^{-1}$, $(l,\sigma) = (1,1)$ *as* $q \ge 2$, $(l,\sigma) = (1,2)$ *as* $q = 1$. *Then under the microhyperbolicity condition (see Definition 20.5.9) with* ℓ *independent on* $\boldsymbol{\tau}$ *one can rewrite* (20.6.7) *as* (20.6.46) *with an error not exceeding* Ch^{1-d}.

Proposition 20.7.9. *Statements of Propositions 20.6.9 and Problem 20.6.10 remain true as* $\epsilon h^{-1} \le \mu \le C_0 h^{-1}$.

Main Theorems

We arrive then to the following theorems:

Theorem 20.7.10. *Let* $\epsilon h^{-1} \le \mu \le C_0 h^{-1}$. *Let conditions* (20.1.1)–(20.1.11), *and* (20.6.72) *be fulfilled.*

Further, let <u>*either*</u> *the microhyperbolicity assumption (see Definition 20.5.9)* <u>*or*</u> $(l,\sigma) = (2,0)$, $f_1 = \mathsf{const}, \dots, f_r = \mathsf{const}$ *and the nondegeneracy assumption* (20.5.10) *be fulfilled.*

Then there exist two framing approximations such that $\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d}$.

Theorem 20.7.11. *In the case of the intermediate-strong magnetic field* $\epsilon h^{-1} \le \mu \le C_) h^{-1}$ *all statements of Theorem 20.6.23 remain true.*

Proofs of Theorems 20.7.10–20.7.11. We leave all easy details to the reader.□

20.7.3 Strong Magnetic Field

Consider now case of strong magnetic field $\mu \ge C_0 h^{-1}$ and a magnetic Schrödinger-Pauli operator.

Then we need to modify condition $(20.1.9)_3$, replacing $(20.1.9)_{1,3}$ by

$$g^{jk}, F_{jk} \in \mathscr{C}^{\bar{l},\bar{\sigma}}, \qquad W = V + \mu h \sum_j f_j \in \mathscr{C}^{l,\sigma} \tag{20.7.2}$$

and assuming that

$$W + 2f_j \mu h \geq \epsilon \qquad \forall j. \tag{20.7.3}$$

Under these assumptions we have really a scalar operator, and should take

$$\varepsilon = Ch|\log h| \tag{20.7.4}$$

and the microhyperbolicity condition (see Definition 20.5.9) transforms into

$$|W| + |\nabla W| \geq \epsilon \tag{20.7.5}$$

while non-degeneracy assumption (20.5.10) transforms into

$$|W| + |\nabla W| \leq \epsilon \implies |\det \mathsf{Hess}\, W| \geq \epsilon. \tag{20.7.6}$$

Skipping calculations (which are not much different from what we did before, so we leave the easy details to the reader) we arrive immediately to

Theorem 20.7.12. *Let $\mu \geq C_0 h^{-1}$ and let conditions* (20.1.1)–(20.1.8), $(20.1.9)_2$, (20.6.72), (20.7.2), *and* (20.7.3) *be fulfilled. Then*

(i) For $q \geq 3$ remainder estimate $\mathsf{R}^{\mathsf{MW}} \leq C\mu^r h^{r+1-d}$ holds.

(ii) For $q = 1, 2$ and the microhyperbolicity condition (20.5.9) *fulfilled, the remainder estimate $\mathsf{R}^{\mathsf{MW}} \leq C\mu^r h^{r+1-d}$ holds.*

(iii) For $q = 2$ and $(l, \sigma) \succeq (2, 1)$ the remainder estimate $\mathsf{R}^{\mathsf{MW}} \leq C\mu^r h^{r+1-d}$ holds.

(iv) For $q = 1$, $(l, \sigma) \succeq (2, 1)$, and non-degeneracy condition (20.5.10) *fulfilled the remainder estimate $\mathsf{R}^{\mathsf{MW}} \leq C\mu^r h^{r+1-d}$ holds.*

(v) As <u>either</u> $q = 2$, $(l, \sigma) \prec (2, 0)$ <u>or</u> $q = 1$ then the remainder estimate

$$\mathsf{R}^{\mathsf{MW}} \leq C\mu^r h^{r-d} \begin{cases} h^{\frac{2l}{l+2}} |\log h|^{-\frac{2\sigma}{l+2}} & \text{as} \quad q = 2, \\ h^{\frac{l}{l+2}} |\log h|^{-\frac{\sigma}{l+2}} & \text{as} \quad q = 1 \end{cases} \tag{20.7.7}$$

holds.

Remark 20.7.13. Under assumption (20.7.3) the main part of asymptotics is

$$h^{-d}\int \mathcal{N}_d^{\mathsf{MW}}(x,\tau)\psi(x)\,dx = \omega_{d-2r}(2\pi)^{-d+r}\mu^r h^{-d+r}\int \Bigl(\tau - \sum_j f_j\mu h - V\Bigr)_+^{\frac{q}{2}} f_1\cdots f_r\sqrt{g}\psi(x)\,dx. \tag{20.7.8}$$

20.8 Degenerating Magnetic Field

Remark 20.8.1. One can get rid of assumption $|V|\ge\epsilon$ as $\mu\le\epsilon h^{-1}$ easily (see Subsubsection 18.9.5.1).

Condition that non-zero eigenvalues of F_k^j are uniformly disjoint from 0 is more subtle. Basically we want to apply arguments of Subsubsection 18.9.5.2 but we should take into account that different eigenvalues have different magnitudes.

Usually we assume that $q=1$ (so at generic point only 1 eigenvalue is 0), but at this moment we just assume

(20.8.1) At each point matrix $F_k^j(x)$ has at least q eigenvalues equal to 0 and let $\pm if_j(x)$ denote other eigenvalues (which may be also vanish at some points[30)]

Let us introduce

$$\gamma(x) = \epsilon_0 \min_j f_j(s) \tag{20.8.2}$$

where in partitions but not in statements we replace γ by $\mathsf{max}(\gamma,\bar\gamma)$ and $\bar\gamma$ will be chosen later.

20.8.1 Weak Magnetic Field

Consider some point y, its $\gamma(y)$ vicinity and classify $f_1,\dots,f_r$ into two groups: those which are greater than $4\nu\gamma$ (we denote them by $f_{r'+1},\dots,f_r$) and those which are less than $\nu\gamma$ (we denote them by $f_1,\dots,f_{r'}$); an appropriate constant

[30)] We will need to assume that these eigenvalues do not vanish in the generic points.

$\nu = \nu(y) \in [c_0, c_1]$ exists with arbitrarily large $c_0 \ge 1$ and $c_1 = c_1(c_0)$. Then the similar inequalities

$$f_j(x) \le 2\nu\gamma(y) \quad \forall j = 1, \dots, r', \qquad f_j \ge 3\nu\gamma(y) \quad \forall j = r'+1, \dots, r \tag{20.8.3}$$

hold in γ-vicinity of y; here $r' = r'(y)$.

We claim that

(20.8.4) As $\gamma(y) \ge C_0\mu^{-1}$ the contribution of $B(y, \gamma(y))$ to the remainder does not exceed

$$C\big(\gamma^{-1} + \mu(\mu h|\log h|)^{\frac{q}{2}}\big)h^{1-d}\gamma^d. \tag{20.8.5}$$

Really, one can use the weak magnetic field approach and take $T_* = \epsilon\mu^{-1}$. Further, one can take $T^* = T^*(y, |\xi'''|) = \epsilon\gamma(y)|\xi'''|$ and after summation with respect to ξ''' we get (20.8.5) as $q \ge 2$; as $q = 1$ we get the same expression albeit with an extra logarithmic factor in the first term and we get rid of it exactly in the same manner as in Subsubsection 20.2.2.2.

Here $C\mu(\mu h|\log h|)^{\frac{q}{2}}h^{1-d}\gamma^d$ estimates a contribution of the complementary zone $\Omega^{\mathsf{c}}_{\mathsf{out}} = \{|\xi'''| \le \bar\rho_1\}$ with $\bar\rho_1 = C\max\big((\mu h|\log h|)^{\frac12}, \mu^{-1}\gamma^{-1}\big)$ as $\bar\rho_1 = C(\mu h|\log h|)^{\frac12} \ge C\mu^{-1}\gamma^{-1}$; otherwise contribution of the complementary zone does not exceed $C\bar\rho_1\mu h^{1-d}\gamma^d \asymp C\gamma^{-1}h^{1-d}$ which is the first term in (20.8.5).

This estimate (20.8.5) could be improved but it does not make any difference. Summation with respect to all balls with $\gamma \gtrsim \bar\gamma$ results in

$$Ch^{1-d}\int_{\{\gamma(x)\ge\bar\gamma\}} \gamma^{-1}\,dx + C\mu(\mu h|\log h|)^{\frac{q}{2}}h^{1-d}; \tag{20.8.6}$$

while contribution of all balls with $\gamma \lesssim \bar\gamma$ does not exceed

$$C\mu h^{1-d}\int_{\{\gamma(x)\le C_0\bar\gamma\}} dx; \tag{20.8.7}$$

Thus the total remainder R^{T} does not exceed

$$Ch^{1-d} + C\mu(\mu h|\log h|)^{\frac{q}{2}}h^{-d}h^{1-d} + Ch^{1-d}\Big(\int_{\{\gamma(x)\ge\bar\gamma\}} \gamma^{-1}\,dx + C\mu\int_{\{\gamma(x)\le C_0\bar\gamma\}} dx\Big) \tag{20.8.8}$$

where we included the first term for "compatibility" only (it does not exceed the second line anyway) and the optimal results are achieved as $\bar{\gamma} \asymp \mu^{-1}$.

We leave to the reader to prove by our standard arguments that the same estimate holds for R^{MW} and $\mathsf{R}^{\mathsf{W}}_{(\infty)}$. Then we arrive to the following

Theorem 20.8.2. *(i) Theorem 20.2.10 remains true without assumption* (20.1.4)*, replaced by* (20.8.1)*, but with the right-hand expression* (20.8.8)*.*

(ii) Under assumption

$$\int_{\mu^{-1}}^{1} \gamma^{-1} d\mu(\gamma) < \infty \tag{20.8.9}$$

with

$$\mu(\gamma) \coloneqq \mathsf{mes}\big(\{x\colon \gamma(x) < \gamma\}\big) \tag{20.8.10}$$

an extra term (the second line in (20.8.8)*) is* $O(h^{1-d})$ *and one can skip it.*

Problem 20.8.3. In our usual manner get rid of logarithmic factor in the second term of (20.8.8).

Problem 20.8.4. Prove that in the generic setting $\mu(\gamma) = O(\gamma^3)$.

20.8.2 Temperate Magnetic Field

General Case

Situation is rather simple if $r = 1$ (or all eigenvalues f_j have the same magnitude $\asymp \gamma$): then we can use the same simple rescaling technique as in Subsubsection 18.9.5.2. However in the general case there are some eigenvalues $\asymp \gamma$ and there are some much larger eigenvalues, may be of magnitude 1.

Notice that after rescaling the eigenvalue of magnitude γ would have magnitude $\mu\gamma^2$ and to deal with this properly we need $\mu\gamma^2 \gg 1$ (or at least $\mu\gamma^2 \ge C_0$) i.e. $\gamma \gg \mu^{-\frac{1}{2}}$ (or at least $\gamma \ge C_0\mu^{-\frac{1}{2}}$).

One possible approach is to apply in the *singular zone* $\{x:\ \gamma(x) \le \bar{\gamma}\}$ the rough estimate $C\mu h^{1-d}\mu(\bar{\gamma})$ which works rather well as

$$\mu(\gamma) = O(\gamma^{\kappa}) \qquad \text{with } \ \kappa > 2; \tag{20.8.11}$$

then we can take $\bar{\gamma} = \mu^{-1/\kappa}$.

In the *regular zone* $\{x\colon \gamma(x)\ge \bar\gamma\}$ one can go to the same canonical form as without degeneration but instead of error $O(\mu^{-m})$ error would become $O(\mu^{-m}\gamma^{-2m})$ in the principal part, and instead of $O(\mu^{-m}h^n)$ error would become $O(\mu^{-m}h^n\gamma^{-2m-n})$ in the lower terms.

Then we conclude that

(20.8.12) As $\gamma(y)\ge C_0\mu^{-\frac12}$ the contribution of $B(y,\gamma(y))$ to the Tauberian remainder does not exceed $O\bigl(h^{1-d}\gamma^{d-1}\bigr)$ as $q\ge 3$, $O\bigl(h^{1-d}\gamma^{d-1}+\mu h^{\frac53-d}\gamma^{d-\frac23}\bigr)$ as $q=2$, and $O\bigl(h^{1-d}\gamma^{d-1}+\mu h^{\frac43-d}\gamma^{d-\frac13}\bigr)$ as $q=1$.

Then summation with respect to partition results in

(20.8.13) Under assumption (20.8.11) Tauberian remainder estimate is $O\bigl(h^{1-d}\bigr)$ as $q\ge 3$, $O\bigl(h^{1-d}+\mu h^{\frac53-d}\bigr)$ as $q=2$, and $O\bigl(h^{1-d}+\mu h^{\frac43-d}\bigr)$ as $q=1$.

Here we took into account that the contribution of the *singular zone* $\{x\colon \gamma(x)\ge \bar\gamma\}$ is $O(\bar\gamma^{\kappa}\mu h^{1-d})=O(h^{1-d})$.

Let us calculate errors. First of all, in the regular zone removal of $O(\mu^{-1}\gamma^{-2})$ terms brings an error $O((\mu^{-1}\gamma^{-2})^{q/2}h^{-d}\gamma^d)$ as $q\ge 2$ (we apply all the arguments used to and then summation with respect to partition results in $O(h^{1-d})$ under assumption (20.8.11). The same is true if we remove $O(\mu^{-2}\gamma^{-4})$ terms as $q=1$.

As $q=1$ and there are unremovable $O(\mu^{-1}\gamma^{-2})$ terms we just go after $\mathsf{R}^{\mathsf{MW}}_{\mathsf{corr}}$; as our goal is to estimate a correction we use the weak magnetic field estimate as $\mu\le h^{-\frac12}$ and also as $\mu\ge h^{-\frac12}$ but $\gamma\le(\mu^2h)^{-\frac12}$; in the latter case summation over such partition elements results in $O(\mu^{\frac12}h^{1-d})$.

Further, as $\gamma\ge(\mu^2h)^{-\frac12}$ and we estimate a correction we follow the arguments of Section 20.6. Finally we use a mollification parameter $\varepsilon=(\mu\gamma^2)^{-1}$ after rescaling or $\varepsilon=(\mu\gamma)^{-1}$ before. Leaving many rather not-very difficult but still delicate details to the reader we formulate the following

Problem 20.8.5. Prove that under assumption (20.8.11) Theorems 20.6.17 and 20.6.19(i),(ii) remain true[31)].

Results under Microhyperbolicity or Non-Degeneracy Assumptions

It becomes a bit difficult to formulate the microhyperbolicity or non-degeneracy assumptions. First we assume that

[31)] Obviously, condition (20.6.79) in Theorem 20.6.19(ii) fails.

(20.8.14) f_j ($j = 2, \dots, r$) are disjoint from 0, $f_1 \asymp \gamma(x) = \mathsf{dist}(x, Y)$ where Y is $\mathscr{C}^{2,1}$-manifold of codimension 3 and $|\nabla f_1| \asymp 1$.

Then without any loss of the generality one can assume that

(20.8.15) At points of Y matrices (g^{jk}) and (F_{jk}) satisfy $g^{jk} = 0$ as $j = 1, 2, 3$, $k = 4, \dots, d$ (and symmetrically) and $F_{jk} = 0$ as $j = 1, 2, 3$, $k = 1, \dots, d$ (and symmetrically).

Let us consider submatrices g'', F'' of these matrices, consisting of elements g^{jk}, F_{jk} with $j, k = 4, \dots, d$. Then canonical form is

$$\mathsf{a}(y, z; \xi'; \zeta_1, \dots, \zeta_r) = \mathsf{g}(y, z; \xi') + f_1(y, z)|\zeta_1|^2 + \mathsf{a}''(y, z; z_2, \dots, \zeta_r) + V(y, z) + \dots \tag{20.8.16}$$

where y are coordinates along Y and $z = (z_1, z_2, z_3)$ are additional coordinates (so $Y = \{z = 0\}$) and

$$|\nabla_z f_1| \asymp 1, \qquad |\nabla_y f_1| = O(|z|), \tag{20.8.17}$$

$\mathsf{g}(y, z; \eta)$ is a non-degenerate quadratic form with respect to η, $\mathsf{a}'(y, z; z_2, \dots, \zeta_r)$ is a Hermitian form with respect to $\zeta_2, \dots, \zeta_r$, .

Further, $f_1^2 = b(y; z) + O(|z|^3)$ where $b(y; z)$ is a non-degenerate quadratic form with respect to z and without any loss of the generality one can assume that $b(y; z) = |z|^2$. Then in the ball $B((y, z), \epsilon_0|z|)$ we can use microhyperbolicity arguments as long as either $|\xi'| \ge \epsilon_0$ or $|\zeta_1| \ge C_0$.

Therefore if we are close to the energy level 0, we need to consider only $|\xi'| \le \epsilon_0$ and $|\zeta_1| \le C_0$ but then the derivatives with respect to z of the first two terms in the right-hand expression of (20.8.16) are $O(|z|)$ and we need to consider only derivatives of

$$\mathsf{a}''(y, z; \zeta_2, \dots, \zeta_r) + V(y, z) \tag{20.8.18}$$

as long as it close to our energy level 0.

Therefore we can apply microhyperbolicity arguments there as well provided ψ is supported in the small tubular vicinity of Y and (20.8.18) satisfies microhyperbolicity assumption with respect to y (as $z = 0$). Leaving many rather not-very difficult but still delicate details to the reader we formulate Problem 20.8.6(i) below.

On the other hand, assume that (20.8.18) is not microhyperbolic. Assume that

(20.8.19) Either $r = 2$ or $f_2|_Y, \dots, f_r|_Y$ are constant.

Further, assume that $V/f_2|_Y$ has only non-degenerate critical points. Without any loss of the generality one can assume that $f_2 = \mathsf{const}$ and there is just one critical point 0. One can see easily that if $\mathsf{g}(y, z; \xi') = \bar{\mathsf{g}}(z; \xi') + O(|y|^2)$ and $b(y; z) = \bar{b}(z) + O(|y|^2)$ then $|\nabla_y \mathsf{a}(y, z; \xi'; \zeta_1, \dots, \zeta_r)| \asymp |y|$ at energy level 0 and we can apply non-degeneracy arguments easily. In the general case however one can see easily that on the energy level 0

$$|\nabla_y \mathsf{a}(y, z; \xi'; \zeta_1, \dots, \zeta_r)| \ge \epsilon |y - \bar{y}(z, \xi', |\zeta_1|)|$$

and again we can apply non-degeneracy arguments. Leaving many rather not-very difficult but still delicate details to the reader we formulate Problem 20.8.6(ii) below.

Problem 20.8.6. Let assumptions (20.8.14)–(20.8.15) be fulfilled and let a cut-off function ψ be supported in the sufficiently small vicinity of Y.

(i) Introduce a microhyperbolicity assumption as a microhyperbolicity assumption for these matrices g'', F'' and V along Y and prove under this assumption the remainder estimate $O(h^{1-d})$.

(ii) Also, as (20.8.19) is fulfilled, formulate a non-degeneracy assumption as non-degeneracy of critical points of $V/f_2|_Y$ and prove under this assumption the remainder estimate $O(h^{1-d})$.

Other Improved Results

Situation becomes even more delicate as only assumptions (20.8.14)–(20.8.15) are fulfilled. Let us assume that either $r = 2$ or $f_2|_Y, \dots, f_r|_Y$ are constant. Then we can use "microhyperbolicity" arguments and take $T_* = \epsilon\mu^{-1}$ unless gradient with respect to z is small enough which is the case as in the decomposition $|\alpha_1 - \bar{\alpha}_1| \ge C|\log h|$ which boils down to an extra factor $C\mu h|\log h|$ in the remainder estimate.

Using rescaling technique we can get of the logarithm improving this factor to $C\mu h$.

Problem 20.8.7. Let assumptions (20.8.14)–(20.8.15), and (20.8.19) be fulfilled and let a cut-off function ψ be supported in the sufficiently small vicinity of Y.

(i) Prove remainder estimates

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d} + C\mu^2 \begin{cases} h^{\frac{8}{3}-d} & \text{as } \ q=2, \\ h^{\frac{7}{3}-d} & \text{as } \ q=1; \end{cases} \tag{20.8.20}$$

(ii) Provided there are no 3-rd- order resonances prove remainder estimates $\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d}$ as $q=2$ and

$$\mathsf{R}^{\mathsf{MW}} \le Ch^{1-d} + C\mu^2 h^{\frac{5}{2}-d} \qquad \text{as } \ q=1. \tag{20.8.21}$$

(iii) Improve estimate (20.8.21) depending on (I,σ).

(iv) Derive estimates for $\mathsf{R}^{\mathsf{W}}_{(\infty)}$ improving those of Subsection 20.8.1; here and in (v) one should consider also $q \ge 3$.

(v) Under microhyperbolicity or non-degeneracy assumptions of Subsubsection 20.8.2.1 further improve these estimates.

20.8.3 Intermediate and Intermediate-Strong Magnetic Field

Intermediate magnetic field basically repeats the intermediate magnetic field case leading us to Part (i) of 20.8.7 below; in the case of the intermediate-strong magnetic field the microhyperbolicity or non-degeneracy conditions should be modified and Problem 20.8.7 should be skipped, leading us to Part (ii) below:

Problem 20.8.8. (i) Solve Problems 20.8.5, 20.8.6, and 20.8.7 in the case of the intermediate magnetic field.

(ii) Solve Problems 20.8.5 and 20.8.6 in the case of the intermediate-strong magnetic field; in this case the microhyperbolicity and non-degeneracy conditions are restricted to $\alpha' = (\alpha_2, \dots, \alpha_r)$ such that

$$|V + \sum_j (2\alpha_j + 1) f_j \mu h| \le \epsilon.$$

20.8.4 Strong Magnetic Field

Theory of the strong magnetic field comes in two flavors: for the Schrödinger operator and for the Schrödinger-Pauli operator.

Schrödinger Operator

In the case of the Schrödinger operator the classically allowed zone is empty unless all f_j vanish at Y; we assume for simplicity hat

$$f_j(x) \asymp \gamma(x) = \mathsf{dist}(x, Y) \qquad \forall j = 1, \dots, r \tag{20.8.22}$$

where Y is $\mathscr{C}^{2,1}$ manifold of codimension 3. Then the classically allowed zone is where after rescaling $\mu \mapsto \mu_{\mathsf{new}} = \mu\gamma^2$ and $h \mapsto h_{\mathsf{new}} = h/\gamma$ we have $\mu_{\mathsf{new}} h_{\mathsf{new}} \lesssim 1$ i.e. $\gamma \le \bar{\gamma} := 1/(\mu h)$. In the border we have $h_{\mathsf{new}} \asymp \mu h^2$, $\mu_{\mathsf{new}} \asymp h_{\mathsf{new}}^{-1}$. To be in the framework of the semiclassical theory we need to assume that $h_{\mathsf{new}} \ll 1$ i.e. $\mu \ll h^{-2}$.

Problem 20.8.9. Under assumption (20.8.22) consider the Schrödinger operator:

(i) Let $\mu \ge C_0 h^{-2}$; then $e(x, y, 0) = O(\mu^{-\infty})$.

(ii) Let $C_0 h^{-1} \le \mu \le C_0 h^{-2}$; then the principal part of the asymptotics is given by a magnetic Weyl expression and has a magnitude of the intermediate-strong magnetic field rescaled case multiplied by $\bar{\gamma}^{3-d}$, namely

$$h_{\mathsf{new}}^{-d} \bar{\gamma}^{3-d} \asymp h^{-d} \bar{\gamma}^3 \asymp \mu^{-3} h^{-3-d} \tag{20.8.23}$$

while the error coincides with one of the intermediate-strong magnetic field case rescaled, multiplied by $\bar{\gamma}^2$:

$$\mathsf{R}^{\mathsf{MW}} \le C h_{\mathsf{new}}^{1-d} \bar{\gamma}^{3-d} \asymp C \mu^{-2} h^{-1-d} \qquad \text{as} \quad q \ge 3, \tag{20.8.24}$$

$$\mathsf{R}^{\mathsf{MW}} \le C h_{\mathsf{new}}^{\frac{q}{3}-d} \bar{\gamma}^{3-d} \asymp C \mu^{-3+\frac{q}{3}} h^{-3+\frac{2q}{3}-d} \qquad \text{as} \quad q \le 3; \tag{20.8.25}$$

in the general case.

(iii) Further, assuming that

$$|f_j - f_k| \asymp \gamma \qquad \forall j \ne k \tag{20.8.26}$$

prove for $(I, \sigma) = (2, 0)$ estimate (20.8.24) as $q = 2$ and

$$\mathsf{R}^{\mathsf{MW}} \le h_{\mathsf{new}}^{\frac{1}{2}-d} \bar{\gamma}^{3-d} \asymp \mu^{-\frac{5}{2}} h^{-2-d} \qquad q = 1; \tag{20.8.27}$$

Also consider $(I, \sigma) \prec (2, 0)$ for $q = 1, 2$ and $(I, \sigma) \succ (2, 0)$ for $q = 1$.

(iv) Furthermore, introduce a notions of microhyperbolicity and non-degeneracy and under these assumptions prove estimate (20.8.24) as $q = 1, 2$.

Schrödinger-Pauli Operator

In the of the Schrödinger-Pauli operator the classically allowed zone does not shrink and we need neither assumption that $\mu \le C_0 h^{-2}$ nor that all f_j vanish on Y; so we return to our previous assumption that only f_1 vanishes on Y.

However in the inner zone $\{x\colon \gamma(x) \le C_0/(\mu h)\}$ we do not have a "scalar operator case" anymore and it can spoil the remainder estimate as $q = 1, 2$. It does not happen, however, it does not happen as $\mu \ge C_0 h^{-2}$ as well. Thus we arrive to Parts (i) and (ii) of Problem 20.8.10 below.

However we are essentially in the "constant multiplicities of f_j" case as f_1 is disjoint from all others and with $f_2, \dots, f_r$ we are restricted to the lowest Landau level. Further, microhyperbolicity with respect to z arguments work unless $\alpha_1 \le C_1$ and therefore we can formulate notions of the microhyperbolicity and non-degeneracy in the terms of $W|_Y$.

Problem 20.8.10. Under assumptions (20.8.14)–(20.8.15) for the Schrödinger-Pauli operator

(i) Prove that the principal part of the asymptotics is as if there was no degeneration at all (i.e. of magnitude $\mu^r h^{r-d}$)).

(ii) Prove that as $q \ge 3$ the remainder estimate is as if there was no degeneration at all (i.e. $O\big(\mu^r h^{1+r-d}\big)$).

(iii) Formulate notions of the microhyperbolicity and non-degeneracy in the terms of $W|Y$ and prove that under either of these assumptions the remainder estimate is also $O\big(\mu^r h^{1+r-d}\big)$.

(iv) In the general case of $q = 1, 2$ as $(I, \sigma) = (2, 0)$ prove that the remainder estimate is as if there was no degeneration at all (i.e. $O\big(\mu^r h^{1+r-d}\big)$ for $q = 2$ and $O\big(\mu^r h^{\frac{1}{2}+r-d}\big)$ for $q = 1$); also consider $(I, \sigma) \prec (2, 0)$ for $q = 1, 2$ and $(I, \sigma) \succ (2, 0)$ for $q = 1$.

Part IX

Magnetic Schrödinger Operator in Dimension 4

Chapter 21

4D-Schrödinger Operator with the Strong Degenerating Magnetic Field

21.1 Introduction

We continue analysis of the Schrödinger operator with the strong degenerating magnetic field, started in Chapter 14. Now we consider 4-dimensional case, assuming that magnetic field is generic degenerated and under certain conditions we derive spectral asymptotics with the principal part $\asymp h^{-4}$ and the remainder estimate $O(\mu^{-1/2}h^{-3})$ where $\mu \gg 1$ is the intensity of the field and $h \ll 1$ is the Plank constant; $\mu h \le 1$.

These asymptotics can contain correction terms of magnitude $\mu^{5/4}h^{-3/2}$ corresponding to the short periodic trajectories.

21.1.1 Framework

We continue analysis of he Schrödinger operator with the strong degenerating magnetic field, started in Chapter 14

$$A = \frac{1}{2}\Big(\sum_{j,k} P_j g^{jk}(x) P_k - V\Big), \qquad P_j = D_j - \mu V_j \tag{21.1.1}$$

where g^{jk}, V_j, V are smooth real-valued functions of $x \in \mathbb{R}^2$ and (g^{jk}) is positive-definite matrix, $0 < h \ll 1$ is a Planck parameter and $\mu \gg 1$ is a

coupling parameter. We assume that A is a self-adjoint operator.

Now let us consider 4-dimensional case, assuming that magnetic field is generic degenerated, i.e. of Martinet-Roussarie type (see J. Martinet [1] and R. Roussarie [1]) which will be described in details in Subsection 21.2.1. Here we just mention that in this case magnetic field F (which currently is considered as a closed 2-form) degenerates on the manifold Σ of dimension 3 and $\dim \operatorname{Ker} F = 2$ at Σ.

Further, $\dim(\operatorname{Ker} F \cap T\Sigma) = 1$ at $\Sigma \setminus \Lambda$ and $\operatorname{Ker} F \subset T\Sigma$ at Λ where $\Lambda \subset \Sigma$ is 1-dimensional manifold; furthermore, an angle between $\operatorname{Ker} F$ and $T\Sigma$ is exactly of magnitude $\operatorname{dist}(x, \Lambda)$. Finally, in an appropriate coordinates $\Sigma = \{x_1 = 0\}$, $\Lambda = \{x_1 = x_3 = x_4 = 0\}$ and *magnetic lines* $\frac{dx}{dt} \in \operatorname{Ker} F \cap T\Sigma$ are circles $\{x_1 = 0, x_2 = \operatorname{const}, x_3^2 + x_4^2 = \operatorname{const}\}$.

Our goal is to find asymptotics of $\int e(x, x, 0)\psi(x)\, dx$ with respect to h, μ, where $e(x, y, \tau)$ is the Schwartz kernel of the spectral projector of A and ψ is a smooth function supported in the vicinity of Σ. I assume that $\mu h \le \operatorname{const}$ (otherwise $e(x, y, 0) = O(\mu^{-\infty})$).

In the next Chapter 22 we are gong to analyze the case of the generic non-degenerating field.

Recall that magnetic field is characterized by $F_{jk} = \partial_j V_k - \partial_k V_j$. However, from the point of view of the classical and quantum dynamics and spectral asymptotics properties of $(F_k^j) = (g^{jl})(F_{lk})$ are more important than those of (F_{jk}). Let $\pm if_1, \pm if_2$ be eigenvalues of (F_k^j), $f_j \ge 0$. Then with the correct choice of notations $f_1 \asymp \operatorname{dist}(x, \Sigma)$ and $f_2 \asymp 1$.

21.1.2 Plan of the Chapter

First of all, in Section 21.2 we study the *weak magnetic case* $\mu \le h^{-\delta}$ with sufficiently small exponent $\delta > 0$ and derive remainder estimate $O(\mu^{-1/2}h^{-3})$ as the main part is given by the standard Weyl formula. We also analyze there the geometry of the degenerate magnetic field and the corresponding classical and quantum dynamics.

Then, assuming that magnetic field is not weak but *moderate* $h^{-\delta} \le \mu \le ch^{-1}$ in the next Section 21.3 we consider the different canonical forms of the operator in question; this canonical forms contain powers of μ^{-1}. There are universal canonical forms and also specific canonical forms as $\operatorname{dist}(x, \Sigma) \gg \mu^{-1/2}$ or as $\operatorname{dist}(x, \Lambda) \gg \mu^{-1/2}$.

Next, in Section 21.4 we derive remainder estimates $O(\mu^{-1/2}h^{-3} + \mu^2 h^{-2})$ in the general case and $O(\mu^{-1/2}h^{-3})$ as $\mu \ge h^{-2/5}$ and some non-degeneracy

condition (depending on the magnitude of μ) is fulfilled. The main part of this estimate is given by the *standard Tauberian formula*

$$h^{-1}\int_{-\infty}^{0}\Bigl(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma u Q_y^t\Bigr)\,d\tau \tag{21.1.2}$$

where u is the Schwartz kernel of $e^{ih^{-1}tA}$, $Q=I$ and as usual $\bar{\chi}_T(t)$ equals 1 as $|t|\le \frac{1}{2}T$ and vanishes as $|t|\ge T$, T is rather arbitrary from interval $[T_0, T_1]$ and the main fight is to make T_0 as small and T_1 as large as possible. More precisely, the main part of asymptotics is given by the sum of expressions (21.1.2) with $Q=Q_{(\iota)}$ making partition of unity and $T=T_\iota\in [T_{(\iota)0}, T_{(\iota)1}]$.

After this in Section 21.5 we calculate expression (21.1.2) in more explicit way and prove main theorems.

Finally, Subsection 21.5.4 is devoted to the analysis of some special cases.

21.2 Weak Magnetic Field

21.2.1 Geometry of Degenerating Magnetic Field

So let $F_{jk}(x)=\partial_j V_k-\partial_k V_j$ be components of the matrix intensity of magnetic field; then $\omega=\sum_{j,k}F_{jk}dx_j\wedge dx_k = d\bigl(\sum_k V_k dx_k\bigr)$ is the corresponding magnetic 2-form. According to J. Martinet [1], the generic form in dimension 4 never degenerates completely and has rank 2 on the submanifold Σ of codimension 1; more precisely, let

$$\Sigma := \bigl\{x : \operatorname{rank}\omega(x)=2\bigr\}; \tag{21.2.1}$$

then

(21.2.2) For a generic 2-form ω, Σ is the smooth manifold of codimension 1 and if $\pm if_1$, $\pm if_2$ are eigenvalues of the corresponding matrix (F_{jk}), $f_2\ge f_1\ge 0$ [1)] then $f_2(x)\ge\epsilon$ and $f_1(x)\asymp \operatorname{dist}(x,\Sigma)$.

Let us consider $\operatorname{Ker}F(x)\cap T_x\Sigma$ in points of Σ. At each point it can be of dimension either 1 or 2; let

$$\Lambda := \bigl\{x\in\Sigma : \dim\bigl(\operatorname{Ker}F(x)\cap T_x\Sigma\bigr)=2\bigr\}; \tag{21.2.3}$$

according to R. Roussarie [1]

[1)] This is a temporary notation, corresponding to Euclidean metrics $g^{jk}=\delta_{jk}$.

(21.2.4) For a generic 2-form ω Λ is the smooth manifold of dimension 1; moreover, in the appropriate coordinates

$$\omega = dx_1 \wedge dx_2 + x_3 dx_2 \wedge dx_3 + d\big(x_1x_3 + x_2x_4 - \frac{1}{3}x_3^3\big) \wedge dx_4. \tag{21.2.5}$$

One can rewrite (21.2.5) as

$$d\Big(\big(x_1 - \frac{1}{2}(x_3^2 + x_4^2)\big)d(x_2 + \frac{1}{2}x_3x_4) + (x_3^2 + x_4^2)x_1 - \frac{1}{4}(x_3^2 + x_4^2)^2\Big);$$

after transformation $x_2 \mapsto x_2 - \frac{1}{2}x_3x_4$ we get instead

$$\begin{aligned}\omega &= dx_1 \wedge dx_2 - x_4 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_4 \\ &\quad + x_3 dx_2 \wedge dx_3 + x_4 dx_2 \wedge dx_4 + 2\big(x_1 - \frac{1}{2}(x_3^2 + x_4^2)\big)dx_3 \wedge dx_4 \\ &= d\big(x_1 - \frac{1}{2}r^2\big) \wedge dx_2 + r^2 dx_1 \wedge d\theta + 2\big(x_1 - \frac{1}{2}r^2\big)rdr \wedge d\theta \\ &= d\Big((x_1 - \frac{1}{2}r^2)dx_2 + (x_1 - \frac{1}{4}r^2)r^2 d\theta\Big)\end{aligned} \tag{21.2.6}$$

where $x_3 = r\cos\theta$, $x_4 = r\sin\theta$. In contrast to the original Roussarie's 2-form (21.2.5) this latter form is obviously (x_3, x_4)-circular symmetric. Then

$$\Sigma = \{x_1 = 0\}, \quad \Lambda = \{x_1 = x_3 = x_4 = 0\} \tag{21.2.7}$$

and *magnetic lines* defined by

$$\frac{dx}{dt} \in \operatorname{Ker} F(x) \cap T_x\Sigma, \quad x \in \Sigma \tag{21.2.8}$$

are helices $\{x_1 = 0, x_3 = r\cos\theta, x_4 = r\sin\theta, x_2 = \mathsf{const} - r^2\theta/2\}$ (with $r = \mathsf{const}$) winding around Λ.

Further, away from Λ one can rewrite

$$\omega = 2x_1 dx_1 \wedge d\theta + dy \wedge dz, \qquad y = x_1 - \frac{1}{2}r^2, \quad z = x_2 - 2y\theta \tag{21.2.9}$$

and therefore there I have just a direct sum of two 2-dimensional magnetic fields; the first one is a generic degenerating field and second one is nondegenerate.

21.2.2 Classical dDynamics

Classical dynamics described by the classical Hamiltonian

$$a(x,\xi) = \frac{1}{2}\Big(\sum_{j,k} g^{jk}(x)p_j p_k - V\Big), \qquad p_j = \xi_j - \mu V_j \tag{21.2.10}$$

corresponding to operator (21.1.1), is much more complicated than the geometry because it also depends on the metrics (g^{jk}) and the scalar potential V; however, it appears that it depends mainly on f_1, f_2 which are eigenvalues of $-i(F_k^j)$ [2)] (where $(F_k^j) = (g^{jl})(F_{lk})$) and V and even on their ratios f_1/V and f_2/V rather than these functions themselves. It also depends on the partition of energy (see below).

Examples

Let us start from considering few examples.

Example 21.2.1. Let F_{jk}, g^{jk} and V are constant (which is not the case in this Chapter). Then $\mathbb{R}^4 = \mathbb{K}_1 \oplus \mathbb{K}_2$ where $\mathbb{K}_j$ here and below are eigenspaces of $-(F_k^j)^2$ corresponding to eigenvalues f_j^2, $\dim \mathbb{K}_j = 2$. Then the kinetic part of the Hamiltonian splits into $a^0 = a_1^0 + a_2^0$ where a_j^0 are Hamiltonians on $\mathbb{K}_j$ and a_j^0 are movement integrals and the classical dynamics splits into two *cyclotron* movements along elliptical orbits in $\mathbb{K}_j$ with the angular velocities μf_j; the sizes of these ellipses are $\asymp (\mu f_j)^{-1}(a_j^0)^{1/2}$.

Example 21.2.2. (i) Let in the framework of the previous example V be variable. Then a_j are no more movement integrals but they evolve with the average speeds[3)] $O(\mu^{-1})$ and in addition to the fast cyclotron movements there appear also slow *drift* movements along $\mathbb{K}_j$ with the speeds of $\asymp \mu^{-1}a_j^0|\nabla_{\mathbb{K}_j}(V/f_j)|$; these drifts are orthogonal to $\nabla_{\mathbb{K}_j}(V/f_j)$. So these drifts depend on the instant energy partition.

(ii) As (g^{jk}) and (F_{jk}) and thus f_j become variable, the spaces $\mathbb{K}_j$ rotate with the average speeds $O(\mu^{-1})$ but the previous statements remain true.

[2)] In contrast to footnote [1)] this is the final definition.

[3)] While the speed of spatial movement is $\asymp 1$ the drift is much slower and one can calculate all functions of x in the instant cyclotron centers.

Example 21.2.3. At points disjoint from Λ one can assume that magnetic field corresponds to $\omega = x_1 dx_1 \wedge dx_2 + dx_3 \wedge dx_4$; we used 2-form (21.2.9) and redefined coordinates $x' = (x_2, x_3, x_4)$ in the obvious way.

Assume that the metrics in these coordinates is Euclidean $g^{jk} = \delta_{jk}$ and that $V = V(x_1, x_2)$. Then $a_1 = a_1^0 - \frac{1}{2}V$ and $a_2 = a_2^0$ are movement integrals as in Example 21.2.1 and the movement splits into two movements again: one of them is the cyclotron movement in (x_3, x_4) and another one is described by Hamiltonian a_1 in coordinates (x_1, x_2). It was studied in details in Chapter 14.

Then there are *outer* and *inner* zones $\mathcal{Z}_{\mathsf{out}}$ and $\mathcal{Z}_{\mathsf{inn}}$ respectively; in the outer zone x_1 should be much larger than the cyclotron radius (associated with a_1) which is $O(\mu^{-1}|x_1|^{-1})$ and this is the case as $\mathcal{Z}_{\mathsf{out}} = \{|x_1| \ge C\mu^{-1/2}\}$. However, now a_1^0 is not necessarily disjoint from 0 (which was an assumption in Chapter 14) and we will need to take it into account and redefine zones.

Still we can conclude from Chapter 14 that if a trajectory $(x(t), \xi(t))$ on the energy level $\le c$ starts from the point $(x(0), \xi(0))$ with $x(0) \in B(0, \frac{1}{2}) \cap \{x : |x_1| = \gamma \ge C^2\mu^{-1/2}\}$, then $x(t) \in B(0, \frac{1}{2}) \cap \{x : C^{-1}\gamma \le |x_1| \le C\gamma\}$ as $|t| \le \epsilon\mu\gamma^2$. Further, the drift speed is $O(\mu^{-1}\gamma^{-2})$; this estimate is sharp as long as a_1^0 is disjoint from 0. So again we have the cyclotron movement with angular velocities $\asymp \mu|x_1|$ and μ and the drift movement mainly along x_2.

On the other hand, if a trajectory $(x(t), \xi(t))$ on the energy level $\le c$ starts from the point $(x(0), \xi(0))$ with $x(0) \in B(0, \frac{1}{2}) \cap \{x : |x_1| \le C\mu^{-1/2}\}$ then $x(t) \in B(0, \frac{1}{2}) \cap \{x : |x_1| \le C^2\mu^{-1/2}\}$ as $|t| \le \epsilon$ and the averaged propagation speed is $O(1)$; this estimate is sharp for the "typical" (in the heuristic sense) trajectory. In this case there are no separate cyclotron and drift movements along $\mathbb{K}_1$ but the cyclotron movement along $\mathbb{K}_2$ remains.

Example 21.2.4. Assume that in some coordinates $(x_1, \dots, x_4)$ 2-form ω is defined by (21.2.6) and also $g^{jk} = \delta_{jk}$. Let $V = \mathsf{const}$. Let us show that the conclusion of the previous example remains true (with the drift directed mainly along magnetic lines).

However, now the angle between $\mathbb{K}_1(x)$ and $T_x\sigma$ is of magnitude r and therefore to prevent cyclotron movements from hitting Σ it is sufficient to assume that $C\mu^{-1}\gamma^{-1}r \le \gamma$ and to keep r of the same magnitude along trajectory it is sufficient to assume that $r \ge C\mu^{-1}\gamma^{-1}$.

Alternatively we can assume that $r \le C\mu^{-1}\gamma^{-1}$ and then it will remain this way and the first inequality should be fulfilled with $r = C\mu^{-1}\gamma^{-1}$. So

we get zone

$$\mathscr{Z}_{\mathsf{out}} = \big\{C\mu^{-1}\gamma^{-1} \le r \le \epsilon\mu\gamma^2\big\}\cup \big\{r \le C\mu^{-1}\gamma^{-1}, C\mu^{-1}\gamma^{-1} \le \epsilon\mu\gamma^2\big\} = \big\{\gamma \ge C\max(\mu^{-2/3}, \mu^{-1/2}r^{1/2})\big\}. \tag{21.2.11}$$

Precise Calculations

Example 21.2.5 (Example 21.2.4 reloaded). Let us instead of rather heuristic arguments of Example 21.2.4 apply precise calculations. With ω defined by (21.2.6) and $g^{jk} = \delta_{jk}$

$$a = \frac{1}{2}\Big(\xi_1^2 + \xi_r^2 + W(x_1, r)\Big) \tag{21.2.12}$$

with

$$W(x_1, r) = \Big(\xi_2 - \mu(x_1 - \frac{1}{2}r^2)\Big)^2 + r^{-2}\Big(\xi_\theta - \mu(x_1 - \frac{1}{2}r^2)r^2\Big)^2 = (r^2+1)\Big(\mu\big(x_1 - \frac{1}{2}r^2\big) - \frac{\xi_\theta - \xi_2 r^2}{r^2+1}\Big)^2 + \frac{1}{r^2(r^2+1)}\big(\xi_\theta - \xi_2 r^2\big)^2. \tag{21.2.13}$$

Then ξ_θ and ξ_2 are movement integrals and the dynamics is restricted to the zone $\mathscr{Y}_{\xi_2,\xi_\theta} = \big\{(x_1, r) : W(x_1, r) \le c\big\}$. While W is nonnegative and convex in the vicinity of $(0,0)$ it is a degenerate as $\xi_2 = \xi_\theta = 0$: $W = \mu^2(1+r^2)\big(x_1 - \frac{1}{2}r^2\big)^2$ and so particle seems to move far away if time is unbounded.

Therefore we will need to bound time (but this bound will be large enough to sustain a nice remainder estimate).

General Case. I

Now we can consider rigorously the general case. Let $\Bbbk_\alpha = (\Bbbk_\alpha^m)$ be eigenvectors of $(F^j_{,k}) = (g^{jl})(F_{lk})$ corresponding to eigenvalues if_α ($\alpha = 1, 2$); then $\Bbbk_\alpha^\dagger$ are eigenvectors corresponding to eigenvalues $-\mathit{if}_\alpha$:

$$\sum_k F^j_{,k}\Bbbk^k_\alpha = \mathit{if}_\alpha \Bbbk^j_\alpha, \qquad \sum_k F^j_{,k}\Bbbk^{\dagger k}_\alpha = -\mathit{if}_\alpha \Bbbk^{\dagger j}_\alpha. \tag{21.2.14}$$

Let us normalize them so that

$$\sum_{j,k} g_{jk}\Bbbk^j_\alpha \Bbbk^k_\beta = 0, \qquad \sum_{j,k} g_{jk}\Bbbk^{\dagger j}_\alpha \Bbbk^{\dagger k}_\beta = 0, \qquad \sum_{j,k} g_{jk}\Bbbk^j_\alpha \Bbbk^{\dagger k}_\beta = 2\delta_{\alpha\beta}. \tag{21.2.15}$$

where the second equations in (21.2.14), (21.2.15) follow from the first ones.

(21.2.16) One can select $\Bbbk_j \in \mathscr{C}^\infty$ satisfying (21.2.14) and (21.2.15) in domain $\Omega = B(0,1) \cap \{|x_1 \le \epsilon\}$ [4)].

Indeed, this statement is true for $\Bbbk_2$; therefore $\mathbb{K}_1 \in \mathscr{C}^\infty$ and (F^j_k) transforms $\mathbb{K}_1 \in \mathscr{C}^\infty$ into itself and is skew-symmetric with respect to (g^{jk}). Then in the orthonormal real base on $\mathbb{K}_1$ $F = \begin{pmatrix} 0 & f_1 \\ -f_1 & 0 \end{pmatrix}$ with $f_1 \in \mathscr{C}^\infty$ and one can select $\Bbbk_1 \in \mathscr{C}^\infty$.

Let

$$(21.2.17)\quad Z_\alpha(x,\xi) = \sum_m \Bbbk^m_\alpha(x) p_m(x,\xi), \quad Z^\dagger_\alpha(x,\xi) = \sum_m \Bbbk^{\dagger m}_\alpha(x) p_m(x,\xi),$$
$$p_m = \xi_m - \mu V_m(x), \qquad \{p_j, p_k\} = -\mu F_{jk}$$

where the second equation follows from the first one. Then since

$$(21.2.18)\quad |p_j(x,\xi)| \le c_1 \text{ on the energy levels } \{(x,\xi) : a(x,\xi) \le c_0\}$$

we arrive to

$$(21.2.19)\quad \{Z_\alpha, Z_\beta\} \equiv \{Z^\dagger_\alpha, Z^\dagger_\beta\} \equiv 0, \quad \{Z_\alpha, Z^\dagger_\beta\} \equiv 2i\mu f_\alpha \delta_{\alpha\beta} \quad \mathsf{mod}\ O(1).$$

Further, note that corrected x_j:

$$(21.2.20)\qquad x'_j = x_j - \mu^{-1} \sum_k \check{F}^{jk} p_k, \quad \text{where} \quad \sum_k \check{F}^{jk} F_{kl} = \delta_{jl}$$

satisfy

$$(21.2.21)\qquad \{x'_j, p_k\} = O(\mu^{-1}\gamma^{-2}),$$
$$(21.2.22)\qquad \{x'_j, x'_k\} = \mu^{-1}\check{F}^{jk} + O(\mu^{-2}\gamma^{-3}).$$

Also note that

$$(21.2.23)\qquad a^0(x,\xi) := \frac{1}{2}\sum_{k,l} g^{kl} p_k p_l = \frac{1}{2}\Bigl(|Z_1|^2 + |Z_2|^2\Bigr)$$

and therefore

$$(21.2.24)\quad \{a^0, Z_\alpha\} \equiv \{a, Z_\alpha\} \equiv -i\mu f_\alpha Z_\alpha, \quad \{a^0, Z^\dagger_\alpha\} \equiv \{a, Z^\dagger_\alpha\} \equiv i\mu f_\alpha Z^\dagger_\alpha$$

[4)] Provided we redefine temporarily f_1 so that $f_1/x_1 > 0$.

and

$$\{a^0, b_\alpha\} \equiv \{a, b_\alpha\} \equiv 0 \quad \mathsf{mod}\ O(1) \tag{21.2.25}$$

where $b_\alpha := |Z_\alpha|^2$.

More precisely, equalities (21.2.19) are fulfilled modulo linear forms[5)] and therefore $\{a^0, b_\alpha\}$ is a cubic form[5)] and $\{a, b_\alpha\}$ is a cubic form[5)] plus a linear form[5)] . Moreover, these cubic terms should contain at least one factor Z_1 or $Z_1^\dagger$ and also at least one factor Z_2 or $Z_2^\dagger$.

(21.2.26) Then correcting b_α ($\alpha = 1, 2$) by cubic terms[5)] plus linear terms[5)], both multiplied by μ^{-1}, one can eliminate from $\{a, b_\alpha\}$ all cubic terms except those with $Z_1 Z_2 Z_2^\dagger$ and $Z_1^\dagger Z_2 Z_2^\dagger$ and all linear terms except those with Z_1 and $Z_1^\dagger$. Then these remaining terms produce $O(\mu^{-1})$ error.

Further, correcting in the zone $\mathcal{Z}_{\mathsf{out}} = \{C\mu^{-1/2} \le |x_1| \le \epsilon\}$ by cubic and linear terms of the above type but with coefficients of the type "$x_1^{-1} \times$ smooth function" one can eliminate the rest of the terms but now instead of $O(\mu^{-1})$ error we get $O(\mu^{-1} x_1^{-2})$ error:

(21.2.27) In the classical dynamics one can correct $b_\alpha = |Z_\alpha|^2$ by $O(\mu^{-1} x_1^{-1})$ so that their propagation speeds would be $O(\mu^{-1} x_1^{-2})$. Therefore as long as $x(t) \in \Omega$ and $|x_1(t)| \asymp \gamma$, $|t| \le T$ the following holds:

$$|Z_\alpha(t)|^2 = |Z_\alpha(0)|^2 + O\big(\mu^{-1}\gamma^{-1} + \mu^{-1}\gamma^{-2}T\big).$$

Similarly, consider $\{a, x_j\}$ which is a linear form[5)]. One can correct x_j in the same way arriving to

(21.2.28) In the classical dynamics one can correct x_j by $O(\mu^{-1} x_1^{-1})$ so that their propagation speeds would be $O(\mu^{-1} x_1^{-2})$. Therefore as long as $x(t) \in \Omega$ and $|x_1(t)| \asymp \gamma$, $|t| \le T$

$$x_j(t) = x_j(0) + O\big(\mu^{-1}\gamma^{-1} + \mu^{-1}\gamma^{-2}T\big).$$

[5)] With respect to $(Z_1, Z_1^\dagger, Z_2, Z_2^\dagger)$ with smooth complex coefficients depending on x.

Moreover, equalities (21.2.25) and (21.2.26) with b_α replaced by x_j corrected yield that

$$\frac{dx}{dt} \in \mathbb{K}_1(x) + O(\mu^{-1}x_1^{-1}) \qquad \text{for } x, \text{ corrected modulo } O(\mu^{-1}x_1^{-1}). \tag{21.2.29}$$

Still it is not yet what we want which is to calculate $\{a, x_j\}$ modulo $O(\mu^{-1}x_1^{-1})$. However, let us consider first

$$b_3' := -\operatorname{Re}\Big(i\mu^{-1}\{Z_1^\dagger, x_1\}\phi(x)^{-1}Z_1\Big) + x_1^2, \qquad \phi = f_1/x_1 \tag{21.2.30}$$

and note that $\{Z_1Z_1^\dagger, b_3'\} = O(\mu^{-1})$. Also note that $\{a, b_3'\}$ is a linear form of $Z_2, Z_2^\dagger$ modulo $O(\mu^{-1})$ and therefore it could be corrected by a linear form of $Z_2, Z_2^\dagger$ multiplied by μ^{-1} so that after correction $\{a, b_3'\} = O(1)$. Therefore we arrive to two following statements:

(21.2.31) In the classical dynamics one can correct $b_3 = x_1^2$ by $O(\mu^{-1})$ so that its propagation speed would be $O(\mu^{-1})$. Therefore

$$x_1^2(t) = x_1^2(0) + O\big(\mu^{-1} + \mu^{-1}T\big)$$

and statements (21.2.27) and (21.2.28) hold without precondition "as long as $x(t) \in \Omega$ and $|x_1(t)| \asymp \gamma$", because it is fulfilled automatically for $x(0) \in B(0, \frac{1}{2}) \cap \{\frac{1}{2}\gamma \le |x_1(0)| \le 2\gamma\}$, $T = \epsilon\mu\gamma^2$, $\gamma \ge \bar\gamma_1 = C\mu^{-1/2}$.

(21.2.32) Also, trajectory originated in $B(0, \frac{1}{2}) \cap \{|x_1| \le \gamma\}$ remains in $B(0,1) \cap \{|x_1| \le C\gamma\}$ as $\gamma = \bar\gamma_1$, $T = \epsilon$.

Furthermore, as long as $|x_1| \le c\gamma$ and $|\{Z_1, x_1\}| \le cr$, correction is $O\big(\mu^{-1}(r+\gamma)\big)$ and therefore $x_1^2(t) = x_1^2(0) + O\big(\mu^{-1}(r+\gamma) + \mu^{-1}T\big)$. Then we arrive to

(21.2.33) As long as $|\{Z_1, x_1\}| \le cr$ along trajectory, statements (21.2.31) and (21.2.32) hold with $\bar\gamma_1 = C\mu^{-1} + C\mu^{-1/2}r^{1/2}$. Moreover, $|\{Z_1, x_1\}(t)| \le cr$ provided it was fulfilled (with constant $c/2$) at $t = 0$ and $T = \epsilon\mu\gamma^2 r$, $r \ge \mu^{-1/2} + \gamma$.

General Case. II

Now let us return to calculation of $\{a, x_j\}$ modulo $O(\mu^{-1})$ where we will correct x_j. Any error, larger than this, comes from eliminating linear terms containing Z_1 or $Z_1^\dagger$, namely $\mathsf{Re}\big(\{Z_1^\dagger, x_j\}Z_1\big)$. This expression was eliminated by adding $\mu^{-1}f_1^{-1}\,\mathsf{Re}(i\{Z_1, x_j\}Z_1^\dagger)$ to x_j which in turn generates an error

$$\mu^{-1}\,\mathsf{Re}\, i\Big(\big\{a, f_1^{-1}\{Z_1, x_j\}\big\}Z_1^\dagger + f_1^{-1}\{Z_1^\dagger, x_j\}\big(i\{a, Z_1^\dagger\} - \mu f_1 Z_1\big)\Big). \tag{21.2.34}$$

In this expression (21.2.34) the part containing factors Z_2 and $Z_2^\dagger$ in the combinations Z_2, Z_2^2, $Z_2^\dagger$, $Z_2^{\dagger\,2}$ can be eliminated by the same way as above; the corresponding correction has an extra factor $O(\mu^{-1})$ and the corresponding error just acquires factor $O(\mu^{-1}x_1^{-1})$; therefore resulting error will be $O\big(\mu^{-2}x_1^{-3}|Z_1| + \mu^{-2}x_1^{-2}\big) = O\big(\mu^{-1} + \mu^{-1}\gamma^{-1}\rho\big)$ provided

$$|Z_1| \le c\rho, \qquad \gamma \ge c\mu^{-1/2}. \tag{21.2.35}$$

This leaves us with expression (which we do not call an error anymore)

$$\frac{1}{2}\Big(-\mu^{-1}x_1^{-2}\phi^{-1}\{Z_1Z_1^\dagger, x_1\}\,\mathsf{Re}(i\{Z_1, x_j\}Z_1^\dagger) + \mu^{-1}x_1^{-1}\,\mathsf{Re}\big(\alpha\{Z_1, x_j\}\big)|Z_2|^2\Big) \tag{21.2.36}$$

with the complex-valued coefficient $\alpha = \alpha(x')$ provided

$$f_1 = \phi(x')x_1 + O(x_1^2), \qquad x' := (x_2, x_3, x_4). \tag{21.2.37}$$

Note that the first term in (21.2.36) is equal to

$$\begin{aligned} &-\mu^{-1}x_1^{-2}\phi^{-1}\cdot\mathsf{Re}(\{Z_1, x_1\}Z_1^\dagger)\cdot\mathsf{Re}(i\{Z_1, x_j\}Z_1^\dagger) \\ &= \frac{1}{4}\mu^{-1}x_1^{-2}\phi^{-1} \\ &\times \mathsf{Re}\, i\Big(2\{Z_1^\dagger, x_1\}\{Z_1^\dagger, x_j\}Z_1^2 + \big(-\{Z_1^\dagger, x_1\}\{Z_1, x_j\} + \{Z_1^\dagger, x_j\}\{Z_1, x_1\}\big)|Z_1|^2\Big) \end{aligned} \tag{21.2.38}$$

and one needs to calculate coefficients at Z_1^2, $|Z_1|^2$ as $x_1 = 0$ only.

Therefore we need to calculate $\Bbbk_1$ as $x_1 = 0$. Let us decompose F_{jk}, g_{jk} and $\Bbbk_1$ into powers of x_1; then $F_{jk}\Bbbk_1 = if_1\Bbbk_1$ implies that

$$F_{jk}^0\Bbbk_1^0 = 0, \qquad (F_{jk}^1 - i\phi g_{jk}^0)\Bbbk_1^0 + F_{jk}^0\Bbbk_1^1 = 0, \tag{21.2.39}$$

where F_{jk} corresponds to symplectic form (21.2.6) and therefore

$$F = \begin{pmatrix} 0 & 1 & -x_4 & x_3 \\ -1 & 0 & x_3 & x_4 \\ x_4 & -x_3 & 0 & 2\bigl(x_1 - \frac{1}{2}(x_3^2 + x_4^2)\bigr) \\ -x_3 & -x_4 & -2\bigl(x_1 - \frac{1}{2}(x_3^2 + x_4^2)\bigr) & 0 \end{pmatrix}. \tag{21.2.40}$$

Then (21.2.39) and (21.2.40) imply for the "differentiation" part of Z_1

$$Z_{1,\ \mathrm{diff}}|_{x_1=0} = \alpha\Bigl(\bigl(r^{-1}\partial_\theta - r\partial_2\bigr) + \beta\bigl(r\partial_1 + \partial_r\bigr)\Bigr) \qquad \text{with} \quad c^{-1} \le |\alpha| \le c,\ c^{-1} \le \operatorname{Re} i\beta \le |\beta| \le c. \tag{21.2.41}$$

Then

$$\operatorname{Re} i\bigl(\{Z_1^\dagger, x_1\}Z_1\bigr)\big|_{x_1=0} = 2|\alpha|^2 \operatorname{Re}(i\beta)\bigl(\partial_\theta - r^2\partial_2\bigr) \tag{21.2.42}$$

is the differentiation along magnetic lines. Further, without any loss of the generality one can assume that $\alpha = 1$.

General Case. III

Consider first $\psi = x_1 - r^2/2$ which is a regular function of x; note that $Z_1\psi = O(x_1)$; therefore ψ can be corrected by $O(\mu^{-1})$ so that $\frac{d}{dt}\psi = O(\mu^{-1})$ and therefore original ψ is preserved modulo $O\bigl(\mu^{-1}(T+1)\bigr)$. Since we already know that x_1 is preserved modulo $O\bigl(\mu^{-1}\gamma^{-1}r(T+1)\bigr)$ we conclude that

Proposition 21.2.6. *Let us consider classical dynamics in* $\Omega = B(0,1) \cap \{|x_1| \le C\epsilon\}$, *originated in* $\Omega' = B(0,\frac{1}{2}) \cap \{|x_1| \le \epsilon\}$.

(i) If in the original point $|x_1| = \gamma$ *with* $\gamma \ge C\max(\mu^{-1}r^{-1}, r^2)$ *then both* $|x_1(t)|$ *and* $r(t)$ *remain of the same magnitudes as* $T = \epsilon\mu\gamma r$.

(ii) If in the original point $r^2 \ge \gamma \ge C\mu^{-1/2}r^{1/2}$ *then both* $|x_1(t)|$ *and* $r(t)$ *remain of the same magnitudes as* $T = \epsilon\mu\gamma^2 r^{-1}$.

(iii) If in the original point $\gamma \ge C\mu^{-2/3}$ *and* $r \le c\mu^{-1}\gamma^{-1}$ *then* $|x_1(t)|$ *remains of the same magnitude while* $r(t) \le 2c\mu^{-1}\gamma^{-1}$ *as* $T = \epsilon$.

(iv) If in the original point $|x_1| \le \gamma = c\mu^{-1/2}r^{1/2}$ *and* $r \ge C\mu^{-1/3}$ *then* $r(t)$ *remains of the same magnitude while* $|x_1(t)| \le C\gamma$ *as* $T = \epsilon$.

(v) If in the original point $|x_1| \le \gamma = c\mu^{-2/3}$ *and* $r \le c\mu^{-1/3}$ *then* $r(t) \le C\mu^{-1/3}$ *and* $|x_1(t)| \le C\gamma$ *as* $T = \epsilon$.

On the other hand, on the energy level τ one can rewrite the last term in (21.2.36) as $\mu^{-1}x_1^{-1}\,\mathsf{Re}(\sigma\{Z_1, x_j\})(\tau + V)$ modulo $O(\mu^{-1} + \mu^{-1}\gamma^{-1}\rho^2)$. Consider (21.2.38). Using the same technique as before one can eliminate

$$\mu^{-1}x_1^{-2}\phi^{-1} \cdot \mathsf{Re}\, i\Big(2\{Z_1^\dagger, x_1\}\{Z_1^\dagger, x_j\}Z_1^2\Big)$$

by $\mu^{-2}x_1^{-3}\phi^{-2}\cdot\mathsf{Re}\Big(\{Z_1^\dagger, x_1\}\{Z_1^\dagger, x_j\}Z_1^2\Big)$ correction (which is $O(\mu^{-2}\gamma^{-3}r\rho^2) = O(\mu^{-1}\gamma^{-1}\rho^2)$) as $\gamma \ge \mu^{-1/2}r^{1/2}$ adding $O(\mu^{-2}\gamma^{-4}r^2\rho^2 + \mu^{-2}\gamma^{-3}\rho^2)$ error; this error is less than $\epsilon\mu^{-1}\gamma^{-2}r\rho^2$ as $\gamma \ge C\mu^{-1/2}r^{1/2}$.

Also, any Z_1 or $Z_1^\dagger$ unbalanced in the product could be treated in the same way as before. Then we arrive to

Proposition 21.2.7. *As* $\gamma \ge C\mu^{-1/2}r^{1/2}$ *the propagation speed with respect to* θ *is*

$$\begin{aligned} (21.2.43)\quad & v(\rho; x, \mu^{-1}) \\ & = \mu^{-1}\gamma^{-2}\kappa(x, \rho^2, \mu)\Big(\rho^2 + r^{-1}\kappa_2(x, \rho^2, \mu)\gamma\Big) + O\Big(\mu^{-2}\gamma^{-3}\rho r^{-1}\Big) \end{aligned}$$

with κ, κ_2 *bounded and* κ *disjoint from* 0.

We also need to consider dynamics of $b_2 = |Z_2|^2$ more precisely. Note that

$$\big\{|Z_2|^2, a\big\} = \frac{1}{2}\,\mathsf{Re}\Big(\Big(\{Z_2, Z_1\}Z_1^\dagger + \{Z_2, Z_1^\dagger\}Z_1 - \{Z_2, V\}\Big)Z_2^\dagger\Big)$$

is a combination of cubic and linear terms (see footnote[5)]) and the only terms where factor Z_1 is not compensated by $Z_1^\dagger$ and v.v. come from

$$(21.2.44)\qquad \{Z_2, Z_1\} = \alpha_{11}Z_2 + \alpha_{12}Z_2^\dagger + \beta_{11}Z_1 + \beta_{12}Z_1^\dagger,$$

$$(21.2.45)\qquad \{Z_2, Z_1^\dagger\} = \alpha_{21}Z_2 + \alpha_{22}Z_2^\dagger + \beta_{21}Z_1 + \beta_{22}Z_1^\dagger;$$

these terms are equal to $\mathsf{Re}(\sigma Z_1)|Z_2|^2$ with $\sigma := \alpha_{11}^{\dagger} + \alpha_{21}$. We need to calculate β as $x_1 = 0$. All other terms could be corrected by $O(\mu^{-1})$ leading to $O(\mu^{-1})$ error while this one is corrected by $\mu^{-1}x_1^{-1}\phi\,\mathsf{Re}(i\beta Z_1)|Z_2|^2$ leading to $O(\mu^{-1}x_1^{-2})$ error; we will calculate this latter error more precisely below. Since

$$\alpha_{11} = (2i\mu f_2)^{-1}\bigl\{\{Z_2, Z_1\}, Z_2^{\dagger}\bigr\},$$
$$\alpha_{21} = (2i\mu f_2)^{-1}\bigl\{\{Z_2, Z_1^{\dagger}\}, Z_2^{\dagger}\bigr\}$$

calculated as $Z_1 = Z_2 = 0$, we conclude that

$$\sigma = (2i\mu f_2)^{-1}\bigl(-\bigl\{\{Z_2, Z_1\}, Z_2^{\dagger}\bigr\} + \bigl\{\{Z_2^{\dagger}, Z_1\}, Z_2\bigr\} = \\ -(i\mu f_2)^{-1}\bigl\{\{Z_2, Z_2^{\dagger}\}, Z_1\bigr\}.$$

Therefore correction is $O(\mu^{-1}\gamma^{-1}\rho)$ and then the part of an error, with factors Z_2 and $Z_2^{\dagger}$ balanced is $O(\mu^{-1}\gamma^{-1} + \mu^{-1}\gamma^{-2}\rho r)$.

Moreover, the part of $\{|Z_2|^2, a\}$ where Z_2 and $Z_2^{\dagger}$ are balanced is $\kappa|Z_2|^2$ with

$$\kappa = (2\mu f_2)^{-1}\,\mathsf{Re}\Bigl(\bigl\{i\{Z_2, Z_2^{\dagger}\}, Z_1\bigr\}\big|_{Z_1=Z_2=0} Z_1^{\dagger}\Bigr)$$

since $i\{Z_2, Z_2^{\dagger}\}$ is real; further, this expression is equal to

$$(2f_2)^{-1}\{f_2, |Z_1|^2\} + O(\rho\gamma) + O(\mu^{-1}\gamma^{-1}) = \\ (2f_2)^{-1}\{f_2, a\} - (2f_2)^{-1}\{f_2, |Z_2|^2\} + O(\rho\gamma) + O(\mu^{-1}\gamma^{-1}).$$

Note that $f_2^{-1}\{f_2, |Z_2|^2\}$ has unbalanced Z_2 or $Z_2^{\dagger}$ and thus is eliminated by $O(\mu^{-1})$ correction. Therefore we arrive to an error $-(2f_2)^{-1}\{f_2, a\}|Z_2|^2$.

Thus

(21.2.46) One can correct $|Z_2|^2$ by $O(\mu^{-1} + \mu^{-1}\gamma^{-1}\rho)$ term so that the propagation speed after correction would be $O(\mu^{-1}\gamma^{-1} + \mu^{-1}\gamma^{-2}\rho r)$.

(21.2.47) Moreover, one can correct $f_2^{-1}|Z_2|^2$ by $(\mu^{-1} + \mu^{-1}\gamma^{-1}\rho^2)$ term so that the propagation speed after correction would be $O(\mu^{-1} + \mu^{-1}\gamma^{-2}\rho^2)$.

We will also need the shorter term classical dynamics result:

Proposition 21.2.8. *Let us consider dynamics described in Proposition 21.2.6(i)–(v). Then* $\mathsf{dist}(x(t),(x(0)) \ge \epsilon\rho t$ *as* $|t| \le T_1 = \epsilon\,\mathsf{min}(\mu^{-1}\gamma^{-1},\rho)$.

Remark 21.2.9. It effectively makes $T_1 = \epsilon\mu^{-1}\gamma^{-1}$ because contribution to the remainder of the subzone $\{|x_1| \asymp \gamma, |Z_1| \le \rho, \rho \le \mu^{-1}\gamma^{-1}\}$ is $O(\mu h^{-3} \times \mu^{-2}\gamma^{-2} \times \gamma) = O(\mu^{-1}h^{-3}\gamma^{-1})$ which after summation over zone $\{\gamma \ge \mu^{-1/2}\}$ results in $O(\mu^{-1/2}h^{-3})$.

21.2.3 Quantum Dynamics. I. Outer Zone

In this subsection we begin to deal with the quantum dynamics and the new parameter h appears; so one can now compare μ and h. Here we assume that the magnetic field is rather weak; more precise requirements should vary from statement to statement but actually we do not care very much now and will just assume as needed that

$$C \le \mu \le h^{-\delta} \tag{21.2.48}$$

with small enough exponent $\delta > 0$; we do not want to push it up so far.

Short sScale Dynamics

As we have mentioned there are few time scales and the shortest one is $\asymp \mu^{-1}$. Let us deal with it first. Scaling $x \mapsto \mu x$, $t \mapsto \mu t$, $h \mapsto \mu h$, $\mu \mapsto 1$ we find ourselves with the standard Schrödinger operator with the propagator which is a standard Fourier integral operator. Returning to the original scale we arrive to

Proposition 21.2.10. *(i) As* $|t| \le T_0 = C\mu^{-1}$ *propagator* $e^{ih^{-1}At}$ *is* h*-FIO corresponding to the classical dynamics* $\Phi_t = e^{tH_a}$ *with Hamiltonian* $a(x,\xi)$.

(ii) In particular, if $\psi \in \mathscr{C}_0^\infty(B(\bar{x},\epsilon\gamma))$

$$e^{2\pi i\mu^{-1}h^{-1}\bar{f}_2^{-1}A}\psi \equiv e^{2\pi i\mu^{-1}h^{-1}\bar{f}_2^{-1}B}\psi \tag{21.2.49}$$

with $\bar{f}_2 = f_2(\bar{x})$ *and the standard* h*-pseudodifferential operator* $\gamma^{-1}B$.

Proof. Let us consider classical dynamics Φ_t first; in (x,p) variables it is described by

$$\frac{dx_j}{dt} = \mu^{-1}\sum_k K_{jk}(x)p_k, \tag{21.2.50}$$

$$\frac{dp_j}{dt} = \sum_k L_{jk}(x)p_k + \mu^{-1}\sum_{kl} L_{jkl}(x)p_k p_l + \mu^{-1}L_j(x) \tag{21.2.51}$$

(after rescaling $t \mapsto \mu^{-1}t$) with uniformly smooth coefficients K_*, L_*. Therefore in these coordinates $\Phi_t : (x,p) \mapsto \big(x+\mu^{-1}X(x,p,t), Y(x,p,t)\big)$. Thus $e^{ih^{-1}At}$ is a product of FIO corresponding to the symplectomorphism Φ_t (a standard quantization, phase function is defined in the standard way and the symbol is just 1) and some PDO.

In particular, one can see easily that $\Phi_t - I = O(\gamma)$ as $t = 2\pi\bar{f}_2^{-1}$; therefore we conclude that $\Phi_t = e^{\gamma H_b}$ with some symbol $b = b(\mu^{-1}x, Z)$. Quantizing it we get that (21.2.49) holds but with an extra PDO factor $\mathcal{Q}$ between FIO and ψ. However, then one can perturb B by operator $\gamma^{-1}hB_1$ so that

$$e^{2\pi i\mu^{-1}h^{-1}\bar{f}_2^{-1}B}\mathcal{Q} \equiv e^{2\pi i\mu^{-1}h^{-1}\bar{f}_2^{-1}(B+\gamma^{-1}hB)};$$

we leave this easy exercise to the reader. □

Proposition 21.2.11. *Let condition*

$$V \ge \epsilon_0 \tag{21.2.52}$$

be fulfilled. Then

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma\big(u\psi Q\big)| \le C'h^s \qquad \forall \tau : |\tau| \le \epsilon \tag{21.2.53}$$

as $\mu \ge K$, $\bar{T} := Ch|\log h| \le T \le 2\pi(1-\epsilon_0)\mu^{-1}\bar{f}_2^{-1}$, $\gamma \le \epsilon$ *where* ψ *is supported in* $B(\bar{x},\gamma)$, γ *is calculated in* $\bar{x}$, Q *is an arbitrary* h*-PDO* [6].

Now let us consider in the outer zone the intermediate scale dynamics:

[6] Recall that $\chi \in \mathscr{C}_0^\infty([-1,-\frac12]\cup[\frac12,1])$, $\bar\chi \in \mathscr{C}_0^\infty([-1,1])$ are our usual cut-off functions, $\chi_T(t) = \chi(t/T)$ etc. and $\Gamma_x(u) = u(x,x)$, $\Gamma u = \int u(x,x)\,dx$.

Proposition 21.2.12. *Consider* $\bar{x} \in \mathcal{Z}_{\mathsf{out}} \coloneqq \{\gamma \ge C\mu^{-1/2}\}$. *Then as* $|t| \le C\mu^{-1}\gamma^{-1}$ $e^{ih^{-1}At}$ *is* h*-FIO corresponding to the classical dynamics* $\Phi_t = e^{tH_a}$ *with Hamiltonian* a *and*

$$e^{ih^{-1}tA}\psi \equiv e^{ih^{-1}t''A}e^{ih^{-1}t'B}\psi \tag{21.2.54}$$

where $t' = 2\pi\mu^{-1}\bar{f}_2^{-1}n$, $n = \bigl\lfloor (2\pi)^{-1}\bar{f}\mu t\bigr\rfloor$, $t'' = t - t'$.

Proof. Formula (21.2.54) follows from (21.2.49) and the fact that $|x_1|$ retains its magnitude in the classical dynamics in the zone $\mathcal{Z}_{\mathsf{out}}$. □

Proposition 21.2.13. *(i) Let* $\mu \le \epsilon h^{-1}|\log h|^{-1}$, $\bar{x} \in \mathcal{Z}_{\mathsf{out}}$ *and* Q *be* h*-PDO with the symbol supported in* $\{|Z_1| \ge \rho\}$ *with*

$$\rho \ge C\mu^{-1}\gamma^{-1} + C(\mu h\gamma|\log h|)^{1/2}. \tag{21.2.55}$$

Then (21.2.53) *holds as* $C\rho^{-2}h|\log h| \le T \le T_1 \coloneqq \epsilon_1\mu^{-1}\gamma^{-1}$.

(ii) In particular, let

$$\rho \ge C\mu^{-1}\gamma^{-1} + C(\mu h|\log h|)^{1/2}. \tag{21.2.56}$$

Then estimate (21.2.53) *holds as* $\bar{T} = Ch|\log h| \le T \le \epsilon_1\mu^{-1}\gamma^{-1}$.

Proof. Let us temporarily direct $\mathbb{K}_1$ as a coordinate plane[7] $\{x_1, x_2\}$ and consider ρ-admissible partition in (ξ_1, ξ_2). Then one can see easily that the propagation along $\mathbb{K}_1$ has the speed $\asymp \rho$. Further, condition (21.2.55) ensures that ρ retains its magnitude since the propagation speed of $|Z_1|^2$ does not exceed $C_0\mu^{-1}\rho^2\gamma^{-2} + C_0\mu^{-1}$ and $|Z_1|^2$ is corrected by $O(\rho\mu^{-1})$. Furthermore, the last term in (21.2.55) ensures the logarithmic uncertainty principle.

Then for time T the shift along $\mathbb{K}_1$ is exactly of magnitude ρT and the logarithmic uncertainty principle $\rho T \times \rho \ge Ch|\log h|$ is fulfilled as long as $T \ge T' = C\rho^{-2}h|\log h|$. This implies Statement (i).

Now it follows from Statement (i) and Proposition 21.2.11 that estimate (21.2.53) holds as $\bar{T} = Ch|\log h| \le T \le T_1 = \epsilon\mu^{-1}\gamma^{-1}$ provided $T' \le \epsilon\mu^{-1}$ which means upgrade of condition (21.2.55) to (21.2.56). □

[7] For $T \le c\mu^{-1}\gamma^{-1}$ and $\gamma \ge C\mu^{-1/2}$ with $C = C(\epsilon, c)$ our propagation is confined to $B(\bar{x}, \epsilon\gamma)$.

Dynamics and Remainder Estimate. I

On the other hand one can see easily that the contribution to the remainder of the rather thin subzone where $|x_1| \asymp \gamma$ and condition (21.2.56) is violated does not exceed $C\mu h^{-3}\big(\mu^{-2}\gamma^{-2} + \mu h|\log h|\big)\gamma$; further, summation of this expression over $\mathcal{Z}_{\mathsf{out}}$ results in $C\mu^{-1/2}h^{-3} + C\mu^2 h^{-2}|\log h|$. Therefore we have proven

Proposition 21.2.14. *Contribution to the remainder of the subzone of* $\mathcal{Z}_{\mathsf{out}}$ *where condition* (21.2.55) *is violated is* $O\Big(\mu^{-1/2}h^{-3} + \mu^2 h^{-2}|\log h|\Big)$. *In particular, this contribution is* $O(\mu^{-1/2}h^{-3})$ *as* $\mu \le C(h|\log h|)^{-2/5}$.

Remark 21.2.15. Actually, one can get rid of logarithmic factors here but under assumption (21.2.48) it is not needed; this estimate (without logarithmic factor) is the best possible if no non-degeneracy condition is assumed.

From now on we will consider only the *main subzone* of $\mathcal{Z}_{\mathsf{out}}$ where condition (21.2.56) is fulfilled and thus estimate (21.2.53) holds with $\bar{T} \le T \le T_1$. We need to estimate its contribution to the remainder estimate.

First of all one needs the quantum version of the results of the previous subsection.

Proposition 21.2.16. *Let condition* (21.2.48) *be fulfilled. Let function* ψ *be properly supported in* $B(\bar{x}, \gamma')$, *and* Q *be properly supported in* $\{|Z_1| \le \rho\}$ *with* $\rho \ge C_0\mu^{-1/2}$ *and*

$$\gamma' \ge C\rho^{-1}h|\log h|. \tag{21.2.57}$$

Further, let $\gamma \ge C_0\mu^{-1/2}$ *and* $T \le T_2 := \epsilon\mu\gamma^2$ *and let* $x'' = (x_3, x_4)$. *Then*

(i) If

$$r = |\bar{x}''| \ge r' := C_0\mu^{-1}\gamma^{-1} + C_0\gamma' \tag{21.2.58}$$

then estimate

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)(1-\psi_1)(u\psi_y Q_y^t)| \le Ch^s \tag{21.2.59}$$

holds provided $\psi_1 = 1$ *in the domain*

$$\Big\{x : C_0^{-1}\gamma \le |x_1| \le C_0\gamma,\ C_0^{-1}r \le |x''| \le C_0 r, \\ \mathsf{dist}(x, \bar{x}) \ge C_0\gamma' + C_0\mu^{-1}\gamma^{-1} + C_0\mu^{-1}(r+\gamma)\gamma^{-2}T\Big\}. \tag{21.2.60}$$

(ii) On the other hand, if condition (21.2.58) *is violated then estimate* (21.2.59) *holds provided* $\psi_1 = 1$ *in the domain*

$$\Big\{x : C_0^{-1}\gamma \le |x_1| \le C_0\gamma,\ |x''| \le C_0 r', \quad \operatorname{dist}(x, \bar{x}) \ge C_0\gamma' + C_0\mu^{-1}\gamma^{-1} + C_0\mu^{-1}(r'+\gamma)\gamma^{-2}T\Big\}. \tag{21.2.61}$$

Proof. Proof is the standard one (see e.g. Chapter 2) based on the analysis of the symbol

$$\varpi\Big(\frac{1}{T}\Big(t \pm \epsilon v^{-1}\big(X(x,\xi) - X(y,\xi)\big)\Big)\Big) \tag{21.2.62}$$

where ϖ is the same function as in Chapter 2, $X(x,\xi)$ is one of the corrected symbols x_1^2, $x_1 - \frac{1}{2}|x''|^2$ and x_j with $j = 1, \dots, 4$ and v is the corresponding speed, namely μ^{-1}, μ^{-1} and $\mu^{-1}x_1^{-2}r$ respectively.

Obviously, it is sufficient to consider T such that the last term in definitions of (21.2.60), (21.2.61) is dominant: $\mu^{-1}(r+\gamma)\gamma^{-2}T \ge \gamma' + \mu^{-1}\gamma^{-1}$. One can see easily that under this condition and (21.2.48) symbol (21.2.62) is quantizable; here also one can be more specific about exponent $\delta > 0$ in condition (21.2.48). □

Now we want to be more precise for smaller r, ρ:

Proposition 21.2.17. *In the framework of Proposition 21.2.16 as* $T \le \epsilon\mu\gamma^2\rho$ *estimate*

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)(1-Q_1)\psi_1(u\psi_y Q_y^t)| \le Ch^s \tag{21.2.63}$$

holds provided $Q_1 = I$ *in the domain* $\{c^{-1}\rho \le |Z_1| \le c\rho\}$ *and estimate* (21.2.59) *holds provided* $\psi_1 = 1$ *in the domain*

$$\Big\{x : C_0^{-1}\gamma \le |x_1| \le C_0\gamma,\ C_0^{-1}r \le |x''| \le C_0 r, \quad \operatorname{dist}(x, \bar{x}) \ge C_0\gamma' + C_0\mu^{-1}\gamma^{-1} + C_0\mu^{-1}\gamma^{-2}(r\rho^2+\gamma)T\Big\} \tag{21.2.64}$$

as $r \ge r'$*, and in the domain*

$$\Big\{x : C_0^{-1}\gamma \le |x_1| \le C_0\gamma,\ |x''| \le C_0 r', \quad \operatorname{dist}(x, \bar{x}) \ge C_0\gamma' + C_0\mu^{-1}\gamma^{-1} + C_0\mu^{-1}\gamma^{-2}(r'\rho^2+\gamma)T\Big\}. \tag{21.2.65}$$

as $r \le r'$.

Proof. Proof is standard, based on the symbol (21.2.62) which is quantizable due to assumption (21.2.48); here again X is one of the corrected symbols $|Z_1|^2$, x' and v is the corresponding upper bound for a speed, namely $\mu^{-1}\gamma^{-2}\rho$, $\mu^{-1}\gamma^{-2}(\rho^2+\gamma)$ respectively. □

Note that under assumption (21.2.48) $r' \asymp \mu^{-1}\gamma^{-1}$ and we want to eliminate the corresponding subzone from the future analysis. Really, let us note that the contribution of the subzone $\{\gamma \le |x_1| \le 2\gamma, |x''| \le r, |Z_1| \le \rho\}$ to the remainder is $O(T_1^{-1}h^{-3}r^2\rho^2\gamma) = O(\mu h^{-3}r^2\rho^2\gamma^2)$ (since we already eliminated subzone where condition (21.2.56) is violated) and plugging $r\rho = \mu^{-1}\gamma^{-1}$ results in $O(\mu^{-1}h^{-3})$. Then summation with respect to γ results in $O(\mu^{-1}h^{-3}|\log\mu|)$. So, we conclude that

(21.2.66) Contribution to the remainder of subzone $\mathcal{Z}_{\mathsf{out}} \cap \{r\rho \le C\mu^{-1}\gamma^{-1}\}$ is $o(\mu^{-1/2}h^{-3})$.

Dynamics and Remainder Estimate. II

From now on we restrict ourselves to the analysis of the subzone $\mathcal{Z}_{\mathsf{out}} \cap \{r\rho \ge C\mu^{-1}\gamma^{-1}\}$.

Now one can see that in $\mathcal{Z}_{\mathsf{out}}$ function $v(\rho; x, \mu^{-1})$ defined by (21.2.43) has no more than one root $\rho \ge C\mu^{-1}r^{-1}\gamma^{-1}$; let us denote it by $w = w(x, \mu^{-1})$. Otherwise (if such root does not exist) we set $w = 0$. One can see easily that

$$v(\rho; x, \mu^{-1}) \asymp \mu^{-1}\gamma^{-2}(\rho^2 - w^2). \tag{21.2.67}$$

If $\rho - w \asymp \Delta$ then the propagation speed for corrected symbol θ is $v(\rho, .)$; then the linear shift for time $\epsilon\mu^{-1}\gamma^{-1}$ is $\asymp r\mu^{-2}\gamma^{-3}\rho\Delta$ (where r appears because θ is an angle). This shift is observable as long as the logarithmic uncertainty principle $r\mu^{-2}\gamma^{-3}\rho\Delta \times \Delta \ge Ch|\log h|$ holds[8] i.e.

$$\Delta \ge C\mu\big(h|\log h|\big)^{1/2}\rho^{-1/2}\gamma^{3/2}r^{-1}. \tag{21.2.68}$$

Since the propagation speed for symbol corrected $|Z_1|^2$ is $O(\mu^{-1}\gamma^{-2}\rho^2)$, the magnitude of Δ is preserved during the time interval $T'' = \epsilon\mu\gamma^2\rho^{-1}\Delta$ which is larger than $C\mu^{-1}\gamma^{-1}$ as

$$\Delta \ge C\mu^{-2}\rho^{-1}\gamma^{-3}. \tag{21.2.69}$$

[8] Now Δ is the scale in ρ.

This condition is stronger than (21.2.68) as $r \ge \mu^{-1}$ and (21.2.48) is fulfilled with small enough $\delta > 0$.

To justify this assertion properly one needs to operate with the quantizable symbols and one can see easily that this is the case under assumptions (21.2.48) with small enough $\delta > 0$ and (21.2.68). Then the contribution to the remainder of the corresponding subzone (where magnitudes of γ, ρ, r and Δ are fixed) does not exceed $Cr^2h^{-3}\rho\gamma\Delta/T'' = C\mu^{-1}h^{-3}r^2\rho^2\gamma^{-1}$.

Let us note that as magnitudes of γ, r, ρ are fixed and Δ ranges from $C\mu^{-2}\rho^{-1}\gamma^{-3}$ dictated by (21.2.69) to $\epsilon\rho$, the number of such elements does not exceed $C\log\mu$. So the summation with respect to this zone results in $C\mu^{-1}h^{-3}r^2\rho^2\gamma^{-1}\log\mu$. Further, as $\Delta \ge \epsilon\rho$, there are no more than C such elements and therefore the contribution to the remainder of the part of the zone

$$\mathcal{Y}_{\gamma,\rho,r} := \Big\{\frac{1}{2}\gamma \le |x_1| \le 2\gamma,\ \frac{1}{2}\rho \le |Z_1| \le 2\rho,\ \frac{1}{2}r \le |x''| \le 2r\Big\} \tag{21.2.70}$$

where condition (21.2.69) is fulfilled does not exceed $C\mu^{-1}h^{-3}r^2\rho^2\gamma^{-1}\log\mu$ as well. Summation with respect to ρ with $\rho \le C(\gamma/r)^{1/2}$ results in $C\mu^{-1}h^{-3}r\log\mu$ and then the summation with respect to all r, γ results in $\mu^{-1}h^{-3}|\log\mu|^2 = o(\mu^{-1/2}h^{-3})$. Therefore contribution to the remainder of the part of the zone $\mathcal{Z}_{\mathsf{out}}$ where $r \ge \mu^{-1}\gamma^{-1}$, $\rho^2 \le C\gamma$ and condition (21.2.69) is fulfilled is $O(\mu^{-1/2}h^{-3})$.

The same arguments one can apply as $\rho^2 \ge C\gamma/r$ but in this case $\Delta = \epsilon\rho$ and condition (21.2.67) is fulfilled automatically, there will be no factor $\log\mu$ and the summation of $C\mu^{-1}h^{-3}r^2\rho^2\gamma^{-1}$ with respect to ρ, r, γ results in $C\mu^{-1/2}h^{-3}$.

Therefore we arrive to

(21.2.71) Contribution to the remainder e of the subzone of $\mathcal{Z}_{\mathsf{out}}$ where condition (21.2.69) is fulfilled is $O(\mu^{-1/2}h^{-3})$.

Finally, contribution to the remainder of the subzone $\mathcal{Y}_{\gamma,\rho}$ with $\gamma \ge C\mu^{-1/2}$ where condition (21.2.69) is violated does not exceed

$$C\big(h^{-3}\rho\gamma \times \mu^{-2}\rho^{-1}\gamma^{-3} \times T_1^{-1}\big) = C\mu^{-1}\gamma^{-1}h^{-3}$$

since $T_1 = \epsilon\mu^{-1}\gamma^{-1}$ and summation with respect to γ results again in $O(\mu^{-1/2}h^{-3})$.

Combining with (21.2.71) we obtain immediately

Proposition 21.2.18. *Under assumption* (21.2.48) *the contribution to the remainder of the zone* $\mathcal{Z}_{\mathsf{out}}$ *is* $O\big(\mu^{-1/2}h^{-3}\big)$.

21.2.4 Quantum Dynamics. II. Inner Zone

Now we need to analyze the contribution to the remainder estimate of the *inner zone* $\mathcal{Z}_{\mathsf{inn}} = \{|x_1| \le \bar{\gamma}_0 = C\mu^{-1/2}\}$ under assumption (21.2.48). Both classical and quantum dynamics are confined to this zone as $T \le \epsilon$ and it starts in $B(0, \frac{1}{2})$; propagation speed with respect to x does not exceed C_0. Note that the correction procedure still works as long as one eliminates only unbalanced factors Z_2, $Z_2^\dagger$.

Dynamics and Remainder Estimate. III

Contribution to the remainder estimate of the subzone $\{|Z_1| \le \rho, |x_1| \le \gamma\}$ does not exceed $CT^{-1}h^{-3}\rho^2 r^2\gamma = C\mu h^{-3}\gamma\rho^2$ with $T = T_0 = \epsilon\mu^{-1}$; plugging $\gamma = c\mu^{-1/2}$ and $\rho = c\mu^{-1/2}$ we get $O(\mu^{-1/2}h^{-3})$ and therefore one needs to consider only the contribution to the remainder estimate of the subzone $\mathcal{Z}_{\mathsf{inn}} \cap \{|Z_1| \ge C\mu^{-1/2}\}$; but then Proposition 21.2.12 remains true here and one can increase $T = T_0$ to $T = T_1 = \epsilon\mu^{-1/2}$ [9)].

Then the contribution to the remainder estimate of the zone $\{|Z_1| \le \rho, |x_1| \le \gamma, |x''| \le r\}$ does not exceed $CT^{-1}h^{-3}\rho^2 r^2\gamma = C\mu^{1/2}h^{-3}\gamma r^2\rho^2$ with $T = \epsilon\mu^{-1/2}$; in particular, the total contribution of the inner zone $\mathcal{Z}_{\mathsf{inn}}$ is $O(h^{-3})$ and one needs to recover factor $\mu^{-1/2}$.

To recover this factor one needs just to increase $T = T_1 = \epsilon\mu^{-1/2}$ to $T = T_2 = \epsilon$. Before doing this just note that the contributions to the remainder estimates of subzones $\mathcal{Z}_{\mathsf{inn}} \cap \{|Z_1| \le c\mu^{-1/4}\}$, $\mathcal{Z}_{\mathsf{inn}} \cap \{|x''| \le c\mu^{-1/4}\}$ and $\mathcal{Z}_{\mathsf{inn}} \cap \{|Z_1| \cdot |x''| \le c\mu^{-1/4}|\log\mu|^{-1}\}$ are $O(\mu^{-1/2}h^{-3})$ and therefore

(21.2.72) One needs to consider only the contribution to the remainder of the subzone $\mathcal{Z}_{\mathsf{inn}} \cap \{|x''| \ge c_1\mu^{-1/4},\ |Z_1| \ge c_1\mu^{-1/4},\ |Z_1|\cdot|x''| \ge c_1\mu^{-1/4}|\log\mu|^{-1}\}$.

Let us recall that the classical dynamics which starts in the subzone $\{|x_1| \le \gamma, |x''| = r \ge C_1\mu^{-1/4}\}$ remains in $\{|x_1| \le C\gamma, \big||x''| - r\big| \le C_1\mu^{-1/2}r^{-1}\}$ as $T = \epsilon$; actually it could be larger as $\rho r \ll 1$.

[9)] As in footnote [6)] one must take $T_1 = \epsilon\min(\mu^{-1/2}, \rho)$ but contribution to the remainder estimate of subzone $\mathcal{Z}_{\mathsf{inn}} \cap \{|Z_1| \le \mu^{-1/2}\}$ is $O(\mu h^{-3}\rho^2\gamma) = O(\mu^{-1/2}h^{-3})$.

Repeating arguments of the proof of Proposition 21.2.16 one can justify it easily for quantum propagation as well:

Proposition 21.2.19. *Under assumption* (21.2.48) *with small enough* $\delta > 0$ *the quantum dynamics which starts in the subzone* $\{|x_1| \le \gamma, |x''| = r \ge C_1\mu^{-1/4}\}$ *remains in* $\{|x_1| \le C\gamma, \big||x''| - r\big| \le C_1\mu^{-1/2}r^{-1}\}$ *as* $T = \epsilon$.

Now let us consider again function $\frac{1}{2}x_1^2$; then

$$\frac{1}{2}\{a, x_1^2\} = \mathsf{Re}\Big(\{Z_1^\dagger, x_1\}x_1Z_1 + \{Z_2^\dagger, x_1\}x_1Z_2\Big) \tag{21.2.73}$$

and thus should be corrected by

$$\mu^{-1}\,\mathsf{Re}\, i\Big(\{Z_1^\dagger, x_1\}f_1^{-1}x_1Z_1 + \{Z_2^\dagger, x_1\}f_2^{-1}x_1Z_2\Big) \tag{21.2.74}$$

which is $O(\mu^{-1}r)$ as $r \ge \mu^{-1/2}$ and leads to the new error

$$\begin{aligned}\mu^{-1}\,\mathsf{Re}\, i\Big(&\big\{a, \{Z_1^\dagger, x_1\}f_1^{-1}x_1\big\}Z_1 + \{Z_1^\dagger, x_1\}f_1^{-1}x_1\big(\{a, Z_1\} + i\mu Z_1\big) + \\ &\big\{a, \{Z_2^\dagger, x_1\}f_2^{-1}x_1\big\}Z_2 + \{Z_2^\dagger, x_1\}f_2^{-1}x_1\big(\{a, Z_2\} + i\mu Z_2\big)\Big).\end{aligned} \tag{21.2.75}$$

This new error is $O(\mu^{-1})$ and thus taking $T = \epsilon r$ one can see that the oscillation of x_1^2 would be $O(\mu^{-1}r)$ which leads to

Proposition 21.2.20. *(i) The classical dynamics which starts in the subzone* $\{|x_1| = \gamma, |x''| = r \ge C_1\mu^{-1/4}\}$ *with* $C_0\mu^{-1/2}r^{1/2} \le \gamma \le C_0\mu^{-1/2}$ *remains in* $\{C^{-1}\gamma \le |x_1| \le C\gamma\}$ *as* $T = \epsilon r$.

(ii) The classical dynamics which starts in the subzone $\{|x_1| \le \gamma, |x''| = r \ge C_1\mu^{-1/4}\}$ *with* $\gamma = C_0\mu^{-1/2}r^{1/2}$ *remains in* $\{|x_1| \le C\gamma\}$ *as* $T = \epsilon r$.

(iii) Under assumption (21.2.48) *with small enough* $\delta > 0$ *statements (i),(ii) remain true for a quantum dynamics as well.*

Therefore one can see that the dynamics in the zones described in Proposition 21.2.20(i),(ii) are different with the dynamics in the former resembling the dynamics in $\mathcal{Z}_{\mathsf{out}}$. Then the arguments of the previous Subsection 21.2.3

work perfectly in the zone described in Proposition 21.2.20(i) tedious but easy[10)] details we leave to the reader:

Proposition 21.2.21. *Under assumption* (21.2.48) *with sufficiently small* $\delta > 0$ *the contribution to the remainder estimate of the zone* $\{C_0\mu^{-1/2}|x''|^{1/2} \le |x_1| \le C_0\mu^{-1/2}\}$ *does not exceed* $C\mu^{-1/2}h^{-3}$.

Thus we are left with the zone

$$\{C_1\mu^{-1/6} \le |x''| \le c, \quad |x_1| \le C_1\mu^{-1/2}|x''|^{1/2}\}. \tag{21.2.76}$$

Again the dynamics in the zones $\mathcal{Y}_{\gamma,\rho,r}$ with $\rho r \le \epsilon\mu\gamma^2$ and $\rho r \ge \epsilon\mu\gamma^2$ are very different with the former one more similar to the dynamics in the outer zone (as long as it remains there).

Furthermore, similarly the propagation speed of the symbol $|Z_1|^2$, corrected by $O(\mu^{-1}\gamma^{-1}r|Z_1| + \mu^{-1})$ does not exceed $C\bigl(\mu^{-1}\gamma^{-2}r|Z_1| + \mu^{-1}\bigr)$ and therefore the propagation speed of $|Z_1|$ does not exceed $C\bigl(\mu^{-1}\gamma^{-2}r + \mu^{-1/2}\bigr)$ and thus magnitude of $|Z_1| \asymp \rho$ is preserved on the time interval $T = \epsilon\bigl(\mu^{-1}\gamma^{-2}r + \mu^{-1/2}\bigr)^{-1}\rho$ as long as $\gamma \ge C(\mu r\rho)^{1/2}$. As a result we arrive to

Proposition 21.2.22. *(i) The classical dynamics which starts in the subzone* $\{|x_1| = \gamma, |x''| = r \ge C_1\mu^{-1/4}, |Z_1| = \rho\}$ *with* $C_0\mu^{-1/2}r^{1/2}\rho^{1/2} \le \gamma \le C_0\mu^{-1/2}$ *remains in* $\{C^{-1}\rho \le |Z_1| \le C\rho\}$ *as* $T = \epsilon\min\Bigl(r, \mu\gamma r\rho^{-1}, \mu^{-1}\rho^{-1}\Bigr)$.

(ii) The classical dynamics which starts in the subzone $\{|x_1| \le \gamma, |x''| = r \ge C_1\mu^{-1/4}, |Z_1| = \rho\}$, *with* $\gamma = C_0\mu^{-1/2}r^{1/2}\rho^{1/2}$ *remains in* $\{C^{-1}\rho \le |Z_1| \le C\rho\}$ *as* $T = \epsilon\min\Bigl(r, \mu\gamma r\rho^{-1}, \mu^{-1}\rho^{-1}\Bigr)$.

(iii) Under assumption (21.2.48) *with small enough* $\delta > 0$ *Statements (i) and (ii) remain true for a quantum dynamics as well.*

Dynamics and Remainder Estimate. IV

Then again arguments of the previous subsection work perfectly in the zone described in Proposition 21.2.22(i); tedious but easy details (see footnote [9)] with the obvious modifications.) I leave to the reader:

[10)] Because one needs to worry only about eliminating factor $\mu^{1/2}$ while in the previous subsection the ofending factor was $\mu^{3/2}$. One can just take $\Delta = C\mu^{-1/2}$. Also one can assume that $|x_1| \ge \hat{\gamma} = c_1\mu^{-3/5}$; otherwise $|x''| \le r = c_2\mu^{-1/5}$ and the contribution to the remainder estimate of this zone would be $O(\mu h^{-3}\gamma^2 r^2) = O(\mu^{-1/2}h^{-3})$.

But then $\rho \ge \mu^{-1}\gamma^{-1}$ in the zone of interest and one can finally upgrade $T_1 = \epsilon\mu^{-1/2}$ to $T_1 = \epsilon\mu^{-1}\gamma^{-1}$ and the crude estimate to $O(\mu h^{-3}\rho^2\gamma^2 r^2)$.

Proposition 21.2.23. *Under assumption* (21.2.48) *with sufficiently small* $\delta > 0$ *the contribution to the remainder of the zone* $\{C_0\mu^{-1/2}|x''|^{1/2}|Z_1|^{1/2} \le |x_1| \le C_0\mu^{-1/2}\}$ *does not exceed* $C\mu^{-1/2}h^{-3}$.

Therefore we are left with the *true inner zone*

$$\mathcal{Z}^0_{\mathsf{inn}} := \{|x_1| \le C_1\mu^{-1/2}|x''|^{1/2}|Z_1|^{1/2}\}. \tag{21.2.77}$$

Similarly to claim (21.2.72)

(21.2.78) One needs to consider only the contribution to the remainder of the subzone $\mathcal{Z}^0_{\mathsf{inn}} \cap \{|x''| \ge c_1\mu^{-1/6},\ |Z_1| \ge c_1\mu^{-1/6},\ |Z_1|\cdot|x''| \ge c_1\mu^{-1/6}|\log\mu|^{-1}\}$.

Let us consider classical dynamics in this zone first; we are interested in the time interval $T \le \epsilon r$. There

$$\{a, Z_1\} = \{Z_1^\dagger, Z_1\}Z_1 + \{Z_2^\dagger, Z_1\}Z_2 + \{Z_2, Z_1\}Z_2^\dagger - \{V, Z_1\}$$

and then we correct Z_1 to

$$\zeta_1 = Z_1 + i\mu^{-1}f_2^{-1}\{Z_2^\dagger, Z_1\}Z_2 - i\mu^{-1}f_2^{-1}\{Z_2, Z_1\}Z_2^\dagger \tag{21.2.79}$$

satisfying

$$\begin{aligned}\{a, \zeta_1\} \equiv\ & \{Z_1^\dagger, Z_1\}Z_1 - \\ & if_2^{-1}\mu^{-1}\Big(\big\{Z_2, \{Z_2^\dagger, Z_1\}\big\} - \big\{Z_2^\dagger, \{Z_2, Z_1\}\big\}\Big)|Z_2|^2 - \{V, Z_1\} = \\ & \{Z_1^\dagger, Z_1\}Z_1 + if_2^{-1}\mu^{-1}\big\{Z_1, \{Z_2, Z_2^\dagger\}\big\}|Z_2|^2 - \{V, Z_1\} \equiv \\ & \{Z_1^\dagger, Z_1\}Z_1 - f_2^{-1}\{Z_1, f_2\}|Z_2|^2 - \{V, Z_1\} \qquad \mathsf{mod}\ O(\mu^{-1/2}).\end{aligned}$$

Since we do not mind to change the speed to one of the same magnitude and we consider only level 0, we can assume without any loss of the generality that

$$f_2 = 1; \tag{21.2.80}$$

then finally $\{a, \zeta_1\} \equiv \{\zeta_1^\dagger\zeta_1 - V, \zeta_1\} \mod O(\mu^{-1/2})$.

Also corrected functions x_j satisfy $\{a, x_j'\} \equiv \{\zeta_1^\dagger\zeta_1 - V, x_j'\} \mod O(\mu^{-1/2})$ and we arrive to the following conclusion:

(21.2.81) Under assumption (21.2.80) the classical dynamics in the zone $\{|x_1| \le C\mu^{-1/2}\}$ in variables x_j, Z_1, Z_2 is described modulo $O(\mu^{-1} + \mu^{-1/2}T)$ by the solution to the short system

$$\frac{dZ_1}{dt} = \frac{1}{2}\{Z_1^\dagger Z_1 - V, Z_1\}, \tag{21.2.82}$$

$$\frac{dx_j}{dt} = \frac{1}{2}\{Z_1^\dagger Z_1 - V, x_j\}, \tag{21.2.83}$$

$$\{Z_1, Z_2\} = O(\mu^{-1}), \quad \{Z_1, Z_2^\dagger\} = O(\mu^{-1}). \tag{21.2.84}$$

Further, for each given initial point one can replace all functions modulo $(x_1 - \frac{1}{2}r^2)$, $(x_2 - \theta r^2)$ so that we get two-dimensional system in variables (r, θ) described by the Hamiltonian $\frac{1}{2}(Z_1^\dagger Z_1 - V)$ with $Z_1, Z_1^\dagger$ satisfying commutator relation

$$\{Z_1, Z_1^\dagger\} = \mu\phi\bar{r}(r - \bar{r}) + O(\mu^{-1})) \tag{21.2.85}$$

which is exactly system studied extensively in Chapter 14 with magnetic field $\phi(r - \bar{r})$ and coupling constant $\bar{\mu} = \mu\bar{r}$. Condition (21.2.77) means exactly that we are in the inner zone $\{|r - \bar{r}| \le C\bar{\mu}^{-1/2}|Z_1|^{1/2}\}$ for this system; since (in contrast to Chapter 14) $|Z_1|$ is not disjoint from 0 anymore one needs to define inner zone in this way.

Then the drift speed with respect to θr is

$$\asymp \Big(\eta - k^*\phi^{-1/2}\rho\Big)\rho r \tag{21.2.86}$$

where constant $k^* \approx 0.66$ is defined in Chapter 14 and η is the corrected symbol $\mu x_1^2/2$ which leads to its propagation speed given by (21.2.75) multiplied by μ.

One can see easily that the fourth term there

$$\{Z_2^\dagger, x_1\}f_2^{-1}x_1\big(\{a, Z_2\} + i\mu Z_2\big)$$

is $O(\mu^{-1/2})$. Further, the third term $\big\{a, \{Z_2^\dagger, x_1\}f_2^{-1}x_1\big\}Z_2$ could be rewritten modulo $O(\mu^{-1/2})$ and terms with unbalanced $Z_2, Z_2^\dagger$ as $|\{Z_2, x_1\}|^2 f_2^{-1}|Z_2|^2$ and disappears after applying $\operatorname{Re} i$.

So we are left with just two terms

$$\operatorname{Re} i\Big(\big\{a, \{Z_1^\dagger, x_1\}\phi^{-1}\big\}Z_1 + \{Z_1^\dagger, x_1\}\phi^{-1}\big(\{a, Z_1\} + i\mu Z_1\big)\Big) \tag{21.2.87}$$

where $\phi = f_1/x_1$ and one can replace a by $Z_1^\dagger Z_1 - V$ (other terms have unbalanced $Z_2, Z_2^\dagger$).

Dynamics and Remainder Estimate. V

So we are *exactly* in the situation described in Chapter 14. Recall that that in Chapter 14 symbol Z_1 could be reduced to

$$Z_1 = e^{i\beta(x)}\Big(\xi_1 + i\alpha(x)\big(\xi_2 - V_2(x)\big)\Big) \tag{21.2.88}$$

with real-valued α, β, V_2 such that $\alpha = 1 + O(x_1)$, $V_2 = x_1^2/2 + O(x_1^3)$.

One can see easily that in this case that corrected μx_1^2 would be $\xi_2 + O(\mu^{-1/2})$; recall that in Chapter 14 exactly deviation of ξ_2 from $k^*\phi^{-1/2}\rho$ was used as the measure of the drift with $\rho = V^{1/2}$. Note that the potential V now should be replaced by $(V - E_2)$ with $E_2 = |Z_2^2|$.

Therefore due to the logarithmic uncertainty principle the violation of periodicity is observable after the first turn as

$$\Big(\eta - k^*\phi^{-1/2}\rho\Big)\rho r\mu^{-1/2} \times \Big(\eta - k^*\phi^{-1/2}\rho\Big) \ge Ch|\log h|; \tag{21.2.89}$$

under assumption (21.2.48) the latter condition is the automatic corollary of

$$\Delta = |\eta - k^*\phi^{-1/2}\rho| \ge C\mu^{-1/2}r^{-1/2}\rho^{-1/2}. \tag{21.2.90}$$

Under this condition the drift speed for the "short" system is greater than $C\epsilon_1\mu^{-1/2}$ which is the larger than the error in the drift speed $c\mu^{-1/2}$ (see (21.2.81).

In this case $T_1 = \epsilon r\Delta$ and the rest is easy but extra logarithmic factor would appear; however, considering the right direction one can take $T_1 = \epsilon r\rho^{1/2}\Delta^{1/2}$ and following Chapter 14 with the standard justification on the quantum level one can prove easily that

(21.2.91) The contribution of the zone where condition (21.2.90) is fulfilled to the remainder is $O(\mu^{-1/2}h^{-3})$.

On the other hand, contribution to the remainder of the zone where this condition is violated would not exceed

$$C\Big(\int \mu^{-1/2}h^{-3}r^{-1/2}\rho^{-1/2}rdrd\rho\Big) = O(\mu^{-1/2}h^{-3})$$

as well.

So, the main result of this section is proven:

Proposition 21.2.24. *Under assumption* (21.2.48) *with small enough* $\delta > 0$ *the remainder estimate is* $O(\mu^{-1/2}h^{-3})$ *while the principal part is given by the standard Weyl formula (which is equal modulo the remainder estimate to magnetic Weyl formula).*

This statement implies trivially

Corollary 21.2.25. *Under assumptions* (21.2.48) *and*

$$V \geq \epsilon_0 > 0. \tag{21.2.92}$$

the following estimate holds:

$$|\int \big(e(x,x,0) - \mathcal{N}^{\mathsf{MW}}(x,0)\big)\psi(x)\,dx| \leq C\mu^{-1/2}h^{-3} + C\mu^2h^{-2} \tag{21.2.93}$$

where

$$\mathcal{N}^{\mathsf{MW}}(x,\tau) = (2\pi)^{-2}\mu^2h^{-2} \sum_{(m,n)\in\mathbb{Z}^{+\,2}} \theta\big(2\tau + V - (2m+1)\mu h f_1 - (2n+1)\mu h f_2\big) f_1 f_2 \sqrt{g}. \tag{21.2.94}$$

21.3 Temperate and Intermediate Magnetic Field. Canonical Forms

Now let us consider the main case

$$h^{-\delta} \leq \mu \leq Ch^{-1} \tag{21.3.1}$$

with an arbitrarily small exponent $\delta > 0$. In this section we consider different canonical forms of the magnetic Schrödinger operator in question (depending on the zone).

21.3.1 Precanonical Form

Precanonical Form. I

Let

$$U_2(x,\xi) = f_2(x)^{-1/2} Z_2(x,\xi) \tag{21.3.2}$$

Then

$$-i\mu^{-1}\{U_2, U_2^\dagger\} \equiv 2 \qquad \mathsf{mod}\ \mathcal{O}_k \tag{21.3.2}_k$$

with $k = 1$ where $\mathcal{O}_k := \mathcal{O}_k(\mu^{-1}Z_1, \mu^{-1}Z_1^\dagger, \mu^{-1}Z_2, \mu^{-1}Z_2^\dagger)$ is the space of the sums of polynomials containing monoms of order k or higher with respect to $(\mu^{-1}Z_1, \mu^{-1}Z_1^\dagger, \mu^{-1}Z_2, \mu^{-1}Z_2^\dagger)$ with the coefficients, smoothly depending on x.

Note that correcting U_2 by $U_2\mathcal{O}_k$ one trades $(21.3.2)_k$ by $(21.3.3)_{k+1}$ and therefore after an appropriate correction $(21.3.2)_k$ holds with arbitrarily large $k = M$.

Thus, correcting U_2:

$$U_2 \mapsto U_2 + \sum_{\substack{k+l+p+q\ge 2\\ k\ge 1}} \alpha_{klpq}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{1-k-l-p-q} \tag{21.3.4}$$

one one can achieve

$$-i\mu^{-1}\{U_2, U_2^\dagger\} \equiv 2 \qquad \mathsf{mod}\ O(\mu^{-M}) \tag{21.3.5}$$

with arbitrarily large M. Note that $\{U_2, Z_1\} \equiv \{U_2^\dagger, Z_1\} \equiv 0$ modulo $\mathcal{O}_1$ both originally and after correction (21.3.4). We claim that

Proposition 21.3.1. *Correcting*

$$Z_1 \mapsto Z_1 + \sum_{\substack{k+l+p+q\ge 2\\ k+l\ge 1}} \alpha'_{klpq}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{1-k-l-p-q}, \tag{21.3.6}$$

$$x_j \mapsto x_j + \sum_{k+l\ge 1} \alpha''_{jklpq}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{-k-l-p-q}, \tag{21.3.7}$$

one can arrange

$$\{U_2, x_j\} \equiv \{U_2, Z_1\} \equiv \{U_2, Z_1^\dagger\} \equiv 0, \tag{21.3.8}$$

$$\{U_2^\dagger, x_j\} \equiv \{U_2^\dagger, Z_1\} \equiv \{U_2^\dagger, Z_1^\dagger\} \equiv 0 \qquad \mathsf{mod}\ O(\mu^{-M}). \tag{21.3.9}$$

Proof. Indeed, correcting in the way described (but with the sum over $k \ge 1$ rather than $k + l \ge 1$) we can achieve (21.3.9). However, *so far corrected symbol x_j is not necessarily real-valued and corrected symbols Z_1 and $Z_1^\dagger$ are not necessarily complex conjugate*.

Then due to (21.3.9), (21.3.5) and Poisson identity $\{U_2^\dagger, \{U_2, y\}\} \equiv 0$ with $y = x_j, Z_1, Z_1^\dagger$. Therefore

$$\{U_2, Z_1\} \equiv \sum_{l+p+q\ge 1} \beta_{lpq}(x) U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{1-l-p-q},$$
$$\{U_2, Z_1^\dagger\} \equiv \sum_{l+p+q\ge 1} \beta'_{lpq}(x) U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{1-l-p-q},$$
$$\{U_2, x_j\} \equiv \sum_{l+p+q\ge 1} \beta''_{jlpq}(x) U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{-l-p-q}$$

where x, Z_1 and $Z_1^\dagger$ are already corrected symbols. Since at this moment x_j are not necessarily real-valued one needs to plug them formally into functions and use a Taylor decomposition with respect to $\mu^{-1}Z_1, \mu^{-1}Z_2, \mu^{-1}U_1, \mu^{-1}U_2$.

Then one can correct x_j, Z_1 and $Z_1^\dagger$ according to formulae (21.3.7), (21.3.8) with summation over $k = 0$ so that (21.3.8) holds.

Note that instead of Z_1 and $Z_1^\dagger$ one could consider $\mathsf{Re}\, Z_1$ and $\mathsf{Im}\, Z_1$ and correct them deriving (21.3.8)–(21.3.9). At this moment corrected symbols x_j, $\mathsf{Re}\, Z_1$ and $\mathsf{Im}\, Z_1$ are not necessarily real-valued. However, U_2 and $U_2^\dagger$ are truly complex conjugate and then (21.3.8)–(21.3.9) for corrected symbols x_j, $\mathsf{Re}\, Z_1$, $\mathsf{Im}\, Z_1$ imply the same equalities for the real parts of them. Let us replace then (corrected) symbols x_j, $\mathsf{Re}\, Z_1$ and $\mathsf{Im}\, Z_1$ by their real parts. After this (corrected) symbols x_j become real-valued and (corrected) symbols $Z_1 = \mathsf{Re}\, Z_1 + i\, \mathsf{Im}\, Z_1$ and $Z_1^\dagger = \mathsf{Re}\, Z_1 - i\, \mathsf{Im}\, Z_1$ become complex conjugate.□

Precanonical Form. II

One can rewrite operator in question in these new variables as

$$\begin{aligned}(21.3.10)\quad 2A \equiv f_2(x) U_2^\dagger U_2 + Z_1^\dagger Z_1 - V(x)& \\ + \sum_{k+l+p+q\ge 3} b_{klpq}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{2-k-l-p-q}& \\ + \sum_{k+l+p+q\ge 3} b'_{klpq}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{-k-l-p-q}&\end{aligned}$$

modulo lower order terms where the first line is the "main part" and the second and third lines are the "perturbation"; here the first part of the "perturbation" comes from $f_2 U_2^\dagger U_2 + Z_1^\dagger Z_1$ and the second part comes from $-V$.

Proposition 21.3.2. *For arbitrarily large M there exist $\epsilon = \epsilon(M) > 0$ and a real valued symbol*

$$\mathcal{L} = \sum_{k+l\ge 1} \ell_{kpqm}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{-1-k-l-p-q-m} \tag{21.3.11}$$

such that in the strip $\{|x_1| < \epsilon\}$ the following equality holds (so far only for the principal symbols of operators in question):

(21.3.12)
$$A^{\#} := e^{-i\mu^{-1}h^{-1}\mathcal{L}} A e^{i\mu^{-1}h^{-1}\mathcal{L}} \equiv \frac{1}{2}\Big(f_2(x^{\#}) U_2^{\#} U_2^{\#\dagger} + Z_1^{\#} Z_1^{\#\dagger} - V(x^{\#})\Big) + \sum_{2k+p+q+2m\ge 3} B_{kpqm}(x^{\#}) \big(U_2^{\#} U_2^{\#\dagger}\big)^k Z_1^{\#\, p} Z_1^{\#\dagger\, q} \mu^{2-2k-p-q-2m}$$

where $e^{-i\mu^{-1}h^{-1}\mathcal{L}}$, $e^{i\mu^{-1}h^{-1}\mathcal{L}}$ are (formal) $\mu^{-1}h$-FIOs and

$$x_j^{\#} = e^{-i\mu^{-1}h^{-1}\mathcal{L}} x_j e^{i\mu^{-1}h^{-1}\mathcal{L}}, \quad U_2^{\#} = e^{-i\mu^{-1}h^{-1}\mathcal{L}} U_2 e^{i\mu^{-1}h^{-1}\mathcal{L}}, \quad Z_1^{\#} = e^{-i\mu^{-1}h^{-1}\mathcal{L}} Z_1 e^{i\mu^{-1}h^{-1}\mathcal{L}} \tag{21.3.13}$$

are linked to the original symbols x_j, U_2, Z_1 by formulae similar to (21.3.4), (21.3.6) *and* (21.3.7) *but containing also factors μ^{-1} exactly on the same role as U_2 or $U_2^{\dagger}$ so that $k+l$ in the condition of summation is replaced by $k+l+m$* [11].

All equalities here hold modulo $O(\mu^{-M})$.

Proof. Proof is standard (see Chapter 19). The problem of the resonances is avoided because we remove only terms $b_*(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q}$ with $k \ne l$ which leads to the denominator $\mu\big((k-l)f_2 + (p-q)f_1\big)$ in the correction term; this denominator is of magnitude $\mu(k-l)$ since $k \ne l$, $f_2 \asymp 1$ and $f_1 \asymp |x_1| \le \epsilon(M)$. □

[11] More precisely,

$$U_2 \mapsto U_2^{\#} = U_2 + \sum_{k+l+p+q+m\ge 2} \alpha_{klpqm}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{1-k-l-p-q-m}, \tag{21.3.4}$$

$$Z_1 \mapsto Z_1^{\#} = Z_1 + \sum_{k+l+p+q+m\ge 2} \alpha'_{klpqm}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{1-k-l-p-q-m}, \tag{21.3.6}$$

$$x_j \mapsto x_j^{\#} = x_j + \sum_{k+l+p+q+m\ge 1} \alpha''_{jklpqm}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{-k-l-p-q-m}. \tag{21.3.7}$$

Now one can upgrade all the above arguments to operators so that equalities (21.3.5), (21.3.8) and (21.3.9) hold for commutators rather than for Poisson brackets and (21.3.12) is replaced by

$$(21.3.12)^* \quad A^{\#} := e^{-i\mu^{-1}h^{-1}\mathcal{L}} A e^{i\mu^{-1}h^{-1}\mathcal{L}} \equiv \frac{1}{2}\Big(f_2^{\mathsf{w}} U_2^* \bullet U_2 + Z_1^* \bullet Z_1 - V^{\mathsf{w}}\Big) + \sum_{2k+p+q+2m+2s\ge 3} B^{\mathsf{w}}_{kpqms} (U_2 \bullet U_2^*)^l Z_1^p Z_1^{*\,q} \mu^{2-2k-2m-p-q-s} h^s$$

where a^{w} is the Weyl quantization of symbol a and one should remember that the corrected symbol x is not just a function of coordinates but of the symbols and therefore the functions of x are replaced by the symbols of $\mu^{-1}h$-PDOs. Here and below $\bullet$ means the symmetrized product: $K \bullet L = \frac{1}{2}(KL + LK)$. Therefore, we arrive to

Proposition 21.3.3. *For arbitrarily large M and arbitrarily small $\delta > 0$ there exist $\epsilon = \epsilon(M, \delta) > 0$ and a real valued symbol*

$$(21.3.11)^* \quad \mathcal{L} = \sum_{k+l+p+q+m+2s\ge 1} \ell_{kpqms}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{-1-k-l-p-q-m-s} h^s$$

such that under assumption (21.3.1) *in the strip $\{|x_1| < \epsilon\}$ formula* $(21.3.12)^*$ *holds where $e^{-i\mu^{-1}h^{-1}\mathcal{L}}$, $e^{i\mu^{-1}h^{-1}\mathcal{L}}$ are $\mu^{-1}h$-FIOs and $x_j^{\#}, U_2^{\#}, Z_1^{\#}$ are full symbols still defined by* (21.3.13) *and they are linked to the original symbols x_j, Z_1, U_2 by formulae similar to* $(21.3.4)^*$,$(21.3.6)^*$,$(21.3.7)^*$ *but containing also factors $(\mu^{-1}h)$ of the "double value" of U_2 or $U_2^\dagger$ or μ^{-1} so that $k+l+m$ in the condition of summation is replaced by $k+l+m+2s$* [12)].

All equalities here and below hold modulo $O(h^M)$.

Precanonical Form. III

Note that due to (21.3.5) for $U_2^{\#}$ there exists $\mu^{-1}h$-FIO $\mathcal{F}$ such that

$$(21.3.14) \qquad \mathcal{F}^*\mathcal{F} \equiv \mathcal{F}\mathcal{F}^* \equiv I \qquad \text{and} \qquad \mathcal{F}^* U_2^{\#} \mathcal{F} \equiv (hD_2 - i\mu x_2).$$

[12)] More precisely

$$(21.3.4)^{**} \quad U_2 \mapsto U_2^{\#} = U_2 + \sum_{k+l+p+q+m+2s\ge 2} \alpha_{klpqms}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{1-k-l-p-q-m-s} h^s,$$

$$(21.3.6)^{**} \quad Z_1 \mapsto Z_1^{\#} = Z_1 + \sum_{k+l+p+q+m+2s\ge 2} \alpha'_{klpqmm}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{1-k-l-p-q-m-s} h^s,$$

$$(21.3.7)^{**} \quad x_j \mapsto x_j^{\#} = x_j + \sum_{k+l+m+2s\ge 1} \alpha''_{jklpqms}(x) U_2^k U_2^{\dagger\, l} Z_1^p Z_1^{\dagger\, q} \mu^{-k-l-p-q-m-s} h^s,$$

Then

$$\mathcal{F}^*(U_2^{\#} \bullet U_2^{\#\dagger})\mathcal{F} \equiv \bigl(h^2D_2^2 + \mu^2 x_2^2\bigr) \tag{21.3.15}$$

and also due to (21.3.8) and (21.3.9) $\mu^{-1}\mathcal{F}^* Z_1^{\#}\mathcal{F}$, $\mathcal{F}^* x_j^{\#}\mathcal{F}$ and $\mathcal{F}^* B_{kpqms}\mathcal{F}$ are $\mu^{-1}h$-PDOs with symbols depending only on x', ξ' ($x' = (x_1, x_3, x_4)$).

Let us redefine $Z_1^{\#} := \mu^{-1}\mathcal{F}^* Z_1^{\#}\mathcal{F}$, $B^{\mathsf{w}}_{kpqms} := \mathcal{F}^* B^{\mathsf{w}}_{kpqms}\mathcal{F}$ and $A^{\#} := \mathcal{F}^* A^{\#}\mathcal{F}$; then also automatically $Z_1^{\#\dagger} := \mu^{-1}\mathcal{F}^* Z_1^{\#\dagger}\mathcal{F}$ is an adjoint operator. Recall that f_2^{w} and B^{w}_{kpqms} are operators with the real-valued symbols and $A^{\#}$ is again defined by (21.3.12)*.

Now one can decompose u into series

$$u(x,y,t) = \sum_{n,n'\in\mathbb{Z}^+} \mu h^{-1} u_{nn'}(x',y',t)\,\Upsilon_n(x_2)\Upsilon_{n'}(y_2), \tag{21.3.16}$$

$$\Upsilon_n(x_2) := \upsilon_n\bigl(\mu^{1/2}h^{-1/2}x_2\bigr)$$

where υ_n are (real-valued and orthonormal) Hermite functions and then replace operator $\bigl(h^2D_2^2 + \mu^2 x_2^2\bigr)$ by $(2n+1)\mu h$, thus reducing operator $A^{\#}$ to the family of 3-dimensional $\mu^{-1}h$-PDOs

$$\mathcal{A}_n := \frac{1}{2}\Bigl(f_2^{\#}(2n+1)\mu h + (Z_1^{\#} \bullet Z_1^{\#\dagger}) - V^{\#}\Bigr) + \tag{21.3.17}$$

$$\sum_{2k+p+q+2m+2s\ge 1} B^{w}_{kpqms} \times \bigl((2n+1)\mu h\bigr)^k Z_1^{\# p} Z_1^{\#\dagger q} \mu^{2-2k-2m-p-q-s} h^s;$$

again the first line is the *main part* of operator and $f^{\#}$, $V^{\#}$ are transformed f_2 and V.

Remark 21.3.4. (i) In this and below formulae one actually needs to consider

$$n \le C_0/(\mu h) + C|\log h|. \tag{21.3.18}$$

Really, we are interested in the domain in the phase space where $\{a \le c\}$ or, equivalently, $\{|U_2| \le c, |Z_1| \le c\}$. Note that condition $\{|U_2| \le \varrho\}$ does not contradict to the logarithmic uncertainty principle as long as $\varrho^2 \ge C\mu h|\log h|$ and thus one can take an upper bound for a as $\varrho^2 = c + C\mu h|\log h|$ which implies (21.3.18). This estimate perfectly suits our purposes as long as

$$h^{-\delta} \le \mu \le \epsilon h^{-1}|\log h|^{-1}. \tag{21.3.19}$$

(ii) From the operator point of view which one needs to apply only as

$$\epsilon h^{-1}|\log h|^{-1} \le \mu \le \epsilon h^{-1} \tag{21.3.20}$$

in the final analysis we need to consider only $n \le C_0/(\mu h)$.

Remark 21.3.5. (i) Obviously $f_j^{\#}$, $V^{\#}$ and $\mu^{-1}Z_1^{\#}$ are $\mu^{-1}h$-PDOs with symbols $f_j \circ \Psi_2$, $V \circ \Psi_2$ and $\mu^{-1}Z_1 \circ \Psi_2$ respectively where $\Psi_2 : \mathbb{R}^6 \ni (x', \xi') \to x \in \mathbb{R}^4$ is some smooth map. Actually we are interested in the symbols of $f_2^{\#}$ and $V^{\#}$ only modulo symbols of $\mu^{-1}Z_1^{\#}$, $\mu^{-1}Z_1^{\#\dagger}$ and therefore actually we are interested only in the map $\Sigma \to \mathbb{R}^4$. However, at this moment we do not have a natural parametrization of Σ.

(ii) Note that $f_1^{\#}$ is not necessarily x_1 anymore. Still since $\nabla \operatorname{Re} Z_1$, $\nabla \operatorname{Im} Z_1$, $\nabla \operatorname{Re} Z_2$, $\nabla \operatorname{Im} Z_2$ and ∇f_1 were linearly independent[13] this linear independence statement is true for $\nabla \operatorname{Re} Z_1^{\#}$, $\nabla \operatorname{Im} Z_1^{\#}$ and $\nabla f_1^{\#}$ [14] as well. Therefore after $\mu^{-1}h$-FIO transformation (in $(x', \mu^{-1}hD')$) one can achieve $f_1^{\#} = x_1$.

So, we arrive to

Proposition 21.3.6. *Let condition* (21.3.1) *be fulfilled. Then by means of* $\mu^{-1}h$*-transform, decomposition* (21.3.16) *and one more* $\mu^{-1}h$*-transform one can reduce operator to the family of* 3*-dimensional operators* (21.3.17) *with* n *satisfying* (21.3.18) *and with*

$$i\mu^{-1}\{Z_1^{\#}, Z_1^{\#\dagger}\} = f_1^{\#} = x_1 \qquad \operatorname{mod}(Z_1, Z_1^{\dagger}). \tag{21.3.21}$$

21.3.2 Canonical Form away from Λ

Canonical Form away from Λ. I

Let us consider first the canonical form in the subdomain, disjoint from Λ which means that

$$|\{Z_1, f_1\}| \ge \epsilon_0. \tag{21.3.22}$$

Let us reduce (21.3.17) to a canonical form (multiplied by an elliptic operator). Note that the multiplication of $Z_1^{\#}$ by symbol β implies the multiplication of $Z_1^{\#\dagger}$ by $\beta^{\dagger}$ and therefore the multiplication of $f_1^{\#}$ by $|\beta|^2$

[13] All the linear independence statements are uniform.

[14] Where in the former case it was $\nabla_{x,\xi}$ and in the latter one $\nabla_{x',\xi'}$; recall that $x' = (x_1, x_3, x_4)$.

modulo $\mathcal{O}_1$ and finally the multiplication of $\{Z_1^\#, f_1^\#\}$ by $|\beta|^2\beta$ modulo $\tilde{\mathcal{O}}_1$ as well where here and below $\tilde{\mathcal{O}}_m$ means the sum of monoms of the type $\alpha_{ljk}\mu^{l-m}x_1^l Z_1^{\#\,j} Z_1^{\#\dagger\,k}$ with $m \ge j+k+l$ and smooth coefficients α_{ljk}. Also to maintain (21.3.12) and (21.3.17) one needs to multiply operators A and $\mathcal{A}_n$ (and thus $f_2^\#$, $V^\#$ and all the perturbation terms) by $|\beta|^2$.

Then picking up

$$\beta = \{Z_1^\#, f_1^\#\}^\dagger \cdot |\{Z_1^\#, f_1^\#\}|^{-4/3}$$

one can achieve

$$\{Z_1^\#, f_1^\#\} = 1 + \alpha \tag{21.3.23}$$

with $\alpha \in \tilde{\mathcal{O}}_m$ and $m = 1$. Consequently, multiplying $Z_1^\#$ by $(1+\beta)$ with $\alpha \in \tilde{\mathcal{O}}_m$ one changes $\{Z_1^\#, f_1^\#\}$ by $(2\beta+\beta^\dagger)\{Z_1^\#, f_1\}$ modulo $\tilde{\mathcal{O}}_{m+1}$ and picking $\beta = \frac{1}{3}\alpha^\dagger - \frac{2}{3}\alpha$ one can achieve (21.3.23) with $\alpha \in \tilde{\mathcal{O}}_M$ with arbitrarily large M (and after asymptotic summation with $M = \infty$). Unfortunately elements of $\tilde{\mathcal{O}}_M$ (in contrast to elements of $\mathcal{O}_M$) are not necessarily negligible unless $|x_1| \le h^\delta$.

However, one can observe easily that multiplying $Z_1^\#$ by $(1+\beta)$ with $\beta \in \mathcal{O}_m \cap \tilde{\mathcal{O}}_\infty$ we change $\{Z_1^\#, f_1^\#\}$ by $\{Z_1^\#, \phi\}$ with

$$\phi = f_1(\beta + \beta^\dagger) + i\mu^{-1}\{Z_1^\#, \beta^\dagger\} - i\mu^{-1}\{Z_1^{\#\dagger}, \beta\}$$

modulo $\mathcal{O}_m \cap \tilde{\mathcal{O}}_\infty$. Obviously there exists $\beta \in \mathcal{O}_1 \cap \tilde{\mathcal{O}}_\infty$ eliminating error α in (21.3.23) modulo $\mathcal{O}_1 \cap \tilde{\mathcal{O}}_\infty$. One can see easily that the only restriction to $\phi \in \mathcal{O}_{m-1} \cap \tilde{\mathcal{O}}_\infty$ is that it must be real valued.

Therefore step by step one can make $\mathsf{Re}\,\alpha \in \mathcal{O}_M \cap \tilde{\mathcal{O}}_\infty$: $\{\mathsf{Re}\, Z_1, f_1\} \equiv 1 \mod \mathcal{O}_M \cap \tilde{\mathcal{O}}_\infty$ with arbitrarily large M. But then picking up

$$f_1^\# = i\mu^{-1}\{Z_1^\#, Z_1^{\#\dagger}\} = 2\mu^{-1}\{\mathsf{Re}\, Z_1^\#, \mathsf{Im}\, Z_1^\#\}$$

we get

$$\big\{\mathsf{Re}\, Z_1^\#, \{\mathsf{Im}\, Z_1^\#, f_1\}\big\} = \big\{\mathsf{Im}\, Z_1^\#, \{\mathsf{Re}\, Z_1^\#, f_1\}\big\} \in \mathcal{O}_{M-1} \cap \tilde{\mathcal{O}}_\infty$$

and since $\{\mathsf{Im}\, Z_1^\#, f_1\} \in \mathcal{O}_{M-1}$ as $f_1 = 0$ and $\{\mathsf{Re}\, Z_1^\#, f_1^\#\} \sim 1$ we derive that $\{\mathsf{Im}\, Z_1^\#, f_1\} \in \mathcal{O}_{M-1} \cap \tilde{\mathcal{O}}_\infty$. Then

$$-i\mu^{-1}\{Z_1^\#, Z_1^{\#\dagger}\} = 2f_1^\#, \qquad \{Z_1^\#, f_1^\#\} \equiv 1 \mod O(\mu^{-M}) \tag{21.3.24}$$

and then after an appropriate $\mu^{-1}h$-FIO transformation

$$Z_1^{\#} \equiv hD_1 + i\bigl(hD_3 - \frac{1}{2}\mu x_1^2\bigr) \tag{21.3.25}$$

(again we are using a linear independence of $\nabla \operatorname{Re} Z_1^{\#}$, $\nabla \operatorname{Im} Z_1^{\#}$ and ∇f_1). However, one can replace operators $\beta(x', \mu^{-1}hD')$ by their decompositions into powers of $\mu^{-1}hD_1 = \frac{1}{2}(Z_1^{\#} + Z_1^{\#\dagger})$ and $\mu^{-1}hD_3 - \frac{1}{2}x_1^2 = \frac{1}{2i}\mu^{-1}(Z_1^{\#} - Z_1^{\#\dagger})$ thus arriving to decomposition (21.3.26) below with all the operators of the $\beta(x_1, x'', \mu^{-1}hD_4; \mu^{-1}h)$ and the following statement has been proven:

Proposition 21.3.7. *Let conditions* (21.3.1) *and* (21.3.22) *be fulfilled. Then by means of $\mu^{-1}h$-transform, decomposition* (21.3.16) *and one more $\mu^{-1}h$-transform one can reduce original operator A to the family of 3-dimensional operators*

$$\mathcal{A}_n = \frac{1}{2}\sigma\Bigl((Z_1^{\#} \bullet Z_1^{\#\dagger}) - W_n^{\#} + \sum_{2k+p+q+2m+2s\ge 3} B^{\mathsf{w}}_{kpqms} \times \bigl((2n+1)\mu h\bigr)^k Z_1^{\#\,p} Z_1^{\#\dagger\,q} \mu^{2-2k-2m-p-q-s} h^s\Bigr) \tag{21.3.26}$$

where symbol $\sigma = |\{Z_1, f_1\}|^{2/3}$ is bounded and disjoint from 0, $Z_1^{\#}$ given by (21.3.25) *and*

(21.3.27) $W_n^{\#}$ *is an operator with the symbol $\sigma^{-1}\bigl(V - f_2(2n+1)\mu h\bigr) \circ \Psi$ where Ψ is the smooth diffeomorphism $\mathbb{R}^4 \to \mathbb{R}^4$, with $D\Psi$ transforming $\operatorname{Span}\bigl((0,1,0;0),(0,0,0;1)\bigr)$ into $\mathbb{K}_2$ and $\operatorname{Span}\bigl((1,0,0,0),(0,0,1,0)\bigr)$ into $\mathbb{K}_1$, and, in particular, $(0,0,1;0)$ into some element of $\mathbb{K}_1$ of $(0,*,*,*)$ form. Further, $|\det D\Psi^{-1}| = f_2$.*

Again the first line in (21.3.26) *is the main part.*

Canonical Form away from Λ. II

Now let us consider domain where condition (21.3.22) is violated. Note first that $\bigl\{(x,\xi) : f_1 = Z_1 = \{Z_1, f_1\} = 0\bigr\}$ was an involutive manifold of codimension 5 in $\mathbb{R}^4 \times \mathbb{R}^4$ with

(21.3.28) ∇f_1, $\nabla \operatorname{Re} Z_1$, $\nabla \operatorname{Im} Z_1$, $\nabla\{\operatorname{Re} Z_1, f_1\}$ and $\nabla\{\operatorname{Im} Z_1, f_1\}$ linearly independent on Λ' [13)].

Also note that

$$\{Z_1, x_1\} = x_3 + ix_4, \quad \{Z_1, x_3\} = x_3 + ix_4, \quad \{Z_1, x_4\} = x_4 - ix_3 \qquad \text{as} \ \ x_1 = 0$$

and

$$\{\alpha^{\#}, \beta^{\#}\} = \{\alpha, \beta\} + i\mu^{-1} f_2^{-1}\Big(\{Z_2^{\dagger}, \alpha\}\{Z_2, \beta\} - \{Z_2, \alpha\}\{Z_2^{\dagger}, \beta\}\Big) + O(\mu^{-2}). \tag{21.3.29}$$

However, $\{Z_2, x_j\}$ are actually arbitrary and therefore while we know that

$$\Lambda^{\#} = \big\{f_1^{\#} = Z_1^{\#} = \{Z_1^{\#}, f_1^{\#}\} = 0\big\} \tag{21.3.30}$$

is a manifold of codimension 5 with

(21.3.31) $\nabla f_1^{\#}$, $\nabla \operatorname{Re} Z_1^{\#}$, $\nabla \operatorname{Im} Z_1^{\#}$, $\nabla\{\operatorname{Re} Z_1^{\#}, f_1\}$ and $\nabla\{\operatorname{Im} Z_1^{\#}, f_1^{\#}\}$ linearly independent on $\Lambda^{\#}$ [13),14)]

but its symplectic structure is not fixed and there is no canonical form uniformly near $\Lambda^{\#}$.

Instead let us consider point $\bar{z}$ ($= (0,0) \in \mathbb{R}^6$ for simplicity of notations) with

$$|f_1| \le \gamma = \epsilon r^2, \quad |\{Z_1, f_1\}| = r \qquad \text{with} \ \ r \ge \mu^{\delta - 1/3}, \tag{21.3.32}$$

and reduce an operator to a canonical form in its $\epsilon(\gamma, r)$-vicinity with respect to $(x_1; x_2, x_3, x_4)$, leaving domain $\{|f_1| \ge \epsilon |\{Z_1, f_1\}|^2\}$ for a later analysis.

Then after reduction of the previous subsection condition (21.3.32) remains valid for $Z_1^{\#}$ and $f_1^{\#}$; therefore now multiplication by a symbol $\beta = \{Z_1^{\#}, f_1^{\#}\}^{\dagger} \cdot |\{Z_1^{\#}, f_1^{\#}\}|^{-4/3} r^{1/3}$ provides a modified equality (21.3.23), namely,

$$\{Z_1^{\#}, f_1^{\#}\} = r + \alpha. \tag{21.3.33}$$

However, β is an uniformly smooth symbol; more precisely $\beta = \hat{\beta}(r^{-1}x', r^{-1}\xi')$ with uniformly smooth symbol $\hat{\beta}$, which means that after original rescaling $x' \mapsto rx'$ [15)] $\beta = \hat{\beta}(x, \mu^{-1} h r^{-2} D')$ and $x_1^{-1}\alpha$ is of the same type.

[15)] Since symplectic structure of $\Lambda^{\#}$ is not defined one cannot confine this non-smoothness to a couple of coordinates.

Continuing as before[16] one can achieve a modified equality (21.3.24), namely,

$$-i\mu^{-1}r^{-1}\{Z_1^{\#}, Z_1^{\#\dagger}\} = 2f_1^{\#}, \qquad \{Z_1^{\#}, f_1^{\#}\} \equiv r \mod O(\mu^{-s}) \tag{21.3.34}$$

where now $\mu^{-1}r^{-1}Z_1$ is $\mu^{-1}hr^{-2}$-PDO and then modified equality (21.3.25)

$$Z_1^{\#} \equiv \bigl(rh_*D_1 + h_*D_4\bigr) + i\bigl(h_*D_3 - \frac{1}{2}\mu r x_1^2\bigr) \tag{21.3.35}$$

where extra terms h_*D_4 and h_*D_3 appear because (original) $\nabla \operatorname{Re} Z_1$ and $\nabla \operatorname{Im} Z_1$ are linearly independent, $h_* = hr^{-1}$.

Then after another rescaling $x_1 \mapsto rx_1$, $D_1 \mapsto r^{-1}D_1$ (21.3.35) becomes

$$Z_1^{\#} \equiv \bigl(h_*D_1 + h_*D_4\bigr) + i\bigl(h_*D_3 - \frac{1}{2}\mu r^3 x_1^2\bigr)$$

and after transformation $(x_1, x_3, x_4; D_1, D_3, D_4) \mapsto (x_1, x_3, x_4{+}x_1; D_1{-}D_4, D_3, D_4)$ becomes

$$Z_1^{\#} = h_*D_1 + i\bigl(h_*D_3 - \frac{1}{2}\mu_* x_1^2\bigr), \qquad h_* = hr^{-1}, \quad \mu_* = \mu r^3 \tag{21.3.36}$$

which is exactly equality (21.3.25) with h and μ replaced by h_* and μ_* respectively.

Observe that operators $\beta(r^{-1}x', \mu^{-1}hr^{-1}D'; \mu^{-1}hr^{-2})$ (with the smooth symbol β) were transformed subsequently into $\beta(x', \mu^{-1}hr^{-1}D'; \mu^{-1}hr^{-2})$, then into $\beta(rx_1, x'', \mu^{-1}hr^{-3}D_1, \mu^{-1}hr^{-2}D''; \mu^{-1}hr^{-2})$ and finally into

$$\beta(rx_1, x'', \mu^{-1}hr^{-3}D_1, \mu^{-1}hr^{-2}D_3, \mu^{-1}hr^{-2}D_4; \mu^{-1}hr^{-3}).$$

Therefore we arrive to (21.3.17)-like decomposition with $Z_1^{\dagger}$ given by (21.3.36) and σ, W_n and B^{w}_{kpqms} of the above type. However, one can replace such operators by their decompositions into powers of $\mu^{-1}hr^{-3}D_1 = \frac{1}{2}\mu^{-1}r^{-2}(Z_1^{\#} + Z_1^{\#\dagger})$ and $\mu^{-1}hr^{-2}D_3 - \frac{1}{2}r^2x_1^2 = \frac{1}{2i}\mu^{-1}r^{-1}(Z_1^{\#} - Z_1^{\#\dagger})$ thus arriving to decomposition (21.3.38) below with all the operators of the simplified type

$$\beta(rx_1, x'', \mu^{-1}hr^{-2}D_4; \mu^{-1}hr^{-3}); \tag{21.3.37}$$

factors $\mu^{-1}hr^{-4}$ appear because one needs to commute $Z_1^{\#}, Z_1^{\#}$ between themselves and with operators of (21.3.37)-type.

So, we finally arrive to

[16] With effective semiclassical parameter $\mu^{-1}hr^{-2}$ rather than $\mu^{-1}h$.

Proposition 21.3.8. *Let conditions* (21.3.1) *and* (21.3.32) *be fulfilled. Then by means of* $\mu^{-1}h$*-FIO transform, decomposition* (21.3.14) *and the series of rescalings and* $\mu^{-1}hr^{-2}$*-FIO transform one can reduce described above original operator* A *to the family of* 3*-dimensional operators*

$$\mathcal{A}_n := \frac{1}{2}\sigma^{\#}\Big((Z_1^{\#}\bullet Z_1^{\#\dagger}) - W_n^{\#} + \sum_{2k+p+q+2m+2s\geq 3} B^{\mathrm{w}}_{kpqms}\times \big((2n+1)\mu h\big)^k Z_1^{\#\,p} Z_1^{\#\dagger\,q}\,\mu^{2-2k-2m-p-q-s}h^s r^{-2p-2q-4m-4s}\Big) \tag{21.3.38}$$

where $\sigma^{\#}$ *is transformed operator with symbol* $\sigma = |\{Z_1, f_1\}|^{2/3}r^{-2/3}$ *which is bounded and disjoint from* 0 *and* $Z_1^{\#}$ *given by* (21.3.35).

Again the first line in (21.3.38) *is the main part.*

Remark 21.3.9. (i) Here operators $\sigma^{\#}$, $W_n^{\#}$ and B^{w}_{kpqms} are of the (21.3.37)-type with uniformly smooth symbol β. Therefore these symbols are quantizable provided under assumption (21.3.32) for sure;

(ii) We leave for a Section 21.4 the the calculation of the symbol of W and the discussion of the corresponding diffeomorphism.

(iii) Reduction has been done in the colored zone on Figure 21.1.

Canonical Form away from Λ. III

Let us look what happens with the cut-off symbol ψ as a result of these transformations. After transformations of Subsubsections 21.3.1.1 and 21.3.1.2 after decomposition (21.3.16) it becomes a matrix operator

$$\sum_{k,l}\phi_{kl}(x', \mu^{-1}hD')\mu^{-k-l}U_2^k U_2^{\dagger l}$$

with basis-shifting (in $\mathscr{L}^2(\mathbb{R})$) operators $\big(2(n+1)\mu h\big)^{-1/2}U_2^{\dagger}$ and $\big(2n\mu h\big)^{-1/2}U_2$.

Then after transformations of Subsubsection 21.3.2.2 this matrix operator becomes

$$\sum_{k,l,p,q,m,s} \phi^{\mathrm{w}}_{klpqms} U_2^k U_2^{\dagger l} Z_1^{\#\,p} Z_1^{\#\dagger\,q}\,\mu^{-k-l-m-p-q-s}h^s r^{-2p-2q-2m-4s} \tag{21.3.39}$$

with the scalar main part of the form (21.3.37).

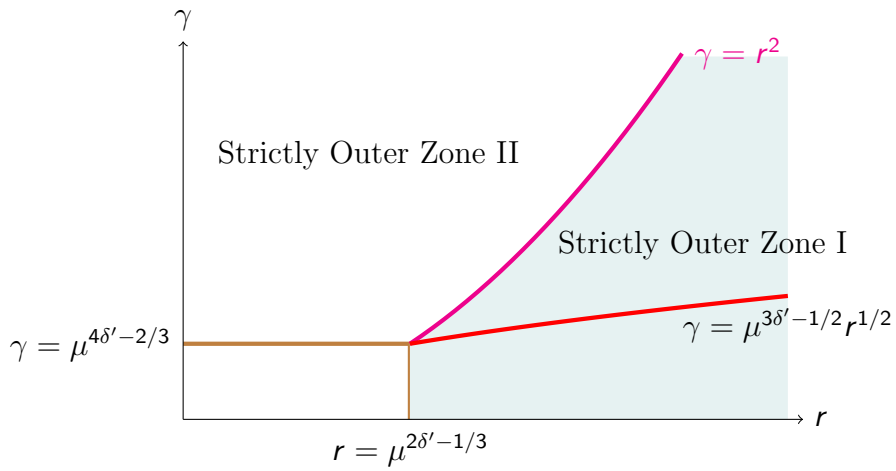

Figure 21.1: Reduction was done in the colored zone.

21.3.3 Canonical Form in the Strictly Outer Zone

Canonical Form in the Strictly Outer Zone. I

Now one can properly redefine the first part of the *outer zone*

$$\mathcal{Z}_{\mathsf{out},I} := \Big\{ C\mu^{-1/2}\mathsf{dist}(x,\Lambda)^{1/2} \le \mathsf{dist}(x,\Sigma) \le \epsilon\, \mathsf{dist}(x,\Lambda)^2 \Big\} \tag{21.3.40}$$

which in the notations $\mathsf{dist}(x,\Sigma) = |x_1|$ and $\mathsf{dist}(x,\Lambda) \asymp r$ is described as $\{\mu^{-1/2} r^{1/2} \le |x_1| \le \epsilon r^2\}$. Note that after rescaling of the previous section $x_1 \mapsto x_1 r^{-2}$, $\mu \mapsto \mu_* = \mu r^3$ it becomes $\{|x_1| \ge C\mu_*^{-1/2}\}$.

In this subsubsection we restrict ourselves to the part of *strictly outer zone*

$$\mathcal{Z}^*_{\mathsf{out},I} := \Big\{ C\mu^{3\delta-1/2}\mathsf{dist}(x,\Lambda)^{1/2} \le \mathsf{dist}(x,\Sigma) \le \epsilon\, \mathsf{dist}(x,\Lambda)^2 \Big\} \tag{21.3.41}$$

which in the same notations is $\{\mu^{-1/2} r^{1/2} \le |x_1| \le \epsilon r^2\}$ and after rescaling becomes $\{|x_1| \ge C\mu_*^{3\delta-1/2}\}$ (may be with a different $\delta > 0$) and provides a nice quantization of all symbols below.

Then since $\mu_* \ge h^{-\delta}$ one can apply the canonical form of the Schrödinger operator with the strong magnetic field. Let us consider an operator obtained after reduction of the previous subsection; there $|x_1| \asymp \gamma_* = \gamma r^{-2}$ if originally $|x_1| \asymp \gamma$. Notation $\gamma = r^2$ was temporary.

Applying rescaling $(x_1, x_3) \mapsto (x_1\gamma_*, x_3\gamma_*)$ one arrives to transformed (21.3.36) of the same type but with h_* and μ_* replaced by $h_*\gamma_*^{-1} = hr\gamma^{-1}$ and $\mu_*\gamma_*^2 = \mu\gamma^2 r^{-1}$ respectively:

$$Z_1^{\#} = \mu\gamma^2 r^{-1}\Big(\hbar D_1 + i\big(\hbar D_3 + \frac{1}{4}x_1^2\big)\Big) \tag{21.3.42}$$

with $|x_1| \asymp 1$ and

$$\hbar = \mu^{-1}hr^2\gamma^{-3} \le c\mu^{1/2}h^{1+3\delta}r^{1/2} \ll ch^{1/2}. \tag{21.3.43}$$

Also $\mu^{-1}hr^{-2}D_3 \mapsto \mu^{-1}h\gamma^{-1}D_3$.

Let us consider the part where $x_1 > 0$; another part is separated and treated exactly in the same manner. Introducing

$$U_1 = x_1^{-1/2}\Big(\hbar D_1 + i\big(\hbar D_3 + \frac{1}{4}x_1^2\big)\Big) \tag{21.3.44}$$

one can see easily that

$$|U_1| \le c\nu^{-1}, \qquad \nu := \mu\gamma^2 r^{-1} \ge h^{-\delta}, \tag{21.3.45}$$

$$\{U_1, U_1^\dagger\} \equiv 1 \qquad \mathsf{mod}\ U_1, U_1^\dagger \tag{21.3.46}$$

where (21.3.45) follows from $|Z_1^{\#}| \le c$ and (21.3.46) is understood in the sense of $\hbar$-symbols.

Therefore there exists $\hbar$-FIO transformation $\mathcal{F}'$ of $(x_1, x_3; \hbar D_1, \hbar D_3)$ thus not affecting $(x_4, \mu^{-1}hr^{-2}D_4)$ such that

$$U_1 \mapsto U_1^{\#} \equiv \hbar D_1 + ix_1 \qquad \mathsf{mod}\ \mathcal{O}_2(U_1, U_1^\dagger); \tag{21.3.47}$$

then operator (21.3.38) is transformed into

$$\begin{aligned}\frac{1}{2}\sigma\sigma_1\Big(&\nu^2 U_1^{\#} \bullet U_1^{\#\dagger} - W_n^{\#\#} + \\ &\sum_{2k+p+q+2m+2s+2l\ge 3} B^{\mathsf{w}}_{kpqmsl}\times \\ &\big((2n+1)\mu h\big)^k\big(\nu U_1^{\#}\big)^p\big(\nu U_1^{\#\dagger}\big)^q\mu^{2-2k-2m-p-q-s}h^s r^{-2p-2q-4m-4s}\hbar^l\Big)\end{aligned} \tag{21.3.48}$$

with $W_n^{\#\#}$ which is the transformation of $x_1^{-1}W_n^{\#}$.

Here all the operators σ, σ_1, $W_n^{\#\#}$ and B^{w}_{kpqmsl} are of the type

$$\beta(x_1, x_3, x_4; \hbar D_1, \hbar D_3, \mu^{-1}hr^{-2}D_4).$$

Note that $\hbar\nu = hr\gamma^{-1} \ll \mu^{1/2}hr^{1/2} \le h^{1/2}$ in the zone in question. One can decompose all such operators into power series with respect to x_1, $\hbar D_1$ and rewrite (21.3.48) with all operators of the type

$$\beta(x_3, x_4; \hbar D_3, \mu^{-1}hr^{-2}D_4), \qquad \hbar = \mu^{-1}hr^2\gamma^{-3} \tag{21.3.49}$$

and with $\bigl(\nu U_1^{\#}\bigr)^p$, $\bigl(\nu U_1^{\#\dagger}\bigr)^q$ replaced by $\bigl(\nu U_1^{\#}\bigr)^{p+p'}\nu^{-p'}$, $\bigl(\nu U_1^{\#\dagger}\bigr)^{q+q'}\nu^{-q'}$ respectively since $\nu \ge h^{-\delta}$.

Repeating arguments of the previous subsections one can transform this operator into

$$\begin{aligned}\frac{1}{2}\sigma\sigma_1\Bigl(&\nu^2 U_1^{\#}\bullet U_1^{\#\dagger} - W_n^{\#\#}+\\ &\sum_{2q+2l+2j\ge 3} B^{\mathrm{w}}_{qjl}\times \bigl(\nu^2 U_1^{\#}\bullet U_1^{\#\dagger}\bigr)^q \nu^{2-2q-2j}\hbar^l+\\ &\sum_{2k+2q+2m+2s+2l\ge 3} B^{\mathrm{w}}_{kqjmsl}\times\\ &\bigl((2n+1)\mu h\bigr)^k\bigl(\nu^2 U_1^{\#}\bullet U_1^{\#\dagger}\bigr)^q \mu^{2-2k-2m-2q-s}h^s r^{-4q-4m-4s}\nu^{-2j}\hbar^l\Bigr)\end{aligned} \tag{21.3.50}$$

where the second line comes from decomposing of $W_n^{\#\#}$ and because it is rescaled before

$$|B_{qjl}| \le C\gamma r^{-1}, \tag{21.3.51}$$

Now one can apply decomposition with respect to $\hbar^{-1/4}v_p(\hbar^{-1/2}x_1)$:

$$u_{nn'}(x', y', t) = \sum_{p,p'\in\mathbb{Z}^+} \hbar^{-1}u_{npn'p'}(x'', y'', t)\, v_p\bigl(\hbar^{-1/2}x_1\bigr)v_{p'}\bigl(\hbar^{-1/2}y_1\bigr) \tag{21.3.52}$$

with $x'' = (x_3, x_4)$. Then the operator (21.3.50) becomes

$$\begin{aligned}\frac{1}{2}\sigma\sigma_1\Bigl(&(2p+1)\mu h\gamma - W_n^{\#\#}+\\ &\sum_{2q+2l+2j\ge 3} B^{\mathrm{w}}_{qjl}\times \bigl((2p+1)\mu h\gamma\bigr)^q \nu^{2-2q-2j}\hbar^l+\\ &\sum_{2k+2q+2m+2s+2l\ge 3} B^{\mathrm{w}}_{kqjmsl}\times\\ &\bigl((2n+1)\mu h\bigr)^k\bigl((2p+1)\mu h\gamma\bigr)^q \mu^{2-2k-2m-2q-s}h^s r^{-4q-4m-4s}\nu^{-2j}\hbar^l\Bigr).\end{aligned} \tag{21.3.53}$$

since $\nu^2\hbar = \mu h\gamma$ and $2k+2q+2m+2s+2j+2l \ge 3$ is equivalent to $k+q+m+s+j+l \ge 2$.

In the decomposition (21.3.53) the role of m, s, j and l is just to bound a magnitude and indicate dependence on μ and h since dependence on r, γ (but not magnitude) is not important. However, $\nu = \mu\cdot\gamma^2 r^{-1} \le \mu r^2$, $\hbar = \mu^{-1}h\cdot r^2\gamma^{-3} \ge \mu^{-1}hr^{-4}$; therefore one can join terms with $s>0$ or $m>0$ to the terms with $s=0$ and $m=0$ without changing $s+l$, $m+j$ thus getting the simplified correction term

$$\sum_{k+q+j+l\ge 2} B^{\mathsf{w}}_{kqjmsl} \times \bigl((2n+1)\mu h\bigr)^k \bigl((2p+1)\mu h\gamma\bigr)^q \mu^{2-2k-2q} h^s r^{-4q}\nu^{-2j}\hbar^l.$$

Therefore we arrive to the following

Proposition 21.3.10. *Let condition* (21.3.1) *be fulfilled. Let us consider zone* $\mathcal{Z}^*_{\mathsf{out},l}$, *described by* (21.3.40).

Then by means of $\mu^{-1}hr^{-2}$*-FIO transforms, decomposition* (21.3.16), *the "special"* $\hbar$*-FIO transform and decomposition* (21.3.52) *one can reduce the original operator* A *to the family of* 2*-dimensional operators*

$$\mathcal{A}_{np} \coloneqq \frac{1}{2}\sigma\sigma_1\Bigl(H_{pn} + \sum_{k+q+j+l\ge 2} B^{\mathsf{w}}_{kqjmsl} \times \bigl((2n+1)\mu h\bigr)^k \bigl((2p+1)\mu h\gamma\bigr)^q \mu^{2-2k-2q} h^s r^{-4q}\nu^{-2j}\hbar^l\Bigr) \tag{21.3.54}$$

with $n \le C_0/(\mu h)$, $p \le C_0/(\mu h\gamma)$ *where all operators are of* (21.3.49)*-type and* σ, σ_1 *are bounded and disjoint from* 0.

Remark 21.3.11. (i) While in the canonical form (21.3.38) only one cyclotron movement was separated, in the canonical form (21.3.54) both of them are separated leaving only pure drift movement which will be studied in the next section.

(ii) We leave for the next Section 21.4 the discussion of the actual elements where reduction was done, corresponding diffeomorphism and the calculation of the symbol of W.

(iii) Cut-off symbol ψ which after previous transformations became (21.3.39) now becomes

$$\sum_{k,i,p,q,s,l} \phi_{kipqsl}\times U_2^{i}U_2^{\dagger\,k}U_1^{\#\,p}U_1^{\#\dagger\,q}\mu^{-i-k-2m-p-q-s}h^{s}r^{-2p-2q-4m-4s}\nu^{-2j-i-k}\hbar^{l} \tag{21.3.55}$$

with (21.3.49)-type operators ϕ_{kipqsl}.

Canonical Form in the Strictly Outer Zone. II

Now our goal is to achieve a similar reduction in the second part of the *outer zone*

$$\mathcal{Z}^{*}_{\mathsf{out},II} = \Big\{ C\,\mathsf{max}\big(\mathsf{dist}(x,\Lambda)^2, \mu^{4\delta-2/3}\big) \le \mathsf{dist}(x,\Sigma) \le \epsilon\Big\}. \tag{21.3.56}$$

Then the previous subsection 21.3.2 becomes irrelevant; however, we can start from (21.3.17). Again without any loss of the generality one can assume that $f_1^{\#} = x_1$. Let us consider a point $\bar{z} = (\bar{x}', \bar{\xi}') \in \mathbb{R}^6$ and an element (21.3.55) with $\gamma = |\bar{x}_1'|$ and $r = \gamma^{1/2}$ (exactly as in the previous subsection it was defined the other way. Without any loss of the generality one can assume that $\bar{z} = (\gamma, 0, \dots, 0)$.

Consider Taylor decomposition (with a remainder) of $\mu^{-1}Z_1^{\#}$ with respect to x', $\mu^{-1}hD'$ and r (in r-vicinity of 0 all of them are $O(r)$); then due to $\{Z_1^{\#}, Z_1^{\#\dagger}\} = O(r)$, $\{Z_1, x_1\} = O(r)$ and without any loss of the generality one can assume that $\mu^{-1}Z_1^{\#} = \mu^{-1}hD_3 + i\mu^{-1}hD_4 + O(r^2)$. Note that $\mu^{-1}hD_1$ could be either with a square (or some higher degree), or with a linear, or with a quadratic factor with respect to other variables.

Also note that the quadratic terms in $O(r^2)$ containing exclusively x_3, x_4 must commute with $D_3 + iD_4$ (otherwise $\{Z_1^{\#}, Z_1^{\#\dagger}\} \sim 2x_1$ would not be possible) and therefore must be of the form $c(x_3 + ix_4)^2$. However, one could remove such terms by $\mu^{-1}h$-PDO affecting only $x'', \mu^{-1}hD''$ and leaving x_1, $\mu^{-1}hD_1$ intact. So, terms in the decomposition must contain either one of the factors x_1, $\mu^{-1}hD_3$, $\mu^{-1}hD_4$, or two factors $\mu^{-1}hD_1$, or three factors x_3, x_4; also x_1 must be accompanied by $O(r)$-type factor.

Then scaling $x'' \mapsto rx'$, $x_1 \mapsto r^2x_1$, $D'' \mapsto r^{-1}D''$, $D_1 \mapsto r^{-2}D_1$ and dividing by r^3 one gets completely legitimate $\mu^{-1}hr^{-4}$-PDO where each factor $\mu^{-1}hD''$ can accommodate division by r^3 and $\mu^{-1}hD_1$ can accommodate division by r^2 but there either at least two such factors, or there is an extra $O(r)$ type factor. This operator, with the semiclassical parameter $\hbar = \mu^{-1}h\gamma^{-2}$ satisfies again $\{Z_1^{\#}, Z_1^{\#\dagger}\} \sim 2x_1$ but must be considered

near point $(1,0,\dots,0)$ and then by the standard arguments one can reduce operator to the canonical form (21.3.54) with all the operators of the type

$$\beta(x_3,x_4;\hbar D_3,\hbar D_4),\qquad \hbar=\mu^{-1}h\gamma^{-2}. \tag{21.3.57}$$

Note that as $r=\gamma^{1/2}$ this is consistent with the definition of $\hbar$ in the zone $\mathcal{Z}_{\mathsf{out},I}$ but form (21.3.57) is more general then (21.3.49).

So, we arrive to

Proposition 21.3.12. *Let condition* (21.3.1) *be fulfilled. Then one can reduce the original operator* A *to the family* (21.3.54) *of* 2*-dimensional operators with* $n\le C_0/(\mu h)$, $p\le C_0/(\mu h\gamma)$ *where all operators are of* (21.3.49)*-type and* σ, σ_1 *are bounded and disjoint from* 0.

Remark 21.3.13. All Statements (i)–(iii) of Remark 21.3.11 remain valid.

Canonical Form in the Strictly Outer Zone. III

We would like to finish this section by the following

Remark 21.3.14. In 2D-case canonical form was derived in Chapter 14

$$\mathcal{A}_p=\sigma\sigma_1\Bigl((2p+1)\mu h\gamma-W^{\#\#}+\sum_{2l+2j+2t\ge 3} B^{\mathsf{w}}_{ljt}\times\bigl((2p+1)\mu h\gamma\bigr)^l\times\mu^{2-2l-2j-t}h^t\gamma^{-4j-3t}\Bigr) \tag{21.3.58}$$

with

$$W^{\#\#}_n=\beta(\gamma^{-1}x_3,\mu^{-1}h\gamma^{-2}D_3). \tag{21.3.59}$$

There instead of 2-parameter family of 2-dimensional PDOs we had only 1-parametric family of 1-dimensional PDOs.

21.4 Remainder Estimates

In this section we derive remainder estimates with the main part in the standard but rather implicit form which is the sum of the Tauberian expressions (21.1.2)

$$h^{-1}\sum_{\iota}\int_{-\infty}^{0}\Bigl(F_{t\to h^{-1}\tau}\bar{\chi}_{T_\iota}(t)\Gamma u Q^t_{\iota y}\Bigr)\,d\tau \tag{21.4.1}$$

with Q_ι partition of unity and an appropriate T_ι where as in previous Chapters $\bar{\chi}_T(t) = \bar{\chi}(t/T)$ and $\bar{\chi}$ is supported in $[-1,1]$ and equal 1 at $[-\frac{1}{2},\frac{1}{2}]$, while the remainder estimate could depend on non-degeneracy condition.

We are going to consider different zones and apply different approaches and use different canonical forms in each of them; let us start from the strictly outer zone.

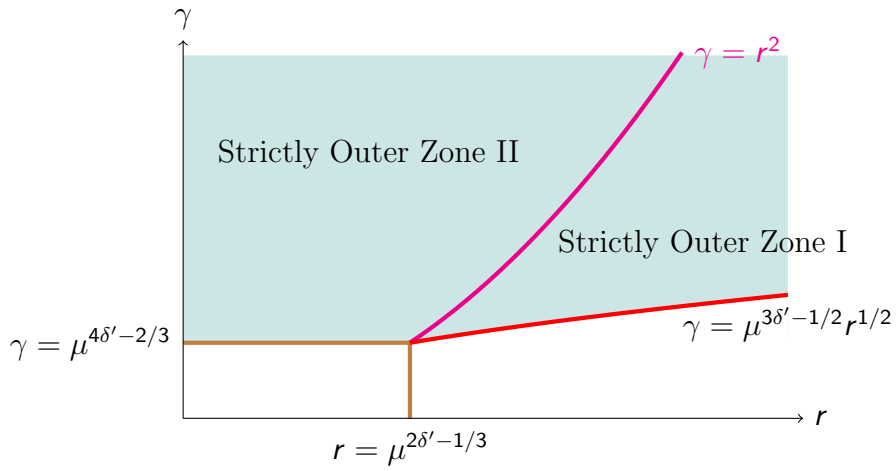

Figure 21.2: Our first target is strictly outer zone.

21.4.1 Estimates in the Strictly Outer Zone

Let us start from the most massive *Strictly Outer Zone I* $\mathcal{Z}^*_{\mathsf{out},I}$ described in (21.3.40) with an arbitrarily small exponent $\delta_0 > 0$ and sufficiently small constant $\epsilon > 0$.

Estimates in the Strictly Outer Zone. I

After rescalings and transformations described in the previous section the original operator *somewhere* there was reduced to the family of two-dimensional $(\hbar_3,\hbar_4)$-PDOs $\mathcal{A}_{pn}(x'',\hbar_3 D_3,\hbar_4 D_4)$ defined by (21.3.54) with

$$\hbar_3 = \mu^{-1} h r^2 \gamma^{-3} \ll \mu^{1/2} h, \qquad \hbar_4 = \mu^{-1} h r^{-2} \ll \mu^{-1/3} h; \tag{21.4.2}$$

one needs to consider only magnetic numbers

$$n \le C_0 (\mu h)^{-1}, \qquad p \le C_0 (\mu h \gamma)^{-1} \qquad \text{as } \mathsf{dist}(x,\Sigma) \asymp \gamma; \tag{21.4.3}$$

these two restrictions mean that energy levels do not exceed ϵ.

These final operators are defined in $B(0,1)$. Therefore

(21.4.4) A contribution of each (pair,element)[17] to to the asymptotics is $O\big(\hbar_3^{-1}\hbar_4^{-1}\big) = O\big(\mu^2 h^{-2}\gamma^3\big)$ [18].

However, one can see easily that this element before the final reduction was the "slab" $B(0,1)\cap\{|x_1-\bar{x}_1|+|x_3-\bar{x}_3|\le \epsilon\gamma_*\}$ with $\gamma_* = \gamma r^{-2}$, $|\bar{x}_1|\asymp \gamma_*$ where here $B(0,1)$ is an image of the original element $B(y',r)\cap \mathcal{Z}_{\mathsf{out},I}$. Therefore the total contribution of the given pair (p,n) over subzone

$$\mathcal{Z}_{(r,\gamma)} = \big\{\gamma \le \mathsf{dist}(x,\Sigma)\le 2\gamma,\ r\le \mathsf{dist}(x,\Lambda)\le 2r\big\}$$

with $r\ge \epsilon\gamma^{1/2}$ to the asymptotics is $O\big(\mu^2h^{-2}\gamma^3\times r^2\gamma^{-1}\big) = O\big(\mu^2h^{-2}\gamma^2r^2\big)$ and the total contribution of this subzone $\mathcal{Z}_{(r,\gamma)}$ to the asymptotics is $O\big(\mu^2h^{-2}\gamma^2r^2\times(\mu h\gamma)^{-1}\times(\mu h)^{-1}\big) = O\big(h^{-4}\gamma r^2\big)$ where the second and the third factors are numbers of permitted indices p and n respectively.

Then

(21.4.5) the total contribution of zone $\mathcal{Z}^*_{\mathsf{out},I}$ to asymptotics is $O(h^{-4})$ [19].

The same approach works in $\mathcal{Z}^*_{\mathsf{out},II}$: here one has the family of two-dimensional $\hbar_3$-PDO $\mathcal{A}_{pn}(x'',\hbar_3D_3,\hbar_3D_4)$ with $\hbar_3 = \mu^{-1}h\gamma^{-2}$. Therefore a contribution of each (pair,element) to to the asymptotics is $O(\hbar_3^{-2}) = O(\mu^2h^{-2}\gamma^4)$. However, there was nondiscriminatory rescaling $(x_3,x_4,\xi_3,\xi_4)\mapsto (r^{-1}x_3,r^{-1}x_4,r^{-1}\xi_3,r^{-1}\xi_4)$ while the number of such balls should be $O(r^{-2})$ since $\mathsf{codim}_\Sigma\Lambda = 2$.

Therefore the total contribution of the given pair (p,n) over subzone

$$\bar{\mathcal{Z}}_{(r,\gamma)} = \big\{\gamma \le \mathsf{dist}(x,\Sigma)\le 2\gamma,\ \mathsf{dist}(x,\Lambda)\le 2r\big\}$$

with $r=\gamma^{1/2}$ to the asymptotics is $O(\mu^2h^{-2}\gamma^3)$, the total contribution of subzone $\bar{\mathcal{Z}}_{(r,\gamma)}$ to the asymptotics is $O(h^{-4}\gamma^2)$ and

(21.4.6) The total contribution of zone $\mathcal{Z}^*_{\mathsf{out},II}$ to asymptotics is $O(h^{-4})$.

Now we need to analyze the contribution of such elements to the remainder estimate.

---

[17] Where here and below a pair is (p,n) and an element means the final element $B(0,1)$.

[18] And for many pairs it is actually of this magnitude.

[19] And is actually of this magnitude under assumption (21.2.92) as $\mu\le\epsilon h^{-1}$. These results are consistent with what one can get without reduction.

Estimates in the Strictly Outer Zone. II

Let us consider some index pair (p,n) and some final $B(0,1)$ element. Assume first that after all these transformations operator $\mathcal{A}_{pn}$ is *regular* on this (pair,element) in the sense of (21.4.7), (21.4.8) and (21.4.10) below

$$\eta_\alpha |\nabla^\alpha_{x'',\xi''} \mathcal{A}_{pn}| \le c_\alpha \ell \qquad \forall \alpha : 0 < |\alpha| \le K, \tag{21.4.7}$$

with $\eta_\alpha = 1$ as $\alpha = (*,0,*,0)$ and $\eta_\alpha = \gamma r^{-2}$ otherwise and ℓ satisfying

$$\ell \ge h^{-K_0}. \tag{21.4.8}$$

Then the propagation speed with respect to $(x_3, \hbar_3 D_3)$ is $O(\ell\hbar_3/h) = O(\ell\mu^{-1}r^2\gamma^{-3})$ and the propagation speed with respect to $(x_4, \hbar_4 D_4)$ is $O(\ell\hbar_4/h \times r^2/\gamma) = O(\ell\hbar_3/h)$ as well. Therefore the dynamics starting in $B(0,\frac12)$ is retained in $B(0,1)$ as $|t| \le T_1$ with

$$T_1 = \epsilon h \ell^{-1}\hbar_3^{-1} \asymp \mu\gamma^3 r^{-2}\ell^{-1}. \tag{21.4.9}$$

Recall that $\hbar_3 \gg \hbar_4$. In the same time under respective assumptions

$$|\nabla_{x_3,\xi_3}\mathcal{A}_{pn}| + \zeta |\nabla_{x_4,\xi_4}\mathcal{A}_{pn}| \asymp \ell, \quad \zeta = \gamma r^{-2} \tag{21.4.10}$$

one can pick up T_0 from the uncertainty principle[20] $\ell\hbar_j h^{-1} T_0 \ge \hbar_j h^{-\delta'}$ [21] or equivalently

$$T_0 = C h^{1-\delta'}\ell^{-1}. \tag{21.4.11}$$

with an arbitrarily small exponent $\delta' > 0$.

Then the contribution of this (pair,element) to the remainder does not exceed

$$C\hbar_3^{-1}\hbar_4^{-1} \times T_0 T_1^{-1} \asymp \hbar_4^{-1} h^{-\delta'} = \mu h^{-1-\delta'} r^2 \tag{21.4.12}$$

where $C\hbar_3^{-1}\hbar_4^{-1} \times T_0$ estimates $\sup_{|\tau|\le\epsilon} |F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma\big(u\psi\big)|$ and factor T_1^{-1} is due to the standard Tauberian approach.

Therefore in comparison with the estimate of the contribution to the asymptotics (21.4.4) the estimate of the contribution to the remainder picked

[20] We do not need the logarithmic uncertainty principle in this place and use the simpler version of the microlocal uncertainty principle.

[21] Alternatively one could scale $x_4 \mapsto (\hbar_4/\hbar_3)^{1/2}x_4$, $\xi_4 \mapsto (\hbar_4/\hbar_3)^{1/2}\xi_1$, $\hbar_4 \mapsto \hbar_3$.

an extra factor $\hbar_3 h^{-\delta'}$ with an arbitrarily small exponent $\delta' > 0$; actually one can reduce δ' to 0 but it is not needed here.

Let us pick up $(p,\mathsf{element})$, then break this element into ζ-subelements with respect to (x_4, ξ_4); then on each such subelement operator $\mathcal{A}_{pn}$ is elliptic[22] $|\mathcal{A}_{pn}| \ge \epsilon\ell$ for all indices n but $C\big(\ell/(\mu h)+1\big)$ of them. Therefore the total contribution to the remainder the pair $(p,\mathsf{element})$ combination such that conditions (21.4.7), (21.4.8) and (21.4.10) are fulfilled with $\ell \le \bar\ell$ does not exceed

$$C\mu h^{-1-\delta'} r^2 \times \big(\bar\ell(\mu h)^{-1}+1\big) = C\big(\bar\ell + \mu h\big) h^{-2-\delta'} r^2. \tag{21.4.13}$$

Then summation over $\mathcal{Z}_{(r,\gamma)}$ results in

$$Ch^{-4}\gamma r^2 \times \mu^{-1} h r^2 \gamma^{-3} \times h^{-\delta'}(\bar\ell + \mu h) = C\big(\bar\ell + \mu h\big)\mu^{-1} h^{-3-\delta'} r^4 \gamma^{-2} \tag{21.4.14}$$

and therefore

(21.4.15) The total contribution to the remainder of all $(\mathsf{pair},\mathsf{element})$ combinations residing in $\mathcal{Z}_{(r,\gamma)} \subset \mathcal{Z}^*_{\mathsf{out},I}$ and satisfying conditions (21.4.7), (21.4.8) and (21.4.10) with $\ell \le \bar\ell$ does not exceed (21.4.14).

It implies

(21.4.16) The total contribution to the remainder of all $(\mathsf{pair},\mathsf{element})$ combinations residing in the *far outer zone*

$$\mathcal{Z}^*_{\mathsf{out},I} \cap \big\{\mathsf{dist}(x,\Sigma) \ge \mu^{-1/4} h^{-\delta'} \mathsf{dist}(x,\Lambda)^2\big\}$$

and satisfying conditions (21.4.7), (21.4.8) and (21.4.10) does not exceed $C\mu^{-1/2}h^{-3}$.

The same arguments work for $\bar{\mathcal{Z}}_{(r,\gamma)}$ with $r = \gamma^{1/2}$ resulting in the estimate $C\mu^{-1}h^{-3-\delta'}$ of its contribution to the remainder; summation over $\mathcal{Z}^*_{\mathsf{out},II}$ results in expression $o(\mu^{-1/2}h^{-3})$. Therefore

[22] See next Subsubsection 21.4.1.3 for justification; to prove (21.4.16) and (21.4.17) which are main results of this subsubsection one does not need these arguments.

(21.4.17) The total contribution to the remainder of all (pair,element) combinations residing in $\mathcal{Z}^*_{\mathsf{out},II}$ and satisfying conditions (21.4.7), (21.4.8) and (21.4.10) with $\zeta = 1$ [23)] does not exceed $C\mu^{-1/2}h^{-3}$.

We consider the exceptional regular elements (i.e. not covered by (21.4.16)) in the next subsubsection. The rest of the analysis of this subsection is devoted to detecting and analysis of the irregular elements.

Estimates in the Strictly Outer Zone. III

Note that

$$H_{pn} = \Big(\phi_1^{-1}\big(V - (2n+1)\mu h f_2 - (2p+1)\mu h f_1\big)\Big) \circ \Psi \tag{21.4.18}$$

with uniformly smooth map Ψ which does not depend on V; this exonerates elliptic arguments of the previous Subsubsection and in particular (21.4.15) since perturbation part satisfies $|\partial_n \mathcal{B}_{pn}| \ll \mu h$; one can consider n and p as continuous parameters.

Proposition 21.4.1. *(i) After reduction in $\mathcal{Z}^*_{\mathsf{out},I}$ symbols $W \circ \Psi$, $f_1 \circ \Psi$, $f_2 \circ \Psi$ satisfy conditions* (21.4.7) *and* (21.4.8) *with $\ell = \gamma r^{-1}$ and indicated above η_α and ζ. Furthermore,*

$$|\nabla_{x_3,\xi_3} f_1 \circ \Psi| \asymp \gamma. \tag{21.4.19}$$

(ii) After reduction in $\mathcal{Z}^*_{\mathsf{out},II}$ symbols $W \circ \Psi$, $f_1 \circ \Psi$, $f_2 \circ \Psi$ satisfy conditions (21.4.7) *and* (21.4.8) *with $\ell = \gamma r^{-1}$; furthermore*

$$|\nabla_{x'',\xi''} f_1 \circ \Psi| \asymp \gamma. \tag{21.4.20}$$

Proof. Proof follows easily from the reductions and is left to the reader. More precise results will be needed and proven later. □

Therefore

(21.4.21) As $p \ge C_0/(\mu h r)$ symbols H_{pq} satisfy conditions (21.4.7), (21.4.8) and (21.4.10 with $\ell = \mu h \gamma p \ge C_0 \gamma / r$.

[23)] One should pick up here $\zeta_1 = 1$ because $\hbar_3 = \hbar_4$.

On the other hand, one can prove easily from the reduction that

(21.4.22) Perturbation symbols $\mathcal{B}_{pn}$ satisfy conditions (21.4.7) and (21.4.8) with ℓ replaced by $\ell' = h^2\gamma^{-2}r^2(p+1)^2 + \mu^{-1}h\gamma^{-3}r^2 \ll \mu h\gamma(p+1)$.

Note that if we restrict ourselves only by $p \le C_0/(\mu hr)$ (condition to be explained later) instead of $p \le C_0/(\mu h\gamma)$, transition from (21.4.13) to (21.4.14) picks up an extra factor γ/r and therefore

(21.4.23) The total contribution to the remainder of all (pair,element) combinations with $p \le C_0/(\mu hr)$, residing in $\mathcal{Z}^*_{\mathsf{out},I}$ and satisfying conditions (21.4.7), (21.4.8) and (21.4.10) does not exceed $C\mu^{-1/2}h^{-3}$.

Now we want to cover indices $p \ge C_0/(\mu hr)$ and elements in $\mathcal{Z}^*_{\mathsf{out},I}$ which are not in the far outer zone and prove that their total contribution to the remainder also does not exceed $C\mu^{-1/2}h^{-3}$. To achieve this we plan to gain a factor γr^{-2} in the estimates of the previous Subsubsection 21.4.1.2, starting from (21.4.12), by increasing T_1 given by (21.4.9) to

$$T_2 = \epsilon r\mu_*\gamma_*^2 = \epsilon\mu\gamma^2. \tag{21.4.24}$$

Note that (21.4.9) appeared because it is a time for which propagation is confined to $B(0,1)$ in the final reduction, for larger t shift with respect to (x_3,ξ_3) would be too large while shift with respect to (x_4,ξ_4) would be still less than ϵ as $|t| \le T_2$.

However, considering propagation after intermediate reduction (i.e. reduction of Subsection 21.3.2) the shift with respect to x_3 would be of magnitude $\mu_*^{-1}\gamma_*^{-2}(\mu h\gamma p)|t|$ which would be observable but less than 1 as $T_1 \le |t| \le T_2$; this propagation just moves to the adjacent "slabs" in the intermediate element $B(0,1)$ corresponding to the different final elements.

Then the contribution of such (pair,element) to the remainder is estimated by a modified expression (21.4.12)

$$C\hbar_3^{-1}\hbar_4^{-1} \times T_0\,T_2^{-1} \asymp \mu h^{-1-\delta'}\gamma\ell^{-1} \asymp \mu h^{-1-\delta'}\gamma, (\mu h\gamma p)^{-1} \tag{21.4.25}$$

which leads to the modified expression (21.4.14)

$$\mu h^{-1-\delta'}\gamma r^{-2} \times (\mu h)^{-1} \times (\mu h\gamma)^{-1}|\log h| \times r^2\gamma^{-1} \tag{21.4.26}$$

where the second, the third and the fourth factors estimate respectively the number of indices n, the sum of $(\mu h\gamma p)^{-1}$ with respect to p and the number of the final elements with given (r,γ).

Then the sum with respect to γ could be estimated by by the same expression with $\gamma = \mu^{-1/2-\delta} r^{1/2}$, which is $C\mu^{-1/2-\delta} h^{-3-\delta'} r^{3/2} |\log h|$; finally summation with respect to r results in $O(\mu^{-1/2} h^{-3})$ as $\delta' > 0$ is small enough.

Therefore we arrive to

(21.4.27) The total contribution to the remainder of all (pair,element) combinations with $p \ge C_0/(\mu hr)$, residing in $\mathcal{Z}^*_{\mathsf{out},I}$ does not exceed $C\mu^{-1/2} h^{-3}$

which together with (21.4.23 concludes the analysis of regular elements; their total contribution to the remainder is $O(\mu^{-1/2} h^{-3})$.

Estimates in the Strictly Outer Zone. IV

Therefore in what follows we are interested only in non-regular elements, with $p \le C_0/(\mu hr)$; then H_{pn} with n violating ellipticity condition satisfies (21.4.7) and (21.4.8) with $\ell = \gamma r^{-1}$ in $\mathcal{Z}^*_{\mathsf{out},I}$ and with $\ell = \gamma r^{-1}$, $\eta_\alpha = (1,1,1,1)$ in $\mathcal{Z}^*_{\mathsf{out},II}$.

First, consider elements in $\mathcal{Z}^*_{\mathsf{out},I}$ with fixed $p \le C_0/(\mu hr)$ and n. After rescaling

$$x_4 \mapsto \gamma r^{-2} x_4, \quad \xi_4 \mapsto \gamma r^{-2} \xi_4 \quad \hbar_4 \mapsto \hbar'_4 := \mu^{-1} h \gamma^{-2} r^2 \tag{21.4.28}$$

symbol $\gamma^{-1} r H_{pn}$ becomes uniformly smooth and therefore one can introduce a scaling function

$$\ell = \ell_{pn}(x'', \xi'') = \epsilon r \gamma^{-1} |\nabla_{x'',\xi''} \mathcal{A}_{pn}| + \bar{\ell}_0, \qquad \bar{\ell}_0 = C\hbar_3^{1/2} h^{-\delta''}; \tag{21.4.29}$$

obviously, $|\nabla \ell| \le \frac{1}{2}$. Let us introduce ℓ-admissible partition of $B(0,1)$. Then the symbols are still quantizable after rescaling $(x'', \xi'') \mapsto \ell^{-1}(x'', \xi'')$, $\hbar_3 \mapsto \ell^{-2}\hbar_3$, $\hbar'_4 \mapsto \ell^{-2}\hbar'_4$.

Further, as $\ell \ge 2\bar{\ell}_0$, the propagation speed does not exceed

$$v \asymp \mu^{-1}\gamma^{-3} r^2 |\nabla \mathcal{A}_{pn}| \asymp \mu^{-1} r \gamma^{-2} \ell \tag{21.4.30}$$

in the corresponding partition subelement[24)] (in the coordinates before rescaling of the subelements) and therefore the dynamics started in $B(z, \ell(z))$ remains in $B(z, 2\ell(z))$ as $|t| \le T_1$ with

$$T_1 = \epsilon \ell v^{-1} = \epsilon \mu r^{-1} \gamma^2. \tag{21.4.31}$$

[24)] We call this partition elements "subelements" because actually $B(0,1)$ itself is a rescaled partition element by itself.

On the other hand, the shift $v|t|$ is observable as $|t| \ge T_0$ with T_0 defined from the uncertainty principle

$$vT_0 \times \ell \ge \hbar_3 h^{-\delta'}. \tag{21.4.32}$$

Indeed, if $|\nabla_{x_3,\xi_3}\mathcal{A}_{pn}| \asymp \ell\gamma$ then the propagation speed with respect to (x_3,ξ_3) is exactly of magnitude v and the uncertainty principle is due to (21.4.32). Otherwise $|\nabla_{x_4,\xi_4}\mathcal{A}_{pn}| \asymp \ell\gamma$ and the propagation speed with respect to (x_4,ξ_4) is exactly of magnitude $v\gamma$ and the uncertainty principle is $\gamma vT_0 \times \ell \ge \hbar_2 h^{-\delta'} r^{-1}$ which is due to (21.4.32) as well.

Thus one can pick up

$$T_0 = \hbar_3 h^{-\delta'} v^{-1}\ell^{-1} \asymp h^{1-\delta'}\gamma^{-1} r\ell^{-2}. \tag{21.4.33}$$

Therefore the contribution of this partition subelement to the remainder does not exceed

$$C\hbar_3^{-1}\hbar_4'^{-1}\ell^4 \times T_0\, T_1^{-1} = C\mu h^{-1-\delta'}\gamma^2 r^{-2}\ell^2 \tag{21.4.34}$$

while in comparison its contribution to the main part of asymptotics does not exceed $C\hbar_3^{-1}\hbar_4'^{-1}\ell^4 = C\mu^2 h^{-2}\gamma^5\ell^4$.

On the other hand, the contributions of the subelement with $\ell \le \bar\ell_0$ to both the main part and the remainder do not exceed

$$C\hbar_3^{-1}\hbar_4'^{-1}\ell^4 = C\mu^2 h^{-2} r^{-4}\gamma^5 \bar\ell_0^4, \tag{21.4.35}$$

which is less than (21.4.34) (with a larger but still arbitrarily small exponent δ').

Let us notice that for given index p and ℓ-element operator $\mathcal{A}_{pn}$ is elliptic as $|n-\bar n|\mu h \ge C_0\ell^2\gamma r^{-1}$ for some $\bar n$; therefore ellipticity fails for no more than $\bigl(C_0\ell^2\gamma r^{-1}(\mu h)^{-1}+1\bigr)$ indices n.

Then the contribution to the remainder of all the indices n (while index p and ℓ-subelement remain fixed) does not exceed

$$C\mu h^{-1-\delta'}\gamma^2\ell^2 \times \bigl(C_0\ell^2\gamma(\mu h r)^{-1}+1\bigr). \tag{21.4.36}$$

For

$$\ell \ge \bar\ell_1 := \epsilon_1 \mu^{1/2} h^{1/2} r^{1/2}\gamma^{-1/2} \tag{21.4.37}$$

this expression is $\asymp Ch^{-2-\delta'}\gamma^3 r^{-1}\ell^4$ and the sum over ℓ-subelements in $B(0,1)$ with ℓ satisfying (21.4.37) results in $Ch^{-2-\delta'}\gamma^3 r^{-1}$.

Then the sum over all such subelements residing in $\mathcal{Z}_{(r,\gamma)}$ does not exceed

$$Ch^{-2-\delta'}\gamma^3 r^{-1}\times r^2\gamma^{-2}\times r^2\gamma^{-1} = Ch^{-2-\delta'}r^3 \tag{21.4.38}$$

where the second factor in the left-hand expression is the number of elements which appeared after rescaling (21.4.28) and the corresponding partition we made in the beginning of Subsubsection and the third factor is the number of the "slabs" corresponding to the final elements after the second transformation in the "intermediate" ball $B(0,1)$ obtained after the first transformation.

Then the sum over the whole $\mathcal{Z}^*_{\mathsf{out},I}$ does not exceed $Ch^{-2-\delta'}|\log h|$. So far index p remains fixed; the sum over $p\le C_0/(\mu h)$ does not exceed $C\mu^{-1}h^{-3-\delta'}|\log h| = O(\mu^{-1/2}h^{-3}$. Thus we arrive to

(21.4.39) The total contribution to the remainder of all (pair, subelement) combinations in zone $\mathcal{Z}^*_{\mathsf{out},I}$ with $\ell\ge \bar\ell_1$ does not exceed $C\mu^{-1/2}h^{-3}$.

Therefore only subelements with $\ell\le \bar\ell_1$ remain to be considered.

For such subelements expression (21.4.36) becomes $C\mu h^{-1-\delta'}\gamma^2\ell^2$; however, since without non-degeneracy condition the number of such subelements in $B(0,1)$ could be $\asymp \ell^{-4}$, the sum would become singular. To overcome this obstacle let us redefine the scaling function (21.4.29) in the manner more suitable for the analysis of elements with $\ell\le\bar\ell_1$:

$$\ell^* = \ell(x'',\xi'') = \epsilon\min_{p,n}\Big(\gamma^{-2}r^2|\nabla_{x'',\xi''}\mathcal{A}_{pn}|^2 + r\gamma^{-1}|\mathcal{A}_{pn}|\Big)^{1/2} + \bar\ell_0, \tag{21.4.40}$$
$$\text{with}\quad \bar\ell_0 = \hbar_3^{1/2}h^{-\delta'} = \mu^{-1/2}h^{1/2-\delta''}\gamma^{-3/2}r.$$

Note that as the minimum in the right-hand expression is achieved for $(p,n)=(\bar p,\bar n)(x'',\xi'')$, for every index p inequality $\ell\ge \ell_p = \ell^*+\epsilon|p-\bar p|\mu hr$ holds as the $n=n(p,(x'',\xi''))$ is selected to break ellipticity condition; this index $\bar n$ is unique.

Then the contribution of all subelements with $\ell^*\asymp\lambda$ in $B(0,1)$ (obtained after rescaling (21.4.28)) to the remainder (after summation with respect to n and $p=\bar p$) does not exceed the sum with respect to p of expressions (21.4.36):

$$C\mu h^{-1-\delta'}\gamma^2\sum_{p\ne\bar p}\big(\lambda+\epsilon|p-\bar p|\mu hr\big)^{-2}; \tag{21.4.41}$$

we included in this sum only p with $|p-\bar{p}| \ge 1$ leaving *special* $((\bar{n},\bar{p})$, subelement) combinations for a special consideration. Expression (21.4.41) obviously does not exceed $C\mu^{-1}h^{-3-\delta'}\gamma^2 r^{-2}$ and therefore the contribution of all such (pair, subelement) combinations residing in $\mathcal{Z}_{(r,\gamma)}$ does not exceed

$$C\mu^{-1}h^{-3-\delta'}r^{-2}\times r^2\gamma^{-2}\times r^2\gamma^{-1} = C\mu^{-1}h^{-3-\delta'}r^2\gamma^{-1}$$

with the same origin of the second and the third factors in the left-hand expression as in (21.4.38).

Finally, after summation over $\mathcal{Z}^*_{\mathsf{out},I}$ one gets $O(\mu^{-1/2}h^{-3})$. Therefore we arrive to

(21.4.42) The total contribution to the remainder of all $((p,n)$, subelement) combinations in zone $\mathcal{Z}^*_{\mathsf{out},I}$ with $\ell \le \bar{\ell}_1$ and $(p,n)\ne(\bar{p},\bar{n})$ does not exceed $C\mu^{-1/2}h^{-3}$.

Estimates in the Strictly Outer Zone. V

So, in what follows we need to consider only *special* $((\bar{n},\bar{p})$, subelement) combinations with $\ell^* \le \bar{\ell}_1$. Exactly for these combinations non-degeneracy condition becomes crucial. In the general case however, $\mathcal{A}_{pn}$ with $(p,n) = (\bar{p},\bar{n})$ could be "flat" and then there would be no difference between estimates of the contribution of such subelement to the remainder and to the main part of the asymptotics which would be $C\hbar_3^{-1}\hbar_4'^{-1} \asymp \mu^2h^{-2}\gamma^5r^{-2}$. Then the contribution of all such (pair, subelement) combinations residing in $\mathcal{Z}_{(r,\gamma)}$ does not exceed

$$C\mu^2h^{-2}\gamma^5r^{-2}\times r^2\gamma^{-2}\times r^2\gamma^{-1} = C\mu^2h^{-2}\gamma^2r^2 \tag{21.4.43}$$

with the same origin of the second and the third factors in the left-hand expression as in (21.4.38).

Finally, after summation over $\mathcal{Z}^*_{\mathsf{out},I}$ one gets $O(\mu^{-1/2}h^{-3})$. Therefore we arrive to

(21.4.44) The total contribution to the remainder of all $((p,n)$, subelement) combinations in zone $\mathcal{Z}^*_{\mathsf{out},I}$ with $\ell \le \bar{\ell}_1$ and $(p,n) = (\bar{p},\bar{n})$ does not exceed $C\mu^2h^{-2}$.

Combining with (21.4.39), (21.4.42) we arrive to

Proposition 21.4.2. *Under assumption* (21.3.1) *the total contribution to the remainder of the zone* $\mathcal{Z}^*_{\mathsf{out},I}$ *does not exceed* $C\mu^2h^{-2} + C\mu^{-1/2}h^{-3}$.

Remark 21.4.3. All the arguments leading to Proposition 21.4.2 are applicable in $\mathcal{Z}^*_{\mathsf{out},II}$ with the following minor modifications and simplifications:

(i) Operators $\mathcal{A}_{pn}$ are $\hbar_3$-PDOs from the very beginning (so there is no need in (21.4.28) rescaling; there are no "slabs" etc, everything is homogeneous.

(ii) A factor r^2 in estimates translates into γ which makes life easier; in particular there is no need in the special propagation analysis leading to (21.4.27).

Therefore

(iii) $O(\mu^{-1/2}h^{-3})$ estimate for the contribution into the remainder of all $((p,n),\mathsf{subelement})$ combinations in zone $\mathcal{Z}^*_{\mathsf{out},II}$ with either $\ell \ge \bar{\ell}_1$ or $\ell \le \bar{\ell}_1$ and $(p,n) \ne (\bar{p},\bar{n})$ and

(iv) $O(\mu^2h^{-2}\gamma^3)$ estimate for the contribution into the remainder of all $((p,n),\ \mathsf{subelement})$ combinations in zone $\mathcal{Z}^*_{\mathsf{out},II} \cap \{\mathsf{dist}(x,\Sigma) \le \gamma\}$ with $\ell \le \bar{\ell}_1$ and $(p,n) = (\bar{p},\bar{n})$.

In particular, we arrive to

Proposition 21.4.4. *Under assumption* (21.3.1) *the total contribution to the remainder of the zone* $\mathcal{Z}^*_{\mathsf{out},II}$ *does not exceed* $C\mu^2h^{-2} + C\mu^{-1/2}h^{-3}$.

Estimates in the Strictly Outer Zone. VI

Estimate (21.2.93) $O\big(\mu^2h^{-2} + \mu^{-1/2}h^{-3}\big)$ is the best possible in the general case (see Subsection 21.5.4) and it coincides with the best possible $O(\mu^{-1/2}h^{-3})$ as $\mu \le h^{-2/5}$; so $\mu \ge h^{-2/5}$ until the end of this subsection.

However, one can do better under some non-degeneracy conditions. Namely, as before one can estimate the contribution of the special pair $((\bar{p},\bar{n}),$ $\mathsf{subelement})$ to the remainder by $C\mu h^{-1-\delta'}\gamma^2\ell^2$ as $\ell \ge \bar{\ell}_0$ (see (21.4.34)) and by $C\mu^2r^{-2}\gamma^5$ (see (21.4.35)) where now ℓ^* defined by (21.4.40) and ℓ as the paragraph below that. Therefore the contribution to the remainder of all the special subelements residing in the final element $B(0,1)$ (obtained after

rescaling and partition (21.4.28)) does not exceed

$$C\mu h^{-1-\delta'}\gamma^2 r^{-2}\int_{\{\bar\ell_0\le \ell^*\le \bar\ell_1\}} \ell^{*\,-2}\,dx''d\xi'' + C\mu^2 h^{-2} r^{-4}\gamma^5 \int_{\{\ell^*\le \bar\ell_0\}} dx''d\xi'', \tag{21.4.45}$$

where we recall that $\bar\ell_0 = \mu^{-1/2}h^{1/2-\delta''}\gamma^{-3/2}r$ and $\bar\ell_1 = \mu^{1/2}h^{1/2}r^{1/2}\gamma^{-1/2}$.

Then the total contribution to the remainder of all the special subelements with Ψ-image residing in $\mathcal{Z}_{(r,\gamma)}\subset \mathcal{Z}^*_{\mathsf{out},I}$ to the remainder does not exceed

$$C\mu h^{-1-\delta'}\gamma^2 r^{-2}\int_{\{\bar\ell_0\le \ell^*\circ\Psi^{-1}\le \bar\ell_1\}\cap \mathcal{Z}_{(r,\gamma)}} \ell^{*\,-2}\,|\det D\Psi|^{-1}\,dx + C\mu^2 h^{-2} r^{-4}\gamma^5 \int_{\{\ell^*\circ\Psi^{-1}\le \bar\ell_0\}\cap \mathcal{Z}_{(r,\gamma)}} |\det D\Psi|^{-1}\,dx.$$

Since

$$|\det D\Psi| \asymp \gamma^4 r^{-4} \tag{21.4.46}$$

one can rewrite this expression as

$$C\mu h^{-1-\delta'}\int_{\{\bar\ell_0\le \ell^*\circ\Psi^{-1}\le \bar\ell_1\}\cap \mathcal{Z}} \gamma^{-2}r^2\ell^{*\,-2}\,dx + C\mu^2 h^{-2}\int_{\{\ell^*\circ\Psi^{-1}\le \bar\ell_0\}\cap \mathcal{Z}} \gamma\,dx \tag{21.4.47}$$

with $\mathcal{Z} = \mathcal{Z}_{(r,\gamma)}$.

We leave to the reader the similar analysis in $\mathcal{Z}^*_{\mathsf{out},II}$ leading to the same estimate for the contribution of $\bar{\mathcal{Z}}_{(r,\gamma)}\subset \mathcal{Z}^*_{\mathsf{out},II}$ with $r=\gamma^{1/2}$:

Proposition 21.4.5. *Under assumption* (21.3.1)

(i) The total contribution of the special subelements with Ψ-image residing in $\mathcal{Z}_{(r,\gamma)}\subset \mathcal{Z}^*_{\mathsf{out},I}$ does not exceed (21.4.47) *with $\mathcal{Z} = \mathcal{Z}_{(r,\gamma)}$.*

(ii) The total contribution of the special subelements with Ψ-image residing in $\bar{\mathcal{Z}}_{(r,\gamma)}\subset \mathcal{Z}^*_{\mathsf{out},II}$ does not exceed (21.4.47) *with $\mathcal{Z} = \bar{\mathcal{Z}}_{(r,\gamma)}$. where $r=\gamma^{1/2}$.*

(iii) The total contribution of all the special subelements (with Ψ*-image residing in* $\mathcal{Z}^*_{\mathsf{out}}$ *does not exceed* (21.4.47) *with* $\mathcal{Z} = \mathcal{Z}^*_{\mathsf{out}}$.

Note that all elements of the matrix $D\Psi$ do not exceed $C\gamma r^{-1}$. Combining with (21.4.47) and (21.4.18) one can conclude that all elements of $D\Psi^{-1}$ do not exceed $C\gamma^{-1}r$. Therefore if ℓ^* was defined by (21.4.40) with H_{pn} instead of $\mathcal{A}_{pn}$ then the following inequality would hold

$$\ell^* \circ \Psi^{-1} \ge L \coloneqq \epsilon|\nabla_{\Sigma}(V/f_2)| + \min_n |\nabla_{\Sigma}(V/f_2) - (2n+1)\mu h|. \tag{21.4.48}$$

Then under assumption

$$|\big(\frac{V}{f_2}\big) - (2n+1)\mu h| + |\nabla_{\Sigma}\big(\frac{V}{f_2}\big)| \ge \epsilon_0 \qquad \forall n \in \mathbb{Z}^+ \tag{21.4.49}$$

$\ell^* \asymp 1$ even if ℓ^* are defined by $\mathcal{A}_{pn}$ and the first term in (21.4.46) does not exceed $C\mu h^{-1-\delta'}\gamma^{-1}r^4$ while the second term vanishes and the total contribution of $\mathcal{Z}^*_{\mathsf{out},I}$ to the remainder does not exceed $C\mu^{3/2}h^{-1}$.

Therefore we arrive to

Proposition 21.4.6. *Under assumptions* (21.3.1) *and* (21.4.49) *the total contribution of* $\mathcal{Z}^*_{\mathsf{out}}$ *to the remainder does not exceed* $C\mu^{-1/2}h^{-3}$.

Furthermore, under assumption (21.4.48) and condition

$$\exists n \in \mathbb{Z}^+ : \ |\frac{V}{f_2} - (2n+1)f_2\mu h| + |\nabla_{\Sigma}\big(\frac{V}{f_2}\big)| \le \epsilon_0 \implies \tag{21.4.50}_q$$

$\mathsf{Hess}_{\Sigma}\big(\frac{V}{f_2}\big)$ has at least q eigenvalues with absolute values greater than ϵ_0,

one can see easily that expression (21.4.47) with $\mathcal{Z} = \mathcal{Z}^*_{\mathsf{out}}$ does not exceed $C\mu^{3/2}h^{-1}$ plus expression

$$C\mu^2h^{-2}\int \gamma \bar{\ell}_0^q \, dx \tag{21.4.51}_q$$

which is

$$\asymp \iint_{\Omega} J_q \, \gamma^{-1} d\gamma \cdot r^{-1} dr, \qquad J_q = h^{-2\delta''}\mu^2h^{-2}(\mu^{-1}hr^2\gamma^{-3})^{q/2}r^2\gamma^2$$

with integral taken over $\Omega = \{\mu^{\delta'-1/2}r^{1/2} \le \gamma \le r^2 \le 1\}$.

Obviously, expression $(21.4.51)_2$ does not exceed J_2 calculated as $\gamma = \mu^{\delta'-1/2}$, $r = 1$ which is $O(\mu^{-1/2}h^{-3})$. Further, expression $(21.4.51)_1$ does not exceed J_1 calculated as $\gamma = r = 1$ which is $C\mu^{3/2}h^{-3/2-\delta''}$ with arbitrarily small $\delta'' > 0$.

Then in the framework of

$$(21.4.52)_q \quad |\nabla_\Sigma(\frac{V}{f_2})| \le \epsilon_0 \implies$$

$\mathsf{Hess}_\Sigma(\frac{V}{f_2})$ has at least q eigenvalues with absolute values greater than ϵ_0 ,

one arrives to the following statement[25]:

Proposition 21.4.7. *Let condition* (21.3.1) *be fulfilled. Then*

(i) Under assumption $(21.4.52)_2$ *the total contribution to the remainder of the zone* $\mathcal{Z}^*_{\mathsf{out}}$ *with does not exceed* $C\mu^{-1/2}h^{-3}$.

(ii) Under assumption $(21.4.52)_1$ *the total contribution to the remainder estimate the zone* $\mathcal{Z}^*_{\mathsf{out}}$ *does not exceed* $C\mu^{3/2}h^{-3/2-\delta} + C\mu^{-1/2}h^{-3}$ *with arbitrarily small* $\delta'' > 0$.

Proof. To get rid of assumption (21.4.28) one needs to take in account perturbation term $\mathcal{B}$ in $\mathcal{A}_{pn}$; then $\ell^* \circ \Psi^{-1} \ge L - |\nabla_\Sigma \mathcal{B}_{pn} \circ \Psi^{-1}|$. Thus it would be enough to replace $\bar\ell_0$ by $\bar\ell_0 + |\nabla_\Sigma \mathcal{B}_{pn} \circ \Psi^{-1}|$ in $(21.4.51)_q$.

There are two leading terms in $\mathcal{B}_{pn}$; the first one is $\mu^{-2}r^2\gamma^{-4}((2p+1)\mu h\gamma)^2$; since $p \le C_0/(\mu hr)$ this term does not exceed $C\mu^{-2}\gamma^{-2}$. The second term is $O(\mu^{-1}hr^2\gamma^{-3})$ but looking its origin one can notice that it must contain $|\nabla_\Sigma(V/f_2)|^2$; then

$$|\nabla_\Sigma \mathcal{B}_{pn} \circ \Psi^{-1}| \le L_1 \coloneqq C\mu^{-2}\gamma^{-3}r.$$

Plugging $\bar\ell_0 = \mu^{-2}\gamma^{-3}r$ into $(21.4.51)_1$ results in $O(\mu^{1/2}h^{-2})$. □

Remark 21.4.8. (i) Note that in the extra terms $O(\mu^2h^{-2})$ and $O(\mu^{3/2}h^{-3/2-\delta''})$ appearing in the general case and under assumption $(21.4.50)_q$ the main contributor is far outer zone where $\gamma \approx 1$ and $r \approx 1$ while in the sharp estimate $O(\mu^{-1/2}h^{-3})$ the main contributor is zone where $\gamma \approx \mu^{-1/2}$ and $r \approx 1$.

[25] Still under assumption (21.4.48).

(ii) Actually one can greatly improve remainder estimate in $(21.4.50)_1$ case using more refined partition-rescaling technique like in the proof of Proposition 21.5.3. We believe that it can be even brought to $O(\mu^{-1/2}h^{-3})$. However, we think that really interesting are only general ($q = 0$) and generic ($q = 3$) cases.

21.4.2 Estimates in the Near Outer Zone

After deriving remainder estimates in $\mathcal{Z}^*_{\mathsf{out}}$ we need to consider the remaining part of $\mathcal{Z}_{\mathsf{out}}$ (which will be the *near outer zone*

$$\mathcal{Z}'_{\mathsf{out}} = \mathcal{Z}_{\mathsf{out}} \cap \{\mathsf{dist}(x, \Sigma) \le \mu^{-1/2}h^{-\delta}\} \tag{21.4.53}$$

and the *inner zone*. However, we also introduce and start from the *inner core* where we are able to apply pretty non-sophisticated approach and still derive proper estimates, just to keep r more disjoint from 0 to be able to use canonical form (21.3.38).

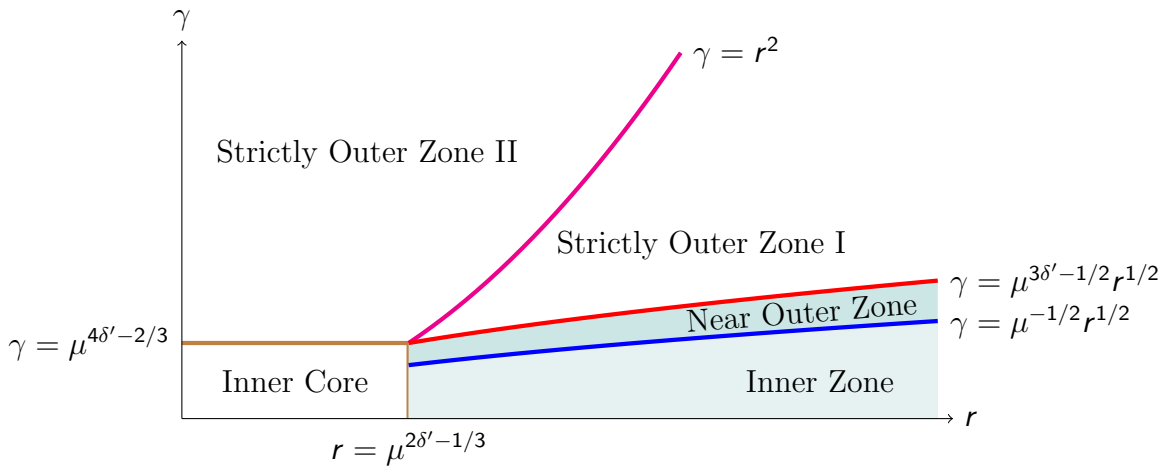

Figure 21.3: Zones.

Estimates in the Near Outer Zone. I

In this subsection our goal is to derive remainder estimate $O(\mu^{-1/2}h^{-3} + \mu^2h^{-2})$. Therefore our approach here would be pretty unsophisticated. Note first that

(21.4.54) The contribution of the domain

$$\bigl\{\mathsf{dist}(x,\Lambda)\le r,\ \mathsf{dist}(x,\Sigma)\le\gamma,\ |Z_1|\le\rho\bigr\} \tag{21.4.55}$$

to the remainder does not exceed $C\mu h^{-3}\gamma r^2\rho^2$ (while the main part of asymptotics is given by (21.4.1) with $T_0\ge Ch|\log h|$);

therefore considering *inner core* defined by (21.4.55) with $r=\mu^{-1/3+2\delta'}$, $\gamma=\mu^{-2/3+4\delta'}$ one gets $O(\mu^{-1/3+6\delta'}h^{-3})$.

This remainder estimate is very rough and should be improved despite rather unambitious goal of this Subsubsection. However, let us observe first that considering the strip $\{|x_1|\le\mu^{-1/2+3\delta'}\}$ and thus taking $r=1$, $\rho=1$ and $\gamma=\mu^{-1/2+3\delta'}$ one would get the remainder estimate $O(\mu^{1/2+3\delta'}h^{-3})$ which is $O(\mu^2h^{-3})$ as long as $\mu\ge h^{-2/3-\delta}$. Further, the contribution of this zone to the asymptotics is $O(h^{-4}\gamma r^2)$ and thus taking $r=1$, $\rho=1$ and $\gamma=\mu^{3\delta'-1/2}$ one would get $O(\mu^{-1/2+3\delta'}h^{-4})$ which is $O(\mu^2h^{-2})$ as $\mu\ge h^{\delta-4/5}$. Therefore in what follows until the end of Subsubsection one can assume that

$$h^{-\delta}\le\mu\le h^{\delta-1}. \tag{21.4.56}$$

Note first that

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma(uQ_y^\dagger)|\le Ch^s \tag{21.4.57}$$

as Q is supported in $\bar{Z}_{(\gamma)}:=\{|x_1|\le\gamma\}$ and $T_0\le T\le T_1$ with $T_0=Ch|\log h|$ and $T_1=\epsilon\mu^{-1}$; this statement is empty in the case of the *very strong magnetic field*

$$\epsilon(h|\log h|)^{-1}\le\mu\le Ch^{-1} \tag{21.4.58}$$

which is not the case now.

However, looking at the propagation in direction $\mathbb{K}_1$ as $|Z_1|\asymp\rho$ one can easily notice that the propagation speed is $\asymp\rho$ and the shift for time T is $\asymp\rho T$; this shift is observable as $\rho^2T\ge Ch|\log h|$. Therefore (21.4.56) holds as $T_0\le T\le T_1$ with $T_0=C\rho^{-2}h|\log h|$ and $T_1=\epsilon\mu^{-1}\gamma^{-1}$. Under assumption $\gamma\ge C_0\mu^{-1/2}$ this short evolution is contained in $\epsilon_1\gamma$-vicinity of the original point[26)].

[26)] One can improve this statement and the estimate following from it.

These two intervals $[Ch|\log h, \epsilon\mu^{-1}]$ and $[C\rho^{-2}h|\log h|, \epsilon\mu^{-1}\gamma^{-1}]$ overlap as

$$\rho \ge (C\mu h|\log h|)^{1/2} \tag{21.4.59}$$

and therefore we arrive to

(21.4.60) Under assumptions (21.4.56) and (21.4.59) estimate (21.4.57) holds as $T \in [T_0, T_1]$ with $T_0 = Ch|\log h|$, $T_1 = \epsilon \min\big(\mu^{-1}\gamma^{-1}, \mu^{-1/2}\big)$.

Therefore the contribution of the domain

$$\big\{\mathsf{dist}(x, \Lambda) \le r,\ \mathsf{dist}(x, \Sigma) \le \gamma,\ |Z_1| \ge C(\mu h|\log h|)^{1/2}\big\} \tag{21.4.61}$$

to the remainder does not exceed $C\mu h^{-3}r^2\gamma(\gamma + \mu^{-1/2})$.

Combining with (21.4.54) we arrive to

Proposition 21.4.9. *Contribution of domain* (21.4.55) *with* $\rho \asymp 1$ *to the remainder does not exceed*

$$C\mu h^{-3}r^2\gamma(\gamma + \mu^{-1/2}) + C\mu^2 h^{-2}\gamma r^2|\log h| \tag{21.4.62}$$

while the main part of asymptotics is given by (21.1.2) *with (any)* $T_0 \ge C\mu h|\log h|$.

Remark 21.4.10. Estimate (21.4.62) is sufficient for our limited goal here. In particular, it covers the inner core giving remainder estimate $O(\mu^{-1/2}h^{-3} + \mu^{2/3+\delta}h^{-2})$ which is not only better than $O(\mu^{-1/2}h^{-3} + \mu^2 h^{-2})$ but than $O(\mu^{-1/2}h^{-3} + \mu^{3/2}h^{-3/2})$.

So, we need to cover the *near outer* and *inner* zones; in both $\gamma \le r^2$ and therefore we can use canonical form (21.3.38). Even without it, according to Section 21.2 the drift speed is $\asymp \mu^{-1}\rho^2 r\gamma^{-2}$ as $\rho^2 r \ge C_0\gamma$.

Then the shift for time T is observable as $\mu^{-1}\rho^2 r\gamma^{-2}T \times \rho \ge Ch|\log h|$; plugging $T = \mu^{-1}\gamma^{-1}$ one gets $(21.4.63)_2$ below and therefore *in the zone* (21.4.57) *one can upgrade* $T_1 = \epsilon\mu^{-1}\gamma^{-1}$ *to* $T_1 = \epsilon\mu\gamma^2$ *provided*

$$\rho^2 r \ge c\gamma, \qquad \rho^3 r \ge C\mu^2\gamma^3 h|\log h|. \tag{21.4.63}$$

However, the contribution to the remainder of the part of $\mathcal{Z}'_{\mathsf{out}}$ where $T_1 = \epsilon\mu\gamma^2$ is $O(\mu^{-1/2}h^{-3})$. Therefore *one needs to consider only part of* $\mathcal{Z}'_{\mathsf{out}}$

where at least one of conditions $(21.4.63)_{1,2}$ *is violated*. At this moment one can ignore subzone $\{\rho \le C(\mu h|\log h|)^{1/2}\}$ and therefore one can upgrade $T_1 = \epsilon\mu^{-1}$ to $T_1 = \epsilon\mu^{-1}\gamma^{-1}$ with no penalty.

Moreover, using $O(\mu h^{-3} r^2\gamma^2\rho^2)$ estimate (with $r = 1, \gamma = h^{-1/2+3\delta'}$) one can see easily that the contribution to the remainder of the zone where condition $(21.4.63)_2$ is violated does not exceed $C\mu^{1/3}h^{-7/3-\delta}$ which is $o(\mu^{-1/2}h^{-3} + \mu^2 h^{-2})$.

Therefore one needs to consider only zone where condition $(21.4.63)_1$ is violated; again due to $O(\mu h^{-3}\gamma^2\rho^2 r^2)$ estimate its contribution to the remainder does not exceed $C\mu^{-1/2}h^{-3-\delta}$ which is only marginally short of what we want (and only the case $\mu \le h^{-2/5-\delta}$ needs to be considered).

The proper estimate of the zone where condition $(21.4.63)_1$ is violated can be done easily on the base of Proposition 21.2.7. Indeed, the drift speed is $\asymp \mu^{-1}(\rho+\rho')\Delta r\gamma^{-2}$ with $\rho' = O(\gamma r^{-1})$, $\Delta = |\rho - \rho'|$ and therefore one can upgrade $T_1 = \epsilon\mu^{-1}\gamma^{-1}$ to $T_1 = \epsilon\mu\gamma^2$ as

$$\mu^{-1}(\rho+\rho')\Delta r\gamma^{-2} \times \mu^{-1}\gamma^{-1} \times \Delta \ge Ch|\log h|$$

with $\rho' = O(\gamma^{1/2}r^{-1/2})$. Therefore one can take

$$\bar{\Delta} = C\mu\gamma^{3/2}r^{-1/2}(h|\log h|)^{1/2}(\rho+\rho')^{-1/2};$$

then as $|\rho - \rho'| \ge \bar{\Delta}$ one can take $T_1 = \epsilon\mu\gamma^2$.

On the other hand, the contribution of the zone $\{(x,\xi) : |\rho-\rho'| \le \bar{\Delta}\} \cap \mathcal{Z}_{(r,\gamma)}$ to the remainder does not exceed

$$C\mu h^{-3} r^2 \rho' \gamma r^2 \bar{\Delta} = C\mu^2 h^{-3} r^2 \gamma^{5/2} r^{3/2} (h|\log h|)^{1/2} \rho'^{1/2}.$$

Since $\rho' \le c(\gamma/r)^{1/2}$ the latter expression does not exceed $C\mu^2 h^{-3}\gamma^{11/4} r^{5/4}(h|\log h|)^{1/2}$. In particular, the contribution to the remainder of the part of $\mathcal{Z}_{(\gamma)}$ where condition $(21.4.63)_1$ is violated does not exceed $C\mu^2 h^{-5/2}\gamma^{11/4}|\log h|^{1/2}$ and therefore the contribution of the corresponding part of $\mathcal{Z}'_{\mathsf{out}}$ does not exceed $C\mu^{5/8}h^{-5/2-\delta}$.

Therefore we arrive to

Proposition 21.4.11. *Under assumption* (21.3.1) *the total contribution to the remainder of zone* $\mathcal{Z}'_{\mathsf{out}}$ *does not exceed* $C(\mu^{-1/2}h^{-3} + \mu^2 h^{-2})$.

Estimates in the Near Outer Zone. II

Now we need to improve the previous estimate under non-degeneracy condition. However, let us summarize the remainder estimate we actually derived:

(i) The inner core intersected with $\{\rho \ge C(\mu h|\log h|)^{1/2}\}$ contributed $O(\mu h^{-3}\bar\gamma_0^2\bar r_0^2) = o(\mu^{-1/2}h^{-3})$ where $\bar\gamma_0 = \mu^{-2/3+4\delta'}$, $\bar r_0 = \mu^{-1/3+2\delta'}$.

(ii) $\mathcal{Z}'_{\mathsf{out}} \cap \{\rho \ge C(\mu h|\log h|)^{1/2}\}$ contributed $O(\mu h^{-3}\bar\gamma_1^2(\mu h\bar\gamma_1)^{2/3}) = O(\mu^{1/3}h^{-7/3-\delta''})$ where $\bar\gamma_1 = \mu^{-1/2+4\delta'}$.

(iii) Zone $\{|x_1| \le \bar\gamma_1\}\cap\{\rho \le C(\mu h|\log h|)^{1/2}\}$ contributed $O(\mu^2h^{-2}\bar\gamma_1|\log h|) = O(\mu^{1/2+\delta''}h^{-2})$.

Each of estimates in (ii), (iii) is less than $O(\mu^{-1/2}h^{-3})$ as $\mu \le h^{-1/2+\delta}$. Therefore in what follows one can assume that

$$h^{-1/2+\delta} \le \mu \le Ch^{-1}. \tag{21.4.64}$$

The better analysis is based on the canonical form (21.3.38). However, this requires the inner core to be treated separately:

Proposition 21.4.12. *Under assumption* (21.4.64) *the contribution of the zone*

$$\mathcal{Z}^0_{(r)} := \{\mathsf{dist}(x,\Sigma) \le r^2,\ \mathsf{dist}(x,\Lambda) \le r\}$$

with $r \ge \mu^{-1/3+2\delta'}$ *does not exceed* $C\mu h^{-3}r^6$ *while the principal part of asymptotics is given by* (21.1.2) *with* $T = \bar T := Ch|\log h|$.

Proof. Looking at the canonical form (21.3.17) and scaling $x'' \mapsto x''r^{-1}$, $x_1 \mapsto x_1r^{-1}$, $\mu \mapsto \mu_* = \mu r^3$, $h \mapsto h_* = hr^{-1}$ one gets 2-dimensional magnetic Schrödinger operator with parameters μ_*, h_*, which also is 1-dimensional $\hbar_4$-PDO with respect to x_4 with $\hbar_4 = \mu^{-1}hr^{-2}$; for such Schrödinger operators one can prove easily that the main part of asymptotics is given by Weyl formula and is of magnitude $\hbar_4^{-1}h_*^{-2} = \mu h^{-3}r^4$ while the remainder estimate gains the factor $\mu_*h_* = \mu hr^2$ and thus is of magnitude $\mu^2h^{-2}r^6$.

Indeed, if $\mu_* = 1$ it would follow from the standard results (and no non-degeneracy condition is needed); as $\mu_* \ge 1$ one needs just apply extra rescaling $x' \mapsto x'\mu_*$ which is also rather standard. □

Corollary 21.4.13. *Therefore even without nondegeneracy conditions contribution of the (extended) inner core $\mathcal{Z}^0_{(r)}$ with $r = \mu^{-1/4}$ to the remainder does not exceed $C\mu^{-1/2}h^{-3}$.*

Thus in what follows we can apply canonical form (21.3.38). Let us fix some index n and some final $B(0,1)$ element where in contrary to Subsection 21.4.1 final means "only after reduction to (21.3.38)". This element is either *regular*, when on the original $B(\bar{x}', \epsilon r)$ element

$$|Vf_2^{-1} - (2n+1)\mu h| + |\nabla_{x'}\big(Vf_2^{-1} - (2n+1)\mu h\big)|^2 \asymp L^2 \ge r^2 \tag{21.4.65}$$

or *singular* when on the original $B(\bar{x}', \epsilon r)$ element

$$|Vf_2^{-1} - (2n+1)\mu h| + |\nabla_{x'}\big(Vf_2^{-1} - (2n+1)\mu h\big)|^2 \le r^2. \tag{21.4.66}$$

Since analysis of the regular elements is similar to the analysis of the singular elements but simpler, we consider only singular elements, leaving regular elements to the reader.

Let us introduce the scaling function with respect to (x_3, x_4, ξ_4)

$$\ell = \ell_n = \epsilon r^{-1}|\nabla_{x_3,x_4,\xi_4} Vf_2^{-1}| + \bar{\ell} \tag{21.4.67}$$

where $\bar{\ell} \ge C\gamma r^{-1}|\log h|$ will be chosen later. With respect to $|Z_1^{\#}|$ we use scaling function $\rho = \epsilon\big(|Z_1| + r^{1/2}\ell\big)$. Then ellipticity of $\mathcal{A}_n$ is not broken unless $\big(Vf_2^{-1} - (2n+1)\mu h\big) \asymp \rho^2$ or $\rho \le cr^{1/2}\ell$.

Then it follows from the standard theory that for u_n holds as Q is supported in the corresponding ℓ-partition element, $T \in [T_0, T_1]$, $\tau \le Cr\ell^2$ with

$$T_0 = Ch\big(\rho^2 + r\ell^2\big)^{-1}|\log h|, \tag{21.4.68}$$

$$T_1 = \epsilon\mu^{-1}\gamma^{-1}, \tag{21.4.69}$$

$$\rho + r^{1/2}\ell \ge C\bar{\rho}_0 := C(\mu h\gamma|\log h|)^{1/2} \tag{21.4.70}$$

and moreover, under these conditions

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(u_n Q y^{\dagger})| \le C\mu h^{-2} r^3 \ell^3 \gamma. \tag{21.4.71}$$

Therefore we arrive to

(21.4.72) While the contribution of this $(\ell;\rho)$-subelement into asymptotics does not exceed $C\mu h^{-3}r^3\big(\rho^2+r\ell^2\big)\ell^3\gamma$, its contribution to the remainder so far does not exceed

$$C\mu h^{-3}r^3\big(\rho^2+r\ell^2\big)\ell^3\gamma\times T_0T_1^{-1}=C\mu^2h^{-2}r^3\ell^3\gamma^2.$$

Note that for each $(\ell;\rho)$-partition subelement in $(x_3,x_4,\xi_4;\xi_1,\xi_3)$ the number of indices n satisfying (21.4.67) and also violating ellipticity does not exceed

$$M:=C_0(\mu h)^{-1}(\rho^2+r\ell^2)+1. \tag{21.4.73}$$

Also note that under assumption $(21.4.50)_q$ the total number of ℓ-subelements is $O(\ell^{q-3})$. Then the total contribution of all ℓ-elements to the asymptotics (as magnitudes of r,γ are given) does not exceed

$$C\mu h^{-3}\gamma r^{(2-q)_+}(\rho^2+r\ell^2)\times\Big(C_0(\mu h)^{-1}(\rho^2+r\ell^2)+1\Big)\times\ell^q \tag{21.4.74}$$

which one can sum nicely to its value at maximal $\ell=1$, $\rho=1$ and then to maximal $r=1$, $\gamma=\mu^{\delta-1/2}$ resulting $Ch^{-4}\mu^{\delta-1/2}$.

On the other hand, the total contribution of ℓ-subelements to the remainder so far does not exceed

$$C\mu^2h^{-2}\gamma^2r^{(2-q)_+}\times\Big(C_0(\mu h)^{-1}(\rho+r\ell)^2+1\Big)\times\ell^q \tag{21.4.75}$$

which one can also sum nicely according to the same principles but the sum is $C\mu h^{-3}\gamma^2$ not estimated that well and ℓ,r close to 1 are problematic. Therefore we want to increase T_1.

Recall that $W_n=\omega\big(Vf_2^{-1}-(2n+1)\mu h\big)$ with $\omega=(\sigma\sigma_1)^{-1}f_2$ and here (see Proposition 21.3.8) $\omega\sim|\{Z_1,f_1\}|^{-2/3}$ rescaled; therefore for "small" ρ calculation of ∇W_n and $\omega\nabla\big(Vf_2^{-1}-(2n+1)\mu h\big)$ are equivalent modulo $\omega\rho^2r^{-1}$. Note that

(21.4.76) As $\big(Vf_2^{-1}-(2n+1)\mu h\big)\asymp\rho^2$ and $\ell\ge C\rho^2r^{-1}$ the propagation speed with respect to x_4 does not exceed $C\mu^{-1}r^{-1}\ell$ and is of this magnitude as $|\partial_{\xi_4}W_n|\asymp\ell$; then the shift with respect to x_4 is observable as

$$\mu^{-1}r^{-1}\ell T\times\ell\ge C\mu^{-1}r^{-2}h|\log h|; \tag{21.4.77}$$

(21.4.78) As $\big(Vf_2^{-1}-(2n+1)\mu h\big)\asymp\rho^2$ and $\ell\ge C\rho^2r^{-1}$ the propagation speed with respect to $\mu^{-1}hD''$ does not exceed $C\mu^{-1}r^{-1}\ell$ and is of this magnitude as $|\partial_{x''}W_n|\asymp\ell$; then the shift with respect to $\mu^{-1}hD''$ is observable under the same condition;

(21.4.79) Finally, in the short-term propagation (as $T \le \epsilon\mu^{-1}\gamma^{-1}$) the propagation speed with respect to (x_1, x_3) is of magnitude $|Z_1| \asymp \rho$ and the shift is observable as $\rho T \times \rho \ge Ch|\log h|$.

On the other hand,

(21.4.80) Let $\rho^2 \ge C\gamma$; then the drift speed (in the rescaled coordinates) with respect to x'' is of magnitude $C\mu^{-1}\gamma^{-2}\rho^2$ and the shift for time T is observable as

$$\mu^{-1}\gamma^{-2}\rho^2 T \times \rho \ge Chr^{-1}|\log h|. \tag{21.4.81}$$

Therefore one can increase $T_1 = \epsilon\mu^{-1}\gamma^{-1}$ provided one of the conditions (21.4.77), (21.4.81) holds with $T = T_1$. Plugging $T_1 = \epsilon\mu^{-1}\gamma^{-1}$ into (21.4.77), (21.4.81) we get

$$\rho \ge \bar{\rho} \coloneqq C\bigl(\mu^2 h\gamma^3 r^{-1}|\log h|\bigr)^{1/3} + C\gamma^{1/2}, \tag{21.4.82}$$

$$\ell \ge \bar{\ell} \coloneqq C\bigl(\mu h\gamma r^{-1}|\log h|\bigr)^{1/2} + C\rho^2 r^{-1} \tag{21.4.83}$$

respectively.

Now consider increased T_1. We claim that

(21.4.84) Under assumption (21.4.83) one can take $T = \epsilon\mu\gamma^2\ell$.

Indeed, then magnitude of $|\nabla_{x_1,x_4,\xi_4} W_n|$ changes by no more than $\epsilon\ell$ as long as $\rho \le \epsilon_1$; however, as $\rho \ge \epsilon$ condition (21.4.82) is fulfilled and one can take $T_2 = \epsilon\mu\gamma^2$.

Therefore the contribution to the remainder of the zone with the given magnitudes of ρ, ℓ, γ, r satisfying (21.4.83) does not exceed expression (21.4.74) multiplied by $T_0 T_2^{-1}$; the resulting expression does not exceed

$$\begin{aligned} C\mu h^{-3}\gamma r^{(2-q)_+}(\rho^2 + r\ell^2) \times \Bigl(C_0(\mu h)^{-1}(\rho + r\ell)^2 + 1\Bigr) \times \ell^q \\ \times h(\rho^2 + r\ell^2)^{-1} \times \mu^{-1}\gamma^{-2}\ell^{-1} \\ = C\mu^{-1}h^{-3}\gamma^{-1}r^{(2-q)_+}\Bigl((\rho + r\ell)^2 + \mu h\Bigr)\ell^{q-1}. \end{aligned} \tag{21.4.85}$$

Then summation with respect ℓ, ρ results in $C\mu^{-1}h^{-3}\gamma^{-1}r$ as $q > 1$ (then one should take q slightly larger than 1). As $q = 1$ one gets $C\mu^{-1/2}h^{-3}(1 + \mu h|\log h|)$ as provided summation is taken over $\rho \le c(\ell + |\log h|^{-1})$; but the latter case is covered by arguments below.

(21.4.86) Under assumptions (21.4.82) and $\rho \ge r^{1/2}\ell$ one can take $T_2 = \epsilon\mu\gamma^2\rho r^{-1/2}$.

Indeed, in this case due to bounds for propagation and drift speeds this guarantees that both $|\nabla_{x_1,x_4,\xi_4} W_n|$ and $W_n \asymp |Z_1|^2$ change by no more than $r^{-1/2}\rho$, $\epsilon_1\rho^2$ respectively during the evolution.

Now instead of (21.4.85) we get

$$\begin{aligned}(21.4.87)\quad C\mu h^{-3}\gamma r^{(2-q)_+}(\rho^2 + r\ell^2) \times \Big(C_0(\mu h)^{-1}(\rho + r\ell)^2 + 1\Big) \times \ell^q \\ \times h(\rho^2 + r\ell^2)^{-1} \times \mu^{-1}\gamma^{-2}\rho^{-1} \\ \le C\mu^{-1}h^{-3}\gamma^{-1}r^{(2-q)_+}\Big((\rho + r\ell)^2 + \mu h\Big)\ell^q\rho^{-1}\end{aligned}$$

and summation with respect to ρ, ℓ, γ, r results in $C\mu^{-1/2}h^{-3}$. Therefore

(21.4.88) The total contribution to the remainder of elements in $\mathcal{Z}''_{\mathsf{out}}$ satisfying (21.4.70) and one of conditions (21.4.82), (21.4.83) does not exceed $C\mu^{-1/2}h^{-3}$.

Estimates in the Near Outer Zone. III

Thus we are left with two types of elements, which are not covered by the above arguments:

(21.4.89) With $\bar\rho_0 \le \rho \le \bar\rho = C(\mu^2 h\gamma^3 r^{-1}|\log h|)^{1/3} + C\gamma^{1/2}$ and $\ell \le \bar\ell \asymp r^{-1}\bar\rho_0 + \rho^2 r^{-1}$;

(21.4.90) With $\rho \le \bar\rho_0 = C(\mu h\gamma)^{1/2}$ and $\ell \le \bar\ell \asymp r^{-1/2}\bar\rho_0$.

For (21.4.89)-type elements $r\ell^2 \le \rho^2$ and expression (21.4.74) becomes

$$C\mu h^{-3}\gamma r^{(2-q)_+}\rho^2 \times \Big((\mu h)^{-1}\rho^2 + 1\Big) \times \bar\ell^q;$$

multiplying it by $T_0 T_1^{-1} = Ch\rho^{-2}|\log h| \cdot \mu\gamma$ one gets

$$C\mu^2 h^{-2}\gamma^2 r^{(2-q)_+} \times \Big((\mu h)^{-1}\rho^2 + 1\Big) \times \bar\ell^q|\log h|,$$

which after summation over ρ becomes

$$(21.4.91)\qquad C\mu^2 h^{-2}\gamma^2 r^{(2-q)_+} \times \Big((\mu h)^{-1}\bar\rho^2 + 1\Big) \times \bar\ell^q|\log h|.$$

Meanwhile for (21.4.90)-type elements expression (21.4.74) becomes

$$C\mu h^{-3}\gamma r^{(2-q)_+}\bar{\rho}_0^2\bar{\ell}^q. \tag{21.4.92}$$

Let us plug $\bar{\rho}$, $\bar{\rho}_0$, $\bar{\ell}$ and γ; one can take

$$\gamma = \mu^{-1/2}r^{1/2},\ \bar{\rho}_0 = (\mu h r^{1/2})^{1/2}, \bar{\rho} = (\mu^{1/2}hr^{1/2})^{1/3} + \mu^{-1/4}r^{1/4} \text{ and}$$
$$\bar{\ell} = (\mu^{1/2}hr^{-1/2})^{1/2} + (\mu^{1/2}h)^{2/3}r^{-2/3} + \mu^{-1/2}r^{-1/2}$$

and compensate this by multiplication the result by $h^{-\delta''}$ with an arbitrarily small exponent $\delta'' > 0$.

Take first $\bar{\ell} = \bar{\ell}_1 := (\mu^{1/2}hr^{-1/2})^{1/2}$; note that then in (21.4.91), (21.4.92) factor $r^{-q/4}$ coming from $\bar{\ell}^q$ is more than compensated by the factor r coming from γ^2 or $\gamma\bar{\rho}_0^2$ respectively. Therefore summation with respect to γ, r results in the same expressions calculated as $r = 1$, $\gamma = \mu^{-1/2}$, $\bar{\ell} = \bar{\rho}_0 = (\mu^{1/2}h)^{1/2}$ and $\bar{\rho} = (\mu^{1/2}h)^{1/3} + \mu^{-1/4}$:

$$Ch^{-3}\Big((\mu^{1/2}h)^{2/3} + \mu^{-1} + \mu h\Big)(\mu^{1/2}h)^{q/2} \times h^{-\delta''}, \tag{21.4.93}$$

$$C\mu^{1/2}h^{-3}(\mu^{1/2}h)^{(2+q)/2} \times h^{-\delta''}. \tag{21.4.94}$$

One can see easily that both these expressions do not exceed

(a) $C\big(\mu^{-1/2}h^{-3} + \mu^{3/2}h^{-3/2}\big)$ as $q = 1$;

(b) $C\big(\mu^{-1/2}h^{-3} + \mu^{1/2+\delta''}h^{-2}\big)$ as $q = 2$;

(c) $C\mu^{-1/2}h^{-3}$ as $q = 3$.

However, one should consider (21.4.91) with $\bar{\ell} = \bar{\ell}_2 := (\mu^{1/2}h)^{2/3}r^{-2/3}$ and $\bar{\ell} = \bar{\ell}_3 := \mu^{-1/2}r^{-1/2}$; as $q = 1$ one can take again $r = 1$ obviously; as $q = 2$ it is not so but one can take then by $q = 9/5$ arriving to $O(\mu^{-1/2}h^{-3})$.

Then combining with (21.4.88) we arrive to

Proposition 21.4.14. *The total contribution to the remainder of the zone* $\mathcal{Z}'_{\mathsf{out}}$ *does not exceed* $C\mu^{-1/2}h^{-3} + C\mu^{3/2}h^{-3/2}$ *under assumption* $(21.4.50)_1$, $C\mu^{-1/2}h^{-3} + C\mu^{3/2}h^{-1-\delta}$ *under assumption* $(21.4.50)_2$ *and* $C\mu^{-1/2}h^{-3}$ *under assumption* $(21.4.50)_3$.

Remark 21.4.15. The main part of this asymptotics is given by expression (21.1.2) with $T = \bar{T}_0$.

21.4.3 Estimates in the Inner Zone

Now we want to consider the *inner zone*

$$\mathcal{Z}_{\mathsf{inn}} := \bigl\{\mathsf{dist}(x,\Sigma) \le \bar{\gamma}_0 = c\mu^{-1/2}\cdot \mathsf{dist}(x,\Lambda)^{1/2}\bigr\}, \tag{21.4.95}$$

in turn consisting of the *inner bulk zone*

$$\mathcal{Z}_{\mathsf{inn},II} := \mathcal{Z}_{\mathsf{inn}} \cap \bigl\{|Z_1|\cdot \mathsf{dist}(x,\Lambda) \le \epsilon\mu\cdot \mathsf{dist}(x,\Sigma)^2\bigr\}, \tag{21.4.96}$$

which mimics the outer (bulk) zone $\mathcal{Z}_{\mathsf{out}}$ and the *true inner zone*

$$\mathcal{Z}_{\mathsf{inn},I} := \mathcal{Z}_{\mathsf{inn}} \cap \bigl\{|Z_1|\cdot \mathsf{dist}(x,\Lambda) \ge \epsilon\mu\cdot \mathsf{dist}(x,\Sigma)^2\bigr\}, \tag{21.4.97}$$

which mimics the inner zone for 2-dimensional Schrödinger operator. One needs to remember that trajectories can leave one of them and enter another one after a while.

Note first that the inner core is already covered by Proposition 21.4.12, so in the definition of the inner zone we can include condition $\mathsf{dist}(x,\Lambda)\ge \mu^{-1/4}$ and therefore use canonical form (21.4.30) there. Actually using the same method as in the proof of Proposition 21.4.12, one can provide estimate $O(\mu h^{-3}r^2\gamma^2) = O(h^{-3}r^3)$ thus covering even $\mathcal{Z}_{\mathsf{inn}}\cap\{\mathsf{dist}(x,\Lambda)\le \mu^{-1/6}\}$ but we do not need this.

Estimates in the Inner Zone. I

We start from the inner bulk zone $\mathcal{Z}_{\mathsf{inn},II}$. Let $|Z_1|\asymp \rho$, $\rho r\le \epsilon\mu\gamma^2$ and condition (21.4.82) be fulfilled. Then exactly as in the outer zone the drift speed is $\asymp \mu^{-1}\rho^2 r\gamma^{-2}\le c\rho$ and evolution speed with respect to hD_4 does not exceed c; then for time $T_1 = \epsilon\rho r$ dynamics remains in $\mathcal{Z}_{\mathsf{inn},II}$. Therefore in the domain where condition (21.4.82) is fulfilled and $\rho\ge C(\mu h|\log h|)^{1/2}$ one can take $T_1=\epsilon\rho r$ and $T_0 = Ch|\log h|$; however, in the estimates this logarithmic factor is not needed.

Therefore the contribution to the remainder of this domain intersected with $\mathcal{Z}_{(r,\gamma,\rho)}$ does not exceed $Ch^{-3}\gamma T_1^{-1}r^2\rho^2 = C\mu^{-1/2}h^{-3}r^2\rho$ and the summation with respect to (r,γ,ρ) results in $O(\mu^{-1/2}h^{-3})$.

On the other hand, consider the domain where condition (21.4.82) is violated, but still $\rho\ge\bar{\rho}_1$. The contribution to the remainder of this domain intersected with $\mathcal{Z}_{(r,\gamma,\rho)}$ does not exceed $C\mu h^{-3}\gamma^2 r^2\rho^2$ where two factors γ come as the width of the strip and the part of $T_1^{-1}\asymp\mu\gamma$. The summation with respect to (r,γ,ρ): $r\rho^2\le c\gamma$ results in $C\mu^{-1/2}h^{-3}$.

Further, consider domain where condition (21.4.83) is violated but still $\rho \ge \bar\rho_1$. In the similar manner one can estimate its contribution to the remainder by $C\mu h^{-3}\gamma_0^2(\mu^{1/2}h|\log h|)^{2/3}$ which is $o(\mu^2h^{-2}+\mu^{-1/2}h^{-3})$ for sure and is $O(\mu^{-1/2}h^{-3})$ as $\mu \le C(h|\log h|)^{-4/5}$.

Finally, contribution to the remainder of the domain where $\rho \le \bar\rho_1$ does not exceed $C\mu\gamma\bar\rho_1^2 = C\mu^{3/2}h^{-2}|\log h|$.

Thus we arrive to

Proposition 21.4.16. *(i) Contribution of $\mathcal{Z}_{\mathsf{inn},II}$ to the remainder is $O(\mu^2h^{-2}+\mu^{-1/2}h^{-3})$ for sure.*

(ii) Furthermore, this contribution is is $O(\mu^{-1/2}h^{-3})$ as $\mu \le C(h|\log h|)^{-1/2}$.

(iii) Again the main part of this asymptotics is given by expression (21.1.2).

Estimates in the Inner Zone. II

Again, the estimate achieved in Proposition 21.4.16 is good in the general case but should be improved under non-degeneracy condition as

$$c(h|\log h|)^{-1/2} \le \mu \le Ch^{-1}. \tag{21.4.98}$$

Again one should consider regular final elements and singular ones and we consider only latter. Further, introducing the scaling function ℓ_n by (21.4.67) we can define T_0 and T_1 by (21.4.68), (21.4.69) as (21.4.70) holds; furthermore, estimate (21.4.71) holds.

Further, if condition (21.4.83) is fulfilled, we can increase T_1 to $T_2 = \epsilon\ell$ since on this time interval both $|\nabla_{x_1,x_4,\xi_4}W_n|$ and $|W_n|$ change by no more than $\epsilon_1 \min(\ell, r)$.

Then repeating the arguments of Subsubsection 21.4.2.2 we find that the contribution of this part of $\mathcal{Z}'_{\mathsf{inn},II}$ to the remainder does not exceed expression similar to (21.4.85) but with $\mu^{-1}\gamma^{-2}$ replaced by 1:

$$\begin{aligned} C\mu h^{-3}\gamma r^{(2-q)_+}(\rho^2+r\ell^2) \times \Big(C_0(\mu h)^{-1}(\rho+r\ell)^2+1\Big) \\ \times \ell^q \times h(\rho^2+r\ell^2)^{-1}\times \ell^{-1} \\ = Ch^{-3}\gamma r^{(2-q)_+}\Big((\rho+r\ell)^2+\mu h\Big)\ell^{q-1}. \end{aligned} \tag{21.4.99}$$

Then the sum with respect to (ℓ,ρ,r)-partition with $\rho \ge \ell$ does not exceed $Ch^{-3}\gamma$ as $q \ge 2$ and $Ch^{-3}\gamma|\log h|$ as $q = 1$ and one can get rid of logarithmic factor exactly as in the analysis in $\mathcal{Z}'_{\mathsf{out},II}$.

Furthermore, if condition (21.4.82) is fulfilled instead of (21.4.83) and $\rho \ge R^{1/2}\ell$, we can increase T_1 to $T_2 = \epsilon\rho$ and the contribution of this part of $\mathcal{Z}'_{\mathsf{inn},II}$ to the remainder does not exceed expression similar to (21.4.87) but with $\mu^{-1}\gamma^{-2}$ replaced by 1:

$$\begin{aligned}(21.4.100)\quad C\mu h^{-3}\gamma r^{(2-q)_+}(\rho^2+r\ell^2)\times \Big(C_0(\mu h)^{-1}(\rho+r\ell)^2+1\Big)\\ \times \ell^q \times h(\rho^2+r\ell^2)^{-1}\times \rho^{-1}\\ = Ch^{-3}\gamma r^{(2-q)_+}\Big(\rho^2+\mu h\Big)\ell^q\rho^{-1}.\end{aligned}$$

Summation with respect to (ℓ,ρ,r)-partition results in $Ch^{-3}\gamma + C\mu h^{-2}\gamma|\log h|$. Therefore the sum with respect to γ-partition with $\rho \ge \ell$ does not exceed $C\mu^{-1/2}h^{-3}$ as $q \ge 2$ and $\mu^{-1/2}h^{-3} + C\mu^{1/2}h^{-2}|\log h|$ as $q = 1$.

On the other hand, due to the same arguments as in Subsubsection 21.4.2.3 the contribution to the remainder of the domain where both conditions (21.4.82) and (21.4.83) are violated but (21.4.70) still holds does not exceed (21.4.91). Then the summation with respect to (ℓ,ρ,r,γ) partition results again in $C\mu h^{-2}(\mu^{1/2}h|\log h|)^{q/2} + O(\mu^{-1/2}h^{-3})$.

Finally, contribution of the domain where (21.4.70) fails is tackled in the same way as in Subsubsection 21.4.2.3 and estimated by (21.4.92).

So, we arrive to the following copy-cat of Proposition 21.4.14 and Remark 21.4.15:

Proposition 21.4.17. *The total contribution to the remainder of the zone* $\mathcal{Z}_{\mathsf{inn},II}$ *does not exceed* $C\mu^{-1/2}h^{-3} + C\mu^{3/2}h^{-3/2}$ *under assumption* $(21.4.50)_1$, $C\mu^{-1/2}h^{-3} + C\mu^{3/2}h^{-1}|\log h|$ *under assumption* $(21.4.50)_2$ *and* $C\mu^{-1/2}h^{-3}$ *under assumption* $(21.4.50)_3$.

Again the main part of this asymptotics is given by expression (21.1.2).

Estimates in the Inner Zone. III

Now let us consider the *true inner zone* $\mathcal{Z}_{\mathsf{inn},I}$ defined by (21.4.99). The crucial difference between this zone and $\mathcal{Z}_{\mathsf{out}} \cap \mathcal{Z}_{\mathsf{inn},II}$ is that in the classical evolution magnitude of $\mathsf{dist}(x,\Sigma)$ is not necessarily preserved[27)].

Recall that this subzone (as the whole inner zone) should be studied as $r = \mathsf{dist}(x,\Lambda) \ge c\mu^{-1/4}$. Also recall that the contribution of

[27)] As $|Z_1|\cdot \mathsf{dist}(x,\Lambda) \ge C\mu\cdot \mathsf{dist}(x,\Sigma)^2$ it is not preserved for sure; in the rest of this zone variation of $\mathsf{dist}(x,\Sigma)$ is of the magnitude of $\mathsf{dist}(x,\Sigma)$.

the subzone $\{|Z_1| \le C(\mu h \log h)^{1/2}\}$ to the remainder does not exceed $C\mu h^{-3}\gamma \times \mu h|\log h| = C\mu^{3/2}h^{-2}|\log h|)$ which is $O(\mu^2 h^{-2})$ for sure and $O(\mu^{-1/2}h^{-3})$ as $\mu \le c(h|\log h|)^{-1/2}$.

Let us introduce $\gamma = \mu^{-1/2}(r\rho)^{1/2}$; then in the classical evolution γ is the magnitude of $\max \operatorname{dist}(x, \Sigma)$ along the cyclotron movement. Note that the contribution of subzone $\{(\mu h|\log h|)^{1/2} \le |Z_1| \le \rho\}$ to the remainder does not exceed $C\mu h^{-3}\gamma^2\rho^2 = Ch^{-3}\rho^3$ which sums to $O(\mu^{-1/2}h^{-3})$ in subzone $\{\rho \le c\mu^{-1/6}|\log h|^{-1/3}\}$.

Let us consider first the general case of $q = 0$ when our goal is remainder estimate $O(\mu^{-1/2}h^{-3}+\mu^2 h^{-2})$. Since in this case the total contribution of the zone $\mathcal{Z}_{\mathsf{inn},I} \cap \{|Z_1| \ge C(\mu h|\log h|)^{1/2}\}$ to the remainder is $O(h^{-3})$, one needs to analyze only case $\mu \le h^{-1/2}$ and only subzone $\{\rho \ge c(\mu^{-1/6} + \mu^{2/3}h^{1/3})\}$.

We are going to prove that the contribution of $\mathcal{Z}_{\mathsf{inn},I}\cap\{\rho \ge C(\mu h|\log h|)^{1/2}\}$ to the remainder does not exceed $C\mu^{-1/2}h^{-3} + C(\mu^{1/2}h|\log h|)^{1/2}h^{-3}$ (see Proposition 21.4.19) but we prefer a bit less direct way to do this; we are concentrating on this zone due to the previous remarks and because we do not have non-degeneracy condition at the moment. Thus one can take $T_0 = Ch|\log h|$ and $T_1 = \epsilon\mu^{-1/2}\gamma^{-1} = \epsilon\mu^{-1}(\rho r)^{-1}$. Let us try to increase it.

According to the series of propositions of Subsubsection 21.2.4 the drift speed in the inner zone is $\kappa r\rho(\xi_3 - k^*\rho) + O(\mu^{-1/2})$. Then the shift during one cyclotron movement is $\asymp r\rho(\xi_3 - k^*\rho)T_1$ provided

$$r\rho|\xi_3 - k^*\rho| \ge C\mu^{-1/2} \tag{21.4.101}$$

and it is observable as long as

$$r\rho|\xi_3 - k^*\rho| \times \mu^{-1/2} \times |\xi_3 - k^*\rho| \ge Ch|\log h|$$

or equivalently

$$\begin{aligned} |\xi_3 - k^*\rho| &\ge C(r\rho)^{-1/2}(\mu h\gamma|\log h|)^{1/2} \\ &= C(r\rho)^{-1/4}(\mu^{1/2}h|\log h|)^{1/2}. \end{aligned} \tag{21.4.102}$$

Both conditions (21.4.101), (21.4.102) are fulfilled as

$$|\xi_3 - k^*\rho| \ge \Delta := c(r\rho)^{-1/4}(\mu^{1/2}h|\log h|)^{1/2} + c\mu^{-1/2}(r\rho)^{-1/2}. \tag{21.4.103}$$

Then as (21.4.103) is fulfilled[28)] one can upgrade T_1 to $T_2 = \epsilon r|\xi_3 - k^*\rho|$ because then not only magnitude of ρ but also of $|\xi_3 - k^*\rho|$ will be pre-

[28)] This makes sense as $\Delta \le \rho$: i.e.

$$\rho \ge \bar\rho_2 := r^{-1/5}(\mu^{1/2}h|\log h|)^{2/5} + \mu^{-1/3}r^{-1/3}; \tag{21.4.104}$$

otherwise one needs to take $\Delta = c\rho$, thus redefining $\Delta := \min(\Delta, \bar\rho_2)$.

served during the evolution. Furthermore, taking the evolution in the right time direction it could be increased to $T_2^* = \epsilon r|\xi_3 - k^*\rho|^{1-\delta'}$ exactly as in Chapter 14. Then the standard arguments imply

(21.4.105) The contribution of the subzone of $\mathcal{Z}_{\mathsf{inn},I}$, where condition (21.4.103) is fulfilled, intersected with $\{\rho \ge C(\mu h|\log h|)^{1/2}\}$, to the remainder does not exceed $C\mu^{1/2}h^{-3}$.

So, one needs to study *near periodic zone*

$$\mathcal{Z}_{\mathsf{per}} = \big\{|\xi_3 - k^*\rho| \le \Delta\big\}, \tag{21.4.106}$$

intersected with $\{|Z_1| \asymp \rho\} \cap \{\mathsf{dist}(x,\Lambda) \asymp r\}$ as long as

$$\rho \ge C(\mu h|\log h|)^{1/2} + \bar\rho_2.$$

Note however, that if $r \le \epsilon$ then $|\nabla\omega| \asymp r^{-1}\omega$ and $|\mathsf{Hess}\,\omega| \asymp r^{-2}\omega$ and therefore in the rescaled coordinates $|\nabla_{x_3,x_4,\xi_4} W_n| \asymp \rho^2$ as $r \le \epsilon$ and $\rho^2 \ge Cr$ and $|\mathsf{Hess}_{x_3,x_4,\xi_4}\, W_n| \asymp \rho^2$ as $r \le \epsilon$ and $\rho \ge Cr$. In the first case periodicity is broken for sure and in the second case periodicity is not broken only as $|\nabla W_n| \le C(\mu\gamma h)^{1/2}$ which in intersection with $\mathcal{Z}_{\mathsf{per}}$ has measure not exceeding $C\mu^{1/2}h\gamma$ instead of $C(\mu^{1/2}h)^{1/2}\gamma$ and since one can take $T_1 = \epsilon\mu^{-1}\gamma^{-1}$ here for sure the contribution of the reduced periodic zone to the remainder does not exceed $Ch^{-4} \times (\mu^{1/2}h|\log h|)\gamma \times \mu h\gamma \le C\mu^{1/2}h^{-2}|\log h|$ thus implying

Proposition 21.4.18. *The contribution of*

$$\mathcal{Z}_{\mathsf{inn},I} \cap \{|Z_1| \ge C(\mu h|\log h|)^{1/2} + Cr\}$$

to the remainder does not exceed $C\mu^{-1/2}h^{-3}$.

Again the main part of this asymptotics is given by expression (21.1.2).

Therefore, only case $\rho \le Cr$ and thus $\gamma \le C\mu^{-1/2}r$ remains to be analyzed.

Proposition 21.4.19. *The contribution of* $\mathcal{Z}_{\mathsf{inn},I} \cap \{|Z_1| \ge C(\mu h|\log h|)^{1/2}\}$ *to the remainder does not exceed* $C\mu^{-1/2}h^{-3} + C(\mu^{1/2}h|\log h|)^{1/2}h^{-3}$.

Again the main part of this asymptotics is given by expression (21.1.2).

Proof. Now we are concentrating only on even more reduced zone. In particular, (21.4.104) implies $r \ge (\mu^{1/2}h|\log h|)^{1/3} + \mu^{-1/4}$.

One can take $T_1 = \epsilon\mu^{-1/2}(r\rho)^{-1/2}$ as condition (21.4.103) fails. Then the contribution of $\mathcal{Z}_{\mathsf{per}}$ to the remainder does not exceed

$$C\mu h^{-3}\gamma^2 r^2\rho\Delta = Ch^{-3}r^3\rho^2\Delta = Ch^{-3}r^3\rho^2\Big(c(r\rho)^{-1/4}(\mu^{1/2}h|\log h|)^{1/2} + c\mu^{-1/2}(r\rho)^{-1/2}\Big). \tag{21.4.107}$$

Obviously when taking sum with respect to (r,ρ) partition the latter expression sums to itself with $\bar{\Delta} = \Delta|_{r=1,\rho=1,\gamma=\mu^{-1/2}}$:

$$\bar{\Delta} = C(\mu^{1/2}h|\log h|)^{1/2} + C\mu^{-1/2}. \tag{21.4.108}$$

Then expression (21.4.107) becomes $O\big(\mu^{-1/2}h^{-3} + h^{-3}(\mu^{1/2}h|\log h|)^{1/2}\big)$.

Finally, one can check easily that the contribution of subzone $\big\{\bar{\rho}_2 \ge \rho \ge C(\mu h|\log h|)^{1/2}\big\}$ is just $O(\mu h^{-3}\gamma^2\bar{\rho}_2^2)$ which is less than this. □

Estimates in the Inner Zone. IV

Again we would like to improve the remainder estimate achieved in Proposition 21.4.19 under non-degeneracy condition $(21.4.50)_q$ with $q \ge 1$ also getting rid of assumption $|Z_1| \ge C(\mu h|\log h|)^{1/2}$. In the analysis below instead of W_n we consider its pullback $\omega(V - (2n+1)f_2\mu h)$ in the rescaled coordinates x' on Σ; it is useful for Hessians but it transforms into properties of W_n since we are interested only in the measures of sets $|\nabla W_n| \asymp \lambda$.

Proposition 21.4.20. *As $\mu \le ch^{-1}|\log h|^{-K}$ the contribution of $\mathcal{Z}_{\mathsf{inn},I} \cap \{|Z_1| \ge Cr\}$ to the remainder does not exceed $C\mu^{-1/2}h^{-3}$.*

Proof. Let us introduce

$$\ell = \ell_n = \epsilon r^{-1}|\nabla_{x_3,x_4,\xi_4} W_n| + \bar{\ell}, \tag{21.4.109}$$

and

$$\varrho = \varrho_n = \epsilon|\xi_3 - k^* W_n^{1/2}| + \Delta. \tag{21.4.110}$$

Then the contribution of (r,ρ,ℓ)-elements to the asymptotics does not exceed (21.4.74)-like expression

$$C\mu h^{-3}\gamma r^2(\rho^2 + r\ell^2) \times \Big(C_0(\mu h)^{-1}(\rho + r\ell^2) + 1\Big); \tag{21.4.111}$$

if $\rho^2 + r\ell^2 \le C\mu h\gamma|\log h|$ then also $r^2 \le C\mu h\gamma|\log h|$ and (21.4.111 does not exceed $C\mu^3 h^{-1}\gamma^3|\log h|^2 \le C\mu^{3/2}h^{-1}|\log h|^2$. So let us consider $\rho^2 + r\ell^2 \ge C\mu h\gamma|\log h|$.

If $\rho^2 \ge Cr$ then periodicity is broken for sure (since $\nabla W_n| \ge \epsilon\rho^2 - Cr$ and one can instantly upgrade $T_0 = Ch|\log h|(\rho^2 + r\ell^2)^{-1}$ to $T_2 = \epsilon$. Otherwise it could be done as $\ell \ge C(\mu h|\log h|)^{1/2}$. Then contribution of such elements to the remainder does not exceed

$$\begin{aligned}(21.4.112)\quad C\mu h^{-3}\gamma r^2(\rho^2 + r\ell^2) \times \Big(C_0(\mu h)^{-1}(\rho + r\ell^2) + 1\Big) \\ \times h|\log h|(\rho^2 + r\ell^2)^{-1} \\ = C\mu h^{-2}\gamma r^2 \times \Big(C_0(\mu h)^{-1}(\rho + r\ell^2) + 1\Big)|\log h|\end{aligned}$$

because $|\operatorname{Hess} W_n| \ge \epsilon\rho^2 - Cr^2$ and one can delete safely logarithmic factor because it is not in the estimate of the Fourier transform.

Otherwise one can take $T_1 = \epsilon\mu^{-1}\gamma^{-1}$ and then one gets

$$\begin{aligned}(21.4.113)\quad C\mu h^{-3}\gamma r^2(\rho^2 + r\ell^2) \times \Big(C_0(\mu h)^{-1}(\rho + r\ell^2) + 1\Big) \\ \times h|\log h|(\rho^2 + r\ell^2)^{-1}\mu\gamma \times \ell\Delta \\ = C\mu^2 h^{-2}\gamma^2 r^2 \times \Big(C_0(\mu h)^{-1}(\rho + r\ell^2) + 1\Big)\bar{\ell}\Delta|\log h| \le Ch^{-3} \times (\mu^{1/2}h) \\ = C\mu^{1/2}h^{-2}|\log h|^2\end{aligned}$$

where one copy of $(\mu^{1/2}h|\log h|)^{1/2}$ comes as the width of zone with respect to ξ_3 and another as the width with respect to $|\nabla W_n|$. □

Proposition 21.4.21. *Let condition* $(21.4.50)_1$ *be fulfilled. Then*

(i) The total contribution of $\mathcal{Z}_{\mathsf{inn},I} \cap \{|Z_1| \le \epsilon r\}$ *to the remainder does not exceed* $C\mu^{-1/2}h^{-3}$ *as* $\mu \le h^{-1}|\log h|^{-K}$.

(ii) The total contribution of $\mathcal{Z}_{\mathsf{inn},I} \cap \{|Z_1| \ge \epsilon r\}$ *to the remainder does not exceed*

$$(21.4.114)\qquad C\mu^{-1/2}h^{-3} + C\mu^{5/4}h^{-3/2}|\log h|^{1/2};$$

in particular it is $O(\mu^{-1/2}h^{-3})$ *as* $\mu \le h^{-6/7}|\log h|^{-2/7}$ [29].

[29] Actually We are going to prove later that one can drop logarithmic factors. In this case estimate cannot be improved without either additional assumptions or correction terms.

Again the main part of this asymptotics is given by expression (21.1.2).

Proof. Note that the most delicate is the case $\epsilon r \le \rho \le cr$ when even condition $(21.4.50)_2$ does not yield $(21.4.50)_1$ for W_n [30)]

(a) Consider first $r \le \epsilon_1$, $\rho \asymp r$, $\gamma \asymp \mu^{-1/2} r^{1/2} \rho^{1/2} \asymp \mu^{-1/2} r$.

Let us introduce a scaling function

$$\ell_n = \epsilon r^{-1} |\nabla W_n| + \frac{1}{2}\bar{\ell}, \qquad \bar{\ell} = C(\mu^{1/2} h |\log h|)^{1/2} \tag{21.4.115}$$

and the corresponding subpartition as $\rho \le cr$. Note that actually $\ell \ge \bar{\ell} = C(\mu\gamma r^{-1} h |\log h|)^{1/2}$ is the condition that for time $T_1 = \epsilon\mu^{-1}\gamma^{-1}$ the shift is observable. Then exactly as before

(21.4.116) The total contribution to the remainder of all ℓ-elements with $\ell \ge \bar{\ell}$, $\rho \asymp r$ does not exceed $C\mu^{-1/2} h^{-3}$.

So only $\ell \le \bar{\ell}$ subelements remain to be considered. Note that after rescaling $|\nabla\omega f_2| \asymp 1$ and therefore

(21.4.117) For each $\bar{\ell}$-subelement there exists number $\bar{n}$ such that $|\nabla\omega(V - (2n+1) f_2 \mu h| \ge \bar{\ell}$ as $|n - \bar{n}| \ge C\bar{\ell}(\mu h)^{-1}$ and therefore over each $\bar{\ell}$-element there are no more than $M = C\bar{\ell}(\mu h)^{-1}$ other appropriate indices.

Therefore the total contribution of them to the remainder does not exceed

$$\begin{aligned} &C\mu h^{-3}\gamma r^2(\rho^2 + r\ell^2) \times \Big(C_0(\mu h)^{-1}\bar{\ell} + 1\Big) \\ &\qquad \times h(\rho^2 + r\ell^2)^{-1} \times \mu\gamma\Delta \asymp C\mu h^{-2} r^3 \times \Big(\bar{\ell}^2(\mu h)^{-1} + \bar{\ell}\Big) \end{aligned} \tag{21.4.118}$$

since $\Delta \asymp \bar{\ell}$.

After summation over r-partition one gets $C\mu^{1/2} h^{-3} |\log h| + C\mu h^{-2}\bar{\ell}$ which is exactly (21.4.114).

[30)] However, $(21.4.50)_1$ with the first and the second order derivatives taken only with respect to Λ implies such condition for W_n thus eliminating vicinity of Λ from the contributing more than $C\mu^{-/2} h^{-3}$ to the remainder.

(b) Consider now $r \asymp 1$, $\rho \asymp 1$, $\gamma \asymp \mu^{-1/2}$. One needs to recover an extra factor μh in comparison with estimate of Proposition 21.4.19 and only case $\mu \ge h^{\delta-2/3}$ needs to be considered.

Let us assume first that

$$|\operatorname{Hess}\Big(\frac{V}{f_2}\Big)| \ge \epsilon_0 \tag{21.4.119}$$

and let us fix

$$\zeta \ge \bar{\zeta} = C\big(\mu^{1/2}h|\log h|\big)^{\frac{1}{3}} + \epsilon\mu h \tag{21.4.120}$$

and examine subzone

$$\Omega_\zeta := \big\{\exists n \in \mathbb{Z}^+ : \ |\operatorname{Hess} W_n| \asymp \zeta\big\}. \tag{21.4.121}$$

Let us introduce ζ-admissible partition in this zone; as $r \le \zeta$ one should replace it by r-admissible partition. Note that one can assume that

(21.4.122) For each element of such partition no more than $M := C_0\zeta(\mu h)^{-1}$ magnetic numbers n satisfy condition $|\operatorname{Hess} W_n| \asymp \zeta$.

Indeed, if $|\nabla\omega f_2| \asymp 1$ then arguments of (a) work. Further, if $|\nabla\omega f_2| \le \epsilon_1$ but $|\nabla f_2^{-1}V| \asymp 1$ then $|\nabla\omega V| \asymp 1$ and then $|\nabla W_n| \asymp 1$ for all $n \le C(\mu h)^{-1}$ and everything is easy. So, let both $|\nabla\omega f_2| \le \epsilon_1$ and $|\nabla f_2^{-1}V| \le \epsilon_1$. Then as $|\operatorname{Hess}(\omega f_2)| \asymp 1$ then (21.4.122) is obviously true.

Finally, if $|\operatorname{Hess}(\omega f_2)| \le \epsilon_1$ then (21.4.119) implies that $|\operatorname{Hess}(\omega V)| \asymp 1$ and then $|\operatorname{Hess} W_n| \asymp 1$ for all $n \le C(\mu h)^{-1}$.

Then for each index n described (21.4.122) all the above arguments could be repeated but with

$$\ell = \zeta^{-1}|\nabla W_n| + \frac{1}{2}\bar{\ell}, \qquad \bar{\ell} := C(\zeta^{-1}\mu h|\log h|)^{1/2} \tag{21.4.123}$$

leading to the contribution to the remainder

$$C\mu^2h^{-3}\gamma^2 \times \zeta(\mu h)^{-1} \times h|\log h| \times \bar{\ell}\Delta \asymp C\mu^{1/2}h^{-2}|\log h|^2 \tag{21.4.124}$$

since $\zeta\bar{\ell} = \Delta$.

Therefore we are left with $\zeta = \bar{\zeta}$ and only with indices n satisfying $|\operatorname{Hess} W_n| \le \bar{\zeta}$. Here one needs to consider only subelements with $\ell \le \bar{\ell}$ i.e. with

$$|\nabla W_n \le C\bar{\zeta}^2. \tag{21.4.125}$$

Thus we are looking on $\bar{\zeta}$ partition elements satisfying this condition.

Assume first that $|\operatorname{Hess}(\omega f_2)| \asymp 1$. Let us introduce scaling function

$$\ell = |\nabla \omega f_2| + \frac{1}{2}\bar{\ell}; \tag{21.4.126}$$

then as $\ell \ge \bar{\ell}$ over each such subelement leave no more than $C\bar{\zeta}^2/(\mu h\ell)$ indices n satisfying (21.4.126) and then (21.4.124) is replaced by

$$C\mu^2 h^{-3}\gamma^2 \times \bar{\zeta}^2\ell^{-1}(\mu h)^{-1} \times h|\log h| \times \Delta \asymp Ch^{-3}\Delta\bar{\zeta}^2\ell^{-1}|\log h|; \tag{21.4.127}$$

alternatively one can replace (21.4.124) by

$$C\mu^2 h^{-3}\gamma^2 \times \bar{\zeta}(\mu h)^{-1} \times h|\log h| \times \ell\Delta \asymp Ch^{-3}\Delta\bar{\zeta}\ell|\log h| \tag{21.4.128}$$

where ℓ is a width of of the strip where $|\nabla\omega f_2| \le \ell$.

Comparing (21.4.127) and (21.4.128) one can see that the best choice of $\bar{\ell}$ is $\bar{\ell} = \bar{\zeta}^{1/2}$; then both of them become $Ch^{-3}\Delta\bar{\zeta}^{3/2}|\log h| \asymp C\mu^{1/2}h^{-3}|\log h|^2$.

On the other hand, if $|\operatorname{Hess}(\omega f_2)| \le \epsilon_1$ then due to the arguments proving (21.4.122) this analysis is not needed.

Furthermore, if (21.4.119) is replaced by condition

$$|\nabla\Big(\frac{V}{f_2}\Big)| \ge \epsilon_0 \tag{21.4.129}$$

the analysis is essentially the same, but simpler, with (21.4.121)–(21.4.122) replaced by

$$\zeta \ge \bar{\zeta} = c(\mu^{1/2}h|\log h|)^{\frac{1}{2}} + \mu h, \tag{21.4.130}$$

$$\Omega_\zeta := \big\{\exists n \in \mathbb{Z}^+ : \ |\nabla W_n| \asymp \zeta\big\}, \tag{21.4.131}$$

respectively and in (21.4.123) $\operatorname{Hess} W_n| \asymp \zeta$ is replaced by $|\nabla W_n| \asymp \zeta$.

(c) Finally as $\rho \le \epsilon r$ condition $(21.4.50)_1$ translates for a similar condition to W_n and then one can apply the arguments similar to those of Proposition 21.4.20 which imply (i). We leave easy details to the reader.

(d) Analysis of $\rho^2 \le c(\mu h|\log h|)^{1/2}$ is also easy. Let us introduce ℓ by (21.4.115). Then as $\rho^2 + r\ell^2 \le C\mu^{1/2}h|\log h|$ factor $(\rho^2 + r\ell^2)$ takes care of everything; otherwise factor Δ is retained but factor $(\rho^2 + r\ell^2)$ translates in the end into extra factor $\mu h|\log h$. We leave easy details to the reader. □

Estimates in the Inner Zone. V

Now, the better remainder estimate should be pursued only as

$$\mu \ge h^{-6/7}|\log h|^{-2/7}. \tag{21.4.132}$$

Further, as $q = 0, 1$ the contribution of $\mathcal{Z}_{\mathsf{per}}$ to the remainder is better than the remainder estimates $O(\mu^{-1/2}h^{-3} + \mu^2 h^{-2})$, $O(\mu^{-1/2}h^{-3} + \mu^{3/2}h^{-2-\delta})$ and therefore only $q = 2, 3$ should be considered.

Moreover, it follows from the proof of Proposition 21.4.21 that if summation is taken over $\mathcal{Z}_{\mathsf{per}} \cap \{\mathsf{dist}(x, \Lambda) \le r\}$ then the second term in (21.4.114) would get an extra factor r^4 (from $r^2\gamma^2$) and therefore only zone $\mathcal{Z}_{\mathsf{per}} \cap \{\mathsf{dist}(x, \Lambda) \ge \bar{r} = \mu^{-1/16}|\log h|^{-1/8}\}$ should be considered.

Moreover, then $\bar{\zeta} \asymp \mu h$ (which is much larger than the first term in (21.4.119) and therefore for each $\bar{\zeta}$-partition subelement only one magnetic number n should be considered.

There is a minor problem as $\mu \ge h^{-1}|\log h|^{-K}$ because then contributions of $\{\rho \le \epsilon r\}$ and $\{\rho \ge Cr\}$ were estimated by $\mu^{1/2}h^{-2}|\log h|^K$; however, in this case there also factor r^2 at least coming if looking at $\mathcal{Z}_{\mathsf{per}} \cap \{\mathsf{dist}(x, \Lambda) \le r\}$ and therefore one needs to consider only $\mathcal{Z}_{\mathsf{per}} \cap \{\mathsf{dist}(x, \Lambda) \ge |\log h|^{-K}\}$. Then in the first case condition $(21.4.50)_2$ supplies factor $(\mu^{1/2}h|\log h)^{1/2}|\log h|^K$ which is more than enough to take care of $|\log h|^K$; however, we will just include both cases in the final analysis.

Then repeating all arguments of Subsubsections 14.3.4.3 and 14.3.4.4 with the obvious modifications, due to the presence of r (which is greater than $\bar{r} \ge \bar{r}$) and of $(x_2, \mu^{-1}hD_2)$ in all the operators, one can prove easily the following

Proposition 21.4.22. *Under assumption* $(21.4.50)_2$ *the total contribution of* $\mathcal{Z}_{\mathsf{inn},I} \cap \{|Z_1| \ge \epsilon\}$ *to the remainder does not exceed* $C\mu^{-1/2}h^{-3}$ *while the*

main part of the asymptotics is given by (21.1.2) *plus a correction term*

$$\int \mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x',\tau)\psi(x')\,dx', \tag{21.4.133}$$

where temporarily

$$\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}} := h^{-1}\int\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\big(\bar{\chi}_{T}(t)-\bar{\chi}_{T_0}(t)\big)\big(\Gamma_x(uQ^t_y)\big)\Big)\,d\tau dx_1 \tag{21.4.134}$$

and in this formula one can choose arbitrarily

$$T_0 \in [Ch|\log h|, \epsilon' \mu^{-1/2}\gamma^{-1}], \tag{21.4.135}$$

$$T \ge Cr^{-1/2}(h|\log h|)^{1/2}\big(\max(\big|\xi_3 - k^*|Z_1|\big|, h^{1/2})\big)^{-1} \tag{21.4.136}$$

and pseudodifferential $Q \equiv I$ *in* $\mathcal{Z}_{\mathsf{per}} \cap \{\mathsf{dist}(x,\Lambda) \ge \bar{r}\}$ *and supported in the same zone, slightly inflated.*

In the next section this correction term will be rewritten in a more explicit form.

Conclusion

So, with the remainder estimates described in Theorems 21.5.23, 21.5.28 below asymptotics are derived where the main parts are given by rather implicit formula (21.5.1) below.

The analysis in framework of Theorem 21.5.30 is similar but much simpler. We leave it to the reader.

21.5 Calculations and Main Results

So, according to the previous Section 21.5, with the remainder estimates described there the main part of the asymptotics of $\int e(x,x,0)\psi(x)\,dx$ is given by

$$\sum_{\iota} h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\bar{\chi}_{T_\iota}(t)\Gamma\big(u\psi Q^t_{\iota y}\big)\Big)\,d\tau \tag{21.5.1}$$

where Q_ι form an appropriate partition of unity and $T_\iota \ge T_{(\iota)0}$ also described there.

In this section we will make all the calculations transforming implicit expression (21.5.1) to a more explicit form, namely to

$$\int \mathcal{N}^{\mathsf{MW}}(x,0)\psi(x)\,dx. \tag{21.5.2}$$

There are many methods to make a reduction and we will employ them depending on the power of the magnetic field and non-degeneracy condition.

Also we will calculate expression(21.4.127) in a more explicit form.

21.5.1 Temperate Magnetic Field

Temperate Magnetic Field. I

Recall that we proved that for $\mu \le h^{-\delta}$ the remainder estimate $O(\mu^{-1/2}h^{-3})$ holds while the main part is given by (21.1.2) with $T = \bar{T} := Ch|\log h|$. Now we want to prove the same under a larger upper bound for μ; further, increasing μ we want to recover a remainder estimate consistent with those in Theorems 21.5.23, 21.5.28, 21.5.30 while keeping the main part given by (21.1.2) with $T = \bar{T}$. To do this we need to estimate properly (21.2.23)-like correction term

$$h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\big(\bar{\chi}_{T'}(t)-\bar{\chi}_{T''}(t)\big)\big(\Gamma u Q_y^t\big)\Big)\,d\tau \tag{21.5.3}$$

with $T' = T_0$ derived in Section 21.4 and $T'' = \bar{T}$ [31]. This will let us to rewrite (21.1.2) as a Weyl expression, may be with $O(\mu^2h^{-2})$ correction. Here $Q = \psi$ but in the analysis below we will need more general PDOs.

This expression (21.5.3) is the sum of expressions of the type

$$h^{-1}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\big(\bar{\chi}_{2T}(t)-\bar{\chi}_{T}(t)\big)\Gamma u Q_y^t\Big)\,d\tau = iT^{-1}\Big(F_{t\to h^{-1}\tau}\chi_T(t)\Gamma u Q_y^t\Big)\Big|_{\tau=0} \tag{21.5.4}$$

with $\chi(t) = t^{-1}\big(\bar{\chi}(\frac{1}{2}t)-\bar{\chi}(t)\big)$ and $T = 2^k T''$, k running from 2 to $\lceil \log_2(T'/T'')\rceil$.

Proposition 21.5.1. *Let assumptions of Theorem 21.5.23 below with*

$$h^{-\delta} \le \mu \le \epsilon(h|\log h|)^{-1} \tag{21.5.5}$$

[31] However, in what follows we will consider some other parameters $T' > T''$.

be fulfilled and let Q *be supported in* $\{\mathsf{dist}(x,\Sigma)\le\gamma\}$ *with* $\gamma\ge\mu^{-1/2}$.

Then expression (21.5.3) *with* $T'=\epsilon\mu^{-1}\gamma^{-1}$ *and* $T''=\bar{T}$ *does not exceed* $C\mu^{-1/2}h^{-3}+C\mu^2h^{-2}$ [32].

Proof. Note first that expression (21.5.4) with $T\in[\bar{T},\epsilon\mu^{-1}]$ is negligible due to condition (21.2.92). So, until the end of the proof we can consider $T''=\epsilon\mu^{-1}$.

Further, note that, as Q is supported in $\{|Z_1|^2\ge C\mu h|\log h|\}$ expression (21.5.4) with $T\in[T'',T']$ is negligible as well. Then for a general PDO Q expression (21.5.4) does not exceed $C\mu^2h^{-2}|\log h|$.

Then proposition is proven as $\mu\le c(h|\log h|)^{-2/5}$ and until the end of the proof we can assume that

$$(h|\log h|)^{2/5}\le\mu\le\epsilon(h|\log h|)^{-1}. \tag{21.5.6}$$

Further, as Q is supported in $\{\mathsf{dist}(x,\Sigma)\le\bar{\gamma}_3=c|\log h|^{-1}\}$, expression (21.5.3) would not exceed $C\mu^2h^{-2}\bar{\gamma}_3|\log h|\asymp\mu^2h^{-2}$ and as Q is supported in $\{|Z_1|^2\le c\mu h\}$ expression (21.5.3) would not exceed $C\mu^2h^{-2}$ as well.

So, the only zone in question is

$$\big\{C_0\mu h\le\rho^2\le C\mu h|\log h|\big\}\cap\big\{\mathsf{dist}(x,\Sigma)\asymp\gamma,|Z_1|\asymp\rho\big\} \tag{21.5.7}$$

with $|\log h|^{-1}\le\gamma\le\epsilon$. In this zone one needs to consider expression (21.5.4).

Using precanonical form and decomposition (21.3.16) one can rewrite (21.5.4) as the sum of the similar expressions but with u,Q,Γ replaced by u_n, Q_n and Γ' respectively:

$$iT^{-1}\Big(F_{t\to h^{-1}\tau}\chi_T(t)\Gamma' u_nQ_{ny}^t\Big)\Big|_{\tau=0}. \tag{21.5.8}$$

From the decomposed problem

$$(hD_t-\mathcal{A}_n)u_{nn'}\equiv 0, \tag{21.5.9}$$

$$u_{nn'}|_{t=0}\equiv\delta_{nn'}\delta(x'-y') \tag{21.5.10}$$

one can prove not only that expression (21.5.8) is negligible as $T\ge T_\rho:=Ch\rho^{-2}|\log h|$ but also that its absolute value does not exceed $C\mu h^{-2}\gamma\rho^2\big(h\rho^{-2}T\big)^{-s}$

[32] Term $C\mu^{-1/2}h^{-3}$ is probably of no need in some estimates here and below but it is in the remainder estimate anyway.

as $t \le T_\rho$. The summation of this expression with respect to T from $\epsilon\mu^{-1}$ to ∞ results in $C\mu h^{-2}\gamma\rho^2\bigl(\mu h\rho^{-2}\bigr)^{-s}$.

Further, summation of the latter expression with respect to n results in

$$C\mu h^{-2}\gamma\rho^2\bigl(\mu h\rho^{-2}\bigr)^{-s} \times \rho^2(\mu h)^{-1} = Ch^{-3}\rho^4\bigl(\mu h\rho^{-2}\bigr)^{-s}$$

since no more than $\rho^2(\mu h)^{-1}$ numbers n violate the ellipticity of $\mathcal{A}_n$.

Finally, the summation with respect to $\rho \ge c(\mu h)^{1/2}$ results in $C\mu^2h^{-2}\gamma$. So, one can conclude that expression (21.5.3) with indicated T' and T'' by absolute value does not exceed $C\mu^2h^{-2}$. □

One can improve the above error estimate under nondegeneracy condition.

Proposition 21.5.2. *Let assumptions of Theorem 21.5.23 below and condition* (21.5.6) *be fulfilled. Further, let Q be supported in $\{\mathsf{dist}(x,\Sigma) \le \gamma\}$ with $\gamma \ge \mu^{\delta-1/2}$. Finally, let condition* $(21.4.52)_q$ *with $q = 1, 2$ be fulfilled.*

Then expression (21.5.3) *with $T' = \epsilon\mu^{-1}\gamma^{-1}$, $T'' = \bar{T}$ and Q supported in $\{\mathsf{dist}(x,\Sigma) \le \gamma\}$ does not exceed*

$$C\mu^{-1/2}h^{-3} + C(\mu h)^{2+q/2}|\log h|^K h^{-4}\gamma. \tag{21.5.11}_q$$

Proof. There is no need to consider the inner core. Further, due to Proposition 21.5.1 the case $\mu \le Ch^{-2/5}$ is already covered. So, let us assume that $\mu \ge h^{-2/5}$.

Consider precanonical form (21.3.17) and define $\ell = |\nabla_v^\#$, $\rho = |Z_1^\#|$. Then even as $\rho \le \bar{\rho} := C(\mu h|\log h|)^{1/2}$ but $\ell \ge \bar{\ell} := \bar{\rho}$ one can still trade $T' = \epsilon\mu^{-1}\gamma^{-1}$ to $T'' = \epsilon\mu^{-1}$ because then the shift for time $T \in [\epsilon\mu^{-1}, T'']$ is still observable. Here perturbation terms and their derivatives are at most $C\mu^{-2} \ll \bar{\ell}$.

On the other hand, the total contribution to the remainder of all partition elements with $\rho \le \bar{\rho}$, $\ell \le \bar{\ell}$, $\mathsf{dist}(x,\Sigma) \le \gamma$ does not exceed $C\bar{\rho}^2\bar{\ell}^{2+q}\gamma h^{-4}$ which is exactly expression $(21.5.11)_q$. □

Remark 21.5.3. While the error estimate of Proposition 21.5.1 is as good as we need in the general case, the error estimate of Proposition 21.5.2 is as good as we need under assumption $(21.4.52)_q$ only under some restrictions to μ and/or γ.

Temperate Magnetic Field. II

Now we want to trade $T'' = \epsilon\mu^{-1}\gamma^{-1}$ to a larger value T_0 which provides the remainder estimate derived in Section 21.4. For this we will need to use some canonical forms and thus different zones will be treated separately.

Let us start from the strictly outer zone.

Proposition 21.5.4. *Let assumptions of Theorem 21.5.23 below and condition* (21.5.5) *be fulfilled and let* Q *be supported in* $\mathcal{Z}^*_{\mathsf{out}} \cap \{\mathsf{dist}(x,\Sigma) \le \gamma\}$ *with* $\gamma \ge C_0\mu^{\delta-1/2}$. *Then*

(i) In the general case expression (21.5.3) *with* $T' = T_0$ *and* $T'' = \bar{T}$ *does not exceed*

$$(21.5.12)_q \quad C\mu^{-1/2}h^{-3} + C(\mu h)^{(4+q)/2}h^{-4}\gamma|\log h|^K + C(\mu h)^{(3+q)/2}h^{-4}\gamma^2|\log h|^K$$

with $q = 0$.

(ii) Under assumption $(21.4.52)_q$ *this expression does not exceed* $(21.5.12)_q$.

Proof. One needs to consider only Q supported in a zone in which in the proofs of Propositions 21.5.1 and 21.5.2 term (21.5.3) with $T' = \epsilon\mu^{-1}$ and $T'' = \bar{T}$ is negligible. Then we are going to prove remainder estimate$(21.5.12)_q$ without the second term.

(a) Assume first that Q is supported in $\mathcal{Z}_{(r,\gamma)} \subset \mathcal{Z}^*_{\mathsf{out},I}$.

Let us consider canonical form (21.3.54) and introduce scaling function $\ell = \ell^*$ by (21.4.40) and ℓ as the paragraph below that that. If $\ell \ge 1$ (i.e. $p \ge Cr^{-1}(\mu h)^{-1}$) we do not partition the "final" ball $B(0,1)$ corresponding to the original $(\gamma;\gamma r^{-1})$ element; otherwise we make ℓ-subpartition.

Then on any given ℓ subpartition element the propagation speed with respect to x_3 does not exceed $\mu^{-1}\gamma^{-2}r\ell_p$; further, the propagation speed is of this magnitude as $|\partial_{\xi_3}H_{pn}| \asymp \ell$; in this case the shift for time T is observable as it satisfies the logarithmic uncertainty principle

$$\mu^{-1}\gamma^{-2}\ell \times T \times \ell \ge Cr^2\mu^{-1}h\gamma^{-3}|\log h|.$$

Plugging $T \asymp \mu^{-1}\gamma^{-1}$ we arrive to

$$\ell \ge \bar{\ell}_1 := C\big(\mu h|\log h|\big)^{1/2}. \tag{21.5.13}$$

Similarly, considering propagations with respect to other variables one arrives to the same conclusion as $|\nabla W_{np}| \asymp \ell$.

Note that for given index p and for index n violating ellipticity $\ell \asymp \ell_p = \bigl(\ell^* + r|p-\bar{p}|\mu h\bigr)$ with some $\bar{p} \le C_0(\mu h)^{-1}$.

Further, on each ℓ^* subpartition element condition (21.5.13) is fulfilled for all indices p as $\ell^* \ge \bar{\ell}_1$ and condition (21.5.13) is violated for no more than $C_0\bar{\ell}_1(r\mu h)^{-1}$ ($\gg 1$) indices p as $\ell^* \le \bar{\ell}_1$. Furthermore, for each such index p ellipticity is violated by no more than $C_0\bigl(\bar{\ell}_1^2\gamma(r\mu h)^{-1}+1\bigr)$ indices n.

Therefore the total contribution to (21.5.3) of all singular subelements (i.e. subelements with $\ell^* \le C_0\bar{\ell}_1$) belonging to $\mathcal{Z}_{(r,\gamma)}$ does not exceed

$$C\mu^2h^{-2}\gamma^2r^2 \times \Bigl(\bar{\ell}_1(r\mu h)^{-1}+1\Bigr) \times \Bigl(\bar{\ell}_1^2\gamma(r\mu h)^{-1}+1\Bigr) \times \ell_1^q; \tag{21.5.14}$$

since the second and the third factors do not exceed $C\bar{\ell}_1(r\mu h)^{-1}$ and $C|\log h|^K$ respectively, we get $Ch^{-4}\bigl(\mu h\bigr)^{(3+q)/2}\gamma^2|\log h|^K$. This expression sums with respect to r,γ to the same expression calculated at the maximal values of γ and $r=1$, and multiplied by $|\log h|$, which is exactly expression $(21.5.12)_q$. We increase K as needed.

(b) Zone $\mathcal{Z}^*_{\mathsf{out},II}$ is treated in the same way. □

While estimates $(21.5.11)_q$ with $q \ge 0$ and even $(21.5.12)_q$ with $q \ge 1$ are sufficient for our needs, $(21.5.12)_0$ is not and needs to be improved to $(21.5.12)_q$ with q arbitrarily close to 1:

Proposition 21.5.5. *Let assumptions of Theorem 21.5.23 below and condition* (21.5.5) *be fulfilled and let Q be supported in $\mathcal{Z}^*_{\mathsf{out}} \cap \{\mathsf{dist}(x,\Sigma) \le \gamma\}$ with $\gamma \ge C_0\mu^{\delta-1/2}$. Then expression* (21.5.3) *with $T' = T_0$ and $T'' = \bar{T}$ does not exceed*

$$C\mu^{-1/2}h^{-3} + C\mu^2h^{-2}\gamma|\log h| + C(\mu h)^{(3+q)/2}h^{-4}\gamma^2|\log h|^K \tag{21.5.15}$$

with $q<1$ arbitrarily close to 1.

Proof. Again, one needs to consider only Q supported in a zone $\{|Z_1| \ge C(\mu h|\log h|)^{1/2}\}$ in which in the proof of Proposition 21.5.1 term (21.5.3) with $T' = \epsilon\mu^{-1}$ and $T'' = \bar{T}$ is negligible. Then we are going to prove (21.5.15) without the second term.

(a) Assume first that Q is supported in $\mathcal{Z}_{(r,\gamma)} \subset \mathcal{Z}^*_{\mathsf{out},I}$. One needs to consider only $p \le C_0(r\mu h)^{-1}$.

Let us introduce the scaling function and ℓ-admissible partition the in the "final" ball $B(0,1)$ corresponding to the original $(\gamma;\gamma r^{-1})$ element:

$$\ell^*_k = \ell_k := \min_{n,p} \epsilon \Big(\zeta^{-1} \sum_{j\le k} |\nabla^j \mathcal{A}_{np}|^{(m+1)/(k+1-j)}\Big)^{1/(m+1)} + \bar\ell_k \tag{21.5.16}$$

with $k = m$, $\zeta = \gamma r^{-1}$ and

$$\bar\ell_k = \big(\mu h |\log h|^K\big)^{1/(k+1)}. \tag{21.5.17}$$

As $m = 1$ this function coincides with ℓ^* given by (21.4.40).

Consider first partition m-*singular* groups $((p,n),\mathsf{element})$ i.e. groups such that

$$|\partial^\alpha \mathcal{A}_{np}| \le c\zeta \ell^{k-|\alpha|} \qquad \forall \alpha : |\alpha| \le k \tag{21.5.18}$$

with $k = m$, $\ell = \bar\ell_m$ and $\zeta = \zeta_m$.

Then contribution of all m-singular groups in $\mathcal{Z}_{(r,\gamma)}$ could be estimated in the same manner as before (see (21.5.14)):

$$C\mu^2 h^{-2} r^2 \gamma^2 \times \Big(\zeta \ell^k (\gamma\mu h)^{-1} + 1\Big) \times \Big(\zeta \ell^{k+1} (\mu h)^{-1} + 1\Big) \tag{21.5.19}$$

with the same k, ℓ and ζ as in (21.5.18). Here the second factor estimate the number of indices p involved and the third factor indicates the number of indices n violating ellipticity of $\mathcal{A}_{pn}$ for given p.

The groups which are not k-singular are k-*regular*, i.e. they satisfy (21.5.18) with some $\ell \ge 2\bar\ell_k$ and also

$$|\partial^\alpha \mathcal{A}_{np}| \asymp \zeta_k \ell^{k-|\alpha|} \qquad \text{for some } \alpha : |\alpha| \le k. \tag{21.5.20}$$

Then one can rescale such element to $B(0,1)$. After this rescaling conditions (21.5.18) and (21.5.20) are fulfilled with $\ell := 1$ and $\zeta := \zeta\ell^k$. So, one can apply the same rescaling and partition to it with $(k-1)$ instead of k and with crucial ℓ (separating $(k-1)$-singular from $(k-1)$-regular) equal to $\bar\ell_{k-1}/\ell_k$. This means that if one returns to the original element, the radius of the k-singular element would be $\bar\ell_k$ every time.

This continuation however, has a new property: every time when regular or singular ℓ'-subelements appear inside some regular elements, their relative densities do not exceed ℓ'/ℓ (all the time we refer to the same scale) and therefore the absolute density of all ℓ'-subelements does not exceed $C\ell'\bar{\ell}_m^{-1}$. Therefore the total contribution of all k-singular elements does not exceed (21.5.19) multiplied by $\bar{\ell}_k\bar{\ell}_m^{-1}$

$$C\bar{\ell}_m^{-1}\mu^2h^{-2}r^2\gamma^2 \times \Big(\zeta\ell^{k+1}(\gamma\mu h)^{-1}+\ell\Big)\times\Big(\zeta\ell^{k+1}(\mu h)^{-1}+1\Big) \tag{21.5.21}$$

with $\ell=\bar{\ell}_k$ and $\zeta=\gamma r^{-1}$ since we know that ζ does not increase (even if it is different in the different elements). Then two last factors do not exceed $|\log h|^K$ for sure and the total contribution of all singular $((p,n),\mathsf{subelement})$ groups does not exceed $\bar{\ell}_m^{-1}\mu^2h^{-2}\gamma^2r^2|\log h|^K$ and summation with respect to (r,γ) results in the same expression with the maximal values of γ and $r=1$.

Microlocally this is sound as logarithmic uncertainty principle

$$\bar{\ell}_k^2 \ge \mu^{-1}hr^2\gamma^{-3}|\log h|^K \tag{21.5.22}$$

and to satisfy it for any $\gamma \gg \mu^{-1/2}r^{1/2}$ one needs to take $k=1$.

This leaves us with 1-regular groups and there are three types of them:

(A) With $\zeta\ell^2 \le \gamma r^{-1}\mu h|\log h|^K$, which are covered by the same estimate (21.5.21).

(B) With $\zeta\ell^2 \ge \gamma r^{-1}\mu h|\log h|^K$; for them the shift $\mu^{-1}\gamma^{-2}r\zeta\ell T$ with $T=\mu^{-1}\gamma^{-1}$ is observable since $\mu^{-1}r^2\gamma^{-3}\zeta\ell\times\mu^{-1}\gamma^{-1}\times\ell \ge Cr^2\mu^{-1}h\gamma^{-3}$.

(C) 0-regular groups but then $\mathcal{A}_{pn}$ is just elliptic on the corresponding elements.

(b) Zone $\mathcal{Z}^*_{\mathsf{out},II}$ is treated in the same way. □

Temperate Magnetic Field. III

Recall that in the general case our purpose is the remainder estimate $O(\mu^{-1/2}h^{-3}+\mu^2h^{-2})$.

Proposition 21.5.6. *Let assumptions of Theorem 21.5.23 below be fulfilled. Then*

(i) As

$$h^{-\delta} \le \mu \le \bar{\mu}_0 := h^{\delta-2/5} \tag{21.5.23}$$

$\int e(x,x,\tau)\psi(x)\,dx$ *is given modulo* $O(\mu^{-1/2}h^{-3})$ *by expression* (21.1.2) *with* $T = \bar{T}$.

(ii) Moreover, as

$$h^{\delta-2/5} \le \mu \le h^{\delta-1} \tag{21.5.24}$$

and $\psi(x)$ *is a nice*[33] *function supported in* $\{|x_1| \le \bar{\gamma}_2 = \mu^{-\delta'}\}$ $\int e(x,x,\tau)\psi(x)\,dx$ *is given modulo* $O(\mu^{-1/2}h^{-3} + \mu^2 h^{-3})$ *by expression* (21.1.2) *with* $T = \bar{T}$.

Proof. Combining Propositions 21.4.2, 21.4.4 and 21.5.5 we conclude that this estimate holds in the framework of Statement (i) for ψ supported in $\mathcal{Z}^*_{\mathsf{out}}$ and in the framework of Statement (ii) for ψ supported in $\mathcal{Z}^*_{\mathsf{out}} \cap \{|x_1| \le \mu^{-\delta}\}$.

In Propositions 21.4.11, 21.4.14, 21.4.16 and 21.4.19 it was essentially proven that this estimate holds for ψ supported in the near outer zone $\mathcal{Z}'_{\mathsf{out}}$, inner core, inner bulk zone $\mathcal{Z}_{\mathsf{inn},II}$ and true inner zone $\mathcal{Z}_{\mathsf{inn},I}$ respectively. □

21.5.2 Final Calculations

Final cCalculations. I

Now we want just to calculate (21.1.2) with $T = Ch|\log h|$. It is well-known (see for example Proposition 13.4.20) that

Proposition 21.5.7. *Let* $\mu \le h^{-1+\delta}$ *with arbitrarily small* $\delta > 0$. *Then under assumption* (21.2.92)

(i) Asymptotics holds

$$h^{-1}\int_{-\infty}^{\lambda} \Bigl(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma_x u Q_y^t\Bigr)\,d\tau \sim \sum_{(n,m)\in\mathbb{Z}^{+2}} \kappa_{n,m,Q}(x,\lambda)h^{-4+2m+2n}\mu^{2n} \tag{21.5.25}$$

as $|\lambda| \le \epsilon$, $\bar{T} \le T \le \epsilon\mu^{-1}$.

[33] F.e. $\psi(x) = \psi_1(x_1/\bar{\gamma}_2)\psi_2(x_2,x_3,x_4)$.

(ii) Moreover, as $Q = Q(x)$ $\kappa_{n,0,Q}(x,.) = \kappa_{n,0,I}(x,.)Q(x)$ *and*

$$Th^{-1}\int_{-\infty}^{\infty} \widehat{\bar{\chi}}\big((\lambda-\tau)Th^{-1}\big)\mathcal{N}^{\mathsf{MW}}(x,\tau)\,d\tau \sim \sum_{n\in\mathbb{Z}^+} \kappa_{n,0,I}(x,\lambda)h^{-4+2m+2n}\mu^{2n} \tag{21.5.26}$$

with the Standard Weyl Expression

$$\kappa_{0,0,I}(x,\lambda)h^{-4} = \mathcal{N}^{\mathsf{W}}(x,\tau) \coloneqq \frac{1}{32\pi^2}(2\tau+V)^2\sqrt{g}. \tag{21.5.27}$$

Combining Propositions 21.5.6(i) and 21.5.7(i) we arrive to

Corollary 21.5.8. *Under assumptions* (21.2.92) *and* (21.5.23) *modified asymptotics* (21.2.93) *(with* $\mathcal{N}^{\mathsf{MW}}$ *replaced by* $\mathcal{N}^{\mathsf{W}}$*) holds.*

To prove Theorem 21.5.23 under assumption (21.5.23) it is sufficient to estimate properly

$$\Big|\int \Big(\mathcal{N}^{\mathsf{MW}}(x,\lambda) - \mathcal{N}^{\mathsf{W}}(x,\lambda)\Big)\psi(x)\,d\tau\Big| \tag{21.5.28}$$

which due to Proposition 21.5.7(ii) is equivalent to

$$\Big|\int \Big(\mathcal{N}^{\mathsf{MW}}(x,\lambda) - Th^{-1}\int_{-\infty}^{\infty} \widehat{\bar{\chi}}\big((\lambda-\tau)Th^{-1}\big)\mathcal{N}^{\mathsf{MW}}(x,\tau)\,d\tau\Big)\psi(x)\,d\tau\Big|. \tag{21.5.29}$$

Proposition 21.5.9. *(i) Under assumptions* (21.2.92) *and* (21.5.23) *both expressions* (21.5.28) *and* (21.5.29) *with* $\lambda = 0$ *do not exceed* $C\mu^{-1/2}h^{-3}$*.*

(ii) Under assumptions (21.2.92) *and* (21.5.24) *both expressions* (21.5.28) *and* (21.5.29) *with* $\lambda = 0$ *and a nice*[33)] *function* ψ *supported in* $\{|x_1| \le \bar{\gamma}_2 = \mu^{-\delta'}\}$ *do not exceed* $C\mu^{-1/2}h^{-3} + C\mu^2h^{-2}$*.*

Proof. Proof is standard, based on the same scaling functions and partition as in the proof of Proposition 21.5.5 as ψ is supported in $\mathcal{Z}^*_{\mathsf{out}}$. These arguments work in the other zones as well since we do not need uncertainty principle anymore. We leave the easy details to the reader. □

Now Propositions 21.5.6(i), 21.5.7 and 21.5.9(i) imply

Corollary 21.5.10. *Under assumptions* (21.2.92) *and* (21.5.23) *asymptotics* (21.2.93) *holds.*

Final Calculations. II

In view of Theorem 21.5.23 below nondegeneracy condition should be used only as

$$ch^{\delta-2/5} \le \mu \le Ch^{-1} \tag{21.5.30}$$

with $\delta = 0$. Then, under assumption $(21.4.52)_1$ the remainder estimate derived in section 21.4 is $O(\mu^{-1/2}h^{-3} + \mu^{3/2}h^{-3/2-\delta})$ i.e. it is $O(\mu^{-1/2}h^{-3})$ as $\mu \le h^{\delta-3/4}$ and then there is no need to invoke $(21.4.52)_q$ with $q \ge 2$. However, it makes life much easier and we will do it right now leaving more difficult analysis for the later; so $\delta > 0$ is arbitrarily small in this Subsubsection.

First of all one needs to prove

Proposition 21.5.11. *Let conditions* (21.2.92) *and* $(21.4.52)_q$ *be fulfilled and* $\mu \le h^{\delta-1}$. *Let* ψ *be a nice function supported in* $\{|x_1| \le \gamma\}$ *with* $\mu^{1/2-\delta} \le \gamma \le \epsilon$. *Then* $\int e(x,x,\tau)\psi(x)\,dx$ *is given by expression* (21.1.2) *with* $T = \bar{T}$ *with an error not exceeding* $(21.5.12)_q$.

Proof. Combining Propositions 21.4.7 and 21.5.4 we conclude that the statement holds for ψ supported in $\mathcal{Z}^*_{\mathsf{out}} \cap \{|x_1| \le \gamma\}$.

In Propositions 21.4.14, 21.4.12, 21.4.17 and 21.4.22 it was essentially proven that this estimate holds for ψ supported in the near outer zone $\mathcal{Z}'_{\mathsf{out}}$, inner core, inner bulk zone $\mathcal{Z}_{\mathsf{inn},II}$ and true inner zone $\mathcal{Z}_{\mathsf{inn},I}$ respectively. □

Propositions 21.5.9 and 21.5.7 imply

Corollary 21.5.12. *Let conditions* (21.2.92) *and* $(21.4.52)_q$ *be fulfilled and* $\mu \le h^{\delta-1}$. *Then*

(i) One can define the main part of asymptotics by (21.1.2) *with* $T = Ch|\log h|$ *(or equivalently by* (21.5.26)*) while the the remainder does not exceed* $(21.5.12)_q$ *with* $\gamma = 1$.

(ii) In particular, the remainder is $O(\mu^{-1/2}h^{-3})$ *as*

$$\mu \le \bar{\mu}_q := h^{-\frac{q+1}{q+4}} |\log h|^{-K} \tag{21.5.31}_q$$

under assumption $(21.4.52)_q$ *with* $q \ge 1$*; in particular* $\bar{\mu}_2 = \mu^{-1/2}|\log h|^{-K}$ *and* $\bar{\mu}_3 = \mu^{-4/7}|\log h|^{-K}$.

In what follows we refer to the case of $(21.5.31)_q$ as a *moderate magnetic field*. While we can get rid of logarithmic factors, there is no point to do it right now.

To prove Theorem 21.5.28 one needs to estimate properly expressions (21.5.28) and (21.5.29).

Proposition 21.5.13. *Let conditions* (21.2.92) *and* $(21.4.52)_q$ *be fulfilled and* $\mu \le h^{\delta-1}$. *Let* ψ *be a nice function supported in* $\{|x_1| \le \gamma\}$ *with* $\mu^{\delta-1/2} \le \gamma \le \epsilon$. *Then*

(i) Both expressions (21.5.28) *and* (21.5.29) *do not exceed* $(21.5.12)_q$.

(ii) In particular, as $\gamma = 1$ *both expressions* (21.5.28) *and* (21.5.29) *do not exceed* $C\mu^{-1/2}h^{-3}$ *as* $\mu \le \mu_q$.

Proof. Proof is standard, based on the same scaling functions and partition as in the proof of Proposition 21.5.4 as ψ is supported in $\mathcal{Z}^*_{\mathsf{out}}$. These arguments work in the other zones as well since we do not need uncertainty principle anymore. We leave the easy details to the reader. □

Corollary 21.5.14. *Theorem 21.5.28 below is proven as* $\mu \le \mu_q$.

Final Calculations. III

Now we want to prove Theorems 21.5.23, 21.5.28 below as $\mu \ge \bar{\mu}_q$ defined by (21.5.23), $(21.5.31)_q$ for $q = 0$ and $q \ge 1$ respectively. To do this we need to consider expression (21.1.2) localized in this singular zone because in the regular zone one can always apply (21.1.2) with $Q = I$ even if the expression is not as explicit as in the case $Q = I$.

Proposition 21.5.15. *One can rewrite with the same error as a remainder estimate derived in Section 21.4 expression* (21.1.2) *as*

$$\int \widehat{\mathcal{N}}^{\mathsf{MW}}_{\bar{Q}}(x,0)\psi(x)\,dx \tag{21.5.32}$$

with

$$\widehat{\mathcal{N}}^{\mathsf{MW}}_{\bar{Q}}(x,0) \coloneqq \mathsf{const}\sum_{n,p}\Big(\theta(\mathcal{A}_{pn})\bar{Q}_{pn}\Big)\Big|_{(x'',\xi'')=\Psi^{-1}(x)} \times f_1(x)f_2(x)\mu^2h^{-2} \tag{21.5.33}$$

with exactly the same constant as in $\mathcal{N}^{\mathsf{MW}}$.

Proof. Proof is rather standard: we consider canonical form of the operator and in this form I apply the method of successive approximations with operator in question $\mathcal{A}_{p,n}(x,\mu^{-1}hD_x)$ and unperturbed operator $\mathcal{A}_{p,n}(y,\mu^{-1}hD_x)$. We leave details repeating those in my multiple papers to the reader. □

Remark 21.5.16. Obviously, defining $\widehat{\mathcal{N}}^{\mathsf{MW}}_{\bar{Q}}$ by (21.5.33) but with $\mathcal{A}_{pn}$ replaced by $\mathcal{A}^0_{pn}$ one would get exactly

$$\mathcal{N}^{\mathsf{MW}}_{\bar{Q}}(x,0) \coloneqq \mathsf{const}\sum_{n,p}\theta\Big(V(x)-(2n+1)\mu hf_2(x)-(2p+1)\mu hf_1(x)\Big)f_1(x)f_2(x)\bar{Q}_{pn}\mu^2h^{-2} \tag{21.5.34}$$

where $\mathcal{A}^0_{pn}$ is obtained from $\mathcal{A}_{pn}$ by replacing perturbation terms $\mathcal{B}_{pn}$ by 0.

While one can take $Q=I$ and then $\bar{Q}_{np}$ will be the diagonal elements of ψ transformed according to (21.3.55) expression (21.5.33) is not very explicit either because of presence of $\mathcal{B}_{pn}$ and the similar terms in (21.3.55). Because of this we want to take Q supported in as small zone as possible; the only restriction so far is that $Q\equiv I$ in the singular zone

$$\Omega_{\mathsf{sing}} \coloneqq \big\{\ell^*\le \bar{\ell}_1 = C(r\mu h|\log h|)^{1/2},\quad |p-\bar{p}|\le r^{-1}\bar{\ell}_1(\mu h)^{-1}, \quad |n-\bar{n}|\le C\gamma r^{-1}\bar{\ell}_1^2(\mu h)^{-1}+1\big\}. \tag{21.5.35}$$

Therefore we will take Q supported in this zone (with increased C).

Furthermore, we actually need to consider not expression (21.5.32) but only its correction with respect to what is given by magnetic Weyl formula; namely

$$\int \Big(\widehat{\mathcal{N}}^{\mathsf{MW}}_{\bar{Q}}(x,0)-\mathcal{N}^{\mathsf{MW}}_{\bar{Q}}(x,0)\Big)\psi(x)\,dx. \tag{21.5.36}$$

Final Calculations. IV

Now our goal is to estimate expression (21.5.36). We do it first under assumption $(21.4.50)_q$ (including formally $q=0$) and in the next Subsubsection 21.5.2.5 we improve case $q=0$ in the same manner as Proposition 21.5.5 improves Proposition 21.5.4.

Proposition 21.5.17. *Let norms of perturbation operators do not exceed* $c\varepsilon$*,*

$$\varepsilon \le \mu h \tag{21.5.37}$$

and Q *be supported in* $\mathcal{Z}^*_{\mathsf{out}}$*. Then expression* (21.5.36) *does not exceed*

$$(21.5.38)_q \qquad \begin{cases} C\varepsilon\gamma^2(\mu h)^{(1+q)/2}h^{-4}|\log h|^K & q\ge 2,\\ C\gamma^2\Big(\varepsilon(\mu h)h^{-4}+C\varepsilon^{1/2}(\mu h)^2h^{-4}\Big)|\log h|^K & q=1,\\ C\varepsilon^{1/2}(\mu h)h^{-4}\gamma^2|\log h|^K + C\gamma^2(\mu h)^2h^{-4} & q=0. \end{cases}$$

Proof. Let us introduce a scaling function by (21.4.40). Further, we replace ℓ by $\mathsf{min}(\ell, r)+\varepsilon^{1/2}$. Then in as $\ell=\ell_{pn}\ge C\bar{\ell}_1$, the contribution of all ℓ_{pn} groups to (21.5.36) does not exceed the left-hand expression of (21.5.14) with (ℓ instead of $\bar{\ell}_1 = C(r\mu h|\log h|)^{1/2}$), multiplied by $C\varepsilon\ell^{-2}$:

$$C\mu^2h^{-2}\gamma^2r^{(2-q)_+}\times\Big(\ell(r\mu h)^{-1}+1\Big)\times\Big(\ell^2\gamma(r\mu h)^{-1}+1\Big)\times\ell^{q-2}\varepsilon; \tag{21.5.39}$$

one can prove it easily by considering such elements and integrating by parts if $\ell\ge c\varepsilon^{1/2}$. Here the third factor does not exceed $|\log h|^K$ for sure as $\ell^2\le \mu h|\log h|^K$ and $\gamma\le r^2$.

Then as $q\ge 2$ (21.5.39) sums with respect to ℓ to its value as ℓ reaches its maximum $\mathsf{min}(\bar{\ell}_1, r)$; then it sums with respect to r to the the first line in $(21.5.38)_q$.

As $q=1$ one gets

$$C(\mu h)h^{-4}r\gamma^2\Big(\varepsilon+(\mu h)\varepsilon\ell^{-1}\Big)|\log h|^K;$$

one can estimate $\ell^{-1}\varepsilon$ by $\varepsilon^{1/2}$ as $\ell\ge\epsilon^{1/2}$ and the summation results in the second line in $(21.5.38)_q$.

The same approach works as $q=0$ as well but results in an unwanted now factor $|\log h|^K$ at $C(\mu h)^2h^{-4}\gamma^2$. However, as $\ell\ge\varepsilon^{1/2}|\log h|^{K_1}$ with the

large enough exponent K_1 one can estimate the fourth factor in (21.5.39) by $|\log h|^{-K_1}$ and all summations will result in $C(\mu h)^2 h^{-4}\gamma^2 |\log h|^{K_0-K_1}$ with K_0 independent on K_1; choosing K_1 large enough one gets $C\mu^2 h^{-2}\gamma^2$.

On the other hand, one can estimate contribution of all groups with $\ell \le \varepsilon^{1/2}|\log h|^{K_1}$ does not exceed the product of the first three factors in (21.5.39) with $\ell = \varepsilon^{1/2}|\log h|_1^K$ (since one does not need to use an integration by parts then) which does not exceed after summation with respect to r, γ the third line in $(21.5.38)_q$. □

Recall that the "perturbation" terms in (21.3.50) are

$$\sum_{2q+2l+2j\ge 3} B^{\mathsf{w}}_{qjl} \times \bigl((2p+1)\mu h\gamma\bigr)^q \nu^{2-2q-2j}\hbar^l + \sum_{2k+2q+2m+2s+2l\ge 3} B^{\mathsf{w}}_{kqjmsl} \times \bigl((2n+1)\mu h\bigr)^k \bigl((2p+1)\mu h\gamma\bigr)^q \times \mu^{2-2k-2m-2q-s} h^s r^{-4q-4m-4s}\nu^{-2j}\hbar^l\Bigr).$$

and due to (21.3.51) their norms do not exceed $C\mu^{-2}r\gamma^{-3}$ and thus one can take

$$\varepsilon = C\min(\mu^{-2}\gamma^{-3}, \mu^{-1-\delta}\gamma^{-1}) \tag{21.5.40}$$

as $|x_1| \asymp \gamma$ since $r \le \min(1, \mu^{1-\delta}\gamma^2)$ in $\mathcal{Z}^*_{\mathsf{out}}$. Then condition (21.5.37) translates into

$$\gamma \ge \bar\gamma_2 := C\min\bigl(\mu^{-1}h^{-1/3}, \mu^{\delta-2}h^{-1}\bigr) = \begin{cases} \mu^{-1}h^{-1/3} & \text{as } \mu \le h^{\delta'-2/3},\\ \mu^{\delta-2}h^{-1} & \text{as } \mu \le h^{\delta'-2/3}. \end{cases} \tag{21.5.41}$$

Final Calculations. V

Consider $q = 0$ first. Then plugging (21.5.40) into $(21.5.38)_0$ and taking the sum with respect to γ one gets the same expression calculated as $\gamma = 1$ i.e. $Ch^{-3}|\log h|^K + C\mu^2 h^{-2}$ which is $O(\mu^2 h^{-2})$ as $\mu \ge h^{-1/2}|\log h|^K$. Therefore due to due to Propositions 21.5.6, 21.5.11, 21.5.15 and 21.5.17 we arrive to

Corollary 21.5.18. *Estimate* (21.2.93) *holds as* $\mu \ge h^{-1/2}|\log h|^K$.

Remark 21.5.19. As $h^{\delta-1} \le \mu \le ch^{-1}$ the contribution of zone $\{|x_1| \le h^{\delta'}\}$ to the asymptotics is $O(h^{-4+\delta'}) = O(\mu^2 h^{-2})$ and one does not need to use Propositions 21.5.6 and 21.5.11.

So estimate (21.2.93) remains to be proven as $h^{\delta-2/5} \le \mu \le h^{-1/2}|\log h|^K$; we will do it in the next Subsubsection.

Consider case $q \ge 1$ now. Plugging (21.5.41) into (the first term of) $(21.5.38)_q$ one gets $C(\mu h)^{(1+q)/2} h^{-4} \min(\mu^{-2}\gamma^{-1}, \mu^{2-\delta}\gamma)$; after summation with respect to $\gamma \ge \bar{\gamma}_2$ one the same expression $(21.5.38)_q$ calculated at

$$\gamma = \bar{\gamma}_3 := \max(\bar{\gamma}_2, \mu^{\delta-1/2}) = \begin{cases} \mu^{-1}h^{-1/3} & \text{as } \mu \le h^{\delta'-2/3}, \\ \mu^{\delta-1/2} & \text{as } \mu \le h^{\delta'-2/3}. \end{cases} \tag{21.5.42}$$

Thus we get remainder estimate $C(\mu h)^{(q-1)/2} h^{-8/3}|\log h|^K = O(\mu^{-1/2})$ as $\mu \le h^{\delta'-2/3}$ and $C(\mu h)^{(q-2/2} h^{-5/2}\mu^{-\delta}$ as $\mu \ge h^{\delta'-2/3}$; the latter expression is $O(\mu^{-1/2}h^{-3})$ for $q \ge 2$.

So, as $q = 2$ one gets $O(\mu^{-1/2}h^{-3})$ as an estimate of the contribution $\mathcal{Z}^*_{\mathsf{out}} \cap \{|x_1| \ge \bar{\gamma}_2\}$ into

$$|\int \bigl(e(x,x,0) - \mathcal{N}^{\mathsf{MW}}(x,0)\bigr)\psi(x)\,dx| \tag{21.5.43}$$

On the other hand, due to Propositions 21.5.9, 21.5.11 and 21.5.13 and their corollaries and also Propositions 21.4.20, 21.4.21, 21.4.22 contribution of zone $\mathcal{Z}^*_{\mathsf{out}} \cap \{|x_1| \le \bar{\gamma}_2\} \cup \mathcal{Z}'_{\mathsf{out}} \cup \mathcal{Z}_{\mathsf{inn}}$ to (21.5.42) is estimated by $(21.5.12)_q$ calculated at $\gamma = \bar{\gamma}_3$. One can see easily that the result is $O(\mu^{-1/2}h^{-3})$ as $\mu \le h^{\delta-2/3}$, $q \ge 2$; as $\mu \ge h^{\delta-2/3}$ one gets

$$C\mu^{-1/2}h^{-3} + C(\mu h)^{(2+q)/2}h^{-3}\mu^{\delta+1/2} + C(\mu h)^{(1+q)/2}h^{-3}\mu^{\delta}$$

where the third term is far less than the second one.

Then for $q = 2$ this result does not exceed $C\mu^{-1/2}h^{-3}$ as $\mu \le h^{\delta'-2/3}$ and Proposition 21.5.20(i) below is proven. On the other hand, as $\mu \ge h^{\delta'-4/5}$ $\bar{\gamma}_2$ is below of the bottom of $\mathcal{Z}^*_{\mathsf{out}}$ (see Figure 21.2) and therefore Proposition 21.5.20(ii) is proven:

Proposition 21.5.20. *Let conditions* (21.2.92) *and* $(21.4.50)_2$ *be fulfilled. Then*

(i) As $\mu \le h^{\delta'-2/3}$ *expression* (21.5.42) *does not exceed* $C\mu^{-1/2}h^{-3}$ *and therefore estimate* (21.5.47) *below holds with* $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}} = 0$.

(ii) As $h^{\delta'-2/3} \le \mu \le ch^{-1}$ *contribution of* $\mathcal{Z}^*_{\mathsf{out}}$ *into the* (21.5.43) *does not exceed* $C\mu^{-1/2}h^{-3}$ *while contribution of* $\mathcal{Z}'_{\mathsf{out}} \cup \mathcal{Z}_{\mathsf{inn}}$ *(including the inner core) does not exceed*

$$C\mu^{-1/2}h^{-3} + C\mu^{5/2+\delta}h^{-1}. \tag{21.5.44}$$

Remark 21.5.21. Plugging (21.5.40) into the second term of $(21.5.38)_1$ one gets after summation its value as $\gamma = 1$ i.e. $C\mu h^{-2}|\log h|^K$. Because this and other ugly terms and because estimate under assumption $(21.4.50)_1$ is not a part of my core theorems, We are no more considering $q = 1$, leaving to the reader either to derive some estimate of (21.5.43) or to derive $O(\mu^{-1/2}h^{-3} + \mu^{3/2}h^{-\delta-3/2})$ estimate of some more complicated expression.

Final Calculations. VI

To finish the proof of Theorem 21.5.23 below we need to prove

Proposition 21.5.22. *Let condition* (21.2.92) *be fulfilled,* $h^{\delta-2/5} \le \mu \le Ch^{-1}$ *and* ψ *be a nice function supported in* $\{\bar\gamma_2 = \mu^{-\delta'} \le |x_1| \le \epsilon\}$ *with small enough exponents* $\delta > 0$, $\delta' > 0$. *Then*

$$|\int \Bigl(\widehat{\mathcal{N}}(x,0) - \mathcal{N}^{\mathsf{MW}}(x,0)\Bigr)\psi(x)\,dx| \le C\mu^2h^{-2}. \tag{21.5.45}$$

Proof. One needs to consider only zone $r \ge \mu^{-\delta'}$. Here and in the assumption $\delta' > 0$ could be taken arbitrarily small.

Let me introduce scaling function ℓ_m as in the proof of Proposition 21.5.5 and the corresponding partition. Then due to the same analysis as in Proposition 21.5.5 the contribution of all the m-regular groups with to the remainder is less than $C\mu^{-1/2}h^{-3} + C\mu^{-2-\delta''}h^{-2}$.

So one needs to consider only m-singular $((p,n),\mathsf{subelement})$ groups. Corresponding subelements are of the size $\bar\ell = (\mu h|\log h|^K)^{1/(m+1)}$ and on each such group $|\partial^2\mathcal{A}_{pn}| \le C(\mu h|\log h|)^{(m-1)/(m+1)}$. Consider $\epsilon\mu h$-subpartition. Then for each element of it and for each index p

$$|V(x) - f_1(x)(2p+1)\mu h - f_2(x)(2n+1)\mu h| \ge \epsilon_0\mu h|n - n(p)|$$

and the same is true for this expression perturbed by $\mathcal{B}_{pn} = O(\varepsilon)$, $\varepsilon = \mu^{-2+4\delta'}$, and also for each element there exists index $\bar{p}$ such that

$$|\nabla\big(V(x) - f_1(x)(2p+1)\mu h - f_2(x)(2n+1)\mu h\big)| \ge \epsilon_0 \mu h |p - \bar{p}|$$

as $n = n(p)$.

Then repeating arguments of the proof of Proposition 21.5.20 one can see easily that the left hand expression of (21.5.45) does not exceed

$$C\mu^{-2}h^{-2}\big(1 + \varepsilon \sum_{k\ge 1} (\mu h k)^{-1}\big) \le C\mu^{-2}h^{-2}$$

since $\varepsilon \ll \mu h$ so perturbation does not violate ellipticity of elliptic $\mathcal{A}_{pn}$. □

Thus we arrive to the first of the main results of this chapter

Theorem 21.5.23. *Let F be of Martinet-Roussarie type and condition* (21.2.92) *$V \ge \epsilon_0 > 0$ be fulfilled. Let ψ be supported in the small vicinity of Σ. Then estimate* (21.2.93)

$$|\int \big(e(x,x,0) - \mathcal{N}^{\mathsf{MW}}(x,0)\big)\psi(x)\,dx| \le C\mu^{-1/2}h^{-3} + C\mu^2 h^{-2} \tag{21.2.93}$$

holds where $\mathcal{N}^{\mathsf{MW}}(x,\tau)$ defined by (21.2.94)

$$\mathcal{N}^{\mathsf{MW}}(x,\tau) = (2\pi)^{-2}\mu^2 h^{-2} \sum_{(m,n)\in\mathbb{Z}^{+\,2}} \theta\Big(2\tau + V - (2m+1)\mu h f_1 - (2n+1)\mu h f_2\Big) f_1 f_2 \sqrt{g} \tag{21.2.94}$$

is Magnetic Weyl Expression.

Remark 21.5.24. Without any additional conditions remainder estimate (21.2.93) is sharp. An extra term $O(\mu^2 h^{-2})$ in its right-hand expression is due to the possibility that for some pair $(m,n) \in \mathbb{Z}^{+\,2}$ function $V - (2m+1)\mu h f_1 - (2n+1)\mu h f_2$ identically vanishes in the vicinity of some point of Σ. However, it is not as bad as it could be in the case of constant commensurable f_1, f_2 in which case $V - (2m+1)\mu h f_1 - (2n+1)\mu h f_2$ could vanish up to $\asymp (\mu h)^{-1}$ pairs $(m,n) \in \mathbb{Z}^{+\,2}$ and the remainder estimate would be the worst possible $O(\mu h^{-3})$.

21.5.3 Stronger Magnetic Field

To prove Theorem 21.5.28 below or slightly worse estimate under assumption $(21.4.50)_2$ one needs to improve estimate (21.5.44) of the contribution of $\mathcal{Z}'_{\mathsf{out}} \cup \mathcal{Z}_{\mathsf{inn}}$ (including the inner core) into remainder as $\mu \ge h^{\delta-2/3}$ and to calculate correction term (21.4.133)–(21.4.134) as $\mu \ge h^{-6/7}|\log h|^{-2/7}$.

Stronger Magnetic Field. I

To improve estimate (21.5.44) of the contribution of $\mathcal{Z}'_{\mathsf{out}} \cup \mathcal{Z}_{\mathsf{inn}}$ (including the inner core) into remainder without calculating the correction term one can notice that this expression is given by (21.1.2) with $T = T_0 = Ch|\log h|(\rho^2+\ell^2)^{-1}$ matching one for 2-dimensional magnetic Schrödinger operator (21.3.26) and therefore the asymptotics with the Weyl expression for (21.3.26) which one can rewrite easily as

$$(21.5.46) \qquad (2\pi)^{-2}\mu h^{-3}\theta\Big(2\tau + V - (2n+1)\mu h f_2 - (2p+1)\mu h f_1\Big) f_1 f_2 \sqrt{g}$$

where we skipped perturbation $\mathcal{B}'_n = O(\mu^{-2})$ which under assumption $(21.4.50)_2$ leads to the relative error $O(\mu^{-2})$.

Observe that each expression (21.5.46) is of magnitude μh^{-3} and after integration over $\big(\mathcal{Z}'_{\mathsf{out}} \cup \mathcal{Z}_{\mathsf{inn}}\big)$ it acquires factor $\gamma = \mu^{-1/2} r^{1/2}$ and after summation over n it acquires factor $(\mu h)^{-1}$; so we get a required expression

$$\int \mathcal{N}^{\mathsf{MW}}(x,\tau)\psi(x)\,dx$$

of magnitude $\mu^{\delta-1/2}h^{-4}$ as ψ is supported in $\big(\mathcal{Z}'_{\mathsf{out}} \cup \mathcal{Z}_{\mathsf{inn}}\big)$, an error has magnitude

$$\mu^{\delta-1/2}h^{-4}r^{5/2} \times \mu^{-2} = \mu^{\delta-5/2}h^{-4} = O(\mu^{-1/2}h^{-3}).$$

Therefore we arrive to our final result:

Theorem 21.5.25. *Let F be of Martinet-Roussarie type and condition* (21.2.92) *be fulfilled. Let ψ be supported in the small vicinity of Σ. Then*

(i) Under assumption $(21.4.50)_3$ *estimate*

$$(21.5.47) \quad \Big|\int \big(e(x,x,0) - \mathcal{N}^{\mathsf{MW}}(x,0)\big)\psi(x)\,dx - \int_\Sigma \mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x',\tau)\psi(x')\,dx'\Big| \le C\mu^{-\frac{1}{2}}h^{-3}$$

holds where dx' *is the standard density on* Σ *and* $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x',\tau)$ *is a correction term (which is* $O(\mu^{5/4}h^{-3/2})$*)*

$$\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x') = (2\pi)^{-\frac{3}{2}}\mu h^{-2}\hbar^{\frac{1}{2}}\kappa^{-\frac{1}{2}}\sum_{n\in\mathbb{Z}^+}\big(V-(2n+1)f_2\mu h\big)^{1/8}\phi^{1/4} \times G\Big(\frac{S_0\big(V-(2n+1)f_2\mu h\big)^{3/4}\phi^{-1/2}}{2\pi\hbar}\Big) f_2\sqrt{g'}\Big|_{\Sigma} \tag{21.5.48}$$

where $g' = g^{11}g$ *and* $\sqrt{g'}\,dx'$ *is a Riemannian density on* Σ*,*

$$\phi = \|\nabla_{\mathbb{K}_1} f_1|_{\mathbb{K}_1}\| \tag{21.5.49}$$

where $\|.\|$ *corresponds to the Riemannian metrics* (g_{jk}) *restricted to* $\mathbb{K}_1$*.*

(ii) Under assumption $(21.4.50)_2$ *the left hand expression of estimate* (21.5.47) *does not exceed* $C\mu^{-1/2}h^{-3}+Ch^{\delta-5/2}$ *with an arbitrarily small exponent* $\delta>0$*.*

Remark 21.5.26. One can see easily that only zone $\{\mathsf{dist}(x,\Lambda)\asymp 1\}$ contributes after integration since otherwise $|\nabla_\Sigma\big(\big(V-(2n+1)f_2\mu h\big)\phi^{-2/3}\big)|$ is disjoint from 0.

Proof of Theorem 21.5.25. So far $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ is defined by (21.4.133)–(21.4.134) and is $O(\mu^{5/4}h^{-3/2}|\log h|^{1/2})$; we want to derive a more explicit representation given by (21.5.48).

First of all note that one can rewrite formula (14.4.63)in the form which does not require $g^{jk}|_\Sigma = \delta_{jk}$:

Lemma 21.5.27. *One can rewrite formula* (14.4.63) *as*

$$\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr},d=2} := (2\pi)^{-3/2}h^{-1}\hbar^{1/2}\kappa^{-1/2}V^{(\nu-1)/4\nu}\phi^{1/2\nu}G\Big(\frac{S_0V^{(\nu+1)/(2\nu)}\phi^{-1/\nu}}{2\pi\hbar}\Big)\sqrt{g'}\Big|_{\Sigma} \tag{21.5.50}$$

where $\hbar=\mu^{1/\nu}h$*,* $g'=g_{22}=g^{11}g$ *so* $\sqrt{g'}\,dx_2$ *is a Riemannian density on* Σ*,*

$$\phi = \big(f\cdot\mathsf{dist}(x,\Sigma)^{1-\nu}\big)\big|_{\Sigma}, \tag{21.5.51}$$

$\mathsf{dist}(.,.)$ *is calculated according to the Riemannian metrics* (g_{jk}) *and function* G *is given by* (14.4.64)*.*

Indeed, in this formula V, ϕ and $\sqrt{g'}\,dx_2$ are invariant with respect to change of the coordinates while multiplication of operator by ω^2 is equivalent to substitution $g^{jk} \mapsto \omega^2 g^k$ and $V \mapsto \omega^2 V$ which leads to $f \mapsto \omega^2 f$, $g' \mapsto \omega^{-2} g'$, $\mathsf{dist}(x,\Sigma) \mapsto \omega^{-2}\mathsf{dist}(x,\Sigma)$ (up to a factor equal 1 at Σ) and $\phi \mapsto \omega^{\nu+1}\phi$; so $V^{(\nu+1)/(2\nu)}\phi^{-1/\nu}$ and $V^{(\nu-1)/4\nu}\phi^{1/2\nu}\sqrt{g'}$ are invariants.

As $\nu = 2$ and $\mu \le Ch^{-1}$ (21.4.51)–(21.4.53) become

$$\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr},d=2} \equiv (2\pi)^{-3/2} h^{-1}\hbar^{1/2}\kappa^{-1/2} V^{1/8}\phi^{1/4} G\Big(\frac{S_0 V^{3/4}\phi^{-1/2}}{2\pi\hbar}\Big)\sqrt{g'}\Big|_{\Sigma} \tag{21.5.52}$$

with an error $O(h^{-1}\hbar) = O(\mu^{-1/2}h^{-1})$ and

$$\phi = \|\nabla f\| = \big(\sum_{j,k} g^{jk}\partial_j f \partial_k f\big)^{1/2}\big|_{\Sigma}, \tag{21.5.53}$$

where norm $\|.\|$ corresponds to the Riemannian metrics (g_{jk}). □

Stronger Magnetic Field. II

In the next few statements we are improving remainder estimate of Theorem 21.5.25.

Theorem 21.5.28. *Let F be of Martinet-Roussarie type, conditions* (21.2.92) *and*

$$|\big(\frac{V}{f_2}\big) - (2n+1)\mu h| + |\nabla_{\Sigma}\big(\frac{V}{f_2}\big)| \le \epsilon_0 \implies |\det \mathsf{Hess}_{\Sigma}\big(\frac{V}{f_2}\big)| \ge \epsilon_0 \quad \forall n \in \mathbb{Z}^+ \tag{21.5.54}$$

be fulfilled. Let ψ be supported in the small vicinity of Σ. Then estimate (21.5.47) *holds*

$$|\int \big(e(x,x,0) - \mathcal{N}^{\mathsf{MW}}(x,0)\big)\psi(x)\,dx - \int_{\Sigma} \mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x',\tau)\psi(x')\,dx'| \le C\mu^{-\frac{1}{2}}h^{-3}. \tag{21.5.47}$$

Remark 21.5.29. (i) The first term in the left-hand expression of (21.5.54) is meaningful only in the case of the *intermediate-strong magnetic field* $\mu h \ge \epsilon$; so as $\mu h \le \epsilon$ condition (21.5.54) is equivalent to

$$|\nabla_{\Sigma}\big(\frac{V}{f_2}\big)| \le \epsilon_0 \implies |\det \mathsf{Hess}_{\Sigma}\big(\frac{V}{f_2}\big)| \ge \epsilon_0. \tag{21.5.55}$$

(ii) One can weaken condition (21.5.54) to $(21.4.50)_q$ with $q = 1, 2, 3$ depending on the magnitude of μ; as $\mu h \le \epsilon$ it is equivalent to $(21.4.52)_q$; obviously (21.5.54), (21.5.55) are equivalent to $(21.4.50)_3$, $(21.4.52)_3$ respectively.

(iii) Under stronger condition (21.4.49) the proof of Theorem 21.5.28 is much easier; as $\mu h \ll 1$ this condition is equivalent to

$$|\nabla_{\Sigma}\Big(\frac{V}{f_2}\Big)| \ge \epsilon_0. \tag{21.5.56}$$

(iv) In the very special cases (see Subsection 21.5.4) one can derive remainder estimate (21.5.47) without condition (21.5.54) or even $(21.4.50)_1$. Then the correction term could be larger, up to $O(\mu^{1/4}h^{-5/2})$.

Stronger Magnetic Field. III

Theorem 21.5.30. *Let F is of Martinet-Roussarie type and condition* (21.4.49) *be fulfilled. Then as ψ is supported in the small vicinity of $\Sigma \cap \{V = 0\}$ asymptotics* (21.5.47) *holds with $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x', \tau) = 0$.*

Proof. This proof cannot be obtained by the simple rescaling since $f_2\mu$ and $f_1\mu/|x_1|$ scale differently. However, arguments of Sections 21.2, 21.4 and 21.5 work with little or no modifications:

(i) *Modifications in Section 21.2*

(a) Propositions 21.2.6–21.2.10 do not require condition (21.2.92) while Proposition 21.2.11 holds under assumption (21.4.49) instead of (21.2.92) where $\bar{T} = Ch|\log h|/\ell$ with

$$\ell = \epsilon \max\big(|x_1|, |V|\big) + \hat{\ell}, \qquad \hat{\ell} = C\mu h|\log h|, \tag{21.5.57}$$

instead of $\bar{T} = Ch|\log h|$.

(b) Further, Propositions 21.2.12 and 21.2.13(i) do not require assumption (21.2.92), and Proposition 21.2.13(ii) holds with modified $\bar{T}$ under assumption (21.4.49). Therefore Proposition 21.2.14 also remains true: one needs to estimate $|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(u\psi_y Q_y^t)|$ as $T = \bar{T}$ and this is done easily as $\ell \ge 3\hat{\ell}$ by rescaling and standard Schrödinger operator analysis; as $\ell \ge 3\hat{\ell}$ it can be done easily by more crude approach as well.

(c) Furthermore, Propositions 21.2.16 and 21.2.17 do not require assumption (21.2.92) and due to the previous modifications arguments of Proposition 21.2.18 lead to the same estimate $O(\mu^{-1/2}h^{-3})$ for the contribution of $\mathcal{Z}_{\mathsf{out}}$ to the remainder estimate under assumption (21.4.49).

(d) Finally, Propositions 21.2.19–21.2.20 also do not require assumption (21.2.92) and Proposition 21.2.21 holds under assumption (21.4.49) instead of (21.2.92); Proposition 21.2.22 also does not require (21.2.92) and Propositions 21.2.23 and 21.2.24 hold under assumption (21.4.49).

(ii) *Modifications in Section 21.3.* Section 21.3 does not require condition (21.2.92) at all.

(iii) *Modifications in Section 21.4*

(a) In Section 21.4 analysis in the strictly outer zone $\mathcal{Z}^*_{\mathsf{out}}$ (Subsection 21.4.1) leading to Propositions 21.4.2 and 21.4.4 does not require condition (21.2.92); the analysis and condition (21.4.49) leads to Proposition 21.4.6 which is the worthy substitution for Proposition 21.4.7.

(b) In the arguments of Subsections 21.4.2 and 21.4.3 in the absence of condition (21.2.92) one again should pick up $\bar{T} = Ch|\log h|/\ell$ with ℓ defined by (21.5.57); in comparison with the arguments of these Subsections the factor ℓ^{-1} appears but it is compensated by a factor $(|W|+\mu h)$ which appears due the change of range of factors ρ^2 or $(2n+1)\mu h$. This would add an extra term $C\mu^{1/2}h^{-2}|\log h|$ to the remainder estimate which is subordinate as $\mu \le C(h|\log h|)^{-1}$.

(c) On the other hand, as $\mu \ge (h|\log h|)^{-1}$ we can take $\epsilon\ell$-admissible partition with respect to x' and note that as $W \le \epsilon_1\mu h$ at some partition element, it will be classically forbidden and therefore its contribution to the asymptotics would be 0; moreover, with the rescaling method I can in the right-hand expression of estimates and using condition (21.4.49) we can take $\bar{T} = Ch$ there thus getting rid of the logarithmic factor:

Proposition 21.5.31. *The total contribution of zones $\mathcal{Z}'_{\mathsf{out}}$ and $\mathcal{Z}_{\mathsf{inn}} \setminus \mathcal{Z}_{\mathsf{per}}$ to the remainder does not exceed $C\mu^{-1/2}h^{-3}$ under assumption* (21.4.49) *while the main part of asymptotics is given by* (21.1.2) *with $T = Ch|\log h|$.*

Moreover, in the vicinity of $\{W = 0\}$ condition (21.4.49) also reads as

$$|\nabla_{\Sigma}\big(V\phi^{-3/2}\big)| \ge \epsilon_0 \tag{21.5.58}$$

which kills periodicity for all n not just for all n but one as it would be the case far from $\{W = 0\}$ thus leading to

Proposition 21.5.32. *The total contribution of zone $\mathcal{Z}_{\mathsf{per}}$ to the remainder does not exceed $C\mu^{-1/2}h^{-3}$ under assumption* (21.4.49) *while the main part of asymptotics is given by* (21.1.2) *with $T = \bar{T}$ modified as above and no correction term is needed.*

(iv) *Modifications in Section 21.5.*

(a) In Section 21.5 condition (21.4.49) allows me to trade easily $T = Ch|\log h|$ to $T = \epsilon\mu^{-1}$ to $T = \epsilon\mu^{-1}\gamma^{-1}$ to $T = \epsilon\mu\gamma^2$ as each next expression is larger than the previous one thus leading to Theorem 21.5.30 as $\mu \le h^{\delta-1}$ covering everything by a moderate magnetic field approach.

(b) Alternatively as $\mu \ge h^{\delta-1}$ the strong magnetic field approach works well: in $\mathcal{Z}^*_{\mathsf{out}}$ one can just skip $O(\mu^{-2}\gamma^{-3})$ perturbation thus replacing $\widehat{\mathcal{N}}^{\mathsf{MW}}$ by $\mathcal{N}^{\mathsf{MW}}$ as in Proposition 21.5.17 with an error well below $\mu^{-1/2}h^{-3}$; similarly in $\mathcal{Z}'_{\mathsf{out}} \cup \mathcal{Z}_{\mathsf{inn}}$ one can just skip $O(\mu^{-2})$ perturbation thus getting magnetic Weyl expression again.

We leave easy details to the reader. □

21.5.4 Remarks

Special Case

Consider the special case of operator A defined as $A_I + A_{II}$ where A_I, A_{II} are operators of type (21.1.1) in variables $x_I = (x_1, x_2)$ and $x_{II} = (x_3, x_4)$ respectively with magnetic intensities $f_1(x_I) = |x_1|$ and $f_2(x_{II}) = 1$:

$$A_I = h^2D_1^2 + \big(hD_2 - \mu x_1^2/2\big)^2 - 1, \tag{21.5.59}$$

and

$$A_{II} = h^2D_3^2 + \big(hD_4A - \mu x_3\big)^2. \tag{21.5.60}$$

One can separate variables and prove spectral asymptotics with the remainder estimate $h^{-2}R_1$ where R_1 is the remainder estimate for operator $A_I + \tau$ for the worst possible $\tau > 0$. According to Chapter 14 $R_1 = O(\mu^{-1/2}h^{-1})$.

Then the main part of asymptotics is exactly as in estimate (21.5.47) with the correction term is given by (21.5.48) and this term is $O(h^{-3}\hbar^{1/2})$ [34].

The leading part of the correction term is rather irregular: it is equal to $(2\pi)^{-3/2}h^{-3}\hbar^{1/2}\kappa^{-1/2}I_n$ with

$$I(\varepsilon,\hbar) := \tag{21.5.61}$$

$$\sum_{n\in\mathbb{Z}^+}\big(V-(2n+1)f_2\varepsilon\big)^{1/8}\phi^{1/4}G\Big(\frac{S_0\big(V-(2n+1)f_2\varepsilon\big)^{3/4}\phi^{-1/2}}{2\pi\hbar}\Big)f_2\sqrt{g'}\Big|_{\Sigma}\varepsilon$$

with $\hbar=\mu^{1/2}h\ll\varepsilon=\mu h\ll 1$.

Obviously that $I_n=O(1)$ and but it is not clear that no better estimate like $I_n=O(\varepsilon^\sigma)$ is possible; one can see easily that $\sup_{\varepsilon'\asymp\varepsilon,\hbar'\asymp\hbar}|I(\varepsilon',\hbar')|\ge c^{-1}\varepsilon$.

We hope that some readers will be able make a numerical experiments.

About Term $C\mu^2h^{-2}$ in the remainder estimate

Is term $O(\mu^{-2}h^{-2})$ really needed in the general case? The answer most likely is positive. Consider operator A defined as A_I+A_{II} where

$$A_I=h^2D_1^2+\big(hD_2-\mu x_1^2/2\big)^2-1-kx_1 \tag{21.5.62}$$

and A_{II} is given by (21.5.60). Then

$$W_{pn} := \tag{21.5.63}$$

$$V-(2n+1)f_2\mu h-(2p+1)f_1\mu h=1+kx_1-(2n+1)\mu h-(2p+1)|x_1|\mu h$$

and as $k=(2p+1)\mu h$, $1=(2n+1)\mu h$ then $\{W_{pn}=0\}=\{x_1>0\}$.

While it does not mean that $\{\mathcal{A}_{pn}=0\}=\{x_1>0\}$ because of perturbation, we believe that perturbing slightly A_I one get achieve the latter result. We leave it to the reader.

21.5.5 Strong and Superstrong Magnetic Field Cases

Consider now the case $\mu h\ge C_0$ with sufficiently large C and operator (21.1.1) with V replaced by $V+\mu h\mathfrak{z}_2f_2+\mu h\mathfrak{z}_1$ [35]:

$$A=\frac{1}{2}\Big(\sum_{j,k}P_jg^{jk}(x)P_k-V-\mu h\mathfrak{z}_2f_2-\mu h\mathfrak{z}_1\Big). \tag{21.5.64}$$

[34] So one loses factor μh in comparison with Theorem 21.5.28 in the estimate of the correction term.

[35] Otherwise we would not get non-trivial results, see Problem 21.5.33(i).

Easy Analysis

We believe that the following problems are not very challenging:

Problem 21.5.33. Prove that

(i) If $\mathfrak{z}_2 < 1$ and $\mu h \ge C_0 = C_0(\mathfrak{z}_2)$, then $e(x,y,0) = O(\mu^{-s})$ for $x, y \in \mathcal{U}$, where $\mathcal{U} = \mathcal{U}(\mathfrak{z}_1, \mathfrak{z}_2)$ is the small vicinity of Σ.

(ii) If $\mathfrak{z}_2 = 1$, $\mathfrak{z}_1 < 1$ and $C_0 h^{-1} \le \mu \le ch^{-2}$, where $C_0 = C_0(\mathfrak{z}_1)$ then $e(x,y,0) = O(\mu^{-s})$ for $x, y : \mathsf{dist}(x,\Sigma) + \mathsf{dist}(y,\Sigma) \ge C(\mu h)^{-1}$.

(iii) If $1 < \mathfrak{z}_2 \notin 2\mathbb{Z}^+ + 1$ and $\mu h \ge C_0 = C_0(\mathfrak{z}_2)$, then in the vicinity of Σ, dividing operator by μh we reduce problem to one with $\mu_{\mathsf{new}} = \mu^{\frac12} h^{-\frac12}$ and $h_{\mathsf{new}} = \mu^{-\frac12} h^{\frac12}$ satisfying conditions of Theorem 21.5.28.

The analysis in the case $\Lambda = \emptyset$ seems to be reasonably challenging:

Problem 21.5.34. Let $\Lambda = \emptyset$.

(i) Investigate the case of $\mu h \ge C_0$, $\mathfrak{z}_2 = 1$, $\mathfrak{z}_1 < 1$. Prove that the the remainder estimates of Chapter 14, multiplied by μh^{-1} also hold: namely,

(ia) Let $C_0 h^{-1} \le \mu \le C_1 h^{-2}$. Then the main part has a magnitude of $\mu^{-1} h^{-3} \times \mu h^{-1} = h^{-4}$, and under appropriate assumptions the remainder is $O(\mu^{-\frac12} h^{-1} \times \mu h^{-1}) = O(\mu^{\frac12} h^{-2})$.

(ib) Let $\mu \ge C_1 h^{-2}$. Then the main part is 0 and the remainder is $O(\mu^{-s})$.

(ii) Investigate the case of $\mu h \ge C_0$, $\mathfrak{z}_2 = 1$, $\mathfrak{z}_1 \ge 1$. Prove that the the remainder estimates of Chapter 14, multiplied by μh^{-1} also hold: namely,

(iia) Let $C_0 h^{-1} \le \mu \le C_1 h^{-2}$. Then the main part has a magnitude of $\mu h^{-1} \times \mu h^{-1} = \mu^2 h^{-2}$, and under appropriate assumptions the remainder is $O(\mu^{-\frac12} h^{-1} \times \mu h^{-1}) = O(\mu^{\frac12} h^{-2})$.

(iib) Let $\mu \ge C_1 h^{-2}$. Then the main part has a magnitude of $\mu^2 h^{-2}$, and under appropriate assumptions the remainder is $O(1 \times \mu h^{-1}) = O(\mu h^{-1})$.

(iii) In the framework of (i) and (ii) write down main term of the asymptotics in the terms of the eigenvalue counting function of operator (14.8.9):

$$L(\eta;\mathfrak{z}) = D_1^2 + \bigl(\eta - x_1^2/2\bigr)^2 - \mu h x_1 \mathfrak{z}. \tag{21.5.65}$$

(iv) Investigate the case of $\mu h \ge C_0$, $\mathfrak{z}_2 = 2\bar{n}+1$, $1 \ge \bar{n} \in \mathbb{Z}^+$. Then we have a mixture of operators from (i) or (ii) (with $n = \bar{n}$) and Problem 21.5.33(iii) (with $n < \bar{n}$).

Difficult Analysis

Assume now that $\Lambda \ne \emptyset$.

The following problems seem to be extremely challenging:

Problem 21.5.35. Let $\Lambda \ne \emptyset$, $\mathfrak{z}_2 = 1$. Prove Statements (ia) and (iia) of Problem 21.5.34.

In particular, use reduction as in Section 21.3 albeit with Figure 21.1 modified by $\mu^{-1} \mapsto \mu^{-\frac{1}{2}} h^{\frac{1}{2}}$.

Problem 21.5.36. Let $\Lambda \ne \emptyset$, $\mathfrak{z}_2 = 1$, $\mathfrak{z}_1 < 1$ and $\mu \ge C_1 h^{-2}$.

(i) Recall that in Chapter 14 important role was played by $W = V/\phi_1$ restricted to Σ, with $\phi_1 = |\{Z_1, f_1\}| \asymp r$ near Λ where $r(x) = \mathsf{dist}(x, \Lambda)$ [36]. Namely, the spectral parameter in the spectral projector of $L(\eta; \mathfrak{z})$ is $V/(\phi_1 \mu h^2)$.

Then $e(x, y, 0) = O(\mu^{-s})$ for all x, y with $r(x) + r(y) \ge C_1/(\mu h^2)$. Therefore domain where $e(x, y, 0)$ is not negligible shrinks not only towards Σ, but also towards Λ. Then the contribution of $\{x\colon r(x) \asymp r\}$ to the main term is $\asymp \mu h^{-3} r^2$ and the magnitude of main term is its value as $r = (\mu h^2)^{-1}$, i.e. $\asymp \mu^{-1} h^{-7}$.

(ii) Prove that the contribution of $\{x\colon r(x) \asymp r\}$ to the remainder is $O(\mu^{\frac{1}{2}} h^{-2} r^{\frac{1}{2}})$ [37] and the remainder is is its value as $r = (\mu h^2)^{-1}$, i.e. $O(h^{-3})$.

(iii) Then the main term is larger than the remainder estimate if and only if $\mu \lesssim h^{-4}$.

(iv) Prove that $e(x, y, 0) = O(\mu^{-s})$ for $\mu \ge C_2 h^{-4}$.

(v) As $\mu \le C_2 h^{-4}$ derive asymptotics with the better remainder esrimate than in Statement (iii) with the main part referring to the toy-model (21.2.41). Do we need another microhyperbolicity type assumption?

Problem 21.5.37. Derive spectral asymptotic if either $\mathfrak{z}_2 > 1$ or $\mathfrak{z}_2 = 1$, $\mathfrak{z}_1 \ge 1$. Prove Statement (iib) of 21.5.34.

[36] Indeed, see (21.2.41) for the toy-model.

[37] Cf. Problem 21.5.34(ia).

Etc

The following problem seems to be interesting as well:

Problem 21.5.38. Consider genuine Schrödinger-Pauli and Dirac operators

(i) In the framework of Chapter 14.

(ii) In the framework of this chapter.

Chapter 22

Generic 4D-Schrödinger Operator with the Strong Magnetic Field

22.1 Introduction

Sharp spectral asymptotics for multidimensional Magnetic Schrödinger were obtained in Chapters 19 and 20 in full- and non-full-rank cases respectively[1)]. The results, as one could expect from the analysis of 2- and 3-dimensional cases, were rather different.

However, there are two problems with these results: first, the rank of the magnetic intensity matrix there was supposed to be constant which is not necessarily the case even if only generic magnetic fields are considered; second, while both the maximal rank[2)] and microhyperbolicity[3)] conditions are stable with respect to the small perturbations, they both are not generic in the sense that even in a small but fixed domain a general Magnetic Schrödinger operator is not necessarily approximated by operators satisfying any of these conditions: exactly in the same way as stationary points are not necessarily removable.

However, the analysis of Chapters 19 and 20 leads us to the following conjecture for even-dimensional Magnetic Schrödinger operators:

[1)] We mean the rank of magnetic intensity matrix (F_{jk}).

[2)] We mean that the rank of (F_{jk}) is $2\lfloor d/2 \rfloor$ at each point.

[3)] Which was assumed in Chapter 19.

Conjecture 22.1.1. *As $\mu \le ch^{-1}$ the main part of asymptotics is given by $\mathcal{N}^{\mathsf{MW}}$ while*

(i) For fixed (g^{jk}) and general $(V_j), V$ the remainder estimate is $O(\mu h^{1-d})$.

(ii) For fixed (g^{jk}) and generic (V_j) (i.e. $(V_1, \dots, V_d) \notin \mathfrak{A}_g$ where $\mathfrak{A}_g$ is nowhere dense closed set) and general V the remainder estimate is $O(\mu^{-1}h^{1-d} + \mu^{d/2}h^{-d/2})$ provided (F_{jk}) has the full rank everywhere and $O(\mu^{-1/2}h^{1-d} + \mu^{d/2}h^{-d/2})$ otherwise.

(iii) For fixed (g^{jk}) and generic (V_j, V) (i.e. $(V_1, \dots, V_d; V) \notin \mathfrak{B}_g$ where $\mathfrak{B}_g$ is nowhere dense closed set) the remainder estimate is $O(\mu^{-1}h^{1-d})$ provided (F_{jk}) has the full rank everywhere and $O(\mu^{-1/2}h^{1-d})$ otherwise.

While Statement (i) is trivial, (ii) and (iii) are rather difficult and our goal is rather limited: to prove them as $d = 4$. For $d = 2$ it was done in Chapter 14. Further, in the small vicinity of the set $\{x : \mathsf{rank}(F_{jk})(x) \le 2\}$ it was done in Chapter 21 Thus, we will need to investigate the case of $\mathsf{rank}(F_{jk}) = 4$ at every point.

So, operator in question is

$$A = \frac{1}{2}\Big(\sum_{j,k} P_j g^{jk}(x) P_k - V\Big), \qquad P_j = D_j - \mu V_j \tag{22.1.1}$$

with smooth[4)] symmetric positive definite matrix (g^{jk}) and smooth real-valued potentials $(V_1, \dots, V_d; V)$ and $\mu \gg 1$, $h \ll 1$. Assuming that A is self-adjoint, let $E(\tau)$ be the spectral projector of A and $e(x, y, \tau)$ be its Schwartz' kernel. These assumptions are fulfilled during the whole Chapter.

Magnetic field is characterized by a skew-symmetric matrix (F_{jk}), $F_{jk} = \partial_j V_k - \partial_k V_j$ and more precisely by $(F^j_k) = (g^{j*})(F_{*k})$ and its eigenvalues $\pm i f_j$, $f_j \ge 0$. As $d = 4$ these are f_1 and f_2.

It is proven in Martinet [1] that

(22.1.2) For generic $(V_1, \dots, V_4)$ f_1 and f_2 do not vanish simultaneously.

Remark 22.1.2. (i) If one of f_1, f_2 vanishes then (for generic $(V_1, \dots, V_4)$ locally situation of Chapter 21 occurs and the remainder estimate is $O(\mu^{-1/2}h^{-3})$ for generic V and $O(\mu^{-1/2}h^{-3} + \mu^2 h^{-2})$ for general V (detailed assumptions

[4)] Smooth means either infinitely smooth or belonging to $\mathscr{C}^K$ with large enough K.

see in Chapter 21) and we exclude this case from the analysis, assuming that

$$f_1 \ge \epsilon_0,\ f_2 \ge \epsilon_0. \tag{22.1.3}$$

(ii) As condition (22.1.4) below is fulfilled and $\mu h \ge c$ then the main part of asymptotics is 0 and the remainder estimate is $O(\mu^{-s})$ with an arbitrarily large exponent s (see f.e. Chapter 13) and therefore in what follows we assume that

$$1 \le \mu \le ch^{-1}. \tag{22.1.4}$$

Our goal is to consider 4-dimensional Schrödinger operator with the generic non-degenerating magnetic field and for a generic potential derive spectral asymptotics with the remainder estimate $O(\mu^{-1}h^{-3})$ and the principal part $\asymp h^{-4}$ where $h \ll 1$ is Planck constant and $\mu \gg 1$ is the intensity of the magnetic field. For general potentials remainder estimate $O(\mu^{-1}h^{-3} + \mu^2 h^{-2})$ will be achieved.

Plan of the Chapter

Section 22.2 is devoted to the geometry (discussion of what is the generic case) and the preliminary analysis in the cases when results of Chapter 19 imply Theorem 22.6.4 immediately.

In Section 22.3 we tackle the weak magnetic field case ($\mu \le h^{-\delta_0}$ with sufficiently small exponent $\delta_0 > 0$) proving the standard Weyl formula with the remainder $O(\mu^{-1}h^{-3})$.

Sections 22.4 and 22.5 are devoted to the case when $h^{-\delta_0} \le \mu \le ch^{-1}$ with arbitrarily small exponent $\bar{\delta} > 0$ and one can reduce operator to the microlocal canonical form. More precisely, in Section 22.4 we prove asymptotics with the announced remainder estimates but with the implicitly given main part

$$h^{-1}\sum_{\iota}\int_{-\infty}^{0}\Big(F_{t\to h^{-1}\tau}\bar{\chi}_{T_\iota}(t)\Gamma u Q_{\iota y}^{t}\Big)\,d\tau \tag{22.1.5}$$

where u is the Schwartz' kernel of the propagator $e^{ih^{-1}tA}$ and Q_ι are pseudodifferential partition elements (see details in my previous papers); in this proof estimate of

$$|\Big(F_{t\to h^{-1}\tau}\bar{\chi}_{T_\iota}(t)\Gamma u Q_{\iota y}^{t}\Big)| \tag{22.1.6}$$

plays the crucial role.

In Section 22.5 we replace (22.1.6) by the standard Magnetic Weyl formula and estimate an error arising from this.

Finally, in Section 22.6 we consider the case when condition

$$V \ge \epsilon_0 > 0 \tag{22.1.7}$$

is violated.

22.2 Geometry and Preliminary Analysis

22.2.1 Geometry

Proposition 22.2.1. *Let either g^{jk} be fixed and then V_j be generic, or, alternatively, V_j be fixed and then g^{jk} be generic. Further, let condition* (22.1.3) *be fulfilled. Then*

(22.2.1) $\Sigma := \{x\colon f_1(x) = f_2(x)\}$ *is a smooth 2-dimensional manifold:* $\Sigma = \{x\colon v_1(x) = v_2(x) = 0\}$ *with appropriate smooth functions v_1 and v_2 such that ∇v_1 and ∇v_2 linearly independent at any point of Σ and*

$$\operatorname{dist}(x,\Sigma) \asymp |f_1 - f_2| = 2(v_1^2 + v_2^2)^{1/2}. \tag{22.2.2}$$

Furthermore, $f_1 + f_2$ and $(f_2 - f_1)^2$ are smooth functions.

(22.2.3) *Consider symplectic form corresponding to F_{jk}*

$$\omega = \frac{1}{2}\sum_{jk} F_{jk}\,dx_j \wedge dx_k; \tag{22.2.4}$$

then $\omega|_\Sigma$ is either non-degenerate or it is generic degenerate[5] *i.e. $\Theta_1 := \{x \in \Sigma,\ \{v_1, v_2\} = 0\}$ is a submanifold of dimension 1 and $\nabla_\Sigma\{v_1, v_2\}$ is disjoint from 0 at Θ_1.*

(22.2.5) *Function $f_1 f_2^{-1}$ has only non-degenerate critical points outside of Σ and near Σ $|\nabla(f_1 f_2^{-1})|$ is disjoint from 0.*

Proof. We leave an easy proof to the reader. □

[5] Exactly as Σ was in Chapter 14.

22.2.2 Microhyperbolicity Condition

Preliminary Analysis

First, let me discuss the case when magnetic intensities f_1 and f_2 are disjoint. Then they are smooth. It still does not exclude third-order resonances $f_1 = 2f_2$ and $f_2 = 2f_1$ [6].

Then there are two cyclotron movements and the drift with the velocity $O(\mu^{-1})$ which we want to calculate. Let

$$p_j = \xi_j - \mu V_j. \tag{22.2.6}$$

Note that

$$x'_j := x_j - \mu^{-1} \sum_l \phi^{jl} p_l, \tag{22.2.7}$$

$$(\phi^{**}) := (F_{**})^{-1} \tag{22.2.8}$$

satisfy

$$\{p_k, x'_j\} = -\mu^{-1} \sum_l \{p_k, \phi^{jl}\} p_l \tag{22.2.9}$$

and therefore

$$\begin{aligned} &\Big\{\frac{1}{2}\sum_{k,m} g^{km} p_k p_m, x'_j\Big\} \\ &\qquad = \frac{1}{2}\sum_{k,m} \{g^{km}, x'_j\} p_k p_m - \mu^{-1} \sum_{k,m,l} g^{km} \{p_k, \phi^{jl}\} p_m p_l \\ &\qquad = \mu^{-1} \sum_{k,m,l} \Big(\frac{1}{2}\phi^{jl}\{p_l, g^{km}\} - \{p_l, \phi^{jk}\} g^{ml}\Big) p_k p_m. \end{aligned} \tag{22.2.10}$$

Proposition 22.2.2. *Let $f_1 \neq f_2$ in Ω; then one can correct*

$$x'_j \mapsto x''_j := x'_j - \frac{1}{2}\mu^{-1} \sum_{k,m} \beta^{jkm} p_k p_m \tag{22.2.11}$$

[6] Higher-order resonances are not a problem according to Chapter 19, at least under microhyperbolicity condition.

so that

$$\Big\{\frac{1}{2}\sum_{k,m} g^{km}p_k p_m, x_j''\Big\} = \mu^{-1}\Big(\{f_1, x_j\}_F b_1 + \{f_2, x_j\}_F b_2\Big) + O(\mu^{-2}) \tag{22.2.12}$$

where in the Birkhoff normal form $a = f_1 b_1 + f_2 b_2 + \dots$ *and*

$$\{w_1, w_2\}_F := \sum_{k,l} \phi^{kl}\partial_k w_1 \cdot \partial_l w_2 \tag{22.2.13}$$

are Poisson brackets associated with symplectic form $\omega = \omega_F$.

Microhyperbolicity Condition. I

Microhyperbolicity condition of Chapter 19 then means for $d = 4$ and $f_1 \neq f_2$ that

$$\begin{aligned}|V - (2p+1)f_1\mu h - (2n+1)f_2\mu h| \le \epsilon \implies \\ |\langle \ell, \nabla\big(V - (2p+1)f_1\mu h - (2n+1)f_2\mu h\big)\rangle| \ge \epsilon\end{aligned} \tag{22.2.14}$$

where ℓ ($|\ell| \asymp 1$) is the *microhyperbolicity direction* (at the given point x) and (n,p) are *magnetic indices*. However, there actually were two microhyperbolicity conditions in Chapter 19: in the weaker condition ℓ could depend not only on x but also on magnetic indices while in the stronger one it was assumed that ℓ is the same for all pairs (n,p) connected by the third order resonance. Respectively, the remainder estimates derived in Chapter 19 were $O(h^{1-d})$ under weaker condition and $O(\mu^{-1}h^{1-d})$ under stronger one.

In the relatively simple case of $d = 4$ the resonance means $kf_1 = lf_2$ where $k, l \in \mathbb{Z}^+$ are coprimes, $(k+l)$ is the *order of the resonance*. So under the stronger microhyperbolicity condition if either $2f_1 - f_2 = 0$ or $f_1 - 2f_2 = 0$ at x then ℓ should not depend on (n,p) satisfying the the left-hand inequality in (22.2.14).

Remark 22.2.3. Case $d = 4$ is relatively simple because there are only two magnetic intensities f_1 and f_2 and thus it is impossible to have collisions between two second-order resonances (of the type $f_j = f_k$), two third order resonances (of the types $f_j = 2f_k$ or $f_j = f_k + f_l$) or the second and the third order resonances.

Further, as $|kf_1 - lf_2| \le \epsilon$ there could be other resonances only of order $m(\epsilon)$ or higher; $m(\epsilon) \to +\infty$ as $\epsilon \to +0$.

So, as $d = 4$ and f_1 is disjoint from f_2 and $\mu h \le \epsilon_1$ (where $\epsilon_1 > 0$ depends on ϵ in the microhyperbolicity condition) both microhyperbolicity conditions are equivalent to

$$|\nabla\big(\alpha \log f_1 + (1-\alpha)\log f_2 - \log V\big)| \ge \epsilon \qquad \forall \alpha : 0 \le \alpha \le 1. \tag{22.2.15}$$

On the other hand, as $\mu h \ge \epsilon_1$ condition (22.2.15) should be checked only at points where *ellipticity condition*

$$|V - (2p+1)f_1\mu h - (2n+1)f_2\mu h| \ge \epsilon \qquad \forall (p,n) \in \mathbb{Z}^{+\,2} \tag{22.2.16}$$

is violated.

Remark 22.2.4. Recall that under condition (22.2.16) on $\mathsf{supp}(\psi)$ the remainder estimate is $O(\mu^{-s}h^s)$ with an arbitrarily large exponent s and the similar result would hold in any dimension provided $\mathsf{rank}(F_{jk}) = d$ at each point.

So, Theorems 19.2.16, 19.2.19, 19.4.17, 19.4.21, 19.5.2, 19.5.3, and 19.5.13 imply

Corollary 22.2.5. *Let $d = 4$ and conditions* (22.1.7), $f_1 \ne f_2$ *and* (22.2.15) *be fulfilled at* $\mathsf{supp}(\psi)$. *Then the remainder is $O(\mu^{-1}h^{-3})$ while the main part of asymptotics is given by the Magnetic Weyl formula.*

Microhyperbolicity Condition. II

This leaves us with bad points where condition (22.2.15) is violated. So let us consider set Λ_α of critical points of

$$\phi_\alpha := \alpha \log f_1 + (1-\alpha)\log f_2 - \log V. \tag{22.2.17}$$

Proposition 22.2.6. *Let g^{jk}, V_j be fixed and let V be generic. Then*

(i) Λ_α is a finite set, continuously depending on $\alpha \in [0,1]$.

(ii) There exist $0 < \alpha_1 < \alpha_2 < \dots < \alpha_J < 1$ such that for $\alpha \notin \{\alpha_1, \dots, \alpha_J\}$ function ϕ_α has non-degenerate critical points, while for $\alpha = \alpha_j$ function ϕ_α has also isolated critical points but in one of them $\mathsf{rank}\,\mathsf{Hess}\,\phi_\alpha = 3$.

(iii) As $\bar{\alpha} \in \{\alpha_1, \dots, \alpha_J\}$ *and* $\bar{x}$ *is a critical point of* $\phi_{\bar{\alpha}}$ *with* $\operatorname{rank} \operatorname{Hess} \phi_{\bar{\alpha}}(\bar{x}) = 3$*, in an appropriate (smooth) coordinate system in vicinity of* $\bar{x} = 0$

$$\phi_\alpha = \frac{1}{3}x_1^3 - (\alpha - \bar{\alpha})x_1 \pm_2 \frac{1}{2}x_2^2 \pm_3 \frac{1}{2}x_3^2 \pm_4 \frac{1}{2}x_4^2 \tag{22.2.18}$$

with independent signs $\pm_k$*.*

Proof. An easy proof is left to the reader. □

Remark 22.2.7. (i) Assume that at some point $\bar{x}$

$$|\nabla(f_1 f_2^{-1})| \ge \epsilon_0. \tag{22.2.19}$$

Then in the vicinity of $\bar{x}$

$$|\nabla\phi_\alpha| \ge \epsilon|\alpha - \bar{\alpha}(x)| \qquad \forall \alpha : 0 \le \alpha \le 1 \tag{22.2.20}$$

with smooth function $\bar{\alpha}(x)$.

(ii) On the other hand, if $\bar{x}$ is an isolated critical point of $f_1 f_2^{-1}$ then as V is generic $|\nabla\phi_\alpha| \ge \epsilon$ for all $\alpha \in [0,1]$ in the vicinity of $\bar{x}$.

22.2.3 Analysis near Σ: Geometry

Preliminary Remarks

Consider now $\Sigma = \{x : f_1 = f_2\}$ assuming that (22.2.1) holds; then the microhyperbolicity condition at point $x \in \Sigma$ means exactly that

$$\frac{1}{2}\ell\big((f_1 + f_2)V^{-1}\big) \ge \big((\ell v_1)^2 + (\ell v_2)^2\big)^{1/2} V^{-1} + \epsilon \tag{22.2.21}$$

where ℓ again is the microhyperbolicity direction, or, equivalently,

$$\Big|\frac{1}{2}\nabla\big((f_1 + f_2)V^{-1}\big) - (\beta_1\nabla v_1 + \beta_2\nabla v_2)V^{-1}\Big| \ge \epsilon \qquad \forall \beta = (\beta_1, \beta_2) \in \mathbb{R}^2 : |\beta| \le 1. \tag{22.2.22}$$

Remark 22.2.8. Note that the microhyperbolicity condition could be violated only at stationary points of $f_1 V^{-1}|_\Sigma$ and only at those of them where

$$\frac{1}{2}\nabla\big((f_1 + f_2)V^{-1}\big) = (\beta_1\nabla v_1 + \beta_2\nabla v_2)V^{-1} \tag{22.2.23}$$

with $\beta_1^2 + \beta_2^2 \le 1$.

Let Σ_0 be the set of stationary points of $f_1 V^{-1}|_\Sigma$; at each point of Σ_0 decomposition (22.2.23) holds; let us denote by Σ_0^+, Σ_0^-, Σ_0^0 the subsets of Σ_0 where $\beta_1^2+\beta_2^2>1$, $\beta_1^2+\beta_2^2<1$ and $\beta_1^2+\beta_2^2=1$ respectively. Then the microhyperbolicity condition holds at Σ_0^+.

So, main theorems of Chapter 19 imply

Corollary 22.2.9. *Let $d=4$ and conditions* (22.2.1) *and* (22.1.7) *be fulfilled. Let ψ be supported in the small vicinity of $\bar{x}\in\Sigma_0^+$. Then the remainder is $O(\mu^{-1}h^{-3})$ while the main part of asymptotics is given by magnetic Weyl formula.*

Geometry. I

One can prove easily two following propositions:

Proposition 22.2.10. *Let $d=4$ and condition* (22.2.1) *be fulfilled. Let V be generic satisfying* (22.1.7)*. Then*

(22.2.24) *Σ_0 consists of the finite number of non-degenerate stationary points of $f_1 V^{-1}|_\Sigma$; these points are generic;*

(22.2.25) *$0<\beta_1^2+\beta_2^2<1$ at each point of $\Sigma_0\setminus\Sigma_0^+$.*

Proposition 22.2.11. *Let $d=4$ and condition* (22.2.1) *be fulfilled. Let $\bar{x}\in\Sigma_0^-$ with $0<\beta_1^2+\beta_2^2<1$ in decomposition* (22.2.23)*. Then in the vicinity of $\bar{x}$*

$$\Lambda:=\bigl\{x: f_1\neq f_2 \text{ and } \exists\alpha\in[0,1]:\quad \nabla\phi_\alpha=0\bigr\}\cup\{\bar{x}\} \tag{22.2.26}$$

is a smooth curve passing through $\bar{x}$ and transversal to Σ; moreover

$$\bar{\alpha}(\bar{x})=\frac{1}{2}\bigl(1\pm(\beta_1^2+\beta_2^2)^{1/2}\bigr). \tag{22.2.27}$$

22.3 Weak Magnetic Field Case

In this section we are going to prove the remainder estimate $O(\mu^{-1}h^{-3})$ for general V as

$$c\le\mu\le h^{-\delta_0} \tag{22.3.1}$$

with small exponent $\delta_0>0$.

22.3.1 Weak Magnetic Field Case. I

Assume first that condition (22.2.19) is fulfilled.

Proposition 22.3.1. *Let conditions* $f_1 \neq f_2$ *and* (22.2.19) *be fulfilled at* $\mathsf{supp}(\psi)$. *Then under condition* (22.3.1) *the remainder is* $O(\mu^{-1}h^{-3})$ *while the main part of asymptotics is given by Weyl formula.*

Proof, Part I. Assume first that there are no resonances of order lesser than M on $\mathsf{supp}(\psi)$:

$$|kf_1 - lf_2| \geq \epsilon, \qquad \forall k, l \in \mathbb{Z}^+ : k + l \leq M. \tag{22.3.2}$$

Then one can reduce operator to the normal form without cubic and "unbalanced" fourth order terms and then along Hamiltonian trajectories

$$\frac{d}{dt}x_j = \frac{\partial}{\partial \xi_j}a, \qquad \frac{d}{dt}\xi_j = -\frac{\partial}{\partial x_j}a \tag{22.3.3}$$

of

$$a(x,\xi) \coloneqq \frac{1}{2}\Big(\sum_{j,k} p_j g^{jk}(x) p_k - V\Big), \qquad p_j = \xi_j - \mu V_j \tag{22.3.4}$$

the following relations hold:

$$\frac{d}{dt}b_j = \{a, b_j\} = O(\mu^{-2}) \tag{22.3.5}$$

where $b_j = \frac{1}{2}(f_j \circ \Psi)(x_j^2 + \xi_j^2) + O(\mu^{-2})$ are quadratic terms in the Birkhoff normal form; see details in Chapter 19 or below.

Therefore in the "corrected" coordinates x'' defined by (22.2.11) along Hamiltonian trajectories

$$\frac{d}{dt}x''_j = \mu^{-1}\big(\beta_{j1}(x'')b_1(x,\xi) + \beta_{j2}(x'')b_2(x,\xi)\big) + O(\mu^{-3}) \tag{22.3.6}$$

and

$$|\frac{d^2}{dt^2}x''| \leq C\mu^{-1}|\frac{d}{dt}x''| + C(\mu^{-4}). \tag{22.3.7}$$

Thus if

$$\mu|\frac{d}{dt}x''| \asymp \rho \geq C\mu^{-2} \tag{22.3.8}$$

then for time $T_1 = \epsilon\mu$ this relation (22.3.8) is retained and also variation of dx''/dt would be less than $C\epsilon\rho\mu^{-1}$. Then there is a fixed direction ℓ, $|\ell| \asymp 1$ such that for $|t| \le T_1$

$$\langle \ell, \frac{d}{dt} x'' \rangle \ge \epsilon_0 \mu^{-1} \rho; \tag{22.3.9}$$

ℓ is a sort of the microhyperbolicity direction. Without any loss of the generality one can assume that $\ell = (1,0,0,0)$ and therefore the shift of x_1'' for time T is exactly of magnitude $\rho\mu^{-1}T$.

According to the logarithmic uncertainty principle this shift is microlocally observable as

$$\rho\mu^{-1}T \times \rho \ge Ch|\log h| \tag{22.3.10}$$

Plugging $T = T_0 = \epsilon\mu^{-1}$ one gets $\rho^2 \ge C\mu^2 h|\log h|$ which would hold for $\rho \ge C\mu^{-2}$ as $\mu \le h^{-\delta}$.

Therefore in this case the contribution of zone (22.3.6) to the remainder does not exceed

$$Ch^{-3}\rho \times T_1^{-1} \asymp C\mu^{-1}h^{-3}\rho$$

where factor ρh^{-3} is due to the estimate of

$$|F_{t\to h^{-1}\tau}\bigl(\bar\chi_{T_0}(t)\Gamma Qu\bigr)| \le C\rho h^{-3}, \tag{22.3.11}$$

Q is the cut-off operator in zone (22.3.8), u is the Schwartz kernel of $e^{-ih^{-1}tA}$ and other standard notations are used.

To prove estimate (22.3.11) let us make ρ-admissible partition in ξ; then one can prove easily that the contribution of each such element to the left hand expression does not exceed $C\rho^3h^{-3}$ because the propagation speed with respect to x is $\asymp 1$ as $|t| \le T_0$ and therefore one can trade T_0 to $\bar T = Ch\rho^{-1}|\log h|$ in the left hand expression of (22.3.11) with a negligible error and in the estimate even to $\bar T = C\rho^{-1}h$ (one can prove easily by the rescaling).

On the other hand, the number of the partition elements for each x (and therefore in its vicinity) so that $hD_t - A$ is not elliptic and (22.3.8) holds is obviously $O(\rho^{-2})$; so the left-hand expression of (22.3.11) does not exceed $C\rho^3h^{-3} \times \rho^{-2}$.

Summation of $O(\mu^{-1}h^{-3})$ with respect to ρ results in $C\mu^{-1}h^{-3}$ and therefore we have proven that

(22.3.12) In the framework of Proposition and (22.3.2) the total contribution of zone

$$\{\mu|\frac{d}{dt}x''| \ge \bar\rho \coloneqq C\mu^{-2}\} \tag{22.3.13}$$

to the remainder is $O(\mu^{-1}h^{-3})$.

On the other hand, contribution of zone

$$\{\mu|\frac{d}{dt}x''| \le \bar\rho = C\mu^{-2}\} \tag{22.3.14}$$

to the remainder does not exceed $C\bar\rho h^{-3}\times T_0^{-1} = C\mu^{-1}h^{-3}$ where for cut-off operator in this zone estimate (22.3.11) holds with $\rho=\bar\rho$.

So, the remainder does not exceed $C\mu^{-1}h^{-3}$ while the main part of the asymptotics is given by the standard Tauberian formula (22.1.5) with $T = Ch|\log h|$. One can easily rewrite (22.1.5) with $T = Ch|\log h|$ as

$$\int \mathcal{N}^{\mathsf{W}}(x,0)\psi(x)\,dx + O(\mu^2h^{-2}).$$

Furthermore, one can replace here $\mathcal{N}^{\mathsf{W}}(x,0)$ by $\mathcal{N}^{\mathsf{MW}}(x,0)$ with the same error. The proof is standard, easy and left to the reader. □

Let us allow resonances.

Proof, Part II. (i) Let us consider now the case when $(kf_1 - lf_2)$ is not disjoint from 0. We will analyze the third order resonance which is worst case scenario leaving the easier case of higher order resonances to the reader.

So, now $(f_1 - 2f_2)$ is not disjoint from 0 (but then one can assume that $(kf_1 - lf_2)$ is as $l \ne 2k$, $k \le M$). Then one can reduce a operator to a (pre)canonical form

$$\frac{1}{2}\Big(f_1(x')Z_1^*Z_1 + f_2(x')Z_2^*Z_2 + \mu^{-1}\operatorname{Re}\beta(x')Z_1^*Z_2^2\Big) + O(\mu^{-2}) \tag{22.3.15}$$

with

$$\{Z_j, Z_k\} \equiv 0, \qquad \{Z_j^*, Z_k\} \equiv 2\mu\delta_{jk} \qquad \mathsf{mod}\ O(\mu^{-s}). \tag{22.3.16}$$

$$\{Z_j, x_k'\} \equiv \{Z_j,^* x_k'\} \equiv 0 \qquad \mathsf{mod}\ O(\mu^{-s}), \tag{22.3.17}$$

$$\{x_j', x_k'\} \equiv \mu^{-1}\phi^{jk}(x') \qquad \mathsf{mod}\ O(\mu^{-2}) \tag{22.3.18}$$

where

$$x' \equiv x \qquad \mathsf{mod}\ O(\mu^{-1}). \tag{22.3.19}$$

Due to (22.2.19) one can assume without any loss of the generality that

$$f_1 - 2f_2 = x_1. \tag{22.3.20}$$

(ii) Now let us consider elements of the partition with

$$|x_1| \asymp \gamma, \qquad \gamma \ge C\mu^{-1}. \tag{22.3.21}$$

Then one can get rid of the cubic term in (22.3.15) by means of the transformation with the generating function $\gamma^{-1}\mu^{-2}\,\mathsf{Re}\,\beta Z_1^* Z_2^2$ with γ-admissible function β (all other cubic terms are "regular" and one can get rid of them in the regular way), leading to the error which, as one can easily check, is the sum of terms of the types $\gamma^{-1}\mu^{-2}\,\mathsf{Re}\,\beta Z_j^* Z_j Z_2^* Z_2$ and also some smaller terms; one can continue this process getting rid of all "unbalanced" terms up to order M. Also one can introduce corrected x'' so that

$$\frac{d}{dt}x_j'' = \mu^{-1}\bigl(\beta_{j1}(x'')b_1 + \beta_{j2}(x'')b_2\bigr) + O(\mu^{-3}\gamma^{-1}) \tag{22.3.22}$$

(compare with (22.3.6). However, this would be slightly short of what is needed and one can improve a term $O(\mu^{-3}\gamma^{-1})$ in (22.3.22) to

$$\sum_{k+l=2} \beta_{kl}(x'')b_1^k b_2^l + O(\mu^{-3-\kappa}\gamma^{-1-\kappa})$$

with $\kappa > 0$. Then the contribution of the zone

$$\bigl\{|x_1| \asymp \gamma,\ \mu|\frac{d}{dt}x''| \le C\mu^{-2-\kappa}\gamma^{-1-\kappa}\bigr\}$$

to the remainder does not exceed $C\mu h^{-3}\gamma \times \mu^{-2-\kappa}\gamma^{-1-\kappa} \asymp C\mu^{-1-\kappa}h^{-3}\gamma^{-\kappa}$ where the first factor γ is the measure. Proof is similar to one in Part I, with estimate (22.3.11) replaced by

$$|F_{t\to h^{-1}\tau}\bigl(\bar\chi_{T_0}(t)\Gamma Qu\bigr)| \le C\rho\gamma h^{-3}, \tag{22.3.23}$$

now Q is the cut-off operator in zone (22.3.8) intersected with $\{|x_1| \asymp \gamma\}$. Obviously summation with respect to $\gamma \ge C\mu^{-1}$ results in $O(\mu^{-1}h^{-3})$.

So one needs to consider the zone where

$$\mu|\frac{d}{dt}x''| \asymp \rho \ge C\mu^{-2-\kappa}\gamma^{-\kappa}, \tag{22.3.24}$$

which implies that

$$|\mu\frac{d}{dt}\big(\beta_{j1}(x'')b_1 + \beta_{j2}(x'')b_2\big)| \le C\rho + C\mu^{-K}\gamma^{-K} \tag{22.3.25}$$

and therefore one can take

$$T_1 = \epsilon\,\mathsf{min}\big(\mu\gamma\rho^{-1},\ \mu,\ \rho\mu^{K}\gamma^{K}\big) \tag{22.3.26}$$

where the restriction $T \le \epsilon\gamma\rho^{-1}$ preserves the magnitude of $|x_1|$.

Therefore the contribution of the zone

$$\big\{|x_1| \asymp \gamma,\ \mu|\frac{d}{dt}x''| \asymp \rho\big\}$$

with $\gamma \ge C\mu^{-1}$, $\rho \ge C\mu^{-2-\kappa}\gamma^{-\kappa}$ to the remainder does not exceed

$$Ch^{-3}\rho\gamma\Big(\mu^{-1}\gamma^{-1}\rho + 1 + \mu^{-K}\gamma^{-K}\Big) \asymp C\mu^{-1}h^{-3}\Big(\rho^2 + \rho\gamma + \mu^{1-K}\gamma^{1-K}\Big). \tag{22.3.27}$$

Obviously, summation with respect to ρ, γ results in $C\mu^{-1}h^{-3}|\log\mu|$.

(iii) Now we want to improve this estimate getting rid of $\log\mu$ factor. Note first that the second term in (22.3.25) sums to $O(\mu^{-1}h^{-3})$, while the first and third terms sum to $O(\mu^{-1}h^{-3})$ over zones complemental to $\{\rho \ge \mathsf{max}(\gamma, |\log\mu|^{-1/2})\}$ and $\{\gamma \le \mathsf{min}(\rho,\ \mu^{-1+\kappa})\}$ respectively.

However, in the first zone one can take $T_1 = \epsilon\mu\gamma^{1-\kappa}$ (rather than $T_1 = \epsilon\mu\gamma$) since in an appropriate time direction $|x_1| \gtrsim \gamma$ for this time. Then an extra factor γ^{κ} in the remainder estimate prevents appearance of the logarithmic factor.

Let us introduce $W(x') = Vf_1^{-1}|_{x_1=0}$ and a scaling function $\zeta = \epsilon|\nabla' W| + \gamma$. Then one can upgrade T_1 to

$$T_1 = \epsilon\,\mathsf{min}\big(\mu\gamma\zeta^{-1},\ \mu,\ \rho^{1-\kappa}\mu^{K}\gamma^{K}\big) \tag{22.3.28}$$

Really, $\mu|\frac{d}{dt}x_1| \le C\zeta$ and one can select direction of time to replace ρ by $\rho^{1-\kappa}$. Then the total contribution to the remainder of all partition elements with $\zeta \le \gamma^\kappa$ is $O(\mu^{-1}h^{-3})$.

On the other hand, as $\zeta \ge \gamma^\kappa$ and $\gamma \le \mu^{-1+\kappa}$ one can obviously take $T_1 = \epsilon\mu\zeta$ and then the total contribution of such partition elements to the remainder does not exceed $C\mu^{-1}\zeta^{-1}h^{-3}\gamma \ll \mu^{-1}h^{-3}$.

(iv) Finally, let us consider now zone where

$$|x_1| \le \gamma = C\mu^{-1}. \tag{22.3.29}$$

Then we are not getting rid of the cubic terms and should take $T_1 = \epsilon\rho$ which one can easily upgrade to $T_1 = \epsilon\rho^{1-\kappa}$ [7] leading to the contribution of this zone to the remainder not exceeding $C\mu^{-1}h^{-3}$ where factor μ^{-1} is the measure of zone defined by (22.3.29). □

22.3.2 Weak Magnetic Field Case. II

Now let us allow $f_1 f_2^{-1}$ to have critical points. However, for the generic magnetic field these critical points are not resonances.

Proposition 22.3.2. *Let conditions* $f_1 \ne f_2$, (22.3.2) *and*

$(22.3.30)_q$ $|\nabla(f_1 f_2^{-1})| \le \epsilon_0 \implies$
$\mathsf{Hess}(f_1 f_2^{-1})$ *has at least* q *eigenvalues with absolute values greater than* ϵ_0

be fulfilled at $\mathsf{supp}(\psi)$ *with* $q \ge 3$. *Then under condition* (22.3.1) *the remainder is* $O(\mu^{-1}h^{-3})$ *while the main part of asymptotics is given by Weyl formula.*

Proof. Let $\gamma = \epsilon|\nabla(f_1 f_2^{-1})| + \bar\gamma$, $\bar\gamma = C\mu^{-1}$. Let us consider γ-admissible partition with respect to x.

[7] One can see easily that in this zone $\frac{d}{dt}b_j = \{a, b_j\} = O(1)$ and $(\frac{d}{dt})^2 b_j = \{a, \{a, b_j\}\} = O(1)$; one can prove that the microhyperbolicity is preserved with respect to the same vector $\ell\rho^{-1}$.

Then similarly to part (i) of the proof of Proposition 22.3.1 one can take $T_1 = \epsilon\mu\rho^{2-\delta'}\gamma$ with ρ defined by (22.3.8). Note that

$$|\alpha\nabla(f_1V^{-1}) + (1-\alpha)\nabla(f_2V^{-1})| \ge \epsilon\gamma|\alpha - \bar{\alpha}(x)| \qquad \forall \alpha : 0 \le \alpha \le 1 \tag{22.3.31}$$

and therefore the measure in ξ-space gets a factor γ^{-1} but the measure in x-space gets a factor γ^q due to condition condition $(22.3.30)_q$.

Then (22.3.11) and (22.3.23) are replaced by the similar estimate with the right hand expression $C\rho\gamma^{q-1}h^{-3}$.

So, the total contribution (ρ,γ)-elements to the remainder does not exceed

$$C\mu^{-1}h^{-3}\rho\gamma^{q-1} \times \rho^{\kappa-1}\gamma^{-1} \asymp C\mu^{-1}h^{-3}\gamma^{q-2}\rho^{\kappa}. \tag{22.3.32}$$

Summation with respect to (ρ,γ) results in $C\mu^{-1}h^{-3}$ as $q \ge 3$ and in $C\mu^{-1}h^{-3}|\log\mu|$ as $q = 2$ and this is the total contribution of the zone $\{\rho\gamma \ge C\mu^{-2}\}$.

On the other hand, contribution of the zone $\{\rho\gamma \le C\mu^{-2}\}$ to the remainder is $O(\mu h^{-3} \times \mu^{-2})$ since its measure is $O(\mu^{-2})$ under condition $(22.3.30)_3$; under condition $(22.3.30)_2$ an extra logarithmic factor appears as well. □

Remark 22.3.3. Probably one can get rid of logarithmic factors as $q = 2$ and derive some estimate as $q = 1$. We leave it to the curious reader.

22.3.3 Weak Magnetic Field Case. III

Now let us consider the vicinity of $\Sigma = \{x\colon f_1(x) = f_2(x)\}$.

Proposition 22.3.4. *Assume that* $|f_1 - f_2| \asymp \operatorname{dist}(x,\Sigma)$ *where* Σ *is a* 2*-dimensional manifold. Let* ψ *be supported in the small vicinity of* Σ*. Then under condition* (22.3.1) *the remainder is* $O(\mu^{-1}h^{-3})$ *while the main part of asymptotics is given by Weyl formula.*

Proof. Let us introduce a scaling function

$$\gamma = \epsilon\operatorname{dist}(x,\Sigma) + \frac{1}{2}\bar{\gamma}, \qquad \bar{\gamma} = C\mu^{-1}. \tag{22.3.33}$$

Then

$$|\alpha\nabla(f_1V^{-1}) + (1-\alpha)\nabla(f_2V^{-1})| \ge \epsilon|\alpha - \bar\alpha(x)| \qquad \forall\alpha : 0\le\alpha\le 1 \tag{22.3.34}$$

holds with γ-admissible $\bar\alpha$ and also (22.3.8) and (22.3.9) hold. Therefore one can take $T_1 = \epsilon\rho^{1-\kappa}\gamma\mu$. Then the contribution of all (γ,ρ) elements to the remainder does not exceed (22.3.32) with $q = 3$ [8] and summation over $\{\gamma \ge C\mu^{-1}, \rho\gamma \ge C\mu^{-2}\}$ results in $O(\mu^{-1}h^{-3})$.

Meanwhile, the contributions of zones $\{x : \gamma \ge C\mu^{-1}, \rho\gamma \le C\mu^{-2}\}$ and $\{x : \gamma \le C\mu^{-1}\}$ to the remainder are $O(\mu h^{-3}\times\mu^{-2}) = O(\mu^{-1}h^{-3})$ since the measures of these zones are $O(\mu^{-2})$. □

Summarizing what is proven one gets

Proposition 22.3.5. *Let (g^{jk}) be fixed and then (V_j) be generic, more precisely:*

(i) Outside of $\Sigma = \{x : f_1 = f_2\}$ critical points of $f_1f_2^{-1}$ satisfy $(22.3.30)_3$ *and* (22.3.2)*.*

(ii) Σ be smooth 2-dimensional manifold and $|f_1 - f_2| \asymp \mathsf{dist}(x,\Sigma)$.

Finally, let V be general but satisfying (22.1.7) *at $\mathsf{supp}(\psi)$. Then under condition* (22.3.1) *the remainder is $O(\mu^{-1}h^{-3})$ while the main part of asymptotics is given by Weyl formula.*

Proposition 22.3.6. *In the framework of Proposition 22.3.5 Weyl and Magnetic Weyl expressions differ by (far less than) $O(\mu^{-1}h^{-3})$.*

Proof. An easy proof is left to the reader. □

Remark 22.3.7. Definitely M in condition (22.3.2) "critical points of $f_1f_2^{-1}$ are not resonances of order not exceeding M" should not be too large and we leave to the curious reader to investigate it.

[8] In comparison with Proposition 22.3.2 there is no factor γ in the right hand expression of (22.3.34) and therefore no factor γ^{-1} in the estimate of the measure in ξ space.

22.4 Temperate Magnetic Field Case: Estimates

22.4.1 Canonical Form

From now one can assume that

$$h^{-\delta_0} \le \mu \le ch^{-1} \tag{22.4.1}$$

with some small fixed exponent $\delta_0 > 0$. Then we can reduce operator to a canonical form (depending on additional assumptions) and also make decomposition with respect to Hermite functions, thus arriving to 2-parametric matrices of 2D $\mu^{-1}h$-PDOs $\mathcal{A}_{pn}(x', \mu^{-1}hD')$ where ere and below $x' = (x_1, x_2)$.

More precisely, assuming that there are no resonances of order not exceeding M:

$$|kf_2 - lf_1| \ge \epsilon \qquad \forall (k,l) \in \mathbb{Z}^{+\,2} : k + l \le M, \tag{22.4.2}$$

a canonical form contains diagonal elements

$$\mathcal{A}_{pn} = \frac{1}{2}\Big(f_1^{\#}(2p+1)\mu h + f_2^{\#}(2n+1)\mu h - V^{\#} + \sum_{l+m+k+j\ge 2} b_{lmkj}\big((2p+1)\mu h\big)^l \big((2n+1)\mu h\big)^m \mu^{3-2l-2m-2k-j} h^j \Big) \tag{22.4.3}$$

with $f_j^{\#} = f_j^{\#}(x', \mu^{-1}hD')$, $V^{\#} = V^{\#}(x', \mu^{-1}hD')$, $b_{lmkj} = b_{lmkj}(x', \mu^{-1}hD')$ while all non-diagonal elements are $O(\mu^{2-M})$.

We will discuss later an alternative form for $M = 2$.

22.4.2 General Estimates at Regular Points

Assume first that there are no resonances of order not exceeding large $M = M(\delta_0)$. Then under condition (22.4.1) perturbation $O(\mu^{2-M})$ is negligible and (22.4.3) is a true diagonal canonical form (with a negligible perturbation).

General Estimates at Regular Points. I

In this case an analysis is easy:

Proposition 22.4.1. *Let there be no resonances of order not exceeding (large enough)* M *and condition* (22.2.19) *be fulfilled at* $\mathsf{supp}(\psi)$. *Then under condition* (22.4.1)

(i) The standard implicit asymptotic formula (22.1.5) *holds with the remainder estimate* $O(\mu^{-1}h^{-3}+\mu^2h^{-2})$.

(ii) In particular, remainder estimate is $O(\mu^{-1}h^{-3})$ *as* $\mu \le h^{-1/3}$.

Proof. (a) Let us for each *pair* (p,n) introduce scaling function

$$\rho_{pn}=\epsilon\big(|\mathcal{A}_{pn}|+|\nabla\mathcal{A}_{pn}|^2\big)^{1/2}+\bar\rho, \qquad \bar\rho = C(\mu^{-1}h|\log h|)^{1/2}+C\mu^{-2} \tag{22.4.4}$$

and a corresponding partition. Then

(22.4.5) The contribution of each *group* $(p,n,\mathsf{element})$ to the main part of asymptotics is $\lesssim \mu^2h^{-2}\rho^4$ [9)].

On the other hand, one can take

$$T_1=\epsilon\mu \tag{22.4.6}$$

since the propagation speed is of magnitude μ^{-1} and also

$$T_0=Ch\rho^{-2}|\log h|. \tag{22.4.7}$$

Note that $T_0\le \epsilon_0\mu^{-1}$ provided

$$\rho\ge\bar\rho_1=C(\mu h|\log h|)^{1/2} \tag{22.4.8}$$

and in this case one can trade T_0 to $\bar T=Ch|\log h|$ in (22.1.5) as $\sum Q_\iota=I$; there will be also the correction term arising from elements failing condition (22.4.8); see Section 22.5. Moreover,

(22.4.9) In the estimate of expression (22.1.6) one can replace T_0 by $T_0^*=Ch\rho^{-2}$.

[9)] And often enough is is of this amplitude, so summation results in the correct magnitude of the main part.

Therefore, the contribution of an element to the remainder does not exceed

$$C\mu^2h^{-2}\rho^4 \times h\rho^{-2} \times \mu^{-1} \times \bigl(\rho^2(\mu h)^{-1}+1\bigr) \times \bigl(\rho(\mu h)^{-1}+1\bigr) \times \rho^{-4} \tag{22.4.10}$$

where $Ch\rho^{-2}|\log h|$ and $Ch\rho^{-2}$ play the roles of T_0 and T_0^* in formulae (22.1.5) and (22.1.6) respectively. Also the numbers of indices n [10)] and p [11)] are estimated by $C\bigl(\rho^2(\mu h)^{-1}+1\bigr)$ and $C\bigl(\rho(\mu h)^{-1}+1\bigr)$ respectively.

One can rewrite expression (22.4.9) as

$$C\mu^{-1}h^{-3}\rho + Ch^{-2}\rho^{-1} + C\mu h^{-1}\rho^{-2}; \tag{22.4.11}$$

then in the zone

$$\bigl\{\rho \ge \bar\rho := C(\mu^{-1}h|\log h|)^{1/2}\bigr\} \tag{22.4.12}$$

the first term sums to $C\mu^{-1}h^{-3}$ while the last two terms sum to their values as $\rho=\bar\rho$, which are $O(h^{-5/2}\mu^{1/2}) = O(\mu^{-1}h^{-3}+\mu^2h^{-2})$ for sure and $O(\mu^2h^{-2})$ respectively.

(b) Meanwhile, with the same main part the total contribution to the remainder of all groups with $\{\rho\le\bar\rho\}$ trivially does not exceed

$$C\mu^2h^{-2}\bigl(\bar\rho^2(\mu h)^{-1}+1\bigr) \times \bigl(\bar\rho(\mu h)^{-1}+1\bigr) \le C\mu h^{-3}\bar\rho + C\mu^2h^{-2}. \tag{22.4.13}$$

Obviously, this expression is $O(\mu^{-1}h^{-3})$ as $\mu \le C(h|\log h|)^{-1/3}$ and $O(\mu^2h^{-2})$ as $\mu \ge Ch^{-1/3}|\log h|^{1/3}$.

(c) To finish the proof we need to reconsider contribution to the remainder of the elements with $\{\rho_{pn}\le\bar\rho\}$ in the border case

$$h^{-1/3}|\log h|^{-1/3} \le \mu \le h^{-1/3}|\log h|^{1/3}. \tag{22.4.14}$$

Let us introduce another scaling function

$$\varrho = \epsilon|\nabla^2\mathcal{A}_{pn}| + \frac{1}{2}\bar\varrho, \qquad \bar\varrho = |\log h|^{-K}, \tag{22.4.15}$$

calculated with (p,n) delivering minimum to ρ_{pn} and let us introduce the corresponding partition.

[10)] For which ellipticity is violated for a given p.

[11)] Such that $\rho_{pn}\asymp\rho$ as n violates ellipticity.

Then for any element with $\varrho \ge \bar{\varrho}$ one can calculate easily that the relative measure of the zone $\{(x', \xi') : \ \min_{p,n} \rho_{pn} \le C\bar{\rho}\}$ is $O(\bar{\rho}|\log h|^K)$ and then the total contribution of zone $\{(x', \xi') : \ \rho \le \bar{\rho},\ \varrho \ge C\bar{\varrho}\}$ to the remainder is much less than $O(\mu^{-1}h^{-3})$.

On the other hand, let us consider elements with $\varrho \le C\bar{\varrho}$. Since the total contribution of subelements with $\rho \le \rho_1^* := C(\mu h)^{1/2}$ to the remainder is estimated properly, one needs to consider only subelements with $\rho_1^* \le \rho \le \bar{\rho}_1$.

But on such subelements $\rho\varrho^{-1}$ is a scaling function as well and using it one can easily decrease T_0 to $Ch|\log h|\bar{\varrho}\rho^{-2}$ leaving $T_1 = \epsilon\mu$; this will add an extra factor $|\log h|^{2-K}$ to the estimate of the contribution of this zone to the remainder and this factor leads to the needed estimate $O(\mu^{-1}h^{-3})$. □

General Estimates at Regular Points. II

Assume now that the is a critical point of $f_1 f_2^{-1}$:

Proposition 22.4.2. *Let there be no resonances of order not exceeding (large enough) M and condition* $(22.3.30)_3$ *be fulfilled at* $\operatorname{supp}(\psi)$. *Then under condition* (22.4.1)

(i) The standard implicit asymptotic formula holds with the remainder estimate $O(\mu^{-1}h^{-3} + \mu^2 h^{-2})$.

(ii) In particular, the remainder estimate is $O(\mu^{-1}h^{-3})$ as $\mu \le Ch^{-1/3}$.

Proof. (a) The arguments of the proof of Proposition 22.4.1 still work without condition (22.2.19) with the exception of the estimate $C\big(\rho(\mu h)^{-1}+1\big)$ of the number of the indices "p".

However, let us introduce another scaling function

$$\gamma := \epsilon_1 |\nabla(f_1 f_2^{-1})| \tag{22.4.16}$$

and if on some group $\rho_{pn} \le \gamma$, then the number of indices "p" should be estimated by $C\big(\rho(\gamma\mu h)^{-1} + 1\big)$; otherwise this number should be estimated by $C\big((\mu h)^{-1} + 1\big)$.

Note that if condition $\rho_{pn} \lesssim \gamma$ (or $\rho_{pn} \gtrsim \gamma$) is fulfilled at some point of ρ-element, then the same condition (with another implicit constant) is fulfilled at any other point of this element.

Anyway, this modification adds no more than one factor γ^{-1} to the estimate, but the factor γ^q comes from condition $(22.3.30)_q$; then summation over γ-partition results in the same estimate $O(\mu^{-1}h^{-3}+\mu^2h^{-2})$ as $q\ge 2$ and in the same estimate but with an extra logarithmic factor as $q=1$. Since we do not use microlocal γ-partition, no estimate of γ from below is needed.

Therefore

(22.4.17) The total contribution of elements with $\rho_{pn}\ge \bar\rho$ to the remainder is $O(\mu^{-1}h^{-3}+\mu^2h^{-2})$.

(b) Meanwhile, the contribution of zone $\{x:\ \rho\le\bar\rho,\ |\nabla(f_1f_2^{-1})|\le\gamma\}$ to the remainder does not exceed $C\mu h^{-3}\bar\rho\gamma^{q-1}+C\mu^2h^{-2}\gamma^q$ which sums to $O(\mu^{-1}h^{-3}+\mu^2h^{-2})$ except in the border case (22.4.14). However, even in this case $q=2$ only a contribution of zone $\{x:|\nabla(f_1f_2^{-1})|\ge|\log h|^{-1}\}$ is not properly estimated. However, one can tackle it by the same arguments as in the part (c) of the proof of Proposition 22.4.1. □

Remark 22.4.3. Probably one can get rid of logarithmic factors as $q=1$.

22.4.3 Sharp Asymptotics at Regular Points

The purpose of this and the next odd-numbered subsections is to consider the case

$$ch^{-1/3}\le\mu\le ch^{-1} \tag{22.4.18}$$

and derive some under non-degeneracy condition remainder estimate $O(\mu^{-1}h^{-3})$.

Sharp Asymptotics at Regular Points. I

Note first that

(22.4.19) In the generic case $\nabla(f_1f_2^{-1})(\bar x)=0$ implies $\nabla(Vf_1^{-1})(\bar x)\ne 0$ and the then microhyperbolicity condition holds with any ℓ such that $\langle\ell,\nabla(Vf_1^{-1})\rangle>0$

and therefore the following statement is generic:

(22.4.20) Let $\nabla(f_1f_2^{-1})(\bar x)=0$ and $\nabla(Vf_1^{-1})(\bar x)\ne 0$. Then for any ψ supported in the small vicinity of $\bar x$ asymptotics with the magnetic Weyl main part and remainder estimate $O(\mu^{-1}h^{-3})$ holds.

Sharp Asymptotics at Regular Points. II

Therefore in what follows one can assume that $\nabla(f_1 f_2^{-1})$ is disjoint from 0. Further, one should consider only vicinities of points where microhyperbolicity condition (22.2.15) is violated.

Proposition 22.4.4. *Let conditions* (22.1.7) *and* (22.4.18) *be fulfilled. Moreover, let us assume that on* $\mathsf{supp}(\psi)$ *there are no resonances of order not exceeding* M *and also* $|\nabla(f_1 f_2^{-1})| \ge \epsilon_0$. *Furthermore, let condition*

$$(22.4.21)_q \quad \nu(\rho) := \mathsf{mes}\big\{(x,\alpha):\ 0\le\alpha\le 1,\ |\nabla\big(\alpha\log f_1 + (1-\alpha)\log f_2 - \log V\big)|\le\rho\big\} = O(\rho^q) \qquad \text{as}\ \ \rho\to+0$$

be fulfilled with $q>1$. *Then asymptotics with the standard implicit main part* (22.1.5) *and the remainder estimate*

$$(22.4.22)_q \qquad O\big(\mu^{-1}h^{-3} + \mu^2 h^{-2}(\mu^{-1}h)^{(q-1)/2}|\log h|^{(q+1)/2}\big)$$

holds.

Proof. As $T = Ch\rho^{-2}|\log h|$ the contribution to the remainder of ρ-elements does not exceed (22.4.10) multiplied by $\nu(\rho)\rho^{-1}$

$$\begin{aligned} C\mu h^{-1}\rho^{-2}\big(\rho^2(\mu h)^{-1}+1\big)\times\big(\rho(\mu h)^{-1}+1\big)\times\nu(\rho)\rho^{-1} &\\ \asymp C\mu h^{-1}\big(\rho^3(\mu h)^{-2}+\rho(\mu h)^{-1}+1\big)\nu(\rho)\rho^{-3} &\\ \le C\mu^{-1}h^{-3}\rho^q + Ch^{-2}\rho^{q-2} + C\mu h^{-1}\rho^{q-3} & \end{aligned}$$

where the last factor in the left-hand expression is the total measure of ρ-elements.

Here the first term in the right-hand expression always sums to $O(\mu^{-1}h^{-3})$ while the second and the third factor sum to their values as $\rho=1$ (i.e. $O(\mu^{-1}h^{-3})$ for sure) plus their values as $\rho=\bar\rho = C(\mu^{-1}h|\log h|)^{1/2}$ which do not exceed the second term in $(22.4.22)_q$. As $q=2,3$ the second or the third term respectively acquires an extra logarithmic factor but it does not change the estimate.

Furthermore, contribution of the zone $\{\rho\le\bar\rho\}$ to the remainder does not exceed $C\mu h^{-3}\nu(\bar\rho)+C\mu^2h^{-2}\nu(\bar\rho)\bar\rho^{-1}$ which is the second term in $(22.4.22)_q$.□

Corollary 22.4.5. *In the framework of Proposition 22.4.4 the remainder estimate* $O(\mu^{-1}h^{-3})$ *holds as* $q>3$.

Remark 22.4.6. In the generic case condition $(22.4.21)_4$ is fulfilled.

Sharp Asymptotics at Regular Points. III

Let us consider a special case

$$\epsilon h^{-1} \le \mu \le c h^{-1}. \tag{22.4.23}$$

Proposition 22.4.7. *Let conditions* (22.1.7) *and* (22.4.23) *be fulfilled. Moreover, let us assume that on* $\mathsf{supp}(\psi)$ *there are no resonances of order not exceeding* M *and also* $|\nabla(f_1 f_2^{-1})| \ge \epsilon_0$. *Furthermore, let condition*

$$\begin{aligned}(22.4.24)_r \quad & |(2p+1)\mu h f_1 V^{-1} + (2n+1)\mu h f_2 V^{-1} - 1| + \\ & |\nabla\big((2p+1)\mu h f_1 V^{-1} + (2n+1)\mu h f_2 V^{-1} - 1\big)| \le \epsilon_0 \implies \\ & \mathsf{Hess}\big((2p+1)\mu h f_1 V^{-1} + (2n+1)\mu h f_2 V^{-1} - 1\big) \ \textit{has at least } r \textit{ eigenvalues} \\ & \textit{with absolute values greater than } \epsilon_0\end{aligned}$$

be fulfilled with $r \ge 1$. *Then asymptotics with the standard implicit main part* (22.1.5) *and the remainder estimate* $(22.4.22)_{r+1}$ *i.e.* $O(h^{-2}+h^{r-4}|\log h|^{(r+2)/2})$ *holds.*

Proof. Follows easily the proof of Proposition 22.4.4. □

Remark 22.4.8. One can get rid of the logarithmic factors in estimate $(22.4.22)_q$ but we do not care since we are interested only in the generic cases $q = 4$, $r = 4$.

22.4.4 General Asymptotics at Resonances

Now assume that there are (k, l) resonances of order $m = k + l \ge 3$. However, considering ϵ_1-vicinity of any point one can assume that

(22.4.25) There are no (k', l')-resonances with $k' + l' \le M$ unless $k'/k = l'/l \in \mathbb{Z}$.

General Asymptotics at Resonances. I

So, let us consider

$$\Xi_{kl} = \{x : \ f_1 k = f_2 l\} \tag{22.4.26}$$

which under condition (22.4.4) is a smooth surface. One can assume without any loss of the generality that the analogue of (22.3.20) holds: $kf_1 - lf_2 = x_1$ while (x_2, ξ_1, ξ_2) are coordinates on Ξ_{kl}. Let us introduce a scaling function

$$\gamma = \epsilon|x_1| + \frac{1}{2}\bar{\gamma}, \qquad \bar{\gamma} = \mu^{-1+\delta} \tag{22.4.27}$$

with arbitrarily small exponent $\delta > 0$.

Proposition 22.4.9. *Let conditions* (22.4.25) *with* $k + l = m \geq 3$ *and* (22.2.19) *be fulfilled at* $\mathsf{supp}(\psi)$. *Then under condition* (22.4.1) *asymptotics with the standard implicit main part* (22.1.5) *and the remainder estimate* $O(\mu^{-1}h^{-3} + \mu^2 h^{-2})$ *holds.*

Proof, Part I. We will give the proof working the worst-case scenario $m = 3$; as $m \geq 4$ one can simplify the proof. In this part we are going to prove that

(22.4.28) The contribution of the zone $\{|x_1| \geq \bar{\gamma}\}$ to the remainder is $O(\mu^{-1}h^{-3} + \mu^2 h^{-2})$.

After reduction to precanonical form with non-diagonal terms corresponding to resonances, in this zone one can get rid of non-diagonal terms (modulo $O(\mu^{-M})$). Let us consider scaling function ρ_{pn} introduced by (22.4.4) for a full symbol of $\mathcal{A}_{pn}$.

(a) Consider first subzone

$$\bigl\{x : \ |x_1| \geq \mathsf{max}(\rho, \bar{\gamma})\bigr\}. \tag{22.4.29}$$

Then one can apply the same arguments as in the proof of Proposition 22.4.1; however, there is a problem[12]: as function of x_1 ρ_{pn} remains γ-admissible only as

$$\gamma \geq \bar{\gamma}_1 := \mu^{(4-2m)/3}. \tag{22.4.30}$$

So far the arguments as in the proof of Proposition 22.4.1 result in[13]

[12] It is not a problem at all as $m = 5$ and rather a marginal problem as $m = 4$ but for $d = 3$ this requires a certain attention.

[13] The border case (22.4.14) (may be with the different powers of logarithm) should be covered only in zone $\{|\log h|^{-K_1} \leq |x_1| \leq \epsilon\}$.

(22.4.31) The contribution of the zone $\{x:\ |x_1|\ge \mathsf{max}(\rho,\bar\gamma_1)\}$ with $\bar\gamma_1 = \mu^{-2/3}$ to the remainder is $O(\mu^{-1}h^{-3}+\mu^2h^{-2})$.

(b) Consider now subzone

$$\{\gamma\ge\bar\gamma,\ \rho\ge\gamma,\}. \tag{22.4.32}$$

Note that here the derivatives of $\mathcal{A}_{pn}$ with respect to x_1 and to (x_2,ξ_1,ξ_2) have different values: while the derivative with respect to x_1 measures speed with respect to ξ_1 and the shift with respect to ξ_1 is quantum observable as $|\nabla_{x_1}\mathcal{A}_{pn}|\times\gamma\ge C\mu^{-1}h|\log h|$ provided γ is the scale with respect to x_1, other derivatives measure speed with respect to (ξ_2,x_1,x_2) and the shift is quantum observable as $|\nabla'\mathcal{A}_{pn}|\times\zeta\ge C\mu^{-1}h|\log h|$, provided ζ is the scale with respect to these variable where here and below $\nabla' := \nabla_{x_2,\xi_1,\xi_2}$.

So, let us introduce the third scaling function

$$\zeta_{pn} = \epsilon\big(|\mathcal{A}_{pn}|+|\nabla'\mathcal{A}_{pn}|^2\big)^{1/2}+(\rho\gamma)^{1/2} \tag{22.4.33}$$

where the last term will be actually included later.

Then one can make a ζ-admissible partition. So we have now (γ,ρ,ζ) elements with $\gamma\le\zeta\le\rho$ where ζ is the scale with respect to (x_2,ξ_1,ξ_2) and γ is the scale with respect to x_1 while ρ at this moment lost its scaling role. However, while ζ is γ-admissible function with respect to x_1, ρ preserves its magnitude in as x_1 varies by $O(\gamma)$ only under condition

$$\rho\ge\mu^{-2}\gamma^{-2}. \tag{22.4.34}$$

We claim that that then

$$T_0 = Ch|\log h|\big(\rho\gamma+\zeta^2\big)^{-1},\qquad T_0^* = Ch\big(\rho\gamma+\zeta^2\big)^{-1} \tag{22.4.35}$$

while

$$T_1 = \epsilon\mu\,\mathsf{min}\big(\zeta\rho^{-1},\gamma\zeta^{-1}\big)\asymp\epsilon\mu\zeta\gamma\big(\rho\gamma+\zeta^2\big)^{-1}. \tag{22.4.36}$$

Really, as $\rho\gamma\asymp\zeta^2$ a propagation speed with respect to ξ_1 is $\asymp\rho$ it is dual to x_1 of the scale γ. Meanwhile, speeds with respect to all other variables are bounded by $C\zeta$ and for given T_1 magnitudes of ζ,γ,ρ are preserved.

On the other hand, as $\rho\gamma\le\epsilon_1\zeta^2$ propagation speed with respect to (x_1,x_2,ξ_2) is $\asymp\zeta$ and they are dual to (ξ_1,ξ_2,x_2) of the scale ζ while propagation speed with respect to ξ_1 is bounded by $C\rho$ and for given T_1 magnitudes of ζ,γ,ρ are preserved.

We leave to the reader the standard justification on the quantum level (based on the energy estimates approach).

Thus $T_0^* T_1^{-1} \asymp \mu^{-1} h \gamma^{-1} \zeta^{-1}$; in virtue of the last term in the definition of ζ one can skip $\rho\gamma$ in $(\zeta^2 + \rho\gamma)$ here and below.

Then the total contribution of all (ρ, ζ, γ) elements to the remainder does not exceed

$$C\mu^2 h^{-2} \times \gamma \times T_0^* T_1^{-1} \times \bigl((\rho\gamma + \zeta^2)(\mu h)^{-1} + 1\bigr) \times \bigl(\rho(\mu h)^{-1} + 1\bigr) \tag{22.4.37}$$

where $C\bigl((\rho\gamma + \zeta^2)(\mu h)^{-1} + 1\bigr) \asymp C\bigl((\zeta^2(\mu h)^{-1} + 1\bigr)$ is an upper bound for a number of indices n violating ellipticity for a given index p.

If one picks up only "1" from both factors with the parentheses in (22.4.37) and replaces $T_0 T_1^{-1}$ by 1, then summation with respect to partitions results in $C\mu^2 h^{-2}$; on the other hand, since $\zeta^2 \le \rho$, one can rewrite the above expression (22.4.37) as

$$C\mu h^{-1} \times \zeta^{-1}\bigl(\zeta^2(\mu h)^{-1} + 1\bigr) \times \rho(\mu h)^{-1} \asymp Ch^{-2}\bigl(\zeta(\mu h)^{-1} + \zeta^{-1}\bigr)\rho. \tag{22.4.38}$$

Then summation with respect to ζ from $(\rho\gamma)^{1/2}$ to ρ results in

$$Ch^{-2}\rho\bigl(\rho(\mu h)^{-1} + \rho^{-1/2}\gamma^{-1/2}\bigr) \asymp C\mu^{-1}h^{-3}\rho^2 + Ch^{-2}\rho^{1/2}\gamma^{-1/2}. \tag{22.4.39}$$

The second term in the right hand expression sums with respect to (ρ, γ) to $Ch^{-2}\bar{\gamma}^{-1/2} \ll C\mu^2 h^{-2}$. However, the first term sums to $C\mu^{-1}h^{-3}|\log h|$ and this logarithmic factor appears due to summation with respect to γ.

To get rid of this factor let us notice that only case $\mu \le h^{-1/3}|\log h|^{1/3}$ needs to be addressed and only zone $\{\zeta \ge \gamma^{\kappa}\}$ should be reconsidered (with an arbitrarily small exponent $\kappa > 0$); in this case $\zeta \gg (\rho\gamma)^{1/2}$. Moreover, only term $Ch^{-4}\rho\zeta^2\gamma \times T_0^* T_1^{-1}$ in (22.4.37) should be reexamined.

However, then one does not need to use the canonical form but rather a weak magnetic field approach and take $T_0^* = h\zeta^{-2}$ and $T_1 = \mu\zeta$ and the contribution of this zone to the term in question does not exceed $C\mu^{-1}h^{-3}\int \zeta^{-1}\,d\gamma \le C\mu^{-1}h^{-3}$. Therefore

(22.4.40) The contribution of the zone $\bigl\{\rho \ge \max(\gamma, \mu^{-2}\gamma^{-2}),\ \gamma \ge \bar{\gamma}\bigr\}$ to the remainder is $O(\mu^{-1}h^{-3} + \mu^2 h^{-2})$.

(c) Now let us consider the remaining part of the zone $\{\epsilon \ge \gamma \ge \bar{\gamma}\}$. In this zone let us introduce the scaling function

$$\eta = \mu^2\gamma^3\rho. \tag{22.4.41}$$

Then ρ as a function of x_1 is η-admissible. Let us modify definition (22.4.33) etc replacing γ by η:

$$\zeta_{pn} = \epsilon\bigl(|\mathcal{A}_{pn}| + |\nabla'\mathcal{A}_{pn}|^2\bigr)^{1/2} + (\rho\gamma)^{1/2} \tag{22.4.33}$$

and then (22.4.35), (22.4.36) are also modified in the same way (anyway, terms $\rho\gamma$ originally and $\rho\eta$ now are not important):

$$T_1 = \epsilon\mu\,\min\bigl(\zeta\rho^{-1}, \eta\zeta^{-1}\bigr) \asymp \epsilon\mu\eta\zeta^{-1} \tag{22.4.35}$$

and $T_0^* T_1^{-1} \asymp \mu^{-1}h\eta^{-1}\zeta^{-1}$.

Then modified (22.4.37) and (22.4.38) expressions

$$\begin{aligned} C\mu h^{-1} \times \zeta^{-1}\bigl(\zeta^2(\mu h)^{-1} + 1\bigr) \times \rho(\mu h)^{-1}\gamma\eta^{-1} \\ \asymp Ch^{-2} \times \bigl(\zeta(\mu h)^{-1} + \zeta^{-1}\bigr)\rho\gamma\eta^{-1} \end{aligned} \tag{22.4.37}$$

estimate contribution of all $(\rho, \gamma, \eta, \zeta)$ elements. Here an (unpleasant) factor $\gamma\eta^{-1}$ appears since the measure of the strip remains γ and γ^{-1} is replaced by η^{-1} in $T_0^* T_1^{-1}$.

Plugging η into (22.4.37)* one gets

$$C\mu^{-3}h^{-3}\gamma^{-2}\zeta + C\mu^{-2}h^{-2}\gamma^{-2}\zeta^{-1} \tag{22.4.42}$$

and the first term sums to $o(\mu^{-1}h^{-3})$.

The second term sums with respect to ζ to its value at the smallest ζ satisfying conditions $T_1 = \mu\eta\zeta^{-1} \ge T_0 = h\zeta^{-2}$ and $\zeta \ge (\rho\eta)^{1/2}$:

$$\zeta = \max\bigl(\mu^{-1}h\eta^{-1}, (\rho\eta)^{1/2}\bigr) = \max\bigl(\mu^{-3}h\rho^{-1}\gamma^{-3}, \rho\mu\gamma^{3/2}\bigr); \tag{22.4.43}$$

so one gets

$$C\mu^{-2}h^{-2}\gamma^{-2}\,\min\bigl(h^{-1}\rho\mu^3\gamma^3, \rho^{-1}\mu^{-1}\gamma^{-3/2}\bigr)$$

which sums with respect to ρ to $C\mu^{-1}h^{-5/2}\gamma^{-5/4}$ and then with respect to γ to $O(\mu^{1/4}h^{-5/2}) \ll \bigl(\mu^{-1}h^{-3} + \mu^2h^{-2}\bigr)$.

On the other hand, if $T_1 \le T_0$ then one just replaces T_1 by T_0 and gets $Ch^{-4}\zeta^2\rho\gamma + C\mu h^{-3}\rho\gamma$ with $\zeta = (\rho\eta)^{1/2}$ i.e.

$$Ch^{-4}(\rho\eta)^{1/2}\rho\gamma + C\mu h^{-3}\rho\gamma \asymp Ch^{-4}\mu\rho^2\gamma^{5/2} + C\mu h^{-3}\rho\gamma \tag{22.4.44}$$

and it happens only if $(\rho\eta)^{1/2} \le \mu^{-1}h\eta^{-1}$ or equivalently $\rho \le \bar{\rho} := \mu^{-2}h^{1/2}\gamma^{-9/4}$. Plugging $\bar{\rho}$ in (22.4.44) we get $C\big(h^{-3}\mu^{-3}\gamma^{-2} + \mu^{-1}h^{-5/2}\gamma^{-5/4}\big)$; summation with respect to γ results in $o(\mu^{-1}h^{-3} + \mu^{1/4}h^{-5/2})$. Therefore

Statement (22.4.28) *is proven.* □

Proof of Proposition 22.4.9, Part II. We are left with the zone $\{|x_1| \le \bar{\gamma}\}$; its contribution to the remainder does not exceed $C\mu h^{-3}\bar{\gamma} = C\mu^{\delta}h^{-3}$ which is $O(\mu^2h^{-2})$ as $\mu \ge h^{-1/2-\delta'}$ and therefore only case $\mu \le h^{-1/2-\delta'}$ needs to be addressed. In this zone we use precanonical form with non-diagonal matrix elements but without singular terms.

Let $\mathcal{A}^0_{pn}$ denote diagonal matrix elements. One can see easily that

$$|\nabla'\mathcal{A}^0_{pn}| \equiv |\nabla'\log(f_jV^{-1})| \qquad \mathsf{mod}\ O(\gamma) \qquad \text{as}\ \ |x_1| \le \gamma \tag{22.4.45}$$

(as $\mathcal{A}_{pn}$ is non-elliptic) does not actually depend on p; let us define

$$\zeta := |\nabla'\log(f_jV^{-1}|\big|_{x_1=0} + (\rho\gamma)^{1/2}. \tag{22.4.46}$$

Let us define ρ_{pn} as before but with $\mathcal{A}^0_{pn}$ instead of $\mathcal{A}_{pn}$.

Note that in contrast to ζ ρ_{pn} strongly depends on p.

Then, as before $\mu^{-1}\rho$ controls the propagation speed with respect to x_1 and thus bounds the propagation speed with respect to ζ while $\mu^{-1}\zeta$ controls the propagation speed with respect to (x_1, x_2, ξ_2) as long as $\zeta \ge C\gamma$, $\rho \ge C\gamma$. Recall that $\gamma \ge C\mu^{-1}$.

However, the propagation speed with respect to ρ is a different matter. Considering commutator of

$$\mathcal{A} = f_1V^{-1}Z_1^*Z_1 + f_2Z_2^*Z_2 + 2\mu^{-1}\,\mathsf{Re}\big(\beta Z_1^*Z_2^2\big) + ...$$

(assuming that resonance is $2f_2 = f_1$) with

$$\partial_{x_1}\mathcal{A}^0 = \varkappa Z_1^*Z_1 + \omega\mathcal{A}^0 + ...\,, \qquad \varkappa = f_1\big(\partial_{x_1}(\log(f_1f_2^{-1})\big) \tag{22.4.47}$$

(with $\dots = O(\gamma)$) one can see easily that

$$\varkappa\mu^{-1}\bigl[\mathcal{A}, \partial_{x_1}\mathcal{A}^0\bigr] \equiv \bigl[\mathsf{Re}\,\beta Z_1^* Z_2^2, Z_1^* Z_1\bigr] = 2\varkappa\,\mathsf{Re}\,\beta Z_1^* Z_2^2 \mod O(\gamma) \tag{22.4.48}$$

which is bounded by 1. Therefore

(22.4.49) The propagation speed with respect to ρ does not exceed 1. Furthermore, ρ is properly defined as $\rho \ge C\mu h|\log h|$ (the logarithmic uncertainty principle).

Without nondegeneracy condition there is not much use of ζ; let us consider elements with $\rho \ge C\gamma$. Due to (22.4.49) we can pick up $T_1 = \epsilon\rho$. Then the contribution to the remainder of all such elements does not exceed

$$C\mu^2 h^{-2}\gamma \times h\zeta^{-2}\rho^{-1}\bigl(\zeta^2(\mu h)^{-1}+1\bigr)\times\bigl(\rho(\mu h)^{-1}+1\bigr) \tag{22.4.50}$$

and as long we include $O(\mu^2 h^{-2})$ into final remainder estimate we can skip "+1" in the last factor (due to the same arguments as before) resulting in

$$C\mu h^{-2}\gamma\zeta^{-2}\bigl(\zeta^2(\mu h)^{-1}+1\bigr) \asymp Ch^{-3}\gamma + C\mu h^{-2}\zeta^{-2}\gamma \le Ch^{-3}\gamma + C\mu h^{-2}\rho^{-1}; \tag{22.4.51}$$

as $\gamma = \bar\gamma$ this expression does not exceed $Ch^{-3}\bar\gamma + C\mu h^{-2}\rho^{-1}$ which sums with respect to ρ to $Ch^{-3}\bar\gamma|\log h| + C\mu h^{-2}\bar\rho^{-1}$ and the last term is $O(\mu^2 h^{-2})$.

On the other hand, in the zone $\{\rho \le \bar\rho\}$ we pick up $T_0^* T_1^{-1} = 1$ and its contribution to the remainder does not exceed $C\bar\rho^2\bar\gamma^2 h^{-4} + C\mu h^{-3}\bar\rho\bar\gamma + C\mu^2 h^{-2}$.

Therefore, as $h^{-1/3-\delta'} \le \mu \le h^{-1/2-\delta'}$, the contribution to the remainder of the zone $\{|x_1| \le \bar\gamma = \mu^{-1+\delta}\}$ with sufficiently small $\delta = \delta(\delta') > 0$ does not exceed $C\mu^2 h^{-2}$. In this case Proposition 22.4.9 is also proven. □

We will need the following

Proposition 22.4.10. *Let conditions* (22.1.7), (22.4.1) *and* $|f_1 - f_2| \ge \epsilon$ *be fulfilled. Let us consider the precanonical form.*

Finally, let $\mathcal{Q}$ *be a* ρ*-admissible partition element element in* (q_1, q_2)*, then quantized as* $\mathcal{Q}\bigl((h^2D_3^2 + \mu^2 x_3^2), (h^2 D_4^2 + \mu^2 x_4^2)\bigr)$ *with*

$$\rho \ge C\mu h|\log h| + C\bar\gamma_0^k \gamma^{1-k} \tag{22.4.52}$$

and ψ be γ-admissible with respect to x_1, either supported in $\{|x_1| \asymp \gamma\}$ as $\gamma > \bar{\gamma}_0 := C\mu^{-1}$ or supported in $\{|x_1| \lesssim \gamma\}$ as $\gamma = \bar{\gamma}_0$. Then

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma(\psi \mathcal{Q} u Q_y^t)| \le Ch^s, \tag{22.4.53}$$

and

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma(\psi \mathcal{Q} u Q_y^t)| \le C\rho\gamma h^{-3} \tag{22.4.54}$$

as $|\tau| \le \epsilon$ and

$$Ch|\log h|\rho^{-1} \le T \le \epsilon\mu^{-1}. \tag{22.4.55}$$

Proof. Consider the propagation with respect to either $(x_3, \mu^{-1}hD_3)$ or $(x_4, \mu^{-1}hD_4)$. Due to (22.1.7) on the energy levels close to 0 at least one of the symbols of $\mu^2 x_j^2 + h^2 D_3^j$ is of magnitude 1 ($j = 3, 4$). The propagation speed with respect to $(x_3, x_4, \mu^{-1}hD_3, \mu^{-1}hD_4)$ is $\asymp 1$. Therefore one can trade $T \le \epsilon\mu^{-1}$ to $\bar{T} = Ch|\log h|$ and in the estimate of Fourier transform to $\bar{T}^* = Ch$. The remaining part of the proof is easy and left to the reader.□

Proof of Proposition 22.4.9, Part III. Therefore only case $h^{-\delta_0} \le \mu \le h^{-1/3-\delta'}$ remains to be addressed where $\delta_0 > 0$ is small and fixed and $\delta' > 0$ is arbitrarily small. It follows from Parts I, II that exponents $\delta > 0$ in the definition of $\bar{\gamma}$ and $\delta' > 0$ are *independently* small.

(a) Let us consider elements with $|\nabla'(V^{-1}f_j)| \asymp \varsigma \ge C\gamma$. Then as $T_0 = Ch|\log h|(\varsigma^2 + \rho\gamma)^{-1} \le \epsilon\rho$, i.e. as

$$\rho \ge \varrho := C\min\bigl(h|\log h|\varsigma^{-2}, (\gamma^{-1}h|\log h|)^{1/2}\bigr) + C\gamma, \tag{22.4.56}$$

ρ is preserved on the time interval T_0 which can be traded to $T_1 = \epsilon\mu\varsigma$. Therefore the contribution to the remainder of all such elements does not exceed

$$C\rho\gamma(\varsigma^2 + \mu h)h^{-4}T_0^* T_1^{-1} \asymp C\mu^{-1}h^{-3}\rho\gamma\varsigma^{-1} + Ch^{-2}\rho\gamma\varsigma^{-3}; \tag{22.4.57}$$

summation with respect to ς, ρ, γ trivially results in $O(\mu^{-1}h^{-3} + \mu^2 h^{-2})$.

On the other hand, one can see easily that if (22.4.56) is violated, then $C\varrho$ remains an upper bound for ρ at time $|t| \le T_0 = Ch|\log h|\varsigma^{-1}$; therefore contribution of such elements to the remainder does not exceed

$$\begin{aligned} &C\rho\gamma(\varsigma^2 + \varrho\gamma + \mu h)h^{-4}T_0^* T_1^{-1} \\ &\qquad \asymp C\mu^{-1}h^{-3}\varrho\gamma\varsigma^{-1} + C\mu^{-1}h^{-3}\varrho^2\gamma^2\varsigma^{-3} + Ch^{-2}\rho\gamma\varsigma^{-3} \end{aligned} \tag{22.4.58}$$

which does not exceed the same expression with $\varsigma = \gamma$ and corresponding ϱ; one can see easily that $\varrho \le \gamma h^{-5\delta'}$ and (22.4.58) is $o(\mu^{-1}h^{-3})$.

Therefore only elements with $|\nabla' V^{-1} f_j| \le C\gamma$ remain to be treated, where either $|x_1| \le \gamma = C\mu^{-1}$ or $C\mu^{-1} \le |x_1| \asymp \gamma \le \mu^{-1+\delta}$.

(b) Let us consider the propagation speed with respect to ρ more precisely. Note that as $|x_1| \asymp \gamma \ge C\mu^{-1}$ one can translate non-diagonal term $\mu^{-1}\operatorname{Re}(\omega Z_1^* Z_2^2)$ into

$$\mu^{-2}x_1^{-1}|\omega|^2(Z_1^* Z_1 - 4z_2^* Z_2)Z_2^* Z_2 + ...$$

with

$$\rho = -\mu^{-2}x_2^{-1}|\omega|^2(Z_1^* Z_1 - 4z_2^* Z_2)Z_2^* Z_2 + ...$$

Then $[a,\rho] = O(\mu^{-s}\gamma^{-s} + \mu^{-1})$ and furthermore along trajectories $[a,\rho](t) = [a,\rho](0) + O((\mu^{-s}\gamma^{-s}|t|)$ (where s is an arbitrarily large exponent and $\delta = \delta(s) > 0$ is small enough) and therefore

(22.4.59) ρ preserves both its sign and the bound from below as $\rho(0) \ge C\mu^{-s}\gamma^{1-s}$ and $|t| \le T_1$ with

$$(22.4.60)\quad T_1 \coloneqq \epsilon\rho^{1/2}\mu^{s/2}\gamma^{s/2} \quad \text{as} \quad \bar\gamma_0 = C\mu^{-1} \le \gamma \le \bar\gamma,\ \rho \ge C\mu^{-s/2}\gamma^{1-s/2}$$

(where $\rho = \rho(0)$). Therefore the contribution of the corresponding strip to the remainder does not exceed

$$G(\rho,\gamma) \coloneqq C\mu^2 h^{-2}\gamma \times h\rho^{-2} \times \bigl((\rho^2+\gamma^2)(\mu h)^{-1}+1\bigr) \times \rho(\mu h)^{-1} \times \rho^{-1/2}\mu^{-s/2}\gamma^{-s/2}.$$

Therefore after summation over $\rho \ge C\mu^{-s/6}\gamma^{1-s/6}$ we arrive to the expression $G(1,\gamma) + G(\mu^{-s/6}\gamma^{1-s/2})$ and for large enough s summation with respect to γ results in $G(1,\mu^{-1}) + G(\mu^{-1}\mu^{-1})$ which is $O(\mu^{-1}h^{-3})$.

Meanwhile, the contribution of the zone $\{|x_1| \asymp \gamma, \rho \le \varrho \coloneqq C\mu^{-s/6}\gamma^{1-s/6}\}$ to the remainder due to Proposition 22.4.10 does not exceed $C\varrho\gamma\mu h^{-3} \asymp C\mu^{-s/6}\gamma^{1-s/6}h^{-3}$ and summation with respect to γ results in $C\mu^{-1}h^{-3}$. Therefore

(22.4.61) Contribution of zone $\{\bar\gamma_0 \le |x_1| \le \bar\gamma\}$ to the remainder is $O(\mu^{-1}h^{-3})$.

(c) Similar arguments work for zones $\{|x_1| \le \bar\gamma_0,\ \rho \ge \varrho := C\mu^{-1}\}$ with $T_1 = C\rho^{1/2}$ and for $\{|x_1| \le \bar\gamma_0,\ \rho \le \varrho\}$ and therefore

(22.4.62) Contribution of the zone $\{|x_1| \le \bar\gamma_0\}$ to the remainder is $O(\mu^{-1}h^{-3})$.

This concludes Part III and the whole proof. □

22.4.5 Sharp Asymptotics at Resonances

In this subsection we are going to prove sharp remainder estimate under generic assumptions to V. We know from Proposition 22.2.6 that in the generic situation critical points of $\phi_\alpha = \alpha \log f_1 + (1-\alpha)\log f_2 - \log V$ are non-degenerate except of discrete values of $\alpha = \alpha_j$. One can prove easily that

(22.4.63) In the generic case degenerate critical points α_j of ϕ_α are not resonances. Then as $\alpha \ne \alpha_j$ the set of critical points is a smooth 1-dimensional curve parametrized by α and resonance surface is 3D surface.

(22.4.64) In the generic case these curve and resonance surface Ξ_{kl} meet at isolated points and are transversal in them.

Then

$$(22.4.65)_{q,r}\quad \tilde\nu(\rho,\zeta,\gamma) := \mathsf{mes}\Big\{(x,\alpha):\ |kf_1 - lf_2| < \gamma,\ |\nabla'\big(\alpha\log f_1 + (1-\alpha)\log f_2 - \log V\big)| \le \zeta,\ |\nabla\big(\alpha\log f_1 + (1-\alpha)\log f_2 - \log V\big)| \le \rho\Big\} \le C\rho^{q-r}\zeta^r\gamma \quad \text{as}\ \ \rho\ge\gamma,\ \rho\ge\zeta\ge(\rho\gamma)^{1/2}$$

with $r = 3$, $q = 4$.

Proposition 22.4.11. *Let us assume that* $|f_1 - f_2| \ge \epsilon_0$ *and* $|\nabla(f_1 f_2^{-1})| \ge \epsilon_0$ *on* $\mathsf{supp}(\psi)$. *Furthermore, let conditions* (22.1.7), $(22.4.21)_q$ *with* $q > 3$ *and* $(22.4.65)_{q,r}$ *be fulfilled with* $r > 2$, $q > 3$. *Then under condition* (22.4.18) *asymptotics with the standard implicit main part* (22.1.5) *and the remainder estimate* $O(\mu^{-1}h^{-3})$ *holds.*

Proof, Part I. We will follow the proof of Proposition 22.4.7 and use the same numbering of its parts.

(a) Repeating arguments leading to the proof of statements of Proposition 22.4.4 one can prove easily the following analogue of (22.4.40):

(22.4.66) Under condition $(22.4.21)_q$ the contribution of the zone $\{x: |x_1| \ge \mathsf{max}(\rho, \mu^{-2/3})\}$ to the remainder does not exceed $(22.4.22)_q$.

(b) Consider zone: $\{\rho \ge \mathsf{max}(\gamma, \mu^{-2}\gamma^{-2},\ \gamma \ge \bar{\gamma}\}$. Then the total contribution of all (ρ,ζ,γ) elements does not exceed expression (22.4.38) multiplied by $\tilde{\nu}(\rho,\zeta,\gamma)\rho^{-1}$:

$$C\mu h^{-1}\big(\zeta(\mu h)^{-1}+\zeta^{-1}\big)\times\big(\rho(\mu h)^{-1}+1\big)\times\rho^{q-r-1}\zeta^{r} \tag{22.4.67}$$

where we used $(22.4.65)_{q,r}$ to estimate $\tilde{\nu}$. Then as $r>1$ summation with respect to ζ results in the same expression as $\zeta=\rho$; further as $q>3$ summation with respect to ρ results in the same expression as $\rho=1$ which is $C\mu^{-1}h^{-3}$ and summation with respect to γ results in $C\mu^{-1}h^{-3}|\log h|$. One can get rid of the logarithmic factor using the same arguments as in the f proof of Proposition 22.4.4.

(c) Consider the remaining part of zone $\{\epsilon \ge \gamma \ge \bar{\gamma}\}$ and introduce scaling function η by (22.4.41). Then one gets $(22.4.37)^*$ modified in the same way as in the f proof of Proposition 22.4.4 and multiplied by $\eta\gamma^{-1}\tilde{\nu}(\rho\gamma)^{-1}$:

$$C\mu h^{-1}\big(\zeta(\mu h)^{-1}+\zeta^{-1}\big)\times\big(\rho(\mu h)^{-1}+1\big)\times\eta\gamma^{-1}\times\rho^{q-r-s-1}\zeta^{r}. \tag{22.4.67}^*$$

Then as in (b) summation with respect to ζ results in its value as $\zeta=\rho$:

$$C\mu h^{-1}\big(\rho(\mu h)^{-1}+\rho^{-1}\big)\times\big(\rho(\mu h)^{-1}+1\big)\times\mu^{2}\varrho^{-2}\gamma^{3}\rho\gamma^{-1}\times\rho^{q-s-1}\gamma^{s-1}$$

and summation with respect to ρ returns the above expression at its largest value which is $\mu^{-2}\gamma^{-2}\varrho^{2}$; the result does not exceed $C\mu^{-1-\delta'}h^{-3}$ and summation with respect to γ returns $o(\mu^{-1}h^{-3})$. □

Remark 22.4.12. (i) Again as the order of resonance $m\ge 4$, analysis of (c) is not needed;

(ii) Furthermore, the rough remainder estimate of zone $\{|x_1| \le \bar\gamma\}$ returns $O(\mu^{-1}h^{-3})$ as $m \ge 5$ and $\mu \le h^{\delta'-1}$ and $O(\mu^{-1+\delta}h^{-3})$ as either $m = 4$ or $m = 5$, $h^{\delta'-1} \le \mu \le ch^{-1}$.

Proof, Part II. (a) Analysis in the zone $\{|x_1| \le \bar\gamma\}$ is now simpler. Note first that the contribution of the zone $\{\zeta \le \bar\gamma\}$ to the remainder does not exceed $C\mu h^{-3}\bar\gamma^{r+1} = O(\mu^{-1}h^{-3})$ as $r > 1$ and $\delta < \delta(r)$.

(b) Consider zone $\{\zeta \ge C\bar\gamma\}$. Then defining T_0, T_0^* by (22.4.35) and T_1 by (22.4.36) one estimates contribution of $(\rho, \bar\gamma, \zeta)$ elements by (22.4.50) multiplied by $\tilde\nu(\rho\bar\gamma)^{-1}$:

$$C\mu^2h^{-2}\bar\gamma \times h\zeta^{-2} \times \bigl(\zeta^2(\mu h)^{-1} + 1\bigr) \times \rho(\mu h)^{-1} \times \bigl(\mu^{-1}\bar\gamma^{-1/2} + \rho^{-1}\bigr) \times \rho^{q-r-2}\zeta^r \tag{22.4.68}$$

and summation with respect to ζ results in its value as $\zeta = \rho$ (now we need $r > 2$)

$$C\mu h^{-2}\bar\gamma \times \bigl(\rho^2(\mu h)^{-1} + 1\bigr) \times \bigl(\mu^{-1}\bar\gamma^{-1/2} + \rho^{-1}\bigr) \times \rho^{q-3}$$

and summation with respect to ρ results in $Ch^{-3}\bar\gamma$ which is marginally worse than $O(\mu^{-1}h^{-3})$. To improve it one can sum to $\zeta \le \mu^{-\kappa}$ and in zone $\{\zeta \ge \mu^{-\kappa}\}$ one can take $T_1 = \epsilon\mu\zeta$. □

22.4.6 General Estimates near Σ

Now let us consider the vicinity of $\Sigma = \{f_1 = f_2\} = \{v_1 = v_2 = 0\}$ where

$$v_1|_\Sigma = v_2|_\Sigma = 0,\ (\nabla v_1)|_\Sigma \text{ and } (\nabla v_2)|_\Sigma \text{ are linearly independent} \tag{22.4.69}$$

and near Σ

$$f_{1,2} = f \pm (v_1^2 + v_2^2)^{1/2}, \qquad f > 0. \tag{22.4.70}$$

This analysis is simpler than near third order resonances because codimension of Σ is now 2 and everywhere factor γ^1 reflecting measure should be replaced by γ^2.

Proposition 22.4.13. *Let condition* (22.2.1)–(22.2.2) *be fulfilled and let* ψ *be supported in the small vicinity of* Σ.

The standard implicit asymptotic formula (22.1.5) *holds with the remainder estimate* $O(\mu^{-1}h^{-3} + \mu^2h^{-2})$.

Proof. (a) Note first that

(22.4.71) The contribution of the zone $\{\mathsf{dist}(x,\Sigma)\le\gamma\}$ to the remainder does not exceed $C\mu h^{-3}\gamma^2$ and as

$$\gamma := \mathsf{dist}(x,\Sigma) \asymp |f_1-f_2| \le \bar\gamma_1 := c\mu^{-1}+c(\mu h)^{1/2} \tag{22.4.72}$$

this contribution is $O(\mu^{-1}h^{-3}+\mu^2h^{-2})$.

On the other hand,

(22.4.73) One can reduce operator to the canonical form without non-diagonal terms as long as

$$\gamma \ge \bar\gamma := \mu^{-1/2}h^{1/2-\delta'}+\mu^{-2}h^{-\delta'}. \tag{22.4.74}$$

The second term in (22.4.74) appears because one needs to get rid of terms $Z_1^iZ_1^{*j}Z_2^kZ_2^{*l}$ with $i+k=j+l$ but $(i,j)\ne(k,l)$ and these terms are of magnitude $O(\mu^{-2})$ unless $i+j+k+l=2$ in which case one just diagonalizes the quadratic form and it is where the first term in in (22.4.74) comes from.

It is important that $\bar\gamma\le\bar\gamma_1$. In the quest for remainder estimate $O(\mu^{-1}h^{-3})$ one would need to take $\bar\gamma=c\mu^{-1}$ and $\bar\gamma\le\bar\gamma_1$ would hold as $\mu\le h^{\delta'-1}$.

(b) Making $\epsilon\gamma$-admissible partition and ρ-admissible subpartition with

$$\rho=\epsilon|\nabla\phi_\alpha|\gamma+\frac{1}{2}\bar\rho,\qquad \bar\rho=\bigl(C\mu^{-1}h|\log h|\bigr)^{1/2} \tag{22.4.75}$$

one can take

$$T_0^*=Ch\rho^{-2}\gamma,\qquad T_1=\epsilon\mu\gamma \tag{22.4.76}$$

and the total contribution to the remainder of (γ,ρ) subelements with $\rho\ge\varrho$ to the remainder does not exceed

$$\begin{aligned} C\mu^2h^{-2}\times\mu^{-1}h\rho^{-2}\times\bigl(\rho^2(\mu h\gamma)^{-1}+1\bigr)\times\bigl(\rho(\mu h\gamma)^{-1}+1\bigr)\times\gamma^2 \\ \asymp C\mu^{-1}h^{-3}\rho+C\rho^{-1}h^{-2}\gamma+C\mu h^{-1}\rho^{-2}\gamma^2 \end{aligned} \tag{22.4.77}$$

where γ^2 is their total measure. The right-hand expression sums with respect to $\rho\in(\bar\rho,\gamma)$ to $G(\gamma):=C\mu^{-1}h^{-3}\gamma+Ch^{-2}\bar\rho^{-1}\gamma+C\mu h^{-1}\bar\rho^{-2}\gamma^2$. Then with respect to γ it sums to $G(1)=O\bigl(\mu^{-1}h^{-3}+\mu^{1/2}h^{-5/2}+\mu^2h^{-2}\bigr)$ where the middle term is less than the sum of two others.

(c) Meanwhile, the total contribution to the remainder of $(\bar\rho,\gamma)$ subelements does not exceed

$$C\mu^2h^{-2}\times\bigl(\bar\rho^2(\mu h\gamma)^{-1}+1\bigr)\times\bigl(\bar\rho(\mu h\gamma)^{-1}+1\bigr)\times\gamma^2\asymp Ch^{-4}\bar\rho^3+C\mu h^{-3}\gamma\bar\rho+C\mu^2h^{-2}\gamma^2. \tag{22.4.78}$$

This expression sums with respect to γ to its value as $\gamma=1$ resulting in

$$C\mu^{-3/2}h^{-5/2}|\log h|^{3/2}+C\mu^{1/2}h^{-5/2}|\log h|^{1/2}+C\mu^2h^{-2}.$$

Note that the first and the third term are properly estimated and the second terms is properly estimated save border case $h^{-1/3}|\log h|^{-K}\le\mu\le h^{-1/3}|\log h|^{K}$ which is treated as in the part (c) of the Proof of Proposition 22.4.1.

(d) Finally, as $\gamma\le\bar\gamma$ one does not need a subpartition; the contribution to the remainder does not exceed $C\mu h^{-3}\bar\gamma^2$ due to an analogue of Proposition 22.4.10 below. □

Proposition 22.4.14. *Proposition 22.4.10 remains true near* Σ *(i.e. without condition* $|f_1-f_2|\ge\epsilon$ *provided at point* $\bar x$ *main part of precanonical form is* $f_1(\mu^2x_3^2+h^2D_3^2)+f_2(\mu^2x_4^2+h^2D_4^2)$.

Proof. Proof basically repeats the one of Proposition 22.4.10. □

22.4.7 Sharp Asymptotics near Σ

Sharp Asymptotics near Σ. I

Now let us improve the results of the previous subsection. Let us observe that

(22.4.79) Microhyperbolicity condition holds at $\bar x\in\Sigma$ iff $\nabla(fV^{-1})$ is not a linear combination of $\nabla(v_1V^{-1})$, $\nabla(v_2V^{-1})$ with coefficients $(\beta_1,\beta_2)\in\mathbb{R}^2\cap B(0,1)$ [14].

In virtue of Chapter 19 we have already

[14] In the uniform sense, i.e. (22.2.22)

Proposition 22.4.15. *If* (22.2.22) *is fulfilled on* $\mathsf{supp}(\psi)$ *then the standard formula holds with* $\bar{T} = Ch|\log h|$.

Now let us analyze the meaning of condition (22.2.22). First of all, it is fulfilled as $\nabla_\Sigma(fV^{-1}) \neq 0$. So one needs to consider only set Σ_0 of the critical points of $fV^{-1}\big|_\Sigma$:

$$\Sigma_0 = \big\{x \in \Sigma,\ \nabla_\Sigma(fV^{-1}) = 0\big\}. \tag{22.4.80}$$

Then

Proposition 22.4.16. *For generic* V

(22.4.81) Σ_0 *consists of separate non-degenerate points.*

(22.4.82) *Magnetic form* ω_f *restricted to* Σ *is the generic closed form on* Σ *and thus degenerates on the smooth curve* $\{\{v_1, v_2\} = 0\}$.

(22.4.83) ω_M *does not degenerate on* Σ_0 *(which is equivalent to* $\{v_1, v_2\} \neq 0$ *on* Σ_0*).*

One can write down many generic properties, but they are overkill.

Sharp Asymptotics near Σ. II

First let us improve the remainder under condition (22.4.84) below (recall that $f = \frac{1}{2}(f_1 + f_2)$):

Proposition 22.4.17. *Let at some point* $\bar{x} \in \Sigma$

$$|\nabla(fV^{-1})| \geq \epsilon_0. \tag{22.4.84}$$

Then as ψ *is supported in the small enough vicinity of* $\bar{x}$ *the standard implicit formula holds with the remainder* $O\big(\mu^{-1}h^{-3} + \mu^{3/2}h^{-3/2}|\log h|\big)$.

Proof. (a) Note that here in contrast to (22.2.22) the complete gradient is considered. One needs to consider the case when microhyperbolicity condition is not fulfilled in $\bar{x} = 0$; then

$$\nabla(fV^{-1}) = \beta_1\nabla(v_1V^{-1}) + \beta_2\nabla(v_2V^{-1}) \qquad \text{at } \bar{x}, \tag{22.4.85}$$
$$\beta = (\beta_1, \beta_2) \in \mathbb{R}^2 : 1 \geq |\beta| \geq \epsilon_0$$

where $|\beta| \ge \epsilon_0$ due to (22.4.84).

Let us consider $\phi_\alpha = \alpha(f_1 V^{-1}) + (1-\alpha)(f_2 V^{-1})$; note that one can extend β_j to the vicinity of $\bar{x}$ so that

$$fV^{-1} = \tilde{f}(w) + \beta_1(w) v_1 V^{-1} + \beta_2(w) v_2 V^{-1} + O(|v|^2)$$

where $w = (w_1, w_2)$ are coordinates on Σ.

Without any loss of the generality one can assume that $\beta_2(w) = 0$; then under conditions (22.4.84), (22.4.85)

$$|\nabla \phi_\alpha| \ge \epsilon_1 \big(|(v_2 - \omega v_1^2)(v_1^2 + v_2^2)^{-1/2}| + |\alpha - \bar{\alpha}| \big) \tag{22.4.86}$$

where $\omega = \omega(w, v_1)$ is a smooth function.

(b) Let us follow the proof of Proposition 22.4.13. In part (b) estimate (22.4.77) (for contribution of all elements with $\rho \ge \varrho$) gains a factor $\rho\gamma^{-1}$ and becomes

$$C\mu^{-1}h^{-3}\rho^2\gamma^{-1} + Ch^{-2} + C\mu h^{-1}\rho^{-1}\gamma. \tag{22.4.77)*}$$

This expression sums with respect to ρ ranging from $\bar{\rho}$ to γ to

$$C\mu^{-1}h^{-3}\gamma + Ch^{-2}|\log h| + C\mu h^{-1}\bar{\rho}^{-1}\gamma.$$

Then summation with respect to γ results in

$$C\mu^{-1}h^{-3} + Ch^{-2}|\log h|^2 + C\mu^{3/2}h^{-3/2}$$

which is less than the announced estimate.

(c) In part (b) estimate (22.4.78) (for the contribution of all elements with $\rho \le \bar{\rho}$) gains a factor $\bar{\rho}\gamma^{-1}$ and becomes

$$Ch^{-4}\bar{\rho}^4\gamma^{-1} + C\mu h^{-3}\bar{\rho}^2 + C\mu^2 h^{-2}\bar{\rho}\gamma. \tag{22.4.78)*}$$

This expression sums with respect to γ to

$$C\mu^{-2}h^{-2}|\log h|^2\bar{\gamma}^{-1} + Ch^{-2}|\log h|^2 + C\mu^{3/2}h^{-3/2}|\log h|^{1/2}$$

which is also less than the announced remainder estimate.

(d) Finally, as $\gamma \le \bar{\gamma}$ one does not need a subpartition; the contribution to the remainder does not exceed $C\mu h^{-3}\bar{\gamma}^2 = C\mu^{-1}h^{-3} + C\mu^{-1/2}h^{\delta-5/2}$ and the second term here is obviously much less than the announced estimate. □

So, condition (22.4.84) is a kind of non-degeneracy condition, improving the remainder estimate.

Sharp Asymptotics near Σ. III

Let us now use the remaining non-degeneracy conditions.

Proposition 22.4.18. *Let conditions* (22.4.18) *and*

$$(22.4.87)_r \quad |(2p+2)\mu hfV^{-1}-1| + |\nabla_\Sigma\big((2p+2)\mu hfV^{-1}-1\big)| \le \epsilon_0 \implies \mathsf{Hess}_\Sigma\big((2p+2)\mu hfV^{-1}-1\big) \text{ has at least } r \text{ eigenvalues with absolute values greater than } \epsilon_0$$

be fulfilled. Then

(i) The total remainder is given by $(22.4.21)_{r+1}$ *while the main part is given by the standard implicit formula* (22.1.5).

(ii) Under condition (22.4.84) *the total remainder is given by* $(22.4.21)_{r+2}$ *as* $r=1$ *while the main part is given by the standard implicit formula* (22.1.5).

(iii) Under conditions (22.4.84) *and* (22.4.83) *the total remainder is* $O(\mu^{-1}h^{-3})$ *as* $r=2$ *while the main part is given by the standard implicit formula* (22.1.5).

Proof. (a) Under condition (22.4.84) the total contribution to the remainder of all (γ,ρ)-elements does not exceed $(22.4.77)^*$ i.e.

$$(22.4.88)_k \qquad C\mu^{-1}h^{-3}\rho^{k+1}\gamma^{-k} + C\rho^{k-1}\gamma^{1-k}h^{-2} + C\mu h^{-1}\gamma^{2-k}\rho^{k-2}$$

with $k=1$ while without it it does not exceed the same expression with $k=0$.

Further, under extra condition $(22.4.87)_r$ this expression acquires factor $C\rho^r\gamma^{-r}$. Therefore again $(22.4.88)_q$ with $q=k+r$ gives a proper estimate for the total contribution of all (γ,ρ)-elements to the remainder.

Consider summation with respect to $\bar\rho \le \rho \le \gamma \le 1$. The first term sums to $C\mu^{-1}h^{-3}$ independently on q; the second term sums to $Ch^{-2}|\log h|^2$ as $q=1$ and to $Ch^{-2}|\log h|$ as $q\ge 2$; the third term sums to $C\mu h^{-1}\bar\rho^{-1}$ as $q=1$, to $C\mu h^{-1}|\log h|^2$ as $q=2$ and to $C\mu h^{-1}|\log h|$ as $q=3$.

Therefore in all cases but one the remainder estimate $(22.4.21)_q$ with $q=k+r$ is proven; this exceptional case is $q=3$, $\mu \ge h^{-1}|\log h|^{-1}$ when the estimate $O(h^{-2}|\log h|)$ is recovered; recall that only zone $\{\rho\ge\bar\rho,\ \gamma\ge\bar\gamma\}$ is covered so far.

(b) To cover the remaining case let us introduce a scaling function $\zeta = \epsilon|\nabla_{\Sigma} f V^{-1}| + \bar{\rho}$ on Σ; one can extend this function to

$$\zeta = \epsilon|\nabla_{\Sigma} f V^{-1}| + \gamma. \tag{22.4.89}$$

Then in arguments above one can replace factor $\rho^2\gamma^{-2}$ by ζ^2 and then contribution of zone $\{\zeta \le \gamma^{\delta'}\}$ would be $O(\mu^{-1}h^{-3})$ as $q = 3$; so only subzone $\{\zeta \ge \gamma^{\delta'}\}$ remains to be treated. In this subzone one can take

$$T_0^* = Ch\zeta^{-2}, \qquad T_1 = \epsilon\mu\gamma \tag{22.4.90}$$

where one can take $T_0^* = Ch\zeta^{-2}$ rather than $T_0^* = Ch\zeta^{-1}\gamma^{-1}$ due to (22.4.83). Then the total contribution of such (γ, ζ) elements to the remainder does not exceed

$$C\mu^2h^{-2}\zeta^2\gamma^2 \times h\zeta^{-2} \times \mu^{-1}\zeta^{-1} \times \big((\zeta^2+\gamma)(\mu h)^{-1}+1\big) \times (\mu h)^{-1}$$

where $\zeta^2\gamma^2$ is the measure and $(\mu h)^{-1}$ and $\big((\zeta^2+\gamma)(\mu h)^{-1}+1\big)$ are estimates of the numbers of indices "p" and corresponding indices "n"; noting that $\zeta^2 \ge \gamma$ one can rewrite this expression as

$$C\mu^{-1}h^{-3}\gamma^2\zeta + Ch^{-2}\gamma^2\zeta^{-1}$$

and the summation with respect to $\zeta \ge \gamma^{\delta'}$ and γ results in $O(\mu^{-1}h^{-3})$.

Thus estimate $(22.4.21)_q$ for contribution of zone $\{\rho \ge \bar{\rho},\ \gamma \ge \bar{\gamma}\}$ is established.

(c) Further, contribution of (γ, ρ) elements with $\rho \le \bar{\rho}$, $\gamma \ge \bar{\gamma}$ to the remainder does not exceed (22.4.77) as $k = 0$ or $(22.4.77)^*$ as $k = 1$ multiplied by $\bar{\rho}^r\gamma^{1-r}$ i.e.

$$Ch^{-4}\bar{\rho}^{q+3}\gamma^{-q} + C\mu h^{-3}\bar{\rho}^{q+1}\gamma^{1-q} + C\mu^2h^{-2}\bar{\rho}^q\gamma^{3-q} \tag{22.4.91}_q$$

and one can check easily that summation with respect to $\gamma \ge \bar{\gamma}$ results in expression not exceeding $(22.4.21)_q$.

(d) Finally, contribution of zone $\{\gamma \le \bar{\gamma}\}$ to the remainder does not exceed $C\mu h^{-3}\bar{\gamma}^2$ which is $O(\mu^{-1}h^{-3})$ unless $\mu \ge h^{2\delta-1}$ and even in this case it is less than $(22.4.21)_1$.

Therefore as $q \geq 2$ we need some better arguments in this zone. Again, one needs to consider only part of it with $\{\zeta \geq \mu^{-\delta'}\}$ as contribution of the zone $\{\zeta \leq \bar{\zeta}\}$ does not exceed $C\mu h^{-3}\bar{\gamma}^2\bar{\zeta}^2$.

As using precanonical form the speed would be $O(\mu^{-1})$ and since only a magnitude of ζ is important for us and also inequality $\zeta \geq \gamma^{\delta'}$, one can take

$$T_0^* = Ch\zeta^{-1}\bar{\gamma}^{-1}, \qquad T_1 = \epsilon\mu\bar{\gamma}^{1-\delta''} \tag{22.4.92}$$

where an appropriate time direction for this T_1 is taken. In these arguments we do not assume (22.4.83) and and thus T_0 is not as it was in (b) (surely, some improvements are possible but not needed).

Then the total contribution to the remainder of ζ-elements does not exceed

$$Ch^{-4}\zeta^r\bar{\gamma}^2 \times h\zeta^{-1}\bar{\gamma}^{-1} \times \mu^{-1}\bar{\gamma}^{\delta''-1} \asymp C\mu^{-1}h^{-3}\zeta^{r-1}\bar{\gamma}^{\delta''}$$

which sums with respect to ζ to $C\mu^{-1}h^{-3}|\log h|\bar{\gamma}^{\delta''} = o(\mu^{-1}h^{-3})$ even as $r = 1$. □

Remark 22.4.19. (i) Since in the generic case in the critical points of $fV^{-1}|_\Sigma$ are non-degenerate (i.e. (22.4.81) is fulfilled) and also (22.4.82), (22.4.83) hold, the remainder estimate is $O(\mu^{-1}h^{-3})$.

(ii) Probably one can get rid of logarithmic factors in the above estimates but we do not care.

22.4.8 Summary

Proposition 22.4.20. *Let (g^{jk}) be fixed and then (V_j) be be generic, more precisely:*

(i) Outside of $\Sigma = \{x : \ f_1 = f_2\}$ critical points of $f_1 f_2^{-1}$ satisfy (22.3.30)$_3$ *and* (22.3.2)*.*

(ii) Σ be smooth 2-dimensional manifold and $|f_1 - f_2| \asymp \mathsf{dist}(x, \Sigma)$.

Further, let V be general but satisfying (22.1.7) *at $\mathsf{supp}(\psi)$.*

Then under condition (22.4.1) *the remainder is $O(\mu^{-1}h^{-3} + \mu^2 h^{-2})$ while the main part of asymptotics is given by implicit formula* (22.1.5)*.*

Proposition 22.4.21. *Let in the framework of Proposition 22.4.20* V *be generic i.e.*

(i) Outside of Σ *and resonances condition* $(22.4.21)_4$ *be fulfilled.*

(ii) Near resonances condition $(22.4.65)_{4,3}$ *be fulfilled.*

(iii) At Σ *conditions* (22.4.81)–(22.4.84) *be fulfilled*[15)]*.*

Let V *satisfy* (22.1.7) *at* $\mathsf{supp}(\psi)$*. Then under condition* (22.4.18) *the remainder is* $O(\mu^{-1}h^{-3})$ *while the main part of asymptotics is given by implicit formula* (22.1.5)*.*

22.5 Calculations

The purpose of this section is to pass from the implicit formula (22.1.5) to more explicit one, namely either (22.6.6) or (22.6.4) below with more or less explicit expression for $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$. It will be done by different methods depending on the magnitude of μ and also non-degeneracy conditions and the methods applied will be used in the classification. Our main concern will be to prove such formula under either no non-degeneracy condition at all or under the generic non-degeneracy condition.

22.5.1 Temperate Magnetic Field

In this subsection we will use formula (22.1.5) with $T_0 \le \epsilon\mu^{-1}$ and derive a remainder estimate. More precisely, if in Section 22.4 T was given by (22.1.5) with $T_0 \ge \epsilon\mu^{-1}$ and we replace it by $T_0 = \epsilon\mu^{-1}$ and estimate an error.

This error actually is the contribution of the affected domain to the remainder with some $T_0 \ll \mu^{-1}$ (usually $T_0 = Ch|\log h|$ under condition (22.1.7)) and $T_1 = \epsilon\mu^{-1}$.

Further, under condition (22.1.7) we can trade $T \le \epsilon\mu^{-1}$ to $T = Ch|\log h|$ and then we can apply the standard approach; we will assume here by default that

$$h^{-\delta_0} \le \mu \le h^{-1+\delta_0} \tag{22.5.1}$$

leaving case $\le h^{-1+\delta_0} \le \mu \le ch^{-1}$ for a separate consideration.

[15)] Actually these conditions should be fulfilled at $\Sigma_0^+ \cup \Sigma_0^0$ only.

Temperate Magnetic Field. I

Let us consider regular points first.

Proposition 22.5.1. *Assume that there are no resonances of order not exceeding* M *and also* $|\nabla(f_1 f_2^{-1})| \ge \epsilon_0$ *on* $\mathsf{supp}(\psi)$. *Further, assume that condition* (22.4.21)$_q$ *with* $q \ge 1$ *is fulfilled. Then under condition* (22.5.1) *the remainder estimate is given by*

$$O\Big(\mu^{-1}h^{-3} + (\mu h)^{(q+2)/2}h^{-4}|\log h|^{q/2}\Big) \tag{22.5.2}_q$$

while the main term of asymptotics given by (22.1.5) *with any* $T \ge \epsilon\mu^{-1}$.

Proof. First of all, observe that as $q = 1$ this remainder estimate is $O\big(\mu^{-1}h^{-3}+(\mu h)^{3/2}h^{-4}|\log h|^{1/2}\big)$ which is no smaller than $O(\mu^{-1}h^{-3}+\mu^2h^{-2})$ given by Proposition 22.4.1 and as $q > 1$ remainder estimate (22.5.2)$_q$ is no smaller than (22.4.22)$_q$ given by Proposition 22.4.4; so both of these propositions could be applied and one needs to estimate a substitution $T_0 \mapsto \epsilon\mu^{-1}$ error.

According to the proofs of these propositions $T_0 = Ch\rho^{-2}|\log h|$ which is less than $\epsilon\mu^{-1}$ as

$$\rho \ge \bar\rho_1 := (C\mu h|\log h|)^{1/2}. \tag{22.5.3}$$

Thus one can take there $T_0 = \epsilon\mu^{-1}$.

On the other hand, contribution of zone $\{\rho \le \bar\rho_1\}$ to the remainder with $T = \epsilon\mu^{-1}$ does no exceed $C\mu h^{-3}\nu(\bar\rho_1) \le C\mu h^{-3}\bar\rho_1^q$ which is exactly the second term in (22.5.2)$_q$. □

Corollary 22.5.2. *Let conditions of Proposition 22.5.1 be fulfilled. Then under assumption* (22.1.7)

(i) The remainder estimate is given by (22.5.2)$_q$ *while the main term of asymptotics given by*

$$\int \mathcal{N}^{\mathsf{MW}}(x,0)\psi(x)\,dx. \tag{22.5.4}$$

(ii) In particular, the remainder estimate is $O(\mu^{-1}h^{-3})$ *as* $\mu \le \bar\mu_q := C(h|\log h|)^{-q/(q+4)}$ *where in the general case* $\bar\mu_1 = C(h|\log h|)^{-1/5}$ *and in the generic case* $\bar\mu_4 = C(h|\log h|)^{-1/2}$.

Proof. According to Proposition 22.5.1 with the remainder estimate $(22.5.2)_q$ the main part of asymptotics is given by (22.1.5) with $T = \epsilon\mu^{-1}$. However, then under condition (22.1.7) one can trade $T = \epsilon\mu^{-1}$ to any $T \ge \bar{T} := Ch|\log h|$. The rest is proven by the standard successive approximation method, applied to the original operator (rather than to the canonical form); one can take as an unperturbed operator the same operator with the coefficients frozen at y which leads to the Weyl expression perturbed by

$$\sum_{m\ge 0, n\ge 1} \kappa_{mn} h^{-4+2m+2n}\mu^{2n}$$

where terms with $n \ge 2$ or $m \ge 1$ do not exceed the announced remainder estimate. Alternatively one can take as an unperturbed operator the same operator with g^{jk}, V frozen at y and with V_j replaced by $V_j(y) + \sum_k (\partial_k V_j)(y)(x_k - y_k)$ which leads directly to Magnetic Weyl expression. Details see in (18.1.28). □

Temperate Magnetic Field. II

Now we want to improve the result in the general case and allow non-degenerate critical points of $f_1 f_2^{-1}$.

Proposition 22.5.3. *Assume that $f_1 \ne f_2$ and there are no resonances of order not exceeding M and also critical points of $f_1 f_2^{-1}$ are non-degenerate on $\mathsf{supp}(\psi)$. Then under condition* (22.5.1) *the remainder estimate is given by* $(22.5.2)_q$ *with any $q < 2$ arbitrarily close to 2 while the main term of asymptotics given by the standard implicit formula with any $T \ge \epsilon\mu^{-1}$.*

Proof. Again, this remainder estimate is no smaller than $O(\mu^{-1}h^{-3} + \mu^2 h^{-2})$ given by Proposition 22.4.2, so again one needs to estimate an error arising from the substitution $T_0 \mapsto \epsilon\mu^{-1}$.

Proof follows ideas of the proof of Proposition 21.5.5. Let us introduce functions ℓ_k, $k = 1, \dots, K$ in the same proof but with $\gamma = 1$ and again let $\bar{\ell}_k = (\mu h|\log h|^J)^{1/(k+1)}$.

(a) Assume first that $|\nabla(f_1 f_2^{-1})| \ge \epsilon_0$. Then the arguments of the mentioned proof survive with this simplification.

(b) Assume now that $f_1 f_2^{-1}$ has one non-degenerate critical point $\bar{x}$. Consider first zone where $\ell_K(x) \le \gamma(x) := \frac{1}{2}|x - \bar{x}|$.

Then the number of indices "p" does not exceed $C\big(\ell_k^K(\mu h\gamma)^{-1}+1\big)$ (where the second term is the smallest one) while the number of indices "n" for each p does not exceed $C\big(\ell_K^{K+1}(\mu h)^{-1}+1\big)$ and the total contribution to the asymptotics of all such elements with $\ell_K \asymp \bar\ell_K$ and fixed magnitude of $\gamma(x)\asymp\gamma$ for some k does not exceed

$$C\mu^2h^{-2}h^{-2}\times\big(\ell_K^{K+1}(\mu h)^{-1}+1\big)\times\ell_k^K(\mu h\gamma)^{-1}\times\gamma^4$$

where γ^4 is the total measure of zone $\{\gamma(x)\asymp\gamma\}$. This expression does not exceed $C\mu^2h^{-2}\bar\ell_K^{-1}\gamma^3|\log h|^J$ and summation with respect to γ results in $C\mu^2h^{-2}\bar\ell_K^{-1}|\log h|^J\lesssim C(\mu h)^{2-\delta}h^{-4}$ as $\delta>1/K$.

On the other hand, repeating arguments of the proof of Proposition 21.5.5 one can see easily that the total contribution to the substitution error (when one replaces $T_0\ge\epsilon\mu^{-1}$ by $T_0=\epsilon\mu^{-1}$) of all elements with $\ell_K\ge C_0\bar\ell_K$ does not exceed $C\mu^{-1}h^{-3}+C\mu^2h^{-2}\bar\ell_K^{-1}$ which results in the same estimate.

(c) Alternatively, assume that $\ell_j\le\gamma\le\ell_{j+1}$ for some $j\le K-1$. Again one needs to consider elements with $\ell_k\asymp\bar\ell_k\le\gamma$ for some $k\le j$. Then in virtue of the same arguments of the proof of Proposition 21.5.5 the contribution to the asymptotics does not exceed

$$C\mu^2h^{-2}\ell_k^{-1}\gamma^{-1}|\log h|^J\times\gamma^4\times\ell_k\gamma^{-1}\asymp C\mu^2h^{-2}|\log h|^J\gamma^2$$

where the last factor $\ell_k\gamma^{-1}$ is the upper bound of the relative measure of ℓ_k elements to γ.

Again summation with respect to γ results in $C\mu^2h^{-2}|\log h|^J$.

(d) Finally, consider elements with $\ell_1\ge\gamma$. Then we redefine $\gamma=\ell_1$ and only $\ell\asymp\bar\ell_1$ should be considered since otherwise $T_0\le\epsilon\mu^{-1}$.

Contribution of such elements to the remainder with $T_0=\epsilon\mu^{-1}$ does not exceed $C\mu h^{-3}\bar\ell_1^4=O(\mu^3h^{-1}|\log h|^J)$. □

Temperate Magnetic Field. III

Now we want to attack resonances.

Proposition 22.5.4. *Assume that that* $|f_1-f_2|\ge\epsilon_0$ *and* $|\nabla(f_1f_2^{-1})|\ge\epsilon_0$ *on* $\mathsf{supp}(\psi)$. *Let consider asymptotics with the main part given by the standard*

implicit formula with any $T \ge \epsilon\mu^{-1}$. *Then under condition* (22.5.1) *the remainder does not exceed* $(22.5.2)'_q$ [16] *with* $q < 2$ *arbitrarily close to* 2.

Proof, Part I. Let us consider first zone $\{|x_1| \ge \bar\gamma\}$ and apply to the canonical form the same method as in the proof of Proposition 22.5.3. Again we assume that there is just one resonance surface $\Xi = \{x_1 = 0\}$.

However, definition of ℓ_k involves derivatives of order k and they are unbounded (with respect to x_1) as $\gamma^{k+1} \le \mu^{4-2m}$. To avoid this problem we rescale first $B(y, \gamma(y))$ into $B(0,1)$ and then apply this method. However, such rescaling would replace ρ by $\rho\gamma$ and ℓ by $\ell\gamma$ and leave $\rho\ell \ge C\mu h|\log h|$ intact. Then the main part of the remainder estimate gets factor γ^{-1} since the number of indices "p" will be $\rho(\mu h\gamma)^{-1}$ because the derivative with respect to x_1 for "p" and "$(p+1)$" would differ by $\mu h\gamma$.

So, the contribution of zone $\{x : \gamma(x) \asymp \gamma\}$ to the error estimate in question does not exceed $C(\mu h)^q h^{-4}$ with $q < 2$ arbitrarily close to 2; factor γ^{-1} discussed above is compensated by the same factor coming from the measure. Summation with respect to γ results in the similar answer (extra factor $|\log h|$ is covered by miniscule decrease of q).

This works as long as $|x_1| \ge \bar\gamma$ with

$$\bar\gamma = \mu^{2-m+\delta'} + \mu^{-2} \tag{22.5.5}$$

where the last term dominates as $m \ge 5$ only and guarantees that $\gamma \ge C\mu^{-1}h^{1-\delta}$.

Therefore,

(22.5.6) Contribution of zone $\{\bar\gamma \le |x_1| \le \epsilon\}$ to the remainder does not exceed $C\mu^{-1}h^{-3} + C(\mu h)^q h^{-4}$.

In particular, as $m = 5$ proposition is proven since the contribution of zone $\{|x_1| \le \bar\gamma$ to the remainder does not exceed $C\mu h^{-3}\bar\gamma$. □

Proof of Proposition 22.5.4, Part II. On the other hand, in zone $\{|x_1| \le \bar\gamma\}$ one can use precanonical form and contribution of the subzone $\{\rho \ge \bar\rho := C\max(\bar\gamma, \mu h|\log h|\bar\gamma^{-1})\}$ to the error is negligible while contribution of the subzone $\{\rho \le \bar\rho\}$ to the remainder does not exceed $C\mu h^{-3}\bar\gamma\bar\rho = C\mu h^{-3}\bar\gamma^2 + C\mu^2 h^{-2}|\log h|$; picking $\bar\gamma$ as $m = 4$ results in the proper estimate then.

[16] Where $'$ denotes that the power of $|\log h|$ could be larger.

However, as $m = 3$ one recovers only remainder estimate $C\mu^{-1+2\delta'}h^{-3} + C(\mu h)^q h^{-4}$ which is the required estimate as $\mu \ge h^{-1/3+\delta'}$ and only marginally worse otherwise. To recover the proper estimate one needs to reexamine the zone $\{|x_1| \le \bar\gamma, \rho \le \bar\gamma\} \setminus \{|x_1| \le \bar\gamma_0, \rho \le \bar\gamma_0\}$ since the contribution to the remainder of the latter does not exceed $C\mu^{-1}h^{-3}$. Without any loss of the generality one can assume that the non-diagonal term does not depend on x_1 since one can remove a term divisible by x_1 by the same method as for $|x_1| \ge \bar\gamma$ was removed the whole term.

In the subzone $\{\max(\rho, \bar\gamma_0) \le |x_1| \le \bar\gamma\}$ one can pick up $T_0 = Ch|\log h|\rho^{-2}$ which is less than $\epsilon\mu^{-1}$ unless $\rho \le \bar\rho := C(\mu h|\log h|)^{1/2}$ (which is less than μ^{-1} and the contribution to the remainder of the subzone $\{|x_1| \le \bar\gamma,\ \rho \le \bar\rho\}$ does not exceed

$$C\mu h^{-4}\bar\rho\bar\gamma = C\mu^{-1}h^{-3} \times \mu^{\delta}(\mu^3 h|\log h|)^{1/2} \le C\mu^{-1}h^{-3}$$

as $\mu \le h^{-1/3+\delta'}$ and $\delta = \delta'$.

In the subzones $\{|x_1| \asymp \gamma \le \rho\}$ with $\bar\gamma_0 \le \gamma \le \bar\gamma_1$ and $\{|x_1| \le \gamma = \bar\gamma_0 \le \rho\}$ one can take $T_0 = Ch|\log h|(\rho\gamma)^{-1}$ which is less than $\epsilon\mu^{-1}$ here for sure. □

Remark 22.5.5. It can happen that T_1 described in Section 22.4 is less than $\epsilon\mu^{-1}$. Then one can take $T_0 \le \epsilon\mu^{-1}$ anyway.

Temperate Magnetic Field. IV

Let us derive sharp remainder estimates in the resonance case:

Proposition 22.5.6. *Assume that that* $|f_1 - f_2| \ge \epsilon_0$ *and also* $|\nabla(f_1 f_2^{-1})| \ge \epsilon_0$ *on* $\mathsf{supp}(\psi)$. *Further, assume that conditions* (22.5.1) *and* $(22.4.21)_q$ *are fulfilled with* $q > 2$ *and that on (any) resonance surface* Ξ

$(22.5.7)_r$ $\mathsf{Hess}_\Xi\big(\mu h f V^{-1} - 1\big)$ *has at least* r *eigenvalues with absolute values greater than* ϵ_0

with $r = q - 1$.

Then the remainder estimate is given by $(22.5.2)'_q$ *while the main term of asymptotics given by the standard implicit formula with any* $T \ge \epsilon\mu^{-1}$.

Proof. Due to Proposition 22.5.3 one should cover only case $\mu \ge h^{-1/3+\delta'}$. Assumptions of proposition imply that condition $(22.4.65)_{r+1,r}$ holds. While

one can apply Proposition 22.4.10 directly only as $r > 2$, the proof of it yields that under condition (22.5.7) with $r = 1, 2$ the remainder estimate $O\big(\mu^{-1}h^{-3} + \mu^{2-r/2}h^{2+r/2}|\log h|^J\big)$ holds, which is not worse than $(22.5.2)_{r+1}$. Let us apply the same partition to zones as in Propositions 22.4.9, 22.4.10.

Then zone

$$\{\gamma \ge \mathsf{max}(\rho, \bar\gamma_2 := C\,\mathsf{max}(\mu^{-2/3}, (\mu h|\log h|)^{1/2}))\}$$

is covered by the arguments used in the proof of Proposition 22.5.1; its contribution to the error does not exceed $(22.5.2)_q$. Meanwhile, in the subzone $\{\rho \ge \gamma \ge \bar\gamma_2)\}$ one can take $T_0 = Ch|\log h|\bar\gamma_2^{-2}$ which is less than $\epsilon\mu^{-1}$.

Further, in the subzone $\{\bar\gamma \le |x_1| \asymp \gamma \le \bar\gamma_2\}$ one can take $T_0 = Ch|\log h|\zeta^{-2}$ which is less than $\epsilon\mu^{-1}$ unless $\zeta \le C(\mu h|\log h|)^{1/2}$. Similarly, in the subzone $\{|x_1| \le \bar\gamma\}$ one can take $T_0 = Ch|\log h|\zeta^{-2}$ as well. Note that the contribution of the zone

$$\{|x_1| \le \bar\gamma_2, \zeta \le \bar\zeta := C(\mu h|\log h|)^{1/2}\}$$

to the remainder does not exceed $C\mu h^{-3}\bar\gamma_2\bar\zeta^r$ which does not exceed the second term in $(22.5.2)_{r+1}$ as $\bar\zeta \asymp \bar\gamma_2$ i.e. $\mu^{-2/3} \le C(\mu h|\log h|)^{1/2}$ i.e. as $\mu \ge C(h|\log h|)^{-3/7}$.

Further, as $\mu \le C(h|\log h|)^{-3/7}$ let us consider zone $\{|x_1| \le \bar\gamma_2\}$ and use the precanonical form here. Then the contribution of the subzone $\{|x_1| \le \bar\gamma_2,\ \rho \le \bar\rho_2,\ \zeta \le \bar\zeta\}$ to the remainder does not exceed $C\mu h^{-3}\bar\gamma_2\bar\rho_2\bar\zeta^r$ which does not exceed the second term in $(22.5.2)_{r+1}$ as $\bar\zeta \ge \bar\gamma_2\bar\rho_2$.

Therefore one can pick up $\bar\rho_2 = \bar\zeta\bar\gamma_2^{-1}$; one can see easily that $\bar\rho_2 \gg \mu^{-2/3}$ and $\bar\rho_2\bar\gamma_2 \gg C\mu h|\log h|$ and then $T_0 \le Ch|\log h|\bar\rho_2^{-1}\bar\gamma_2^{-1} \ll \epsilon\mu^{-1}$ and the contribution of the zones $\{|x_1| \le \bar\gamma_2,\ \rho \ge \bar\rho_2\}$ and $\{|x_1| \le \bar\gamma_2,\ \zeta \ge \bar\zeta\}$ to the error is negligible. □

Temperate Magnetic Field. V

Finally, let us consider the vicinity of Σ.

Proposition 22.5.7. *Let conditions* (22.2.1)–(22.2.2), (22.5.1) *be fulfilled and let* ψ *be supported in the small vicinity of* Σ.

Then the remainder estimate is given by $(22.5.2)'_q$ *with any* $q < 2$ *arbitrarily close to* 2 *while the main term of asymptotics given by the standard implicit formula with any* $T \ge \epsilon\mu^{-1}$.

Proof. Again this remainder estimate is no smaller than $O(\mu^{-1}h^{-3}+\mu^2h^{-2})$ delivered by Proposition 22.4.13.

Applying the same approach as in the Part I of the proof of Proposition 22.5.4 (which is applicable as $\gamma \ge \bar{\gamma}_2 := \mathsf{max}\big((\mu h|\log h|)^{1/2}, \mu^{-1}\big)$ now) we conclude after summation with respect to γ that the contribution of the zone $\{x : \gamma(x) \asymp \gamma\}$ (where $\gamma(x) = \mathsf{dist}(x, \Sigma)\}$) to the error does not exceed $C(\mu h)^q h^{-4}\gamma$ with $q < 2$ arbitrarily close to 1. Here an extra factor γ appears because the measure of $\{x : \gamma(x) \asymp \gamma\}$ is $\asymp \gamma^2$ rather than $\asymp \gamma$ as it was before. Then after summation we conclude that

(22.5.8) The contribution of the zone $\{x : \mathsf{max}((\mu h, \mu^{-1}) \le \gamma(x) \le \gamma\}$ to the error is $O\big((\mu h)^q h^{-4}\gamma\big)$. In particular, it is $O\big((\mu h)^q h^{-4}\big)$ as $\gamma = \epsilon$.

Meanwhile, the contribution of the zone $\{x : \gamma(x) \le \bar{\gamma}_2\}$ to the error does not exceed $C\mu h^{-3}\bar{\gamma}_2^2 \asymp (\mu^{-1}h^{-3}+\mu^2h^{-2})$. □

Temperate Magnetic Field. VI

Let us improve the above estimate under generic conditions.

Proposition 22.5.8. *Let conditions* (22.2.1)–(22.2.2), (22.5.1) *and* $(22.4.87)_r$ *with* $r = 1, 2$ *be fulfilled. Let* ψ *be supported in the small enough vicinity of* Σ. *Then*

(i) The total remainder is given by $(22.5.2)'_{r+1}$ *while the main part is given by the standard implicit formula with any* $T \ge \epsilon\mu^{-1}$.

(ii) Under condition (22.4.84) *the total remainder is given by* $(22.5.2)'_{r+2}$ *as* $r = 0, 1$ *while the main part is given by the standard implicit formula with any* $T \ge \epsilon\mu^{-1}$.

(iii) Under conditions (22.4.84) *and* (22.4.83) *the total remainder is given by* $(22.5.2)'_{r+2}$ *as* $r = 2$ *while the main part is given by the standard implicit formula with any* $T \ge \epsilon\mu^{-1}$.

Proof. Again let us note that the remainder estimate given by Proposition 22.4.17 as $r = 0$ and Proposition 22.4.18 as $r = 1, 2$ is no worse than one announced here.

In the zone $\{\gamma \ge \mathsf{max}(\rho, (\mu h|\log h)^{1/2}, \mu^{-1})\}$ the same arguments as in the proof of Proposition 22.5.3 are applied, and conditions $(22.4.87)_r$ and

(22.4.84) add factors $(\mu h|\log h|)^{r/2}$ and $(\mu h|\log h|)^{1/2}\gamma^{-1}$ respectively to the measure of the zone $\{\rho \le (\mu h|\log h|)^{1/2}$.

Then after summation with respect to γ one arrives to the estimate $(22.5.2)'_{r+k+1}$ of the contribution of this zone to the error estimate and $k=1$ under condition (22.4.84) and $k=0$ otherwise.

Meanwhile, $T_0 \le \epsilon\mu^{-1}$ in the zone $\{\rho \ge \gamma \ge C\max((\mu h|\log h|)^{1/2}, \mu^{-1})\}$.

So, one needs to consider zone $\{\gamma \le C\bar\gamma_2\}$. In this zone condition (22.4.74) has no value. Contribution of this zone to the remainder obviously does not exceed $(22.5.2)'_2$. Therefore case $r+k\le 1$ is covered.

Let us introduce ζ as before. As *either* $\zeta\le\gamma$ *or* condition (22.4.83) is fulfilled, one can take $T_0 = Ch\zeta^{-2}$ and then only the subzone $\{\zeta \le C(\mu h|\log h|)^{1/2}\}$ should be considered; then condition $(22.4.87)_r$ adds an extra factor $(\mu h|\log h|)^{r/2}$ to the measure and to the estimate which becomes $(22.5.2)'_{r+2}$.

That leaves us with the analysis of the zone $\{\zeta\ge\gamma\}\cap\{\gamma\le\bar\gamma_2\}$ and only without condition (22.4.83), in which case only estimate $(22.5.2)'_4$ should be proven under condition $(22.4.87)_1$. However, then one can take $T_0 = C\mu h|\log h|\gamma^{-1}\zeta^{-1}$ and then only the subzone $\zeta \le C\mu h|\log h|\gamma^{-1}$ remains to be considered. Its contribution to the remainder does not exceed $C\mu h^{-3}\times\mu h|\log h|\gamma^{-1}\times\gamma^2$ which sums with respect to γ ranging from $\bar\gamma_1 := \max(C\mu h|\log h|, \mu^{-1})$ to $\bar\gamma_2$ to $C\mu^2h^{-2}|\log h|\bar\gamma_2$ which is properly estimated. Meanwhile, contribution of the zone $\{x : \gamma(x)\le\bar\gamma_1\}$ to the remainder does not exceed $C\mu h^{-3}\bar\gamma_1^2$ and is properly estimated as well. □

Conclusion

Corollary 22.5.9. *Let conditions of one of Propositions 22.5.3–22.5.8 be fulfilled. Further, let condition* (22.1.7) *be fulfilled. Then*

(i) The remainder estimate is $(22.5.2)'_q$ *where*

(a) $q<2$ *is arbitrarily close to* 2 *in the general case (Propositions 22.5.3, 22.5.4, 22.5.7) and*

(b) q *is described in the corresponding case (Propositions 22.5.1, 22.5.6, 22.5.8),*

while the main part of asymptotics is given by (22.5.4).

(ii) In particular, the remainder estimate is $O(\mu^{-1}h^{-3})$ *as* $\mu\le\bar\mu_q := Ch^{-q/(q+4)}|\log h|^{-K}$. *In particular*

(a) in the general case $\bar{\mu}_1 = Ch^{\delta-1/3}$

(b) in the generic case $\bar{\mu}_4 = h^{-1/2}|\log h|^{-K}$.

Proof. Proof coincides with the proof of Corollary 22.5.2. □

Then we get immediately

Corollary 22.5.10. *(i) Theorem 22.6.3 below is proven with* $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}} = 0$ *as* $\mu \le h^{\delta-1/3}$.

(ii) Theorem 22.6.4 below is proven as $\mu \le Ch^{-1/2}|\log h|^{-K}$.

22.5.2 Intermediate Magnetic Field

In this Subsection we assume that

$$h^{-1/3} \le \mu \le ch^{-1} \tag{22.5.9}$$

sometimes making separate considerations for the case of the *intermediate strong* magnetic field

$$h^{-1}|\log h|^{-K} \le \mu \le ch^{-1} \tag{22.5.10}$$

as needed. Also in the general case we consider a special range $h^{-1/3+\delta} \le \mu \le h^{-1/3-\delta}$.

Strong Magnetic Field. I

We start from the regular points.

Proposition 22.5.11. *Assume that there are no resonances of order not exceeding* M *and also* $|\nabla(f_1 f_2^{-1})| \ge \epsilon_0$ *on* $\mathsf{supp}(\psi)$. *Further, assume that condition* $(22.4.21)_q$ *with* $q \ge 1$ *is fulfilled and*

$$\mu^{-1}h|\log h| \le \varepsilon \le \mu h \tag{22.5.11}$$

where $\varepsilon := \mu^{-2}$ *here.*

Then the remainder does not exceed

$$(22.5.12)_q \quad C\mu^{-1}h^{-3} + Ch^{-4}(\mu h|\log h|)^{q/2}\varepsilon + \begin{cases} 0 & q \ge 3, \\ \mu^2 h^{-2}\varepsilon^{(q-1)/2} & 2 \le q < 3, \\ \bigl(\mu^2 h^{-2}\varepsilon^{(q-1)/2} + \mu h^{-3}\varepsilon^{q/2}\bigr) & 1 \le q < 2 \end{cases}$$

while the main term of asymptotics given by (22.5.4).

Proof. The proof follows the sequence of the proofs of Propositions 21.5.15, 21.5.17 with $\varepsilon = \mu^{-2}$ and 21.5.20(i):

(a) First, using the method of successive approximations, we rewrite the implicit formula as expression (22.5.4) plus a (temporary) *correction term*

$$\sum_{\iota} \int \Big(\widehat{\mathcal{N}}^{\mathsf{MW}}_{\bar{Q}_\iota}(x,0) - \mathcal{N}^{\mathsf{MW}}_{\bar{Q}_\iota}(x,0) \Big)\, dx \tag{22.5.13}$$

with an integrand defined by (21.5.32):

$$\widehat{\mathcal{N}}^{\mathsf{MW}}_{\bar{Q}}(x,0) \coloneqq \mathsf{const} \sum_{n,p} \Big(\theta(\mathcal{A}_{pn})\bar{Q} \Big)\Big|_{(x'',\xi'')=\Psi^{-1}(x)} \times f_1(x) f_2(x) \mu^2 h^{-2} \tag{22.5.14}$$

while

$$\mathcal{N}^{\mathsf{MW}}_{\bar{Q}}(x,0) \coloneqq \mathsf{const} \sum_{n,p} \theta\Big(V(x) - (2n+1)\mu h f_2(x) - (2p+1)\mu h f_1(x) \Big) f_1(x) f_2(x) \bar{Q} \mu^2 h^{-2} \tag{22.5.15}$$

where Q_ι cover zone with $T_0 \ge \epsilon\mu^{-1}$ (i.e. $\rho \le C(\mu h|\log h|)^{1/2}$); here const meant the same constant as in the definition of $\mathcal{N}^{\mathsf{MW}}$.

(b) Then we need to estimate expression (22.5.13). Now, however, one needs only analysis which was used in Chapter 21 in the strictly outer zone. The crucial moment is the estimate of the correction term as in Proposition 21.5.17 which was

$$C\mu^2 h^{-2} \times h\rho^{-2} \times \varepsilon h^{-1} \times \big(\rho^2(\mu h)^{-1} + 1\big) \times \big(\rho(\mu h)^{-1} + 1\big)\rho^{q-1} \asymp C\mu^2 h^{-2} \varepsilon \big(\rho^2(\mu h)^{-1} + 1\big) \times \big(\rho(\mu h)^{-1} + 1\big)\rho^{q-3} \tag{22.5.16}$$

and one needs to sum this expression with respect to ρ ranging from $\varepsilon^{1/2}$ ($= \mu^{-1/2}$ as $\varepsilon = \mu^{-2}$) to $\bar{\rho} \coloneqq C(\mu h|\log h|)^{1/2}$. Then

(A) As $q > 3$ expression (22.5.16) sums to its value as $\rho = \bar{\rho}$, which is exactly $C\varepsilon(\mu h|\log h|)^{q/2} h^{-4}$.

(B) As $1 < q \le 3$ an extra term due to summation of $C\mu^2h^{-2}\varepsilon\rho^{q-3}$ appears in the estimate. As $q < 3$ this results in the value of this term as $\rho = \varepsilon^{1/2}$ which is $C\mu^2h^{-2}\varepsilon^{(q-1)/2}$ while as $q = 3$ it results in $C\mu^2h^{-2}\varepsilon \log(\mu h|\log h|\varepsilon^{-1})$.

(C) Further, as $1 \le q \le 2$ one should take in account also $C\mu h^{-3}\varepsilon\rho^{q-2}$ which sums to $C\mu h^{-3}\varepsilon^{q/2}$ as $1 < q < 2$ and $C\mu h^{-3}\varepsilon|\log h|$ as $q = 2$.

(c) Finally, in the zone $\{\rho \le \varepsilon^{1/2}\}$ one simply considers its contribution to the remainder with $T = \epsilon\mu^{-1}$ rather than to the correction:

$$C\mu^2h^{-2} \times \bigl(\rho^2(\mu h)^{-1} + 1\bigr) \times \bigl(\rho(\mu h)^{-1} + 1\bigr)\rho^{q-1}\big|_{\rho=\varepsilon^{1/2}} \asymp C\mu h^{-3}\varepsilon^{q/2} + C\mu^2h^{-2}\varepsilon^{(q-1)/2}. \tag{22.5.17}$$

This concludes the proof. □

To cover properly the case of the *intermediate strong* magnetic field (22.5.10) let us rewrite the implicit formula as

$$\int \widehat{\mathcal{N}}_l^{\mathsf{MW}}(x,0)\,dx \tag{22.5.18}$$

and then as $q > 3$ rewrite it as the three term decomposition with respect to ε; then the third (remainder) term is $o(\mu^{-1}h^{-3})$ while the second term is given by a two-dimensional Riemannian sum proportional μ^{-2} with the steps $2f_1\mu h$ and $2f_2\mu h$.

Replacing this Riemannian sum by an integral one can see easily that again with an error $o(\mu^{-1}h^{-3})$ the second term is $\varkappa\mu^{-2}h^{-4}$ which disagrees with expression as $\mu \le h^{-1}|\log h|^{-K}$ unless $\varkappa = 0$. Combining with the previous proposition we arrive to

Proposition 22.5.12. *In the framework of Proposition 22.5.11 with* $q > 3$ *asymptotics with the main part* (22.5.4) *and* $O(\mu^{-1}h^{-3})$ *remainder holds.*

Combining Propositions 22.5.11 and 22.5.12, Corollary 22.5.2 and results of Section 22.3 we conclude that

(22.5.19) In the framework of Proposition 22.5.11 in the generic case $q > 3$ estimate $O(\mu^{-1}h^{-3})$ is proven for the complete range of μ ($1 \le \mu \le ch^{-1}$). Meanwhile, in the general case $q = 1$ estimate $O(\mu^2h^{-2})$ is proven as $h^{-1/2}|\log h|^K \le \mu \le ch^{-1}$.

Intermediate Magnetic Field. II

Now we want to improve result in the general case.

Proposition 22.5.13. *Assume that $f_1 \neq f_2$ and there are no resonances of order not exceeding M and also critical points of $f_1 f_2^{-1}$ are non-degenerate on $\mathsf{supp}(\psi)$. Then as $\mu \geq h^{-1/3}$ the remainder estimate is given by* $(22.5.12)_q$ *with any $q < 2$ arbitrarily close to 2 while the main term of asymptotics given by* (22.5.4).

Proof. Proof follows ideas of the proof of Proposition 21.5.22. Let us introduce functions ℓ_k, $k = 1, \dots, K$ in the same proof but with $\gamma = 1$ and again let $\bar{\ell}_k = (\mu h |\log h|^J)^{1/(k+1)}$.

(a) Assume first that $|\nabla(f_1 f_2^{-1})| \geq \epsilon_0$. Then the arguments of the mentioned proof survive with this simplification.

(b) Assume now that $f_1 f_2^{-1}$ has one non-degenerate stationary point $\bar{x}$. Assume first that $\ell_K \leq \gamma$ where $\gamma = \frac{1}{2}|x - \bar{x}|$. Then the number of indices "p" does not exceed $C\ell_k^k/(\mu h \gamma)$ and the contribution of all such elements with $\ell_k \asymp \bar{\ell}_k$ for some $k = 1, \dots, (K-1)$ to the correction term does not exceed $C(\mu h)^{1-\delta} \cdot \mu^{-2} h^{-4} \gamma^{-1} \times \gamma^4 |\log h|^J$ where γ^4 is the total measure of such elements, $\delta = 1/K$ and J is large enough. Summation with respect to γ results in the announced estimate.

On the other hand, the total contribution to the correction of all elements with $\ell_k \geq C_0 \bar{\ell}_k$ does not exceed $C\mu^{-1}h^{-3} + C\mu h^{-3} \cdot \mu^{-2} \bar{\ell}_K^{-1}$.

(c) Alternatively, assume that $\ell_j \leq \gamma \leq \ell_{j+1}$. Again one needs to consider elements with $\ell_k \asymp \bar{\ell}_k \leq \gamma$ for some $k \leq j$. Then the contribution to the correction term does not exceed $C\mu h^{-3} \times \mu^{-2}\gamma^{-1} \times \gamma^4 \times \gamma^{-1} |\log h|$ where the second factor γ^{-1} appears since the relative measure of ℓ_k elements to γ elements does not exceed $\ell_k \gamma^{-1}$. Again summation with respect to γ results in $C\mu^{-1}h^{-3}|\log h|^J$.

(d) Finally, consider elements with $\ell_1 \geq \gamma$. Then we redefine $\gamma = \ell_1$ and only $\ell_1 \asymp \bar{\ell}_1$ should be considered. Contribution of such elements to the correction does not exceed $Ch^{-4} \times \mu^{-2} \bar{\ell}_1^4 = O(\mu^{-1}h^{-3})$. □

Therefore we arrive to

(22.5.20) In the general case asymptotics with the main part (22.5.4) and the remainder estimate $O(\mu^{-1}h^{-3}+\mu^2h^{-2})$ holds unless

$$h^{-1/3+\delta}\le \mu \le h^{-1/3-\delta} \tag{22.5.21}$$

in which case the remainder estimate is $O(\mu^{-1}h^{-3-\delta})$ containing an extra factor $h^{-\delta}$; $\delta>0$ is an arbitrarily small exponent.

Intermediate Magnetic Field. III

Let us recover remainder estimate $O(\mu^{-1}h^{-3}+\mu^2h^{-2})$ in the latter case (22.5.21) (introducing some correction term). Using the same arguments as above we can purge from $\mathcal{A}_{pn}$ in (22.5.14) all terms which are even marginally less than μ^{-2}; it includes all higher order terms and also the difference between $\mu^{-2}\mathcal{B}_{pn}$ and $\mu^{-2}\mathcal{B}_{\bar p,\bar n}$ where $\bar p=\alpha/(2f_1\mu h)$ and $\bar n=(1-\alpha)/(2f_2\mu h)$, $\alpha=\alpha(x)$ is the minimizer of $|\nabla\phi_\alpha|^2$.

Then $\mathcal{B}_{pn}$ becomes $\mathcal{B}_{\bar p,\bar n}=\omega(x)$ [17]. However, let us include this modified term $\mu^{-2}\mathcal{B}_{pn}$ for all p, n and not only for those for which $T_0\ge\epsilon\mu^{-1}$. Then one needs to correct this alternation by the term $\varkappa\mu^{-2}h^{-4}$ with $\varkappa$ selected so it would result in the correction term 0, if one replaces the Riemann sum by the corresponding integral because it would provide the result for $\mu\ge h^{-1/3-\delta}$ and it should agree with the results of the previous subsubsection. We leave the easy but tedious details to the reader. Then we arrive to

Proposition 22.5.14. *Assume that $f_1\ne f_2$ and there are no resonances of order not exceeding M and also the critical points of $f_1f_2^{-1}$ are non-degenerate on $\mathsf{supp}(\psi)$. Then for μ satisfying* (22.5.21) *the remainder estimate is $O(\mu^{-1}h^{-3}+\mu^2h^{-2})$ while the main part of the asymptotics is given by*

$$\int \Bigl(\mathcal{N}^{\mathsf{MW}}(x,0)+\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x,0)\Bigr)\psi(x)\,dx \tag{22.5.22}$$

with

$$\begin{aligned}&\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x,\tau)= \\ &(2\pi)^{-2}\mu^2h^{-2}\sum_{(p,n)\in\mathbb{Z}^{+2}}\Bigl(\theta\bigl(2\tau+V-(2p+1)\mu hf_1-(2n+1)\mu hf_2-\omega(x)\mu^{-2}\bigr)- \\ &\qquad\theta\bigl(2\tau+V-(2p+1)\mu hf_1-(2n+1)\mu hf_2\bigr)\Bigr)f_1f_2\sqrt{g} \\ &\qquad+(4\pi)^{-2}\mu^{-2}h^{-4}(2\tau+V)\omega\sqrt{g}.\end{aligned} \tag{22.5.23}$$

[17] Which is the smooth function outside of the critical points of $f_1f_2^{-1}$.

Remark 22.5.15. (i) Here

$$\int \mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x,\tau)\psi(x)\,dx = O(\mu^{-1}h^{-3-\kappa})$$

with an arbitrarily small exponent $\kappa > 0$ [18].

(ii) Furthermore, under nondegeneracy condition

$$\sum_{|\beta|\le K} |\nabla^\beta \phi_\alpha| \ge \epsilon_0 \qquad \forall x\ \forall \alpha \in [0,1] \tag{22.5.24}$$

with arbitrarily large K one can skip a correction term without deteriorating remainder estimate unless $h^{-1/3}|\log h|^{-J} \le \mu \le h^{-1/3}|\log h|^{J}$ and with the remainder estimate $O(h^{-8/3}|\log h|^{J})$ in this border case [19]. We suspect that one can get rid of logarithmic factors and to prove an estimate $O(\mu^{-1}h^{-3} + \mu^2 h^{-2})$ even in the border case.

(iii) However, we could not find any example demonstrating that this correction term is not superficial and without (22.5.24) one cannot skip it without penalty. Clarification of this would be interesting.

On the other hand, correction term in Chapter 21 was not superficial for sure.

Intermediate Magnetic Field. IV

Now we want to attack resonances. Let us start from the generic case. First, after rescaling again in the same manner as before one can see easily that the contribution of zone $\{|x_1 \asymp \gamma\}$ to an error does not exceed $C\mu h^{1-d}\varepsilon^{1-\kappa}$ where factor γ^{-1} appearing from the calculation of the number of indices "p" is compensated by factor γ appearing from the measure. Then this contribution does not exceed $C\mu h^{-3}(\mu^{4-2m}\gamma^{-1})^{(1-\kappa)}$ if only terms originated from non-diagonal terms are removed and summation with respect to $\gamma \ge \bar\gamma$ results in the value of this expression as $\gamma = \bar\gamma$ and as $m \ge 4$ it is $O(\mu^{-1}h^{-3} + \mu^2 h^{-2})$. Recall that $\bar\gamma = \mu^{\delta-2}$ as $m = 4$ and $\bar\gamma = C\mu^{-2}$ as $m \ge 5$.

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[18] It follows from Proposition 22.5.3.

[19] Notice that under condition (22.5.24) $\bar\ell_K \asymp 1$ in the proof of Propositions 22.5.3, 22.5.13.

On the other hand, in the zone $\{|x_1| \le \bar\gamma\}$ one can apply Proposition 22.4.10 and get $C(\mu h)^{3/2}h^{-4}\bar\gamma|\log h|^J$ which is $o(\mu^{-1}h^{-3}+\mu^2h^{-2})$ as well (as $m\ge 4$).

Now after singular terms from operator are removed it can be treated as if there was no resonance resulting in two following statements:

Proposition 22.5.16. *Assume that $f_1\ne f_2$ and $|\nabla(f_1f_2^{-1})|\ge\epsilon_0$ on $\operatorname{supp}(\psi)$. Furthermore, assume that there are no third-order resonances on $\operatorname{supp}(\psi)$.*

Then

(i) As $\mu\ge h^{-1/3-\delta}$ the remainder estimate is $O(\mu^2h^{-2})$ while the main term of asymptotics given by (22.5.4)*.*

(ii) As $h^{-1/3+\delta}\le\mu\le h^{-1/3-\delta}$ statement of Remark 22.5.15(i) holds; furthermore, under condition (22.5.24) *one can skip correction term with $O(\mu^2h^{-2}|\log h|^J)$ penalty*[20)]*.*

Intermediate Magnetic Field. V

Consider resonances of order 3 now. As $\mu\ge h^{-1/3-\delta}$ in the virtue of the above arguments, the contribution of the zone

$$\bigl\{|x_1|\ge\bar\gamma_3\coloneqq\mu^{-3}h^{-1-\delta}\bigr\} \tag{22.5.25}$$

to the correction does not exceed $C\mu^2h^{-2}$.

Moreover, the contribution of the zone

$$\bigl\{|x_1|\le\bar\gamma_2\coloneqq(\mu h)^{1/2}|\log h|^{-1/2}\bigr\} \tag{22.5.26}$$

to the correction also does not exceed $C\mu^2h^{-2}$. However, zone

$$\bigl\{\bar\gamma_2\le|x_1|\le\bar\gamma_3\bigr\} \tag{22.5.27}$$

needs to be reexamined; here $\bar\gamma_2\ge\bar\gamma$ as $\mu\ge h^{-1/3-\delta}$ and as $\mu\ge h^{-3/7-\delta'}$ zone (22.5.27) disappears. In virtue of arguments of Subsection 22.5.2 the contribution of this zone does not exceed $C\mu^2h^{-2-\kappa}$ with arbitrarily small $\kappa>0$ and now we want to improve it marginally.

We claim that

[20)] Which is probably superficial.

Proposition 22.5.17. *For any $\delta > 0$ there exists $K = K(\delta)$ such that if $\mu^{-1+\delta} \le \gamma \le \mu^{-\delta}$ and $B(y, \gamma(y))$ with $\gamma(y) \asymp \gamma$ is rescaled to $B(0,1)$ then either*

$$\sum_{|\alpha|\le K} |\partial_z^\alpha \mathcal{A}_{pn}| \asymp \varsigma \qquad \forall z \in B(0,1) \tag{22.5.28}$$

with some ς or

$$\sum_{|\alpha|\le K} |\partial_z^\alpha \mathcal{A}_{pn}| \le \varsigma \qquad \forall z \in B(0,1) \tag{22.5.29}$$

with $\varsigma = \mu h^{1-\kappa}\gamma$ where (continuous) parameters p and n are selected so

$$\mathcal{A}_{pn}(y) = \partial_{x_1}\mathcal{A}_{pn}(y) = 0. \tag{22.5.30}$$

Proof. Note that before rescaling $\mathcal{A}_{pn} = \mathcal{A}^0_{pn} + x_1^{-1}\mu^{-2}\mathcal{B}_{pn} + ...$ with smooth symbols. Then decomposing all smooth symbols into Taylor series at y one can prove proposition easily since high powers of there contain high powers of γ and thus are small. □

Proposition 22.5.18. *In the framework of Proposition 22.5.17 the contribution of the zone $\{\mu^{-1+\delta} \le |x_1| \le \mu^{-\delta}\}$ to the error is $O(\mu^{-1}h^{-3} + \mu^2 h^{-2}|\log h|^J)$.*

Proof. Proof repeats those which we had before. One needs to construct ℓ_k corresponding to $\varsigma\mathcal{A}_{pn}$ and then $(\mu h|\log h|)$ as minimal value for ℓ_k^{k+1} should be replaced by $\varsigma^{-1}(\mu h|\log h|)$ and there will be no final division by $\bar\ell_K$ which would be $\asymp 1$. That will give $C\mu^2h^{-2} \times |\log h| \times |\log h|\gamma^{-1} \times \gamma$ where the factors γ^{-1} and γ appear from division by $(\mu h\gamma)$ and the measure.

Then summation with respect to γ results in an extra $|\log h|$ factor. □

Remark 22.5.19. (i) Under condition

$$\sum_{|\beta|\le K} |\nabla_\Xi^\beta V f_1^{-1}| \ge \epsilon_0 \qquad \forall x \tag{22.5.31}$$

with arbitrarily large K one can skip a correction term without deteriorating remainder estimate unless $h^{-1/3}|\log h|^{-J} \le \mu \le h^{-1/3}|\log h|^J$ and with the remainder estimate $O(h^{-8/3}|\log h|^J)$ in this border case. We suspect that one can get rid of logarithmic factors and to prove an estimate $O(\mu^{-1}h^{-3}+\mu^2h^{-2})$ even in the border case.

(ii) One can easily construct a correction term in the case of third-order resonances but an expression seems rather too complicated. So we leave it to the curious reader.

Intermediate Magnetic Field. VI

Let us consider the generic case now:

Proposition 22.5.20. *Assume that that* $|f_1-f_2|\ge \epsilon_0$ *and also* $|\nabla(f_1f_2^{-1})|\ge \epsilon_0$ *on* $\mathsf{supp}(\psi)$. *Further let us assume that conditions* $(22.4.21)_4$ *and* $(22.5.7)_3$ *are fulfilled.*

Then the remainder is $O(\mu^{-1}h^{-3})$ *while the main term of asymptotics given by* (22.5.4).

Proof. Again it is sufficient to consider the case of the single resonance surface $\Xi=\{x_1=0\}$.

(a) Combining in the zone $\{|x_1|\asymp \gamma\ge \bar\gamma:=\mu^{\delta-1}\}$ arguments of the proofs of Proposition 22.5.6 and 22.5.11 one can estimate contribution of it to the correction terms by $C(\mu h|\log h|)^{q/2}h^{-4}\gamma\times \mu^{-2}\gamma^{-1}$ which after summation with respect to γ results in $C(\mu h|\log h|)^{q/2}\mu^{-2}h^{-4}|\log h|$, which in turn is $O(\mu^{-1}h^{-3})$ as long as $\mu\le h^{-1}|\log h|^{-J}$.

(b) In the zone $\{|x_1|\le \bar\gamma\}$ one can use arguments of the proofs of Proposition 22.5.7 and estimate the contribution of this zone to the correction term arising as $T\ge T_0$ is replaced by $T=\epsilon\mu^{-1}$ by $C(\mu h|\log h|)^{5/2}h^{-4}\bar\gamma$ which is $O(\mu^{-1}h^{-3})$ as $\mu\le h^{-3/5+\delta'}$.

(c) Let $\mu\ge h^{-5/3+\delta'}$. In the zone $\{|x_1|\le \bar\gamma\}$ one can consider the precanonical form and then the contribution of this zone to the correction term does not exceed $C(\mu h|\log h|)^{3/2}\mu^{-1}h^{-4}\bar\gamma$ which is $O(\mu^{-1}h^{-3})$ as $\mu\le h^{-1+\delta'}$.

The direct calculation shows that the first approximation term actually vanishes as it comes from the "main perturbation term" and is odd with respect to x_1, and its estimate contains an extra factor $\bar\gamma$ otherwise.

So the correction terms, associated with zone $\{|x_1|\le \bar\gamma\}$, do not exceed

$$C(\mu h)^{3/2}\mu^{-1}h^{-4}\bar\gamma^2+C(\mu h)^{1/2}\mu^{-2}h^{-4}\bar\gamma=o(\mu^{-1}h^{-3})$$

(d) Finally, arguments of (a) should be slightly improved as $\mu\ge h^{-1}|\log h|^{-J}$. Namely the source of term containing logarithmic factor is the only perturbation of the type $C\mu^{-2}x_1^{-1}B(x',\mu^{-1}hD)$ but then if ψ is even with respect

to x_1 the results of calculation will be 0 and if ψ contains factor x_1 it would compensate x_1^{-1} and no logarithm would appear. □

Intermediate Magnetic Field. VII

Now we need to consider the vicinity of Σ. Let us start from the general case first.

Then scaling $x \to x/\gamma$, $h \to h/\gamma$, $\mu \to \mu\gamma$ and applying the same arguments as in the resonance case we estimate the contribution of the zone $\{\gamma(x) \asymp \gamma\}$ to the correction term by

$$C\mu h^{-3}\gamma^{-1} \times (\mu^{-2}\gamma^{-1})^{1-\kappa} \times \gamma^2$$

which in comparison to resonance case gains an extra factor γ and thus sums to its values as $\gamma = 1$, which is $C\mu^{-1+\kappa}h^{-3}$ which in turn is $O(\mu^2 h^{-2})$ as $\mu \ge h^{-1/3-\delta'}$. We ignore again term which sums to $C\mu^2 h^{-2}$ in the end of the day.

Meanwhile, contribution of the zone $\{\gamma(x) \le \bar\gamma\}$ does not exceed $C\mu h^{-3}\bar\gamma^2 = O(\mu^2 h^{-2})$.

Thus we arrive to the following statement:

Proposition 22.5.21. *Statement of Proposition 22.5.3 remains true in the vicinity of* Σ.

Remark 22.5.22. (i) Under condition

$$\sum_{|\beta|\le K} |\nabla_\Sigma^\beta V f_1^{-1}| \ge \epsilon_0 \qquad \forall x \tag{22.5.32}$$

with arbitrarily large K one can skip a correction term without deteriorating remainder estimate unless $h^{-1/3}|\log h|^{-J} \le \mu \le h^{-1/3}|\log h|^J$ and with the remainder estimate $O(h^{-8/3}|\log h|^J)$ in this border case. We suspect that one can get rid of logarithmic factors and to prove an estimate $O(\mu^{-1}h^{-3}+\mu^2 h^{-2})$ even in the border case.

(ii) One can easily construct a correction term in the case $\Sigma \ne \emptyset$ resonances but an expression seems rather too complicated. So we leave it to the curious reader.

Intermediate Magnetic Field. VIII

Finally let us consider the generic case near Σ. Using the same arguments as before we arrive to

Proposition 22.5.23. *Under non-degeneracy conditions* (22.4.83) *and* (22.4.84) *for ψ supported in the vicinity of Σ, the remainder estimate is $O(\mu^{-1}h^{-3})$ while the main part is given by the magnetic Weyl formula.*

22.6 Vanishing V Case

22.6.1 Generic Case

In the generic case, as $V = 0$, $|\nabla V| \ge \epsilon$ and then the microhyperbolicity condition of Chapter 13 holds and then

Proposition 22.6.1. *Assume that*

$$|V| + |\nabla V| \ge \epsilon. \tag{22.6.1}$$

Then in any dimension without no condition to F_{jk} other than $|\det(F_{jk})| \ge \epsilon$ for ψ supported in the small vicinity of the surface

$$\Delta := \{x : \ V(x) = 0\} \tag{22.6.2}$$

the asymptotics with the main part given by Magnetic Weyl expression and the remainder estimate $O(\mu^{-1}h^{1-d})$ holds.

This covers the generic case completely.

22.6.2 General Case

However, the general case is more complicated. Let us introduce a scaling function

$$\ell = \epsilon\big(|V| + |\nabla V|^2\big)^{1/2} + \frac{1}{2}\bar{\ell}, \qquad \bar{\ell} = \epsilon_0 \max\big((\mu h)^{1/2}, \mu^{-1}\big) \tag{22.6.3}$$

with dominating the second and the first terms in $\bar{\ell}$ as $\mu \le h^{-1/3}$ and $\mu \ge h^{-1/3}$ respectively.

Let us apply scaling $x \mapsto x\ell$, $h \mapsto h\ell^{-2}$, $\mu \mapsto \mu$, $V \mapsto V\ell^{-2}$: and consider the following cases:

(a) Let $\ell \le \epsilon_0(\mu h)^{1/2}$. Then we are in the classically forbidden zone and the contributions of ℓ-element to the principal part and the remainder estimate are 0 and negligible respectively.

(b) Let $\ell \ge \bar{\ell}$ and $|\nabla V| \asymp \ell$. Then after rescaling we are in the framework of Subsection 22.6.1 and the contribution of ℓ-element to the remainder is $O(\mu^{-1}h^{-3}\ell^6)$ and the total contribution of all ℓ-elements to the remainder is $O(\mu^{-1}h^{-3}\ell^2)$ and we are done here.

(c) Also, the contribution of all elements with $\ell \asymp \mu^{-1}$ to the remainder is $O(\mu h^{-3}\bar{\ell}^2)$ which is $O(\mu^{-1}h^{-3})$. So case $\ell \asymp \bar{\ell}$ is completely covered.

(d) Let us consider $V \asymp \ell^2$, $\ell \ge \bar{\ell}$; then condition $V \asymp 1$ is recovered after rescaling but now conditions to F_{jk} could fail; actually these condition do not fail completely, but are replaced by somewhat weaker condition with extra factors ℓ or ℓ^2 in the estimates from below. However, in the weak magnetic field case this weakened condition is enough; we leave the easy and tedious details to the reader:

Proposition 22.6.2. *As F_{jk} is generic and $\mu \le h^{-\delta}$ the remainder estimate $O(\mu^{-1}h^{-3})$ holds.*

Now one can assume that $h^{-\delta} \le \mu \le ch^{-1}$. Then since in Section 22.4 no condition "$|V| \ge \epsilon$" was required, the remainder estimate is $O(\mu^{-1}h^{-3} + \mu^2 h^{-2})$ but the principal part is given by the implicit formula (22.1.5) rather than by the magnetic Weyl expression (22.5.4) and now we need to modify arguments of Section 22.5 to pass from (22.1.5) to (22.5.4).

Note that in this implicit formula one always can take $T_0 = \epsilon\mu^{-1}$ with an arbitrarily small constant $\epsilon > 0$. Then in the virtue of arguments of the proof of Propositions 22.5.3 , 22.5.4 and 22.5.8 we get the remainder estimate $O(\mu^{-1}h^{-3} + \mu^2 h^{-2-\delta})$ with an arbitrarily small exponent $\delta > 0$.

However, in the contrast to the analysis in Subsection 22.5.1, without condition "$|V| \ge \epsilon$" it does not translates into $T_0 = Ch|\log h|$ but rather into $T_0 = Ch\ell^{-2}|\log h|$ after the above partition is applied. Nevertheless this last implicit formula (22.1.5) translates (with the same error) into Weyl or Magnetic Weyl formula. Therefore Theorem 22.6.3 below with the correction term 0 is proven for $\mu \le h^{-1/3-\delta}$ with an arbitrarily small exponent $\delta > 0$.

The similar arguments work as $\mu \ge h^{-1/3-\delta'}$ since one again refers to any $T_0 \le \epsilon\mu^{-1}$. So, Both Theorem 22.6.3 and Theorem 22.6.4 below hold without condition $|V| \ge \epsilon$.

Theorem 22.6.3. *Let* (g^{jk}) *be fixed and* $(V_1, \dots, V_4)$ *be generic. Further, let conditions* (22.1.3) *and* (22.1.4) *be fulfilled and* ψ *be a smooth function. Then*

$$|\int \big(e(x,x,0) - \mathcal{N}^{\mathsf{MW}}(x,0)\big) - \mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x,0)\big)\psi(x)\,dx| \le C\mu^{-1}h^{-3} + C\mu^2 h^{-2} \tag{22.6.4}$$

where

$$\mathcal{N}^{\mathsf{MW}}(x,\tau) = (2\pi)^{-2}\mu^2 h^{-2} \sum_{(m,n)\in\mathbb{Z}^{+\,2}} \theta\big(2\tau + V - (2m+1)\mu h f_1 - (2n+1)\mu h f_2\big) f_1 f_2 \sqrt{g} \tag{22.6.5}$$

is Magnetic Weyl Expression, $g = \mathsf{det}(g^{jk})^{-1}$, $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}$ *is a correction term, defined by* (22.5.23).

Theorem 22.6.4. *Let* (g^{jk}) *be fixed and* $(V_1, \dots, V_4; V)$ *be generic. Further, let conditions* (22.1.3) *and* (22.1.4) *be fulfilled and* ψ *be a smooth function. Then*

$$|\int \big(e(x,x,0) - \mathcal{N}^{\mathsf{MW}}(x,0)\big)\psi(x)\,dx| \le C\mu^{-1}h^{-3}. \tag{22.6.6}$$

Remark 22.6.5. (i) More precise conditions describing conditions for $(V_1, \dots, V_d)$ in Theorem 22.6.3 and for $(V_1, \dots, V_d; V)$ in Theorem 22.6.4 were formulated in Section 22.5.

(ii) We were able to prove that one can skip $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x,0)$ and preserve remainder estimate (22.6.4) unless $h^{-1/3+\delta} \le \mu \le h^{-1/3-\delta}$; in the latter case the remainder estimate (22.6.5) should be replaced by $h^{-8/3-\delta}$; however, we suspect that one can always skip $\mathcal{N}^{\mathsf{MW}}_{\mathsf{corr}}(x,0)$ and preserve remainder estimate (22.6.4).

Part X

Eigenvalue Asymptotics for Schrödinger and Dirac Operators with the Strong Magnetic Field

Chapter 23

Eigenvalue asymptotics. 2D Case

23.1 Introduction

In this chapter we obtain eigenvalue asymptotics for 2D-Schrödinger, Schrödinger-Pauli and Dirac operators in the situations in which the role of the magnetic field is important. We have seen in Chapters 13 and 17 that these operators are essentially different and they also differ significantly from the corresponding 3D-operators.

While we are trying to emulate results of Chapter 11, we find ourselves now in the very different situation. Indeed, for operators we study the remainder estimates in the local spectral asymptotics under non-degeneracy assumptions are better than for similar operators with the magnetic field. However, as we seen in Chapters 14 and 15 these remainder estimates deteriorate if the magnetic field degenerates or there is a boundary. This significantly limits our ability to consider the cases when magnetic field asymptotically degenerates at the point of singularity (finite or infinite) along some directions, or domains with the boundary.

We start from Section 23.2 in which we consider the case when the spectral parameter is fixed ($\tau = \mathsf{const}$) and study asymptotics with respect to μ, h exactly like in Section 11.1 we considered asymptotics with respect to h. However, since now we have two parameters, we need to consider an interplay between them: while always $h \to +0$, we cover $\mu \to +0$, μ remains disjoint from 0 and ∞ and $\mu \to \infty$, which in turn splits into subcases

$\mu h \to 0$, μh remains disjoint from 0 and ∞ and $\mu h \to \infty$.

In Section 23.3 we consider asymptotics with $\mu = h = 1$ and with the spectral parameter τ tending to $+\infty$ for the Schrödinger and Schrödinger-Pauli operators and to $\pm\infty$ for the Dirac operator. We consider bounded domains with the singularity at some point and external domains with the singularity at infinity. In the latter case the specifics of the 2D magnetic Schrödinger and Schrödinger-Pauli operators manifest itself in the better remainder estimate (and in the larger principal part for the Schrödinger-Pauli operator) than in the non-magnetic case. Furthermore, in contrast to the non-magnetic case there are non-trivial results for the Dirac operator. This happens only in even dimensions.

In Section 23.4 we consider asymptotics with the singularity at infinity and $\mu = h = 1$ and with τ tending to $+0$ for the Schrödinger and Schrödinger-Pauli operators and to $\pm(M - 0)$ for the Dirac operator. Again the specifics of the 2D magnetic Schrödinger and Schrödinger-Pauli operators manifest itself in the better remainder estimate (and in the larger principal part for the Schrödinger-Pauli operator) than in non-magnetic case.

It includes the most interesting case (see Subsection 23.4.1) when magnetic field is either constant or stabilizes fast at infinity and potential decays at infinity. As we know if magnetic field and potential were constant then the operator would have purely point spectrum of infinite multiplicity and each eigenvalue (Landau level) would be disjoint from the rest. Now we have a sequences of eigenvalues tending to the Landau level either from below, or from above, or from both sides and we are interested in their asymptotics. In contrast to the rest of the section we consider multidimensional case as well. In contrast to Section 11.6 there are non-trivial results for fast decaying potentials as well.

In Section 23.5 we consider asymptotics with respect to μ, h, τ, like in Section 11.7 again with significant differences mentioned above.

Finally, in Appendix 23.A.1 the self-adjointness of the 2D-Dirac operator with a very singular magnetic field is proven.

The Schrödinger operator theory is more extensive than the Dirac operator theory: there are many more meaningful problems and questions for the Schrödinger operator than for the Dirac operator. Also, the 2D-theory is more extensive and provides more accurate remainder estimates than the 3D-theory considered in the next Chapter 24. These circumstances are not due to the technical difficulties but are instead due to the fact that the spectrum of the Schrödinger operator is discrete more often than the

spectrum of the Dirac operator in dimension $d = 2$ the spectrum is discrete more often than in dimension $d = 3$.

23.2 Asymptotics with Fixed Spectral Parameter

In this section we consider asymptotics with a fixed spectral parameter for 2-dimensional magnetic Schrödinger, Schrödinger-Pauli and Dirac operators and discuss some of the generalizations[1)].

As in Chapters 9–12 we will introduce a semiclassical zone and a singular zone, where $\rho\gamma \ge h$ and $\rho\gamma \le h$ respectively. In the semiclassical zone we apply asymptotics of Chapters 13–22 (but mainly of 13 and 19–in the multidimensional case). In the singular zone we need to apply estimates for a number of eigenvalues; usually it would be sufficient to use non-magnetic estimate[2)] for number of eigenvalues which trivially follows from standard one but if needed one can use more delicate estimates.

23.2.1 Schrödinger Operator

Estimates of the Spectrum

Consider first the Schrödinger operator (13.2.1)

$$A = \sum_{j,k} P_j g^{jk} P_k + V, \qquad \text{with} \quad P_j = hD_j - \mu V_j \tag{23.2.1}$$

where g^{jk}, V_j, V satisfy (13.2.2) and (13.2.4) i.e.

$$\epsilon|\xi|^2 \le \sum_{j,k} g^{jk}\xi_j\xi_k \le c|\xi|^2 \qquad \forall \xi \in \mathbb{R}^d. \tag{23.2.2}$$

Without any loss of the generality we can fix $\tau = 0$ and then in the important function $V_{\mathsf{eff}} F_{\mathsf{eff}}^{-1}$ the parameters μ and h enter as factors. Thus, we treat the operator (13.2.1) assuming that it is self-adjoint.

[1)] Mainly to higher dimensions with full-rank magnetic field.

[2)] With V modified accordingly; for example, for the Schrödinger and Schrödinger-Pauli operators V_- is replaced by $C\big((1-\epsilon)V - C_\epsilon \mu^2 |\vec{V}|^2\big)_-$.

We make assumptions similar to those of Chapter 9

$$(23.2.3)_{1-3} \quad |D^\alpha g^{jk}| \le c\gamma^{-|\alpha|}, \quad |D^\alpha F_{jk}| \le c\rho_1\gamma^{-|\alpha|}, \quad |D^\alpha V| \le c\rho^2\gamma^{-|\alpha|}$$

where scaling function $\gamma(x)$ and weight functions $\rho(x), \rho_1(x)$ satisfy the standard assumptions $(9.1.6)_1$ and $(9.1.6)_2$. Then

$$(23.2.4)_{1,2} \qquad \mu_{\text{eff}} = \mu\rho_1\gamma\rho^{-1}, \qquad h_{\text{eff}} = h\rho^{-1}\gamma^{-1}.$$

Let us introduce a *semiclassical zone*

$$(23.2.5) \qquad X' = \{x\colon \rho\gamma \ge h\}$$

and a *singular zone*

$$(23.2.6) \qquad X'' = \{x\colon \rho\gamma \le 2h\}.$$

Further, let us introduce two other overlapping zones

$$(23.2.7) \qquad X'_1 = \{x \in X_{\text{scl}}\colon \mu\rho_1 \le 2c\rho\gamma^{-1}\}$$

and

$$(23.2.8) \qquad X'_2 = \{x \in X'\colon \mu\rho_1 \ge c\rho\gamma^{-1}\}$$

where the magnetic field $\mu_{\text{eff}} = \mu\rho_1\rho^{-1}\gamma$ is *normal* ($\mu_{\text{eff}} \le 2c$) and where it is *strong* ($\mu_{\text{eff}} \ge c$) respectively. We assume that

$$(23.2.9) \qquad |F| \ge \epsilon\rho_1 \qquad \text{in} \quad X'_2$$

where F_{jk} and F are the tensor and scalar intensities of the magnetic field.

Moreover, let us assume that

$$(23.2.10)_1 \qquad B\big(x, \gamma(x)\big) \subset X \qquad \forall x \in X'_{2-},$$

and

$$(23.2.10)_2 \qquad u\big|_{\partial X \cap B\left(x,\gamma(x)\right)} = 0 \qquad \forall x \in X' \quad \forall u \in \mathfrak{D}(A)$$

where

$$(23.2.11) \qquad X'_{2-} = \{x \in X'_2\colon V + \mu hF \ge \epsilon\rho^2\}$$

and

$$(23.2.12) \qquad X'_{2+} = \{x \in X'\colon V + \mu hF \le 2\epsilon\rho^2\}.$$

Finally, let the standard boundary regularity condition be fulfilled:

(23.2.13) For every $y \in X$, $\partial X \cap B(y, \gamma(y)) = \{x_k = \phi_k(x_{\hat{k}})\}$ with

$$|D^\alpha \phi_k| \le c\gamma^{-|\alpha|}$$

and $k = k(y)$.

Due to $(23.2.10)_1$ we need this condition only as $y \in X'_1 \cup X'_{2-}$. Finally, in X'_{2-} let one of the following non-degeneracy conditions

$$|V + (2n+1)\mu hF| + |\nabla VF^{-1}|\gamma \ge \epsilon\rho^2 \qquad \forall n \in \mathbb{Z}^+, \tag{23.2.14}$$

$$|V + (2n+1)\mu hF| + |\nabla VF^{-1}|\gamma \le \epsilon\rho^2\epsilon \implies \det \mathsf{Hess}(VF^{-1}) \ge \epsilon\rho^4\rho_1^{-2}\gamma^{-4} \qquad \forall n \in \mathbb{Z}^+, \tag{23.2.15}$$

$$|V + (2n+1)\mu hF| + |\nabla VF^{-1}|\gamma \le \epsilon\rho^2\epsilon \implies |\det \mathsf{Hess}(VF^{-1})| \ge \epsilon\rho^4\rho_1^{-2}\gamma^{-4} \qquad \forall n \in \mathbb{Z}^+, \tag{23.2.16}$$

be fulfilled.

Recall that according to Chapter 13 the contribution of the partition element $\psi \in \mathscr{C}_0^K(B(y, \frac{1}{2}\gamma(y))$ to the principal part of asymptotics is

$$\mathcal{N}^-(\mu, h) = \mathcal{N}^{\mathsf{MW}-}(\mu, h) := h^{-2} \int \mathcal{N}^{\mathsf{MW}}(x, \mu h)\psi(x)\, dx \tag{23.2.17}$$

with $\mathcal{N}^{\mathsf{MW}}(x, \mu h)$ given by (13.2.9) with $d = 2$.

On the other hand, its contribution to the remainder does not exceed $Ch^{-1}\rho\gamma$ if $\mu\rho_1\gamma \le c\rho$ and it does not exceed $C\mu^{-1}h^{-1}\rho^2\rho_1^{-1}$ if $\mu\rho_1\gamma \ge c\rho$ but $\mu h\rho_1 \le \rho^2$, $y \in X'_+$ and assumption (23.2.15) is fulfilled[3)], and it does not exceed $C(\mu^{-s}\rho_1^{-s}\gamma^{-2s})$ if $C\mu\rho_1\gamma \ge \rho$, $\mu h\rho_1 \le \rho^2$, $y \in X'_{2-}$.

Then we get estimate of N^- from below by the magnetic Weyl approximation $\mathcal{N}^-(\mu, h)$ minus corresponding remainder, and also from above

[3)] It does not exceed the same expression with an extra logarithmic factor under assumption (23.2.16) but logarithmic factor could be skipped if we add corrections at the points with negative $\det \mathsf{Hess}(VF^{-1})$.

by magnetic Weyl approximation plus corresponding remainder, provided $X = X'$ (so, there is no singular zone $X'' = \emptyset$):

$$h^{-d}\int_{X'} \mathcal{N}^{\mathsf{MW}}(x,\mu h)\,dx - CR_1 \le \mathsf{N}^-(0) \le h^{-d}\int_{X'} \mathcal{N}^{\mathsf{MW}}(x,\mu h)\,dx + CR_1 + C'R_2 \tag{23.2.18}$$

with

$$R_1 = \mu^{-1}h^{1-d}\int_{X'_+} \rho^d \rho_1^{-1}\gamma^{-2}\,dx, \tag{23.2.19}$$

$$R_2 = \mu h^{s-d}\int_{X'_-} \rho_1 \rho^{d-s-1}\gamma^{1-s}\,dx \tag{23.2.20}$$

provided

$$\mu\rho_1\gamma \ge \rho \tag{23.2.21}$$

where the latter condition could be assumed without any loss of the generality, C' depens also on s and ϵ.

We leave to the reader the following not very challenging set of problems:

Problem 23.2.1. (i) Consider the case $X'' \neq \emptyset$ and prove the estimate from above with an extra term C_1R_0 like in estimate (9.1.29) (Theorem 9.1.7) with R_0 defined in the same way as in Section 9.1.

(ii) Consider multidimensional case; then we need to impose more sophisticated non-degeneracy assumptions (see Chapter 19).

(iii) Incorporate results of Chapters 14, 15, 18, 19 (in non-smooth settings), 21 and 22.

(iv) Using arguments and methods of Chapter 10 and results of Chapter 17 consider Dirac operator (in the full-rank case).

Basic Results

In what follows $h \to +0$ and the semiclassical zone X' expands to X while μ is either bounded (then we can assume that the zone of the strong magnetic field X'_2 is fixed) or tends to ∞ (then X'_2 expands to X). We assume that all

conditions of the previous subsection are fulfilled with $\mu = h = 1$ but we will assume them fulfilled in the corresponding zones.

The other important question is whether $\mu h \to 0$, remains bounded and disjoint from 0 or tends to ∞.

Finally, we should consider the singular zone X''. In order to avoid this task we assume initially that

$$\rho_1\gamma^2 + \rho\gamma \ge \epsilon. \tag{23.2.22}$$

Then we obtain the following assertion from the arguments of the previous Subsection 23.2.1.1:

Theorem 23.2.2. *Let $d = 2$ and let the Schrödinger operator A satisfy conditions* (23.2.15), (23.2.2), $(23.2.3)_{1-3}$, (23.2.9) *in the corresponding regions where ρ, ρ_1, γ satisfy* $(9.1.6)_1$–$(9.1.6)_4$, $(23.2.10)_{1,2}$, (23.2.22).

Let $\rho_1 \ge \epsilon\rho\gamma^{-1}$ and

$$\rho^2\rho_1^{-1}\gamma^{-2} \in \mathscr{L}^1\bigl(X \cap \{V + tF \le \epsilon\rho^2\}\bigr), \tag{23.2.23}$$

$$\rho_1^{-s}\gamma^{-2s-d} \in \mathscr{L}^1(X) \tag{23.2.24}$$

with some $s \ge 0$ and $t \ge 0$. Then for $h \to +0$, $1 \le \mu$ such that $\mu h \le t$ the "standard" asymptotics

$$\mathsf{N}^-(\mu, h) = \mathcal{N}^-(\mu, h) + O(\mu^{-1}h^{-1}) \tag{23.2.25}$$

holds with

$$\mathcal{N}^-(\mu, h) := h^{-d}\int \mathcal{N}^{\mathsf{MW}}(x, \mu h)\, dx. \tag{23.2.26}$$

Power Singularities

Example 23.2.3. (i) Let 0 be an *inner singular point*[4] and let conditions of Theorem 23.2.2 be fulfilled with $\gamma = \epsilon|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$ and let $m_1 < \min(m-1, 2m)$ [5),6)].

[4] I.e. $0 \in \bar{X}$ is an isolated point of $\mathbb{R}^2 \setminus X$.

[5] One can construct such potential (V_1, V_2) easily; f.e. $V_1 = -x_2|x|^{m_1}$, $V_2 = x_1|x|^{m_1}$ for $m_1 \ne 2$; for $m_1 = -2$ one needs to multiply V_1, V_2 by $\log|x|$.

[6] To have the non-degeneracy condition (23.2.15) fulfilled in the vicinity of 0 we assume that

$$|\nabla VF^{-1}| \ge C\rho^2\rho_1^{-1}\gamma^{-1} \qquad \text{as } |x| \le \epsilon; \tag{23.2.27}$$

in Statement (ii) one should replace $|x| \le \epsilon$ by $|x| \ge c$.

Let $V + \mu hF \ge \epsilon\rho^2$ on $(\partial X \setminus 0) \cup \{x\colon |x| \ge c\}$.

Then conditions (23.2.23), (23.2.24) are fulfilled automatically and asymptotics (23.2.25)–(23.2.26) holds for μ disjoint from 0 and $h \to +0$.

Further,

$$\mathcal{N}^-(\mu, h) = \begin{cases} O(h^{-2}) & m > -1, \\ O\big(h^{-2}(|\log(\mu h)| + 1)\big) & m = -1, \\ O\big(h^{-2}(\mu h)^{2(m+1)/(2m-m_1)}\big) & m < -1. \end{cases} \tag{23.2.28}$$

Furthermore, one can replace "$= O$"with "$\asymp$" if <u>either</u> $m > -1$, $\mu h \le t$ and

$$\{X \setminus 0, V \le -tF - \epsilon\} \ne \emptyset \tag{23.2.29}$$

<u>or</u> $m \le -1$ and

$$V \le -\epsilon\rho^2 \qquad \text{in} \quad \Gamma \cap \{|x| \le \epsilon\} \subset X \tag{23.2.30}$$

where Γ is an open non-empty sector (cone) with vertex at 0, and $\mu h \le t$ with a small enough constant $t > 0$.

(ii) Let infinity be an *inner singular point*[7)] and let conditions of Theorem 23.2.2 be fulfilled with $\gamma = \epsilon\langle x\rangle$, $\rho\langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$ and let $m_1 > \max(m-1, 2m)$[5),6)]. Let $V + \mu hF \ge \epsilon\rho^2$ on ∂X.

Then conditions (23.2.23) and (23.2.24) are fulfilled automatically and asymptotics (23.2.25)–(23.2.26) holds for μ disjoint from 0 and $h \to +0$.

Further,

$$\mathcal{N}^-(\mu, h) = \begin{cases} O(h^{-2}) & m < -1, \\ O\big(h^{-2}(|\log(\mu h)| + 1)\big) & m = -1, \\ O\big(h^{-2}(\mu h)^{2(m+1)/(2m-m_1)}\big) & m > -1. \end{cases} \tag{23.2.28$'$}$$

Furthermore, one can replace "$= O$" by "$\asymp$" if <u>either</u> $m < -1$, $\mu h \le t$ and (23.2.29) is fulfilled <u>or</u> $m \le -1$,

$$V \le -\epsilon\rho^2 \qquad \text{in} \quad \Gamma \cap \{|x| \ge c\} \subset X \tag{23.2.30$'$}$$

where Γ is an open non-empty sector (cone) with vertex at 0, and $\mu h \le t$ with a small enough constant $t > 0$.

[7)] I.e. $\mathbb{R}^2 \setminus X$ is compact.

(iii) One can easily see that for $m > -1$ in (i), $m < -1$ in (ii)

$$\mathcal{N}^{-}(\mu, h) = \mathcal{N}^{\mathsf{W}-}(\mu, h) + O\big(h^{-2}(\mu h)^{(2m+2)/(2m-m_1)}\big) \tag{23.2.31}$$

under condition (23.2.15) where $\mathcal{N}^{\mathsf{W}-}$ is the standard Weyl expression[8]. Therefore for $\mu \le h^p$ with $p = (2-m_1)/(4m-m_1+2)$ the asymptotics remain true with $\mathcal{N}^{-}$ replaced by $\mathcal{N}^{\mathsf{W}-}$.

On the other hand, under condition (23.2.30) or (23.2.30)$^{\#}$ (respectively) $\mathcal{N}^{\mathsf{W}-} - \mathcal{N}^{\mathsf{MW}-} \ge \epsilon h^{-2}(\mu h)^{(2m+2)/(2m-m_1)}$. Thus, for $\mu > h^p$ one cannot replace $\mathcal{N}^{-}$ by $\mathcal{N}^{\mathsf{W}-}$ and preserve the remainder estimates. Note that p is not necessarily negative in our conditions! Therefore, due to the singularity, the magnetic field can be essential even for a fixed μ. In what follows we leave this type of analysis to the reader.

Example 23.2.4. (i) Assume now that $m_1 \ge \min(m-1, 2m)$, $m_1 \ne 2m$[9], while all other assumptions of Example 23.2.3(i) are fulfilled. Since we want to have a finite $\mathcal{N}^{-}$ and (23.2.30) to be fulfilled we need to assume that $m > -1$. Then $\mathcal{N} \asymp h^{-2}$. Let us calculate the remainder estimate.

(a) Assume first that $m-1 \le m_1 < 2m$; then $\int \rho_1^{-1}\rho^2\,dx < \infty$ and then contribution of the semiclassical zone $X' = \{x\colon |x| \ge r_0 = h^{1/(m+1)}\}$ to the remainder is $O(\mu^{-1}h^{-1})$ and we need to estimate the contribution of the singular zone $X'' = \{x\colon |x| \le r_0\}$. Without any loss of the generality we can assume that $|\vec{V}| \le C\rho_1\gamma$[10]. Then CLR[11] implies that the contribution of X'' to the asymptotics does not exceed $Ch^{-2}\int_{X''}(\rho^2 + \mu^2\rho_1^2\gamma^2)\,dx \asymp C$.

(b) Let now $m_1 > 2m$; then we need to consider zones

$$X_2' = \{x\colon |x| \ge r_1 = \mu^{-1/(m_1+1-m)}\}$$

where $\mu_{\mathsf{eff}} \ge 1$,

$$X_1' = \{x\colon r_0 \le |x| \le r_1\}$$

where $\mu_{\mathsf{eff}} \le 1$ and $h_{\mathsf{eff}} \le 1$ and X''. Contributions of X_2', X_1' to the remainder do not exceed $K_1 := C\mu^{-1}h^{-1}\int_{X_2'}\rho_1^{-1}\rho^2\,dx$ and $K_1 := Ch^{-1}\int_{X_1'}\rho\gamma^{-1}\,dx$ respectively.

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[8] Rather than the magnetic Weyl expression $\mathcal{N}^{\mathsf{MW}-}$.

[9] Otherwise we could not satisfy(23.2.27).

[10] One can prove it easily taking $V_1 = -\partial_2\phi$, $V_2 = \partial_1\phi$ with ϕ solving $\Delta\phi = F$.

[11] Sure, CLR does not hold for $d = 2$ but we can use more complicated Rozenblioum's estimate exactly like in Section 11.1.

Obviously, $K_1 \asymp K_2 \asymp Ch^{-1}r_1^{m+1} \asymp Ch^{-1}\mu^{-(m+1)/(m_1+m-1)}$ for $m_1 > 2m$.

Finally, contribution of X'' to the asymptotics is $O(1)$.

Thus as $h \to +0$, μ is disjoint from 0 and μh is disjoint from infinity, we have asymptotics

$$\mathsf{N}^-(\mu, h) = \mathcal{N}^-(\mu, h) + \begin{cases} O(h^{-1}\mu^{-1}) & m_1 < 2m, \\ O(h^{-1}\mu^{-(m+1)/(m_1+1-m)}) & m_1 > 2m. \end{cases} \tag{23.2.32}$$

(ii) Assume now that $m_1 \le \max(m-1, 2m)$ while all other assumptions of Example 23.2.3(ii) are fulfilled and $m < -1$. Again, considering cases

(a) $2m < m_1 < m-1$ and

(b) $m_1 < 2m$

we arrive to the asymptotics

$$\mathsf{N}^-(\mu, h) = \mathcal{N}^-(\mu, h) + \begin{cases} O(h^{-1}\mu^{-1}) & m_1 > 2m, \\ O(h^{-1}\mu^{-(m+1)/(m_1+1-m)}) & m_1 < 2m. \end{cases} \tag{23.2.32}^{\#}$$

Consider now fast increasing μ so that $\mu h \to \infty$. We will get non-trivial results only when domain defined by $\mu_{\mathsf{eff}} h_{\mathsf{eff}} \le C_0$ shrinks but remains non-empty which happens only in the frameworks of subcases (b) of Example 23.2.4.

Example 23.2.5. (i) In the framework of Example 23.2.4(i) with $m_1 > 2m$ consider $\mu h \to \infty$. Then the allowed domain is

$$\{x\colon |x| \lesssim r_2 = (\mu h)^{-1/(m_1-2m)}\} \tag{23.2.33}$$

and we have $r_0 \le r_1 \le r_2$ if $\mu \lesssim h^{-(m_1+1-m)/(m+1)}$ while for $\mu \gtrsim h^{-(m_1+1-m)/(m+1)}$ inequalities go in the opposite direction.

Therefore as $h \to +0$, $ch^{-1} \le \mu \le h^{-(m_1+1-m)/(m+1)}$ asymptotics (23.2.32) holds and one can see easily that $\mathcal{N}^-(\mu, h) \asymp h^{-2}r_2^{2m+2}$:

$$\mathcal{N}^-(\mu, h) \asymp \mu^{-2(m+1)/(m_1-2m)} h^{-2(m_1+1-m)/(m_1-2m)}. \tag{23.2.34}$$

(ii) Similarly, in the framework of Example 23.2.4(ii) with $m_1 < 2m$ asymptotics (23.2.32)$^{\#}$ and (23.2.34) hold as $h \to +0$, $ch^{-1} \le \mu \le h^{-(m_1+1-m)/(m+1)}$.

Let us consider $\mu \to \mu_0$ where $\mu_0 \ge 0$ is fixed.

Example 23.2.6. (i) Let us consider singularity at 0. In this case we are in the framework of Section 11.1 provided $m > -1$, $m_1 > -2$. So we need to consider the case when either $m \le -1$ or $m_1 \le -2$ and $m_1 \ne 2m$.

(a) The contribution of the semiclassical zone to the remainder is $O(h^{-1})$ only if $m > -1$.

(b) Assume now that $m \le -1$; then we need to assume that $m_1 < 2m$ and we have a normal magnetic field zone $X_1' = \{x\colon |x| \ge r_1 = \mu^{-1/(m_1-m+1)}\}$, strong magnetic field zone $X_2' = \{x\colon r_0 = (\mu h)^{-1/(m_1-2m)} \le |x| \le r_1\}$, and forbidden zone $X_3' = \{x\colon |x| \le r_1\}$. One can check that $r_0 < r_1$. Then the contribution to the remainder of X_1' and X_2' are both $O(h^{-1}r_1^{m+1}) = O(h^{-1}\mu^{-(m+1)/(m_1-m+1)})$ as $m < -1$ while for $m = -1$ contribution of X_1' is $O(h^{-1}(|\log\mu| + 1))$ and of X_2' is $O(h^{-1})$. Contribution of X_3' is smaller. So we get a remainder estimate

$$\mathsf{N}^-(\mu,h) = \mathcal{N}^-(\mu,h) + \begin{cases} O(h^{-1}), & m > -1,\\ O(h^{-1}(|\log\mu| + 1)) & m = -1,\\ O(h^{-1}\mu^{-(m+1)/(m_1-m+1)}) & m < -1. \end{cases} \tag{23.2.35}$$

Meanwhile, one can prove easily that

$$\mathcal{N}^-(\mu,h) \asymp \begin{cases} h^{-2} & m > -1,\\ h^{-2}(|\log\mu h| + 1) & m = -1,\\ h^{-2(m_1-m+1)/(m_1-2m)}\mu^{-2(m+1)/(m_1-2m)} & m < -1. \end{cases} \tag{23.2.36}$$

under assumption (23.2.30).

(ii) Similarly, consider the singularity at infinity. Then

(a) Let $m < -1$; then the remainder estimate is $O(h^{-1})$.

(b) Let $m \ge -1$, $m_1 > 2m$. Then

$$\mathsf{N}^-(\mu,h) = \mathcal{N}^-(\mu,h) + \begin{cases} O(h^{-1}) & m < -1,\\ O(h^{-1}(|\log\mu| + 1)) & m = -1,\\ O(h^{-1}\mu^{-(m+1)/(m_1-m+1)}) & m > -1. \end{cases} \tag{23.2.35}^{\#}$$

and

$$(23.2.36)^{\#}\quad \mathcal{N}^{-}(\mu,h)\asymp \begin{cases} h^{-2} & M<-1,\\ h^{-2}(|\log \mu h|+1) & m=-1,\\ h^{-2(m_1-m+1)/(m_1-2m)}\mu^{-2(m+1)/(m_1-2m)} & m>-1. \end{cases}$$

under assumption $(23.2.30)^{\#}$.

Improved Remainder Estimates

Let us improve remainder estimates (23.2.32), $(23.2.32)^{\#}$, (23.2.35), $(23.2.35)^{\#}$ under certain non-periodicity-type assumptions.

Example 23.2.7. (i) In the framework of Example 23.2.6(i) with $m>-1$ the contribution to the remainder of the zone $\{x\colon |x|\le \varepsilon\}$ does not exceed σh^{-1} with $\sigma=\sigma(\varepsilon)\to 0$ as $\varepsilon\to +0$. Then the standard arguments imply that under the standard non-periodicity assumption for Hamiltonian billiards[12)] with the Hamiltonian

$$(23.2.37)\qquad a(x,\xi,\mu_0)=\sum_{j,k} g^{jk}(\xi_j-\mu_0 V_j)(\xi_k-\mu_0 V_k)+V(x)$$

the improved asymptotics

$$(23.2.38)\qquad \mathsf{N}^{-}(\mu,h)=\mathcal{N}^{-}(\mu,h)+\kappa_1 h^{-1}+o(h^{-1})$$

holds as $h\to +0$, $\mu\to\mu_0$ where $\kappa_1 h^{-1}$ is the contribution of ∂X calculated as $\mu=\mu_0$. However, in the general case we cannot replace $\mathcal{N}^{\mathsf{MW}-}$ by $\mathcal{N}^{\mathsf{W}-}$ even if $\mu_0=0$.

(ii) Similarly in the framework of Example 23.2.6(ii) with $m<-1$ under the standard non-periodicity assumption for Hamiltonian billiards[12)] with the Hamiltonian (23.2.37) asymptotics (23.2.38) holds as $h\to +0$, $\mu\to\mu_0$.

(iii) In the framework of Example 23.2.6(i) with $m<-1$ the contributions to the remainder of the zones $\{x\colon |x|\le \varepsilon r_1\}$ and $\{x\colon |x|\ge \varepsilon^{-1} r_1\}$ do not exceed $\sigma h^{-1} r_1^{m+1}$ with $\sigma=\sigma(\varepsilon)\to 0$ as $\varepsilon\to +0$. After scaling $x\mapsto x r_1^{-1}$ etc

[12)] On the energy level 0.

the magnetic field in the zone $\{x\colon \varepsilon r_1 \le |x| \le \varepsilon^{-1} r_1\}$ becomes disjoint from 0 and ∞.

Assume that g^{jk}, V_j, V stabilize to positively homogeneous of degrees $0, m_1+1, 2m$ functions g^{jk0}, V_j^0, V^0 as $x \to 0$:

$$D^\alpha\bigl(g^{jk} - g^{jk0}\bigr) = o\bigl(|x|^{-|\alpha|}\bigr), \tag{23.2.39}$$

$$D^\alpha\bigl(V_j - V_j^0\bigr) = o\bigl(|x|^{m_1+1-|\alpha|}\bigr), \tag{23.2.39}$$

$$D^\alpha\bigl(V - V^0\bigr) = o\bigl(|x|^{2m-|\sigma|}\bigr) \qquad \forall \alpha : |\alpha| \le 1. \tag{23.2.39}$$

Then the standard arguments imply that under the standard non-periodicity assumption for Hamiltonian trajectories[12),13)] with the Hamiltonian

$$a^0(x,\eta) = \sum_{j,k} g^{jk0}(\eta_j - V_j^0)(\eta_k - V_k^0) + V_0 \tag{23.2.40}$$

the improved asymptotics

$$\mathsf{N}^-(\mu, h) = \mathcal{N}(\mu, h) + o\bigl(h^{-1}\mu^{-(m+1)/(m_1+1-m)}\bigr) \tag{23.2.41}$$

holds as $h \to +0$, $\mu \to 0$.

(iv) Similarly in the framework of Example 23.2.6(ii) with $m > -1$ let stabilization conditions $(23.2.39)^{\#}_{1-3}$ be fulfilled. Then under the standard non-periodicity assumption for Hamiltonian trajectories[12),13)] with the Hamiltonian (23.2.40) asymptotics (23.2.41) holds as $h \to +0$, $\mu \to 0$.

(v) In the framework of Example 23.2.6(i) with $m = -1$ the contributions to the remainder of the zone $\{x : |x| \le r_1\}$ does not exceed Ch^{-1}, while the contributions to the remainder of the zones $\{x : r_1 \le |x| \le r_1\mu^{-\delta}\}$ and $\{x : |x| \ge \mu^\delta\}$ do not exceed $C\delta h^{-1}|\log\mu|$ respectively. So, only zone $\{x : r_1\mu^{-\delta} \le |x| \le \mu^\delta\}$ should be treated. After rescaling magnetic field in this zone is small.

Let stabilization conditions $(23.2.39)_{1,3}$ be fulfilled. Then the standard arguments imply that under the standard non-periodicity assumption for Hamiltonian trajectories[12),13)] with the Hamiltonian

$$a^0(x,\eta) = \sum_{j,k} g^{jk0}\eta_j\eta_k + V_0 \tag{23.2.42}$$

[13)] In $T^*(\mathbb{R}^2 \setminus 0)$.

the improved asymptotics

$$\mathsf{N}^-(\mu,h) = \mathcal{N}(\mu,h) + o(h^{-1}|\log\mu|) \tag{23.2.43}$$

holds as $h\to+0$, $\mu\to 0$.

(vi) Similarly in the framework of Example 23.2.6(ii) with $m=-1$ let stabilization conditions $(23.2.39)^{\#}_{1,3}$ be fulfilled. Then under the standard non-periodicity assumption for Hamiltonian trajectories[12),13)] with the Hamiltonian (23.2.42) asymptotics (23.2.43) holds as $h\to+0$, $\mu\to 0$.

Consider now case of $\mu\to\infty$; we would like to improve estimates (23.2.32) for $m_1\ge 2m$ and $(23.2.32)^{\#}$ for $m_1\le 2m$. Recall that these estimates hold provided $\mu\lesssim h^{-(m_1+1-m)/(m+1)}$.

Example 23.2.8. (i) In the framework of Example 23.2.4(i) with $m_1>2m$ let stabilization conditions $(23.2.39)_{1-3}$ be fulfilled. Then using arguments of Example 23.2.7(iii) one can prove easily that under the the standard non-periodicity assumption for Hamiltonian trajectories[12),13)] with the Hamiltonian (23.2.40) asymptotics (23.2.41) holds as $h\to+0$, $\mu\to\infty$ provided $\mu = o\bigl(h^{-(m_1+1-m)/(m+1)}\bigr)$.

(ii) Similarly in the framework of Example 23.2.4(ii) with $m_1<2m$ let stabilization conditions $(23.2.39)^{\#}_{1-3}$ be fulfilled. Then under the the standard non-periodicity assumption for Hamiltonian trajectories[12),13)] with the Hamiltonian (23.2.40) asymptotics (23.2.41) holds as $h\to+0$, $\mu\to\infty$ provided $\mu = o\bigl(h^{-(m_1+1-m)/(m+1)}\bigr)$.

Degenerations

Consider magnetic field with degeneration.

Example 23.2.9. (i) Let 0 be an inner singular point of X and all assumptions of Example 23.2.3 (i) be fulfilled, except (23.2.9) $F\asymp\rho_1$ which is replaced now by

$$|F| + |\nabla F|\gamma \asymp \rho_1. \tag{23.2.44}$$

let $\Sigma=\{x\colon F_{12}=0\}$ be the manifold of degeneration, $\mathcal{Z}=\{x\colon |F|\le\epsilon\rho_1\}$ be the vicinity of the degeneration. Then we need to refer to Chapter 14.

To have $\mu_{\text{eff}} \gg 1$, and also $\mu_{\text{eff}} \ge Ch_{\text{eff}}^{-2}$ near singularity with as before $\mu_{\text{eff}} = \mu|x|^{m_1+1-m}$, $h_{\text{eff}} = h|x|^{-m-1}$ we assume that $m_1 < 3m+1$ if $m \le -1$. Assume that $m_1 < \min(m-1, 3m+1)$.

We preserve the non-degeneracy assumption (23.2.27) in $X \setminus \mathcal{Z}$ and replace it by[14)]

$$|\nabla_\Sigma W| \asymp \rho^2 \rho_1^{-\frac{2}{3}} \gamma^{-\frac{1}{3}} \tag{23.2.45}$$

where $W : |W| \lesssim \rho^2 \rho_1^{-\frac{2}{3}} \gamma^{\frac{2}{3}}$ is introduced according to Chapter 14. Then in addition to $m_1 \ne 2m$ we assume also that $m_1 \ne 3m+1$.

Recall that for the Schrödinger operator instead of $C\mu_{\text{eff}}^{-1} h_{\text{eff}}^{-1}$ the local remainder estimate now is $C\mu_{\text{eff}}^{-\frac{1}{2}} h_{\text{eff}}^{-1} \asymp C\mu^{-\frac{1}{2}} h^{-1} \gamma^{\frac{1}{2}(-m_1+3m+1)}$ and summation with respect to partition of unity returns $O(\mu^{-\frac{1}{2}} h^{-1})$ as $h \to +0$, μ is disjoint from 0 and $\mu^2 h$ is disjoint from infinity.

One can prove easily that $\mathcal{N}^-(\mu, h) \asymp h^{-2}$ as $\mu \lesssim h^{-1}$ and $\mathcal{N}^-(\mu, h) \asymp \mu^{-1} h^{-3}$ as $h^{-1} \lesssim \mu \lesssim h^{-2}$ exactly like in the regular case.

(ii) Similar results hold in the case when infinity is an inner singular point, $m_1 > \max(m-1,\, 3m+1)$.

The following problems are not very challenging but interesting:

Problem 23.2.10. (i) Consider the case of 0 being an inner singular point, $m > -1$, $m_1 \ge m-1$. The threshold value is $m_1 = 3m+1$. Assume that $h \to +0$.

(a) Consider cases $1 \lesssim \mu \lesssim h^{-1}$ and $h^{-1} \lesssim \mu \lesssim h^{-2}$ (cf. Example 23.2.4). Prove that in the case $m-1 \le m_1 < 3m+1$ the remainder estimate is $O(-\mu^{\frac{1}{2}} h^{-1})$. Derive the remainder estimate in the case $m_1 > 3m+1$. Calculate the magnitude of $\mathcal{N}^-$.

(b) Consider case $m_1 > 3m+1$ and $\mu \gtrsim h^{-2}$ (cf. Example 23.2.5). Derive the remainder estimate. Calculate the magnitude of $\mathcal{N}^-$.

(c) Consider $\mu \to 0$ (cf. Example 23.2.6 and possibly Example 23.2.7). Derive the remainder estimate. Calculate the magnitude of $\mathcal{N}^-$.

[14)] We need it mainly to avoid correction terms.

(ii) Similarly consider the case of 0 being an inner singular point, $m > -1$, $m_1 \le m-1$.

Example 23.2.11. Stronger asymptotic degenerations are not easily accessible. F.e. even if Chapter 14 provides us with tools to consider $F_{12} = x_1^n |x|^{m_1-n}$ with $n = 2 \ge$, it does not provide us with a tool to deal with the "small perturbations" like $F_{12} = x_1^n |x|^{m_1-n} + b x_1^{n-2} |x|^{m_2-n+2}$ with $m_2 > m_1$ ($m_2 < m_1$), if we consider vicinity of 0 (infinity respectively).

We want to recover whatever remainder estimates are possible.

(i) Case $n = 2$ and $b > 0$ should be the easiest as then the perturbation helps: the first rescaling is the standard $x \mapsto 2(x-y)r^{-1}$, $r = |y|$ in $B(y, \frac{1}{2}r)$, transforming it to $B(0,1)$, and as before $h \mapsto h_{\text{eff}} = hr^{-m-1}$, $\mu \mapsto \mu_{\text{eff}} = \mu r^{m_1+1}$ and perturbing field has the strength $\varepsilon\mu$ with $\varepsilon = r^{m_2-m_1}$.

Then the second rescaling $x \to (x-y)\gamma^{-1}$ with $\gamma = |F|^{1/2}$ as long as $|F| \ge \bar\gamma = C_0 \max(\mu^{-1/3}, \varepsilon^{\frac{1}{2}})$ and $\bar\gamma$ otherwise. Therefore the contribution of $B(y, \gamma(y))$ to the remainder does not exceed $C\mu^{-1}h^{-1}\gamma^{-2}$ and the summation over partition results in $C\mu^{-1}h^{-1}\bar\gamma^{-3}$, i.e. $Ch^{-1}\min(1, \mu^{-1}\varepsilon^{-\frac{3}{2}})$.

We leave to the reader to plug $h_{\text{eff}}, \mu_{\text{eff}}$ instead of μ, h and take a summation over the primary partition.

(ii) Cases $n = 3$ and $b > 0$ and $n = 2, 3$ and $b < 0$ are harder but for m_2 close enough to m_1 we refer to Chapter 14 after the second rescaling.

Then the second rescaling is the same $x \to (x-y)\gamma^{-1}$ with $\gamma = |F|^{1/n}$ as long as $|F| \ge \bar\gamma = C_0 \max(\mu^{-1/(n+1)}, \varepsilon^{\frac{1}{2}})$ and $\bar\gamma$ otherwise. Repeating the above arguments we conclude that the contribution of this zone is Ch^{-1} for $\varepsilon \le \mu^{-2/(n+1)}$ and $Ch^{-1}\mu^{-1}\varepsilon^{-(n+1)/2}$ for $\varepsilon \le \mu^{-2/(n+1)}$.

In the former case we are done, in the latter we need to explore zone $|F| \le \varepsilon^{n/2}$. Rescaling we get $\mu' = \mu\varepsilon^{(n+1)/2}$ and $h' = h\varepsilon^{-1/2}$. Then we can refer to Chapter 14 rather than Chapter 13 and the contribution of $B(y, \varepsilon^{1/2})$ to the remainder does not exceed $C\mu^{-1/2}h^{-1}\varepsilon^{-(n-1)/4}$ and summation over this zone results in $C\mu^{-1/2}h^{-1}\varepsilon^{-(n+1)/4}$, which is greater than the contribution of the previous zone.

Again, we leave to the reader to plug $h_{\text{eff}}, \mu_{\text{eff}}$ instead of μ, h and take a summation over the primary partition.

Power sSingularities. II

Let us modify our arguments for the case $\rho_3 < 1$. Namely, in addition to $(23.2.3)_{1-3}$ we assume that

$$(23.2.46)_{1,2} \qquad |D^\alpha g^{jk}| \le c\rho_2\gamma^{-|\alpha|}, \qquad |D^\alpha F_{jk}| \le c\rho_2\rho_1\gamma^{-|\alpha|},$$

$$(23.2.46)_3 \qquad |D^\alpha \frac{V}{F}| \le c\rho_3\rho^2\rho_1^{-1}\gamma^{-|\alpha|} \qquad \forall \alpha : 1 \le |\alpha| \le K$$

with $\rho_3 \le \rho_2 \le 1$ in the corresponding regions where $\rho, \rho_1, \gamma, \rho_3$ are scaling functions.

Recall that in Chapter 13 operator was reduced to the canonical form with the term, considered to be negligible, of magnitude $\rho_2\mu_{\text{eff}}^{-2N}$. In this case impose non-degeneracy assumptions

$$(23.2.47)^* \quad \bigl|V + (2n+1)\mu hF\bigr| \le \epsilon_0\rho_3\rho^2, \quad n \in \mathbb{Z}^+ \implies |\nabla v^*| \ge \epsilon_0\rho_3\rho^2\rho_1^{-1}\gamma^{-1}$$

and

$$(23.2.48) \qquad \rho_3 \ge C_0\rho_2(\mu\rho_1\gamma\rho^{-1})^{-N}$$

where v^* is what this reduction transforms VF^{-1} to and (23.2.48) means that "negligible" terms do not spoil $(23.2.47)^*$.

Then according to Chapter 13 the contribution of $B(x,\gamma)$ to the Tauberian remainder does not exceed

$$(23.2.49) \qquad C\bigl(\rho_3\mu_{\text{eff}}^{-1}h_{\text{eff}}^{-1} + 1\bigr)$$

while the approximation error does not exceed

$$(23.2.50) \qquad C\rho_2\rho_3^{-1}\mu_{\text{eff}}^{1-2N}h_{\text{eff}}^{-1}\underbrace{\bigl(\rho_3\mu_{\text{eff}}^{-1}h_{\text{eff}}^{-1} + 1\bigr)}\min\bigl((\mu_{\text{eff}}h_{\text{eff}}\rho_3^{-1})^s, 1\bigr)$$

with arbitrarily large s and therefore selected factor could be skipped.

Example 23.2.12. (i) Let 0 be an inner singular point[4)] and let assumptions $(23.2.46)_{1-3}$ be fulfilled with $\gamma = \epsilon|x|$, $\rho = |x|^m\bigl(\bigl|\ln|x|\bigr| + 1\bigr)^\alpha$, $\rho_1 = |x|^{2m}\bigl(\bigl|\ln|x|\bigr| + 1\bigr)^\beta$, $\rho_2 = 1$ and $\rho_3 = \bigl(\bigl|\ln|x|\bigr| + 1\bigr)^{-1}$.

Assume that

(23.2.51) Either $m < -1$ and $\beta > 2\alpha$ or $m = -1$ and $\beta > \max(\alpha, 2\alpha)$.

Then (23.2.48) is fulfilled with $N = 1$ and we can replace (23.2.47)* by

$$\text{(23.2.47)} \quad \bigl|V + (2n+1)\mu h F\bigr| \le \epsilon_0\rho_3\rho^2, \quad n \in \mathbb{Z}^+ \implies |\nabla V F^{-1}| \ge \epsilon_0\rho_3\rho^2\rho_1^{-1}\gamma^{-1}$$

Then (23.2.49) becomes $C\bigl(\mu^{-1}h^{-1}|\log r|^{2\alpha-\beta-1} + 1\bigr)$ and for $s = 2$ the summation over zone $\{\mu_{\text{eff}} h_{\text{eff}} \lesssim 1\} = \{\mu h|\log r|^{\beta-2\alpha} \lesssim 1\}$ results in $C\bigl((\mu h)^{-1} + (\mu h)^{-1/(\beta-2\alpha)}\bigr)$.

Meanwhile, (23.2.50) with $N = 1$ becomes $C(\mu h)^s|\log r|^{s(\beta-2\alpha+1)+2}$ in the zone $\{\mu h|\log r|^{\beta-2\alpha+1} \le 1\}$ and the summation over this zone results in $C(\mu h)^{-3/(\beta-2\alpha+1)}$.

On the other hand, (23.2.50) with $N = 1$ becomes $C\mu^{-1}h^{-1}|\log r|^{1-\beta+2\alpha}$ in the zone $\{\mu h|\log r|^{\beta-2\alpha+1} \ge 1,\ \mu h|\log r|^{\beta-2\alpha} \le 1\}$ and the summation over this zone results in $O(\mu^{-1}h^{-1})$ as $\beta > 2\alpha+2$, $C(\mu h)^{-2/(\beta-2\alpha)}$ as $\beta < 2\alpha+2$ and $C\mu^{-1}h^{-1}|\log(\mu h)|$ as $\beta = 2\alpha+2$. Therefore, we conclude that the remainder is $O(R)$ with

$$\text{(23.2.52)} \qquad R := \begin{cases} (\mu h)^{-1} & \beta > 2\alpha+2, \\ (\mu h)^{-1}|\log(\mu h)| & \beta = 2\alpha+2, \\ (\mu h)^{-2/(\beta-2\alpha)} & 2\alpha < \beta < 2\alpha+2. \end{cases}$$

In particular, for $\beta \ge 2\alpha+2$ asymptotics (23.2.25)–(23.2.26) hold for $h \to +0$, μ disjoint from 0 and μh disjoint from infinity[15)].

One can see easily that under condition (23.2.30) for $\mu h \le t$ with small enough $t > 0$

$$\text{(23.2.53)}_1 \qquad \mathcal{N}^-(\mu, h) \asymp h^{-2}\bigl|\log(\mu h)\bigr| \qquad \text{for } \ m = -1, \alpha = -\frac{1}{2},$$

$$\text{(23.2.53)}_2 \qquad \mathcal{N}^-(\mu, h) \asymp h^{-2}(\mu h)^{(2\alpha+1)/(\beta-2\alpha)} \qquad \text{for } \ m = -1, \alpha > -\frac{1}{2}$$

and

$$\text{(23.2.53)}_3 \quad \epsilon h^{-2}\exp\bigl(\epsilon(\mu h)^{1/(2\alpha-\beta)}\bigr) \le \mathcal{N}^-(\mu, h) \le C_1 h^{-2}\exp\bigl(C_1(\mu h)^{1/(2\alpha-\beta)}\bigr)$$

[15)] Provided $V + \mu h F \ge \epsilon\rho^2$ on ∂X.

for $m<-1$ where $C_1>\epsilon>0$ are constants.

On the other hand, for $m=-1$, $\alpha<-\frac12$ the equivalence $\mathcal{N}^-(\mu,h)\asymp h^{-2}$ holds.

(ii) Let infinity be an inner singular point[7)] and let $(23.2.46)_{1-3}$ be fulfilled with $\gamma=\epsilon\langle x\rangle$, $\rho=\langle x\rangle^m\bigl(\log\langle x\rangle+1\bigr)^{\alpha}$, $\rho_1=\langle x\rangle^{2m}\bigl(\log\langle x\rangle+1\bigr)^{\beta}$, $\rho_2=1$, $\rho_3=\bigl(\log\langle x\rangle+1\bigr)^{-1}$. Assume that

$(23.2.51)^{\#}$ Either $m>-1$ and $\beta>2\alpha$ or $m=-1$ and $\beta>\max(\alpha,2\alpha)$.

Then all the statements of (i) remain true with the obvious modification: condition (23.2.30) should be replaced by $(23.2.30)^{\#}$ and estimates $(23.2.53)_3$ hold for $m>-1$.

Let us investigate further the case of $\beta\le 2\alpha+2$. Assume the following non-degeneracy assumption

$$-\langle x,\nabla\rangle VF^{-1}\ge\epsilon|\log|x||^{2\alpha-\beta-1}\qquad\text{for }\ |x|\le\epsilon. \tag{23.2.54}$$

In fact, we need to check it for $V^*=V+\mu^{-2}\omega_1$, but this condition for V and V^* are equivalent as long as $\mu_{\mathsf{eff}}^{-2}\ge C_0\rho_3$. This condition is fulfilled automatically for all r provided either $m<-1$ or $\beta>\alpha+\frac12$.

Consider first the semiclassical error. Recall that the contribution of the partition element does not exceed (23.2.49) and only summation over zone where $\rho_3\lesssim\mu_{\mathsf{eff}}h_{\mathsf{eff}}\lesssim 1$ could bring an error, exceeding $C\mu^{-1}h^{-1}$.

Assume first that $m<-1$. Then in this zone $(\mu h)^{-1/(2m+2)}|\log\mu h|^a\lesssim r\lesssim(\mu h)^{-1/(2m+2)}|\log\mu h|^b$ and then its contribution is $C|\log(\mu h)|=O(\mu^{-1}h^{-1})$.

Consider the case $m=-1$. In this case the problematic zone is $\mathcal{Z}:=\{(\mu h)^{-1/(\beta-2\alpha+1)}\lesssim|\log r|\lesssim(\mu h)^{-1/(\beta-2\alpha)}\}$. Observe that $\int_{\mathcal{Z}}\gamma^{-2}\,dx\asymp(\mu h)^{-1/(\beta-2\alpha)}$ which is $O(\mu^{-1}h^{-1})$ provided $\beta\ge 2\alpha+1$.

We can improve these arguments in the following way. Let us observe that the zone $\{|\log r|\le C_0(\mu h)^{-1/(\beta-2\alpha)}\}$ consists of the *spectral strips*

$$\Pi_n=\{x\colon \bigl|v^*+(2n+1)\mu h\bigr|\le C_1|\log r|^{-1}\} \tag{23.2.55}$$

with $v^*:=(V+\omega_1\mu^{-2})F^{-1}$ with $n=0,\dots,n_0=C_1(\mu h)^{-1/(\beta-2\alpha+1)}$, separated by the *lacunary strips*.

Condition (23.2.54) provides that for fixed n and $x|x|^{-1}$ (thus, only $|x|$ varies) on Π_n the inequality $\bigl|\mathfrak{d}\log|x|\bigr| \le C_1$ holds where $\mathfrak{d}f$ means the oscillation of f (on Π_n) and the constant C_1 does not depend on $n, x|x|^{-1}, \mu, h$.

Therefore one can easily see that the contribution of each strip Π_n to the semiclassical remainder does not exceed C and their total contribution does not exceed $Cn_0 = O(\mu^{-1}h^{-1})$.

Meanwhile, contribution of each lacunary strip does not exceed the contribution of the adjacent spectral strip, multiplied by $C\rho_3^{-1} \times (\mu_{\text{eff}}^{-1}h_{\text{eff}})^s$ and one can see easily that their total contribution is $O(\mu^{-1}h^{-1})$. Therefore

(23.2.56) If in the framework of Example 23.2.12(i) either $m < -1$ or $m = -1, \beta > \alpha + \frac{1}{2}$ then the semiclassical error is $O(\mu^{-1}h^{-1})$.

The same arguments work for the approximation error (23.2.50) with $N = 1$: the contribution of the spectral strip gains factor ρ_3 and the contribution of the lacunary strip is 0. Then the total approximation error does not exceed "improved" (23.2.52):

$$R := \begin{cases} (\mu h)^{-1} & \beta > 2\alpha + 1, \\ (\mu h)^{-1}|\log(\mu h)| & \beta = 2\alpha + 1, \\ (\mu h)^{-1/(\beta - 2\alpha)} & 2\alpha < \beta < 2\alpha + 1. \end{cases} \tag{23.2.57}$$

Then we arrive to

Example 23.2.13. (i) Let all the assumptions of Example 23.2.12(i) be fulfilled, in particular (23.2.51). Moreover, let us assume that (23.2.54) is fulfilled and either $m < -1$ or $\beta > \alpha + \frac{1}{2}$. Then for $h \to +0$, μ disjoint from 0 and μh disjoint from infinity the asymptotics

$$\mathsf{N}^-(\mu, h) = \mathcal{N}^-(\mu, h) + O(R) \tag{23.2.58}$$

holds with R defined by (23.2.57).

(ii) Let all the assumptions of Example 23.2.12(ii) be fulfilled. Moreover, let us assume that condition

$$\langle x, \nabla\rangle VF^{-1} \ge \epsilon|\log x|^{2\alpha-\beta-1} \qquad \text{for } |x| \ge c. \tag{23.2.54}^{\#}$$

is fulfilled and either $m > -1$ or $\beta \ge \alpha + \frac{1}{2}$. Then for $h \to +0$, μ disjoint from 0 and μh disjoint from infinity the asymptotics (23.2.58) holds. with R defined by (23.2.57).

If we want to improve the remainder estimate in the case $\beta \le 2\alpha+1$ we should take $N=2$, which would make our formulae more complicated but still "computable". Then expression (23.2.50) with $N=2$ acquires (in comparison with the same expression with $N=1$) the factor $\mu^{-2}r^{-2(m+1)}|\log r|^{-2(\beta-\alpha)}$ and becomes

$$\mu^{-3}r^{-2(m+1)}|\log r|^{-3\beta+4\alpha+1}. \tag{23.2.59}$$

Further, we can get gain a factor $|\log r|^{-1}$ dues to above arguments concerning spectral and lacunary zones. Then after the summation we get $O(\mu^{-3}h^{-1})$ if either $m<-1$ or $m=-1$, $3\beta>4\alpha+1$. However if $m=-1$ this is the case due to $\beta>2\alpha$ and $\beta\ge\alpha+\frac12$. Then we arrive to

Example 23.2.14. (i) Let all the assumptions of Example 23.2.12(i) be fulfilled, in particular (23.2.51). Moreover, let us assume that (23.2.54) is fulfilled and either $m<-1$ or $\beta>\alpha+\frac12$.

Then for $h\to+0$, μ disjoint from 0 and μh disjoint from infinity the asymptotics

$$\mathsf{N}^-(\mu,h)=\mathcal{N}^{-\prime}(\mu,h)+O(\mu^{-1}h^{-1}) \tag{23.2.60}$$

holds where

$$\mathcal{N}^{-\prime}(\mu,h)=\int h^{-2}\mathcal{N}^{\mathsf{MW}\prime}(x,\mu h)\,dx \tag{23.2.61}$$

and $\mathcal{N}^{\mathsf{MW}\prime}$ is defined by (13.5.133) with $\psi=1$.

(ii) Let all the assumptions of Example 23.2.12(ii) be fulfilled. Moreover, let us assume that condition $(23.2.54)^{\#}$ is fulfilled and either $m>-1$ or $\beta>\alpha+\frac12$. Then for $h\to+0$, μ disjoint from 0 and μh disjoint from infinity the asymptotics 23.2.60–23.2.61 holds.

Remark 23.2.15. Asymptotics (23.2.60)–(23.2.61) still holds if $m=-1$ and $2\alpha<\beta\le\alpha+\frac12$ (and therefore $\alpha<\frac12$) under assumption (23.2.62) below.

Indeed, recall that we need $\beta>\alpha+\frac12$ only to ensure that inequality (23.2.54) for V implies the same inequality for V^*. This conclusion should be checked as $|\log r|\le C_0(\mu h)^{-1/(\beta-2\alpha)}$ only. Observe that this conclusion still holds in the case under consideration if $\mu^{-2}(\mu h)^{(2\beta-2\alpha-1)/(\beta-2\alpha)}\le\epsilon$, i.e.

$$\mu\ge Ch^{(2\beta-2\alpha-1)/(1-2\alpha)}. \tag{23.2.62}$$

Then the semiclassical error does not exceed $C\mu^{-1}h^{-1}$ and an approximation error does not exceed $C\mu^{-3}h^{-1}(\mu h)^{-(4\alpha-3\beta+1)/(\beta-2\alpha)}$ (multiplied by $|\log(\mu h)|$ if $4\alpha-3\beta+1=0$). One can prove easily that it is less than $C\mu^{-1}h^{-1}$ under assumption (23.2.62).

We leave to the reader

Problem 23.2.16. (i) Consider cases of $\mu\to\mu_0>0$ and $\mu\to\mu_0=0$ like in Example 23.2.6.

(ii) Consider cases of $m>-1$ and singularity at 0, $m<-1$ and singularity at infinity, and $m=-1$, $\alpha<0$ and singularity either at 0 or at infinity. In these three cases assumption $\beta>2\alpha$ is not necessary; therefore, one needs to consider also the case $\beta<2\alpha$.

(iii) Furthermore, in the framework of (ii) and $\beta<2\alpha$ consider the case of $\mu h\to\infty$ like in Example 23.2.5.

Exponential Singularities

Consider now singularities of the exponential type. The following example follows immediately from Theorem 23.2.2:

Example 23.2.17. (i) Let 0 be an inner singular point[4)] and let conditions of Theorem 23.2.2 be fulfilled with $\gamma=\epsilon|x|^{1-\beta}$, $\rho=\exp(a|x|^{\alpha})$, $\rho_1=\exp(b|x|^{\beta})$ with either $0>\alpha>\beta$, $a>0$, $b>0$ or $0>\alpha=\beta$, $b>2a>0$.

Let $V+\mu hF\ge\epsilon\rho^2$ on $(\partial X\setminus 0)\cup\{x\colon |x|\ge c\}$. Then conditions (23.2.23) and (23.2.24) are fulfilled automatically and for $\mu\ge 1$, $h\to+0$ asymptotics (23.2.25)–(23.2.26) hold.

Moreover, under condition (23.2.30) for $\alpha=\beta$, $\mu h<t$ with a small enough constant $t>0$

$$\mathcal{N}^-(\mu,h)\asymp h^{-2}(\mu h)^{-2a/(b-2a)}|\log\mu h|^{\frac{2}{\alpha}-1}; \tag{23.2.63}$$

otherwise the left-hand expression is "O" only.

(ii) Let infinity be an inner singular point[7)] and let conditions of Theorem 23.2.2 be fulfilled with $\gamma=\epsilon\langle x\rangle^{1-\beta}$, $\rho=\exp(a\langle x\rangle^{\alpha})$, $\rho_1=\exp(b\langle x\rangle^{\beta})$ where either $0<\alpha<\beta$, $a>0$, $b>0$ or $0<\alpha=\beta$, $b>2a>0$.

Let $V + \mu hF \ge \epsilon\rho^2$ on ∂X. Then conditions (23.2.23)–(23.2.24) are fulfilled automatically and asymptotics (23.2.25)–(23.2.26) hold.

Moreover, under condition (23.2.30)$^{\#}$ for $\alpha = \beta$, $\mu h < t$ with a small enough constant $t > 0$ (23.2.45) holds; otherwise the left-hand expression is "O" only.

We leave to the reader the following

Problem 23.2.18. Calculate magnitude of $\mathcal{N}^-(\mu, h)$ as $\alpha < \beta$ in Example 23.2.17.

Example 23.2.19. (i) Let 0 be an inner singular point[4]. Assume that conditions of Theorem 23.2.2 are fulfilled with $\gamma = \epsilon|x|^{1-\beta}$, $\rho = |x|^m \exp(|x|^\beta)$, $\rho_1 = |x|^{m_1} \exp(2|x|^\beta)$, $\rho_2 = 1$, $\rho_3 = |x|^{-\beta}$ and let $\beta < 0$, $m_1 < 2m$.

Let $V + \mu hF \ge \epsilon\rho^2$ on $(\partial X \setminus 0) \cup \{x\colon |x| \ge c\}$. Then conditions (23.2.24), (23.2.48) are fulfilled automatically.

Moreover, condition

$$\int \rho_3^{-1}\rho_1^{-1}\rho^2 \, dx < \infty \tag{23.2.64}$$

is fulfilled provided $2m+3\beta > m_1$; then asymptotics (23.2.25)–(23.2.26) holds. Moreover, under condition (23.2.30) while we cannot calculate magnitude of $\mathcal{N}^-(\mu, h)$ itself, we conclude that

$$\log(\mathcal{N}^-(\mu, h)) \asymp (\mu h)^{\beta/(2m-m_1)}). \tag{23.2.65}$$

(ii) Let infinity be an inner singular point[7]. Assume that conditions of Theorem 23.2.2 are fulfilled with $\gamma = \epsilon_0\langle x\rangle^{1-\beta}$, $\rho = \langle x\rangle^m \exp(\langle x\rangle^\beta)$, $\rho_1 = \langle x\rangle^{m_1} \exp(2\langle x\rangle^\beta)$, $\rho_2 = 1$, $\rho_3 = \langle x\rangle^{-\beta}$ where $\beta > 0$, $m_1 > 2m$.

Let $V + \mu hF \ge \epsilon\rho^2$ on ∂X. Then condition (23.2.64) is fulfilled provided $2m+3\beta < m_1$; then asymptotics (23.2.25)–(23.2.26) holds. Moreover, under condition (23.2.30)$^{\#}$ estimates (23.2.65) hold.

Consider case of $2m + 3\beta \le m_1$, $2m + 3\beta \ge m_1$ in the frameworks of Statements (i) and (ii) respectively. One can get some remainder estimates albeit less precise. We want to improve them using the same technique as in Subsubsection *23.2.1.6. Power sSingularities. II.* However now things are a bit simpler. First of all, condition $\mu_{\text{eff}}^{-2} \le \rho_3$ is fulfilled automatically.

Contribution of the zone $\mathcal{Z}_1 = \{r \ge (\mu h)^{1/(-\beta-m_1+2m)}\}$ where $\rho_3 \ge \mu_{\text{eff}} h_{\text{eff}}$ to the semiclassical error does not exceed $(\mu h)^{-1} \int_{\mathcal{Z}_1} \rho_3 \rho_1^{-1} \rho^2 \gamma^{-2}\, dx$. Contribution of the zone $\mathcal{Z}_2 = \{(\mu h)^{1/(-m_1+2m)} \le r \le (\mu h)^{1/(-\beta-m_1+2m)}\}$ where $\rho_3 \le \mu_{\text{eff}} h_{\text{eff}} \lesssim 1$ to the semiclassical error does not exceed $\int_{\mathcal{Z}_2} \rho_3 \gamma^{-2}\, dx$ due to the same "spectal and lacunary strips" arguments. Then the semiclassical error is does not exceed $C(\mu h)^{-1}$ provided $m_1 < 2m + \beta$.

Further, contributions of these zones $\mathcal{Z}_j$ to the approximation error with $N = 1$ do not exceed $C\mu^{-1}h^{-1} \int_{\mathcal{Z}_j} \rho_1^{-1} \rho^2 \gamma^{-2}\, dx$ and the approximation error is $O(\mu^{-1}h^{-1})$ provided $m_1 < 2m + 2\beta$.

On the other hand, one can see easily that the approximation error with $N = 2$ is $O(\mu^{-1}h^{-1})$ for sure. We arrive to the following

Example 23.2.20. (i) Let all the assumptions of Example 23.2.19(i) be fulfilled. Moreover, let us assume that condition

$$-\langle x, \nabla\rangle VF^{-1} \ge \epsilon |x|^{2m-m_1} \qquad \text{for } |x| \le \epsilon. \tag{23.2.66}$$

is fulfilled.

Then for $h \to +0$, μ disjoint from 0 and μh disjoint from infinity the asymptotics (23.2.25)–(23.2.26) holds provided $m_1 < 2m + 2\beta$ and asymptotics (23.2.60)–(23.2.61) holds with $\mathcal{N}^{\mathsf{MW}\prime}$ defined by (13.5.133) with $\psi = 1$ provided $m_1 < 2m + \beta$.

(ii) Let all the assumptions of Example 23.2.12(ii) be fulfilled. Moreover, . Moreover, let us assume that condition

$$\langle x, \nabla\rangle VF^{-1} \ge \epsilon |x|^{2m-m_1} \qquad \text{for } |x| \ge \epsilon. \tag{23.2.66}^{\#}$$

is fulfilled.

Then for $h \to +0$, μ disjoint from 0 and μh disjoint from infinity the asymptotics (23.2.25)–(23.2.26) holds provided $m_1 > 2m + 2\beta$ and asymptotics (23.2.60)–(23.2.61) holds with $\mathcal{N}^{\mathsf{MW}\prime}$ defined by (13.5.133) with $\psi = 1$ provided $m_1 > 2m + \beta$.

Example 23.2.21. (i) In the framework of Example 23.2.17(i) assume that (23.2.66) is fulfilled. Moreover, let

$$|D^\sigma g^{jk}| \le c|x|^{-|\sigma|}, \qquad |D^\sigma \log F| \le c|x|^{\beta-|\sigma|}, \qquad |D^\sigma V/F| \le c|x|^{2m-m_1-|\sigma|} \qquad \forall \sigma : 1 \le |\sigma| \le 2. \tag{23.2.67}$$

Then for $h \to +0$, $1 \le \mu = O(h^{-1})$ asymptotics (23.2.25)–(23.2.26) holds provided $m_1 < 2m + \beta$.

Indeed, one can easily see that under condition (23.2.67)

$$|\omega_1| \le C_1 F^{-1} |x|^{\beta+4m-2m_1-2} = \omega_1^* \tag{23.2.68}$$

while the general theory yields only that

$$|\omega_1| \le C_1 F^{-1} |x|^{2\beta+4m-2m_1-2}.$$

Then the error $|\mathcal{N}^-(\mu, h) - \mathcal{N}^{-\prime}(\mu, h)|$ does not exceed

$$C\mu^{-1}h^{-1} \int_{\{\rho_1\mu h \le \rho_3\rho^2\}} \rho_1^2 \rho^{-2} \rho_3^{-1} \omega_1^* \, dx + Ch^{-2} \int_{\{\rho_3\rho^2 \le \rho_1\mu h \le C_0\rho^2\}} \rho_1 (\omega_1^* \mu^{-2} + \chi_\Pi) \, dx$$

where χ_Π is the characteristic function of the set

$$\Pi = \bigcup_{n\in\mathbb{Z}^+} \{x\colon \big|v + (2n+1)\mu h\big| \le C_1 \mu^{-2} \omega_1^*\}.$$

Applying estimate (23.2.68) and condition (23.2.66) one can prove that this error is $O(\mu^{-1}h^{-1})$.

(ii) Similarly, in the framework of Example 23.2.17(ii) assume that $(23.2.66)^\#$ is fulfilled. Moreover, let

$$|D^\sigma g^{jk}| \le c\langle x\rangle^{-|\sigma|}, \qquad |D^\sigma \log F| \le c\langle x\rangle^{\beta-|\sigma|} \qquad |D^\sigma V/F| \le c\langle x\rangle^{2m-m_1-|\sigma|} \qquad \forall \sigma : 1 \le |\sigma| \le 2. \tag{23.2.67}^\#$$

Then for $h \to +0$, $1 \le \mu = O(h^{-1})$ asymptotics (23.2.25)–(23.2.26) holds provided $m_1 > 2m + \beta$.

We leave to the reader:

Problem 23.2.22. Consider cases of $\mu \to \mu_0 > 0$ and $\mu \to \mu_0 = 0$ like in Example 23.2.6.

23.2.2 Schrödinger-Pauli Operators

Consider now Schrödinger-Pauli operators, either genuine (0.34) or generalized (13.6.3). The principal difference is that now F does not "tame" singularities of V, on the contrary, it needs to be "tamed" by itself. As a result there are fewer examples than for the Schrödinger. Also we do not have a restriction $\mu_{\text{eff}} h_{\text{eff}} = O(1)$ which we had in the most of the previous Subsection 23.2.1. Then we need to add 1 and $\mu_{\text{eff}} h_{\text{eff}}^{-1} = \mu h^{-1} \rho_1 \gamma^2$ to the contributions of this element to the remainder estimate and $\mathcal{N}^-(\mu, h)$ respectively.

Because of this, here we do not consider an abstract theorem like Theorem 23.2.2, but go directly to the examples.

Example 23.2.23[16]. (i) Let 0 be an inner singular point[4] and $\gamma = \epsilon|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$ and let $m > -1$, $2m \ne m_1 > -2$. Let our usual non-degeneracy assumptions be fulfilled. Then, in comparison with the theory of the previous Subsection 23.2.1, we need to consider also the the zone $\mu_{\text{eff}} h_{\text{eff}} \gtrsim 1$.

Its contribution to the remainder does not exceed $C \int \gamma^{-2}\, dx$ while its contribution to $\mathcal{N}^-(\mu, h)$ does not exceed $C\mu h^{-1} \int \rho_1\, dx$. Indeed, contributions of each γ-element do not exceed C and $C\mu_{\text{eff}} h_{\text{eff}}^{-1} \asymp C\mu h^{-1} \rho_1 \gamma^2$.

(a) Let $m_1 > 2m$; then this zone is disjoint from 0 and this expression is $O(1)$. Furthermore, contribution of the zone $\{r \le \mu^{-1/(m_1+1-m)}\}$ where $\mu_{\text{eff}} \lesssim 1$ to the remainder does not exceed $Ch^{-1} \int \rho\gamma^{-1}\, dx$, taken over this zone, and it is $\asymp h^{-1}\mu^{-(m+1)/(m_1+1-m)}$ and we arrive to (23.2.70)–(23.2.71) below; cf. (23.2.32).

(b) Consider now the case $m_1 < 2m$. In this case zone where $\mu_{\text{eff}} h_{\text{eff}} \gtrsim 1$ is not disjoint from 0 and we need to consider also a singular zone $\{r \le \bar{r} := (\mu^{-1}h)^{-1/(m_1-2m)}\}$ where $\mu_{\text{eff}}^{-1} h_{\text{eff}} \gtrsim 1$.

However due to the variational estimates as before

(23.2.69) For the disk $\{x\colon |x| \le 2\bar{r}\}$ with the Dirichlet boundary conditions

$$\mathsf{N}^-(\mu, h) \le Ch^{-2}\bar{r}^{2m+2} + C(\mu h^{-1})^2 \bar{r}^{2(m_1+2)}$$

which is $O(1)$ due to the choice of $\bar{r}$.

[16] Cf. Example 23.2.3.

We leave to the reader to prove (23.2.69) and (23.2.69)$^{\#}$ below and to justify the final result using methods of Section 10.1:

$$\mathsf{N}^-(\mu,h) = \mathcal{N}^-(\mu,h) + \begin{cases} O(h^{-1}\mu^{-1}+1) & m_1 < 2m, \\ O(h^{-1}\mu^{-(m+1)/(m_1+1-m)})+1 & m_1 > 2m. \end{cases} \tag{23.2.70}$$

with

$$\mathcal{N}^-(\mu,h) \asymp h^{-2} + \mu h^{-1}. \tag{23.2.71}$$

(ii) Let infinity be an inner singular point[7)] and $\gamma = \epsilon\langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$ and let $m < -1$, $2m \ne m_1 < -2$. Let Let our usual non-degeneracy assumptions be fulfilled. Then again we need to consider cases (a) $m_1 < 2m$ and (b) $m_1 > 2m$ and in the latter case we need to consider contribution of zone $\{r \le \bar{r} := (\mu^{-1}h)^{-1/(m_1-2m)}\}$.

However due to variational estimates

(23.2.69)$^{\#}$ For the zone $\{x\colon |x| \ge \frac{1}{2}\bar{r}\}$ with the Dirichlet boundary conditions

$$\mathsf{N}^-(\mu,h) \le Ch^{-2}\bar{r}^{2m+2} + C(\mu h^{-1})^2\bar{r}^{2(m_1+2)}$$

which is again $O(1)$ due to the choice of $\bar{r}$. Then

$$\mathsf{N}^-(\mu,h) = \mathcal{N}^-(\mu,h) + \begin{cases} O(h^{-1}\mu^{-1}+1) & m_1 > 2m, \\ O(h^{-1}\mu^{-(m+1)/(m_1+1-m)})+1 & m_1 < 2m. \end{cases} \tag{23.2.70}$$

and (23.2.71) hold.

We leave to the reader the following problems:

Problem 23.2.24. (i) As 0 is an inner point consider both Schrödinger and Schrödinger-Pauli operators as

(a) $\gamma = |x|$, $\rho = |x|^m$, $m > -1$ and $\rho_1 = |x|^{-2}|\log|x||^{\beta}$ [17)].

[17)] One should take $\beta < -1$ for the Schrödinger-Pauli operator.

(b) $\gamma = |x|$, $\rho = |x|^{-1}|\log|x||^{\alpha}$, $\alpha < -\frac{1}{2}$ [18] and $\rho_1 = |x|^{m_1}|$, $m_1 > -2$.

(ii) As infinity is an inner point consider both Schrödinger and Schrödinger Pauli operators as

(a) $\gamma = |x|$, $\rho = |x|^{m}$, $m < -1$ and $\rho_1 = |x|^{-2}|\log|x||^{\beta}$ [17].

(b) $\gamma = |x|$, $\rho = |x|^{-1}|\log|x||^{\alpha}$, $\alpha < -\frac{1}{2}$ [18] and $\rho_1 = |x|^{m_1}$, $m_1 > -2$.

For the Schrödinger operator in cases (a) non-trivial results could be also obtained even as $\mu h \to +\infty$.

Problem 23.2.25. Let either 0 or infinity be an inner singular point.

Using the same arguments and combining them with the arguments of Subsubsection *23.2.1.6. Power sSingularities. II* consider both Schrödinger-Pauli and Schrödinger operators with $\gamma = |x|$, $\rho = |x|^{-1}|\log|x||^{\alpha}$, $\rho_1 = |x|^{-2}|\log|x||^{\beta}$ [17].

Take into account whether $\beta > 2\alpha$ or $\beta < 2\alpha$. For the Schrödinger operator in case $\beta < 2\alpha$ non-trivial results could be obtained even as $\mu h \to +\infty$.

23.2.3 Dirac Operator

Preliminaries

Let us now consider the generalized magnetic Dirac operator (17.1.1)

$$A = \frac{1}{2}\sum_{l,j} \sigma_l\big(\omega^{jl}P_j + P_j\omega^{jl}\big) + \sigma_0 M + I\cdot V, \qquad P_j = hD_j - \mu V_j \tag{23.2.72}$$

where $\sigma_0, \sigma_1, \sigma_2$ are 2×2-matrices.

We are interested in $\mathsf{N}(\tau_1,\tau_2)$, the number of eigenvalues in (τ_1,τ_2) [19] with $\tau_1 < \tau_2$, fixed in this subsection.

The theory of the Dirac operator is more complicated than the theory of the Schrödinger operator because it is different in the cases when $V \pm M$ [20] and F tend to 0 and (or) ∞ and thus singularities at 0 and at infinity should be treated differently.

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[18] One needs to consider cases $\alpha < -1$, $\alpha = -1$, $-1 < \alpha < -\frac{1}{2}$ separately.

[19] Assuming that this interval does not contain essential spectrum; otherwise $\mathsf{N}(\tau_1,\tau_2) := \infty$. It is more convenient for us to exclude both ends of the segment.

[20] If $V \pm M \in [\tau_1+\epsilon, \tau_2-\epsilon]$ for both signs $\pm$, then infinity is not a singular point.

Furthermore, the theory of the magnetic Dirac operator is even more complicated. Indeed, the "pointwise Landau levels" are $V \pm \bigl(M^2 + 2j\mu hF\bigr)^{\frac{1}{2}}$ with $j = 0, 1, 2, ...$ but one of those is, in fact, excepted: with the same sign as ςF_{12} and $j = 0$. Recall that $\varsigma = \pm 1$ is defined by (17.1.14)

$$\sigma_0\sigma_1\sigma_2 = \varsigma i. \tag{23.2.73}$$

Without any loss of generality one can assume that

$$\varsigma = 1, \qquad F_{12} > 0. \tag{23.2.74}$$

Then the Landau levels (at the point x) are $V + \bigl(M^2 + 2j\mu hF\bigr)^{\frac{1}{2}}$ with $j = 1, 2, 3, ...$ and $V - \bigl(M^2 + 2j\mu hF\bigr)^{\frac{1}{2}}$ with $j = 0, 1, 2, ...$ But then the negative V could be "tamed" by a larger F. This allows us to get meaningful results in the situations impossible for non-magnetic Dirac operator: V as singular at 0 as $|x|^m$ with $m \le -1$ or $V + M$ as singular at infinity as $|x|^{2m}$ with $m \ge -1$ (as $M > 0$) provided it is negative there. Furthermore, if $F \to \infty$ as $|x| \to \infty$, we can get meaningful results even if $M = 0$.

On the other hand, we often *should prove* that the Dirac operator is essentially self-adjoint while the Schrödinger operator is *obviously* semibounded and therefore essentially self-adjoint. We do this in Appendix 23.A.1.

Therefore, we treat the operator given by (23.2.72) under the following assumptions

$$|D^\alpha \omega^{jk}| \le c\gamma^{-|\alpha|}, \qquad |D^\alpha F| \le c\rho_1\gamma^{-|\alpha|}, \tag{23.2.75}_{1-2}$$

$$|D^\alpha V| \le c\min\bigl(\rho, \frac{1}{M}\rho^2\bigr)\gamma^{-|\alpha|} \qquad (\alpha \ne 0) \quad \forall \alpha : |\alpha| \le K, \tag{23.2.75}_3$$

$$(V - \tau_2 - M)_+ \le c\min\bigl(\rho, \frac{1}{M}\rho^2\bigr), \tag{23.2.75}_{4+}$$

$$(V - \tau_1 + M)_- \le c\min\bigl(\rho, \frac{1}{M}\rho^2\bigr) \tag{23.2.75}_{4-}$$

and also $(23.2.3)_{1-3}$, (23.2.2) for $g^{jk} = \sum_{l,r} \omega^{jl}\omega^{kr}\delta_{lr}$ and (23.2.9) (with $F > 0$). In what follows $(23.2.75)_4$ means the pair of conditions $(23.2.75)_{4\pm}$.

Moreover, let condition (23.2.13) be fulfilled and

$$\bar{X}'' \cap \partial X = \emptyset, \tag{23.2.76}$$

$$|V_j| \le c\rho, \quad |D_j\omega^{kl}| \le c\rho \qquad \text{in} \quad X''. \tag{23.2.77}$$

Finally, we assume that

(23.2.78) Either $\partial X = \emptyset$ or $\mu = O(1)$ and $\partial X \cap X_2' = \emptyset$ (in what follows).

Asymptotics. I

Example 23.2.26[21]. Let condition (23.2.74) be fulfilled.

(i) Let 0 be an inner singular point and let all the above conditions, be fulfilled with $\gamma = \epsilon_0|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$, $m_1 < \min(m-1,\, 2m)$.

Further, let

$$-V \ge \epsilon \min(\rho,\, \frac{\rho^2}{M}) \qquad \forall x : |x| \le \epsilon \tag{23.2.79}$$

and let non-degeneracy condition

$$\begin{aligned} &\bigl|V + \varsigma\bigl(M^2 + 2j\mu hF\bigr)^{\frac12} - \tau_\iota\bigr| \le \epsilon\rho \implies \\ &\qquad |\nabla v_\iota| \ge \epsilon\rho_1\rho^2\gamma^{-1} \qquad \text{or} \qquad \det \operatorname{Hess} v_\iota \ge \epsilon\rho_1^{-2}\rho^4\gamma^{-4} \qquad \text{or} \\ &\qquad |\nabla V| \ge \epsilon\rho\gamma^{-1} \qquad \text{and} \quad M \le \epsilon'\rho \quad \forall (j,\varsigma) \ne (0,1) \end{aligned} \tag{23.2.80}$$

be fulfilled with a small enough constant $\epsilon' = \varepsilon(c, \epsilon, \epsilon_0) > 0$; this assumption is a part of condition (23.2.80))[22].

(a) Let $m < 0$, $\tau_1 < \tau_2$. Then for $h \to +0$, $1 \le \mu = O(h^{-1})$ asymptotics (23.2.25)–(23.2.26) holds with $\mathcal{N}$ defined by $(17.1.12)_2$ with $d = 2$. Moreover,

$$\mathcal{N}(\tau_1, \tau_2, \mu, h) \asymp \begin{cases} h^{-2} & m \ge -1, \\ h^{-2}(\mu h)^{2(m+1)/(2m-m_1)} & m < -1. \end{cases} \tag{23.2.81}$$

(b) Let $m > 0$, $M > 0$, $\tau_1 \in (-M, M)$, $\tau_2 = M$. Then for $h \to +0$, $1 \le \mu = O(h^{-1})$ asymptotics (23.2.25)–(23.2.26) holds with $\mathcal{N}$ defined by $(17.1.12)_2$ with $d = 2$. Moreover, equivalence (23.2.28) holds.

(ii) Let infinity be an inner singular point and let all the above conditions be fulfilled with $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$, $m_1 > \max(m-1,\, 2m)$.

Further, let

$$-V \ge \epsilon \min(\rho,\, \frac{\rho^2}{M}) \qquad \forall x : |x| \ge c \tag{23.2.79}^{\#}$$

and let non-degeneracy condition (23.2.80) be fulfilled.

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[21] Cf. Example 23.2.3.

[22] Recall that $v_\iota = F^{-1}\bigl((V-\tau_\iota)^2 - M^2\bigr)$ and the conditions should be fulfilled for both $\iota = 1, 2$.

(a) Let $m > 0$, $\tau_1 < \tau_2$. Then for $h \to +0$, $1 \le \mu = O(h^{-1})$ asymptotics (23.2.25)–(23.2.26) holds with $\mathcal{N}$ defined by $(17.1.12)_2$ with $d = 2$. Moreover,

$$\mathcal{N}^-(\tau_1, \tau_2, \mu, h) \asymp h^{-2}(\mu h)^{2(m+1)/(2m-m_1)}. \tag{23.2.81}^{\#}$$

(b) Let $m < 0$, $M > 0$, $\tau_1 \in (-M, M)$, $\tau_2 = M$. Then for $h \to +0$, $1 \le \mu = O(h^{-1})$ asymptotics (23.2.25)–(23.2.26) holds with $\mathcal{N}$ defined by $(17.1.12)_2$ with $d = 2$. Moreover, equivalence $(23.2.28)^{\#}$ holds.

Example 23.2.27[23)]. Let condition (23.2.74) be fulfilled.

(i) Let 0 be an inner singular point and let all the above conditions, be fulfilled with $\gamma = \epsilon_0|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$, $m_1 > 2m$, $m > -1$.

Further, let assumption (23.2.79) and non-degeneracy condition (23.2.80) be fulfilled. Let either

(a) $m < 0$, $\tau_1 < \tau_2$ or

(b) $m > 0$, $M > 0$, $\tau_1 \in (-M, M)$, $\tau_2 = M$.

Then for $h \to +0$, $1 \le \mu$ asymptotics (23.2.32), (23.2.26) holds with $\mathcal{N}$ defined by $(17.1.12)_2$ with $d = 2$.

Moreover, $\mathcal{N}(\tau_1, \tau_2, \mu, h) \asymp h^{-2}$ as $\mu h \lesssim 1$ and (23.2.34) holds as $\mu h \gtrsim 1$.

(ii) Let infinity be an inner singular point and let all the above conditions be fulfilled with $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$, $m_1 < 2m$, $m < -1$.

Further, let assumption $(23.2.79)^{\#}$ and non-degeneracy condition (23.2.80) be fulfilled. Let $M > 0$, $\tau_1 \in (-M, M)$, $\tau_2 = M$. Then for $h \to +0$, $1 \le \mu$ asymptotics $(23.2.32)^{\#}$, (23.2.26) holds with $\mathcal{N}$ defined by $(17.1.12)_2$ with $d = 2$.

Moreover, $\mathcal{N}(\tau_1, \tau_2, \mu, h) \asymp h^{-2}$ as $\mu h \lesssim 1$ and (23.2.34) holds as $\mu h \gtrsim 1$.

We leave to the reader the following problems:

Problem 23.2.28. Modify for the Dirac operator under assumption (23.2.79) or $(23.2.79)^{\#}$ (as 0 or infinity is an inner singular point, respectively) in the frameworks of

[23)] Cf. Examples 23.2.4 and 23.2.5.

(a) Examples 23.2.6, 23.2.7, 23.2.8 , 23.2.9, 23.2.11 (power singularities) and Problem 23.2.10,

(b) Examples 23.2.12, 23.2.13, 23.2.14 and Problem 23.2.16 (power-logarithmic singularities),

(c) Examples 23.2.17, 23.2.19, 23.2.20, 23.2.21 and Problems 23.2.18, 23.2.22 (exponential singularities).

Asymptotics. II

While under assumptions (23.2.79), (23.2.79)$^{\#}$ the Dirac operator behaves as the Schrödinger operator, under the same assumptions albeit with an opposite sign the the Dirac operator behaves as the Schrödinger-Pauli operator.

Example 23.2.29[24]. Let condition (23.2.74) be fulfilled.

(i) Let 0 be an inner singular point and let all the above conditions, be fulfilled with $\gamma = \epsilon_0|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$, $m_1 > 2m$, $m > -1$.

Further, let assumption

$$V \geq \epsilon \min(\rho, \frac{\rho^2}{M}) \qquad \forall x : |x| \leq \epsilon \tag{23.2.82}$$

and non-degeneracy condition (23.2.80) be fulfilled.

(a) $m < 0$, $\tau_1 < \tau_2$ or

(b) $m > 0$, $M > 0$, $\tau_1 = -M$, $\tau_2 \in (-M, M)$.

Then for $h \to +0$, $1 \leq \mu$ asymptotics (23.2.32), (23.2.26) holds with $\mathcal{N}$ defined by $(17.1.12)_2$ with $d = 2$.

Moreover, $\mathcal{N}(\tau_1, \tau_2, \mu, h) \asymp \mu h^{-1} + h^{-2}$.

(ii) Let infinity be an inner singular point and let all the above conditions be fulfilled with $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$, $m_1 < 2m$, $m < -1$.

Further, let assumption

$$V \geq \epsilon \min(\rho, \frac{\rho^2}{M}) \qquad \forall x : |x| \geq c \tag{23.2.82}$$

[24] Cf. Example 23.2.23.

and non-degeneracy condition (23.2.80) be fulfilled. Let $M > 0$, $\tau_1 = -M$, $\tau_2 \in (-M, M)$. Then for $h \to +0$, $1 \le \mu$ asymptotics $(23.2.32)^{\#}$, (23.2.26) holds with $\mathcal{N}$ defined by $(17.1.12)_2$ with $d = 2$.

Moreover, $\mathcal{N}(\tau_1, \tau_2, \mu, h) \asymp \mu h^{-1} + h^{-2}$.

Problem 23.2.30. (a) Modify Problem 23.2.27 under assumptions (23.2.82) or $(23.2.82)^{\#}$ (if 0 or infinity is an inner singular point, respectively).

(b) Consider degenerations like in Subsubsection *23.2.1.5. Degenerations*.

Finally, we leave to the reader

Problem 23.2.31. Generalize results of this section to the even-dimensional full-rank case. In particular, consider power singularities[25)].

23.3 Asymptotics of Large Eigenvalues

In this section we consider the case when $\mu = h = 1$ are fixed and we consider the asymptotics of the eigenvalues, tending to $+\infty$ and for Dirac operator also to $-\infty$.

Here we consider the case of the spectral parameter tending to $+\infty$ (and for the Dirac operator we consider $\tau \to -\infty$ as well).

23.3.1 Singularities at the Point

We consider series of example with singularities at the point.

Schrödinger Operator

Example 23.3.1. (i) Let X be a compact domain and conditions $(23.2.3)_{1-3}$ be fulfilled with $\gamma = \epsilon|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$ with $m_1 < 2m$. Let

$$(23.3.1)_{1,2} \qquad |F| \ge \epsilon_0 \rho_1, \qquad |\nabla F| \ge \epsilon_0 \rho_1 \gamma^{-1} \qquad \text{for } |x| \le \epsilon.$$

[25)] Note that for $m_1 \ne -2$ one can construct V_j positively homogeneous of degrees m_1+1 such that F_{jk} is non-degenerate. F.e. one can take $V_{2j-1} = \frac{1}{2} x_{2j} |x|^{m_1}$, $V_{2j} = -\frac{1}{2} x_{2j-1} |x|^{m_1}$, $j = 1, 2, \dots, d/2$, in which case $f_1 = |(m_1+2)/2| \cdot |x|^{m_1}$, $f_2 = \dots = f_{d/2} = |x|^{m_1}$. See for details Appendix 24.A.3.

Then for the Schrödinger operator as $\tau \to +\infty$

$$\mathsf{N}^-(\tau) = \mathcal{N}^-(\tau) + O(\tau^{(d-1)/2}) \tag{23.3.2}$$

while $\mathcal{N}^-(\tau) \asymp \tau^{d/2}$.

Indeed, we need to consider only case $m_1 \le -2$ (otherwise it is covered by Section 11.2). Assume for simplicity, that $V = 0$ (modification in the general case is trivial). Then we need to consider zones $\mathcal{X}_1 = \{x\colon |x| \ge \tau^{1/2(m_1+1)}\}$ where $\mu_{\mathsf{eff}} = |x|^{m_1+1}\tau^{-1/2} \le 1$ and $\mathcal{X}_2 = \{x\colon |x| \le \tau^{1/2(m_1+1)}\}$ where $\mu_{\mathsf{eff}} \ge 1$. Meanwhile $h_{\mathsf{eff}} = \tau^{-1/2}|x|^{-1}$.

Contribution to the remainder of the γ-element from $\mathcal{X}_1$ does not exceed $Ch_{\mathsf{eff}}^{1-d} = C\tau^{(d-1)/2}\gamma^{d-1}$ while contribution to the remainder of the γ-element from $\mathcal{X}_2$ does not exceed $C\mu_{\mathsf{eff}}^{-1}h_{\mathsf{eff}}^{1-d} = C\tau^{d/2}\gamma^{d-2-m_1} \le C\tau^{(d-1)/2}\gamma^{d-1}$ and the rest is easy.

(ii) Under proper assumptions the same proof is valid in the full-rank even-dimensional case.

(iii) The similar proof is valid for $d = 3$ and under proper assumptions it is valid in the maximal-rank odd-dimensional case (while contribution to the remainder of γ-element from $\mathcal{X}_2$ is $O(\tau^{(d-1)/2}\gamma^{d-1})$).

(iv) On the other hand, without assumption $(23.3.1)_2$ contribution to the remainder γ-element from $\mathcal{X}_2$ is $O(\tau^{(d-2)/2}\gamma^{d+m_1})$ but we need to consider only $\gamma \gtrsim \tau^{1/m_1}$ (otherwise $\mu_{\mathsf{eff}}h_{\mathsf{eff}} \ge C_0$) and we arrive to the remainder estimates $O(\tau^{(d-1)/2})$ as $m_1 \ge -2d$ and $O(\tau^{d/2+d/m_1})$ as $m_1 \le -2d$.

Definitely these arguments are far from optimal for $d = 3$ or in the maximal-rank odd-dimensional case; we leave this case to the next Chapter 24.

Let us note that the case $m < -1$, $m_1 \ge m-1$ is covered by Chapter 11; we need to assume that

$$V \ge \epsilon_0\rho^2 \qquad \text{as } |x| \le \epsilon. \tag{23.3.3}$$

Example 23.3.2. (i) Let X be a compact domain and conditions $(23.2.3)_{1-3}$, $(23.3.1)_1$ be fulfilled with $\gamma = \epsilon|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$ and with $m < -1$, $2m \le m_1 < m-1$. Then we need to assume that

$$V + F \ge \epsilon_0\rho^2 \qquad \text{as } |x| \le \epsilon \tag{23.3.3}'$$

which for $m_1 > 2m$ is equivalent to (23.3.3).

We need also to have some non-degeneracy assumption. Assume that[26)]

$$\tau \ge V + F, \qquad |\nabla(\tau - V)F^{-1}|\gamma \le \epsilon_0 \tau F^{-1} \implies |\nabla^2(\tau - V)F^{-1}|\gamma^2 \ge \epsilon_0 \tau F^{-1} \qquad \text{as } |x| \le \epsilon. \tag{23.3.4}$$

Then asymptotics (23.3.2) holds while $\mathcal{N}^-(\tau) \asymp \tau^{d/2}$.

Indeed, it follows from Chapter 13 (see condition (13.4.54)) that in this case the contribution to the remainder of the γ-element from $\mathcal{X}_2$ is also $O(h_{\mathsf{eff}}^{-d+1})$.

(ii) Under proper assumptions the same proof is valid in the maximal-rank multidimensional case.

(iii) Without assumption (23.3.4) we can apply arguments of Example 23.3.1(iv), however cutting of as $|x| \le \tau^{1/(2m)}$ and we arrive to the remainder estimate $O(\tau^{(d-1)/2})$ as $d + m_1 \ge m$ and $O(\tau^{d(1/2+1/(2m))+(m_1-2m)/(2m)})$ as $d + m_1 \le m$.

Again, these arguments are far from optimal for $d = 3$ or in the maximal-rank odd-dimensional case.; we leave this case to the next Chapter 24.

Schrödinger-Pauli Operator

Next, consider Schrödinger-Pauli operators. We will need to impose (23.3.3) and the related non-degeneracy assumption

$$|\nabla V| \ge \epsilon_0 \rho^2 \gamma^{-1} \qquad \text{for } |x| \le \epsilon. \tag{23.3.5}$$

Example 23.3.3. (i) Let X be a compact domain, $d = 2$. Let conditions $(23.2.3)_{1-3}$, $(23.3.1)_{1,2}$, (23.3.3) and (23.3.5) be fulfilled with $\gamma = \epsilon$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$.

Let <u>either</u> $m_1 < \min(2m, -2)$ <u>or</u> $2m \le m_1 < m-1$ and condition (23.3.4) be fulfilled.

Then for the Schrödinger-Pauli operator as $\tau \to +\infty$ asymptotics (23.3.2) holds while

$$\mathcal{N}^-(\tau) \asymp \tau^{(m_1+2)/(2m)} + \tau. \tag{23.3.6}$$

[26)] One can check easily that this condition or a similar condition in the multidimensional case is fulfilled provided F and V stabilize as $|x| \to 0$ to V^0 and F^0, positively homogeneous of degrees $2m$ and m_1 respectively.

Indeed, if $m_1 \ge 2m$ then no modification to the arguments of Examples 23.3.1 and 23.3.2 is needed; if $m_1 < 2m$ we also need to consider the zone $\{\tau^{1/(2m)} \lesssim |x| \lesssim \tau^{1/m_1}\}$. The contribution of the corresponding partition element to the principal part of the asymptotics is $O(r^{m_1}\gamma^2)$ ($r = |x| \asymp \gamma$) while its contribution to the remainder is $O(\tau^{1/2}\gamma + 1)$.

(ii) One can generalize this example to the even-dimensional full-rank case; then

$$\mathcal{N}^-(\tau) \asymp \tau^{(m_1+2)d/(4m)} + \tau^{d/2} \tag{23.3.7}$$

and the remainder is $O(R)$ with

$$R = \tau^{(m_1+2)(d-2)/(4m)} + \tau^{(d-1)/2}. \tag{23.3.8}$$

Dirac Operator

Finally, consider Dirac operator. We want to consider either $\mathsf{N}(0,\tau)$ with $\tau \to +\infty$ and $\mathsf{N}(\tau,0)$ as $\tau \to -\infty$.

Example 23.3.4. Let X be a compact domain, $d = 2$ and conditions $(23.2.75)_{1-2}$, $(23.3.1)_{1,2}$ and

$$|D^\alpha V| \le c\rho\gamma^{-|\alpha|} \tag{23.3.9}$$

be fulfilled with $\gamma = \epsilon|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$ and $m_1 < \min(2m, -2)$. Let assumption (23.2.74) be fulfilled as $|x| \le \epsilon$.

(i) Let $V < M$ in the vicinity of 0. Then for the Dirac operator asymptotics

$$\mathsf{N}(0,\tau) = \mathcal{N}(0,\tau) + O(\tau) \tag{23.3.10}$$

holds as $\tau \to +\infty$ and $\mathcal{N}(0,\tau) \asymp \tau^2$.

Indeed, assumption $V < M$ guarantees that $V - (M^2 + 2jF)^{\frac12}$ with $j = 0, 1, ...$ do not contribute.

(ii) Let $V^2 < M^2 + F$ in the vicinity of 0 and

$$V \le -\epsilon_0\rho, \qquad |\nabla V| \ge \epsilon_0\rho\gamma^{-1} \qquad \text{as } |x| \le \epsilon. \tag{23.3.11}_2$$

Then for the Dirac operator asymptotics

$$\mathsf{N}(\tau,0) = \mathcal{N}(\tau,0) + O(|\tau|^{\frac12}) \tag{23.3.12}$$

and

$$\mathcal{N}(\tau,0) \asymp \tau^2 + |\tau|^{(m_1+2)/m} \tag{23.3.13}$$

holds as $\tau \to -\infty$.

Indeed, assumption $V^2 < M^2 + F$ guarantees that $V + (M^2 + 2jF)^{\frac12}$ with $j = 1, \dots$ do not contribute.

We leave to the reader

Problem 23.3.5. (i) Extend results of Example 23.3.4(i) to the case $2m \le m_1 < m-1$.

(ii) Expand results of Example 23.3.4(ii) to the case $m_1 = 2m < -2$.

In both Statements (i) and (ii) one needs to formulate the analogue of the non-degeneracy assumption (23.3.4).

(iii) Consider the full-rank even-dimensional case.

Miscellaneous Singularities

Consider now miscellaneous singularities in the point, restricting ourselves to $d = 2$:

Example 23.3.6. Let X be a compact domain, $d = 2$ and conditions $(23.2.3)_{1-3}$, $(23.3.1)_{1,2}$ and (23.3.3) be fulfilled with $\gamma = \epsilon|x|$, $\rho = |\log|x||^{\sigma}$, $\rho_1 = |x|^{m_1}$, $\sigma > 0$, $m_1 < -2$.

Then for the Schrödinger-Pauli operator asymptotics (23.3.2) holds and

$$\log(\mathcal{N}^-(\tau)) \asymp \tau^{-(m_1+2)/2\sigma}. \tag{23.3.14}$$

Example 23.3.7. Let X be a compact domain, $d = 2$ and conditions $(23.2.3)_{1-3}$, $(23.3.1)_{1,2}$ be fulfilled with $\gamma = \epsilon|x|^{1-\beta}$, $\rho_1 = \exp(b|x|^{\beta})$, $\beta < 0$, $b > 0$ and with $\rho = \exp(a|x|^{\alpha})$ where <u>either</u> $\beta < \alpha < 0$ <u>or</u> $\beta = \alpha$ and $b > 2a$.

(i) Then for Schrödinger operator asymptotics (23.3.2) holds and $\mathcal{N}^-(\tau) \asymp \tau$ as $\tau \to +\infty$.

Indeed, as $\beta > -1$ it is easy, and as $\beta \le -1$ one can apply the same arguments as in Example 23.3.23 below (in which case Remark 23.3.11(iii) does not apply.

(ii) Let also conditions $(23.2.3)_{1,3}$, (23.3.3) and (23.3.5) be fulfilled with $\gamma = \epsilon|x|^{1-\alpha}$. Then for the Schrödinger-Pauli operator asymptotics (23.3.2) holds with

$$\mathcal{N}^-(\tau) \asymp \tau^{b/2a} |\log \tau|^{(2-\beta)/\beta} \qquad \text{as } \alpha = \beta, \tag{23.3.15}$$

$$\log(\mathcal{N}^-(\tau)) \asymp |\log \tau|^{\beta/\alpha} \qquad \text{as } \beta < \alpha < 0. \tag{23.3.16}$$

Indeed, in this case the forbidden zone is $\{x\colon |x| \le r^*\}$ with $r_* = \epsilon|\log\tau|^{1/\alpha}$ and contribution of the zone $\{x\colon |x| \ge r_*\}$ to the "extra" remainder does not exceed $\int r^{-1-2\beta} r\,dr$, taken over this zone, which is $r_*^{-2\beta} \asymp |\log\tau|^{2\beta/\alpha}$.

Example 23.3.8. and conditions $(23.2.3)_{1-3}$, $(23.3.1)_{1,2}$ be fulfilled with $\gamma = \epsilon|x|^{1-\beta}$, $\rho_1 = \exp(b|x|^\beta)$, $\beta < 0$, $b > 0$ and with $\rho = |x|^{2m}$.

(i) Then Schrödinger operator is covered by the previous Example 23.3.7..

(ii) Let conditions $(23.2.3)_{1,3}$, (23.3.3), (23.2.23) be fulfilled with $\gamma = \epsilon|x|$, $\rho = |x|^m$, $m < 0$.

Then for the Schrödinger-Pauli operator the following asymptotics holds:

$$\mathsf{N}^-(\tau) = \mathcal{N}^-(\tau) + O(\tau^{1/2} + \tau^{\beta/(2m)}) \tag{23.3.17}$$

holds with

$$\log(\mathcal{N}^-(\tau)) \asymp \tau^{\beta/(2m)}. \tag{23.3.18}$$

Indeed, in this case $r_* = \tau^{1/(2m)}$ (cf. Example 23.3.7(i)).

Indeed, the contributions of the zone where $\mu_{\mathsf{eff}} h_{\mathsf{eff}} \le C$ to the main term of the asymptotics and to the remainder estimates are $\int_{\mathcal{X}} \rho_1 r\,dr$ and $\int_{\mathcal{Z}} \gamma^{-2} r\,dr$ where $\mathcal{X} = \{x\colon : V(x) \le \tau\}$ and $\mathcal{Z}$ is a γ-vicinity of ∂X; so we get (23.3.18) (we cannot get magnitude of $\mathcal{N}^-(\tau)$ itself precisely) and $\bar r$ with $\bar r = \tau^{1/(2m)}$.

The following problem seems to be very challenging:

Problem 23.3.9. Using the fact that singularities propagate along the drift lines, and the length of the drift line is $\asymp \bar r$ rather than $\asymp \bar\gamma = \bar r^{1-\beta}$ prove that the contribution of $\mathcal{Z}$ to the remainder is in fact $O(1)$ and thus improve the remainder estimate (23.3.17) to $O(\tau^{1/2})$.

Problem 23.3.10. Extend results of Examples 23.3.6, 23.3.7 and 23.3.8 to the Dirac operator.

Remark 23.3.11. (i) Observe that the contribution to the remainder of the zone $\{x\colon |x|\le \varepsilon\}$ does not exceed $\varepsilon^{\sigma}\tau^{(d-1)/2}$ with $\sigma>0$ in the frameworks of Example 23.3.1(i), Example 23.3.1(iv) with $m_1>-2d$, Example 23.3.2(i), Example 23.3.2(iii) with $d+m_1>m$ and Example 23.3.3.

The same is correct in Example 23.3.6 and Examples 23.3.7 and 23.3.8 with $0>\beta>-1$.

Therefore, in these cases under the standard non-periodicity condition to the geodesic flow with reflections from ∂X the asymptotics

$$\mathsf{N}(\tau)=\mathcal{N}(\tau)+\varkappa_1\tau^{\frac12}+o\bigl(\tau^{\frac12}\bigr) \tag{23.3.19}$$

holds with the standard coefficient $\varkappa_1$.

(ii) The similar statement (with τ replaced by τ^2) is true in the framework of Example 23.3.4.

(iii) Since we used local estimates $O(h_{\mathsf{eff}}^{-1})$ rather than $O(\mu_{\mathsf{eff}}^{-1}h_{\mathsf{eff}}^{-1})$ as $\mu_{\mathsf{eff}}\ge 1$ (the latter gave us no advantage) we do not need 0 to be an inner singular point; the same results hold for $0\in\partial X$ under Dirichlet or Neumann boundary condition.

Problem 23.3.12. In the frameworks of Examples 23.3.1(i), 23.3.1(iv), 23.3.2(i), 23.3.2(iii) and 23.3.3 estimate $|\mathcal{N}^-(\tau)-\kappa_0\tau^{d/2}|$.

Furthermore, in the frameworks of Examples 23.3.4 (i), (ii) estimate $|\mathcal{N}(0,\tau)-\kappa_0\tau^2|$ and $|\mathcal{N}(\tau,0)-\kappa_0\tau^2|$ respectively.

Finally, consider the case when the singularity is located on the curve.

Example 23.3.13. Let X be a compact domain, $d=2$ and conditions $(23.2.3)_{1-3}$, $(23.3.1)_{1,2}$ be fulfilled with $\gamma=\epsilon\delta(x)$, $\rho_1=\delta(x)^m$, $\rho_1=\delta(x)^{m_1}$ with $m_1<\min(2m,-2)$ where $\delta(x)=\mathsf{dist}(x,L)$, $m<0$, L is <u>either</u> a closed curve ($q=1$) <u>or</u> a closed set of Minkowski dimension $q<1$.

(i) Then for the Schrödinger operator asymptotics (23.3.2) holds for $\tau\to+\infty$ and $\mathcal{N}^-(\tau)\asymp\tau$.

Indeed, using the same arguments as before we can get a remainder estimate $O(\tau^{1/2})$ if $q<1$ and $O(\tau^{\frac12}|\log\tau|)$ if $q=1$ but in the latter case we can get rid of logarithm using standard propagation arguments.

(ii) Let also conditions (23.3.3) and (23.3.5) be fulfilled with $m > -1$. Then for the Schrödinger-Pauli operator asymptotics holds

$$\mathsf{N}^-(\tau) = \mathcal{N}^-(\tau) + O(\tau^{1/2} + \tau^{(q-2)/(2m)}) \tag{23.3.20}$$

holds while

$$\mathcal{N}^-(\tau) \asymp \tau + \tau^{(m_1+q)/(2m)}. \tag{23.3.21}$$

Indeed, in this case a forbidden zone is $\{x\colon \delta(x) \le \delta_* = \epsilon\tau^{1/(2m)}\}$ and contributions of the zone $\{x\colon \delta(x) \ge \delta_*\}$ to the main term of the asymptotics and the remainder are $\asymp \delta_*^{m_1+q}$ and δ_*^{q-2} respectively.

Problem 23.3.14. (i) Explore, if we can using propagation arguments improve remainder estimate (23.3.20).

(ii) Extend results of Example 23.3.13 to different types of the singularities along L and/or Dirac operator.

23.3.2 Singularities at Infinity

Let us consider *unbounded domains:*

Power Singularities: Schrödinger Operator

Let us start from the power singularities.

Example 23.3.15[27]. (i) Let X be a connected exterior domain[28] with $\mathscr{C}^K$ boundary, $d = 2$. Let conditions (23.2.2) and $(23.2.3)_{1-3}$ be fulfilled with $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$, $m_1 > 2m$. Further, let

$$|F| \ge \epsilon_0\rho_1, \qquad |\nabla F| \ge \epsilon_0\rho_1\gamma^{-1} \qquad \text{for } |x| \ge c. \tag{23.3.1}$$

Then for the Schrödinger operator the following asymptotics holds:

$$\mathsf{N}^-(\tau) = \mathcal{N}^-(\tau) + O(\tau^{(m_1+2)/2(m_1+1)}) \tag{23.3.22}$$

with

$$\mathcal{N}^-(\tau) \asymp \tau^{(m_1+2)/m_1}. \tag{23.3.23}$$

---

[27] Cf. Example 23.3.1.

[28] I.e. a domain with compact complement $\complement X$ in $\mathbb{R}^d$.

Indeed, there will be zone $X_1' = \{|x| \le \tau^{1/2(m_1+1)}\}$ with $\mu_{\text{eff}} = |x|^{m_1+1}\tau^{-1/2} \lesssim 1$ and a zone $X_2' = \{|x| \ge \tau^{1/2(m_1+1)}\}$ with $\mu_{\text{eff}} \ge 1$. Contribution of the partition element in X_1' to the remainder is $O(h_{\text{eff}}^{-1}) = O(\tau^{1/2}\gamma)$ and the total contribution of X_1' is $O(\tau^{(m_1+2)/2(m_1+1)})$. On the other hand, contribution of the partition element in X_2' to the remainder is $O(\mu_{\text{eff}}^{-1}h_{\text{eff}}^{-1}) = O(\tau\gamma^{-m_1})$ and the total contribution of this zone is $O(\tau^{(m_1+2)/2(m_1+1)})$ again.

(ii) Under proper assumptions the similar asymptotics holds in the full-rank even-dimensional case:

$$\mathsf{N}^-(\tau) = \mathcal{N}^-(\tau) + O(\tau^{(d-1)(m_1+2)/2(m_1+1)}) \tag{23.3.24}$$

with

$$\mathcal{N}^-(\tau) \asymp \tau^{d(m_1+2)/(2m_1)}. \tag{23.3.25}$$

Example 23.3.16[29]. (i) Let X be a connected exterior domain[28] with $\mathscr{C}^K$ boundary. Let conditions (23.2.2), $(23.2.3)_{1-3}$ and $(23.3.1)_1^{\#}$ be fulfilled with $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$, $m > 0$, $m - 1 < m_1 \le 2m$.

Then we need to assume that

$$V + F \ge \epsilon_0\rho^2 \qquad \text{as} \quad |x| \ge c \tag{23.3.3}^{\#\prime}$$

which for $m_1 < 2m$ is equivalent to

$$V \ge \epsilon_0\rho^2 \qquad \text{as} \quad |x| \ge c \tag{23.3.3}^{\#}$$

We need also to have some non-degeneracy assumption. Assume that (cf. (23.2.16))[30]

$$\begin{aligned} &\tau \ge V + F, \qquad |\nabla(\tau - V)F^{-1}|\gamma \le \epsilon_0\tau F^{-1} \\ &\qquad \implies |\det \mathsf{Hess}(\tau - V)F^{-1}|\gamma^2 \ge \epsilon_0\tau F^{-1} \qquad \text{as} \quad |x| \ge c. \end{aligned} \tag{23.3.27}$$

---

[29] Cf. Example 23.3.2.

[30] One can see easily, that if F, V stabilize at infinity to functions F^0, V^0, positively homogeneous of degrees $m_1 \ge 0$, $2m$ respectively, then even stronger non-degeneracy assumption (cf. (23.2.14)) holds:

$$\tau \ge V + F \implies |\nabla(\tau - V)F^{-1}|\gamma \ge \epsilon_0\tau \qquad \text{as} \quad |x| \ge c. \tag{23.3.26}$$

On the other hand, for $m_1 < 0$ condition (23.3.26) is fulfilled if $w(\theta) = F^{0\,-2m}V^{0\,m_1}$ has only nodegenerate critical points on $\mathbb{S}^1$.

Then for the Schrödinger operator asymptotics

$$\mathsf{N}^-(\tau) = \mathcal{N}^-(\tau) + O(R), \tag{23.3.28}$$

holds with

$$R = \begin{cases} \tau^{(m_1+2)/2(m_1+1)} & m_1 > 0, \\ \tau |\log \tau|^2 & m_1 = 0, \\ \tau^{1-m_1/(2m)} |\log \tau| & m-1 < m_1 < 0, \end{cases} \tag{23.3.29}$$

and

$$\mathcal{N}^-(\tau) \asymp \tau^{(m+1)/m}. \tag{23.3.30}$$

Indeed, the contribution of γ-element in X_2' (see Example 23.3.15) to the remainder does not exceed $C\mu_{\mathsf{eff}}^{-1} h_{\mathsf{eff}}^{-1} |\log \mu_{\mathsf{eff}}| = C\tau r^{-m_1} |\log r r_*^{-1}|$, $r_* = \tau^{1/(2(m_1+1)}$. Then summation with respect to partition returns R; we need to take into account that $\{|x| \ge C\tau^{1/(2m)}\}$ is a forbidden zone. Meanwhile, contribution of X_1' does not exceed $C\tau^{(d-1)/2} r_*^{d-1}$.

(ii) As $m_1 < 0$ we can get rid of the logarithmic factor in the remainder estimate, if we define $\mathcal{N}^-(\tau)$ by the corrected magnetic Weyl formula; similarly, for $m_1 = 0$ we can then replace $|\log \tau|^2$ by $|\log \tau|$.

Also for $m_1 = 0$ we can $|\log \tau|^2$ by $|\log \tau|$ under assumption

$$\begin{aligned} &\tau \ge V + F, \qquad |\nabla(\tau - V)F^{-1}|\gamma \le \epsilon_0 \tau F^{-1} \\ &\qquad \implies \det \mathsf{Hess}(\tau - V)F^{-1}\gamma^2 \ge \epsilon_0 \tau F^{-1} \qquad \text{as } |x| \ge c. \end{aligned} \tag{23.3.31}$$

(iii) Under proper assumptions the similar asymptotics holds in the full-rank even-dimensional case:

$$R = \begin{cases} \tau^{(d-1)(m_1+2)/2(m_1+1)} & m_1 > d-2, \\ \tau^{d/2} |\log \tau|^2 & m_1 = d-2, \\ \tau^{d/2+(d-2-m_1)/(2m)} |\log \tau| & m-1 < m_1 < d-2, \end{cases} \tag{23.3.32}$$

with

$$\mathcal{N}^-(\tau) \asymp \tau^{d(m_1+2)/(2m_1)}. \tag{23.3.33}$$

Moreover, we can get rid of one logarithmic factor under assumption similar to (23.3.31).

Let is improve (23.3.29) using arguments associated with long-range dynamics:

Example 23.3.17. Let $m_1 > 0$ in the framework of Examples 23.3.15(i) or 23.3.16(i). Moreover, let the stabilization conditions

$$(23.3.34)_1 \qquad D^\sigma(g^{jk} - g^{jk0}) = o(|x|^{-|\sigma|}),$$

$$(23.3.34)_2 \qquad D^\sigma(V_j - V_j^0) = o(|x|^{m_1+1-|\sigma|}) \qquad \forall \sigma : |\sigma| \le 1$$

be fulfilled for $|x| \to \infty$ with the positively homogeneous functions $g^{jk0}, V_j^0 \in \mathscr{C}^K(\mathbb{R}^2 \setminus 0)$ of degrees 0 and $m_1 + 1$ respectively. Let the standard non-periodicity condition be fulfilled for the Hamiltonian

$$(23.3.35) \qquad H^0 = |x|\Big(\sum_{j,k} g^{jk0}(\xi_j - V_j^0)(\xi_k - V_k^0) - 1\Big)$$

on the energy level 0. Then for the Schrödinger operator as $\tau \to +\infty$ the following asymptotics holds:

$$(23.3.36) \qquad \mathsf{N}^-(\tau) = \mathcal{N}^-(\tau) + o\big(\tau^{\frac{1}{2}(m_1+2)/(m_1+1)}\big).$$

Indeed, let us observe that for $m_1 > 0$ the main contribution to the remainder estimate (23.3.29) is given by the zone $\{\varepsilon \le |x|\tau^{-1/2(m_1+1)} \le \varepsilon^{-1}\}$ with an arbitrarily small constant $\varepsilon > 0$. In this zone the magnetic field is normal, $\mu_{\mathsf{eff}} \lesssim 1$ (for every fixed $\varepsilon > 0$) and $V \ll \tau$, $|\nabla V| \ll \tau|x|^{-1}$ Applying the improved Weyl asymptotics here we obtain (23.3.36).

Remark 23.3.18. Similar improvements are possible in the full-rank even-dimensional case.

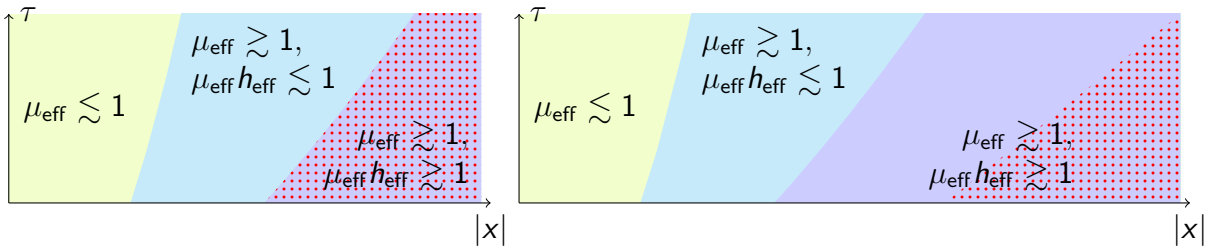
Power Singularities: Schrödinger-Pauli Operator

Next, consider Schrödinger-Pauli operators. We will need to impose $(23.3.3)^\#$ and the related non-degeneracy assumption

$$(23.3.5)^\# \qquad |\nabla V| \ge \epsilon_0 \rho^2 \gamma^{-1} \qquad \text{for } |x| \ge c.$$

Example 23.3.19[31]. Let $(23.3.3)^\#$ and $(23.3.5)^\#$ be fulfilled. Then for the Schrödinger-Pauli operator

[31] Cf. Example 23.3.3.



(a) Schrödinger operator (b) Schrödinger-Pauli operator

Figure 23.1: $m_1 > 2m$; dots show the forbidden zone

(i) In the framework of Examples 23.3.15 and 23.3.16 asymptotics (23.3.22) and (23.3.29)–(23.3.29) hold, respectively.

(ii) In the framework of Example 23.3.19(i), asymptotics (23.3.36) holds.

(iii) Further,

$$\mathcal{N}^-(\tau) \asymp \tau^{(m+1)/m} + \tau^{(m_1+2)/(2m)}. \tag{23.3.37}$$

(iv) Finally, under proper assumptions one can consider the full-rank even-dimensional case and prove asymptotics with the remainder estimate $O(R)$, with $R := R_1 + R_2$ where R_1 is the remainder estimate for the Schrödinger operator,

$$R_2 = \tau^{(m_1+2)(d-2)/(4m)} \tag{23.3.38}$$

and

$$\mathcal{N}^-(\tau) \asymp \tau^{d(m+1)/(2m_1)} + \tau^{d(m_1+2)/(4m)}. \tag{23.3.39}$$

Power Singularities: Dirac Operator

Finally, consider the Dirac operators. We want to explore either $\mathsf{N}(0,\tau)$ with $\tau \to +\infty$ and $\mathsf{N}(\tau,0)$ as $\tau \to -\infty$.

Example 23.3.20. Let X be a connected exterior domain with $\mathscr{C}^K$ boundary, $d = 2$. Let conditions (23.2.2), $(23.2.75)_{1-2}$, $(23.3.1)^{\#}_{1,2}$ and (23.3.9) be fulfilled with $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$, $m_1 > \max(2m,-2)$. Further, let assumption (23.2.74) be fulfilled as $|x| \le \epsilon$.

(i) Let $V < M$ in the vicinity of infinity. Then as $\tau \to +\infty$ for the Dirac operator asymptotics

$$\mathsf{N}^-(\tau) = \mathcal{N}^-(\tau) + O(\tau^{(m_1+2)/(m_1+1)}) \tag{23.3.40}$$

holds with

$$\mathcal{N}^-(\tau) \asymp \tau^{2(m_1+2)/m_1}. \tag{23.3.41}$$

(ii) Let $V^2 < M^2 + F$ in the vicinity of infinity and

$$V \le -\epsilon_0\rho, \qquad |\nabla V| \ge \epsilon_0\rho\gamma^{-1} \qquad \text{as } |x| \ge c. \tag{23.3.11}$$$^{\#}_{2}$

Then as $\tau \to -\infty$ for the Dirac operator asymptotics (23.3.40) holds with

$$\mathcal{N}^-(\tau) \asymp \tau^{2(m+1)/m} + \tau^{2(m_1+2)/(2m)}. \tag{23.3.42}$$

We leave to the reader

Problem 23.3.21. (i) Using arguments of Example 23.3.16 extend results of Example 23.3.20(i) to the case $2m \ge m_1 > m-1$.

(ii) Using arguments of Examples 23.3.16 and 23.3.19 expand results of Example 23.3.20(ii) to the case $m_1 = 2m < -2$.

In both Statements formulate the analogue of the non-degeneracy assumption (23.3.4)$^{\#}$.

(iii) Consider the full-rank even-dimensional case.

Problem 23.3.22. Extend to the Dirac operator results of Example 23.3.17; one still defines Hamiltonian H^0 by (23.3.35).

Exponential Singularities

Consider now an exponential growth at infinity.

Example 23.3.23. Let X be a connected exterior domain with $\mathscr{C}^K$ boundary. Let conditions (23.2.2), (23.2.3)$_{1-3}$, (23.3.1)$^{\#}_{1,2}$ be fulfilled with $\gamma = \epsilon_0\langle x\rangle^{1-\beta}$, $\rho = \exp(a\langle x\rangle^{\alpha})$, $\rho_1 = \exp(b\langle x\rangle^{\beta})$, $\beta > 0$ and either $\beta > \alpha$ or $\beta = \alpha$ and $b > 2a > 0$.

(i) Then for the Schrödinger operator the following asymptotics holds:

$$\mathsf{N}^-(\tau) = \mathcal{N}^-(\tau) + O\bigl(\tau^{\frac{1}{2}}|\log\tau|^{1/\beta}\bigr) \tag{23.3.43}$$

with

$$\mathcal{N}^-(\tau) \asymp \tau|\log\tau|^{2/\beta}. \tag{23.3.44}$$

Indeed, using described γ, consider zone $\{\bar{r} \le |x| \le C|\log\tau|^{1/\beta}\}$, where $\mu_{\mathsf{eff}} = F\gamma\tau^{-1/2} \gtrsim 1$ and $F \le c\tau$; here $\bar{r}$ is defined by $\bar{r}^{1-\beta}\exp(b\bar{r}^{\beta}) = \tau^{1/2}$. Then contribution of γ-element to the remainder does not exceed $C\mu_{\mathsf{eff}}^{-1}h_{\mathsf{eff}}^{-1} = \tau\exp(-br^{\beta})$ with $r = |x|$ and the total contribution of this zone does not exceed CR with

$$R = \int_{\bar{r}} \tau\exp(-br^{\beta})r^{-2+2\beta}r\,dr,$$

which is equal to the integrand, multiplied by $r^{1-\beta}$ and calculated as $r = \bar{r}$: $R \asymp \tau\exp(-b\bar{r}^{\beta})\bar{r}^{\beta} = \tau^{1/2}\bar{r}$ with $\bar{r} \asymp |\tau|^{1/\beta}$.

On the other hand, consider zone $\{|x| \le \bar{r}\}$, where we can redefine $\gamma = \bigl(\tau^{1/2}r^{(1-\beta)\delta}\exp(-br^{\beta})\bigr)^{1/(1+\delta)}$ with $\delta > 0$; then its contribution to the remainder does not exceed CR with

$$R = \int^{\bar{r}} \tau^{1/2}\bigl(\tau^{1/2}r^{(1-\beta)\delta}\exp(-br^{\beta})\bigr)^{-1/(1+\delta)}r\,dr,$$

which is also equal to the integrand, multiplied by $r^{1-\beta}$ and calculated as $r = \bar{r}$: $R \asymp \tau^{1/2}\bigl(\tau^{1/2}\bar{r}^{(1-\beta)\delta}\exp(-b\bar{r}^{\beta})\bigr)^{-1/(1+\delta)}\bar{r}^{2-\beta} = \tau^{1/2}\bar{r}$ again.

(ii) Let also conditions $(23.2.3)_{1,3}$, $(23.3.3)^{\#}$ and $(23.3.5)^{\#}$ be fulfilled with $\gamma = \epsilon|x|^{1-\alpha}$. Then for the Schrödinger-Pauli operator asymptotics (23.3.43) holds with

$$\mathcal{N}^-(\tau) \asymp \tau^{b/(2a)}|\log\tau|^{(2-\beta)/\beta} \qquad \beta = \alpha, \tag{23.3.45}$$

$$\log(\mathcal{N}^-(\tau)) \asymp |\log\tau|^{\beta/\alpha} \qquad \beta > \alpha. \tag{23.3.46}$$

Indeed, in this case the forbidden zone is $\{x\colon |x| \ge r_* = c|\log\tau|^{1/\alpha}$ (cf. Example 23.3.7 (ii)).

Example 23.3.24. Let X be a connected exterior domain with $\mathscr{C}^K$ boundary. Let conditions (23.2.2), $(23.2.3)_{1-3}$, $(23.3.1)^{\#}_{1,2}$ be fulfilled with $\gamma = \epsilon_0\langle x\rangle^{1-\beta}$,

$\rho = \exp(a\langle x\rangle^{\alpha})$, $\rho_1 = \exp(b\langle x\rangle^{\beta})$, $\beta > 0$ and $\beta = \alpha$ and $2a > b > a > 0$. Let conditions $(23.3.3)^{\#}$ and $(23.3.5)^{\#}$ be also fulfilled.

Then for both Schrödinger and Schrödinger-Pauli operators (23.3.43) and (23.3.44) hold. Indeed, the forbidden zone is the same as in the previous example.

Example 23.3.25. Let X be a connected exterior domain with $\mathscr{C}^K$ boundary. Let conditions (23.2.2), $(23.2.3)_{1-3}$, $(23.3.1)^{\#}_{1,2}$ be fulfilled with $\gamma = \epsilon_0\langle x\rangle^{1-\beta}$, $\rho_1 = \exp(b\langle x\rangle^{\beta})$, $\beta > 0$ and $\rho = \langle x\rangle^{m}$, $m > 0$.

(i) Then the Schrödinger operator is covered by Example (i).

(ii) Let also conditions $(23.2.3)_{1,3}$, $(23.3.3)^{\#}$ and $(23.3.5)^{\#}$ be fulfilled with $\gamma = \epsilon|x|$. Then for the Schrödinger-Pauli operator the following asymptotics holds holds:

$$\mathsf{N}^{-}(\tau) = \mathcal{N}^{-}(\tau) + O(\tau^{1/2}|\log\tau|^{1/\beta} + \tau^{\beta/(2m)}) \tag{23.3.47}$$

and

$$\log(\mathcal{N}^{-}(\tau)) \asymp |\tau|^{\beta/(2m)}. \tag{23.3.48}$$

Indeed, in this case the forbidden zone is $\{x\colon |x| \ge r_* = c|\tau|^{1/(2m)}$ (cf. Example 23.3.8 (ii)).

The following problem seems to be very challenging:

Problem 23.3.26 [32]**.** Using the fact that singularities propagate along the drift lines, and the length of the drift line is $\asymp \bar{r}$ rather than $\asymp \bar{\gamma} = \bar{r}^{1-\beta}$ prove that the contribution of $\mathcal{Z}$ to the remainder is in fact $O(1)$ and thus improve the remainder estimate (23.3.17) to $O(\tau^{1/2})$.

Problem 23.3.27. (i) Consider Dirac operator in the same settings as in Example 23.3.23.

(ii) Since for Schrödinger, Schrödinger-Pauli and Dirac operators the main contribution to the remainder is delivered by the zone where $\varepsilon \le \mu_{\mathsf{eff}} \le \varepsilon^{-1}$ (and $V \ll \tau$), while the contribution of the zones of the zones where $\mu_{\mathsf{eff}} \le \varepsilon$ and $\mu_{\mathsf{eff}} \ge \varepsilon^{-1}$ do not exceed $\sigma(\varepsilon)\tau|\log\tau|^{1/\beta}$ with $\sigma = o(1)$ as $\varepsilon \to 0$, derive remainder estimate $o(\tau|\log\tau|^{1/\beta})$ under non-periodicity condition for the Hamiltonian similar to (23.3.35).

[32] Cf. Problem 23.3.9

(iii) Consider Schrödinger-Pauli and Dirac operators in the same settings as in Example 23.3.11 albeit with V of the logarithmic growth at infinity (i.e. with $\rho = |\log|x||^{\alpha}$, $\gamma = \epsilon|x|$).

More challenging is the following

Problem 23.3.28. (i) In the frameworks of Examples 23.3.1, 23.3.2, 23.3.3 and 23.3.4 allow degenerations of F.

(ii) In the frameworks of Examples 23.3.2, 23.3.3 and 23.3.4 allow degenerations of V.

23.4 Asymptotics of Small Eigenvalues

Now we consider external domains and asymptotics of eigenvalues tending to some finite limit.

23.4.1 Operators Stabilizing at Infinity

We begin with the analysis of the Schrödinger operator A defined by (23.2.1) under assumption (23.2.2) assuming that

$$(23.4.1)_{1-3} \qquad \mathsf{g} \to \mathsf{g}_{\infty}, \quad \mathsf{F} \to \mathsf{F}_{\infty}, \quad V \to 0 \qquad \text{as } |x| \to \infty.$$

Recall that $\mathsf{F} \coloneqq (F_{jk})$ with $F_{jk} = \partial_k V_j - \partial_j V_k$, $\mathsf{g} \coloneqq (g^{jk})$.

We start from the theorem, describing the essential spectrum of A:

Theorem 23.4.1. *Let X be an exterior domain*[33)]*. with $\mathscr{C}^K$ boundary. Let the Schrödinger operator A satisfy conditions* (23.2.1)*,* (23.2.2)*, and* $(23.4.1)_{1-3}$*. Then*

(i) If $\operatorname{rank} \mathsf{F}_{\infty} = 2r = d$ *then*

$$(23.4.2) \qquad \operatorname{Spec}_{\mathsf{ess}}(A) = \{\sum_j \mathfrak{z}_j f_{\infty,j} : \mathfrak{z} = (\mathfrak{z}_1, \dots, \mathfrak{z}_r) \in (2\mathbb{Z}^+ + 1)^r\}$$

where $\pm i f_{\infty,j}$ are eigenvalues of $g_{\infty}\mathsf{F}_{\infty}$, $f_{\infty,j} > 0$, $j = 1, \dots, r$.

[33)] I.e. with a compact complement. If $X \neq \mathbb{R}^d$, then the appropriate boundary condition are given on ∂X such that operator is self-adjoint. In other words, infinity is an isolated singular point; see [7)].

(ii) If $\mathsf{rank}\,\mathsf{F}_\infty = 2r < d$ *then* $\mathsf{Spec}_{\mathsf{ess}}(A) = [f^*, \infty)$ *with* $f_* = f_{\infty,1} + ... + f_{\infty,r}$.

Proof. Indeed, one can see easily that $\mathsf{Spec}_{\mathsf{ess}}(A)$ coincides with $\mathsf{Spec}(A_\infty)$ where A_∞ is a toy-model operator in $\mathbb{R}^d$ with $\mathsf{g} = \mathsf{g}_\infty$, $\mathsf{F} = \mathsf{F}_\infty$ and $V = 0$. For such operator we calculated spectrum in Theorem 13.2.1. □

Remark 23.4.2. (i) Similarly, for Schrödinger-Pauli operator $\mathsf{Spec}_{\mathsf{ess}}(A)$ is defined by (23.4.2) albeit with $\mathfrak{z}$ running $(2\mathbb{Z}^+)^r$ if $\mathsf{rank}\,\mathsf{F}_\infty = 2r = d$ and $\mathsf{Spec}_{\mathsf{ess}}(A) = [0, \infty)$ if $\mathsf{rank}\,\mathsf{F}_\infty = 2r < d$.

(ii) Further, for the Dirac operator $\mathsf{Spec}_{\mathsf{ess}}(A)$ also coincides with $\mathsf{Spec}(A_\infty)$, calculated in Theorem 17.1.2.

In this section we assume that

$$\mathsf{rank}\,F_\infty = d; \tag{23.4.3}$$

very different and a more complicated case of $\mathsf{rank}\,F_\infty < d$ is left for the next Chapter 24.

According to Theorem 23.4.1(i) under assumption (23.4.3) the essential spectrum consists of separate points, which are points of the pure point spectrum (of infinite multiplicity) of the limiting operator A_∞. We are interested in the asymptotics of eigenvalues of A tending to some fixed $\tau^* \in \mathsf{Spec}_{\mathsf{ess}}(A)$. Namely, let us introduce

$$\mathsf{N}^-(\eta) = \mathsf{N}(\tau^* - \epsilon, \tau^* - \eta) \tag{23.4.4}_-$$

and

$$\mathsf{N}^+(\eta) = \mathsf{N}(\tau^* + \eta, \tau^* + \epsilon) \tag{23.4.4}_+$$

with a small constant $\epsilon > 0$ and a small parameter $\eta \to +0$. We also introduce

$$\mathfrak{W} := \{\mathfrak{z} \in (2\mathbb{Z}^+ + 1)^r : \sum_j \mathfrak{z}_j f_{\infty,j} = \tau^*\}. \tag{23.4.5}$$

To characterize the rate of the decay at infinity we assume that

$$|\nabla^\alpha(\mathsf{g} - \mathsf{g}_\infty)| = o(\rho^2\gamma^{-|\alpha|}), \qquad |\nabla^\alpha(\mathsf{F} - \mathsf{F}_\infty)| = o(\rho^2\gamma^{-|\alpha|}),$$
$$|\nabla^\alpha V| = O(\rho^2\gamma^{-|\alpha|}) \qquad \text{as } |x| \to \infty \quad \forall \alpha. \tag{23.4.6}_{1-3}$$

Theorem 23.4.3. *Let X be a connected exterior domain with $\mathscr{C}^K$ boundary. Let the Schrödinger operator A satisfy conditions* (23.2.1), (23.2.2) *and* $(23.4.6)_{1-3}$ *with scaling functions*[34] *such that $\gamma \to \infty$ and $\rho \to 0$ as $|x| \to \infty$.*

Let $\operatorname{rank} \mathsf{F}_\infty = 2r = d$. Moreover let

$$(23.4.7)_\mp \qquad \mp V \ge -\epsilon\rho^2 \implies |\nabla V| \ge \epsilon_0 \rho^2 \gamma^{-1} \qquad \text{as} \quad |x| \ge c.$$

(i) Then

$$(23.4.8) \qquad |\mathsf{N}^\mp(\eta) - \mathcal{N}^\mp(\eta)| \le C \int_{\mathcal{Z}(\eta)} \gamma^{-2}\, dx + C \int \gamma^{-s}\, dx$$

where

$$(23.4.9) \qquad \mathcal{N}^\mp(\eta) \coloneqq (2\pi)^{-r} \sum_{\mathfrak{z}\in\mathfrak{Z}} \int_{\{x:\, \mp V_\mathfrak{z}(x) \ge \eta\}} f_1 f_2 \cdots f_r \sqrt{g}\, dx$$

$g = \det \mathsf{g}^{-1}$, $\pm i f_j$ *are eigenvalues of* gF, $f_j > 0$, $j = 1, \dots, r$, *and*

$$(23.4.10) \qquad V_\mathfrak{z}(x) \coloneqq V(x) + \sum_j \mathfrak{z}_j (f_j(x) - f_{\infty,j}),$$

$\mathcal{Z}(\eta)$ *is $\epsilon\gamma$-vicinity*[35] *of* $\Sigma(\eta) = \{x:\ \mp V_\mathfrak{z}(x) = \eta\}$.

(ii) Further, under assumption

$$(23.4.11)_\mp \qquad \mp V \ge \epsilon_0 \rho^2$$

$\tau^* \pm 0$ *is not a limit point of the discrete spectrum.*

Proof. Indeed, in the zones $\mathcal{Z}(\eta)$ and

$$(23.4.12) \qquad \Omega(\eta) \coloneqq \{x:\, |\mp V(x) - \eta| \ge \epsilon(\rho^2 + \eta)\},$$

it suffices to make γ-admissible partition of unity and observe that after rescaling $B(x, \gamma(x)) \mapsto B(0,1)$ we have $\mu \mapsto \mu_{\mathsf{new}} = \mu\gamma\rho^{-1}$, $h \mapsto h_{\mathsf{new}} = h\gamma^{-1}\rho^{-1}$ and therefore $\mu h \mapsto \mu h/\rho^2$, $\mu^{-1}h \mapsto \mu^{-1}h\gamma^{-2}$ and before rescaling $\mu = h = 1$.

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[34] Recall that this means that $|\nabla\gamma| \le c$ and $|\nabla\rho \le c\rho\gamma^{-1}$.

[35] I.e. $\mathcal{Z}(\eta) = \bigcup_{x\in\Sigma(\eta)} B(x, \epsilon\gamma(x)))$.

Applying Theorem 13.5.32 for $d = 2$ and Theorem 19.6.25 for $d \ge 4$ we estimate contribution of $\mathcal{Z}(\eta)$ to the remainder by the first term in the right-hand expression of (23.4.8).

Further, applying Theorem 13.6.6 for $d = 2$ and similar results of Section 19.6 for $d \ge 4$ case we estimate contribution of $\Omega(\eta) \cap \{\rho^2 \ge \eta\}$ to the remainder by the second term in the right-hand expression of (23.4.8).

In the same way we estimate contribution of $\Omega(\eta) \cap \{\rho^2 \le \eta\}$ to the remainder by the second term in the right-hand expression of (23.4.8) albeit now we use scale $\mu \mapsto \mu_{\mathsf{new}} = \mu\gamma_\eta\eta^{-\frac12}$, $h \mapsto h_{\mathsf{new}} = h\gamma^{-1}\eta^{-\frac12}$. □

We discuss possible generalizations later; right now we want just get two simple corollaries which follow immediately from Theorem 23.4.3:

Example 23.4.4. (i) In the framework of Theorem 23.4.3 with $\gamma = \langle x\rangle$, $\rho = \langle x\rangle^m$, $m < 0$

$$|\mathsf{N}^{\mp}(\eta) - \mathcal{N}^{\mp}(\eta)| \le C \begin{cases} |\log\eta| & \text{for } d = 2,\\ \eta^{(d-2)/(2m)} & \text{for } d \ge 4 \end{cases} \tag{23.4.13}$$

with $\mathcal{N}^{\mp}(\eta) = O(\eta^{d/(2m)})$. Further, $\mathcal{N}^{\mp}(\eta) \asymp \eta^{d/(2m)}$ if condition $(23.4.11)_{\mp}$ is fulfilled in some non-empty cone.

(ii) Furthermore, if condition $(23.4.11)_{\mp}$ is fulfilled, then for $d = 2$

$$\mathsf{N}^{\mp}(\eta) = \mathcal{N}^{\mp}(\eta) + O(1). \tag{23.4.14}$$

Example 23.4.5. (i) In the framework of Theorem 23.4.3 with $\gamma = \langle x\rangle^{1-\sigma}$, $\rho \le \exp(-\epsilon\langle x\rangle^{\sigma})$, $0 < \sigma < 1$

$$\mathsf{N}^{\mp}(\eta) = \mathcal{N}^{\mp}(\eta) + O(|\log\eta|^{2+(d-2)/\sigma}) \tag{23.4.15}$$

with $\mathcal{N}^{\mp}(\eta) = O(|\log\eta|^{d/\sigma})$. Further, $\mathcal{N}^{\mp}(\eta) \asymp |\log\eta|^{d/\sigma}$ if condition $(23.4.11)_{\mp}$ is fulfilled in some non-empty cone and $\rho \ge \exp(-c\langle x\rangle^{\sigma})$.

(ii) Furthermore, if condition $(23.4.11)_{\mp}$ is fulfilled then the remainder estimate (23.4.15) could be improved to

$$\mathsf{N}^{\mp}(\eta) = \mathcal{N}^{\mp}(\eta) + O(|\log\eta|^{2+(d-2)/\sigma}). \tag{23.4.16}$$

The following problem seems to be very challenging:

Problem 23.4.6 [36]**.** Let $d = 2$, $\mathsf{log}(\pm V(x)) = |x|^{\beta}\phi(x)$ where $|\nabla\phi| \le C$. Then the drift line is of the length asymp r. Improve the remainder estimate (23.4.15) to $O(1)$.

Remark 23.4.7. (i) We need conditions $(23.4.6)_{1-3}$ only for $|\alpha| \le 3$ due to Section 19.6 and we need "o" in this condition only for $|\alpha| \le 1$. Further, if ϵ_0 in conditions $(23.4.7)_{\mp}$ and $(23.4.11)_{\mp}$ is fixed we can replace "$= o(\rho^2\gamma^{-|\alpha|})$" by "$\le \epsilon_1\rho^2\gamma^{-|\alpha|}$" with $\epsilon_1 = \epsilon_1(\epsilon_0)$.

(ii) If $\#\mathfrak{W} = 1$ we can have "O" but replace V in $(23.4.7)_{\mp}$ and $(23.4.11)_{\mp}$ by $V_{\mathfrak{z}}$.

We leave to the reader the series of the following not challenging but interesting problems:

Problem 23.4.8. (i) Consider even faster decaying $\rho \le \mathsf{exp}(-|x|\gamma^{-1}(|x|))$ with monotone increasing $\gamma(t)$ such that $\gamma'(t) = o(\gamma(t)t^{-1})$ and $\gamma(t) \to \infty$ as $t \to \infty$ and prove the remainder estimate

(a) $O(t^d\gamma(t)^{-2})$ in the general case and

(b) $O(t^{d-1}\gamma(t)^{-1})$ under assumption $(23.4.11)_{\mp}$

while $\mathcal{N}^{\mp}(\eta) = O(t^d)$ in the general case and $\mathcal{N}^{\mp}(\eta) \asymp t^d$ under assumption $(23.4.11)_{\mp}$ fulfilled some non-empty cone Γ as $|x| \ge c$. Here $t = t(\eta)$ recovered from $t\gamma(t)^{-1} \asymp |\log\eta|$.

While proof of Theorem 23.4.3 provides proper estimates of the contributions to the remainder of the zones $\mathcal{Z}(\eta)$ and $\Omega^{+}(\eta) \setminus \mathcal{Z}(\eta)$ it fails in the zone $\Omega^{-}(\eta) \setminus \mathcal{Z}(\eta)$ where $\Omega^{\pm}(\eta) := \{x\colon |V_{\mathfrak{z}}(x)| \gtrless \eta\}$. However one can use here $\gamma_\eta = \frac{1}{2}(r - r(\eta))$ instead of γ.

(ii) For example, consider $\gamma(t) = (\mathsf{log}_{(n)}\, t)^{\sigma}$, where $\mathsf{log}_{(n)}\, t$ is n-tuple logarithm[37] with $\sigma > 0$. Then $t(\eta) = |\log\eta||\mathsf{log}_{(n+1)}\,\eta|^{\sigma}$.

(iii) Consider even $\mathsf{exp}(-c\varepsilon|x|) \le \rho \le \mathsf{exp}(-\varepsilon|x|)$, $\gamma = \varepsilon^{-1}$ with sufficiently small $\varepsilon \le \varepsilon(c, \epsilon_0, \Gamma)$ and condition $(23.4.11)_{\mp}$ fulfilled in some non-empty cone Γ. Then

$$\mathsf{N}^{\mp}(\eta) \asymp \varepsilon^{-d}|\log\eta|^{d}. \tag{23.4.17}$$

[36] Cf. Problem 23.3.9.

[37] I.e. $\mathsf{log}_{(1)}\, t = \log t$ and $\mathsf{log}_{(n)}\, t = \mathsf{log}(\mathsf{log}_{(n-1)}\, t)$.

Remark 23.4.9. Asymptotics in the case of $\rho \le \exp(-\epsilon_0|x|)$ or even compactly supported V is out of reach of our methods.

Amazingly such asymptotics (without remainder estimate) were obtained in papers M. Melgaard and G. Rozenblum [1], G. Rozenblum and G. Tashchiyan [1, 2] G. Raikov and S. Warzel [1] by completely different methods.

Problem 23.4.10. Consider slowly decreasing potentials with $\gamma \asymp |x|$ and $\rho = |\log_{(n)} |x||^{-\sigma}$ with $\sigma > 0$.

In this case we need to replace assumptions $(23.4.6)_{1-3}$ with $|\alpha| \ge 1$ and $(23.4.7)_{\mp}$ by

$$(23.4.6)'_{1-3} \quad |\nabla^\alpha (g - g_\infty)| = o(\varrho\rho^2\gamma^{-|\alpha|}), \qquad |\nabla^\alpha (F - F_\infty)| = o(\varrho\rho^2\gamma^{-|\alpha|}),$$
$$|\nabla^\alpha V| = O(\varrho\rho^2\gamma^{-|\alpha|}) \qquad \text{as } |x| \to \infty \quad \forall \alpha : |\alpha| \ge 1.$$

and

$$(23.4.7)'_{\mp} \qquad \mp V \ge -\epsilon\rho^2 \implies |\nabla V| \ge \epsilon_0 \varrho\rho^2\gamma^{-1} \qquad \text{as } |x| \ge c.$$

respectively where ϱ is another γ-admissible scaling function; $\varrho \le 1$.

Here again we apply Theorems 13.5.32 and 13.7.6 for $d = 2$ and results of Section 19.6 for $d \ge 4$.

The first of the following problems seems to be challenging enough while the second one is rather easy:

Problem 23.4.11. Using results of Chapter 15 consider 2-dimensional domains X with are γ-admissible boundaries, f.e. domains which are conical outside of the ball $B(0, c)$. Neumann boundary conditions would be especially interesting and challenging.

Problem 23.4.12. Generalize results of this subsection to genuine Schrödinger-Pauli and Dirac operators. While in the former case no modifications is needed (except the Landau levels), in the latter case we need to consider two cases

(a) $M^2 + 2jF_\infty > 0$ and potential $V \sim \rho^2$ at infinity.

(b) $M^2 + 2jF_\infty = 0$ and potential $V \sim \rho$ at infinity.

It is so because $M^2 + 2jF_\infty$ plays a role of the mass.

23.4.2 Operators Stabilizing at Infinity. II

Assume now that g and F stabilize at infinity to $\mathsf{g}_\infty = \mathsf{g}_\infty(\theta)$, $\mathsf{F}_\infty = \mathsf{F}_\infty(\theta)$, positively homogeneous of degree 0, and $V \to 0$. Then one can see easily that the for the Schrödinger and Schrödinger-Pauli operators essential spectrum of A consists of (possibly overlapping) *spectral bands* $\Pi_{\mathfrak{z}}$

$$\mathsf{Spec}_{\mathsf{ess}}(A) = \bigcup_{\mathfrak{z}\in\mathfrak{Z}} \Pi_{\mathfrak{z}}, \qquad \Pi_{\mathfrak{z}} := \bigl\{\sum_j \mathfrak{z}_j f_{\infty,j}(\theta), : \theta \in [0, 2\pi]\bigr\}. \tag{23.4.18}$$

with the *spectral gaps* between them.

In particular, for the Schrödinger operator all spectral bands in the generic case have non-zero width while for the Schrödinger-Pauli operator $\Pi_0 = \{0\}$. Then under proper assumptions for the eigenvalues tending to $+0$ or -0 the results of the previous Subsection 23.4.1 hold. In this subsection we are interested in the asymptotics of the eigenvalues tending to the border between a spectral gap and a spectral bund of non-zero width.

Further, for $d = 2$ in the generic case there could be an infinite number of spectral gaps, but for $d \ge 4$ there is only finite number of them.

Similarly, for the Dirac operator essential spectrum consists of the spectral bunds, one of them consisting of a single point M or $-M$.

Theorem 23.4.13. *Let S be a Schrödinger or Schrödinger-Pauli operator. Let conditions* $(23.4.6)_{1-3}$ *are fulfilled with g_∞, F_∞ positively homogeneous of degree 0, $\gamma = \epsilon|x|$, $\rho = |x|^m$, $m < 0$. Let $x = r\theta$ with $r = |x|$, and $\theta \in \mathbb{S}^{d-1}$.*

Assume for simplicity[38] *that*

$$\tau^* = \textstyle\sum_j \mathfrak{z}_j f_{\infty,j}(\theta) \text{ if and only if } \mathfrak{z} = \bar{\mathfrak{z}} \text{ and } \theta = \bar{\theta}, \tag{23.4.19}$$

$$f_{\infty,1}(\theta), \dots, f_{\infty,r}(\theta) \text{ are disjoint,} \tag{23.4.20}$$

$$\pm \sum_j \bar{\mathfrak{z}}_j f_{\infty,j}(\theta) \ge \epsilon|\theta - \bar{\theta}|^{2n} \tag{23.4.21}_{\pm}$$

and

$$|\partial^\alpha \sum_j \bar{\mathfrak{z}}_j f_{\infty,j}(\theta)| \le c_\alpha |\theta - \bar{\theta}|^{2n-|\alpha|} \qquad \forall \alpha : |\alpha| \le 2n. \tag{23.4.22}$$

[38] Otherwise we will get the sums of asymptotics.

Under assumption $(23.4.21)_+$ *let* $\mathsf{N}^-(\eta) \coloneqq \mathsf{N}(\tau^* - \epsilon, \tau^* - \eta)$ *and under assumption* $(23.4.21)_-$ *let* $\mathsf{N}^+(\eta) \coloneqq \mathsf{N}(\tau^* + \eta, \tau^* + \epsilon)$.

(i) Let

$$(23.4.23)_{\mp,1,2} \qquad \mp V(r\bar{\theta}) \ge \epsilon r^{2m}, \quad \mp \partial_r V(r\bar{\theta}) \ge \epsilon r^{2m-1} \qquad \text{as} \quad r \ge c$$

and $m + n > 0$. *Then as* $\eta \to +0$

$$\mathsf{N}^{\mp}(\eta) = \mathcal{N}^{\mp}(\eta) + O(\eta^{((m+n)(d-3)+n)/(2mn)} \tag{23.4.24}$$

with

$$\mathcal{N}^{\mp}(\eta) = (2\pi)^{-r} \int_{\{x:\, \mp V_\mathfrak{z} \ge \eta\}} f_1 \cdots f_r \, dx \asymp \eta^{((m+n)(d-1)+n)/(2mn)}, \tag{23.4.25}$$

$$V_\mathfrak{z}(x) \coloneqq V(x) + \sum_j \bar{\mathfrak{z}}_j (f_j(x) - f_{\infty,j}(\bar{\theta})). \tag{23.4.26}$$

(ii) On the other hand, under assumption $(23.4.23)_\pm$ $\mathsf{N}^{\mp}(\eta) = O(1)$.

Proof. Assume that $\bar{\theta} = (1, 0, \dots, 0)$, $x = \{x_1; x'\} = (x_1; x_2, \dots, x_d)$. Observe that outside of $\mathcal{X}(\eta) = \{x\colon |x'| \le c r^{1+m/n},\ 0 < x_1 = r \le c\eta^{1/(2m)}\}$ is a forbidden zone and one can prove easily that its contribution to the remainder is $O(1)$. On the other hand, contribution to the remainder of $\gamma'(r)$-partition element in $\mathcal{X}'(\tau)$ is $O(\gamma^{d-2})$ and the total remainder does not exceed $\int_{\{r \le c\eta^{1/(2m)}\}} {\gamma'}^{d-3}(r)\, dr$ which results in (23.4.24). □

The following problem is rather easy:

Problem 23.4.14. Derive similar results for the Dirac operator.

The following problem looks challenging:

Problem 23.4.15. Investigate what happens if $m + n \le 0$. Our methods provide only $\mathsf{N}^{\mp}(\eta) = O(\eta^{-1/(2n)})$. Probably methods of Section 12.3 could provide an answer.

23.4.3 Case $F \to \infty$ as $|x| \to \infty$

In this subsection we consider cases of $F \to \infty$ and $V \to 0$ as $|x| \to \infty$. In this case the Schrödinger operator does not have any essential spectrum at all and thus is not the subject of our analysis, while for the Schrödinger-Pauli and Dirac operators essential spectrum consists of just one point: 0 and $\pm M$

respectively (see Theorem 17.1.2 to find out which; if $d = 2$ it is determined by signs of F_{12} and ς). Again due to the specifics of the problem we can consider the multidimensional case without any modifications.

It turns out that for $d = 2$ the remainder estimate is as in the previous Subsection 23.4.1, while the magnitude of the principal part is larger but it is still given by the same formula).

Let for the Schrödinger-Pauli operator $\mathsf{N}^-(\eta)$ be a number of eigenvalues in $(-\epsilon, -\eta)$ and $\mathsf{N}^+(\eta)$ be a number of eigenvalues in (η, ϵ).

Theorem 23.4.16 [39]**.** *(i) Let X be a connected exterior domain with $\mathscr{C}^K$ boundary. Let conditions* (23.2.2), $(23.2.3)_{1-3}$ *and* $(23.4.7)_{\mp}$ *be fulfilled with scaling functions γ, ρ and ρ_1, $\rho \to 0$, $\rho_1 \to \infty$ and $\rho_1\gamma^2 \to \infty$ as $|x| \to \infty$. Assume that*

$$|\mathsf{F}^{-1}| \le c\rho_1^{-1} \qquad \text{for } |x| \ge c \tag{23.4.27}$$

and

(23.4.28) *For each $j \ne k$ <u>either</u> $f_j = f_k$ <u>or</u> $|f_j - f_k| \ge \epsilon\rho_1$ for all $|x| \ge c$.*

Then for the Schrödinger-Pauli operator

$$\mathsf{N}^{\mp}(\eta) = \mathcal{N}^{\mp}(\eta) + O(R) \tag{23.4.29}$$

where

$$\mathcal{N}^{\mp}(\eta) := (2\pi)^{-r} \int_{\{x:\, \mp V(x) \ge \eta\}} f_1 f_2 \cdots f_r \sqrt{g}\, dx \tag{23.4.30}$$

$$R = C \int_{\mathcal{Z}(\eta)} \rho_1^{r-1}\gamma^{-2}\, dx + C \int \rho_1^{r-s}\gamma^{-2s}\, dx \tag{23.4.31}$$

holds, $r = d/2$, $\mathcal{Z}(\eta)$ is $\epsilon\gamma$-vicinity[35] *of $\Sigma(\eta) = \{x:\, \mp V(x) = \eta\}$.*

(ii) Further, under assumption $(23.4.11)_{\mp}$ *0 ± 0 is not a limit point of the discrete spectrum.*

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[39] Cf. Theorem 23.4.3.

[40] Cf. Example 23.4.4.

Example 23.4.17[40]. (i) In the framework of Theorem 23.4.16 with $\gamma = \langle x\rangle$, $\rho = \langle x\rangle^{m}$, $\rho_1 = \langle x\rangle^{m_1}$, $m < 0 < m_1$ estimate (23.4.29) holds with

$$R = \begin{cases} |\log \eta| & \text{for } d = 2, \\ \eta^{(d-2)k/(2m)} \qquad k = (m_1+2)/2 & \text{for } d \ge 4 \end{cases} \tag{23.4.32}$$

and with $\mathcal{N}^{\mp}(\eta) = O(\eta^{(dk/(2m)})$. Further, $\mathcal{N}^{\mp}(\eta) \asymp \eta^{dk/(2m)}$ if condition $(23.4.11)_{\mp}$ is fulfilled in some non-empty cone.

(ii) Furthermore, if condition $(23.4.11)_{\mp}$ is fulfilled, then even for $d = 2$ $R = 1$.

Example 23.4.18. (i) Let conditions (23.2.2), $(23.2.3)_2$, (23.4.24), (23.4.28) be fulfilled with $\gamma = \langle x\rangle^{1-\beta}$, $\rho = \langle x\rangle^{m}$, $\rho_1 = \exp(b\langle x\rangle^{\beta})$, $m < 0$, $\beta > 0$.

Further, let conditions $(23.2.3)_{1,3}$ and $(23.4.7)_{\mp}$ be fulfilled with $\rho = \langle x\rangle^{m}$, $\gamma = \epsilon\langle x\rangle$. Then the following asymptotics holds:

$$\mathsf{N}^{\mp}(\eta) = \mathcal{N}^{\mp}(\eta) + O(\eta^{(d-2+2\beta)/(2m)}) \tag{23.4.33}$$

and

$$\log(\mathcal{N}^{\pm}(\eta)) = O(\eta^{\beta/(2m)}). \tag{23.4.34}$$

(ii) Let conditions (23.2.2), $(23.2.3)_{1-3}$ and $(23.4.7)_{\mp}$ be fulfilled with $\gamma = \langle x\rangle^{1-\alpha}$, $\rho = \exp(a\langle x\rangle^{\alpha})$, $\rho_1 = \langle x\rangle^{m_1}$, $a < 0$, $\alpha > 0$, $m_1 > 0$, $m_1 + 2(1-\alpha) > 0$. Then following asymptotics holds:

$$\mathsf{N}^{\mp}(\eta) = \mathcal{N}^{\mp}(\eta) + O(|\log \eta|^{(d-2+2\alpha)/\alpha}) \tag{23.4.35}$$

and

$$\mathcal{N}^{\pm}(\eta) = O(|\log \eta|^{(d+m_1)/\alpha}). \tag{23.4.36}$$

(iii) Moreover, if condition $(23.4.11)_{\mp}$ is fulfilled, then the remainder estimate (23.4.35) could be improved to

$$\mathsf{N}^{\mp}(\eta) = \mathcal{N}^{\mp}(\eta) + O(|\log \eta|^{(d-2+\alpha)/\alpha}) \tag{23.4.37}$$

(iv) Let conditions (23.2.2) and $(23.2.3)_{1-3}$ be fulfilled with $\gamma = \langle x\rangle^{1-\sigma}$, $\rho = \exp(a\langle x\rangle^{\alpha})$, $\rho_1 = \exp(b\langle x\rangle^{\beta})$, $a < 0 < b$, $\alpha > 0$, $\beta > 0$, $\sigma = \max(\alpha, \beta)$.

Further, let conditions $(23.2.3)_{1,3}$ and $(23.4.7)_{\mp}$ be fulfilled with $\gamma = \langle x\rangle^{1-\alpha}$, $\rho = \exp(a\langle x\rangle^{\alpha})$. Then the following asymptotics holds:

$$\mathsf{N}^{\pm}(\eta) = \mathcal{N}^{\mp}(\eta) + O(|\log\eta|^{(d-2+2\sigma)/\alpha}) \tag{23.4.38}$$

and

$$\log(\mathcal{N}^{\pm}(\eta)) = O(|\log\eta|^{\beta/\alpha}). \tag{23.4.39}$$

(v) Moreover, if condition $(23.4.11)_{\mp}$ is fulfilled, then the remainder estimate (23.4.38) could be improved to

$$\mathsf{N}^{\mp}(\eta) = \mathcal{N}^{\mp}(\eta) + O(|\log\eta|^{(d-2+2\sigma-\alpha)/\alpha}) \tag{23.4.40}$$

(vi) Furthermore, if condition $(23.4.11)_{\mp}$ is fulfilled in some non-empty cone then there is "$\asymp \cdot$" rather than "$= O(.)$" in (23.4.34), (23.4.36) and (23.4.39).

Problem 23.4.19. Again, one can hope to improve estimates (23.4.37) and (23.4.40) in the same way as specified in Problem 23.4.6.

We leave to the reader

Problem 23.4.20[41]. Consider the Dirac operator. In this case $\mathsf{N}^{-}(\eta)$ is a number of eigenvalues in $(\pm M - \epsilon, \pm M - \eta)$ and $\mathsf{N}^{+}(\eta)$ is a number of eigenvalues in $(\pm M + \eta, \pm M + \epsilon)$, $0 < \eta < \epsilon$ and $\pm M$ is a point of the essential spectrum.

We need to distinguish two cases

(a) $M > 0$ and potential $V \sim \rho^2$ at infinity.

(b) $M = 0$ and potential $V \sim \rho$ at infinity.

23.4.4 Case $F \to 0$ as $|x| \to \infty$

In this subsection we consider cases of $F \to 0$ and $V \to 0$ as $|x| \to \infty$. In this case the essential spectra of the Schrödinger and Schrödinger-Pauli operators are $[0, \infty)$; however, as $V = o(F)$ as $|x| \to \infty$ the Schrödinger operator has only a finite number of the negative eigenvalues and thus is not a subject of our analysis while the Schrödinger-Pauli operator is.

Further, the Dirac operator has its essential spectrum $(-\infty, -M]\cup[M, \infty)$ and we need to assume that $M > 0$.

[41] Cf. Problem 23.4.12.

It turns out that the remainder estimate is as in Subsection 23.4.1, while the magnitude of the principal part is smaller but it is still given by the same formula).

Theorem 23.4.21 [42]**.** *(i) Let X be a connected exterior domain with $\mathscr{C}^K$ boundary. Let conditions* (23.2.2)*,* $(23.2.3)_{1-3}$*,* (23.2.19)*,* (23.4.24) *and* (23.4.28) *and* $(23.4.7)_{\mp}$ *(with sign "$-$") be fulfilled with scaling functions γ, ρ and ρ_1, $\rho \to 0$, $\rho_1 \to 0$ and $\rho_1\gamma^2 \to \infty$, $\rho_1\rho^{-2} \to \infty$ as $|x| \to \infty$. Then for the Schrödinger-Pauli operator* (23.4.29)–(23.4.32) *holds*[43]*.*

(ii) Further, under assumption $(23.4.11)_{\mp}$ *(with sign "$-$") $0-0$ is not a limit point of the discrete spectrum.*

Example 23.4.22 [44]. (i) Let conditions of Theorem 23.4.21 be fulfilled with $\gamma = \langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$, $m < 0$, $\max(2m, -2) < m_1 < 0$.

Then estimate (23.4.29) [43] holds with R defined by (23.4.32) and with $\mathcal{N}^-(\eta) = O(\eta^{dk/(2m)})$. Further, $\mathcal{N}^-(\eta) \asymp \eta^{dk/(2m)}$ if condition $(23.4.11)_-$ is fulfilled in some non-empty cone.

(ii) Furthermore, if condition $(23.4.11)_-$ is fulfilled, then even for $d = 2$ $R = 1$.

Example 23.4.23. (i) Let conditions of Theorem 23.4.21 be fulfilled with $\gamma = \epsilon\langle x\rangle$, $\rho_1 = \langle x\rangle^{-2}|\log\langle x\rangle|^{\beta}$, $\beta > 0$. Let either $\rho = \langle x\rangle^m$ with $m < -1$ or $\rho = \langle x\rangle^{-1}|\log\langle x\rangle|^{\alpha}$ with $2\alpha < \beta$. Then the remainder estimate is $O(R)$ with R defined by (23.4.32) and $\mathcal{N}^-(\eta) = O(S)$ with

$$S = \begin{cases} |\log\eta|^{\beta+1} & d = 2, \\ |\eta^{(d-2)/(2m)}|\log\eta|^{\beta} & d \ge 4. \end{cases} \tag{23.4.41}$$

Further, $\mathcal{N}^-(\eta) \asymp S$ if condition $(23.4.11)_-$ is fulfilled in some non-empty cone.

(ii) Furthermore, if condition $(23.4.11)_-$ is fulfilled, then even for $d = 2$ $R = 1$.

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[42] Cf. Theorems 23.4.3 and 23.4.16.

[43] With the sign "$-$".

[44] Cf. Examples 23.4.4 and 23.4.17.

Example 23.4.24[45]. (i) Let conditions of Theorem 23.4.21 be fulfilled with $\gamma = \langle x\rangle^{1-\alpha}$, $\rho = \exp(a\langle x\rangle^{\alpha})$, $\rho_1 = \langle x\rangle^{m_1}$, $a < 0$, $\alpha > 0$, $m_1 < 0$, $m_1 + 2 - 2\alpha > 0$. Then the remainder estimate is $O(R)$ with R defined by (23.4.35) and (23.4.36) holds.

(ii) Further, if condition $(23.4.11)_{\mp}$ is fulfilled, then the remainder estimate (23.4.35) could be improved to (23.4.37).

(iii) Furthermore, if condition $(23.4.11)_{-}$ is fulfilled in some non-empty cone then there is $\asymp$ in (23.4.36).

Problem 23.4.25. Again, one can hope to improve estimates (23.4.37) and (23.4.40) in the same way as specified in Problem 23.4.6.

We also leave to the reader

Problem 23.4.26[46]. Consider in this framework the Dirac operator. In this case $\mathsf{N}^{-}(\eta)$ is a number of eigenvalues in $(\pm M - \epsilon, \pm M - \eta)$ and $\mathsf{N}^{+}(\eta)$ is a number of eigenvalues in $(\pm M + \eta, \pm M + \epsilon)$, $0 < \eta < \epsilon$ and $\pm M$ is a point of the accumulation of the discrete spectrum.

We need to assume that $M > 0$ and potential $V \sim \rho^2$ at infinity.

Consider now the case when condition $\rho^2 = o(\rho_1)$ as $|x| \to \infty$ is not fulfilled. Then the results will be similar to those of Section 23.3.

Example 23.4.27[47]. (i) Let X be a connected exterior domain[28] with $\mathscr{C}^K$ boundary and $d = 2$. Let conditions (23.2.2), $(23.2.3)_{1-3}$ and $(23.3.1)_1^{\#}$ be fulfilled with $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^{m}$, $\rho_1 = \langle x\rangle^{m_1}$, $-1 < m < 0$, $m - 1 < m_1 \le 2m$.

Assume that conditions (23.3.27) and (23.3.28) are fulfilled. Then for the Schrödinger operator asymptotics

$$\mathsf{N}^{-}(\eta) = \mathcal{N}^{-}(\eta) + O(\eta^{1-m_1/(2m)}) \tag{23.4.42}$$

holds as $\eta \to +0$ with

$$\mathcal{N}^{-}(\eta) = O(\eta^{(m+1)/m}). \tag{23.4.43}$$

Indeed, it follows from the arguments of Example 23.3.15; we need to take into account that $\{|x| \ge C\tau^{1/(2m)}\}$ is a forbidden zone.

[45] Cf. Example 23.4.18.

[46] Cf. Problem 23.4.12 and 23.4.20.

[47] Cf. Example 23.3.16.

(ii) Similar results hold in the full-rank even-dimensional case:

$$\mathsf{N}^-(\eta) = \mathcal{N}^-(\eta) + O(\eta^{1-m_1/(2m)+(d-2)(m+1)/(2m)}) \tag{23.4.44}$$

holds as $\eta \to +0$ with

$$\mathcal{N}^-(\eta) = O(\eta^{d(m+1)/(2m)}). \tag{23.4.45}$$

We leave to the reader:

Problem 23.4.28. Consider the case of $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^{-2}|\log x|^{\alpha}$, $\rho_1 = \langle x\rangle^{-2}|\log x|^{\beta}$, $2\alpha \ge \beta > \alpha$.

We also leave to the reader

Problem 23.4.29[48]. Consider in this framework the Dirac operator with $M > 0$. In this case both points $M - 0$ and $-M + 0$ could be limits of the discrete spectrum simultaneously.

23.5 Multiparameter Asymptotics

In this section we consider asymptotics with respect to three parameters μ, h and τ; here spectral parameter τ tends either to $\pm\infty$ or to the border of the essential spectrum or to $-\infty$ (for Schrödinger and Schrödinger-Pauli operators) or to the border of the spectrum. In two last cases presence of $h \to +0$ is crucial. We consider here only $d = 2$ and $h \ll 1$.

23.5.1 Asymptotics of Large Eigenvalues

In this subsection $\tau \to +\infty$ for the Schrödinger and Schrödinger-Pauli operators and $\tau \to \pm\infty$ for the Dirac operator. We consider the Schrödinger and Schrödinger-Pauli operators, leaving the Dirac operator to the reader.

Example 23.5.1. Assume first that $\psi \in \mathscr{C}_0^\infty$ and there are no singularities on $\mathsf{supp}(\psi)$. We consider

$$\mathsf{N}^-_\psi(\tau) = \int e(x,x,\tau)\psi(x)\,dx. \tag{23.5.1}$$

Then for scaling $A \mapsto \tau^{-1}A$ leads to $h \mapsto h\tau^{-1/2}$ and $\mu \to \mu\tau^{-1/2}$.

[48] Cf. Problems 23.4.12, 23.4.20 and 23.4.26.

(i) If $\mu \lesssim \tau^{1/2}$ then we can apply the standard theory with the "normal" magnetic field; we need to assume that $h \ll \tau^{1/2}$ and we need neither condition $d = 2$, nor $F \geq \epsilon_0$, nor $\partial X = \emptyset$; the principal part of the asymptotics has magnitude $h^{-d}\tau^{d/2}$ and the remainder estimate is $O\big(h^{1-d}\tau^{(d-1)/2}\big)$ which one can even improve to $o\big(h^{1-d}\tau^{(d-1)/2}\big)$ under proper non-periodicity assumption.

(ii) Let $\mu \gtrsim \tau^{1/2}$, $\mu h \lesssim \tau$. Then we can apply the standard theory with the "strong" magnetic field; we assume that $d = 2$, $\partial X = \emptyset$ and $F \geq \epsilon_0$. Then the principal part of the asymptotics has magnitude $h^{-2}\tau$ and under non-degeneracy assumptions

$$\nabla F = 0 \implies \det \mathsf{Hess}\, F \geq \epsilon \tag{23.5.2}$$

and

$$\nabla F = 0 \implies |\det \mathsf{Hess}\, F| \geq \epsilon \tag{23.5.3}$$

fulfilled on $\mathsf{supp}(\psi)$ the remainder estimate is $O(\mu^{-1}h^{-1}\tau)$ and $O(\mu^{-1}h^{-1}\tau(|\log(\mu\tau^{-1/2})| + 1)$ respectively[49]. Without non-degeneracy assumption the remainder estimate is $O(\mu h^{-1})$.

(iii) If $\mu \gtrsim \tau^{1/2}$, $\mu h \geq c\tau$ than $\mathsf{N}^-(\tau) = 0$ for the Schrödinger operator; for the Schrödinger-Pauli operator the principal part of the asymptotics has magnitude μh^{-1} and under non-degeneracy assumptions (23.5.2) and (23.5.3) the remainder estimate is $O(1)$ and $O(\log \mu)$ respectively (or better for τ belonging to the spectral gap).

Example 23.5.2[50]. Let X be a connected exterior domain with $\mathscr{C}^K$ boundary. Let conditions (23.2.2), $(23.2.3)_{1-3}$, $(23.3.1)^{\#}_{1,2}$, be fulfilled with $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$, $m_1 > 2m \geq 0$. Consider the Schrödinger operator and assume that

$$\tau \geq \mu h, \qquad \tau^{2m-m_1} \leq \epsilon(\mu h)^{2m}. \tag{23.5.4}_{1,2}$$

Then

$$\mathcal{N}^-(\tau, \mu, h) \asymp \tau^{(2+m_1)/m_1} h^{-2(1+m_1)/m_1} \mu^{-2/m_1}. \tag{23.5.5}$$

[49] In the letter case logarithmic factor could be removed by adding a correction term.
[50] Cf. Example 23.3.15.

(i) Further, if $\tau \gtrsim \mu^2$, then the zone of the strong magnetic field $\mu_{\mathsf{eff}} = \mu\langle x\rangle^{m_1+1}\tau^{-1/2} \geq C$ is contained in $\{x\colon |x| \geq c\}$ and here we have non-degeneracy condition fulfilled. Then the remainder estimate is $O(R)$ with

$$R = \tau^{(m_1+2)/2(m_1+1)}\mu^{-1/(m_1+1)}h^{-1}, \tag{23.5.6}$$

which could be improved under non-periodicity assumption; see Example 23.3.18.

(ii) On the other hand, if $\mu^2 \gg \tau$, then the contribution of the zone $\{x\colon |x| \geq c\}$ to the remainder is $O(\mu^{-1}h^{-1}\tau)$. The contribution of the zone $\{x\colon |x| \leq c\}$ to the remainder is $O(\mu^{-1}h^{-1}\tau)$ provided $X = \mathbb{R}^2$ and non-degeneracy assumption (23.5.2) is fulfilled (etc) and $O(\mu h^{-1})$ in the general case.

(iii) Let us replace $(23.5.4)_2$ by the opposite inequality, and assume $(23.3.3)^{\#}$. Then (23.5.5) is replaced by $\mathcal{N}^-(\tau,\mu,h) \asymp h^{-2}\tau^{(m+1)/m}$. Let us discuss R.

(a) If $\mu\tau^{(m_1+1-m)/(2m)} \lesssim 1$, then $\mu_{\mathsf{eff}} \lesssim 1$ as $|x| \lesssim \tau^{1/(2m)}$ and $R = h^{-1}\tau^{(m+1)/(2m)}$.

(b) If $\mu\tau^{(m_1+1-m)/(2m)} \gtrsim 1$, but $\mu^2\tau \lesssim 1$, then R is given by (23.5.6).

(c) If $\mu^2 \gg \tau$, then we are in the framework of (ii).

Example 23.5.3[51]. In the framework of Example 23.5.2 for the Schrödinger-Pauli operator under assumption $(23.3.3)^{\#}$ the remainder estimate is the same as in Statement (iii) while

$$\mathcal{N}^-(\tau,\mu,h) \asymp h^{-2}\tau^{(m+1)/m} + \mu h^{-1}\tau^{(m_1+2)/(2m)}. \tag{23.5.7}$$

We leave to the reader

Problem 23.5.4. Consider the Schrödinger and Schrödinger-Pauli operators

(i) In the same framework albeit with condition $m_1 > 2m$ replaced by $2m \geq m_1 \geq 0$. Assume that $(23.3.3)^{\#}$ is fulfilled.

Then magnitude of $\mathcal{N}^-(\tau,\mu,h)$ is described in Examples 23.5.2 and 23.5.3. Under proper non-degeneracy assumption (which we leave to the reader to formulate) derive the remainder estimate.

[51] Cf. Example 23.3.19.

(ii) In the same framework as in (i) albeit in with $m_1 < 0$ (magnetic field is stronger in the center but there is no singularity), in which case the center can become a classically forbidden zone.

(iii) With other types of the behaviour at infinity.

Problem 23.5.5. For the Dirac operators derive similar results as $\tau \to \pm\infty$.

23.5.2 Asymptotics of the Small Eigenvalues

In this subsection for the Schrödinger and Schrödinger-Pauli operators we consider asymptotics of eigenvalues tending to -0.

Example 23.5.6[52]. Let X be a connected exterior domain with $\mathscr{C}^K$ boundary. Let conditions (23.2.2), $(23.2.3)_{1-3}$, $(23.3.1)_1^{\#}$ be fulfilled with $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$, $-1 < m < 0$, $m_1 > m-1$.

Consider the Schrödinger operator and assume that

$$(23.5.8)_{1,2} \qquad 1 \ge \mu h, \qquad |\tau|^{2m-m_1} \le \epsilon(\mu h)^{2m}.$$

Then $\mathcal{N}^-(\tau,\mu,h) = O(h^{-2}|\tau|^{(m+1)/m})$ as $\tau \to -0$ with "$\asymp$" instead of "$= O$" if condition $(23.4.11)_{\mp}$ (with the sign "$-$") fulfilled in some non-empty cone.

Further, under non-degeneracy assumption (23.3.27) the contribution to the remainder of zone $\{x\colon |x| \ge c\}$ is $O(R)$ with

(i) If $\mu|\tau|^{(m_1+1-m)/(2m)} \lesssim 1$ then $R = h^{-1}|\tau|^{(m+1)/(2m)}$.

(ii) Let $\mu|\tau|^{(m_1+1-m)/(2m)} \gtrsim 1$. Then

(a) If $m_1 < 2m$ then $R = \mu^{-1}h^{-1}|\tau|^{(2m-m_1)/(2m)}$.

(b) If $m_1 = 2m$ then $R = \mu^{-1}h^{-1}|\log\mu|$.

(c) If $m_1 > 2m$ then $R = h^{-1}\mu^{(m+1)/(m_1+1-m)}$ for $\mu \lesssim 1$ and $R = \mu^{-1}h^{-1}$ for $\mu \gtrsim 1$.

Example 23.5.7[53]. In the framework of Example 23.5.6 for the Schrödinger-Pauli operator under assumption $(23.3.3)^{\#}$ the contribution to the remainder of the zone $\{x\colon |x| \ge c\}$ the same as in Example 23.5.6(ii) while

$$(23.5.9) \qquad \mathcal{N}^-(\tau,\mu,h) = O(h^{-2}|\tau|^{(m+1)/m} + \mu h^{-1}|\tau|^{(m_1+2)/(2m)})$$

[52] Cf. Example 23.4.22.

[53] Cf. Example 23.4.22.

$\asymp$ if condition $(23.4.11)_{\mp}$ (with the sign "$-$") fulfilled in some non-empty cone.

Problem 23.5.8. Consider the Schrödinger and Schrödinger-Pauli operators if

(i) If condition $(23.5.8)_1$ is violated (then there could be a forbidden zone in the center).

(ii) $m_1 \le m-1$.

(iii) With other types of the behaviour at infinity.

Problem 23.5.9. Consider the Schrödinger and Schrödinger-Pauli operators in the framework of Subsections 23.4.1 and 23.4.2 if

(i) $\mu h = 1$; then the essential spectrum does not change.

(ii) $\mu h \to \infty$; then only point 0 of the essential spectrum is preserved for the Schrödinger-Pauli operators, while others go to $+\infty$. Consider $\mathsf{N}^{\pm}(\eta)$ with $\eta \to 0$.

(iii) $\mu h \to 0$; then only point 0 of the essential spectrum is preserved for the Schrödinger-Pauli operators, while others move towards it. Consider $\mathsf{N}^{-}(\eta)$ with $\eta \to 0$ for both Schrödinger and Schrödinger-Pauli operators.

Problem 23.5.10. (i) Consider the Schrödinger-Pauli operators in the framework of Subsection 23.4.3.

(ii) Consider the Schrödinger and Schrödinger-Pauli operators in the framework of Subsection 23.4.4.

Problem 23.5.11. For the Dirac operators derive similar results as $M \ne 0$ and $\tau \to M-0$ and $-M+0$; or as $M=0$ albeit $m_1 \ge 0$.

23.5.3 Case of $\tau \to +0$

In this subsection $\tau \to +0$ for the Schrödinger and Schrödinger-Pauli operators and $\tau \to \pm M \pm 0$ for the Dirac operator. Consider the Schrödinger and Schrödinger-Pauli operators first.

Example 23.5.12[54]. Let $V > 0$ everywhere except $V(0) = 0$. Let conditions (23.2.2), $(23.2.3)_{1-3}$, $(23.3.1)^{\#}_{1,2}$, be fulfilled with $\gamma = \epsilon_0|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$, $m_1 > 2m \ge 0$. Consider the Schrödinger operator and assume that $\tau \to +0$.

(i) Let

$$(23.5.10)_{1,2} \qquad \mu \ll \tau^{(m_1+2)/2} h^{-(m_1+1)}, \qquad \tau^{2m-m_1} \le \epsilon(\mu h)^{2m}.$$

Then (23.5.5) holds.

Then the remainder estimate is $O(R)$ with defined by (23.5.6) which could be improved under non-periodicity assumption; see Example 23.3.18.

(ii) Let us replace $(23.5.10)_2$ by the opposite inequality, and assume $(23.3.3)^{\#}$. Then (23.5.5) is replaced by $\mathcal{N}^-(\tau, \mu, h) \asymp h^{-2}\tau^{(m+1)/m}$. Let us discuss R.

(a) If $\mu\tau^{(m_1+1-m)/(2m)} \lesssim 1$, then $\mu_{\text{eff}} \lesssim 1$ as $|x| \lesssim \tau^{1/(2m)}$ and $R = h^{-1}\tau^{(m+1)/(2m)}$.

(b) If $\mu\tau^{(m_1+1-m)/(2m)} \gtrsim 1$, but $\mu^2\tau \lesssim 1$, then R is given by (23.5.6).

Example 23.5.13[55]. In the framework of Example 23.5.12 for the Schrödinger-Pauli operator under assumption $(23.3.3)^{\#}$ the remainder estimate is the same as in Statement (ii) while (23.5.7) holds.

We leave to the reader

Problem 23.5.14[56]. Consider the Schrödinger and Schrödinger-Pauli operators

(i) In the same framework albeit with condition $m_1 > 2m$ replaced by $2m \ge m_1 \ge 0$. Assume that $(23.3.3)^{\#}$ is fulfilled.

Then magnitude of $\mathcal{N}^-(\tau, \mu, h)$ is described in Examples 23.5.12 and 23.5.13. Under proper non-degeneracy assumption (which we leave to the reader to formulate) derive the remainder estimate.

(ii) In the same framework as in (i) albeit in with $m_1 < 0$ (magnetic field is stronger in the center but there is no singularity), in which case the center can become a classically forbidden zone.

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[54] Cf. Example 23.5.2.
[55] Cf. Example 23.5.3.
[56] Cf. Problem 23.5.4.

(iii) With other types of the behaviour at infinity.

Problem 23.5.15. For the Dirac operators derive similar results as $\tau \to \pm(M+0)$.

23.5.4 Case of $\tau \to -\infty$

In this subsection for the Schrödinger and Schrödinger-Pauli operators we consider asymptotics with $\tau \to -\infty$.

In this subsection for the Schrödinger and Schrödinger-Pauli operators we consider asymptotics of eigenvalues tending to -0.

Example 23.5.16[57]. Let $X \ni 0$ and let conditions (23.2.2), $(23.2.3)_{1-3}$, $(23.3.1)_1^{\#}$ be fulfilled with $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$, $-1 < m < 0$, $m_1 > m-1$.

Consider the Schrödinger operator and assume that

$$(23.5.11)_{1,2} \qquad h \ll |\tau|^{(m+1)/(2m)}, \qquad |\tau|^{m_1-2m} \le \epsilon(\mu h)^{2m}.$$

Then $\mathcal{N}^-(\tau,\mu,h) = O(h^{-2}|\tau|^{(m+1)/m})$ as $\tau \to -0$ with $\asymp$ if condition $(23.4.11)_{\mp}$ (with the sign "$-$") fulfilled in some non-empty cone.

(i) If $\mu|\tau|^{(m_1+1-m)/(2m)} \lesssim 1$ then $R = h^{-1}|\tau|^{(m+1)/(2m)}$.

(ii) Let $\mu|\tau|^{(m_1+1-m)/(2m)} \gtrsim 1$. Then

(a) If $m_1 < 2m$ then $R = \mu^{-1}h^{-1}|\tau|^{(2m-m_1)/(2m)}$.

(b) If $m_1 = 2m$ then $R = \mu^{-1}h^{-1}|\log\mu|$.

(c) If $m_1 > 2m$ then $R = h^{-1}\mu^{(m+1)/(m_1+1-m)}$ for $\mu \lesssim 1$ and $R = \mu^{-1}h^{-1}$ for $\mu \gtrsim 1$.

Example 23.5.17[58]. In the framework of Example 23.5.16 for the Schrödinger-Pauli operator under assumption $(23.3.3)^{\#}$ the contribution to the remainder of the zone $\{x\colon |x| \ge c\}$ the same as in Example 23.5.16(ii) while

$$(23.5.12) \qquad \mathcal{N}^-(\tau,\mu,h) = O\bigl(h^{-2}|\tau|^{(m+1)/m} + \mu h^{-1}|\tau|^{(m_1+2)/(2m)}\bigr)$$

$\asymp$ if condition $(23.4.11)_{\mp}$ (with the sign "$-$") fulfilled in some non-empty cone.

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[57] Cf. Example 23.4.22.

[58] Cf. Example 23.4.22.

Problem 23.5.18. Consider the Schrödinger and Schrödinger-Pauli operators if

(i) $m_1 \le m-1$.

(ii) With other types of the behaviour at 0.

23.A Appendices

23.A.1 Self-Adjointness of the Dirac Operator

The Dirac operators treated in this Chapter are surely self-adjoint in the case of an exterior domain with the singularity at infinity. However, the same fact should be proven for an interior domain with singular points. We consider a single singular point at 0.

Theorem 23.A.1. *Let $X \subset \mathbb{R}^2/(\bar\gamma_1\mathbb{Z}\times\bar\gamma_2\mathbb{Z})$ $(0<\bar\gamma_j\le\infty)$ be an open domain. Let conditions* (23.2.2), $(23.2.3)_{1-3}$, (23.2.13) *be fulfilled. Further, let*

$$(23.A.1)_{1,2} \qquad F\ge\epsilon\rho_1 \qquad |V|\ge\epsilon\rho as\ \ |x|\le\epsilon.$$

Let us assume that there exists a neighborhood of ∂X, denoted by Y, such that for every $r>0$ the inequalities

$$(23.A.2)_{1-3} \qquad \rho_1\ge\varepsilon\rho^2,\quad (\rho\gamma)^s\ge\varepsilon\rho_1,\quad \rho\ge 1,$$

$$(23.A.3) \qquad \bigl|V+\sqrt{2j\mu hF}\bigr|\ge\varepsilon\sqrt{\rho}_1-\varepsilon^{-1} \qquad \forall j\in\mathbb{Z}^+\setminus 0,$$

$$(23.A.4) \qquad |V|\ge\varepsilon\rho-\varepsilon^{-1}$$

are fulfilled on $Y\cap X\cap\{|x|\le r\}$ with appropriate $\varepsilon=\varepsilon(r)>0$.

Then for $\mu>0$, $h>0$ the operator A with domain $\mathfrak{D}(A)=\mathscr{C}_0^1(X,\mathbb{H})$ is essentially self-adjoint in $\mathscr{L}^2(X,\mathbb{H})$.

Proof. Let us consider the adjoint operator A^*. This operator is defined by the same formula with $\mathfrak{D}(A^*)=\{u\in\mathscr{L}^2, Au\in\mathscr{L}^2\}$ with Au calculated as a distribution. We should prove that $\mathsf{Ker}(A^*\pm iI)=0$ for both signs. So, here and below let $u\in\mathscr{L}^2$ and $(A^*\pm iI)u=0$ for some sign.

The microlocal canonical form of Section 17.2 yields the inequality

$$\|\rho v\| \le M\bigl(\|Av\| + \|v\| + \|\gamma^{-1}v\|\bigr) \qquad \forall v \in \mathscr{C}_0^1\bigl(B(y, \frac{1}{2}\gamma(y))\bigr) \qquad \forall y \in X_r = X \cap \{|x| \le r\} \tag{23.A.5}$$

with a constant $M = M(r)$; all the constants now depend on μ and h.

Let us prove that $(\rho\gamma)^n u \in \mathscr{L}^2(X_r)$ for every $r > 0$ by induction on n. This is true for $n = 0$ by the assumption $u \in \mathscr{L}^2$.

Let $(\rho\gamma)^n u \in \mathscr{L}^2(X_r)$ for some n; then $v = \gamma(\rho\gamma)^n u$ also belongs to $\mathscr{L}^2(X_r)$ and

$$Av = [A, \gamma(\rho\gamma)^n]u \mp iv.$$

One can easily see that $[A, \gamma(\rho\gamma)^n]$ is a matrix-valued function the matrix norm of which does not exceed $M(\rho\gamma)^n$. Therefore, taking a γ-admissible partition of unity in a neighborhood of X_r we see that $\sum_\nu \|A\psi_\nu v\|^2 < \infty$ and therefore (23.A.5) yields that $\sum_\nu \|\rho\psi_\nu v\|^2 < \infty$. Therefore $\rho v = (\rho\gamma)^{n+1}u \in \mathscr{L}^2(X_r)$ and the induction step is complete.

Thus we have proven that $(\rho\gamma)^n u \in \mathscr{L}^2(X_r)$ for every $r > 0$. Then $\rho_1 u \in \mathscr{L}^2(X_r)$ by $(23.A.2)_2$. The ellipticity of A yields that $D_j u \in \mathscr{L}^2(X_r)$. Therefore, if $\psi \in \mathscr{C}_0^1\bigl(\mathbb{R}^2/(\bar\gamma_1\mathbb{Z} \times \bar\gamma_2\mathbb{Z})\bigr)$ then $\psi u \in \mathfrak{D}(\bar A)$ where $\bar A$ is the closure of A. Moreover,

$$(\bar A \pm iI)\psi u = [\bar A, \psi]u.$$

Calculating the real part of the inner product with $i\psi u$ we obtain the inequality

$$\|\psi u\| \le \max|\nabla\psi^2| \cdot \|u\|^2.$$

Let us take $\psi = \psi^0(x/r)$ where $\psi^0 \in \mathscr{C}_0^1$ is a fixed function equal to 1 in a neighborhood of 0. Then for $r \to +\infty$ we see that the left-hand expression of this inequality tends to $\|u\|^2$ and the right-hand expression tends to 0. Therefore $u = 0$. □

Theorem 23.A.2. *Let all of the conditions of Theorem 23.A.1 excluding condition* $(23.A.2)_{1-3}$ *be fulfilled with* $\gamma = \epsilon_0|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$, $m_1 \le 2m$, $m_1 < -2$. *Then the Dirac operator has a self-adjoint extension for* $\mu > 0$, $h > 0$.

Proof. For $m < -1$ condition $(23.A.2)_{1-3}$ is also fulfilled and therefore the operator A is essentially self-adjoint. So, let us treat the case $m \ge -1$. Let us consider the Dirac operator A_D with the potential $V_t = V + tW$ where $t > 0$, W is a potential which is regular away from $\{x = 0\}$, and $W = \pm|x|^{m'}$ in a neighborhood of $x = 0$ with $-1 > m' > \frac{1}{2}m_1$. This operator is essentially self-adjoint by Theorem 23.A.1. For $m < 0$ let us choose the sign of W coinciding with the sign of V on a neighborhood of 0 (condition $(23.A.1)_2$ yields that this is possible).

For $m \ge 0$ let us choose an appropriate interval $[\tau_1, \tau_2]$ with $\tau_1 < \tau_2$. Then applying the results of Section 23.3 we see that the number of eigenvalues of the operator $\bar{A}_t$ lying in the interval $[\tau_1, \tau_2]$ is bounded uniformly with respect to $t > 0$. Then there exists a sequence $t_k \to +0$ such that there exist $\tau_1' < \tau_2'$ which do not depend on k and such that $[\tau_1', \tau_2'] \cap \mathsf{Spec}(\bar{A}_t) = \emptyset$ for $t = t_k$.

Let $\tau = \frac{1}{2}(\tau' + \tau_2')$; then all the operators $(\bar{A}_t - \tau)^{-1}$ are uniformly bounded and $\|A_t u\| \ge \epsilon\|u\|$ $\forall u \in \mathfrak{D}(A_t)$ for $t = t_k$. Then the same is true for A. Therefore A has a self-adjoint extension $\tilde{A}$ satisfying the same estimate.

Comments

In addition to papers, mentioned in Remark 23.4.9 I would like also mention S. Solnyshkin [1], A. Sobolev [1–4], Y. Colin de Verdiere [8, 9], A. Morame [3, 4], A. Morame &J. Nourrigat [1] and H. Tamura [6–9], M. Birman & G. Raikov [1], G. Raikov [2–9].

Chapter 24

Eigenvalue asymptotics. 3D Case

24.1 Introduction

In this chapter we obtain eigenvalue asymptotics for 3D-Schrödinger, Schrödinger-Pauli and Dirac operators in the situations in which the role of the magnetic field is important. We have seen in Chapters 13 and 17 that these operators are essentially different and they also differ significantly from the corresponding 2D-operators which we considered in the previous Chapter 23.

Now we *usually* find ourselves in the situation much closer to Chapter 11 than Chapter 23 was. Indeed, our local asymptotics now are the same as without magnetic field, under very week non-degeneracy assumptions. We also allow boundaries and a singular points, finite or infinite, belonging to the boundary.

We start from Section 24.2 in which we consider the case when the spectral parameter is fixed ($\tau = \mathsf{const}$) and study asymptotics with respect to μ, h exactly like in Section 11.1 we considered asymptotics with respect to h. However, since now we have two parameters, we need to consider an interplay between them: while always $h \to +0$, we cover $\mu \to +0$, μ remains disjoint from 0 and ∞ and $\mu \to \infty$, which in turn splits into subcases $\mu h \to 0$, μh remains disjoint from 0 and ∞ and $\mu h \to \infty$.

In Section 24.3 we consider asymptotics with $\mu = h = 1$ and with τ tending to $+\infty$ for the Schrödinger and Schrödinger-Pauli operators and to $\pm\infty$ for the Dirac operator. We consider bounded domains with the

singularity at some point and unbounded domains with the singularity at infinity.

In Section 24.4 we consider asymptotics with the singularity at infinity and $\mu = h = 1$ and with τ tending to $+0$ for the Schrödinger and Schrödinger-Pauli operators and to $\pm(M - 0)$ for the Dirac operator.

It includes the most interesting case (see Subsection 23.4.2) when magnetic field is either constant or stabilizes fast at infinity and potential fast decays at infinity in the direction of magnetic field. In this case we consider a reduced one-dimensional operator which has just one negative eigenvalue $\Lambda(x')$ and it turns out that the asymptotics of the eigenvalues tending to the bottom of the continuous spectrum for 3D-operator coinsides with the asymptotics obtained in Subsection 23.4.1 for 2D-operator with the potential $\Lambda(x')$. In contrast to the rest of the section we consider multidimensional case as well.

In Section 24.5 we consider asymptotics with respect to μ, h, τ, like in Sections 11.7 and 23.5 again with significant differences mentioned above.

Finally, in Appendices 24.A.1 and 24.A.2 we investigate 1D-Schrödinger operators and in Appendix 24.A.3 we construct examples of vector potentials with different rates of growth of the magnetic field at infinity.

24.2 Asymptotics with Fixed Spectral Parameter

In this section we consider asymptotics with a fixed spectral parameter for 3-dimensional magnetic Schrödinger, Schrödinger-Pauli and Dirac operators and discuss some of the generalizations[1)].

As in Chapters 9–12 and 23 we will introduce a semiclassical zone and a singular zone, where $\rho\gamma \ge h$ and $\rho\gamma \le h$ respectively. In the semiclassical zone we apply asymptotics of Chapters 13, 18 and 20–in the multidimensional case. In the singular zone we need to apply estimates for a number of eigenvalues; usually it would be sufficient to use non-magnetic estimate[2)] for number of eigenvalues which trivially follows from standard one but if needed one can use more delicate estimates.

[1)] Mainly to higher dimensions with maximal-rank magnetic field.

[2)] With V modified accordingly; for example, for the Schrödinger and Schrödinger-Pauli operators V_- is replaced by $C\big((1-\epsilon)V - C_\epsilon \mu^2 |\vec{V}|^2\big)_-$.

24.2.1 Schrödinger Operator

Estimates of the Spectrum

Consider first the Schrödinger operator (13.2.1) where g^{jk}, V_j, V satisfy (13.2.2) and (13.2.4) i.e.

$$\epsilon|\xi|^2 \le \sum_{j,k} g^{jk}\xi_j\xi_k \le c|\xi|^2 \qquad \forall \xi \in \mathbb{R}^d. \tag{24.2.1}$$

Without any loss of the generality we can fix $\tau = 0$ and then in the important function $V_{\mathsf{eff}}F_{\mathsf{eff}}^{-1}$ the parameters μ and h enter as factors. Thus, we treat the operator (13.2.1) assuming that it is self-adjoint.

We make assumptions the same assumptions 23.2.3 i.e.

$$|D^\alpha g^{jk}| \le c\gamma^{-|\alpha|}, \quad |D^\alpha F_{jk}| \le c\rho_1\gamma^{-|\alpha|}, \quad |D^\alpha V| \le c\rho^2\gamma^{-|\alpha|} \tag{24.2.2}$$

where scaling function $\gamma(x)$ and weight functions $\rho(x), \rho_1(x)$ satisfy the standard assumptions $(9.1.6)_1$ and $(9.1.6)_2$. Then

$$\mu_{\mathsf{eff}} = \mu\rho_1\gamma\rho^{-1}, \qquad h_{\mathsf{eff}} = h\rho^{-1}\gamma^{-1}. \tag{24.2.3}$$

Let us introduce a *semiclassical zone* $X' = \{x\colon \rho\gamma \ge h\}$ and a *singular zone* $X'' = \{x\colon \rho\gamma \le 2h\}$ by (23.2.5) and (23.2.6) respectively.

Further, let us introduce two other overlapping zones $X_1' = \{x \in X_{\mathsf{scl}}\colon \mu\rho_1 \le 2c\rho\gamma^{-1}\}$ and $X_2' = \{x \in X'\colon \mu\rho_1 \ge c\rho\gamma^{-1}\}$ where the magnetic field $\mu_{\mathsf{eff}} = \mu\rho_1\rho^{-1}\gamma$ is *normal* ($\mu_{\mathsf{eff}} \le 2c$) and where it is *strong* ($\mu_{\mathsf{eff}} \ge c$) respectively (see (23.2.7) and (23.2.8). We also assume that

$$|F| \ge \epsilon\rho_1 \qquad \text{in} \quad X_2' \tag{24.2.4}$$

where F_{jk}, F^j and F are the tensor, vector (as $d = 3$) and scalar intensities of the magnetic field respectively. Moreover, let us assume that

$$u\big|_{\partial X \cap B\left(x,\gamma(x)\right)} = 0 \qquad \forall x \in X_2' \quad \forall u \in \mathfrak{D}(A); \tag{24.2.5}$$

we do not need $(23.2.10)_1$ since in 3D the boundary does not lead to the deterioration of the remainder estimate. We define $X_-' = \{x \in X_2'\colon V + \mu hF \ge \epsilon\rho^2\}$ and $X_{2+}' = \{x \in X'\colon V + \mu hF \le 2\epsilon\rho^2\}$ by (23.2.10) and (23.2.11) respectively.

Finally, let the standard boundary regularity condition be fulfilled:

(24.2.6) For every $y \in X$, $\partial X \cap B(y, \gamma(y)) = \{x_k = \phi_k(x_{\hat{k}})\}$ with

$$|D^\alpha \phi_k| \le c\gamma^{-|\alpha|}$$

and $k = k(y)$.

Recall that according to Chapter 13 the contribution of the partition element $\psi \in \mathscr{C}_0^K(B(y, \frac{1}{2}\gamma(y))$ to the principal part of asymptotics is

$$\mathcal{N}^-(\mu, h) = \mathcal{N}^{\mathsf{MW}-}(\mu, h) \coloneqq h^{-3} \int \mathcal{N}^{\mathsf{MW}}(x, \mu h)\psi(x)\, dx \tag{24.2.7}$$

with $\mathcal{N}^{\mathsf{MW}}(x, \mu h)$ given by (13.2.9) with $d = 3$.

On the other hand, its contribution to the remainder does not exceed $Ch^{-2}\rho^2\gamma^2$ if $\mu\rho_1\gamma \le c\rho$ or $\mu\rho_1\gamma \ge c\rho$ but $\mu h\rho_1 \le \rho^2$, $y \in X'_+$ and non-degeneracy assumption

$$\sum_{\alpha:|\alpha|\le k} |\nabla^\alpha (VF^{-1} + (2n+1)\mu h)|\gamma^{|\alpha|} \ge \epsilon\rho^2\rho_1^{-1} \qquad \forall n \in \mathbb{Z}^+ \tag{24.2.8}$$

is fulfilled[3], and it does not exceed $C(\mu^{-s}\rho_1^{-s}\gamma^{-2s})$ if $C\mu\rho_1\gamma \ge \rho$, $\mu h\rho_1 \le \rho^2$, $y \in X'_{2-}$.

Then we get estimate of N^- from below by the magnetic Weyl approximation $\mathcal{N}^-(\mu, h)$ minus corresponding remainder, and also from above by magnetic Weyl approximation plus corresponding remainder, provided $X = X'$ (so, there is no singular zone $X'' = \emptyset$):

$$h^{-d} \int_{X'} \mathcal{N}^{\mathsf{MW}}(x, \mu h)\, dx - CR_1 \le \mathsf{N}^-(0) \le h^{-d} \int_{X'} \mathcal{N}^{\mathsf{MW}}(x, \mu h)\, dx + CR_1 + C'R_2 \tag{24.2.9}$$

with

$$R_1 = \mu^{-1}h^{1-d} \int_{X'_+} \rho^{d-1}\gamma^{-1}\, dx, \tag{24.2.10}$$

$$R_2 = \mu h^{s-d} \int_{X'_-} \rho_1\rho^{d-s-1}\gamma^{1-s}\, dx \tag{24.2.11}$$

[3] It does not exceed the same expression plus $\mu h^{-1-\delta}\rho_1\rho^\delta\gamma^{2+\delta}$ in the general case; here $\delta > 0$ is arbitrarily small but K in $(24.2.2)_{1-3}$ depends on it.

provided

$$\mu\rho_1\gamma \ge \rho \tag{24.2.12}$$

where the latter condition could be assumed without any loss of the generality, C' depens also on s and ϵ [4].

We leave to the reader the following not very challenging set of problems:

Problem 24.2.1. (i) Consider in the current framework Problems 23.2.1(i), (ii), (iv).

(ii) Using results of Subsubsection *13.8.2.1 Case* $d = 3$ replace condition $(24.2.8)_k$ by $(13.8.19)_m$.

In what follows $h \to +0$ and the semiclassical zone X' expands to X while μ is either bounded (then we can assume that the zone of the strong magnetic field X_2' is fixed) or tends to ∞ (then X_2' expands to X). We assume that all conditions of the previous subsection are fulfilled with $\mu = h = 1$ but we will assume them fulfilled in the corresponding zones.

The other important question is whether $\mu h \to 0$, remains bounded and disjoint from 0 or tends to ∞.

Finally, we should consider the singular zone X''. In order to avoid this task we assume initially that

$$\rho_1\gamma^2 + \rho\gamma \ge \epsilon. \tag{24.2.13}$$

Power Singularities

Example 24.2.2 [5]. (i) Let X be a compact domain, $0 \in \bar{X}$ and let conditions $(24.2.8)_k$, (24.2.1), $(24.2.2)_{1-3}$, (24.2.4) and (24.2.6) be fulfilled with $\gamma = \epsilon_0|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$, $m_1 < 2m \le -2$ [6),7)].

Then, for the Schrödinger operator the following asymptotic holds as $h \to +0$, μh bounded:

$$\mathsf{N}^-(\mu, h) = \mathcal{N}^-(\mu, h) + \begin{cases} O(h^{-2}(\mu h)^{2(m+1)/(2m-m_1)}) & m < -1, \\ O(h^{-2}(|\log \mu h| + 1)) & m = -1 \end{cases} \tag{24.2.14}$$

---

[4] Cf. (23.2.18)–(23.2.21).

[5] Cf. Example 23.2.3.

[6] Such potential (V_1, V_2, V_3) exists, see Appendix 24.A.3.

[7] The non-degeneracy condition $(24.2.8)_2$ is fulfilled in the vicinity of 0 if V, F stabilize as $x \to 0$ to V^0, F^0 positively homogeneous of degrees $2m, m_1$ respectively.

with

$$\mathcal{N}^-(\mu,h)=\begin{cases} O(h^{-3}(\mu h)^{2(m+1)/(2m-m_1)}) & m<-1,\\ O(h^{-3}(|\log \mu h|+1)) & m=-1.\end{cases} \tag{24.2.15}$$

Furthermore, one can replace in (24.2.15) "$=O$" with "$\asymp$" if

$$V\le -\epsilon\rho^2 \qquad \text{in} \quad \Gamma\cap\{|x|\le\epsilon\}\subset X \tag{24.2.16}$$

where Γ is an open non-empty sector (cone) with vertex at 0, and $\mu h\le t$ with small enough $t>0$.

(ii) Let X be unbounded domain and let conditions $(24.2.8)_k$, (24.2.1), $(24.2.2)_{1-3}$, (24.2.4) and (24.2.6) be fulfilled with $\gamma=\epsilon_0\langle x\rangle$, $\rho\langle x\rangle^m$, $\rho_1=\langle x\rangle^{m_1}$, $m_1>2m\ge -2$ [6),6)].

Then for the Schrödinger operator asymptotics (24.2.14) holds as $h\to+0$, μh bounded.

Further, (24.2.15) holds and one can replace "$=O$" by "$\asymp$" if

$$V\le -\epsilon\rho^2 \qquad \text{in} \quad \Gamma\cap\{|x|\ge c\}\subset X \tag{24.2.16)$^\#$}$$

where Γ is an open non-empty sector (cone) with vertex at 0, and $\mu h\le t$ with small enough $t>0$.

Example 24.2.3[8)]. (i) Assume now that $m>-1$ while all other assumptions of Example 24.2.2(i) are fulfilled. Then $\mathcal{N}^-(\mu,h)=O(h^{-3})$. Let us calculate the remainder estimate and prove that

$$\mathsf{N}^-(\mu,h)=\mathcal{N}^-(\mu,h)+O(h^{-2}). \tag{24.2.17}$$

Obviously, the contribution of the regular zone $X'=\{x\colon |x|\ge r^*=h^{1/(m+1)}\}$ is $O(h^{-2})$ and we need to consider the contribution of the singular zone $X''=\{x\colon |x|\le r_2\}$.

(a) Assume first that $m_1\ge m-1$. If $\mu r_1^{m_1+1-m}\le c$, then by virtue of CLR we estimate contribution of X''

$$Ch^{-3}\int_{X''}\bigl(r^{3m}+\mu^3 r^{3(m_1+1)}\bigr)\,dx\asymp h^{-3}\bigl(r_1^{3m+3}+\mu^3 r_1^{3m_1+6}\bigr)=O(1). \tag{24.2.18}$$

[8)] Cf. Example 23.2.4.

Indeed, we can take $\vec{V} = O(r^{m_1+1})$.

On the other hand, if $\mu r_1^{m_1+1-m} \ge c$, the same estimate would work for $X''' = \{x\colon |x| \le r_{**} = \mu^{-1/(m_1+1-m)}\}$ while contribution of $X'' \setminus X'''$ does not exceed due to Chapter 13

$$C \int_{X''\setminus X'''} (\mu r^{m_1+1-m})^{-s} r^{-3}\, dx = O(|\log h|) \tag{24.2.19}$$

(actually it is $O(1)$ if $m_1 > m-1$).

(b) Let now $m_1 < m-1$. If $\mu r_1^{m_1+1-m} \ge c$, then we can apply estimate (24.2.19) in the whole zone X''.

On the other hand, if $\mu r_1^{m_1+1-m} \le c$, then we can apply estimate (24.2.19) in the zone X''' and estimate (24.2.18) in $X'' \setminus X''$.

(ii) Assume now that $m < -1$ while all other assumptions of Example 24.2.2(ii) are fulfilled. Again, considering cases

(a) $m_1 \le m-1$ and

(b) $m_1 > m-1$

we arrive to the asymptotics (24.2.17) and $\mathcal{N}^-(\mu, h) = O(h^{-3})$.

Consider now fast increasing μ so that $\mu h \to \infty$. We will get non-trivial results only when domain defined by $\mu_{\text{eff}} h_{\text{eff}} \le C_0$ shrinks but remains non-empty which happens only if $m_1 > 2m$, $m_1 < 2m$ in the frameworks of Example 24.2.3(i) and (ii) respectively.

Example 24.2.4[9]. (i) In the framework of Example 24.2.3(i) with $m_1 > 2m$ consider $\mu h \to \infty$. Then the allowed domain is

$$\{x\colon |x| \lesssim r_2 = (\mu h)^{-1/(m_1-2m)}\} \tag{24.2.20}$$

and we have $r_1 \le r_2$ if $\mu \lesssim h^{-(m_1+1-m)/(m+1)}$ while for $\mu \gtrsim h^{-(m_1+1-m)/(m+1)}$ inequalities go in the opposite direction.

Therefore as $h \to +0$, $ch^{-1} \le \mu \le h^{-(m_1+1-m)/(m+1)}$ asymptotics

$$\mathsf{N}^-(\mu, h) = \mathcal{N}^-(\mu, h) + O(\mu^{-2(m+1)/(m_1-2m)} h^{-2(m_1+1-m)/(m_1-2m)}) \tag{24.2.21}$$

holds and one can see easily that $\mathcal{N}^-(\mu, h) \asymp h^{-2} r_2^{2m+2}$:

$$\mathcal{N}^-(\mu, h) \asymp \mu^{-3(m+1)/(m_1-2m)} h^{-3(m_1+1-m)/(m_1-2m)}. \tag{24.2.22}$$

[9] Cf. Example 23.2.5.

(ii) Similarly, in the framework of Example 24.2.3(ii) with $m_1 < 2m$ asymptotics (24.2.21) and (24.2.22) hold as $h \to +0$, $ch^{-1} \le \mu \le h^{-(m_1+1-m)/(m+1)}$.

Improved Remainder Estimates

Let us improve remainder estimates under certain non-periodicity-type assumptions.

Example 24.2.5[10]. (i) In the case of the singularity at 0 with $m > -1$ the contribution to the remainder of the zone $\{x\colon |x| \le \varepsilon\}$ does not exceed σh^{-1} with $\sigma = \sigma(\varepsilon) \to 0$ as $\varepsilon \to +0$. Then the standard arguments imply that under the standard non-periodicity assumption for Hamiltonian billiards[11] with the Hamiltonian

$$a(x,\xi,\mu_0) = \sum_{j,k} g^{jk}(\xi_j - \mu_0 V_j)(\xi_k - \mu_0 V_k) + V(x) \tag{24.2.23}$$

the improved asymptotics

$$\mathsf{N}^-(\mu,h) = \mathcal{N}^-(\mu,h) + \kappa_1 h^{-2} + o(h^{-2}) \tag{24.2.24}$$

holds as $h \to +0$, $\mu \to \mu_0$ where $\kappa_1 h^{-2}$ is the contribution of ∂X calculated as $\mu = \mu_0$.

(ii) Similarly in the case of the singularity at infinity with $m < -1$ under the standard non-periodicity assumption for Hamiltonian billiards[11] with the Hamiltonian (24.2.23) asymptotics (24.2.24) holds as $h \to +0$, $\mu \to \mu_0$.

(iii) In the case of the singularity at 0 with $m < -1$ (and thus $m_1 < 2m$) and $h \to 0$, $\mu h \to 0$ the contributions to the remainder of the zone $\{x\colon |x| \ge \varepsilon^{-1} r_2\}$ with $r_2 = (\mu h)^{-1/(2m-m_1)}$ do not exceed $\sigma h^{-2} r_2^{m+1}$ with $\sigma = \sigma(\varepsilon) \to 0$ as $\varepsilon \to +0$. After scaling $x \mapsto x r_2^{-1}$ etc the magnetic field in the zone $\{x\colon |x| \le \varepsilon^{-1} r_2\}$ becomes strong.

Assume that g^{jk}, V_j, V stabilize to positively homogeneous of degrees $0, m_1+1, 2m$ functions g^{jk0}, V_j^0, V^0 as $x \to 0$: namely, assume that $(23.2.39)_{1-3}$ are fulfilled. Then the standard arguments imply that under the standard non-periodicity assumption for 1-dimensional Hamiltonian

[10] Cf. Example 23.2.7.

[11] On the energy level 0.

movement along magnetic lines (see Subsection 13.7.2) the improved asymptotics

$$\mathsf{N}^-(\mu,h)=\mathcal{N}(\mu,h)+o\bigl(h^{-2}(\mu h)^{-2(m+1)/(m_1-2m)}\bigr) \tag{24.2.25}$$

holds as $h\to+0$, $\mu h\to 0$. Furthermore, if 0 is not an inner point and domain stabilizes to the conical X^0 near it, we need to consider movement along magnetic lines with reflection at ∂X^0 and include into asymptotics the term $\kappa_1 h^{-2}(\mu h)^{-2(m+1)/(m_1-2m)}$ which comes out from the contribution of ∂X^0:

$$\mathsf{N}^-(\mu,h)=\mathcal{N}(\mu,h)+\kappa_1 h^{-2}(\mu h)^{-2(m+1)/(m_1-2m)}+ o\bigl(h^{-2}(\mu h)^{-2(m+1)/(m_1-2m)}\bigr) \tag{24.2.26}$$

(iv) In the case of the singularity at $0\in X$ with $m_1<2m=-2$ the main contribution to the remainder comes from the zone $\{x\colon \varepsilon^{-1}r_2\le |x|\le \varepsilon\}$. After rescaling magnetic field in this zone is strong.

Let stabilization conditions $(23.2.39)_{1,3}$ be fulfilled. Then the standard arguments imply that under the same non-periodicity assumption as in (iii) the improved asymptotics

$$\mathsf{N}^-(\mu,h)=\mathcal{N}(\mu,h)+o(h^{-2}|\log(\mu h)|) \tag{24.2.27}$$

holds as $h\to+0$, $\mu h\to 0$. Furthermore, if 0 is not an inner point and domain near it stabilizes to the conical X^0 near it,

$$\mathsf{N}^-(\mu,h)=\mathcal{N}(\mu,h)+\kappa_1 h^{-2}\log(\mu h)+o(h^{-2}|\log(\mu h)|); \tag{24.2.28}$$

again, an extra term comes out from the contribution of ∂X^0.

Remark 24.2.6. Statements, similar to (iii), (iv) but with the singularity at infinity seem to have impossible conditions.

Power Singularities. II

Let us modify our arguments for the case $\rho_3<1$. Namely, in addition to $(24.2.2)_{1-3}$ we assume that

$$|D^\alpha g^{jk}|\le c\rho_2\gamma^{-|\alpha|},\qquad |D^\alpha F_{jk}|\le c\rho_2\rho_1\gamma^{-|\alpha|}, \tag{24.2.29}_{1,2}$$

$$|D^\alpha \frac{V}{F}|\le c\rho_3\rho^2\rho_1^{-1}\gamma^{-|\alpha|}\qquad \forall \alpha\colon 1\le|\alpha|\le K \tag{24.2.29}_{3}$$

with $\rho_3 \le \rho_2 \le 1$ in the corresponding regions where $\rho, \rho_1, \gamma, \rho_3$ are scaling functions.

Recall that in Chapter 13 operator was reduced to the canonical form with the term, considered to be negligible, of magnitude $\rho_2\mu_{\mathsf{eff}}^{-2N}$. In this case impose non-degeneracy assumptions

$$(24.2.30)^*_k \qquad \sum_{\alpha:|\alpha|\le k} |\nabla^\alpha (v^* + (2n+1)\mu h)|\gamma^{|\alpha|} \ge \epsilon\rho_3\rho^2 \qquad \forall n \in \mathbb{Z}^+$$

and

$$\rho_3 \ge C_0\rho_2(\mu\rho_1\gamma\rho^{-1})^{-N} \tag{24.2.31}$$

where v^* is what this reduction transforms VF^{-1} to and (24.2.31) means that "negligible" terms do not spoil $(24.2.30)^*_k$.

Then according to Chapter 13 the contributions of $B(x,\gamma)$ to the both to the Tauberian remainder and an approximation error do not exceed $C\rho^2\gamma^{-1}h^{-2}$.

Example 24.2.7[12]. (i) Let $0 \in \bar{X}$ be a singular point and let assumptions $(24.2.29)_{1-3}$ be fulfilled with $\gamma = \epsilon_0|x|$, $\rho = |x|^{-m}\bigl(\bigl|\ln|x|\bigr| + 1\bigr)^{\alpha}$, $\rho_1 = |x|^{-2m}\bigl(\bigl|\ln|x|\bigr| + 1\bigr)^{\beta}$, $\rho_2 = 1$ and $\rho_3 = \bigl(\bigl|\ln|x|\bigr| + 1\bigr)^{-1}$.

Assume that

$$m = -1, \beta > \max(\alpha, 2\alpha). \tag{24.2.32}$$

Then (24.2.31) is fulfilled with $N = 1$ and we can replace $(24.2.30)^*_k$ by

$$(24.2.30)_m \qquad \sum_{\alpha:|\alpha|\le k} |\nabla^\alpha (VF^{-1} + (2n+1)\mu h)|\gamma^{|\alpha|} \ge \epsilon\rho_3\rho^2 \qquad \forall n \in \mathbb{Z}^+.$$

Therefore, we conclude that the remainder is $O(R)$ with

$$R := \begin{cases} h^{-2} & \alpha < -\dfrac{1}{2}, \\ h^{-2}|\log(\mu h)| & \alpha = -\dfrac{1}{2}, \\ h^{-2}(\mu h)^{-(2\alpha+1)/(\beta-2\alpha)} & 2\alpha < \beta < 2\alpha + 2. \end{cases} \tag{24.2.33}$$

[12] Cf. Example 23.2.12.

One can see easily that under condition (24.2.16) for $\mu h \le t$ with small enough $t > 0$

$$\mathcal{N}^-(\mu, h) \asymp \begin{cases} h^{-3} & \alpha < -\frac{1}{3}, \\ h^{-3}(|\log(\mu h)| + 1) & \alpha = -\frac{1}{3}, \\ h^{-3}(\mu h)^{-(3\alpha+1)/(\beta-2\alpha)} & \alpha > -\frac{1}{3}. \end{cases} \tag{24.2.34}$$

(ii) Let infinity be a singular point and let conditions $(24.2.29)_{1-3}$ be fulfilled with $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^m\big(\log\langle x\rangle + 1\big)^\alpha$, $\rho_1 = \langle x\rangle^{2m}\big(\log\langle x\rangle + 1\big)^\beta$, $\rho_2 = 1$, $\rho_3 = \big(\log\langle x\rangle + 1\big)^{-1}$. Assume that (24.2.32) is fulfilled.

Then all the statements of (i) remain true with the obvious modification: condition (24.2.16) should be replaced by $(24.2.16)^\#$.

Problem 24.2.8. [12] Consider

(i) Case of a singular point at 0, $m < -1$, $\beta > 2\alpha$.

(ii) Case of a singular point at infinity, $m < -1$, $\beta > 2\alpha$.

In both cases $\log(\mathcal{N}^-(\mu, h)h^3) \asymp (\mu h)^{-1/(\beta-2\alpha)}$ and $\log(\mathcal{N}^-(\mu, h)/R^{3/2}) \sim 1$.

Exponential Singularities

Consider now singularities of the exponential type.

Since we assume only that either $0 \in \bar{X}$ or X is a unbounded domain and $\gamma(x) \ll |x|$, it can happen that

$$\operatorname{mes}(\{x \in X\colon |x| \le r\}) \asymp r^n \tag{24.2.35}$$

with $n \ne d$ (more precisely, $n \ge d$ if 0 is a singular point and $n \le d$ is infinity is a singular point). See Figure 11.10 for unbounded domains; for singularity at 0 it is a spike:

Example 24.2.9 [13]. (i) Let $0 \in \bar{X}$ be a singular point and let our standard assumptions be fulfilled with $\gamma = \epsilon_0|x|^{1-\beta}$, $\rho = \exp(a|x|^\beta)$, $\rho_1 = \exp(b|x|^\beta)$ with $\beta < 0$, $b > 2a > 0$.

[13] Cf. Example 23.2.17.

Figure 24.1: Domains of the type (24.2.35)

Then for $\mu h < t$ with a small enough constant $t > 0$

$$\mathsf{N}^-(\mu, h) = \mathcal{N}^-(\mu, h) + O(h^{-2}(\mu h)^{-2a/(b-2a)} |\log(\mu h)|^{(n-1)/\beta}), \tag{24.2.36}$$

$$\mathcal{N}^-(\mu, h) = O(h^{-3}(\mu h)^{-3a/(b-2a)} |\log(\mu h)|^{n/\beta-1}). \tag{24.2.37}$$

(ii) Let infinity be a singular point and let our standard assumptions be fulfilled with $\gamma = \epsilon_0\langle x\rangle^{1-\beta}$, $\rho = \exp(a\langle x\rangle^{\beta})$, $\rho_1 = \exp(b\langle x\rangle^{\beta})$ where $\beta > 0$, $b > 2a > 0$.

Then asymptotics (24.2.36)–(24.2.37) holds.

We leave to the reader the following

Problem 24.2.10. (i) Consider the case of a singular point at 0 and $\gamma = \epsilon_0|x|^{1-\beta}$, $\rho = |x|^m \exp(|x|^{\beta})$, $\rho_1 = |x|^{m_1}\exp(2|x|^{\beta})$, $\rho_2 = 1$, $\rho_3 = |x|^{-\beta}$, $\beta < 0$, $m_1 < 2m$.

(ii) Consider the case of a singular point at infinity and $\gamma = \epsilon_0\langle x\rangle^{1-\beta}$, $\rho = \langle x\rangle^m \exp(\langle x\rangle^{\beta})$, $\rho_1 = \langle x\rangle^{m_1}\exp(2\langle x\rangle^{\beta})$, $\rho_2 = 1$, $\rho_3 = \langle x\rangle^{-\beta}$, $\beta > 0$, $m_1 > 2m$.

We leave to the reader:

Problem 24.2.11. Consider cases of $\mu \to \mu_0 > 0$ and $\mu \to \mu_0 = 0$.

24.2.2 Schrödinger-Pauli Operators

Consider now Schrödinger-Pauli operators, either genuine (0.34) or generalized (13.6.3). The principal difference is that now F does not "tame" singularities of V, on the contrary, it needs to be "tamed" by itself. As a

result there are fewer examples than for the Schrödinger. Also we do not have a restriction $\mu_{\text{eff}} h_{\text{eff}} = O(1)$ which we had in the most of the previous Subsection 24.2.1.

Example 24.2.12[14]. (i) Let 0 be a singular point and let our standard assumptions be fulfilled with $\gamma = \epsilon_0|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$ and let $m > -1$, $2m \ne m_1 > -2$. Then, in comparison with the theory of the previous Subsection 24.2.1, we need to consider also the the zone $\mu_{\text{eff}} h_{\text{eff}} \gtrsim 1$, and also a singular zone where $h_{\text{eff}} \gtrsim 1$.

The contribution to the remainder of the regular part (defined by $\mu_{\text{eff}} h_{\text{eff}} \gtrsim 1$, $h_{\text{eff}} \lesssim 1$) does not exceed $C\mu h^{-1} \int \rho_1 \gamma^{-1}\, dx$ while its contribution to $\mathcal{N}^-(\mu, h)$ does not exceed $C\mu h^{-2} \int \rho_1 \rho\, dx$. Indeed, contributions of each γ-element do not exceed $C\mu_{\text{eff}} h_{\text{eff}}^{-1} \asymp C\mu h^{-1}\rho_1\gamma^2$ and $C\mu_{\text{eff}} h_{\text{eff}}^{-2} \asymp C\mu h^{-2}\rho_1\rho\gamma^3$.

Consider now the singular zone $\{x\colon |x| \le h^{1/(m+1)}\}$. We claim that

(24.2.38) Contribution of the singular zone to the asymptotics does not exceed $C(\mu h + 1)$.

Indeed, if $\mu h \le 1$ it suffices to apply CLR as in 2D-case. If $\mu h \ge 1$, then CLR returns $\mu^{3/2} h^{1/2}$ and we need to be more tricky. Without any loss of the generality we can assume that $V_1 = 0$ and then apply (variant of) CLR for 1D-operator $h^2 D_1^2 - |x|^{2m}$ and for 2D-operator $h^2|D|^2 - \mu^2|x|^{m_1} - |x|^{2m}$. We leave details to the reader.

$$N^-(\mu, h) = \mathcal{N}^-(\mu, h) + O(\mu h^{-1} + h^{-2}) \tag{24.2.39}$$

with

$$\mathcal{N}^-(\mu, h) = O(\mu h^{-2} + h^{-3}) \tag{24.2.40}$$

where under condition (24.2.16) fulfilled in non-empty cone "$= O(\cdot)$"could be replaced by "$\asymp \cdot$".

(ii) Let infinity be a singular point and let our standard assumptions be fulfilled with $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$ and let $m < -1$, $2m \ne m_1 < -2$. Then the same arguments yield (24.2.39)–(24.2.40) again.

We leave to the reader the following problems:

[14] Cf. Example 24.2.2.

Problem 24.2.13. (i) As 0 is a singular point consider both Schrödinger and Schrödinger-Pauli operators as

(a) $\gamma = |x|$, $\rho = |x|^m$, $m > -1$ and $\rho_1 = |x|^{-2}|\log|x||^{\beta}$ [15].

(b) $\gamma = |x|$, $\rho = |x|^{-1}|\log|x||^{\alpha}$, $\alpha < -\frac{1}{2}$ and $\rho_1 = |x|^{m_1}|$, $m_1 > -2$.

(ii) As infinity is a point consider both Schrödinger and Schrödinger Pauli operators as

(a) $\gamma = |x|$, $\rho = |x|^m$, $m < -1$ and $\rho_1 = |x|^{-2}|\log|x||^{\beta}$ [15].

(b) $\gamma = |x|$, $\rho = |x|^{-1}|\log|x||^{\alpha}$, $\alpha < -\frac{1}{2}$ [15] and $\rho_1 = |x|^{m_1}$, $m_1 > -2$.

For the Schrödinger operator in cases (a) non-trivial results could be obtained even as $\mu h \to +\infty$.

Problem 24.2.14. Let either 0 or infinity be a singular point.

Using the same arguments and combining them with the arguments of Subsubsection *24.2.1.6. Power Singularities. II* consider both Schrödinger-Pauli and Schrödinger operators with $\gamma = |x|$, $\rho = |x|^{-1}|\log|x||^{\alpha}$, $\rho_1 = |x|^{-2}|\log|x||^{\beta}$, $\alpha < -\frac{1}{2}$, $\beta < -1$.

For the Schrödinger operator in case $\beta < 2\alpha$ non-trivial results could be obtained even as $\mu h \to +\infty$.

The following problem seems to be rather challenging:

Problem 24.2.15. Investigate $\rho_1 = |x|^{-2}$.

24.2.3 Dirac Operator

Preliminaries

Let us now consider the generalized magnetic Dirac operator (17.1.1) either

$$A = \frac{1}{2}\sum_{l,j} \sigma_l\big(\omega^{jl}P_j + P_j\omega^{jl}\big) + \sigma_0 M + I\cdot V, \qquad P_j = hD_j - \mu V_j \tag{24.2.41}$$

where $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ are 4×4-matrices and $M > 0$ or

$$A = \frac{1}{2}\sum_{l,j} \sigma_l\big(\omega^{jl}P_j + P_j\omega^{jl}\big) + I\cdot V, \qquad P_j = hD_j - \mu V_j \tag{24.2.42}$$

[15] One should take $\beta < -1$ for the Schrödinger-Pauli operator.

where $\sigma_1, \sigma_2, \sigma_3$ are 2×2-matrices and $M = 0$.

We are interested in $\mathsf{N}(\tau_1, \tau_2)$, the number of eigenvalues in (τ_1, τ_2) [16] with $\tau_1 < \tau_2$, fixed in this subsection.

In contrast to 2D-case relations similar to (23.2.73)–(23.2.74) do not matter. Tthe Landau levels (at the point x) are $V \pm \bigl(M^2 + 2j\mu hF\bigr)^{\frac{1}{2}}$ with $j = 0, 1, 2, 3, ...$ and magnetic Dirac operator *always* behaves like Schrödinger-Pauli operator.

Therefore, we treat the operator given by (24.2.41) under the following assumptions

$$(24.2.43)_{1-2} \qquad |D^\alpha \omega^{jk}| \le c\gamma^{-|\alpha|}, \qquad |D^\alpha F| \le c\rho_1\gamma^{-|\alpha|},$$

$$(24.2.43)_{3} \qquad |D^\alpha V| \le c\min\bigl(\rho, \frac{1}{M}\rho^2\bigr)\gamma^{-|\alpha|} \qquad (\alpha \ne 0) \quad \forall \alpha\colon |\alpha| \le K,$$

$$(24.2.43)_{4+} \qquad (V - \tau_2 - M)_+ \le c\min\bigl(\rho, \frac{1}{M}\rho^2\bigr),$$

$$(24.2.43)_{4-} \qquad (V - \tau_1 + M)_- \le c\min\bigl(\rho, \frac{1}{M}\rho^2\bigr)$$

and also $(24.2.2)_{1-3}$, (24.2.1) for $g^{jk} = \sum_{l,r} \omega^{jl}\omega^{kr}\delta_{lr}$ and (24.2.4) (with $F > 0$). In what follows $(24.2.43)_4$ means the pair of conditions $(24.2.43)_{4\pm}$.

Moreover, let condition (24.2.6) be fulfilled and

$$(24.2.44) \qquad \bar{X}'' \cap \partial X = \emptyset,$$

$$(24.2.45) \qquad |V_j| \le c\rho, \quad |D_j\omega^{kl}| \le c\rho \qquad \text{in} \quad X''.$$

Finally, we assume that

(24.2.46) Either $\partial X = \emptyset$ or $\mu = O(1)$ and $\partial X \cap X_2' = \emptyset$ (in what follows).

Asymptotics

Example 24.2.16 [17]. (i) Let 0 be an inner singular point and let all the above conditions, be fulfilled with $\gamma = \epsilon_0|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$, $m_1 > 2m$, $m > -1$.

[16] Assuming that this interval does not contain essential spectrum; otherwise $\mathsf{N}(\tau_1, \tau_2) := \infty$. It is more convenient for us to exclude both ends of the segment.

[17] Cf. Example 24.2.12.

Further, let assumption

$$|V| \ge \epsilon \min(\rho, \frac{\rho^2}{M}) \qquad \forall x\colon |x| \le \epsilon \tag{24.2.47}$$

and non-degeneracy condition $(24.2.8)_k$ be fulfilled.

(a) $m < 0$, $\tau_1 < \tau_2$ or

(b) $m > 0$, $M > 0$, $\tau_1 = -M$, $\tau_2 \in (-M, M)$.

Then for $h \to +0$, $1 \le \mu$ asymptotics

$$\mathsf{N}(\tau_1, \tau_2; \mu, h) = \mathcal{N}(\tau_1, \tau_2; \mu, h) + O(\mu h^{-1} + h^{-2}) \tag{24.2.48}$$

holds with $\mathcal{N}$ defined by $(17.1.12)_1$ with $d = 3$, $r = 1$. Moreover, $\mathcal{N}(\tau_1, \tau_2, \mu, h) \asymp \mu h^{-2} + h^{-3}$.

(ii) Let infinity be an inner singular point and let all the above conditions be fulfilled with $\gamma = \epsilon_0 \langle x \rangle$, $\rho = \langle x \rangle^m$, $\rho_1 = \langle x \rangle^{m_1}$, $m_1 < 2m$, $m < -1$.

Further, let assumption

$$|V| \ge \epsilon \min(\rho, \frac{\rho^2}{M}) \qquad \forall x\colon |x| \ge c \tag{24.2.47}^{\#}$$

and non-degeneracy condition $(24.2.8)_k$ be fulfilled. Let $M > 0$, $\tau_1 = -M$, $\tau_2 \in (-M, M)$. Then for $h \to +0$, $1 \le \mu$ asymptotics (24.2.48) holds with $\mathcal{N}$ defined by $(17.1.12)_1$ with $d = 3$, $r = 1$. Moreover, $\mathcal{N}(\tau_1, \tau_2, \mu, h) \asymp \mu h^{-2} + h^{-3}$.

Problem 24.2.17. Generalize results of this section to the odd-dimensional maximal-rank case. In particular, consider power singularities.

24.3 Asymptotics of Large Eigenvalues

In this section we consider the case when μ and h are fixed and we consider the asymptotics of the eigenvalues, tending to $+\infty$ and for Dirac operator also to $-\infty$.

Here we consider the case of the spectral parameter tending to $+\infty$ (and for the Dirac operator we consider $\tau \to -\infty$ as well).

24.3.1 Singularities at the Point

We consider series of example with singularities at the point.

Schrödinger Operator

Example 24.3.1. (i) Let X be a compact domain, $0 \in \bar{X}$ and let conditions $(24.2.2)_{1-3}$ and (24.2.6) be fulfilled with $\gamma = \epsilon|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$, $m_1 < 2m$. Let

$$|F| \ge \epsilon_0 \rho_1 \qquad \text{for } |x| \le \epsilon. \tag{24.3.1}$$

Then for the Schrödinger operator as $\tau \to +\infty$

$$\mathsf{N}^-(\tau) = \mathcal{N}^-(\tau) + O(\tau^{(d-1)/2}) \tag{24.3.2}$$

while $\mathcal{N}^-(\tau) \asymp \tau^{d/2}$.

Indeed, we need to consider only case $m_1 \le -2$ (otherwise it is covered by Section 11.2). Assume for simplicity, that $V = 0$ (modification in the general case is trivial). Recall that $\mu_{\mathsf{eff}} = |x|^{m_1+1}\tau^{-1/2}$ and $h_{\mathsf{eff}} = \tau^{-1/2}|x|^{-1}$. Then under week non-degeneracy assumption[18)] contribution to the remainder of any element with $\mu_{\mathsf{eff}} h_{\mathsf{eff}} \le c$ does not exceed $C h_{\mathsf{eff}}^{1-d} = C\tau^{(d-1)/2}\gamma^{d-1}$ while contribution to the remainder of the γ-element $\mu_{\mathsf{eff}} h_{\mathsf{eff}} \ge c$ does not exceed $C\mu_{\mathsf{eff}}^{-s} h_{\mathsf{eff}}^{-d-s}$ and the rest is easy. Without non-degeneracy assumption we need to add to the contribution of the γ-element $C\mu_{\mathsf{eff}} h_{\mathsf{eff}}^{1-d-\delta}$, which after summation results in $O(\tau^{(d-1-\delta')/2})$.

(ii) Under proper assumptions the same proof is valid in the odd-dimensional maximal-rank case.

Let us note that the case $m < -1$, $m_1 \ge m-1$ is covered by Chapter 11; we need to assume that

$$V \ge \epsilon_0 \rho^2 \qquad \text{as } |x| \le \epsilon. \tag{24.3.3}$$

Example 24.3.2. (i) Let X be a compact domain, $0 \in \bar{X}$ and let conditions $(24.2.2)_{1-3}$, (24.2.6) and $(24.3.1)_1$ be fulfilled with $\gamma = \epsilon|x|$, $\rho = |x|^m$,

[18)] Which we do not, however, assume fulfilled at this example.

$\rho_1 = |x|^{m_1}$ and with $m < -1$, $2m \le m_1 < m-1$. Then we need to assume that

$$V + F \ge \epsilon_0 \rho^2 \qquad \text{as } |x| \le \epsilon \tag{24.3.3}'$$

which for $m_1 > 2m$ is equivalent to (24.3.3).

By the same reason as in the previous Example 24.3.1 we do not need any non-degeneracy assumption. Then asymptotics (24.3.2) holds while $\mathcal{N}^-(\tau) \asymp \tau^{d/2}$.

(ii) Under proper assumptions the same proof is valid in the odd-dimensional maximal-rank case.

Schrödinger-Pauli Operator

Next, consider Schrödinger-Pauli operators. We will need to impose (24.3.3) and the related non-degeneracy assumption

$$\sum_{\alpha: 1 \le |\alpha| \le k} |\nabla^\alpha V| \gamma^{|\alpha|} \ge \epsilon \rho^2 \qquad \text{as } |x| \le \epsilon \tag{24.3.4}_k$$

Example 24.3.3. (i) Let X be a compact domain, $0 \in \bar{X}$ and let conditions $(24.2.2)_{1-3}$, (24.2.6), (24.3.1), (24.3.3) and $(24.3.4)_k$ be fulfilled with $\gamma = \epsilon|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$, $m < -1$.

Then for the Schrödinger-Pauli operator as $\tau \to +\infty$ asymptotics

$$\mathsf{N}^-(\tau) = \mathcal{N}^-(\tau) + O(\tau^{(m_1+2)/(2m)} + \tau) \tag{24.3.5}$$

holds while

$$\mathcal{N}^-(\tau) \asymp \tau^{(m_1+m+3)/(2m)} + \tau^{3/2}. \tag{24.3.6}$$

Indeed, if $m_1 \ge 2m$ then no modification to the arguments of Examples 24.3.1 and 24.3.2 is needed; if $m_1 < 2m$ we also need to consider the zone where $\mu_{\mathsf{eff}} h_{\mathsf{eff}} \gtrsim 1$.

The contribution of the corresponding partition element to the principal part of the asymptotics is $\mu_{\mathsf{eff}} h_{\mathsf{eff}}^{-2}$ while its contribution to the remainder is $O(\mu_{\mathsf{eff}} h_{\mathsf{eff}}^{-1})$; also, zone $\{x\colon |x| \le \epsilon \tau^{1/(2m)}\}$ is forbidden.

(ii) Without condition $(24.3.4)_k$ one needs to add $\mu_{\mathsf{eff}} h_{\mathsf{eff}}^{-1-\delta}$ to the contribution of the partition element to the remainder; it does not affect the remainder estimate $O(\tau)$ if $m_1 > 2m+2$; if $m_1 \le 2m+2$ we arrive to the remainder estimate $O(\tau^{(m_1+2)/(2m)+\delta})$.

(iii) One can generalize this example to the odd-dimensional maximal-rank case; then

$$\mathcal{N}^-(\tau) \asymp \tau^{(m_1+m+3)(d-1)/(4m)} + \tau^{d/2} \tag{24.3.7}$$

and the remainder is $O(R)$ with

$$R = \tau^{(m_1+2)(d-1)/(4m)} + \tau^{(d-1)/2}. \tag{24.3.8}$$

Problem 24.3.4. (i) Investigate the case of $\gamma = \epsilon|x|$, $\rho = |x|^{-1}|\log|x||^{\alpha}$, $\alpha > 0$, $\rho_1 = |x|^{m_1}$, $m_1 < -2$.

(ii) Investigate the case of $\gamma = \epsilon|x|$, $\rho = |x|^{-1}|\log|x||^{\alpha}$, $\rho_1 = |x|^{-2}|\log|x||^{\beta}$, $\beta > \alpha > 0$.

The following problem seems to be challenging; we don't know even if $\mathsf{N}^-(\tau) < \infty$ for $\tau > 0$.

Problem 24.3.5. (i) Investigate the case of $\gamma = \epsilon|x|$, $\rho = |x|^{-1}|\log|x||^{\alpha}$, $\alpha \le 0$, $\rho_1 = |x|^{m_1}$, $m_1 < -2$.

(ii) Investigate the case of $\gamma = \epsilon|x|$, $\rho = |x|^{m}$, $\rho_1 = |x|^{m_1}|$, $m_1 < -2$, $-1 < m < 0$.

Miscellaneous Singularities

Consider now miscellaneous singularities in the point.

Example 24.3.6. Let $0 \in \bar{X}$ be a compact domain, and let conditions $(24.2.2)_{1-3}$, (24.2.6) and (24.3.1) be fulfilled with $\gamma = \epsilon|x|^{1-\beta}$, $\rho_1 = \exp(b|x|^{\beta})$, $\beta < 0$, $b > 0$ and with $\rho = \exp(a|x|^{\beta})$ where $b > 2a$. Assume also that

$$\sum_{\alpha:1\le|\alpha|\le k} |\nabla^{\alpha} F|\gamma^{|\alpha|} \ge \epsilon\rho_1 \qquad \text{as } |x| \le \epsilon. \tag{24.3.9}_k$$

(i) Then for the Schrödinger operator $\mathsf{N}^-(\tau) = \mathcal{N}^-(\tau)+O(R)$ with $\mathcal{N}^-(\tau) \asymp \tau^{3/2}$ and

$$R = \begin{cases} \tau & \beta \ge -2, \\ \tau|\log\tau|^{(2+\beta)/\beta} & \beta < -2. \end{cases} \tag{24.3.10}$$

Indeed, this is trivial unless $\beta > -2$. If $\beta \le -2$ we need to take into account that the forbidden zone where $\mu_{\mathsf{eff}} h_{\mathsf{eff}} \ge C_0$ is $\{z\colon |x| \le \epsilon_0 |\log\tau|^{1/\beta}\}$. Then we get the remainder $O(\tau r_*^{2+\beta})$ for $\beta < -2$ and $O(\tau|\log r_*|)$ for $\beta = -2$.

Consider $\beta = -2$. Since $\mu_{\mathsf{eff}} \lesssim 1$ in the zone $\{x\colon |x| \ge r^* := C|\log\tau|^{1/\beta}\}$, we can take $\gamma \asymp |x|^{1-\beta'}$ with $\beta' > -2$ there and the contribution of this zone to the remainder is $O(\tau)$. Meanwhile, the contribution of the zone $\{x\colon r_* \le |x| \le r^*\}$ to the remainder is $O(\tau|\log(r_*/r^*)|) = O(\tau)$.

(ii) Let conditions (24.3.3) and $(24.3.4)_k$ be fulfilled. Then for the Schrödinger-Pauli operator asymptotics $\mathsf{N}^-(\tau) = \mathcal{N}^-(\tau) + O(R)$ holds with

$$R = \tau^{b/2a}|\log\tau|^{2/\beta} \tag{24.3.11}$$

and

$$\mathcal{N}^-(\tau) \asymp \tau^{b/2a+1/2}|\log\tau|^{(3-\beta)/\beta}. \tag{24.3.12}$$

(iii) On the other hand, let $a < b \le 2a$ and conditions (24.3.3) and $(24.3.4)_k$ Be fulfilled. Then for both Schrödinger and Schrödinger-Pauli operators $\mathcal{N}^-(\tau) \asymp \tau^{3/2}$ and R is defined by (24.3.10).

Moreover, for $b < 2a$ we do not need the non-degeneracy assumption $(24.3.9)_k$ because in the allowed zone $\{x\colon V(x) \le C\tau\}$ we have $\mu_{\mathsf{eff}} \le h_{\mathsf{eff}}^{\delta-1}$.

Problem 24.3.7. Extend results of Example 24.3.6 to the Dirac operator.

Remark 24.3.8. (i) Observe that the contribution to the remainder of the zone $\{x\colon |x| \le \varepsilon\}$ does not exceed $\varepsilon^\sigma \tau^{(d-1)/2}$ with $\sigma > 0$ in the frameworks of some above examples (sometimes under certain additional assumptions).

Therefore, in these cases under the standard non-periodicity condition to the geodesic flow with reflections from ∂X the asymptotics

$$\mathsf{N}(\tau) = \mathcal{N}(\tau) + \varkappa_1 \tau + o(\tau) \tag{24.3.13}$$

holds with the standard coefficient $\varkappa_1$.

(ii) The similar statement (with τ replaced by τ^2) is true for the Dirac operator.

Problem 24.3.9. In the frameworks of the examples above estimate $|\mathcal{N}^-(\tau) - \kappa_0\tau^{d/2}|$.

Finally, consider the case when the singularity is located on the curve or a surface (or a more general set).

Example 24.3.10. Let $0 \in \bar{X}$ be a compact domain, and let conditions $(24.2.2)_{1-3}$, (24.2.6) and (24.3.1) be fulfilled with $\gamma = \epsilon_0\delta(x)$, $\rho_1 = \delta(x)^m$, $\rho_1 = \delta(x)^{m_1}$ with $m_1 < \min(2m, -2)$ where $\delta(x) = \operatorname{dist}(x, L)$, $m < 0$, L is a set of Minkowski codimension $p > 1$ or a smooth surface; in the latter case we assume also that $(24.3.9)_k$ is fulfilled.

(i) Then for the Schrödinger operator asymptotics (24.3.2) holds for $\tau \to +\infty$ and $\mathcal{N}^-(\tau) \asymp \tau^{3/2}$.

Indeed, using the same arguments as before we can get a remainder estimate $O(\tau)$ if $p > 1$ or $O(\tau|\log\tau|)$ if $p = 1$ but in the latter case we can get rid of logarithm using standard propagation arguments.

(ii) Let also conditions (24.3.3) and $(24.3.4)_k$ be fulfilled. Then for the Schrödinger-Pauli operator asymptotics

$$\mathsf{N}^-(\tau) = \mathcal{N}^-(\tau) + O(\tau + \tau^{(m_1+2-p)/(2m)}) \tag{24.3.14}$$

holds while

$$\mathcal{N}^-(\tau) \asymp \tau + \tau^{(m_1+m+3-p)/(2m)}. \tag{24.3.15}$$

Problem 24.3.11. Extend results of Example 24.3.10 to different types of the singularities along L.

24.3.2 Singularities at iInfinity

Let us consider *unbounded domains:*

Power sSingularities: Schrödinger Operator

Let us start from the power singularities.

Example 24.3.12[19]. (i) Let X be an unbounded domain. Let conditions (24.2.1), (24.2.6), $(24.2.2)_{1-3}$, (24.3.1) and $(24.3.9)_k$ be fulfilled with $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$, $m_1 > 2m$.

Then for the Schrödinger operator the following asymptotics holds:

$$\mathsf{N}^-(\tau) = \mathcal{N}^-(\tau) + O(\tau^{(m_1+2)/m_1}) \tag{24.3.16}$$

with

$$\mathcal{N}^-(\tau) \asymp \tau^{3(m_1+2)/(2m_1)}. \tag{24.3.17}$$

The proof is standard.

(ii) Under proper assumptions the similar asymptotics holds in the odd-dimensional maximal-rank case:

$$\mathsf{N}^-(\tau) = \mathcal{N}^-(\tau) + O(\tau^{(d-1)(m_1+2)/(2m_1)}) \tag{24.3.18}$$

with

$$\mathcal{N}^-(\tau) \asymp \tau^{d(m_1+2)/(2m_1)}. \tag{24.3.19}$$

Example 24.3.13[20]. (i) Let X be an unbounded domain. Let conditions (24.2.1), (24.2.6), $(24.2.2)_{1-3}$ and $(24.3.1)^\#$ be fulfilled with $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$, $m > 0$, $m - 1 < m_1 \le 2m$.

Let us assume that <u>either</u> $m_1 = 2m$ and

$$V + F \ge \epsilon_0\rho^2 \qquad \text{as} \quad |x| \ge c \tag{24.3.3}^{\#\prime}$$

<u>or</u> $m_1 < 2m$ and

$$V \ge \epsilon_0\rho^2 \qquad \text{as} \quad |x| \ge c \tag{24.3.3}^{\#}$$

Assume that if $m_1 = 2m$ then

$$\tau \ge V + F \implies \sum_{\alpha:1\le|\alpha|\le k} |\nabla(\tau - V)F^{-1}|\gamma^{|\alpha|} \ge \epsilon_0\tau\rho_1^{-1} \qquad \text{as} \quad |x| \ge c. \tag{24.3.20}_k$$

---

[19] Cf. Example 24.3.1.

[20] Cf. Example 24.3.2.

Then for the Schrödinger operator asymptotics

$$\mathsf{N}^-(\tau) = \mathcal{N}^-(\tau) + O(\tau^{(d-1)(m+1)/(2m)}), \tag{24.3.21}$$

holds with

$$\mathcal{N}^-(\tau) \asymp \tau^{d(m+1)/(2m)}. \tag{24.3.22}$$

(ii) Under proper assumptions the similar asymptotics holds in the odd-dimensional maximal-rank case.

Power Singularities: Schrödinger-Pauli Operator

Next, consider Schrödinger-Pauli operators. We will need to impose $(24.3.3)^{\#}$ and the related non-degeneracy assumption

$$\sum_{\alpha:1\le|\alpha|\le k} |\nabla^\alpha V|\gamma^{|\alpha|} \ge \epsilon\rho_1 \qquad \text{as } |x| \ge \epsilon. \tag{24.3.4}$$

Example 24.3.14[21]. Let $(24.3.3)^{\#}$ and $(24.3.4)^{\#}_k$ be fulfilled. Then for the Schrödinger-Pauli operator

(i) In the framework of Example 24.3.12

$$\mathsf{N}^-(\tau) = \mathcal{N}^-(\tau) + O(\tau^{(m_1+2)/(2m)}) \tag{24.3.23}$$

with

$$\mathcal{N}^-(\tau) \asymp \tau^{(m_1+m+3)/(2m)}. \tag{24.3.24}$$

(ii) In the framework of Example 24.3.13

$$\mathsf{N}^-(\tau) = \mathcal{N}^-(\tau) + O(\tau^{(m+1)/m}) \tag{24.3.25}$$

with

$$\mathcal{N}^-(\tau) \asymp \tau^{3(m+1)/(2m)}). \tag{24.3.26}$$

(iii) Finally, under proper assumptions one can consider the odd-dimensional maximal-rank case and prove asymptotics with the remainder estimate $O(R)$, with

$$R(\tau) = \begin{cases} \tau^{(d-1)(m_1+2)/(4m)} & m_1 > 2m, \\ \tau^{(d-1)(m+1)/(2m)} & m_1 \le 2m, \end{cases} \tag{24.3.27}$$

[21] Cf. Example 24.3.3.

and

$$\mathcal{N}^-(\tau) \asymp \begin{cases} \tau^{(d-1)(m_1+2)/(4m)+(m+1)/(2m)} & m_1 > 2m, \\ \tau^{d(m+1)/(2m)} & m_1 \le 2m, \end{cases} \tag{24.3.28}$$

Exponential Singularities

Consider now an exponential growth at infinity.

Example 24.3.15. Let X be an unbopunded domain. Let conditions (24.2.1), $(24.2.2)_{1-3}$, (24.2.4), (24.2.6) and $(24.3.9)_k$ be fulfilled with $\gamma = \epsilon_0\langle x\rangle^{1-\beta}$, $\rho = \exp(a\langle x\rangle^{\alpha})$, $\rho_1 = \exp(b\langle x\rangle^{\beta})$, $\beta > 0$ and either $\beta > \alpha$ or $\beta = \alpha$ and $b > 2a > 0$.

(i) Then for the Schrödinger operator the following asymptotics holds:

$$\mathsf{N}^-(\tau) = \mathcal{N}^-(\tau) + O\bigl(\tau|\log\tau|^{(2+\beta)/\beta}\bigr) \tag{24.3.29}$$

with

$$\mathcal{N}^-(\tau) \asymp \tau^{3/2}|\log\tau|^{3/\beta}. \tag{24.3.30}$$

(ii) Let $\alpha = \beta$ and conditions $(24.3.3)^{\#}$ and $(24.3.4)^{\#}_k$ be fulfilled. Then for the Schrödinger-Pauli operator asymptotics holds

$$R(\tau) = \tau^{b/2a}|\log\tau|^{2/\beta} \tag{24.3.31}$$

and

$$\mathcal{N}^-(\tau) \asymp \tau^{(b+a)/2a}|\log\tau|^{3/\beta}. \tag{24.3.32}$$

(iii) On the other hand, let $\beta = \alpha$, $a < b \le 2a$ and conditions $(24.3.3)^{\#}$ and $(24.3.4)^{\#}_k$ Be fulfilled. Then for both Schrödinger and Schrödinger-Pauli operators $\mathcal{N}^-(\tau) \asymp \tau^{3/2}$ and R is defined by (24.3.10).

Moreover, for $b < 2a$ we do not need the non-degeneracy assumption $(24.3.9)_k$ because in the allowed zone $\{x\colon V(x) \le C\tau\}$ we have $\mu_{\mathsf{eff}} \le h_{\mathsf{eff}}^{\delta-1}$.

We leave to the reader

Problem 24.3.16. (i) Consider the Schrödinger-Pauli operators in the same settings as in Example 24.3.15, albeit with ρ of the power growth at infinity.

(ii) Consider the Schrödinger-Pauli operators in the same settings as in Example 24.3.14 albeith with V of the logarithmic growth at infinity (i.e. with $\rho = |\log|x||^{\alpha}$, $\gamma = \epsilon|x|$).

24.4 Asymptotics of Small Eigenvalues

In this section we need to consider first miscellaneous asymptotics (cf. Subsections 23.4.3 and 23.4.4) and only after case of F stabilizing at infinity (cf. Subsections 23.4.1 and 23.4.2).

24.4.1 Miscellaneous Asymptotics

Case $F \gtrsim 1$ as $|x| \to \infty$

In this subsection we consider cases of either $F \to \infty$ or $F \asymp 1$ and $V \to 0$ as $|x| \to \infty$. If $F \to \infty$ the Schrödinger operator either does not have any essential spectrum "at infinity"[22]; if $F \asymp 1$ we do not assume any stabilization conditions so far. Anyway, the Schrödinger operator is not the subject of our analysis, while for the Schrödinger-Pauli and Dirac operators essential spectrum "at infinity" equals $[0,\infty)$ and $(-\infty, M] \cup [M,\infty)$ respectively[23].

Again due to the specifics of the problem we can consider the multidimensional case with minimal modifications. In this case we assume that

(24.4.1) $\mathsf{rank}\, F = 2p$ as $|x| \ge c$ and $f_1 \asymp f_2 \asymp ... \asymp f_p \asymp \rho_1$ as $|x| \ge c$

and

(24.4.2) For each $j \ne k$ either $f_j = f_k$ or $|f_j - f_k| \ge \epsilon\rho_1$ for all $|x| \ge c$.

Let for the Schrödinger-Pauli operator $\mathsf{N}^-(\eta)$ be a number of eigenvalues in $(-\epsilon, -\eta)$ and $\mathsf{N}^+(\eta)$ be a number of eigenvalues in (η, ϵ).

Theorem 24.4.1[24]**.** *Let X be an unbounded domain. Let conditions* (24.2.1), $(24.2.2)_{1-3}$, (24.4.1), (24.4.2) *and* $(24.3.4)_k^{\#}$ *be fulfilled with scaling functions γ, ρ and ρ_1, $\rho \to 0$, $\rho_1 \gtrsim 1$, $\rho_1\gamma/\rho \to \infty$ and $\rho\gamma \to \infty$ as $|x| \to \infty$. Then for the Schrödinger-Pauli operator the following asymptotics holds*

[22] More precisely, the lowest Landau level tends to $+\infty$ as $|x| \to \infty$.

[23] Again, understood in the sense of the Lanfdau levels.

[24] Cf. Theorem 23.4.16.

$\mathsf{N}^-(\eta) = \mathcal{N}^-(\eta) + O(R)$ *where*

$$\mathcal{N}^-(\eta) := (2\pi)^{-d+p}\varpi_{d-2p}\int_{\{x:\,-V(x)\ge\eta\}} f_1 f_2\cdots f_p(-V-\eta)_+^{(d-2p)/2}\sqrt{g}\,dx \tag{24.4.3}$$

and

$$R = C\int_{\{x:\,-V(x)\ge\eta\}} \rho_1^p \rho^{d-2p-1}\gamma^{-1}\,dx + C\int \rho_1^{p-s}\gamma^{-2s}\,dx. \tag{24.4.4}$$

Example 24.4.2. Let X be an unbounded domain. Let conditions (24.2.1), $(24.2.2)_{1-3}$, (24.2.6), (24.4.1), (24.4.2) and $(24.3.4)_k^{\#}$ be fulfilled with scaling functions $\gamma = \epsilon\langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$, $-1 < m < 0 \le m_1$.

Then for the Schrödinger-Pauli operator

$$R = \eta^{(m_1+2)(d-1)/(4m)} \tag{24.4.5}$$

and

$$\mathcal{N}^-(\eta) = O(\eta^{(m_1+2)(d-1)/(4m)+(m+1)/(2m)}). \tag{24.4.6}$$

Further, we can replace "$= O(\cdot)$" by "$\asymp \cdot$ if condition $V \le -\epsilon\rho^2$ is fulfilled in some non-empty cone.

We leave to the reader

Problem 24.4.3. (i) Consider the case of $\gamma = \epsilon\langle x\rangle$, $\rho = \langle x\rangle^{-1}|\log\langle x\rangle|^\alpha$, $\rho_1 = \langle x\rangle^{m_1}$, $m_1 \ge 0$, $\alpha > 0$.

(ii) Consider the case of $\gamma = \epsilon\langle x\rangle$, $\rho = |\log\langle x\rangle|^\alpha$, $\rho_1 = \langle x\rangle^{m_1}$, $m_1 \ge 0$, $\alpha < 0$.

(iii) Consider the case of $\gamma = \langle x\rangle^{1-\beta}$, $\rho_1 = \exp(b\langle x\rangle^\beta)$, $\beta > 0$ while conditions to V, g^{jk} are fulfilled with $\gamma = \epsilon\langle x\rangle$, and either $\rho = \langle x\rangle^{-1}|\log\langle x\rangle|^\alpha$, $\alpha > 0$ or $\rho = |\log\langle x\rangle|^\alpha$, $\alpha < 0$.

Problem 24.4.4. Consider the Dirac operator. In this case $\mathsf{N}^-(\eta)$ is a number of eigenvalues in $(M-\epsilon, M-\eta)$ and $\mathsf{N}^+(\eta)$ is a number of eigenvalues in $(-M+\eta, -M+\epsilon)$, $0 < \eta < \epsilon$, $M > 0$.

Case $F \to 0$ as $|x| \to \infty$

In this subsection we consider cases of $F \to 0$ and $V \to 0$ as $|x| \to \infty$. In this case the essential spectra of the Schrödinger and Schrödinger-Pauli operators are $[0, \infty)$; however, as $V = o(F)$ as $|x| \to \infty$ the Schrödinger operator has only a finite number of the negative eigenvalues and thus is not a subject of our analysis while the Schrödinger-Pauli operator is.

Further, the Dirac operator has its essential spectrum $(-\infty, -M] \cup [M, \infty)$ and we need to assume that $M > 0$.

Theorem 24.4.5 [25]**.** *Let X be an unbounded domain. Let conditions* (24.2.1), $(24.2.2)_{1-3}$, (24.2.6), (24.4.1), (24.4.2) *and* $(24.3.4)_k^{\#}$ *be fulfilled with scaling functions γ, ρ and ρ_1, $\rho \to 0$, $\rho_1 \to 0$, $\rho_1 \rho/\rho^2 \ge C_0$ and $\rho\gamma \to \infty$ as $|x| \to \infty$.*

Then for the Schrödinger-Pauli operator $\mathsf{N}^-(\eta) = \mathcal{N}^-(\eta) + O(R)$ with $\mathcal{N}^-(\eta)$ and R defined by (24.4.3)– (24.4.4).

Example 24.4.6. Let conditions of Theorem 24.4.5 be fulfilled with $\gamma = \langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$, $m < 0$, $\max(2m, -2) < m_1 < 0$.

Then all statements of the Example 24.4.2 remain true.

Example 24.4.7. Let conditions of Theorem 24.4.5 be fulfilled with $\gamma = \epsilon\langle x\rangle$, $\rho_1 = \langle x\rangle^{-2} |\log\langle x\rangle|^{\beta}$, $\rho = \langle x\rangle^{-1} |\log\langle x\rangle|^{\alpha}$ with $0 < 2\alpha < \beta$. Then

$$R = |\log \eta|^{(\beta p + \alpha(d-2p-1)+1)/2} \tag{24.4.7}$$

and $\mathcal{N}^-(\eta) = O(S)$ with

$$S = |\log \eta|^{(\beta p + \alpha(d-2p)+1)/2} \tag{24.4.8}$$

Further, $\mathcal{N}^-(\eta) \asymp S$ if condition $V \le -\epsilon\rho^2$ is fulfilled in some non-empty cone.

We also leave to the reader

Problem 24.4.8. Consider in this framework the Dirac operator. In this case $\mathsf{N}^-(\eta)$ is a number of eigenvalues in $(M - \epsilon, M - \eta)$ and $\mathsf{N}^+(\eta)$ is a number of eigenvalues in $(-M + \eta, -M + \epsilon)$, $0 < \eta < \epsilon$. We need to assume that $M > 0$ and potential $V \sim \rho^2$ at infinity.

[25] Cf. Theorem 24.4.1.

Consider now the case when condition $\rho^2 = o(\rho_1)$ as $|x| \to \infty$ is not fulfilled. Then the results will be similar to those of Section 24.3.

Example 24.4.9. (i) Let X be an unbounded domain with $\mathscr{C}^K$ boundary. Let conditions (24.2.1), $(24.2.2)_{1-3}$, (24.2.6), (24.4.1), (224.4.2) and $(24.3.4)_k^{\#}$ be fulfilled with scaling functions $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$, $-1 < m < 0$, $m - 1 < m_1 < 2m$.

Then for the Schrödinger and Schrödinger-Pauli operators asymptotics

$$R = \eta^{(m+1)(d-1)/2m} \tag{24.4.9}$$

and

$$S = \eta^{(m+1)(d-1)/2m} \tag{24.4.10}$$

(ii) Similar results hold if $m_1 = 2m$ but as $p = 1$ one needs to assume a non-degeneracy assumption (we leave it to the reader) and for the Schrödinger operator $\mathcal{N}^-(\eta) \asymp S$ provided $V + F \le -\epsilon\rho^2$ in some non-empty cone.

We leave to the reader:

Problem 24.4.10. Consider the case of $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^{-2}|\log x|^{\alpha}$, $\rho_1 = \langle x\rangle^{-2}|\log x|^{\beta}$, $2\alpha \ge \beta > \alpha$.

Problem 24.4.11. Consider in this framework the Dirac operator with $M > 0$.

24.4.2 Case $\operatorname{rank} F_\infty = d - 1$. Fast Decaying Potential

Consider case of stabilization at infinity (cf. Subsection 23.4.1)

$$\mathsf{g} \to \mathsf{g}_\infty, \quad \mathsf{F} \to \mathsf{F}_\infty, \quad V \to 0 \qquad \text{as} \quad |x| \to \infty. \tag{24.4.11}_{1-3}$$

Recall that $\mathsf{F} := (F_{jk})$ with $F_{jk} = \partial_k V_j - \partial_j V_k$, $\mathsf{g} := (g^{jk})$.

Assume now that

$$\operatorname{rank} \mathsf{F}_\infty = 2p \text{ while } d = 2p + 1 \tag{24.4.12}$$

and potential V decays faster than in the previous subsection–at least in the direction of the magnetic field.

Preliminary aAnalysis

Observe first that

(24.4.13) If F and g are constant than without any loss of the generality we can assume that $g^{jk} = \delta_{jk}$ and $\mathsf{Ker}\,\mathsf{F} = \mathbb{R}^{d-2p} \times \{0\}$.

Indeed we can achieve it by a linear change of the coordinates. In the general case under assumption (24.4.12) we can assume that

$$\mathsf{Ker}\,\mathsf{F} = \mathbb{R} \times \{0\} \qquad \text{as} \quad |x| \ge c \tag{24.4.14}$$

and

$$V_1 = 0 \qquad \text{as} \quad |x'| \ge c \tag{24.4.15}$$

where $x = (x_1; x')$. Indeed, we can achieve (24.4.14) by the change of the coordinate system which straightens magnetic lines[26)] and we can achieve 24.4.15) by the gauge transformation.

These two assumptions imply $V_j = v_j(x')$ for $j = 2, \dots, d$ and together with stabilization as $x_1 \to \infty$ we conclude that F is constant. Without any loss of the generality we can assume that

(24.4.16) $\mathsf{g}_\infty = \delta_{jk}$, $F_{jk} = 0$ unless either $j = 2l$, $k = 2l+1$ when $F_{jk} = f_{\infty,l}$, or $j = 2l+1$, $k = 2l$ when $F_{jk} = -f_{\infty,l}$.

(24.4.17) By means of the allowed change of the coordinates[26)] on each magnetic line[27)] $\{x\colon x' = y'\}$ with $|y'| \ge c$ we can achieve

$$g^{jd} = 0 \qquad j = 0, \dots, d. \tag{24.4.18}$$

Remark 24.4.12. One can prove easily that in this reduction V is perturbed by $O(\rho^4\gamma^{-2})$ which would not affect the principal part and an error estimate.

[26)] After this we are allowed only changes $x \mapsto y$ with $y' = y'(x')$ and $y_d = y_d(x)$.

[27)] But not necessarily on all of them in simultaneously.

As (24.4.14), (24.4.15) and (24.4.18) [28] are fulfilled consider 1D-operator on $\mathbb{R} \ni z$

$$\mathcal{L}(y') \coloneqq D_z g^{11}(z;y') D_z + V^*(z;y'), \tag{24.4.19}$$

$$V^*(x) \coloneqq V(x) + \sum_j (f_j(x) - f_\infty). \tag{24.4.20}$$

Let us consider operator for which (24.4.14) and (24.4.15) are fulfilled allowing instead some anisotropy:

$$|\nabla^\alpha (\mathsf{g} - \mathsf{g}_\infty)| = o(\rho^2 \gamma^{-|\alpha'|} \gamma_1^{-\alpha_1}), \tag{24.4.21}_1$$

$$|\nabla^\alpha V| = O(\rho^2 \gamma^{-|\alpha'|} \gamma_1^{-\alpha_1}), \tag{24.4.21}_2$$

$$|\nabla^\alpha F_{jk}| = O(\rho^2 \gamma^{-|\alpha'|} \gamma_1^{-\alpha_1}) \qquad \text{as } |x| \to \infty \quad \forall \alpha \tag{24.4.21}_3$$

with scaling functions

$$\gamma_1(x') \ge 1, \qquad \gamma = \gamma(x') \to \infty \quad \text{as } |x'| \to \infty \tag{24.4.22}_{1-2}$$

$$\rho(x) = \varrho(x') \varrho_1(x_1/\gamma_1) \tag{24.4.23}$$

such that

$$|\nabla \gamma| \le \frac{1}{2}, \qquad |\nabla \gamma_1| \le \frac{1}{2} \gamma_1 \gamma^{-1}, \tag{24.4.24}$$

$$|\varrho_1| \le 1, \qquad \int_{\mathbb{R}} |t| \varrho_1^2(t)\, dt < \infty, \tag{24.4.25}$$

$$|\nabla \varrho| \le \varrho \gamma^{-1}, \qquad |\nabla \varrho_1| \le \varrho_1 \gamma^{-1} \tag{24.4.26}$$

$$\zeta \coloneqq \varrho^2 \gamma_1 \to 0 \qquad \text{as } |x'| \to \infty. \tag{24.4.27}$$

In virtue of Proposition 24.A.1 operator $\mathcal{L}(x')$ has a finite number of negative eigenvalues for all x' and no more than one negative eigenvalue as $|x'| \ge c$; further, under assumption

$$W(x') \coloneqq -\frac{1}{2} \int_{\mathbb{R}} V^*(x_1;x')\, dx_1 > 0 \quad \text{and} \quad W(x') \asymp \zeta \quad \text{as } |x'| \ge c, \tag{24.4.28}$$

[28] Only as $x' = y'$.

there is exactly one negative eigenvalue $\lambda(x')$ and

$$\nabla^{\alpha'}\bigl(\lambda(x')+W(x')^2\bigr)=o(\zeta^2\gamma^{-|\alpha'|}) \tag{24.4.29}$$

while

$$\nabla^{\alpha'}W=O(\zeta\gamma^{-|\alpha'|}) \tag{24.4.30}$$

and

$$\|\nabla^{\alpha}v\|=O(\gamma^{-|\alpha'}\gamma_1^{-\alpha_1}). \tag{24.4.31}$$

Here $v=v(x_1;x')$ is a corresponding eigenfunction.

Main Theorem

The principal result of this subsection is the following theorem:

Theorem 24.4.13. *Let conditions* $(24.4.11)_{1-3}$, (24.4.14), (24.4.15), $(24.4.21)_{1-3}$, $(24.4.22)_{1-2}$, (24.4.24)–(24.4.28) *be fulfilled. Moreover, let*

$$|\nabla W|\ge \epsilon_0\zeta^2\gamma^{-1} \qquad \textit{as}\quad |x|\ge c. \tag{24.4.32}$$

Then

$$|\mathsf{N}^-(\eta)-\mathcal{N}^-(\eta)|\le C\int_{\mathcal{Z}(\eta)}\gamma^{-2}\,dx',+C\int\gamma^{-s}\,dx' \tag{24.4.33}$$

where

$$\mathcal{N}^-(\eta)\coloneqq(2\pi)^{-r}\int_{\{x:\,-\lambda\ge\eta\}} f_{\infty,1}f_{\infty,2}\cdots f_{\infty,r}\,dx' \tag{24.4.34}$$

$\mathcal{Z}(\eta)$ *is* $\epsilon\gamma$*-vicinity of* $\Sigma(\eta)=\{x':\,-\lambda(x')=\eta\}$*; cf.* [34] *of Chapter 23.*

Proof. (a) We know that

$$\mathsf{N}^-(\eta)=\mathsf{N}^-(A-f^*_\infty+\eta)=\mathsf{N}^-(\mathsf{A}_\eta)=\mathsf{Tr}(E_\eta(0)), \tag{24.4.35}$$

where

$$\mathsf{A}_\eta\coloneqq J^{-\frac12}(A-f^*_\infty+\eta)J^{-\frac12} \tag{24.4.36}$$

is a self-adjoint operator in $\mathscr{L}^2(\mathbb{R}^d)$ and $E_\eta(\tau)$ is the spectral projector of this operator and $J\asymp\rho^2$ is such that

$$|\nabla^{\alpha}J|=O(J\gamma^{-|\alpha'|}\gamma_1^{-|\alpha_1|}) \qquad \forall\alpha. \tag{24.4.37}$$

We consider only $\eta > 0$ and for any fixed $\eta > 0$ and τ this projector is finite-dimensional and its Schwartz kernel belongs to $\mathscr{S}(\mathbb{R}^{2d})$ uniformly on $\tau \le \tau_0$. Let us note that

$$((A-f_\infty^*)v, v) \ge (1-\epsilon)((A_\infty - f_\infty^*)v, v) - C\|\rho v\|^2 \qquad \forall v \in \mathscr{C}_0^2(\mathbb{R}^d) \tag{24.4.38}$$

where $(A_\infty - f_\infty^*)$ is non-negative.

Indeed, without any loss of the generality one can assume that

$$A_\infty = D_1^2 + \sum_{1\le j\le p} \big(D_{2j}^2 + (D_{2j+1} + f_j x_{2j})^2\big). \tag{24.4.39}$$

Then

$$A_\infty - f_\infty = D_1^2 + \sum_{1\le j\le p} Z_j^* Z_j \tag{24.4.40}$$

with

$$Z_j = iD_{2j} + (D_{2j+1} + f_j x_{2j}). \tag{24.4.41}$$

On the other hand, $A - A_\infty$ is a linear combination of D_1^2, $D_1 Z_j$, $D_1 Z_j^*$, $Z_j^* Z_k$, $Z_j Z_k^*$, $Z_j Z_k$, $Z_j^* Z_k^*$, Z_j, Z_j^* and 1 with the coefficients β_* satisfying

$$|\nabla^\alpha \beta_*| = O(\rho^2 \gamma^{-|\alpha'|}\gamma_1^{-\alpha_1}) \qquad \forall \alpha. \tag{24.4.42}$$

Then extra terms in (Av, v) do not exceed

$$C\Big(\|\rho D_1 v\|^2 + \sum_j \|\rho Z_j v\|^2 + \sum_j \|\rho Z_j^* v\|^2 + \|\rho v\|^2\Big),$$

where obviously

$$\|\rho Z_j^* v\|^2 \le C\Big(\|\rho Z_j v\|^2 + \|\rho v\|^2\Big).$$

This inequality (24.4.38) immediately yields estimates

$$\|D_1 J^{-\frac12} E_\eta(\tau)\| \le C, \qquad \|Z_j J^{-\frac12} E_\eta(\tau)\| \le C \quad j = 1, \dots, p, \tag{24.4.43}_{1,2}$$

$$\|J^{-\frac12} E_\eta(\tau)\| \le C\eta^{-\frac12}, \qquad \|E_\eta(\tau)\| \le 1 \tag{24.4.43}_{3,4}$$

and therefore

$$\|Z_j^* J^{-\frac12} E_\eta(\tau)\| \le C\eta^{-\frac12}, \quad \|Z_j^* E_\eta(\tau)\| \le C \tag{24.4.43}_{5,6}$$

for operator norms where here and below $\tau \le \tau_0$.

Then one can prove easily that

(24.4.44) Let Q be a product of several factors D_1, $Z_\bullet$ and $Z_\bullet^*$. Then $\|QJ^{-\frac12}E_\eta(\tau)\| \le C$ provided there more of factors D_1, $Z_\bullet$ than of $Z_\bullet^*$, and $\|QJ^{-\frac12}E_\eta(\tau)\| \le C\eta^{-\frac12}$, $\|QE_\eta(\tau)\| \le C$ provided there as many of factors D_1, $Z_\bullet$ as of $Z_\bullet^*$.

Then this claim remains true for Q replaced by $Q' := D_{x'}^\alpha Q$ for any α and then in virtue of the embedding theorem this is also true for the operator norm from $\mathscr{L}^2(\mathbb{R}^d) \mapsto \mathscr{L}^2(\mathbb{R}^d)$ replaced by the operator norm from $\mathscr{L}^2(\mathbb{R}^d) \mapsto \mathscr{L}^2(\mathbb{R})$ taken over any magnetic line $\{x\colon x' = y'\}$ uniformly with respect to y'. In particular,

$$\|D_1 J^{-\frac12}E_\eta(\tau)v\|_{\mathscr{L}^2(\mathbb{R})} \le C\|v\| \qquad \text{and} \qquad \|E_\eta(\tau)v\|_{\mathscr{L}^2(\mathbb{R})} \le C\|v\|$$

and therefore

$$|\bigl(E_\eta(\tau)v\bigr)(x)| \le \underbrace{CJ^{\frac12}(x)\Bigl(J_0^{-\frac12}(x')\gamma_1^{-1} + \langle x_1\rangle^{\frac12}\Bigr)}\,\|v\|$$

with $J_0(x') = \max_{x_1} J(x_1, x')$. So, we estimated the operator norm of $w \to (E_\eta(\tau)w)(x)$ from $\mathscr{L}^2(\mathbb{R}^d)$ to $\mathbb{C}$; therefore

$$|e_\eta(x,x,\tau)| \le CJ(x)\Bigl(J_0^{-1}(x')\gamma_1^{-1} + \langle x_1\rangle\gamma_1^2\Bigr)$$

and therefore $|\int e_\eta(x,x,\tau)\,dx_1| \le C(1+\varrho^2(x')\gamma_1^2) \le C$ due to properties of J, ϱ and ϱ_1. Therefore we have proven:

Proposition 24.4.14. *In the framework of Theorem 24.4.13 and the definitions of J and $e_\eta(x,y,\tau)$,*

$$\int_{|x'|\le r} e_\eta(x,x,\tau)\,dx = O(1)$$

for all $\eta > 0$, $\tau \le \tau_0$ and for any fixed r and τ_0.

Recall that $e_\eta(x,y,\tau)$ is the Schwartz kernel of $E_\eta(\tau)$. So, we only need to treat the contribution of the zone $\{x\colon |x'| \ge r\}$.

(b) Let us fix $y' \in \mathbb{R}^d$ and consider $\psi(x')$, $\psi \in \mathscr{C}_0^K\bigl(B(y', \frac12\gamma)\bigr)$ with $\gamma = \gamma(y')$ such that $|D^\alpha\psi| \le c\gamma^{-|\alpha'|}$ $\forall \alpha' : |\alpha'| \le K$. We want to derive asymptotics of

$$\int \psi(x')e_\eta(x,x,0)\,dx = \int \psi\,\mathsf{Tr}_{\mathbb{H}}\bigl(e_\eta(x',x',0)\bigr)\,dx' = \mathsf{Tr}(\psi E_\eta(0)), \tag{24.4.45}$$

where $\mathsf{e}_\eta(x',y',\tau)$ is the family of operators in $\mathbb{H}$ with Schwartz kernel $e_\eta(x,y,\tau)$. Let us rescale $x'_{\mathsf{new}} = (x'-y')\gamma^{-1}$, $x_{1\mathsf{new}} = x_1\gamma_1^{-1}$. Then we obtain the standard LSSA problem for an operator with an operator-valued symbol, with the semiclassical parameters $h=\gamma^{-1}$ and $h_1=\gamma_1^{-1}$ and with magnetic field intensity parameter $\mu=\gamma$. Recall that the rescaled operator is

$$h_1^2D_1^2 + \sum_{1\le j\le p}\Bigl(h^2D_{2j}^2 + \bigl(hD_{2j+1} - h^{-1}f_jx_{2j}\bigr)^2\Bigr) + \rho^2A', \tag{24.4.46}$$

where $A'=a'(x,h_1D_1,hD',h)$ is an operators with uniformly smooth symbol a' (we consider it more carefully later). Let $U(x,y,t)$ be the Schwartz kernel of the operator $\exp(ih^{-2}t\mathsf{A}_\eta)$. Later we rescale t. Then

$$(h^2D_t - \mathsf{A}_\eta)U=0, \qquad U|_{t=0} = \gamma^{2p}\delta(x-y)I \tag{24.4.47}$$

So let ψ be a γ-admissible partition element. It follows from (24.4.44) that the operator norm (from $\mathscr{L}^2(\mathbb{R}^d)$ to $\mathscr{L}^2(\mathbb{R}^d)$) of $Q\psi E_\eta(\tau)Q^*$ does not exceed C for the operators Q which are products of several factors h_1D_1, $Z_\bullet$ and $Z_\bullet^*$ [29] and there are more factors of h_1D_1, $Z_\bullet$ and than of $Z_\bullet^*$. Then the operator norm of $F_{t\to h^{-2}\tau}\chi_T(t)Q\psi UQ^*$ does not exceed CT for the operators Z listed above where $\chi\in \mathscr{C}_0^K(\mathbb{R})$ is fixed and $T\ge T_0$ with constant $T_0>0$.

Let us apply the transformation $\mathcal{T}=\mathcal{T}_0^{-1}\mathcal{T}_1\mathcal{T}_0$ where $\mathcal{T}_0 = F_{x'''\to h^{-2}\xi'''}$, $x'=(x'',x''')$, $x''=(x_2,x_4,\dots,x_{d-1})$, $x'''=(x_3,x_5,\dots,x_d)$ and the same partition for $\xi'=(\xi'',\xi''')$, and

$$\mathcal{T}_1v(x_2,\xi_3,x_4,\dots,\xi_d) = v(x_2-f_1^{-1}\xi_3,\xi_3,\dots,x_{d-1}-f_p^{-1}\xi_d,\xi_d). \tag{24.4.48}$$

Then instead of hD_{2j} and $(hD_{2j+1}-h^{-1}f_jx_{2j})$ we obtain hD_{2j} and $-h^{-1}x_{2j+1}$ respectively. Let Ψ be the corresponding linear symplectic transformation. Let $\bar U = \mathcal{T}_x\psi' U^t\mathcal{T}_y$ where ψ' is supported in $B(0,1-\epsilon)$ [30] and equals 1 in $B(0,1-2\epsilon)$.

Let us decompose $U(x,y,t)$ in terms of the functions

$$\Upsilon_\varsigma(x'') = h^{-\frac12}v_{\varsigma_1}(x_2h^{-1})h^{-\frac12}v_{\varsigma_2}(x_4h^{-1})\cdots h^{-\frac12}v_{\varsigma_p}(x_{2p}h^{-1}) \tag{24.4.49}$$

and $\Upsilon_\nu(y'')$:

$$\bar U(x,y,t) = \sum_{\varsigma,\nu\in\mathbb{Z}^+}\Upsilon_\varsigma(x'')\Upsilon_\nu(y'')U_{\varsigma\nu}(x_1,x''';y_1,y''';t). \tag{24.4.50}$$

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[29] Due to (24.4.41) now $Z_j = ihD_{2j} + (hD_{2j+1}+h^{-1}f_jx_{2j})$.

[30] We shifted the coordinate system so that our partition element is supported there.

We make the same decomposition for $E(x,y,\tau)$. Then the above estimates yield that

(24.4.51) The operator norm of $F_{t\to h^{-2}\tau}\chi_T(t)U_{\varsigma\nu}$ does not exceed CT.

Next, the standard ellipticity arguments show that

(24.4.52) The operator norm of[31)] $F_{t\to h^{-2}\tau}\chi_T(t)J^{-\frac12}(x_1)U_{\varsigma\nu}$ does not exceed $C\varrho^{|\varsigma|-1}T$ for $\varsigma\neq 0$, and also the operator norm of $F_{t\to h^{-2}\tau}\chi_T(t)J^{-\frac12}(y_1)U_{\varsigma\nu}$ does not exceed $C\varrho^{|\nu|-1}T$ for $\nu\neq 0$, and, finally, the operator norm of $F_{t\to h^{-2}\tau}\chi_T(t)J^{-\frac12}(x_1)J^{-\frac12}(y_1)U_{\varsigma\nu}$ does not exceed $C\varrho^{|\nu|+|\varsigma|-2}T$ for $\nu\neq 0$ and $\varsigma\neq 0$ and the same is true if we apply $D^k_{x_1}$ and $D^l_{y_1}$.

Moreover, for $\varsigma=\nu=0$ we have

(24.4.53) The operator norm of $F_{t\to h^{-2}\tau}\chi_T(t)D^k_{x_1}J^{-\frac12}(x_1)U_{00}$ does not exceed CT for $k\geq 1$, and also the operator norm of $F_{t\to h^{-2}\tau}\chi_T(t)D^l_{y_1}J^{-\frac12}(y_1)U_{00}$ does not exceed CT for $l\geq 1$, and, finally, the operator norm of

$$F_{t\to h^{-2}\tau}\chi_T(t)D^k_{x_1}D^l_{y_1}J^{-\frac12}(x_1)J^{-\frac12}(y_1)U_{00}$$

does not exceed CT for $k\geq 1$ and $l\geq 1$.

Then

$$\mathsf{Tr}(\psi E)=\sum_{\varsigma,\nu} h^{|\varsigma-\nu|}\,\mathsf{Tr}(\psi_{\varsigma\nu}E_{\varsigma,\nu})+O(h^s) \tag{24.4.54}$$

where we have the original expression on the left-hand side, $\psi_{\varsigma\nu}=\psi_{\varsigma\nu}(x''',h^2D''',h^2)$, $\psi_{\varsigma\varsigma}(x''',\xi''',0)=\psi(x''',\xi''')$. Moreover, $\mathsf{supp}(\psi_{\varsigma\nu})\subset\mathsf{supp}(\psi)$ and one can replace $\psi_{\varsigma\nu}-\delta_{\varsigma\nu}\psi$ by a linear combination of the derivatives of ψ of non-zero order.

[31)] In the obvious situations we do not distinguish operators and their Schwartz kernels.

(c) It follows from Proposition 24.A.1 that operator $(A - f^*_\infty)J^{-1}$ is elliptic outside of $\mathcal{Z}(\eta)$ and then one can prove easily that the total contribution of $\mathbb{R}^d \setminus \mathcal{Z}(\eta)$ to the remainder does not exceed

$$C \int \gamma^{-s}\, dx', \tag{24.4.55}$$

while its contribution to the principal part of asymptotics is given by the Tauberian expression.

(d) From now on ψ' is a partition element in $\mathcal{Z}(\eta)$. Recall that

$$(h^2 D_t - J^{-\frac{1}{2}} A J^{-\frac{1}{2}}) = 0, \qquad U|_{t=0} = \delta(x-y). \tag{24.4.56}$$

and the dual equation with respect to y. Then using ellipticity arguments we can express $U_{\varsigma\nu}$ with $|\varsigma| + |\nu| \ge 1$ via U_{00} via some (h_1, h^2)-pseudodifferential operators (with respect to (x_1, x''')) and h_1^2 and then plugging back into equation we get

$$(h^2 D_t - J_0^{-\frac{1}{2}} A_0 J_0^{-\frac{1}{2}}) = 0, \qquad U_{00}|_{t=0} = M\delta(x-y). \tag{24.4.57}$$

where A_0 differs from $h_1^2 D_1 + V^* + \eta$, with $V^* = V + f^* - f^*_\infty$ by $o(\rho^2)$; from the beginning we could assume that $g^{11} = 1$. Here J_0 and A_0 are (h_1, h^2)-pseudodifferential operators (with respect to (x_1, x'''))-pseudodifferential operators. Let $\hbar := h^2$. Let us observe that in virtue of Proposition 24.A.1 the operator $J_0^{-\frac{1}{2}} A_0 J_0^{-\frac{1}{2}}$ has discrete spectrum in $\mathbb{H}$ and all the eigenvalues of this operator excluding at most one are positive and uniformly disjoint from 0 and there is one (the lowest) eigenvalue $\Lambda = \Lambda(x''', \xi''', \eta)$ which is $O(1)$; moreover, due to (24.4.32) it satisfies the microhyperbolicity condition

$$|\Lambda| + |\nabla \Lambda| \asymp 1. \tag{24.4.58}$$

Then there exists a symbol $q(x''', \xi''', \hbar) : \mathbb{H} \to \mathbb{C} \oplus \mathbb{H}$ such that for the operator $Q = q(x''', \hbar D''', \hbar)$ and for $U' = QU_{00}Q^*$ we obtain separate equations for all four blocks of $U' = \begin{pmatrix} U'_{00} & U'_{01} \\ U'_{10} & U'_{11} \end{pmatrix}$. Moreover, for the blocks U'_{10} and U'_{11} the equations are elliptic for $\tau \le \epsilon_1 \hbar$ and for U'_{10} and U'_{11} this is true for the dual equations. Therefore $U' \equiv \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}$ with $u = U'_{00}$ and

$$F_{t\to\hbar^{-1}\tau}\chi_T(t)\,\mathsf{Tr}(\psi U) \equiv F_{t\to\hbar^{-1}\tau}\chi_T(t)\,\mathsf{Tr}(\psi'' u) \tag{24.4.59}$$

for $\tau \le \epsilon_0 h^2$, $T \in (h^{-\delta}, \epsilon_0)$ where $\psi'' = \psi''(x''', h'''D''', \hbar)$. We have an equation for u:

$$(\hbar D_t - \Lambda)u = 0 \tag{24.4.60}$$

where Λ is an $\hbar$-pseudodifferential operator. More precisely: due to the microhyperbolicity we conclude that

(24.4.61) The contribution of the partition element to the final answer is given by Tauberian expression with $T = \hbar^{1-\delta}$ with an error $O(1)$.

Therefore, the total contribution of $\mathcal{Z}(\eta)$ to the remainder does not exceed $C \int_{\mathcal{Z}(\eta)} \gamma^{-2}\, dx$ (in the original coordinates).

(e) Employing the method of the successful approximations and picking $\psi = 1$, and we conclude that the final answer is given by (24.4.34) since since $\Lambda < 0 \iff \lambda < -\eta$. We leave easy details to the reader. □

Remark 24.4.15. If $V^*(x) \asymp |x|^{-2}$ then (formally) $W(x') \asymp |x'|^{-1}$ and $\lambda(x') \asymp |x'|^{-2}$ and $\mathcal{N}(\eta) \asymp \eta^{-r}$ as follows from the results of Subsection 24.4.1.

Generalizations

Remark 24.4.16. (i) For the Schrödinger-Pauli operator Theorem 24.4.13 obviously holds albeit with $f^* = f^*_\infty = 0$.

(ii) The same is true for the Dirac operator. The proof is essentially the same. We need to assume that the mass $M \neq 0$, otherwise the spectral gap $(-M, M)$ is empty. Then we consider $\mathsf{N}^-(\eta) = \mathsf{N}(0, M-\eta)$ and $\mathsf{N}^+(\eta) = \mathsf{N}(-M+\eta, 0)$. Instead of 0 we can take any $\bar\tau \in (-M, M)$ which preserves the result modulo $O(1)$.

Let us consider $\mathsf{N}^-(\eta)$. Modulo $O(1)$ it equals to $\tilde{\mathsf{N}}(\eta; -\epsilon_2, 0)$, where $\tilde{\mathsf{N}}(\eta; \tau_1, \tau_2)$ is the number of eigenvalues of the problem

$$(A - M + \eta)v + \tau \mathsf{J} v = 0 \tag{24.4.62}$$

belonging to the interval (τ_1, τ_2) and $\mathsf{J} = \frac{1}{2}(I + \sigma_0)J$ where J was introduced in the proof of Theorem 24.4.13. This problem is equivalent to the problem

$$(\mathcal{A}_\eta - \tau J)w = 0, \qquad \mathcal{A}_\eta = \mathcal{L}^*(2M - \eta - V)^{-1}\mathcal{L} + V + \eta, \tag{24.4.63}$$

where we assume that

$$\sigma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \sigma_j = \begin{pmatrix} 0 & \sigma_j' \\ \sigma_j' & 0 \end{pmatrix}, \quad \sigma_j'^* = \sigma_j' \ (j = 1, \dots, d) \tag{24.4.64}$$

and

$$\mathcal{L} = \sum_{1\le j\le d} \frac{1}{2}(P_k\omega^{jk} + \omega^{jk}P_k)\sigma_j'. \tag{24.4.65}$$

One can easily transform $\mathcal{A}_\eta$ to the form of the Schrödinger-Pauli operator with the metric $\tilde{g}^{jk} = (2M - \eta - V)^{-1}g^{jk}$.

Example 24.4.17. (i) In the standard isotropic case $\gamma_1 = \gamma = \langle x' \rangle$ and as $\rho(x) = \langle x \rangle^l$ with $l < -2$; then for W defined by (24.4.28) satisfies similar conditions with $m = 2l + 1$ and thus we are in the framework of Example 23.4.4.

(ii) However, we can also consider $\rho(x) = \langle x \rangle^l \langle x' \rangle^p$ with $l < -2$; then $m = 2l + 2p + 1$.

(iii) We can also consider faster and slower decaying potentials, as soon as W^2 satisfies conditions imposed on V in Subsection 24.4.1. See Example 23.4.5, Problems 23.4.8 and 23.4.10.

Example 24.4.18. Let us consider the case when at infinity f_j stabilize not to $f_\infty = \mathsf{const}$ but to $f_\infty(\theta)$, $\theta = x'/|x'| \in \mathbb{S}^{d-2}$.

Then as long as other assumptions are fulfilled, we arrive to asymptotics described in Theorem 23.4.13; again, instead of $V(x')$ we have $\Lambda(x')$ in the conditions and in the expression for $\mathcal{N}^-(\eta)$.

Remark 24.4.19. Let us consider an auxiliary operator with potential V which is $\asymp |x_1|^{-2}$ as $x_1 \to \infty$. One can easily prove (see Proposition 24.A.3 below) that if

$$V^* \ge -\frac{1}{4}|x|^{-2} \qquad \forall x\colon |x| \ge C, \tag{24.4.66}$$

then the number of negative eigenvalues is finite and there is no more than one negative eigenvalue if this inequality holds for all x. Moreover, under the conditions

$$V^* \ge (\epsilon - \frac{1}{4})|x|^{-2} \qquad \forall x\colon |x| \ge C \tag{24.4.66}^*$$

and (24.A.12) with arbitrarily small $\epsilon > 0$ all the statements of Proposition 24.A.1 remain true. Furthermore, under condition (24.4.66)*

$$\langle \mathsf{a}v, v\rangle \ge \frac{\epsilon}{2}\|\langle x\rangle^{-1}v\|^2 - C\|\langle x\rangle^{-s}v\|^2 \qquad \forall v \tag{24.4.67}$$

with arbitrarily large s. Therefore we can cover the case

$$\rho(x) = \langle x\rangle^{-2}\langle x'\rangle^{p+2}, \qquad p < -1 \tag{24.4.68}$$

provided

$$V^* \ge (\epsilon - \frac{1}{4})|x|^{-2} \qquad \forall x\colon |x_1| \ge c|x'| \tag{24.4.69}$$

with arbitrarily small $\epsilon > 0$. The remainder estimate is the same $O(1)$ as above. The details are left to the reader.

Remark 24.4.20. Let $\mathsf{rank}\,\mathsf{F}(x) = 2r \le d-2$ (as $|x| \ge c$). Then the auxiliary operator is $(d-2r)$-dimensional and does not have negative eigenvalues at all in the assumptions of this subsection.

Then one can prove easily that $\mathsf{N}^-(\eta) = O(1)$. In particular, if $\gamma_1 = \gamma = \langle x'\rangle$ and $\rho = \langle x'\rangle^m$, $\mathsf{N}^-(\eta) = O(1)$ for $m < -1$ (and even for $m = -1$ under assumption (24.4.69) but there is a non-trivial asymptotics for $m > -1$; see Subsection 24.4.3 below.

Possible Generalizations

Consider the case when condition (24.4.28) is not fulfilled. We believe that while the Part (i) is not extremely challenging, the Part (ii) is:

Problem 24.4.21. (i) Prove that the main part of the asymptotics is still given by (24.4.34).

(ii) Prove that (24.4.33) still holds.

24.4.3 Case $\mathsf{rank}\, F_\infty = d-1$. Slow Decaying Potential

Now we consider the case as in the previous Subsection 24.4.2 but we assume that the potential V which either decays slower than x_1^{-2} or as x_1^{-2} but fails condition (24.4.66)*. We only sketch the main arguments.

Main Theorem (Statement)

We assume that

$$(24.4.70)_1 \quad |\nabla^\alpha (\mathsf{g} - \mathsf{g}_\infty)| = o(\rho^2 \langle x'\rangle^{-|\alpha'|} \langle x\rangle^{-\alpha_1}),$$

$$(24.4.70)_2 \quad |\nabla^\alpha V| = O(\rho^2 \langle x'\rangle^{-|\alpha'|} \langle x\rangle^{-\alpha_1}),$$

$$(24.4.70)_3 \quad |\nabla^\alpha (F_{jk} - F_{\infty,jk})| = O(\rho^2 \langle x'\rangle^{-|\alpha'|} \langle x\rangle^{-\alpha_1}) \qquad \text{as } |x| \to \infty \quad \forall \alpha$$

where

$$\rho = \langle x\rangle^{-q} \langle x'\rangle^{m+q} \tag{24.4.71}$$

$q > 0$, $m < 0$. Further, as $m + q = 0$ we assume in addition that

$$(24.4.72)_1 \quad |\nabla^\alpha (\mathsf{g} - \mathsf{g}_\infty)| = O(\rho^2 \langle x'\rangle^{2-|\alpha'|} \langle x\rangle^{-2-\alpha_1}),$$

$$(24.4.72)_2 \quad |\nabla^\alpha V| = O(\rho^2 \langle x'\rangle^{2-|\alpha'|} \langle x\rangle^{-2-\alpha_1}),$$

$$(24.4.72)_3 \quad |\nabla^\alpha (F_{jk} - F_{\infty,jk})| = O(\rho^2 \langle x'\rangle^{2-|\alpha'|} \langle x\rangle^{-2-\alpha_1}) \quad \text{as } |x| \to \infty \quad \forall \alpha' : |\alpha'| \ge 1.$$

Theorem 24.4.22. *Let conditions* $(24.4.70)_{1-3}$ *be fulfilled. Let one of two assumptions be fulfilled:*

(i) $m + q < 0$ *and*

$$-\langle x', \nabla' V^*\rangle \ge \epsilon \rho^2 \qquad \forall x\colon |x'| \ge C_0. \tag{24.4.73}$$

(ii) $m + q = 0$, *conditions* $(24.4.72)_{1-3}$ *be also fulfilled*

$$-\langle x', \nabla' V^*\rangle \ge \epsilon \rho^2 \langle x'\rangle^2 \langle x\rangle^{-2} \qquad \forall x\colon |x'| \ge C_0. \tag{24.4.74}$$

Then

$$\mathsf{N}^-(\eta) = \mathcal{N}^-(\eta) + O(R(\eta)) \tag{24.4.75}$$

with

$$\mathcal{N}^-(\eta) = (2\pi)^{-r} \int \boldsymbol{n}(x',\eta) f_{\infty,1} f_{\infty,2} \cdots f_{\infty,r}\, dx' \tag{24.4.76}$$

and

$$R(\eta) = \int_{\Lambda(\eta)} \bigl(\boldsymbol{n}(x',\eta)+1\bigr)\langle x'\rangle^{-2}\, dx' + \int \gamma^{-s}\, dx' \tag{24.4.77}$$

where $\boldsymbol{n}(x',\eta)$ *is the number of eigenvalues of the operator* $\mathcal{L}(x')$ *which are less than* $-\eta$ *and* $\Lambda(\eta)$ *is* $\epsilon\gamma$*-vicinity of* $\Sigma(\eta) = \{x' \colon \boldsymbol{n}(x',\eta) > 0\}$, $\gamma = \langle x'\rangle$ *and* $\mathcal{L}(x')$ *is defined by* (24.4.19)–(24.4.20).

Proof of Theorem 24.4.22: Propagation of Singularities

Again let us consider the number of negative eigenvalues of operator A_η, defined by (24.4.36) with

$$J(x) = j(x')\langle x\rangle^{-2p}\langle x'\rangle^{2p} \qquad p = \begin{cases} q & \text{if } \ m+q<0,\\ q+1 & \text{if } \ m+q=0 \end{cases} \tag{24.4.78}$$

and γ-admissible function $j(x')$. Let $\rho = \langle x'\rangle^{2m}$. As in the previous Subsection 24.4.2 we consider A_η as operator in $\mathscr{L}^2(\mathbb{R}^{2r},\mathbb{H})$ with $\mathbb{H} = \mathscr{L}^2(\mathbb{R},\mathbb{C})$. As usual after proper scaling $h = \rho^{-1}\gamma^{-1}$ and $\mu = \rho^{-1}\gamma$.

Again let consider the corresponding propagator. Our first goal is to estimate the propagation speed with respect to x' from above and then to estimate it under the microhyperbolicity condition also from below.

Proposition 24.4.23. *Let assumptions* $(24.4.70)_{1-3}$ *be fulfilled with* $m<0$ *and* $0<q\le 1$. *Further, let for* $m+q=0$ *assumptions* $(24.4.72)_{1-3}$ *be fulfilled as well.*

Then the propagation speed with respect to x' *does not exceed* $C_0 j^{-1}\rho^2\gamma^{-1}$ *(before scaling) and therefore singularity initially supported in* $B(y',\frac{1}{2}\bar\gamma)$ *is confined to* $B(y',\gamma)$ *for* $T \le T^* = \epsilon j \rho^{-2}\gamma^2$ *calculated at* y'*:*

$$\bigl\|\!\bigl| F_{t\to h^{-1}\tau}\bar\chi_T(t)\psi(x')(1-\psi_0(y'))u(x,y,t)\bigr|\!\bigr\| \le CT\gamma^{-s} \tag{24.4.79}$$

provided $\psi \in \mathscr{C}_0^\infty B(y',\frac{1}{2}\gamma)$, $\psi_0 \in \mathscr{C}_0^\infty B(y',\gamma)$, $\psi_0 = 1$ *in* $B(y',\gamma)$, $\gamma = \gamma(y')$, $|\tau|\le\epsilon$, $\|\!|\cdot|\!\|$ *is a standard operator norm from* $\mathscr{L}^2(\mathbb{R}^d)$ *to* $\mathscr{L}^2(\mathbb{R}^d)$.

Proposition 24.4.24. *In the framework of Proposition 24.4.23 assume that the microhyperbolicity assumption* (24.4.73) *is fulfilled for* $m+q<0$ *and the microhyperbolicity assumption* (24.4.74) *is fulfilled for* $m+q=0$. *Then the propagation speed with respect to* x' *in an appropriate direction is greater than* $\epsilon_1 j^{-1}\rho^2\gamma^{-1}$ *(before scaling) and therefore*

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma'(u\psi)|\le C\gamma^{d-1}(T/T_*)^{-s}\qquad \text{for } |\tau|\le\epsilon \tag{24.4.80}$$

where, as usual, $\chi\in \mathscr{C}_0^\infty([-1,-\frac12]\cup[\frac12,1])$, $\psi\in \mathscr{C}_0^\infty(B(x',\frac12\gamma(x'))$ *and* $T\in[T_*,T^*]$, $T_*=cj^{-1}\rho^{-2}$ *and* $\gamma=\gamma(y')$.

Proofs of Propositions 24.4.23 and 24.4.24. Standard proofs are left to the reader. We just observe the following:

If $m+q<0$ then due to $(24.4.70)_{1-3}$ and (24.4.78) the drift speed with respect to x' does not exceed $cj^{-1}\rho^2\gamma^{-1}$ (Proposition 24.4.23) and is exactly of this magnitude due to (24.4.73) (Proposition 24.4.23). This is also true for $m+q=0$ due to $(24.4.72)_{1-3}$ and (24.4.78), and (24.4.74). □

Proof of Theorem 24.4.22: Estimates

Now we can use the standard successive approximations method leading us to the estimate

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma'(u\psi)|\le C\gamma^{d-1} \tag{24.4.81}$$

and then to estimate

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma(u\psi)|\le C\gamma^{d-1}N \tag{24.4.82}$$

as $T_*\le T\le T^*$ and $|\tau|\le\epsilon$, where $N=\sup_{B(y',(1+\epsilon)\gamma)}\boldsymbol{n}(x',\eta)$. The latter estimate leads us to the estimate (24.4.77) to the Tauberian error and in the conjugation with successive approximation method to the estimate (24.4.77) itself.

Discussion

Remark 24.4.25. If we replace $\boldsymbol{n}(x',\eta)$ by the corresponding 1D-Weyl expression, we will get

$$\mathsf{N}^-(\eta)=\mathcal{N}^{\mathsf{MW}\,-}(\eta)+O(R(\eta)+R_1(\eta)) \tag{24.4.83}$$

with

$$\mathcal{N}^{\mathsf{MW}-}(\eta) = (2\pi)^{-r} \int (-V^* - \eta)_+^{\frac{1}{2}} / \sqrt{g^{11}} f_{\infty,1} f_{\infty,2} \cdots f_{\infty,r}\, dx \tag{24.4.84}$$

and

$$R_1(\eta) = \int_{\Lambda(\eta)} dx'. \tag{24.4.85}$$

Remark 24.4.26. (i) Obviously $V(x) = -\langle x\rangle^{-2q} U(x')$ with U positively homogeneous of degree $2(m+q)$ satisfies $(24.4.70)_{1-3}$, and for $q+m=0$ it also satisfies $(24.4.72)_{1-3}$.

(ii) Furthermore, if $U \asymp \langle x'\rangle^{2(m+q)}$ this V satisfies (24.4.73) if $q+m<0$ and (24.4.74) if $q+m=0$.

(iii) On the other hand, it does not satisfy (24.4.73) if $q+m>0$.

Example 24.4.27. Let us evaluate magnitudes of $\mathcal{N}^-(\eta)$, $R(\eta)$ and $R_1(\eta)$, defined by (24.4.76), (24.4.77) and (24.4.85). To do this consider $\boldsymbol{n}(x',\eta)$. We are interested only in the case of $0 < q \le 1$ since $q>1$ combined with $m+q \le 0$ would imply $m<-1$ and this is covered by previous Subsection 24.4.2. One can see easily that in this case Theorem 24.4.22 is a special case of Theorem 24.4.13 as $\boldsymbol{n}(x',\eta) \le 1$ for $|x'| \ge c$ and $\boldsymbol{n}(x',\eta) \le C$ as $|x'| \le c$.

Assume that $0<q<1$. Then $\boldsymbol{n}(x',\eta) \asymp \eta^{(q-1)/(2q)} \gamma^{(m+q)/q}$ with $\gamma = \langle x'\rangle$ as $\gamma \le \epsilon \min(\bar\gamma_1, \bar\gamma_2)$ with

$$\bar\gamma_1 = \eta^{(1-q)/(2(m+q))}, \qquad \bar\gamma_2 = \eta^{1/(2m)} \tag{24.4.86}$$

and relying upon Proposition 24.A.4 we conclude that

$$\boldsymbol{n}(x',\eta) \asymp \eta^{(q-1)/(2q)} \gamma^{(m+q)/q} \qquad \text{as } \ \gamma \le \epsilon \min(\bar\gamma_1, \bar\gamma_2) \tag{24.4.87}$$

provided $m+q<0$. On the other hand, obviously $\boldsymbol{n}(x',\eta)=0$ as $\gamma \ge C\bar\gamma_2$. Recall that $m<0$. Observe that $\bar\gamma_1 \ge \bar\gamma_2$ for $\eta \le 1$ if and only if $m \in [-1,0)$.

(a) Let $m \in [-1,0)$. Then (24.4.87) holds as $\gamma \le \epsilon\bar\gamma_2$ and we conclude that

$$\mathcal{N}^-(\eta) \asymp \eta^{(d+m)/(2m)}, \quad R(\eta) \asymp \eta^{(d+m-2)/(2m)}, \quad R_1(\eta) = \eta^{(d-1)/(2m)}. \tag{24.4.88}_{1-3}$$

(b) Let $m < -1$. Then (24.4.87) holds as $\gamma \le \epsilon\bar{\gamma}_1$ and contributions of the zone $\mathcal{X}_0 = \{x\colon |x'| \lesssim \bar{\gamma}_1\}$ to $\mathcal{N}^-(\eta)$ and $R(\eta)$ are respectively of the magnitudes $\eta^{(1-q)(d-1)/2(m+q)}$ and $\eta^{(1-q)(d-3)/2(m+q)}$.

We need to consider the zone $\mathcal{X}_1 := \{x\colon \bar{\gamma}_1 \lesssim |x'| \lesssim \bar{\gamma}_2\}$ separately.

(c) If $1 > q > \frac{1}{2}$ we in virtue of Proposition 24.A.6 $\boldsymbol{n}(x',\eta) \asymp 1$ if $\gamma \le \epsilon\bar{\gamma}_3$ and $\boldsymbol{n}(x',\eta) = 0$ if $\gamma \ge C\bar{\gamma}_3$ with $\bar{\gamma}_3 = \eta^{1/2(2m+1)}$; one can see easily that $\bar{\gamma}_1 \le \bar{\gamma}_3 \le \bar{\gamma}_2$. Then contribution of $\mathcal{X}_1$ to $\mathcal{N}^-(\eta)$ and $R(\eta)$ are of magnitudes $\bar{\gamma}_3^{d-1}$ and $\bar{\gamma}_3^{d-3}$ respectively except the case $d = 3$ when the contribution to $R(\eta)$ is of magnitude $|\log(\eta)|$.

Combining with Statement (b) we conclude that $\mathcal{X}_1$ contributes more to $\mathcal{N}^-(\eta)$ and $R(\eta)$ than $\mathcal{X}_0$ and therefore

$$\mathcal{N}^-(\eta) \asymp \eta^{(d-1)/2(2m+1)}, \quad R(\eta) \asymp \eta^{(d-3)/2(2m+1)} \tag{24.4.89}_{1,2}$$

except the case $d = 3$ when $R(\eta)$ is of magnitude $|\log(\eta)|$. Meanwhile, $R_1(\eta)$ is of the same magnitude as $\mathcal{N}^-(\eta)$.

(d) If $0 < q \le \frac{1}{2}$ then in virtue of Proposition 24.A.8 with $\varepsilon = \gamma^{2(m+q)}$ we conclude that $\boldsymbol{n}(x',\eta) \asymp 1$ if $\gamma \le \epsilon\bar{\gamma}_1$ and $\boldsymbol{n}(x',\eta) = 0$ if $\gamma \ge C\bar{\gamma}_1$. Combining with Statement (b) we conclude that in this case

$$\mathcal{N}^-(\eta) \asymp \eta^{(1-q)(d-1)/2(m+q)}, \quad R(\eta) \asymp \eta^{-(q-1)(d-3)/2(m+q)}. \tag{24.4.90}_{1,2}$$

Meanwhile, $R_1(\eta)$ is of the same magnitude as $\mathcal{N}^-(\eta)$.

Problem 24.4.28. As $d = 3$ derive remainder estimate $O(1)$ in the framework of Example 24.4.27(c). This analysis should be done in the zone $\mathcal{X}_1$ and we need to consider the spectral stripes $\Lambda_k := \{x'\colon \lambda_k(x') \asymp \eta\}$ and zone $\mathcal{X}_1 \setminus (\Lambda_1 \cup \Lambda_2 \cup ... \cup \Lambda_K)$ separately.

Example 24.4.29. (a) If $q > 1$ then $\boldsymbol{n}(x',\eta) \le C\gamma^{m+1}$ with $m + 1 < 0$ since we need to assume that $m + q \le 0$ in virtue Remark 24.4.26(iii) which puts us in the framework of Subsection 24.4.2.

(b) If $q = 1$, $m < -1$ then $\boldsymbol{n}(x',\eta) = 0$ for $|x'| \ge c$ which leads only to the estimate $\mathsf{N}^-(\eta) \le C|\log\eta|$ rather than asymptotics.

(c) Thus, consider $q = 1$, $m = -1$. Then $\boldsymbol{n}(x', \eta) \lesssim \gamma^{m+1} |\log(\gamma/\bar{\gamma})|$ and $\boldsymbol{n}(x', \eta) = 0$ if $\gamma \ge C\bar{\gamma}$, $\bar{\gamma} = \eta^{-1/2}$.

This leads to

$$\mathcal{N}^-(\eta) \lesssim \eta^{-(d-1)/2}, \qquad R(\eta) \asymp \eta^{-(d-3)/2} \tag{24.4.91}_{1,2}$$

again except $d = 3$, in which case $R(\eta) \asymp |\log \eta|^2$. However, under assumption $V \le -C_0 \langle x \rangle^{-2}$ with sufficiently large C_0, we conclude that $\boldsymbol{n}(x', \eta) \asymp \gamma^{m+1} |\log(\gamma/\bar{\gamma})|$ if $\gamma \le \epsilon\bar{\gamma}$ and then $\mathcal{N}^-(\eta) \asymp \eta^{-(d-1)/2}$.

Problem 24.4.30. Let us consider the case $m + q > 0$. In this case we do not have a microhyperbolicity condition and we can apply only more simple and less precise approach of Subsection 24.4.1. So, we leave to the reader to derive the remainder estimate in the following cases:

(a) Let $0 < q < 1$ [32]. Then, exactly as in Example 24.4.27(a) (24.4.87) holds if $\gamma \le \epsilon\bar{\gamma}_2 = \eta^{1/(2m)}$ and $\boldsymbol{n}(x', \eta) = 0$ if $\gamma \ge \bar{\gamma}_2$ and therefore $(24.4.88)_1$ holds.

(b) Let $q = 1$ [32]. Then we have the same magnitude of $\boldsymbol{n}(x', \eta)$ as in Example 24.4.29(c) and $\mathcal{N}^-(\eta) \asymp \eta^{(d+m)/(2m)}$.

(c) Let $q > 1$. Then for $m < -1$ we are in the framework of Subsection 24.4.2, and for $m = -1$ we are either in the framework of that Subsection or close to it.

So, let us assume that $m \in (-1, 0)$. In this case $\boldsymbol{n}(x', \eta) \lesssim \gamma^{m+1}$, $\boldsymbol{n}(x', \eta) \asymp \gamma^{m+1}$ for $\gamma \le \epsilon\bar{\gamma}_2$ and $\boldsymbol{n}(x', \eta) = 0$ for $\gamma \ge C\bar{\gamma}_2$. Then $(24.4.88)_1$ holds.

Finally, we leave to the reader

Problem 24.4.31. Consider the case of $\operatorname{rank} \mathsf{F} = d - 2r \le d - 2$. To do so we need to modify Theorem 24.4.22 in rather obvious way, in this case $x' = (x''; x') = (x_1, \dots, x_n; x_{n+1}, \dots, x_d)$ with $n = d - 2r$ and $\mathcal{L}(x')$ is n-dimensional Schrödinger operator.

[32] And then $m \in (-1, 0)$.

24.5 Multiparameter Asymptotics

In this section we consider asymptotics with respect to three parameters μ, h and τ; here spectral parameter τ tends either to $\pm\infty$ or to the border of the essential spectrum or to $-\infty$ (for Schrödinger and Schrödinger-Pauli operators) or to the border of the spectrum. In two last cases presence of $h \to +0$ is crucial. We consider here only $d = 2$ and $h \ll 1$.

24.5.1 Asymptotics of Large Eigenvalues

In this subsection $\tau \to +\infty$ for the Schrödinger and Schrödinger-Pauli operators and $\tau \to \pm\infty$ for the Dirac operator. We consider the Schrödinger and Schrödinger-Pauli operators, leaving the Dirac operator to the reader.

Example 24.5.1. Assume first that $\psi \in \mathscr{C}_0^\infty$ and there are no singularities on $\mathsf{supp}(\psi)$. We consider

$$\mathsf{N}_\psi^-(\tau) = \int e(x,x,\tau)\psi(x)\,dx. \tag{24.5.1}$$

Then for scaling $A \mapsto \tau^{-1}A$ leads to $h \mapsto h_{\mathsf{eff}} = h\tau^{-1/2}$ and $\mu \to \mu_{\mathsf{eff}} = \mu\tau^{-1/2}$.

(i) If $\mu \lesssim \tau^{1/2}$ then we can apply the standard theory with the "normal" magnetic field; we need to assume that $h \ll \tau^{1/2}$ and we need neither condition $d = 3$, nor $F \ge \epsilon_0$, nor $\partial X = \emptyset$; the principal part of the asymptotics has magnitude $h^{-d}\tau^{d/2}$ and the remainder estimate is $O\big(h^{1-d}\tau^{(d-1)/2}\big)$ which one can even improve to $o\big(h^{1-d}\tau^{(d-1)/2}\big)$ under proper non-periodicity assumption.

(ii) Let $\mu \gtrsim \tau^{1/2}$, $\mu h \lesssim \tau$. Then we can apply the standard theory with the "strong" magnetic field; we assume that $d = 3$ and $F \ge \epsilon_0$. Then the principal part of the asymptotics has magnitude $h^{-3}\tau^{3/2}$ and under weak non-degeneracy assumption (which is needed, only in the case $\mu_{\mathsf{eff}} \le h_{\mathsf{eff}}^{\delta-1}$) fulfilled on $\mathsf{supp}(\psi)$ the remainder estimate is $O(h^{-2}\tau)$ and marginally worse without non-degeneracy assumption

(iii) If $\mu \gtrsim \tau^{1/2}$, $\mu h \ge c\tau$ than $\mathsf{N}^-(\tau) = 0$ for the Schrödinger operator; for the Schrödinger-Pauli operator the principal part of the asymptotics has magnitude μh^{-1} and under weak non-degeneracy assumptions the remainder estimate is $O(\mu h^{-1})$.

Example 24.5.2[33]. Let X be an unbounded domain. Let conditions (24.2.1), (24.2.6), $(24.2.2)_{1-3}$, (24.3.1) and $(24.3.9)_k$ be fulfilled with $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$, $m_1 > 2m$. Consider the Schrödinger operator and assume that

$$(24.5.2)_{1,2} \qquad \tau \ge \mu h, \qquad \tau^{2m-m_1} \le \epsilon(\mu h)^{2m}.$$

Then

$$\mathcal{N}^-(\tau,\mu,h) \asymp \tau^{3(2+m_1)/(2m_1)} h^{-3(1+m_1)/m_1}\mu^{-3/m_1}. \tag{24.5.3}$$

(i) Further, if $\tau \gtrsim \mu^2$, then the zone of the strong magnetic field $\mu_{\mathsf{eff}} = \mu\langle x\rangle^{m_1+1}\tau^{-1/2} \ge C$ is contained in $\{x\colon |x| \ge c\}$ and here we have non-degeneracy condition fulfilled. Then the remainder estimate is $O(R)$ with

$$R = \tau^{(2+m_1)/m_1} h^{-2(1+m_1)/m_1}\mu^{2/m_1}. \tag{24.5.4}$$

(ii) On the other hand, if $\mu^2 \gg \tau$, then the contribution of the zone $\{x\colon |x| \ge c\}$ to the remainder is $O(R)$ with R defined by (24.5.4). The contribution of the zone $\{x\colon |x| \le c\}$ to the remainder is $O(h^{-2}\tau)$ under weak non-degeneracy assumption fulfilled there.

(iii) Let us replace $(24.5.2)_2$ by the opposite inequality, and assume $(24.3.3)^\#$. Then (24.5.3) is replaced by $\mathcal{N}^-(\tau,\mu,h) \asymp h^{-3}\tau^{3(m+1)/(2m)}$ while under non-degeneracy assumption

$$R = h^{-2}\tau^{(m+1)/m}. \tag{24.5.5}$$

Example 24.5.3[34]. In the framework of Example 24.5.2 for the Schrödinger-Pauli operator under assumption $(23.3.3)^\#$

$$\mathcal{N}^-(\tau,\mu,h) \asymp h^{-3}\tau^{3(m+1)/(2m)} + \mu h^{-2}\tau^{(m_1+m+3)/(2m)}. \tag{24.5.6}$$

$$R = h^{-2}\tau^{(m+1)/m} + \mu h^{-1}\tau^{(m_1+2)/(2m)} \tag{24.5.7}$$

We leave to the reader

Problem 24.5.4. Consider the Schrödinger and Schrödinger-Pauli operators with other types of the behaviour at infinity.

Problem 24.5.5. For the Dirac operators derive similar results as $\tau \to \pm\infty$.

[33] Cf. Example 24.3.12.

[34] Cf. Example 24.3.14.

24.5.2 Asymptotics of the Small Eigenvalues

In this subsection for the Schrödinger and Schrödinger-Pauli operators we consider asymptotics of eigenvalues tending to -0.

Example 24.5.6[35]. Let X be an unbounded domain. Let conditions (24.2.1), $(24.2.3)_{1,2}$, $(24.3.1)_1^{\#}$ be fulfilled with $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$, $-1 < m < 0$, $m_1 > m-1$.

Consider the Schrödinger operator and assume that

$$(24.5.8)_{1,2} \qquad 1 \ge \mu h, \qquad |\tau|^{2m-m_1} \le \epsilon(\mu h)^{2m}.$$

Then $\mathcal{N}^-(\tau) = O(h^{-3}|\tau|^{3(m+1)/(2m)})$ as $\tau \to -0$ with "$\asymp$" instead of "$= O$" if condition $V \le -\epsilon\rho^2$ fulfilled in some non-empty cone.

Further, under the non-degeneracy assumption $(24.3.4)_k^{\#}$ the contribution to the remainder of the zone $\{x\colon |x| \ge c\}$ is $O(h^{-2}|\tau|^{(m+1)/m})$.

Example 24.5.7[36]. In the framework of Example 24.5.6 for the Schrödinger-Pauli operator under assumption $(24.3.3)^{\#}$ the contribution to the remainder of the zone $\{x\colon |x| \ge c\}$ is $O(R)$ with

$$R = h^{-2}|\tau|^{(m+1)/m} + \mu h^{-1}|\tau|^{(m_1+2)/(2m)}, \tag{24.5.9}$$

while

$$\mathcal{N}^-(\tau) = O\bigl(h^{-3}|\tau|^{3(m+1)/(2m)} + \mu h^{-2}|\tau|^{(m_1+m+3)/(2m)}\bigr) \tag{24.5.10}$$

with "$\asymp$" instead of "$= O$" if condition $V \le -\epsilon\rho^2$ fulfilled in some non-empty cone.

Problem 24.5.8. Consider the Schrödinger and Schrödinger-Pauli operators if

(i) If condition $(24.5.8)_1$ is violated (then there could be a forbidden zone in the center).

(ii) With other types of the behaviour at infinity.

Problem 24.5.9. Consider the Schrödinger and Schrödinger-Pauli operators in the framework of Subsections 24.4.2 and 24.4.3 if

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[35] Cf. Example 24.4.2.

[36] Cf. Example 24.4.2.

(i) $\mu h = 1$; then the essential spectrum does not change.

(ii) $\mu h \to \infty$; then the limit of the essential spectrum is $\emptyset$ (for the Schr]"odinger operator) and $[0,\infty)$ (for the Schrödinger-Pauli operator). Consider $\mathsf{N}^-(\eta)$ with $\eta \to 0$.

(iii) $\mu h \to 0$; then the limit of the essential spectrum is $[0,\infty)$ (for both the Schrödinger and Schrödinger-Pauli operators). Consider $\mathsf{N}^-(\eta)$ with $\eta \to 0$.

Problem 24.5.10. For the Dirac operators derive similar results as $M \neq 0$ and $\tau \to M - 0$ and $-M + 0$.

24.5.3 Case of $\tau \to +0$

In this subsection $\tau \to +0$ for the Schrödinger and Schrödinger-Pauli operators and $\tau \to \pm M \pm 0$ for the Dirac operator. Consider the Schrödinger and Schrödinger-Pauli operators first.

Example 24.5.11[37]. Let $V > 0$ everywhere except $V(0) = 0$. Let conditions (24.2.1), $(24.2.2)_{1-3}$ and (24.3.1) be fulfilled with $\gamma = \epsilon_0|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$, $m_1 > 2m \geq 0$. Consider the Schrödinger operator and assume that $\tau \to +0$.

(i) Let

$$(24.5.11)_{1,2} \qquad \mu \ll \tau^{(m_1+2)/2} h^{-(m_1+1)}, \qquad \tau^{2m-m_1} \leq \epsilon(\mu h)^{2m}.$$

Then (24.5.3) holds while the remainder estimate is $O(R)$ with defined by (24.5.4).

(ii) Let us replace $(24.5.11)_2$ by the opposite inequality, and assume $(24.3.3)^{\#}$. Then (24.5.3) is replaced by $\mathcal{N}^-(\tau) \asymp h^{-3}\tau^{3(m+1)/(2m)}$ while $R = h^{-2}\tau^{(m+1)/2m}$.

Example 24.5.12[38]. In the framework of Example 24.5.11 for the Schrödinger-Pauli operator under assumption $(24.3.3)^{\#}$ both (24.5.6) and (24.5.7) hold.

We leave to the reader

[37] Cf. Example 24.5.2.
[38] Cf. Example 24.5.3.

Problem 24.5.13[39]. Consider the Schrödinger and Schrödinger-Pauli operators

(i) In the same framework albeit with condition $m_1 > 2m$ replaced by $2m \ge m_1 \ge 0$. Assume that $(24.3.3)^{\#}$ is fulfilled.

Then magnitude of $\mathcal{N}^-(\tau)$ is described in Examples 24.5.11 and 24.5.12. Under proper non-degeneracy assumption (which we leave to the reader to formulate) derive the remainder estimate.

(ii) In the same framework as in (i) albeit in with $m_1 < 0$ (magnetic field is stronger in the center but there is no singularity), in which case the center can become a classically forbidden zone.

(iii) With other types of the behaviour at infinity.

Problem 24.5.14. For the Dirac operators derive similar results as $\tau \to \pm(M+0)$.

24.5.4 Case of $\tau \to -\infty$

In this subsection for the Schrödinger and Schrödinger-Pauli operators we consider asymptotics with $\tau \to -\infty$.

In this subsection for the Schrödinger and Schrödinger-Pauli operators we consider asymptotics of eigenvalues tending to -0.

Example 24.5.15[40]. Let $X \ni 0$ and let conditions (24.2.1), $(24.2.2)_{1-3}$, (24.3.1) and a week non-degeneracy assumption be fulfilled with $\gamma = \epsilon_0\langle x\rangle$, $\rho = \langle x\rangle^m$, $\rho_1 = \langle x\rangle^{m_1}$, $-1 < m < 0$, $m_1 > m-1$.

Consider the Schrödinger operator and assume that

$$(24.5.12)_{1,2} \qquad h \ll |\tau|^{(m+1)/(2m)}, \qquad |\tau|^{m_1-2m} \le \epsilon(\mu h)^{2m}.$$

Then $\mathcal{N}^-(\tau) = O(h^{-3}|\tau|^{3(m+1)/(2m)})$ as $\tau \to -0$ with "$\asymp$" instead of "$= O$" if condition $V \le -\epsilon\rho^2$ as $|x| \le \epsilon$ fulfilled in some non-empty cone and $R = h^{-2}|\tau|^{(m+1)/m}$.

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[39] Cf. Problem 24.5.4.

[40] Cf. Example 24.4.2.

Example 24.5.16[41]. In the framework of Example 24.5.15 for the Schrödinger-Pauli operator under assumption (24.3.3)$^{\#}$

$$\mathcal{N}^-(\tau) = O\big(h^{-3}|\tau|^{3(m+1)/(2m)} + \mu h^{-2}|\tau|^{(m_1+m+3)/(2m)}\big) \tag{24.5.13}$$

and

$$R = h^{-2}|\tau|^{(m+1)/m} + \mu h^{-1}|\tau|^{(m_1+2)/(2m)}\big) \tag{24.5.14}$$

with "$\asymp$" instead of "$= O$" if condition $V \le -\epsilon\rho^2$ as $|x| \le \epsilon$ fulfilled in some non-empty cone.

Problem 24.5.17. Consider the Schrödinger and Schrödinger-Pauli operators if with other types of the behaviour at 0.

24.A Appendices

24.A.1 1D Schrödinger Operator

Operators of the type we consider here studied by many authors. Related statements could be found in many books, including Chapter XIII, Part 2 of N. Danford and J. T. Schwarz [1], M. S. Birman and M. Z. Solomyak [23] and V. Maz'ya and I. Verbitsky [1].

Proposition 24.A.1. *Let us consider the operator*

$$\boldsymbol{a}_\varepsilon = D_t g_\varepsilon(t) D_t + \varepsilon^{-1} V_\varepsilon(t) \tag{24.A.1}$$

in $\mathbb{H} = \mathscr{L}^2(\mathbb{R})$ *with* $D = D_t$, $V_\varepsilon(x) = V(\frac{x}{\varepsilon})$, *etc.*, $t \in \mathbb{R}$,

$$\epsilon_0 \le g \le c, \quad |V| \le \rho^2, \quad 0 \le \rho \le c, \quad \|\rho\|_{\mathscr{L}^1} + \|\rho^2 t\|_{\mathscr{L}^1} \le c. \tag{24.A.2}$$

Then

(i) The number of negative eigenvalues of the operator $\boldsymbol{a}$ *does not exceed* C_0 *for* $|\varepsilon| \le 1$.

(ii) The number of negative eigenvalues of the operator $\boldsymbol{a}$ *does not exceed* 1 *for* $|\varepsilon| \le \epsilon$ *with a small enough constant* $\epsilon > 0$.

[41] Cf. Example 24.4.2.

(iii) Further, let us assume that

$$W = -\frac{1}{2}\int_{-\infty}^{+\infty} V(t)\,dt \geq \epsilon_0. \tag{24.A.3}$$

Then for $\varepsilon \in (0, \epsilon]$ *there is exactly one negative eigenvalue* $\lambda(\varepsilon)$ *and*

$$-\epsilon_2 \geq \lambda \geq -c_1. \tag{24.A.4}$$

(iv) Furthermore, let us assume that (24.A.3) *holds and*

$$|g - 1| \leq c\rho. \tag{24.A.5}$$

Then

$$|\lambda + W^2| \leq C_0\varepsilon. \tag{24.A.6}$$

(v) Moreover, let us assume that (24.A.3) *holds and that* g *and* V *depend on the parameter* $z \in \Omega$ *and*

$$|D_z^\alpha g| \leq c, \quad |D_z^\alpha V| \leq c\rho^2 \qquad \forall \alpha : |\alpha| \leq K. \tag{24.A.7}$$

Then

$$|D_z^\alpha \lambda| \leq C_0; \tag{24.A.8}$$

moreover, under the condition

$$|D_z^\alpha g| \leq c\rho \qquad \forall \alpha : |\alpha| \leq K \tag{24.A.9}$$

we obtain that

$$|D_z^\alpha(\lambda + W^2)| \leq C_0\varepsilon. \tag{24.A.10}$$

(vi) Finally, let $v \in \mathbb{H}$, $\|v\| = 1$ *be an appropriate eigenfunction of* $\boldsymbol{a}$ *with eigenvalue* λ*. Then*

$$\|D_z^\alpha v\| \leq C_0, \qquad \|D_z^\alpha D_t v\| \leq C_0, \qquad \|D_z^\alpha v\|_\infty \leq C_0, \tag{24.A.11}_{1-3}$$

where $\|.\|_p$ *means the* $\mathscr{L}^p$*-norm and we skip* $p = 2$ *in this notation.*

Proof. Statement (i) follows from the fact that the operator $\rho^s(D_3^2 + 1)^{-s}$ is compact in $\mathbb{H}$ for any $s > 0$.

In order to prove Statement (ii) let us consider the quadratic form $Q(u) = \langle \boldsymbol{a}u, u\rangle$ on the subspace $\mathbb{H}_1 = \{u \in \mathbb{H}, \int_{-\varepsilon}^{\varepsilon} u\,dt = 0\}$ of codimension 1 [42] Obviously $|u(t)| \le 3(|t|+\varepsilon)^{\frac12}\|D_t u\|$ for $u \in \mathbb{H}_1$ and therefore

$$|\langle V_\varepsilon u, u\rangle| \le C_0\varepsilon^2\|D_t^2 u\|^2$$

in virtue of (24.A.2). Then (24.A.2) yields that the quadratic form $Q(u)$ is positive definite on $\mathbb{H}_1$ for $|\varepsilon| \le \sigma_2$ and therefore $\boldsymbol{a}$ has no more than one negative eigenvalue λ. Moreover, for arbitrary $u \in \mathbb{H}$ the inequality $|u(t)| \le \sigma\|D_t u\| + C_\sigma\|u\|$ with arbitrarily small $\epsilon > 0$ yields that

$$|\langle V_\varepsilon u, u\rangle| \le c_0\sigma\varepsilon\|D_t u\|^2 + C_\sigma\varepsilon\|u\|^2$$

and hence $Q(u)$ is uniformly semibounded from below and therefore

$$\lambda \ge -C_0. \tag{24.A.12}$$

Let v be the corresponding eigenfunction with $\|v\| = 1$ (if there exists a negative eigenvalue). Then obviously $|v(t)| \le C_0$ and then $|D_t v(t)| \le C_0$ and hence $|v(t) - v(0)| \le C_0|t|$. Then (24.A.2), (24.A.5) yield that

$$|Q(v) - \bar{Q}(v)| \le C_1\varepsilon$$

for the quadratic form $\bar{Q}(u) = \|D_t u\|^2 - W|u(0)|^2$. Therefore $\lambda \ge \bar{\lambda} - C_1\varepsilon$ where $\bar{\lambda}$ is the lower bound of $\bar{Q}(u)\|u\|^{-2}$ at $\mathbb{H}$.

One can apply the same arguments to the eigenvalues and eigenfunctions of $\bar{Q}$; as a result we obtain that $\bar{\lambda} \ge \lambda - C_1\varepsilon$. On the other hand, one can see easily that $\bar{\lambda} = -\frac14 W^2$ if $W > 0$ (otherwise $\bar{Q}$ is non-negative definite) and $\bar{v}(t) = \frac{W}{2}\exp(-\frac12 W|x|)$ and hence we obtain that for $W > 0$ there is a negative eigenvalue and (24.A.6) holds.

Moreover, if (24.A.5) is violated then one can treat the quadratic form $C_0\|D_t^2\| + \varepsilon^{-1}(V_\varepsilon u, u)$ instead of the original one and since (24.A.4) holds for this form it remains true for the original one.

So all the statements excluding those associated with derivatives on z are proven[43].

[42] One can consider the subspace $\{u \in \mathfrak{D}(\boldsymbol{a}), u(0) = 0\}$ as well.

[43] One can easily prove that for $W < 0$ and small enough ε the operator $\boldsymbol{a}$ is non-negative definite. We think that it would be nice to treat the case $W = 0$. However we are not an expert here.

The proof of (24.A.8), (24.A.11) is standard, due to K. O. Friedrichs [1]. Let these estimates be proven for $|\alpha| \le n$; then applying the operator ∂_z^α with $|\alpha| = n$ to the equation

$$(a - \lambda)v = 0 \tag{24.A.13}$$

we obtain

$$\sum_{\beta \le \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!}(a-\lambda)^{(\alpha-\beta)}v^{(\beta)} = 0 \tag{24.A.14}$$

with $u^{(\alpha)} = \partial_z^\alpha u$.

Let us multiply this equation by v. Then terms with $v^{(\alpha)}$ disappear and we obtain terms with $|\beta| < n$

$$\langle g_\varepsilon^{(\alpha-\beta)} D_t v^{(\beta)}, D_t v\rangle, \quad \varepsilon^{-1}\langle V_\varepsilon^{(\alpha-\beta} v^{(\beta)}, v\rangle, \quad \lambda^{(\alpha-\beta)}\langle v^{(\beta)}, v\rangle. \tag{24.A.15}$$

Terms of the first and second types are bounded in virtue of $(24.A.11)_2$, $(24.A.11)_3$ respectively for $|\beta| < n$. Terms of the third type are bounded for $\beta \ne 0$ by (24.A.8), $(24.A.11)_1$. Therefore the remaining term

$$-\lambda^{(\alpha)}\|v\|^2$$

should also be bounded and (24.A.8) holds for $|\alpha| = n$. Let us consider equation (24.A.14); we now multiply it by $w = v^{(\alpha)}$. We obtain terms with $|\beta| \le n$

$$\langle g_\varepsilon^{(\alpha-\beta)} v^{(\beta)}, w\rangle, \quad \varepsilon^{-1}\langle V_\varepsilon^{(\alpha-\beta} v^{(\beta)}, w\rangle, \quad \lambda^{(\alpha-\beta)}\langle v^{(\beta)}, w\rangle.$$

For $|\beta| < n$ terms of the first and second type do not exceed $C\|D_t w\|$ and $C\|w\|_\infty$ due to $(24.A.11)_2$, $(24.A.11)_3$. Finally, terms of the third type for $|\beta| < n$ do not exceed $C\|w\|$ due to (24.A.8) and $(24.A.11)_1$. Thus

$$|\langle (a-\lambda)w, w\rangle| \le C\|D_t w\| + C\|w\|$$

because $\|w\|_\infty \le C\|D_t w\| + C\|w\|$. Taking into account that $\lambda \le -\epsilon_0$ we obtain from this inequality that

$$\|D_t w\|^2 + \|w\|^2 \le C\varepsilon^{-1}|\langle V_\varepsilon w, w\rangle| + C. \tag{24.A.16}$$

Let us assume that $v(0) = 1$. Surely, we should reject the condition $\|v\| = 1$ but our above arguments yield that $\|v\| \asymp |v(0)|$. Then $w(0) = 0$ and

$|w(t)| \le |t|^{\frac12}\|D_t w\|$ and therefore $|\langle V_\varepsilon w, w\rangle| \le C\varepsilon^2\|D_t w\|$ and therefore (24.A.16) yields $(24.A.11)_{1,2}$ for $|\alpha| = n$; $(24.A.11)_3$ follows from these estimates.

In order to prove (24.A.10) let us note that equation (24.A.14) and $(24.A.11)_{1-3}$ yield that

$$\|D_z^\alpha D_t v\|_\infty \le C_0 \tag{24.A.11}_4$$

and therefore under condition (24.A.9) terms of the first type in (24.A.15) do not exceed $C\varepsilon$. Moreover,$(24.A.11)_4$ yields that

$$|v(t) - v(0)| \le C_0|t|, \qquad |D_z^\alpha v| \le C_0|t| \quad \forall \alpha \ne 0$$

(because of condition $v(0) = 1$) and therefore terms of the second type in (24.A.15) do not exceed $C\varepsilon$ for $\beta \ne 0$. Moreover, the error does not exceed $C\varepsilon$ if we replace $v(t)$ by $v(0)$ in this term with $\beta = 0$; we then obtain $W^{(\alpha)}|v(0)|^2$ and under additional the restriction $W = \mathsf{const}$ this term vanishes. Then induction on n yields that $|\lambda^{(\alpha)}| \le C_0\varepsilon$ under this restriction. So under this restriction (24.A.10) holds. However one can reduce the general case to the case $W = 1$ by introducing $t' = tW^{-1}$ and multiplying a by W^2. □

Remark 24.A.2. Applying the above results one can find v in the form

$$v = \exp\Big(\int_0^t \phi_\varepsilon(t')dt'\Big)\cdot\big(1 + \varepsilon^2\psi_\varepsilon + ...\big) \tag{24.A.17}$$

where the number of terms depends on m and $\lambda = -W^2 + \mu\varepsilon + \cdots$ with $\partial_t\phi = V$ and one can obtain $\mu \ne 0$ in the generic case; so estimate (24.A.6) is the best possible estimates without this correction term. Therefore (24.4.33) remains true with $\lambda(x')$ replaced by $-W(x')^2$ provided $m \le -2$ and $\rho(x) = \langle x\rangle^m$, $\gamma(x) = \gamma_1(x) = \langle x\rangle$.

For $m > -2$ this is correct with the remainder estimate $O\big(\eta^{(m+2)(2m+1)^{-1}}\big)$ coinciding with the principal part for $m = -1$ (in the framework of Remark 24.4.19).

Proposition 24.A.3. *(i) Under condition* (24.4.66) *the operator $\mathcal{L}$ has a finite number of negative eigenvalues.*

(ii) Moreover, if this condition is fulfilled for all x then there is at most one negative eigenvalue.

(iii) On the other hand, if

$$W \le -(\frac{1}{4}+\epsilon)|x|^{-2} \qquad \forall x\colon x \ge C \tag{24.A.18}$$

then there is an infinite number of negative eigenvalues.

Proof. To prove Statements (i) and (ii) one needs to prove the estimate

$$\|u'\|^2 \ge \frac{1}{4}\||x|^{-1}u\|^2 \qquad \forall u : u(0)=0 \tag{24.A.19}$$

where $\|u\|$ is the $L^2(\mathbb{R}^+)$-norm. However, the left side is equal to

$$\|x^{\frac{1}{2}}\bigl(ux^{-\frac{1}{2}}\bigr)' + \frac{1}{2}x^{-1}u\|^2 = \|x^{\frac{1}{2}}\bigl(ux^{-\frac{1}{2}}\bigr)'\|^2 + \frac{1}{4}\||x|^{-1}u\|^2$$

provided $u = o(x^{\frac{1}{2}})$ as $x \to 0$.

To prove Statement (iii) it is sufficient to prove that the inequality $\|u'\|^2 \le (\frac{1}{4}+\epsilon)\||x|^{-1}u\|^2$ is fulfilled on some subspace of $\mathscr{L}^2([1,\infty))$ of infinite dimension. It is sufficient to prove that for any n this inequality is fulfilled on some function supported in $[L^n, L^{n+1}]$ with sufficiently large L.

Further, due to homogeneity it is sufficient to consider only $n=0$. Substituting $u = x^{\frac{1}{2}}v$, $x = e^t$ we obtain that it is sufficient to fulfill the inequality $\|v'\|^2 \le \epsilon\|v\|^2$ with some v such that $v(0) = v(\log L) = 0$. But this is obvious provided L is large enough. □

24.A.2 1D Schrödinger Operator. II

We consider operator

$$\boldsymbol{b}_\varepsilon = D^2 + \varepsilon V(x) \tag{24.A.20}$$

with

$$|V| \le \rho^2 = \langle x\rangle^{-2q}, \qquad V \le -\epsilon_0\rho^2 \qquad \text{for } |x| \ge c, \tag{24.A.21}$$

$0 < q \le 1$, and $\varepsilon > 0$.

We are interested in $\boldsymbol{n}_\varepsilon(\eta)$, the number of eigenvalues of $\boldsymbol{b}_\varepsilon$ which are less than $-\eta$. Consider first the corresponding Weyl's expression

$$\mathsf{n}^{\mathsf{W}}_\varepsilon(\eta) := (2\pi)^{-1}\int (\varepsilon V(x)-\eta)_+^{\frac{1}{2}}\,dx. \tag{24.A.22}$$

Proposition 24.A.4. *(i) If* $\mathsf{n}_\varepsilon^W(\eta) \ge C_0$ *then* $\boldsymbol{n}_\varepsilon(\eta) \asymp \mathsf{n}^{\mathsf{W}}(\varepsilon,\eta)$.

(ii) If $\mathsf{n}^W(\varepsilon,\eta) \le C_0$ *then* $\boldsymbol{n}\varepsilon(\eta) \le C_1$.

Remark 24.A.5. Obviously $\mathsf{n}_\varepsilon^W(\eta) \asymp \varepsilon^{(2q-1)/2q}\eta^{-(1-q)/2q}$ and $\mathsf{n}_\varepsilon^W(\eta) \le C_0$ if and only if $\eta \ge c_0\varepsilon^{(2q-1)/(1-q)}$.

Proof of Proposition 24.A.3. One can easily prove Statement (i) using our semiclassical theory.

On the other hand, one can easily prove Statement (ii) using variational methods, and covering $\mathbb{R}$ by a finite number of intervals $[L_k, L_{k+1}]$ and $[-L_{k+1}, -L_k,]$ with $k = 1, \dots, n-1$, $[L_n, \infty]$ and $[-\infty, -L_n]$ and $[-L_0, L_0]$ such that $L_{k+1} = \epsilon_0 L_k^q \varepsilon^{-1/2}$, $L_0 = 1$, $L_n \ge c_0\eta^{-1/2q}\varepsilon^{1/2q}$.

We leave the easy details to the reader. □

Now we need to figure out when $\boldsymbol{n}_\varepsilon(\eta) \ge 1$. To do so we need to evaluate the lowest eigenvalue $\lambda(\varepsilon) < 0$ of operator (24.A.20).

Proposition 24.A.6. *Let* $V \in \mathscr{L}^1(\mathbb{R})$ *and* $W > 0$. *Then*

$$\lambda(\varepsilon) = -\varepsilon^2(W^2 + o(1)) \qquad \textit{textas} \quad \varepsilon \to 0 \tag{24.A.23}$$

with W *defined by* (24.A.3).

Remark 24.A.7. (i) Since after scaling $x \mapsto x/\varepsilon$ and multiplication by ε^{-1} with ε operator (24.A.20) becomes (24.A.1), this is consistent with (24.A.6).

(ii) For $V = -\langle x\rangle^{-2q}$ we have $W < \infty$ if and only if $q > \frac{1}{2}$.

Proof of Proposition 24.A.6. (a) We can apply Proposition 24.A.1 for $V \ge 0$, $V = 0$ as $|t| \ge \varepsilon^{-1}$. Therefore $\boldsymbol{b}'_\varepsilon = D^2 + V'(x) \ge -C\varepsilon^2$ where $V' = V(x)$ as $|x| \le t$, $V(x) = 0$ as $|x| \ge t$. Indeed it is true for V replaced by $-C\rho(x)^2$.

Consider $t_n = 2^n$ and consider $V_0(x) = V(x)$ as $|x| \le t_0$, $V_n(x) = V(x)$ as $t_{n-1} \le |x| \le t_n$ and vanishing on all other segments. Consider

$$\sigma_n > 0, \qquad \sum_n \sigma_n \le 1 \tag{24.A.24}$$

Then

$$\boldsymbol{b}_\varepsilon \ge \sum_n \boldsymbol{b}_{n,\varepsilon}, \qquad \boldsymbol{b}_{n,\varepsilon} = \sigma_n D^2 + \varepsilon V_n(x).$$

Scaling $x \mapsto x/t_n$ we have $\boldsymbol{b}_{n,\varepsilon} \mapsto \sigma_n t_n^{-2}\big[D^2 + \sigma_n^{-1}\varepsilon t_n^2 \rho(t_n)^2 U_n(x)\big]$, with $U_n(x) = \rho(t_n)^{-2} V_n(x/t_n)$ and if $\sigma_n^{-1}\varepsilon t_n^2\rho(t_n)^2 \le 1$ we can apply the above estimate to the operator in the brackets. On the other hand, it is greater than $-C\varepsilon\sigma_n^{-1}\rho(t_n)^2$ and we can apply this estimate even without this condition; so we arrive to $\boldsymbol{b}_{n,\varepsilon} \ge \sigma_n^{-1}\varepsilon^2 t_n^2 \rho(t_n)^4$ and therefore $]\boldsymbol{b}_\varepsilon \ge -C\varepsilon^2$ provided

$$\sum_n \sigma_n^{-1}\varepsilon^2 t_n^2 \rho(t_n)^4 \le C_0. \tag{24.A.25}$$

Picking up $\sigma_n = \epsilon_0 t_n \rho(t_n)^2$ we satisfy both (24.A.24) and (24.A.25).

(b) Consider now t such that $t\rho(t)^2 \le \delta^2$. Then $|W_1 - W| \le C\delta^2$ with $W_1 = -\frac{1}{2}\int_{-t}^{t} V(x)\,dx$. Therefore $\boldsymbol{b}_\varepsilon = \boldsymbol{b}_{1,\varepsilon} + \boldsymbol{b}_{2,\varepsilon}$ with $\boldsymbol{b}_{1,\varepsilon} = (1-\delta)D^2 + \varepsilon V_1(x)$, $\boldsymbol{b}_{2,\varepsilon} = \delta D^2 + \varepsilon V_2(x)$. Applying Proposition 24.A.1 to $\boldsymbol{b}_{1,\varepsilon}$ and the results of Part (a) to $\boldsymbol{b}_{2,\varepsilon}$ we conclude that $\boldsymbol{b}_\varepsilon \ge -\varepsilon^2(W^2 + C\delta) \implies \lambda(\varepsilon) \ge -\varepsilon^2(W^2 + C\delta)$.

Similarly, $\boldsymbol{b}_{3,\varepsilon} = (1+\delta)D^2 - V_1(x) \ge \boldsymbol{b}_\varepsilon + \boldsymbol{b}_{4,\varepsilon}$ and applying Proposition 24.A.1 to $\boldsymbol{b}_{3,\varepsilon}$ and the results of Part (a) to $\boldsymbol{b}_{4,\varepsilon}$ we conclude that $\lambda(\varepsilon) \le -\varepsilon^2(W^2 - C\delta)$.

Since we can take $\delta > 0$ arbitrarily small we arrive to (24.A.23). □

Let $0 < q \le \frac{1}{2}$. Then the integral defining W in (24.A.23), diverges (logarithmically, as $q = \frac{1}{2}$).

Proposition 24.A.8. *Let* $0 < q < \frac{1}{2}$. *Then*

(i) $\lambda \ge -\varepsilon^{1/(1-q)}$.

(ii) Assume that $V(x) \sim V^0(x)$ *as* $|x| \to \infty$ *where* $V^0(x) = V_\pm |x|^{-2q}$ *as* $\pm x > 0$. *Let either* $V_+ < 0$ *or* $V_- < 0$ *and let* $\mu < 0$ *be the lowest eigenvalue of the operator* $\boldsymbol{a}^0 = D^2 + V^0(x)$. *Then*

$$\lambda = \varepsilon^{1/(1-q)}(\mu + o(1)) \qquad \text{as} \quad \varepsilon \to 0. \tag{24.A.26}$$

Proof. Observe first that for $0 < q < \frac{1}{2}$ operator $\boldsymbol{a}^0$ is properly defined and semibounded from below and in the framework of Statement (ii) it has an infinite number of negative eigenvalues.

(i) Replacing V by $-C|x|^{-2q}$ and using scaling $x \mapsto c\varepsilon^{-1/2(1-q)}x$ we arrive to operator $\varepsilon^{1/(1-q)}\boldsymbol{a}^0 \ge -C\varepsilon^{1/(1-q)}$. Thus we arrive to Statement (i).

(ii) Observe that in the framework of Statement (ii)

$$b_\varepsilon \ge b_{1,\varepsilon} + b_{2,\varepsilon}$$

with

$$b_{1,\varepsilon} = (1-\delta)D^2 - \varepsilon(V^0(x) + \delta|x|^{-2q}), \qquad b_{1,\varepsilon} = \sigma D^2 - \varepsilon U(x)$$

with arbitrarily small $\sigma > 0$ and U supported in $[-t, t]$ with $t = t(\delta)$. Then $b_{1,\varepsilon} \ge (\mu_1 - C\delta)\varepsilon^{1/(1-q)}$, $b_{2,\varepsilon} \ge C(t,\delta)\varepsilon^2$ and therefore $\lambda(\varepsilon) \ge (\mu_1 - 2C\delta)\varepsilon^2$.

Similarly, one can prove that $\lambda(\varepsilon) \le (\mu_1 + 2C\delta)\varepsilon^2$. Since we can take $\delta > 0$ arbitrarily small we arrive to (24.A.26). □

Problem 24.A.9. (i) Using arguments of the proof of Part (a) of Proposition 24.A.6 prove that if $\int_{\mathbb{R}} \rho^2(x)\,dx = \infty$ then $\lambda(\varepsilon) \ge -C\eta$ where $\eta = \eta(\varepsilon)$ is defined from

$$\eta^{\frac12} = \varepsilon \int_{x:\ \varepsilon\rho(x)^2 \ge \eta} \rho^2(x)\,dx \tag{24.A.27}$$

which is consistent with $\varepsilon^{1/(1-q)}$ in the framework of Proposition 24.A.8 but also works for $\rho(x) = |x|^{-q}|\log|x||^p$ with <u>either</u> $0 < q < \frac12$ <u>or</u> $q = \frac12$, $p \ge -\frac12$.

(ii) Derive asymptotics of $\lambda(\varepsilon)$ in the framework of Statement (i); the most interesting and difficult case seems to be $q = \frac12$.

(iii) Provide a better error estimate in (24.A.23) and (24.A.26).

24.A.3 Examples of Vector Potential

In this Appendix we prove that the results of this Chapter are meaningful. The only questionable part of their conditions is the existence of vector potentials with the given properties of the scalar intensities F and the doubts are only in the three-dimensional case. The construction of a conformal asymptotic Euclidean metric tensor[44] g^{jk} and the construction of a scalar potential V are obvious in all cases and are left to the reader. Analysis of the two-dimensional case is also obvious. In what follows $g^{jk} = \delta^{jk}$.

[44] I.e., a tensor $g^{jk} = \delta^{jk}(1+\varphi)$ with $|1+\varphi| \ge \epsilon > 0$ such that $D^\alpha\varphi = o(\gamma^{-|\alpha|})$ as $|x| \to \infty$ or $|x| \to 0$ for all α.

In what follows $\Lambda_{(2n)}$ be a block-diagonal $2n\times 2n$-matrix with n diagonal 2×2-blocks $\lambda = \left(\begin{smallmatrix} 0 & 1\\ -1 & 0\end{smallmatrix}\right)$ (and non-diagonal blocks 0). Further, let $\Lambda_{(2n+1)}$ be a block-diagonal $(2n+1)\times(2n+1)$-matrix with n diagonal 2×2-blocks λ and 1×1-block and all non-diagonal blocks equal 0.

Lemma 24.A.10. *(i) Let $d = 2n$ and $V_j = (\Lambda x)_j\sigma(|x|)$. Then eigenvalues of (F_{jk}) are $\pm i f_1(x), \dots, \pm i f_n(x)$ with*

$$(24.A.28)_{1,2}\quad f_1(x) = 2\sigma(|x|) + |x|\sigma'(|x|), \qquad f_2(x) = \dots = f_n(x) = 2\sigma(|x|).$$

(ii) Let $d = 2n+1$ and $V_j = (\Lambda x)_j\sigma(|x|)$. Then eigenvalues of (F_{jk}) are $\pm i f_1(x), \dots, \pm i f_n(x), 0$ with $f_2, \dots, f_n$ defined by $(24.A.28)_{1,2}$ and

$$(24.A.29)\qquad f_1(x)^2 = (2\sigma(|x|) + |x'|^2|x|^{-1}\sigma'(|x|))^2 + x_d^2|x'|^2|x|^{-4}\sigma(|x|)'^2,$$

$x' = (x_1, \dots, x_{d-1})$.

Proof. (i) Without any loss of the generality one can assume that $x_j = 0$ for $j = 3, \dots, d$ since we can always reach it by a rotation, commuting with Λ. Then $F_{jk} = 2\sigma\Lambda_{jk}$ if either $j \ge 3$ or $k \ge 3$ and $F_{jk} = (2\sigma + |x|\sigma')\Lambda_{jk}m$ for $j, k \le 2$ which implies Statement (i).

(ii) Without any loss of the generality one can assume that $x_j = 0$ for $j = 3, \dots, d-1$. Then again $F_{jk}(x) = 2\sigma\Lambda_{jk}$ if either $j = 3, \dots, d-1$ or $k = 3, \dots, d-1$ and therefore again $f_2 = \dots = f_n$ are defined by $(24.A.28)_2$.

Meanwhile $F_{12} = (2\sigma + |x'|^2|x|^{-1}\sigma')$, $F_{13} = x_2x_d|x|^{-1}\sigma'$, $F_{12} = -x_1x_d|x|^{-1}\sigma'$, and $f_1^2 = F_{12}^2 + F_{13}^2 + F_{23}^2$ which implies (24.A.29). □

We start from power singularities:

Example 24.A.11. (i) In the framework of Lemma 24.A.10(i) with $\sigma = |x|^m$ $f_1 = (2+m)|x|^m$. In particular, $|f_1| \asymp |x|^m$ if $m \ne -2$.

(ii) In the framework of Lemma 24.A.10(ii) with $\sigma = |x|^m$

$$(24.A.30)\qquad f_1^2 = \big((2 + m|x'|^2|x|^{-2})^2 + m^2|x'|^2x_d^2|x|^{-4}\big)|x|^{2m}.$$

In particular, $|f_1| \asymp |x|^m$ if $m \ne -2$.

Example 24.A.12. In the framework of Lemma 24.A.10(ii) with $\sigma = |x|^m$ let us define $V_d = a|x|^{m+1}$. Again, without any loss of the generality we can assume that $x_j = 0$, $j = 3, \dots, d-1$. In this case the only F_{jk} to change are F_{1d}, F_{2d} (and F_{d1}, F_{d2}) and therefore $f_2, \dots, f_n$ are still defined by $(24.A.28)_2$. One can prove easily that

$$f_1^2 = \bigl((2+m|x'|^2|x|^{-2})^2 + m^2x_d^2|x'|^2|x|^{-4} + (m+1)^2a^2|x'|^2|x|^{-2}\bigr)|x|^{2m}. \tag{24.A.31}$$

In particular, $f_1 \asymp |x|^m$ if $a \neq 0$.

Example 24.A.13. (i) In the framework of Lemma 24.A.10(i) with $\sigma = \langle x\rangle^m$ $f_1 = (2+m|x|\langle x\rangle^{-2})\langle x\rangle^m$. In particular, $|f_1| \asymp |x|^m$ if $m > -2$.

(ii) In the framework of Lemma 24.A.10(ii) with $\sigma = \langle x\rangle^m$

$$f_1^2 = \bigl((2+m|x'|^2\langle x\rangle^{-2})^2 + m^2|x'|^2x_d^2\langle x\rangle^{-4}\bigr)\langle x\rangle^{2m}. \tag{24.A.32}$$

In particular, $f_1 \asymp \langle x\rangle^m$ if $m > -2$.

(iii) In the framework of Lemma 24.A.10(ii) with $\sigma = \langle x\rangle^m$ and $V_d = a\langle x\rangle^{m+1}$

$$f_1^2 = \bigl((2+m|x'|^2\langle x\rangle^{-2})^2 + m^2x_d^2|x'|^2\langle x\rangle^{-4} + (m+1)^2a^2|x'|^2\langle x\rangle^{-2}\bigr)|x|^{2m}. \tag{24.A.33}$$

In particular, $f_1 \asymp \langle x\rangle^m$ if $a \neq 0$.

Consider now power-log singularities.

Example 24.A.14. (i) In the framework of Lemma 24.A.10(i) with $\sigma = |x|^m\ell(x)^\beta$, $\ell(x) = |\log|x|| + C_0$ (with sufficiently large C_0) $f_1 \asymp \sigma$ if $m \neq -2$ and $f_1 = \beta|x|^{-2}\ell^{\beta-1}$ if $m = -2$.

(ii) In the framework of Lemma 24.A.10(ii) with $\sigma = |x|^m\ell(x)^\beta$, $\ell(x) = |\log|x|| + C_0$ (with sufficiently large C_0) $f_1 \asymp \sigma$ if $m \neq -2$.

(iii) In the framework of Lemma 24.A.10(ii) with $\sigma = \langle x\rangle^m$ and $V_d = a|x|^{m+1}\ell(x)^\beta$ $f_1 \asymp \sigma$ if $a \neq 0$.

Example 24.A.15. (i) In the framework of Lemma 24.A.10(i) with $\sigma = \langle x\rangle^m \ell(x)^\beta$, $\ell(x) = \log\langle x\rangle + C_0$ (with sufficiently large C_0) $f_1 \asymp \sigma$ if $m > -2$.

(ii) In the framework of Lemma 24.A.10(ii) with $\sigma = \langle x\rangle^m \ell(x)^\beta$, $\ell(x) = \log\langle x\rangle + C_0$ (with sufficiently large C_0) $f_1 \asymp \sigma$ if $m > -2$.

(iii) In the framework of Lemma 24.A.10(ii) with $\sigma = \langle x\rangle^m \ell(x)^\beta$ and $V_d = a\langle x\rangle^{m+1}\ell(x)^\beta$ $f_1 \asymp \sigma$ if $a \neq 0$.

Consider now exponential potentials.

Example 24.A.16. Let $d = 2$. Then

(i) $V_1 = -x_2|x|^m \exp(|x|^\beta)$, $V_2 = x_1|x|^m \exp(|x|^\beta)$ with $\beta > 0$ provide $f \asymp |x|^{m+\beta-1}\exp(|x|^\beta)$ as $|x| \geq c$.

(ii) The same example with $\beta < 0$ provide $f \asymp |x|^{m+\beta-1}\exp(|x|^\beta)$ as $|x| \leq \epsilon$.

For $d = 3$ we need to be more crafty.

Example 24.A.17. Let $d = 3$.

(i) Consider

$$V_1 = \exp(\nu(x))\cos\big(\psi(x)\big)|x|^m, \tag{24.A.34}_1$$

$$V_2 = \exp(\nu(x))\sin\big(\psi(x)\big)|x|^m, \qquad V_3 = 0 \tag{24.A.34}_{2,3}$$

with $\nu(x) = |x|^\beta$, $\beta > 0$. Then

$$|\nabla^\alpha V_j| \leq c_\alpha \exp(a|x|^\beta)|x|^{(\beta-1)|\alpha|+m} \qquad \forall \alpha \tag{24.A.35}$$

as $|x| \gtrsim 1$ provided $|\nabla\psi| \lesssim |x|^{\beta-1}$. Moreover, one can see easily that

$$F \geq (\epsilon_0 b - C)|x|^{\beta-1+m}\exp(|x|^\beta) \tag{24.A.36}$$

provided $|\partial\psi| \geq b|x|^{\beta-1}$ as $|x| \geq c$. One can take

$$\psi(x', x_3) = \int_0^{x_3} (|x'|^2 + y^2)^{(\beta-1)/2}\, dy, \qquad x' = (x_1, x_2)$$

satisfying these restrictions.

(ii) Similarly, for $\beta < 0$ this constructions work for $|x| \leq \epsilon$.

Consider now quasihomogeneous case. In what follows $L = (l_1, l_2)$ for $d = 2$, $L = (l_1, l_2, l_3)$ for $d = 3$

$$[x]_L = \Big(\sum_j x_j^{2n/l_j}\Big)^{1/2n} \tag{24.A.37}$$

is L-quasihomogeneous length, and n is large so functions are smmoth in $\mathbb{R}^d \setminus 0$.

Example 24.A.18. (i) Let $d = 2$ and $L = (l_1, l_2)$ with $1 = l_1 < l_2$. Let

$$V_1 = -ax_2[x]_L^m, \qquad V_2 = x_1[x]_L^m. \tag{24.A.38}$$

Then $F \asymp [x]_L^m$ for $[x]_L \ge c$ provided $m \ne -(1 + l_2)$ and a is properly chosen.

(ii) Let $d = 2$ and $L = (l_1, l_2)$ with $1 = l_1 > l_2 > 0$. Let V_1, V_2 are defined by (24.A.38). Then $F \asymp [x]_L^m$ for $[x]_L \le \epsilon$ provided $m \ne -(1 + l_2)$ and a is properly chosen.

Example 24.A.19. (i) Let $d = 3$ and $L = (l_1, l_2, l_3)$ with $1 = l_1 \le l_2 \le l_3$. Let

$$V_1 = 0, \qquad V_2 = x_1[x]_L^m, \qquad V_3 = x_1[x]_L^{m+1}, \tag{24.A.39}$$

$$V_1 = -x_2[x]_L^m, \qquad V_2 = x_1[x]_L^m, \qquad V_3 = 0 \tag{24.A.40}$$

if $m \ne -1$, $m = -1$ respectively. Then $F \asymp [x]_L^m$ for $[x]_L \ge c$.

(ii) Let $d = 2$ and $L = (l_1, l_2)$ with $1 = l_1 \ge l_2 \ge l_3 > 0$. Let V_1, V_2, V_3 are defined by (24.A.39) or (24.A.39) if $m \ne -1$, $m = -1$ respectively. Then $F \asymp [x]_L^m$ for $[x]_L \le \epsilon$.

Example 24.A.20. Let $d = 3$, $X = (\mathbb{R}^2 \setminus 0) \times \mathbb{R}/\mathbb{Z} \ni (x', x_3) = (x_1, x_2, x_3)$, $m \ne -2$ and

$$V_1 = x_2|x'|^m, \quad V_2 = -x_1|x'|^m, \quad V_3 = a|x'|^{m+2} \tag{24.A.41}$$

positively homogeneous on x' of degrees $m + 1$, $m + 1$, $m + 2$ respectively[45]. Then $F^3 \ne 0$, $|x'| = \mathsf{const}$ along integral curves of the vector field $\frac{1}{F^3}(F^1, F^2, F^3)$ and for irrational a/π these curves are not closed.

[45] Then F^1, F^2, F^3 are positively homogeneous of degrees $m+1$, $m+1$, m respectively.

Microlocal Analysis, Sharp Spectral Asymptotics and Applications. V.

Applications to Quantum Theory and Miscellaneous Problems

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Part XI

Application to Multiparticle Quantum Theory

Chapter 25

No Magnetic Field Case

25.1 Introduction

The purpose of this Part is to apply semiclassical methods developed in the previous parts to the theory of heavy atoms and molecules. Because of this we combine our semiclassical methods with the traditional methods of that theory, mainly function-analytic.

In this Chapter we consider the case without magnetic field. Next chapters will be devoted to the cases of the self-generated magnetic field, strong external magnetic field and the combined external and self-generated fields. Basically this Chapter should be considered as an introduction.

We explore the ground state energy, an excessive negative charge, ionization energy and excessive negative charge when atoms can still bind into molecules.

25.1.1 Framework

Let us consider the following operator (quantum Hamiltonian)

$$\mathsf{H} = \mathsf{H}_N := \sum_{1\le j\le N} H_{V,x_j} + \sum_{1\le j<k\le N} |x_j - x_k|^{-1} \tag{25.1.1}$$

on

$$\mathfrak{H} = \bigwedge_{1\le n\le N} \mathsf{H}, \qquad \mathsf{H} = \mathscr{L}^2(\mathbb{R}^d, \mathbb{C}^q) \tag{25.1.2}$$

with

$$H_V = D^2 - V(x) \tag{25.1.3}$$

describing N same type particles in the external field with the scalar potential $-V$ (it is more convenient but contradicts notations of the previous chapters), and repulsing one another according to the Coulomb law.

Here $x_j \in \mathbb{R}^d$ and $(x_1, \dots, x_N) \in \mathbb{R}^{Nd}$, potential $V(x)$ is assumed to be real-valued. Except when specifically mentioned we assume that

$$V(x) = \sum_{1\le m\le M} \frac{Z_m}{|x - \mathsf{y}_m|} \tag{25.1.4}$$

where $Z_m > 0$ and y_m are charges and locations of nuclei.

Mass is equal to $\frac{1}{2}$ and the Plank constant and a charge are equal to 1 here. The crucial question is the quantum statistics.

(25.1.5) We assume that the particles (electrons) are *fermions*. This means that the Hamiltonian should be considered on the *Fock space* $\mathfrak{H}$ defined by (25.1.2) of the functions antisymmetric with respect to all variables $(x_1, \varsigma_1), \dots, (x_N, \varsigma_N)$.

Here $\varsigma \in \{1, \dots, q\}$ is a *spin variable*.

Remark 25.1.1. (i) Meanwhile for *bosons* one should consider this operator on the space of symmetric functions. The results would be very different from what we will get here. Since our methods fail in that framework, we consider only fermions here.

(ii) In this Chapter we do not have magnetic field and we can assume that $q = 1$; for $q \ge 1$ no modifications of our arguments is required and results are the same albeit with different numerical coefficients. In the next chapters we introduce magnetic field (external or self-generated) we will be interested in $d = 3$, $q = 2$ and

$$H_{V,A} = \big((i\nabla - A)\cdot \boldsymbol{\sigma}\big)^2 - V(x) \tag{25.1.6}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, σ_k are Pauli matrices.

Let us assume that

(25.1.7) Operator H is self-adjoint on $\mathfrak{H}$.

As usual we will never discuss this assumption.

25.1.2 Problems to Consider

We are interested in the *ground state energy* $\mathsf{E} = \mathsf{E}_N$ of our system i.e. in the lowest eigenvalue of the operator $\mathsf{H} = \mathsf{H}_N$ on $\mathfrak{H}$:

$$\mathsf{E} := \inf \operatorname{Spec} \mathsf{H} \qquad \text{on } \mathfrak{H}; \tag{25.1.8}$$

more precisely we are interested in the asymptotics of $\mathsf{E}_N = \mathsf{E}(\underline{\mathsf{y}}; \underline{Z}; N)$ as V is defined by (25.1.4) and $N \asymp Z := Z_1 + Z_2 + ... + Z_M \to \infty$ and we are going to prove that[1)] E is equal to *Thomas-Fermi energy* $\mathcal{E}^{\mathsf{TF}}$ with Scott and Dirac-Schwinger corrections and with $o(Z^{\frac{5}{3}})$ error.

Here we use notations $\underline{\mathsf{y}} = (\mathsf{y}_1, \dots, \mathsf{y}_M)$, $\underline{Z} = (Z_1, \dots, Z_M)$.

We are also interested in the asymptotics for the *ionization energy*

$$\mathsf{I}_N := \mathsf{E}_{N-1} - \mathsf{E}_N. \tag{25.1.9}$$

It is well-known (see G. Zhislin [1]) that $\mathsf{I}_N > 0$ as $N \le Z$ (i.e. molecule can bind at least Z electrons) and we are interested in the following question: estimate *maximal excessive negative charge*

$$\max_{N:\ \mathsf{I}_N > 0} (N - Z) \tag{25.1.10}$$

i.e. *how many extra electrons can bind a molecule?*.

All these questions so far were considered in the framework of the fixed positions $\mathsf{y}_1, \dots, \mathsf{y}_M$ but we can also consider

$$\widehat{\mathsf{E}} = \widehat{\mathsf{E}}_N = \widehat{\mathsf{E}}(\underline{\mathsf{y}}; \underline{Z}; N) = \mathsf{E} + U(\underline{\mathsf{y}}; \underline{Z}) \tag{25.1.11}$$

with

$$U(\underline{\mathsf{y}}; \underline{Z}) := \sum_{1 \le m < m' \le M} \frac{Z_m Z_{m'}}{|\mathsf{y}_m - \mathsf{y}_{m'}|} \tag{25.1.12}$$

and

$$\widehat{\mathsf{E}}(\underline{Z}; N) = \inf_{\mathsf{y}_1, \dots, \mathsf{y}_M} \widehat{\mathsf{E}}(\underline{\mathsf{y}}; \underline{Z}; N) \tag{25.1.13}$$

and replace I_N by $\widehat{\mathsf{I}}_N = -\widehat{\mathsf{E}}_N + \widehat{\mathsf{E}}_{N-1}$ and modify all our questions accordingly. We call these frameworks *fixed nuclei model* and *free nuclei model* respectively.

In the free nuclei model we can consider two other problems:

[1)] Under reasonable assumption $|\mathsf{y}_m - \mathsf{y}_{m'}| \gg Z^{-\frac{1}{3}}$ for all $m \ne m'$.

(a) Estimate from below *minimal distance between nuclei* i.e.

$$a := \min_{1\le m<m'\le M} |\mathsf{y}_m - \mathsf{y}_{m'}|$$

for which such minimum is achieved;

(b) Estimate *maximal excessive positive charge*

$$\max_N \bigl\{ Z - N \colon \widehat{\mathsf{E}} < \min_{\substack{N_1,\dots,N_M:\\ N_1+\dots N_M=N}} \sum_{1\le m\le M} \mathsf{E}(Z_m; N_m)\bigr\} \tag{25.1.14}$$

for which molecule does not disintegrates into atoms[2)].

25.1.3 Thomas-Fermi Theory

The first approximation is the Thomas-Fermi theory. Let us introduce the *spacial density* of the particle with the state $\Psi \in \mathfrak{H}$:

$$\rho(x) = \rho_\Psi(x) = N \int |\Psi(x, x_2, \dots, x_N)|^2 \, dx_2 \cdots dx_N \tag{25.1.15}$$

where $|\cdot|$ means a norm in $\mathbb{C}^{Nq}$ and antisymmetricity of Ψ implies that it does not matter what variable x_j is replaced by x while in the general case one should sum on $j = 1, \dots, N$. Let us write the Hamiltonian, describing the corresponding "quantum liquid":

$$\mathcal{E}(\rho) = \int \tau(\rho(x))\, dx - \int V(x)\rho(x)\, dx + \frac{1}{2}\mathsf{D}(\rho,\rho), \tag{25.1.16}$$

with

$$\mathsf{D}(\rho,\rho) = \iint |x-y|^{-1}\rho(x)\rho(y)\, dxdy \tag{25.1.17}$$

where τ is the energy density of a gas of noninteracting electrons. Namely,

$$\tau(\rho) = \sup_{w\ge 0}\bigl(\rho w - P(w)\bigr) \tag{25.1.18}$$

[2)] One can ask the same question about disintegration into smaller molecules but our methods are too crude to distinguish between such questions.

is the Legendre transform of the *pressure* $P(w)$ given by the formula

$$P(w) = \varkappa_1 w_+^{\frac{d}{2}+1}, \qquad \varkappa_1 = 2(2\pi)^{-d}(d+2)^{-1}\varpi_d q. \tag{25.1.19}$$

The classical sense of the second and the third terms in the right-hand expression of (25.1.16) is clear and the density of the kinetic energy is given by $\tau(\rho)$ in the semiclassical approximation (see Remark 25.1.2). So, the problem is

(25.1.20) Minimize functional $\mathcal{E}(\rho)$ defined by (25.1.16) under restrictions:

$$\rho \ge 0, \qquad \int \rho\, dx \le N. \tag{25.1.21}_{1,2}$$

The solution if exists is unique because functional $\mathcal{E}(\rho)$ is strictly convex (see below). The existence and the property of this solution denoted further by ρ^{TF} is known in the series of physically important cases.

Remark 25.1.2. If w is the negative potential then

$$\operatorname{tr} e(x,x,0) \approx P'(w) \tag{25.1.22}$$

defines the density of all non-interacting particles with negative energies at point x and

$$\int_{-\infty}^{0} \tau\, d_\tau \operatorname{tr} e(x,x,\tau) dx \approx -\int P(w)\, dx \tag{25.1.23}$$

is the total energy of these particles; here $\approx$ means "in the semiclassical approximation".

We consider in the case of $d = 3$ a large (heavy) molecule with potential (25.1.4). It is well-known[3] that

Proposition 25.1.3. *(i) For $V(x)$ given by* (25.1.4) *minimization problem* (25.1.20) *has a unique solution $\rho = \rho^{\mathsf{TF}}$; then denote $\mathcal{E}^{\mathsf{TF}} := \mathcal{E}(\rho^{\mathsf{TF}})$;*

(ii) Equality in $(25.1.21)_2$ *holds if and only if $N \le Z := \sum_m Z_m$.*

[3] E. Lieb, "Thomas-Fermi and related theories of atoms and molecules", [4], pp. 263–301.

(iii) Further, ρ^{TF} does not depend on N as $N \ge Z$.

(iv) Thus

$$\int \rho^{\mathsf{TF}}\,dx = \min(N, Z), \qquad Z := \sum_{1\le m\le M} Z_m. \tag{25.1.24}$$

25.1.4 Main Results Sketched and Plan of the Chapter

In the first half of the Chapter we derive asymptotics for ground state energy and justify Thomas-Fermi theory.

First of all, in Section 25.2 we reduce the calculation of E to calculation of $\mathsf{N}_1(H_W - \nu)$ and to estimate for $\mathsf{D}\big(e(x,x,\nu) - \rho, e(x,x,\nu) - \rho\big)$ where $\mathsf{N}_1(H_W - \nu) = \mathsf{Tr}((H_W - \nu)^-)$ is the sum of the negative eigenvalues of operator $H_W - \nu$ $H_W = D^2 - W$, $W = W^{\mathsf{TF}}$, $\rho = \rho^{\mathsf{TF}}$ are Thomas-Fermi potential and Thomas-Fermi density respectively (or their appropriate approximations), ν is either λ_N (N-th eigenvalue of H_W) or its appropriate approximation and $e(x,y,\nu)$ is the Schwartz kernel of $E(\nu)$ which is the spectral projector of H_W.

Section 25.3 is devoted to the systematic presentation of the Thomas-Fermi theory.

Further, in Section 25.4 we apply our standard semiclassical arguments and calculate $\mathsf{N}_1(H_W - \nu)$ and estimate $\mathsf{D}\big(e(x,x,\nu) - \rho, e(x,x,\nu) - \rho\big)$ and also $|\lambda_N - \nu|$ where now ν is the chemical potential (which is the Thomas-Fermi approximation to λ_N). As a result under appropriate restrictions to N, Z and

$$a := \min_{1\le m<m'\le M} |\mathsf{y}_j - \mathsf{y}_k| \gg Z^{-\frac{1}{3}} \tag{25.1.25}$$

we prove that

$$\mathsf{E} = \mathcal{E}^{\mathsf{TF}} + \mathsf{Scott} + \mathsf{Dirac} + \mathsf{Schwinger} + o(Z^{\frac{5}{3}}) \tag{25.1.26}$$

and

$$\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}},\, \rho_\Psi - \rho^{\mathsf{TF}}) = o(Z^{\frac{5}{3}}) \tag{25.1.27}$$

where

$$\mathsf{Scott} = \frac{q}{2} \sum_{1\le m\le M} Z_m^2 \tag{25.1.28}$$

$$\mathsf{Dirac} = -\bigl(\frac{6}{\pi q}\bigr)^{\frac{1}{3}} \int \rho^{\mathsf{TF}\,\frac{4}{3}}\, dx, \tag{25.1.29}$$

$$\mathsf{Schwinger} = \bigl(\frac{q}{6\pi^2}\bigr)^{\frac{2}{3}} \int^{\mathsf{TF}\,\frac{4}{3}}\, dx \tag{25.1.30}$$

and Ψ is the ground state.

Remark 25.1.4. (i) Actually we will recover even slightly better remainder estimate $O(Z^{\frac{5}{3}-\delta})$ in (25.1.26) and (25.1.27) as $a \ge Z^{-\frac{1}{3}+\delta_1}$.

(ii) Condition $a \gtrsim Z^{-\frac{1}{3}}$ bans nuclei to be so close that the repulsion energy between them be much larger than the total energy of all the electrons. Estimates in case when this condition is violated will be also proven;

(iii) Keeping in mind that there is no binding in Thomas-Fermi theory (and this statement could be quantified) one gets immediately that in the free nuclei model $a \ge Z^{-\frac{5}{21}}$ and therefore remainder estimate $O(Z^{\frac{5}{3}-\delta})$ holds.

(iv) Due to scaling in the Thomas-Fermi theory (see Proposition 25.3.3 and Appendix 25.A.4) $\mathcal{E}^{\mathsf{TF}} \sim q^{\frac{2}{3}} Z^{\frac{7}{3}} = q^3 (q^{-1}Z)^{\frac{3}{7}}$, $\mathsf{Scott} \sim qZ^2 = q^3(q^{-1}Z)^2$, Dirac is $\sim q^{\frac{1}{3}} Z^{\frac{5}{3}} = q^2 (q^{-1}Z)^{\frac{5}{3}}$ and $\mathsf{Schwinger}$ is $\sim q^{\frac{4}{3}} Z^{\frac{5}{3}} = q^3 (q^{-1}Z)^{\frac{5}{3}}$.

In the second half of the Chapter we apply estimate (25.1.27) to investigate negatively and positively charged systems. In Section 25.5 we consider negatively charged systems and derive an upper estimate for the excessive negative charge $(N - Z)$ such that $\mathsf{I}_N > 0$ and ionization energy I_N itself.

In Section 25.6 we derive upper and lower estimates for $\mathsf{I}_N + \nu$ and an upper estimate for the excessive positive charge for which in the framework of the free nuclei model $a < \infty$.

25.2 Reduction to Semiclassical Theory

To justify the heuristic formula $\mathsf{E} \sim \mathcal{E}^{\mathsf{TF}} = \mathcal{E}(\rho^{\mathsf{TF}})$ and to find an error estimate let us deduce the lower and upper estimates for E.

25.2.1 Lower Estimate

For the lower estimate we apply the *electrostatic inequality* due to E. H. Lieb:

$$\sum_{1\le j<k\le N}\int |x_j-x_k|^{-1}|\Psi(x_1,\dots,x_N)|^2\,dx_1\cdots dx_N \ge \frac{1}{2}\mathsf{D}(\rho_\Psi,\rho_\Psi)-C\int \rho_\Psi^{\frac{4}{3}}(x)\,dx \tag{25.2.1}$$

with ρ_Ψ defined by (25.1.15).

Remark 25.2.1. Inequality (25.2.1 holds for all (not necessarily antisymmetric) functions Ψ with $\|\Psi\|_{L^2(\mathbb{R}^{3N})}=1$.

Therefore

$$\langle \mathsf{H}_N\Psi,\Psi\rangle \ge \sum_{1\le j\le N}\langle H_{V,x_j}\Psi,\Psi\rangle+\frac{1}{2}\mathsf{D}\big(\rho_\Psi,\rho_\Psi\big)-C\int\rho_\Psi^{\frac{4}{3}}(x)\,dx = \sum_{1\le j\le N}\langle H_{W,x_j}\Psi,\Psi\rangle+\frac{1}{2}\mathsf{D}\big(\rho_\Psi-\rho,\rho_\Psi-\rho\big)-\frac{1}{2}\mathsf{D}\big(\rho,\rho\big)-C\int\rho_\Psi^{\frac{4}{3}}(x)\,dx \tag{25.2.2}$$

where $\langle\cdot,\cdot\rangle$ means the inner product in $\mathfrak{H}$ and H_W is one-particle Schrödinger operator with the potential

$$W=V-|x|^{-1}*\rho, \tag{25.2.3}$$

where ρ is an arbitrary chosen real-valued non-negative function.

The physical sense of the second term in W is transparent: it is a potential created by a charge $-\rho$. Skipping the positive second term in the right-hand expression of (25.2.2) and believing that the last term is not very important for the ground state function Ψ [4)] we see that we need to estimate from below the first term.

Here assumption that Ψ is antisymmetric is crucial. Namely, for general (or symmetric–does not matter) Ψ the best possible estimate is $N\lambda_1$ where λ_1 is the lowest eigenvalue of H_W (we always assume that there is sufficiently many eigenvalues under the bottom of the essential spectrum of H_W) and we

[4)] When we derive also an upper estimate for E we will get an upper estimate for this term as a bonus.

cannot apply semiclassical theory. However, for antisymmetric Ψ situation is rather different.

Namely, let $\lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$ be negative eigenvalues of H_W (on $\mathsf{H} = \mathscr{L}^3(\mathbb{R}^3, \mathbb{C}^q)$. Then the first term in the right-hand expression of (25.2.2) is bounded from below by

$$\sum_{1\le j\le N} \lambda_j = \mathsf{N}_1(H_W - \bar\lambda) + \bar\lambda N \tag{25.2.4}$$

where $\mathsf{N}(B)$, $\mathsf{N}_1(B) = \mathsf{Tr}(B^-)$ are the number and the sum of all the negative eigenvalues of operator B respectively such that $\mathsf{Spec}_{\mathsf{ess}}(B) \subset \mathbb{R}^+$ provided $\bar\lambda = \lambda_N < 0$; the latter assumption is equivalent to

$$\mathsf{N}(H_W) \ge N. \tag{25.2.5}$$

Applying the semiclassical approximation (which needs to be justified!) one gets

$$\mathsf{N}_1(H_W - \bar\lambda) = \mathcal{N}_1(H_W - \bar\lambda) + \mathsf{error}_1 \tag{25.2.6}$$

with

$$\mathcal{N}_1(H_W - \bar\lambda) := -\int P\big(W(x) + \bar\lambda\big)\, dx \tag{25.2.7}$$

and error_0 *an error in the semiclassical approximation* for $\mathsf{N}_1(H_W - \bar\lambda)$. Therefore the lower estimate for the ground state energy is

$$\mathsf{E} \ge -\int P(W + \bar\lambda)\, dx + \bar\lambda N - \frac{1}{2}\mathsf{D}(\rho,\rho) - \mathsf{error} \tag{25.2.8}$$

where error now includes both an estimate for $\int \rho_\Psi^{\frac{4}{3}}\, dx$ and the semiclassical remainder estimate.

Furthermore, applying a semiclassical approximation for the number $\mathsf{N}(H_W - \bar\lambda)$ of eigenvalues below $\bar\lambda$ (and this number should be approximately N) one gets an equality

$$N = \mathcal{N}(H_W - \bar\lambda) + \mathsf{error}_0 \tag{25.2.9}$$

with

$$\mathcal{N}(H_W - \bar\lambda) := \int P'\big(W(x) + \bar\lambda\big)\, dx \tag{25.2.10}$$

and error_0 *an error in the semiclassical approximation* for $\mathsf{N}(H_W - \bar\lambda)$.

To get the best possible lower estimate one should pick up ρ delivering maximum to the functional

$$-\int P\big(W(x)+\nu\big)\,dx + \nu N - \frac{1}{2}\mathsf{D}(\rho,\rho) \tag{25.2.11}$$

($\nu = \bar\lambda$ here) under assumptions $(25.1.21)_{1,2}$ and (25.2.3) as we skip all the errors.

One can see that the optimal choice is the *Thomas-Fermi potential* W^{TF} *and density* ρ^{TF}. The above arguments are very standard in MQT with $\rho = \rho^{\mathsf{TF}}$, $W = W^{\mathsf{TF}}$ [5)] from the very beginning.

On the other hand, let us consider the Euler-Lagrange equation for $\rho = \rho^{\mathsf{TF}}$ under condition $\int \rho\,dx = N$:

$$\tau'(\rho) - W = \nu \quad (\rho > 0), \qquad W = V - |x|^{-1} * \rho \tag{25.2.12}$$

with the Lagrange factor ν [6)]. Expressing ρ and integrating we get

$$N = \mathcal{N}(H_W - \nu) = \int P'\big(W(x)+\nu\big)\,dx. \tag{25.2.13}$$

Comparing (25.2.12) and (25.2.13) we get that with some error $\bar\lambda \sim \nu$. Substituting to the first term in (25.2.6) $\bar\lambda = \nu$ and $\nu - W = -\tau'_B(\rho)$ we get the lower estimate $\mathsf{E} \ge \mathcal{E}^{\mathsf{TF}} - \mathsf{error}$.

Remark 25.2.2. (i) Instead of (25.2.6) we will use a better estimate[7)];

(ii) To minimize errors we will also recalculate (25.2.4) effectively replacing $\bar\lambda$ by ν:

$$\sum_{1\le j\le N} \lambda_j = \sum_{1\le j\le N} (\lambda_j - \nu) + \nu N \ge \mathsf{N}_1(H_W - \nu) + \nu N. \tag{25.2.14}$$

The advantage is that we even do not mess up with the semiclassical asymptotics for $\mathsf{N}(H_W - \nu)$. Further, one can replace here N by $\int \rho^{\mathsf{TF}}\,dx$: (these quantities fail to be equal only for $N > Z$ i.e. for $\nu = 0$).

[5)] Or some their close approximations.

[6)] Called *chemical potential* and in contrast to $\bar\lambda$ belonging to Thomas-Fermi theory.

[7)] With $\mathcal{E}^{\mathsf{TF}}$ replaced by $\mathcal{E}^{\mathsf{TF}} + \mathsf{Scott} + \mathsf{Schwinger}$ and with much smaller error_1 than for a simple semiclassical approximation.

(iii) Recall that we assumed that $\lambda_N < 0$ i.e. (25.2.5) holds. In the opposite case

$$\mathsf{N}(H_W) < N. \tag{25.2.15}$$

we estimate the first term in the right-hand expression of (25.2.2) from below by $\mathsf{N}_1(H_W)$ i.e. we will get the same formula but with $\nu = 0$.

25.2.2 Upper Estimate

To get the upper estimate one takes a test function $\Psi(x_1, \dots, x_N)$ which is not a ground state here but an antisymmetrization with respect to $(x_1, \dots, x_N)$ of the product $\phi_1(x_1) \cdots \phi_N(x_N)$ where $\phi_1, \dots, \phi_N$ are orthonormal eigenfunctions of H_W corresponding to eigenvalues $\lambda_1, \dots, \lambda_N$, provided $\lambda_N < 0$. Namely this function minimizes the first term in the right-hand expression of (25.2.2).

One can write

$$\Psi = \frac{1}{N!} \det\bigl(\phi_i(x_j)\bigr)_{i,j=1,\dots,N} \tag{25.2.16}$$

and it is called the *Slater determinant*. Obviously, $\|\Psi\| = 1$ and

$$\rho_\Psi(x) = \operatorname{tr} e_N(x,x) \tag{25.2.17}$$

where

$$e_N(x,y) = \sum_{1\le j\le N} \phi_j(x)\phi_j^\dagger(y) \tag{25.2.18}$$

is the Schwartz kernel of the projector to the subspace spanned on $\{\phi_j\}_{1\le j\le N}$.

Remark 25.2.3. If $q \ge 2$ then $\phi_j = \phi_j(x,\varsigma)$ and $\Psi(x_1,\varsigma_1;\dots;x_N,\varsigma_N)$ is an antisymmetrization with respect to $(x_1,\varsigma_1;\dots;x_N,\varsigma_N)$ of the product $\phi_1(x_1,\varsigma_1)\cdots\phi_N(x_N,\varsigma_N)$.

Easy calculations show that

$$\begin{aligned}\langle \mathsf{H}\Psi, \Psi\rangle = &\sum_{1\le j\le N} \lambda_j + \frac{1}{2}\mathsf{D}(\rho_\Psi - \rho, \rho_\Psi - \rho)\\ &- \frac{1}{2}\mathsf{D}(\rho,\rho) - \frac{1}{2}\iint |x-y|^{-1} \operatorname{tr} e_N^\dagger(x,y) e_N(x,y)\, dxdy.\end{aligned} \tag{25.2.19}$$

The first term in the right-hand expression again is equal to the middle expression in (25.2.4) which does not exceed

$$\mathsf{N}_1(H_W-\nu)+\nu N+|\lambda_N-\nu|\cdot|\mathsf{N}(H_W-\nu)-N|. \tag{25.2.20}$$

Really, we need to consider (non-zero) terms which do not cancel in

$$\sum_{j\le N}(\lambda_j-\nu)-\sum_{\lambda_j<\nu}(\lambda_j-\nu) \tag{25.2.21}$$

and their absolute value does not exceed $|\lambda_N-\nu|$ while their number does not exceed $|\mathsf{N}(H_W-\nu)-N|$.

Again, discounting all the errors and considering semiclassical approximation (including $\rho_\Psi(x)\sim P'\big(W(x)+\nu\big)\big)$ we arrive to a functional

$$-\int P\big(W(x)+\nu\big)\,dx+\nu N-\frac{1}{2}\mathsf{D}(\rho,\rho)+ \frac{1}{2}\mathsf{D}(P'(W+\nu)-\rho,P'(W+\nu)-\rho) \tag{25.2.22}$$

which needs to be minimized under assumptions $(25.1.21)_{1,2}$ and (25.2.3). This functional differs from (25.2.11) which was minimized by the last term. One can prove that (25.2.22) minimizes as $\rho=\rho^{\mathsf{TF}}$, $W=W^{\mathsf{TF}}$ and ν is a chemical potential. So again we may pick them (or their appropriate approximations) up from the very beginning.

Therefore in addition to a semiclassical error_1 [7] of the previous subsection we need to consider also semiclassical errors

$$\mathsf{D}\big(\mathsf{tr}\,e(x,x,\nu)-P'(W+\nu),\mathsf{tr}\,e(x,x,\nu)-P'(W+\nu)\big), \tag{25.2.23}$$

$$\mathsf{D}\big(e(x,x,\nu)-e_N(x,x),e(x,x,\nu)-e_N(x,x)\big) \tag{25.2.24}$$

where $e(x,y,\nu)$ is the Schwartz kernel of the spectral projector $\theta(\tau-H_W)$ of H_W,

$$N(H_W-\nu)-\int P'(W+\nu)\,dx \tag{25.2.25}$$

and

$$\lambda_N-\nu. \tag{25.2.26}$$

Remark 25.2.4. (i) Recall that we assumed that $\lambda_N < 0$ i.e. (25.2.5) holds. In the opposite case (25.2.15) selecting appropriate $\phi_j(x)$ with $j = \mathsf{N}(H_W) + 1, \dots, N$ we with arbitrarily small error estimate the first term in the right-hand expression of (25.2.2) from above by $\mathsf{N}_1(H_W)$ as i.e. we will get the same formula but with $\nu = 0$ and we also will need to estimate (25.2.23) with $\nu = 0$.

(ii) To make this case compatible with the case (25.2.5) we will need to estimate $|\nu|$ (and $|N - Z|$) under assumption (25.2.15); we will also compare $\mathcal{E}^{\mathsf{TF}}$ calculated for such ν (or, equivalently, N as they are connected) and $\nu = 0$ (and $N = Z$).

(iii) Sure ρ^{TF} and W^{TF} depend on ν (or N) but we will prove that for $N - Z$ relatively small we can do all calculations as $\nu = 0$ (and $N = Z$).

(iv) If we are interested in the estimate for $\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}})$ where Ψ is the ground state, we do not need to calculate a semiclassical error in $\mathsf{N}_1(H_W - \nu)$. In fact, we can simply stick with $\mathsf{N}_1(H_W - \bar{\lambda})$ with $\bar{\lambda} = \lambda_N$ under assumption (25.2.5) and $\bar{\lambda} = 0$ otherwise. As a result in certain cases our estimate for $\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}})$ will be better than the error in an approximation for E and we need the former rather than the latter for the results of the second half of this Chapter. Especially significant the difference will be when we introduce magnetic field.

25.2.3 Remarks and Dirac Correction

Now almost everything is in framework of the theory we developed; the only missing is an estimate

$$\int \rho_\Psi^{\frac{4}{3}} dx \le CZ^{\frac{5}{3}} \tag{25.2.27}$$

for a reasonable candidate Ψ to the ground state; one can find it in E. Lieb's *Selecta*[3)].

However if we want a more sharp asymptotics with Dirac–Schwinger terms, we need a remainder estimate $o(Z^{\frac{5}{3}})$ or better; luckily there is improved electrostatic inequality due to Theorem 1, G. Graf and J. P. Solovej [1] (see also V. Bach [1]).

Theorem 25.2.5. *Let $N \ge \epsilon Z$. Then for the ground state Ψ*

$$\mathsf{E}_{\mathsf{HF}} \ge \mathsf{E} \ge \mathsf{E}_{\mathsf{HF}} - CZ^{\frac{5}{3}-\delta} \tag{25.2.28}$$

and

$$\mathsf{E} \ge \mathsf{E}_{\mathsf{DS}} - CZ^{\frac{5}{3}-\delta} \tag{25.2.29}$$

with some exponent $\delta > 0$ where

$$\mathsf{E}_{\mathsf{HF}} \coloneqq \inf_{\Psi} E_{\mathsf{HF}}(\Psi), \tag{25.2.30}$$

where in (25.2.30) Ψ *runs through Slater determinants*[8] *and*

$$E_{\mathsf{HF}}(\Psi) \coloneqq \sum_{1\le j\le N} \langle H_{V,x_j}\Psi, \Psi\rangle + \frac{1}{2}\mathsf{D}\big(\rho_\Psi, \rho_\Psi\big) - \frac{1}{2}\iint |x-y|^{-1}\,\mathsf{tr}\, e_N^\dagger(x,y)e_N(x,y)\,dxdy, \tag{25.2.31}$$

$$E_{\mathsf{DS}} \coloneqq \sum_{j:1\le j\le N;\lambda_j<0} \lambda_j - \frac{1}{2}\mathsf{D}(\rho^{\mathsf{TF}},\, \rho^{\mathsf{TF}}) - \kappa_{\mathsf{Dirac}} \int {\rho^{\mathsf{TF}}}^{\frac{4}{3}}\,dx, \tag{25.2.32}$$

$\kappa_{\mathsf{Dirac}} = (6/\pi q)^{\frac{1}{3}}$ *is a Dirac constant.*

Here (25.2.28)–(25.2.31) are (1.15), (1.16), (1.8), (1.6) respectively and (25.2.32) is a combination of (1.12) and (25.3.30) of this paper[9]. Actually we need only (25.2.29) and (25.2.32).

As we are going to prove that the last terms in (25.2.31) and (25.2.19) coincide modulo $O(Z^{\frac{5}{3}-\delta})$ we made a necessary step completely.

25.3 Thomas-Fermi Theory

Thomas-Fermi theory is well-developed in the no-magnetic-field case. We cannot suggest any better reading than E. Lieb's *Selecta*[3].

In the Thomas-Fermi theory N is a real nonnegative number (not necessarily an integer).

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[8] Albeit not necessarily of eigenfunctions of H_W. See Appendix 25.A.4.

[9] We do not have a coefficient $\frac{1}{2}$ in the definition of $\mathsf{D}(.,.)$ but G. Graf and J. P. Solovej [1] have.

25.3.1 Existence

Let us recall that in order to get the best lower estimate (neglecting semiclassical errors) one needs to maximize

$$\Phi_*(W+\nu) \coloneqq -\int P(W+\nu)\,dx - \frac{1}{8\pi}\|\nabla(W-V)\|^2 \tag{25.3.1}$$

given by (25.2.11) where we used equalities

$$\mathsf{D}(\rho,\rho) = -(\rho, W-V) = \frac{1}{4\pi}\|\nabla(W-V)\|^2, \tag{25.3.2}$$

$$\rho \coloneqq \frac{1}{4\pi}\Delta(W-V), \tag{25.3.3}$$

$\|.\|$ means $\mathscr{L}^2$-norm and $W \to 0$ as $|x| \to \infty$.

On the other hand, to get the best possible upper estimate (neglecting semiclassical errors) one needs to minimize

$$\Phi^*(\rho',\nu) \coloneqq \int \bigl(\tau(\rho') - V\rho'\bigr)\,dx + \frac{1}{2}\mathsf{D}(\rho',\rho') - \nu\int \rho'\,dx \tag{25.3.4}$$

where

$$\rho' \coloneqq P'(W+\nu) \tag{25.3.5}$$

and $\tau(\rho)$ the Legendre transformation (25.1.18) of P. Recall that according to (25.1.19)

$$P(w) = \frac{q}{15\pi^2}w_+^{\frac{5}{2}}, \qquad P'(w) = \frac{q}{6\pi^2}w_+^{\frac{3}{2}} \tag{25.3.6}_{1,2}$$

and therefore

$$\tau(\rho) = \frac{3}{5}\bigl(6\pi^2 q^{-1}\bigr)^{\frac{2}{3}}\rho^{\frac{5}{3}}. \tag{25.3.7}$$

Proposition 25.3.1. *In our assumptions for any fixed $\nu \le 0$*

(i) $\Phi_*(W+\nu)$ is a strictly concave functional.

(ii) $\Phi^*(\rho)$ is a strictly convex functional.

(iii) $\Phi_*(W+\nu) \le \Phi^*(\rho,\nu)$ for any $\rho \ge 0$ and W.

(iv) These extremal problems have a common solution W and ρ and

$$\rho = \frac{1}{4\pi}\Delta(W-V) = P'(W+\nu), \tag{25.3.8}$$

$$W = o(1) \qquad \text{as } |x| \to \infty. \tag{25.3.9}$$

(v) On the other hand, solution of (25.3.8)–(25.3.9) *is the solution of the both extremal problems.*

(vi) Neither of these problem has a solution for $\nu > 0$.

(vii) Function

$$\mathcal{N}(\nu) = \int P'(W+\nu)\,dx \tag{25.3.10}$$

is continuous and monotone increasing at $(-\infty, 0]$ *with* $\mathcal{N}(\nu) \to 0$ *as* $\nu \to -\infty$ *and* $\mathcal{N}(0) = Z$.

(viii) For ν *and* N *linked by* $N = \mathcal{N}(\nu)$ *solutions of the problem above coincide with* ρ^{TF}, W^{TF} *of the problem* (25.1.20) *and one can skip condition* $(25.1.20)_2$ *for* $N \ge Z$ *and*

$$\mathcal{E}^{\mathsf{TF}} = \Phi(W^{\mathsf{TF}}+\nu) + \nu N = \Phi^*(\rho^{\mathsf{TF}}, \nu) + \nu N. \tag{25.3.11}$$

Proof. The proof of Statements (i) and (ii) is obvious; therefore both problems have unique solutions. Comparing Euler-Lagrange equations we get that these solutions coincide which yields Statements (iv) and (iii).

Proof of Statements (v)–(viii) is also rather obvious. □

Proposition 25.3.2. *For arbitrary* W *the following estimates hold with absolute constants* $\epsilon_0 > 0$ *and* C_0*:*

$$\epsilon_0 \mathsf{D}(\rho-\rho^{\mathsf{TF}}, \rho-\rho^{\mathsf{TF}}) \le \Phi_*(W^{\mathsf{TF}}+\nu) - \Phi_*(W+\nu) \le C_0 \mathsf{D}(\rho-\rho', \rho-\rho') \tag{25.3.12}$$

and

$$\epsilon_0 \mathsf{D}(\rho'-\rho^{\mathsf{TF}}, \rho'-\rho^{\mathsf{TF}}) \le \Phi^*(\rho,\nu) - \Phi^*(\rho^{\mathsf{TF}}, \nu) \le C_0 \mathsf{D}(\rho-\rho', \rho-\rho') \tag{25.3.13}$$

with $\rho = \frac{1}{4\pi}\Delta(W-V)$, $\rho' = P'(W+\nu)$.

Proof. This proof is rather obvious as well. □

25.3.2 Properties

Proposition 25.3.3. *The solution of the Thomas-Fermi problem has following scaling properties*

$$W^{\mathsf{TF}}(x;\, \underline{Z};\, \underline{y};\, N;\, q) = q^{\frac{2}{3}} N^{\frac{4}{3}} W^{\mathsf{TF}}(q^{\frac{2}{3}} Z^{\frac{1}{3}} x;\, N^{-1}\underline{Z};\, q^{\frac{2}{3}} N^{\frac{1}{3}}\underline{y};\, 1;\, 1), \tag{25.3.14}$$

$$\rho^{\mathsf{TF}}(x;\, \underline{Z};\, \underline{y};\, N;\, q) = N^2 q^2 \rho^{\mathsf{TF}}(q^{\frac{2}{3}} Z^{\frac{1}{3}} x;\, N^{-1}\underline{Z};\, q^{\frac{2}{3}} N^{\frac{1}{3}}\underline{y};\, 1;\, 1), \tag{25.3.15}$$

$$\mathcal{E}^{\mathsf{TF}}(\underline{Z};\, \underline{y};\, N;\, q) = q^{\frac{2}{3}} N^{\frac{7}{3}} \mathcal{E}^{\mathsf{TF}}(N^{-1}\underline{Z};\, q^{\frac{2}{3}} N^{\frac{1}{3}}\underline{y};\, 1;\, 1), \tag{25.3.16}$$

$$\nu^{\mathsf{TF}}(\underline{Z};\, \underline{y};\, N;\, q) = q^{\frac{2}{3}} N^{\frac{4}{3}} \nu^{\mathsf{TF}}(N^{-1}\underline{Z};\, q^{\frac{2}{3}} N^{\frac{1}{3}}\underline{y};\, 1;\, 1) \tag{25.3.17}$$

where $\nu^{\mathsf{TF}} = \nu$ is the chemical potential; recall that $\underline{Z} = (Z_1, \dots, Z_M)$ and $\underline{y} = (y_1, \dots, y_M)$ are arrays and parameter q also enters into Thomas-Fermi theory.

Proof. Proof is trivial by scaling. □

Since we can exclude q by scaling, we do not indicate dependence on it anymore. The following properties of Thomas-Fermi potential and density in the case of the single atom ($M = 1$) are well-known:

Proposition 25.3.4. *Let $M = 1$. Then the solution of the Thomas-Fermi problem has the following properties:*

(i) $W^{\mathsf{TF}}(x; Z_m, y_m; N)$ and $\rho^{\mathsf{TF}}(x; Z_m, y_m; N)$ are spherically symmetric (with respect to y_m) and are non-increasing convex functions of $|x - y_m|$.

(ii) If $N = Z_m$, then

$$W^{\mathsf{TF}} \asymp \min\bigl(Z_m|x - y_m|^{-1}, |x - y_m|^{-4}\bigr), \tag{25.3.18}$$

$$\rho^{\mathsf{TF}} \asymp \min\bigl(Z_m^{\frac{3}{2}}|x - y_m|^{-\frac{3}{2}}, |x - y_m|^{-6}\bigr) \tag{25.3.19}$$

with the threshold at $|x - y_m| \asymp r_m^* = Z_m^{-\frac{1}{3}}$ when $W^{\mathsf{TF}} \asymp Z_m^{\frac{4}{3}}$ and $\rho^{\mathsf{TF}} \asymp Z_m^2$.

(iii) If $\epsilon Z_m \le N < Z_m$, then

$$-\nu \asymp |Z_m - N|^{\frac{4}{3}} \tag{25.3.20}$$

and (25.3.18) *holds as $|x - y_m| \le \bar{r}_m$ where*

$$\bar{r}_m = -\nu^{-1}|Z_m - N|^{-1} \asymp |Z_m - N|^{-\frac{1}{3}} \tag{25.3.21}$$

for atoms denotes the exact radius of the support of ρ^{TF} (see Statement (iv).

(iv) On the other hand,

$$W^{\mathsf{TF}} = (Z_m - N)|x - \mathsf{y}_m|^{-1} \qquad \text{as} \quad |x - \mathsf{y}_m| \ge \bar{r}_m \tag{25.3.22}$$

and $\rho^{\mathsf{TF}} = 0$ *as* $|x - \mathsf{y}_m| \ge \bar{r}_m$*;*

(v) Meanwhile, $W^{\mathsf{TF}} \asymp -\nu$ *and* $\rho^{\mathsf{TF}} = O\big(|Z_m - N|^2\big)$ *as* $|x - \mathsf{y}_m| \asymp \bar{r}_m$*.*

(vi) Finally,

$$-\langle x - \mathsf{y}_m, \nabla W\rangle \asymp W. \tag{25.3.23}$$

Consider now the molecular case ($M \ge 2$):

Proposition 25.3.5. *(i) Let* $M \ge 2$*. Then*

$$\nu \asymp |Z - N|^{\frac{4}{3}} \tag{25.3.24}$$

and

$$\sum_m \epsilon W_m^{\mathsf{TF}}(c(x - \mathsf{y}_m)) \le W^{\mathsf{TF}} \le c \sum_m W_m^{\mathsf{TF}}(\epsilon(x - \mathsf{y}_m)) \tag{25.3.25}$$

where $Z = Z_1 + \ldots + Z_M$ *and* W_m^{TF} *denotes an atomic Thomas-Fermi potential with the charge* Z_m *located at* 0 *and the same chemical potential* ν*. Here* ϵ *and* c *depend only on* M*.*

(ii) In particular, if $N < Z$ *and* $|x - \mathsf{y}_m| \ge c\bar{r}_m$ *for all* $m = 1, \ldots, M$*, then*

$$W^{\mathsf{TF}}(x) \asymp \sum_m |Z - N||x - \mathsf{y}_m|^{-1} \tag{25.3.26}$$

and

$$\rho^{\mathsf{TF}}(x) = 0. \tag{25.3.27}$$

Proof. Proof is due to the comparison arguments of E. Lieb, J. P. Solovej and J. Yngvason [1,3]. □

Proposition 25.3.6. *(i) Let* $N \le Z$ *and*

$$\ell(x) = \frac{1}{2} \min_m |x - \mathsf{y}_m|, \tag{25.3.28}$$

$$\zeta(x) = \big(W^{\mathsf{TF}}(x)\big)^{\frac{1}{2}}. \tag{25.3.29}$$

Then

$$\zeta(x) \le \bar{\zeta}(x) = C \begin{cases} Z^{\frac12}\ell(x)^{-\frac12} & \text{as } \ell(x) \le Z^{-\frac13}, \\ \ell(x)^{-2} & \text{as } Z^{-\frac13} \le \ell(x) \le |Z-N|^{-\frac13}, \\ |Z-N|^{\frac12}\ell(x)^{-\frac12} & \text{as } \ell(x) \ge |Z-N|^{-\frac13}; \end{cases} \tag{25.3.30}$$

$\zeta(x)$ and $\bar{\zeta}(x)$ are both ℓ-admissible and

$$\bigl|D^\alpha W^{\mathsf{TF}}(x)\bigr| \le C_\alpha \zeta(x)^2\, \ell(x)^{-|\alpha|} \qquad \forall \alpha : |\alpha| \le 3, \tag{25.3.31}$$

and

$$|D^\alpha W^{\mathsf{TF}}(x) - D^\alpha W^{\mathsf{TF}}(y)| \le C_3 \zeta(x)^2 \ell(x)^{-\frac72} |x-y|^{\frac12} \qquad \forall x, y : |x-y| \le \epsilon \ell(x) \tag{25.3.32}$$

(ii) Unless $\zeta(x) \asymp (-\nu)^{\frac12}$ estimates (25.3.32) *hold for all α.*

Proof. This proof is rather obvious corollary of the Euler-Lagrange equation. □

Remark 25.3.7. Let

$$Z_m \asymp N \qquad \forall m. \tag{25.3.33}$$

Then $\zeta(x) \asymp \bar{\zeta}(x)$.

Theorem 25.3.8 [10]**.** *Consider $\mathcal{E}^{\mathsf{TF}}$ and*

$$\widehat{\mathcal{E}}^{\mathsf{TF}} := \mathcal{E}^{\mathsf{TF}} + U, \tag{25.3.34}$$

$$U = U(Z_1, \dots, Z_M; \mathsf{y}_1, \dots, \mathsf{y}_M) = \sum_{1 \le m < m' \le M} \frac{Z_m Z_{m'}}{|\mathsf{y}_m - \mathsf{y}_{m'}|}. \tag{25.3.35}$$

Select a nucleus y_m and a unit vector n such that

$$\langle \mathsf{y}_k - \mathsf{y}_m, \mathsf{n}\rangle \le 0 \qquad \forall k \tag{25.3.36}$$

and plug $\mathsf{y}_m + \alpha\mathsf{n}$ instead of y_m into $\mathcal{E}^{\mathsf{TF}}$ and into $\widehat{\mathcal{E}}^{\mathsf{TF}}$ [11]*. Then*

[10] Theorem 1 of R. Benguria [1]; see proof in Appendix 25.A.6.

[11] So all other nuclei are confined in half-space and y_m moves away outside.

(i) $\widehat{\mathcal{E}}^{\mathsf{TF}}_{\alpha}$ *is a non-increasing function of* $\alpha \ge 0$*;*

(ii) $\mathcal{E}^{\mathsf{TF}}_{\alpha}$ *is a non-decreasing function of* $\alpha \ge 0$*;*

(iii) For fixed $\alpha > 0$ $\widehat{\mathcal{E}}^{\mathsf{TF}}_{\alpha} - \widehat{\mathcal{E}}^{\mathsf{TF}}_{0}$ *is non-decreasing function of* N*;*

(iv) For fixed $\alpha > 0$ $\mathcal{E}^{\mathsf{TF}}_{\alpha} - \mathcal{E}^{\mathsf{TF}}_{0}$ *is non-decreasing function of* N*.*

Equality

$$\nu = \frac{\partial \mathcal{E}^{\mathsf{TF}}}{\partial N} \tag{25.3.37}$$

implies that (iii) is equivalent to

$$\nu_{\alpha} \ge \nu_0. \tag{25.3.38}$$

Theorem 25.3.9 [12]**.** *(i) For fixed* $Z_1, \dots, Z_M; \mathsf{y}_1, \dots, \mathsf{y}_M$ *and* $N = Z$

$$\lambda^7 \widehat{\mathcal{E}}^{\mathsf{TF}}(\underline{Z}; \lambda \underline{y}; N) = \widehat{\mathcal{E}}^{\mathsf{TF}}(\lambda^3 \underline{Z}; \underline{\mathsf{y}}; \lambda^3 N) \tag{25.3.39}$$

is positive non-decreasing function of $\lambda > 0$ *and has a finite limit as* $\lambda \to +\infty$*.*

(ii) This limit does not depend on $Z_1, \dots, Z_M$*.*

Remark 25.3.10. One can observe easily that the same scaling property without assumption $N = Z$ holds for $\mathcal{E}^{\mathsf{TF}}$ and U as well.

These two theorems and remark imply immediately

Proposition 25.3.11. *Let assumption* (25.3.33) *be fulfilled. Then*

$$\widehat{\mathcal{E}}^{\mathsf{TF}}(\underline{Z}; \underline{\mathsf{y}}; N) - \min_{\substack{N_1,\dots,N_M: \\ N_1+\dots N_M = N}} \sum_{1 \le m \le M} \mathcal{E}^{\mathsf{TF}}(Z_m; N_m) \ge \epsilon \min(a^{-7}, Z^{\frac{7}{3}}) \tag{25.3.40}$$

where

$$a = \frac{1}{2} \min_{m<m'} |\mathsf{y}_m - \mathsf{y}_{m'}|. \tag{25.3.41}$$

[12] (1.8)–(1.9) of H. Brezis and E. Lieb [1].

Proof. In virtue of theorem 25.3.8(i) it suffices to prove proposition for $M = 2$ (all other nuclei could be pulled to infinity), and in virtue of theorem 25.3.8(iii) it suffices to prove proposition for $Z = N$.

Then the proof is due to theorem 25.3.9(i) and (25.3.33), which provides uniformity. □

Remark 25.3.12. In virtue of (25.3.37) the minimum (with respect to $N_1, \dots, N_M$) in the sum in the right hand expression is reached when $\nu_j = \nu_k$ for all j, k. The same is true for a system of isolated molecules.

Proposition 25.3.13 (Non-Binding Theorem). *Let $\mathcal{Q}$ denote Thomas-Fermi excess energy which is the left-hand expression of* (25.3.40)*. Then*

$$\mathsf{D}(\rho^{\mathsf{TF}} - \bar{\rho}^{\mathsf{TF}},\, \rho^{\mathsf{TF}} - \bar{\rho}^{\mathsf{TF}}) \le C\mathcal{Q}, \qquad \bar{\rho}^{\mathsf{TF}} := \sum_{1\le m\le M} \rho_m^{\mathsf{TF}}. \tag{25.3.42}$$

Proof. We follow "non-binding" proof due to Baxter with small modification.

According to Baxter's Lemma (see Proposition 25.A.5) there exist g, $0 \le g \le \rho^{\mathsf{TF}}$ and $h = \rho^{\mathsf{TF}} - g$ such that $g * |x|^{-1} = V_1$ a.e. when $h > 0$ and $g * |x|^{-1} \le V_1$ a.e. when $h = 0$. Here $V_m = Z_m|x - \mathsf{y}_m|^{-1}$.

Let $\alpha = \int g\,dx$, $\beta = \int h\,dx$ and let $\mathcal{E}_1^{\mathsf{TF}}$, $\mathcal{E}_{(1)}^{\mathsf{TF}}$ be Thomas-Fermi energies for the first atom and for the rest of molecule respectively and ρ_1^{TF}, $\rho_{(1)}^{\mathsf{TF}}$ be corresponding Thomas-Fermi densities. Then

$$\min_{N_1+N'\le N}\Bigl(\mathcal{E}_1^{\mathsf{TF}}(N_1) + \mathcal{E}_{(1)}^{\mathsf{TF}}(N')\Bigr) \le \mathcal{E}_1(\alpha) + \mathcal{E}_{(1)}(\beta)$$

with

$$\begin{aligned}\mathcal{E}_1(\alpha) &\le \mathcal{E}_1(g) - \epsilon\mathsf{D}(\rho_1^{\mathsf{TF}} - g,\, \rho_1^{\mathsf{TF}} - g)\\ &= \int \tau(g)\,dx - \int V_1 g\,dx + \frac{1}{2}\mathsf{D}(g,g) - \epsilon\mathsf{D}(\rho_1^{\mathsf{TF}} - g,\, \rho_1^{\mathsf{TF}} - g)\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_{(1)}(\beta) &\le \mathcal{E}_{(1)}(h) - \epsilon\mathsf{D}(\rho_{(1)}^{\mathsf{TF}} - h,\, \rho_{(1)}^{\mathsf{TF}} - h)\\ &= \int \tau(h)\,dx - \int (V - V_1)h\,dx + \frac{1}{2}\mathsf{D}(h,h) - \epsilon\mathsf{D}(\rho_{(1)}^{\mathsf{TF}} - h,\, \rho_{(1)}^{\mathsf{TF}} - h)\end{aligned}$$

and therefore

$$\mathcal{E}_1(g)+\mathcal{E}_{(1)}(h)=\int \bigl(\tau(g)+\tau(h)-V(g+h)\bigr)\,dx + \frac{1}{2}\mathsf{D}(g+h,g+h)+\int (V-V_1)g\,dx+\int V_1 h\,dx-\mathsf{D}(g,h)$$

with

$$\int \bigl(\tau(g)+\tau(h)-V(g+h)\bigr)\,dx+\frac{1}{2}\mathsf{D}(g+h,g+h) \le \int \bigl(\tau(g+h)-V(g+h)\bigr)\,dx+\frac{1}{2}\mathsf{D}(g+h,g+h)=:\mathcal{E}(g+h)=\mathcal{E}(\rho) \tag{25.3.43}$$

due to superadditivity of $\tau(\cdot)$ (that is $\tau(g+h)\ge\tau(g)+\tau(h)$ as $h\ge 0, h\ge 0$). Further,

$$\int (V-V_1)g\,dx=\int \sum_{2\le m\le M}\frac{Z_m}{|x-y_m|}g(x)\,dx = \int \sum_{2\le m\le M} Z_m\delta(x-y_m)(|x|^{-1}*g)\,dx \le \int \sum_{2\le m\le M}\delta(x-y_m)V_1(x)\,dx = \sum_{2\le m\le M}\frac{Z_1Z_m}{|\mathsf{y}_1-\mathsf{y}_m|}$$

due to properties g and

$$\int V_1 h\,dx-\mathsf{D}(g,h)=\int (V_1-|x|^{-1}*g)h\,dx=0$$

due to properties g, h.

Therefore

$$\min_{N_1+N'\le N}\Bigl(\mathcal{E}_1^{\mathsf{TF}}(N_1)+\mathcal{E}_{(1)}^{\mathsf{TF}}(N')\Bigr)\le \mathcal{E}(g+h)+\sum_{2\le m\le M}\frac{Z_1Z_m}{|\mathsf{y}_1-\mathsf{y}_m|} - \epsilon\mathsf{D}(\rho_1^{\mathsf{TF}}-g,\,\rho_1^{\mathsf{TF}}-g)-\epsilon\mathsf{D}(\rho_{(1)}^{\mathsf{TF}}-h,\,\rho_{(1)}^{\mathsf{TF}}-h). \tag{25.3.44}$$

Then

$$\mathsf{D}(\rho^{\mathsf{TF}}-\rho_1^{\mathsf{TF}}-\rho_{(1)}^{\mathsf{TF}},\rho^{\mathsf{TF}}-\rho_1^{\mathsf{TF}}-\rho_{(1)}^{\mathsf{TF}})\le 2\mathsf{D}(g-\rho_1^{\mathsf{TF}},g-\rho_1^{\mathsf{TF}})+2\mathsf{D}(h-\rho_{(1)}^{\mathsf{TF}},h-\rho_{(1)}^{\mathsf{TF}})\le C\mathcal{Q}. \tag{25.3.45}$$

Using induction with respect to M we arrive to (25.3.42). □

Problem 25.3.14. Find the stronger lower bound in (25.3.40) as $N < Z$. Would be the left-hand expression $\asymp \min(a^{-7} + |Z-N|^2 a^{-1}, Z^{\frac{7}{3}})$?

25.4 Application of Semiclassical Methods

25.4.1 Asymptotics of the Trace

In this subsection we calculate asymptotics of $\mathsf{Tr}((H_W - \nu)^-)$. Here we need to consider both inner and outer zones.

An *inner zone* (near nucleus y_m) is a ball where $V_j = Z_m|x-\mathsf{y}_m|^{-1}$ dominates $W - V_m$. For a single nucleus ($M = 1$) it is defined by

$$|x-\mathsf{y}_m| \le \epsilon Z_m^{-\frac{1}{3}} \tag{25.4.1}$$

but in the case $M \ge 2$ there are another restrictions

$$|x-\mathsf{y}_m| \le \epsilon \min_{m'\neq m}\Big(Z_m(Z_m+Z_{m'})^{-1}|\mathsf{y}_m-\mathsf{y}_{m'}|\Big) \tag{25.4.2}$$

and

$$|x-\mathsf{y}_m| \le Z_m\nu^{-1} \tag{25.4.3}$$

but we shrink this zone to

$$|x-\mathsf{y}_m| \le r_m \coloneqq \epsilon \min\big(Z_m Z^{-1}a, Z_m^{-\frac{1}{3}}\big). \tag{25.4.4}$$

Let us consider contribution of the zone $\mathcal{X}_m$ described by (25.4.4) to $\mathsf{N}_1(H_W-\nu) = \mathsf{Tr}((H_W-\nu)^-)$, both into the principal part of asymptotics and the remainder. Let ψ_m be a partition element concentrated in $\mathcal{X}_m$ and equal to 1 in $\{x\colon |x-\mathsf{y}_m| \le \frac{1}{2}r_m\}$. Then, according to Theorem 12.6.8,

$$\mathsf{Tr}\big((H_W-\nu)^-\psi_m\big) = \int \mathsf{Weyl}_1(x)\psi_m(x)\,dx + \mathsf{Scott}_m + O(R_m) \tag{25.4.5}$$

where $\mathsf{Weyl}_1(x)$ and Scott_m are calculated for the case $q = 1$ and then multiplied by q [13]:

$$\mathsf{Weyl}_1(x) \coloneqq -\frac{q}{15\pi^2}\big(W(x)+\nu\big)_+^{\frac{5}{2}} \tag{25.4.6}$$

[13] As operator $H_w - \nu$ is nothing but q copies of $\Delta - (W+\nu)$.

while R_m is $C\zeta^2(\zeta\ell) = C\zeta^3\ell$ calculated on its border i.e.

$$R_m = CZ_m^{\frac{5}{3}} + CZ_m^{\frac{3}{2}} r_m^{-\frac{1}{2}} \le CZ^{\frac{5}{3}} + CZ^{\frac{3}{2}} a^{-\frac{1}{2}}. \tag{25.4.7}$$

Really, one needs just to rescale $x \mapsto (x - \mathsf{y}_m) r_m^{-1}$ and $\tau \mapsto \tau Z_m^{-1} r_m$ and introduce a semiclassical parameter $\hbar = Z_m^{\frac{1}{2}} r_m^{\frac{1}{2}}$.

Remark 25.4.1. (i) Clearly, these arguments work only if $r_m \ge Z_m^{-1}$ (i.e. $Z_m^2 \ge a^{-1}Z$).

(ii) On the other hand, if $Z_m^2 \ge a^{-1}Z$ but $a \ge Z^{-1}$ we define $r_m = a^{\frac{1}{2}} Z^{-\frac{1}{2}}$ and we do not include Scott_m [14] into the principal expression; moreover, in this case we include $\mathcal{X}_m$ into a singular zone and use variational methods to estimate its contribution into the principal part of asymptotics; it will not exceed $CZa^{-1} \le CZ^{\frac{3}{2}} a^{-\frac{1}{2}}$.

(iii) Furthermore, if $a \le Z^{-1}$, we set $r_m = Z^{-1}$ and we do not include any Scott_m [14] into the principal part of asymptotics and include all $\mathcal{X}_m$ into singular zones; using variational methods we estimate their contributions into the principal part of asymptotics by CZ^2.

Therefore we conclude that

(25.4.8) The total contribution of all inner zones into remainder does not exceed the right-hand expression of (25.4.7) as $a \ge Z^{-1}$ and CZ^2 as $a \le Z^{-1}$.

Let us consider contributions of the *outer zone* $\mathcal{X}_0$ which is complimentary to the union of all inner zones. Then

$$\mathsf{Tr}\big((H_W - \nu)^- \psi_0\big) = \int \mathsf{Weyl}_1(x)\psi_0(x)\,dx + O(R_0) \tag{25.4.9}$$

with $\mathsf{Weyl}_1(x)$ defined by (25.4.6) where

$$R_0 = \int_{\mathcal{X}_0} C\bar{\zeta}(x)^3 \ell^{-2}\,dx \tag{25.4.10}$$

[14] However it will be less than the remainder estimate, so we can include it into the principal part of asymptotics anyway.

and

$$\bar{\zeta} = \begin{cases} Z^{\frac{1}{2}}\ell^{-\frac{1}{2}} & \text{as } \ell \le Z^{-\frac{1}{3}}, \\ \ell^{-2} & \text{as } \ell \ge Z^{-\frac{1}{3}}. \end{cases} \tag{25.4.11}$$

We justify (25.4.9)–(25.4.11) a bit later by an appropriate partition of unity. One can see easily that the contribution of the zone $\{x\colon \ell(x) \le Z^{-\frac{1}{3}}\}$ into expression (25.4.10) does not exceed the same expression as in (25.4.8) and that the contribution of the zone $\{x\colon \ell(x) \ge Z^{-\frac{1}{3}}\}$ into (25.4.10) does not exceed $CZ^{\frac{5}{3}}$. Then we arrive to

Theorem 25.4.2. *Let $W = W^{\mathsf{TF}}$, $N \asymp Z$, $W = W^{\mathsf{TF}}$. Then*

$$\mathsf{Tr}\big((H_W - \nu)^-\big) = \int \mathsf{Weyl}_1(x)\,dx + \sum_{1\le m\le M} \mathsf{Scott}_m + O(R), \tag{25.4.12}$$

with

$$R := \begin{cases} CZ^{\frac{5}{3}} + CZ^{\frac{3}{2}}a^{-\frac{1}{2}} & \text{as} \quad a \ge Z^{-1}, \\ CZ^2 & \text{as} \quad a \le Z^{-1}. \end{cases} \tag{25.4.13}$$

Proof. (i) Consider $\nu = 0$ first. Then we just apply ℓ-admissible partition of unity. Sure, $\zeta\ell \le 1$ as $\ell \ge 1$ but we can deal with it either by taking $\zeta = \ell^{-1}$ here or considering it as a singular zone and applying here variational estimate as well.

(ii) Variational estimate works for $\nu < 0$ as well; furthermore, zone $\{x\colon W(x) \le (1-\epsilon)\nu, \ell \ge |\nu|^{-\frac{1}{2}}\}$ is classically forbidden.

(iii) However as $\nu \le -c$ we have a little problem as W^{TF} is not very smooth, it is only $\mathscr{C}^{\frac{7}{2}}$ as $W \asymp -\nu$. The best[15)] way to deal with it is to take $\varepsilon\ell$-mollification with $\varepsilon = \hbar^{1-\delta}$, $\hbar = (\zeta\ell)^{-1}$, use rough microlocal analysis of Section 4.6 and the bracketing; it will bring an approximation error not exceeding $C\varepsilon^{\frac{7}{2}}\hbar^{-3}|\nu|$ which does not exceed $|\nu| = O(|Z-N|^{\frac{4}{3}})$. We leave easy details to the reader. □

Now we arrive to the lower estimate for E:

[15)] From the point of view of generalization to the case when magnetic field is present and it is not too weak, so magnetic version of W^{TF} has multiple singularities.

Corollary 25.4.3. *Let $N \asymp Z$. Then*

$$\mathsf{E} \geq \mathcal{E}^{\mathsf{TF}} + \mathsf{Scott} - CR \tag{25.4.14}$$

with R defined by (25.4.13).

Proof. We know from Subsection 25.2.1 that

$$\mathsf{E} \geq \mathsf{N}_1(H_W - \nu) + \nu N - \frac{1}{2}\mathsf{D}(\rho,\rho) - CZ^{\frac{5}{3}} \tag{25.4.15}$$

as W is given by (25.2.3). In virtue of Theorem 25.4.2

$$\mathsf{E} \geq \int \mathsf{Weyl}_1(x)\,dx + \nu \int \mathsf{Weyl}(x)\,dx - \frac{1}{2}\mathsf{D}(\rho,\rho) + \mathsf{Scott} - CR \tag{25.4.16}$$

where we also plugged instead of N as $N < Z$ (and $\nu < 0$)

$$N = \int \mathsf{Weyl}(x)\,dx \tag{25.4.17}$$

with

$$\mathsf{Weyl}(x) = \frac{q}{6\pi^2}\big(W(x) + \nu\big)_+^{\frac{3}{2}}. \tag{25.4.18}$$

One can check easily that three first terms in the right-hand expression of (25.4.18) constitute exactly $\Phi_*(W+\nu)$ coinciding with $\mathcal{E}^{\mathsf{TF}}$ as $W = W^{\mathsf{TF}}$.□

Remark 25.4.4. (i) As $a \leq Z^{-\frac{1}{3}}$ using the same method one can prove a slightly better remainder estimate–with $Z^{\frac{3}{2}}a^{-\frac{1}{2}}$ replaced by

$$\sum_{1\leq m<m'\leq M} \min\Big((Z_m + Z_{m'})^{\frac{3}{2}}|x_m - \mathsf{y}_{m'}|^{-\frac{1}{2}}, (Z_m + Z_{m'})^2\Big) \tag{25.4.19}$$

allowing to lighter nuclei to be closer one to another.

(ii) To improve this estimate further, allowing lighter nuclei to be closer to heavier, ones one needs to improve Theorem 12.6.8 which seems to be too difficult task for a such little gain.

25.4.2 Upper Estimate for E

Recall that in virtue of Subsection 25.2.2

$$\mathsf{E} \le \mathsf{N}_1(H_W-\nu)+\nu N - \frac{1}{2}\mathsf{D}(\rho,\rho)+ |\lambda_N-\nu|\cdot|\mathsf{N}(H_W-\nu)-N| + \frac{1}{2}\mathsf{D}(\operatorname{tr} e_N(x,x)-\rho, \operatorname{tr} e_N(x,x)-\rho\big) \tag{25.4.20}$$

and thus we need to estimate two last terms in the right-hand expression.

Estimating $|\lambda_N-\nu|$

First, we need to estimate $|\lambda_N-\nu|$. We will use the heuristic equality $\mathsf{N}(H_W-\lambda_N)\approx N$ or more precisely two inequalities

$$\mathsf{N}(H_W-\lambda_N-0)\le N\le \mathsf{N}(H_W-\lambda_N+0) \tag{25.4.21}$$$_{1,2}$

and equality

$$\int \mathsf{Weyl}(x)\,dx = \min(N,Z) \tag{25.4.22}$$

where the right inequality (25.4.21)$_2$ is valid only if $\lambda_N<0$ i.e. $\mathsf{N}(H_W)\ge N$.

Case $\lambda_N<\nu$. Then we will use (25.4.21)$_2$ and to calculate $\mathsf{N}(H_W-\lambda_N+0)$ we will use semiclassical approximation:

$$\mathsf{N}(H_W-\lambda_N+0)=\int \mathsf{Weyl}(x,\lambda_N)\,dx + O(R_0) \tag{25.4.23}$$

with the semiclassical error

$$R_0=\int \zeta^2\ell^{-1}\,dx \tag{25.4.24}$$

with the integral not exceeding

$$CZ\int_{\{|x|\le Z^{-\frac{1}{3}}\}} |x|^{-2}\,dx + C\int_{\{|x|\ge Z^{-\frac{1}{3}}\}} |x|^{-5}\,dx \asymp CZ^{\frac{2}{3}};$$

again we need to consider separately the case of $N\ge Z$ and $\nu=0$, when integral (25.4.24) is taken over $\mathbb{R}^3$, and the case of $N<Z$ and $\nu<0$, when this integral should be taken over $\{x\colon \ell(x)\le C(Z-N)^{-\frac{1}{3}}\}$; in the latter

case to cover non-smoothness we consider an approximation (mollification) of W and an approximation error $\varepsilon^{\frac{7}{2}}\hbar^{-3} \ll 1$. Therefore

$$\int \mathsf{Weyl}(x,\lambda_N)\,dx \ge N - CZ^{\frac{2}{3}}. \tag{25.4.25}$$

Comparing with (25.4.22) we conclude that

$$\int \Big(\big(W(x)+\nu\big)_+^{\frac{3}{2}} - \big(W(x)+\lambda_N\big)_+^{\frac{3}{2}}\Big)\,dx \le CZ^{\frac{2}{3}}. \tag{25.4.26}$$

Here an integrand is non-negative (since $\lambda_N < \nu$). Further, one can see easily that the main contribution to this integral is delivered by the zone $\{x\colon \ell(x) \asymp |\lambda_N|^{-\frac{1}{4}}\}$ [16] and the whole integral is $\asymp |\lambda_N|^{-\frac{1}{4}}|\nu-\lambda_N|$. Therefore (25.4.26) yields that

$$|\lambda_N-\nu| \le C\big(|\nu|+|\lambda_N-\nu|\big)^{\frac{1}{4}}Z^{\frac{2}{3}},$$

which is equivalent to

$$|\lambda_N-\nu| \le CZ^{\frac{8}{9}} + C|\nu|^{\frac{1}{4}}Z^{\frac{2}{3}} \asymp CZ^{\frac{8}{9}} + C(Z-N)_+^{\frac{1}{3}}Z^{\frac{2}{3}} \le CZ. \tag{25.4.27}$$

In particular,

(25.4.28) If $|\nu| \le c_0 Z^{\frac{8}{9}}$ (i.e. $(Z-N)_+ \le c_1 Z^{\frac{2}{3}}$), then $|\lambda_N| \le C_0 Z^{\frac{8}{9}}$

and also

$$|\lambda_N-\nu|\cdot|\mathsf{N}(H_W-\nu)-N| \le CZ\cdot Z^{\frac{2}{3}} = CZ^{\frac{5}{3}}. \tag{25.4.29}$$

Case $\lambda_N \ge \nu$. Then we will use the left inequality $(25.4.21)_1$, but if $\mathsf{N}(H_W) < N$ then integral (25.4.24) is diverging.

To avoid all related difficulties we will consider first the case when we necessarily conclude that $|\lambda_N| \ge (1-\epsilon)|\nu|$. To do so observe that even if the main contribution to the integral (25.4.26) [17] is delivered by the zone $\{x\colon \ell(x) \asymp (Z-N)|\lambda_N|^{-1} \asymp |\nu|^{\frac{3}{4}}|\lambda_N|^{-1}\}$, we will ignore this observation and consider a larger zone $\{x\colon \ell(x) \le C_0|\nu|^{-\frac{1}{4}}\}$ instead of $\{x\colon \ell(x) \le C_0|\nu|^{\frac{3}{4}}|\lambda_N|^{-1}\}$ [18].

One can prove easily that

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[16] Provided $|\lambda_N| \le C_0 Z^{\frac{4}{3}}$. On the other hand, if $|\lambda_N| \ge C_0 Z^{\frac{4}{3}}$ one can see easily that (25.4.26) is $\asymp Z$ because $|\nu| \le c_0 Z^{\frac{4}{3}}$ due to $N \asymp Z$.

[17] Now an integrand is non-positive.

[18] Obviously these zones coincides as $|\nu| \asymp |\lambda_N|$.

(25.4.30) The contribution of the zone $\{x\colon \ell(x) \le C_0|\nu|^{-\frac14}\}$ to the semiclassical remainder, when calculating $\mathsf{N}(H_W - \lambda_N)$, does not exceed $CZ^{\frac23}$.

Therefore we arrive to the estimate

$$|\lambda_N - \nu| \le C|\nu|^{\frac14} Z^{\frac23} \asymp C(Z-N)_+^{\frac13} Z^{\frac23} \le CZ \tag{25.4.31}$$

(cf. (25.4.27)). In particular, we conclude that

(25.4.32) If $|\nu| \ge CZ^{\frac89}$ (i.e. $(Z-N) \ge CZ^{\frac23}$), then $|\lambda_N| \asymp |\nu|$

(cf. (25.4.28)). Then one can easily recover (25.4.27) completely. Since $|\mathsf{N}(H_W - \nu) - N| \le CZ^{\frac23}$ we arrive to (25.4.29).

Case $|\nu| \le \eta = CZ^{\frac89}$. This case (i.e. $(Z-N) \le \eta^{\frac34} = CZ^{\frac23}$) is the most important one. The easiest way to tackle it is to pick up ρ^{TF} and W^{TF} calculated as if $\nu = 0$ i.e. $Z = N$; that means the change of the test function Ψ in the upper estimate.

We need to modify an upper estimate of $\sum_{1\le j\le N} \lambda_j$. To do this we note that

$$\lambda_N \ge -\eta \tag{25.4.33}$$

and

(25.4.34) A number of eigenvalues between λ_N (if $\lambda_N < 0$) and 0 does not exceed $CZ^{\frac23} + C\eta^{\frac34}$,

which can be proven easily by our standard methods. Therefore

$$\sum_{1\le j\le N} \lambda_j \le \mathsf{Tr}(H_W^-) + C\eta\big(Z^{\frac23} + \eta^{\frac34}\big) \tag{25.4.35}$$

and the last term is less than $CZ^{\frac53}$ as long as $\eta \le CZ^{\frac{20}{21}}$ which is fulfilled.

In this last case[19)] we arrive to an upper estimate with $\mathcal{E}^{\mathsf{TF}}$ calculated as if $\nu = 0$ i.e. $\mathcal{E}^{\mathsf{TF}}(\underline{Z}; \underline{y}; Z)$ which is less than $\mathcal{E}^{\mathsf{TF}}(\underline{Z}; \underline{y}; N)$. Actually the difference between these two is $\asymp |\nu|^{\frac74} \le |\eta|^{\frac74}$.

[19)] Pending an analysis of the next subsubsection.

Estimating D-Term

We need to estimate the last term in the right-hand expression of (25.4.20); we estimate it by

$$\begin{aligned}(25.4.36)\quad & C_0\mathsf{D}\big(\operatorname{tr} e(x,x,\nu)-P'(W+\nu),\operatorname{tr} e(x,x,\nu)-P'(W+\nu)\big)\\ &+C_0\mathsf{D}\big(\operatorname{tr} e(x,x,\nu)-\operatorname{tr} e_N(x,x),\operatorname{tr} e(x,x,\nu)-\operatorname{tr} e_N(x,x)\big)\\ &+C_0\mathsf{D}\big(\rho-P'(W+\nu),\rho-P'(W+\nu)\big),\end{aligned}$$

where the last term vanishes as $\rho=\rho^{\mathsf{TF}}$, $W=W^{\mathsf{TF}}$; however, for some technical reasons we want to avoid this assumption.

Estimating the First Term. To estimate the first term in (25.4.36) we apply the semiclassical asymptotics

$$\operatorname{tr} e(x,x,\nu)=\mathsf{Weyl}(x)+O(\zeta^2\ell^{-1}), \tag{25.4.37}$$

where $\mathsf{Weyl}(x)=P'(W(x)+\nu)$ and therefore this term does not exceed

$$\iint \zeta(x)^2\zeta(y)^2\ell(x)^{-1}\ell(y)^{-1}|x-y|^{-1}\,dxdy. \tag{25.4.38}$$

Estimating this integral by the double sum of the integrals over domains $\{(x,y)\colon \ell(x)\le Z^{-\frac{1}{3}},\ell(y)\le Z^{-\frac{1}{3}}\}$ and $\{(x,y)\colon \ell(x)\ge Z^{-\frac{1}{3}},\ell(y)\ge Z^{-\frac{1}{3}}\}$ we get

$$CZ^2\iint_{\{|x|\le Z^{-\frac{1}{3}},|y|\le Z^{-\frac{1}{3}}\}} |x|^{-2}|y|^{-2}|x-y|^{-1}\,dxdy \tag{25.4.39}$$

and

$$C\iint_{\{|x|\ge Z^{-\frac{1}{3}},|y|\ge Z^{-\frac{1}{3}}\}} |x|^{-3}|y|^{-3}|x-y|^{-1}\,dxdy \tag{25.4.40}$$

respectively, and rescaling we get the same integrals but both with the "threshold" 1 rather than $Z^{-\frac{1}{3}}$ and both with factor $Z^{\frac{5}{3}}$ rather than Z^2 or 1 respectively; one can see easily that both obtained integrals (without factor $Z^{\frac{5}{3}}$) are $\asymp 1$. Therefore, expression (25.4.38) is $O(Z^{\frac{5}{3}})$.

Therefore we proved

Proposition 25.4.5. *As $W=W^{\mathsf{TF}}$ the first term in* (25.4.36) *does not exceed $CZ^{\frac{5}{3}}$.*

Remark 25.4.6. (i) For $N \ge Z$ and $\nu = 0$ we used that $\zeta \le C_1\ell^{-2}$ for $\ell \ge Z^{-\frac{1}{3}}$.

(ii) For $N < Z$ and $\nu < 0$ we used that zone $\{x\colon \ell \ge C(Z-N)^{-\frac{1}{3}}\}$ is classically forbidden ($W+\nu<0$ there) and therefore integral is taken over zone $\{x\colon \ell \le C(Z-N)^{-\frac{1}{3}}\}$ where $\zeta \le C_1\ell^{-2}$.

(iii) As $N<Z$ W^{TF} is not very smooth near $W+\nu=0$ but one can handle it by rescaling arguments.

(iv) Alternatively (preferably[15)]) one can replace W^{TF} by its mollification $W^{\mathsf{TF}}_\varepsilon$.

Remark 25.4.7. Estimating this term, and also the second D-term (in the next paragraph) we need to estimate the contribution of the singular zone $\{x\colon \ell(x) \le \bar{r} = Z^{-1}\}$ where effective semiclassical parameter is less than 1. We claim that there

$$e(x,x,\lambda) \le CZ^3 \qquad \text{for } \lambda \le cZ^2. \tag{25.4.41}$$

Indeed, it is true if $\ell(x)\ge 1$. Also operator H is bounded from below by $-CZ^2$. And finally, in the ball of $B(\mathsf{y}_m, \epsilon Z^{-1})$ operator Δ is larger than $Z|x-\mathsf{y}_m|^{-1}$. We leave the easy details to the reader.

Therefore the contribution of this zone into N-term is $O(CZ^3\bar{r}^3) = O(1)$, into both D-terms is $O(Z^6\bar{r}^5) = O(Z)$, and into T-term is $O(Z^5\bar{r}^3) = O(Z^2)$.

Estimating the Second Term. Consider now the second term in (25.4.36). Due to the arguments of the previous paragraph modulo $O(Z^{\frac{5}{3}})$ one can rewrite it as

$$C_1\mathsf{D}\big(P'(W+\lambda_N)-P'(W+\nu), P'(W+\lambda_N)-P'(W+\nu)\big). \tag{25.4.42}$$

Really, if we replace $\mathsf{tr}\, e(x,x,\nu)$ and $\mathsf{tr}\, e_N(x,x)$ by $\mathsf{Weyl}(x,\nu)$ and $\mathsf{Weyl}(x,\lambda_N)$ respectively we make a semiclassical errors estimated by $Z^{\frac{5}{3}}$ provided either $\lambda_N < \nu$ or $\lambda_N \asymp \nu$ which is always the case unless $|\nu| \le CZ^{\frac{8}{9}}$ but in this case we just "cheat" resetting everything to the case $\nu = 0$.

Let us estimate (25.4.42). According to the previous Subsubsection 25.4.2.1 there are two cases:

(i) $N \ge Z - CZ^{\frac{2}{3}}$, in which case $|\nu| \le CZ^{\frac{8}{9}}$ and $|\lambda_N| \le CZ^{\frac{8}{9}}$, and (25.4.42) does not exceed

$$C|\nu-\lambda_N|^2 \iint_{\{\ell(x)\le L,\ell(y)\le L\}} |x-y|^{-1}\zeta(x)\zeta(y)\,dxdy + C\iint_{\{\ell(x)\ge L,\ell(y)\ge L\}} |x-y|^{-1}\zeta(x)^3\zeta(y)^3\,dxdy$$

with $L = |\lambda_N - \nu|^{-\frac{1}{4}}$; one can calculate easily that both terms are of the magnitude $C|\lambda_N-\nu|^{\frac{7}{4}} \le CZ^{\frac{14}{9}} \ll Z^{\frac{5}{3}}$.

(ii) $N \le Z - CZ^{\frac{2}{3}}$, in which case (25.4.42) does not exceed

$$C|\nu-\lambda_N|^2 \iint_{\{\ell(x)\le L,\ell(y)\le L\}} |x-y|^{-1}\zeta(x)\zeta(y)\,dxdy$$

with $L = |\nu|^{-\frac{1}{4}}$ and this integral does not exceed $|\lambda_N-\nu|^2|\nu|^{-\frac{1}{4}}$ which due to (25.4.27) does not exceed $C(Z-N)^{\frac{1}{3}}Z^{\frac{4}{3}} \le CZ^{\frac{5}{3}}$.

Estimating the Third Term. The third term in (25.4.36) does not exceed

$$C_1\mathsf{D}\big(\rho-\rho^{\mathsf{TF}},\,\rho-\rho^{\mathsf{TF}}\big)+ C_1\mathsf{D}\big(P'(W+\nu)-P'(W^{\mathsf{TF}}+\nu),\,P'(W+\nu)-P'(W^{\mathsf{TF}}+\nu)\big)$$

and we leave to the reader an easy proof that both terms here are $O(Z^{\frac{5}{3}-\delta})$ as W is a described mollification of W^{TF}.

Finally, a Theorem

As we finished an upper estimate for E we arrive to the following main result:

Theorem 25.4.8. *Let $N \asymp Z$ and let Ψ be a ground state. Then*

$$|\mathsf{E}-\mathcal{E}^{\mathsf{TF}}-\mathsf{Scott}| \le C\begin{cases} Z^{\frac{5}{3}}+a^{-\frac{1}{2}}Z^{\frac{3}{2}} & \text{as}\quad a\ge Z^{-1},\\ Z^2 & \text{as}\quad a\le Z^{-1},\end{cases} \tag{25.4.43}$$

where a is the minimal distance between nuclei.

25.4.3 Improved Asymptotics

So far as $a \ge Z^{-\frac{1}{3}}$ we recovered only $O(Z^{\frac{5}{3}})$ for both error estimate in E and (as a coproduct, see Subsection 25.4.4) for $\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}})$.

Our purpose is to improve them to $o(Z^{\frac{5}{3}})$ (or slightly better) as $a \gg Z^{-\frac{1}{3}}$ and recover the Schwinger and Dirac terms. To do so in the lower estimate for E one just need an improved electrostatic inequality (see Theorem 25.2.5) and also improved semiclassical estimates in $\mathsf{Tr}((H_W - \nu)^-)$ and $\mathsf{N}(H_W - \nu)$.

For the improved upper estimate we will need also to improve estimate $\frac{1}{2}\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}})$ for the test function Ψ and apply an estimate

$$|\frac{1}{2}\iint |x-y|^{-1} \operatorname{tr} e_N(x,y)e_N^\dagger(x,y)\,dxdy + \kappa_{\mathsf{Dirac}} \int \rho^{\mathsf{TF}\,\frac{4}{3}}(x)\,dx| \le Z^{\frac{5}{3}-\delta} \tag{25.4.44}$$

which is due to Theorem 6.4.17[20].

Remark 25.4.9. (i) Only contributions to the remainder of zones

$$\{x\colon Z^{-\frac{1}{3}-\delta_1} \le |x - \mathsf{y}_m| \le Z^{-\frac{1}{3}} b^{\delta_1}\} \tag{25.4.45}$$

with $b := \min(aZ^{\frac{1}{3}}, 1)$ (where now we assume that $b \ge 1$) should be considered because the contributions of both zones $\{x\colon \ell(x) \le Z^{-\frac{1}{3}} b^{-\delta_1}\}$ and $\{x\colon \ell(x) \le Z^{-\frac{1}{3}} b^{\delta_1}\}$ are $O(Z^{\frac{5}{3}} b^{-\delta})$.

(ii) Only m with $Z_m \ge Z b^{-\delta_2}$ should be considered because contributions of other m are are $O(Z^{\frac{5}{3}} b^{-\delta})$ as well.

Proposition 25.4.10. *Let ϑ be a (small) parameter such that $b^{-\delta_2} \le \vartheta \le 1$; consider m with $Z_m \ge Z\vartheta^{\delta_1}$. Let $\phi(x) = \psi(r^{-1}(x - \mathsf{y}_m))$ with $\psi \in \mathscr{C}_0^\infty(B(0,1))$ and r defined below.*

(i) Further, let $(Z-N)^{-\frac{1}{3}} \ge r = Z^{-\frac{1}{3}}\vartheta^{-1}$. Then inequalities

$$\Big|\int \phi(x) \int_{-\infty}^{\lambda} \big(e(x,x,\lambda') - \mathsf{Weyl}(x,\lambda')\big)\,dxd\lambda' - \mathsf{Scott} - \mathsf{Schwinger}\Big| \le CZ^{\frac{5}{3}}\vartheta^{\delta}, \tag{25.4.46}$$

[20] This theorem implies the above estimate with $\delta = 1$ which is definitely overkill for our purposes.

$$\Big|\int \phi(x)\big(e(x,x,\lambda)-\mathsf{Weyl}(x,\lambda)\big)\,dx\Big| \le CZ^{\frac{2}{3}}\vartheta^{\delta} \tag{25.4.47}$$

and

$$\mathsf{D}\Big(\phi\big(e(x,x,\lambda)-\mathsf{Weyl}(x,\lambda)\big),\phi\big(e(x,x,\lambda)-\mathsf{Weyl}(x,\lambda)\big)\Big)\le CZ^{\frac{5}{3}}\vartheta^{\delta} \tag{25.4.48}$$

hold with some exponent $\delta>0$ for all $\lambda\le 0$ and for ϕ which is r-admissible.

(ii) On the other hand, let $(Z-N)^{-\frac{1}{3}}\le r$. Let W be a constructed above mollification of W^{TF}. Then

(a) Estimates (25.4.46)–(25.4.48) *hold for all $\lambda\le\nu$.*

(b) Further, estimates (25.4.47), (25.4.48) *hold for $\lambda\in[\nu,0]$ such that $\mathsf{N}(H-\lambda)\le N$.*

(c) Furthermore, in this last case

$$|\lambda-\nu|\le CZ^{\frac{2}{3}}(Z-N)^{\frac{1}{3}}\vartheta^{\delta}. \tag{25.4.49}$$

To prove these statements we need to study behavior of the Hamiltonian trajectories. First we want to prove that in the indicated zone W^{TF} is a weak perturbation of W_m^{TF} which is a single atom Thomas-Fermi potential with Z_m and with $\nu_m=\nu$.

Proposition 25.4.11. *In the framework of Proposition 25.4.10 in $B(\mathsf{y}_m,r)$*

$$|D^{\alpha}(W^{\mathsf{TF}}-W_m^{\mathsf{TF}})|\le c_{\alpha}W_m^{\mathsf{TF}}|x-\mathsf{y}_m|^{-|\alpha|}\vartheta^{\delta}. \tag{25.4.50}$$

This estimate holds for all α as $W^{\mathsf{TF}}/(-\nu)$ is disjoint from 1; otherwise it holds for $|\alpha|\le 3$ and

$$|D^{\alpha}(W^{\mathsf{TF}}-W_m^{\mathsf{TF}})(x)-D^{\alpha}(W^{\mathsf{TF}}-W_m^{\mathsf{TF}})(y)|\le cW_m^{\mathsf{TF}}|x-\mathsf{y}_m|^{-\frac{7}{2}}|x-y|^{\frac{1}{2}}\vartheta^{\delta} \tag{25.4.51}$$

for $|\alpha|=3$ and $|x-\mathsf{y}_m|\asymp|y-y_m|\asymp(Z-N)^{-\frac{1}{3}}$.

Proof. An easy proof based on the variational approach is left to the reader.□

Next, let us consider a manifold $\Sigma_\lambda = \{(x,\xi)\colon H(x,\xi) = \lambda\}$ with the classical Hamiltonian $H(x,\xi) = |\xi|^2 - W(x)$, and let us introduce a measure μ_λ with the density $dxd\xi : dH$ on Σ_λ; this measure is invariant with respect to the Hamiltonian flow with the Hamiltonian $H(x,\xi)$. Note that $\mu_\lambda(\Sigma_\lambda) \asymp Z$.

Proposition 25.4.12. *In the framework of Proposition 25.4.10 there exists a set $\Sigma'_{\lambda,\vartheta} \subset \Sigma_\lambda$ such that*

$$\mu_\lambda(\Sigma'_{\lambda,\vartheta}) \le C\vartheta^\delta Z \tag{25.4.52}$$

and through each point (x,ξ) belonging to $\vartheta(Z^{-\frac{1}{3}}, Z^{\frac{2}{3}})$-vicinity of $\Sigma_\lambda \setminus \Sigma'_{\lambda,\vartheta}$ in $T^*\mathbb{R}^3$ there passes a Hamiltonian trajectory $(x(t),\xi(t))$ of H of the length $T = Z^{-1}\vartheta^{-\delta}$ along which

$$\bigl|D^\alpha_{(xZ^{\frac{1}{3}},\xi Z^{-\frac{2}{3}})}\bigl(x(t)Z^{\frac{1}{3}}, \xi(t)Z^{-\frac{2}{3}}\bigr)\bigr| \le C\vartheta^{-K} \qquad \forall \alpha : |\alpha| \le k \tag{25.4.53}$$

and

$$|x(t) - x(0)|Z^{\frac{1}{3}} + |\xi(t) - \xi(0)|Z^{-\frac{2}{3}} \ge \vartheta^K |t| Z, \tag{25.4.54}$$

where m is arbitrary and K, δ depend on k.

Proof. We will just sketch the proof.

(i) Consider first the case $M = 1$. Then the the classical dynamical system is completely integrable since both the angular momentum and the energy are preserved.

First, let us include into $\Sigma'_{\lambda,\vartheta}$ all the points (x,ξ) with trajectories not residing over $\{\vartheta^{\delta'} Z^{-\frac{1}{3}} \le \ell(x) \le Z^{-\frac{1}{3}}\vartheta^{-\delta'}\}$ for all $t : |t| \le T$ [21].

Further, if $(Z - N)_+ \ge Z\vartheta^{\delta''}$ we also include into $\Sigma'_{\lambda,\vartheta}$ all the points (x,ξ) with the trajectories not residing over $B\bigl(\mathsf{y}_1, (1-\vartheta^{\delta'})\bar r\bigr)$; recall that for atoms $\bar r$ is an *exact* radius of $\mathsf{supp}(\rho^{\mathsf{TF}})$. Then (25.4.52)–(25.4.53) hold and we reduce δ, δ' if necessary.

It is known that not all the Hamiltonian trajectories are closed (see Appendix 25.A.2). Then one can prove easily that adding to $\Sigma'_{\lambda,\vartheta}$ the set satisfying (25.4.52) we can fulfill (25.4.52) and (25.4.54) as well[22]. One can find the similar arguments in the proof of Theorem 7.4.12.

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[21] I.e. trajectories, leaving this "comfort zone" for some $t : |t| \le T$.

[22] It is important that in the classical dynamics is completely integrable.

(ii) The general case $M \ge 2$ is due of this particular one, Proposition 25.4.11 and trivial perturbation arguments. □

Proof of Proposition 25.4.10. Now estimates (25.4.46) and (25.4.47) follow from Propositions 25.4.11 and 25.4.12 and the standard arguments. Note that if $(Z-N)_+ \ge Z\vartheta^{\delta''}$ we need to mollify W^{TF} in the standard way.

To prove estimate (25.4.48) we can use decomposition (16.4.1) and apply to the contribution of zone $\{(x,y)\colon |x-y| \ge Z^{-\frac{1}{2}}\vartheta^{\delta'}\}$ the same standard arguments. Meanwhile one can notice that the contribution of the zone $\{(x,y)\colon |x-y| \ge Z^{-\frac{1}{2}}\vartheta^{\delta'}\}$ is $O(Z^{\frac{5}{3}-\delta})$. □

Combining all these improvements we arrive to

Theorem 25.4.13. *Let $N \asymp Z$. Let $a \ge Z^{-\frac{1}{3}}$ and let Ψ be a ground state. Then*

$$|\mathsf{E} - \mathcal{E}^{\mathsf{TF}} - \mathsf{Scott} - \mathsf{Dirac} - \mathsf{Schwinger}| \le CZ^{\frac{5}{3}}\bigl(Z^{-\delta} + (aZ^{\frac{1}{3}})^{-\delta}\bigr). \tag{25.4.55}$$

25.4.4 Corollaries and Discussion

Estimates for $\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}})$

Recall that in the lower estimate there was in the left-hand expression a non-negative term $\frac{1}{2}\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}})$ which we so far just dropped. Then in the frameworks of Theorems 25.4.8 and 25.4.13 we conclude that this term does not exceed the right-hand expressions of (25.4.43) and (25.4.55) respectively.

However, we can do better in the case $a \le Z^{-\frac{1}{3}}$. Indeed, recall that the term $a^{-\frac{1}{2}}Z^{\frac{3}{2}}$ in the remainder estimate (25.4.43) appeared *only* because we replaced $\mathsf{Tr}((H_W-\nu)^-)$ by its Weyl approximation (with correction terms) which we by no means need for estimating this term since $\mathsf{Tr}((H_W-\nu)^-)$ was present in both lower and upper estimates. Then we arrive to the following

Theorem 25.4.14. *Let $N \asymp Z$ and let Ψ be a ground state. Then*

$$\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}}) \le CQ := C\begin{cases} Z^{\frac{5}{3}} & \text{if} \quad a \le Z^{-\frac{1}{3}},\\ Z^{\frac{5}{3}}\bigl(Z^{-\delta} + (aZ^{\frac{1}{3}})^{-\delta}\bigr) & \text{if} \quad a \ge Z^{-\frac{1}{3}}. \end{cases} \tag{25.4.56}$$

Estimates for Distance between Nuclei in the Free Nuclei Model

Let us estimate from below the distance between nuclei in the stable molecule in the free nuclei model (with the full energy optimized with respect to the position of the nuclei).

Theorem 25.4.15. *Let $M \ge 2$ and let condition* (25.3.33) *be fulfilled. Assume that*

$$\mathsf{E}(\underline{Z};\underline{\mathsf{y}};N) + \sum_{1\le m<m'\le M} Z_jZ_k|\mathsf{y}_m-\mathsf{y}_{m'}|^{-1} \le \min_{\substack{N_1,\dots,N_M:\\ N_1+\dots+N_M=N}} \sum_{1\le m\le M} \mathsf{E}(Z_m;N_m) \tag{25.4.57}$$

Then

$$|\mathsf{y}_m-\mathsf{y}_{m'}| \ge \epsilon Z^{-\frac{5}{21}+\delta_1} \qquad \forall m\ne m' \tag{25.4.58}$$

and

$$|\widehat{\mathsf{E}}^{\mathsf{TF}}(\underline{Z};N) - \widehat{\mathcal{E}}^{\mathsf{TF}}(\underline{Z};N) - \mathsf{Scott} - \mathsf{Dirac} - \mathsf{Schwinger}| \le CZ^{\frac{5}{3}-\delta}. \tag{25.4.59}$$

Proof. Note first that $|\mathsf{E}| \le CZ^{\frac{7}{3}}$ and in virtue of Theorem 25.4.8 we can replace E by $\mathcal{E}^{\mathsf{TF}}$ with $O(Z^2)$ error:

$$\mathcal{E}^{\mathsf{TF}}(\underline{Z};\underline{\mathsf{y}};N) + \sum_{1\le m<m'\le M} Z_jZ_k|\mathsf{y}_m-\mathsf{y}_{m'}|^{-1} \le \min_{\substack{N_1,\dots,N_M:\\ N_1+\dots+N_M=N}} \sum_{1\le m\le M} \mathcal{E}^{\mathsf{TF}}(Z_m;N_m) - CZ^2; \tag{25.4.60}$$

which in virtue of Proposition 25.3.11 is impossible unless $a^{-7} \le CZ^2$ i.e. $a \ge \epsilon Z^{-\frac{2}{7}} \gg Z^{-\frac{1}{3}}$.

Then, again in virtue of Theorem 25.4.8 we can replace E by $\mathcal{E}^{\mathsf{TF}} + \mathsf{Scott}$ with $O(Z^{\frac{5}{3}})$ error. Let us take into account that for molecule Scott equals the sum of Scott_m. Therefore in (25.4.60) we can replace CZ^2 by $CZ^{\frac{5}{3}}$. Applying again Proposition 25.3.11 we conclude that $a \ge \epsilon Z^{-\frac{5}{21}}$.

Let us improve this estimate further. In virtue of Theorem 25.4.13 we can replace E by $\mathcal{E}^{\mathsf{TF}} + \mathsf{Scott} + \mathsf{Dirac} + \mathsf{Schwinger}$ with $O(Z^{\frac{5}{3}-\delta_2})$ error. However one needs to compare Dirac–Schwinger correction for the molecule with the sums of such corrections for separate atoms:

Proposition 25.4.16. *If* $a \ge Z^{-\frac{1}{3}+\delta_1}$ *and*

$$\mathcal{E}^{\mathsf{TF}}(\underline{Z};\underline{\mathsf{y}};N) = \sum_{1\le m\le M} \mathcal{E}^{\mathsf{TF}}(Z_m;N_m) + O(Z^{\frac{7}{3}-\delta_1}) \tag{25.4.61}$$

where $N = N_1 + ... + N_M$, *then*

$$\int \rho^{\mathsf{TF}\,\frac{4}{3}}\,dx = \sum_{1\le m\le M} \int (\rho_m^{\mathsf{TF}})^{\frac{4}{3}}\,dx + O(Z^{\frac{5}{3}-\delta_2}), \tag{25.4.62}$$

where $\rho_m^{\mathsf{TF}} = \rho^{\mathsf{TF}}(x-\mathsf{y}_m;Z_m;N_m)$ *are atomic Thomas-Fermi densities.*

Proof of this Proposition 25.4.16 will be provided immediately. Therefore Dirac and Schwinger correction terms for a molecule are equal to the sums of Dirac_m and $\mathsf{Schwinger}_m$ correction terms respectively with $O(Z^{\frac{5}{3}-\delta_2})$ error and in (25.4.60) we can replace CZ^2 by $CZ^{\frac{5}{3}-\delta_2}$.

Applying Proposition 25.3.11 again we conclude that $a \ge Z^{\frac{5}{21}+\delta}$. □

Proof of Proposition 25.4.16. Note that the left-hand expression of (25.3.42) is equal to

$$\|\nabla\big(W^{\mathsf{TF}} - \bar{W}^{\mathsf{TF}}\big)\|^2 \qquad \text{with} \quad \bar{W}^{\mathsf{TF}} \coloneqq \sum_{1\le m\le M} W_m^{\mathsf{TF}} \tag{25.4.63}$$

and therefore this expression is less than $CQ \le CZ^{\frac{7}{3}-\delta_1}$ as well. Then since $a \ge Z^{-\frac{1}{3}+\delta_1}$ we conclude that if we restrict the norm to the zone $\{x : |x-\mathsf{y}_m| \le Z^{-\frac{1}{3}+\delta}\}$, we can replace $\bar{\rho}^{\mathsf{TF}}$ and $\bar{W}^{\mathsf{TF}}$ by ρ_m^{TF} and W_m^{TF} respectively in (25.3.42), (25.4.63).

Using Thomas-Fermi equations we conclude that in this zone

$$|D^\alpha(W^{\mathsf{TF}} - W_m^{\mathsf{TF}})| \le C_\alpha W_m^{\mathsf{TF}} \ell^{-|\alpha|} Z^{-\delta_2} \qquad \forall \alpha : |\alpha| \le 2 \tag{25.4.64}$$

and then $|(\rho^{\mathsf{TF}\,\frac{4}{3}} - (\rho_m^{\mathsf{TF}})^{\frac{4}{3}}| \le C(\rho_m^{\mathsf{TF}})^{\frac{4}{3}} Z^{-\delta_2}$ which implies (25.4.62) because $\int (\rho_m^{\mathsf{TF}})^{\frac{4}{3}}\,dx \asymp Z^{\frac{5}{3}}$ and contributions of zone $\{x : \ell(x) \ge Z^{-\frac{1}{3}+\delta}\}$ to each integral is $O(Z^{\frac{5}{3}-\delta})$. □

The following problem seems to be very challenging:

Problem 25.4.17. In the framework of the free nuclei model consider the case when assumption (25.3.33) is violated, i.e. when some nuclei are much lighter than the others. We do not need this assumption for

Theorems 25.4.8, 25.4.13 or 25.4.14 but we need to estimate from below the minimal distance between nuclei a.

We cannot do this without some generalization of Proposition 25.3.11, which we definitely do not expect to survive in its current form without assumption (25.3.33). It would be unrealistic to expect any estimate from below for $a_m := \min_{m' \neq m} |y_m - y_{m'}|$ without some estimate from below for Z_m.

The following problem seems to be reasonably challenging:

Problem 25.4.18. (i) Let us discuss the case $M = 2$ and $Z_2 \ll Z_1$, $a \le Z^{-\frac{1}{3}}$. Then there is an unpleasant remainder $O(a^{-\frac{1}{2}} Z^{\frac{3}{2}})$ in the trace asymptotics. Let us discuss how one can improve it.

Let us observe that in $\mathbb{R}^3 \setminus B(y_1, Cb)$ we have $W_1^{\mathsf{TF}} \gg W_2^{\mathsf{TF}}$, where $b = \min(Z_2^{-\frac{1}{3}+\delta}, aZ_2Z^{-1})$. One can expect that we can modify W^{TF} in $B(y_1, Cr_2)$ to W so that the dynamical systems corresponding to Hamiltonians H_W and H_{W_1} would be close; then for H_W we would be able to recover trace asymptotics with the remainder estimate $O(Z^{\frac{5}{3}-\delta})$.

Meanwhile, if $b \ge Z_2^{-1}$, i.e. $a \ge Z_2^{-2}Z$, the contribution of $B(y_1, Cr_2)$ to the trace remainder would be $O(b^{-\frac{1}{2}} Z_2^{\frac{3}{2}}) = O(Z_2^{\frac{5}{3}-\delta}) + O(a^{-1/2} Z_2 Z^{1/2})$ and we would improve $O(a^{-\frac{1}{2}} Z^{\frac{3}{2}})$ to $O(Z^{\frac{5}{3}-\delta} + a^{-\frac{1}{2}} Z_2 Z^{\frac{1}{2}})$.

On the other hand, if $b \le Z_2^{-1}$, i.e. $a \le Z_2^{-2}Z$, the contribution of $B(y_1, Cr_2)$ to the trace remainder would be $O(Z_2^2)$, so we would get the remainder $O(Z^{\frac{5}{3}-\delta} + Z_2^2)$.

In particular, we get $O(Z^{\frac{5}{3}-\delta})$ provided $Z_2 \le Z^{\frac{5}{6}-\delta}$.

(ii) Generalize for $M \ge 2$.

25.5 Negatively Charged Systems

In this section we consider the case $N \ge Z$ and provide upper estimates for excessive negative charge $(N - Z)$ if $I_N > 0$ and for ionization energy $I_N := E_{N-1} - I_N$.

25.5.1 Estimates of the Correlation Function

First of all, we provide some estimates which will be used for both negatively and positively charged systems. Let us consider the ground-state function $\Psi(x_1, \varsigma_1; \dots; x_N, \varsigma_N)$ and the corresponding density $\rho_\Psi(x)$.

The crucial role plays estimate (25.4.56) $\mathsf{D}\bigl(\rho_\psi - \rho^{\mathsf{TF}},\, \rho_\psi - \rho^{\mathsf{TF}}\bigr) \le CQ$ of Theorem 25.4.14 and the difference between upper and lower bounds for E_N (with $\mathsf{N}_1(H_W - \nu) + \nu N$ not replaced by its semiclassical approximation).

Let us consider the *classical density* of the electron system

$$\varrho_{\underline{x}}(x) = \sum_{1\le j\le N} \delta(x - x_j) \tag{25.5.1}$$

and the *smeared classical density*

$$\varrho_{\underline{x},\varepsilon}(x) = \varrho_{\underline{x}} * \phi_\varepsilon = \sum_{1\le j\le N} \phi_\varepsilon(x - x_j), \tag{25.5.2}$$

where ε will be chosen later; here $(\underline{x}, \underline{\varsigma}) = (x_1, \varsigma_1; \dots, x_N, \varsigma_N) \in (\mathbb{R}^3 \times \mathbb{C}^q)^N$,

(25.5.3) $\phi_\varepsilon(z) = \varepsilon^{-3}\phi(z\varepsilon^{-1})$, $\phi \in \mathscr{C}_0^\infty\bigl(B(0, \frac12)\bigr)$ is a spherically symmetric, non-negative function such that $\int \phi(x)\,dx = 1$.

Then $\int \phi_\varepsilon(x) f(x)\,dx = f(0) + O(\varepsilon^2)$ for any $f \in \mathscr{C}^2$. Let us consider

$$K_{N,\varepsilon}(\underline{x}) := \frac12 \mathsf{D}\bigl(\varrho_{\underline{x},\varepsilon}(\cdot) - \rho_\varepsilon(\cdot),\, \varrho_{\underline{x},\varepsilon}(\cdot) - \rho_\varepsilon(\cdot)\bigr), \tag{25.5.4}$$

where $\rho := \rho^{\mathsf{TF}}$ and $\rho_\varepsilon = \phi_\varepsilon * \rho$.

Using Newton's screening theorem[23)] we conclude that

$$\sum_{1\le j<k\le N} |x_j - x_k|^{-1} \ge \frac12 \mathsf{D}\bigl(\varrho_{\underline{x},\varepsilon}(\cdot),\, \varrho_{\underline{x},\varepsilon}(\cdot)\bigr) - C\varepsilon^{-1}N \tag{25.5.5}$$

where the last term estimates

$$\sum_{1\le j\le N} \mathsf{D}\bigl(\phi_\varepsilon(x - x_j), \phi_\varepsilon(x - x_j)\bigr).$$

On the other hand,

$$\frac12 \mathsf{D}\bigl(\varrho_{\underline{x},\varepsilon}(\cdot),\, \varrho_{\underline{x},\varepsilon}(\cdot)\bigr) = \int \varrho_{\underline{x},\varepsilon}(|x|^{-1} * \rho_\varepsilon)\,dx + K_{N,\varepsilon}(\underline{x}) - \frac12 \mathsf{D}(\rho_\varepsilon, \rho_\varepsilon) \tag{25.5.6}$$

[23)] That uniformly charged sphere $S(0, r)$ creates potential $v(x) = -q\min(|x|^{-1}, r^{-1})$ where q is the total charge of the sphere.

and therefore

$$\mathsf{H}_N \geq \sum_{1\leq j\leq N} H_{W_\varepsilon}(x_j) + K_{N,\varepsilon}(\underline{x}) - \frac{1}{2}\mathsf{D}(\rho_\varepsilon,\rho_\varepsilon) - C\varepsilon^{-1}N \tag{25.5.7}$$

on $(\mathscr{L}^2(\mathbb{R}^3,\mathbb{C}^q))^N$ where W_ε is the *smeared potential*:

$$W_\varepsilon(x) = V(x) - |x|^{-1} * \rho_\varepsilon. \tag{25.5.8}$$

Observe that the smeared potential does not depend on $\underline{x}$ and is defined via ρ rather than ρ_Ψ.

Also let us define

$$N_{x,\varepsilon}(x_2,\dots,x_N) \coloneqq \sum_{2\leq j\leq N} (\chi_x * \phi_\varepsilon)(x_j) \tag{25.5.9}$$

and

$$\bar{N}_{x,\varepsilon} \coloneqq \int \rho_\varepsilon(y)\chi_x(y)dy \tag{25.5.10}$$

with $\chi_x(y) \coloneqq \chi(x,y)$, $\chi \in \mathscr{C}^\infty(\mathbb{R}^6)$.

Furthermore, let us consider function $\theta \in \mathscr{C}^\infty(\mathbb{R}^3)$ such that

$$0 \leq \theta \leq 1. \tag{25.5.11}$$

Finally, consider

$$\mathscr{J}_\varepsilon \coloneqq \Big|\int \Big(\rho^{(2)}_\Psi(x,y) - \rho_\varepsilon(y)\rho_\Psi(x)\Big)\theta(x)\chi(x,y)\,dxdy\Big| \tag{25.5.12}$$

where $\rho^{(2)}_\Psi(\cdot,\cdot)$ is the *quantum correlation function,*

$$\rho^{(2)}_\Psi(x,y) \coloneqq N(N-1)\int |\Psi(x,y,x_3,\dots,x_N)|^2\, dx_3\cdots dx_N, \tag{25.5.13}$$

$$\int \rho^{(2)}_\Psi(x,y)dy = (N-1)\rho_\Psi(x), \tag{25.5.14}$$

$$\Theta = \Theta_\Psi \coloneqq \int \theta(x)\rho_\Psi(x)dx. \tag{25.5.15}$$

Obviously

$$\begin{aligned}\mathcal{J}_\varepsilon &\le \int \rho^{(2)}_\Psi(x,y)\big|(\chi_x * \phi_\varepsilon)(y) - \chi_x(y)\big|\theta(x)\,dxdy \\ &\quad + N\int |\Psi(x,x_2,\dots,x_N)|^2\big|N_{x,\varepsilon}(x_2,\dots,x_N) - \bar N_{x,\varepsilon}\big|\theta(x)\,dxdx_2\cdots dx_N \\ &\le CN\varepsilon\|\nabla_y\chi\|_{\mathscr{L}^\infty}\Theta \\ &\qquad + \Big(N\int |\Psi(x,x_2,\dots,x_N)|^2|N_{x,\varepsilon} - \bar N_{x,\varepsilon}|^2\theta(x)\,dxdx_2\cdots dx_N\Big)^{\frac12}\Theta^{\frac12}\end{aligned}$$

where we used Cauchy-Schwartz inequality.

Since

$$\begin{aligned}(25.5.16)\quad &N_{x,\varepsilon}(x_2,\dots,x_N) - \bar N_{x,\varepsilon} \\ &\qquad = \int \Big(\sum_{2\le j\le N}\phi_\varepsilon(y-x_j) - \rho_\varepsilon(y)\Big)\chi(x,y)\,dxdy,\end{aligned}$$

we again from Cauchy-Schwartz inequality conclude that

$$(25.5.17)\qquad \big|N_{x,\varepsilon} - \bar N_{x,\varepsilon}|^2 \le C\|\nabla_y\chi\|^2_{\mathscr{L}^2}\cdot K_{N-1,\varepsilon}(x_2,\dots,x_N).$$

Note that

$$\begin{aligned}(25.5.18)\quad &\sum_{1\le j\le N}\langle \mathsf{H}_N\theta^{\frac12}(x_j)\Psi, \theta^{\frac12}(x_j)\Psi\rangle = \\ &\qquad \mathsf{E}_N\int\int \theta(x)\rho_\Psi(x)\,dx + \int |\nabla\theta^{\frac12}|^2(x)\rho_\Psi(x)\,dx\end{aligned}$$

and then (25.5.7) yields that

$$\begin{aligned}(25.5.19)\quad &\mathsf{E}_N\int\theta(x)\rho_\Psi(x)\,dx \ge -\int |\nabla\theta^{\frac12}(x)|^2\rho_\Psi(x)\,dx \\ &\qquad + \sum_{1\le j,k\le N}\langle H_{W_{\varepsilon,x_k}}\theta^{\frac12}(x_j)\Psi,\theta^{\frac12}(x_j)\Psi\rangle \\ &+ N\int K_{N,\varepsilon}(\underline{x})\theta(x_1)|\Psi(x_1,\dots,x_N)|^2\,dx_1\dots dx_N - \frac12\mathsf{D}(\rho_\varepsilon,\rho_\varepsilon)\Theta - C\varepsilon^{-1}N\Theta.\end{aligned}$$

Also note that

(25.5.20) The sum of $(N-1)$ lowest eigenvalues of H_{W_ε} on $\mathcal{H}$ is greater than $\big(\nu N + \mathsf{N}_1(H_{W_\varepsilon} - \nu)\big)$.

Then the second term in the right-hand expression of (25.5.19) is bounded from below by $\big(\nu N + \mathsf{N}_1(H_{W_\varepsilon} - \nu)\big)\Theta$, while the left-hand expression is $\mathsf{E}_N\Theta$. Therefore assembling terms proportional to Θ we conclude that

$$
\begin{aligned}
(25.5.21)\quad (1 + C\varepsilon^{-1}N)S_\varepsilon\Theta + \int |\nabla\theta^{\frac12}(x)|^2\rho_\Psi(x)\,dx& \\
\geq \sum_j \langle H_{W_\varepsilon,x_j}\theta^{\frac12}(x_j)\Psi, \theta^{\frac12}(x_j)\Psi\rangle& \\
+ N\int K_{N,\varepsilon}(x_1,\dots,x_N)\theta(x_1)|\Psi(x_1,\dots,x_N)|^2\,dx_1\cdots dx_N&
\end{aligned}
$$

with

$$
S_\varepsilon = \mathsf{E}_N - \nu N - \mathsf{N}_1(H_{W_\varepsilon} - \nu) + \frac{1}{2}\mathsf{D}(\rho_\varepsilon, \rho_\varepsilon). \tag{25.5.22}
$$

Due to the non-negativity of operator D_x^2, the second term in the right-hand expression of (25.5.21) is greater than $-CT_\varepsilon\Theta$ with

$$
T_\varepsilon = \sup_{\mathsf{supp}(\theta)} W_\varepsilon; \tag{25.5.23}
$$

so we arrive to

$$
\begin{aligned}
(25.5.24)\quad N\int K_{N,\varepsilon}(x_1,\dots,x_N)\theta(x_1)|\Psi(x_1,\dots,x_N)|^2\,dx_1\cdots dx_N \leq& \\
C\big(S_\varepsilon + T_\varepsilon + \varepsilon^{-1}N\big)\Theta + R&
\end{aligned}
$$

with

$$
R = \int |\nabla\theta^{\frac12}|^2\rho_\Psi\,dx. \tag{25.5.25}
$$

Combining this inequality with (25.5.13), (25.5.17) we conclude that

$$
\begin{aligned}
(25.5.26)\quad \mathcal{J}_\varepsilon \leq C\sup_x \|\nabla_y\chi_x\|_{\mathscr{L}^2(\mathbb{R}^3)}\Big(\big(S_\varepsilon + T_\varepsilon + \varepsilon^{-1}N\big)\Theta + R\Big)^{\frac12}\Theta^{\frac12}& \\
+ C\varepsilon N\|\nabla_y\chi\|_{\mathscr{L}^\infty}\Theta&
\end{aligned}
$$

due to obvious estimate

$$K_{N-1,\varepsilon}(x_2,\dots,x_N) \le 2K_{N,\varepsilon}(x_1,\dots,x_N) + \varepsilon^{-1}N. \tag{25.5.27}$$

Now we want to estimate S_ε from above and for this we need an upper estimate for E_N. Recall that due to the arguments of Subsection 25.2.2

$$\mathsf{E}_N \le \mathsf{N}_1(H_{W_\varepsilon}-\nu)+\nu N \\ + \frac{1}{2}\mathsf{D}(\mathsf{tr}(e_\varepsilon(x,x,\nu))-\rho_\varepsilon,\ \mathsf{tr}(e_\varepsilon(x,x,\nu))-\rho_\varepsilon) - \frac{1}{2}\mathsf{D}(\rho_\varepsilon,\rho_\varepsilon)$$

where $e_\varepsilon(x,y,\tau)$ is the Schwartz kernel of the spectral projector of H_{W_ε}. Therefore $S_\varepsilon \le CQ$ provided we manage to prove that

(25.5.28) Right-hand expressions in (25.2.23)–(25.2.26) satisfy the same estimates as before if we plug W_ε instead of W.

So, we need to calculate both semiclassical errors (which are calculated for $W_\varepsilon = V - |x|^{-1} * \rho_\varepsilon$ with $\rho_\varepsilon = \rho^{\mathsf{TF}} * \phi_\varepsilon$)exactly in the same way as for $W = W^{\mathsf{TF}}$) and the principal parts, and in calculations of

$$\mathsf{D}\big(P'(W_\varepsilon+\nu)-\rho,\ P'(W_\varepsilon+\nu)-\rho\big) \tag{25.5.29}$$

and

$$\mathsf{D}(\rho_\varepsilon-\rho,\ \rho_\varepsilon-\rho) \tag{25.5.30}$$

Observe that

$$|\rho_\varepsilon-\rho| \le C\begin{cases} Z^{\frac{3}{2}}\ell^{-\frac{3}{2}} & \text{for } \ell \le 2\varepsilon, \\ \varepsilon^2 Z^{\frac{3}{2}}\ell^{-\frac{7}{2}} & \text{for } 2\varepsilon \le \ell \le Z^{-\frac{1}{3}}, \quad \ell \not\asymp \bar r, \\ \varepsilon^2\ell^{-8} & \text{for } \ell \ge Z^{-\frac{1}{3}}, \qquad \ell \not\asymp \bar r, \\ \varepsilon^{\frac{3}{2}}\ell^{-\frac{15}{2}} & \text{for } \ell \asymp \bar r \end{cases} \tag{25.5.31}$$

which implies that expression (25.5.30) does not exceed $\varepsilon^2 Z^3$. Recall that $\bar r = (Z-N)_+^{-\frac{1}{3}}$ if $N \asymp Z$.

On the other hand, $W_\varepsilon - W = -|x|^{-1} * (\rho_\varepsilon - \rho) = (|x|^{-1} - |x|^{-1} * \phi_\varepsilon) * \rho$ where $\rho = \rho^{\mathsf{TF}}$, which implies

$$|W_\varepsilon - W| \le C\begin{cases} \varepsilon^{\frac{1}{2}} Z^{\frac{3}{2}} & \text{for } \ell \le 2\varepsilon, \\ \varepsilon^2 Z^{\frac{3}{2}}\ell^{-\frac{3}{2}} & \text{for } 2\varepsilon \le \ell \le Z^{-\frac{1}{3}}, \\ \varepsilon^2\ell^{-6} & \text{for } \ell \ge Z^{-\frac{1}{3}} \end{cases} \tag{25.5.32}$$

and then expression (25.5.29) does not exceed $\varepsilon^2 Z^3$.

Furthermore, the smearing error in the principal part of the asymptotics of $\int e(x,x,\tau)\,dx$, namely

$$|\int \big(P'(W_\varepsilon+\nu)-P'(W+\nu)\big)\,dx|, \tag{25.5.33}$$

does not exceed $C\varepsilon^2 Z^{\frac{5}{3}}$.

Choosing $\varepsilon = Z^{-\frac{2}{3}}$ we estimate these two terms by $CZ^{\frac{5}{3}}$, and simultaneously $\varepsilon^{-1}N \asymp Z^{\frac{5}{3}}$.

With this choice a smearing error does not exceed $CZ^{\frac{3}{2}}\varepsilon^{\frac{3}{2}} = O(Z^{\frac{1}{2}})$ which is less than the semiclassical error. Then $S \le CQ$. So, we derived estimate (25.5.34) below for $\mathcal{J}_\varepsilon$. Since

$$|\mathcal{J}_\varepsilon - \mathcal{J}| \le C\varepsilon Z \|\nabla_y \chi_x\|_{\mathscr{L}^\infty}\Theta$$

we arrive to the following

Proposition 25.5.1. *If θ, χ are as above, $\varepsilon \le Z^{-\frac{2}{3}}$ then*

$$\mathcal{J} = |\int \Big(\rho^{(2)}_\Psi(x,y)-\rho(y)\rho_\Psi(x)\Big)\theta(x)\chi(x,y)\,dxdy| \le \tag{25.5.34}$$

$$C\sup_x \|\nabla_y\chi_x\|_{\mathscr{L}^2(\mathbb{R}^3)}\Big((\bar{Q}+\varepsilon^{-1}N+T)^{\frac{1}{2}}\Theta + R^{\frac{1}{2}}\Theta^{\frac{1}{2}}\Big) + C\varepsilon N\|\nabla_y\chi\|_{\mathscr{L}^\infty}\Theta$$

with $\Theta = \Theta_\Psi$ defined by (25.5.15) *and T, R defined by* (25.5.23), (25.5.25) *respectively, and*

$$\bar{Q} = \max(Q,\, Z^{\frac{5}{3}}). \tag{25.5.35}$$

Remark 25.5.2. (i) The estimates for expression (25.5.29), (25.5.30) and (25.5.33) imply also that the smearing error in the principal part of asymptotics of the energy, namely

$$|\int \big(P(W_\varepsilon+\nu)-P(W+\nu)\big)\,dx| \tag{25.5.36}$$

does not exceed $C\varepsilon^2 Z^2$; to prove this directly is also possible simce $|x|^{-1}$ is harmonic function as $x \ne 0$.

(ii) Estmate of (25.5.30) by $C\varepsilon^2 Z^3$ cannot be improved.

(iii) In what follows $N \asymp Z$ and $\varepsilon = Z^{-\frac{2}{3}}$.

25.5.2 Excessive Negative Charge

We will prove the following

Theorem 25.5.3. *Let condition* (25.3.33) *be fulfilled. Then*

(i) Let $\mathsf{I}_N > 0$. *For a single atom and for molecule the fixed nuclei model the following estimate holds*

$$(N-Z)_+ \le CZ^{\frac{5}{7}}. \tag{25.5.37}$$

(ii) In particular, for molecule in the free nuclei model let $\widehat{\mathsf{I}}_N > 0$. *Then estimate* (25.5.37) *holds.*

Proof. We will follow C. L. Fefferman and L. A. Seco [1], L. A. Seco, I. M. Sigal and J. P. Solovej [1] and, in the case of several nuklei, J. P. Solovej [1].

(a) Let us select $\theta = \theta_{(b)}(x)$

$$\begin{aligned} &\mathsf{supp}(\theta) \subset \{x:\ \ell(x) \ge b\}, \qquad \mathsf{supp}(1-\theta) \subset \{x:\ \ell(x) \le 2b\}, \\ &0 \le \theta \le 1, \qquad |\nabla\theta^{\frac12}| + |\nabla(1-\theta)^{\frac12}| \le Cb^{-1}. \end{aligned} \tag{25.5.38}$$

Observe that $\mathsf{H}_N\Psi = \mathsf{E}_N\Psi$ yields

$$\begin{aligned} &\mathsf{E}_N \int \rho_\Psi(x)\varphi(x)\theta(x)\,dx = \sum_j \langle \Psi, \varphi(x_j)\theta(x_j)\mathsf{H}_N\Psi\rangle = \\ &\sum_j \langle \varphi(x_j)^{\frac12}\theta^{\frac12}(x_j)\Psi, \varphi(x_j)^{\frac12}\theta^{\frac12}(x_j)\mathsf{H}_N\Psi\rangle - \sum_j \|\nabla\big(\theta^{\frac12}(x_j)\varphi(x_j)^{\frac12}\big)\Psi\|^2 \end{aligned} \tag{25.5.39}$$

where we pick up

$$\varphi(x) = ZV(x)^{-1} \asymp \ell(x). \tag{25.5.40}$$

Isolating the contribution of j-th electron in j-th term we get

$$\begin{aligned} &\mathsf{E}_N \int \rho_\Psi(x)\varphi(x)\theta\,dx \ge \mathsf{E}_{N-1} \int \rho_\Psi(x)\varphi(x)\theta(x)\,dx + \\ &\sum_j \langle \Psi, \varphi(x_j)\theta(x_j)\Big(-V(x_j) + \sum_{k:k\ne j} |x_j - x_k|^{-1}\Big)\Psi\rangle - \sum_j \|\nabla\big(\theta^{\frac12}(x_j)\varphi(x_j)^{\frac12}\big)\Psi\|^2 \end{aligned} \tag{25.5.41}$$

due to non-negativity of operator D_x^2.

Now let us select b to be able to calculate the magnitude of $\Theta = \Theta_\Psi$ defined by (25.5.1). Observe that

$$|\int (1-\theta(x))\big(\rho_\Psi(x)-\rho(x)\big)\,dx| \tag{25.5.42}$$

$$\le C\mathsf{D}(\rho_\Psi-\rho,\rho_\Psi-\rho)^{\frac12}\|\nabla(1-\theta)^{\frac12}\| \asymp C\mathsf{D}(\rho_\Psi-\rho,\rho_\Psi-\rho)^{\frac12} b^{\frac12} \le CQ^{\frac12}b^{\frac12},$$

$$\int \big(\rho_\Psi(x)-\rho(x)\big)\,dx = N-Z \tag{25.5.43}$$

and

$$\int \theta(x)\rho(x)\,dx \asymp b^{-3} \tag{25.5.44}$$

as long as

$$\epsilon_1 Z^{-\frac13} \le b \le \epsilon (Z-N)_+^{-\frac13} \tag{25.5.45}$$

because $\rho(x) \asymp |x|^{-6}$ as long as $\epsilon_1 Z^{-\frac13} \le |x| \le c_0(Z-N)_+^{-\frac13}$.

Therefore

$$\int \theta(x)\rho_\Psi(x)\,dx \asymp b^{-3} + (N-Z)_+ \tag{25.5.46}$$

provided

$$b \le \epsilon_1 \min\big((Q^{-\frac17}+Q^{-1}(N-Z)_+^2),\,(Z-N)_+^{-\frac13}\big). \tag{25.5.47}$$

Therefore let us pick up $b = \epsilon_1 Q^{-\frac17}$; it does not conflict with (25.5.45) provided

$$N \ge Z - C_0 Q^{\frac37} \tag{25.5.48}$$

and then

$$\Theta \asymp Q^{\frac37} + Q^{-3}(N-Z)_+^6. \tag{25.5.49}$$

Then the same arguments imply that the last term in the right-hand expression of (25.5.41) does not exceed $Cb^{-1}\Theta$; therefore we arrive to inequality

$$\begin{aligned} \mathsf{I}_N \int \rho_\Psi(x)\varphi(x)\theta(x)\,dx \le Z\int \theta(x)\rho_\Psi(x)\,dx \\ - \iint \rho_\Psi^{(2)}(x,y)\varphi(x)|x-y|^{-1}\theta(x)\,dxdy + Cb^{-1}\Theta = \\ = Z\Theta_\Psi - \iint \rho_\Psi^{(2)}(x,y)\varphi(x)|x-y|^{-1}\big(1-\theta(y)\big)\theta(x)\,dxdy \\ -\frac12\iint \rho_\Psi^{(2)}(x,y)\big(\varphi(x)+\varphi(y)\big)|x-y|^{-1}\theta(y)\theta(x)\,dxdy + Cb^{-1}\Theta. \end{aligned} \tag{25.5.50}$$

(b) Let

$$\theta_0+\theta_1=1,\qquad 0\le\theta_0\le 1,\qquad |\nabla\theta_m^{\frac12}|\le C\beta^{-1}, \tag{25.5.51}$$

$$\mathsf{supp}(\theta_1)\subset\{x\colon \ell(x)\le 2\beta\},\ \mathsf{supp}(\theta_0)\subset\{x\colon \ell(x)\ge\beta\},\ \text{and}\ \beta=\tfrac14 b. \tag{25.5.52}$$

Then

$$\begin{aligned}\iint \rho^{(2)}_\Psi(x,y)\varphi(x)|x-y|^{-1}\theta(x)(1-\theta(y))\,dxdy&\\ \ge \iint \rho^{(2)}_\Psi(x,y)\varphi(x)|x-y|^{-1}\theta(x)\theta_1(y)\,dxdy&\\ \ge \iint \rho_\Psi(x)\rho(y)\varphi(x)|x-y|^{-1}\theta(x)\theta_1(y)\,dxdy&\\ - C\bigl(\beta^{\frac12}\bar{Q}^{\frac12}+\beta^{-1}Z^{\frac13}\bigr)\Theta_\Psi&\end{aligned} \tag{25.5.53}$$

where we applied Proposition 25.5.1 with

$$\chi(x,y)=\theta_1(y)\varphi(x)|x-y|^{-1}$$
$$\implies \|\nabla_y\chi\|_{\mathscr{L}^\infty(\mathbb{R}^3_y)}\le C\beta^{-1},\quad \|\nabla_y\chi\|_{\mathscr{L}^2(\mathbb{R}^3_y)}\le C\beta^{\frac12}.$$

Further, for $\ell(x)\ge b$

$$\begin{aligned}\int \rho(y)|x-y|^{-1}\theta_1(y)\,dy&\\ = \int \rho(y)|x-y|^{-1}\,dy - \int \rho(y)|x-y|^{-1}\theta(x)\theta_0(y)\,dy&\\ = V(x)-W(x)+O(\ell(x)^{-1}b^{-3}) = V(x)+O(\ell(x)^{-1}b^{-3})&\end{aligned} \tag{25.5.54}$$

because $W(x)=O(\ell(x)^{-4})$ and $\rho(y)=O(\ell(y)^{-6})$ as $\nu=0$ and $\ell(x)\gtrsim b$, $\ell(y)\gtrsim Z^{-\frac13}$.

Since $\varphi(x)=ZV(x)^{-1}\asymp\ell(x)$ we conclude that

(25.5.55) The first line in the right-hand expression of (25.5.50) does not exceed $\bigl(Z-N+Cb^{-3}+Cb^{\frac12}\bar{Q}^{\frac12}\bigr)\Theta_\Psi$.

(c) Consider the second line. Observe that

$$\varphi(x)+\varphi(y)=\varphi(x)\varphi(y)\sum_{1\le m\le M} z_m|x-\mathsf{y}_m|^{-1}|y-\mathsf{y}_m|^{-1}\big(|x-\mathsf{y}_m|+|y-\mathsf{y}_m|\big) \\ \ge \sum_{1\le m\le M}\varphi(x)\varphi(y)z_m|x-y|$$

with $z_m = Z_m Z^{-1}$ and therefore this term does not exceed

(25.5.56)
$$-\frac{1}{2}\sum_{1\le m\le M} z_m\iint \rho^{(2)}_{\Psi}(x,y)|x-\mathsf{y}_m|^{-1}|y-\mathsf{y}_m|^{-1}\varphi(x)\varphi(y)\theta(x)\theta(y)\,dxdy \\ \le -\frac{1}{2}\sum_{1\le m\le M} z_m\Big\{\Big(\int \rho_{\Psi}(x)\varphi(x)\theta(x)|x-\mathsf{y}_m|^{-1}\,dx\Big)^2 \\ -\int \rho_{\Psi}(x)\varphi^2(x)\theta(x)^2|x-\mathsf{y}_m|^{-2}\,dx\Big\} \\ \le -\epsilon_1\Big(\int \rho_{\Psi}(x)\theta(x)\,dx\Big)^2 + C_1\int \rho_{\Psi}(x)\theta(x)^2\,dx$$

where the first inequality is due to inequality $\langle F^2\rangle \ge \langle F\rangle^2$ where $\langle .\rangle$ is an average with respect to density $\rho^{(2)}_{\Psi}(x,y)$ and equality $\int \rho^{(2)}_{\Psi}(x,y)\,dy = (N-1)\rho_{\Psi}(x)$.

Therefore

(25.5.57) The second line in the right-hand expression of (25.5.50) does not exceed $\big(-\epsilon_2\Theta_{\Psi} + Cb^{-1}\big)\Theta_{\Psi}$.

Since the left-hand expression of (25.5.50) is greater than $\epsilon b\mathsf{I}_N\Theta_{\Psi}$ and since Θ_{Ψ} is disjoint from 0 we conclude that

$$b\mathsf{I}_N \le C(Z-N) - \epsilon_2\Theta_{\Psi} + Cb^{\frac{1}{2}}\bar{Q}^{\frac{1}{2}} + Cb^{-3} \tag{25.5.58}$$

and therefore due to assumption $\mathsf{I}_N > 0$ and (25.5.46)

$$N - Z \le C(b\bar{Q})^{\frac{1}{2}} + Cb^{-3} \tag{25.5.59}$$

which implies (25.5.37) under optimal choice $b = \epsilon\bar{Q}^{\frac{1}{7}} = \epsilon Z^{-\frac{5}{21}}$, satisfying (25.5.47) unless (25.5.37) is fulfilled. □

25.5.3 Estimate for Ionization Energy

Continuing arguments of the proof of Theorem 25.5.3, we see that estimate (25.5.58) implies that under conditions (25.5.47) and (25.5.48) with $Q = Z^{\frac{5}{3}}$

$$\mathsf{I}_N \le Cb^{-1}(Z-N)_+ + Cb^{-\frac{1}{2}}\bar{Q}^{\frac{1}{2}} \le C_1(Z-N)_+^{\frac{4}{3}} + C_1 Q^{\frac{1}{14}}\bar{Q}^{\frac{1}{2}} \tag{25.5.60}$$

where we took the largest $b = \epsilon_1 \min\bigl(Q^{-\frac{1}{7}}, (Z-N)_+^{-\frac{1}{3}}\bigr)$ such that $\Theta_\Psi > 0$.

Then we arrive to the following

Theorem 25.5.4. *Let condition* (25.3.33) *be fulfilled.*

(i) Then the following estimate holds:

$$\mathsf{I}_N \le C(Z-N)_+^{\frac{4}{3}} + CZ^{\frac{20}{21}}. \tag{25.5.61}$$

(ii) Further, under assumption

$$a := \min_{1\le m<m'\le M} |\mathsf{y}_m - \mathsf{y}_{m'}| \ge Z^{-\frac{1}{3}} \tag{25.5.62}$$

the following estimate holds

$$\mathsf{I}_N \le C(Z-N)_+^{\frac{4}{3}} + CZ^{\frac{20}{21}}\bigl(Z^{-\delta} + (aZ^{\frac{1}{3}})^{-\delta}\bigr). \tag{25.5.63}$$

(iii) Furthermore, for $M \ge 2$ in the framework of free nuclei model (assuming that the molecule exists; see Theorem 25.6.3)

$$\widehat{\mathsf{I}}_N \le CZ^{\frac{20}{21}-\delta'}. \tag{25.5.64}$$

Remark 25.5.5. (i) Classical theorem of G. Zhislin [1] implies that the system can bind at least Z electrons; the proof is based on the demonstration that the energy of the system with $N < Z$ electrons plus one electron on the distance r is increasing as $r \to +\infty$ because potential created by the system with $N < Z$ electrons behaves as $(Z-N)|x|^{-1}$ as $|x| \to \infty$.

(ii) If we assume that $N \ge 2Z$ then the proof of Theorem 25.5.3 implies that $I_N = 0$; so we actually do not need assumption (25.3.33) in Theorems 25.5.3 and 25.5.4.

(iii) In the proof of Theorem 25.5.4(ii) we use that under assumption (25.5.62) $Q = Z^{\frac{5}{3}}\bigl(Z^{-\delta} + (aZ^{\frac{1}{3}})^{-\delta}\bigr)$.

(iv) In the proof of Theorems 25.5.3(ii) and 25.5.4(iii) we note that tearing of one electron in free nuclei model is easier than in the fixed nuclei model.

(v) We are interested in the cases when the first term in the right-hand expressions (25.5.61) and (25.5.63) does not exceed the second ones.

It would be interesting to make the following improvements:

Problem 25.5.6. Improve Proposition 25.5.1 so that we could have $\bar{Q} := \mathsf{max}(Q, Z^{\frac{5}{3}-\delta})$ rather than $\bar{Q} := \mathsf{max}(Q, Z^{\frac{5}{3}})$. This would lead to the improvement of Theorem 25.5.3 and similar theorems in the forthcoming chapters.

25.6 Positively Charged Systems

In this section we consider the case of positively charged system with violated assumption (25.5.61) i.e. with

$$Z - N \ge C_0 Q^{\frac{3}{7}} \tag{25.6.1}$$

with sufficiently large constant C_0 and the same Q as before.

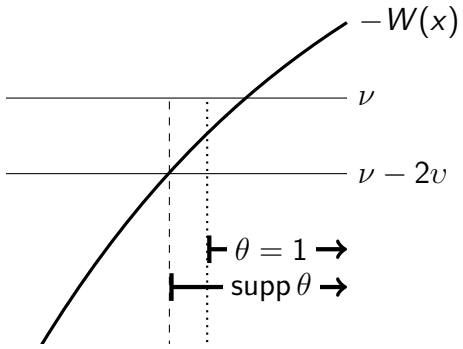
First let us find asymptotics of the ionization energy; the principal term will be $-\nu$ but we need to estimate a remainder.

25.6.1 Estimate from above for Ionization Energy

In this subsection we will prove that under assumption (25.6.1)

$$\mathsf{I}_N \le -\nu + C\upsilon \tag{25.6.2}$$

with $\upsilon \ll |\nu|$ to be defined below. Similar estimates from below will be proven in the next Subsection 25.6.2.



As $M = 1$ construction is well-known: let us pick up function θ such that $\theta = 1$ as $|x-\mathsf{y}_m| \ge \bar{r}-\beta$ and $\theta = 0$ as $|x-\mathsf{y}_m| \le \bar{r}-2\beta$ where $\bar{r}$ is an exact radius of support ρ^{TF}. Here $\beta \ll \bar{r}$. As $M \ge 1$ let us pick instead

$$\theta(x) = f^2\big(\upsilon^{-1}[W(x)+\nu]\big) \tag{25.6.3}$$

where $f \in \mathscr{C}^\infty(\mathbb{R})$, supported in $(-\infty, 2)$ and equal 1 in $(-\infty, 1)$, $\upsilon \le \nu$.

(a) We will assume first that

$$a \ge \bar{r} = C_1(Z-N)^{-\frac{1}{3}} \tag{25.6.4}$$

with sufficiently large C_1; we will discuss droppin g this assumption later. Then as we know that ρ^{TF} is supported in $c\bar{r}$-vicinity of nuclei, we conclude that "atoms" are rather disjoint.

One can see easily that then as long as $W+\nu \approx 0$

$$|\nabla W| \asymp \bar{r}^{-5}; \tag{25.6.5}$$

then the width of the zone $\{x\colon 0 \le W(x)+\nu \le 2\upsilon\}$ is $\asymp \upsilon|\nabla W|^{-1} \asymp \beta = \upsilon\bar{r}^5$ and

$$\Theta^{\mathsf{TF}} := \int \rho^{\mathsf{TF}}\theta(x)\,dx \asymp \upsilon^{\frac{3}{2}} \times \beta\bar{r}^2 = \upsilon^{\frac{5}{2}}\bar{r}^7 \tag{25.6.6}$$

while $\|\nabla\theta\| \asymp \beta^{-\frac{1}{2}}\bar{r} = \upsilon^{-\frac{1}{2}}\bar{r}^{-\frac{3}{2}}$ for θ supported in $\{x\colon 0 \le W(x)+\nu \le 3\upsilon\}$ and equal 1 in $\{x\colon \upsilon \le W(x)+\nu \le 2\upsilon\}$. On the other hand

$$\begin{aligned} &|\int (\rho_\Psi - \rho^{\mathsf{TF}})\theta\,dx| = \frac{1}{4\pi}|\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}},\, \Delta\theta)| \\ &\qquad \le \mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}},\, \rho_\Psi - \rho^{\mathsf{TF}})^{\frac{1}{2}}\mathsf{D}(\Delta\theta,\, \Delta\theta)^{\frac{1}{2}} \le CQ^{\frac{1}{2}}\|\nabla\theta\|_{\mathscr{L}^2} \le C\beta^{-\frac{1}{2}}\bar{r}Q^{\frac{1}{2}}. \end{aligned} \tag{25.6.7}$$

Therefore to ensure that $\Theta = \Theta_\Psi$ has the same magnitude (25.6.6) we pick up the smallest υ such that $\upsilon^{\frac{5}{2}}\bar{r}^7 \ge C\upsilon^{-\frac{1}{2}}\bar{r}^{-\frac{3}{2}}Q^{\frac{1}{2}}$ i.e.

$$\upsilon := C_2\bar{r}^{-\frac{17}{6}}Q^{\frac{1}{6}} \qquad (\iff \beta = C_2\bar{r}^{\frac{13}{6}}Q^{\frac{1}{6}}); \tag{25.6.8}$$

then

$$\Theta \asymp \bar{r}^{-\frac{1}{12}} Q^{\frac{5}{12}} \tag{25.6.9}$$

Then (25.5.15) is fulfilled. Note that $\upsilon \lesssim \nu = \bar{r}^{-4}$ iff $(Z-N) \gtrsim Q^{\frac{3}{7}}$ exactly as we assumed. Then inequality (25.5.50) is replaced by

$$\begin{aligned} \mathsf{I}_N \int \ell(x)\rho_\Psi(x)\theta(x)\,dx \le \int \theta(x)V(x)\ell(x)\rho_\Psi(x)\,dx \\ - \int \rho^{(2)}_\Psi(x,y)\ell(x)|x-y|^{-1}\theta(x)\,dxdy + C\beta^{-2}\bar{r}\Theta \end{aligned} \tag{25.6.10}$$

where $C\beta^{-2}\bar{r}\Theta$ estimates the last term in the right-hand expression of (25.5.41). Then

$$\begin{aligned} \mathsf{I}_N \int \ell(x)\rho_\Psi(x)\theta(x)\,dx \le \int \theta(x)V(x)\ell(x)\rho_\Psi(x)\,dx \\ - \int \Big(\rho^{(2)}_\Psi(x,y) - \rho_\Psi(x)\rho(y)\Big)\ell(x)|x-y|^{-1}\theta(x)\,dxdy \\ - \int \rho_\Psi(x)\rho(y)\ell(x)|x-y|^{-1}\theta(x)\,dxdy + C\beta^{-2}\bar{r}\Theta. \end{aligned} \tag{25.6.11}$$

Let us estimate from above

$$\begin{aligned} - \int \Big(\rho^{(2)}_\Psi(x,y) - \rho_\Psi(x)\rho(y)\Big)\ell(x)|x-y|^{-1}\theta(x)\,dxdy \le \\ - \int \Big(\rho^{(2)}_\Psi(x,y) - \rho_\Psi(x)\rho(y)\Big)\big(1-\omega(x,y)\big)\ell(x)|x-y|^{-1}\theta(x)\,dxdy \\ + \int \rho_\Psi(x)\rho(y)\omega(x,y)\ell(x)|x-y|^{-1}\theta(x)\,dxdy \end{aligned} \tag{25.6.12}$$

with $\omega = \omega_\gamma$: $\omega = 0$ for $|x-y| \ge 2\gamma$ and $\omega = 1$ for $|x-y| \le \gamma$, $\gamma \ge \beta$.

To estimate the first term in the right-hand expression one can apply Proposition 25.5.1 with

$$\chi(x,y) = \big(1-\omega(x,y)\big)\ell(x)|x-y|^{-1}.$$

In this case $\|\nabla_y \chi\|_{\mathscr{L}^2} \asymp C\bar{r}\gamma^{-\frac{1}{2}}$, $\|\nabla_y \chi\|_{\mathscr{L}^\infty} \asymp \bar{r}\gamma^{-2}$ and plugging $P = \beta^{-2}\Theta$ and $T = |\nu|$, $\varepsilon = Z^{-\frac{2}{3}}$ we conclude that this term does not exceed

$$C\bar{r}\big(\gamma^{-\frac{1}{2}}\bar{Q}^{\frac{1}{2}} + CZ^{\frac{2}{3}}\gamma^{-2}\big)\Theta, \qquad \bar{Q} = \max(Q, Z^{\frac{5}{3}}). \tag{25.6.13}$$

Consider the second term in the right-hand expression of (25.6.12). Note that

$$\int \rho(y)\omega(x,y)|x-y|^{-1}\,dy \le C(\bar r^{-\frac{15}{2}}\gamma^{\frac72}+\upsilon^{\frac32}\gamma^2)$$

since $\rho(y)\asymp \big(W(y)+\nu\big)^{\frac32}$ and $|\nabla W|\asymp \bar r^{-5}$; then this term does not exceed $C(\bar r^{-\frac{15}{2}}\gamma^{\frac72}+\upsilon^{\frac32}\gamma^2)\bar r\Theta$.

Adding to (25.6.13) we get

$$C\Big(\gamma^{-\frac12}\bar Q^{\frac12}+CZ^{\frac23}\gamma^{-2}+\bar r^{-\frac{15}{2}}\gamma^{\frac72}+\upsilon^{\frac32}\gamma^2\Big)\bar r\Theta.$$

Optimizing with respect to $\gamma=\bar Q^{\frac15}\upsilon^{-\frac35}$ we get $C\upsilon^{\frac{3}{10}}\bar Q^{\frac25}\bar r\Theta$ and (25.6.11) becomes

$$\begin{aligned}(25.6.14)\quad (\mathsf{I}_N+\nu)\int \ell(x)\rho_\psi(x)\theta(x)\,dx \le\\ \int \theta(x)\big(W(x)+\nu\big)\ell(x)\rho_\psi(x)\,dx + C\upsilon^{\frac{3}{10}}\bar Q^{\frac25}\bar r\Theta \le\\ C\big(\upsilon+\upsilon^{\frac{3}{10}}\bar Q^{\frac25}\big)\bar r\Theta\,,\end{aligned}$$

where we took into account that $V-|x|^{-1}*\rho=W$, and that $W+\nu\le 3\upsilon$ on $\mathsf{supp}(\theta)$.

Since a factor at $(\mathsf{I}_N+\nu)$ in the left-hand expression of (25.6.14) is obviously $\asymp \bar r\Theta$ we arrive to $(\mathsf{I}_N+\nu)\le C\upsilon+\upsilon^{\frac{3}{10}}\bar Q^{\frac25}$.

Recalling definition (25.6.8) of υ we arrive to an upper estimate in Theorem 25.6.1 below:

$$(25.6.15)\qquad \mathsf{I}_N+\nu\le C\upsilon = C(Z-N)^{\frac{17}{18}}Q^{\frac16}.$$

Indeed, it is true if $Q\ge Z^{\frac53}$ (we are interested in the general approach) and because then $\upsilon\ge \upsilon^{\frac{3}{10}}Q^{\frac25}$ due to assumption (25.6.1).

On the other hand, if $Q=Z^{\frac53}(aZ^{\frac13})^{-\delta}$, $aZ^{\frac13}\ge 1$, then we get an extra term $CQ^{\frac{1}{20}}(Z-N)^{\frac15}Z^{\frac23}$ but we can skip it decreasing unspecified exponent $\delta>0$ in the definition of Q.

(b) To get rid of assumption (25.6.4) we observe that we used only its corollary (25.6.5). If we do not have such assumption instead of equality

(25.6.6) we have inequality $\Theta^{\mathsf{TF}} \gtrsim \upsilon^{\frac{5}{2}}\bar{r}^7$ (but probably equality still holds). However we have a problem to estimate $|\Theta - \Theta^{\mathsf{TF}}|$: namely, we need to estimate the second factor in the product $\mathsf{D}(\rho - \rho^{\mathsf{TF}}, \rho - \rho^{\mathsf{TF}})^{\frac{1}{2}}\|\nabla\theta^{\frac{1}{2}}\|$ in (25.6.7).

Observe that we select f in (25.6.3) such that $|f'| \le c\delta^{-1}f^{1-\sigma/2}$ with arbitrarily small $\delta > 0$ and constant c which does not depend on δ.

$$b^2\|\nabla\theta^{\frac{1}{2}}\|^2 \le C\delta^{-2}\iint_{\mathcal{Z}} \theta^{1-\delta}\,dx \le C\delta^{-2}\Big(\int_{\mathcal{Z}} \theta\,dx\Big)^{1-\delta}\Big(\int_{\mathcal{Z}} 1\,dx\Big)^{\delta} \le C\delta^{-2}\upsilon^{-\frac{3}{2}(1-\delta)}\big(\Theta^{\mathsf{TF}}\big)^{1-\delta}\bar{r}^{3\delta}$$

with $\mathcal{Z} = \mathsf{supp}(\nabla\theta)$ and $b = \upsilon\bar{r}^5$ and an error is less than $\epsilon\Theta^{\mathsf{TF}}$ with $\Theta^{\mathsf{TF}} \ge \epsilon\upsilon^{\frac{5}{2}}\bar{r}^7$ as long as

$$\upsilon = CQ^{\frac{1}{6}}\bar{r}^{-\frac{17}{6}} \times \sigma^{-\frac{1}{3}}\bar{r}^{-7\sigma}Q^{-\sigma}$$

with $\sigma = \delta/(36 + 6\delta)$ and C which does not depend on σ. Optimizing the right-hand expression by $\sigma > 0$ we arrive to estimate (25.6.15) with marginally larger error:

$$\mathsf{I}_N + \nu \le C(Z - N)^{\frac{17}{18}}Q^{\frac{1}{6}} \times \log^{\frac{1}{3}}\big((Z - N)Q^{-\frac{3}{7}}\big). \tag{25.6.16}$$

Estimate T could be done in the same manner but here slight increase of it does not matter.

25.6.2 Estimate from below for Ionization Energy

In this subsection we will prove that under assumption (25.6.1)

$$\mathsf{I}_N \ge -\nu - C\upsilon \tag{25.6.17}$$

with $\upsilon \ll |\nu|$ to be defined below.

Let $\Psi = \Psi_N(x_1, \dots, x_N)$ be the ground state for N electrons, $\|\Psi\| = 1$; consider an antisymmetric test function

$$\tilde{\Psi} = \tilde{\Psi}(x_1, \dots, x_{N+1}) = \Psi(x_1, \dots, x_N)u(x_{N+1}) - \sum_{1\le j\le N} \Psi(x_1, \dots, x_{j-1}, x_{N+1}, x_{j+1}, \dots, x_N)u(x_j). \tag{25.6.18}$$

Then

$$\mathsf{E}_{N+1}\|\tilde\Psi\|^2 \le \langle \mathsf{H}_{N+1}\tilde\Psi,\tilde\Psi\rangle = N\langle \mathsf{H}_{N+1}\Psi u,\tilde\Psi\rangle = N\langle \mathsf{H}_N\Psi u,\tilde\Psi\rangle + N\langle H_{V,x_{N+1}}\Psi u,\tilde\Psi\rangle + N\langle \sum_{1\le i\le N}|x_i-x_{N+1}|^{-1}\Psi u,\tilde\Psi\rangle = (\mathsf{E}_N-\nu')\|\tilde\Psi\|^2 + N\langle H_{W+\nu',x_{N+1}}\Psi u,\tilde\Psi\rangle + N\langle\big(\sum_{1\le i\le N}|x_i-x_{N+1}|^{-1}-(V-W)(x_{N+1})\big)\Psi u,\tilde\Psi\rangle$$

and therefore

$$N^{-1}(\mathsf{I}_{N+1}+\nu')\|\tilde\Psi\|^2 \ge -\langle H_{W+\nu',x_{N+1}}\Psi u,\tilde\Psi\rangle - \langle\big(\sum_{1\le i\le N}|x_i-x_{N+1}|^{-1}-(V-W)(x_{N+1})\big)\Psi u,\tilde\Psi\rangle \tag{25.6.19}$$

with $\nu'\ge\nu$ to be chosen later. One can see easily that

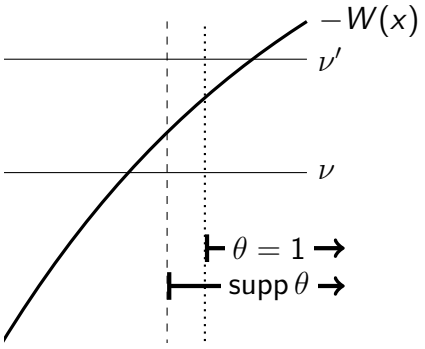
$$N^{-1}\|\tilde\Psi\|^2 = \|\Psi\|^2\cdot\|u\|^2 - N\int \Psi(x_1,\dots,x_{N-1},x)\Psi^\dagger(x_1,\dots,x_{N-1},y)u(y)u^\dagger(x)\,dx_1\cdots dx_{N-1}\,dxdy \tag{25.6.20}$$

where † means a complex or Hermitian conjugation.

Note that every term in the right-hand expression in (25.6.19) is the sum of two terms: one with $\tilde\Psi$ replaced by $\Psi(x_1,\dots,x_N)u(x_{N+1})$ and another with $\tilde\Psi$ replaced by $-N\Psi(x_1,\dots,x_{N-1},x_{N+1})u(x_N)$. We call these terms *direct* and *indirect* respectively.

Obviously, in the direct and indirect terms u appears as $|u(x)|^2\,dx$ and as $u(x)u^\dagger(y)\,dxdy$ respectively, multiplied by some kernels.

Recall that u is an arbitrary function. Let us take $u(x)=\theta^{\frac12}(x)\phi_j(x)$ where ϕ_j are orthonormal eigenfunctions of $H_{W+\nu}$ and $\theta(x)$ is β-admissible function supported in $\{x:\ -\upsilon\ge W(x)+\nu\ge\frac23\nu\}$ and equal 1 in $\{x:\ -2\upsilon\ge W(x)+\nu\ge\frac12\nu\}$, satisfying (25.5.11), and $\beta=\upsilon\bar r^5$.



Let us substitute it into (25.6.19), multiply by $\varphi(\lambda_j L^{-1})$ and take sum with respect to j.

We get the same expressions with $|u(x)|^2\,dx$ and $u(x)u^\dagger(y)\,dxdy$ replaced by $F(x,x)\,dx$ and $F(x,y)\,dxdy$ respectively with

$$F(x,y)=\int \varphi(\lambda L^{-1})\,d_\lambda e(x,y,\lambda). \tag{25.6.21}$$

Here $\varphi(\tau)$ is a fixed $\mathscr{C}^\infty$ non-negative function equal to 1 as $\tau\le \frac{1}{2}$ and equal to 0 as $\tau\ge 1$ and $L=\nu'-\nu=6\upsilon$.

Under described construction and procedures the direct term generated by $N^{-1}\|\tilde{\Psi}\|^2$ is

$$\int \theta(x)\varphi(\lambda L^{-1})\,d_\lambda e(x,x,\lambda)\,dx \tag{25.6.22}$$

and applying the semiclassical approximation we get

$$\Theta \coloneqq \int \varphi(\lambda L^{-1})\,d_\lambda P'(W+\nu-\lambda)\,dx. \tag{25.6.23}$$

(a) Observe that under assumptions (25.3.33) and (25.6.4) [24] the remainder estimate is $C\hbar^{-1}\bar{r}^2b^{-2}=C\upsilon^{\frac{1}{2}}\bar{r}^2b^{-1}=C\upsilon^{-\frac{1}{2}}\bar{r}^{-3}$; one can prove it easily by partition of unity on $\mathsf{supp}(\theta)$ and applying the semiclassical asymptotics with effective semiclassical parameter $\hbar=1/(\upsilon^{\frac{1}{2}}b)=\upsilon^{-\frac{3}{2}}\bar{r}^{-5}$.

On the other hand, indirect term generated by $N^{-1}\|\tilde{\Psi}\|^2$ is

$$-N\int \theta^{\frac{1}{2}}(x)\theta^{\frac{1}{2}}(y)\Psi(x_1,\dots,x_{N-1},x)\Psi^\dagger(x_1,\dots,x_{N-1},y)\times F(x,y)\,dxdydx_1\cdots dx_{N-1} \tag{25.6.24}$$

and since the operator norm of $F(.,.,.)$ is 1 the absolute value of this term does not exceed

$$N\int \theta(x)|\Psi(x_1,\dots,x_{N-1},x)|^2\,dx=\int \theta(x)\rho_\Psi(x)\,dx\le \int \theta(x)\rho^{\mathsf{TF}}(x)\,dx+CQ^{\frac{1}{2}}\|\nabla\theta^{\frac{1}{2}}\| \tag{25.6.25}$$

where we applied (25.5.6). Observe that $\rho^{\mathsf{TF}}=0$ on $\mathsf{supp}(\theta)$ and under assumption (25.6.4) [24] $\|\nabla\theta^{\frac{1}{2}}\|\asymp b^{-\frac{1}{2}}\bar{r}\asymp\zeta^{-\frac{1}{2}}\bar{r}^{-\frac{3}{2}}$.

[24] Or rather its corollary (25.6.5).

Recall that $P'(W^{\mathsf{TF}}+\nu)=\rho^{\mathsf{TF}}$. We will take $\nu'=\nu+L$ large enough to keep Θ larger than all the remainders including those due to replacement W by W^{TF} and ρ_Ψ by ρ^{TF} in the expression above. One can see easily that

$$\Theta \asymp \hbar^{-3}\times b^{-2}\bar r^2 \asymp \upsilon^{\frac{3}{2}} b\bar r^2 \asymp \upsilon^{\frac{5}{2}}\bar r^7. \tag{25.6.26}$$

Therefore

(25.6.27) If $\upsilon = C_0\bar r^{-\frac{17}{6}}Q^{\frac{1}{6}}$ and $Z-N\ge C_0 Z^{\frac{5}{7}}$ the total expression generated by $N^{-1}\|\tilde\Psi\|^2$ is greater than $\epsilon\Theta$ with $\Theta=\upsilon^{\frac{5}{2}}\bar r^7$.

Now let us consider direct terms in the right-hand expression of (25.6.19). The first of them is

$$\begin{aligned}
&-\int \theta^{\frac{1}{2}}(x)\varphi(\lambda L^{-1})\,d_\lambda\big(H_{W+\nu',x}\theta^{\frac{1}{2}}(x)e(x,y,\lambda)\big)_{y=x}\,dx = \\
&\qquad -\int \theta(x)\varphi(\lambda L^{-1})\,d_\lambda\big(H_{W+\nu',x}e(x,y,\lambda)\big)_{y=x}\,dx \\
&\qquad\qquad -\frac{1}{2}\int \varphi(\lambda L^{-1})[[H_W,\theta^{\frac{1}{2}}],\theta^{\frac{1}{2}}]\,d_\lambda e(x,x,\lambda)\ge \\
&\int \theta(x)(\nu'-\nu-\lambda)\varphi(\lambda L^{-1})\,d_\lambda e(x,x,\lambda)\,dx - C\int |\nabla\theta^{\frac{1}{2}}|^2 e(x,x,\nu')dx.
\end{aligned} \tag{25.6.28}$$

Note that the absolute value of last term in the right-hand expression of (25.6.28) does not exceed $Cb^{-1}\bar r^2 L^{\frac{3}{2}}\asymp C\upsilon^{\frac{1}{2}}\bar r^{-3}\ll C\upsilon\Theta$.

The second direct term in the right-hand expression is

$$\begin{aligned}
&-\int \theta(x)\Big(\rho_\Psi * |x|^{-1} - (V-W)(x)\Big)F(x,x)\,dx = \\
&\qquad -\mathsf{D}\big(\rho_\Psi-\bar\rho,\theta(x)F(x,x)\big)\ge \\
&-C\mathsf{D}\big(\rho_\Psi-\rho,\rho_\Psi-\rho\big)^{\frac{1}{2}}\cdot \mathsf{D}\Big(\theta^{\frac{1}{2}}F(x,x),\theta^{\frac{1}{2}}F(x,x)\big)\Big)^{\frac{1}{2}} \ge -CQ^{\frac{1}{2}}\bar r^{\frac{15}{2}}\upsilon^{\frac{5}{2}}
\end{aligned} \tag{25.6.29}$$

provided $V-W=|x|^{-1}*\rho$ with $\mathsf{D}(\rho-\rho^{\mathsf{TF}},\,\rho-\rho^{\mathsf{TF}})\le CQ$; the absolute value of this term is $\ll\upsilon\Theta$.

Further, the first indirect term in the right-hand expression of (25.6.19) is

$$
\begin{aligned}
(25.6.30)\quad & -N\int \theta^{\frac12}(y)\Psi(x_1,\dots,x_{N-1},x)\Psi^{\dagger}(x_1,\dots,x_{N-1},y)\times \\
&\qquad \varphi(\lambda L^{-1})\,d_\lambda\big(H_{W+\nu',x}\theta^{\frac12}(x)e(x,y,\lambda)\big)\,dxdydx_1\cdots dx_{N-1} = \\
& -N\int \theta^{\frac12}(y)\theta^{\frac12}(x)\Psi(x_1,\dots,x_{N-1},x)\Psi^{\dagger}(x_1,\dots,x_{N-1},y)\times \\
&\qquad \varphi(\lambda L^{-1})(\nu'-\nu-\lambda)\,d_\lambda e(x,y,\lambda)\,dxdydx_1\cdots dx_{N-1} \\
& -N\int \theta^{\frac12}(y)\Psi(x_1,\dots,x_{N-1},x)\Psi^{\dagger}(x_1,\dots,x_{N-1},y)\times \\
&\qquad \varphi(\lambda L^{-1})[H_{W,x},\theta^{\frac12}(x)]\,d_\lambda e(x,y,\lambda)\,dxdydx_1\cdots dx_{N-1}.
\end{aligned}
$$

Note that one can rewrite the sum of the first terms in the right-hand expressions in (25.6.28) and (25.6.30) as $\sum_j \varphi(\lambda_j L^{-1})(\nu'-\nu-\lambda_j)\|\widehat{\Psi}_j\|^2$ with

$$
\widehat{\Psi}_j(x_1,\dots,x_{N-1}) \coloneqq \int \Psi(x_1,\dots,x_{N-1},x)\theta^{\frac12}(x)\phi_j(x)\,dx
$$

and therefore this sum is non-negative.

One can prove easily that the absolute value of the second term in (25.6.30) is less than

$$
C\upsilon^{\frac12}b^{-1}\int \rho_\Psi(y)\theta^{\frac12}(y)\,dy \le C\upsilon^{-\frac12}\bar{r}^{-5}\Theta \ll \upsilon\Theta.
$$

Therefore

(25.6.31) The sum of the first direct and indirect terms in the right-hand expression of (25.6.19) is greater than $-C\upsilon\Theta$.

Finally, we need to consider the second indirect term generated by the right-hand expression of (25.6.19)

$$
\begin{aligned}
(25.6.32)\quad & -\int\Big(\sum_{1\le i\le N}|y-x_i|^{-1}-(V-W)(y)\Big)\times \\
&\qquad \Psi(x_1,\dots,x_N)\Psi^{\dagger}(x_1,\dots,x_{N-1},y)\theta^{\frac12}(x_N)\theta^{\frac12}(y)F(x_N,y)\,dx_1\cdots dx_N dy =
\end{aligned}
$$

$$= -\int \Big(|y|^{-1} * \rho_{\underline{x}}(y) - (V-W)(y)\Big)\Psi(x_1,\dots,x_N)\Psi^\dagger(x_1,\dots,x_{N-1},y)\times \theta^{\frac12}(x_N)\theta^{\frac12}(y)F(x_N,y)\,dx_1\cdots dx_N dy$$

$$-\int \Big(\sum_{1\le i\le N}|y-x_i|^{-1} - |y|^{-1} * \rho_{\underline{x}}(y)\Big)\Psi(x_1,\dots,x_N)\Psi^\dagger(x_1,\dots,x_{N-1},y)\times \theta^{\frac12}(x_N)\theta^{\frac12}(y)F(x_N,y)\,dx_1\cdots\, dx_N dy;$$

recall that $\rho_{\underline{x}}$ is a smeared density, $\underline{x}=(x_1,\dots,x_N)$.

Since $|y|^{-1} * \rho_{\underline{x}}(y) - (V-W)(y) = |y|^{-1} * (\rho_{\underline{x}} - \rho)$, the first term in the right-hand expression is equal to

$$\int \theta^{\frac12}(x_N)\Psi(x_1,\dots,x_N)\times \mathsf{D}_y\Big(\rho_{\underline{x}}(y)-\rho(y), F(x_N,y,\lambda)\theta^{\frac12}(y)\Psi(x_1,\dots,x_{N-1},y)\Big)\,dx_1\cdots dx_N \tag{25.6.33}$$

and its absolute value does not exceed

$$\Big(N\int \mathsf{D}\big(\rho_{\underline{x}}(\cdot)-\rho(\cdot),\rho_{\underline{x}}(\cdot)-\rho(\cdot)\big)|\Psi(x_1,\dots,x_N)|^2\theta(x_N)\,dx_1\cdots dx_N\Big)^{\frac12} \times N^{-\frac12}\Big(\int \mathsf{D}_y\Big(F(x_N,y)\theta^{\frac12}(y)\Psi(x_1,\dots,x_{N-1},y), F(x_N,y)\theta^{\frac12}(y)\Psi(x_1,\dots,x_{N-1},y)\Big)\,dx_1\cdots dx_N\Big)^{\frac12}. \tag{25.6.34}$$

Due to estimate (25.5.24) and definition (25.5.4) as the first factor in (25.6.34) does not exceed $\big((Q+T+\varepsilon^{-1}N)\Theta+R\big)^{\frac12}$, where we assume that $\varepsilon \le Z^{-\frac23}$ and $\Theta \asymp b^{-\frac12}Q^{\frac12}\bar r$ is now an upper estimate for $\int \theta(y)\rho_\Psi(y)\,dy$-like expressions; due to our choice of υ it coincides with $\Theta = \upsilon^{\frac52}\bar r^7$.

Then according to (25.5.25) $R \asymp Cb^{-2}\Theta \ll Q\Theta$ and according to (25.5.23) $T \ll Q$ and therefore in all such inequalities we may skip R and T terms; so we get $C(Q+\varepsilon^{-1}N)^{\frac12}\Theta^{\frac12}$.

Meanwhile, the second factor in (25.6.34) (without square root) is equal to

$$N^{-1}\int L^{-2}\varphi'(\lambda L^{-1})\varphi'(\lambda' L^{-1})|y-z|^{-1}\underbrace{e(x_N,y,\lambda)}\,\theta^{\frac12}(y)\Psi(x_1,\dots,x_{N-1},y)\times \underbrace{e(x_N,z,\lambda')}\,\theta^{\frac12}(z)\Psi^\dagger(x_1,\dots,x_{N-1},z)\,dydz\,dx_1\cdots dx_{N-1}\underbrace{dx_N}\,d\lambda d\lambda';$$

after integration by x_N we get instead of marked terms $e(y,z,\lambda)$ (recall that $e(.,.,.)$ is the Schwartz kernel of projector and we keep $\lambda<\lambda'$) and then integrating with respect to λ' we arrive to

$$N^{-1}\int |y-z|^{-1}F(y,z)\theta^{\frac12}(y)\Psi(x_1,\dots,x_{N-1},y)\times \theta^{\frac12}(z)\Psi^\dagger(x_1,\dots,x_{N-1},z)\,dydz\,dx_1\cdots dx_{N-1}\,,$$

where now F is defined by (25.6.21) albeit with φ^2 instead of φ. This latter expression does not exceed

$$N^{-1}\iint |y-z|^{-1}|F(y,z)|\theta^{\frac12}(y)|\Psi(x_1,\dots,x_{N-1},y)|^2\times dydz\,dx_1\cdots dx_{N-1}. \tag{25.6.35}$$

Then due to Proposition 25.A.3 expression $\int |y-z|^{-1}|F(y,z)|\,dz$ does not exceed $Cb^{-1}\hbar^{-1}\asymp \upsilon^{\frac12}$, and thus expression (25.6.35) does not exceed $CZ^{-2}\upsilon^{\frac12}\Theta$ and therefore the second factor in (25.6.34) does not exceed $CN^{-1}\upsilon^{\frac14}\Theta^{\frac12}$ and the whole expression (25.6.34) does not exceed

$$C(Q+\varepsilon^{-1}N)^{\frac12}\Theta^{\frac12}\times N^{-1}\upsilon^{\frac14}\Theta^{\frac12}= CN^{-1}(Q+\varepsilon^{-1}N)^{\frac12}\upsilon^{\frac14}\Theta\,.$$

Therefore we conclude that

(25.6.36) As $\varepsilon\ge Z^{-1}\upsilon^{-\frac32}$ the first term in the right-hand expression of (25.6.32) does not exceed $C\upsilon\Theta$.

Further, we need to estimate the second term in the right-hand expression of (25.6.32). It can be rewritten in the form

$$\sum_{1\le i\le N}\int U(x_i,y)\Psi(x_1,\dots,x_N)\Psi^\dagger(x_1,\dots,x_{N-1},y)\theta^{\frac12}(x_N)\theta^{\frac12}(y)\times F(x_N,y)\,dx_1\cdots dx_N dy\,, \tag{25.6.37}$$

where $U(x_i,y)$ is the difference between potential generated by the charge $\delta(x-x_i)$ and the same charge smeared; note that $U(x_i,y)$ is supported in $\{(x_i,y)\colon |x_i-y|\le\varepsilon\}$. Let us estimate the i-th term in this sum with $i<N$ first; multiplied by $N(N-1)$, it does not exceed

(25.6.38)
$$N\biggl(\int |U(x_i,y)|^2|\Psi(x_1,\dots,x_N)|^2\theta^{\frac12}(x_N)\theta^{\frac12}(y)|F(x_N,y)|\,dx_1\cdots dx_N dy\biggr)^{\frac12}\times$$
$$N\biggl(\int \omega(x_i,y)|\Psi(x_1,\dots,x_{N-1},y)|^2\theta^{\frac12}(x_N)\theta^{\frac12}(y)|F(x_N,y)|\,dx_1\cdots dx_N dy\biggr)^{\frac12};$$

here ω is ε-admissible and supported in $\{(x_i,y)\colon |x_i-y|\le 2\varepsilon\}$ function. Due to Proposition 25.A.3 in the second factor $\int \theta^{\frac12}(x_N)|F(x_N,y)|\,dx_N\le C$ and therefore the whole second factor does not exceed

$$C\underbrace{\biggl(\int \theta^{\frac12}(x)\omega(x,y)\rho^{(2)}_\Psi(x,y)\,dxdy\biggr)}^{\frac12} \tag{25.6.39}$$

where we replaced x_i by x. According to Proposition 25.5.1 in the selected expression one can replace $\varrho^{(2)}_\Psi(x,y)$ by $\rho_\Psi(x)\rho(y)$ with an error which does not exceed

$$C\Bigl(\sup_x \|\nabla_y\chi_x\|_{\mathscr{L}^2(\mathbb{R}^3)}\bigl(Q+\varepsilon^{-1}N\bigr)^{\frac12}+C\varepsilon N\|\nabla_y\chi\|_{\mathscr{L}^\infty}\Bigr)\Theta,$$

which, as we plug $\sup_x \|\nabla_y\chi_x\|_{\mathscr{L}^2(\mathbb{R}^3)}\asymp\varepsilon^{\frac12}$, $\|\nabla_y\chi\|_{\mathscr{L}^\infty}\asymp\varepsilon^{-1}$ becomes $CN\Theta$.

Meanwhile, let us consider

$$\int |U(x_i,y)|^2\theta^{\frac12}(y)|F(x_N,y)|\,dy. \tag{25.6.40}$$

Again due to Proposition 25.A.3 it does not exceed

$$C\upsilon^{\frac32}\int |U(x_i,y)|^2\theta^{\frac12}(y)\bigl(|x_N-y|\upsilon^{\frac12}+1\bigr)^{-s}\,dy$$

and this integral should be taken over $B(x_i,\varepsilon)$, with $|U(x_i,y)|\le|x_i-y|^{-1}$, so (25.6.40) does not exceed $C\varepsilon\upsilon^{\frac32}\omega'(x_i,x_N)$ with $\omega'(x,y)=\bigl(1+\upsilon^{\frac12}|x-y|\bigr)^{-s}$

(provided $\varepsilon \le \upsilon^{-\frac12}$ which will be the case). Therefore the first factor in (25.6.38) does not exceed

$$C\varepsilon^{\frac12}\upsilon^{\frac34}\Bigl(\underbrace{\int \theta^{\frac12}(x)\omega'(x,y)\rho^{(2)}_\Psi(x,y)\,dxdy}\Bigr)^{\frac12}. \tag{25.6.41}$$

Therefore in selected expression one can replace $\rho^{(2)}_\Psi(x,y)$ by $\rho_\Psi(x)\rho(y)$ with an error which does not exceed what we got before but with ε replaced by $\upsilon^{-\frac12}$, i.e. also $CN\Theta$.

However in both selected expressions replacing $\rho^{(2)}_\Psi(x,y)$ by $\rho_\Psi(x)\rho(y)$ we get just 0. Therefore expression (25.6.38) does not exceed $C\varepsilon^{\frac12}\upsilon^{\frac34}Z\Theta$ which does not exceed $C\upsilon\Theta$ provided $\varepsilon \le C\upsilon^{\frac12}Z^{-2}$.

So, we have two restrictions to ε from above: the last one and $\varepsilon \le Z^{-\frac23}$ and one can see easily that both of them are compatible with restriction to ε in (25.6.36) provided $\upsilon \ge C_0Q^{\frac37}$, $Q \ge Z^{\frac53-\delta}$.

Finally, consider term in (25.6.37) with $i = N$ (multiplied by N):

$$N\int U(x_N,y)|\Psi(x_1,\dots,x_N)|^2\theta^{\frac12}(x_N)\theta^{\frac12}(y)F(x_N,y)\,dx_1\cdots dx_N dy; \tag{25.6.42}$$

due to Cauchy inequality it does not exceed

$$\begin{aligned} &N\Bigl(\int |x_N-y|^{-2}|\Psi(x_1,\dots,x_N)|^2\theta^{\frac12}(x_N)\theta^{\frac12}(y)\,dx_1\cdots dx_N dy\Bigr)^{\frac12}\times\\ &\qquad N\Bigl(\int |F(x_N,y)|^2|\Psi(x_1,\dots,x_N)|^2\theta^{\frac12}(x_N)\theta^{\frac12}(y)\,dx_1\cdots dx_N dy\Bigr)^{\frac12}, \end{aligned} \tag{25.6.43}$$

where both integrals are taken over $\{|x_N-y|\le\varepsilon\}$ and integrating with respect to y there we get that it does not exceed

$$C\varepsilon^{\frac12}\Theta^{\frac12}\times \upsilon^{\frac34}\varepsilon^{\frac32}\Theta^{\frac12} = C\upsilon^{\frac34}\varepsilon^2\Theta \ll \upsilon\Theta.$$

Therefore the right-hand expression in (25.6.19) is $\ge -C\upsilon\Theta$ and recalling that $\nu'-\nu = O(\upsilon)$ we recover an lower estimate in Theorem 25.6.1 below under assumption (25.6.4):

$$\mathsf{I}_N + \nu \ge -C\upsilon = -C(Z-N)^{\frac{17}{18}}Q^{\frac16}. \tag{25.6.44}$$

(b) In the general case, following arguments of the part (a) of Theorem 25.6.1 we recover estimate

$$\mathsf{I}_N + \nu \ge C\upsilon \ge -C(Z-N)^{\frac{17}{18}} Q^{\frac{1}{6}} \times \log^{\frac{1}{3}}\bigl((Z-N)Q^{-\frac{3}{7}}\bigr). \tag{25.6.45}$$

Combining with a lower estimate (25.6.15) or (25.6.16) and recalling estimate (25.4.5) for Q we arrive to

Theorem 25.6.1. *Let condition* (25.3.33) *be fulfilled and let* $N \le Z - C_0 Q^{\frac{3}{7}}$ *with* $Q \le C_1 Z^{\frac{5}{3}}$. *Then in the framework of fixed nuclei model*

(i) In the general case $Q = Z^{\frac{5}{3}}$

$$|\mathsf{I}_N + \nu| \le C(Z-N)^{\frac{17}{18}} Z^{\frac{5}{18}} \times \log^{\frac{1}{3}}\bigl((Z-N)Q^{-\frac{3}{7}}\bigr). \tag{25.6.46}$$

(ii) Under assumption (25.6.4) *as* $Q = Z^{\frac{5}{3}}$

$$|\mathsf{I}_N + \nu| \le C(Z-N)^{\frac{17}{18}} Z^{\frac{5}{18}}. \tag{25.6.47}$$

(iii) Further, as

$$a := \min_{1\le m<m'\le M} |\mathsf{y}_m - \mathsf{y}_{m'}| \ge Z^{-\frac{1}{3}}$$

we can take $Q = Z^{\frac{5}{3}}\bigl(Z^{-\delta} + (aZ^{\frac{1}{3}})^{-\delta}\bigr)$ *and then both estimates* (25.6.46) *and* (25.6.47) *hold with an extra factor* $\bigl(Z^{-\delta'} + (aZ^{\frac{1}{3}})^{-\delta'}\bigr)$ *in the right-hand expressions.*

Remark 25.6.2. (i) In virtue of Theorem 25.6.3 stable molecules do not exist in the free nuclei model as $Z - N \ge CQ^{\frac{3}{7}}$ and in the atomic case $\widehat{\mathsf{I}}_N = \mathsf{I}_N$.

(ii) Observe that the relative error in the estimates (25.6.46) and (25.6.47) is $\asymp \upsilon/\nu$; in particular, in (25.6.47) it is $\asymp (Z-N)^{-\frac{7}{18}} Q^{\frac{1}{6}}$ and in (25.6.46) it acquires factor $\log^{\frac{1}{3}}\bigl((Z-N)Q^{-\frac{3}{7}}\bigr)$. Both of these expressions are $\lesssim 1$ due to assumption (25.6.1) and they are $\asymp 1$ as $Z - N \asymp Q^{\frac{3}{7}}$.

25.6.3 Estimate for Excessive Positive Charge

To estimate excessive positive charge when molecules can still exist in free nuclei model we apply arguments of section 5 of B. Ruskai and J. P. Solovej [1]. In view of Theorem 25.4.15 it is sufficient to consider the case

$$a := \min_{1\le m<m'\le M} |\mathsf{y}_m - \mathsf{y}_{m'}| \ge C_0\bar{r}. \tag{25.6.48}$$

Therefore in Thomas-Fermi theory ρ^{TF} is supported in separate "atoms".

Let us consider a-admissible functions $\theta_m(x)$, supported in $B(\mathsf{y}_m, \frac{1}{3}a)$ for $m = 1, \dots, M$ and in $\{|x - \mathsf{y}_m| \ge \frac{1}{4}a \ \ \forall m = 1, \dots, M\}$ for $m = 0$, such that

$$\theta_0^2 + \dots + \theta_M^2 = 1. \tag{25.6.49}$$

Then for the ground state Ψ

$$\mathsf{E}_N = \langle \mathsf{H}\Psi, \Psi\rangle = \sum_{\alpha} \langle \theta_\alpha \mathsf{H} \theta_\alpha \Psi, \Psi\rangle - \sum_{\alpha, j} \|(\nabla_j \theta_\alpha)\Psi\|^2 \tag{25.6.50}$$

with the sum over of $(M+1)$-cluster decompositions $\alpha = (\alpha_0, \dots, \alpha_M)$ of $\{1, \dots, N\}$ and $\theta_\alpha(\underline{x}) = \prod_{0\le m\le M}\prod_{i\in\alpha_m}\theta_m(x_i)$; $j = 1, \dots, N$. Then for any given α

$$\mathsf{H} = \sum_{0\le m\le M} \mathsf{H}_{\alpha_m} + J_\alpha \tag{25.6.51}$$

with the *cluster Hamiltonians* H_{α_m}, involving only potential of m-th nucleus (no nucleus potential as $m = 0$) and only electrons belonging to α_m and therefore satisfying

$$\mathsf{H}_{\alpha_m} \ge \mathsf{E}_{\mathsf{atom}}(N_m(\alpha), Z_m), \qquad \mathsf{H}_{\alpha_0} \ge 0, \tag{25.6.52}$$

and with the *intercluster Hamiltonian* (actually, just potentials)

$$\begin{aligned} J_\alpha = \sum_{1\le m\le M}\sum_{i\notin\alpha_m} -Z_m|x_i - \mathsf{y}_m|^{-1} \\ + \sum_{m<l}\sum_{i\in\alpha_m, j\in\alpha_l} |x_i - x_j|^{-1} + \sum_{m<l} Z_m Z_l |\mathsf{y}_l - \mathsf{y}_m|^{-1}. \end{aligned} \tag{25.6.53}$$

Let us note that

$$\sum_{\alpha} \theta_\alpha^2 J_\alpha = \sum_{0\le m<l\le M} J_{ml} \tag{25.6.54}$$

with J_{ml} given by (32)–(33) of B. Ruskai–J. P. Solovej [1] if $m, l > 0$ and $m = 0$ respectively:

$$J_{ml} = Z_m Z_l |\mathsf{y}_m - \mathsf{y}_l|^{-1} - Z_m \sum_i \theta_l^2(x_i)|x_i - \mathsf{y}_m|^{-1} - Z_l \sum_i \theta_m^2(x_i)|x_i - \mathsf{y}_l|^{-1} + \sum_{i \neq j} \theta_m^2(x_i)\theta_l^2(x_j)|x_i - x_j|^{-1}, \tag{25.6.55}$$

and

$$J_{0l} = \sum_i \theta_0^2(x_i)\Big(-Z_l|x_i - \mathsf{y}_l|^{-1} + \sum_j \theta_l^2(x_j)|x_i - x_j|^{-1}\Big). \tag{25.6.56}$$

Then we recover (35) of B. Ruskai–J. P. Solovej [1]

$$\langle J_{ml}\Psi, \Psi\rangle = Z_m Z_l |\mathsf{y}_m - \mathsf{y}_l|^{-1} - Z_l \int \rho_\Psi(x)\theta_m^2(x)|x - \mathsf{y}_l|^{-1}\,dx - Z_m \int \rho_\Psi(x)\theta_l^2(x)|x - \mathsf{y}_m|^{-1}\,dx + \int \rho_\Psi^{(2)}(x,y)\theta_m^2(x)\theta_l^2(y)\,dxdy. \tag{25.6.57}$$

Applying estimate (25.4.56) (replacing first θ_m with $m = l, \dots, M$ by $\tilde\theta_m$ supported in $B(\mathsf{y}_l, c\bar r)$ and estimating an error), we conclude that

$$\int \rho_\Psi(x)\theta_m^2(x)|x - \mathsf{y}_l|^{-1} = \int \rho^{\mathsf{TF}}(x)\theta_m^2(x)|y - \mathsf{y}_l|^{-1}\,dx + O(Y)|\mathsf{y}_m - \mathsf{y}_l|^{-1} \tag{25.6.58}$$

and

$$\int \rho_\Psi(x)\theta_m^2(x)dyx = \mathsf{N}_m^{\mathsf{TF}} + O(Y) \tag{25.6.59}$$

with

$$N_m^{\mathsf{TF}} = \int \rho^{\mathsf{TF}}(x)\theta_m^2(x)\,dx, \tag{25.6.60}$$

where here and below $Y = (Q\bar r)^{\frac12}$, $\bar Y = (\bar Q \bar r)^{\frac12}$ (compare with (36)–(37) of B. Ruskai–J. P. Solovej [1]) which yields

$$\int \rho_\Psi(x)\big(1 - \sum_{1\le m\le M} \tilde\theta_m(x)\big)\,dx \le CY \tag{25.6.61}$$

and we prove that (25.6.59) holds for θ_m as well (compare with (38) of B. Ruskai–J. P. Solovej [1]).

Let us estimate the last term in (25.6.57) by Proposition 25.5.1 with $\theta(x) = \tilde{\theta}_m^2(x)$ and $\chi(x,y) = \tilde{\theta}_l^2(y)|x-y|^{-1}$,

$$\begin{aligned}(25.6.62)\quad \int \rho_\Psi^{(2)}(x,y)\theta_m^2(x)\theta_l^2(y)|x-y|^{-1}\,dxdy \geq \\ \int \rho_\Psi^{(2)}(x,y)\theta_m^2(x)\tilde{\theta}_l^2(y)|x-y|^{-1}\,dxdy \geq \\ \int \rho_\Psi(x)\rho^{\mathsf{TF}}(y)\theta_m^2(x)\tilde{\theta}_l^2(y)|x-y|^{-1}\,dxdy - \\ C\bar{Y}\int \rho_\Psi(x)\theta_m^2(x)|\mathsf{y}_m-\mathsf{y}_l|^{-1}\end{aligned}$$

because $\|\nabla_y \chi_x\|_{\mathscr{L}^2} \leq C\bar{r}^{\frac{1}{2}}|\mathsf{y}_m-\mathsf{y}_l|^{-1}$ and

$$R = \int |\nabla\theta_m|^2\rho_\Psi(x)\,dx \leq Ca^{-2}Y.$$

Repeating the same trick with replacing $\rho_\Psi(x)$ by $\rho^{\mathsf{TF}}(x)$ we get that it is larger than

$$(25.6.63)\quad \int \rho^{\mathsf{TF}}(x)\rho^{\mathsf{TF}}(y)\theta_m^2(x)\theta_l^2(y)\ |x-y|^{-1}\,dxdy - CN\bar{Y}|\mathsf{y}_m-\mathsf{y}_l|^{-1}.$$

Then we conclude that

$$\begin{aligned}(25.6.64)\quad \langle J_{ml}\Psi,\Psi\rangle \geq \int \big(Z_m\delta(x-\mathsf{y}_m)-\rho^{\mathsf{TF}}(x)-CY\big) \\ \times\big(Z_l\delta(y-\mathsf{y}_l)-\rho^{\mathsf{TF}}(x)-CY\big)\theta_m^2(x)\theta_l^2(y)|x-y|^{-1}\,dxdy \\ - CN\bar{Y}|\mathsf{y}_m-\mathsf{y}_l|^{-1}\end{aligned}$$

and finally

$$\begin{aligned}(25.6.65)\quad &\langle J_{ml}\Psi,\Psi\rangle \\ &\geq (Z_m-N_m^{\mathsf{TF}}-CY)\cdot(Z_l-N_l^{\mathsf{TF}}-Y)|\mathsf{y}_l-\mathsf{y}_m|^{-1} - C_1N\bar{Y}|\mathsf{y}_m-\mathsf{y}_l|^{-1}\end{aligned}$$

(compare with (39) of B. Ruskai–J. P. Solovej [1]) provided

$$\begin{aligned}(25.6.66)\quad \Big|\int \rho^{\mathsf{TF}}(y)|x-y|^{-1}\tilde{\theta}_m(x)\,dy - N_m^{\mathsf{TF}}|x-\mathsf{y}_m|^{-1}\Big| \leq \\ C(Z-N)|x-\mathsf{y}_m|^{-1}\end{aligned}$$

for $|x - y_m| \ge C\bar{r}$. Similarly, under the same condition (25.6.66)

$$\langle J_{0l}\Psi, \Psi\rangle \ge -CY(Z-N) - C_1 N\bar{Y}a^{-1} \tag{25.6.67}$$

(compare with (40) of B. Ruskai–J. P. Solovej [1]).

Let us note that the absolute value of the last term in the right-hand expression of (25.6.50) does not exceed $Ca^{-2}Y$ due to (25.6.59). Now stability condition yields

$$J = \sum_{0\le m<l\le M} J_{ml} \le CYa^{-2} + C(Z-N)Ya^{-1} + C_1 N\bar{Y}a^{-1}. \tag{25.6.68}$$

This inequality, (25.6.48) and Proposition 25.6.5 below yield that

$$Z - N \le CY + CN^{\frac12}\bar{Y}^{\frac12} \le C_1 N^{\frac12}\bar{Y}^{\frac12} \le C_2 N^{\frac{11}{6}}(Z-N)^{\frac16} \tag{25.6.69}$$

because $\bar{r} \asymp (Z-N)^{-\frac13}$ and $\bar{Q} \asymp Z^{\frac53}$ thus proving the following theorem:

Theorem 25.6.3. *Let condition* (25.3.33) *be fulfilled. Then in the framework of free nuclei model with $M \ge 2$ the stable molecule does not exist unless*

$$Z - N \le Z^{\frac{11}{13}}. \tag{25.6.70}$$

Remark 25.6.4. Unfortunately, we do not prove that molecules exist for any values of $Z_1, \dots, Z_m$ and N. We are not aware of any rigorous result of this type in the frameworks of our models.

Proposition 25.6.5. *Let* (25.6.48) *be fulfilled. Then inequality* (25.6.66) *holds and*

$$J \ge \epsilon(Z-N)^2 a^{-1}. \tag{25.6.71}$$

Proof. Note first that

$$\mathcal{E}^{\mathsf{TF}} \le \mathcal{E}(\bar{\rho}) = \sum_j \mathcal{E}(\rho_j^{\mathsf{TF}}) + J^{\mathsf{TF}}(\bar{\rho}) \le \sum_j \mathcal{E}^{\mathsf{TF}}(\rho_j^{\mathsf{TF}}) + C(Z-N)^2 a^{-1} \tag{25.6.72}$$

with $\bar{\rho} = \sum_j \rho_j^{\mathsf{TF}}$ while

$$\sum_j \mathcal{E}(\rho_j^{\mathsf{TF}}) \le \sum_j \mathcal{E}(\theta_j \rho^{\mathsf{TF}}); \tag{25.6.73}$$

then

$$J^{\mathsf{TF}}(\rho^{\mathsf{TF}}) \le C(Z-N)^2 a^{-1} \tag{25.6.74}$$

and using (25.6.70) we conclude that

$$\mathsf{D}(\rho^{\mathsf{TF}} - \bar{\rho}, \rho^{\mathsf{TF}} - \bar{\rho}) \le C(Z-N)^2 a^{-1}. \tag{25.6.75}$$

Further,

$$\Lambda := \sum_j \mathsf{D}(\rho^{\mathsf{TF}}\theta_j - \rho_j^{\mathsf{TF}}, \rho^{\mathsf{TF}}\theta_j - \rho_j^{\mathsf{TF}}) \le \mathsf{D}(\rho^{\mathsf{TF}} - \bar{\rho}, \rho^{\mathsf{TF}} - \bar{\rho}) + C\bar{r}a^{-1}\Lambda \tag{25.6.76}$$

and combining with (25.6.74) we conclude that

$$\Lambda \le C(Z-N)^2 a^{-1} \tag{25.6.77}$$

due to (25.6.48). Combining with $\rho_j^{\mathsf{TF}} * |x|^{-1} = N_j^{\mathsf{TF}} |x - \mathsf{y}_j|^{-1}$ for $|x - \mathsf{y}_j| \ge r_j$ we arrive to (25.6.66). Further,

$$J^{\mathsf{TF}}(\rho^{\mathsf{TF}}) \ge J^{\mathsf{TF}}(\bar{\rho}) - C\Lambda^{\frac{1}{2}}\bar{r}\,\frac{1}{2}(Z-N)a^{-\frac{3}{2}} - C\Lambda\bar{r}^{-1} \tag{25.6.78}$$

which together with (25.6.76) and (25.6.77) yields (25.6.71). □

25.A Appendices

25.A.1 Electrostatic Inequalities

We know already that there are two sources of errors in the lower estimate: due to electrostatic inequality (25.2.1) and semiclassical errors. For the first error in the case $\vec{B} = \mathsf{const}$ E. Lieb, J. P. Solovej and J. Yngvason [3] provide the (almost) perfect estimate; the reader can find the proof based on the magnetic Lieb–Thirring inequality (and this inequality as well) in that paper (p. 122) which in the case of $\vec{B} = 0$ becomes

Theorem 25.A.1. *For the ground state* Ψ *of* (25.1.1) *with potential* (25.1.4)

$$\int \rho_\Psi^{\frac{4}{3}}\,dx \le CZ^{\frac{5}{6}}N^{\frac{1}{2}}\big(Z+N\big)^{\frac{1}{3}} \tag{25.A.1}$$

otherwise.

In particular for $c^{-1}N \le Z \le cN$ *the right-hand expression does not exceed* $CZ^{\frac{5}{3}}$.

On the other hand, for $B = 0$ there is a more precise inequality due to V. Bach [1] and G. Graf and J. P. Solovej [1]:

Theorem 25.A.2. *For the ground state* Ψ *of* (25.1.1) *with potential* (25.1.4)

$$\langle H\Psi,\Psi\rangle \ge \mathsf{N}_1(A-\nu)+\nu N-\frac{1}{2}\iint |x-y|^{-1}|e(x,y,\nu)|^2\,dxdy - CN^{\frac{5}{3}-\delta} \tag{25.A.2}$$

with some exponent $\delta > 0$.

We will discuss magnetic field case in more details in the Appendix to the next Chapter 26.

25.A.2 Hamiltonian Trajectories

We are going to prove that for $W = W^{\mathsf{TF}}$ in $M = 1$ case the generic trajectory on the energy level ν is not periodic. We use some ideas from V. Arnold [2], pages 37–38. Recall that in this case $W = W(r)$ ($r = |x-\mathsf{y}_1|$) and angular momentum $\vec{M}$ is a motion integral. Then any trajectory lies on some plane and if $M = |\vec{M}| > 0$ it lies in $\{0 < r < \bar{r}\}$ where W is analytic and $W(\bar{r}) = -\nu$.

(25.A.3) Let us assume that all the trajectories on the energy level ν are periodic.

Then the rotation number

$$\Phi = \int_{r_1}^{r_2} \frac{M\,dr}{r^2\sqrt{2\big(W(r)+\nu\big)-M^2r^{-2}}} \tag{25.A.4}$$

showing the increment of the polar angle over a half-trajectory should be πk^{-1} with $k \in \mathbb{N}$ and should not depend on M where $r_1 \le r_2$ are roots of $\big(W(r)+\nu\big) - M^2 r^{-2} = \nu$. So, Φ should be the same for all trajectories on the energy level ν. One can see easily that for $M \to 0$ Φ tends to those of the Coulomb potential. So, $\Phi = 2\pi$ for all trajectories on the energy level ν.

Let r_0 be a root of $F(r) = 2\big(W(r)+\nu\big) + r^4\big(W'(r)\big)^2 = 0$. One can see easily that $F(\bar r) > 0$, $F(r) \to \infty$ as $r \to 0$ and $F'(r) < 0$ because

$$W' < 0, \qquad W'' + 2rW' = r^{-1}(r^2 W')' = r\Delta W > 0. \tag{25.A.5}$$

So, the unique root exists. Then $r = r_0$ is a circular motion with $M = -r_0^3 W'(r_0)$.

Then (V. Arnold [2], problem 2 at page 37)

$$\Phi \to \pi \sqrt{W'/(3W' + rW'')^{-1}}\big|_{r=r_0} = \Phi_0$$

for trajectories tending to circular. However, $3W' + rW'' > W'$ due to (25.A.5) and then $\Phi_0 > \pi$. Contradiction to assumption (25.A.3).

25.A.3 Some Spectral Function Estimates

Proposition 25.A.3. *For Schrödinger operator with $W \in \mathscr{C}^\infty$ and for $\phi \in \mathscr{C}_0^\infty([-1,1])$ the following estimate holds for any s:*

$$|F(x,y)| \le Ch^{-3}\big(1 + h^{-1}|x-y|\big)^{-s}, \tag{25.A.6}$$

$$F(x,y) := \int \phi(\lambda)\, d_\lambda e(x,y,\lambda). \tag{25.A.7}$$

Proof. Let $u(x,y,t) = \int e^{-ih^{-1}t\lambda}\, d_\lambda e(x,y,\lambda)$ be the Schwartz's kernel of $e^{-ih^{-1}Ht}$.

Fix y. Note first that $\mathscr{L}^2$-norm[25)] of $\phi(hD_t)\chi(t)\omega(x)u(x,y,t)$ is less than Ch^s as $\chi \in \mathscr{C}_0^\infty([-\epsilon,\epsilon])$ and $\omega \in \mathscr{C}^\infty$ supported in $\{|x-y| \ge \epsilon_1\}$ ($\epsilon_1 = C\epsilon$) due to the finite speed of propagation of singularities.

We conclude then that $\mathscr{L}^2$-norm of $\phi(hD_t)\chi(t)\omega(x)u(x,y,t)$ is less than Ch^s for $\omega \in \mathscr{C}^\infty$ supported in $\{x\colon |x-y| \ge C\}$.

Then $\mathscr{L}^2$-norm of $\partial_t^l \nabla^\alpha \phi(hD_t)\chi(t)\omega(x)u$ does not exceed Ch^s. Then due to imbedding inequality $\mathscr{L}^\infty$-norm of $\phi(hD_t)\chi(t)\omega(x)u$ does not exceed

[25)] With respect to x, t here and below.

Ch^s. Setting $t = 0$ and using this inequality and $|F(x,y)| \le Ch^{-3}$ (due to Chapter 4) we get that $|F(x,y)| \le Ch^s$ for $|x-y| \ge \epsilon_0$;

Now let us consider general $|x-y| = r \ge Ch$. Rescaling $x-y \mapsto (x-y)r^{-1}$ we need to rescale $h \mapsto hr^{-1}$ and rescaling above inequality and keeping in mind that $F(x,y)$ is a density with respect to x we get $|F(x,y)| \le Ch^s r^{-3-s}$ which is equivalent to (25.A.6)–(25.A.7). □

25.A.4 Calculation of Scott, Dirac and Schwinger correction terms

Consider multiparticle Schrödinger operator (with a single nucleus, but for multiple nuclei our result will be the same) in $\bigwedge_{1\le n\le N} \mathscr{L}^2(\mathbb{R}^3, \mathbb{C}^q)$:

$$\mathsf{H} := \sum_{1\le n\le N} \Big[-\frac{\hbar^2}{2\mathsf{m}}\Delta_{x_n} - \frac{Z\mathsf{e}^2}{|x_n|}\Big] + \sum_{1\le n<n'\le N} \frac{\mathsf{e}^2}{|\mathsf{x}_n - \mathsf{x}_{n'}|}; \tag{25.A.8}$$

dividing by e^2 we get

$$\mathsf{G} := \sum_{1\le n\le N} \Big[-\gamma\Delta_{x_n} - \frac{Z}{|x_n|}\Big] + \sum_{1\le n<n'\le N} \frac{1}{|\mathsf{x}_n - \mathsf{x}_{n'}|}; \tag{25.A.9}$$

with

$$\gamma = \frac{\hbar^2}{2\mathsf{m}\mathsf{e}^2}. \tag{25.A.10}$$

First we consider case $Z = \gamma = q = 1$.

Scott correction term

To calculate Scott correction term we need to calculate the limit as $\eta \to +0$ of

$$\mathsf{Tr}[(H+\eta)^-] + \frac{1}{15\pi^2}\int (V-\eta)_+^{5/2}\,dx \tag{25.A.11}$$

with

$$H = -\Delta - V(x), \qquad V(x) = |x|^{-1} \tag{25.A.12}$$

where the second term in (25.A.10) is the Weyl expression for the first one. Since eigenvalues of H are $-\frac{1}{4n^2}$ of multiplicity n^2, the first term in (25.A.11) is equal to

$$\sum_{n:\ 4n^2\eta<1} \Big(-\frac{1}{4n^2}+\eta\Big)n^2 = -\frac{1}{4}n+\frac{1}{6}n(n+1)(2n+1)\eta \tag{25.A.13}$$

as

$$\eta = \frac{1}{4(n+\alpha)^2}, \qquad 0\le \alpha \le 1 \tag{25.A.14}$$

while the second term in (25.A.10) is equal to

$$\frac{4}{15\pi}\int_0^{\eta^{-1}} (\eta^{-1}-r)^{5/2} r^{-1/2}\,dr = \frac{4}{15\pi}\eta^{-1/2}B(7/2,1/2) = \frac{1}{12}\eta^{-1/2}. \tag{25.A.15}$$

Plugging (25.A.14) into (25.A.13) and (25.A.15) and calculating the limit of (25.A.11) we conclude thar $\kappa_{\mathsf{Scott}} = \frac{1}{2}$.

Dirac correction term

To calculate Dirac correction term we need to calculate a constant, as $d=1$ and $q=1$. To do this it is sufficient to calculate expression (25.A.18) below where $e(x,y,\tau)$ is a Schwartz kernel of the projector for $-\Delta-1$, i.e.

$$e(x,y,0) = \frac{1}{8\pi^3}\int_{|\xi|\le 1} e^{i\langle x-y,\xi\rangle}\,d\xi \tag{25.A.16}$$

and we can assume that $x-y=(r,0,0)$; then

$$e(x,y,0) = \frac{1}{8\pi^3}\int_{|\xi|\le 1} e^{ir\xi_1}d\xi_1 d\xi' = \frac{1}{8\pi^2}\int_{-1}^{1}\cos(r\xi_1)(1-\xi_1^2)\,d\xi_1 = \frac{1}{2\pi^2 r^3}\big(\sin(r)-r\cos(r)\big). \tag{25.A.17}$$

Then

$$\int |x-y|^{-1}|e(x,y,0)|^2\,dy = \frac{1}{\pi^3}\int_0^\infty r^{-5}\big(\sin(r)-r\cos(r)\big)^2\,dr\,.$$

Integrating by parts we get

$$\begin{aligned}&\frac{1}{8\pi^3 r^4}\Big(-2r^2+2r\sin(2r)+\cos(2r)-1\Big)\Big|_{r=0}^{r=\infty}\\ &\qquad=-\frac{1}{2r^4}\Big(-2r^2+2r\big(2r-\frac{1}{6}(2r)^3\big)-\frac{1}{2}(2r)^2+\frac{1}{24}(2r)^4\Big)\Big|_{r=0}\\ &\qquad\qquad=-\frac{1}{8\pi^3 r^4}\Big(-\frac{1}{6}(2r)^4+\frac{1}{24}(2r)^4\Big)=\frac{1}{6\pi^3}.\end{aligned}$$

Therefore,

$$\mathsf{Dirac}=\frac{1}{6\pi^3}(W^{\mathsf{TF}}-\nu)_+^2\,dx=\big(\frac{6}{\pi}\big)^{\frac{1}{3}}\int \rho^{\mathsf{TF}}\,dx \tag{25.A.18}$$

because for $q=\gamma=1$

$$\rho^{\mathsf{TF}}=\frac{q}{6\pi^2}(W^{\mathsf{TF}}-\nu)_+^{3/2} \tag{25.A.19}$$

where ν is a chemical potential.

Schwinger correction term

Let $H=-h^2\Delta+V$ and $a=|\xi|^2-V(x)$ its Weyl symbol. Consider modulo $O(h^4)$ symmetric product

$$\frac{1}{2}\Big(((\tau-a)^{-1})^{\mathsf{w}}(\tau-a)^{\mathsf{w}}+(\tau-a)^{\mathsf{w}}((\tau-a)^{-1})^{\mathsf{w}}\Big)=I-\frac{h^2}{4}B^{\mathsf{w}}$$

with

$$B=\sum_{\alpha\colon|\alpha|=2}\frac{1}{\alpha!}\Big((\tau-a)^{(\alpha)}((\tau-a)^{-1})_{(\alpha)}+(\tau-a)_{(\alpha)}((\tau-a)^{-1})^{(\alpha)}\Big),$$

where b^{w} means Weyl quantization of b, and we used the simple observation that $(\tau-a)^{(\alpha)}_{(\beta)}=0$ as $\alpha\neq 0,\beta\neq 0$

$$\begin{aligned}&=\frac{1}{2}\sum_{j,k}\Big((\tau-a)^{(jk)}((\tau-a)^{-1})_{(jk)}+(\tau-a)_{(jk)}((\tau-a)^{-1})^{(jk)}\Big)\\ &=2\sum_j\Big((\tau-a)^{-2}V_{(jj)}-(\tau-a)^{-3}V_{(j)}^2\Big)+4\sum_{j,k}(\tau-a)^{-3}V_{(jk)}\xi_j\xi_k.\end{aligned}$$

Therefore Weyl symbol of $(\tau-H)^{-1}$ is equal to

$$(\tau-a)^{-1}+\frac{h^2}{4}(\tau-a)^{-1}B$$

and Weyl symbol of the symmetric product $\frac{1}{2}\bigl(\psi(x)(\tau-H)^{-1}+\tau-H)^{-1}\psi(x)\bigr)$ is calculated in the same way

$$(\tau-a)^{-1}\psi+\frac{h^2}{4}(\tau-a)^{-1}\psi B-\frac{h^2}{4}\sum_{\alpha\colon|\alpha|=2}\bigl((\tau-a)^{-1}\bigr)^{(\alpha)}\psi_{(\alpha)}\,,$$

but integration with respect to ξ (since we integrate by parts) kills the last term and we get that the coefficient at $h^2\psi$ is

$$\begin{aligned}&\frac{1}{2}\sum_j\int\Bigl((\tau-a)^{-3}V_{(jj)}-(\tau-a)^{-4}V_{(j)}^2\Bigr)\,d\xi+i\sum_{j,k}\int(\tau-a)^{-4}V_{(jk)}\xi_j\xi_k\,d\xi\\&=\int\Bigl(\frac{1}{2}(\tau-a)^{-3}\Delta V-\frac{1}{2}(\tau-a)^{-4}|\nabla V|^2+\frac{1}{3}(\tau-a)^{-4}|\xi|^2\Delta V\Bigr)\,d\xi\\&=i\partial_\tau^2\int\frac{1}{4}(\tau-a)^{-1}\Delta V\,d\xi+\partial_\tau^3\int(\tau-a)^{-1}\Bigl(\frac{1}{12}|\nabla V|^2-\frac{1}{18}|\xi|^2\Delta V\Bigr)\,d\xi.\end{aligned}$$

Taking real residue we get

$$\begin{aligned}&\partial_\tau^3\int\frac{1}{4}\theta(\tau-a)\Delta V\,d\xi+\partial_\tau^4\int\theta(\tau-a)\Bigl(\frac{1}{12}|\nabla V|^2-\frac{1}{18}|\xi|^2\Delta V\Bigr)\,d\xi\\&=4\pi\partial_\tau^3\Bigl[\frac{1}{6}(\tau+V)_+^{3/2}\Delta V+\frac{1}{18}\partial_\tau(\tau+V)_+^{3/2}|\nabla V|^2-\frac{1}{45}\partial_\tau(\tau+V)_+^{5/2}\Delta V\Bigr]\\&=\frac{\pi}{3}\partial_\tau^2\Bigl[(\tau+V)_+^{1/2}\Delta V+\frac{1}{2}(\tau+V)_+^{-1/2}|\nabla V|^2\Bigr]=\frac{2\pi}{9}\partial_\tau^2\Delta\Bigl[(\tau+V)_+^{3/2}\Bigr]\\&\qquad\qquad=\frac{\pi}{3}\partial_\tau\Delta\Bigl[(\tau+V)_+^{1/2}\Bigr].\end{aligned}$$

Multiplying by $\frac{1}{8\pi^3}\psi$ and integrating with respect to x and τ, plugging $V=W^{\mathsf{TF}}$, $\tau=\nu$ and $\psi=(W^{\mathsf{TF}}-V)$, we get

$$\begin{aligned}\mathsf{Schwinger}&=\frac{\pi}{24\pi^3}\int\Delta(W^{\mathsf{TF}}-V)\cdot W^{\mathsf{TF}\,1/2}\,dx\\&=\frac{1}{24\pi^2}\int 4\pi\rho^{\mathsf{TF}}(6\pi^2\rho^{\mathsf{TF}})^{\frac{1}{3}}\,dx=(6\pi^2)^{-\frac{2}{3}}\int\rho^{\mathsf{TF}\,\frac{4}{3}}\,dx.\end{aligned}\tag{25.A.20}$$

General case

Scaling $x\mapsto\gamma x$ and multiplying by γ we get from operator (25.A.10) the same operator with $\gamma=1$. This preserves Z and we have a scaling $\mathsf{E}\mapsto\gamma^{-1}\mathsf{E}$.

Meanwhile

$$\rho^{\mathsf{TF}}(x) \mapsto \gamma^3 \rho^{\mathsf{TF}}(\gamma x)$$

preserves $N = \int \rho^{\mathsf{TF}}(x)\,dx$) and

$$\int \rho^{\mathsf{TF}\,\frac{4}{3}}\,dx \mapsto \gamma \int \rho^{\mathsf{TF}\,\frac{4}{3}}\,dx, \qquad \int \rho^{\mathsf{TF}\,\frac{5}{3}}\,dx \mapsto \gamma^2 \int \rho^{\mathsf{TF}\,\frac{5}{3}}\,dx,$$
$$\int V\rho^{\mathsf{TF}}\,dx \mapsto \gamma^{-1} \int V\rho^{\mathsf{TF}}\,dx, \qquad \mathsf{D}(\rho^{\mathsf{TF}}, \rho^{\mathsf{TF}}) \mapsto \gamma^{-1}\mathsf{D}(\rho^{\mathsf{TF}}, \rho^{\mathsf{TF}})$$

Therefore

(25.A.21) $\mathcal{E}^{\mathsf{TF}}_{q,Z}$, $\mathsf{Scott}_{q,Z}$, $\mathsf{Dirac}_{q,Z}$ and $\mathsf{Schwinger}_{q,Z}$ could be obtained from the corresponding expressions for $\gamma = 1$ by multiplication by γ^{-1}, while $\int \rho^{\mathsf{TF},\,\frac{4}{3}}\,dx$ and $\int \rho^{\mathsf{TF},\,\frac{5}{3}}\,dx$ are obtained from the corresponding expression by multiplication by γ and γ^2 respectively.

Next we want to look on dependence on q. In Thomas-Fermi theory we have

$$\Delta(W^{\mathsf{TF}} - V) = 4\pi\rho^{\mathsf{TF}} = \frac{2q}{3\pi} W^{\mathsf{TF}\,3/2}, \qquad \int \rho^{\mathsf{TF}} = N/Z. \tag{25.A.22}$$

Therefore

(25.A.23) $\mathcal{E}^{\mathsf{TF}}_{q,Z}$ could be obtained from the corresponding expression for $q = 1$ by scaling with $\gamma = q^{\frac{2}{3}}$ and multiplication by $q^{\frac{1}{3}}$, while $\int \rho^{\mathsf{TF},\,\frac{4}{3}}\,dx$ and $\int \rho^{\mathsf{TF},\,\frac{5}{3}}\,dx$ is obtained from the corresponding expression by multiplication by $q^{\frac{2}{3}}$ and $q^{\frac{4}{3}}$.

Since Scott, Dirac and $\mathsf{Schwinger}$ also are given by Weyl expressions through W^{TF} with factor q we arrive to

Theorem 25.A.4. *As* $\mathsf{e} = 1$

$$\mathcal{E}^{\mathsf{TF}} = \int \left(\kappa\rho^{\mathsf{TF}\,\frac{5}{3}} - V\rho^{\mathsf{TF}}\right) dx + \frac{1}{2}\mathsf{D}(\rho^{\mathsf{TF}}, \rho^{\mathsf{TF}}) \tag{25.A.24}$$

$$\kappa = \frac{6^{\frac{4}{3}}}{15}\pi^{\frac{2}{3}} q^{-\frac{1}{3}}\gamma^{-3},$$

$$\text{Scott} = \frac{1}{2}\gamma^{-1}q \sum_{1\le m\le M} Z_m^2\,, \tag{25.A.25}$$

$$\text{Dirac} = -\big(\frac{6}{\pi q}\big)^{\frac{1}{3}}\gamma^{-2}\int \rho^{\mathsf{TF}\frac{4}{3}}\,dx, \tag{25.A.26}$$

$$\text{Schwinger} = \big(\frac{q}{6\pi^2}\big)^{\frac{2}{3}}\gamma^{-2}\int \rho^{\mathsf{TF}\frac{4}{3}}\,dx. \tag{25.A.27}$$

25.A.5 Baxter's Lemma

Proposition 25.A.5 (Baxter's Lemma) [26]**.** *Let $V \in \mathscr{D}$ and let $\rho(x) \ge 0$ be a given function with $\psi_\rho := |x|^{-1} * \rho \in \mathscr{D}$. Assume $\rho \in \mathscr{L}^p$ for some $p > \frac{3}{2}$ and $\mathsf{D}(\rho,\,\rho) < \infty$. Then there exists g with $0 \le g(x) \le \rho(x)$ such that $\psi_g := |x|^{-1} * g$ satisfies $\psi_g = V(x)$ a.e. when $g(x) < \rho(x)$ and $\psi_g(x) \le V(x)$ a.e.*

Proof. Baxter proves this when p and g are measures. We following E. H. Lieb give a simpler proof for functions. Consider

$$\mathcal{E}(g) := \frac{1}{2}\mathsf{D}(g,g) - \int Vg\,dx \qquad \text{and} \qquad E = \inf_{0\le g\le\rho} \mathcal{E}(g).$$

Let g^n be a minimizing sequence. There exists a subsequence that converges weakly in $\mathscr{L}^p$ to some g and, by Mazur's theorem (1933), there exists a sequence h^n of convex combinations of the g^n that converges strongly to g in $\mathscr{L}^p$. Then a subsequence of the h^n converges a.e. to g.

Clearly, $0 \le h^n(x) \le \rho(x)$. Since $\mathcal{E}(g)$ is convex (this is crucial) $\mathcal{E}(h^n) \to E$ but, by dominated convergence, $\mathcal{E}(h^n) \to \mathcal{E}(g)$. So g minimizes $\mathcal{E}$ and satisfies (a.e.): $V(x) = \psi_g(x)$ when $0 < g(x) < \rho(x)$ and $\psi_g(x) \ge V(x)$ when $g(x) = 0$ and $\rho(x) > 0$.

We have to eliminate the possibility $f(x) = \psi_g(x) - V(x) > 0$ when $g(x) = 0$. We claim $\psi_g \in \mathscr{D}$ and hence $f(x)$ is continuous and goes to zero at infinity. Since $g \le \rho$ we have $\psi_g \le \psi_\rho$ and therefore $\psi_g \to 0$ at infinity.

To examine the continuity at $x = 0$, write $\psi_g = h + (\psi_g - h)$ where $h = |x|^{-1} * (\chi g)$ where χ is the characteristic function of the ball $B(0,1)$. Clearly $\psi_g - h$ is continuous at $x = 0$. Moreover, $\chi g \in \mathscr{L}^p \cap \mathscr{L}^1$ so $h \in \mathscr{D}$ by Lemma 3.1 of *Selecta*. It is here that $p > \frac{3}{2}$ is used.

[26] See J. R. Baxter [1] and E. H. Lieb *Selecta* "Thomas-Fermi and related theories of atoms and molecules", [4], pp. 263–301.

Now, since $f \in \mathscr{D}$, $\Omega = \{\{x\colon f(x) > 0\}$ is open and, since $x \in \Omega \implies g(x) = 0$, f is is subharmonic on Ω. But $f(x)$ vanishes on the boundary of Ω and also at infinity, so Ω is empty. □

25.A.6 Dependence of Thomas-Fermi energy of the nuclear coordinates

Our main goal is to prove Theorem 25.3.8. We follow R. Benguria [1].

Regularized Thomas-Fermi theory

Let ϕ be a non-negative real-valued spherically symmetric function belonging to $\mathscr{C}_0^\infty(\mathbb{R}^+)$, monotone non-increasing function of $|x|$, supported in $B(0,1)$ and $\int \phi(x)\,dx = 1$ and let $\phi_\varepsilon = \varepsilon^{-3}\phi(\varepsilon^{-1}x)$.

For $0 < \varepsilon < a$ (recall that a is the minimal distance between nuclei), define

$$V_\varepsilon(x) = (\phi_\varepsilon * V)(x) \tag{25.A.28}$$

(i.e., V_ε is the Coulomb potential corresponding to smeared nuclei; we have imposed an upper bound on a so that the smeared nuclei do not overlap). Since $V \in \mathscr{L}^1_{\mathsf{loc}}$ $V_\varepsilon \in \mathscr{C}_0^\infty(\mathbb{R}^3)$.

Let us define a regularized Thomas-Fermi theory by the functional

$$\Phi^*_\varepsilon(\rho, \mathsf{y}_1, \dots, \mathsf{y}_M) = \int \tau(\rho)\,dx - \int V_\varepsilon(x)\rho\,dx + \frac{1}{2}\mathsf{D}(\rho,\,\rho) + U_\varepsilon, \tag{25.A.29}$$

$$U_\varepsilon = \sum_{1\le m<m'\le M} Z_m Z_{m'} \int |x - y - \mathsf{y}_m + \mathsf{y}_{m'}|^{-1}\phi_\varepsilon(x)\phi_\varepsilon(y)\,dxdy. \tag{25.A.30}$$

That is, the functional is of exactly the same form as the TF functional, the only change being in that we have smeared the singularities of V(x). We have also changed the constant term U by U_ε which represents the interaction between the smeared nuclei.

The minimization problem associated with Φ^*_s is contained in the family of variational principles studied in E. H. Lieb and B. Simon [1] and H. Brezis and E. H. Lieb [1]. We summarize below those properties of this regularized Thomas-Fermi theorem:

(i) This problem, has a unique solution for $0 < N \le Z$. For $N > Z$ there is no solution. That Z is the largest N for which there is a solution follows from H. Brezis and E. H. Lieb [1], Theorem 3(c), since

$$\int -\Delta V_\varepsilon \, dx = \int (-\Delta V_\varepsilon)_+ \, dx = \sum_{1 \le m \le M} Z_m = Z.$$

(ii) The unique minimizing ρ_ε satisfies the Euler equation

$$\tau'(\rho_\varepsilon) = (W_\varepsilon + \nu)_+ \tag{25.A.31}$$

for some $\nu \le 0$, where

$$W_\varepsilon = V_\varepsilon - |x|^{-1} * \rho_\varepsilon \tag{25.A.32}$$

(iii) $W_\varepsilon(x)$ and thus $\rho_\varepsilon(x)$ are continuous everywhere and they go to zero at infinity. This follows from (25.A.32), since $V_\varepsilon(x)$ is continuous everywhere and goes to zero at infinity and so does $|x|^{-1} * \rho_\varepsilon$ (see E. H. Lieb and B. Simon [1], Theorem II.25) because $\rho_\varepsilon \in \mathscr{L}^{\frac{5}{3}} \cap \mathscr{L}^1$ and $|x|^{-1} \in \mathscr{L}^{\frac{5}{2}} + \mathscr{L}^4$.

Let us define the total energy of the regularized Thomas-Fermi theory by

$$\widehat{\mathcal{E}}_\varepsilon(N, \mathsf{y}_1, \dots, \mathsf{y}_M) = \min\Big(\Phi_\varepsilon(\rho, \mathsf{y}_1, \dots, \mathsf{y}_M) \colon \int \rho_\varepsilon \, dx = N\Big)$$

and the corresponding electronic contribution by

$$\mathcal{E}_\varepsilon(N, \mathsf{y}_1, \dots, \mathsf{y}_M) = \widehat{\mathcal{E}}_\varepsilon(N, \mathsf{y}_1, \dots, \mathsf{y}_M) - U_\varepsilon(\mathsf{y}_1, \dots, \mathsf{y}_M).$$

We are interested in the dependence of $\widehat{\mathcal{E}}_\varepsilon$ and $\mathcal{E}_\varepsilon$ on the nuclear coordinates; in particular let us keep $\mathsf{y}_2, \dots, \mathsf{y}_M$ fixed and study the change on $\widehat{\mathcal{E}}_\varepsilon$ and $\mathcal{E}_\varepsilon$ under the shift $\mathsf{y}_1 \mapsto \mathsf{y}_1 + \alpha \mathsf{n}$ with $\alpha \ge 0$ and $\mathsf{n} \in \mathbb{R}^3$. Denote by

$$\widehat{\mathcal{E}}_\varepsilon(N, \alpha) = \widehat{\mathcal{E}}_\varepsilon(N, \mathsf{y}_1 + \alpha \mathsf{n}, \mathsf{y}_2, \dots, \mathsf{y}_M)$$

and

$$\mathcal{E}_\varepsilon(N, \alpha) = \mathcal{E}_\varepsilon(N, \mathsf{y}_1 + \alpha \mathsf{n}, \mathsf{y}_2, \dots, \mathsf{y}_M).$$

In the same way let us introduce V^α and $V^\alpha_\varepsilon = \phi_\varepsilon * V^\alpha$. We have the following

Theorem 25.A.6 [27]**.** *Let ρ_ε^α be the unique minimizing ρ for $\Phi_\varepsilon^*(\rho)$ with y_1 replaced by $\mathsf{y}_1 + \alpha\mathsf{n}$ and $\int \rho_\varepsilon^\alpha \, dx = N$. Then, the functions $\alpha \to \widehat{\mathcal{E}}_\varepsilon(N, \alpha)$ and $\alpha \to \mathcal{E}_\varepsilon(N, \alpha)$ are continuously differentiable and*

$$\frac{\partial \mathcal{E}_\varepsilon(N,\alpha)}{\partial \alpha} = -\int \frac{\partial V_\varepsilon^\alpha(x)}{\partial \alpha} \rho_\varepsilon^\alpha(x)\, dx \tag{25.A.33}$$

$$= Z_1 \int (\mathsf{n} \cdot \nabla \phi_\varepsilon(y)) \psi^\alpha(y + \mathsf{y}_1 + \alpha\mathsf{n})\, dy \tag{25.A.34}$$

and

$$\frac{\partial \widehat{\mathcal{E}}_\varepsilon(N,\alpha)}{\partial \alpha} = -\, Z_1 \int (\mathsf{n} \cdot \nabla \phi_\varepsilon(y)) \varphi^\alpha(y + \mathsf{y}_1 + \alpha\mathsf{n})\, dy \tag{25.A.35}$$

where $\psi^\alpha := |x|^{-1} * \rho^\alpha$ and $\varphi^\alpha = V_\varepsilon^\alpha - \psi^\alpha$.

Proof. Let us first prove that the mapping α to the right-hand expression of (25.A.33) is continuous. We need only prove continuity at $\varepsilon = 0$. Note that $V_\varepsilon^\alpha \to V_\varepsilon$ in $\mathscr{L}^{\frac{5}{2}} + \mathscr{L}^4$ (Lemma 25.A.12(i)), and therefore $\rho^\alpha \to \rho^0$ in $\mathscr{L}^{\frac{5}{3}}$ because of Theorem II.15 in [1]. Since $\|\rho^\alpha\|_{\mathscr{L}^1} \le N$ we also have $\rho^\alpha \to \rho^0$ in weak $\mathscr{L}^p$ for any p: $1 < p < \frac{5}{3}$. Let $Y^\alpha := \dfrac{\partial V_\varepsilon^\alpha(x)}{\partial \alpha}$, we have

$$\int Y^\alpha \rho^\alpha \, dx - \int Y^0 \rho^0 \, dx = \int (Y^\alpha - Y^0)\rho^\alpha \, dx + \int Y^0(\rho^\alpha - \rho^0)\, dx. \tag{25.A.36}$$

The first term in the right side of (25.A.36) goes to 0 as $\alpha \to 0^+$ because $Y^\alpha \to Y^0$ in $\mathscr{L}^{\frac{5}{2}} + \mathscr{L}^4$ and ρ^α is bounded in $\mathscr{L}^{\frac{5}{3}} \cap \mathscr{L}^{\frac{4}{3}}$. Also the second term in in the right side of (25.A.36) goes to 0 as $\alpha \to 0^+$ because $Y^0 \in \mathscr{L}^{\frac{5}{2}} + \mathscr{L}^4$ and $\rho^\alpha \to \rho^0$ in weak $\mathscr{L}^{\frac{4}{3}}$. Hence continuity of the right-hand expression has been established.

Now, we prove differentiability of $\widehat{\mathcal{E}}_\varepsilon(N, \alpha)$. Again we need only prove it at $\alpha > 0$. For $\alpha > 0$, $\widehat{\mathcal{E}}_\varepsilon(N, \alpha) - \widehat{\mathcal{E}}_\varepsilon(N, 0)$ does not exceed

$$\Big(\Phi_\varepsilon^*(\rho^0, \mathsf{y}_1 + \alpha\mathsf{n}, \mathsf{y}_2, \dots, \mathsf{y}_M) - U_\varepsilon(y_1 + \alpha\mathsf{n}, \mathsf{y}_2, \dots, \mathsf{y}_M\Big) - \Big(\Phi_\varepsilon^*(\rho^0, \mathsf{y}_1, \mathsf{y}_2, \dots, \mathsf{y}_M) - U_\varepsilon(y_1 \mathsf{y}_2, \dots, \mathsf{y}_M\Big)$$

[27] A Feynman-Heltman theorem for regularized Thomas-Fermi.

by the minimization property of ρ^α, and it is equal to

$$-\int \Big(V_\varepsilon^\alpha - V_\varepsilon\Big)\rho^\alpha \, dx. \tag{25.A.37}$$

Also, by the minimization property of ρ^0, $\widehat{\mathcal{E}}_\varepsilon(N,\alpha) - \widehat{\mathcal{E}}_\varepsilon(N,0)$ is larger than or equal to

$$-\int \Big(V_\varepsilon^\alpha - V_\varepsilon\Big)\rho^0 \, dx. \tag{25.A.38}$$

Now, $\alpha^{-1}\big(V_\varepsilon^\alpha - V_\varepsilon\big) \to \frac{V_\varepsilon^\alpha}{\partial\alpha}\big|_{\alpha=0}$ in $\mathscr{L}^{\frac{5}{2}} + \mathscr{L}^4$ (see Lemma 25.A.12(ii)) and $\rho^0 \in \mathscr{L}^{\frac{5}{3}} \cap \mathscr{L}^1$. Hence by Hölder inequality, expression (25.A.37) divided by α converges to $-\int \frac{V_\varepsilon^\alpha}{\partial\alpha}\big|_{\alpha=0}\rho^0 \, dx$. Similarly, one can prove that expression (25.A.38) divided by α converges to the same limit as $\alpha \to 0^+$ and, therefore,

$$\lim_{\alpha\to 0^+}\Big(\widehat{\mathcal{E}}_\varepsilon(N,\alpha) - \widehat{\mathcal{E}}_\varepsilon(N,0)\Big) = -\int \frac{V_\varepsilon^\alpha}{\partial\alpha}\big|_{\alpha=0}\rho^0 \, dx. \tag{25.A.39}$$

A similar argument controls the same limit as $\alpha \to 0^-$. Thus $\widehat{\mathcal{E}}_\varepsilon(N,\alpha)$ is continuously differentiable in α and its derivative is given by (25.A.33).

In order to prove (25.A.34), we just compute

$$\frac{\partial V_\varepsilon^\alpha}{\partial\alpha}(x) = Z_1 \int \phi_\varepsilon(y)\mathsf{n}\cdot\nabla_y|x-y-\mathsf{y}_1-\alpha\mathsf{n}|^{-1}\, dy$$

and after introducing this expression in (25.A.33), we integrate by parts.

Finally, since $U_\varepsilon(\mathsf{y}_1+\alpha\mathsf{n},\mathsf{y}_2,\dots,\mathsf{y}_M)$ is continuously differentiable with respect to α (see Lemma 25.A.12(iii)), $\alpha \to \mathcal{E}_\varepsilon(N,\alpha)$ is continuously differentiable.

Moreover,

$$\begin{aligned}\frac{\partial U_\varepsilon(\mathsf{y}_1+\alpha\mathsf{n},\mathsf{y}_2,\dots\mathsf{y}_M)}{\partial\alpha} \\ = -Z_1\int (\mathsf{n}\cdot\nabla\phi_\varepsilon(y))\, V_\varepsilon^\alpha(y+\mathsf{y}_1+\alpha\mathsf{n})\, dy\end{aligned} \tag{25.A.40}$$

and adding the right sides of (25.A.34) and (25.A.40) we get the right side of (25.A.35). □

Proof of Theorem 25.3.8

We will need some geometric preliminaries. For n, a unit vector in $\mathbb{R}^3$ and $d \geq 0$, let

$$L(\mathsf{n}, d) = \{x \in \mathbb{R}^3 \colon x \cdot \mathsf{n} = d\} \quad \text{and} \quad L^{\pm}(\mathsf{n}, d) = \{x \in \mathbb{R}^3 \colon x \cdot \mathsf{n} \lessgtr d\}.$$

For any function $f \colon \mathbb{R}^3 \to \mathbb{R}$ we define

$$f^+(x) = f(x), \quad \text{and} \quad f^-(x) = f(x + 2(d - x \cdot \mathsf{n})\mathsf{n}),$$

for every $x \in L^+(\mathsf{n}, d) \cup L(\mathsf{n}, d)$, i.e., $f^-(x)$ is just the reflection of f through the plane $L(\mathsf{n}, d)$.

In order to prove Theorem 25.3.8 we will need the following:

Lemma 25.A.7 [28]**.** *Let* n *and* d *be such that* $\mathsf{y}_m \in L^+(\mathsf{n}, d) \cup L(\mathsf{n}, d)$ *for all* $1 \leq m \leq M$ *and choose*

$$\varepsilon < \min_{1 \leq m \leq M} \big(\mathsf{dist}(\mathsf{y}_m,\, L(\mathsf{n}, d)) \colon \mathsf{y}_m \notin L(\mathsf{n}, d)\big)$$

this condition on a *insures that the smeared nuclei with* $\mathsf{y}_m \notin L(\mathsf{n}, d)$ *are supported on* $L^+(\mathsf{n}, d)$*).*

Let $\rho_\varepsilon(x)$ *be the unique solution to the regularized TF equation* (25.A.31) *and* W_ε *the corresponding potential* (25.A.32)*. Then*

(i) $h_\varepsilon(x) \coloneqq W_\varepsilon^+(x) - W_\varepsilon^-(x) > 0$ *for all* $x \in L^+(\mathsf{n}, d)$*.*

(ii) For each $x \in L^+(\mathsf{n}, d)$*,* $h_\varepsilon(x)$ *strictly decreases when* N *increases, and*

(iii) $\rho_\varepsilon^+(x) - \rho_\varepsilon^-(x) \geq 0$ *for all* $x \in L^+(\mathsf{n}, d)$*.*

Proof. (i) Certainly $h_\varepsilon(x) = 0$ for $x \in \partial L^+(\mathsf{n}, d) = L(\mathsf{n}, d)$ and at infinity. Let $D = \{x \in L^+(\mathsf{n}, d) \colon h_\varepsilon(x) < 0\}$. Since $W_\varepsilon(x)$ is continuous everywhere, D is open. On D, $-(4\pi)^{-1}\Delta h_\varepsilon \geq \rho_\varepsilon^-(x) - \rho_\varepsilon^+(x) > 0$ since, through (25.A.31), $\varphi_\varepsilon^-(x) > \varphi_\varepsilon^+(x)$ implies $\rho_\varepsilon^-(x) - \rho_\varepsilon^+(x) > 0$. Thus, $h_\varepsilon(x)$ is superharmonic on D, so D is empty.

By the strong maximum principle, $h_\varepsilon(x) > 0$, in fact, for $x \in L^+(\mathsf{n}, d)$.

[28] Lemma 3 of R. Benguria [1].

(ii) Let $N' < N$ with corresponding $h'_\varepsilon(x)$ and $h_\varepsilon(x)$. We want to prove that $B := \{x \in L^+(\mathsf{n}, d)\colon h_\varepsilon(x) - h'_\varepsilon(x) > 0\}$ is empty. Since the h_ε and h'_ε are continuous, B is open and $h_\varepsilon - h'_\varepsilon = 0$ on $L(\mathsf{n}, d)$ and at infinity. We have

$$-(4\pi)^{-1}\Delta(h_\varepsilon - h'_\varepsilon) = -b_+^{\frac{3}{2}} + c_+^{\frac{3}{2}} + p_+^{\frac{3}{2}} - q_+^{\frac{3}{2}} =: r$$

with

$$b = W_\varepsilon^+ - \mu, \quad b = W_\varepsilon^- - \nu, \quad p = W'^{+_\varepsilon} - \nu', \quad q = W_\varepsilon'^{-s} - \nu'.$$

By (i) (E. H. Lieb and B. Simon [1], Theorem V.9), $b > c \geq q$ and $b \geq p > q$, for all $x \in L^+(\mathsf{n}, d)$. On B, $b + q > c + p$. Thus $r < 0$ on B, whence $h_\varepsilon(x) - h'_\varepsilon(x)$ is subharmonic on B and therefore B is empty. Again, once can prove the stronger result that $h_\varepsilon(x) - h'_\varepsilon(x) < 0$ for $x \in L^+(\mathsf{n}, d)$. T

(iii) Trivially, (i) implies (iii) through the Tomas-Fermi equation (25.A.31). □

Remark 25.A.8. The proof of Lemma 25.A.7(i) is analogous to the proof of Teller's lemma (E. H. Lieb and B. Simon [1], Theorem V.8).

By the same argument one can prove the following lemma:

Lemma 25.A.9 [29]**.** *Let* n *and* d *be such that* $\mathsf{y}_m \in L^+(\mathsf{n}, d)$ *for all* $m = 1, \dots, M$*, let* $\rho(x)$ *be Thomas-Fermi density and* V *be the Coulomb potential. Then*

$$\rho^+(x) > \rho^-(x) \qquad \forall x \in L^+(\mathsf{n}, d)$$

and

$$V^+(x) > V^-(x) \qquad \forall x \in L^+(\mathsf{n}, d).$$

We will now prove the analog of Theorem 25.3.8 for the regularized Thomas-Fermi model:

Theorem 25.A.10 [30]**.** *Fix* Z_m*,* $m = 1, \dots, M$ *and* $N \leq Z$*. Fix*m *and assume that* y_m *is such that*

$$\mathsf{y}_{m'} \in L^+(\mathsf{n}, \mathsf{n} \cdot \mathsf{y}_m) \cup L(\mathsf{n}, \mathsf{n} \cdot \mathsf{y}_m) \qquad \forall m' \neq m \tag{25.A.41}$$

[29] Lemmas 4, 5 of R. Benguria [1].
[30] Theorem 6 of R. Benguria [1].

for some fixed unit vector $\mathsf{n} \in \mathbb{R}^3$. *Let*

$$\mathcal{E}_\varepsilon(N, \alpha) \coloneqq \mathcal{E}_\varepsilon(N, \mathsf{y}_1, \dots, \mathsf{y}_{m-1}\mathsf{y}_m + \alpha\mathsf{n}, \mathsf{y}_{m+1}, \dots, \mathsf{y}_M)$$

and

$$\widehat{\mathcal{E}}_\varepsilon(N, \alpha) \coloneqq \widehat{\mathcal{E}}_\varepsilon(N, \mathsf{y}_1, \dots, \mathsf{y}_{m-1}\mathsf{y}_m + \alpha\mathsf{n}, \mathsf{y}_{m+1}, \dots, \mathsf{y}_M).$$

Then

(i) $\widehat{\mathcal{E}}_\varepsilon(N, \alpha)$, *is a monotonic non-increasing function of* $\alpha \geq 0$;

(ii) $\mathcal{E}_\varepsilon(N, \alpha)$ *is a monotonic non-decreasing function of* $\alpha \geq 0$;

(iii) For fixed $\alpha > 0$, $\widehat{\mathcal{E}}_\varepsilon(N, \alpha) - \widehat{\mathcal{E}}_\varepsilon(N, 0)$ *is non-decreasing in* N,

(iv) For fixed $\alpha > 0$ $\mathcal{E}_\varepsilon(N, \alpha) - \mathcal{E}_\varepsilon(N, 0)$ *is non-decreasing in* N.

Proof. (i) By the Feynman-Hellman Theorem 25.A.6 [see (25.A.35)],

$$\frac{\partial \widehat{\mathcal{E}}_\varepsilon(N, \alpha)}{\partial \alpha} = -Z_1 \int (\mathsf{n} \cdot \nabla \phi_\varepsilon(y)) \varphi^\alpha(y + \mathsf{y}_1 + \alpha\mathsf{n})\, dy. \tag{25.A.42}$$

We can write the right side of (25.A.42) with $\alpha = 0$ as

$$-Z_m \int_{y \in L^+(\mathsf{n}, \mathsf{n}\cdot\mathsf{y}_m)} \big(\mathsf{n} \cdot \nabla \phi_\varepsilon(y - \mathsf{y}_m)\big) \varphi_\varepsilon(y)\, dy \tag{25.A.43}$$

because $\mathsf{n} \cdot \nabla \phi_\varepsilon(y - \mathsf{y}_m)$ only changes sign under a reflection (of y) through $L(\mathsf{n}, \mathsf{n} \cdot \mathsf{y}_m)$. Here $\varphi_\varepsilon^\pm$ are defined as in (25.A.41) with respect to the plane $L(\mathsf{n}, \mathsf{n} \cdot \mathsf{y}_m)$.

Note that $\mathsf{n} \cdot \nabla \phi_\varepsilon(y - \mathsf{y}_m) \geq 0$ for all $y \in L^+(\mathsf{n}, \mathsf{n} \cdot \mathsf{y}_m) \cup L(\mathsf{n}, \mathsf{n} \cdot \mathsf{y}_m)$ because $\phi(y)$ is spherically symmetric and decreasing by $|y|$ and $\varphi_\varepsilon^+(y) \geq \varphi_\varepsilon^-(y)$ for all $y \in L^+(\mathsf{n}, \mathsf{n} \cdot \mathsf{y}_m) \cup L(\mathsf{n}, \mathsf{n} \cdot \mathsf{y}_m)$ (Lemma 25.A.7(i)), expression (25.A.43) is non-positive and Statement (i) follows.

(ii) By integration by parts one can write (25.A.33) as

$$\frac{\partial \mathcal{E}_\varepsilon(N, \alpha)}{\partial \alpha} = Z_j \int p_j(x) \rho_\varepsilon(x)\, dx$$

with

$$p_m(x) = \mathsf{n} \cdot \nabla_v x \int \phi_\varepsilon(x - y) |y - \mathsf{y}_m|^{-1}\, dy.$$

As before, we have as $\alpha = 0$

$$\frac{\partial \mathcal{E}_\varepsilon(N,\alpha)}{\partial \alpha} = Z_m \int_{x \in L^+(\mathsf{n},\mathsf{n}\cdot \mathsf{y}_m)} p_j(x)\big(\rho_\varepsilon^+(x) - \rho_\varepsilon^-(y)\big)\, dx \tag{25.A.44}$$

because $p_m(x)$ only changes sign under a reflection (of x) through $L(\mathsf{n}, \mathsf{n}\cdot \mathsf{y}_m)$. Moreover, $p_m(x) \geq 0$ for all $x \in L^+(\mathsf{n}, \mathsf{n}\cdot \mathsf{y}_m) \cup L(\mathsf{n}, \mathsf{n}\cdot \mathsf{y}_m)$ again because $\phi_\varepsilon * |x - \mathsf{y}_m|^{-1}$ is spherically symmetric and decreasing by $|x|$). Hence, Statement (ii) follows from (25.A.44) and Lemma 25.A.7(iii).

(iii) Write $\widehat{\mathcal{E}}_\varepsilon(N,\alpha) - \widehat{\mathcal{E}}_\varepsilon(N,0) = \int_0^\alpha \frac{\partial \widehat{\mathcal{E}}_\varepsilon(N,\alpha)}{\partial \alpha}\, d\alpha$. Now, for every fixed α integrand is an increasing function of N because of (25.A.43) and Lemma 25.A.7(ii). This proves Statement (iii). Obviously Statements (iii) and (iv) are equivalent. □

After all these preliminaries we can go to the

Proof of Theorem 25.3.8. Due to Lemma 25.A.12(iv), $V_\varepsilon \to V$ as $s \to 0^+$ in $\mathscr{L}^{\frac{5}{2}} + \mathscr{L}^4$.

Therefore, Theorem II.15 in E. H. Lieb and B. Simon [1] implies that, for fixed N, $\mathcal{E}_\varepsilon(N, \mathsf{y}_1, \dots, \mathsf{y}_M) \to \mathcal{E}_\varepsilon(N, \mathsf{y}_1, \dots, \mathsf{y}_M)$ as $\varepsilon \to 0^+$. Hence Statements (ii) and (iv) of Theorem 25.3.8 follow from Statements (ii) and (iv) of Theorem 25.A.10.

Now, by Lemma 25.A.12(v) $U_\varepsilon \to U$ as $\varepsilon \to 0^+$, hence for fixed N, $\widehat{\mathcal{E}}_\varepsilon(N, \mathsf{y}_1, \dots, \mathsf{y}_M) \to \widehat{\mathcal{E}}_\varepsilon(N, \mathsf{y}_1, \dots, \mathsf{y}_M)$ as $\varepsilon \to 0^+$. Hence Statements (i) and (iii) of Theorem 25.3.8 follow from Statements (i) and (iii) of Theorem 25.A.10. □

Electronic contribution to the Born-Oppenheimer energy

The electronic contribution to the Born-Oppenheimer energy for a system of $N = 1$ electron and M (fixed) nuclei located at $\mathsf{y}_m \in \mathbb{R}^3$, with positive charge Z_m is defined by E. H. Lieb and B. Simon [2]

$$\mathsf{E}(\mathsf{y}_1, \dots, \mathsf{y}_M) \coloneqq \inf \mathsf{Spec}(H) \tag{25.A.45}$$

with

$$H = -\Delta - V(x), \tag{25.A.46}$$

The behavior of $\mathsf{E}(\mathsf{y}_1, \dots, \mathsf{y}_M)$ as a function of the nuclear coordinates has been studied in E. H. Lieb and B. Simon [2] and T. Hoffman-Ostenhof [1]. In

particular, it has been shown that $\mathsf{E}(\mathsf{y}_1, \dots, \mathsf{y}_M)$ is monotonic non-decreasing under the dilation $\mathsf{y}_m \mapsto t\mathsf{y}_m$ $(t > 1)$, for all m. Here we add the following result

Theorem 25.A.11 [31]**.** *Fix Z_m, $m = 1, \dots, M$. Fix m and assume that y_m is such that*

$$(\mathsf{y}_m - \mathsf{y}_{m'}) \cdot \mathsf{n} \geq 0 \qquad \forall m' \neq m \tag{25.A.47}$$

for some fixed unit vector $\mathsf{n} \in \mathbb{R}^3$. Then $\mathsf{E}(\mathsf{y}_1, \dots, \mathsf{y}_{m-1}, \mathsf{y}_m + \alpha\mathsf{n}, \mathsf{y}_{m+1}, \dots \mathsf{y}_M)$ is a monotonic non-decreasing function of $\alpha \geq 0$.

Auxiliary statements

Let $f \in \mathscr{L}^{\frac{5}{2}} + \mathscr{L}^4)$ and define $f_\varepsilon = \phi_\varepsilon * f$, with ϕ_ε defined above.

Lemma 25.A.12 [32]**.** *(i) Let $f_\varepsilon^\alpha := f_\varepsilon(x - \alpha\mathsf{y})$, $\alpha > 0$, $\mathsf{y} \in \mathbb{R}^3$. Then $f_\varepsilon^\alpha \to f_\varepsilon$ in $(\mathscr{L}^{\frac{5}{2}} + \mathscr{L}^4)$ as $\alpha \to 0^+$.*

(ii) $\alpha^{-1}(f_\varepsilon^\alpha - f_\varepsilon) \to \frac{\partial f_\varepsilon^\alpha}{\partial \alpha}\big|_{\alpha=0}$.

(iii) $U_\varepsilon^\alpha := U_\varepsilon(\mathsf{y}_1, \dots, \mathsf{y}_{m-1}, \mathsf{y}_m + \alpha\mathsf{n}, \mathsf{y}_{m+1}, \dots \mathsf{y}_M)$ is continuousy differentiable by α and

$$\frac{\partial U_\varepsilon^\alpha}{\partial \alpha} = -Z_m \int (\mathsf{n} \cdot \phi_\varepsilon(y)) V_\varepsilon^\alpha(y + \mathsf{y}_m + \alpha\mathsf{n})\, dy. \tag{25.A.48}$$

(iv) $f_\varepsilon \to f$ in $(\mathscr{L}^{\frac{5}{2}} + \mathscr{L}^4)$ as $\varepsilon \to 0^+$.

(v) $U_\varepsilon \to U$ as $\varepsilon \to 0^+$.

Comments

There are papers of physicists. L.H. Thomas and E. Fermi have suggested in 1927 that a large Coulomb system (atom or molecule) in the ground state looks like a classical gas but with the Pauli principle, so, leading to the first term $\mathcal{E}^{\mathsf{TF}}$ in the asymptotics. The second term of asymptotics was conjectured by J. M.C. Scott in 1952 as a contribution of those electrons which move very close to the nuclei. Next terms, Dirac and Schwinger corrections were conjectured in 1930 and 1980, respectively.

[31] Theorem 7 of R. Benguria [1].

[32] Lemmas A.1–A.5 of R. Benguria [1].

The mathematical rigorous papers one can separated into several groups:

First, there are papers concerning only the Thomas–Fermi model (so, studying the Thomas-Fermi equation, may be, with some modifications, without any consideration of the quantum mechanical model, even if the latter was a source for the former). Most notably H. Brezis, H. and E. Lieb E. [1], R. Benguria [1], R. Benguria, R. and Lieb E. H. [1].

The second group consists of the papers, justifying Thomas–Fermi model as an approximation to the quantum mechanical model: E. H. Lieb and B. Simon, [1], where the leading term was derived; also certain properties of the the Thomas–Fermi model were established.

Next, W. Hughes [1] and H. Siedentop and R. Weikart [1–3] justified the Scott correction term in the atomic case, while V. Ivrii and M. Sigal [1] justified it in the molecular case.

Then, C. Fefferman and L. Seco [2] justified Dirac and Schwinger correction terms in the atomic case, while V. Ivrii justified them in [21] (even in the case of the relatively weak magnetic field). J. P. Solovej, J. P., T. Ø. Sørensen and W. L. Spitzer [1] recovered Scott correction term in the relativistic case (under assumption preventing relativistic instability).

The third group consists of the papers, related to the ground state energy problem: B. Ruskai and J. P. Solovej [1], J. P. Solovej [1] and L. A. Seco, I. M. Sigal, and J. P. Solovej [1].

Finally, we already mentioned papers which provided the solid functional-analytical base for all this construction. Pretty complete survey could be found in C. L. Fefferman, V. Ivrii, L. A. Seco, and I. M. Sigal [1].

Addendum

Further development (with references) could be found in several articles (in the same volume of our online monograph)

1. *Upper estimates for the electronic density in heavy atoms and molecules*,
2. *Scott and Thomas-Fermi approximations to electronic density* (survey),
3. *Asymptotics of the ground state energy in the relativistic settings*,
4. *Asymptotics of the ground state energy in the relativistic settings and with self-generated magnetic field*

and references there.

Important Remarks

Sections 25.5 and 25.6 were rewritten. The reason is that the printed version of this monograph contains crucial errors in the proofs of Theorem 25.5.2 (see Theorem 25.5.3 here) and Theorem 25.6.3.

1. The first error in the proof of Theorem 25.5.2 occurred only in the case $M \geq 2$. It has been corrected.

2. The second one has not been corrected and we cannot prove the estimate of the excessive negative charge better than $O(Z^{\frac{5}{7}})$ even for the case $M = 1$. The reason is that we need to use $Q \geq Z^{\frac{5}{3}}$ in Proposition 25.5.1.

3. The error in the proof of Theorem 25.6.3 has been fixed but this Theorem became much weaker, with the estimate $O(Z^{\frac{11}{13}})$ instead of $O(Z^{\frac{5}{7}-\delta})$.

All these errors propagated to Sections 26.7 and 26.8 will be corrected as well. Some other improvement were done: added proof of Theorem 25.3.8 in Appendix 25.A.6.

Chapter 26

The Case of External Magnetic Field

26.1 Introduction

In this Chapter we repeat analysis of the previous Chapter 25 but in the case of the constant external magnetic field[1)].

26.1.1 Framework

Let us consider the following operator (quantum Hamiltonian)

$$\mathsf{H} = \mathsf{H}_N \coloneqq \sum_{1\le j\le N} H_{A,V,x_j} + \sum_{1\le j<k\le N} |x_j - x_k|^{-1} \tag{26.1.1}$$

on

$$\mathfrak{H} = \bigwedge_{1\le n\le N} \mathsf{H}, \qquad \mathsf{H} = \mathscr{L}^2(\mathbb{R}^d, \mathbb{C}^q) \tag{26.1.2}$$

with

$$H_{V,A} = \big((i\nabla - A)\cdot\boldsymbol{\sigma}\big)^2 - V(x) \tag{26.1.3}$$

describing N same type particles in the external field with the scalar potential $-V$ and vector potential $A(x)$, and repulsing one another according to the Coulomb law.

[1)] Actually we need a magnetic field either sufficiently weak or close to a constant on the very small scale.

Here $x_j \in \mathbb{R}^d$ and $(x_1, \dots, x_N) \in \mathbb{R}^{dN}$, potentials $V(x)$ and $A(x)$ are assumed to be real-valued. Except when specifically mentioned we assume that

$$V(x) = \sum_{1\le k\le M} \frac{Z_m}{|x - \mathsf{y}_m|} \tag{26.1.4}$$

where $Z_m > 0$ and y_m are charges and locations of nuclei. Here $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_d)$, σ_k are $q \times q$-Pauli matrices.

So far in comparison with the previous Chapter 25 we only changed (25.1.3) to (26.1.3) introducing magnetic field. Now spin enters not only in the definition of the space but also into operator through matrices σ_k. Since we need $d = 3$ Pauli matrices it is sufficient to consider $q = 2$ but we will consider more general case as well (but q should be even).

Remark 26.1.1. In the case of the the constant magnetic field $\nabla \times A$

$$H_{A,V} = \bigl(-i\nabla - A(x)\bigr)^2 + \boldsymbol{\sigma}\cdot(\nabla \times A) - V(x) \tag{26.1.5}$$

In the case $d = 2$ this operator downgrades to

$$H_{A,V} = \bigl(-i\nabla - A(x)\bigr)^2 + \sigma_3(\nabla \times A) - V(x) \tag{26.1.6}$$

Again, let us assume that

(26.1.7) Operator H is self-adjoint on $\mathfrak{H}$.

As usual we will never discuss this assumption.

26.1.2 Problems to Consider

As in the previous Chapter we are interested in the *ground state energy* $\mathsf{E} = \mathsf{E}_N$ of our system i.e. in the lowest eigenvalue of the operator $\mathsf{H} = \mathsf{H}_N$ on $\mathfrak{H}$:

$$\mathsf{E} := \inf \operatorname{Spec} \mathsf{H} \qquad \text{on } \mathfrak{H}; \tag{26.1.8}$$

more precisely, we are interested in the asymptotics of $\mathsf{E}_N = \mathsf{E}(\underline{\mathsf{y}}; \underline{Z}; N)$ as V is defined by (26.1.4) and $N \asymp Z := Z_1 + Z_2 + \dots + Z_M \to \infty$ and we are going

to prove that[2] E is equal to *Magnetic Thomas-Fermi energy* $\mathcal{E}_B^{\mathsf{TF}}$, possibly with the Scott , Dirac and Schwinger correction terms given by (25.1.28), (25.1.29) and (25.1.30) with $q = 2$ respectively and with an appropriate error.

We are also interested in the asymptotics for the *ionization energy*

$$\mathsf{I}_N := \mathsf{E}_{N-1} - \mathsf{E}_N \tag{26.1.9}$$

and we also would like to estimate *maximal excessive negative charge*

$$\max_{N:\, \mathsf{I}_N>0} (N - Z). \tag{26.1.10}$$

All these questions so far were considered in the framework of the fixed positions $\mathsf{y}_1, \dots, \mathsf{y}_M$ but we can also consider

$$\widehat{\mathsf{E}} := \widehat{\mathsf{E}}_N = \widehat{\mathsf{E}}(\underline{\mathsf{y}}; \underline{Z}; N) = \mathsf{E} + U(\underline{\mathsf{y}}; \underline{Z}) \tag{26.1.11}$$

with

$$U(\underline{\mathsf{y}}; \underline{Z}) := \sum_{1\le m<m'\le M} \frac{Z_m Z_{m'}}{|\mathsf{y}_m - \mathsf{y}_{m'}|} \tag{26.1.12}$$

and

$$\widehat{\mathsf{E}}(\underline{Z}; N) = \inf_{\mathsf{y}_1,\dots,\mathsf{y}_M} \widehat{\mathsf{E}}(\underline{\mathsf{y}}; \underline{Z}; N) \tag{26.1.13}$$

and replace I_N by $\widehat{I}_N = \widehat{\mathsf{E}}_{N-1} - \widehat{\mathsf{E}}_N$ and modify all our questions accordingly. We call these frameworks *fixed nuclei model* and *free nuclei model* respectively.

In the free nuclei model we can consider two other problems:

(a) Estimate from below *minimal distance between nuclei* i.e.

$$\min_{1\le m<m'\le M} |\mathsf{y}_m - \mathsf{y}_{m'}|$$

for which such minimum is achieved.

(b) Estimate *maximal excessive positive charge*

$$\max_N \bigl\{(Z - N)\colon \widehat{\mathsf{E}} < \min_{\substack{N_1,\dots,N_M:\\ N_1+\dots N_M=N}} \sum_{1\le m\le M} \mathsf{E}(Z_m; N_m)\bigr\} \tag{26.1.14}$$

for which molecule does not disintegrates into atoms.

[2] Under reasonable assumption to the minimal distance between nuclei.

26.1.3 Magnetic Thomas-Fermi Theory

As in the previous Chapter 25 the first approximation is the Hartree-Fock (or Thomas-Fermi) theory. Let us introduce the *spacial density* of the particle with the state $\Psi \in \mathfrak{H}$:

$$\rho(x) = \rho_\Psi(x) = N \int |\Psi(x, x_2, \dots, x_N)|^2 \, dx_2 \cdots dx_N. \tag{26.1.15}$$

Let us write the Hamiltonian, describing the corresponding "quantum liquid":

$$\mathcal{E}_B(\rho) = \int \tau_B(\rho(x)) \, dx - \int V(x)\rho(x) \, dx + \frac{1}{2}\mathsf{D}(\rho, \rho), \tag{26.1.16}$$

with

$$\mathsf{D}(\rho, \rho) = \iint |x - y|^{-1} \rho(x)\rho(y) \, dxdy \tag{26.1.17}$$

where τ_B is the energy density of a gas of noninteracting electrons:

$$\tau_B(\rho) = \sup_{w \geq 0} \bigl(\rho w - P_B(w)\bigr) \tag{26.1.18}$$

is the Legendre transform of the *pressure* $P_B(w)$ given by the formula

$$P_B(w) = \varkappa_1 B \Bigl(\frac{1}{2} w_+^{\frac{d}{2}} + \sum_{j \geq 1} (w - 2jB)_+^{\frac{d}{2}}\Bigr) \tag{26.1.19}$$

with $\varkappa_1 = (2\pi)^{-1} q, (3\pi^2)^{-1} q$ for $d = 2, 3$ respectively.

The classical sense of the second and the third terms in the right-hand expression of (26.1.16) is clear and the density of the kinetic energy is given by $\tau_B(\rho)$ in the semiclassical approximation (see remark 26.1.2). So, the problem is

(26.1.20) Minimize functional $\mathcal{E}_B(\rho)$ defined by (26.1.16) under restrictions:

$$\rho \geq 0, \qquad \int \rho \, dx \leq N. \tag{26.1.21}_{1,2}$$

The solution if exists is unique because functional $\mathcal{E}_B(\rho)$ is strictly convex (see below). The existence and the property of this solution denoted further by ρ_B^{TF} is known in the series of physically important cases.

Remark 26.1.2. If w is the negative potential then

$$\operatorname{tr} e(x,x,0) \approx P'_B(w) \tag{26.1.22}$$

defines the density of all non-interacting particles with negative energies at point x and

$$\int_{-\infty}^{0} \tau\, d_\tau \operatorname{tr} e(x,x,\tau) dx \approx -\int P_B(w)\, dx \tag{26.1.23}$$

is the total energy of these particles; here $\approx$ means "in the semiclassical approximation".

We consider in the case of $d = 3$ a large (heavy) molecule with potential (25.1.4). It is well-known[3] that

Proposition 26.1.3. *(i) For $V(x)$ given by* (26.1.4) *minimization problem* (26.1.20) *has a unique solution $\rho = \rho_B^{\mathsf{TF}}$; then denote $\mathcal{E}_B^{\mathsf{TF}} := \mathcal{E}_B(\rho_B^{\mathsf{TF}})$.*

(ii) Equality in $(26.1.21)_2$ *holds if and only if $N \le Z := \sum_m Z_m$.*

(iii) Further, ρ^{TF} does not depend on N as $N \ge Z$.

(iv) Thus

$$\int \rho_B^{\mathsf{TF}}\, dx = \min(N,Z), \qquad Z := \sum_{1\le m\le M} Z_m. \tag{26.1.24}$$

26.1.4 Main Results Sketched and Plan of the Chapter

In the first half of this Chapter we derive asymptotics for ground state energy and justify Thomas-Fermi theory. As construction of Section 25.2 works with minimal modifications (see Section 26.6) in the magnetic case as well we start immediately from magnetic Thomas-Fermi theory in Section 26.2.

We discover that there are three different cases: *a moderate magnetic field case* $B \ll Z^{\frac{4}{3}}$ when $\mathcal{E}_B^{\mathsf{TF}} \asymp Z^{\frac{5}{3}}$ and $\mathcal{E}_B^{\mathsf{TF}} = \mathcal{E}_0^{\mathsf{TF}}(1+o(1))$, *a strong magnetic field case* $B \gg Z^{\frac{4}{3}}$ when $\mathcal{E}_B^{\mathsf{TF}} \asymp B^{\frac{2}{5}} Z^{\frac{9}{5}}$ and $\mathcal{E}_B^{\mathsf{TF}} = \bar{\mathcal{E}}_B^{\mathsf{TF}}(1+o(1))$ where $\bar{\mathcal{E}}_B^{\mathsf{TF}}$

[3] Section IV of E. H. Lieb, J. P. Solovej and J. Yngvason [3].

is Thomas-Fermi potential derived as $P_B(w) = \frac{1}{2}\varkappa_1 w^{\frac{d}{2}}$ (cf. (26.1.19)), and *an intermediate case* $B \sim Z^{\frac{4}{3}}$.

Then we apply semiclassical methods (like in Section 25.4) albeit now our analysis is way more complicated due to two factors: the semiclassical theory of the magnetic Schrödinger operator is more difficult than the corresponding theory for the non-magnetic Schrödinger operator and also Thomas-Fermi potential W^{TF} is not very smooth in the magnetic case, so we need to approximate it by a smooth one (on a microscale).

We discover that both semiclassical methods and Thomas-fermi theory are relevant only if $B \ll Z^3$. The case of the superstrong magnetic field $B \gg Z^3$ was considered in E. H. Lieb, J. P. Solovej and J. Yngvason [1].

First of all, in Section 26.3 we consider the case $M = 1$; then the Thomas-Fermi potential W_B^{TF} is non-degenerate and in this case we derive sharp spectral asymptotics.

Next, in Section 26.4 we consider the case $M \geq 2$ but we analyze only zone $\{W_B^{\mathsf{TF}} + \nu \gtrsim B\}$ where ν is a chemical potential and B is an intensity of the magnetic field. A certain weaker non-degeneracy condition is satisfied due to the Thomas-Fermi equation and we derive almost sharp spectral asymptotics.

Furthermore, in Section 26.5 we analyze in the case $M \geq 2$ the *boundary strip* $\{W^{\mathsf{TF}} + \nu \lesssim B\}$ containing the boundary of $\mathsf{supp}(\rho_B^{\mathsf{TF}})$; this is the most difficult case to analyze and our remainder estimates are not sharp unless $N \geq Z - CZ^{\frac{2}{3}}$.

Finally, in Section 26.6 we derive asymptotics of the ground state energy. Their precision (or lack of it) follows from the precision of the corresponding semiclassical results; so our results in the case $M = 1$ are sharp, but our results in the case $M \geq 2$ (especially if $N \leq Z - CZ^{\frac{2}{3}}$) are not.

In the second half of this Chapter we consider related problems. In Section 26.7 (cf. Section 25.5) we consider negatively charged systems ($N \geq Z$) and estimate both ionization energy I_N and excessive negative charge $(N - Z)_+$ [4)].

In Section 26.8 (cf. Section 25.6) we consider positively charged systems ($N \leq Z$) and estimate the remainder $|\mathsf{I}_N + \nu|$ in the formula $\mathsf{I}_N \approx -\nu$; for $M \geq 2$ we also consider a free nuclei model and estimate from below the distance between nuclei and an excessive positive charge $(Z - N)_+$ when atoms can be bound into molecule[4)].

[4)] In the (magnetic) Thomas-Fermi theory both answers are 0.

Appendices contain some auxiliary material, most notably, electrostatic inequalities in Appendix 26.A.1 and also Zhislin's theorem (that system can bind at least Z electrons) in Appendix 26.A.4–all in the case of magnetic field.

26.2 Magnetic Thomas-Fermi Theory

26.2.1 Framework and Existence

The Thomas-Fermi theory is well developed in the magnetic case as well albeit in the lesser degree than in the non-magnetic one. The most important source now is Section IV of E. H. Lieb, J. P. Solovej and J. Yngvason [3].

Again as in the previous Chapter 25 to get the best lower estimate for the ground state energy (neglecting semiclassical errors) one needs to maximize functional $\Phi_{B,*}(W+\nu)$ defined by (25.3.1) albeit with the pressure $P_B(w)$ given for $d=2,3$ by (26.1.19). Formulae (25.3.2) and (25.3.3) also remain valid.

Further, to get the best upper estimate (neglecting semiclassical errors) one needs to minimize functional $\Phi^*_B(\rho',\nu)$ defined by (25.3.4) where (25.3.4) remains valid with P replaced by P_B and respectively $\tau(\rho')$ replaced by $\tau_B(\rho')$ which is Legendre transformation of P_B (see (26.1.18)).

Since P_B is given by much more complicated expression (26.1.19) rather than $(25.3.6)_1$, and respectively

$$P'_B(w) = \frac{d}{2}\varkappa_1 B\Big(\frac{1}{2}w_+^{\frac{d}{2}-1} + \sum_{j\ge 1}(w-2jB)_+^{\frac{d}{2}-1}\Big) \tag{26.2.1}$$

(cf. $(25.3.6)_2$), there is no explicit expression for τ_B similar to (25.3.7).

Remark 26.2.1. (i) $B(x) = |\nabla \times A(x)|$.

(ii) From now on we will assume that $d=3$.

(iii) P_B is a strictly convex function and therefore τ_B is also a strictly convex function[5)].

[5)] As $d=2$, P_B is a convex and piecewise linear function and therefore τ_B is also a convex function.

(iv) $P_B(w) \to P_0(w)$, $P'_B(w) \to P'_0(w)$ and $\tau_B(\rho) \to \tau_0(\rho)$ as $B \to 0$ where (without subscript "0") the limit functions have been defined by $(25.3.6)_{1,2}$ and (25.3.7) respectively.

Remark 26.2.2. (i) Alternatively we minimize $\mathcal{E}_B(\rho) = \Phi^*_B(\rho, 0)$ under assumptions

$$(26.2.2)_{1,2} \qquad \rho \ge 0, \qquad \int \rho \, dx \le N.$$

(ii) So far in comparison with the previous Chapter 25 we changed only definition of $P_B(w)$ and $\tau_B(\rho)$ respectively. Note that $P_B(w)$ belongs to $\mathscr{C}^{\frac{d}{2}+1}$ (as $d = 2, 3$) as function of w; this statement will be quantified later.

(iii) While not affecting existence (with equality in $(26.2.2)_1$ iff $N \le Z$) and uniqueness of solution, it affects other properties, especially as $B \ge Z^{\frac{4}{3}}$.

Proposition 26.2.3. *In our assumptions for any fixed $\nu \le 0$ Statements (i)–(viii) of Proposition 25.3.1 hold.*

Proof. The proof is the same as of Proposition 25.3.1. The proof that threshold $\nu = 0$ matches to $N = Z$ are theorems 4.9 and 4.10 of Section IV of E. H. Lieb, J. P. Solovej and J. Yngvason [3]. □

Note that (25.3.8)–(25.3.9) and (25.3.10) become

$$\rho = \frac{1}{4\pi}\Delta(W - V) = P'_B(W + \nu), \tag{26.2.3}$$

$$W = o(1) \qquad \text{as} \quad |x| \to \infty \tag{26.2.4}$$

and

$$\mathcal{N}(\nu) = \int P'_B(W + \nu) \, dx \tag{26.2.5}$$

respectively.

Similarly, Proposition 25.3.2 remains true:

Proposition 26.2.4. *For arbitrary W the following estimates hold with absolute constants $\epsilon_0 > 0$ and C_0:*

$$\epsilon_0 \mathsf{D}(\rho - \rho^{\mathsf{TF}}, \rho - \rho^{\mathsf{TF}}) \le \Phi_{B,*}(W^{\mathsf{TF}} + \nu) - \Phi_{B,*}(W + \nu) \le C_0 \mathsf{D}(\rho - \rho', \rho - \rho') \tag{26.2.6}$$

and

$$\epsilon_0 \mathsf{D}(\rho' - \rho^{\mathsf{TF}}, \rho' - \rho^{\mathsf{TF}}) \le \Phi^*_B(\rho, \nu) - \Phi^*_B(\rho^{\mathsf{TF}}, \nu) \le C_0 \mathsf{D}(\rho - \rho', \rho - \rho') \tag{26.2.7}$$

with $\rho = \frac{1}{4\pi}\Delta(W - V)$, $\rho' = P'_B(W + \nu)$.

Proof. This proof is rather obvious as well. □

26.2.2 Properties

Proposition 26.2.5. *The solution of the magnetic Thomas-Fermi problem has the following scaling properties*

$$W^{\mathsf{TF}}(x;\, \underline{z};\, \underline{y};\, B;\, N;\, q) = q^{\frac{2}{3}} N^{\frac{4}{3}} W^{\mathsf{TF}}(q^{\frac{2}{3}} N^{\frac{1}{3}} x;\, N^{-1}\underline{z};\, q^{\frac{2}{3}} N^{\frac{1}{3}} \underline{y};\, q^{-\frac{2}{3}} N^{-\frac{4}{3}} B;\, 1; 1), \tag{26.2.8}$$

$$\rho^{\mathsf{TF}}(x;\, \underline{z};\, \underline{y};\, B;\, N;\, q) = q^2 N^2 \rho^{\mathsf{TF}}(q^{\frac{2}{3}} N^{\frac{1}{3}} x;\, N^{-1}\underline{z};\, q^{\frac{2}{3}} N^{\frac{1}{3}} \underline{y};\, q^{-\frac{2}{3}} N^{-\frac{4}{3}} B;\, 1; 1), \tag{26.2.9}$$

$$\mathcal{E}^{\mathsf{TF}}(\underline{z};\, \underline{y};\, B;\, N;\, q) = q^{\frac{2}{3}} N^{\frac{7}{3}} \mathcal{E}^{\mathsf{TF}}(N^{-1}\underline{z};\, q^{\frac{2}{3}} N^{\frac{1}{3}} \underline{y};\, q^{-\frac{2}{3}} N^{-\frac{4}{3}} B;\, 1;\, 1), \tag{26.2.10}$$

$$\nu^{\mathsf{TF}}(\underline{z};\, \underline{y};\, B;\, N;\, q) = q^{\frac{2}{3}} N^{\frac{4}{3}} \nu^{\mathsf{TF}}(N^{-1}\underline{z};\, q^{\frac{2}{3}} N^{\frac{1}{3}} \underline{y};\, q^{-\frac{2}{3}} N^{-\frac{4}{3}} B;\, 1;\, 1) \tag{26.2.11}$$

where $\nu^{\mathsf{TF}} = \nu$ *is the chemical potential; recall that* $\underline{z} = (z_1, \dots, z_M)$ *and* $\underline{y} = (y_1, \dots, y_M)$ *are arrays and parameter* q *also enters into Thomas-Fermi theory.*

In particular, ν^{TF} *and* B *scale the same way.*

Proof. Proof is trivial by scaling. □

Now one can guess that there are two cases $B \ll Z^{\frac{4}{3}}$ and $B \gg Z^{\frac{4}{3}}$ (recall that $N \asymp Z$) in which magnetic Thomas-Fermi theory looks very different (and also an intermediate case $B \sim Z^{\frac{4}{3}}$). To explain this difference let us consider one atom case:

First of all recall that if $B = 0$ and $N = Z$ theory (as $M = 1$) has just one parameter and we can get rid of it by rescaling; $W^{\mathsf{TF}} \asymp Z\ell^{-1}$ as $\ell \lesssim Z^{-\frac{1}{3}}$ and $W^{\mathsf{TF}} \asymp \ell^{-4}$ as $\ell \gtrsim Z^{-\frac{1}{3}}$. Then

$$\mathsf{W}^{\mathsf{TF}\,\frac{3}{2}} \ell^3 \asymp Z^{\frac{3}{2}} \ell^{\frac{3}{2}}, \quad \mathsf{W}^{\mathsf{TF}\,\frac{5}{2}} \ell^3 \asymp Z^{\frac{5}{2}} \ell^{\frac{1}{2}}$$

and

$$\mathsf{W}^{\mathsf{TF}}\tfrac{3}{2}\ell^3 \asymp \ell^{-3}, \quad \mathsf{W}^{\mathsf{TF}}\tfrac{5}{2}\ell^3 \asymp \ell^{-7}$$

respectively where the first factors are spacial densities of the charge and (negative) Thomas-Fermi energy respectively and therefore zone $\ell \asymp Z^{-\frac{1}{3}}$ provides the main contributions into both.

Therefore, if in this *main zone* $B \ll W^{\mathsf{TF}} \asymp Z^{\frac{4}{3}}$ we guess that the magnetic theory is similar to non-magnetic one, and actually it is true.

However, let us study an atomic case rigorously. Let $M = 1$, $\mathsf{y}_m = 0$ and $N \le Z$. Then

(26.2.12) W_B^{TF} is a spherically symmetric, and it is monotone non-increasing function of $|x|$; $W_B^{\mathsf{TF}} \to +0$ as $|x| \to \infty$;

$$W_B^{\mathsf{TF}}(x) \le -\nu \implies W_B^{\mathsf{TF}} = |x|^{-1}(Z - N). \tag{26.2.13}$$

Indeed, (26.2.12) is obvious and (26.2.13) follows from it and Newton screening theorem.

Two propositions below treat cases $B \lesssim Z^{\frac{4}{3}}$ and $B \gtrsim Z^{\frac{4}{3}}$ respectively; in the former case there is another fork: $B \lesssim (Z - N)_+^{\frac{4}{3}}$ and $B \gtrsim (Z - N)_+^{\frac{4}{3}}$.

Proposition 26.2.6. *Let $M = 1$, $\mathsf{y}_m = 0$, $N \asymp Z_m$ and $B \le Z^{\frac{4}{3}}$.*

(i) Then

$$W_B^{\mathsf{TF}} \le \min(Z|x|^{-1}, C|x|^{-4}) \tag{26.2.14}$$

and

$$\rho_B^{\mathsf{TF}} \le C\min(Z^{\frac{3}{2}}|x|^{-\frac{3}{2}} + BZ^{\frac{1}{2}}|x|^{-\frac{1}{2}}, |x|^{-6} + B|x|^{-2}). \tag{26.2.15}$$

(ii) There exists

$$\bar{r}_m \asymp \min\bigl(B^{-\frac{1}{4}}, (Z - N)_+^{-\frac{1}{3}}\bigr) \tag{26.2.16}$$

such that $W_B^{\mathsf{TF}} \gtrless -\nu$ as $|x| \lesseqgtr \bar{r}_m$ and then $\rho_B^{\mathsf{TF}} = 0$ iff $x \ge \bar{r}_m$.

(iii) (26.2.14) *and* (26.2.15) *become equivalencies ($\asymp$) as $|x| \le (1 - \epsilon)\bar{r}_m$.*

(iv) $B \le (Z-N)_+^{\frac{4}{3}}$ *implies* $\bar{r}_m \asymp (Z-N)_+^{-\frac{1}{3}}$, $\nu \asymp (Z-N)_+^{\frac{4}{3}}$ *and*

$$W^{\mathsf{TF}} + \nu \asymp (Z-N)_+^{\frac{5}{3}}(\bar{r}_m - |x|), \qquad -\partial_{|x|} W^{\mathsf{TF}} \asymp (Z-N)_+^{\frac{5}{3}} \quad \text{if } (1-\epsilon)\bar{r}_m \le |x| \le \bar{r}_m. \tag{26.2.17}$$

(v) $B \ge (Z-N)_+^{\frac{4}{3}}$ *implies* $\bar{r}_m \asymp B^{-\frac{1}{4}}$, $\nu \asymp (Z-N)_+ B^{\frac{1}{4}} \lesssim B$ *and*

$$W^{\mathsf{TF}} + \nu \asymp B^2(\bar{r}_m - |x|)^4 + B^{\frac{1}{2}}(Z-N)_+(\bar{r}_m - |x|) \qquad -\partial_{|x|} W^{\mathsf{TF}} \asymp B^2(\bar{r}_m - |x|)^3 + B^{\frac{1}{2}}(Z-N)_+ \quad \text{as } (1-\epsilon)\bar{r}_m \le |x| \le \bar{r}_m. \tag{26.2.18}$$

Proposition 26.2.7. *Let* $M = 1$, $\mathsf{y}_m = 0$, $N \asymp Z_m$ *and* $B \ge Z^{\frac{4}{3}}$.

(i) Then

$$W_B^{\mathsf{TF}} \le Z|x|^{-1} \tag{26.2.19}$$

and

$$\rho_B^{\mathsf{TF}} \le CZ^{\frac{3}{2}}|x|^{-\frac{3}{2}} + CBZ^{\frac{1}{2}}|x|^{-\frac{1}{2}}. \tag{26.2.20}$$

(ii) There exist $\bar{r}_m$ *and* $\bar{r}'_m$,

$$\bar{r}_m \asymp B^{-\frac{2}{5}}Z^{\frac{1}{5}}, \qquad \bar{r}'_m \asymp B^{-1}Z_m, \tag{26.2.21}$$

such that $W_B^{\mathsf{TF}} \gtrless B$ *as* $|x| \lessgtr \bar{r}_m$, $W_B^{\mathsf{TF}} \gtrless -\nu$ *as* $|x| \lessgtr \bar{r}'_m$ *and then* $\rho_B^{\mathsf{TF}} = 0$ *iff* $x \ge \bar{r}_m$.

(iii) (26.2.19)–(26.2.20) *become equivalencies* ($\asymp$) *as* $|x| \le (1-\epsilon)\bar{r}_m$.

(iv) $\nu \asymp (Z-N)_+ B^{\frac{2}{5}} Z^{-\frac{1}{5}} \lesssim B$ *and*

$$W^{\mathsf{TF}} + \nu \asymp B^2(\bar{r}_m - |x|)^4 + \bar{r}_m^{-2}(Z-N)_+(\bar{r}_m - |x|) \tag{26.2.22}$$

and

$$-\partial_{|x|} W^{\mathsf{TF}} \asymp B^2(\bar{r}_m - |x|)^3 + \bar{r}_m^{-2}(Z-N)_+ \tag{26.2.23}$$

as $(1-\epsilon)\bar{r}_m \le |x| \le \bar{r}_m$.

Proofs of Propositions 26.2.6 and 26.2.7. Proofs easily follow from equation and "boundary conditions" satisfied by $w(r)$ where $r = |x|$:

$$w'' + 2r^{-1}w = P'_B(w+\nu), \tag{26.2.24}$$

$$w = r^{-1}Z_m + O(1) \qquad \text{as} \quad r \to 0, \tag{26.2.25}$$

$$w(\bar{r}_m) = -\nu, \qquad w'(\bar{r}_m) = \nu\bar{r}_m^{-1} \tag{26.2.26}$$

where $\nu = -(Z_m - N)_+\bar{r}_m^{-1}$. □

Corollary 26.2.8. *Let $M = 1$, $\mathsf{y}_m = 0$ and $N \asymp Z_m$. Then*

(i) $W_B^{\mathsf{TF}} \lesssim B$ if $|x| \ge \bar{r}'_m$ where $\bar{r}'_m \asymp B^{-1}Z_m$ as $B \ge Z_m^{\frac{4}{3}}$ and $\bar{r}'_m \asymp B^{-\frac{1}{4}}$ as $B \le cZ_m^{\frac{4}{3}}$.

(ii) As $B \lesssim Z_m^{\frac{4}{3}}$ the main contribution to both the charge and the Thomas-Fermi energy is delivered by zone $\{x\colon |x| \asymp r_m^*\}$ with $r_m^* = Z_m^{-\frac{1}{3}}$; in particular, then $\mathcal{E}_B^{\mathsf{TF}} \asymp \mathcal{E}^{\mathsf{TF}} \asymp Z_m^{\frac{7}{3}}$; further, in this case $W_B^{\mathsf{TF}} \asymp W^{\mathsf{TF}}$ in the zone $\{x\colon |x| \lesssim \epsilon\bar{r}_m\}$.

(iii) Further, $\mathcal{E}_B^{\mathsf{TF}} \sim \mathcal{E}^{\mathsf{TF}}$ as $B \ll Z_m^{\frac{4}{3}}$; furthermore, in this case $W_B^{\mathsf{TF}} \sim W^{\mathsf{TF}}$ in the zone $\{x\colon |x| \ll \bar{r}_m\}$.

(iv) On the other hand, as $B \ge Z^{\frac{4}{3}}$, the main contributions to the total charge and energy are delivered by $\{x\colon |x| \asymp \bar{r}_m\}$ and in particular $\rho_m \asymp BZ_m^{\frac{1}{2}}\bar{r}_m^{\frac{5}{2}}$ and

$$\mathcal{E}_B^{\mathsf{TF}} \asymp BZ_m^{\frac{3}{2}}\bar{r}_m^{\frac{3}{2}} \asymp B^{\frac{2}{5}}Z_m^{\frac{9}{5}}. \tag{26.2.27}$$

Recall that $\bar{r}_m \asymp B^{-\frac{1}{4}}$ as $B \le Z_m^{\frac{4}{3}}$ and $\bar{r}_m \asymp B^{-\frac{2}{5}}Z_M^{\frac{1}{5}}$ as $B \ge Z_M^{\frac{4}{3}}$. Note that Proposition 25.3.5 (comparing W^{TF} for molecule with the sum of those for single atoms) still holds. Therefore we conclude that

Corollary 26.2.9. *(i) Assume that*

$$Z_1 \asymp Z_2 \asymp \ldots \asymp Z_M. \tag{26.2.28}$$

Then all statements of corollary 26.2.8 remain true for $M \ge 2$ with $|x|$ and Z_m replaced by $\ell(x)$ and Z and $\bar{r}_m$, $\bar{r}'_m$, r_m^* by $\bar{r}$, $\bar{r}'$, r^* respectively.

(ii) In the general case global statements remain true, pointwise statements remain true without modification only as $\ell(x) = \ell_m(x) := |x - \mathsf{y}_m|$ *with* $Z_m \asymp Z$.

Remark 26.2.10. (i) Also holds Proposition 25.3.13 as it uses only super-additivity of $\tau(\rho)$ and $\tau_B(\rho)$ is also super-additive (this follows from convexity of $\tau_B(\rho)$ and equality $\tau_B(0) = 0$).

(ii) However there is a significant difference: if there is no magnetic field atoms really repulse one another on any distances and we can attribute it to either excessive positive charge as $N < Z$ or their infinite spatial size as $N = Z$. However with magnetic field atoms have a finite size even as $N = Z$ and they do not repulse one another on the large distances. In particular, Proposition 26.2.11 below holds.

Proposition 26.2.11. *Let* $N = Z$ *and*

$$|\mathsf{y}_m - \mathsf{y}_{m'}| \ge \bar{r}_m + \bar{r}_{m'} \qquad \forall m:\ 1 \le m < m' \le M. \tag{26.2.29}$$

Then

$$\mathcal{E}_B^{\mathsf{TF}}(\underline{Z}, \underline{\mathsf{y}}, B, Z) = \sum_{1\le m\le M} \mathcal{E}_B^{\mathsf{TF}}(Z_m, \mathsf{y}_m, B, Z_m) \tag{26.2.30}$$

and

$$\rho_B^{\mathsf{TF}}(x, \underline{Z}, \underline{\mathsf{y}}, B, Z) = \sum_{1\le m\le M} \rho_B^{\mathsf{TF}}(x, Z_m, \mathsf{y}_m, B, Z_m). \tag{26.2.31}$$

Proposition 26.2.12. *(i)* ν *is monotone increasing function of* N.

(ii) $W_B(x)$ *is monotone non-increasing function of* N.

(iii) $W_B(x) + \nu$ *is monotone non-decreasing function of* N *; in particular* ρ_B *can only increase as* N *increases.*

(iv) ν *is monotone non-increasing function of* Z_m.

(v) $W_B(x)$ *is monotone non-decreasing function of* Z_m.

Proof. (i) Statement (i) follows from the strict convexity of $\mathcal{E}(\rho)$: consider two solutions with corresponding subscripts. Then $\mathcal{E}(\rho) - \mathcal{E}(\rho_j) > \nu_j(N - N_j)$ for any non-negative $\rho \ne \rho_j$ and $N = \int \rho\, dx$.

In particular, $\mathcal{E}(\rho_1) - \mathcal{E}(\rho_2) > \nu_2(N_1 - N_2)$ and $\mathcal{E}(\rho_2) - \mathcal{E}(\rho_1) > \nu_1(N_2 - N_1)$ and then $(\nu_1 - \nu_2)(N_1 - N_2) > 0$.

(ii) Indeed, consider $N_1 < N_2$ and in the definition of W_2 slightly decrease $Z_1, \dots, Z_M$ thus replacing them by $Z'_1, \dots, Z'_M$. Then $W_1 > W_2$ for large $|x|$, $W_1 - W_2 \to +\infty$ as $x \to \mathsf{y}_m$ and therefore if Statement (ii) fails, then $W_1 - W_2$ reaches non-positive minimum at some regular point $\bar{x}$; at this point $W_1 \le W_2$ and

$$0 \le \frac{1}{4\pi}\Delta(W_1 - W_2) = P'(W_1 + \nu_1) - P'(W_2 + \nu_2).$$

This is possible only if at this point $W_2 + \nu_2 \le 0$ and $W_1 + \nu_1 < 0$. Then in the small vicinity $\Delta(W_1 - W_2) \le 0$ and $\bar{x}$ cannot be a point of minimum unless $W_1 - W_2 = \mathsf{const}$ there. Then any point of this vicinity is also a point of minimum and then due to standard analytic arguments $W_1 - W_2 = \mathsf{const}$ everywhere which is impossible.

So, $W_1(x; Z_1, \dots, Z_M) > W_2(x; Z'_1, \dots, Z'_M)$. Taking limit as $Z'_m \to Z_m$ we arrive to $W_1(x; Z_1, \dots, Z_M) \ge W_2(x; Z_1, \dots, Z_M)$.

(iii) Proof of Statement (iii) is similar but roles of W_1 and W_2 are played by $W_2 + \nu_2$ and $W_1 + \nu_1$ respectively.

(iv) Let $Z_{m,2} > Z_{m,1}$ for all m. Assume that $\nu_2 > \nu_1$. Then similar arguments prove that $W_2 + \nu_2 \ge W_1 + \nu_1$ and thus $\rho_2 \ge \rho_1$ everywhere which is impossible unless there are just identical equalities as $W_2 + \nu_2 > 0$, which is impossible.

(v) Finally, after Statement (iv) was established, the same arguments prove Statement (v). □

As far as we know Theorem 1 of R. Benguria [1] (see Theorem 25.3.8) has not been proven in the case of magnetic field; however one can see easily that arguments of of R. Benguria's proof (see Appendix 25.A.6) remain partially valid and we arrive to

Theorem 26.2.13. *Statements (i) and (ii) of Theorem 25.3.8*[6] *hold in the case of the constant external magnetic field.*

Remark 26.2.14. On the other hand, proofs of Statements (iii) and (iv) of Theorem 25.3.8 (that for fixed $\alpha > 0$ $\widehat{\mathcal{E}}^{\mathsf{TF}}_\alpha - \widehat{\mathcal{E}}^{\mathsf{TF}}_0$ and $\mathcal{E}^{\mathsf{TF}}_\alpha - \mathcal{E}^{\mathsf{TF}}_0$ are non-decreasing function of N) fail; more precisely, the proof of Lemma 25.A.7(ii) fails.

[6] That $\widehat{\mathcal{E}}^{\mathsf{TF}}_\alpha$ is a non-increasing function of $\alpha \ge 0$ and $\mathcal{E}^{\mathsf{TF}}_\alpha$ is a non-decreasing function of $\alpha \ge 0$.

Problem 26.2.15. (i) Investigate how $\mathsf{supp}(\rho_B^{\mathsf{TF}})$ depends on B and on Z in the atomic case $M = 1$.

(ii) More generally, investigate how $\mathsf{supp}(\rho_B^{\mathsf{TF}})$ depends on B and on Z in the case $M \ge 2$.

26.2.3 Positive Ions

In view of Remark 26.2.10 we need to consider repulsion of positive ions in more details. Our purpose is to prove

Theorem 26.2.16. *Let condition* (26.2.28) *be fulfilled. Then the energy excess is estimated from below*

$$\mathcal{Q} := \widehat{\mathcal{E}}_B^{\mathsf{TF}} - \sum_{1\le m\le M} \mathcal{E}_{B,m}^{\mathsf{TF}} \ge \epsilon (Z-N)_+^2 a^{-1}. \tag{26.2.32}$$

Note first that

$$\begin{aligned}\mathsf{D}\bigl(\rho_{B(\nu)}^{\mathsf{TF}} - \rho_{B,0}^{\mathsf{TF}},\, \rho_{B(\nu)}^{\mathsf{TF}} - \rho_{B(0)}^{\mathsf{TF}}\bigr) + {}& \\ \int \bigl(P'_B(W_{B(\nu)}^{\mathsf{TF}} + \nu) - P'_B(W_{B(0)}^{\mathsf{TF}})\bigr)\bigl(W_{B(\nu)}^{\mathsf{TF}} + \nu - W_{B(0)}^{\mathsf{TF}})\bigr)\,dx = {}& \\ \nu \int \bigl(P'_B(W_{B(\nu)}^{\mathsf{TF}} + \nu) - P'_B(W_{B(0)}^{\mathsf{TF}} + 0)\bigr)\,dx & \end{aligned} \tag{26.2.33}$$

with the right-hand expression equal $\nu(N-Z) \asymp (Z-N)^2 \bar{r}^{-1}$ and due to monotonicity $P'_B(w)$ we conclude that

Proposition 26.2.17. *Let condition* (26.2.28) *be fulfilled. Then*

$$\mathsf{D}\bigl(\rho_{B(\nu)}^{\mathsf{TF}} - \rho_{B(0)}^{\mathsf{TF}},\, \rho_{B,\nu}^{\mathsf{TF}} - \rho_{B(0)}^{\mathsf{TF}}\bigr) \le C(Z-N)^2 \bar{r}^{-1}. \tag{26.2.34}$$

Proof of Theorem 26.2.16. Step 1. Note first that due to non-negativity of the expression

$$\widehat{\mathcal{E}}_B^{\mathsf{TF}}(\underline{Z}, \underline{y}, N) - \min_{N_1+N'=N} \bigl(\mathcal{E}_B^{\mathsf{TF}}(Z_1, N_1) - \widehat{\mathcal{E}}_B^{\mathsf{TF}}(\underline{Z}', \underline{y}', N')\bigr) \tag{26.2.35}$$

(see proof of Proposition 25.3.13 which persists even if there is constant magnetic field, see Remark 26.2.10) it is sufficient to prove theorem only for $M = 2$. From now on we assume that $M = 2$.

Step 2. According to Proposition 25.3.13

$$\mathsf{D}(\rho_B^{\mathsf{TF}} - \rho_{B,1}^{\mathsf{TF}} - \rho_{B,2}^{\mathsf{TF}}) \le C\mathcal{Q}. \tag{26.2.36}$$

Therefore due to superadditivity τ_B

$$\mathcal{Q} \ge -\int V_1\rho_{B,2}^{\mathsf{TF}}\,dx - \int V_2\rho_{B,1}^{\mathsf{TF}}\,dx + \mathsf{D}(\rho_{B,2}^{\mathsf{TF}}, \rho_{B,1}^{\mathsf{TF}}) + Z_1Z_2a^{-1} - C\mathcal{Q} \tag{26.2.37}$$

and it is sufficient to prove the same estimate from below for the right-hand expression without the last term. However this is easy if $a \ge \bar{r}_1 + \bar{r}_2$ since $V_m = |x - \mathsf{y}_m|^{-1}Z_m$ and $\rho_{B,m}^{\mathsf{TF}} = \rho_{B,m}^{\mathsf{TF}}(|x - \mathsf{y}_m|)$ are spherically symmetric functions[7)].

Therefore for $a \ge \bar{r}_1 + \bar{r}_2$ inequality (26.2.32) has been proven and in what follows we can assume that $a \le \bar{r}_1 + \bar{r}_2$. Further, applying Theorem 26.2.13 we conclude then that

(26.2.38) Inequality (26.2.32) holds for $a \ge \epsilon\bar{r}$.

Step 3. Recall that the bulk of electrons are in the zone $\{\ell(x) \asymp r^*\}$ [8)]. Based on this one can prove easily that as $a \le \epsilon\bar{r}$ the right-hand hand expression of (26.2.37) is greater than $(1-\epsilon_1)a^{-1}Z_1Z_2$ and therefore

(26.2.39) As $B \ge Z^{\frac{4}{3}}$ and $a \le \epsilon r^*$ we have $\mathcal{Q} \ge (1-\epsilon_1)a^{-1}Z_1Z_2$

and combining with (26.2.38) we conclude that (26.2.32) holds for $B \gtrsim Z^{\frac{4}{3}}$ and for $B \lesssim Z^{\frac{4}{3}}$ we need to consider the case $\epsilon_0 r^* \le a \le \epsilon\bar{r}$ with arbitrarily small constant ϵ.

Replacing then P_B by P_0 and noting that an error will not exceed $C_0\bar{r}B^2 \le C_1\epsilon a^{-7}$ while $\mathcal{Q} \ge \epsilon_0 a^{-7}$ for $B = 0$ we conclude that (26.2.32) holds as $\epsilon_0 r^* \le a \le c\bar{r}$ and $(Z - N) \le C_2 a^{-3}$.

Finally, as $(Z - N) \ge C_2 a^{-3}$ we see that $\bar{r} \le C_0(Z-N)^{-\frac{1}{3}} \le \epsilon a$ and (26.2.32) holds again. □

[7)] However this is not true in general as $a < \bar{r}_1 + \bar{r}_2$. Really, consider $N_m = Z_m$ and uniformly charged spheres. Then the right-hand expression of (26.2.37) is 0 as $a \ge \bar{r}_1 + \bar{r}_2$ and is negative and decays as a decays from $\bar{r}_1 + \bar{r}_2$ to $\max(\bar{r}_1, \bar{r}_2)$ and it increases again as a decays from $\max(\bar{r}_1, \bar{r}_2)$ to 0.

[8)] I.e. zone $\{c(\epsilon)^{-1}r^* \le \ell(x) \le c(\epsilon)r^*\}$ contains at least $(1-\epsilon)N$ electrons.

Even if we do not need it for our purposes we want to consider the repulsion of too close neutral atoms:

Theorem 26.2.18. *Let condition* (26.2.28) *be fulfilled and* $N = Z$.

(i) If $B \le Z^{\frac{4}{3}}$ *and* $a \le \epsilon \bar{r}$ *then*

$$\mathcal{Q} \asymeq \begin{cases} a^{-1}Z^2 & a \le Z^{-\frac{1}{3}} \\ a^{-7} & a \ge Z^{-\frac{1}{3}}. \end{cases} \tag{26.2.40}$$

(ii) If $B \le Z^{\frac{4}{3}}$, $a \ge \epsilon \bar{r}$ *and*

$$0 < \max_{1 \le m < m' \le M} \bigl(\bar{r}_m + \bar{r}_{m'} - |\mathsf{y}_m - \mathsf{y}_{m'}|\bigr) = \varepsilon \bar{r} \tag{26.2.41}$$

with exact radius $\bar{r}_m$ *of* m*-th atom (that is,* $\mathsf{supp}(\rho^{\mathsf{TF}}_{B,m} = B((\mathsf{y}_m, \bar{r}_m))$ *and* $\varepsilon \in (0, c)$ *then*

$$\mathcal{Q} \asymeq B^{\frac{7}{4}} \varepsilon^7. \tag{26.2.42}$$

(iii) If $B \ge Z^{\frac{4}{3}}$, $a \le \epsilon \bar{r}$ *then*

$$\mathcal{Q} \asymeq a^{-1} Z^2. \tag{26.2.43}$$

(iv) If $B \ge Z^{\frac{4}{3}}$, $a \ge \epsilon \bar{r}$ *and* (26.2.41) *holds with* $\varepsilon \in (0, c)$ *then*

$$\mathcal{Q} \asymeq B^{\frac{2}{5}} Z^{\frac{9}{5}} \varepsilon^7. \tag{26.2.44}$$

Proof. (a) Since $\mathcal{E}^{\mathsf{TF}}_B \asymeq \mathcal{E}^{\mathsf{TF}}_{B,m} \asymeq \mathsf{max}(Z^{\frac{7}{3}}; B^{\frac{2}{5}} Z^{\frac{9}{5}})$ even without assumption $N = Z$ we conclude that $\mathcal{Q} \asymeq a^{-1}Z^2$ as long as $a \le \epsilon_0 r^* = \epsilon_0 \,\mathsf{min}(Z^{-\frac{1}{3}}; B^{-\frac{2}{5}} Z^{\frac{1}{5}})$ which proves Statement (i) as $a \le \epsilon_0 Z^{-\frac{1}{3}}$ and also Statement (iii).

(b) Observe that $\mathcal{E}^{\mathsf{TF}}_B = \mathcal{E}^{\mathsf{TF}}_0 + O(B^{\frac{7}{4}})$ and $\mathcal{E}^{\mathsf{TF}}_{B,m} = \mathcal{E}^{\mathsf{TF}}_{0,m} + O(B^{\frac{7}{4}})$ as $a \ge \epsilon_0 Z^{-\frac{1}{3}}$ where $\mathcal{E}^{\mathsf{TF}}_0$ and $\mathcal{E}^{\mathsf{TF}}_{0,m}$ are non-magnetic Thomas-Fermi energies and therefore $\mathcal{Q} \coloneqq \mathcal{Q}_B = \mathcal{Q}_0 + O(B^{\frac{7}{4}})$ and since $\mathcal{Q}_0 \asymeq a^{-7}$, we arrive to $\mathcal{Q}_B \asymeq a^{-7}$ as $\epsilon_0 Z^{-\frac{1}{3}} \le a \le \epsilon B^{-\frac{1}{4}}$.

(c) Thus we need to prove Statements (ii) and (iv). Assume first that $\varepsilon \in (0, \epsilon]$ with sufficiently small constant $\epsilon > 0$.

$$
\begin{aligned}
(26.2.45)\quad & -\int P_B(W)\,dx - \frac{1}{2}\mathsf{D}(\rho,\,\rho) \le \mathcal{E}_B^{\mathsf{TF}} \\
& \le -\int P_B(W)\,dx - \frac{1}{2}\mathsf{D}(\rho,\,\rho) + \frac{1}{2}\mathsf{D}(P_B'(W)-\rho,\,P_B'(W)+\rho)
\end{aligned}
$$

with arbitrary ρ and $W = V - |x|^{-1} * \rho$. Let us plug $\rho = \sum_{1\le m\le M}\rho_m$ with $\rho_M \coloneqq \rho_{B,m}^{\mathsf{TF}}$; then $W = \sum_{1\le m\le M} W_m$, $W_m = W_{B,m}^{\mathsf{TF}}$. Let us compare different terms in the left and right side expressions of (26.2.45) and the corresponding sums of these terms for separate atoms.

First, consider

$$
\frac{1}{2}\mathsf{D}(\rho_1+\dots+\rho_m,\,\rho_1+\dots+\rho_m) - \sum_{1\le m\le M}\frac{1}{2}\mathsf{D}(\rho_m,\,\rho_m) = \sum_{1\le m<m'\le M}\mathsf{D}(\rho_m,\,\rho_{m'})
$$

and as $m \ne m'$

$$
\begin{aligned}
(26.2.46)\quad & \mathsf{D}(\rho_m,\,\rho_{m'}) \\
& = \frac{Z_m Z_{m'}}{|\mathsf{y}_m - \mathsf{y}_{m'}|} - \int_{\mathcal{Z}_{mm'}}\Big(\int_{\mathcal{Y}_{m'}\setminus B(\mathsf{y}_{m'},|x-\mathsf{y}_{m'}|)}\rho_{m'}(y)\,dy\Big)\frac{\rho_m(x)}{|x-\mathsf{y}_{m'}|}\,dx
\end{aligned}
$$

with $\mathcal{Z}_{mm'} = \mathcal{Y}_m \cap \mathcal{Y}_{m'}$, $\mathcal{Y}_m = B(\mathsf{y}_m, \bar{r}_m)$. Indeed, using Newton's screening theorem and that ρ_m and $\rho_{m'}$ are spherically symmetric about y_m and $\mathsf{y}_{m'}$ correspondingly, and $\mathsf{y}_m \notin B(\mathsf{y}_{m'}, \bar{r}_{m'})$ and $\mathsf{y}_{m'} \notin B(\mathsf{y}_m, \bar{r}_m)$ as $\varepsilon \le \epsilon$ we have

$$
\begin{aligned}
&\iint_{(\mathcal{Y}_m\setminus\mathcal{Z}_{mm'})\times\mathcal{Y}_{m'}}\frac{\rho_m(x)\rho_{m'}(y)}{|x-y|}\,dxdy + \iint_{\mathcal{Z}_{mm'}\times\mathcal{Y}_{m'}}\frac{\rho_m(x)\rho_{m'}(y)}{|x-y|}\,dxdy \\
&\qquad = \int_{\mathcal{Y}_m\setminus\mathcal{Z}_{mm'}}\frac{\rho_m(x)Z_{m'}}{|x-\mathsf{y}_{m'}|}\,dx + \iint_{\mathcal{Z}_{mm'}\times\mathcal{Y}_{m'}}\frac{\rho_m(x)\rho_{m'}(y)}{|x-y|}\,dxdy \\
&= \int_{\mathcal{Y}_m}\frac{\rho_m(x)Z_{m'}}{|x-\mathsf{y}_{m'}|}\,dx - \int_{\mathcal{Z}_{mm'}}\frac{\rho_m(x)Z_{m'}}{|x-\mathsf{y}_{m'}|}\,dx + \iint_{\mathcal{Z}_{mm'}\times\mathcal{Y}_{m'}}\frac{\rho_m(x)\rho_{m'}(y)}{|x-y|}\,dxdy \\
&\qquad = \frac{Z_m Z_{m'}}{|\mathsf{y}_m-\mathsf{y}_{m'}|} - \int_{\mathcal{Z}_{mm'}}\Big(\frac{Z_{m'}}{|x-\mathsf{y}_{m'}|} - \int_{\mathcal{Y}_{m'}}\frac{\rho_{m'}(y)}{|x-y|}\,dy\Big)\rho_m(x)\,dx
\end{aligned}
$$

and the last expression is equal to the right-hand expression of (26.2.46).

Next, consider

$$\int \Big(P_B(W_1 + ... + W_m) - \sum_{1\le m\le M} P_B(W_m)\Big)\, dx \tag{26.2.47}$$

and the last term in the right-hand expression of (26.2.45); one can see easily that they are given by integrals over $\mathcal{Z} = \bigcup_{1\le m<m'\le M} \mathcal{Z}_{mm'}$ and $\mathcal{Z}\times\mathcal{Z}$ respectively.

Remark 26.2.19. (A) The last term in the right-hand expression of (26.2.46) is $O(S\varepsilon^7)$ with S defined by

$$G \coloneqq \begin{cases} B & \text{if } B \le Z^{\frac{4}{3}}, \\ B^{\frac{2}{5}} Z^{\frac{4}{5}} & \text{if } B \ge Z^{\frac{4}{3}} \end{cases}, \qquad S \coloneqq \begin{cases} B^{\frac{7}{4}} & \text{if } B \le Z^{\frac{4}{3}}, \\ B^{\frac{2}{5}} Z^{\frac{9}{5}} & \text{if } B \ge Z^{\frac{4}{3}}. \end{cases} \tag{26.2.48}$$

The last term in the right-hand expression of (26.2.46) is $\asymp S\varepsilon^7$ provided

$$0 < \bar{r}_m + \bar{r}_{m'} - |\mathsf{y}_m - \mathsf{y}_{m'}| \asymp \varepsilon \bar{r}. \tag{26.2.49}$$

(B) Expression (26.2.47) is $O(S\varepsilon^8)$.

(C) The last term in the right-hand expression of (26.2.45) is $O(S\varepsilon^{\frac{15}{2}})$.

Indeed, in all integrands in (A)–(C) $W_m = O(G\varepsilon^4)$, $\rho_m = P'_B(W_m) = O(BG^{\frac{1}{2}}\varepsilon^2)$, and the same is true for $W_{m'}$ and $\rho_{m'}$. Further, the volume of $\mathcal{Z}_{mm'}$ is $\asymp \bar{r}^3\varepsilon^2$ and the volume of $\mathcal{Y}_{m'} \setminus B(\mathsf{y}_{m'}, |x-\mathsf{y}_{m'}|)$ is $O(\bar{r}^3\varepsilon)$ and $\int_{\mathcal{Z}_{mm'}} |x-y|^{-1}\, dxdy = O(\bar{r}^5\varepsilon^{\frac{15}{2}})$.

Furthermore, under assumption (26.2.49) there are $\mathcal{W}_{mm'} \subset \mathcal{Z}_{mm'}$ and $\mathcal{W}_{m'} \subset \mathcal{Y}_{m'} \setminus B(\mathsf{y}_{m'}, |x-\mathsf{y}_{m'}|)$ of the volumes $\asymp \bar{r}^3\varepsilon^2$ and $\asymp \bar{r}^3\varepsilon$ correspondingly such that there $\rho_m \asymp \rho_{m'} \asymp G\varepsilon^2$.

Finally, in the integrand in (B) $P_B(W_m) = O(BG^{\frac{3}{2}}\varepsilon^6)$.

Since

$$\widehat{\mathcal{E}}^{\mathsf{TF}}_B = \mathcal{E}^{\mathsf{TF}}_B + \sum_{1\le m<m'\le M} \frac{Z_m Z_{mm'}}{|\mathsf{y}_m - \mathsf{y}_{m'}|} \tag{26.2.50}$$

and

$$\mathcal{Q} = \widehat{\mathcal{E}}^{\mathsf{TF}}_B - \sum_{1\le m\le M} \mathcal{E}^{\mathsf{TF}}_{B,m} \tag{26.2.51}$$

we conclude that $\mathcal{Q} \asymp S\varepsilon^7$ as $\varepsilon \in (0, \epsilon]$ with sufficiently small constant $\epsilon > 0$, and in this case Statements (ii) and (iv) of the theorem are proven.

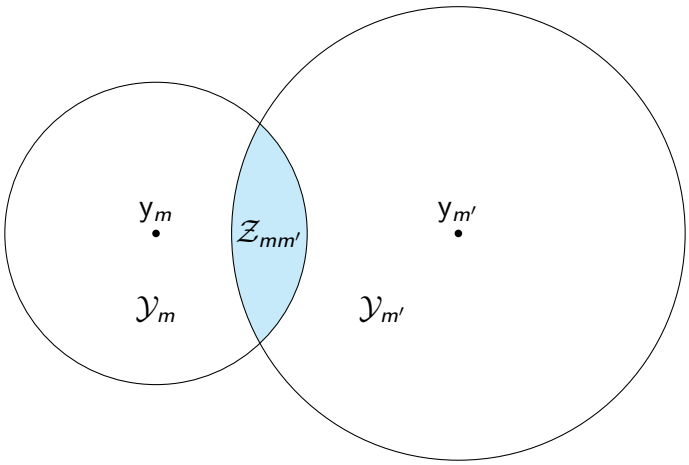


Figure 26.1: The thickness of the "lens" $\mathcal{Z}_{mm'}$ is $\asymp \varepsilon\bar{r}$ and its width is $\asymp \varepsilon^{\frac{1}{2}}\bar{r}$

(d) The above arguments simplified imply that $\mathcal{Q} \lesssim S$ as $\varepsilon \in (0, c]$ with arbitrarily large constant $c > 0$.

Let us estimate $\mathcal{Q}$ from below, assuming that $\varepsilon \in [\epsilon, c)$ with small constant $\epsilon > 0$ from Part (c). Due to Theorem 26.2.13 (Benguria's theorem in the case of magnetic field, Statement refthm-25.3.8-i) one can move y_m to y'_m, $m = 1, \dots, M$ so that $\widehat{\mathcal{E}}^{\mathsf{TF}}_B(\mathsf{y}_1, \dots, \mathsf{y}_M) \geq \widehat{\mathcal{E}}^{\mathsf{TF}}_B(\mathsf{y}'_1, \dots, \mathsf{y}'_M)$ and for $\mathsf{y}'_1, \dots, \mathsf{y}'_M$ assumption (26.2.41) holds with $\varepsilon = \epsilon$.

Then it follows from Part (c) that (26.2.42) and (26.2.44) hold for the new arrangement as $B \leq Z^{\frac{4}{3}}$ and $B \geq Z^{\frac{4}{3}}$ respectively, which mean exactly that $\mathcal{Q} \gtrsim S$.

Remark 26.2.20. These two figures show thresholds for $B \leq Z^{\frac{4}{3}}$ and $B \geq Z^{\frac{4}{3}}$ respectively:

26.3 Applying Semiclassical Methods: $M = 1$

26.3.1 Heuristics

Let us consider first a mock proof of our main results; we deal here *as if* W^{TF}_B *was very smooth* which it is not the case; however later we will show that its smoothness is sufficient to employ arguments of Chapter 18 rather

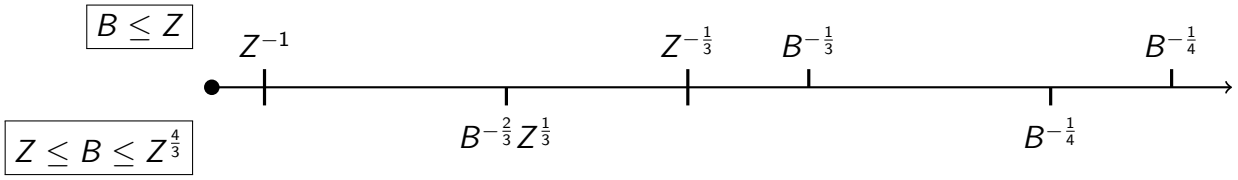


Figure 26.2: Let $B \le Z^{\frac{4}{3}}$. Then at $\{\ell \asymp Z^{-\frac{1}{3}}\}$ are contained both the bulk of charge and the bulk of energy, $\ell \asymp \mathsf{min}((Z-N)_+^{-\frac{1}{3}}, B^{-\frac{1}{4}}\}$ is the border of $\mathsf{supp}(\rho_B^{\mathsf{TF}})$; $\ell \asymp Z^{-1}$ is the Scott distance; here $h \asymp 1$. Further, $\ell \asymp B^{-\frac{1}{3}}$ if $B \le Z$ and $\ell \asymp B^{-\frac{2}{3}}Z^{\frac{1}{3}}$ if $Z \le B \le Z^{\frac{4}{3}}$ separates $\{\mu \lesssim 1\}$ (on the left) and $\{\mu \gtrsim 1\}$ (on the right).

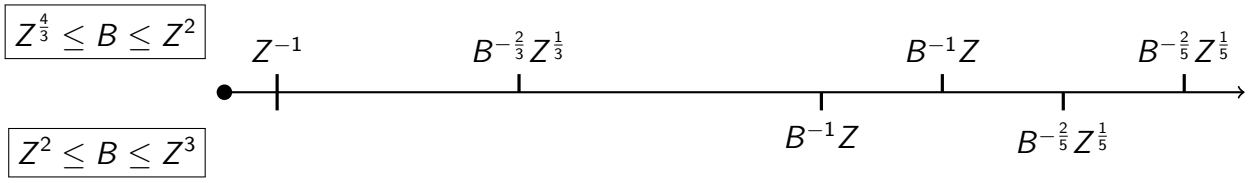


Figure 26.3: Let $Z^{\frac{4}{3}} \le B \le Z^3$. Then at $\{\ell \asymp Z^{-\frac{2}{5}}B^{\frac{1}{5}}\}$ are contained both the bulk of charge and the bulk of energy and it is also the border of $\mathsf{supp}(\rho_B^{\mathsf{TF}})$; $\ell \asymp Z^{-1}$ is the Scott distance; here $h \asymp 1$. Further, if $Z^{\frac{4}{3}} \le B \le Z^2$ $\ell \asymp B^{-\frac{2}{3}}Z^{\frac{1}{3}}$ separates $\{\mu \lesssim 1\}$ (on the left) and $\{\mu \gtrsim 1\}$ (on the right) and $\ell \asymp B^{-1}Z$ separates $\{\mu h \lesssim 1\}$ (on the left) and $\{\mu h \gtrsim 1\}$ (on the right).

than those of Chapter 13. We also will deal as if non-degeneracy conditions were satisfied leaving them also to more rigorous arguments below.

It will allow us to establish our target remainder estimates which we will be able to prove rigorously for $M = 1$ (in this section) while for $M \ge 2$ (in the next two sections) our results will be not that good.

Total Charge

Consider

$$\int e(x,x,\nu)\psi(x)\,dx, \tag{26.3.1}$$

first with γ-admissible $\psi(x)$, where $\gamma \le \epsilon\ell$. Recall that $\ell(x) = \mathsf{min}_m |x - y_m|$ is the distance to the nearest nucleus.

General Arguments. The main part of the semiclassical expression for (26.3.1) is of magnitude $h'^{-3}+\mu' h'^{-2} \asymp \zeta^3\gamma^3 + B\zeta\gamma^3$ with $h'=1/(\zeta\gamma)$ and $\mu'=B\gamma/\zeta$.

Indeed, let us rescale $x\mapsto x/\gamma$ and $\tau\mapsto \tau/\zeta^2$ which leads to $h=1\mapsto h'$ and $B\mapsto \mu'$. In particular, for $\gamma\asymp \ell$ we get

$$\zeta^3\ell^3 + B\zeta\ell^3. \tag{26.3.2}$$

Meanwhile, the remainder in the semiclassical expression for (26.3.1) does not exceed $Ch'^{-2}+C\mu' h'^{-1}\asymp \zeta^2\gamma^2+B\gamma^2$ (gaining factor h' in comparison to the main part; here we need the smoothness and if $\mu'\ge h'^{\delta-1}$ we also need the non-degeneracy); for $\gamma\asymp\ell$ we get

$$\zeta^2\ell^2 + B\ell^2. \tag{26.3.3}$$

Sure, we ignored the fact that $h'\le 1$ does not necessarily hold even if $\gamma\asymp\ell$ but we believe that the contributions to the main part and the remainder of these zones will be less than of zone where this inequality holds, provided $B\ll Z^3$.

Finally, let us sum expressions (26.3.2) and (26.3.3) with respect to ℓ-partition.

Moderate Magnetic Field. Consider the case $B\le Z^{\frac43}$ first. Then for $\ell\le Z^{-\frac13}$ we plug $\zeta = Z^{\frac12}\ell^{-\frac12}$ into (26.3.2) and (26.3.3) resulting in

$$Z^{\frac32}\ell^{\frac32} + BZ^{\frac12}\ell^{\frac52} \qquad \text{and} \qquad Z\ell + B\ell^2 \tag{26.3.4}_{0,1}$$

in the main part and in the remainder respectively and the summation over zone $\{x\colon \ell(x)\le Z^{-\frac43}\}$ results in the same expressions with $\ell = Z^{-\frac13}$, i. e. in $Z+BZ^{-\frac13}\asymp Z$ and $Z^{\frac23}+BZ^{-\frac23}\asymp Z^{\frac23}$ respectively.

On the other hand, for $\ell\ge Z^{-\frac13}$ we plug $\zeta=\ell^{-2}$ into (26.3.2) and (26.3.3) resulting in

$$\ell^{-3}+B\ell \qquad \text{and} \qquad \ell^{-2}+B\ell^2; \tag{26.3.5}_{0,1}$$

then summation over zone $\{x\colon Z^{-\frac13}\le \ell(x)\le \bar r = B^{-\frac14}\}$ results in $Z+B^{\frac34}\asymp Z$ and $Z^{\frac23}+B^{\frac12}\asymp Z^{\frac23}$ respectively.

Strong Magnetic Field. Consider the case $B \ge Z^{\frac{4}{3}}$ now. Then the threshold $Z^{-\frac{1}{3}}$ disappears and we sum expressions $(26.3.4)_{0,1}$ over zone $\{x\colon \ell(x) \le \bar{r} \coloneqq Z^{\frac{1}{5}}B^{-\frac{2}{5}}\}$, resulting in $Z^{\frac{9}{5}}B^{-\frac{3}{5}}+Z \asymp Z$ and $Z^{\frac{6}{5}}B^{-\frac{2}{5}}+Z^{\frac{2}{5}}B^{\frac{1}{5}} \asymp Z^{\frac{2}{5}}B^{\frac{1}{5}}$ respectively.

Therefore, for both cases $B \lessgtr Z^{\frac{4}{3}}$ we arrive to

(26.3.6) The total charge is $\mathsf{min}(N, Z)$ (due to the choice of ν) with the remainder estimate $O\big(\mathsf{max}(Z^{\frac{2}{3}}, Z^{\frac{2}{5}}B^{\frac{1}{5}})\big)$ which is $O(Z^{\frac{2}{3}})$ if $B \le Z^{\frac{4}{3}}$ and $O(Z^{\frac{2}{5}}B^{\frac{1}{5}})$ if $Z^{\frac{4}{3}} \le B \le Z^3$.

Remark 26.3.1. Remainder is less than the main part if $Z^{\frac{2}{5}}B^{\frac{1}{5}} \lesssim Z$ i.e. $B \le Z^3$. It means exactly that $\zeta\ell \ge 1$ if $\ell = \bar{r}$ (in the case $B \ge Z^{\frac{4}{3}}$), or, in other words that $h \lesssim 1$. The same is true for all other semiclassical asymptotics below.

If $B \ll Z^3$ we arrive to asymptotics, ig $B \lesssim Z^3$ we have estimates and in the case of superstrong magnetic field $B \gg Z^3$ Thomas-Fermi theory is not valid for our main model.

Semiclassical D-Term

Consider now the semiclassical D-term

$$\mathsf{D}\big(e(x,x,\nu) - \rho_B^{\mathsf{TF}}(x),\ e(x,x,\nu) - \rho_B^{\mathsf{TF}}(x)\big). \tag{26.3.7}$$

General Arguments. We do not have appropriate asymptotics for $e(x,x,\nu)$ in the case of the magnetic field[9)] but we apply Fefferman–de Llave decomposition (16.4.1):

$$|x-y|_\gamma^{-1}(x,y) \coloneqq |x-y|\varphi(\gamma^{-1}|x-y|) = \gamma^{-4}\int \psi_{1,\gamma}(x,z)\psi_{2,\gamma}(y,z)\,dz \tag{26.3.8}$$

where $\varphi \in \mathscr{C}^\infty([1,2])$.

Therefore contribution of $B(z,\gamma) \times B(z',\gamma)$ with $3\gamma \le |z-z'| \le 4\gamma$, $\gamma \le \epsilon\ell(z)$ to such term does not exceed $C\big(\zeta^2\gamma^2 + B\gamma^2\big)^2\gamma^{-1}$. There are $\asymp \ell^3\gamma^{-3}$ of such pairs with $\ell(x) \asymp \ell$ and their total contribution does not exceed $C\big(\zeta^2 + B\big)^2\ell^3$.

[9)] Unless we really assume that W is smooth and apply results sections 16.6–16.9.

Now we need to sum over $\gamma^{-1}\,d\gamma$ which does not look good because it leads to the logarithmic divergency but there is a simple remedy: we treat this way only pairs $t\ell \le |z-z'| \le \ell$ and apply for pairs with $|z-z'| \le t\ell$ pointwise asymptotics; then we get

$$C(\zeta^2+B)^2\ell^3\big(1+(\log B\ell/\zeta)_+\big); \tag{26.3.9}$$

to get rid of this logarithmic factor we apply more delicate arguments similar to those of Subsection 16.10.3.

Thus, ignoring this logarithmic factor we conclude that the contribution of all pairs (z,z') with $\ell(z)\asymp\ell(z')\asymp\ell$ does not exceed $C(\zeta^2+B)^2\ell^3$ while contribution of all pairs (z,z') with $\ell(z)\asymp\ell_1\not\asymp\ell(z')\asymp\ell_2$ does not exceed $C(\zeta_1^2+B)(\zeta_2^2+B)\ell_1^2\ell_2^2(\ell_1+\ell_2)^{-1}$.

Finally let us sum these expressions over partitions of unity.

Moderate Magnetic Field. Consider the case $B\le Z^{\frac{4}{3}}$. Then summation over zone $\{\ell_1\le Z^{-\frac{1}{3}},\ \ell_2\le Z^{-\frac{1}{3}}\}$ results in $CZ^{\frac{5}{3}}$ and the same is also true for summation over zone $\{Z^{-\frac{1}{3}}\le\ell_1\le B^{-\frac{1}{4}},\ Z^{-\frac{1}{3}}\le\ell_2\le B^{-\frac{1}{4}}\}$.

Obviously, in such estimates, if there is a fixed number of zones, we do not need to sum over "mixed" pairs when z and z' belong to different zones.

Strong Magnetic Field. Consider the case $B\ge Z^{\frac{4}{3}}$. Then summation over zone $\{\ell_1\le Z^{\frac{1}{5}}B^{-\frac{2}{5}},\ \ell_2\le Z^{\frac{1}{5}}B^{-\frac{2}{5}}\}$ results in $CZ^{\frac{3}{5}}B^{\frac{4}{5}}$.

Therefore, for both cases $B\lessgtr Z^{\frac{4}{3}}$ we arrive to

(26.3.10) Term (26.3.7) does not exceed $C\max(Z^{\frac{5}{3}}, Z^{\frac{3}{5}}B^{\frac{4}{5}})$ which is $CZ^{\frac{5}{3}}$ if $B\le Z^{\frac{4}{3}}$ and $CZ^{\frac{3}{5}}B^{\frac{4}{5}}$ if $Z^{\frac{4}{3}}\le B\le Z^3$.

Remark 26.3.2[10]. Estimating this term, and also the second D-term (in the next paragraph) we need to estimate the contribution of the singular zone $\{x\colon \ell(x)\le \bar r = Z^{-1}\}$ where effective semiclassical parameter is less than 1. We claim that there

$$e(x,x,0)\le C(BZ+Z^3)\qquad \text{for}\quad \lambda\le cZ^2. \tag{26.3.11}$$

Indeed, it is true if $\ell(x)\ge 1$. Also operator H is bounded from below by $-CZ^2$. And finally, in the ball of $B(\mathsf{y}_m,\epsilon Z^{-1})$ operator Δ is larger than $Z|x-\mathsf{y}_m|^{-1}$. We leave the easy details to the reader.

[10] Cf. Remark 25.4.7.

Therefore for $B \le Z^2$ the contribution of this zone into N-term is $O(CZ^3\bar{r}^3) = O(1)$, into both D-terms is $O(Z^6\bar{r}^5) = O(Z)$, and into T-term is $O(Z^5\bar{r}^3) = O(Z^2)$ exactly as in Chapter 25. On the other hand, for $Z^2 \le B \le Z^3$ the contribution of this zone into N-term is $O(CBZ\bar{r}^3) = O(BZ^{-2})$, into both D-terms is $O(B^2Z^2\bar{r}^5) = O(B^2Z^{-3})$, and into T-term is $O(BZ^3\bar{r}^3) = O(B)$.

$|\lambda_N - \nu|$ and another D-Term

Consider two other non-trace terms in the upper estimate.

Moderate Magnetic Field. In the case $B \le Z^{\frac{4}{3}}$ we established the remainder in the total charge $O(Z^{\frac{2}{3}})$. Then using our standard arguments we conclude easily that $|\lambda_N - \nu| = O(Z)$ and then

$$|\lambda_N - \nu| \cdot |\mathsf{N}(\nu) - N| \le CZ^{\frac{5}{3}} \tag{26.3.12}$$

and

$$\begin{aligned}\mathsf{D}\big(&P'_B(W_B^{\mathsf{TF}}(x) + \lambda_N) - P'_B(W_B^{\mathsf{TF}}(x) + \nu),\\ &P'_B(W_B^{\mathsf{TF}}(x) + \lambda_N) - P'_B(W_B^{\mathsf{TF}}(x) + \nu)\big) \le CZ^{\frac{5}{3}};\end{aligned} \tag{26.3.13}$$

combining with the estimate of the previous subsubsection we conclude that

$$\mathsf{D}(\rho_\Psi - \rho_B^{\mathsf{TF}}, \rho_\Psi - \rho_B^{\mathsf{TF}}) \le CQ = O(Z^{\frac{5}{3}}), \tag{26.3.14}$$

exactly as in (25.4.56).

Strong Magnetic Field. Let now $Z^{\frac{4}{3}} \le B \le Z^3$. Then we established the remainder in the total charge $O(Z^{\frac{2}{5}}B^{\frac{1}{5}})$ and for the semiclassical D-term we established estimate $O(Z^{\frac{3}{5}}B^{\frac{4}{5}})$. Therefore to estimate

$$|\lambda_N - \nu| \cdot |\mathsf{N}(\nu) - N| \le CZ^{\frac{3}{5}}B^{\frac{4}{5}} \tag{26.3.15}$$

as well we want to prove that

$$|\lambda_N - \nu| = O(Z^{\frac{1}{5}}B^{\frac{3}{5}}). \tag{26.3.16}$$

Observe that $|\nu| \lesssim Z\bar{r}^{-1} \asymp Z^{\frac{4}{5}}B^{\frac{2}{5}} \le CB$. Therefore if $|\lambda_N - \nu| \le \frac{1}{2}|\nu|$ we conclude that

$$|\int \bigl(P'_B(W^{\mathsf{TF}}_B(x)+\lambda_N) - P'_B(W^{\mathsf{TF}}_B(x)+\nu)\bigr)\,dx| \ge \epsilon|\lambda_N-\nu|B\int (W+\nu)^{-\frac{1}{2}}_+\,dx \tag{26.3.17}$$

with the integral taken over zone $\{x\colon W(x)+\nu \ge |\lambda_N-\nu|\}$.

One can see easily that as $|\lambda_N-\nu| \le \epsilon|\nu|$ the right-hand expression of (26.3.17) is larger than $\epsilon|\lambda_N-\nu|\cdot Z^{\frac{1}{5}}B^{-\frac{2}{5}}$ and it must be less than $CZ^{\frac{2}{5}}B^{\frac{1}{5}}$: $|\lambda_N-\nu|Z^{\frac{1}{5}}B^{-\frac{2}{5}} \le CZ^{\frac{2}{5}}B^{\frac{1}{5}}$ which implies (26.3.16).

Let us estimate the left-hand expression of (26.3.13). For this, however, estimate (26.3.16) is insufficient. We consider here only the atomic case. Then using (26.2.22)–(26.2.23) one can prove easily that the right-hand expression of (26.3.17) is of magnitude

$$|\lambda_N-\nu|\cdot B\bar{r}^2 \times \underbrace{(|\nu|\bar{r}^{-1})^{-\frac{1}{3}}B^{-\frac{1}{3}}} \asymp |\lambda_N-\nu|\cdot|\nu|^{-\frac{1}{3}}Z^{\frac{7}{15}}B^{-\frac{4}{15}}$$

provided $|\lambda-N| \le \epsilon\nu$, where the selected factor is just $\int (B^2z^4+|\nu|\bar{r}^{-1})^{-\frac{1}{2}}_+\,dz$ (appearing due to (26.2.22)–(26.2.23)). Comparing with $Z^{\frac{2}{5}}B^{\frac{1}{5}}$ we conclude that

(a) If $|\nu| \ge C_1Z^{-\frac{1}{10}}B^{\frac{7}{10}}$ $(= C_1Z^{\frac{2}{5}}B^{\frac{3}{5}} \times Z^{-\frac{3}{10}}B^{\frac{1}{10}})$ then

$$|\lambda_N-\nu| \le C|\nu|^{\frac{1}{3}}Z^{-\frac{1}{15}}B^{\frac{7}{15}} \tag{26.3.18}$$

which is less than $\epsilon|\nu|$ and coincides with (26.3.16) as $(Z-N)_+ \asymp Z$.

(b) If $|\nu| \ge C_1Z^{-\frac{1}{10}}B^{\frac{7}{10}}$ then $|\lambda_N-\nu| \le C_2Z^{-\frac{1}{10}}B^{\frac{7}{10}}$.

In the former case one can prove easily that the left-hand expression of (26.3.13) does not exceed $CZ^{\frac{3}{5}}B^{\frac{4}{5}}$.

In the latter case (exactly as in Subsection 25.4.2) we consider Thomas-Fermi theory with $\nu = 0$ i.e. $N = Z$ and also prove that that

(26.3.19) The left-hand expression of (26.3.16) does not exceed $Q = CZ^{\frac{3}{5}}B^{\frac{4}{5}}$.

In particular, we slightly improve estimate (26.3.15) to $|\nu|^{\frac{1}{3}}Z^{\frac{3}{5}}B^{\frac{2}{3}}$ as well (if $(Z-N) \ll Z$).

Therefore in our framework we estimated all non-trace terms in the upper estimate by $CZ^{\frac{3}{5}}B^{\frac{4}{5}}$ and therefore "proved" estimate

$$\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}_B, \rho_\Psi - \rho^{\mathsf{TF}}_B) \le CQ = O(Z^{\frac{3}{5}}B^{\frac{4}{5}}). \tag{26.3.20}$$

Trace

Consider now $\mathsf{Tr}((H_{A,W}-\nu)^-)$. This term is of magnitude $\int(\zeta^5+B\zeta^3)\,dx$ and one can see easily that it is $\asymp Z^{\frac{7}{3}}$ for $B\le Z^{\frac{4}{3}}$ and $\asymp B^{\frac{2}{5}}Z^{\frac{9}{5}}$ for $Z^{\frac{4}{3}}\le B\le Z^3$.

Meanwhile, consider the remainder. Again for simplicity consider only the atomic case. If $B\le Z$ the contribution of the zone $\{x\colon \ell(x)\le Z^{-\frac{1}{3}}\}$ is $O(Z^{\frac{5}{3}})$ (we need to include Scott correction term in the main part) while the contribution of the zone $\{x\colon \ell(x)\ge Z^{-\frac{1}{3}}\}$ does not exceed

$$C\int(\zeta^3+B\zeta)\ell^{-2}\,dx \tag{26.3.21}$$

taken over this zone and it is $\asymp Z^{\frac{5}{3}}$ as well.

If $Z\le B\le B^2$ the contribution of the zone $\{x\colon \ell(x)\le b\coloneqq B^{-\frac{2}{3}}Z^{\frac{1}{3}}\}$ is $O(b^{-\frac{1}{2}}Z^{\frac{3}{2}})=O(Z^{\frac{4}{3}}B^{\frac{1}{3}})$ and we need to include Scott correction term. Meanwhile, the contribution of the zone $\{x\colon \ell(x)\ge b\}$ does not exceed integral (26.3.21) taken over this zone which is $\asymp Z^{\frac{4}{3}}B^{\frac{1}{3}}+Z^{\frac{3}{5}}B^{\frac{4}{5}}$ where the last term coincides with estimate for (26.3.7) if $B\ge Z^{\frac{4}{3}}$ and does not exceed $CZ^{\frac{5}{3}}$ if $B\le Z^{\frac{4}{3}}$.

Finally, if $Z^2\le B\le B^3$ we need to reset $b=Z^{-1}$ because $h=1/(\zeta\ell)$ becomes $\gtrsim 1$ inside. Then we do not need Scott correction term and the contributions of the zone $\{x\colon \ell(x)\le b\}$ to both the main part and the remainder do not exceed $C\int(\zeta^5+B\zeta^3)\,dx\asymp Z^2+B\asymp B$.

Further, the contribution of the zone $\{x\colon \ell(x)\ge b\}$ to the remainder does not exceed integral (26.3.21) taken over this zone which results in $CB+CZ^{\frac{3}{5}}B^{\frac{4}{5}}$ and the second term dominates due to assumption $B\ll Z^3$. Thus we arrive to

(26.3.22) The main therm in $\mathsf{Tr}((H_{A,W}-\nu)^-)$ is of magnitude $Z^{\frac{7}{3}}$ for $B\le Z^{\frac{4}{3}}$ and $B^{\frac{2}{5}}Z^{\frac{9}{5}}$ for $Z^{\frac{4}{3}}\le B\le Z^3$, while the remainder estimate is $O(Z^{\frac{5}{3}})$ for $B\le Z$, $O(Z^{\frac{4}{3}}B^{\frac{1}{3}})$ for $Z\le B\le Z^{\frac{4}{3}}$, and $O(Z^{\frac{4}{3}}B^{\frac{1}{3}}+Z^{\frac{3}{5}}B^{\frac{4}{5}})$ for $Z^{\frac{4}{3}}\le B\le Z^3$.

If $B\le Z^{\frac{7}{4}}$ we need to include into main part Scott correction term.

Discussion

Now let us formulate our expectations:

Remark 26.3.3. We expect

(i) Estimate (26.3.14) for $B \le Z^{\frac{4}{3}}$ and estimate (26.3.20) for $Z^{\frac{4}{3}} \le B \le Z^3$.

(ii) Furthermore, since for $B \le Z^{\frac{4}{3}}$ the main contribution to all terms needed to derive this estimate is delivered by the zone $\{x\colon \ell(x) \approx Z^{-\frac{1}{3}}\}$ and the effective magnetic field is $\mu = B\ell/\zeta \approx BZ^{-1}$ we expect improved to "o" (or better) estimate (26.3.14) if $B \ll Z$ and $a \gg Z^{-\frac{1}{3}}$ [11)].

(iii) Statement, similar to (ii) should be also true for the trace term; however then we need to include the Schwinger term.

(iv) The remainder estimate for the ground state energy is maximum of the remainder estimate for the non-trace and trace terms; therefore we expect the same remainder estimate as in (26.3.22); Statement, similar to (ii) should be also correct for the ground state energy. However then we need to include both Schwinger and Dirac terms.

(v) We expect the described remainder estimate of the trace term and the ground state energy if a is large enough; otherwise it should contain term $O(a^{-\frac{1}{2}}Z^{\frac{3}{2}})$ ifs $B \le Z^{\frac{7}{4}}$ and $a \ge Z^{-1}$ (and in this case we include Scott correction term).

Remark 26.3.4. The other difference between cases $B \le Z^{\frac{4}{3}}$ and $B \ge Z^{\frac{4}{3}}$ is that $\mu h = B\zeta^{-2} \lesssim 1$ in the former case if $\ell(x) \le \bar{r}$; however in the latter case it happens only if $\ell(x) \le B^{-1}Z$ but in the zone $\{x\colon B^{-1}Z \le \ell(x) \le B^{-\frac{2}{5}}Z^{\frac{1}{5}}\}$ an opposite inequality holds.

26.3.2 Smooth Approximation

An approach described in Subsection 26.3.1 hits two obstacles: the non-smoothness of W_B^{TF} and its possible degeneration i.e. ∇W_B^{TF} is not disjoint from 0. However non-smoothness of W_B^{TF} is due to the non-smoothness of P_B. So we want to consider first the zone where we can just replace $P_B(W+\nu)$ by $P(W+\nu)$ and therefore W_B^{TF} by some smooth function W which does not necessary coincides with W^{TF}.

Trivial Arguments

Obviously we can do this as an effective magnetic field $\mu = B\ell/\zeta \lesssim 1$. In this case we do not need assumption $W + \nu \asymp \zeta^2$ and therefore we can

[11)] Recall that $a = \min_{1\le m<m'\le M} |\mathsf{y}_m - \mathsf{y}_{m'}|$ is the minimal distance between nuclei.

take $\zeta = \ell^{-4}$ as $B \lesssim Z^{\frac{4}{3}}$ and $\ell \gtrsim Z^{-\frac{1}{3}}$ and $\zeta = Z^{\frac{1}{2}}\ell^{-\frac{1}{2}}$ in all other cases. Therefore zone in question is

$$\mathcal{X}_1 \coloneqq \{x\colon \ell(x) \le r_1\} \tag{26.3.23}$$

$$\text{with}\quad r_1 = \begin{cases} B^{-\frac{1}{3}} & \text{if}\ \ 1 \le B \lesssim Z, \\ B^{-\frac{2}{3}} Z^{\frac{1}{3}} & \text{if}\ \ Z \lesssim B \lesssim Z^2. \end{cases}$$

In this zone $\mathcal{X}_1$ for such modified W we can unleash the full power of the same smooth theory as in Section 25.4 and prove easily the following

Proposition 26.3.5. *Let $1 \le B \le Z^2$. Then*

(i) A contribution of zone $\mathcal{X}_1$ defined by (26.3.23) *to*

$$\int \Big(e(x,x,\nu) - P'(W(x)+\nu)\Big)\,dx \tag{26.3.24}$$

does not exceed $CZ^{\frac{2}{3}}$ while its contribution to

$$\mathsf{D}\Big(e(x,x,\nu) - P'(W(x)+\nu), e(x,x,\nu) - P'(W(x)+\nu)\Big) \tag{26.3.25}$$

does not exceed $CZ^{\frac{5}{3}}$, and its contribution to

$$\int \Big(e_1(x,x,\nu) + P(W(x)+\nu)\Big)\,dx - \mathsf{Scott} \tag{26.3.26}$$

does not exceed $CZ^{\frac{5}{3}} + Ca^{-\frac{1}{2}} Z^{\frac{3}{2}} + CZ^{\frac{4}{3}} B^{\frac{1}{3}}$ [11], [12].

(ii) Further, if $B \ll Z$ and $a \gg Z^{-\frac{1}{3}}$ we can recover for these contributions estimates $CZ^{\frac{2}{3}}\upsilon$, $CZ^{\frac{5}{3}}\upsilon$ and $CZ^{\frac{5}{3}}\upsilon$ respectively with

$$\upsilon \coloneqq Z^{-\delta} + (aZ^{\frac{1}{3}})^{-\delta} + (BZ^{-1})^{\delta} \tag{26.3.27}$$

where expression (26.3.26) *should be modified to*

$$\int \big(e_1(x,x,\nu) - P(W(x)+\nu)\big)\,dx - \mathsf{Scott} - \mathsf{Schwinger}. \tag{26.3.28}$$

Furthermore in this case contribution of $\mathcal{X}_1$ to

$$\frac{1}{2}\int \mathsf{tr}\big(e^{\dagger}(x,y,\nu)e(x,y,\nu)\big)\,dxdy - \mathsf{Dirac} \tag{26.3.29}$$

does not exceed $CZ^{\frac{5}{3}-\delta}$.

[11] If $a \lesssim Z^{-1}$ we skip Scott and reset $a = Z^{-1}$ in the remainder estimate which become CZ^2.

Remark 26.3.6. (i) So far we should use $P(.)$ instead of $P_B(.)$ but we will prove that the same results would hold for P_B as well.

(ii) In the next subsubsections we expand this zone to one defined by $\mu \le h^{-\frac{1}{3}}$ [13)] but for trace term we still need a separate analysis as $\mu \lesssim 1$.

(iii) The same estimates hold if we replace in all expressions (26.3.24)–(26.3.28) P by P_B.

(iv) We assumed that $B \le Z^2$ since otherwise $h \gtrsim 1$ not only in $\mathcal{X}_1$ but even in $\{x\colon W^{\mathsf{TF}}(x) \ge B\}$.

(v) Note that if $r_1 \gtrsim (Z-N)_+^{-\frac{1}{3}}$ this zone (and the whole analysis) could be cut short since outside zone in question $W+\nu \ge 0$. From Chapter 25 we already know how to deal with such irregularities.

(vi) We need to assume that $a \ge Z^{-\frac{1}{3}}$ and to include the second term $(aZ^{\frac{1}{3}})^{-\delta}$ in the definition of υ only as we estimate the trace term (26.3.26).

Remark 26.3.7. (i) If either $a \ll Z^{-\frac{1}{3}}$ or $B \gg Z$ we estimated (26.3.5) by $Ca^{-\frac{1}{2}}Z^{\frac{3}{2}} + CZ^{\frac{4}{3}}B^{\frac{1}{3}}$. While the first term does not bother us since assumption $a \ll \min(Z^{-\frac{1}{3}}, B^{-\frac{1}{4}})$ is unrealistic, the second term is troublesome. Let us assume that $a \ge Z^{-\frac{1}{3}}$.

We can marginally improve this estimate of expression (26.3.26) to $CZ^{\frac{5}{3}} + o\bigl(Z^{\frac{4}{3}}B^{\frac{1}{3}}\bigr)$.

First, observe, that this term $CZ^{\frac{3}{2}}B^{\frac{1}{3}}$ appears as $b^{-\frac{1}{2}}Z^{\frac{3}{2}}$ with $b = B^{-\frac{2}{3}}Z^{\frac{2}{3}} \ll 1$. Therefore we need to estimate this way only contribution of the zone $\mathcal{Y} := \{x\colon b^{\delta} \le \ell(x)Z^{\frac{1}{3}} \le b^{-\delta}\}$ and it is sufficient to investigate the corresponding classical dynamics in the zone $\mathcal{Y}_1 := \{x\colon b^{2\delta} \le \ell(x)Z^{\frac{1}{3}} \le b^{-2\delta}\}$.

Indeed, to recover estimate we have now, we used a classical dynamics on $\Sigma := \{(x,\xi)\colon H(x,\xi)=0\}$ for time $T(x) = Z^{-1}\ell(x)^{\frac{3}{2}}$.

Further, one can see easily that along classical trajectories, starting in $\Sigma|_{\mathcal{Y}}$, $\ell(x) \le b^{-2\delta}$ for time $T = b^{-\sigma}T_0$ with $\sigma = \sigma(\delta) > 0$.

On the other hand, the invariant measure of $\Sigma_r = \{(x,\xi) \in \Sigma, \ell(x) \asymp r\}$ is $\asymp r^2 Z$ and since the spatial speed there is $O(Z^{\frac{1}{2}}r^{-\frac{1}{2}})$ we conclude that

[13)] Or even to $\mu \le h^{-\frac{3}{5}}$ under non-degeneracy assumption (26.3.35) with $\gamma = \ell$, in particular, in the atomic case.

(26.3.30) The invariant measure of the points in $\Sigma|_{\mathcal{Y}}$, such that the classical trajectories starting from them do not remain in $\mathcal{Y}_1$ for time $T = b^{-\sigma}T_0$, does not exceed $b^{2+\sigma}Z^{\frac{1}{3}}$.

(ii) Now it is sufficient to explore the classical dynamics with the Hamiltonian, corresponding to the Coulomb potential and constant magnetic field, and to prove that

(26.3.31) The invariant measure of the periodic points Σ is 0.

To do so, we need to prove that there are non-periodic trajectories, which do not hit an origin. It is sufficient to consider trajectories belonging to the plane $\{z = 0\}$; we assume that magnetic intensity is $(0, 0, B)$. See Part (iii).

(iii) To improve this estimate further we need to investigate the classical dynamics in more details, and it seems to be a daunting, if not impossible task. Indeed, while in 2D the system is completely integrable[14)], it does not seem so in 3D as we know only two first integrals, energy E and $M_z = (x\dot{y} - y\dot{x}) + \frac{1}{2}B(x^2 + y^2)$.

Formal Expansion

Now we want to expand zone $\mathcal{X}_1$. Note first that

$$P'_B(W + \nu) - P'(W + \nu) = O(B^{\frac{3}{2}}) \tag{26.3.32}$$

and

$$P_B(W + \nu) - P(W + \nu) - \kappa_1 B^2(W + \nu)^{\frac{1}{2}} = O(B^{\frac{5}{2}}). \tag{26.3.33}$$

Really, one can consider $P'_B(w)$ and $P_B(w)$ as Riemann sums for integrals $P'(w)$ and $P(w)$ respectively; see Appendix 26.A.3 for details.

However under non-degeneracy assumption $|\nabla W| \asymp \zeta^2\ell^{-1}$ we can do better with the integrated expressions.

[14)] And easily solvable in the polar coordinates since $E = \frac{1}{2}\dot{r}^2 + V^*(r)$ with effective potential $V^*(r) = \frac{1}{2}r^2(M_z r^{-2} - B/2)^2 - r^{-1}$, and the corrected angular momentum $M_z = r^2\dot{\theta} + \frac{1}{2}Br^2$. One can see easily that $V^*(r) \to +\infty$ as $r \to +0$ or $r \to +\infty$ and has a single nondegenerate minimum. Therefore along each trajectory r oscillates between $r_{\min}$ and $r_{\max}$. If all trajectories on the energy level E were periodic, then the number of oscillations was constant for increment θ equal to $2\pi n$ with some $n \in \mathbb{Z}^+$. But this is definitely not the case since the number of oscillations tends to ∞ as $M_z/B \to +\infty$.

Proposition 26.3.8. *Assume that in* $B(z,\gamma)$

$$|\nabla^\alpha W| \le C_\alpha \zeta^2 \gamma^{-|\alpha|} \qquad \forall \alpha : |\alpha| \le n, \tag{26.3.34}$$

$$|\nabla W| \ge \epsilon \zeta^2 \gamma^{-1}, \tag{26.3.35}$$

and

$$B \le \zeta^2. \tag{26.3.36}$$

Then

$$\int \phi(x)\Big(P'_B(W(x)+\nu) - \tilde{P}'_B(W(x)+\nu)\Big)\,dx = O(B^2\zeta^{-1}\gamma^3), \tag{26.3.37}$$

$$\int \phi(x)\Big(P_B(W(x)+\nu) - \tilde{P}'_B(W(x)+\nu)\Big)\,dx = O(B^5\zeta^{-5}\gamma^3) \tag{26.3.38}$$

and

$$\mathsf{D}\Big(\phi(x)\big(P'_B(W(x)+\nu) - \tilde{P}'_B(W(x)+\nu)\big), \phi(x)\big(P'_B(W(x)+\nu) - \tilde{P}'_B(W(x)+\nu)\big)\Big) = O(B^5\zeta^{-4}\gamma^5) \tag{26.3.39}$$

with

$$\tilde{P}_B(w) \coloneqq P(w) + \big(\kappa_1 P''(w)B^2 + \kappa_2 P^{IV} B^4\big)\cdot\big(1-\varphi(w/B)\big) \tag{26.3.40}$$

where $\varphi \in \mathscr{C}^\infty([-2,2])$, $\varphi = 1$ *on* $[-1,1]$.

Proof. Rescaling $x \mapsto x\gamma^{-1}$, $w \mapsto w\zeta^{-2}$ and therefore $B \mapsto \beta = B\zeta^{-2}$ one can reduce the case to $\gamma = \zeta = 1$, $\beta \le 1$ [15]. Then estimates (26.3.37) and (26.3.38) are trivially proven by (multiple) integration by parts which integrates P_β on each step increasing its smoothness[16].

To prove estimate (26.3.39) we apply decomposition (26.3.8). Integration by parts shows that (26.3.37) with t-admissible function ϕ is $O(\beta^3 t^{\frac{3}{2}})$ if $t \ge \beta$ and therefore the contribution of the zone $\{(x,y)\colon |x-y| \asymp t\}$ is $O(\beta^6 t^3 \times t^{-4})$. Then the total contribution of the zone $\{(x,y)\colon |x-y| \ge \beta\}$ is $O(\beta^5)$. Meanwhile a total contribution of the zone $\{(x,y)\colon |x-y| \le \beta\}$ is $O(\beta^3 \times \beta^2)$.

[15] Recall that $\beta = \mu h$ with $\mu = B\gamma/\zeta$ and $h = 1/\zeta\gamma$.

[16] In fact one can prove then estimates $O(\beta^s)$ but adding correction terms $\sum \kappa_k \beta^{2k}$. However this improvement is not carried on to (26.3.39) in full.

Therefore we expect that the zone $\mathcal{X}_1$ defined by $\mu \lesssim 1$ could be expanded to the zone $\mathcal{X}_1'$ defined by $\mu \lesssim h^{-\frac{1}{3}}$ [13] or even larger[17]; furthermore, under assumption $|\nabla W| \asymp \zeta^2\ell^{-1}$ we can define $\mathcal{X}_1'$ by $\mu \lesssim h^{-\frac{3}{5}}$ or even larger[17].

Expansion: Justification

Now however we need to deal with $e(x,x,\nu)$ rather than $P_B'(W(x)+\nu)$ (etc).

Proposition 26.3.9. *Assume that in $B(z,\gamma)$ conditions* (26.3.34), $\zeta\gamma \ge 1$ *and*

$$B \le c\zeta^2(\zeta\gamma)^{-\delta} \tag{26.3.41}$$

are fulfilled. Then for γ-admissible ϕ

$$\int \phi(x)\Big(e(x,x,\nu) - \tilde{P}_B'(W(x)+\nu)\Big)\,dx = O(\zeta^2\gamma^2), \tag{26.3.42}$$

$$\int \phi(x)\Big(e_1(x,x,\tau) - \tilde{P}_B(W(x)+\nu)\Big)\,dx = O(\zeta^3\gamma) \tag{26.3.43}$$

and

$$\mathsf{D}\Big(\phi(x)\big(e(x,x,\nu) - \tilde{P}_B'(W(x)+\nu)\big), \phi(x)\big(e(x,x,\nu) - \tilde{P}_B'(W(x)+\nu)\big)\Big) = O(\zeta^4\gamma^3). \tag{26.3.44}$$

Proof. Estimates (26.3.42) and (26.3.43) are due to Chapter 13. Really, rescale $x \mapsto x\gamma^{-1}$, $\tau \mapsto \tau\zeta^{-2}$ and $h = 1 \mapsto h = \gamma^{-1}\zeta^{-1}$, $B \mapsto \mu = B\gamma\zeta^{-1}$.

To prove (26.3.44) let us apply decomposition (26.3.8); then according to (26.3.42) $J(t)$ (defined as expression (26.3.42) with $t\gamma$-admissible ϕ_t) does not exceed $C\zeta^2\gamma^2t^2$ as long as $t\zeta\gamma \ge 1$; therefore contribution of zone $\{(x,y)\colon |x-y| \asymp t\}$ into the left-hand expression of (26.3.44) does not exceed $C(\zeta^2\gamma^2t^2)^2 \times t^{-4}\gamma^{-1} \asymp C\zeta^4\gamma^3$.

Then summation over $t \ge \mu^{-1} = B^{-1}\gamma^{-1}\zeta$ returns $C\zeta^4\gamma^3\int t^{-1}\,dt \asymp C\zeta^4\gamma^3\log\mu$ (we assume that $\mu \ge 2$; case $\mu \lesssim 2$ has been covered already). So, the total contribution of zone $\{(x,y)\colon |x-y| \ge \mu^{-1}\}$ does not exceed $C\zeta^4\gamma^3\log\mu$.

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[17] We do not need for each ℓ have a sharp remainder estimates but need only them to sum to a sharp estimate.

Let us get rid of the logarithmic factor. Returning back to $B(z,t)$ stretched to $B(0,1)$ one can see easily that conditions of Proposition 13.7.25 are fulfilled as well with $T = \min\bigl(t^{-\delta}, h^{-\delta}t^{\delta}\bigr)$ and thus $|J(t)| \le C(ht^{-1})^{-2}T^{-1} \le Ch^{-2}t^2\bigl(t^{\delta} + h^{\delta}t^{-\delta}\bigr)$. Plugging into (26.3.8) we get

$$Ch^{-4}\gamma^{-1}\int_{\mu^{-1}}^{1} t^{-1}\bigl(t^{\delta} + h^{\delta}t^{-\delta}\bigr)\,dt \asymp Ch^{-4}\gamma^{-1} = C\zeta^4\gamma^3.$$

On the other hand, in zone $t \le \mu^{-1}$ we use the trivial estimate

$$e(x,x,\nu) - P'(W(x)+\nu) = O(\mu\zeta^2\gamma^2)$$

(due to simple rescaling $x \mapsto \mu x$) and its contribution to the left-hand expression of (26.3.44) does not exceed $C(\mu\zeta^2\gamma^2)^2 \times \mu^{-2}\gamma^{-1} \asymp C\zeta^4\gamma^3$. □

Combining with estimates (26.3.32) and (26.3.33) we arrive to Statement (i) below; combining with Proposition 26.3.7 to Statement (ii):

Corollary 26.3.10. *Assume that in $B(z,\gamma)$ conditions* (26.3.34) *and $\zeta\gamma \ge 1$ are fulfilled. Let ϕ be γ-admissible function.*

(i) Let

$$B \le c\zeta^{\frac{4}{3}}\gamma^{-\frac{2}{3}}; \tag{26.3.45}$$

then

$$\int \phi(x)\Bigl(e(x,x,\nu) - \tilde{P}'_B(W(x)+\nu)\Bigr)\,dx = O(\zeta^2\gamma^2), \tag{26.3.46}$$

$$\int \phi(x)\Bigl(e_1(x,x,\tau) - \tilde{P}_B(W(x)+\nu)\Bigr)\,dx = O(\zeta^3\gamma) \tag{26.3.47}$$

and

$$\mathsf{D}\Bigl(\phi(x)\bigl(e(x,x,\nu) - \tilde{P}'_B(W(x)+\nu)\bigr), \phi(x)\bigl(e(x,x,\nu) - \tilde{P}'_B(W(x)+\nu)\bigr)\Bigr) = O(\zeta^4\gamma^3). \tag{26.3.48}$$

(ii) Let assumption (26.3.35) *be fulfilled and*

$$B \le c\zeta^{\frac{8}{5}}\gamma^{-\frac{2}{5}}. \tag{26.3.49}$$

Then (26.3.46)–(26.3.48) *hold.*

26.3.3 Rough Approximation

Unless our analysis has been cut short with $r_1 \gtrsim (Z-N)_+^{-\frac{1}{3}}$, we need to consider the zone $\{x\colon \ell(x) \ge r_1\}$ with redefined r_1, so that this zone is described by $\mu \gtrsim h^{-\frac{1}{3}}$ or $\mu \gtrsim h^{-\frac{3}{5}}$ in the general or non-degenerate (i.e. satisfying assumption (26.3.35)) cases respectively.

In this zone the replacement of P_B by P and thus W_B^{TF} by some smooth function leads to the error which is too large. Therefore instead in this zone we consider $\varepsilon\ell$-mollification of W_B^{TF} with $\varepsilon \ll 1$ (after rescaling $x \mapsto x/\ell$). In contrast to potentials considered in Chapter 18 function W_B^{TF} is more regular.

Properties of Mollification

First, recall regularity properties of W_B^{TF}:

Proposition 26.3.11. *W_B^{TF} have the following properties:*

$$|\nabla^\alpha W_B^{\mathsf{TF}}(x)| \le c_\alpha \zeta(x)^2 \ell(x)^{-|\alpha|} \qquad \forall \alpha : |\alpha| \le 2, \tag{26.3.50}$$

$$\begin{aligned} |\nabla^\alpha \bigl(W_B^{\mathsf{TF}}(x) - W_B^{\mathsf{TF}}(y)\bigr)| \le {}& \\ & c_0 B\ell(x)^{-\frac{5}{2}}|x-y|^{\frac{1}{2}} + c_0\zeta(x)^2\ell(x)^{-3}|x-y| \\ & \forall |\alpha| = 2 \quad \forall x,y\colon |x-y| \le \epsilon\ell(x) \end{aligned} \tag{26.3.51}$$

where we recall

$$\zeta(x) = \min\bigl(Z^{\frac{1}{2}}\ell(x)^{-\frac{1}{2}}, \ell(x)^{-2}\bigr) \qquad \text{if } \ B \le Z^{\frac{4}{3}}, \tag{26.3.52}$$

$$\zeta(x) = Z^{\frac{1}{2}}\ell(x)^{-\frac{1}{2}} \qquad \text{if } \ B \ge Z^{\frac{4}{3}}; \tag{26.3.53}$$

Proof. This proof is rather obvious corollary of the Thomas-Fermi equation (26.2.3). See also arguments below. □

Let us consider $B(z,\ell(z))$ with $\zeta^2 \gtrsim B$ and rescale $x \mapsto x\ell^{-1}$, $W \mapsto w = \zeta^{-2}(W+\nu)$ (where we included ν for a convenience). After such rescaling $w \in \mathscr{C}^{\frac{5}{2}}$ uniformly, but there is more: Thomas-Fermi equation (26.2.3) translates into

$$\frac{1}{4\pi}\Delta w = \ell^2 P'_\beta(w) = \ell^2\zeta P'(w) + \ell^2\zeta\bigl(P'_\beta(w) - P'(w)\bigr) \tag{26.3.54}$$

with $\beta = B\zeta^{-2}$; observe that $P'_B(W)$ is positively homogeneous of degree 3 with respect to (W, B).

Note that parameter $\eta := \zeta\ell^2 \lesssim 1$ and $\eta \asymp 1$ if and only if $B \lesssim Z^{\frac{4}{3}}$ and $\ell \gtrsim Z^{-\frac{4}{3}}$ (in which case $\zeta \asymp \ell^{-2}$).

Also note that the first term and the second terms in the right-hand expression of (26.3.54) belong to $\mathscr{C}^{\frac{5}{2}}$ and $\beta\eta\mathscr{C}^{\frac{1}{2}}$ respectively uniformly[18] and

$$\beta\eta = \beta\zeta\ell^2 = B\zeta^{-1}\ell^2, \qquad \eta := \zeta\ell^2 \qquad \text{if } \beta \lesssim 1. \tag{26.3.55}$$

Because of this $w \in \mathscr{C}^{\frac{9}{2}} \oplus \beta\eta\mathscr{C}^{\frac{5}{2}}$ again uniformly. Iterating, we conclude that $w \in \mathscr{C}^n \oplus \beta\eta\mathscr{C}^{\frac{5}{2}}$ with arbitrarily large exponent n.

On the other hand, if $B \gtrsim \zeta^2$ (i.e. $\beta \gtrsim 1$) without invoking P'_B one can prove easily that $w \in \eta\mathscr{C}^{\frac{5}{2}}$ with

$$\eta := \beta\zeta\ell^2 = B\zeta^{-1}\ell^2 \qquad \text{if } \beta \gtrsim 1. \tag{26.3.55}'$$

Therefore we have proven

(26.3.56) $w \in \mathscr{C}^n \oplus \beta\eta\mathscr{C}^{\frac{5}{2}}$ with arbitrarily large exponent n as $\beta \lesssim 1$ and $w \in \eta\mathscr{C}^{\frac{5}{2}}$ as $\beta \gtrsim 1$

and one can see easily that

(26.3.57) Parameter $\eta = B\zeta^{-1}\ell^2$ is $O(1)$ and $\eta \asymp 1$ iff <u>either</u> $B \le Z^{\frac{4}{3}}$ and $\ell \asymp B^{-\frac{1}{4}}$ <u>or</u> $B \ge Z^{\frac{4}{3}}$ and $\ell \asymp B^{-\frac{2}{5}}Z^{\frac{1}{5}}$ (i.e. near border of $\mathsf{supp}(\rho_B^{\mathsf{TF}})$, uncut by ν).

Remark 26.3.12. It may seem strange to define η differently as $\beta \lesssim 1$ and $\beta \gtrsim 1$ but there is a good reason for this when we consider the case of $M \ge 2$. Anyway, η is the magnitude of the right-hand expression of (26.3.54).

Proposition 26.3.13. *(i) Let w_ε be a ε-mollification of w with $\varepsilon \lesssim \min(\beta, h^\delta)$ (recall that $h = 1/(\zeta\ell)$). Then if $\beta \lesssim 1$ the following estimates hold:*

$$|\nabla^\alpha(w - w_\varepsilon)| \le c_\alpha\beta\eta\varepsilon^{\frac{5}{2}-|\alpha|} \qquad \forall \alpha : |\alpha| \le 2, \tag{26.3.58}$$

$$|P_\beta(w) - P_\beta(w_\varepsilon)| \le c\beta\eta\varepsilon^{\frac{5}{2}} \tag{26.3.59}$$

and

$$|P'_\beta(w) - P'_\beta(w_\varepsilon)| \le c\beta^{\frac{3}{2}}\eta^{\frac{1}{2}}\varepsilon^{\frac{5}{4}} + c\beta\eta\varepsilon^{\frac{5}{2}}; \tag{26.3.60}$$

[18] I.e. norms do not depend on any parameters.

(ii) On the other hand, if $\beta \gtrsim 1$ *the right-hand expressions of* (26.3.58)–(26.3.60) *should be replaced by the similar expressions albeit without* β*:*

$$|\nabla^{\alpha}(w - w_{\varepsilon})| \le c_{\alpha}\eta\varepsilon^{\frac{5}{2}-|\alpha|} \qquad \forall \alpha : |\alpha| \le 2, \tag{26.3.58}'$$

$$|P_{\beta}(w) - P_{\beta}(w_{\varepsilon})| \le c\eta\varepsilon^{\frac{5}{2}} \tag{26.3.59}'$$

and

$$|P'_{\beta}(w) - P'_{\beta}(w_{\varepsilon})| \le c\eta^{\frac{1}{2}}\varepsilon^{\frac{5}{4}}. \tag{26.3.60}'$$

(iii) Further, under assumption $|\nabla w| \asymp 1$ *in both cases*

$$|\int \phi(x)\big(P_{\beta}(w) - P_{\beta}(w_{\varepsilon})\big)\,dx| \le c\eta\varepsilon^{\frac{7}{2}}, \tag{26.3.61}$$

$$|\int \phi(x)\big(P'_{\beta}(w) - P'_{\beta}(w_{\varepsilon})\big)\,dx| \le c\eta\varepsilon^{\frac{9}{4}} \tag{26.3.62}$$

and

$$\mathsf{D}\big(\phi(P'_{\beta}(w) - P'_{\beta}(w_{\varepsilon})),\, \phi(P'_{\beta}(w) - P'_{\beta}(w_{\varepsilon}))\big) \le c\eta^{2}\varepsilon^{\frac{9}{2}}. \tag{26.3.63}$$

Proof. Proof of Statement (i) is trivial; in particular, we observe that $\eta\varepsilon^{\frac{5}{2}} \lesssim \beta$.

Proof of Statement (iii) is also easy since then w_{ε} is different from w on the set of measure $\asymp \beta^{-1}\varepsilon$ if $\beta \le C_0$ and on the set of measure $\asymp \varepsilon$ if $\beta \ge C_0$. Actually w is uniformly smooth if $\beta \gtrsim 1$ and $\ell(x) \le \epsilon\bar{r}$ and we do not need any mollification here.

One definitely can improve estimates (26.3.61)–(26.3.63) but we do not need it. □

Consider now the analytical expressions and estimate the semiclassical errors.

Remark 26.3.14. (i) From now on until the end of this Section we assume that $M = 1$ to avoid possible degenerations.

(ii) Recall that we can reduce operator with mollified potential to a canonical form provided $\varepsilon \ge C(\mu^{-1}h)^{\frac{1}{2}}|\log\mu|$ (see Section 18.7). However here we will have a much better estimate since we will take $\varepsilon \ge h^{\frac{2}{3}-\delta}$.

Charge Term

Let us consider the *charge term* i.e. expression $\int e(x,x,\nu)\,dx = (\mathsf{Tr}\,\theta(\nu - H))$.

Regular Zone. Then the results of Section 18.9 implies that as

$$W+\nu \asymp \zeta^2 \tag{26.3.64}$$

and

$$|\nabla W| \asymp \zeta^2\ell^{-1} \tag{26.3.65}$$

contribution of the ball $B(x,\ell(x))$ to expression (26.3.24) does not exceed $C(1+\mu h)h^{-2} \asymp C\zeta^2\ell^2 + CB\ell^2$ exactly as in the mock proof.

Then summation with respect to ℓ-partition in this zone results in $CB^{\frac{2}{3}}$ as $B\le Z$, $CZ^{\frac{2}{3}}$ as $Z\le B\le Z^{\frac{4}{3}}$ and $CB^{\frac{1}{5}}Z^{\frac{2}{5}}$ as $Z^{\frac{4}{3}}\le B\le Z^3$.

Remark 26.3.15. (i) Condition (26.3.64) is fulfilled as $\ell(x)\le \epsilon\bar{r}$.

(ii) Further, since $M=1$ both conditions (26.3.64) and (26.3.65) are fulfilled if $|x|\le (1-\epsilon)\bar{r}_m$ (we pick up $\mathsf{y}_m=0$ and $\bar{r}_m$ exact radius of $\mathsf{supp}(\rho_B^{\mathsf{TF}})$).

Border Strip. Now we need to consider the contribution of the border strip $\mathcal{Y}:=\{x\colon \gamma(x)\le\epsilon\}$ with $\gamma(x)=\epsilon(\bar{r}-|x|)\bar{r}^{-1}$ and $\bar{r}:=\bar{r}_m$. Here $\ell\asymp\bar{r}$, $\zeta\asymp\bar{\zeta}$ with

$$\bar{r}\asymp \begin{cases} (Z-N)_+^{-\frac{1}{3}} & \text{if } B\le (Z-N)_+^{\frac{4}{3}},\\ B^{-\frac{1}{4}} & \text{if } (Z-N)_+^{\frac{4}{3}}\le B\le Z^{\frac{4}{3}},\\ Z^{\frac{1}{5}}B^{-\frac{2}{5}} & \text{if } Z^{\frac{4}{3}}\le B\le CZ^3 \end{cases} \tag{26.3.66}$$

and

$$\bar{\zeta}\asymp \begin{cases} (Z-N)_+^{\frac{2}{3}} & \text{if } B\le (Z-N)_+^{\frac{4}{3}},\\ B^{\frac{1}{2}} & \text{if } (Z-N)_+^{\frac{4}{3}}\le B\le Z^{\frac{4}{3}},\\ Z^{\frac{2}{5}}B^{\frac{1}{5}} & \text{if } Z^{\frac{4}{3}}\le B\le CZ^3 \end{cases} \tag{26.3.67}$$

and scaling we get $\mu=B\bar{r}\bar{\zeta}^{-1}$ and $h=\bar{\zeta}^{-1}\bar{r}^{-1}$ here.

Let us consider first the case $\nu=0$. Then both conditions (26.3.64) and (26.3.65) are fulfilled albeit with $\ell_1=\gamma(x)\bar{r}$ and $\varsigma(x)=\bar{\zeta}\gamma(x)^2$ instead of ℓ and ζ.

Thus if $\varsigma\ell_1\ge 1$ (i.e. $\gamma\ge\bar{\gamma}:=h^{\frac{1}{3}}$), the contribution of the ball $B(x,\gamma(x)\bar{r})$ to the remainder does not exceed $C\mu h^{-1}\gamma^2$ [19] and therefore the total contribution of zone $\mathcal{Y}_1:=\{x\colon \bar{\gamma}\le\gamma(x)\le\epsilon\bar{r}\}$ to the remainder does not

[19] Really, after additional rescaling $x\mapsto x\gamma^{-1}$, $w\mapsto w\gamma^{-2}$ we have $\mu_1=\mu\gamma^{-1}$, $h_1=h\gamma^{-3}$ and $\mu_1h_1^{-1}=\mu h^{-1}\gamma^2$.

exceed

$$C\mu h^{-1}\int \gamma(x)^{-1}\,dx \asymp C\mu h^{-1}|\log h| = CB\bar{r}^2|\log h| \tag{26.3.68}$$

which is $O(Z^{\frac{2}{3}})$ as long as $B \le Z^{\frac{4}{3}}(\log Z)^{-2}$.

Further, the same approach works if $|\nu| \lesssim \bar{\zeta}^2\bar{\gamma}^3 \asymp \bar{\zeta}^2 h \asymp \bar{\zeta}\bar{r}^{-1}$ which is equivalent to $(Z-N)_+ \le \bar{\zeta}$ (then $|\nabla W| \asymp \varsigma^2\ell_1^{-1}$ if $\gamma(x) \ge \bar{\gamma}$) and also if this condition is violated but $|\nu| \le \bar{\zeta}^2$; in the latter case we need to pick up $\bar{\gamma} = \bar{\gamma}_1 := |\nu|^{\frac{1}{3}}\bar{\zeta}^{-\frac{2}{3}}$.

To get rid of the logarithmic factor let us consider propagation. Recall that it goes along magnetic lines i.e. that (x_1, x_2) remains constant. Let us consider propagation in the direction in which $|x_3|$ increases (i.e. $\gamma(x)$ decays); we do not need to consider zone $\mathcal{Y}_1 \cap \{|x_3| \le Z^{-\delta}\bar{r}\}$ since contribution of this zone (26.3.68) is $o(B\bar{r}^2)$.

One can see easily that we can follow dynamics which does not return for a time $T_1^*(x) := T_1(x)(\gamma(x)/\bar{\gamma})^{\delta}$ where $T(x) \asymp \ell_1^{-1}\varsigma^{-1} \asymp \bar{r}\bar{\zeta}^{-1}\gamma^{-1}$ is a time required for this dynamics to pass though $B(x, \ell_1(x))$. Therefore one can replace (26.3.68) by

$$C\mu h^{-1}\int_{\{x:\,\gamma(x)\ge\bar{\gamma}\}} \bar{\gamma}^{\delta}\gamma(x)^{-1-\delta}\,dx \asymp C\mu h^{-1} = CB\bar{r}^2. \tag{26.3.69}$$

Further, as $|\nu| \ge B\bar{r}$ we need also to consider zone $\mathcal{Y}_0 := \{x:\,\gamma(x) \le \bar{\gamma}\}$. In this zone we take $\ell_1 = \bar{\ell}_1 = \bar{\gamma}\bar{r}$ and $\varsigma = \bar{\varsigma} = (|\nu|\bar{\gamma})^{\frac{1}{2}}$ with $\ell_1\varsigma \ge 1$ and since $|\nabla W| \asymp \varsigma^2\ell_1^{-1}$, contribution of $B(x, \ell_1(x))$ to the remainder does not exceed $CB\bar{\ell}_1^2$ and the total contribution of $\mathcal{Y}_2$ does not exceed $CB\bar{r}^2$ which what exactly we achieved for zone $\mathcal{Y}_1$ after we got rid of logarithm. We take mollification parameter $\varepsilon = \bar{\varsigma}^{-1}Z^{\delta}$ [20].

Furthermore, zone $\mathcal{Y}_3 = \{x:\,|x| \ge \bar{r} + \bar{\ell}_1\}$ is classically forbidden. So we can take here

$$\ell_1(x) = \epsilon(|x| - \bar{r}), \qquad \varsigma(x) = \min\bigl(\bar{\varsigma}\bar{\ell}_1^{-\frac{1}{2}}\ell_1(x)^{\frac{1}{2}}, |\nu|^{\frac{1}{2}}\bigr) \tag{26.3.70}$$

and prove easily that its contribution also does not exceed $CB\bar{r}^2$.

Returning to the case $|\nu| \lesssim \bar{\zeta}$ we see that the contribution of zone $\mathcal{Y}_2$ to the remainder does not exceed $CB\bar{r}^2$ because effective semiclassical

[20] One can see easily that the resulting errors in the expressions(26.3.24) and (26.3.25)–(26.3.26) will not violate our claims.

parameter here is $h_1 = 1$ and non-degeneracy condition is of no concern for us. We take mollification parameter $\varepsilon = \bar{\varsigma}^{-1} Z^{\delta}$ [20].

Moreover, we can modify W in $\mathcal{Y}_2$ (make it negative there) so that this zone would be classically forbidden with ℓ_1, ς defined by (26.3.70) with $|\nu|$ replaced by $\bar{\zeta}$.

Finally in the case $B \le |\nu|$ (i. e. $B \le C(Z-N)_+^{\frac{4}{3}}$ we can apply the above arguments with $\bar{\gamma} = 1$ and arrive to the same result. Therefore we proved in all cases

(26.3.71) If $M = 1$ the total contribution of the border strip $\mathcal{Y}$ to the remainder in the charge term is $O(B\bar{r}^2)$ which does not exceed $CB^{\frac{1}{2}}$ as $B \le Z^{\frac{4}{3}}$ and $CB^{\frac{1}{5}} Z^{\frac{2}{5}}$ if $Z^{\frac{4}{3}} \le B \le Z^3$.

Conclusion. If $Z^2 \le B \le Z^3$ we need to estimate also contribution of the *inner core* $\mathcal{X}_0 := \{x\colon \ell(x) \le CZ^{-1}\}$. By means of variational methods we will prove (see Corollary 26.A.5)

(26.3.72) If $Z^2 \le B \le Z^3$ the contributions of $\mathcal{X}_0$ to both $\int e(x,x,\nu)\,dx$ and $\int P'_B(W(x)+\nu)\,dx$ do not exceed CBZ^{-2}.

Then we arrive to the following

Proposition 26.3.16. *Let $M = 1$. Then*

(i) For constructed above potential W expression (26.3.24) *does not exceed $CZ^{\frac{2}{3}} + CB^{\frac{1}{5}} Z^{\frac{2}{5}}$.*

(ii) If $B \le Z$ expression (26.3.24) *does not exceed $C(B+1)^{\delta} Z^{\frac{2}{3}-\delta}$.*

Trace Term

Let us consider the *trace term* i.e. expression $\int e_1(x,x,\nu)\,dx = \mathsf{Tr}((H-\nu)^-)$.

Regular Zone. Here again let us consider first zone where $|x| \le (1-\epsilon)\bar{r}$. Then the contribution of $B(x, \ell(x))$ to the Tauberian remainder[21] does not exceed $C\zeta^2(h^{-1}+\mu) \asymp C\zeta^3\ell + CB\zeta\ell$ as in the mock proof and the summation over zone results in $CZ^{\frac{5}{3}} + CZ^{\frac{4}{3}} B^{\frac{1}{3}} + CZ^{\frac{3}{5}} B^{\frac{4}{5}}$.

[21] We will consider a bit later transition from the Tauberian expression to the magnetic Weyl expression.

Border Strip. Again in zone $\mathcal{Y}_1$ contribution of $B(x,\gamma(x))$ does not exceed $CB\varsigma\ell_1$ and the summation over this zone returns

$$CB\int \varsigma\ell_1^{-2}\,dx \tag{26.3.73}$$

and plugging $\ell_1 = \bar{r}\gamma$ and $\varsigma = \bar{\zeta}\gamma^2$ results in $CB^{\frac{5}{4}}$ as $B \lesssim Z^{\frac{4}{3}}$ and $CB^{\frac{4}{5}}Z^{\frac{3}{5}}$ otherwise. The analysis of zone $\mathcal{Y}_0$ if there $\mathcal{Y}_2 = \emptyset$ is also easy.

Consider zones $\mathcal{Y}_2$ and $\mathcal{Y}_0$. The same arguments as before imply that their contributions to the remainder do not exceed $CB\bar{r}^2\bar{\varsigma}\bar{\ell}_1^{-1}$ which is what we got before.

Justification: from Tauberian to Magnetic Weyl Expression.

Case $\mu h \le C_0$. We need to prove that with the announced error we can replace the Tauberian expression by magnetic Weyl one. Note that the canonical form of $\zeta^{-2}H_{A,W}$ as described in Sections 13.3 and 18.7 is

$$\begin{aligned}\mathcal{H} = \mathcal{H}_0 + {}& \\ & \mu^{-2}\omega_1(x_1,\mu^{-1}hD_1,x_3) + \mu^{-2}\omega_2(x_1,\mu^{-1}hD_1,x_3)(x_2^2+\mu^2h^2D_2^2) \\ & + \mu^{-1}h\omega_3(x_1,\mu^{-1}hD_1,x_3) + O\big(\mu^{-3}h(\gamma+\mu^{-1})^{-\frac{3}{2}} + \mu^{-4}\big)\end{aligned} \tag{26.3.74}$$

with

$$\mathcal{H}_0 = h^2D_3^2 - (x_2^2+\mu^2h^2D_2^2 \pm \mu h) + w(x_1,\mu^{-1}hD_1,x_3) \tag{26.3.75}$$

and

$$\gamma = \epsilon \min_j |w - 2j\mu h| \tag{26.3.76}$$

where we used the fact that $w \in \mu h\mathscr{C}^{\frac{5}{2}} + \mathscr{C}^n$, $\mu^{-3}h = \mu^{-4}\cdot\mu h$. Here we have signs "+" and "−" on $q/2$ of the diagonal elements equally.

Then the Tauberian expression is

$$\begin{aligned}\mathsf{const}\cdot\mu h^{-2}\int \sum_{j\ge 0}\big(w - 2j\mu h - \mu^{-2}\omega_1 - 2j\mu^{-1}h\omega_2\big)_+^{\frac{3}{2}}\times & \\ \big(\psi + \mu^{-2}\psi_1 + 2j\mu^{-1}h\psi_2\big)\,dx & \end{aligned} \tag{26.3.77}$$

where term with $j=0$ enters with the weight $\frac{1}{2}$ and an error does not exceed

$$Ch^{-3}\Big(\mu^{-4}+\mu^{-3}h\int(\gamma+\mu^{-1})^{-\frac{3}{2}}\,dx\Big)\asymp C\mu^{-\frac{7}{2}}h^{-3}$$

because an integral does not exceed $C\mu^{\frac{1}{2}}(\mu h)^{-1}$; since $\mu\ge h^{-\frac{3}{5}}$ this error does not exceed $Ch^{-\frac{9}{10}}$ which is better than $O(h^{-1})$.

On the other hand, if we consider the difference between (26.3.77) and the same expression with $\omega_1=\omega_2=\psi_1=\psi_2=0$ and consider it as a Riemannian sum and replace it by an integral we get $G\mu^{-2}h^{-3}$ with an error not exceeding $C\mu^{-4}(\mu h)^{\frac{1}{2}}h^{-3}$ which is even less. Therefore (26.3.77) becomes

$$\int P_{\mu h}(w)\psi\,dx+G\mu^{-2}h^{-3}$$

and comparing with the result if $\mu\asymp h^{-\frac{3}{5}}$ when we get the same answer albeit with $G=0$ we conclude that G *must* be 0. This concludes the justification in $\mathcal{X}_2$.

Case $\mu h\ge C_0$. In this case we need a simplified version of (26.3.74) $\mathcal{H}=\mathcal{H}_0+O(\mu^{-1}h)$ and we need to consider only $j=0$ and replacing $\mathcal{H}$ by $\mathcal{H}_0$ brings and error $C\mu h^{-2}\times\mu^{-1}h=O(h^{-1})$. This takes care of $\mathcal{X}_2$ and after scaling of $\mathcal{Y}$.

Conclusion. As $Z^2\le B\le Z^3$ we need to estimate also contribution of $\mathcal{X}_0=\{x\colon \ell(x)\le CZ^{-1}\}$. By means of variational methods we will prove (see Corollary 26.A.5)

(26.3.78) For $Z^2\le B\le Z^3$ the contributions of $\mathcal{X}_0$ to both $\int e_1(x,x,\nu)\,dx$ and $\int P'_B(W(x)+\nu)\,dx$ do not exceed CB.

Then we arrive to the following

Proposition 26.3.17. *Let $M=1$. Then*

(i) For constructed above potential W expression (26.3.26) *does not exceed $CZ^{\frac{5}{3}}+CZ^{\frac{4}{3}}B^{\frac{1}{3}}+CZ^{\frac{3}{5}}B^{\frac{4}{5}}$.*

(ii) If $B\le Z$ expression (26.3.26) *does not exceed $C(B+1)^{\delta}Z^{\frac{5}{3}-\delta}$ (but one should subtract a Schwinger term from the trace).*

Semiclassical D-Term: Local Theory

Unfortunately, we do not have any non-smooth theory (cf. Section 16.8) here so far but actually we almost do not need it since singularities are rather rare. Let us introduce a scaling function (26.3.76) and consider

$$J_\lambda(z) = \int \phi_{z,\lambda}(x)\big(e(x,x,\tau) - P_\beta(w(x)+\tau)\big)\,dx \tag{26.3.79}$$

with $\phi_{z,\lambda}(x) = \phi(\lambda^{-1}(x-z))$ and $\lambda \le \gamma(z)$. Scaling $x \mapsto \lambda^{-1}(x-z)$ we have $\mu \mapsto \mu' = \lambda\mu$ and $h \mapsto h' = \lambda^{-1}h$.

Then, according to Section 13.5

$$|J_\lambda(z)| \le Ch'^{-2}(1+\mu'h') \asymp C\lambda^2 h^{-2}(1+\mu h) \tag{26.3.80}$$

as long as $\lambda \ge h$.

Really, a transition from the Tauberian decomposition to magnetic Weyl one in this case is easy: skipping all perturbation terms $O(\mu^{-2}+\mu^{-1}h)$ in (26.3.74) and also setting $\psi_1 = \psi_2 = 0$ results in an error $O\big(\mu^{-2}h^{-3}+h^{-1}\big)$ in (26.3.77)-like expression albeit with the power $\frac12$ rather than $\frac32$ and without integration:

$$\mathsf{const}\cdot \mu h^{-2}\sum_{j\ge 0}\big(w - 2j\mu h - \mu^{-2}\omega_1 - 2j\mu^{-1}h\omega_2\big)_+^{\frac12}\times \big(\psi + \mu^{-2}\psi_1 + 2j\mu^{-1}h\psi_2\big); \tag{26.3.81}$$

scaling produces expression smaller than (26.3.80).

Let us apply this estimate (26.3.80) to the Fefferman–de Llave decomposition (26.3.8).

Case $\mu \le C_0 h^{-1}$.

(i) Consider first a pair (z,z') such that $|z'-z''| \le \epsilon_0\gamma(z')$; then also $|z'-z''| \le \epsilon_0\gamma(z'')$ and we take $\lambda = \epsilon|z'-z''|$.

Then in the virtue of (26.3.80) the total contribution to D-term of all such pairs belonging to $B(z,\gamma(z))$, and with $|z'-z''| \asymp \lambda$ does not exceed

$$C\gamma^3\lambda^{-3}\times\lambda^{-1}\times\lambda^2h^{-4}(1+\mu h)^2 \asymp C\gamma^3h^{-4}(1+\mu h)^2 \tag{26.3.82}$$

where $C\gamma^3\lambda^{-3}$ estimate the number of such pairs, λ^{-1} the inverse distance between them, and $C\lambda^2h^{-2}(1+\mu h)$ is the right-hand expression of (26.3.80).

Then summation over $\lambda \in (\mu^{-1}, \gamma)$ results in $C\gamma^3(1+\mu h)^2 h^{-4}|\log(\mu\gamma)|$.

Further, summation over all balls $B(z,\gamma) \subset B(0,1)$ with $\gamma(z) \asymp \gamma$ results in $C(\mu h)^{-1}\gamma h^{-4}|\log(\mu\gamma)|$ since there are $\asymp (\mu h)^{-1}\gamma^{-2}$ such balls due to non-degeneracy assumption $|\nabla w| \asymp 1$. Summation over $\gamma \in (\mu^{-1}, \mu h)$ results in $Ch^{-4}|\log(\mu^2 h)|$.

As $\lambda \le \mu^{-1}$ we can apply standard non-magnetic methods without Fefferman–de Llave decomposition (26.3.8). Coefficients are smooth after scaling as long as $\varepsilon \ge \mu^{-1}$.

(ii) Consider *disjoint* pairs (z', z'') with $|z'-z''| \ge \max\bigl(\gamma(z'), \gamma(z'')\bigr)$. Here estimate (26.3.80) is not sufficient and it should be replaced by

$$|J_\gamma(z)| \le C\lambda^3 h^{-2}(1+\mu h) \tag{26.3.83}$$

as long as

$$\gamma \ge h^{\frac{2}{3}-\delta}. \tag{26.3.84}$$

Really, the shift for time T with respect to ξ_3 is $\asymp T$ provided $|\nabla_{x_3} w| \asymp 1$ and this shift is observable if $T \times \gamma \gtrsim h^{1-\delta}$. Similarly, in the canonical form the shift for time T with respect to $\mu^{-1}\xi_i$ is $\asymp \mu^{-1}T$ provided $|\nabla_{x_i} w| \asymp 1$ and this shift is observable if $\mu^{-1}T \times \gamma \gtrsim \mu^{-1}h^{1-\delta}$. In both cases shift with $T \in (\gamma^{\frac12}, \epsilon_0)$ is observable under assumption (26.3.84) and therefore we can extend $T \asymp \gamma^{\frac12}$ to $T \asymp 1$.

Note that for $\varepsilon \ge h^{\frac{2}{3}-\delta}$ assumption (26.3.84) is fulfilled automatically. Then contribution of each disjoint pair to D-term does not exceed

$$Ch^{-4}(1+\mu h)^2\gamma(z')^3\gamma(z'')^3|z'-z''|^{-1}$$

and the total contribution does not exceed

$$Ch^{-4}(1+\mu h)^2 \iint |z'-z''|^{-1}\,dz'dz'' \asymp Ch^{-4}(1+\mu h)^2.$$

(iii) To shed of logarithm in (i) we need a slightly better estimate than (26.3.80). The same arguments as in Part (ii) result in

$$|J_\lambda(z)| \le C\lambda^2 h^{-2}(1+\mu h)\cdot(1+\lambda\gamma/h)^{-\delta}. \tag{26.3.85}$$

Really, we just advance from time $T \asymp \lambda$ to $T \asymp \lambda(1+\lambda\gamma/h)^{\delta}$.

Then the same factor is acquired by the right-hand expression of (26.3.82) and the summation with respect to $\lambda \in (h\gamma^{-1}, \gamma)$ results in $C\gamma^3(1+\mu h)^2h^{-4}$ but summation with respect to $\lambda \in (\mu^{-1}, h\gamma^{-1})$ results

$$C\gamma^3(1+\mu h)^2h^{-4}\bigl(1+|\log(\mu h\gamma^{-1})|\bigr).$$

Further, summation over all balls $B(z,\gamma) \subset B(0,1)$ with $\gamma(z) \asymp \gamma$ results in $C(\mu h)^{-1}\gamma(1+\mu h)^2h^{-4}\bigl(1+|\log(\mu h\gamma^{-1})|\bigr)$ and, finally, summation over $\gamma \lesssim \mu h$ results in $Ch^{-4}(1+\mu h)^2$.

Note that in all cases perturbation terms in (26.3.74) and (26.3.81) result in the error not exceeding the announced one.

Case $\mu \ge C_0h^{-1}$. So far factor $(1+\mu h)$ was for a compatibility only. Now it is important.

Exactly the same arguments work as $\mu \ge C_0h^{-1}$ with a minor modifications:

(a) $\gamma(x)$ now is defined by (26.3.76) with $j=0$ and its upper bound is 1 rather than μh.

(b) Also the number of γ-balls is $\asymp \gamma^{-2}$ rather than $\asymp (\mu h)^{-1}\gamma^{-2}$;

(c) λ now runs from h to γ in (i) and (iii).

(d) We need to estimate contribution of pairs (z',z'') with $|z'-z''| \le h$. One can see easily that $e(x,x,\tau) \le \mu h^{-2}$ and therefore the total contribution of these pairs does not exceed $C\mu^2h^{-4}\iint |z'-z''|^{-1}\,dz'dz'' \asymp C\mu^2h^{-4}\times h^2 \asymp C\mu^2h^{-2}$.

Therefore we have the following

Proposition 26.3.18. *As $|\nabla w| \asymp 1$ and $\varepsilon \ge h^{\frac{2}{3}-\delta}$ in $B(0,1)$ and $\phi \in \mathscr{C}^\infty(B(0,\frac{1}{2}))$*

$$\mathsf{D}\Bigl(\phi\bigl(e(x,x,\tau)-P'_\beta(w(x)+\tau)\bigr),\phi\bigl(e(x,x,\nu)-P'_\beta(w(x)+\tau)\bigr)\Bigr) \le C(1+\mu h)^2h^{-4}. \tag{26.3.86}$$

Remark 26.3.19. One can see easily that one can select $\varepsilon \ge h^{\frac{2}{3}-\delta}$ such that expressions (26.3.61), (26.3.62) and (26.3.62) will be respectively $O(h^{2+\delta})$, $O(h^{1+\delta})$ and $O(h^{2+\delta})$.

Semiclassical D-Term: Global Theory

Regular Zone. The above results allow us to consider a total contribution of zone $\mathcal{X}_2$ into semiclassical D-term. As before let us consider ℓ-admissible partition of unity there and apply it to Fefferman–de Llave decomposition (26.3.8). Then the total contribution of the elements which are not disjoint does not exceed

$$\sum_n \ell_n^{-1}\bigl(1+B\zeta_n^{-2}\bigr)^2\ell_n^4\zeta_n^4 \asymp \int \underbrace{\bigl(\zeta^4+B^2\bigr)\ell^3}\ \ell^{-1}d\ell \tag{26.3.87}$$

where $\bigl(1+B\zeta_n^{-2}\bigr)^2$ and $\ell_n^4\zeta_n^4$ are $(1+\mu h)$ and h^{-4} respectively and ℓ_n^{-1} is a scaling factor.

Then if $\zeta^2 = Z\ell^{-1}$, an integral equals to the value of the selected expression as ℓ reaches its maximum, i.e. for $\ell = Z^{-\frac{1}{3}}$ for $B \le Z^{\frac{4}{3}}$ and $\ell = Z^{\frac{1}{5}}B^{-\frac{2}{5}}$ for $Z^{\frac{4}{3}} \le B \le Z^3$ and we arrive to $CZ^{\frac{5}{3}}$ and $CZ^{\frac{3}{5}}B^{\frac{4}{5}}$ respectively.

On the other hand, if $\zeta^2 = \ell^{-4}$ an integral equals to the value of the selected expression as ℓ reaches its minimum, i.e. for $\ell = Z^{-\frac{1}{3}}$ and only in the case $B \le Z^{\frac{4}{3}}$ and we arrive to $CZ^{\frac{5}{3}}$ again.

Furthermore, the total contribution of the disjoint elements does not exceed

$$\sum_{n,p} |z_n-z_p|^{-1}\bigl(1+B\zeta_n^{-2}\bigr)\bigl(1+B\zeta_p^{-2}\bigr)\ell_n^2\zeta_n^2\ell_p^2\zeta_p^2 \asymp \iint \underbrace{(\ell+\ell')^{-1}\bigl(\zeta^2+B\bigr)\ell^2\bigl(\zeta'^2+B\bigr)\ell'^2}\ \ell^{-1}d\ell\,\ell'^{-1}d\ell'. \tag{26.3.88}$$

Then if $\zeta^2 = Z\ell^{-1}$ and $\zeta'^2 = Z\ell'^{-1}$ an integral equals to the value of the selected expression as both ℓ and ℓ' reach their maxima, and we arrive to $CZ^{\frac{5}{3}}$ and $CZ^{\frac{3}{5}}B^{\frac{4}{5}}$ respectively.

On the other hand, if $\zeta^2 = \ell^{-4}$ and $\zeta'^2 = \ell'^{-4}$ (we do not need to consider mixed pair) an integral equals to the value of the selected expression as both ℓ and ℓ' reach their minima, and we arrive to $CZ^{\frac{5}{3}}$.

Therefore (combining with Proposition 26.A.5 as $Z^2 \le B \le Z^3$) we arrive to

Proposition 26.3.20. *Let $M = 1$. Then*

(i) The total contribution of the zone $\{x\colon \ell(x) \le (1-\epsilon)\bar{r}\}$ *to the semiclassical* D*-term does not exceed* $CZ^{\frac{5}{3}}$ *and* $CZ^{\frac{3}{5}}B^{\frac{4}{5}}$ *for* $B \le Z^{\frac{4}{3}}$ *and for* $Z^{\frac{4}{3}} \le B \le Z^3$ *respectively.*

(ii) If $B \le Z$ *this contribution does not exceed* $C(B+1)^{\delta}Z^{\frac{5}{3}-\delta}$.

Border Strip. Border strip $\mathcal{Y} = \{x\colon (1-\epsilon)\bar{r} \le \ell(x) \le (1+\epsilon)\bar{r}\}$ is more subtle. Here we need to use the same $\ell_1(x) = \epsilon_0(\bar{r} - |x|)$ partition as before.

Remark 26.3.21. $\mathcal{Y}$ is already covered by our arguments if $\bar{r} \asymp (Z-N)_+^{-\frac{1}{3}}$.

Close Elements. Consider first contribution of elements which are not disjoint. It is given by the left-hand expression of (26.3.87) with ℓ, ζ replaced by $\ell_1(x) = \bar{r}\gamma(x)$ and $\varsigma(x) = \bar{\zeta}\gamma(x)^2$ respectively. However since the layer $\{x\colon \gamma(x) \asymp \gamma\}$ contains $\asymp \gamma^{-2}$ elements the right-hand expression should be replaced by

$$\int \underbrace{B^2\bar{r}^3\gamma}\, \gamma^{-1}d\gamma \asymp B^2\bar{r}^3$$

since $\varsigma^2 \le B$; so we arrive to $O\bigl(\max(B^{\frac{5}{4}}, Z^{\frac{3}{5}}B^{\frac{4}{5}})\bigr)$.

Meanwhile for $\mathcal{Y}_2$ we have $\gamma(x) = \bar{\gamma} \le 1$ and $\varsigma(x) = \bar{\varsigma} = \bar{\zeta}\bar{\gamma}^2$ and its contribution does not exceed what we got for $\mathcal{Y}_1$.

Disjoint Elements. Consider contribution of the disjoint elements. It is given by the left-hand expression of (26.3.88) with ℓ, ζ replaced by γ and ς respectively. Note that $\sum_{n,p} |z_n - z_p|^{-1} \asymp \bar{r}^{-1}\gamma^{-2}\gamma'^{-2}$ where we sum with respect to all pairs with $\gamma_n \asymp \gamma$ and $\gamma_p \asymp \gamma'$. Therefore the right-hand expression should be replaced by

$$\int \underbrace{\bar{r}^3B^2}\, \gamma^{-1}d\gamma\, \gamma'^{-1}d\gamma' \tag{26.3.89}$$

which leads to $C\bar{r}^3B^2|\log(\bar{r}^{-1}\bar{\gamma})|^2$ which differs from what we got before by a logarithmic factor. To get rid of it we will use exactly the same trick as in Paragraph *26.3.3.2.2. Border Strip.* proving Proposition 26.3.16 because considering disjoint pairs we consider the same objects as there. Then instead of (26.3.89) we arrive to

$$\int \underbrace{\bar{r}^3B^2\gamma^{-\delta}\gamma'^{-\delta}\bar{\gamma}^{2\delta}}\, \gamma^{-1}d\gamma\, \gamma'^{-1}d\gamma'$$

which results in $C\bar{r}^3B^2$.

Meanwhile for $\mathcal{Y}_2$ we have $\gamma(x) = \bar{\gamma} \le \bar{r}$ and $\varsigma(x) = \bar{\varsigma} = \bar{\zeta}\bar{\gamma}^2$ and its contribution does not exceed what we got for $\mathcal{Y}_1$.

Conclusion. Finally, analysis in the outer zone is trivial. Therefore we arrive

Proposition 26.3.22. *Let $M = 1$. Then for constructed above potential W*

(i) Expression (26.3.25) *does not exceed $CZ^{\frac{5}{3}} + CZ^{\frac{3}{5}}B^{\frac{4}{5}}$.*

(ii) If $B \le Z$ expression (26.3.25) *does not exceed $C(B+1)^{\delta}Z^{\frac{5}{3}-\delta}$.*

26.4 Applying Semiclassical Mmethods: $M \ge 2$

Let us consider now the molecular case ($M \ge 2$). The major problem is that the non-degeneracy condition $|\nabla w| \asymp 1$ is not necessarily fulfilled. Therefore we need to find an alternative approach to the zone where $\mu \ge h^{-\frac{1}{3}}$) (with $\mu = B\ell\zeta^{-1}$ and $h = 1/\ell\zeta$). Recall that it consists of three smaller zones: zone $\mathcal{X}_2 := \{\mu h \le C_0,\ W_B^{\mathsf{TF}} \ge \epsilon_0\zeta^2\}$ [22], zone $\mathcal{X}_3 := \{\mu h \ge C_0,\ W_B^{\mathsf{TF}} \ge \epsilon_0\zeta^2\}$ [23], and the (most difficult) *boundary strip* $\mathcal{Y} = \{W_B^{\mathsf{TF}} \le \epsilon_0\zeta^2\}$, which we leave for the next Section 26.5.

26.4.1 Scaling Functions in Zone $\mathcal{X}_2$

Step 1. We will use the scaling method in this zone; the good news is that W_B^{TF} is sufficiently regular after a proper rescaling and also sufficiently non-degenerate. Recall that after we rescale $x \mapsto \bar{\ell}^{-1}(x-\bar{x})$, $\tau \mapsto \bar{\zeta}^{-2}\tau$ in the ball $B(\bar{x}, \frac{1}{2}\bar{\ell})$ with $\bar{\ell} = \ell(\bar{x})$, $\bar{\zeta} = Z^{\frac{1}{2}}\bar{\ell}^{-1}$, the rescaled potential $w = \bar{\zeta}^{-2}W_B^{\mathsf{TF}}$ satisfies in $B(0,2)$ equation

$$\frac{1}{4\pi}\Delta w = \eta P'_\beta(w) \qquad \text{with } \eta = \bar{\zeta}\bar{\ell}^2 \lesssim 1, \quad \beta = \mu h = B\bar{\zeta}^{-2} \le 1 \tag{26.4.1}$$

and therefore in $B(0,1)$

$$w = -\frac{1}{4\pi}\int |x-z|^{-1}\eta P'_\beta(w(z))\phi(z)\,dz + w' \tag{26.4.2}$$

with $\phi \in \mathscr{C}_0^\infty(B(0,\frac{5}{6}))$ and $w' \in \mathscr{C}_0^\infty(B(0,\frac{3}{4}))$.

[22] Only if $B \le C_1Z^2$; this zone disappears for $C_1Z^2 \le B \lesssim Z^3$.

[23] Only if $Z^{\frac{4}{3}} \lesssim B \lesssim Z^3$; this zone disappears for $B \lesssim Z^{\frac{4}{3}}$.

Remark 26.4.1. (i) If $\zeta^2 \asymp Z\ell^{-1}$ then $\eta = Z^{\frac12}\ell^{\frac32} \le 1$; this happens for $B \le Z^{\frac43}$, $\ell \lesssim Z^{-\frac13}$ and for $B \ge Z^{\frac43}$, $\ell \lesssim B^{-\frac13}$.

(ii) If $\zeta^2 \asymp \ell^{-2}$ then $\eta \asymp 1$; this happens only for $B \le Z^{\frac43}$, $\ell \gtrsim Z^{-\frac13}$.

Let us introduce a function

$$\gamma_0(x) = \Big(\min_j |w - 2j\beta|^3 + |\nabla w|^4 + |\nabla^2 w|^6 + |\nabla^3 w'|^{12}\Big)^{\frac{1}{12}}. \tag{26.4.3}$$

Remark 26.4.2. We cannot replace w' by w in the last term because $w \in \mathscr{C}^{\frac52}$ only rather than $\mathscr{C}^3$.

Proposition 26.4.3. *$\gamma_0(x)$ is a scaling function i.e. $|x-y| \le \epsilon\gamma_0(x) \implies \gamma_0(y) \asymp \gamma_0(x)$.*

Proof. (a) If w belonged to $\mathscr{C}^4$ (and we would put w instead of w' in the last term of (26.4.3)) then we would just prove that $|\nabla\gamma_0| \le c$. Here we should be more subtle. We need to prove that if

$$\min_j |w - 2j\beta| \le \gamma_0^4, \qquad |\nabla w| \le \gamma_0^3, \tag{26.4.4}_{1,2}$$

$$|\nabla^2 w| \le \gamma_0^2, \qquad |\nabla^3 w'| \le \gamma_0 \tag{26.4.4}_{3,4}$$

at point x, then at point y the same inequalities hold with γ_0 replaced by $c\gamma_0$ [24]. Definitely this is true for $(26.4.4)_4$ since w' is smooth.

(b) Consider $|\nabla^2 w|$. Consider $|\Lambda_\alpha w(y) - \Lambda_\alpha w(x)|$ with $\Lambda_\alpha = \nabla^\alpha - \frac13\delta_{ij}\Delta$, $\alpha = (i,j)$. Then due to (26.4.2)

$$|\Lambda_{\alpha,y} w(y) - \Lambda_{\alpha,x} w(x)| \le \\ |\eta \int \big(\Lambda_{\alpha,x}|x-z|^{-1} - \Lambda_{\alpha,y}|y-z|^{-1}\big) P'_\beta(w(z))\phi(z)\,dz| + \epsilon_1\gamma_0^2 \tag{26.4.5}$$

where the last term estimates $|\Lambda_{\alpha,y} w'(y) - \Lambda_{\alpha,x} w'(x)|$ and we used $(26.4.4)_4$. Integrals here are understood in the sense of the principal value (vrai) and $\epsilon_1 = \epsilon_1(\epsilon) \to +0$ as $\epsilon \to +0$.

Note that the integral in expression (26.4.5) does not change if we add to $P'_\beta(w)$ any constant with respect to z.

[24] We need to prove a bit of converse as well; see Part (c).

Let us consider first this integral over $\{z\colon |x-z|\ge 2\gamma_0\}$, provided $\gamma_0\ge |x-y|$, and note that this integral does not exceed

$$\eta\Big|\int |x-z|^{-4}|x-y|\times \\ \big(|\nabla w(x)|\cdot|x-z|+|\nabla^2 w(x)|\cdot|x-z|^2+|x-z|^{\frac{5}{2}}\big)^{\frac12}\,dz\Big|\le C\epsilon_1\eta.$$

Consider now integral over zone $\{z\colon |x-z|\le 2\gamma_0\}$:

$$|\eta\int \Lambda_{\alpha,x}|x-z|^{-1}\cdot\big(P'_\beta(w(z))-P'_\beta(w(x))\big)\,dz|$$

and note that it also does not exceed $\epsilon_1\eta$. Further, t same arguments work for this integral with x replaced by y but still integrated over zone $\{z\colon |x-z|\le 2\gamma_0\}$ (one needs to remember that $|\nabla w(x)-\nabla w(y)|=O(\gamma_0)$ and $|\nabla^2 w(x)-\nabla^2 w(y)|=O(\gamma_0^{\frac12})$).

Furthermore, the same arguments work also for these expressions integrated over $\{z\colon |y-z|\le 2\gamma_0\}$ and we are left with

$$|\eta\int \omega(x,y,z)\big(P'_\beta(w(x))-P'_\beta(w(y))\big)\,dz|$$

integrated over zone $\{z\colon |x-z|\asymp\gamma_0\}$ and $\omega\asymp\gamma_0^{-3}$ and one can estimate it by $\epsilon_1\eta$ easily in the same way. Therefore since $|\Delta w(x)-\Delta w(y)|$ does not exceed $\epsilon_1\eta$ we conclude that $|\nabla^2 w(x)-\nabla^2 w(y)|$ does not exceed $\epsilon_1(\eta+\gamma_0^2)$.

However (26.4.2) implies that $\eta\le c\gamma_0^2$ since $P'_\beta(w)\asymp 1$ in $\mathcal{X}_2$ and therefore $|\nabla^2 w(x)-\nabla^2 w(y)|\le\epsilon_1\gamma_0^2$ in $B(x,\gamma_0)$.

Finally, combining this inequality with $(26.4.4)_2$ we conclude that $|\nabla w(x)-\nabla w(y)|\le\epsilon_1\gamma_0^3$ in $B(x,\gamma_0)$; finally, combining with $(26.4.4)_1$ we conclude that $|w(x)-w(y)|\le\epsilon_1\gamma_0^4$ in $B(x,\gamma_0)$.

(c) Therefore $(26.4.4)_{1-4}$ are fulfilled in $y\in B(x,\epsilon\gamma_0)$ with γ_0 replaced by $\gamma_0(1+C\epsilon_1)$. Further, if we redefine γ_0 as the minimal scale such that inequalities $(26.4.4)_{1-4}$ are fulfilled in x, then $(26.4.4)_{1-4}$ fail in $y\in B(x,\epsilon\gamma_0)$ with γ_0 replaced by $\gamma_0(1-C\epsilon_1)$. Therefore with appropriate $\epsilon>0$ we conclude that $\frac12\le\gamma_0(x)/\gamma_0(y)\le 2$.

Obviously, $\gamma_0\asymp\gamma_{0\mathsf{old}}$ where $\gamma_{0\mathsf{old}}$ was defined by (26.4.3) and therefore the same conclusion also holds for $\gamma_{0\mathsf{old}}$. □

Now let us reintroduce the scaling function

$$(26.4.3)^* \quad \gamma_0(x) = \epsilon\Big(\min_j |w - 2j\beta|^3 + |\nabla w|^4 + |\nabla^2 w|^6 + |\nabla^3 w'|^{12}\Big)^{\frac{1}{12}} + C_0 h^{\frac{1}{3}} + C_0 \eta^{\frac{1}{2}}.$$

Then

(26.4.6) $|x-y| \le 2\gamma_0(x) \implies \gamma_0(y) \asymp \gamma_0(x)$ and $(26.4.4)_{1-4}$ hold (with some constant factor in the right-hand expression).

Consider $B(\bar{x}, \bar{\gamma}_0)$, $\bar{\gamma}_0 = \gamma_0(\bar{x})$, and scale again $x \mapsto \bar{\gamma}_0^{-1}(x - \bar{x})$, $\tau \mapsto \bar{\gamma}_0^{-4}\tau$ and respectively $w \mapsto w_1 = \bar{\gamma}_0^{-4}(w - 2\bar{\jmath}\beta))$, $h \mapsto h_1 = h\bar{\gamma}_0^{-3}$. Further, since after rescaling $|\Delta w_1| = O(\eta\bar{\gamma}_0^{-2})$ we set $\eta \mapsto \eta_1 = \eta\bar{\gamma}_0^{-2}$.

Due to cut-off in the end of $(26.4.3)^*$ $h_1 \lesssim 1$ and $\eta_1 \lesssim 1$. If $\bar{\gamma}_0 \asymp h^{\frac{1}{3}}$ then $h_1 \asymp 1$ and we are done. If $\bar{\gamma}_0 \asymp \eta^{\frac{1}{2}}$ then $\eta_1 \asymp 1$ and we proceed to *Step 3.*.

Step 2. So, let $\eta_1 \ll 1$. Let us introduce a scaling function in $B(0,1)$ obtained after the previous rescaling:

$$(26.4.7) \qquad \gamma_1(x) = \epsilon\Big(\min_j |w_1 + 2(\bar{\jmath} - j)\beta\bar{\gamma}_0^{-4}|^2 + |\nabla w_1|^3 + |\nabla^2 w_1|^6\Big)^{\frac{1}{6}}.$$

Then

$$(26.4.8)_{1-2} \qquad \min_j |w_1 + 2(\bar{\jmath} - j)\beta\bar{\gamma}_0^{-4}| \le C_0\gamma_1^3, \qquad |\nabla w_1| \le C_0\gamma_1^2,$$

$$(26.4.8)_3 \qquad |\nabla^2 w_1| \le C_0\gamma_1.$$

Remark 26.4.4[25]. Since now we do not have the third derivative in (26.4.7), we do not need in w_1' in the definition of γ_1, only in the proof of Proposition 26.4.5 below.

Proposition 26.4.5. *(i) $\gamma_1(x)$ is a scaling function:* $|x-y| \le 2\gamma_1(x) \implies \gamma_1(y) \asymp \gamma_1(x)$.

(ii) If $\eta_1 \le \epsilon_0$ then

$$(26.4.9) \qquad |\nabla^2 w_1(x) - \nabla^2 w_1'(x)| \le \epsilon_2\gamma_1, \qquad w_1' = (w' - \bar{\jmath}\beta)\gamma_0^{-4}.$$

[25] Cf. Remark 26.4.2.

Proof. Proof is similar but simpler than one of Proposition 26.4.3; it is based on the rescaled version of (26.4.1)–(26.4.2):

$$\frac{1}{4\pi}\Delta w_1 = \eta_1 P_\beta(w_1\bar\gamma_0^4 + 2\bar\jmath\beta), \qquad \eta_1 = \eta\bar\gamma_0^{-2} \le 1 \tag{26.4.10}$$

and therefore in $B(0,1)$

$$w_1 = -\frac{1}{4\pi}\int |x-z|^{-1}\eta_1 P'_\beta(w_1(z)\bar\gamma_0^4 + 2\bar\jmath\beta)\phi(z)\,dz + w'_1. \tag{26.4.11}$$

□

Now let us reintroduce the scaling function

$$\gamma_1(x) = \epsilon\Big(\min_j |w_1 + 2(\bar\jmath - j)\beta\bar\gamma_0^{-4}|^2 + |\nabla w_1|^3 + |\nabla^2 w_1|^6\Big)^{\frac16} + C_0 h_1^{\frac25}. \tag{26.4.7)*}$$

Then

(26.4.12) $|x-y| \le 2\gamma_0(x) \implies \gamma_0(y) \asymp \gamma_0(x)$ and $(26.4.8)_{1-3}$ hold (with some constant factor in the right-hand expression).

Let us consider $\bar x \in B(0,1)$ (it is a new point), $B(\bar x, \bar\gamma_1)$, $\bar\gamma_1 = \gamma_1(\bar x)$, and scale again $x \mapsto \bar\gamma_1^{-1}(x-\bar x)$, $\tau \mapsto \bar\gamma_1^{-3}\tau$ and respectively $w_1 \mapsto w_2 = \bar\gamma_1^{-3} w_1$, $h_1 \mapsto h_2 = h_1\bar\gamma_1^{-\frac52}$.

If $\bar\gamma_1 \asymp h_1^{\frac25}$ then $h_2 \asymp 1$ and we are done. If $|\nabla w_2| \asymp 1$ we are done as well.

Step 3. So, consider the remaining case $|\nabla^2 w_2| \asymp 1$. Then we introduce the scaling function (now, we have no doubt that this is a scaling function):

$$\gamma_2(x) = \epsilon\Big(\min_j |w_2 + 2(\bar\jmath - j)\beta\bar\gamma_0^{-4}\bar\gamma_1^{-3}| + |\nabla w_2|^2\Big)^{\frac12} + C_0 h_2^{\frac12}. \tag{26.4.13}$$

Let us consider $\bar x \in B(0,1)$ (it is a new point), $B(\bar x, \bar\gamma_2)$, $\bar\gamma_2 = \gamma_2(\bar x)$, and scale again $x \mapsto \bar\gamma_2^{-1}(x-\bar x)$, $\tau \mapsto \bar\gamma_2^{-2}\tau$ and respectively $w_2 \mapsto w_3 = \bar\gamma_2^{-2} w_2$, $h_2 \mapsto h_3 = h_2\bar\gamma_2^{-2}$.

If $\bar\gamma_2 \asymp h_1^{\frac12}$ then $h_3 \asymp 1$ and we are done. If $|\nabla w_3| \asymp 1$ we are done as well.

Step 4. Finally, introduce

$$\gamma_3(x) = \epsilon \min_j |w_3 + 2(\bar{j} - j)\beta\bar{\gamma}_0^{-4}\bar{\gamma}_1^{-3}\bar{\gamma}_2^{-2}| + Ch_3^{\frac{2}{3}}. \tag{26.4.14}$$

26.4.2 Zone $\mathcal{X}_2$: Semiclassical N-Term

Now we apply scaling the arguments using scaling functions γ_{1-3} constructed above.

We revert our steps. While we call γ_{1-3} *relative scaling functions* let us introduce *absolute scaling functions* $\alpha_0(x) = \gamma_0(x)$, $\alpha_1(x) = \gamma_0(x)\gamma_1(x)$, $\alpha_2(x) = \gamma_0(x)\gamma_1(x)\gamma_2(x)$, and $\alpha_3(x) = \gamma_0(x)\gamma_1(x)\gamma_2(x)\gamma_3(x)$ [26].

We need first

Proposition 26.4.6. *Consider $B(0,1)$ and assume that in it*

$$|\nabla w| \asymp \theta, \tag{26.4.15}$$

and

$$|\nabla^2 w| \le c\theta. \tag{26.4.16}$$

Let

$$\gamma(x) := \epsilon \min_j |w - 2\mu hj|\theta^{-1} + \hbar^{\frac{2}{3}} \tag{26.4.17}$$

and

$$\varepsilon \ge \hbar^{\frac{2}{3}-\delta}, \qquad \hbar := h\theta^{-\frac{1}{2}}. \tag{26.4.18}$$

Let $\varphi \in \mathscr{C}_0^\infty([-\epsilon_0, \epsilon_0])$. Then for $\alpha \le \bar{\gamma} := \gamma(\bar{x})$,

$$\Big| \int \phi_\alpha(x)\Big(e_\varphi(x,x,\tau) - P'_{\mu h,\varphi}\big(w(x)+\tau\big)\Big)\,dx\Big| \le C\mu h^{-1}\alpha^3 + C\mu h^{-1}\alpha^3\bar{\gamma}^{-1}(\hbar\bar{\gamma}^{-\frac{1}{2}}\alpha^{-1})^s \tag{26.4.19}$$

with

$$e_\varphi(x,y,\tau) := \varphi\big(h^2 D_{x_3}^2(\mu h)^{-1}\big)e(x,y,\tau) \tag{26.4.20}$$

and Weyl expression

$$P'_{\beta,\varphi}(w+\tau) = \mathsf{const}\sum_j \beta(w+\tau-2j\mu h)^{\frac{1}{2}}\varphi(w+\tau-2j\mu h) \tag{26.4.21}$$

[26] So far we ignore the very first scaling $x \mapsto (x-\bar{x})\ell^{-1}$. Therefore really absolute scaling functions would be $\alpha_j\ell$.

with the standard constant where for each x only one term is present in this sum. Here we take large $s > 0$ as $\hbar \le \bar\gamma\alpha$ and $s = 0$ otherwise.

Proof. The proof is standard and based on the standard reduction to the canonical form, standard estimates for $U(x,y,t)$ a Schwartz kernel of propagator $e^{i\hbar^{-1}\theta^{-1}tH}$:

$$|F_{t\to\hbar^{-1}\tau}\int \bar\chi_T(t)\phi_\alpha(x)U_\varphi(x,x,t)\,dx| \le C\mu h^{-1}\alpha^3 \tag{26.4.22}$$

for $T \asymp 1$ and

$$|F_{t\to\hbar^{-1}\tau}\int \phi_\alpha(x)\big(\bar\chi_T(t)-\bar\chi_{\bar T}(t)\big)U_\varphi(x,x,t)\,dx| \le C\mu h^{-1}\alpha^3(\hbar\bar\gamma^{-\frac12}\alpha^{-1})^s \tag{26.4.23}$$

with $\bar T = \epsilon\gamma^{\frac12}$ where U_φ is defined similarly to (26.4.20).

Here obviously we can skip in (26.4.19) all perturbation terms in the argument and in ϕ_α transformed. □

Then plugging into (26.4.19) $\alpha = \gamma$ $(= \bar\gamma)$, we have factor $(h\gamma^{-\frac32})^s$ in the second term.

There are two cases: $\theta \le \mu h$ and $\theta \ge \mu h$.

In the former case $\theta \le \mu h$, taking the sum over γ-partition of 1-element we estimate the same expressions with ϕ_1 instead of ϕ_γ by their right-hand expressions integrated over $\gamma^{-3}\,d\gamma$ which returns $C\mu h^{-1}$.

On the other hand, in the latter case $\theta \ge \mu h$, let $\lambda = \mu h\theta^{-1}$. Taking the sum over γ_3-partition of λ-element ϕ_λ by the right-hand expressions which returns $C\mu h_2^{-1}\lambda^2$. In this case summation over λ-partition return $C\theta h^{-2}$.

In both cases we arrive to the following estimate:

$$|\int \phi(x)\Big(e_\varphi(x,x,\tau) - P'_{\mu h,\varphi}\big(w(x)+\tau\big)\Big)\,dx| \le C\theta h^{-2} + C\mu h^{-1}. \tag{26.4.24}$$

Applying this estimate after α_2-scaling we conclude that the left-hand expression with $\phi = \phi_{\alpha_2}$ (in the non-scaled settings) does not exceed $C\theta h^{-2}\alpha_2^2 + C\mu h^{-1}\alpha_2^2$. Here the first term is $O(h^{-2}\alpha_2^3)$ and the summation over 1-element returns $O(h^{-2})$.

Consider the second term $C\mu h^{-1}\alpha_2^2 = C\mu h^{-1}\gamma_0^2\gamma_1^2\gamma_2^2$. Then summation over α_2-partition of α_1-element returns

$$C\mu h^{-1}\gamma_0^2\gamma_1^2 \int \gamma_2^2 \times \gamma_2^{-3}\, dx \asymp C\mu h^{-1}\gamma_0^2\gamma_1^2(1+|\log \hat{\gamma}_2|)$$

where $\hat{\gamma}_k$ is a minimal value of γ_k over γ_{k-1}-element. However in fact there will be no logarithmic factor because in virtue of equation (26.4.1) there is a positive eigenvalue of $\mathsf{Hess}\, w_2$ of the maximal size (cf. Section 5.2.1). Therefore, in fact, we have $C\mu h^{-1}\gamma_0^2\gamma_1^2$.

Now summation over α_1-partition of α_0-element returns

$$C\mu h^{-1}\gamma_0^2 \int \gamma_1^2 \times \gamma_1^{-3}\, dx \asymp C\mu h^{-1}\gamma_0^2(1+|\log \hat{\gamma}_1|).$$

Finally, summation over α_0-partition of 1-element returns

$$C\mu h^{-1} \int \gamma_0^2(1+|\log \hat{\gamma}_1|) \times \gamma_0^{-3}\, dx \asymp C\mu h^{-1}\hat{\gamma}_0^{-1}(1+|\log \check{\gamma}_1|),$$

where $\check{\gamma}_k$ is an absolute minimum of γ_k. However $\gamma_0^2 \ge \eta$ and $\gamma_1 \ge \eta$ and therefore expression above does not exceed $C\mu h^{-1}\eta^{-\frac{1}{2}}(1+|\log \eta|)$.

Remark 26.4.7. Recall that we estimated only the cut-off expression. To calculate the full expression we need to calculate also the contribution of the zone $\{\xi_3^3 \ge \mu h\}$. However this is easy.

Really, instead of $\varphi(h^2D_3^2/(\mu h))$ consider $\varphi'(h^2D_3^2/\theta)$ with $\varphi' \in \mathscr{C}^\infty([1,4])$ and $\epsilon\mu h \le \theta \le 1$. Without any scaling one can prove easily that such modified expression (26.4.24) does not exceed $C\theta h^{-2}$. We leave easy details to the reader.

Therefore plugging $\theta = 2^n\mu h$ and taking a sum over $n = 0, \dots, \lfloor |\log_2 \mu h| \rfloor$ we get the required expressions. Also note that in such expressions we need to consider perturbed argument $w + \mu^{-2}\omega_1 + j\mu^{-1}h\omega_2$ (all other terms which are $O(\mu^{-4} + \mu^{-\frac{3}{2}}h)$ could be skipped and also a perturbed function transformed).

Remark 26.4.8. (i) However we need to get rid of these perturbations for $\theta \le \mu h^{1-\delta}$ only. Indeed, for $\theta \ge \mu h^{1-\delta}$ we need canonical form *only* to study propagation and calculations could be performed without it. But then

getting rid of the perturbation is trivial provided this perturbation does not exceed $C\mu h^{1+\delta}$ which is the case if

$$\mu \ge h^{-\frac{1}{3}-\delta}. \tag{26.4.25}$$

(ii) Note that in the smooth approximation contributions of $\mathcal{X}_1$ is always less than $CZ^{\frac{2}{3}-\delta_1}$, $C\max(Z^{\frac{5}{3}}, Z^{\frac{3}{5}}B^{\frac{4}{5}})Z^{-\delta_1}$ or $C\max(Z^{\frac{5}{3}}, Z^{\frac{3}{5}}B^{\frac{4}{5}})Z^{-\delta_1} + CB^{\frac{1}{2}}Z^{\frac{7}{6}-\delta_1}$ respectively with an exception of the first two and only in the case of the threshold value $\ge Z^{-\frac{1}{3}-\delta_2}$. However in this case $\eta \ge Z^{-\delta_3}$ and the errors of the smooth approximation approach in fact are less than $CZ^{\frac{2}{3}-\delta_4}$, $CZ^{\frac{5}{3}-\delta_4}$ as well. Therefore there are in fact no exception.

(iii) It is important to have $\varepsilon \le \mu h$ and with $\varepsilon = \hbar^{\frac{2}{3}-\delta}$, $\hbar = h\theta^{-\frac{1}{2}}$ it means $\mu \ge h^{-\frac{1}{3}-\delta}\theta^{\frac{1}{3}-\frac{1}{2}\delta}$ which is due to (26.4.25).

Therefore we conclude that in the completely non-scaled settings with $\phi = \phi_\ell(x)$

$$|\int \phi(x)\Big(e(x,x,\tau) - P'_B\big(W(x)+\tau\big)\Big)| \le C\zeta^2\ell^2 + CB\ell\zeta^{-\frac{1}{2}}(1+|\log \ell^2\zeta|) \tag{26.4.26}$$

where the first term is Ch^{-2} and the second term is $C\mu h^{-1}\eta^{-\frac{1}{2}}(1+|\log\eta|)$; recall that $h^{-1} \asymp \ell\zeta$, $\mu \asymp B\ell\zeta^{-1}$ and $\eta \asymp \ell^2\zeta$. In comparison with the non-degenerate case $|\nabla W_B^{\mathsf{TF}}| \asymp \zeta^2\ell^{-1}$ we acquired the last term.

Assume first that condition (26.2.28) is fulfilled. Then

(i) If $B \le Z^{\frac{4}{3}}$, $\ell \le Z^{-\frac{1}{3}}$ we have $\zeta = Z^{\frac{1}{2}}\ell^{-\frac{1}{2}}$ and the right-hand expression of (26.4.26) returns $CZ\ell + CB\ell^{\frac{5}{4}}Z^{-\frac{1}{4}}$ and the summation with respect to ℓ results in its value as $\ell = Z^{-\frac{1}{3}}$ i.e. $CZ^{\frac{2}{3}} + CBZ^{-\frac{2}{3}}$ with the dominating first term.

(ii) If $B \le Z^{\frac{4}{3}}$, $\ell \ge Z^{-\frac{1}{3}}$ we have $\zeta = \ell^{-2}$ and the right-hand expression of (26.4.26) returns $C\ell^{-2} + CB\ell^2$. We need to sum as long as $\mu h \le 1$ i.e. $Z^{-\frac{1}{3}} \le \ell \le B^{-\frac{1}{4}}$ and the summation returns $CZ^{\frac{2}{3}} + CB^{\frac{1}{2}}$ with the dominating first term.

(iii) If $Z^{\frac{4}{3}} \le B \le Z^2$, $\ell \le B^{-1}Z$ we have $\zeta = Z^{\frac{1}{2}}\ell^{-\frac{1}{2}}$ and the right-hand expression of (26.4.26) returns $CZ\ell + CBZ^{-\frac{1}{4}}\ell^{\frac{5}{4}}$. Then summation results in $CZ^2B^{-1} + CZ^2B^{-\frac{1}{4}}Z \lesssim Z^{\frac{2}{3}}$.

If assumption (26.2.28) is not fulfilled we can estimate in the first term of the right-hand expression of (26.4.26) parameter ζ from above by $\min(Z^{\frac12}\ell^{-\frac12}, \ell^{-2})$ and in the second term from below by $\zeta_m = \min(Z_m^{\frac12}\ell_m^{-\frac12}, \ell_m^{-2})$ if $\ell = \ell_m := |x - y_m|$ and repeat all above arguments.

Therefore we arrive to the Statement (i) of Proposition 26.4.9 below. Furthermore, note that for $B \le Z$ the zone $\mathcal{X}_2$ is contained in the zone $\{x\colon \ell(x) \ge B^{-\frac{3}{10}} \ge Z^{-\frac{3}{10}}\}$ (really, $\mu \ge h^{-\frac13}$ in $\mathcal{X}_2$) and we arrive to the Statement (ii) below.

Proposition 26.4.9. *(i) For $B \le Z^2$ the contribution of zone $\mathcal{X}_2$ to the expression*

$$\int \bigl(e(x,x,\nu) - P'_B(W(x)+\nu)\bigr)\,dx \tag{26.4.27}$$

does not exceed $CZ^{\frac23}$.

(ii) For $B \le Z$ the contribution of zone $\mathcal{X}_2$ to the expression (26.4.27) *does not exceed $CZ^{\frac23-\delta}$.*

26.4.3 Zone $\mathcal{X}_2$: Semiclassical D-Term

Further, we need to estimate the *semiclassical* D*-term*

$$\mathsf{D}\Bigl(\phi_\alpha[e(x,x,\nu) - P'_B(W(x)+\nu)], \phi_\alpha[e(x,x,0) - P'_B(W(x)]\Bigr) \tag{26.4.28}$$

where $\phi_\alpha(x)$ is an α-admissible function. Again we revert our steps.

Consider $B(x,\bar\alpha_3)$ and apply Fefferman–de Llave decomposition (26.3.8). Then in the framework of Proposition 26.4.6 contribution of pairs $B(x,\alpha)$ and $B(y,\alpha)$ with $3\alpha \le |x-y| \le 4\alpha$ does not exceed the right-hand expression of (26.4.19) squared and multiplied by α^{-4}, where $C\alpha^{-3}$ estimates the number of the pairs and α^{-1} is the inverse distance. At this moment we discuss a cut-off version of (26.4.28) i.e. with $e_\varphi(.,.,.)$ and $P'_{\mu h,\varphi}(.)$. So, we have

$$C\mu^2 h^{-2}\bigl(1+\gamma_3^{-1}(h_2\gamma_3^{-\frac12}\alpha^{-1})^s\bigr)^2\alpha^4.$$

Then integrating this expression with respect to $\alpha^{-1}\,d\alpha$ with $\alpha \le \gamma_3$ we arrive to $C\mu^2h^{-2}\bigl(\gamma_3^4 + h_2^2\gamma_3\bigr)$.

Therefore we conclude that

(26.4.29) A cut-off version of expression (26.4.28) with $\alpha = \alpha_3$ does not exceed $C\mu^2 h^{-2}\bigl(\gamma_3^4 + h_2^2\gamma_3\bigr)\alpha_2^3$

The first term here $C\mu^2 h^{-2}\gamma_3^4\alpha_2^3$ does not exceed $C\mu^2 h^{-2}\alpha_3^3$ (recall that $\alpha_j = \alpha_{j-1}\gamma_j$) and the summation over α_3-partition of 1-element returns $C\mu^2 h^{-2}$.

Consider the second term $C\mu^2 h^{-2}h_2^2\gamma_3\alpha_2^3$; its summation with respect to α_3-partition of α_2-element returns $C\mu^2 h^{-2}\alpha_2^3 h_2^2 \int \gamma_3^{-2}\,d\gamma_3 \lesssim C\mu^2 h^{-2}\alpha_2^3$ (really, recall that according to (26.4.14) $\gamma_3 \ge h_3^{\frac{2}{3}}$) and then the summation over α_2-partition of 1-element returns $C\mu^2 h^{-2}$.

Consider $B(x,\bar\alpha_2)$ and apply Fefferman–de Llave decomposition (26.3.8). There are two kinds of pairs:

(a) those with $|x-y| \ge \epsilon(\alpha_3(x)+\alpha_3(y))$ for all (x,y) and

(b) those with $|x-y| \le \min(\alpha_3(x),\alpha_3(y))$ for all (x,y).

The total contribution of the pairs of the second type (i.e. summation is taken over *all* pairs of α_3-elements in $B(0,1)$) as we already know is $O(\mu^2 h^{-2})$. Meanwhile according to the analysis in the previous Subsection 26.4.2 a contribution of one pair of kind (a) does not exceed

$$C\underbrace{\bigl(h^{-2}+\mu h^{-1}\gamma_0^{-1}\gamma_1^{-1}\gamma_2^{-1}\gamma_3^{-1-\delta}\bar\gamma_3^{\delta}\bigr)\alpha_3^3}_{\text{at } x}\times \underbrace{\bigl(h^{-2}+\mu h^{-1}\gamma_0^{-1}\gamma_1^{-1}\gamma_2^{-1}\gamma_3^{-1-\delta}\bar\gamma_3^{\delta}\bigr)\alpha_3^3}_{\text{at } y}\times|x-y|^{-1}$$

where each of two first factors is just an estimate of the integral (26.4.24) calculated over corresponding domain. If we take the first term in the first factor and sum over α_3-partition of 1-element we get only the second factor multiplied by μh^{-1} and then summation was done in the previous subsection. Similarly we can deal with the first term in the second factor. On the other hand, if we take only second factors and sum over pairs of α_3-subelements of the same α_2-element we get

$$C\mu^2 h^{-2}\gamma_0^{-2}\gamma_1^{-2}\gamma_2^{-2}\alpha_2^5 \asymp C\mu^2 h^{-2}\alpha_2^3.$$

Then summation with respect to α_2-partition of 1-element returns $C\mu^2 h^{-2}$.

Consider now $B(\bar x,\bar\alpha_1)$ and apply here Fefferman-de Llave decomposition (26.3.8). There are two kinds of pairs:

(a) those with $|x-y| \ge \epsilon(\alpha_2(x)+\alpha_2(y))$ for all (x,y) and

(b) those with $|x-y| \le \mathsf{min}(\alpha_2(x),\alpha_2(y))$ for all (x,y).

According to the above analysis the total contribution of the pairs of the second type (i.e. the summation is taken over *all* pairs of α_2-elements in $B(0,1)$) as we already know is $O(\mu^2 h^{-2})$. Meanwhile according to the analysis in the previous Subsection 26.4.2 a contribution of one pair of kind (a) does not exceed

$$C\underbrace{\bigl(h^{-2}+\mu h^{-1}\gamma_0^{-1}\gamma_1^{-1}\gamma_2^{-1}\bigr)\alpha_3^3}_{\text{at } x}\times\underbrace{\bigl(h^{-2}+\mu h^{-1}\gamma_0^{-1}\gamma_1^{-1}\gamma_2^{-1}\bigr)\alpha_3^3}_{\text{at } y}\times|x-y|^{-1}$$

and here again we can "forget" about the first terms in each factor. Then the summation with respect to pairs of α_2-subelements of the same α_1-element results in $C\mu^2h^{-2}\gamma_0^{-2}\gamma_1^{-2}\alpha_1^5 \asymp C\mu^2h^{-2}\alpha_1^3$ where we avoid logarithmic factor in virtue of the same positive eigenvalue of $\mathsf{Hess}\, w$. Summation with respect to α_1-admissible partition of 1-element returns $C\mu^2h^{-2}$.

Consider now $B(\bar{x},\bar{\alpha}_0)$ and apply here Fefferman–de Llave decomposition. Again there are two kinds of pairs and the total contributions of the pairs of the second kind we already calculated and contribution of the pairs of α_1-subelements of the same 1-element does not exceed

$$C\mu^2h^{-2}\gamma_0^{-2}(1+|\log\hat{\gamma}_1|)^2\alpha_0^5 \lesssim C\mu^2h^{-2}(1+|\log\eta|)^2\alpha_0^3$$

and the summation with respect to α_0-partition of 1-element returns $C\mu^2h^{-2}(1+|\log\eta|)^2$.

Finally, consider $B(\bar{x},1)$ and apply here Fefferman-de Llave decomposition. Again there are two kinds of pairs and the total contributions of the pairs of kind (b) we already estimated while the total contribution of the pairs of kind (a) does not exceed $Ch^{-4}+C\mu^2h^{-2}\eta^{-1}(1+|\log\eta|)^2$ where we recalled the forgotten terms.

Again, this is estimate for cut-off expression. Going to uncut expression we repeat the same trick as before but as we deal with D-term we need to consider "mixed" pairs when one "factor" comes with θ and another with θ' but then contribution of such pair does not exceed $C(\nu h^{-4})^{\frac12}(\nu' h^{-4})^{\frac12}$. Easy details are left to the reader.

Therefore returning to the original scale we conclude that the contribution of ℓ-layer to (26.4.28) does not exceed

$$C\zeta^4\ell^3 + CB^2\ell\zeta^{-1}(1+|\log\ell^2\zeta|)^2 \tag{26.4.30}$$

which is exactly the right-hand expression of (26.4.26) squared and multiplied by ℓ^{-1} due to scaling.

Remark 26.4.10. In comparison with the non-degenerate case $|\nabla W_B^{\mathsf{TF}}| \asymp \zeta^2\ell^{-1}$ we acquired the last term.

Assume first that condition (26.2.28) is fulfilled. Then

(i) For $B \le Z^{\frac{4}{3}}$, $\ell \le Z^{-\frac{1}{3}}$ we have $\zeta = Z^{\frac{1}{2}}\ell^{-\frac{1}{2}}$ and expression (26.3.42) returns $CZ^2\ell + CB^2\ell^{\frac{3}{2}}Z^{-\frac{1}{2}}$ and the summation with respect to ℓ results in its value as $\ell = Z^{-\frac{1}{3}}$ i.e. $CZ^{\frac{5}{3}} + CB^2Z^{-1}$ with the dominating first term.

(ii) For $B \le Z^{\frac{4}{3}}$, $\ell \ge Z^{-\frac{1}{3}}$ we have $\zeta = \ell^{-2}$ and expression (26.3.42) returns $C\ell^{-5} + CB^2\ell^3$. We need to sum as long as $\mu h \le 1$ i.e. $Z^{-\frac{1}{3}} \le \ell \le B^{-\frac{1}{4}}$ and the summation returns $CZ^{\frac{5}{3}} + CB^{\frac{5}{4}}$ with the dominating first term.

(iii) For $Z^{\frac{4}{3}} \le B \le Z^2$, $\ell \le B^{-1}Z$ we have $\zeta = Z^{\frac{1}{2}}\ell^{-\frac{1}{2}}$ and expression (26.3.42) returns $CZ^2\ell + CB^2Z^{-\frac{1}{2}}\ell^{\frac{3}{2}}$. Then the summation results in $CZ^3B^{-1} + CB^{\frac{1}{2}}Z \lesssim Z^{\frac{3}{5}}B^{\frac{4}{5}}$.

Sure, we need to consider also mixed pairs of the layers and their contributions are

$$C\bigl(\zeta^2\ell^2 + CB\ell\zeta^{-\frac{1}{2}}(1+|\log \ell^2\zeta|)\bigr)\times\bigl(\zeta'^2\ell'^2 + CB\ell'\zeta'^{-\frac{1}{2}}(1+|\log \ell'^2\zeta'|)\bigr)\times(\ell+\ell')^{-1}$$

and the summation with respect to ℓ and ℓ' returns the same expression as above.

If assumption (26.2.28) is not fulfilled we use the same trick as in the previous Subsection 26.4.2. Therefore we arrive to the Statement (i) of Proposition 26.4.11 below. Applying the same arguments as in the proof of Proposition 26.4.9 we arrive to the Statement (ii):

Proposition 26.4.11. *(i) For $B \le Z^2$ the contribution of the zone $\mathcal{X}_2\times\mathcal{X}_2$ to expression* (26.4.28) *does not exceed $C\max(Z^{\frac{5}{3}}, Z^{\frac{3}{5}}B^{\frac{4}{5}})$.*

(ii) For $B \le Z$ contribution of the zone $\mathcal{X}_2 \times \mathcal{X}_2$ to expression (26.4.28) *does not exceed $CZ^{\frac{5}{3}-\delta}$.*

26.4.4 Semiclassical T-Term

Semiclassical T-Term: Zone $\mathcal{X}_1$ Extended

First let us cover zone $\mathcal{X}_1$ extended.

What is $\mathcal{X}_1$ Extended? To define this zone $\mathcal{X}_1' := \{x\colon \ell(x) \le r\}$, where we define W using P rather than P_B let first us analyze the precise extension in the framework of N- and D-terms. For N-term we have approximation error and corresponding D-term not exceeding respectively $C(\mu h)^2\eta^{-\frac12}h^{-3} = CB^2\zeta^{-\frac32}\ell^2$ and this expression squared and multiplied by ℓ^{-1} i.e. $CB^4\zeta^{-3}\ell^3$. Finally, both expressions are summed to their values as $\ell = r$. Recall that either $\zeta = Z^{\frac12}\ell^{-\frac12}$ or $\zeta = \ell^{-2}$.

(i) Consider first $B \le Z^{\frac43}$. Then we want these errors not to exceed respectively $CZ^{\frac23}$ and $CZ^{\frac53}$. Obviously, if $r \ge Z^{-\frac13}$ the first condition is more restrictive. In this case plugging $\zeta = r^{-2}$ and we set $CB^2r^5 = Z^{\frac23}$ i.e. $r = B^{-\frac25}Z^{\frac{2}{15}}$. Then $r \ge Z^{-\frac13}$ as long as $B \le Z^{\frac76}$.

Then $\mu = Br^3 = B^{-\frac15}Z^{\frac{6}{15}} \ge Z^{\frac16}$ and $h = B^{-\frac25}Z^{\frac{2}{15}} \ge Z^{-\frac13}$; and one can see easily that $\mu \gtrsim h^{-\frac12}$ provided $\ell(x) \ge r$.

(ii) Consider next $Z^{\frac76} \le B \le Z^{\frac43}$. Then for $r \le Z^{-\frac13}$ we have $\zeta = Z^{\frac12}r^{-\frac12}$. In this case the second requirement is more restrictive and we set $B^4Z^{-\frac32}r^{\frac92} = Z^{\frac53}$, i.e. $r = B^{-\frac89}Z^{\frac{19}{27}}$. Then $\mu = B^{-\frac13}Z^{\frac59}$ and $h = Z^{-\frac12}r^{-\frac12} = B^{\frac49}Z^{-\frac{23}{27}}$ and $\mu \ge h^{-\frac37}$; this is better than $h^{-\frac13}$.

However, we can do better than this: observe that $\mu h \le \eta$ if and only if $r \ge B^2Z^{-3}$ i.e. $B \le Z^{\frac{50}{39}}$, which is greater than $Z^{\frac76}$ but less than $Z^{\frac43}$, so we test μ and h in this case: $\mu = Z^{\frac{5}{39}}$ and $h = Z^{\frac{11}{39}}$ and $\mu \ge h^{-\frac{5}{11}}$ provided $\ell(x) \ge r$.

If $B \ge Z^{\frac{5}{11}}$ we will use another estimate for D-term: namely it does not exceed $(\mu h)^3h^{-6}r^{-1} = B^3r^5$ and we want it not to exceed $Z^{\frac53}$, so $r = Z^{\frac13}B^{-\frac35}$ (which is still less than $Z^{-\frac13}$) and $\mu = B^{\frac{1}{10}}$ and $h = B^{\frac{3}{10}}Z^{-\frac23}$ and we test it as $B = Z^{\frac43}$ when $\mu = B^{\frac{1}{10}}$ and $h = B^{-\frac15}$, so exponent $-\frac{5}{11}$ fits again.

(iii) Finally, if $Z^{\frac43} \le B \le Z^3$ then the error D-term does not exceed B^3r^5 and we want it not to exceed $B^{\frac45}Z^{\frac35}$. So, we pick up $r = B^{-\frac{11}{25}}Z^{\frac{3}{25}}$, $\mu = B^{\frac{17}{50}}Z^{-\frac{8}{25}}$, $h = B^{\frac{11}{50}}Z^{-\frac{14}{25}}$ and exponent $-\frac{5}{11}$ fits again.

When we can Use the same Method for T-Term? As far as semi-classical T-expression is concerned an approximation error of such approach in the localized and scaled settings is $C(\mu h)^{\frac52}h^{-3}$ [27] which is $O(h^{-1})$ only

[27] If instead of $P_B(W)$ we use $P(W)+\frac12 P''(W)B^2$ rather than $P(W)$. This modification does not affect our previous arguments.

if $\mu \ge h^{-\frac{1}{5}}$. One can extend it to $\mu \ge h^{-\frac{1}{5}-\delta}$ using the same trick as in Remark 26.4.8 but we need to do better than this.

On the other hand, observe that in fact an approximation error does not exceed[27] $C(\mu h)^3\eta^{-\frac{1}{2}}h^{-3} \asymp CB^3\ell^2\zeta^{-\frac{7}{2}}$ in the localized scaled settings. The simple proof is left to the reader. This is translated into $CB^3\ell^2\zeta^{-\frac{3}{2}}$ into unscaled settings. Summation with respect to $\ell \le r$ returns its value as $\ell = r$.

So we get CB^3r^5 as $B \le Z^{\frac{4}{3}}$ and $r \ge Z^{-\frac{1}{3}}$. Consider first $B \le Z$. In this case we want $CB^3r^5 \le CZ^{\frac{5}{3}}$ and we pick up $r = B^{-\frac{3}{5}}Z^{\frac{1}{3}}$ which is greater than $Z^{-\frac{1}{3}}$ provided $B \le Z^{\frac{10}{9}}$. Then $\mu = Br^3 = B^{-\frac{4}{5}}Z \ge Z^{\frac{1}{5}}$ and $h = r = B^{-\frac{3}{5}}Z^{\frac{1}{3}} \ge Z^{-\frac{4}{15}}$ and $\mu \ge h^{-\frac{3}{4}}$.

If $Z \le B \le Z^{\frac{4}{3}}$ but still $r \ge Z^{-\frac{1}{3}}$ we want $CB^3r^5 \le CZ^{\frac{4}{3}}B^{\frac{1}{3}}$[28] and we pick up $r = B^{-\frac{8}{15}}Z^{\frac{4}{15}}$ and we want it to be greater than $Z^{-\frac{1}{3}}$ i.e. $B \le Z^{\frac{9}{8}}$. Then $\mu = Br^3 = B^{-\frac{3}{5}}Z^{\frac{4}{5}} \ge Z^{\frac{1}{8}}$ and $h = B^{-\frac{8}{15}}Z^{\frac{4}{15}} \ge B^{-\frac{1}{3}}$ and $\mu \ge h^{-\frac{3}{8}}$. It is not as good as $\mu \ge h^{-\frac{3}{7}}$.

Then we use the smooth canonical form. In the operator perturbation terms have factors μ^{-2}, μ^{-4} etc and we can use the standard approach to get rid of $\mu^{-4} \le \mu h$, so we need to consider only μ^{-2}.

However let before scaling the second derivative of W be of magnitude θ; then after scaling it becomes of magnitude $\theta' = \gamma_0^2\gamma_1^2\theta$ and then the perturbation is of magnitude $\theta\mu^{-2}$ but contribution of the error will be (after we compare the true Riemann sum and the corresponding integral and their difference $Ch^{-3}\nu\mu^{-2}(\mu h)^2(\theta')^{-\frac{1}{2}}\times\alpha_1^3 \le C\theta^{\frac{1}{2}}h^{-1}\gamma_0^{-1}\gamma_1^{-1}\alpha_1^3 \le Ch^{-1}\gamma_1^{-\frac{1}{2}}\alpha_1^3$ where we used that $\theta \le C\gamma_0^2\gamma_1$. Then summation over α_1-partition of α_0 element returns $Ch^{-1}\alpha_0^3$ and the summation over γ_0-partition returns Ch^{-1} as desired. Therefore we covered zone $\mathcal{X}_1$ for T-term.

Semiclassical T-Term: Zone $\mathcal{X}_2$

Tauberian Estimate. Tauberian estimate for cut-off expression is rather simple:

$$C\mu h^{-1}\gamma_0^{-1}\gamma_1^{-1}\gamma_2^{-1}\times h\gamma_0^{-3}\gamma_1^{-\frac{5}{2}}\gamma_1^{-2}\times\gamma_0^4\gamma_1^3\gamma_2^2\alpha_2^3 \asymp C\mu\gamma_1^{-\frac{1}{2}}\gamma_2^{-1}\alpha_2^3$$

which nicely sums to $C\mu$ without logarithm due to the same positive eigenvalue arguments as before; for θ-cut-off with $\theta \ge \mu h$ we get the same albeit

[28] Because the semiclassical remainder estimate is not better than this. Actually, due to Remark 26.3.7 we can do marginally better than this, but we leave this analysis to the reader.

with γ_j defined by the same formula albeit with $(w_j - 2j\mu h s_j^{-1})$ replaced by θs_j^{-1} where s_j^{-1} means the scale; and this should be multiplied by $\theta/(\mu h)$. The result nicely sums to Ch^{-1}. This is what was required.

Magnetic Weyl Expression. Now we will get the same answer albeit $C\mu^{-4}$ term will be supplemented by $C\mu^{-\frac{3}{2}}h$ which in cut-off sum adds $C\mu^{-\frac{3}{2}}h \times \mu h^{-2} \le Ch^{-1}$.

We can use the standard approach, with an error $C\mu^{-\frac{3}{2}}h \times \theta/(\mu h) \times \mu h^{-2} \asymp C\theta\mu^{-\frac{3}{2}}h^{-2}$ which means that we can take $\theta = \mu^{\frac{3}{2}}h$ which is sufficient to deal with $\theta \ge C\mu^{\frac{3}{2}}h$; in particular, for $\mu \ge h^{\frac{2}{3}}$ we are done. But for $\theta \ge \mu h^{1-\delta}$ we can apply the weak magnetic field approach, which is sufficient. So we arrive to inequality

$$|\int_{-\infty}^{\tau}\int \phi(x)\Big(e_\varphi(x,x,\tau) - P'_{\beta,\varphi}\big(w(x)+\tau\big)\Big)\,dx\,d\tau| \le Ch^{-1} \tag{26.4.31}$$

and therefore we arrive to

Proposition 26.4.12. *(i) If $B \le Z^2$ the contribution of zone $\mathcal{X}_2$ to the expression*

$$\int_{-\infty}^{\tau}\int \phi(x)\Big(e_\varphi(x,x,\tau) - P'_{B,\varphi}\big(W(x)+\tau\big)\Big)\,dx\,d\tau \tag{26.4.32}$$

does not exceed $C\max\big((Z+B)^{\frac{1}{3}}Z^{\frac{4}{3}}, Z^{\frac{3}{5}}B^{\frac{4}{5}}\big)$.

(ii) If $B \le Z$ the contribution of zone $\mathcal{X}_2$ to expression (26.4.32) *does not exceed $CZ^{\frac{5}{3}-\delta}$.*

Mollification Errors. Further, we need to estimate

$$\int \phi(x)\Big(P'_B(W(x)+\tau)) - P'_B(W_B^{\mathsf{TF}}(x)+\tau))\Big)\,dx, \tag{26.4.33}$$

$$\int \phi(x)\Big(P_B(W(x)+\tau)) - P_B(W_B^{\mathsf{TF}}(x)+\tau))\Big)\,dx, \tag{26.4.34}$$

$$\mathsf{D}\Big(\phi(x)\big(P'_B(W(x)+\tau)) - P'_B(W_B^{\mathsf{TF}}(x)+\tau)\big),\ \phi(x)\big(P'_B(W(x)+\tau)) - P'_B(W_B^{\mathsf{TF}}(x)+\tau)\big)\Big) \tag{26.4.35}$$

and

$$\|\phi(x)\nabla\big(W(x)-W_B^{\mathsf{TF}}(x)\big)\|^2. \tag{26.4.36}$$

We start from local versions (so in fact we dealing with w and w_β^{TF}).

Obviously after all rescalings $\hbar = h\gamma_0^{-3}\gamma_1^{-\frac{5}{2}}\gamma_2^{-2}$ and therefore $\varepsilon = \hbar^{\frac{3}{2}} = h^{\frac{3}{2}}(\gamma_0^{-3}\gamma_1^{-\frac{5}{2}}\gamma_2^{-2})^{-\frac{3}{2}}$ where we set $\delta = 0$ but we will show that we have a reserve to set it as $\delta > 0$ if we want to estimate (26.4.33) by h and (26.4.34)–(26.4.36) by h^2.

We claim that

$$|w - w_\beta^{\mathsf{TF}}| \le C\varsigma := C\beta\eta\big(\gamma_0^4\gamma_1^3\gamma_2^2\varepsilon\big)^{\frac{5}{2}}. \tag{26.4.37}$$

and

$$|\nabla(w - w_\beta^{\mathsf{TF}})| \le C\varsigma_1 := C\beta\eta\big(\gamma_0^4\gamma_1^3\gamma_2^2\varepsilon\big)^{\frac{3}{2}}. \tag{26.4.38}$$

Indeed, it follows from equation (26.4.2).

Then the contribution of α_2-element to (26.4.34) does not exceed $C\varsigma\varepsilon\alpha_2^3$ as measure of zone of α_2-element where $w \ne w_\beta^{\mathsf{TF}}$ is $O(\varepsilon\alpha_2^3)$. One can see easily that $\varsigma\varepsilon = O(h^{\frac{7}{3}})$ and therefore $C\varsigma\varepsilon\alpha_2^3 = O(h^{\frac{7}{3}}\alpha_2^3)$ and the summation over α_2-partition of 1-element returns $O(h^{\frac{7}{3}})$.

Modulo above calculations the contribution of α_2-element to (26.4.33) does not exceed $C\beta\varsigma^{\frac{1}{2}}\varepsilon\alpha_3^2$. One can check easily that $\varsigma\varepsilon = O(h^{\frac{3}{2}}\gamma_2^{-\frac{1}{2}})$ and therefore $C\varsigma^{\frac{1}{2}}\varepsilon\alpha_2^3 = O(h^{\frac{7}{3}}\alpha_1^3\gamma_2^{\frac{5}{2}})$ and the summation over α_2-partition of α_1-element returns $O(h^{\frac{3}{2}}\alpha_1^3)$ and then the summation over α_1-partition of 1-element returns $O(h^{\frac{3}{2}})$.

Similarly, expression (26.4.35) with $\phi = \phi_{\alpha_2}$ does not exceed $C\varsigma\varepsilon_2\alpha_2^5 \le Ch^3\alpha_2^4$ and the summation over α_2-partition of 1-element returns $O(h^3)$. However we need to consider disjoint pairs of α_2-elements belonging to given α_1-element and their contribution does not exceed

$$Ch^3\int \gamma_{2x}^{-\frac{1}{2}}\gamma_{2y}^{-\frac{1}{2}}|x-y|^{-1}\,dxdy \le Ch^2\alpha_1^5$$

and then summation over α_1-partition of 1-element returns $O(h^{\frac{3}{2}})$. We need also to consider disjoint pairs of α_1-elements belonging to given 1-element and their contribution does not exceed $Ch^3\int |x-y|^{-1}\,dxdy = O(h^3)$.

Finally, contribution of α_2-element to (26.4.36) does not exceed $C\varsigma_1^2\varepsilon\alpha_2^3$ and one can check easily that this does not exceed $C\alpha_1^3\gamma_2^{\frac{8}{3}}h^{\frac{8}{3}}$ and the summation over α_2-partition of α_1-element returns $C\alpha_1^3h^{\frac{8}{3}}$; then summation over 1-partition of 1-element returns $O(h^{\frac{8}{3}})$.

So, the scaled versions of (26.4.33) and (26.4.34)–(26.4.36) do not exceed Ch and Ch^2 respectively. Then the original versions of (26.4.33), (26.4.34), (26.4.34), and (26.4.36) do not exceed respectively $C\zeta^3\ell^3\times(\zeta\ell)^{-1}=C\zeta^2\ell^2$, $C\zeta^5\ell^3\times(\zeta\ell)^{-2}=C\zeta^3\ell$, $C\zeta^6\ell^5\times(\zeta\ell)^{-2}=C\zeta^4\ell^3$, and $C\zeta^4\ell\times(\zeta\ell)^{-2}=C\zeta^2\ell^{-1}\le C\zeta^4\ell^3$.

Leaving the easy details to the reader we arrive to

Proposition 26.4.13. *(i) Contribution of zone $\mathcal{X}_2$ to the mollification error* (26.4.33) *does not exceed $CZ^{\frac{2}{3}}$.*

(ii) Contribution of zone $\mathcal{X}_2$ to the mollification error (26.4.34) *does not exceed $CZ^{\frac{5}{3}}+o(Z^{\frac{4}{3}}B^{\frac{1}{3}})$.*

(iii) Contributions of zone $\mathcal{X}_2$ to the mollification errors (26.4.35) *and* (26.4.36) *do not exceed $CZ^{\frac{5}{3}}$.*

and

Proposition 26.4.14. *Let $B\le Z$. Then*

(i) Contribution of zone $\mathcal{X}_2$ to the mollification error (26.4.33) *does not exceed $CB^{\delta}Z^{\frac{2}{3}-\delta}$.*

(ii) Contributions of zone $\mathcal{X}_2$ to the mollification errors (26.4.35)–(26.4.35) *do not exceed $CB^{\delta}Z^{\frac{5}{3}-\delta}$.*

Remark 26.4.15. Consider the mollification parameter in "absolute" scale (i.e. ℓ-scale): $\varepsilon=\gamma_0\gamma_1\gamma_2\gamma(h/\gamma_0^3\gamma_1^{\frac{5}{2}}\gamma_2^2)^{\frac{2}{3}-\delta}$. One can see easily that $\varepsilon\ge h^{\frac{2}{3}-\delta}\ge(\mu^{-1}h)^{\frac{1}{2}-\delta_1}$ which makes reduction possible.

Remark 26.4.16. All statements of Propositions 26.4.13 and 26.4.14 are valid for semiclassical errors as well except statements, concerning T-term; tose should include also terms $Ca^{-\frac{1}{2}}Z^{\frac{3}{2}}$ for $a\le Z^{-\frac{1}{3}}$ and $Ca^{-\delta}Z^{\frac{5}{3}+\frac{1}{3}\delta}$ for $a\ge Z^{-\frac{1}{3}}$.

26.4.5 Zone $\mathcal{X}_3$

Zone $\mathcal{X}_3$ defined by $\mu h \ge C_0$, $h \le 1$, $\{x\colon \ell(x) \le \epsilon_0 \bar{r}\}$ appears only as $Z^{\frac{4}{3}} \le B \le Z^3$. In this zone W_B^{TF} is smooth and no mollification is necessary. Further, in this zone the canonical form contains only one number $j = 0$ and $|D^\alpha W| \le C_\alpha \zeta^2 \ell^{-|\alpha|}$ and $W \asymp \zeta^2$.

Therefore we have non-degeneracy condition fulfilled and applying the standard theory we conclude that in the scaled version contribution of $B(0,1)$ to the semiclassical errors in N- and T-terms and into D-term are $C\mu h^{-1}$, $C\mu$ and $C\mu^2 h^{-2}$ respectively.

In the unscaled version they become $CB\ell^2$, $CB\ell\zeta \le CBZ^{\frac{1}{2}}\ell^{\frac{1}{2}}$ and $CB^2\ell^3$ and after summation (where for D-term we need to consider mixed contribution of different layers) we arrive to the same expressions calculated as $\ell = \bar{r} = B^{-\frac{2}{5}} Z^{\frac{1}{5}}$ i.e. $CB^{\frac{1}{5}} Z^{\frac{2}{5}}$, $CB^{\frac{4}{5}} Z^{\frac{3}{5}}$ and $CB^{\frac{4}{5}} Z^{\frac{3}{5}}$ respectively. Thus we have proven

Proposition 26.4.17. *Let $Z^{\frac{4}{3}} \le B \le Z^3$. Then*

(i) Contribution of zone $\mathcal{X}_3$ to the N*-error does not exceed $CZ^{\frac{2}{5}} B^{\frac{1}{5}}$.*

(ii) Contributions of zone $\mathcal{X}_3$ to the T*-error and* D*-term do not exceed $CZ^{\frac{3}{5}} B^{\frac{4}{5}}$.*

26.5 Semiclassical Analysis in the Boundary Strip for $M \ge 2$

To finish our analysis we need to get the same estimates as before in the *boundary strip*

$$\mathcal{Y} := \{x\colon W(x) + \nu \le \epsilon G,\ \epsilon \bar{r} \le \ell(x) \le c\bar{r}\} \tag{26.5.1}$$

with

$$G := \begin{cases} (Z-N)_+^{\frac{4}{3}} & \text{for } B \le (Z-N)_+^{\frac{4}{3}}, \\ B & \text{for } (Z-N)_+^{\frac{4}{3}} \le B \le Z^{\frac{4}{3}}, \\ Z^{\frac{4}{5}} B^{\frac{2}{5}} & \text{for } B \ge Z^{\frac{4}{3}}. \end{cases} \tag{26.5.2}$$

which coincides with (26.2.48) as $B \ge (Z-N)^{\frac{4}{3}}$. Recall that $\bar{r} = (Z-N)_+^{-\frac{1}{3}}$, $\bar{r} = B^{-\frac{1}{4}}$ and $\bar{r} = B^{-\frac{2}{5}} Z^{\frac{1}{5}}$ in these three cases respectively. Analysis of

the *external zone* $\mathcal{X}_4 := \{x\colon \ell(x) \ge C_1\bar{r}\}$ will be trivial and *inner zone* $\{x\colon W(x)+\nu \ge \epsilon G\}$ has been covered already.

26.5.1 Properties of W_B^{TF} if $N = Z$

Let us explore properties of W_B^{TF} in $\mathcal{Y}$ if $N = Z$ [29] Let us rescale $x \mapsto x' = x\bar{r}^{-1}$, $W \mapsto w = G^{-1}W$ and define $h = G^{-\frac12}\bar{r}^{-1}$, $\mu = G^{-\frac12}B\bar{r}$. Then

(a) In the case $B \le Z^{\frac43}$ we need to rescale $w(x') = B^{-1}W_B^{\mathsf{TF}}(x'\bar{r})$ and take $h = B^{-\frac14} \le 1$, $\mu = B^{\frac14} \ge 1$, $\mu h = 1$.

(b) On the other hand, for $B \ge Z^{\frac43}$ one should set $w(x') = \bar{r}Z^{-1}W_B^{\mathsf{TF}}(x'\bar{r})$ and $h = (Z\bar{r})^{-\frac12} = (BZ^{-3})^{\frac15} \le 1$, $\mu = BZ^{-\frac12}\bar{r}^{\frac32} = B^{\frac25}Z^{-\frac15} \ge 1$, $\mu h = B^{\frac35}Z^{-\frac45} \ge 1$ ($\mu h \asymp 1$ iff $B \lesssim Z^{\frac43}$, $h \asymp 1$ iff $B \asymp Z^3$).

We will use now only rescaled coordinates unless the opposite is specified. Then in $\mathcal{Y}$ rescaled

$$\Delta w = \kappa w_+^{\frac12}, \qquad \kappa = 12, \qquad w \to \theta = \nu\bar{\zeta}^{-2} \quad \text{as} \quad |x| \to \infty, \tag{26.5.3}$$

with $\bar{\zeta} := G^{\frac12}$ where one can always get $\kappa = 12$ after rescaling $w \mapsto 144\kappa^{-2}w$.

Proposition 26.5.1. *Let $Z = N$. Then in $\mathcal{Y}$ after rescaling*

$$|D^\alpha w| \le C_\alpha w\gamma^{-|\alpha|} \qquad \forall \alpha \tag{26.5.4}$$

with the scaling function $\gamma = w^{\frac14}$ and

$$|\nabla w^{\frac14}| \le 1 + Cw^t \tag{26.5.5}$$

with some constant C and exponent $t > 0$.

Proof. (a) Rescaling $x \mapsto x\bar{r}^{-1}$ we get an equations $(26.5.3)_0 = (26.5.3)$ with $\theta = 0$. We know that $W = 0$ for $\ell(x) \ge c\bar{r}$; so after rescaling $w = 0$ for $\ell(x) \ge c$. On the other hand, $w \asymp 1$ as $\ell(x) \le \epsilon$ (uniformly with respect to all the parameters).

Let us consider solution of the equation

$$\Delta w_s = 12 w_+^s \tag{26.5.6}$$

[29] I.e. $\nu = 0$ and $G = B\bar{r}^{-4}$.

in $\Omega = \{w \le \epsilon, \ell \le c\}$ with the boundary condition $w_s = w$ at $\partial\Omega$; $s > \frac{1}{2}$.

Note first that $w_s \ge 0$. Really, w_s is the solution of the variational problem to minimize

$$\|\nabla w\|^2 + 24(s+1)^{-1} \int w_+^{s+1}\, dx \tag{26.5.7}$$

and one makes this functional only less replacing w by w_+.

Further, the standard maximum principle arguments show that $w_s \searrow$ as $s \searrow$ [30]. Obviously $w_s \searrow w$ and $w_s \to w$ in $\mathscr{C}^\infty$ in $\{x\colon w(x) > 0\}$ as $s \searrow \frac{1}{2}$.

We claim that

$$w_s \in \mathscr{C}^{4s+2}. \tag{26.5.8}$$

To prove (26.5.8) note first that $w \in \mathscr{C}^{2-\delta'}$ uniformly with respect to all the parameters for any $\delta' > 0$. Then $w_s^s \in \mathscr{C}^{s-\delta}$ and then (26.5.6) yields that $w_s \in \mathscr{C}^{2+s-\delta}$ as soon as $s - \delta \notin \mathbb{Z}$. Then since $w_s \ge 0$ we get $|\nabla w_s| \le c w_s^{\frac{1}{2}}$ and so $w_s^s \in \mathscr{C}^{s-\frac{1}{2}}$. Then equation (26.5.6) again yields that $w_s \in \mathscr{C}^{s+\frac{3}{2}}$.

Now we need more subtle arguments. First, for $|y| = 1$

$$w_s(x+ty) = w_s(x) + t(\nabla w_s)_x \cdot y + \frac{1}{2}(\nabla^2 w_s)_x(y)t^2 + O\big(t^3\big). \tag{26.5.9}$$

Then the lowest eigenvalue ς of $\nabla^2 w_s$ at x should be greater than $-C w_s^{\frac{1}{3}}$. Indeed, otherwise we can take y as the corresponding eigenvector and t with $|t| = \epsilon\varsigma$ and with a sign making second term non-positive and get $w_s(x+ty) < 0$.

This lower estimate for eigenvalues of $\nabla^2 w_s$ and equation (26.5.6) yield that $|\nabla^2 w_s| \le C w_s^{\frac{1}{3}}$. But then $|\nabla w_s| \le C w_s^{\frac{2}{3}}$. Really, otherwise picking $y = |\nabla w_s|^{-1}\nabla w_s$ with $|t| = \epsilon |\nabla w_s|^{\frac{1}{2}}$ and an appropriate sign we would get $w_s(x+ty) < 0$.

These estimates yield that $w_s(x') \asymp w_s(x)$ in $B\big(x, \gamma(x)\big)$ with $\gamma(x) = \epsilon w_s^{\frac{1}{3}}$. Then $w_s^s \in \mathscr{C}^{\frac{3}{2}}$. In fact, let us consider $f = w_s^{s-1}\nabla w$ and $|f(x) - f(x')|$. Let

[30] If $\Delta w_i = f_i(w_i)$ in Ω, $f_i(w) \nearrow$ as $w \nearrow$ and $f_1(w) \ge f_2(w)$ then $\Delta(w_1 - w_2) > 0$ as $w_1 > w_2$ and then $w_1 - w_2$ does not reach maximum inside Ω.

us consider first $|x-x'| \ge \frac{1}{3}\bigl(\gamma(x)+\gamma(x')\bigr)$; since $|f(x)| \le \gamma(x)^{\frac{1}{2}}$ at each point we get that $|f(x)-f(x') \le |x-x'|^{\frac{1}{2}}$.

On the other hand, for $|x-x'| \le \frac{1}{3}\bigl(\gamma(x)+\gamma(x')\bigr)$ we conclude that $\gamma(x) \asymp \gamma(x')$ and $|f(x)-f(x')| \le |\nabla f|\cdot|x-x'| \le |x-x'|^s$ due to inequality $|\nabla f| \le |\nabla^2 w_s| w_s^{s-1} + |\nabla w_s|^2 w_s^{s-2} \le C\gamma^{s-1}$.

Therefore $w_s^s \in \mathscr{C}^{s+1}$ and equation (26.5.6) yields that $w_s \in \mathscr{C}^{3+s}$.

(b) In the next round we assume that $w_s \in \mathscr{C}^{4+s-\delta}$ with *some* $\delta \in (0,1)$. Then

$$
\begin{aligned}
(26.5.10)\quad & w_s(x+ty) \le \\
& w_s(x) + t(\nabla w_s)_x \cdot y + \frac{1}{2}(\nabla^2 w_s)_x(y)t^2 + \frac{1}{6}(\nabla^3 w_s)_x(y)t^3 + C|t|^p
\end{aligned}
$$

with $p = \min(4, 4+s-\delta)$.

We claim now that the lowest possible eigenvalue ς of $\bigl(\nabla^2 w_s\bigr)_x$ is greater than $-Cw_s^{(p-2)/p}$. Really, otherwise let us pick up y as the corresponding eigenvector, t with $|t| = \epsilon|\varsigma|^{1/(p-2)}$ and with a sign making expression

$$
t(\nabla w_s)_x \cdot y + \frac{1}{6}t^3(\nabla^3 w_s)_x(y)
$$

non-positive and get $w_s(x+ty) < 0$ again. Now equation (26.5.6) yields that inequality

$$
(26.5.11)_k \qquad |\nabla^k w_s| \le C w_s^{(p-k)/p}
$$

holds with $k=2$.

Further, we claim that this inequality holds with $k=1,3$. Indeed, if one or both of these inequalities are violated then let us take corresponding y and t with

$$
|t| = \epsilon\Bigl(|\nabla w_s|^{1/(p-1)} + |\nabla^3 w_s(y)|^{1/(p-3)}\Bigr)
$$

(calculated on y); replacing ϵ by 2ϵ if necessary we get

$$
|t(\nabla w_s)_x \cdot y + \frac{1}{6}t^3(\nabla^3 w_s)_x(y)\bigr| \ge \epsilon_0 |t(\nabla w_s)_x \cdot y| + |\frac{1}{6}t^3(\nabla^3 w_s)_x(y)|
$$

and choosing an appropriate sign of t we get $w(x+ty) < 0$.

Therefore inequalities $(26.5.11)_{1-3}$ *hold*. The same arguments as above with $\gamma = w_s^{1/p}$ lead us to $w^s \in \mathscr{C}^{ps}$ and then equation (26.5.6) yields that $w_s \in \mathscr{C}^{ps+2}$. So, now we came back with δ replaced by $\delta' = 2 + s - ps$ and one can see easily that if $\delta > s$ then $\delta' = s + (2 - 4s) + (\delta - s)s$ and after few repeats $\delta < s$. Then we get (26.5.8). Unfortunately, constants depend on s due to the fact that $\Delta w \in \mathscr{C}^2$ fails to yield $w \in \mathscr{C}^4$.

(c) Now we are going to finish the proof of (26.5.4). Let us consider w_s again and let $\gamma = \gamma_{s,\delta} = w_s^{1/(4-\delta)}$. Due to the previous inequalities $\gamma \in \mathscr{C}^1$. We claim that $|\nabla\gamma|$ is bounded uniformly with respect to s, δ. Note first that $\Delta\gamma^{4-\delta} = \gamma^{(4-\delta)s}$ implies that

$$a|\nabla\gamma|^2 + b\gamma\Delta\gamma = \gamma^\sigma \tag{26.5.12}$$

with $a = \frac{1}{12}(4-\delta)(3-\delta)$, $b = \frac{1}{12}(4-\delta)$, and $\sigma = 4s - 2 + (1-s)\delta$. Let $\psi = |\nabla\gamma|^2$; obviously ψ is uniformly bounded at $\partial\Omega$. Let us consider maximum of ψ reached inside Ω. At the point of maximum

$$\sum_i \gamma_{x_i x_j}\gamma_{x_i} = 0 \tag{26.5.13}$$

and

$$\frac{1}{2}\Delta\psi = \sum_{i,j}\gamma^2_{x_i x_j} + \sum_i \gamma_{x_i}\big(\Delta\gamma\big)_{x_i} = \sum_{i,j}\gamma^2_{x_i x_j} + b^{-1}\sum_i \gamma_{x_i}\Big(\gamma^{-1}\big(\gamma^\sigma - a|\nabla\gamma|^2\big)\Big)_{x_i}.$$

Due to (26.5.12) and due to (26.5.13) this expression is equal to

$$\sum_{i,j}\gamma^2_{x_i x_j} - b^{-1}\gamma^{-2}|\nabla\gamma|^2\big(\gamma^\sigma - a|\nabla\gamma|^2\big) + b^{-1}\sigma\gamma^{\sigma-2}|\nabla\gamma|^2$$

and therefore at an inner point of minimum $a|\nabla\gamma|^2 \le \gamma^\sigma$. So, $|\nabla\gamma| \le C$ is proven and for $s \searrow \frac{1}{2}$, $\delta \searrow 0$ we get that $|\nabla w^{\frac{1}{4}}| \le C$.

Let us pick $\gamma(x) = \epsilon' w^{\frac{1}{4}}(x)$; then $|\nabla w| \le \frac{1}{2}$ and $w(x) \asymp w(\mathsf{x})$ in $B\big(x, \gamma(\mathsf{x})\big)$. This and equation (26.5.6) easily yield (26.5.4).

To prove inequality (26.5.5) let us consider w_s again and let us take now $\psi = |\nabla\gamma|^2 - F\gamma^{2t}$ with $t > 0$; obviously ψ is non-positive at $\partial\Omega$ for

sufficiently large F. Let us consider maximum of ψ reached inside Ω. At the point of maximum

$$\sum_i \gamma_{x_i x_j}\gamma_{x_i} - Ft\gamma^{2t-2}\gamma_{x_j} = 0 \tag{26.5.13$'$}$$

and the same arguments as before (plus inequality $|\nabla\gamma| \le C_0$) show that at an inner point of maximum $a|\nabla\gamma|^2 \le \gamma^\sigma + CtF\gamma^{2t}$ where C does not depend on F and small $t > 0$. Then at this point $\psi \le 1$ for small enough $t > 0$ and as $s \to \frac{1}{2}$ and $\delta \to 0$ we get (26.5.5). □

The following statement heavily uses estimate (26.5.5):

Proposition 26.5.2. *The following estimate holds*

$$\mathsf{D}(\gamma^{-1+s}, \gamma^{-1+s}) \le Cs^{-2} \tag{26.5.14}$$

with some constant C which does not depend on $s \in (0,1)$ where we set $\gamma^{-1+s} := w_+^{\frac{1}{4}(-1+s)}$ (i.e. it is 0 as $w \le 0$).

Proof. As in the notations of the proof of Proposition 26.5.1 $\delta = 0$ and $s = \frac{1}{2}$ we have (26.5.12) with $a = 1$, $b = 3$ and $\sigma = 0$:

$$\frac{1}{3}\gamma\Delta\gamma + |\nabla\gamma|^2 = 1. \tag{26.5.15}$$

Then

$$\gamma^{-1+s} = \gamma^{-1+s}|\nabla\gamma|^2 + \frac{1}{3}\gamma^s\Delta\gamma = (1 - \frac{s}{3})\gamma^{-1+s}|\nabla\gamma|^2 + \frac{1}{3(1+s)}\Delta\gamma^{1+s}$$

and

$$\begin{aligned}\mathsf{D}(\gamma^{-1+s}, \gamma^{-1+s}) \le (1 - \frac{s}{3})\mathsf{D}(\gamma^{-1+s}|\nabla\gamma|^2, \gamma^{-1+s}) + C \le \\ (1 - \frac{s}{3})\mathsf{D}(\gamma^{-1+s}, \gamma^{-1+s}) + C\mathsf{D}(\gamma^{-1+t+s}, \gamma^{-1+s}) + C\end{aligned}$$

due to (26.5.5) and this yields

$$\mathsf{D}(\gamma^{-1+s}, \gamma^{-1+s}) \le Cs^{-2}\mathsf{D}(\gamma^{-1+t+s}, \gamma^{-1+t+s}) + Cs^{-1}.$$

Substituting $s + mt$ instead of s, $0 \le m \le Ct^{-1}$ we recover (26.5.14). □

26.5.2 Analysis in the Boundary Strip $\mathcal{Y}$ for $N \ge Z$

We consider now the case of if $N \ge Z$ (i.e. $\nu = 0$ and $G = B\bar{r}^{-4}$).

It is really easy to construct the proper potential in this case: we just take

$$w_\varepsilon = w\phi_\varepsilon, \qquad \phi_\varepsilon = f\bigl(w\varepsilon^{-4}\bigr) \tag{26.5.16}$$

with $f \in \mathscr{C}^\infty((\frac12,\infty))$, $\mathsf{supp}(f) \subset (\frac12,\infty)$, $0 \le f \le 1$, $f(t) = 1$ for $t > 1$. Note that due to (26.5.3)

$$\begin{aligned}&\mathsf{D}(\gamma^{-1}\phi_\varepsilon, \gamma^{-1}\varphi_\varepsilon) \le C\varepsilon^{-2s}\mathsf{D}(\gamma^{s-1},\gamma^{s-1}) \le Cs^{-2}\varepsilon^{-2s},\\ &\mathsf{D}(1-\phi_\varepsilon, 1-\phi_\varepsilon) \le C\varepsilon^{2-2s}\mathsf{D}(\gamma^{s-1},\gamma^{s-1}) \le Cs^{-2}\varepsilon^{2-2s};\end{aligned}$$

then minimizing with respect to s $(= |\log\varepsilon|^{-1})$ the right-hand expression we conclude that

$$\mathsf{D}\bigl(\gamma^{-1}\phi_\varepsilon, \gamma^{-1}\varphi_\varepsilon\bigr) + \varepsilon^{-2}\mathsf{D}\bigl(1-\phi_\varepsilon, 1-\phi_\varepsilon\bigr) \le C\bigl(1+|\log\varepsilon|\bigr)^2 \tag{26.5.17}$$

and therefore

$$\int \gamma^{-1}\phi_\varepsilon\,dx + \varepsilon^{-1}\int (1-\phi_\varepsilon)\,dx \le C\bigl(1+|\log\varepsilon|\bigr). \tag{26.5.18}$$

Remark 26.5.3. (i) Recall that all these integrals are taken over domain $\{x\colon w(x) > 0\}$. To avoid possible troubles we pick $\varepsilon = h^{\frac13}$ and set in the zone $\{x\colon w(x) \le C_0 h^{\frac43}\}$

$$\gamma(x) = \mathsf{dist}(x, \{w \ge 2C_0h^{\frac43}\}),$$

$$w_\varepsilon = \begin{cases} -\gamma^4\phi'_\varepsilon & \text{for } \gamma \le \varepsilon,\\ -\varepsilon^4 & \text{for } \gamma \ge \varepsilon\end{cases} \tag{26.5.16}'$$

with $\phi'_\varepsilon = f(\gamma\varepsilon^{-1})$ and then in the complemental domain $\{x\colon w(x) \le -\varsigma^2\}$ our assumptions are fulfilled with $\varsigma = \varepsilon^2$ and $\varsigma\gamma = \gamma^3 \ge h$.

(ii) Further, for $\varepsilon = h^{\frac13-\delta}$ with sufficiently small exponent $\delta > 0$ it does not break estimate for mollification error in T-term.

(iii) Furthermore, for $t > \varepsilon$

$$\mathsf{mes}(\{x\colon \gamma(x) \le t\}) \le Ct^3\varepsilon^{-3}\,\mathsf{mes}(\{x\colon \gamma(x) \le \epsilon\varepsilon\})$$

and therefore

$$h^s \int \gamma^{-1-s}\varsigma^{-s}\,dx \le C\varepsilon^{-1}\,\mathsf{mes}(\{x\colon \gamma(x)\le \epsilon\varepsilon\}) \le CL \coloneqq C(1+|\log h|)$$

for sufficiently large s.

Using these estimates and the last remark we can prove easily

Proposition 26.5.4. *Let $N \ge Z$. Then*

(i) Contribution of $\mathcal{Y}\cup\mathcal{X}_4$ with external zone $\mathcal{X}_4 \coloneqq \{x\colon w(x)=0\}$ *to mollification and semiclassical errors in* N*-term do not exceed $CT_0\varepsilon^3(1+|\log\varepsilon|)$ and $R_0(1+|\log\varepsilon|)$ respectively with*

$$(26.5.19)_1 \quad T_0 = B^{\frac34},\qquad R_0 = B^{\frac12},\qquad T = B^{\frac74},\qquad R = B^{\frac54} \qquad \text{for } B \le Z^{\frac43}$$

and

$$(26.5.19)_2 \quad T_0 = Z,\qquad R_0 = B^{\frac15}Z^{\frac25},\qquad T = Z^{\frac95}B^{\frac25},\qquad R = Z^{\frac35}B^{\frac45} \qquad \text{for } Z^{\frac43} \le B \le Z^3.$$

(ii) Contribution of $\mathcal{Y}\cup\mathcal{X}_4$ to mollification and semiclassical D*-terms do not exceed $CT\varepsilon^6(1+|\log\varepsilon|)^2$ and $R(1+|\log\varepsilon|)^2$ respectively.*

(iii) Contribution of $\mathcal{Y}\cup\mathcal{X}_4$ to both mollification and semiclassical errors in T*-term do not exceed $CT\varepsilon^7(1+|\log\varepsilon|)$ and CR respectively.*

Proof. Really, estimates for mollification errors and terms immediately follow from the inequality

$$\mathsf{mes}(\{x\colon w(x)\le \varepsilon^4\}) \le C\varepsilon(1+|\log\varepsilon|) \tag{26.5.20}$$

which is due to (26.5.18).

Let us consider semiclassical errors and terms.

(i) Let us consider N-term first. Let us consider all possible balls and their contributions: the contribution of each ball $B\big(x,\gamma(x)\big)$ to the semiclassical error does not exceed $C\mu h^{-1}\gamma^2 \asymp CB\bar{r}^2\gamma^2$ and the total contribution does not exceed

$$CR_0 \int \gamma(x)^{-1}\,dx \le CR_0\big(1+|\log\varepsilon|\big) \tag{26.5.21}$$

where $R_0 = B\bar{r}^2$; recall that $\gamma(x)\ge\varepsilon$.

(ii) Consider semiclassical D-term. Let us consider all possible balls and their contributions: the similar arguments with the analysis of disjoint balls of different types and with analysis of the intersecting balls (of the same type) lead us to the proper estimate of the contribution of $\mathcal{Y}_4 \cup \mathcal{X}_4$ to semiclassical D-term: namely, it does not exceed $CR_0^2\bar{r}^{-1}\bigl(1+|\log\varepsilon|\bigr)^2$ (i.e. expression (26.5.21) squared and multipled by $C\bar{r}^{-1}$) where $R_0^2\bar{r}^{-1} \asymp R$.

(iii) Consider T-term. Let us consider all possible balls and their contributions. Contribution of each ball $B\bigl(x,\gamma(x)\bigr)$ to the semiclassical error does not exceed $C\bar{\zeta}^2\mu\varsigma^2\gamma \asymp CB\bar{\zeta}^2\bar{r}\varsigma\gamma^2$ and the total contribution does not exceed

$$CR\int \varsigma(x)\gamma(x)^{-2}\,dx \asymp CR \tag{26.5.22}$$

where $R = B\bar{\zeta}^2\bar{r}$ and $\varsigma(x) \asymp \gamma(x)^2$. □

Then picking appropriate $\varepsilon = h^{\frac{1}{3}}$ we arrive to

Corollary 26.5.5. *Let $N \ge Z$. Then*

(i) Contributions of $\mathcal{Y}\cup\mathcal{X}_4$ to all errors in N-terms do not exceed CR_0L with $L = (1+|\log BZ^{-3}|)$.

(ii) Contribution of $\mathcal{Y}\cup\mathcal{X}_4$ to all D-terms do not exceed CRL^2.

(iii) Contribution of $\mathcal{Y}\cup\mathcal{X}_4$ to all errors in T-terms do not exceed CR.

We will sum contributions of all zones to errors in Propositions 26.5.14 and 26.5.17 below.

Remark 26.5.6. Could we get rid of the logarithmic factors i.e. make $L = 1$ as it was in the case $M = 1$?

(i) With the mollification errors we need to replace (26.5.20) by

$$\mathsf{mes}\bigl(\{x\colon w(x) \le \varepsilon^4\}\bigr) \le C\varepsilon; \tag{26.5.23}$$

(ii) With the semiclassical terms our arguments here are insufficient even if we established (26.5.23); we need extra propagation arguments in the direction of decaying w along magnetic lines–exactly as in the case $M = 1$. Surely there could be points where such arguments do not work; f.e. consider

$M = 2$ and nuclei so that $|\mathsf{y}_1 - \mathsf{y}_2|$ is slightly less than $\bar{r}_1 + \bar{r}_2$ where $\bar{r}_{1,2}$ are precise radii of support. Then w reaches its minimum at $\mathcal{Y}$.

So, we need to prove that the measure of such points is sufficiently small (f.e. less than $C|\log BZ^{-3}|^{-1}$).

Unfortunately, we do not know how to make the above remark work and we suggest

Problem 26.5.7. Follow through the discussed plan. For $M = 2$ it could be easier due to the rotational symmetry of the potential W_B^{TF}.

26.5.3 Analysis in the Boundary Strip $\mathcal{Y}$ for $N < Z$

Now let us consider the case of $N < Z$ (i.e. $\nu < 0$).

Case $B \ge (Z-N)_+^{\frac{4}{3}}$

We start from the case $B \ge (Z-N)_+^{\frac{4}{3}}$ when $\bar{r} = \min(B^{-\frac{1}{4}}, Z^{\frac{1}{5}}B^{-\frac{2}{5}})$ matching cases $B \lesssim Z^{\frac{4}{3}}$ and $Z^{\frac{4}{3}} \lesssim B \lesssim Z^3$.

Remark 26.5.8. (i) The results of the previous Subsection 26.4.2 remain true as long as $|\nu|G^{-1} \le C_0 h^{\frac{4}{3}}$; in other words, as $(Z-N)_+ \le C_0 G\bar{r}h^{\frac{4}{3}}$. Plugging $\bar{r}$, G and h, we rewrite it as

$$(Z-N)_+ \le C_0 \min\bigl(B^{\frac{5}{12}},\, Z^{\frac{1}{5}}B^{\frac{4}{15}}\bigr) \tag{26.5.24}$$

matching cases $B \lesssim Z^{\frac{4}{3}}$ and $Z^{\frac{4}{3}} \lesssim B \lesssim Z^3$.

(ii) Therefore in this Subsection we assume that condition (26.5.24) fails. Let $\theta = |\nu|G^{-1} \asymp (Z-N)_+ \cdot \max\bigl(B^{-\frac{3}{4}},\, Z^{-1}\bigr)$, also matching cases $B \lesssim Z^{\frac{4}{3}}$ and $Z^{\frac{4}{3}} \lesssim B \lesssim Z^3$.

Proposition 26.5.9. *Consider dependence of $W_B^{\mathsf{TF}} = W_{B(\nu)}^{\mathsf{TF}}(x)$ on ν. Then*

(i) $W_{B(\nu)}^{\mathsf{TF}}(x) + \nu$ is non-decreasing with respect to ν at each point x.

(ii) $W_{B(\nu)}^{\mathsf{TF}}(x)$ is non-increasing with respect to ν at each point x.

(iii) In particular, $W_{B(\nu)}^{\mathsf{TF}}(x) + \nu \nearrow W_{B(0)}^{\mathsf{TF}}(x)$ and $W_{B(\nu)}^{\mathsf{TF}}(x) \searrow W_{B(0)}^{\mathsf{TF}}(x)$ at each point x as $\nu \nearrow 0$.

Proof. (i) Consider $W_j := W^{\mathsf{TF}}_{B(\nu_j)} + \nu_j$ with $0 > \nu_1 > \nu_2$. One can prove easily that $W^{\mathsf{TF}}_B - V$ is a continuous function and since

$$W_1 - W_2 \to \nu_1 - \nu_2 \qquad \text{as } \ell(x) \to \infty \tag{26.5.25}$$

with $\nu_1 - \nu_2 > 0$ we conclude that $W_1 \ge W_2$ at each point x (which is exactly our Statement (i)) unless $W_1 - W_2$ achieves a negative minimum at some point x^*:

(a) Let $x^* \ne \mathsf{y}_m$; then $\Delta(W_1 - W_2)(x^*) = P'_B(W_1) - P'_B(W_2) \le 0$ because $W_1 < W_2$ at x^* and therefore x^* cannot be such point.

(b) Let $x^* = \mathsf{y}_m$. From Thomas-Fermi equations for $W_{1,2}$ one can prove easily that

$$(W_1 - W_2)(x) = (W_1 - W_2)(\mathsf{y}_m) + L_m(x - \mathsf{y}_m) + \kappa_m |x - y_m|^{\frac{3}{2}} (W_1 - W_2)(\mathsf{y}_m) + O(|x - \mathsf{y}_m|^2)$$

near y_m where $L_m(x)$ is a linear function and $\kappa_m > 0$ and therefore if $(W_1 - W_2)(\mathsf{y}_m) < 0$, y_m cannot be a minimum point either.

(ii) So, $W_1 \ge W_2$ and therefore $\Delta(W_1 - W_2)(x^*) = P'_B(W_1) - P'_B(W_2) \ge 0$ and $W_1 - W_2$ is a subharmonic function. Then due to (26.5.25) we conclude that $W_1 - W_2 \le \nu_1 - \nu_2$ i.e. $W^{\mathsf{TF}}_{B,(\nu_1)} \le W^{\mathsf{TF}}_{B,(\nu_2)}$ at each point.

(iii) Statement (iii) follows from Statements (i) and (ii). □

From Statements (i) and (iii) we conclude immediately that

Corollary 26.5.10. *(i)* $\rho^{\mathsf{TF}}_{B(\nu)}(x)$ *is non-decreasing with respect to* ν *at each point* x.

(ii) $\rho^{\mathsf{TF}}_{B(\nu)}(x) \nearrow \rho^{\mathsf{TF}}_{B(0)}(x)$ *at each point* x *as* $\nu \nearrow 0$.

Therefore in the zone $\{x \in \mathcal{Y}\colon W^{\mathsf{TF}}_{B(\nu)} \ge (1+\epsilon)|\nu|\}$ we can apply the same (γ, ς) scaling with $\varsigma = \gamma^2$ defined for $\nu = 0$. Indeed, we know that there $W^{\mathsf{TF}}_{B(\nu)} + \nu \asymp W^{\mathsf{TF}}_{B(0)} \asymp \varsigma^2$ and $\varsigma = \gamma^2$.

Then Thomas-Fermi equation (26.2.3) implies that

$$|\nabla^\alpha W^{\mathsf{TF}}_{B(\nu)}| \le C_\alpha \varsigma^2 \gamma^{-|\alpha|} \qquad \forall \alpha \tag{26.5.26}$$

and then we arrive to the Statement (i) in Proposition 26.5.11 below. On the other hand, in the zone $\{x\colon W^{\mathsf{TF}}_{B(\nu)} \le (1-\epsilon)|\nu|\}$ we can apply the

same arguments but this zone is classically forbidden and we arrive to Statement (ii) below. In both cases $\varsigma\gamma \ge h$ (where in the latter case γ is the distance from x to $W^{\mathsf{TF}}_{B(\nu)}$ (scaled) and $\varsigma = |\theta|^{\frac12}$ in virtue of Remark 26.5.8.

Proposition 26.5.11. *Let* <u>*either*</u> $B \le Z^{\frac43}$ *and* $|\nu|^{\frac34} \ge Z^{\frac23}$ <u>*or*</u> $Z^{\frac43} \le B \le Z^3$ *and* $|\nu|^{\frac34} \ge B^{\frac12}$. *Then*

(i) Contributions of zone $\{x\colon W^{\mathsf{TF}}_B(x) \ge (1+\epsilon_0)|\nu|\}$ *to the semiclassical errors in* N- *and* T*-terms and into semiclassical* D*-term do not exceed* CR_0L, CR *and* CRL^2 *respectively.*

(ii) Contributions of zone $\{x\colon W^{\mathsf{TF}}_B(x) \le (1-\epsilon_0)|\nu|\}$ *to the semiclassical errors in* N- *and* T*-terms and into semiclassical* D*-term do not exceed* CR_0L, CR *and* CRL^2 *respectively.*

Remark 26.5.12. Here actually we can replace L by $L_* = 1 + |\log\theta|$ with

$$\theta = |\nu| G^{-1} \asymp \begin{cases} (Z-N)_+ B^{-\frac34} & \text{if } (Z-N)_+^{\frac43} \le B \le Z^{\frac43}, \\ (Z-N)_+ Z^{-1} & \text{if } Z^{\frac43} \le B \le Z^3; \end{cases} \tag{26.5.27}$$

Therefore we need to explore the following zone

$$\mathcal{Y}^* := \{x\colon (1-\epsilon_0)|\nu| \le W^{\mathsf{TF}}_B(x) \le (1+\epsilon_0)|\nu|\}$$

in the framework of Proposition 26.5.11. In virtue of Remark 26.5.8 $\hbar \lesssim 1$ where

$$\hbar = h\theta^{-\frac34} \asymp \begin{cases} (Z-N)_+^{-\frac34} B^{\frac{5}{16}} & \text{if } B \le Z^{\frac43}, \\ (Z-N)_+^{-\frac34} Z^{\frac{3}{20}} B^{\frac15} & \text{if } Z^{\frac43} \le B \le Z^3. \end{cases} \tag{26.5.28}$$

Let us rescale the ball $B(.,\alpha)$ to $B(.,1)$ by $x \mapsto x\alpha^{-1}$ with $\alpha = \theta^{\frac14}$ (after we already rescaled $x \mapsto x\bar r^{-1}$). After this let us introduce scaling function γ_0 by (26.4.3). Then let us introduce consequently scaling functions γ_1 by (26.4.7), γ_2 by (26.4.13) and γ_3 by (26.4.14) [31].

Consider contributions of different balls in this hierarchy into semiclassical and approximation errors in N- and T-terms and into D semiclassical and approximation D-terms.

[31] With $j = \bar\jmath = 0$ and corrected as in $(26.4.3)^*$ and $(26.4.7)^*$.

(i) Consider first semiclassical error in N-term. Due to Chapter 18 the contribution of α_j element does not exceed $C B\ell_j^2 = C B\bar{r}^2\alpha_j^2$ for $j = 3, 2$, where recall that $\alpha_j = \gamma_0 \cdots \gamma_j$.

Then for $j = 2$ we have $C B\bar{r}^2\alpha_2^2 = C B\bar{r}^2\alpha_1^2\gamma_1^2$ and therefore we estimate the contribution of α_1 element by $C B\bar{r}^2\alpha_1^2 \int \gamma_2^{-1}\, dx$ [32], which also results in $C B\bar{r}^2\alpha_1^2$ but with the logarithmic factor. However we can get rid of this factor due to a simple observation:

(26.5.29) If $\gamma_2 \le \epsilon$ then $\mathsf{Hess}(w_1)$ has at least two eigenvalues of magnitude 1 due to $|\Delta w_1| \le \epsilon_1$.

Then the contribution of α_0 element does not exceed $C B\bar{r}^2\alpha_0^2 \int \gamma_1^{-1}\, dx$ [32]; we claim that it is $C B\bar{r}^2\alpha_0^2$. Indeed, we need to consider only points with $\gamma_1 \le \epsilon$ and there we use a similar observation:

(26.5.30) If $\gamma_1 \le \epsilon$ then $|\nabla^3 w'| \asymp 1$ and also $|\nabla^3 w' - e\otimes e\otimes e| \ge c^{-1}$ for any $e \in \mathbb{R}^3$ due to $|\partial_j \Delta w_1| \le \epsilon_1$; here $\nabla^3 w'$ is a 3-tensor of the third derivatives of w'.

Further, the contribution of α element does not exceed $C B\bar{r}^2\alpha^2 \int \gamma_0^{-1}\, dx$ [32]; since $\gamma \ge \bar{\gamma}_0 = \hbar^{\frac{1}{3}}$, we estimate it by $C B\bar{r}^2\alpha^2\hbar^{-\frac{1}{3}}$.

Finally, since $\bar{r}^{-1}\mathcal{Y}^*$ is covered by no more than $C L_*\alpha^{-2}$ such elements[33], we conclude that

(26.5.31) The total contribution of $\mathcal{Y}^*$ into the semiclassical (and also approximation) errors in N-term does not exceed $C B\bar{r}^2\hbar^{-\frac{1}{3}}L_*$, where $L_* := (1 + |\log\theta|)$.

Plugging values of $\hbar$ and θ, we arrive to expression (26.5.33) in Proposition 26.5.13(i) below.

(ii) Similarly, in virtue of Subsubsection *26.3.1.2. Semiclassical* D*-Term* we know the that the contribution of the non-disjoint pair of α_j-elements to the semiclassical D-term does not exceed $C B^2\ell^3 = C B^2\bar{r}^3\alpha_j^3$ for $j = 3, 2$.

---

[32] With the integral calculated in the scaled coordinates.

[33] Indeed, due to Subsection 26.5.1 $\mathsf{mes}(\bar{r}^{-1}\mathcal{Y}^*) \le C\alpha L_*$.

Therefore the contribution of all non-disjoint pairs of α_2 subelements to the same expression for α_1 element does not exceed $CB^2\bar{r}^3\alpha_1^3$. Adding all disjoint pairs, we get[32)]

$$CB^2\bar{r}^3\alpha_1^3 \iint |x-y|^{-1}\gamma_2(x)^{-1}\gamma_2(y)^{-1}\,dxdy. \tag{26.5.32}$$

Then using the results of Part (i) together with observation (26.5.29) we arrive to $CB^2\bar{r}^3\alpha_1^3$. So, contribution of the non-disjoint pair of α_1-elements to the semiclassical D-term does not exceed $CB^2\ell^3 = CB^2\bar{r}^3\alpha_1^3$.

Further, continuing in the same manner, we estimate the contribution of the non-disjoint pair of α_0-elements by $CB^2\bar{r}^3\alpha_0^3$.

Furthermore, in the same manner we estimate the contribution of the non-disjoint pair of α-elements by expression (26.5.32) with γ_2 replaced by γ_0, which does not exceed $CB^2\bar{r}^3\alpha^3\hbar^{-\frac{2}{3}}$.

Finally, adding contribution of all non disjoint pairs and using results of Part (i), we conclude that the total contribution of $\mathcal{Y}^*\times\mathcal{Y}^*$ into the semiclassical (and also approximation) D-terms does not exceed the final expression we recovered there, squared and multiplied by $\bar{r}^{-1}$, i.e. $CB^2\bar{r}^3\hbar^{-\frac{2}{3}}L_*^2$.

Plugging values of $\hbar$ and θ we arrive to expression (26.5.34) in Proposition 26.5.13(ii) below.

(iii) Due to Chapter 18 the contribution of α_j element to the semiclassical error in T-term does not exceed $CB\ell_*\zeta_*$ as $j=3,2$. Note that $\zeta = G^{\frac12}\gamma_0^2\gamma_1^{\frac32}\gamma_2\gamma_3^{\frac12}$ and $\zeta = G^{\frac12}\gamma_0^2\gamma_1^{\frac32}\gamma_1$ for $j=3,2$. Here we took $\theta=\alpha=1$ thus covering the whole zone $\mathcal{Y}$.

Then the contribution of α_2-element does not exceed $CBG^{\frac12}\bar{r}\gamma_0^3\gamma_1^{\frac52}\gamma_2^2$. Further, the contribution of α_1-element does not exceed $CBG^{\frac12}\bar{r}\gamma_0^3\gamma_1^{\frac52}\int\gamma_2^{-1}\,dx$ [32)], resulting in $CBG^{\frac12}\bar{r}\gamma_0^3\gamma_1^{\frac52}$ in virtue of the same observation (26.5.29).

Further, the contribution of α_0-element does not exceed $CBG^{\frac12}\bar{r}\gamma_0^3\int\gamma_1^{-\frac12}$ resulting in $CBG^{\frac12}\bar{r}\gamma_0^3$ in virtue of the same observation (26.5.30).

Finally, the total contribution of $\mathcal{Y}$ does not exceed $CBG^{\frac12}\bar{r} = CB^2\bar{r}^3 = \max(B^{\frac54}, Z^{\frac35}B^{\frac45})$.

Therefore we arrive to

Proposition 26.5.13. *In the framework of Proposition 26.5.13 there exists potential W_ε such that*

(i) Contributions of $\mathcal{Y}^*$ to both semiclassical and approximation errors for N*-term do not exceed*

$$(Z-N)_+^{\frac14} L \times (B^{\frac{19}{48}};\ Z^{\frac{7}{20}} B^{\frac{2}{15}}), \tag{26.5.33}$$

where here and below we list different values for $(Z-N)_+^{\frac43} \le B \le Z^{\frac43}$ and for $Z^{\frac43} \le B \le Z^3$.

(ii) Contributions of $\mathcal{Y}^* \times \mathcal{Y}^*$ to both semiclassical and approximation D*-terms do not exceed*

$$(B^{\frac{25}{24}};\ Z^{\frac12} B^{\frac23}). \tag{26.5.34}$$

(iii) Contributions of $\mathcal{Y}^*$ to both semiclassical and approximation errors for T*-term do not exceed*

$$(B^{\frac54};\ Z^{\frac35} B^{\frac45}). \tag{26.5.35}$$

Case $B \le (Z-N)_+^{\frac43}$

Now let us consider the case $B \le (Z-N)_+^{\frac43}$. In this case the *boundary strip*

$$\mathcal{Y} := \{x\colon |W(x)+\nu| \le \epsilon|\nu|\} \tag{26.5.36}$$

consists of two subzones

$$\mathcal{Y}_1 := \{x\colon \epsilon B \le |W(x)+\nu| \le \epsilon|\nu|\} \tag{26.5.37}$$

and

$$\mathcal{Y}^* := \{x\colon |W(x)+\nu| \le \epsilon B\}. \tag{26.5.38}$$

Applying arguments of Section 26.4 (more precisely, analysis in zones $\mathcal{X}_1$, $\mathcal{X}_1$ extended and $\mathcal{X}_2$) one can prove easily that

Proposition 26.5.14. *Let $B \le (Z-N)_+^{\frac43}$. Then*

(i) Contributions of $\mathcal{Y}_1$ into semiclassical and approximation errors in N*-term do not exceed $C|\nu|\bar{r}^2 \asymp C(Z-N)_+^{\frac23}$.*

(ii) Contributions of $\mathcal{Y}_1 \times \mathcal{Y}_1$ *into semiclassical and approximation* D*-terms do not exceed* $C|\nu|^2\bar{r}^3 \asymp C(Z-N)_+^{\frac{5}{3}}$.

(iii) Contribution of $\mathcal{Y}_1$ *into semiclassical and approximation errors in* T*-term do not exceed* $C|\nu|^{\frac{3}{2}}\bar{r} \asymp C(Z-N)_+^{\frac{5}{3}}$.

Proof. We leave easy details to the reader. □

On the other hand, applying arguments of the previous Subsubsection *26.5.3.1. Case* $B \ge (Z-N)_+^{\frac{4}{3}}$ with $\theta = 1$, $\hbar = |\nu|^{-\frac{1}{2}}\bar{r}^{-1} \asymp (Z-N)_+^{-\frac{1}{3}}$ one can prove easily the following

Proposition 26.5.15. *Let* $B \le (Z-N)_+^{\frac{4}{3}}$. *Then*

(i) Contributions of $\mathcal{Y}_2$ *into semiclassical and approximation errors in* N*-term do not exceed* $C(Z-N)_+^{-\frac{5}{9}}B$.

(ii) Contributions of $\mathcal{Y}_2 \times \mathcal{Y}_2$ *into semiclassical and approximation* D*-terms do not exceed* $C(Z-N)_+^{-\frac{7}{9}}B^2$.

(iii) Contribution of $\mathcal{Y}_2$ *into semiclassical and approximation errors in* T*-term do not exceed* $C(Z-N)_+^{\frac{5}{3}}$.

Proof. We leave easy details to the reader.

26.5.4 Summary

Adding contributions of all other zones we arrive to

Proposition 26.5.16. *Let* $M \ge 2$. *Then for the constructed potential* W

(i) Total semiclassical and approximation errors in N*-term do not exceed*

$$C\begin{cases} CZ^{\frac{2}{3}} + (Z-N)_+^{-\frac{5}{9}}B & \text{if } B \le (Z-N)_+^{\frac{4}{3}},\\ Z^{\frac{2}{3}} + B^{\frac{1}{2}}L + (Z-N)_+^{\frac{1}{4}}B^{\frac{19}{48}}L_* & \text{if } (Z-N)_+^{\frac{4}{3}} \le B \le Z^{\frac{4}{3}},\\ Z^{\frac{2}{5}}B^{\frac{1}{5}}L + (Z-N)_+^{\frac{1}{4}}Z^{\frac{7}{20}}B^{\frac{2}{15}}L_* & \text{if } Z^{\frac{4}{3}} \le B \le Z^3 \end{cases} \tag{26.5.39}$$

where $L_* = (1 + |\log\theta|)$ *with* $\theta = |\nu|G^{-1} = (Z-N)_+ \cdot \max(B^{-\frac{3}{4}}, Z^{-1})$ *and* $L = (1 + |\log BZ^3|)$.

(ii) Both semiclassical and approximation D*-terms do not exceed*

$$C\begin{cases} Z^{\frac{5}{3}} + (Z-N)_+^{-\frac{7}{9}} B^2 & \text{if } B \le (Z-N)_+^{\frac{4}{3}}, \\ Z^{\frac{5}{3}} + B^{\frac{5}{4}} L^2 + (Z-N)_+^{\frac{1}{2}} B^{\frac{25}{24}} L_*^2 & \text{if } (Z-N)_+^{\frac{4}{3}} \le B \le Z^{\frac{4}{3}}, \\ Z^{\frac{3}{5}} B^{\frac{4}{5}} L^2 + (Z-N)_+^{\frac{1}{2}} Z^{\frac{1}{2}} B^{\frac{2}{3}} L_*^2 & \text{if } Z^{\frac{4}{3}} \le B \le Z^3. \end{cases} \tag{26.5.40}$$

(iii) Total approximation error in T*-term does not exceed*

$$CQ \coloneqq C \max\bigl(Z^{\frac{5}{3}},\, Z^{\frac{3}{5}} B^{\frac{4}{5}}\bigr) = C \begin{cases} Z^{\frac{5}{3}} & \text{if } B \le Z^{\frac{4}{3}}, \\ Z^{\frac{3}{5}} B^{\frac{4}{5}} & \text{if } Z^{\frac{4}{3}} \le B \le Z^3. \end{cases} \tag{26.5.41}$$

(iv) Total semiclassical error in T*-term does not exceed*

$$CQ + CZ^{\frac{4}{3}} B^{\frac{1}{3}} + CZ^{\frac{3}{2}} a^{-\frac{1}{2}} \tag{26.5.42}$$

provided $a \ge Z^{-1}$*; for* $a \le Z^{-1}$ *the last term should be replaced by* CZ^2*.*

Also we arrive to

Proposition 26.5.17. *Let* $M \ge 2$*,* $B \le Z$ *and* $a \ge Z^{-\frac{1}{3}}$*. Then for the constructed potential* W

(i) Total semiclassical and approximation errors in N*-term do not exceed* $CZ^{\frac{2}{3}}\bigl((BZ^{-1})^{\delta} + (aZ^{\frac{1}{3}})^{-\delta} + Z^{-\delta}\bigr)$*.*

(ii) Both semiclassical and approximation D*-terms and semiclassical and approximation errors in* T*-term do not exceed* $CZ^{\frac{5}{3}}\bigl((BZ^{-1})^{\delta} + (aZ^{\frac{1}{3}})^{-\delta} + Z^{-\delta}\bigr)$*.*

26.6 Ground State Energy

26.6.1 Lower Estimates

Now the lower estimates for the ground state energy E_N are already proven: in virtue of the analysis given in Subsection 25.2.1 we know that

$$\mathsf{E}_N \ge \Phi_*(W) + \Bigl(\mathsf{Tr}(H_{A,W} - \nu)^- + \int P_B(W+\nu)\, dx\Bigr) - \nu N \tag{26.6.1}$$

for *arbitrary* potential W and $\nu \le 0$; picking Thomas-Fermi potential $W = W_B^{\mathsf{TF}}$ and chemical potential ν, we arrive to estimate (26.6.2) below with $W = W_B^{\mathsf{TF}}$ and $Q = 0$.

However we use slightly different potential W and arrive to estimate (26.6.2) below where CQ, defined by (26.5.42), estimates an approximation error; replacing T-term by its semiclassical approximation and applying Proposition 26.5.16(iii) and 26.5.17(ii), we arrive to estimates (26.6.3)–(26.6.6) below:

Proposition 26.6.1. *Let $B \le Z^3$. Then*

(i) The following estimate holds with an approximate potential W we constructed:

$$E^{\mathsf{TF}} \ge \mathcal{E}^{\mathsf{TF}} + \Bigl(\mathsf{Tr}((H_{A,W}-\nu)^-) + \int P_B(W+\nu)\,dx\Bigr) - CQ \tag{26.6.2}$$

with Q defined by (26.5.41)*; further, for $W = W_B^{\mathsf{TF}}$ this estimate holds with $Q = 0$.*

(ii) The following estimates hold for $M = 1$ and $M \ge 2$ respectively

$$E^{\mathsf{TF}} \ge \mathcal{E}^{\mathsf{TF}} + \mathsf{Scott} - CQ - CZ^{\frac{4}{3}}B^{\frac{1}{3}} \tag{26.6.3}$$

and

$$E^{\mathsf{TF}} \ge \mathcal{E}^{\mathsf{TF}} + \mathsf{Scott} - CQ - CZ^{\frac{4}{3}}B^{\frac{1}{3}} - CZ^{\frac{3}{2}}a^{-\frac{1}{2}} \tag{26.6.4}$$

provided $a \ge Z^{-1}$ [34] *and $B \le Z^2$; on the other hand, if $a \le Z^{-1}$, we can skip* Scott *and replace the last term in (26.4.2) by CZ^2.*

(iii) As $B \le Z$ the following estimates hold for $M = 1$ and $M \ge 2$, $a \ge Z^{-\frac{1}{3}}$ respectively

$$E^{\mathsf{TF}} \ge \mathcal{E}^{\mathsf{TF}} + \mathsf{Scott} + \mathsf{Dirac} + \mathsf{Schwinger} - CZ^{\frac{5}{3}}\bigl(Z^{-\delta} + (BZ^{-1})^{\delta}\bigr) \tag{26.6.5}$$

and

$$E^{\mathsf{TF}} \ge \mathcal{E}^{\mathsf{TF}} + \mathsf{Scott} + \mathsf{Dirac} + \mathsf{Schwinger} - CZ^{\frac{5}{3}}\bigl(Z^{-\delta} + (BZ^{-1})^{\delta} + (aZ^{\frac{1}{3}})^{-\delta}\bigr). \tag{26.6.6}$$

[34] Recall that a is the minimal distance between nuclei.

26.6.2 Upper Estimate: General Scheme

On the other hand, the upper estimate is more demanding. Recall that, according to Subsection 25.2.2, for the upper estimate in addition to the trace we need to estimate also $|\lambda_N - \nu|$ where $\lambda_N < 0$ is N-th eigenvalue of $H_{A,W}$ and $\lambda_N = 0$ if $H_{A,W}$ has less than N negative eigenvalues, and the product

$$|\lambda_N - \nu| \cdot |\mathsf{N}(H_{A,W}) - N| \tag{26.6.7}$$

and also three D-terms: two of them are semiclassical:

$$\mathsf{D}\Big(e(x,x,\lambda) - P'_B(W(x)+\lambda),\, e(x,x,\lambda) - P'_B(W(x)+\lambda)\Big) \tag{26.6.8}_{1,2}$$

with $\lambda = \nu$ and $\lambda = \lambda_N$ and also

$$\mathsf{D}\Big(P'_B(W(x)+\lambda_N) - P'_B(W(x)+\nu),\, P'_B(W(x)+\lambda_N) - P'_B(W(x)+\nu)\Big). \tag{26.6.9}$$

For this purpose our tool will be semiclassical estimates for two semiclassical N-terms

$$\int \Big(e(x,x,\lambda) - P'_B(W(x)+\lambda)\Big)\,dx \tag{26.6.10}_{1,2}$$

also with $\lambda = \nu$ and $\lambda = \lambda_N$ and also estimate from below for the third N-term

$$\Big|\int \Big(P'_B(W(x)+\lambda_N) - P'_B(W(x)+\nu)\Big)\,dx\Big|. \tag{26.6.11}$$

26.6.3 Upper Estimate as $M = 1$

Estimate for $|\lambda_N - \nu|$

We start from the easier case $M = 1$. Exactly as in Subsection 25.2.2 we have two cases: in the first case $|\nu|$ is small enough so we construct W^{TF} with $\nu = 0$ and estimate $|\lambda_N|$, and in the second case we prove that $\lambda_N \asymp \nu$ and estimate $|\lambda_N - \nu| \le \epsilon|\nu|$.

Proposition 26.6.2. *Let $M = 1$, $B \le Z^3$.*

(i) Assume first that

$$(Z-N)_+ \le K \coloneqq C_0 \max\bigl(Z^{\frac{2}{3}},\, Z^{\frac{2}{5}} B^{\frac{1}{5}}\bigr) \tag{26.6.12}$$

and let us construct W *as if* $\nu = 0$ *i.e.* $N = Z$. *Then*

$$|\lambda_N| \le C_1 \max\bigl(Z^{\frac{8}{9}},\, B^{\frac{2}{3}}\bigr). \tag{26.6.13}$$

(ii) Assume now that

$$(Z-N)_+ \ge K = C_0 \max\bigl(Z^{\frac{2}{3}},\, Z^{\frac{2}{5}} B^{\frac{1}{5}}\bigr) \tag{26.6.14}$$

with sufficiently large C_0. *Then* $\lambda_N \asymp \nu$ *and*

$$|\lambda_N - \nu| \le C_1 \max\bigl(Z^{\frac{2}{3}},\, B^{\frac{1}{2}}\bigr) |\nu|^{\frac{1}{4}}. \tag{26.6.15}$$

Proof. (i) In the framework of Statement (i) assume first that $B \ge (Z-N)_+^{\frac{4}{3}}$. One can see easily that then

(26.6.16) Expression (26.6.11) is

$$\asymp B|\lambda_N|^{\frac{1}{2}} \times \underbrace{\Bigl(\frac{|\lambda_N|}{G}\Bigr)^{\frac{1}{4}} \bar{r}^3}_{} \asymp |\lambda_N|^{\frac{3}{4}} \min\bigl(1,\, B^{-\frac{3}{10}} Z^{\frac{2}{5}}\bigr)$$

where $(|\lambda_N|/G)^{\frac{1}{4}}\bar{r}$ is a width of the zone where $0 < W \le -\lambda_N$ and the selected factor is the volume of this zone. Indeed, $W \asymp (\bar{r} - |x|)_+^4 \bar{r}^{-4} G$ for $|x| \asymp \bar{r}$.

However this expression (26.6.11) should be less than $C \max\bigl(Z^{\frac{2}{3}},\, Z^{\frac{2}{5}} B^{\frac{1}{5}}\bigr)$ which is exactly an error estimate in the semiclassical expression for N. Thus

$$|\lambda_N|^{\frac{3}{4}} \min\bigl(1,\, Z^{\frac{2}{5}} B^{-\frac{3}{10}}\bigr) \le C \max\bigl(Z^{\frac{2}{3}},\, Z^{\frac{2}{5}} B^{\frac{1}{5}}\bigr) \tag{26.6.17}$$

where everywhere the first and the second cases are as $B \le Z^{\frac{4}{3}}$ and $Z^{\frac{4}{3}} \le B \le Z^3$ respectively. The last inequality is equivalent to (26.6.13).

On the other hand, if $B \le (Z-N)_+^{\frac{4}{3}}$, inequality (26.6.17) is replaced by $|\lambda_N|^{\frac{3}{4}} \le C Z^{\frac{2}{3}}$ which coincides with (26.6.17) with B reset to $(Z-N)_+^{\frac{4}{3}}$ and also with the same inequality derived for $B = 0$ in Subsection 25.4.2; therefore (26.6.13) holds in this case as well.

(ii) One can prove easily that

(26.6.18) If condition (26.6.14) is fulfilled, and expression (26.6.11) does not exceed the semiclassical error estimate $C_0 \mathsf{max}(Z^{\frac{2}{3}}, Z^{\frac{2}{5}}B^{\frac{1}{5}})$, then $\lambda_N \asymp \nu$ and, furthermore, expression (26.6.11) is

$$\asymp B|\lambda_N - \nu| \int P''_B(W+\nu)\,dx \asymp B|\lambda_N - \nu| \int (W+\nu)_+^{-\frac{1}{2}}\,dx, \tag{26.6.19}$$

which for $B \ge (Z-N)_+^{\frac{4}{3}}$ is

$$\asymp B\bar{r}^3|\lambda_N - \nu| \cdot |\nu|^{-\frac{1}{2}}\Big(\frac{|\nu|}{G}\Big)^{\frac{1}{4}} \asymp |\lambda_N - \nu| \cdot |\nu|^{-\frac{1}{4}}\,\mathsf{min}\big(1,\, B^{-\frac{3}{10}}Z^{\frac{2}{5}}\big) \tag{26.6.20}$$

and this should be less than $C\,\mathsf{max}\big(Z^{\frac{2}{3}},\, Z^{\frac{2}{5}}B^{\frac{1}{5}}\big)$, and this implies (26.6.15).

On the other hand, if $B \le (Z-N)_+^{\frac{4}{3}}$, then the right-hand expression of (26.6.19) is $\asymp |\lambda_N - \nu|\bar{r} \asymp |\lambda_N - \nu|(Z-N)_+^{-\frac{1}{3}}$ and this should be less than $CZ^{\frac{2}{3}}$, and this implies (26.6.15) in this case as well. □

Proposition 26.6.2 immediately implies

Corollary 26.6.3. *In the frameworks of Proposition 26.6.2(i), (ii),*

$$|\lambda_N - \nu| \cdot \mathsf{N}([\lambda_N, \nu]) \le CQ, \tag{26.6.21}$$

where $\mathsf{N}(\lambda_N, \nu)$ *is the number of (non-zero) eigenvalues on interval* $[\lambda_N, \nu]$ *or* $[\nu, \lambda_N]$ [35].

Estimate for D-Terms

Proposition 26.6.4. *In the frameworks of Proposition 26.6.2(i),(ii) expressions*

$$\mathsf{D}\big(e(x,x,\lambda) - P'_B(W+\lambda),\, e(x,x,\lambda) - P'_B(W+\lambda)\big) \tag{26.6.22}$$

with $\lambda = \nu$ [35] *and with* $\lambda = \lambda_N$ *and*

$$\mathsf{D}\big(P'_B(W+\nu) - P'_B(W+\lambda),\, P'_B(W+\nu) - P'_B(W+\lambda_N)\big) \tag{26.6.23}$$

with $\lambda = \lambda_N$ *do not exceed* $C\,\mathsf{max}(Z^{\frac{5}{3}}, Z^{\frac{3}{5}}B^{\frac{4}{5}})$.

[35] Recall, that the frameworks of Proposition 26.6.2(i) we pick up $\nu = 0$.

Proof. Recall that we already derived in Section 26.3 this estimate for D-term (26.6.22) with $\lambda = \nu$. Further, the same estimate for this term with $\lambda = \lambda_N$ can be proven exactly in the same way; we leave easy details to the reader.

Furthermore, one can derive the same estimate for D-term (26.6.23) using Proposition 26.6.2; again we leave easy details to the reader. □

Remark 26.6.5. Let $B \le Z$. Then in (26.6.12)–(26.6.15) and therefore also in (26.6.19) and in Proposition 26.6.4 one can replace C_0 and C by $C_0\varepsilon$ and $C\varepsilon$ respectively with the small parameter ε: $\max\bigl(Z^{-\delta}, (BZ^{-1})^{\delta}\bigr) \le \varepsilon \le 1$.

Summary

Then following the scheme of Subsection 25.4.4 we arrive to upper estimates in Theorem 26.6.6 below (lower estimates have been proven in Proposition 26.6.1). Furthermore, based on estimates (26.6.2) and (26.6.24) and the fact, that the left-hand term in (26.6.28) should fit into the "gap" between them (see Section 25.2), we also arrive to Theorem 26.6.7 below:

Theorem 26.6.6. *Let $M = 1$, $B \le Z^3$. Then*

(i) The following estimate holds:

$$E^{\mathsf{TF}} \le \mathcal{E}^{\mathsf{TF}} + \Bigl(\mathsf{Tr}\bigl((H_{A,W} - \nu)^-\bigr) + \int P_B\bigl(W^{\mathsf{TF}}(x) + \nu\bigr)\,dx\Bigr) + CQ \tag{26.6.24}$$

with $Q = \max\bigl(Z^{\frac{5}{3}}, Z^{\frac{3}{5}}B^{\frac{4}{5}}\bigr)$.

(ii) The following estimate holds:

$$E^{\mathsf{TF}} \le \mathcal{E}^{\mathsf{TF}} + \mathsf{Scott} + CQ + CZ^{\frac{4}{3}}B^{\frac{1}{3}}. \tag{26.6.25}$$

Here for $Z^2 \le B \le Z^3$ one can skip Scott.

(iii) If $B \le Z$, then

$$E^{\mathsf{TF}} \ge \mathcal{E}^{\mathsf{TF}} + \mathsf{Scott} + \mathsf{Dirac} + \mathsf{Schwinger} + CZ^{\frac{5}{3}}\bigl(Z^{-\delta} + (BZ^{-1})^{\delta}\bigr). \tag{26.6.26}$$

Theorem 26.6.7. *Let $M = 1$, $B \le Z^3$. Then*

(i) The following estimate holds:

$$\mathsf{D}(\rho_\Psi - \rho_B^{\mathsf{TF}}, \rho_\Psi - \rho_B^{\mathsf{TF}}) \le CQ. \tag{26.6.27}$$

(ii) If $B \le Z$, then

$$\mathsf{D}(\rho_\Psi - \rho_B^{\mathsf{TF}}, \rho_\Psi - \rho_B^{\mathsf{TF}}) \le CZ^{\frac{5}{3}}\bigl(Z^{-\delta} + (BZ^{-1})^{\delta}\bigr). \tag{26.6.28}$$

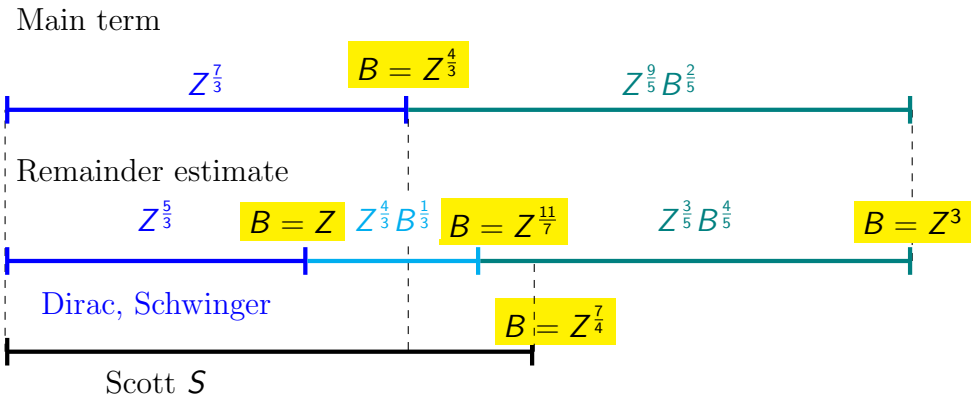

Figure 26.4: This figure illustrates the remainder estimate for E_N. Thresholds $B = Z^\star$ are shown in the yellow boxes.

26.6.4 Upper Estimate as $M \ge 2$

Estimate for $|\lambda_N - \nu|$

Again we need to consider two cases: *almost neutral molecules* (systems) when $(Z-N)_+ \le C_0K$ with K slightly redefined below and we can set $\nu = 0$ in the definition of Thomas-Fermi potential and establish estimate for $|\lambda_N|$ (and for optimal ν we have the same estimate for both $|\nu|$ and λ_N) and *not almost neutral molecules* (systems) when $(Z-N)_+ \ge C_0K$ and we can prove that $|\lambda_N| \asymp |\nu|$ and estimate $|\lambda_N - \nu|$.

Proposition 26.6.8 [36]**.** *Let $M \ge 1$, $B \le Z^3$ and condition* (26.2.28) *be fulfilled.*

(i) Assume first that

$$(Z-N)_+ \le K := C_0 \begin{cases} Z^{\frac{2}{3}} + B^{\frac{1}{2}}L & \text{if } B \le Z^{\frac{4}{3}}, \\ Z^{\frac{1}{5}}B^{\frac{4}{15}}L & \text{if } Z^{\frac{4}{3}} \le B \le Z^3 \end{cases} \tag{26.6.29}$$

[36] Cf. Proposition 26.6.2.

and let us construct W *as if* $\nu = 0$ *i.e.* $N = Z$. *Then*

$$|\lambda_N| \le C_1 \begin{cases} Z^{\frac{8}{9}} + B^{\frac{2}{3}} L^{\frac{4}{3}} & \text{if } B \le Z^{\frac{4}{3}}, \\ Z^{\frac{1}{5}} B^{\frac{4}{15}} L^{\frac{4}{3}} & \text{if } Z^{\frac{4}{3}} \le B \le Z^3; \end{cases} \tag{26.6.30}$$

recall that $L = |\log BZ^{-3}|$.

(ii) Assume now that

$$(Z - N)_+ \ge K \tag{26.6.31}$$

with sufficiently large C_0 *in the definition of* K. *Then* $\lambda_N \asymp \nu$ *and moreover*

$$|\lambda_N - \nu| \le C \max\bigl(Z^{\frac{2}{3}}, B^{\frac{1}{2}} L_1\bigr) |\nu|^{\frac{1}{4}}, \tag{26.6.32}$$

where

$$L_1 = \begin{cases} 1 & \text{if } B \le (Z - N)_+^{\frac{4}{3}}, \\ |\log((Z - N)_+ / B^{\frac{3}{4}}| & \text{if } (Z - N)_+^{\frac{4}{3}} \le B \le Z^{\frac{4}{3}}, \\ |\log(Z - N)_+ / B^{\frac{4}{5}} Z^{\frac{3}{5}}|) & \text{if } Z^{\frac{4}{3}} \le B \le Z^3. \end{cases} \tag{26.6.33}$$

(iii) For $M = 1$ *one can take* $L = L_1 = 1$.

Proof. We will apply arguments slightly more sophisticated than the obvious ones, used in the proof of Proposition 26.6.2. These better arguments will allow us to derive slightly better estimates for $|\lambda_N - \nu|$ as $(Z - N)_+ \ge CK$, and for threshold K itself.

Recall that estimates for $|\lambda_N - \nu|$ are derived by comparison of expression (26.6.11) and the semiclassical errors for the number of eigenvalues below $\lambda = \nu$ and $\lambda = \lambda_N$: expression (26.6.11) should be less than the sum of these semiclassical errors.

Consider contribution of each ball

$$B(x, \ell(x)) \subset \mathcal{Y} = \{x \colon \min_m |x - \mathsf{y}_m| \ge \epsilon \bar{r}\} \tag{26.6.34}$$

to semiclassical errors as $\lambda = \nu$ and $\lambda = \lambda_N$ and compare it with its contribution to (26.6.11):

(a) Each ball contributes no more than $CB\ell^2$ to the first error (with $\lambda = \nu$) where due to our choice $\zeta\ell \ge 1$.

(b) Further, each ball with $\zeta \ge C_1|\lambda_N - \nu|^{\frac12}$ contributes no more than $CB\ell^2$. On the other hand, each ball with $\zeta \le C_1|\lambda_N - \nu|^{\frac12}$ contributes no more than $CB\ell^{3-\sigma}|\lambda_N - \nu|^{\sigma/2}$ to the second error (with $\lambda = \lambda_N$); here $\sigma = \frac13$ is due to rescaling.

(c) Meanwhile, each ball with $\zeta \ge C_1|\lambda_N - \nu|^{\frac12}$ contributes no less than $\epsilon_0 B|\lambda_N - \nu|\zeta^{-1}\ell^3$, and each ball with $\zeta \ge C_1|\lambda_N - \nu|^{\frac12}$ contributes no less than $\epsilon_0|\lambda_N - \nu|^{\frac12}\ell^3$ to expression (26.6.11) and it is larger than the contributions of this ball to each of semiclassical errors (multiplied by C) as long as

$$(26.6.35)_{1,2} \qquad \zeta^2 \ge |\lambda_N - \nu| \ge C_2\zeta\ell^{-1}, \qquad |\lambda_N - \nu| \ge C_2\ell^{-2}.$$

Obviously in Statements (i), (ii) we can assume that

(26.6.36) Inequalities (26.6.30) and (26.6.32) respectively (with C replaced by arbitrarily large C_3) are violated.

(i)(a) Assume first that $(Z-N)_+^{\frac43} \le B \le Z^3$. Then in the framework of assumption $\zeta = B^2\ell^4$ with minimal $\ell = B^{-\frac13}$ and therefore $(26.6.35)_{1,2}$ are fulfilled for $\ell \le C_2B^{-1}|\lambda_N|$. Therefore we need to account for the semiclassical errors contributed by an inner shell (not exceeding $C\max(Z^{\frac23}, B^{\frac12})$) and by zone $\mathcal{Y} \cap \{\ell \ge C_2B^{-1}|\lambda_N|\}$; there $\zeta \ge C_1|\lambda_N|^{\frac12}$ and therefore its contribution does not exceed $CB\int \ell(x)^{-1}\,dx$ with integral over this zone and it does not exceed $CB\bar{r}^2L$.

So, these truncated semiclassical errors do not exceed $C\max(Z^{\frac23}, B\bar{r}^2L)$. Meanwhile, expression (26.6.11) is no less than $CB^{\frac12}\bar{r}^2|\lambda_N|^{\frac34}$. Therefore comparing these two expressions as $B \le Z^{\frac43}$ and as $Z^{\frac43} \le B \le Z^3$ we arrive to (26.6.30).

(b) Consider the remaining case $B \le (Z-N)_+^{\frac43}$. Semiclassical arguments remain valid while estimate of (26.6.11) from below by $\epsilon_0|\lambda_N|^{\frac34}$ also could be proven easily.

(ii)(a) Again, assume first that $(Z-N)_+^{\frac43} \le B \le Z^3$. Again, in the calculation of the truncated semiclassical errors we integrate over zone $\{\ell \ge C_2B^{-1}|\lambda_N - \nu|\}$ where $\zeta \ge C_1|\lambda_N|^{\frac12}$ and therefore its contribution does not exceed $CB\int \ell(x)^{-1}\,dx$ with integral over this zone and it does not exceed $CB\bar{r}^2L_1$ [37].

[37] In Statement (i) this leads only to insignificant improvement.

Again, expression (26.6.21) is larger than the expressions afterwards and comparing with the semiclassical error estimate we arrive to (26.6.32).

(b) Consider the remaining case $B \le (Z-N)_+^{\frac{4}{3}}$. Semiclassical arguments remain valid while estimate of (26.6.11) from below by $\epsilon_0|\lambda_N - \nu|\cdot|\lambda_N|^{-\frac{1}{4}}$ also could be proven easily.

(iii) Recall that for $M = 1$ the semiclassical error estimate hold with $L = L_1 = 1$. □

Then we arrive immediately to

Corollary 26.6.9. *In the framework of Proposition 26.6.8* $|\lambda_N-\nu|\cdot \mathsf{N}([\lambda_N,\nu])$ *does not exceed expression* (26.5.40).

Estimate for D-Terms for Almost Neutral sSystems

We need to estimate the semiclassical error D-term (26.6.22) with $\lambda = \lambda_N$ because for $\lambda = \nu$ we already estimated it, and also we need to estimate another D-term (26.6.23). We start from the latter one. Recall that under assumption (26.6.29) we take $\nu = 0$. The trivial estimate is based on

$$|P'_B(W) - P'_B(W+\lambda)| \le CW^{\frac{1}{2}}|\lambda| + CBW^{-\frac{1}{4}}|\lambda|^{\frac{3}{4}}, \tag{26.6.37}$$

leading to

$$J \le C\mathsf{D}(W^{\frac{1}{2}},\, W^{\frac{1}{2}})|\lambda|^2 + CB^2|\lambda|^{\frac{3}{2}}\mathsf{D}(W^{-\frac{1}{4}}\theta,\, W^{-\frac{1}{4}}\theta) \tag{26.6.38}$$

where here and below J is expression (26.6.23), θ is a characteristic function of the domain $\{x\colon \gamma(x) \ge h^{\frac{1}{3}}\}$ and we can ignore the contribution of the zone $\{x\colon \gamma(x) \le h^{\frac{1}{3}}\}$. Really, the contribution of this zone does not exceed a semiclassical error estimate $R \coloneqq C\max(Z^{\frac{5}{3}},\, Z^{\frac{3}{5}}B^{\frac{4}{5}}L^2)$.

Note that even without assumption (26.6.29)

$$\mathsf{D}(W^{\frac{1}{2}},\, W^{\frac{1}{2}}) \asymp (B^{-\frac{1}{4}};\, B^{-\frac{8}{5}}Z^{\frac{9}{5}}) \qquad \text{for } \ B \le Z^{\frac{4}{3}},\, Z^{\frac{4}{3}} \le B \le Z^3 \tag{26.6.39}$$

respectively and (26.6.30) implies that the first term in the right-hand expression of (26.6.38) is much less than R.

Meanwhile, under assumption (26.6.29)

$$\mathsf{D}(W^{-\frac{1}{4}}\theta, W^{-\frac{1}{4}}\theta) \asymp B^{-1}\mathsf{D}(\ell^{-1}\theta, \ell^{-1}\theta) \asymp B^{-1}\bar{r}^3\mathsf{D}(\gamma^{-1}\theta, \gamma^{-1}\theta) \tag{26.6.40}$$

where in the right-hand expression D and γ, θ are in the scale $x \mapsto x\bar{r}^{-1}$ and then $\mathsf{D}(\gamma^{-1}\theta, \gamma^{-1}\theta) \asymp L^2$ so the second term in (26.6.38) does not exceed $CB\bar{r}^3|\lambda_N|^{\frac{3}{2}}L^2$ which due to (26.6.30) does not exceed

$$R := C\max\bigl(Z^{\frac{5}{3}},\, Z^{\frac{3}{5}}B^{\frac{4}{5}}L^4\bigr). \tag{26.6.41}$$

Consider now term (26.6.22) with $\lambda = \lambda_N$. Let us consider zones $\Omega_1 := \{x\colon |\lambda - \nu| \lesssim \zeta\ell^{-1}\}$ and $\Omega_2 := \{x\colon |\lambda - \nu| \gtrsim \zeta\ell^{-1}\}$.

Note that the contribution to the term in question of each pair of balls contained in $\Omega_1 \times \Omega_1$ does not exceed estimate for the same term with $\lambda = \nu$; really, after rescaling $x \mapsto x/\ell$ and $\tau \mapsto \tau/\zeta^2$ we conclude that the difference between energy levels does not exceed local semiclassical parameter $C/(\zeta\ell)$.

Therefore the total contribution of $\Omega_1 \times \Omega_1$ to this term does not exceed $C\max\bigl(Z^{\frac{5}{3}},\, Z^{\frac{3}{5}}B^{\frac{4}{5}}L^2\bigr)$.

On the other hand, the contribution to the term in question of each pair of balls contained in $\Omega_2 \times \Omega_2$ does not exceed its contribution to (26.6.25) and therefore the total contribution of $\Omega_2 \times \Omega_2$ to this term does not exceed expression (26.6.41). Thus, term (26.6.22) with $\lambda = \lambda_N$ does not exceed (26.6.41).

Therefore we arrive immediately to

Theorem 26.6.10. *Let $M \ge 2$, $B \le Z^3$ and condition* (26.2.28) *be fulfilled. Then under assumption* (26.6.29)

(i) The following estimate holds:

$$E^{\mathsf{TF}} \le \mathcal{E}^{\mathsf{TF}} + \Bigl(\mathsf{Tr}\bigl((H_{A,W} - \nu)^-\bigr) + \int P_B(W^{\mathsf{TF}} + \nu)\,dx\Bigr)+ C\max\bigl(Z^{\frac{5}{3}},\, Z^{\frac{3}{5}}B^{\frac{4}{5}}L^4\bigr). \tag{26.6.42}$$

(ii) If $a \ge Z^{-1}$ then the following estimate holds:

$$E^{\mathsf{TF}} \le \mathcal{E}^{\mathsf{TF}} + \mathsf{Scott} + C\max\bigl(Z^{\frac{5}{3}},\, Z^{\frac{3}{5}}B^{\frac{4}{5}}L^4\bigr) + CZ^{\frac{4}{3}}B^{\frac{1}{3}} + Ca^{-\frac{1}{2}}Z^{\frac{3}{2}}; \tag{26.6.43}$$

if $a \le Z^{-1}$ one should replace the last term in the right-hand expression by CZ^2 and skip Scott.

(iii) If $B \le Z$ and $a \ge Z^{-\frac{1}{3}}$

$$E^{\mathsf{TF}} \le \mathcal{E}^{\mathsf{TF}} + \mathsf{Scott} + \mathsf{Dirac} + \mathsf{Schwinger}+ CZ^{\frac{5}{3}}\bigl(Z^{-\delta} + (BZ^{-1})^{\delta} + (aZ^{\frac{1}{3}})^{-\delta}\bigr). \tag{26.6.44}$$

Here proof of Statement (iii) is due to the same arguments as in the case $B = 0$. Combining with the estimate from below we also conclude that

Theorem 26.6.11. *(i) In the framework of Theorem 26.6.10 the following estimate holds:*

$$\mathsf{D}\big(\rho_\Psi - \rho^{\mathsf{TF}},\, \rho_\Psi - \rho^{\mathsf{TF}}\big) \le Q := C\max\big(Z^{\frac{5}{3}};\, Z^{\frac{3}{5}}B^{\frac{4}{5}}L^4\big). \tag{26.6.45}$$

(ii) In the framework of Theorem 26.6.10(iii) the following estimate holds:

$$\mathsf{D}\big(\rho_\Psi - \rho^{\mathsf{TF}},\, \rho_\Psi - \rho^{\mathsf{TF}}\big) \le CZ^{\frac{5}{3}}\big(Z^{-\delta} + (BZ^{-1})^{\delta} + (aZ^{\frac{1}{3}})^{-\delta}\big). \tag{26.6.46}$$

Estimate for D-Terms for Positively Charged sSystems

Let assumption (26.6.31) be fulfilled. Let $W = W_\nu$ and $\ell = \ell_\nu$ be a potential and a scaling function (used to derive semiclassical remainder estimates) for this $\nu < 0$ (and $N < Z$) while W_0 and ℓ_0 be a potential and a scaling function for $\nu = 0$ (and $N = Z$).

Let us start from rather trivial arguments. Note that

$$|P'_B(W+\lambda) - P'_B(W+\nu)| \le CW^{\frac{1}{2}}|\lambda-\nu|\theta_1 + CB|\lambda-\nu|^{\frac{1}{2}}\theta_2, \tag{26.6.47}$$

where θ_1 and θ_2 are characteristic functions of $\mathcal{Y}_1 = \{x\colon W(x) + \nu \ge C_0|\nu|\}$ and $\mathcal{Y}_2 = \{x\colon 0 < W(x) + \nu \le C_0|\nu|\}$ respectively. Let

$$J_k := \mathsf{D}\big([P'_B(W+\lambda_N) - P'_B(W+\nu)]\theta_k,\, [P'_B(W+\lambda_N) - P'_B(W+\nu)]\theta_k\big).$$

Then in virtue of (26.6.47)

$$J_1 \le C\mathsf{D}(W^{\frac{1}{2}}\theta_1, W^{\frac{1}{2}}\theta_1)|\lambda_N - \nu|^2.$$

Note that[38)]

$$\mathsf{D}(W^{\frac{1}{2}}\theta_1,\, W^{\frac{1}{2}}\theta_1) \asymp \big((Z-N)_+^{-\frac{1}{3}};\, B^{-\frac{1}{4}};\, Z^{\frac{9}{5}}B^{-\frac{8}{5}}\big).$$

Then, using inequality (26.6.32) one can prove easily that $J_1 \le CZ^{\frac{5}{3}}$.

However estimate for a contribution of zone $\mathcal{Y}_2$ is much worse:

$$J_2 \le CB^2\mathsf{D}(\theta_2,\theta_2)|\lambda_N - \nu| \le CB^2\bar{r}^3(|\nu|/B^2)^{\frac{1}{2}}L_1^2|\lambda_N - \nu|, \tag{26.6.48}$$

[38)] In our three cases $B \le (Z-N)_+^{\frac{4}{3}}$, $(Z-N)_+^{\frac{4}{3}} \le B \le Z^{\frac{4}{3}}$, and $Z^{\frac{4}{3}} \le B \le Z^3$ respectively.

where for $B \le (Z-N)_+^{\frac{1}{3}}$ we should replace $(|\nu|/B^2)^{\frac{1}{2}}L_1^2$ by $\bar{r}^2$. Then, using (26.6.32), we conclude that for $(Z-N)_+^{\frac{4}{3}} \lesssim B \lesssim Z^3$

$$J_2 \le CB\bar{r}^3|\nu|^{\frac{3}{4}}L_1^2 \max(Z^{\frac{2}{3}}, B^{\frac{1}{2}}) \asymp CB(Z-N)_+^{\frac{3}{4}}\bar{r}^{\frac{9}{4}} \max(Z^{\frac{2}{3}}, B^{\frac{1}{2}}L_1)L_1^2,$$

and therefore we arrive to the last two cases below; the first case is proven similarly:

$$J_2 \le C \begin{cases} (Z-N)_+^{-\frac{4}{3}} Z^{\frac{2}{3}} B^2, \\ (Z-N)_+^{\frac{3}{4}} B^{\frac{7}{16}} \max(Z^{\frac{2}{3}}, B^{\frac{1}{2}}L_1)L_1^2, \\ (Z-N)_+^{\frac{3}{4}} Z^{\frac{9}{20}} B^{\frac{3}{5}} L_1^3 \end{cases} \tag{26.6.49}$$

in our three cases.

This is really shabby estimate. To improve it let us observe that

(26.6.50) If estimate $|\lambda_N - \nu| \le C \max(B^{\frac{2}{3}}, (Z-N)_+^{\frac{8}{9}})$ holds, then J_2 does not exceed (26.5.40)

and therefore we can assume that

$$|\lambda_N - \nu| \ge C \max((Z-N)_+^{\frac{8}{9}};\, B^{\frac{2}{3}}). \tag{26.6.51}$$

Let us estimate the truncated semiclassical error[39)].

Proposition 26.6.12. *(i) Let* $(Z-N)_+^{\frac{4}{3}} \le B \le Z^3$ *and*

$$C_0 B^{\frac{2}{3}} \le |\lambda_N - \nu| \le C_1 B^{\frac{1}{2}}|\nu|^{\frac{1}{4}}. \tag{26.6.52}$$

Then the truncated semiclassical error in N*-term does not exceed*

$$F := CZ^{\frac{2}{3}} + CB\bar{r}^2(|\nu|/B^2)^{\frac{1}{4}}L \times (B^{-1}|\lambda_N - \nu|)^{-1}. \tag{26.6.53}$$

(ii) Let $(Z-N)_+^{\frac{4}{3}} \le B \le Z^3$ *and*

$$C_1 B^{\frac{1}{2}}|\nu|^{\frac{1}{4}} \le |\lambda_N - \nu| \le C_1 B^{\frac{1}{2}}|\nu|^{\frac{1}{4}}L. \tag{26.6.54}$$

Then the truncated semiclassical error does not exceed

$$F := CZ^{\frac{2}{3}} + CB\bar{r}^2 L. \tag{26.6.55}$$

[39)] I.e. contribution to such error of the zone, where it exceeds the contribution to the principal part.

(iii) Let $B \le (Z-N)_+^{\frac{4}{3}}$ *and*

$$(Z-N)_+^{\frac{8}{9}} \le |\lambda_N - \nu| \le C_1(Z-N)_+. \tag{26.6.56}$$

Then the truncated semiclassical error does not exceed

$$F := CZ^{\frac{2}{3}} + C(Z-N)_+^{\frac{5}{3}}|\lambda_N - \nu|^{-1}. \tag{26.6.57}$$

(iv) Let $B \le (Z-N)_+^{\frac{4}{3}}$ *and*

$$C_0(Z-N)_+ \le |\lambda_N - \nu| \le C_1 Z^{\frac{2}{3}}(Z-N)_+^{\frac{1}{3}}. \tag{26.6.58}$$

Then the truncated semiclassical error does not exceed $F := CZ^{\frac{2}{3}}$.

Proof. The easy proof, which uses arguments of the proof of Proposition 26.6.8, is left to the reader. □

Proposition 26.6.13. *In the framework of Proposition 26.6.12(i)–(iv) term* (26.6.25) *does not exceed*

$$CF^{\frac{5}{3}}(B|\lambda_N - \nu|^{\frac{1}{2}})^{\frac{1}{3}} + (26.5.40)$$

with F *defined in the corresponding cases in Proposition 26.6.12.*

Proof. Using Proposition 26.6.8 one can prove easily that

(26.6.59) Contribution of $\{x\colon \ell(x) := \min_m |x - y_m| \le \epsilon\bar{r}\}$ to (26.6.25) does not exceed $CZ^{\frac{5}{3}}$.

Now we need to estimate the excess of expression (26.6.25) over semiclassical D-term (with $\lambda = \nu$), which has been estimated by (26.5.40). To do so we need to estimate

$$\mathsf{D}\big([P_B(W+\nu) - P_B(W+\lambda)]\theta,\, [P_B(W+\nu) - P_B(W+\lambda)]\theta\big) \tag{26.6.60}$$

which is the contribution of the domain $\Omega' := \{x\colon \ell(x) \le CB^{-1}|\lambda - \nu|\}$ where θ is the characteristic function of Ω'. Recall that in the complimentary domain $|P_B(W+\nu) - P_B(W+\lambda)| \le C\ell^{-1}$. Let us consider

$$\mathsf{D}\big([P_B(W+\nu) - P_B(W+\lambda)]\theta_0,\, [P_B(W+\nu) - P_B(W+\lambda)]\theta_0\big) \tag{26.6.61}$$

and

$$\mathsf{D}\bigl([P_B(W+\nu)-P_B(W+\lambda)]\theta_t,\,[P_B(W+\nu)-P_B(W+\lambda)]\theta_{t'}\bigr), \tag{26.6.62}$$

where θ_0 is a characteristic function of

$$\Omega'_0 \coloneqq \{x\colon \ell(x) \le t_0 \coloneqq (|\lambda_N-\nu|B^{-2})^{\frac14}\}$$

and θ_t is a characteristic function of $\Omega'_t \coloneqq \{x\colon t \le \ell(x) \le 2t\}$ with $t \ge t' \ge t_0$.

Observe that, when calculating expression (26.6.11), the contribution of Ω'_0 is $\asymp B|\lambda_N-\nu|^{\frac12}\,\mathsf{mes}(\Omega'_0)$, and therefore due to Proposition 26.6.12

$$\mathsf{mes}(\Omega'_0) \le CF\bigl(B|\lambda_N-\nu|^{\frac12}\bigr)^{-1}, \tag{26.6.63}$$

while term (26.6.61) is

$$\asymp B^2|\lambda_N-\nu|\mathsf{D}(\theta_0,\theta_0) \le CB^2|\lambda_N-\nu|\bigl(\mathsf{mes}(\Omega_0)\bigr)^{\frac53} \le CF^{\frac53}\bigl(B|\lambda_N-\nu|^{\frac12}\bigr)^{\frac13},$$

where the middle inequality

$$\mathsf{D}(\chi_G,\chi_G) \le C\bigl(\mathsf{mes}(G)\bigr)^{\frac53} \tag{26.6.64}$$

is well known[40)] and the last one is due to (26.6.63); χ_G denotes characteristic function of G.

Similarly, when calculating expression (26.6.11), one can see easily that the contribution of Ω'_t is $\asymp B|\lambda_N-\nu|(B^2t^4)^{-1}\,\mathsf{mes}(\Omega'_t)$ and therefore

$$\mathsf{mes}(\Omega'_t) \le CF|\lambda_N-\nu|^{-1}t^2, \tag{26.6.65}$$

while term (26.6.62) is $\asymp |\lambda_N-\nu|^2t^{-2}t'^{-2}\mathsf{D}(\theta_t,\theta_{t'})$, which does not exceed

$$C|\lambda_N-\nu|^2t^{-2}t'^{-2}\,\mathsf{mes}(\Omega_t)\,\mathsf{mes}(\Omega_{t'})\bigl[\max\bigl(\mathsf{mes}(\Omega_t),\,\mathsf{mes}(\Omega_{t'})\bigr)\bigr]^{-\frac13} \tag{26.6.66}$$

due to inequality

$$\mathsf{D}(\chi_G,\chi_{G'}) \le C\,\mathsf{mes}(G)\,\mathsf{mes}(G')\bigl[\max\bigl(\mathsf{mes}(G),\,\mathsf{mes}(G')\bigr)\bigr]^{-\frac13}, \tag{26.6.67}$$

which trivially follows from the obvious inequality $\mathsf{D}(\chi_G,\delta_z) \le C(\mathsf{mes}(G))^{\frac23}$, where $\delta_z(x)=\delta(x-z)$.

[40)] Really, among uniform solids of equal mass and density the ball has the least potential energy; then $C=\frac15(12\pi)^{\frac13}$.

Due to (26.6.65) expression (26.6.66) does not exceed $CF^{\frac{5}{3}}t^{-\frac{2}{3}}$; recall that $t \ge t'$. Since summation with respect to $t \ge t'$ and then with respect to $t' \ge t_0$ returns $CF^{\frac{5}{3}}t_0^{-\frac{2}{3}}$, we conclude that term (26.6.61) with θ_0 replaced by θ'' (the characteristic function of $\{x\colon \ell(x) \ge t_0\}$) also does not exceed $CF^{\frac{5}{3}}(B|\lambda_N - \nu|^{\frac{1}{2}})^{\frac{1}{3}}$. □

So, we have now two estimates for an excess of expression (26.6.25) over (26.5.40): one estimate is

$$CF^{\frac{5}{3}}(B|\lambda_N - \nu|^{\frac{1}{2}})^{\frac{1}{3}} \tag{26.6.68}$$

with $F = F(|\lambda_N - \nu|)$ derived in Proposition 26.6.12 and another one is due to (26.6.48). Let us consider the best of them. Note that estimate (26.6.68) consists of two terms each due to the corresponding term in the definition of F. The second term in the framework of Proposition 26.6.12(i) is

$$C\bigl(B^{\frac{3}{2}}\bar{r}^2|\nu|^{\frac{1}{4}}L|\lambda_N - \nu|^{-1}\bigr)^{\frac{5}{3}}\bigl(B|\lambda_N - \nu|^{\frac{1}{2}}\bigr)^{\frac{1}{3}} \asymp B^{\frac{17}{6}}\bar{r}^{\frac{10}{3}}|\nu|^{\frac{5}{12}}L^{\frac{5}{3}}|\lambda_N - \nu|^{-\frac{3}{2}}.$$

Then, taking minimum of this expression and $CB\bar{r}^3|\nu|^{\frac{1}{2}}L^2|\lambda_N - \nu|$, we see that this minimum does not exceed

$$C\bigl(B^{\frac{17}{6}}\bar{r}^{\frac{10}{3}}|\nu|^{\frac{5}{12}}L^{\frac{5}{3}}\bigr)^{\frac{2}{5}}\bigl(B\bar{r}^3|\nu|^{\frac{1}{2}}L^2\bigr)^{\frac{3}{5}} \asymp CB^{\frac{5}{3}}\bar{r}^{\frac{8}{3}}(Z-N)_+^{\frac{7}{15}}L^{\frac{28}{15}} \asymp$$
$$C\begin{cases}(Z-N)_+^{\frac{7}{15}}BL^{\frac{28}{15}} & \text{if } (Z-N)_+^{\frac{4}{3}} \le B \le Z^{\frac{4}{3}},\\ (Z-N)_+^{\frac{7}{15}}Z^{\frac{8}{15}}B^{\frac{3}{5}}L^{\frac{28}{15}} & \text{if } Z^{\frac{4}{3}} \le B \le Z^3,\end{cases}$$

which is achieved for $|\lambda_N - \nu| \asymp B^{\frac{11}{15}}\bar{r}^{\frac{2}{15}}|\nu|^{-\frac{1}{30}}L^{-\frac{2}{15}}$. One can see easily that this expression does not exceed (26.5.40).

Therefore in the framework of Proposition 26.6.12(i)(ii) we can select $F = (Z^{\frac{2}{3}} + B\bar{r}^2L)$ according to (26.6.55), arriving to

$$C(Z^{\frac{2}{3}} + B\bar{r}^2L)^{\frac{5}{3}}B^{\frac{1}{3}}|\lambda_N - \nu|^{\frac{1}{6}} \le C(Z^{\frac{2}{3}} + B\bar{r}^2L)^{\frac{5}{3}}B^{\frac{1}{3}}(Z^{\frac{2}{3}} + B^{\frac{1}{2}}L_1)^{\frac{1}{6}}|\nu|^{\frac{1}{24}},$$

which we can rewrite (slightly increasing powers of logarithms) as two last cases in expression

$$C\begin{cases}(Z-N)_+^{\frac{1}{18}}Z^{\frac{11}{9}}B^{\frac{1}{3}} & \text{if } B \le (Z-N)_+^{\frac{4}{3}},\\ (Z-N)_+^{\frac{1}{24}}(Z^{\frac{11}{9}} + B^{\frac{11}{12}}L^4)B^{\frac{11}{32}} & \text{if } (Z-N)_+^{\frac{4}{3}} \le B \le Z^{\frac{4}{3}},\\ (Z-N)_+^{\frac{1}{24}}Z^{\frac{79}{120}}B^{\frac{23}{30}} & \text{if } Z^{\frac{4}{3}} \le B \le Z^3.\end{cases} \tag{26.6.69}$$

In the framework of Proposition 26.6.12(iii) one should replace $(\nu/B^2)^{\frac{1}{4}}$ by $\bar{r}$ and L by 1, so $B^{\frac{3}{2}}\bar{r}^2|\nu|^{\frac{1}{4}}L|\lambda_N-\nu|^{-1}\mapsto B^2\bar{r}^3|\lambda_N-\nu|^{-1}$; further, one should preserve $B|\lambda_N-\nu|^{\frac{1}{2}}$ and therefore the second term becomes

$$\bigl(B^2\bar{r}^3|\lambda_N-\nu|^{-1}\bigr)^{\frac{5}{3}}\bigl(B|\lambda_N-\nu|^{\frac{1}{2}}\bigr)^{\frac{1}{3}}\asymp B^{\frac{11}{3}}\bar{r}^5|\lambda_N-\nu|^{-\frac{3}{2}}$$

and taking minimum of it and (26.6.48) we again get a term lesser than (26.5.40).

Meanwhile, the first term becomes $Z^{\frac{10}{9}}B^{\frac{1}{3}}|\lambda_N-\nu|^{\frac{1}{6}}\le (Z-N)_+^{\frac{1}{18}}Z^{\frac{11}{9}}B^{\frac{1}{3}}$ occupying the first line in (26.6.69).

Therefore we have proven

Proposition 26.6.14. *If $M\ge 2$, $B\le Z^3$ all three* D*-terms do not exceed* (26.5.40) + (26.6.69).

Summary

Therefore all error terms in the upper estimate do not exceed (26.5.40) and we arrive to

Theorem 26.6.15. *Let $M\ge 2$, $B\le Z^3$. Then*

(i) The following estimate holds:

$$E^{\mathsf{TF}}\le \mathcal{E}^{\mathsf{TF}}+\Bigl(\mathsf{Tr}\bigl((H_{A,W}-\nu)^-\bigr)+\int P_B(W^{\mathsf{TF}}+\nu)\,dx\Bigr)+ (26.5.40)+(26.6.69). \tag{26.6.70}$$

(ii) The following estimate holds for $a\ge Z^{-1}$:

$$E^{\mathsf{TF}}\le \mathcal{E}^{\mathsf{TF}}+\underbrace{\mathsf{Scott}+CZ^{\frac{4}{3}}B^{\frac{1}{3}}+a^{-\frac{1}{2}}Z^{\frac{3}{2}}}+(26.5.40)+(26.6.69); \tag{26.6.71}$$

for $a\le Z^{-1}$ one should replace selected terms by CZ^2.

(iii) If $B\le Z$ and $a\ge Z^{-\frac{1}{3}}$

$$E^{\mathsf{TF}}\ge \mathcal{E}^{\mathsf{TF}}+\mathsf{Scott}+\mathsf{Dirac}+\mathsf{Schwinger}+ CZ^{\frac{5}{3}}\bigl(Z^{-\delta}+(BZ^{-1})^{\delta}+(aZ^{\frac{1}{3}})^{-\delta}\bigr). \tag{26.6.72}$$

We also arrtive to

Theorem 26.6.16. *(i) In the framework of Theorem 26.6.15(i) the following estimate holds:*

$$\mathsf{D}\bigl(\rho_\psi - \rho^{\mathsf{TF}}, \rho_\psi - \rho^{\mathsf{TF}}\bigr) \le (26.5.40) + (26.6.69). \tag{26.6.73}$$

(ii) In the framework of Theorem 26.6.15(iii) (albeit without assumption $a \ge Z^{-\frac{1}{3}}$) the following estimate holds:

$$\mathsf{D}\bigl(\rho_\psi - \rho^{\mathsf{TF}}, \rho_\psi - \rho^{\mathsf{TF}}\bigr) \le CQ \coloneqq CZ^{\frac{5}{3}}\bigl(Z^{-\delta} + (BZ^{-1})^{\delta}\bigr). \tag{26.6.74}$$

Remark 26.6.17. In virtue of Remark 26.3.7 we can replace term $CZ^{\frac{4}{3}}B^{\frac{1}{3}}$ to $o(Z^{\frac{4}{3}}B^{\frac{1}{3}})$. This is also true in the case of the better estimates $M = 1$.

We leave to the reader the following easy problem:

Problem 26.6.18. Investigate conditions to $(Z-N)_+$ so that terms (26.5.40) and (26.6.69) do not spoil the upper estimate for E_N or $\mathsf{D}(\rho_\Psi - \rho_B^{\mathsf{TF}},\ \rho_\Psi - \rho_B^{\mathsf{TF}})$.

26.7 Negatively Charged Systems

In this section we following Section 25.5 consider the case $N \ge Z$ and provide upper estimates for the excessive negative charge $(N-Z)$ if $\mathsf{I}_N > 0$ and for the ionization energy I_N.

26.7.1 Estimates of the Correlation Function

First of all we provide some estimates which will be used for both negatively and positively charged systems. Let us consider the ground-state function $\Psi(x_1, \varsigma_1; \dots; x_N, \varsigma_N)$ and the corresponding density $\rho_\Psi(x)$. Again the crucial role play estimates[41)]

$$\mathsf{D}\bigl(\rho_\Psi - \rho_B^{\mathsf{TF}},\ \rho_\Psi - \rho_B^{\mathsf{TF}}\bigr) \le Q \tag{26.7.1}$$

[41)] Namely, estimate (26.6.27) of Theorem 26.6.7 for $M = 1$, and similar estimates (26.6.45) of Theorem 26.6.11 and (26.6.73) of Theorem 26.6.16 for $M \ge 2$.

For $B \le Z$ and $a \ge Z^{-\frac{1}{3}}$, we use estimate (26.6.74) in all cases.

where Q is just the right-hand expression of the corresponding estimate. Recall that the same estimate holds also for difference between upper and lower bounds for E_N (with $\mathsf{Tr}((H_W-\nu)^-)+\nu N$ not replaced by its semiclassical approximation).

Remark 26.7.1. (i) Similarly to Subsection 25.5.1 $\rho := \rho_B^{\mathsf{TF}}$, $W = W_B^{\mathsf{TF}}$, $\rho_\varepsilon := \phi_\varepsilon * \rho$, $W_\varepsilon = |x|^{-1} * \rho_\varepsilon$ with ϕ_ε defined in Proposition 25.5.1.

(ii) All arguments and conclusions of Subsection 25.5.1 up to but excluding estimates (25.5.31)–(25.5.32) are not related to the Schrödinger operator and remain true.

So we need to calculate both the semiclassical errors and the principal parts. Note that *all semiclassical errors for W_ε do not exceed those obtained for W we selected*. Consider approximations errors in the principal part, namely

$$\mathsf{D}\bigl(P'(W_\varepsilon+\nu)-P'(W+\nu),\, P'(W_\varepsilon+\nu)-P'(W+\nu)\bigr) \tag{26.7.2}$$

and

$$\mathsf{D}(\rho_\varepsilon-\rho,\,\rho_\varepsilon-\rho) \tag{26.7.3}$$

since we already estimated terms $\mathsf{D}\bigl(P'(W+\nu)-\rho,\, P'(W+\nu)-\rho\bigr)$ and $\mathsf{D}(\rho_\Psi-\rho,\,\rho_\Psi-\rho)$ by Q.

Note that while since now $\rho=\rho_B^{\mathsf{TF}}$ is only $\mathscr{C}^{\frac12}$ function (in the corresponding scale) and estimate (25.5.31) fails, estimate (25.5.32) survives after minor modification:

$$|W-W_\varepsilon| \le C(1+\ell\varepsilon^{-1})^{-2}\zeta^2 \tag{26.7.4}$$

with the corresponding scaling function ζ and then

$$\begin{aligned}|P'(W_\varepsilon+\nu)-P'(W+\nu)|&\\ \le C(1+\ell\varepsilon^{-1})^{-2}\zeta^3 + C(1+\ell\varepsilon^{-1})^{-1}\zeta B&\end{aligned} \tag{26.7.5}$$

and therefore expression (26.7.2) does not exceed $C\bigl(Z^3\varepsilon^2+ZB^2\varepsilon^2\bar r^2\bigr)$ and it does not exceed $C\max(Z^{\frac53}, B^{\frac45}Z^{\frac35})$ for

$$\varepsilon = \min(Z^{-\frac23},\, Z^{\frac25}B^{-\frac45}) \tag{26.7.6}$$

and this does not exceed $C\bar{Q}$, where as before $\bar{Q} = \mathsf{max}(Q,\, Z^{\frac{5}{3}})$.

Further, consider expression (26.7.3); it is equal to $|(W_\varepsilon - W, \rho_\varepsilon - \rho)|$ and one can prove easily the same estimate for it:

$$\mathsf{D}(\rho_\varepsilon - \rho,\, \rho_\varepsilon - \rho) \le C\bar{Q}. \tag{26.7.7}$$

Furthermore, under this restriction we estimate an error in the principal part of asymptotics of $\int e(x,x,\lambda)\,dx$

$$|\int \big(P'(W_\varepsilon + \nu) - P'(W+\nu)\big)\,dx| \le C\big(Z^{\frac{3}{2}}\varepsilon^{\frac{3}{2}} + Z^{\frac{1}{2}}B\varepsilon\bar{r}^{\frac{3}{2}}\big), \tag{26.7.8}$$

which is less than the semiclassical error.

Then $S \coloneqq S_\varepsilon \le C\bar{Q}$ with S_ε defined by (25.5.22). So, the following proposition is proven:

Proposition 26.7.2 [42]**.** *If θ, χ are as in Subsection 25.5.1, then estimate* (25.5.34) *holds, namely,*

$$\mathcal{J} = |\int \Big(\rho^{(2)}_\Psi(x,y) - \rho(y)\rho_\Psi(x)\Big)\theta(x)\chi(x,y)\,dxdy| \le \tag{26.7.9}$$

$$C\sup_x \|\nabla_y \chi_x\|_{\mathscr{L}^2(\mathbb{R}^3)}\Big((\bar{Q} + \varepsilon^{-1}N + T)^{\frac{1}{2}}\Theta + R^{\frac{1}{2}}\Theta^{\frac{1}{2}}\Big) + C\varepsilon N\|\nabla_y\chi\|_{\mathscr{L}^\infty}\Theta$$

with $\Theta = \Theta_\Psi$ defined by (25.5.15) *and T and R defined by* (25.5.23) *and* (25.5.25) *respectively with ε, defined by* (26.7.7)*.*

Recall that $\rho^{(2)}_\Psi(x,y)$ defined by (25.5.13) is the *quantum correlation function*. Observe that the threshold value again is $B \asymp Z^{\frac{4}{3}}$.

26.7.2 Excessive Negative Charge

We are going to prove the following theorem, similar to Theorem 25.5.3:

Theorem 26.7.3. *Assume that $\mathsf{I}_N > 0$. Then in the fixed nuclei model*

(i) For $B \le Z^{\frac{20}{21}}$ the following estimate holds:

$$N - Z \le CZ^{\frac{5}{7}}. \tag{25.5.37}$$

[42] Cf. Proposition 25.5.1.

(ii) If $Z^{\frac{20}{21}} \le B \le Z^3$ *then*

$$N - Z \le C(\bar{r}Q)^{\frac{1}{2}}. \tag{26.7.10}$$

In particular for $M = 1$

$$N - Z \le C \begin{cases} B^{-\frac{1}{8}} Z^{\frac{5}{6}} & as \quad Z^{\frac{20}{21}} \le B \le Z^{\frac{4}{3}}, \\ B^{\frac{1}{5}} Z^{\frac{2}{5}} & as \quad Z^{\frac{4}{3}} \le B \le Z^3 \end{cases} \tag{26.7.11}$$

and for $M \ge 2$

$$N - Z \le C \begin{cases} B^{-\frac{1}{8}} Z^{\frac{5}{6}} + B^{\frac{1}{2}} L & as \quad Z^{\frac{20}{21}} \le B \le Z^{\frac{4}{3}}, \\ B^{\frac{1}{5}} Z^{\frac{2}{5}} L & as \quad Z^{\frac{4}{23}} \le B \le Z^3 \end{cases} \tag{26.7.12}$$

with $L = |\log(BZ^{-3})|$.

Proof. Let us follow the proof of Theorem 25.5.3 with the necessary modifications:

(a) Let us select $\theta = \theta_{(b)}$ according to (25.5.38):

$$\begin{aligned} &\operatorname{supp}(\theta) \subset \{x : \ \ell(x) \ge b\}, \qquad \operatorname{supp}(1-\theta) \subset \{x : \ \ell(x) \le 2b\}, \\ &0 \le \theta \le 1, \qquad |\nabla \theta^{\frac{1}{2}}| + |\nabla (1-\theta)^{\frac{1}{2}}| \le Cb^{-1}. \end{aligned} \tag{25.5.38}$$

Note that again equation $\mathsf{H}_N \Psi = \mathsf{E}_N \Psi$ yields identity (25.5.39), where we pick up again according to (25.5.40) $\varphi = ZV(x)^{-1} \asymp \ell(x)$ and isolating the contribution of j-th electron in j-th term we get inequality (25.5.41):

$$-\mathsf{I}_N \int \rho_\Psi(x) \ell(x) \theta \, dx \ge \tag{25.5.41}$$

$$\sum_j \langle \Psi, \ell(x_j)\theta(x_j) \Big(-V(x_j) + \sum_{k:k\ne j} |x_j - x_k|^{-1} \Big) \Psi \rangle - \sum_j \|\nabla \big(\theta^{\frac{1}{2}}(x_j) \ell(x_j)^{\frac{1}{2}} \big) \Psi\|^2$$

now due to the non-negativity of operator $\big((D_x - A(x)) \cdot \boldsymbol{\sigma}\big)^2$.

Now let us again select b to be able to calculate the magnitude of Θ_Ψ. Note that inequality (25.5.42) holds. Then repeating our arguments, leading to (25.5.46) we see that there are two cases:

(i) $B \lesssim Z^{\frac{20}{21}}$ which is equivalent to $\bar{Q}^{\frac{1}{7}} \lesssim \bar{r}$ in which case we take $b = \epsilon \bar{Q}^{\frac{1}{7}} = \epsilon Z^{\frac{5}{21}}$ and then (25.5.46) holds. Then we can apply all arguments of the proof of Theorem 25.5.3 and arrive to the same estimate; thus we arrive to Statement (i).

(ii) $Z^{\frac{20}{21}} \lesssim B \le Z^3$ which is equivalent to $\bar{Q}^{\frac{1}{7}} \lesssim \bar{r}$ in which case we take $b = C_0 \bar{r}$ and then $\Theta_\Psi \asymp (N - Z)_+$ provided (26.7.10) with large enough constant C is violated. We need to prove that it is incompatible with $\mathsf{I}_N > 0$. We will do it below.

(b) So we take $b = C_0 \bar{r}$ and then, assuming that (26.7.10) with large enough constant C fails, we have $\Theta_\Psi \asymp (N - Z)_+ > 0$ and

$$\rho_B^{\mathsf{TF}}(x) = W_B^{\mathsf{TF}}(x) = 0 \qquad \text{on} \quad \mathsf{supp}(\theta_0) \tag{26.7.13}$$

where θ_0 and θ_1 satisfy (25.5.51) and $\varphi(x) = ZV(x)^{-1}$ again.

Then the proof goes even simpler: while arriving to (25.5.53)

$$\begin{aligned} \iint \rho_\Psi^{(2)}(x,y)\varphi(x)|x-y|^{-1}\theta(x)(1-\theta(y))\,dxdy & \\ \ge \iint \rho_\Psi^{(2)}(x,y)\varphi(x)|x-y|^{-1}\theta(x)\theta_1(y)\,dxdy & \\ \ge \iint \rho_\Psi(x)\rho(y)\varphi(x)|x-y|^{-1}\theta(x)\theta_1(y)\,dxdy & \\ - C\bigl(\beta^{\frac12}\bar{Q}^{\frac12} + \beta^{-1}Z^{\frac13}\bigr)\Theta_\Psi & \end{aligned} \tag{25.5.53}$$

due to (26.7.13) we can simply skip $\theta_1(y)$ in the right-hand expression and observe that $\int \rho(y)\varphi(x)|x-y|^{-1} = V(x)$.

Thus we arrive to the following statement, similar to (25.5.55):

(26.7.14) The first line in the right-hand expression of (25.5.50) does not exceed $\bigl(Z - N + Cb^{\frac12}\bar{Q}^{\frac12} Cb^{-1}Z^{\frac13}\bigr)\Theta_\Psi$.

(c) Then continuing arguments of Part (c) of the proof of Theorem 25.5.3 we arrive to estimate

$$N - Z \le C(b\bar{Q})^{\frac12} + Cb^{-1}Z^{\frac13} \tag{26.7.15}$$

which differs from (25.5.59) by the last term $Cb^{-1}Z^{\frac13}$ instead of Cb^{-3}. With $b = C_0\bar{r}$ the second term is smaller and we arrive to (26.7.11).

(d) Recalling that $\bar{Q} = \max(Z^{\frac{5}{3}}, B^{\frac{4}{5}}Z^{\frac{3}{5}})$ for $M = 1$ and $\bar{Q} = \max(Z^{\frac{5}{3}} + B, B^{\frac{4}{5}}Z^{\frac{3}{5}})$ for $M = 1$ we arrive to (26.7.11); on the other hand, $\bar{Q} = \max(Z^{\frac{5}{3}} + B^{\frac{5}{4}}L^2, B^{\frac{4}{5}}Z^{\frac{3}{5}}L^2)$. for $M \ge 2$ and we arrive to (26.7.12). □

26.7.3 Estimate for Ionization Energy

Let us estimate the ionization energy, assuming that

(26.7.16) $(Z - N)_+$ does not exceed $C(b\bar{Q})^{\frac{1}{2}}$ with b defined in the proof of Theorem 26.7.3 [43].

Assume, however, first that $\nu = 0$ (in particular it happens as $N \ge Z$). Few cases are possible:

(i) $B \le Z^{\frac{20}{21}}$ and $(Z - N)_+ \le C_0 Z^{\frac{5}{7}}$. In this case we act exactly as in Subsection 25.5.2: we pick up $b = \epsilon Z^{-\frac{5}{21}}$ with a small enough constant $\epsilon' > 0$; then

$$|\int \theta_b(x)\big(\rho_\Psi - \rho\big)\,dx| \le Cb^{\frac{1}{2}}Q^{\frac{1}{2}}, \tag{26.7.17}$$

while

$$\int \theta_b(x)\rho\,dx \asymp b^{-3} \tag{26.7.18}$$

and therefore

$$\Theta := \int \theta_b(x)\rho_\Psi\,dx \asymp b^{-3} \tag{26.7.19}$$

and

$$|\int \theta(x)\big(\rho_\Psi - \rho\big)\,dx| \le \epsilon''\Theta. \tag{26.7.20}$$

Then all corresponding estimates of Subsection 25.5.3 yield that $\mathsf{I}_N \le CZ^{\frac{20}{21}}$.

In all other cases one needs to replace θ_b by a function which is not b-admissible, exactly like we did in Section 25.6.

(ii) Let $Z^{\frac{20}{21}} \le B \le Z^3$ and $M = 1$. Let here $\bar{r}$ be the exact radius of $\mathsf{supp}(\rho)$, $\rho = \rho_B^{\mathsf{TF}}$ and $W = W_B^{\mathsf{TF}}$, which were obtained in the Thomas-Fermi theory with $\nu = 0$. Recall that $\bar{r} \asymp \max(B^{-\frac{1}{4}}; B^{-\frac{2}{5}}Z^{\frac{1}{5}})$ and $\bar{Q} \asymp \max(Z^{\frac{5}{3}}; B^{\frac{4}{5}}Z^{\frac{3}{5}})$.

[43] That is (25.5.37), or (26.7.20) in the framework of the corresponding Statement.

Also recall that $W \asymp Gt^4$ and $\rho \asymp BG^{\frac12}$ for $r = (1-t)\bar r$ with $1-\epsilon \le t \le 1$, where here and below

$$G \coloneqq \min(B;\, B^{\frac25} Z^{\frac45}). \tag{26.7.21}$$

We take in this case $\bar r t$-admissible function θ, equal 0 for $|x-y| \le \bar r(1-t)$ and equal 1 for $|x-y| \ge \bar r(1-\frac12 t)$. Then

(26.7.22) In all the above estimates one needs to replace $Cb^{-1}\Theta_{b(1-\epsilon)}$ by $C\bar r^{-1}t^{-1}\Theta'$ with Θ' defined by θ' which is also $\bar r t$-admissible and equal 1 in $\epsilon \bar r t$-vicinity of $\operatorname{supp}(\theta)$.

Then estimates (26.7.17)–(26.7.20) are replaced by

$$|\int \theta(x)\big(\rho_\Psi - \rho\big)\,dx| \le CQ^{\frac12} \times \|\nabla\theta\| \asymp Ct^{-\frac12}\bar r^{\frac12} Q^{\frac12} \tag{26.7.23}$$

while

$$\int \theta(x)\rho\,dx \asymp BG^{\frac12}\bar r^3 t^3 \tag{26.7.24}$$

and therefore

$$\Theta \coloneqq \int \theta(x)\rho_\Psi\,dx \asymp BG^{\frac12}\bar r^3 t^3. \tag{26.7.25}$$

Therefore (26.7.20) holds provided the right-hand expression of (26.7.23) does not exceed the right-hand expression of (26.7.25), multiplied by ϵ:

$$t = t_* \coloneqq C_0 B^{-\frac27} G^{-\frac17} \bar r^{-\frac57} Q^{\frac17} = C_1 \max(B^{-\frac14} Z^{\frac{5}{21}};\, B^{\frac{2}{35}} Z^{-\frac{6}{35}}) \tag{26.7.26}$$

where we picked up the smallest possible value of t. Note that

(26.7.27) $t \asymp 1$ as either $B \asymp Z^{\frac{20}{21}}$ or $B \asymp Z^3$.

Further, let us estimate from above

$$\begin{aligned} \mathcal{I}' = -\int \Big(\rho_\Psi^{(2)}(x,y) - \rho_\Psi(x)\rho(y)\Big)\ell(x)|x-y|^{-1}\theta(x)\,dxdy \le \\ -\int \Big(\rho_\Psi^{(2)}(x,y) - \rho_\Psi(x)\rho(y)\Big)\big(1-\omega_\tau(x,y)\big)\ell(x)|x-y|^{-1}\theta(x)\,dxdy + \\ \int \rho_\Psi(x)\rho(y)\omega_\tau(x,y)\ell(x)|x-y|^{-1}\theta(x)\,dxdy \end{aligned} \tag{26.7.28}$$

with $\omega_\tau(x,y)=0$ as $|x-y|\ge 2\tau\bar r$ and $\omega_\tau(x,y)=1$ as $|x-y|\le \tau\bar r$, with $\tau\in(t,1)$.

Then due to Proposition 26.7.2 with

$$\chi(x,y)=\bigl(1-\omega_\tau(x,y)\bigr)|x-y|^{-1}$$

the first term in the right-hand expression does not exceed $C\bar r^{\frac12}\tau^{-\frac12}\bar Q^{\frac12}\Theta$ since $\|\nabla_y\chi_x\|_{\mathscr{L}^2(\mathbb{R}^3)}\asymp(\bar r\tau)^{-\frac12}$ and also one can prove easily that all other terms in $\Bigl(\bigl(Q+\varepsilon^{-1}N+T\bigr)^{\frac12}\Theta+R^{\frac12}\Theta^{\frac12}\Bigr)$ do not exceed $CQ^{\frac12}\Theta$.

Meanwhile, the second term in in the right-hand expression of (26.7.28) does not exceed $CBG^{\frac12}\tau^2\times\bar r^3\tau^2\times\Theta$ because $\rho(y)\le CBG^{\frac12}\tau^2$ if $|x-y|\le 2\tau\bar r$, $x\in\mathsf{supp}(\theta)$, and therefore $\int\rho(y)\omega_\tau(x,y)\,dy\le CBG^{\frac12}\tau^4$.

Minimizing their sum

$$C\Bigl(\bar r^{\frac12}\tau^{-\frac12}Q^{\frac12}+BG^{\frac12}\bar r^3\tau^4\Bigr)\Theta$$

with respect to $\tau\ge t$ [44)], we arrive to estimate

$$\mathscr{I}'\le C\bar r^{\frac79}Q^{\frac49}B^{\frac19}G^{\frac{1}{18}}\Theta.$$

Then exactly as in the proof of Theorem 25.5.4 we have estimate

$$\bar r\,\mathsf{I}_N\le C(Z-N)_+ + C\bar r^{\frac79}Q^{\frac49}B^{\frac19}G^{\frac{1}{18}}, \tag{26.7.29}$$

and therefore we arrive to the estimate

$$\mathsf{I}_N\le C\bar r^{-1}(Z-N)_+ + C\bar r^{-\frac29}Q^{\frac49}B^{\frac19}G^{\frac{1}{18}}, \tag{26.7.30}$$

which implies (26.7.32) under assumptions (26.7.31) and $N\ge Z$.

Furthermore, estimate (26.7.34) under assumptions (26.7.33) and $N\ge Z$ is due to the fact that for $B\le Z$ one can upgrade $Q=Z^{\frac53}$ to $Q=Z^{\frac53}\bigl(B^\delta Z^{-\delta}+Z^{-\delta}\bigr)$.

Theorem 26.7.4. *Let* $M=1$ *and assumption* (25.3.33) *be fulfilled.*

[44)] One can see easily that minimum is achieved as $\tau\asymp t^{\frac79}$.

(i) Then for $B \le Z^3$ and

$$(Z-N)_+ \le C_0 \begin{cases} Z^{\frac{5}{7}} & \text{if } B \le Z^{\frac{20}{21}}, \\ B^{-\frac{1}{8}} Z^{\frac{5}{6}} & \text{if } Z^{\frac{20}{21}} \le B \le Z^{\frac{4}{3}}, \\ B^{\frac{1}{5}} Z^{\frac{2}{5}} & \text{if } Z^{\frac{4}{3}} \le B \le Z^3 \end{cases} \tag{26.7.31}$$

the following estimate holds

$$\mathsf{I}_N \le C \begin{cases} Z^{\frac{20}{21}}, & \\ B^{\frac{2}{9}} Z^{\frac{20}{27}} & \text{if } Z^{\frac{20}{21}} \le B \le Z^{\frac{4}{3}}, \\ B^{\frac{26}{45}} Z^{\frac{4}{15}} & \text{if } Z^{\frac{4}{3}} \le B \le Z^3. \end{cases} \tag{26.7.32}$$

(ii) Furthermore for $B \le Z$ under assumption (26.7.31)

$$(Z-N)_+ \le C_0 \begin{cases} Z^{\frac{5}{7}-\delta} & \text{if } B \le Z^{\frac{20}{21}}, \\ B^{-\frac{1}{8}+\delta} Z^{\frac{5}{6}-\delta} & \text{if } Z^{\frac{20}{21}} \le B \le Z \end{cases} \tag{26.7.33}$$

the following estimate holds

$$\mathsf{I}_N \le C \begin{cases} Z^{\frac{20}{21}-\delta'} & \text{if } B \le Z^{\frac{20}{21}}, \\ B^{\frac{2}{9}+\delta'} Z^{\frac{20}{27}-\delta'} & \text{if } Z^{\frac{20}{21}} \le B \le Z. \end{cases} \tag{26.7.34}$$

Proof in the general settings. To prove estimates (26.7.32) and (26.7.34) in the general settings (i.e. without assumption $N \ge Z$) observe that for $N < Z$

$$\begin{aligned} \mathsf{D}(\rho_N^{\mathsf{TF}} - \rho_Z^{\mathsf{TF}},\, \rho_N^{\mathsf{TF}} - \rho_Z^{\mathsf{TF}}) &\le C(Z-N)^2 \bar{r}^{-1} \asymp \\ & C \max\bigl((Z-N)^{\frac{7}{3}};\, C(Z-N)^2 B^{\frac{1}{4}};\, C(Z-N)^2 B^{\frac{2}{5}} Z^{-\frac{1}{5}}\bigr) \end{aligned} \tag{26.7.35}$$

(where subscript here denotes the number of electrons rather than the intensity of the magnetic field) because the same estimate holds for $\mathcal{E}_N^{\mathsf{TF}} - \mathcal{E}_Z^{\mathsf{TF}}$:

$$0 \le \mathcal{E}_N^{\mathsf{TF}} - \mathcal{E}_Z^{\mathsf{TF}} \le C(Z-N)^2 \bar{r}^{-1}, \tag{26.7.36}$$

which itself follows from

$$\frac{\partial \mathcal{E}^{\mathsf{TF}}}{\partial N} = \nu \asymp (Z-N)\bar{r}^{-1}. \tag{26.7.37}$$

Therefore to preserve our estimates we need to assume that the right-hand expression of (26.7.35) does not exceed Q; this assumption is equivalent to $(Z-N)_+ \le \min(Z^{\frac{5}{7}}; Z^{\frac{5}{6}}B^{-\frac{1}{8}})$ for $B \le Z^{\frac{4}{3}}$ which is exactly the first and the second cases in (26.7.31) (and these cases in (26.7.33) appear in the same way), and to $(Z-N)_+ \le CB^{\frac{1}{5}}Z^{\frac{2}{5}}$ for $Z^{\frac{4}{3}} \le B \le Z^3$, which is exactly the third case in (26.7.35).

Also there is a term $C(Z-N)_+\bar{r}^{-1}$ in the estimate of I_N. However, under assumption (26.7.35) this term does not exceed the right hand expression of (26.7.35) or (26.7.37), in fact coincides with it only in the first case. □

Consider now $M \ge 2$. Assume that $B \ge Z^{\frac{20}{21}}$ since the opposite case has been analyzed already.

Let us pick up $\bar{r}t$-admissible function θ such that $\theta = 1$ if $W \le C_0Gt^4$ and $\theta = 0$ if $W \ge 2C_0Gt^4$. In this case ($M \ge 2$) we can claim only that $\|\nabla\theta\| \le Ct^{-\frac{1}{2}}\bar{r}^{\frac{1}{2}}|\log t|^{\frac{1}{2}}$ and therefore we arrive to the following modifications of (26.7.23)–(26.7.26):

$$|\int \theta(x)\big(\rho_\Psi - \rho\big)\,dx| \le Ct^{-\frac{1}{2}}|\log t|^{\frac{1}{2}}\bar{r}^{\frac{1}{2}}\bar{Q}^{\frac{1}{2}}, \tag{26.7.23)'}$$

while

$$BG^{\frac{1}{2}}\bar{r}^3t^3 \lesssim \int \theta(x)\rho\,dx \lesssim BG^{\frac{1}{2}}|\log t|\bar{r}^3t^3 \tag{26.7.24)'}$$

and therefore

$$\Theta := \int \theta(x)\rho_\Psi\,dx \gtrsim BG^{\frac{1}{2}}\bar{r}^3t^3 \tag{26.7.25)'}$$

for

$$t \ge t_* := C_0B^{-\frac{2}{7}}G^{-\frac{1}{7}}\bar{r}^{-\frac{5}{7}}\bar{Q}^{\frac{1}{7}}|\log t|^{\frac{2}{7}}. \tag{26.7.26)'}$$

Now we need to look more carefully at $\bar{Q}$, especially because while it may contain "rogue" factor L or L^2, it can also be large as $(Z-N)_+$ is large. Fortunately, this is not the case in the current framework:

Proposition 26.7.5. *(i) Under condition* (26.7.41) *below* $\bar{Q}$ *is as in the case* $N = Z$ *i.e.*

$$\bar{Q} = \begin{cases} Z^{\frac{5}{3}} + B^{\frac{5}{4}}L^2 & \text{if } B \le Z^{\frac{4}{3}}, \\ B^{\frac{4}{5}}Z^{\frac{3}{5}}L^2 & \text{if } Z^{\frac{4}{3}} \le B \le Z^3. \end{cases} \tag{26.7.38}$$

(ii) Furthermore, if $B \le Z$ *and* $a \ge Z^{-\frac{1}{3}}$ *under condition* (26.7.43) *below* $\bar{Q}$ *is exactly as in the case* $N = Z$*, i.e.*

$$\bar{Q} = Z^{\frac{5}{3}}\bigl(Z^{-\delta} + (aZ^{\frac{1}{3}})^{-\delta} + (BZ^{-1})^{\delta}\bigr). \tag{26.7.39}$$

Proof. One can either derive it from the existing estimates or just repeat estimates with $\nu = 0$ adding $(Z-N)_+^2 \bar{r}^{-1}$ to $\bar{Q}$. We leave easy details to the reader.

Therefore all the above arguments could be repeated with this new expression $\bar{Q}$ which also acquires factor $|\log t|$ (due to this factor in the estimate of $\|\nabla\theta\|$ and this factor boils to $L_1^{\frac{1}{2}}$ with

$$L_1 = \begin{cases} |\log BZ^{-\frac{20}{21}}| + 1 & Z^{\frac{20}{21}} \le B \le Z^{\frac{4}{3}}, \\ |\log BZ^{-3}| + 1 & Z^{\frac{4}{3}} \le B \le Z^3. \end{cases} \tag{26.7.40}$$

Therefore we arrive to

Theorem 26.7.6. *Let* $M \ge 2$*. Then*

(i) For

$$(Z-N)_+ \le C_0 \begin{cases} Z^{\frac{5}{7}} & \text{if } B \le Z^{\frac{20}{21}}, \\ Z^{\frac{5}{6}} B^{-\frac{1}{8}} + B^{\frac{1}{2}} L & \text{if } Z^{\frac{20}{21}} \le B \le Z^{\frac{4}{3}}, \\ B^{\frac{1}{5}} Z^{\frac{2}{5}} L & \text{if } Z^{\frac{4}{3}} \le B \le Z^3 \end{cases} \tag{26.7.41}$$

the following estimate holds

$$\mathsf{I}_N \le C L_1^{\frac{2}{9}} \begin{cases} Z^{\frac{20}{21}} & \text{if } B \le Z^{\frac{20}{21}}, \\ Z^{\frac{20}{27}} B^{\frac{2}{9}} + B^{\frac{7}{9}} L^{\frac{8}{9}} & \text{if } Z^{\frac{20}{21}} \le B \le Z^{\frac{4}{3}} \\ Z^{\frac{4}{15}} B^{\frac{26}{45}} L^{\frac{8}{9}} & \text{if } Z^{\frac{4}{3}} \le B \le Z^3. \end{cases} \tag{26.7.42}$$

(ii) Furthermore, for $B \le Z$*,* $a \ge Z^{-\frac{1}{3}}$ *and*

$$(Z-N)_+ \le C_0 \varsigma^{\delta} \begin{cases} Z^{\frac{5}{7}} & \text{if } B \le Z^{\frac{20}{21}}, \\ Z^{\frac{5}{6}} B^{-\frac{1}{8}} & \text{if } Z^{\frac{20}{21}} \le B \le Z, \end{cases} \tag{26.7.43}$$

with

$$\varsigma = Z^{-1} + BZ^{-1} + a^{-1}Z^{\frac{1}{3}} \tag{26.7.44}$$

the following estimate holds

$$\mathsf{I}_N \le CL_1^{\frac{2}{9}}\varsigma^{\delta'} \begin{cases} Z^{\frac{20}{21}} & \text{if } B \le Z^{\frac{20}{21}}, \\ Z^{\frac{20}{27}}B^{\frac{2}{9}} & \text{if } Z^{\frac{20}{21}} \le B \le Z. \end{cases} \tag{26.7.45}$$

26.8 Positively Charged Systems

Now let us estimate from above and below the ionization energy in the case when $N < Z$ and condition (26.7.31) (if $M = 1$) or (26.7.41) (if $M \ge 2$) fails. We also estimate excessive the positive charge in the case of $M \ge 2$ and free nuclei model. We will follow arguments of the corresponding three subsections of Section 25.6.

26.8.1 Upper Estimate for Ionization Energy: $M = 1$

Consider first the case of $M = 1$. Then for $B = 0$ arguments are well-known (see Section 25.6) but we repeat them for $B > 0$: we pick up β-admissible function θ such that $\theta = 1$ if $|x - \mathsf{y}_1| \ge \bar{r} - \beta$ and $\theta = 0$ if $|x - \mathsf{y}_1| \le \bar{r} - 2\beta$ where $\bar{r}$ is an exact radius of support of ρ^{TF} (see the very beginning of Subsection 25.6.1) and $\beta \ll \bar{r}$. Recall that

$$\bar{r} \asymp \begin{cases} (Z-N)^{-\frac{1}{3}} & \text{if } B \le Z^{\frac{20}{21}}, \\ \min\bigl((Z-N)^{-\frac{1}{3}}, B^{-\frac{1}{4}}\bigr) & \text{if } Z^{\frac{20}{21}} \le B \le Z^{\frac{4}{3}}, \\ B^{-\frac{2}{5}}Z^{\frac{1}{5}} & \text{if } Z^{\frac{4}{3}} \le B \le Z^{3}, \end{cases} \tag{26.8.1}$$

where in the first case we used that $Z - N \ge Z^{\frac{5}{7}}$ while in the second case both subcases $(Z-N)^{-\frac{1}{3}} \gtrless B^{-\frac{1}{4}}$ are possible.

We can assume without any loss of the generality that $\mathsf{y}_1 = 0$. Now in the spirit of Subsection 25.6.1 we need to select as we did in Subsection 26.7.3 the smallest β such that

$$\Theta^{\mathsf{TF}} := \int \theta(x)\rho^{\mathsf{TF}}(x)\,dx \ge C\beta^{-\frac{1}{2}}\bar{r}\bar{Q}^{\frac{1}{2}} \tag{26.8.2}$$

implying that

$$\Theta_\Psi \coloneqq \int \theta(x)\rho_\Psi(x)\,dx \asymp \Theta^{\mathsf{TF}}, \tag{26.8.3}$$

where the right-hand expression of (26.8.2) estimates $|\int \theta(x)(\rho^{\mathsf{TF}}-\rho_\Psi)\,dx|$ (recall that it does not exceed $\|\nabla\theta\|\cdot \mathsf{D}(\rho^{\mathsf{TF}}-\rho_\Psi, \rho^{\mathsf{TF}}-\rho_\Psi)^{\frac12}$). Again as in Subsection 25.6.1 $\rho^{\mathsf{TF}}=\rho^{\mathsf{TF}}_N$ is calculated for the actual value of $N<Z$.

Then, following Subsubsection 25.6.1, eventually we arrive to estimate (25.6.10), namely:

$$\begin{aligned}\mathsf{I}_N \int \ell(x)\rho_\Psi(x)\theta(x)\,dx \le{}& \int \theta(x)V(x)\ell(x)\rho_\Psi(x)\,dx\\ &- \int \Big(\rho^{(2)}_\Psi(x,y)-\rho_\Psi(x)\rho(y)\Big)\ell(x)|x-y|^{-1}\theta(x)\,dxdy\\ &- \int \rho_\Psi(x)\rho(y)\ell(x)|x-y|^{-1}\theta(x)\,dxdy + C\beta^{-2}\bar r\Theta,\end{aligned} \tag{26.8.4}$$

and then estimate from above the second term in the right-hand expression

$$\begin{aligned}&- \int \Big(\rho^{(2)}_\Psi(x,y)-\rho_\Psi(x)\rho(y)\Big)\ell(x)|x-y|^{-1}\theta(x)\,dxdy \le\\ &\quad - \int \Big(\rho^{(2)}_\Psi(x,y)-\rho_\Psi(x)\rho(y)\Big)\big(1-\omega(x,y)\big)\ell(x)|x-y|^{-1}\theta(x)\,dxdy\\ &\qquad + \int \rho_\Psi(x)\rho(y)\omega(x,y)\ell(x)|x-y|^{-1}\theta(x)\,dxdy\end{aligned} \tag{26.8.5}$$

with $\omega=\omega_\gamma$: $\omega=0$ if $|x-y|\ge 2\gamma$ and $\omega=1$ if $|x-y|\le\gamma$, $\gamma\ge\beta$ (see (25.6.12)). To estimate the first term in the right-hand expression of (26.8.5) one can apply Proposition 25.5.1. In this case $\|\nabla_y\chi\|_{\mathscr{L}^2}\asymp C\bar r\gamma^{-\frac12}$, $\|\nabla_y\chi\|_{\mathscr{L}^\infty}\asymp \bar r\gamma^{-2}$ and plugging $R=\beta^{-2}\Theta$ and $T=|\nu|$, $\varepsilon=Z^{-\frac23}$ we conclude that this term does not exceed (25.6.13)

$$C\bar r\big(\gamma^{-\frac12}Q^{\frac12}+Z^{\frac13}\gamma^{-2}\big)\Theta \tag{26.8.6}$$

(if $Q\ge Z^{\frac53}$; otherwise here we should reset here $Q\coloneqq Z^{\frac53}$).

Note that if $0\le \bar r-|x|\asymp\beta$ then

$$W+\nu\asymp \upsilon \coloneqq \max\Big\{\big(\frac{|\nu|\beta}{\bar r}\big);\ G\big(\frac{\beta}{\bar r}\big)^4\Big\}, \tag{26.8.7}$$

with G defined by (26.2.48) and therefore

$$\rho \asymp \max\Big\{\big(\frac{|\nu|\beta}{\bar r}\big)^{\frac32};\ B\big(\frac{|\nu|\beta}{\bar r}\big)^{\frac12};\ BG^{\frac12}\big(\frac{\beta}{\bar r}\big)^2\Big\} \tag{26.8.8}$$

where the first and the second clauses are forks of the first clause in (26.8.7) since in the second clause automatically $W+\nu \le B$ for $0 \le \bar r - |x| \lesssim \beta$; therefore

$$\int \rho(x)\theta(x)\,dx \asymp \max\Big\{\big(\frac{|\nu|\beta}{\bar r}\big)^{\frac32};\ \ B\big(\frac{|\nu|\beta}{\bar r}\big)^{\frac12};\ BG^{\frac12}\big(\frac{\beta}{\bar r}\big)^2\Big\}\beta\bar r^2, \tag{26.8.9}$$

and therefore (26.8.2) holds if and only if

$$\max\Big\{\big(\frac{|\nu|}{\bar r}\big)^{\frac32}\beta^3;\ \ B\big(\frac{|\nu|}{\bar r}\big)^{\frac12}\beta^2;\ BG^{\frac12}\big(\frac{1}{\bar r}\big)^2\beta^{\frac72}\Big\}\bar r \ge CQ^{\frac12}; \tag{26.8.10}$$

then

$$\beta = \min\Big\{Q^{\frac16}|\nu|^{-\frac12}\bar r^{\frac16};\ B^{-\frac12}Q^{\frac14}|\nu|^{-\frac14}\bar r^{-\frac14};\ B^{-\frac27}G^{-\frac17}Q^{\frac17}\bar r^{\frac27}\Big\} \tag{26.8.11}$$

and in the corresponding cases

$$\upsilon = \Big\{Q^{\frac16}|\nu|^{\frac12}\bar r^{-\frac56};\ B^{-\frac12}Q^{\frac14}|\nu|^{\frac34}\bar r^{-\frac54};\ B^{-\frac87}G^{\frac37}Q^{\frac47}\bar r^{-\frac{20}{7}}\Big\}. \tag{26.8.12}$$

Observe, however, that for $B \lesssim Q^{\frac47}$ and $|\nu| \lesssim Q^{\frac47}$ we do not need these arguments; simpler arguments of Subsection 25.5.3 show that in this case $|\mathsf{I}_N| \le CQ^{\frac47}$.

On the other hand, for $B \lesssim Q^{\frac47}$ but $|\nu| \gtrsim Q^{\frac47}$, we pick $\gamma = Q^{\frac18}|\nu|^{-\frac{15}{32}}$, like in Subsection 25.6.1, and observe that $|\nu|\bar r^{-1}\gamma \gtrsim B$ and therefore we conclude that $\mathsf{I}_N + \nu \le CQ^{\frac16}|\nu|^{\frac{17}{24}}$, exactly like in that subsection. Therefore we arrive to

Proposition 26.8.1. *Let $B \le C_0 Z^{\frac{20}{21}}$. Then*

(i) If $|\nu| \le C_0 Z^{\frac{20}{21}}$, then estimate $\mathsf{I}_N \le CZ^{\frac{20}{21}}$ holds like in the case $B=0$.

(ii) If $|\nu| \ge C_0 Z^{\frac{20}{21}}$, then estimate $\mathsf{I}_N + \nu \le CZ^{\frac{5}{18}}|\nu|^{\frac{17}{24}}$ holds like in the case $B=0$.

Therefore in what follows we assume that $B \ge Q^{\frac47}$. One can see easily that then $\beta \le \bar r$.

Meanwhile, the same arguments imply that the second term in the right-hand expression of (26.8.5) is of magnitude

$$\mathsf{max}\Big\{\big(\frac{|\nu|\gamma}{\bar{r}}\big)^{\frac{3}{2}};\ \ B\big(\frac{|\nu|\gamma}{\bar{r}}\big)^{\frac{1}{2}};\ BG^{\frac{1}{2}}\big(\frac{\gamma}{\bar{r}}\big)^{2}\Big\}\gamma^{2}$$

and we need to minimize

$$\gamma^{-\frac{1}{2}}Q^{\frac{1}{2}}+\mathsf{max}\Big\{\big(\frac{|\nu|\gamma}{\bar{r}}\big)^{\frac{3}{2}},\ \ B\big(\frac{|\nu|\gamma}{\bar{r}}\big)^{\frac{1}{2}};\ BG^{\frac{1}{2}}\big(\frac{\gamma}{\bar{r}}\big)^{2}\Big\}\gamma^{2},$$

which is achieved when

$$\gamma^{-\frac{1}{2}}Q^{\frac{1}{2}}\asymp \mathsf{max}\Big\{\big(\frac{|\nu|\gamma}{\bar{r}}\big)^{\frac{3}{2}};\ \ B\big(\frac{|\nu|\gamma}{\bar{r}}\big)^{\frac{1}{2}};\ BG^{\frac{1}{2}}\big(\frac{\gamma}{\bar{r}}\big)^{2}\Big\}\gamma^{2}.$$

Let us compare this equation with equation to β. It is the same albeit with factor $\bar{r}^2$ rather than γ^2. Therefore if $\gamma \ge \bar{r}$ then $\gamma \le \beta \le \bar{r}$ which is a contradiction. Thus $\gamma \le \bar{r}$ but then $\gamma \ge \beta$.

Therefore we conclude that this term does not exceed

$$\varsigma := \mathsf{max}\Big\{Q^{\frac{7}{16}}\big(\frac{|\nu|}{\bar{r}}\big)^{\frac{3}{16}};\ Q^{\frac{5}{12}}B^{\frac{1}{6}}\big(\frac{|\nu|}{\bar{r}}\big)^{\frac{1}{12}};\ Q^{\frac{4}{9}}B^{\frac{1}{9}}G^{\frac{1}{18}}\bar{r}^{-\frac{2}{9}}\Big\}, \tag{26.8.13}$$

and to estimate $\mathsf{I}_N+\nu$ we need just to compute its sum with υ defined by (26.8.12).

Therefore we conclude that

$$\mathsf{I}_N+\nu \le C(\upsilon+\varsigma). \tag{26.8.14}$$

Remark 26.8.2. Observe that

$$\upsilon(Z,B,|\nu|)=Z^{\frac{20}{21}}\upsilon(1,Z^{-\frac{20}{21}}B,|\nu|Z^{-\frac{20}{21}}|\nu|) \qquad \text{if } \ Z^{\frac{20}{21}}\le B\le Z^{\frac{4}{3}} \tag{26.8.15}$$

and

$$\upsilon(Z,BZ^{-3},|\nu|)=Z^{2}\upsilon(1,Z^{-3}B,|\nu|Z^{-2}|\nu|) \quad \text{if } \ Z^{\frac{4}{3}}\le B\le Z^{3}, \tag{26.8.16}$$

and ς has the same scaling properties.

Therefore we can make all calculations with $Z=1$ and then scale. Leaving easy calculations to the reader, we arrive to

Proposition 26.8.3. *(i) For* $Z^{\frac{20}{21}} \le B \le Z^{\frac{4}{3}}$

$$I_N + \nu \le C \begin{cases} Z^{\frac{5}{18}}|\nu|^{\frac{17}{24}} & \text{if } |\nu| \ge Z^{-\frac{20}{51}} B^{\frac{24}{17}}, \\ Z^{\frac{5}{12}} B^{-\frac{1}{2}} |\nu|^{\frac{17}{16}} & \text{if } B \le |\nu| \le Z^{-\frac{20}{51}} B^{\frac{24}{17}}, \\ Z^{\frac{5}{48}} B^{-\frac{3}{16}} |\nu|^{\frac{3}{4}} & \text{if } Z^{\frac{5}{12}} B^{\frac{9}{16}} \le |\nu| \le B, \\ Z^{\frac{25}{36}} B^{\frac{3}{16}} |\nu|^{\frac{1}{12}} & \text{if } Z^{\frac{5}{9}} B^{\frac{5}{12}} \le |\nu| \le Z^{\frac{5}{9}} B^{\frac{9}{16}}, \\ Z^{\frac{20}{27}} B^{\frac{2}{9}} & \text{if } |\nu| \le Z^{\frac{5}{9}} B^{\frac{5}{12}}. \end{cases} \tag{26.8.17}$$

(ii) In particular,

$$I_N \le C Z^{\frac{20}{27}} B^{\frac{2}{9}} \qquad \text{if } |\nu| \le Z^{\frac{20}{27}} B^{\frac{2}{9}}. \tag{26.8.18}$$

(iii) For $Z^{\frac{4}{3}} \le B \le Z^3$

$$I_N + \nu \le C \begin{cases} Z^{\frac{7}{30}} B^{\frac{8}{15}} |\nu|^{\frac{1}{12}} & \text{if } |\nu| \ge Z^{\frac{2}{5}} B^{\frac{8}{15}}, \\ Z^{\frac{4}{15}} B^{\frac{26}{45}} & \text{if } |\nu| \le Z^{\frac{2}{5}} B^{\frac{8}{15}}. \end{cases} \tag{26.8.19}$$

(iv) In particular,

$$I_N \le C Z^{\frac{4}{15}} B^{\frac{26}{45}} \qquad \text{if } |\nu| \le Z^{\frac{4}{15}} B^{\frac{26}{45}}. \tag{26.8.20}$$

Remark 26.8.4. Recall that $Q = Z^{\frac{5}{3}}\bigl(B^{\delta}+1\bigr)Z^{-\delta}$ if $B \le Z$; therefore we can add factor $\bigl(B^{\delta'}+1\bigr)Z^{-\delta'}$ in all estimates of Propositions 26.8.1 and 26.8.3.

26.8.2 Lower Estimate for Ionization Energy: $M = 1$

Now let us derive an estimate $I_N + \nu$ from below. Let $\Psi = \Psi_N(x_1, \dots, x_N)$ be the ground state for N electrons, $\|\Psi\| = 1$; consider an antisymmetric *test function*

$$\tilde{\Psi} = \tilde{\Psi}(x_1, \dots, x_{N+1}) = \Psi(x_1, \dots, x_N)u(x_{n+1}) - \sum_{1 \le j \le N} \Psi(x_1, \dots, x_{j-1}, x_{N+1}, x_{j+1}, \dots, x_N)u(x_j) \tag{26.8.21}$$

Then exactly as in Subsection 25.6.2

$$\mathsf{E}_{N+1}\|\tilde\Psi\|^2 \le \langle \mathsf{H}_{N+1}\tilde\Psi,\tilde\Psi\rangle = N\langle \mathsf{H}_{N+1}\Psi u,\tilde\Psi\rangle = N\langle \mathsf{H}_N\Psi u,\tilde\Psi\rangle + N\langle H_{V,x_{N+1}}\Psi u,\tilde\Psi\rangle + N\langle \sum_{1\le i\le N}|x_i-x_{N+1}|^{-1}\Psi u,\tilde\Psi\rangle = (\mathsf{E}_N-\nu')\|\tilde\Psi\|^2 + N\langle H_{W+\nu',x_{N+1}}\Psi u,\tilde\Psi\rangle + N\langle\big(\sum_{1\le i\le N}|x_i-x_{N+1}|^{-1}-(V-W)(x_{N+1})\big)\Psi u,\tilde\Psi\rangle$$

and therefore

$$N^{-1}(\mathsf{I}_{N+1}+\nu')\|\tilde\Psi\|^2 \ge -\langle H_{W+\nu',x_{N+1}}\Psi u,\tilde\Psi\rangle - \langle\big(\sum_{1\le i\le N}|x_i-x_{N+1}|^{-1}-(V-W)(x_{N+1})\big)\Psi u,\tilde\Psi\rangle \tag{26.8.22}$$

and

$$N^{-1}\|\tilde\Psi\|^2 = \|\Psi\|^2\cdot\|u\|^2 - N\int \Psi(x_1,\dots,x_{N-1},x)\Psi^\dagger(x_1,\dots,x_{N-1},y)u(y)u^\dagger(x)\,dx_1\cdots dx_{N-1}\,dxdy \tag{26.8.23}$$

as in (25.6.19) and (25.6.20) respectively where † means a complex or Hermitian conjugation and $\nu'\ge\nu$ to be chosen later.

Note that every term in the right-hand expression in (26.8.22) is the sum of two terms: one with $\tilde\Psi$ replaced by $\Psi(x_1,\dots,x_N)u(x_{N+1})$ and another with $\tilde\Psi$ replaced by $-N\Psi(x_1,\dots,x_{N-1},x_{N+1})u(x_N)$. We call these terms, as in Subsection 25.6.2, *direct* and *indirect* respectively.

Obviously, in the direct and indirect terms u appears as $|u(x)|^2\,dx$ and as $u(x)u^\dagger(y)\,dxdy$ respectively multiplied by some kernels.

Recall that u is an arbitrary function. Let us take $u(x)=\theta^{\frac12}(x)\phi_j(x)$ where ϕ_j are orthonormal eigenfunctions of $H_{W+\nu}$ and $\theta(x)$ is β-admissible function which is supported in $\{x:\ -\upsilon\ge W(x)+\nu\ge\frac23\nu\}$ and equal 1 in $\{x:\ -2\upsilon\ge W(x)+\nu\ge\frac12\nu\}$, satisfying (25.5.11), and υ is related to β as in the previous Section 26.7:

$$\upsilon = C\max(\nu\bar r^{-1}\beta;\ G\bar r^{-4}\beta^4). \tag{26.8.24}$$

Let us substitute it into (26.8.22), multiply by $\varphi(\lambda_j L^{-1})$ and take the sum with respect to j; then we get the same expressions with $|u(x)|^2\,dx$ and $u(x)u^\dagger(y)\,dxdy$ replaced by $F(x,x)\,dx$ and $F(x,y)\,dxdy$ respectively with

$$F(x,y)=\int \varphi(\lambda L^{-1})\,d_\lambda e(x,y,\lambda). \tag{26.8.25}$$

Here $\varphi(\tau)$ is a fixed $\mathscr{C}^\infty$ non-negative function equal to 1 for $\tau\le \frac{1}{2}$ and equal to 0 for $\tau\ge 1$ and $L=\nu'-\nu=6\upsilon$.

Under described construction and procedures the direct term generated by $N^{-1}\|\tilde\Psi\|^2$ is

$$\int \theta(x)\varphi(\lambda L^{-1})\,d_\lambda e(x,x,\lambda)\,dx. \tag{26.8.26}$$

Then, applying semiclassical approximation, we get

$$\Theta_\Psi := \int \varphi(\lambda L^{-1})\,d_\lambda P'_B(W+\nu-\lambda)\,dx. \tag{26.8.27}$$

Consider the remainder estimate. Assume that $M=1$ (case $M\ge 2$ will be considered later). Then since $L=C_1\upsilon$ the remainder does not exceed

$$Ch^s(\mu h+1)\beta^{-2}\bar r^2, \tag{26.8.28}$$

where

$$h=1/(\upsilon^{\frac{1}{2}}\beta) \tag{26.8.29}$$

and

$$\mu=B\beta\upsilon^{-\frac{1}{2}}; \tag{26.8.30}$$

one can prove it easily by partition of unity on $\mathsf{supp}(\theta)$ and applying semiclassical asymptotics with effective semiclassical parameter h and magnetic parameter μ.

On the other hand, the indirect term generated by $N^{-1}\|\tilde\Psi\|^2$ is

$$-N\int \theta^{\frac{1}{2}}(x)\theta^{\frac{1}{2}}(y)\Psi(x_1,\dots,x_{N-1},x)\Psi^\dagger(x_1,\dots,x_{N-1},y)\times F(x,y)\,dxdydx_1\cdots dx_{N-1}, \tag{26.8.31}$$

and since the operator norm of $F(.,.,.)$ is 1, the absolute value of this term does not exceed

$$N\int \theta(x)|\Psi(x_1,\dots,x_{N-1},x)|^2\,dx = \int \theta(x)\rho_\Psi(x)\,dx \le \int \theta(x)\rho^{\mathsf{TF}}(x)\,dx + CQ^{\frac12}\|\nabla\theta^{\frac12}\| \tag{26.8.32}$$

where $\rho^{\mathsf{TF}}=0$ on $\mathsf{supp}(\theta)$ and $\|\nabla\theta^{\frac12}\| \asymp \beta^{-\frac12}\bar r$.

Recall that $P'(W^{\mathsf{TF}}+\nu)=\rho^{\mathsf{TF}}$. We will take $\nu'=\nu+L$ to keep Θ_Ψ larger than all the remainders including those due to replacement W by W^{TF} and ρ by ρ^{TF} in the expression above. One can observe easily that then β should satisfy (26.8.10); let us define β and then υ by (26.8.11) and (26.8.12) respectively. Then

$$\Theta_\Psi \asymp \bigl(\upsilon^{\frac32}+B\upsilon^{\frac12}\bigr)\beta\bar r^2. \tag{26.8.33}$$

Therefore

(26.8.34) Let $h\le\epsilon_0$ (i.e. $\upsilon^{\frac12}\beta\ge C_0$), and β,υ be defined by (26.8.11) and (26.8.12) respectively. Then expression (26.8.33) is larger than $C_0\beta^{-\frac12}Q^{\frac12}$ and the total expression generated by $N^{-1}\|\tilde\Psi\|^2$ is greater than $\epsilon\Theta$ with $\Theta=\Theta_\Psi$ defined by (26.8.33).

Now let us consider the direct terms in the right-hand expression of (26.8.22). The first of them is like in (25.6.28)

$$\begin{aligned} &-\int \theta^{\frac12}(x)\varphi(\lambda L^{-1})\,d_\lambda\bigl(H_{W+\nu',x}\theta^{\frac12}(x)e(x,y,\lambda)\bigr)_{y=x}\,dx = \\ &\qquad -\int \theta(x)\varphi(\lambda L^{-1})\,d_\lambda\bigl(H_{W+\nu',x}e(x,y,\lambda)\bigr)_{y=x}\,dx \\ &\qquad\qquad -\frac12\int \varphi(\lambda L^{-1})[[H_W,\theta^{\frac12}],\theta^{\frac12}]\,d_\lambda e(x,x,\lambda) \ge \\ &\int \theta(x)(\nu'-\nu-\lambda)\varphi(\lambda L^{-1})\,d_\lambda e(x,x,\lambda)\,dx - C\int |\nabla\theta^{\frac12}|^2 e(x,x,\nu')dx. \end{aligned} \tag{26.8.35}$$

Observe that the absolute value of last term in the right-hand expression of (26.8.35) does not exceed $C\beta^{-1}\bar r^2\bigl(\upsilon^{\frac32}+B\upsilon^{\frac12}\bigr)\asymp\beta^{-2}\Theta$.

The second direct term in the right-hand expression of (26.8.22) is like in (25.6.29)

$$\begin{aligned}(26.8.36)\quad &-\int \theta(x)\Big(\rho_\Psi * |x|^{-1} - (V-W)(x)\Big)F(x,x)\,dx = \\ &\qquad -\mathsf{D}\big(\rho_\Psi - \bar\rho, \theta(x)F(x,x)\big) \ge \\ &-C\mathsf{D}\big(\rho_\Psi-\rho,\rho_\Psi-\rho\big)^{\frac12}\cdot \mathsf{D}\Big(\theta^{\frac12}F(x,x),\theta^{\frac12}F(x,x)\big)\Big)^{\frac12} \ge -CQ^{\frac12}\bar r^{-\frac12}\Theta,\end{aligned}$$

provided $V-W = |x|^{-1}*\rho$ with $\mathsf{D}(\rho-\rho^{\mathsf{TF}},\rho-\rho^{\mathsf{TF}})\le CQ$.

Further, the first indirect term in the right-hand expression of (26.8.22) is like in (25.6.30)

$$\begin{aligned}(26.8.37)\quad &-N\int \theta^{\frac12}(y)\Psi(x_1,\dots,x_{N-1},x)\Psi^\dagger(x_1,\dots,x_{N-1},y)\times \\ &\qquad \varphi(\lambda L^{-1})\,d_\lambda\big(H_{W+\nu',x}\theta^{\frac12}(x)e(x,y,\lambda)\big)\,dxdydx_1\cdots dx_{N-1} = \\ &-N\int \theta^{\frac12}(y)\theta^{\frac12}(x)\Psi(x_1,\dots,x_{N-1},x)\Psi^\dagger(x_1,\dots,x_{N-1},y)\times \\ &\qquad \varphi(\lambda L^{-1})(\nu'-\nu-\lambda)\,d_\lambda e(x,y,\lambda)\,dxdydx_1\cdots dx_{N-1} \\ &-N\int \theta^{\frac12}(y)\Psi(x_1,\dots,x_{N-1},x)\Psi^\dagger(x_1,\dots,x_{N-1},y)\times \\ &\qquad \varphi(\lambda L^{-1})[H_{W,x},\theta^{\frac12}(x)]\,d_\lambda e(x,y,\lambda)\,dxdydx_1\cdots dx_{N-1}.\end{aligned}$$

Observe that one can rewrite the sum of the first terms in the right-hand expressions in (26.8.35) and (26.8.37) as $\sum_j \varphi(\lambda_j L^{-1})(\nu'-\nu-\lambda_j)\|\widehat\Psi_j\|^2$ with

$$\widehat\Psi_j(x_1,\dots,x_{N-1}) \coloneqq \int \Psi(x_1,\dots,x_{N-1},x)\theta^{\frac12}(x)\phi_j(x)\,dx$$

and therefore this sum is non-negative.

One can see easily that the absolute value of the second term in the right-hand expression of (26.8.37) does not exceed

$$\int \rho_\Psi(y)\theta^{\frac12}(y)\,dy \times \beta^{-1}\int \theta_1(x)e(x,x,\nu')\,dx \asymp C\Theta\times C\big(\upsilon^{\frac32}+B\upsilon\big)\bar r^2 \asymp C\beta^{-\frac32}\bar r Q^{\frac12}\Theta$$

due the choice of β. This is larger than the absolute value of the right-hand expression in (26.8.36). Therefore (cf. 25.6.31) we conclude that

(26.8.38) The sum of the first direct and indirect terms in the right-hand expression of (26.8.22) is greater than $-C\beta^{-\frac{3}{2}}\bar{r}Q^{\frac{1}{2}}\Theta$.

Finally, we need to consider the second indirect term generated by the right-hand expression of (26.8.22):

$$
\begin{aligned}
(26.8.39)\quad & -\int \Big(\sum_{1\le i\le N} |y-x_i|^{-1} - (V-W)(y)\Big)\times \\
& \Psi(x_1,\dots,x_N)\Psi^{\dagger}(x_1,\dots,x_{N-1},y)\theta^{\frac{1}{2}}(x_N)\theta^{\frac{1}{2}}(y)F(x_N,y)\,dx_1\cdots dx_N dy = \\
& -\int \Big(|y|^{-1} * \varrho_{\underline{x}}(y) - (V-W)(y)\Big)\Psi(x_1,\dots,x_N)\Psi^{\dagger}(x_1,\dots,x_{N-1},y)\times \\
& \qquad \theta^{\frac{1}{2}}(x_N)\theta^{\frac{1}{2}}(y)F(x_N,y)\,dx_1\cdots dx_N dy \\
& -\int \Big(\sum_{1\le i\le N} |y-x_i|^{-1} - |y|^{-1} * \varrho_{\underline{x}}(y)\Big)\Psi(x_1,\dots,x_N)\Psi^{\dagger}(x_1,\dots,x_{N-1},y)\times \\
& \qquad \theta^{\frac{1}{2}}(x_N)\theta^{\frac{1}{2}}(y)F(x_N,y)\,dx_1\cdots\, dx_N dy;
\end{aligned}
$$

recall that $\varrho_{\underline{x}}$ is a smeared density, $\underline{x} = (x_1,\dots,x_N)$.

Since $|y|^{-1} * \varrho_{\underline{x}}(y) - (V-W)(y) = |y|^{-1} * (\varrho_{\underline{x}} - \rho)$, the first term in the right-hand expression is equal to

$$
\begin{aligned}
(26.8.40)\quad & \int \theta^{\frac{1}{2}}(x_N)\Psi(x_1,\dots,x_N)\times \\
& \mathsf{D}_y\Big(\varrho_{\underline{x}}(y)-\rho(y), F(x_N,y,\lambda)\theta^{\frac{1}{2}}(y)\Psi(x_1,\dots,x_{N-1},y)\Big)\,dx_1\cdots dx_N
\end{aligned}
$$

and its absolute value does not exceed

$$
\begin{aligned}
(26.8.41)\quad & \Big(N\int \mathsf{D}\big(\varrho_{\underline{x}}(\cdot)-\rho(\cdot), \varrho_{\underline{x}}(\cdot)-\rho(\cdot)\big)|\Psi(x_1,\dots,x_N)|^2\theta(x_N)\,dx_1\cdots dx_N\Big)^{\frac{1}{2}}\times \\
& N^{-\frac{1}{2}}\Big(\mathsf{D}_y\Big(F(x_N,y,\lambda)\theta^{\frac{1}{2}}(y)\Psi(x_1,\dots,x_{N-1},y), \\
& \qquad F(x_N,y,\lambda)\theta^{\frac{1}{2}}(y)\Psi(x_1,\dots,x_{N-1},y)\Big)\,dx_1\cdots dx_N\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Recall that the first factor is equivalently defined by (25.5.4) and therefore due to estimate (25.5.24) it does not exceed $\bigl((Q+T+\varepsilon^{-1}N)\Theta+R\bigr)^{\frac12}$, where we assume that $\varepsilon\le Z^{-\frac23}$ and $\Theta\asymp \beta\bigl(\upsilon^{\frac32}+B\upsilon^{\frac12}\bigr)\bar r^2\beta \asymp \beta^{-\frac12}\bar r Q^{\frac12}$ is now an upper estimate for $\int \theta(y)\rho_\Psi(y)\,dy$-like expressions.

Then, according to (25.5.25), $R\asymp C\beta^{-2}\Theta \ll Q\Theta$ and, according to (25.5.23), $T\ll Q$ and therefore in all such inequalities we may skip R and T terms; so we get $C(Q+\varepsilon^{-1}N)^{\frac12}\Theta^{\frac12}$.

Meanwhile, the second factor in (26.8.41) (without square root) is equal to

$$N^{-1}\int L^{-2}\varphi'(\lambda L^{-1})\varphi'(\lambda' L^{-1})|y-z|^{-1}\underbrace{e(x_N,y,\lambda)}\,\theta^{\frac12}(y)\Psi(x_1,\dots,x_{N-1},y)\times \underbrace{e(x_N,z,\lambda')}\,\theta^{\frac12}(z)\Psi^\dagger(x_1,\dots,x_{N-1},z)\,dydz\,dx_1\cdots dx_{N-1}\underbrace{dx_N}\,d\lambda d\lambda';$$

after integration with respect to x_N we get instead of the marked terms $e(y,z,\lambda)$ (recall that $e(.,.,.)$ is the Schwartz kernel of the projector and we keep $\lambda<\lambda'$) and then, integrating with respect to λ' we arrive to

$$N^{-1}\int |y-z|^{-1}F(y,z)\theta^{\frac12}(y)\Psi(x_1,\dots,x_{N-1},y)\times \theta^{\frac12}(z)\Psi^\dagger(x_1,\dots,x_{N-1},z)\,dydz\,dx_1\cdots dx_{N-1},$$

where now F is defined by (26.8.25) albeit with φ^2 instead of φ. This latter expression does not exceed

$$N^{-1}\iint |y-z|^{-1}|F(y,z)|\theta^{\frac12}(y)|\Psi(x_1,\dots,x_{N-1},y)|^2\times dydz\,dx_1\cdots dx_{N-1}. \tag{26.8.42}$$

Then due to Proposition 26.A.6 expression $\int |y-z|^{-1}|F(y,z)|\,dz$ does not exceed $C\beta^{-1}(h^{-1}+\mu)\asymp \upsilon^{\frac12}+B\upsilon^{-\frac12}$, and thus expression (26.8.42) does not exceed $CZ^{-2}\bigl(\upsilon^{\frac12}+B\upsilon^{-\frac12}\bigr)\Theta$. Therefore the second factor in (26.8.41) does not exceed $CN^{-1}\bigl(\upsilon^{\frac14}+B^{\frac12}\upsilon^{-\frac14}\bigr)\Theta^{\frac12}$ and the whole expression (26.8.41) does not exceed

$$C(Q+\varepsilon^{-1}N)^{\frac12}\Theta^{\frac12}\times N^{-1}\bigl(\upsilon^{\frac14}+B^{\frac12}\upsilon^{-\frac14}\bigr)\Theta^{\frac12}= CN^{-1}(Q+\varepsilon^{-1}N)^{\frac12}\bigl(\upsilon^{\frac14}+B^{\frac12}\upsilon^{-\frac14}\bigr)\Theta.$$

Finally we arrive to

Proposition 26.8.5 [45]**.** *Let*

$$\upsilon \ge \max\bigl(Z^{-\frac{4}{3}}Q^{\frac{2}{3}};\, Z^{-\frac{4}{5}}Q^{\frac{2}{5}}B^{\frac{2}{5}}\bigr) \tag{26.8.43}$$

and

$$\varepsilon \ge Z^{-1}\max(\upsilon^{-\frac{3}{2}}, B\upsilon^{-\frac{5}{2}}). \tag{26.8.44}$$

Then the first term in the right-hand expression of (26.8.39) *does not exceed* $C\upsilon\Theta$.

Further, we need to estimate the second term in the right-hand expression of (26.8.39). It can be rewritten in the form

$$\sum_{1\le i\le N}\int U(x_i,y)\Psi(x_1,\dots,x_N)\Psi^\dagger(x_1,\dots,x_{N-1},y)\theta^{\frac{1}{2}}(x_N)\theta^{\frac{1}{2}}(y)\times F(x_N,y)\,dx_1\cdots dx_N dy, \tag{26.8.45}$$

where $U(x_i,y)$ is the difference between two potentials, one generated by the charge $\delta(x-x_i)$ and another by the same charge smeared; note that $U(x_i,y)$ is supported in $\{(x_i,y)\colon |x_i-y|\le\varepsilon\}$. Let us estimate the i-th term in this sum with $i<N$ first. Multiplied by $N(N-1)$, it does not exceed

$$N\biggl(\int |U(x_i,y)|^2|\Psi(x_1,\dots,x_N)|^2\theta^{\frac{1}{2}}(x_N)\theta^{\frac{1}{2}}(y)|F(x_N,y)|\,dx_1\cdots dx_N dy\biggr)^{\frac{1}{2}}\times N\biggl(\int \omega(x_i,y)|\Psi(x_1,\dots,x_{N-1},y)|^2\theta^{\frac{1}{2}}(x_N)\theta^{\frac{1}{2}}(y)|F(x_N,y)|\,dx_1\cdots dx_N dy\biggr)^{\frac{1}{2}} \tag{26.8.46}$$

here ω is ε-admissible and supported in $\{(x_i,y)\colon |x_i-y|\le 2\varepsilon\}$ function.

Due to Proposition 26.A.6 in the second factor

$$\int \theta^{\frac{1}{2}}(x_N)|F(x_N,y)|\,dx_N \le C(1+\mu h) \asymp C(1+B\upsilon^{-1})$$

and therefore the whole second factor does not exceed

$$C\Bigl(\int \theta^{\frac{1}{2}}(x)\omega(x,y)\varrho^{(2)}_\Psi(x,y)\,dxdy\Bigr)^{\frac{1}{2}}(1+B\upsilon^{-1})^{\frac{1}{2}}, \tag{26.8.47}$$

[45] Cf. claim (25.6.36).

where we replaced x_i by x. According to Proposition 25.5.1 in the selected expression one can replace $\rho^{(2)}_\Psi(x,y)$ by $\rho_\Psi(x)\rho(y)$, with an error which does not exceed

$$C\Big(\sup_x \|\nabla_y \chi_x\|_{\mathscr{L}^2(\mathbb{R}^3)}\big(Q+\varepsilon^{-1}N\big)^{\frac12} + C\varepsilon N\|\nabla_y\chi\|_{\mathscr{L}^\infty}\Big)\Theta.$$

When we plug $\sup_x \|\nabla_y\chi_x\|_{\mathscr{L}^2(\mathbb{R}^3)} \asymp \varepsilon^{\frac12}$, $\|\nabla_y\chi\|_{\mathscr{L}^\infty}\asymp \varepsilon^{-1}$ this expression becomes $CN\Theta$.

Meanwhile, consider

$$\int |U(x_i,y)|^2\theta^{\frac12}(y)|F(x_N,y)|\,dy. \tag{26.8.48}$$

Again, due to Proposition 26.A.6, it does not exceed

$$C\big(\upsilon^{\frac32}+B\upsilon^{\frac12}\big)\int |U(x_i,y)|^2\theta^{\frac12}(y)\big(|x_N-y|\upsilon^{\frac12}+1\big)^{-s}\,dy$$

and this integral should be taken over $B(x_i,\varepsilon)$, with $|U(x_i,y)|\le |x_i-y|^{-1}$, so (26.8.48) does not exceed

$$C\varepsilon\big(\upsilon^{\frac32}+B\upsilon^{\frac12}\big)\omega'(x_i,x_N)$$

with $\omega'(x,y)=\big(1+\upsilon^{\frac12}|x-y|\big)^{-s}$ (provided $\varepsilon\le \upsilon^{-\frac12}$ which will be the case). Therefore the first factor in (26.8.46) does not exceed

$$C\varepsilon^{\frac12}\big(\upsilon^{\frac34}+B^{\frac12}\upsilon^{\frac14}\big)\Big(\underbrace{\int \theta^{\frac12}(x)\omega'(x,y)\rho^{(2)}_\Psi(x,y)\,dxdy}\Big)^{\frac12}. \tag{26.8.49}$$

Therefore in the selected expression one can replace $\rho^{(2)}_\Psi(x,y)$ by $\rho_\Psi(x)\rho(y)$ with an error which does not exceed what we got before but with ε replaced by $\upsilon^{-\frac12}$, i.e. also $CN\Theta$.

However, in both selected expressions, (26.8.47) and (26.8.49), replacing $\rho^{(2)}_\Psi(x,y)$ by $\rho_\Psi(x)\rho(y)$ we get just 0. Therefore expression (26.8.46) does not exceed $C\varepsilon^{\frac12}\big(\upsilon^{\frac34}+B^{\frac12}\upsilon^{\frac14}\big)Z\Theta$, which, in turn, does not exceed $C\upsilon\Theta$ provided $\varepsilon\le C\upsilon^{\frac12}\big(1+B\upsilon^{-1}\big)^{-1}Z^{-2}$.

So, we have two restriction to ε from above: the last one and $\varepsilon\le Z^{-\frac23}$ and one can see easily that both of them are compatible with restriction to ε in (26.8.43); also we can see easily that condition (26.8.43) is weaker than

$\upsilon \ge \{Z^{\frac{20}{21}} : Z^{\frac{20}{27}}B^{\frac{2}{9}} : Z^{\frac{4}{15}}B^{\frac{26}{45}}\}$ if $\{B \le Z^{\frac{20}{21}};\ Z^{\frac{20}{21}} \le B \le Z^{\frac{4}{3}};\ Z^{\frac{4}{3}} \le B \le Z^{3}\}$ respectively.

Finally, consider term in (26.8.45) with $i = N$ (multiplied by N):

$$N \int U(x_N, y)|\Psi(x_1, \dots, x_N)|^2 \theta^{\frac{1}{2}}(x_N)\theta^{\frac{1}{2}}(y)F(x_N, y)\, dx_1 \cdots dx_N dy \tag{26.8.50}$$

due to Cauchy inequality it does not exceed

$$N\Big(\int |x_N - y|^{-2}|\Psi(x_1, \dots, x_N)|^2 \theta^{\frac{1}{2}}(x_N)\theta^{\frac{1}{2}}(y)\ dx_1 \cdots dx_N dy\Big)^{\frac{1}{2}} \times N\Big(\int |F(x_N, y)|^2|\Psi(x_1, \dots, x_N)|^2 \theta^{\frac{1}{2}}(x_N)\theta^{\frac{1}{2}}(y)\ dx_1 \cdots dx_N dy\Big)^{\frac{1}{2}} \tag{26.8.51}$$

where both integrals are taken over $\{|x_N - y| \le \varepsilon\}$. Integrating with respect to y there we get that it that it does not exceed

$$C\varepsilon^{\frac{1}{2}}\Theta^{\frac{1}{2}} \times \big(\upsilon^{\frac{3}{4}} + B\upsilon^{\frac{1}{4}}\big)\varepsilon^{\frac{3}{2}}\Theta^{\frac{1}{2}} = C\big(\upsilon^{\frac{3}{4}} + B\upsilon^{\frac{1}{4}}\big)\varepsilon^{2}\Theta \ll \upsilon\Theta.$$

Therefore the right-hand expression in (26.8.22) is $\ge -C\upsilon\Theta$ and recalling that $\nu' - \nu = O(\upsilon)$ we recover a lower estimate $\mathsf{I}_N + \nu \ge -C\upsilon$ in Theorem 26.8.6 below. Here υ must be found from (26.8.11)–(26.8.12) and must satisfy $\upsilon \le |\nu|$.

Combining this estimate with the estimate from the above, derived in Proposition 26.8.3 we arrive to

Theorem 26.8.6. *Let $M = 1$. Let condition* (26.2.28) *be fulfilled. Then*

(i) For $B \le Z^{\frac{20}{21}}$ and $|\nu| \ge Z^{\frac{20}{21}}$

$$|\mathsf{I}_N + \nu| \le CZ^{\frac{5}{18}}|\nu|^{\frac{17}{24}}. \tag{26.8.52}$$

(ii) For $Z^{\frac{20}{21}} \le B \le Z^{\frac{4}{3}}$ and $|\nu| \ge Z^{\frac{20}{27}}B^{\frac{2}{9}}$ estimate (26.8.17) *from above and a similar estimate rom below hold:*

$$\mathsf{I}_N + \nu \ge -C \begin{cases} Z^{\frac{5}{18}}|\nu|^{\frac{17}{24}} & \text{if } \ B \le Z^{\frac{5}{18}}|\nu|^{\frac{17}{24}}, \\ Z^{\frac{5}{12}}B^{-\frac{1}{2}}|\nu|^{\frac{17}{16}} & \text{if } \ Z^{\frac{5}{18}}|\nu|^{\frac{17}{24}} \le B \le |\nu|, \\ Z^{\frac{5}{12}}B^{-\frac{3}{16}}|\nu|^{\frac{3}{4}} & \text{if } \ |\nu| \le B \le Z^{-\frac{20}{7}}|\nu|^{4}, \\ Z^{\frac{20}{21}} & \text{if } \ Z^{-\frac{20}{7}}|\nu|^{4} \le B. \end{cases} \tag{26.8.53}$$

(iii) For $Z^{\frac{20}{21}} \le B \le Z^3$ *and* $|\nu| \ge Z^{\frac{4}{15}} B^{\frac{26}{45}}$ *estimate* (26.8.19) *from above and a similar estimate from below hold:*

$$I_N + \nu \ge -C \begin{cases} Z^{\frac{7}{30}} B^{\frac{8}{15}} |\nu|^{\frac{1}{12}} & \text{if } |\nu| \ge Z^{\frac{2}{5}} B^{\frac{8}{15}}, \\ Z^{\frac{4}{15}} B^{\frac{26}{45}} & \text{if } |\nu| \le Z^{\frac{2}{5}} B^{\frac{8}{15}}. \end{cases} \tag{26.8.54}$$

Remark 26.8.7. Recall that $Q = Z^{\frac{5}{3}}\bigl(B^{\delta}+1\bigr)Z^{-\delta}$ as $B \le Z$; therefore we can add factor $\bigl(B^{\delta'}+1\bigr)Z^{-\delta'}$ in all estimates of Theorem 26.8.6.

26.8.3 Estimates for Ionization Energy: $M \ge 2$

Recall that for $M \ge 2$ we have only estimate (26.6.73):

$$\mathsf{D}(\rho_\Psi - \rho,\, \rho_\Psi - \rho) \le \bar{Q}) := (26.5.40) + (26.6.69).$$

Then exactly the same arguments lead us to the following theorem (we leave all details to the reader):

Theorem 26.8.8. *Let* $M \ge 2$. *Then*

(i) Estimate $\mathsf{I}_N + \nu \le C(\upsilon + \varsigma)$ *holds with* υ *and* ς *defined by* (26.8.11)–(26.8.12) *and* (26.8.13) *albeit with* Q *replaced by* $\bar{Q}$.

(ii) Estimate $\mathsf{I}_N + \nu \ge -C\upsilon$ *holds with* υ *defined by* (26.8.11)–(26.8.12) *albeit with* Q *replaced by* $\bar{Q}$.

Therefore the case when $\bar{Q} \le Q$ is not affected. One can see easily that it happens for sure as $B \le Z^{\frac{20}{17}} L^{-\kappa}$ where $\kappa > 0$ is some exponent.

We leave to the reader

Problem 26.8.9. (i) Find explicit formula for $\upsilon + \varsigma$ and υ.

(ii) Find $\nu^* = \nu^*(Z, B)$ and $\nu_* = \nu_*(Z, B)$ such that $|\nu| \lesssim \upsilon + \varsigma$ iff and only if $\nu \lesssim \nu^*$ and $|\nu| \lesssim \upsilon$ iff and only if $\nu \lesssim \nu_*$.

26.8.4 Free Nuclei Model

In this subsection we consider two extra problems appearing in the free nuclei model–estimate the minimal distance between nuclei and the maximal excessive positive charge when system does not break apart. We also slightly improve estimates for the maximal negative charge and for the ionization energy.

Preliminary Arguments

Recall that we assume that

$$\mathsf{Q} := \widehat{\mathsf{E}} - \sum_{1\le m\le M} \mathsf{E}_m < 0 \tag{26.8.55}$$

where

$$\widehat{\mathsf{E}} = \mathsf{E} + \sum_{1\le m<m'\le M} \frac{Z_m Z_{m'}}{|\mathsf{y}_m - \mathsf{y}_{m'}|}. \tag{26.8.56}$$

We apply estimate from below for $\widehat{\mathsf{E}}$ delivered by Proposition 26.6.1(ii), and estimates from above for E_m, delivered by Theorem 26.6.6(ii); then

$$\widehat{\mathcal{E}}^{\mathsf{TF}} + \mathsf{Scott} - \sum_{1\le m\le M} \Bigl(\mathcal{E}_m^{\mathsf{TF}} - \mathsf{Scott}_m\Bigr) \le CQ + CZ^{\frac{4}{3}}B^{\frac{1}{3}} + Ca^{-\frac{1}{2}}Z^{\frac{3}{2}}$$

or, equivalently, due to equality $\mathsf{Scott} = \sum_{1\le m\le M} \mathsf{Scott}_m$ and non-binding theorem

$$0 \le \mathcal{Q} := \widehat{\mathcal{E}}^{\mathsf{TF}} - \sum_{1\le m\le M} \mathcal{E}_m^{\mathsf{TF}} \le CQ + CZ^{\frac{4}{3}}B^{\frac{1}{3}} + Ca^{-\frac{1}{2}}Z^{\frac{3}{2}}. \tag{26.8.57}$$

Assume that assumption (26.2.28) is fulfilled. Then $\mathcal{Q} \asymp \mathcal{E}^{\mathsf{TF}}$ for $a \le \epsilon r^*$ with $r^* = \min(Z^{-\frac{1}{3}}; B^{-\frac{2}{5}}Z^{\frac{1}{5}})$ and therefore

(26.8.58) In the free nuclei model $a \ge \epsilon r^*$.

Then the last term in (26.8.57) is not needed.

Remark 26.8.10. (i) Obviously, the second term $CZ^{\frac{4}{3}}B^{\frac{1}{3}}$ in the right-hand expression of (26.8.57) matters only if $Z \le B \le Z^{\frac{11}{7}}$; however we will show that it could be skipped even in this case.

(ii) All these estimates hold also for $\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}})$ because this term is present in the estimate from below.

Minimal Distance

We are going to improve (26.8.58). Consider the case $B \le Z^{\frac{4}{3}}$ first. Then since $\mathcal{Q} \ge \epsilon_0 a^{-7}$ for $\epsilon r^* \le a \le \epsilon \bar{r}$ (where in this case $r^* = Z^{-\frac{1}{3}} \le \bar{r} = B^{-\frac{1}{4}}$), we conclude that $a \gtrsim Z^{-\frac{5}{21}}$ provided $B \le Z^{\frac{20}{21}}$. Then we conclude that $a \ge Z^{-\frac{5}{21}}$ as long as it is less than $\epsilon\bar{r}$ and we arrive to Statement (i) of Theorem 26.8.11 below:

Theorem 26.8.11. *Let condition* (26.2.28) *be fulfilled. Then in the free nuclei model*

(i) For $B \le Z^{\frac{20}{21}}$ *the minimal distance satisfies*

$$a \ge \epsilon Z^{-\frac{5}{21}} \tag{26.8.59}$$

(ii) For $Z^{\frac{20}{21}} \le B \le Z^3$ *the distances satisfy*

$$|\mathsf{y}_m - \mathsf{y}_{m'}| \ge \bar{r}_m + \bar{r}_{m'} - C\varepsilon\bar{r} \qquad \forall m \ne m' \tag{26.8.60}$$

with

$$\varepsilon = C(S^{-1}Q)^{\frac{1}{7}} \tag{26.8.61}$$

with S *defined by* (26.2.48) *and* Q *defined by* (26.6.45)*, where* $\bar{r}_m$ *denote the exact radii of* $\mathsf{supp}(\rho_m^{\mathsf{TF}})$*. In particular up to a logarithmic factor*

$$\varepsilon = C \begin{cases} B^{-\frac{1}{4}+\delta} Z^{\frac{5}{21}} & Z^{\frac{5}{21}} \le B \le Z^{\frac{4}{3}}, \\ B^{\frac{2}{35}} Z^{-\frac{6}{35}} & Z^{\frac{4}{3}} \le B \le Z^3. \end{cases} \tag{26.8.62}$$

(iii) In both cases

$$\mathsf{D}(\rho_B^{\mathsf{TF}} - \rho_{B,1}^{\mathsf{TF}} - \dots - \rho_{B,M}^{\mathsf{TF}};\ \rho_B^{\mathsf{TF}} - \rho_{B,1}^{\mathsf{TF}} - \dots - \rho_{B,M}^{\mathsf{TF}}) \le CQ. \tag{26.8.63}$$

Proof. To prove Statement (ii) we will apply Theorem 26.2.18 and to prove Statement (iii) we will apply Theorem 26.2.13 and to do this we need to prove that

$$\mathcal{Q} \coloneqq \widehat{\mathcal{E}}_B^{\mathsf{TF}} - \sum_{1 \le m \le M} \mathcal{E}_{B,m}^{\mathsf{TF}} \le CQ \tag{26.8.64}$$

assuming that

$$\widehat{\mathsf{E}} - \sum_{1 \le m \le M} \mathsf{E}_m \le 0. \tag{26.8.65}$$

(i) We claim first that

$$a \ge \epsilon r^* = \epsilon \min(Z^{-\frac{1}{3}};\ B^{-\frac{1}{4}};\ B^{-\frac{2}{5}} Z^{\frac{1}{5}}). \tag{26.8.66}$$

Indeed, it follows from the fact that $\mathsf{E} \asymp \mathsf{E}_m \asymp \max(Z^{\frac{7}{3}}; B^{\frac{2}{5}} Z^{\frac{9}{5}})$ and if (26.8.65) is violated then $\widehat{\mathsf{E}} \ge \epsilon_1 a^{-1} Z^2$ which would imply $\widehat{E} \ge \epsilon_2 a^{-1} Z^2$ which contradicts (26.8.65).

Then $\mathsf{E} \ge \mathcal{E}^{\mathsf{TF}}_B - CQ - CB^{\frac{1}{3}}Z^{\frac{4}{3}}$ and $\mathsf{E}_m \le \mathcal{E}^{\mathsf{TF}}_{B,m} + CQ + CB^{\frac{1}{3}}Z^{\frac{4}{3}}$ and then (26.8.65) implies that

$$\mathcal{Q} \le CQ + CB^{\frac{1}{3}}Z^{\frac{4}{3}} \tag{26.8.67}$$

which together with (26.8.65) implies

$$a \ge \epsilon \bar{r} = \epsilon \min(Z^{-\frac{5}{21}};\, B^{-\frac{1}{4}};\, B^{-\frac{2}{5}}Z^{\frac{1}{5}}). \tag{26.8.68}$$

(ii) In order to get rid of the last term in the right-hand expression of (26.8.67) (which is less than the first term unless $Z \le B \le Z^{\frac{11}{7}}$) we apply now estimate from below for $\widehat{\mathsf{E}}$ delivered by Proposition 26.6.1(i), and estimates from above for E_m, delivered by Theorem 26.6.6(i); then instead of this term and the difference between Scott correction terms for E and for $\sum_{1\le m\le M} \mathsf{E}_m$ (which is equal to 0 under assumption (26.8.66)) we have

$$\Bigl(\mathsf{Tr}((H_{A,W}-\nu)^-) + \int P_B(W^{\mathsf{TF}}+\nu)\,dx\Bigr) - \sum_{1\le m\le M}\Bigl(\mathsf{Tr}((H_{A,W_m}-\nu')^-) + \int P_B(W^{\mathsf{TF}}+\nu')\,dx\Bigr), \tag{26.8.69}$$

where W and W_m are approximations of W^{TF}_B and $W_{B,m}$ respectively and we know that $\nu' = \nu_1 = ... = \nu_M$.

Let us apply partition of unity $\phi_0 + \phi_1 + ... + \phi_M = 1$ with ϕ_m supported $B(\mathsf{y}_m, \epsilon 2\bar{r})$ in equal 1 in $B(\mathsf{y}_m, \epsilon\bar{r})$ as $m = 1, ..., M$ where we decrease constant $\epsilon > 0$ as needed.

Then our standard methods imply that the absolute values of

$$\mathsf{Tr}\bigl((H_{A,W}-\nu)^-\phi_0\bigr) + \int P_B(W^{\mathsf{TF}}+\nu)\phi_0(x)\,dx, \tag{26.8.70}$$

and

$$\mathsf{Tr}\bigl((H_{A,W_m}-\nu')^-\phi_{m'}\bigr) + \int P_B(W^{\mathsf{TF}}_m+\nu')\phi_{m'}\,dx \tag{26.8.71}$$

with $m = 1, ..., M$, $m' = 0, 1, ..., M$, $m' \ne m$ do not exceed CQ [46]. Therefore we need to estimate an absolute value of

$$\mathsf{Tr}\bigl(\bigl[(H_{A,W}-\nu)^- - (H_{A,W_m}-\nu')^-\bigr]\phi_m\bigr) + \int \bigl(P_B(W^{\mathsf{TF}}+\nu) - P_B(W^{\mathsf{TF}}_m+\nu')\bigr)\phi_m\,dx. \tag{26.8.72}$$

[46] Recall that W and W_m are approximations to W^{TF} and W^{TF}_m.

To do this recall that due to Theorem 26.2.13

$$\mathsf{D}(\rho-\rho_1-...-\rho_M,\, \rho-\rho_1-...-\rho_M)\le C\mathcal{Q}; \tag{26.8.73}$$

one can prove easily that there also

$$|W-W_m|\le CT := C\mathcal{Q}^{\frac12}\bar r^{-\frac12}+C(Z-N)\bar r^{-1} \tag{26.8.74}$$

and, moreover,

$$|\nabla(W-W_m)|\le CT\bar r^{-1}= C\mathcal{Q}^{\frac12}\bar r^{-\frac32}+C(Z-N)\bar r^{-2}, \tag{26.8.75}$$

$$|\nabla^2(W-W_m)|\le CT\bar r^{-2}= C\mathcal{Q}^{\frac12}\bar r^{-\frac52}+C(Z-N)\bar r^{-3}. \tag{26.8.76}$$

Then using our standard methods one can prove easily that an absolute value of expression (26.8.72) with ϕ_m replaced by ℓ-admissible function ψ_m does not exceed

$$CTh^{-2}(1+\mu h) \tag{26.8.77}$$

with our standard

$$h=Z^{-\frac12}r^{-\frac12},\qquad \mu = BZ^{-\frac12}r^{\frac32} \tag{26.8.78}$$

if <u>either</u> $B\le Z^{\frac43}$, $\ell\asymp r\le r^*=Z^{-\frac13}$ <u>or</u> $Z^{\frac43}\le B\le Z^3$, $\ell\asymp r\le \bar r$ and with

$$h=r,\qquad \mu = Br^3 \tag{26.8.79}$$

if $B\le Z^{\frac43}$ and $r^*\lesssim \ell\asymp \bar r$.

Plugging (26.8.78) and (26.8.79) and summing over partition we arrive to $CTZ^{\frac23}$ as $Z^{\frac{20}{21}}\le B\le Z^{\frac43}$ and $CTZ^{\frac25}B^{\frac15}$ as $Z^{\frac43}\le B\le Z^3$.

Plugging $T=(Z-N)\bar r^{-1}$ we get expressions which are much smaller than $\epsilon(Z-N)^2\bar r^{-1}$ due to (26.8.68); plugging $T=\mathcal{Q}^{\frac12}\bar r^{-\frac12}$ we get terms smaller than $\epsilon'\mathcal{Q}+C(\epsilon')B^{\frac14}Z^{\frac43}$ if $B\le Z^{\frac43}$ and $\epsilon'\mathcal{Q}+C(\epsilon')Z^{\frac35}B^{\frac45}$ if $Z^{\frac43}\le B\le Z^3$; here $\epsilon'>0$ is arbitrarily small and thus term (26.8.72) does not make any difference.

Then we arrive to $\mathcal{Q}\le C\mathcal{Q}$ which is Statement (iii). Finally, using Theorem 26.2.18(i) we arrive to Statement (ii). □

Let $B \le Z$ and $a \ge \epsilon \bar{r}$. Then in the proof of Theorem 26.8.11 we can apply improved remainder estimates, namely $Q = (rZ^{\frac{1}{3}})^{-\delta'} = Z^{\frac{5}{3}-\delta}(1+B^{\delta})$ with r defined from $\mu = O(h^{\delta'})$ with $\mu = Br/r^{-2}$, $h = r$, and we conclude that

$$\mathcal{Q} \le CQ = CZ^{\frac{5}{3}-\delta}(1+B^{\delta}) \tag{26.8.80}$$

and we arrive to

Theorem 26.8.12. *Let condition* (26.2.28) *be fulfilled. Then in the free nuclei model*

(i) For $B \le Z^{\frac{20}{21}(1-\delta)}$ *the minimal distance satisfies*

$$a \ge \epsilon Z^{-\frac{5}{21}(1-\delta)} \tag{26.8.81}$$

and

$$\mathsf{D}(\rho_B^{\mathsf{TF}} - \rho_{B,1}^{\mathsf{TF}} - ... - \rho_{B,M}^{\mathsf{TF}};\, \rho_B^{\mathsf{TF}} - \rho_{B,1}^{\mathsf{TF}} - ... - \rho_{B,M}^{\mathsf{TF}}) \le CZ^{\frac{5}{3}(1-\delta)}. \tag{26.8.82}$$

(ii) For $Z^{\frac{20}{21}(1-\delta)} \le B \le Z$ *the distances satisfy* (26.8.60) *with*

$$\varepsilon = B^{-\frac{1}{4}+\frac{1}{7}\delta_1} Z^{\frac{5}{21}-\frac{1}{7}\delta_1} \tag{26.8.83}$$

and

$$\mathsf{D}(\rho_B^{\mathsf{TF}} - \rho_{B,1}^{\mathsf{TF}} - ... - \rho_{B,M}^{\mathsf{TF}};\, \rho_B^{\mathsf{TF}} - \rho_{B,1}^{\mathsf{TF}} - ... - \rho_{B,M}^{\mathsf{TF}}) \le CB^{\delta_1} Z^{\frac{5}{3}-\delta_1}. \tag{26.8.84}$$

Since $\mathcal{Q} \ge \epsilon a^{-1}(Z-N)^2$ we arrive to

Proposition 26.8.13. *Let condition* (26.2.28) *be fulfilled. Then*

(i) If $(Z-N) \ge C(Qa)^{\frac{1}{2}}$ *where* Q *is our remainder estimate in the ground state energy, then in free nuclei model minimal distance between nuclei must be at least* a*.*

(ii) In particular, if $(Z-N) \ge C_1(Q\bar{r})^{\frac{1}{2}}$ *then in free nuclei model minimal distance between nuclei must be at least* $C_0\bar{r}$ *and molecule consists of separate atoms.*

Problem 26.8.14. Can we using the fact that as $B \ge Z$ overlapping of $B(\mathsf{y}_m, \bar{r}_m)$ and $B(\mathsf{y}_{m'}, \bar{r}_{m'})$ is relatively small, improve Q to $\max(Z^{\frac{5}{3}};\, B^{\frac{4}{5}}Z^{\frac{3}{5}})$ that is without factor L^4?

Estimate of Excessive Positive Charge

To estimate excessive positive charge when molecules can still exist in free nuclei model we apply arguments of section 5 of B. Ruskai and J. P. Solovej [1]. In view of Proposition 26.8.13 for $(Z-N)$ violating (26.8.88) below it is sufficient to assume that (25.6.48) is satisfied:

$$a := \min_{1\le m<m'\le M} |\mathsf{y}_m - \mathsf{y}_{m'}| \ge C_0\bar{r}. \tag{26.8.85}$$

i.e. in Thomas-Fermi theory ρ^{TF} is supported in the separate "atoms". Indeed, it is the case if $C_0Z^{\frac{20}{21}} \le B \le Z^3$ but also it is so if $B \le C_0Z^{\frac{20}{21}}$ and $(Z-N)_+ \ge C_1Z^{\frac{5}{7}}$ since then $\bar{r} \asymp (Z-N)_+^{-\frac{1}{3}}$.

Like in Subsection 25.6.3 consider a-admissible functions $\theta_m(x)$, supported in $B(\mathsf{y}_m, \frac{1}{3}a)$ as $m = 1, \dots, M$ and in $\{|x-\mathsf{y}_{m'}| \ge \frac{1}{4}a \ \ \forall m' = 1, \dots, M\}$ as $m = 0$, and satisfying (25.6.49):

$$\theta_0^2 + \dots + \theta_M^2 = 1. \tag{26.8.86}$$

Then for the ground state Ψ equalities (25.6.50)–(25.6.51) hold with *cluster Hamiltonians* H_{α_m} with the *cluster Hamiltonians* H_{α_m}, involving only potential of m-th nucleus (no nucleus potential as $m = 0$) and only electrons belonging to α_m and with the *intercluster Hamiltonian* J_α defined by (25.6.53) and satisfying (25.6.54) with J_{ml} defined by (25.6.55)–(25.6.56). Furthermore, equality (25.6.57) holds.

Applying Proposition 25.5.1 and estimate (25.4.56) (replacing first θ_k with $k = 1, \dots, M$ by $\tilde{\theta}_k$, supported in $B(\mathsf{y}_k, c\bar{r})$, and estimating the resulting error), we conclude that equalities (25.6.58)–(25.6.59) hold with $Y = Q^{\frac{1}{2}}\bar{r}^{\frac{1}{2}}$ since $\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}}) \le CQ$.

The last term in (25.6.57) is estimated by Proposition 25.5.1 and estimate (26.8.73) instead of (25.4.56) and the same replacement trick; so we arrive to (25.6.62) and repeating the same trick we get that it is larger than (25.6.63).

Again let us note that the absolute value of the last term in the right-hand expression of (25.6.57) does not exceed $Ca^{-2}Y$ due to (25.6.59). Now stability condition yields that (26.6.61) must be fulfilled.

Then we conclude that (25.6.65) and (25.6.67) and then (25.6.68) hold with J_{ml} defined by (25.6.55) and (25.6.55) provided (25.6.66) is fulfilled as $|x-\mathsf{y}_m| \ge C\bar{r}$.

This inequality, (26.8.85) and Proposition 25.6.5 (which is the special case of Theorem 26.2.13) yield that

$$Z - N \le C\bar{Y} + C(Z\bar{Y})^{\frac{1}{2}}, \qquad \bar{Y} = (\bar{r}\bar{Q})^{\frac{1}{2}}. \tag{26.8.87}$$

Now we need to consider twocases:

(a) $B \le Z^{\frac{4}{3}}$; then $\bar{r} = \min\big((Z-N)^{-\frac{1}{3}};\, B^{-\frac{1}{4}}\big)$ and we conclude that

$$(Z-N) \le C \min\big(Z^{\frac{11}{13}};\, B^{-\frac{1}{16}} Z^{\frac{11}{12}}\big)$$

with a threshold at $B = Z^{\frac{44}{39}}$.

(b) $Z^{\frac{4}{3}} \le B \le Z^3$; then plugging $\bar{r} = B^{-\frac{2}{5}} Z^{\frac{1}{5}}$ and $\bar{Q} = B^{\frac{4}{5}} Z^{\frac{3}{5}}$ we arrive to $Z - N \le C B^{\frac{1}{10}} Z^{\frac{7}{10}}$.

Then we arrive to the following

Theorem 26.8.15 [47]**.** *Let condition* (26.2.28) *be fulfilled.*

Then in the framework of the free nuclei model with $M \ge 2$ the stable molecule does not exist unless

$$(Z-N) \le C_1 \begin{cases} Z^{\frac{11}{13}} & \text{if } B \le Z^{\frac{44}{39}}, \\ B^{-\frac{1}{16}} Z^{\frac{11}{12}} & \text{if } Z^{\frac{44}{39}} \le B \le Z^{\frac{4}{3}}, \\ B^{\frac{1}{10}} Z^{\frac{7}{10}} & \text{if } Z^{\frac{4}{3}} \le B \le Z^3. \end{cases} \tag{26.8.88}$$

26.A Appendices

26.A.1 Electrostatic Inequalities

There are two kinds of electrostatic inequalities: those which hold for any fermionic state Ψ and those which hold only for the ground-state (or near ground state) Ψ. Inequalities of the first kind do not depend on the quantum Hamiltonian and they are (25.2.1) repeated here:

$$\sum_{1\le j<k\le N} \int |x_j - x_k|^{-1} |\Psi(x_1, \dots, x_N)|^2 \, dx_1 \cdots dx_N \ge \frac{1}{2}\mathsf{D}(\rho_\Psi, \rho_\Psi) - C \int \rho_\Psi^{\frac{4}{3}}(x)\, dx \tag{26.A.1}$$

[47] Cf. Theorem 25.6.3.

and (26.A.5) below.

Inequalities of the second kind are for $B = 0$:

$$\sum_{1\le j<k\le N}\int |x_j - x_k|^{-1}|\Psi(x_1,\dots,x_N)|^2\,dx_1\cdots dx_N \ge \frac{1}{2}\mathsf{D}(\rho_\Psi,\rho_\Psi) - CZ^{\frac{5}{3}}, \tag{26.A.2}$$

and more precise one (26.A.26) below.

For $\vec{B} = \mathsf{const}$ there is an inequality established in E. Lieb, J. P. Solovej and J. Yngvason [3] (p. 122):

Theorem 26.A.1. *Let $\vec{B} = \mathsf{const}$. Then for the ground state Ψ*

$$\int \rho_\Psi^{\frac{4}{3}}\,dx \le CZ^{\frac{5}{6}}N^{\frac{1}{2}}(Z+N)^{\frac{1}{3}}\bigl(1+BZ^{-\frac{4}{3}}\bigr)^{\frac{2}{5}}; \tag{26.A.3}$$

In particular, for $c^{-1}N \le Z \le cN$ the right-hand expression does not exceed

$$CZ^{\frac{5}{3}}\bigl(1+BZ^{-\frac{4}{3}}\bigr)^{\frac{2}{5}} \asymp C\begin{cases} Z^{\frac{5}{3}} & \text{if } B \le Z^{\frac{4}{3}},\\ Z^{\frac{17}{15}}B^{\frac{2}{5}} & \text{if } B \ge Z^{\frac{4}{3}}.\end{cases} \tag{26.A.4}$$

We want to establish inequality, similar to (25.A.2), but in the magnetic case. We will use for this the following

Theorem 26.A.2 [48]**.** *Fix $0 < \delta \le 1/6$. Then for any density matrix F and any density $\rho_0(x) \ge 0$ the following inequality holds*

$$\sum_{1\le j<k\le N}\int |x_j - x_k|^{-1}|\Psi(x_1,\dots,x_N)|^2\,dx_1\cdots dx_N \ge \mathsf{D}(\rho_0,\rho_\gamma) - \frac{1}{2}\mathsf{D}(\rho_0,\rho_0) - \frac{1}{2}\sum_{\varsigma,\varsigma'}\iint |F(x,\varsigma;y,\varsigma')|^2|x-y|^{-1}\,dxdy - C\|\rho\|_{5/3}^{5/6}\cdot\|\rho\|_1^{1/6+\delta}\cdot\upsilon(\gamma,F)^{1/3-\delta}, \tag{26.A.5}$$

where $\rho = \rho_0 + \rho_F + \rho_\Psi$, $\upsilon(\gamma,F) \coloneqq \mathsf{Tr}(\gamma(I-F))$ and

$$\gamma = \gamma_\Psi(x,y) = N\int \Psi(x,x_2,\dots,x_N)\Psi^\dagger(y,x_2,\dots,x_N)\,dx_2\cdots x_N \tag{26.A.6}$$

is two-point one particle density.

[48] Lemma 6 of G. Graf and J. P. Solovej [1].

Recall that $\|.\|_p$ denotes $\mathscr{L}^p$-norm.

There is a connection between (26.A.1) and (26.A.5): if we set $F = 0$, we get $\varsigma = \|\rho\|_1$ and the last term in (26.A.5) becomes $\|\rho\|_{5/3}^{5/6} \cdot \|\rho\|_1^{1/2}$. On the other hand, $\|\rho\|_{4/3}^{4/3} \le \|\rho\|_{5/3}^{5/6} \cdot \|\rho\|_1^{1/2}$, so (26.A.5) is slightly deteriorated (26.A.1) with $F = 0$ but with "free" ρ_0.

Let us follow G. Graf and J. P. Solovej [1] further albeit in the case of magnetic field. Let us estimate first $\|\rho\|_{5/3}$.

If $\rho = \rho^{\mathsf{TF}}$ direct calculations show that for $N \asymp Z$

$$\int \rho^{\mathsf{TF}}\,dx = \min(Z, N), \tag{26.A.7}$$

$$\int {\rho^{\mathsf{TF}}}^{\frac{4}{3}}\,dx \asymp C{\rho^*}^{\frac{4}{3}} {r^*}^3 = CZ^{\frac{5}{3}}\bigl(1 + BZ^{-\frac{4}{3}}\bigr)^{\frac{2}{5}}, \tag{26.A.8}$$

$$\int {\rho^{\mathsf{TF}}}^{\frac{5}{3}}\,dx \asymp C{\rho^*}^{\frac{5}{3}} {r^*}^3 = CZ^{\frac{7}{3}}\bigl(1 + BZ^{-\frac{4}{3}}\bigr)^{\frac{4}{5}} \tag{26.A.9}$$

with

$$r^* = \min(Z^{-\frac{1}{3}}, B^{-\frac{2}{5}}Z^{\frac{1}{5}}) \asymp Z^{-\frac{1}{3}}\bigl(1 + BZ^{-\frac{4}{3}}\bigr)^{-\frac{2}{5}}, \tag{26.A.10}$$

$$\rho^* = \min(N, Z){r^*}^{-3} \tag{26.A.11}$$

and we use $\|\rho\|_{5/3}^{5/6} \cdot \|\rho\|_1^{1/2} \asymp \|\rho\|_{4/3}^{4/3}$ for $\rho = \rho^{\mathsf{TF}}$.

If $\rho = \rho_\Psi$ we use magnetic Lieb–Thirring inequality (see f.e. Theorem 2.2 in L. Erdös [1])

$$\mathsf{Tr}(H_{A,W}^-) \ge -C\int P_B(W)\,dx \tag{26.A.12}$$

and therefore

$$\begin{aligned}\langle \mathsf{H}\Psi, \Psi\rangle \ge \mathsf{Tr}((H_{A,W})^-) + \int W\rho_\Psi\,dx \\ - \int V\rho_\Psi\,dx + \frac{1}{2}\mathsf{D}(\rho_\Psi, \rho_\Psi) - C\|\rho_\Psi\|_{4/3}^{4/3},\end{aligned} \tag{26.A.13}$$

which due to (26.A.12) is greater than

$$\begin{aligned}\int \bigl(-CP_B(W) + W\rho_\Psi\bigr)\,dx - \\ \int V\rho_\Psi\,dx + \frac{1}{2}\mathsf{D}(\rho_\Psi, \rho_\Psi) - C\|\rho_\Psi\|_{4/3}^{4/3} \ge \\ 3\epsilon_0\int \tau_B(\rho_\Psi)\,dx - \int V\rho_\Psi\,dx + \frac{1}{2}\mathsf{D}(\rho_\Psi, \rho_\Psi) - C\|\rho_\Psi\|_{4/3}^{4/3}\end{aligned} \tag{26.A.14}$$

where we picked up $W : CP'_B(W) = \rho_\Psi$.

The first two terms in the right-hand expression are estimated from below by

$$2\epsilon_0 \int \tau_B(\rho_\Psi)\,dx - C\int P_B(V)\phi\,dx - C\int V\rho_\Psi(1-\phi)\,dx,$$

where $\mathsf{supp}(\phi) \subset \{x\colon \ell(x) \le 2r^*\}$ and $\mathsf{supp}(1-\phi) \subset \{x\colon \ell(x) \ge r^*\}$.

One can see easily that the absolute value of the second term is $\asymp Z^{\frac{7}{3}}\bigl(1+BZ^{-\frac{4}{3}}\bigr)^{\frac{2}{5}}$, while the absolute value of the third term does not exceed $CZ\int V(1-\phi)\,dx \asymp CZ^2 r^{*\,-1}$ which does not exceed the same expression $Z^{\frac{7}{3}}\bigl(1+BZ^{-\frac{4}{3}}\bigr)^{\frac{2}{5}}$. Therefore

$$\langle \mathsf{H}\Psi,\Psi\rangle + C_1 Z^{\frac{7}{3}}\bigl(1+BZ^{-\frac{4}{3}}\bigr)^{\frac{2}{5}} \ge 2\epsilon_0\int \tau_B(\rho_\Psi)\,dx + \frac{1}{2}\mathsf{D}(\rho_\Psi,\rho_\Psi) - C\|\rho_\Psi\|_{4/3}^{4/3}. \tag{26.A.15}$$

Note that $\|\rho_\Psi\|_{4/3}^{4/3}$, calculated over domain $\{x\colon \rho_\Psi(x) \ge B^{\frac{4}{3}}\}$, does not exceed $C\|\rho_\Psi\|_{5/3}^{5/6}\cdot\|\rho\|_1^{\frac{1}{2}}$ with norms, calculated over the same domain, which does not exceed $CT^{\frac{1}{2}}Z^{\frac{1}{2}}$ with $T = \int \tau_B(\rho_\Psi)\,dx$.

Meanwhile, $\|\rho_\Psi\|_{4/3}^{4/3}$, calculated over domain $\{x\colon \rho_\Psi(x) \le B^{\frac{4}{3}}\}$, does not exceed $C\|\rho\|_3^{\frac{1}{2}}\cdot\|\rho\|_1^{\frac{5}{6}}$ with norms, calculated over the same domain, which does not exceed $CZ^{\frac{1}{2}}B^{\frac{1}{3}}T^{\frac{1}{6}}$.

Therefore

$$\|\rho_\Psi\|_{4/3}^{4/3} \le CT^{\frac{1}{2}}Z^{\frac{1}{2}} + CZ^{\frac{5}{6}}B^{\frac{1}{3}}T^{\frac{1}{6}} \tag{26.A.16}$$

and therefore (26.A.15) implies that if

$$\langle \mathsf{H}\Psi,\Psi\rangle \le C_1 Z^{\frac{7}{3}}\bigl(1+BZ^{-\frac{4}{3}}\bigr)^{\frac{2}{5}}, \tag{26.A.17}$$

then

$$T = \int \tau_B(\rho_\Psi)\,dx \le C_2 Z^{\frac{7}{3}}\bigl(1+BZ^{-\frac{4}{3}}\bigr)^{\frac{2}{5}} \tag{26.A.18}$$

and

$$\|\rho_\Psi\|_{4/3}^{4/3} \le CZ^{5/3}\bigl(1+BZ^{-\frac{4}{3}}\bigr)^{\frac{2}{5}}; \tag{26.A.19}$$

taking $F = 0$ we arrive to (26.A.3) if $N \asymp Z$.

However, on our preparatory step we need to estimate also $\|\rho_\Psi\|_{5/3}^{5/3}$ and due to (26.A.18) we need to consider only norms over $\{x\colon \rho_\Psi(x) \ge B^{\frac{4}{3}}\}$. Then $\|\rho_\Psi\|_{5/3}^{5/3} \le C\|\rho_\Psi\|_{4/3}^{16/15} \cdot \|\rho_\Psi\|_3^{3/5}$ and plugging the same estimates (26.A.19), (26.A.19) we conclude that

$$\|\rho_\Psi\|_{5/3}^{5/3} \le CZ^{7/3}\bigl(1 + BZ^{-\frac{4}{3}}\bigr)^{\frac{4}{5}}. \tag{26.A.20}$$

Now we assume that $B \le Z^3$, take $F = e(x,y,\nu)$, where $e(x,y,\nu)$ is the Schwartz kernel of spectral projector for potential W, approximating W^{TF} and $\nu \le 0$ is a chemical potential. One can prove easily that $\|\rho_F\|_{5/3}^{5/3}$ satisfies the same estimate and we need to estimate $\mathsf{Tr}\bigl(\gamma_\Psi(I - E(\mu))\bigr)$.

Consider

$$\begin{aligned} N\langle H_{A,W(x_1)}\Psi,\Psi\rangle - \mathsf{Tr}(HE(\nu)) - \alpha\,\mathsf{Tr}\bigl(\gamma_\Psi(I-E(\nu))\bigr) &\\ \ge \int_{\beta<0} \beta\, d_\beta\, \mathsf{Tr}\, E(\beta) - \int_{\beta\le\nu} (\beta - \nu + \alpha)\, d_\beta\, \mathsf{Tr}\, E(\beta) &\\ = -\int_{\nu-\alpha<\beta<\nu} (\beta-\nu+\alpha) d_\beta E(\beta) &\\ = -\alpha E(\nu) + \int_{\nu-\alpha<\beta<\nu} E(\beta)\, d\beta. & \end{aligned} \tag{26.A.21}$$

We can replace $E(\beta)$ by $\int P'(W+\beta)\,dx$ with a resulting error $O(Z\alpha\hbar^\delta)$, $\hbar \coloneqq BZ^{-3}$. Then the right-hand expression becomes

$$\begin{aligned} -L(\alpha) \coloneqq \int \Bigl(-\alpha P_B'(W+\nu) + \int_0^\alpha P_B'(W+\nu-\beta)\,d\beta\Bigr)\,dx = &\\ -\int_0^\alpha (\alpha-\beta)\Bigl(\int P_B''(W+\nu-\beta)\,dx\Bigr)\,d\beta. & \end{aligned} \tag{26.A.22}$$

Therefore

$$\begin{aligned} \alpha\Bigl(\mathsf{Tr}\bigl(\gamma_\Psi(I-E(\mu))\bigr) - CZ\hbar^\delta\Bigr) \le &\\ \underbrace{N\langle H_{A,W(x_1)}\Psi,\Psi\rangle - \mathsf{Tr}(HE(\nu))}\, + L(\alpha). & \end{aligned} \tag{26.A.23}$$

Note that adding to the selected terms $-\frac{1}{2}\mathsf{D}(\rho^{\mathsf{TF}},\rho^{\mathsf{TF}})$ we obtain exactly the snippet, occurring in the lower estimate of E_N, but in virtue of the upper

estimate it should not exceed $Q = CZ^{\frac{5}{3}}\big(1 + BZ^{-\frac{4}{3}}\big)^{\frac{2}{5}} \le CZ^{\frac{5}{3}} + Ch^2Z^{\frac{7}{3}}$, and therefore, plugging $\alpha = Z^{\frac{4}{3}}\hbar^{\delta}$, we conclude that

$$\mathsf{Tr}\big(\gamma_\Psi(I - E(\mu)\big) \le Z\hbar^{\delta} \tag{26.A.24}$$

provided we prove that

$$L(\alpha) \le Q \qquad \text{for } \ \alpha = Z^{\frac{4}{3}}\hbar^{\delta}. \tag{26.A.25}$$

Therefore modulo proof of (26.A.25) we arrive to the estimate (26.A.26) below:

Theorem 26.A.3. *Let $N \asymp Z$ and $B \le Z$. Then for the ground state energy*

$$\sum_{1\le j<k\le N} \int |x_j - x_k|^{-1} |\Psi(x_1, \dots, x_N)|^2 \, dx_1 \cdots dx_N \ge \tag{26.A.26}$$
$$\frac{1}{2}\mathsf{D}(\rho^{\mathsf{TF}}, \rho^{\mathsf{TF}}) + \mathsf{Dirac} - CZ^{\frac{5}{3}-\delta}(1 + B^{\delta})$$

To prove (26.A.25) we note that $0 \le P''_B(w) \asymp w^{\frac{1}{2}} + Bw^{-\frac{1}{2}}$. One can prove easily then that $L(\alpha) \le C\alpha^{\frac{7}{4}} + CB^{\frac{1}{4}}\alpha^{\frac{3}{2}}$, which obviously implies (26.A.25).

26.A.2 Very Strong Magnetic Field Case

Let us consider now case $Z^2 \le B \le Z^3$.

Proposition 26.A.4. *Consider the Schrödinger operator $H_{A,W}$ with a constant magnetic field of intensity B and potential W: $W \le Z|x|^{-1}$. Let $\phi(x) \coloneqq \phi_r(x)$ be r-admissible function. Then if $Z^2 \lesssim B \lesssim Z^3$ and $r \asymp Z^{-1}$*

$$|e(x, y, 0)| \le CZB \qquad \textit{in } \ B(0, r) \tag{26.A.27}$$

and

(26.A.28) *All eigenvalues are $\ge -CZ^2$.*

Proof. Without any loss of the generality one can assume that

$$H_{A,W} = D_3^2 + D_2^2 + (D_1 - Bx_2)^2 - W. \tag{26.A.29}$$

Consider $f \in \mathscr{L}^2$; then $\|E(\lambda)f\| \le \|f\|$ and then one can prove easily (26.A.28) and inequality

$$\|H_{A,0}E(\lambda)f\| \le (CZ^2 + \lambda_+)\|f\|. \tag{26.A.30}$$

Indeed, $\frac{1}{2}D_3^2 + CZ^2 \ge W$ in the operator sense.

Then (26.A.30) implies that in $B(0,r)\times B(y',r')$ with $r' = B^{-\frac{1}{2}}$

$$|P^\alpha E(\lambda)f| \le CZ^{\alpha_3}B^{\frac{1}{2}|\alpha'|} \qquad \forall \alpha : |\alpha| \le 2\ \forall \lambda \le Z^2$$

with $P = (D_1 - Bx_2, D_2, D_3)$ and therefore $\|E(\lambda)f\|_{\mathscr{C}} \le CZ^{\frac{1}{2}}B^{\frac{1}{2}}\|f\|$. Then $\|E(x,.,\lambda)\|_{\mathscr{L}^2_y}\| \le CZ^{\frac{1}{2}}B^{\frac{1}{2}}$.

Repeating the same arguments with respect to y we arrive to estimate (26.A.27). □

The following corollary follows immediately:

Corollary 26.A.5. *In the framework of Proposition 26.A.4 with $\phi \in \mathscr{L}^\infty(B(0,r))$, $\|\phi\|_{\mathscr{L}^\infty} \le 1$*

$$|\int \phi(x)e(x,x,0)\,dx| \le CZ^{-2}B, \tag{26.A.31}$$

$$\mathsf{D}\big(\phi(x)e(x,x,0), \phi(x)e(x,x,0)\big) \le CZ^{-3}B^2 \tag{26.A.32}$$

and

$$|\int_{-\infty}^0 \int \phi(x)e(x,x,\tau)\,d\tau dx| \le CB. \tag{26.A.33}$$

26.A.3 Riemann Sums and Integrals

If $f \in \mathscr{C}^\infty(\mathbb{R}^+)$ and fast decays at $+\infty$, then

$$f(0)h + \sum_{n\ge 1} 2f(2nh)h \sim \int_0^\infty f(t)\,dt + \sum_{m\ge 1}\kappa_m f^{(2m-1)}(0)h^{2m}, \tag{26.A.34}$$

$$\sum_{n\ge 0} 2f((2n+1)h)h \sim \int_0^\infty f(t)\,dt + \sum_{m\ge 1}\kappa'_m f^{(2m-1)}(0)h^{2m} \tag{26.A.35}$$

as $h \to +0$. The proofs of both formulae follow from the Taylor's decomposition and observation that the odd powers of h should disappear. Taking

$f(t) = e^{-tz/h}$ with $\operatorname{Re} z > 0$ we arrive to

$$1 - \frac{\cosh(z)}{\sinh(z)} z \sim \sum_{m\ge 1} \kappa_m z^{2m}, \tag{26.A.36}$$

$$1 - \frac{1}{\sinh(z)} z \sim \sum_{m\ge 1} \kappa'_m z^{2m} \tag{26.A.37}$$

for $|z| \ll 1$. In particular, $\kappa_1 = \frac{1}{3}$ and $\kappa'_1 = -\frac{1}{6}$.

26.A.4 Some Spectral Ffunction Estimates

Proposition 26.A.6. *For the Schrödinger operator with $A, W \in \mathscr{C}^\infty$ and for $\phi \in \mathscr{C}_0^\infty([-1,1])$ the following estimate holds for any s:*

$$|F(x,y)| \le C(\mu h + 1) h^{-3} \big(1 + h^{-1}|x-y|\big)^{-s} \tag{26.A.38}$$

where

$$F(x,y) := \int \phi(\lambda)\, d_\lambda e(x,y,\lambda). \tag{26.A.39}$$

Proof. Let $u(x,y,t) = \int e^{-ih^{-1}t\lambda}\, d_\lambda e(x,y,\lambda)$ be the Schwartz's kernel of $e^{-ih^{-1}Ht}$.

Let us fix y. Note first that $\mathscr{L}^2$-norm[49)] of $\phi(hD_t)\chi(t)\omega(x)u(x,y,t)$ is less than Ch^s for $\chi \in \mathscr{C}_0^\infty([-\epsilon,\epsilon])$ and $\omega \in \mathscr{C}^\infty$ supported in $\{x\colon |x-y| \ge \epsilon_1\}$ (with $\epsilon_1 = C\epsilon$) due to the finite speed of propagation of singularities.

We conclude then that $\mathscr{L}^2$-norm of $\phi(hD_t)\chi(t)\omega(x)u(x,y,t)$ does not exceed $C(\mu h+1)h^s$ for $\omega \in \mathscr{C}^\infty$ supported in $\{x\colon |x-y| \ge C\}$.

Then $\mathscr{L}^2$-norm of $\partial_t^l \nabla^\alpha \phi(hD_t)\chi(t)\omega(x)u$ does not exceed $C(\mu h+1)h^s$. Therefore due to imbedding inequality $\mathscr{L}^\infty$-norm of $\phi(hD_t)\chi(t)\omega(x)u$ also does not exceed $C(\mu h+1)h^s$. Setting $t=0$ and using this inequality and estimate $|F(x,y)| \le C(\mu h+1)h^{-3}$ (due to Chapter 7), we conclude that $|F(x,y)| \le C(\mu h+1)h^s$ for $|x-y| \ge \epsilon_0$.

Now let us consider general x with $|x-y| = r \ge Ch$. Then rescaling $(x-y) \mapsto (x-y)r^{-1}$ we need also to rescale $h \mapsto hr^{-1}$, $\mu \mapsto \mu r$ and rescaling the above inequality and keeping in mind that $F(x,y)$ is a density with respect to x, we conclude that $|F(x,y)| \le Ch^s r^{-3-s}$ which is equivalent to (26.A.38)–(26.A.39). □

[49)] With respect to x, t here and below.

26.A.5 Zhislin's Theorem for Constant Magnetic Field

We provide just a scheme to prove Zhislin's theorem in the case of the constant magnetic field. In this analysis $\underline{Z}$, $\underline{y}$, N and B are constant.

Proposition 26.A.7. *Let $\Psi = \Psi_N$ be the ground state with the energy $\mathsf{E}_N < \mathsf{E}_{N-1}$. Then*

(i) $\Psi \in \mathscr{C}^1$ and $\Psi = O(e^{-\epsilon|\underline{x}|})$ as $|\underline{x}| \to \infty$.

(ii) Let $N < Z$. Then $V_\Psi - V \in \mathscr{C}^2$ and $V_\Psi = (Z-N)|x|^{-1} + O(|x|^{-2})$, $\nabla V_\Psi = (Z-N)|x|^{-2} + O(|x|^{-3})$ as $|x| \to \infty$.

Proof. Obvious proof is left to the reader. □

Theorem 26.A.8 (Zhislin's theorem). $\mathsf{E}_{N+1} < \mathsf{E}_N$ *for $N < Z$.*

Proof. We can assume that $\mathsf{E}_N < 0$ and the ground state energy exists. Really, it is true for some $N < Z$ and if we prove that then automatically $\mathsf{E}_{N+1} < \mathsf{E}_N$, then it would be true for $(N+1)$ as well, so we may go by induction.

Consider $\Psi = \Psi_N(x_1, \dots, x_N)$ and also $\tilde{\Psi}_{N+1}$, which is an antisymmetrized $\Psi_N(x_1, \dots, x_N)u(x_{N+1})$ (cf. (26.8.21)):

$$\tilde{\Psi} = \tilde{\Psi}(x_1, \dots, x_{N+1}) = \Psi(x_1, \dots, x_N)u(x_{n+1}) - \sum_{1 \le j \le N} \Psi(x_1, \dots, x_{j-1}, x_{N+1}, x_{j+1}, \dots, x_N)u(x_j). \tag{26.A.40}$$

Then like in the estimate of the ionization energy (cf. (26.8.22)–(26.8.23)):

$$N^{-1}\mathsf{I}_{N+1}\|\tilde{\Psi}\|^2 \ge -\langle H_{V,x_{N+1}}\Psi u, \tilde{\Psi}\rangle - \langle \sum_{1 \le i \le N} |x_i - x_{N+1}|^{-1}\Psi u, \tilde{\Psi}\rangle \tag{26.A.41}$$

and

$$N^{-1}\|\tilde{\Psi}\|^2 = \|\Psi\|^2 \cdot \|u\|^2 - N\int \Psi(x_1, \dots, x_{N-1}, x)\Psi^\dagger(x_1, \dots, x_{N-1}, y)u(y)u^\dagger(x)\, dx_1 \cdots dx_{N-1}\, dxdy. \tag{26.A.42}$$

Now let us consider u supported in $\{x\colon \frac{1}{2}a \le |x| \le 3a\}$ with a to be chosen later. Then in virtue of Proposition 26.A.7(i) modulo $O(e^{-\epsilon_1 a})$ we can replace in the right-hand expressions $\tilde{\Psi}$ by $\Psi_N(x_1,\dots,x_N)u(x_{N+1})$ resulting in $-\langle H_W u, u\rangle$ and $\|u\|^2$ respectively with $W = V_\Psi$ defined in Proposition 26.A.7(ii).

Therefore all we need to prove this theorem is to be able to select u with $\|u\| \asymp 1$, supported in $\{x\colon a \le |x| \le 3a\}$ and with $\langle H_W u, u\rangle \le -\epsilon_0 a^{-1}$.

In virtue of Proposition 26.A.7(ii) $V_\Psi \ge \epsilon_0 a^{-1}$ in $\{x\colon a \le |x| \le 3a\}$ and therefore we can replace W by $\epsilon_0 a^{-1}$. Without any loss of the generality one can assume that $A = (Bx_2, 0, 0)$. Recall that for the linear vector-potential $\vec{A}$ operator $H_0 = ((i\nabla - A)\cdot\boldsymbol{\sigma})^2$ is a direct sum of $H_0^+ = (i\nabla - A)^2 + B$ and $H_0^- = (i\nabla - A)^2 - B$; so we can consider only the latter. Note that $H_0^- = (i\partial_1 - Bx_2)^2 - \partial_2^2 - \partial_3^2$ and $H_0^- v = 0$ with $v = \exp(-\frac{1}{2}B(x_2-a)^2 + iBax_1)$.

Then $u = v(x)\chi(r^{-1}(x-\bar{x}))$ with $\chi \in \mathscr{C}^\infty(B(0,1))$, $\chi = 1$ in $B(0,\frac{1}{2})$, $\bar{x} = (0, 2a, 0)$, $r = \frac{1}{3}a$ is a required function. □

Comments

We already mentioned papers E. H. Lieb, J. P. Solovej and J. Yngvason [1,3] where asymptotics of the ground state energy were derived in the cases $B \ll Z^3$ and $B \gg Z^3$ respectively. Intermediate case $B \sim Z^3$ was covered also in E. H. Lieb, J. P. Solovej and J. Yngvason [1]. Even without remainder estimates certain results concerning ionization energy and maximal possible positive and negative charges were also derived.

Remainder estimates in the case $B \ll Z^3$ were derived by V. Ivrii in [20,21]. Unfortunately there are gaps in the proofs of the second paper in the case of $M \ge 2$ and large $Z - N > 0$ which I was unable to fill.; so our results in this case are not as sharp as they supposed to be.

Important Remarks

Sections 26.7 and 26.8 were rewritten. The reason is that the printed version of this monograph contains crucial errors in the proof of Theorem 26.7.4 (see Theorem 26.7.3 here); errors 1 and 2 from the list below propagated from Section 25.6.

1. The first one occurred only in the case $M \ge 2$. It has been corrected.

2. The second one has not been corrected and we cannot prove the estimate of the excessive negative charge better than $O(Z^{\frac{5}{7}})$ even for the case $M = 1$.

3. The error in the proof of Theorem 26.8.15 has been fixed but this Theorem became much weaker.

4. The proof of Proposition 26.8.11 (Theorem 26.8.11 here) there was incomplete; it has been completed and now we have a way better estimate than proposed in Problem 26.8.14 there.

Chapter 27

The Case of Self-Generated Magnetic Field

27.1 Introduction

We are going to replace Schrödinger operator without magnetic field as in Chapter 25 or with a constant magnetic field as in Chapter 26 by Schrödinger operator

$$H = H_{A,V} = \big((D - A)\cdot\boldsymbol{\sigma}\big)^2 - V(x) \tag{27.1.1}$$

with unknown magnetic field A but then to add to the ground state energy of the atom (or molecule) the energy of magnetic field (see selected term in (27.1.2) thus arriving to

$$\mathsf{E}(A) = \inf \mathsf{Spec}(\mathsf{H}_{A,V}) + \underbrace{\alpha^{-1}\int |\nabla\times A|^2\,dx} \tag{27.1.2}$$

with N-particle quantum Hamiltonian $\mathsf{H}_{A,V}$ defined by (26.1.1) and a parameter $\alpha \in (0, \kappa^* Z^{-1}]$ with small constant $\kappa^* > 0$.

Then finally

$$\mathsf{E}^* = \inf_{A\in\mathscr{H}^1_0} \mathsf{E}(A) \tag{27.1.3}$$

defines a ground state energy with a *self-generated magnetic field*[1)].

[1)] This notion was introduced in series of papers L. Erdös, S. Fournais and J. P. Solovej [1,3,4]; see also L. Erdös and J. P. Solovej [1].

First of all we are lacking so far a semiclassical local theory and we are developing it in Section 27.2 where we consider one-particle quantum Hamiltonian

$$H = H_{A,V} = \big((hD - A)\cdot\boldsymbol{\sigma}\big)^2 - V(x) \tag{27.1.4}$$

but instead of $\inf \mathsf{Spec}(\mathsf{H}_{A,V})$ we consider $\mathsf{Tr}^-(H_{A,V})$ which as we already know is what replaces $\inf \mathsf{Spec}(\mathsf{H}_{A,V})$ if electrons do not interact (then if electrons interact we will need to replace V by W which includes a potential generated by the electron cloud and justify this by estimating an error).

We define the energy of the magnetic field as in (27.1.2) but with κ replaced by κh^{d-1} (here $d \ge 2$ is arbitrary) and we prove that for $d = 2, 3$ in this framework a self-generated magnetic field is weak and the asymptotics with the remainder $O(h^{2-d})$ (or even $o(h^{2-d})$ under standard assumption of the global nature) is exactly as for $\kappa = 0$ (i.e. with $A = 0$). In the latter case asymptotics includes the Schwinger correction term $\varkappa_2 h^{-1}$.

Then in Section 27.3 we consider operator with potential having Coulomb-type singularities and combining results and arguments of Sections 27.2 and 12.6 prove for $d = 3$ that

$$\mathsf{Tr}^-(H_{A,V}) = \mathsf{Weyl}_1 + 2S(\kappa)h^{-2} + O\big(\kappa|\log\kappa|^{\frac{1}{3}}h^{-\frac{4}{3}} + h^{-1}\big) \tag{27.1.5}$$

provided $\kappa < \kappa^*$ (which is a small constant) and there is just one singularity; when there are several singularities with a minimal distance $a \gg 1$ between them we prove that

$$\mathsf{Tr}^-(H_{A,V}) = \mathsf{Weyl}_1 + 2S(\kappa)h^{-2} + O\big(\kappa|\log\kappa|^{\frac{1}{3}}h^{-\frac{4}{3}} + h^{-1} + \kappa a^{-3}h^{-2}\big). \tag{27.1.6}$$

If $\kappa \ll h^{\frac{1}{3}}|\log h|^{-\frac{1}{3}}$ then under standard assumption about trajectories we can upgrade this asymptotics to even sharper with the remainder estimate $o(h^{-1})$ and with the Schwinger correction term.

Further, in Section 27.4 we apply these results to provide estimates from above and below for the total energy (27.1.3). As a byproduct we also estimate $\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}},\ \rho_\Psi - \rho^{\mathsf{TF}})$ where Ψ is a ground state for a near-minimizer A.

This estimate enables us in Section 27.5 to derive upper estimates for the excessive negative charge, estimates or asymptotics for the ionization energy, and in the free nuclei model also for the minimal distance between nuclei and (in the case of molecule) for the excessive positive charge.

27.2 Local Semiclassical Trace Asymptotics

27.2.1 Toy-Model

Statement of the Pproblem

Let us consider operator (27.1.1) in $\mathbb{R}^d$ with $d = 3$ where A, V are real-valued functions and $V \in \mathscr{L}^{\frac{5}{2}} \cap \mathscr{L}^4$, $A \in \mathscr{H}^1_0$. Then this operator is self-adjoint. We are interested in $\mathsf{Tr}^-(H_{A,V})$ (the sum of all negative eigenvalues of this operator). Let

$$\mathsf{E}^* := \inf_{A\in\mathscr{H}^1_0(B(0,r))} \mathsf{E}(A), \tag{27.2.1}$$

$$\mathsf{E}(A) := \Bigl(\mathsf{Tr}^-\, H_{A,V} + \kappa^{-1}h^{-2}\int |\partial A|^2\, dx\Bigr) \tag{27.2.2}$$

with $\partial A = (\partial_i A_j)$ a matrix; here and below r is a parameter and constants do not depend on it.

The estimate from above is delivered by $A = 0$ and Weyl formula with an error $O(h^{-1})$ as $V \in \mathscr{C}^{2,1}$ [2)]

$$\mathsf{E}^* \le \mathsf{Weyl}_1 + O(h^{-1}); \tag{27.2.3}$$

where

$$\mathsf{Weyl}(\tau) = \frac{1}{3\pi^2}h^{-3}\int (V+\tau)_+^{\frac{3}{2}}\, dx, \tag{27.2.4}$$

and

$$\mathsf{Weyl}_1 = \int_{-\infty}^{0} \tau \mathsf{d}_\tau \mathsf{Weyl}(\tau) = -\frac{2}{15\pi^2}\int V_+^{\frac{5}{2}}\, dx. \tag{27.2.5}$$

Also for estimates $o(h^{-2})$ we need to include into Weyl_1 the corresponding boundary term. Now our goal is to provide an estimate from below

$$\mathsf{E}^* \ge \mathsf{Weyl}_1 - O(h^{-1}); \tag{27.2.6}$$

We will use also $\mathsf{Weyl}(x,\tau)$ and $\mathsf{Weyl}_1(x)$ defined the same way albeit without integration with respect to x.

[2)] Recall that this means that the second derivatives of V are continuous with the continuity modulus $O(|\log|x-y||^{-1})$, see Section 4.6. If there is a boundary it does not pose any problem provided it is in the classically forbidden region.

Preliminary Analysis

So, let us estimate $\mathsf{E}(A)$ from below. First we need the following really simple

Proposition 27.2.1. *Let $V \in \mathscr{L}^{\frac{5}{2}} \cap \mathscr{L}^4$. Then*

$$\mathsf{E}^* \geq -Ch^{-3} \tag{27.2.7}$$

and either

$$\frac{1}{\kappa h^2} \int |\partial A|^2 \, dx \leq Ch^{-3} \tag{27.2.8}$$

or $\mathsf{E}(A) \geq ch^{-3}$.

Proof. Using the Magnetic Lieb–Thirring inequality (5) of E. H. Lieb, M. Loss, M. and J. P. Solovej [1])

$$\int \operatorname{tr} e_1(x,x,\tau)\, dx \geq - Ch^{-3} \int V_+^{\frac{5}{2}}\, dx - Ch^2 \Big(h^{-2} \int |\partial A|^2\, dx\Big)^{\frac{3}{4}} \Big(h^{-8} \int V_+^4\, dx\Big)^{\frac{1}{4}}, \tag{27.2.9}$$

we conclude that for any $\delta > 0$

$$\mathsf{E}(A) \geq -Ch^{-3} - C\delta^3 h^{-3} + \big(\kappa^{-1}h^{-2} - \delta^{-1}h^{-1}\big) \int |\partial A|^2\, dx \tag{27.2.10}$$

which implies both assertions of this proposition. □

Proposition 27.2.2. *Let $V_+ \in \mathscr{L}^{\frac{5}{2}} \cap \mathscr{L}^4$, $\kappa \leq ch^{-1}$ and*

$$V \leq -C^{-1}(1+|x|)^{\delta} + C. \tag{27.2.11}$$

Then there exists a minimizer A.

Proof. Consider a minimizing sequence A_j. Without any loss of the generality one can assume that $A_j \to A_\infty$ weakly in $\mathscr{H}^1$ and in $\mathscr{L}^6$ and strongly in $\mathscr{L}^p_{\mathsf{loc}}$ with any $p < 6$ [3]. Then A_∞ is a minimizer.

Really, due to (27.2.8) and (7.2.11) negative spectra of $H_{A_j,V}$ are discrete and the number of negative eigenvalues is bounded by $N = N(h)$. Consider

[3] Otherwise we select a converging subsequence.

ordered eigenvalues $\lambda_{j,k}$ of $H_{A_j,V}$. Without any loss of the generality one can assume that $\lambda_{j,k}$ have limits $\lambda_{\infty,k} \le 0$ (we go to the subsequence if needed).

We claim that $\lambda_{\infty,k}$ are also eigenvalues and if $\lambda_{\infty,k} = ... = \lambda_{\infty,k+r-1}$ then it is eigenvalue of at least multiplicity r. Indeed, let $u_{j,k}$ be corresponding eigenfunctions, orthonormal in $\mathscr{L}^2$. Then in virtue of A_j being bounded in $\mathscr{L}^6$ and $V \in \mathscr{L}^4$ we can estimate

$$\|Du_{j,k}\| \le K\|u_{j,k}\|_6^{1-\sigma} \cdot \|u_{j,k}\|^{\sigma} \le K\|Du_{j,k}\|^{1-\sigma} \cdot \|u_{j,k}\|^{\sigma}$$

with $\sigma > 0$ which implies $\|Du_{j,k}\| \le K$. Also assumption (27.2.11) implies that $\|(1+|x|)^{\delta/2}u_{j,k}\|$ are bounded and therefore without any loss of the generality one can assume that $u_{j,k}$ converge strongly.

Then

$$\lim_{j\to\infty} \mathsf{Tr}^-(H_{A_j,V}) \ge \mathsf{Tr}^-(H_{A_\infty,V}), \tag{27.2.12}$$

$$\liminf_{j\to\infty} \int |\partial A_j|^2 \,dx \ge \int |\partial A_\infty|^2 \,dx \tag{27.2.13}$$

and therefore $\mathsf{E}(A_\infty) \le \mathsf{E}^*$. Then A_∞ is a minimizer and there are equalities in (27.2.12)–(27.2.13) and, in particular, there no negative eigenvalues of $H_{A_\infty,V}$ other than $\lambda_{\infty,k}$. □

Remark 27.2.3. We do not know if a minimizer is unique. Also we do not impose here any restrictions on r, K (which may depend on h) in (27.2.11) or $\kappa > 0$. From now on until the further notice let $A = A_h$ be a minimizer.

Proposition 27.2.4. *In the framework of Proposition 27.2.2 let A be a minimizer. Then*

$$\frac{2}{\kappa h^2}\Delta A_j(x) = \Phi_j := \tag{27.2.14}$$
$$-\mathsf{Re}\,\mathsf{tr}\Big[\sigma_j\Big((hD-A)_x \cdot \boldsymbol{\sigma} e(x,y,\tau) + e(x,y,\tau)\,{}^t(hD-A)_y \cdot \boldsymbol{\sigma}\Big)\Big]\Big|_{y=x}$$

where $A = (A_1, A_2, A_3)$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and $e(x,y,\tau)$ is the Schwartz kernel of the spectral projector $\theta(-H)$ of $H = H_{A,V}$ and tr is a matrix trace.

Proof. Consider variation δA of A and variation of $\mathsf{Tr}^-(H) = \mathsf{Tr}(H^-)$ where $H^- = H\theta(-H)$ is a negative part of H. Note that the spectral projector of H is

$$\theta(\tau - H) = \frac{1}{2\pi i}\int_{-\infty}^{\tau} \mathsf{Res}_{\mathbb{R}}(\tau - H)^{-1} \tag{27.2.15}$$

and therefore

$$\delta \operatorname{Tr}\big(\theta(\tau - H)\big) = \frac{1}{2\pi i}\int_{-\infty}^{\tau} \operatorname{Res}_{\mathbb{R}} \operatorname{Tr}\big((\tau - H)^{-1}(\delta H)(\tau - H)^{-1}\big) =$$
$$\frac{1}{2\pi i}\int_{-\infty}^{\tau} \operatorname{Res}_{\mathbb{R}} \operatorname{Tr}\big((\delta H)(\tau - H)^{-2}\big) = -\partial_\tau \frac{1}{2\pi i}\int_{-\infty}^{\tau} \operatorname{Res}_{\mathbb{R}} \operatorname{Tr}\big((\delta H)(\tau - H)^{-1}\big) =$$
$$-\partial_\tau \operatorname{Tr}\big((\delta H)\theta(\tau - H)\big).$$

Plugging it into

$$\operatorname{Tr}^-(H) = \int_{-\infty}^{0} \tau d_\tau \operatorname{Tr}\big(\theta(\tau - H)\big) = -\int_{-\infty}^{0} \operatorname{Tr}\big(\theta(\tau - H)\, d\tau\big) \tag{27.2.16}$$

and integrating with respect to τ we arrive after simple calculations to

$$\delta \operatorname{Tr}^-(H - \tau) = \operatorname{Tr}\big((\delta H)\theta(\tau - H)\big) = \sum_j \int \Phi_j(x)\delta A_j(x)\, dx \tag{27.2.17}$$

where $\Phi(x)$ is the right-hand expression of (27.2.14). Therefore

$$\delta \mathsf{E}(A) = \sum_j \int \big(\Phi_j(x) - \frac{2}{\kappa h^2}\Delta A_j(x)\big)\delta A_j(x)\, dx \tag{27.2.18}$$

which implies (27.2.14). □

Proposition 27.2.5. *If for* $\kappa = \kappa^*$

$$\mathsf{E}^* \geq \mathsf{Weyl}_1 - CM \tag{27.2.19}$$

with $M \geq Ch^{-1}$ *then for* $\kappa \leq \kappa^*(1 - \epsilon_0)$

$$\frac{1}{\kappa h^2}\int |\partial A|^2\, dx \leq C_1 M. \tag{27.2.20}$$

Proof. Proof is obvious based also on the upper estimate $\mathsf{E}^* \leq \mathsf{Weyl}_1 + Ch^{-1}$. □

Estimates

Proposition 27.2.6. *Let estimate* (27.2.20) *be fulfilled and let*

$$\varsigma = \kappa M h \le c. \tag{27.2.21}$$

Then for $\tau \le c$

(i) Operator norm in $\mathscr{L}^2$ *of* $(hD)^k \theta(\tau - H)$ *does not exceed* C *for* $k = 0, 1, 2$.

(ii) Operator norm in $\mathscr{L}^2$ *of* $(hD)^k \big((hD - A)\cdot \sigma\big)\theta(\tau - H)$ *does not exceed* C *for* $k = 0, 1$.

Proof. (i) Let $u = \theta(\tau - H)f$. Then $\|u\| \le \|f\|$ and since

$$\|A\|_{\mathscr{L}^6} \le C\|\partial A\| \le C(\kappa M)^{\frac12} h \tag{27.2.22}$$

we conclude that

$$\begin{aligned}\|hDu\| \le \|(hD - A)u\| + \|Au\| \le \|(hD - A)u\| + C\|A\|_{\mathscr{L}^6}\cdot \|u\|_{\mathscr{L}^3} \le \\ \|(hD - A)u\| + C(\kappa M)^{\frac12} h \|u\|^{1/2}\cdot \|u\|^{1/2}_{\mathscr{L}^6} \le \\ \|(hD - A)u\| + C(\kappa M h)^{\frac12} \|u\|^{1/2}\cdot \|hDu\|^{1/2} \le \\ \|(hD - A)u\| + \frac{1}{2}\|hDu\| + C\kappa M h\|u\|;\end{aligned}$$

therefore due to (27.2.21)

$$\|hDu\| \le 2\|(hD - A)u\| + C\kappa M h\|u\|. \tag{27.2.23}$$

On the other hand, for $B = \nabla \times A$ and $\tau \le c$

$$\begin{aligned}\|(hD - A)u\|^2 \le C\|u\|^2 + (h|B|u, u) \le C\|u\|^2 + h\|B\|\cdot \|u\|^2_{\mathscr{L}^4} \le \\ C\|u\|^2 + C(\kappa M)^{\frac12} h^2 \|u\|\cdot \|u\|_{\mathscr{L}^6} \le C\|u\|^2 + C(\kappa M)^{\frac12} h\|u\|\cdot \|hDu\| \le \\ C(1 + \kappa M h^2 + \kappa^{\frac32} M^{\frac32} h^2)\|u\|^2 + \frac{1}{2}\|(hD - A)u\|^2\end{aligned}$$

and due to (27.2.23) we conclude that

$$\|(hD - A)u\| \le C\|u\| \quad \text{and} \quad \|hDu\| \le C(1 + \kappa M h)\|u\| \tag{27.2.24}$$

provided $\kappa M h^{1+\delta} \le c$ for sufficiently small $\delta > 0$. Therefore under assumption (27.2.21) for $k = 0, 1$ statement (i) is proven.

Further, since $(hD)^2 = (hD - A)^2 + A(hD - A) + AhD - h[D, A]$ we in the same way as before (and using (27.2.24)) conclude that

$$\|(hD)^2 u\| \le C\|u\|^2 + \frac{1}{4}\|hD(hD - A)u\| + \frac{1}{4}\|h^2 D^2 u\|$$

and therefore

$$\|h^2 D^2 u\| \le C\|u\|^2 + C\|AhDu\|$$

and repeating the same arguments we get $\|h^2 D^2 u\| \le C\|u\|$; so for $k = 2$ Statement (i) is also proven.

(ii) Statement (ii) is proven in the same way. □

Corollary 27.2.7. *Let* (27.2.20) *and* (27.2.21) *be fulfilled. Then for* $\tau \le c$

$$e(x, x, \tau) \le C h^{-3} \tag{27.2.25}$$

and

$$|((hD - A) \cdot \boldsymbol{\sigma}) e(x, y, \tau)|_{x=y}| \le C h^{-3}. \tag{27.2.26}$$

Proof. Let us prove that

(27.2.27) Operator norms from $\mathscr{L}^2$ to $\mathscr{C}$ of both operators $\theta(\tau - H)$ and $((hD - A) \cdot \boldsymbol{\sigma})\theta(\tau - H)$ do not exceed $Ch^{-\frac{3}{2}}$.

Indeed, Proposition 27.2.6 and embedding theorem imply that the operator norm of $\theta(\tau - H)$ from $\mathscr{L}^2$ to C does not exceed $Ch^{-\frac{3}{2}}$. Then due to interpolation operator norms of $\theta(\tau - H)$ from $\mathscr{L}^2$ to $\mathscr{L}^3$ and $\mathscr{L}^6$ do not exceed $Ch^{-\frac{1}{2}}$ and Ch^{-1} respectively.

Let $v = ((hD - A) \cdot \boldsymbol{\sigma})u$, $u = \theta(\tau - H)f$, $\|f\| = 1$. We know that $\mathscr{L}^6$-norms of Hu and u do not exceed Ch^{-1}, and then it is true for $((hD - A) \cdot \boldsymbol{\sigma})u\|$ as well.

Then $\|Au\|_{\mathscr{L}^3} \le C\|A\|_{\mathscr{L}^6}\|f\|_{\mathscr{L}^6} \le Ch^{-\frac{1}{2}}$ and $\|Du\|_{\mathscr{L}^3} \le Ch^{-\frac{3}{2}}$ which together with $\|u\|_{\mathscr{L}^2} \le C$ implies that $\|u\|_{\mathscr{L}^p} \le Ch^{-3/2+3/p}$ for $2 \le p < \infty$. Then $\|Au\|_{\mathscr{L}^4} \le C\|A\|_{\mathscr{L}^6}\|u\|_{\mathscr{L}^{12}} \le Ch^{-\frac{1}{4}}$ and $\|Du\|_{\mathscr{L}^4} \le Ch^{-\frac{5}{4}}$. Together with $\|u\| \le C$ it implies $\|u\| \le Ch^{-\frac{3}{2}}$. So, (27.2.27) has been proven.

Then the same estimate for holds adjoint operators which imply both statements of the corollary. □

Corollary 27.2.8. *Let* (27.2.20) *and* (27.2.21) *be fulfilled and* A *be a minimizer. Then*

$$\|\partial A\|_{\mathscr{C}^{1-\delta}} \le C\kappa h^{-1} \tag{27.2.28}$$

and

$$\|\partial A\|_{\mathscr{L}^\infty} \le C'_\delta h^{-\frac{4}{5}-\delta} \tag{27.2.29}$$

where $\mathscr{C}^\theta$ *is the scale of Hölder spaces and* $\delta > 0$ *is arbitrarily small.*

Proof. Really, due to (27.2.14) minimizer A satisfies $\|\Delta A\|_{\mathscr{L}^\infty} \le C\kappa h^{-1}$. Also we know that $\|\partial A\| \le C(\kappa M h^2)^{\frac12} \le Ch^{\frac12}$ due to (27.2.21). Then (27.2.28) holds due to the standard properties of the elliptic equations[4)].

Therefore if at some point y we have $|\partial A(y)| \gtrsim \mu$, it is true in its $\epsilon(\mu h\kappa^{-1})^{1-\delta}$-vicinity (provided $\mu \le \kappa h^{-1}$) and then

$$\|\partial A\|^2 \gtrsim \mu^2(\mu h\kappa^{-1})^{3(1-\delta)}$$

and we conclude that

$$\mu^2(\mu h\kappa^{-1})^{3(1-\delta)} \le C\kappa h^2 M \iff \mu^{5-3\delta} \le C\kappa^{4-3\delta}h^{-1+3\delta}M$$

and one can see easily that (27.2.29) holds due to (27.2.21) and assumption $h^{-1} \le M \lesssim h^{-3}$.

On the other hand, if $\mu \ge \kappa h^{-1}$ then we need to take ϵ-vicinity and then $\mu^2 \le C\kappa M h^2 \le Ch^{\frac12}$ where we used (27.2.21) again. Therefore (27.2.29) has been proven. □

Remark 27.2.9. (i) It is not clear if it is possible to generalize this theory to arbitrary $d \ge 2$ with the magnetic field energy given by

$$\frac{1}{\kappa h^{d-1}} \int \left(|\partial A|^2 - |\nabla \cdot A|^2\right) dx. \tag{27.2.30}$$

Surely one should use generalized Pauli matrices σ_j in the definition of the operator: for $d = 2$ one can prove that $\mathsf{E}(A)$ is bounded from below and a minimizer exists; for $d = 4$ one can prove that $\mathsf{E}(A)$ is bounded from below if $\kappa \le \epsilon_0 h$; especially problematic is the case $d \ge 5$ since then $A \in \mathscr{H}^1$ does not guarantee enough regularity.

(ii) Therefore while arguments of Subsection 27.2.2 below remain valid for $d \ge 4$, so far they remain conditional (if a minimizer exists and satisfies some crude estimates).

[4)] Actually we can slightly improve this statement.

27.2.2 Microlocal Analysis Unleashed

Sharp Estimates

Now we can unleash the full power of microlocal analysis but we need to extend it to our framework. It follows by induction from (27.2.28)–(27.2.29) and the arguments we used to derive these estimates that

$$\|\partial A\|_{\mathscr{C}^{n-\delta}} \le C_n \kappa h^{-1-n} \qquad \forall n \in \mathbb{Z}^+, \tag{27.2.31}$$

so A is "smooth" in $\varepsilon = h$ scale while for rough microlocal analysis as in Section 2.3 one needs at least $\varepsilon = Ch|\log h|$. We consider in this section arbitrary $d \ge 2$; see however Remark 27.2.9.

Proposition 27.2.10. *For a commutator of a pseudodifferential operator with a smooth symbol and $\mathscr{C}^{\theta+1}$-function $A(x)$ a usual commutator formula holds modulo $O(h^{\theta+1}\|\partial A\|_\theta)$ for any non-integer $\theta > 0$ where*

$$\|f\|_\theta := \begin{cases} \displaystyle\sum_{\alpha:|\alpha|=\theta} \sup_x |\partial^\alpha f(x)| & \theta \in \mathbb{Z}^+, \\ \displaystyle\sum_{\alpha:|\alpha|=\lfloor\theta\rfloor} \sup_{x\neq y} |x-y|^{\lfloor\theta\rfloor-\theta}\cdot|\partial^\alpha f(x)-\partial^\alpha f(y)| & \theta \notin \mathbb{Z}^+. \end{cases} \tag{27.2.32}$$

Proof. Easy proof is left to the reader. □

Proposition 27.2.11. *Assume that*

$$\|\partial V\|_{\mathscr{C}(B(0,2))} \le C_0 \tag{27.2.33}$$

and

$$\mu := \|\partial A\|_{\mathscr{C}(B(0,2))} \le C_0. \tag{27.2.34}$$

Let $U(x,y,t)$ be the Schwartz kernel of $e^{ih^{-1}tH_{A,V}}$. Then for $T \asymp 1$

(i) Estimate

$$\|F_{t\to h^{-1}\tau}\chi_T(t)(hD_x)^\alpha(hD_y)^\beta\psi_1(x)\psi_2(y)U\| \le Ch^s \tag{27.2.35}$$

holds for all $\alpha : |\alpha| \le 2$, $\beta : |\beta| \le 2$, s and all $\psi_1, \psi_2 \in \mathscr{C}_0^\infty(B(0,1))$, such that $\operatorname{dist}(\operatorname{supp}(\psi_1), \operatorname{supp}(\psi_2)) \ge C_0 T$ and $\tau \le c_0$; here $\|.\|$ means an operator norm from $\mathscr{L}^2$ to $\mathscr{L}^2$.

(ii) Estimate

$$\|F_{t\to h^{-1}\tau}\chi_T(t)(hD_x)^\alpha(hD_y)^\beta\varphi_1(hD_x)\varphi_2(hD_y)U\| \le Ch^s + Ch^\theta\big(\|\!|A|\!\|_{\theta+1} + \|\!|V|\!\|_{\theta+1}\big) \tag{27.2.36}$$

holds for all $\alpha : |\alpha| \le 2$, $\beta : |\beta| \le 2$, s *and all* $\varphi_1, \varphi_2 \in \mathscr{C}_0^\infty$, *such that* $\mathsf{dist}(\mathsf{supp}(\varphi_1), \mathsf{supp}(\varphi_2)) \ge C_0T$, *and* $\tau \le c_0$.

(iii) If also in $B(0,2)$

$$\epsilon_0 \le |V| \le c \tag{27.2.37}$$

then for a small constant $T = \epsilon$ *estimate*

$$\|F_{t\to h^{-1}\tau}\chi_T(t)(hD_x)^\alpha(hD_y)^\beta U\| \le Ch^s + Ch^\theta\big(\|\!|A|\!\|_{\theta+1} + \|\!|V|\!\|_{\theta+1}\big) \tag{27.2.38}$$

holds for all $\alpha : |\alpha| \le 2$, $\beta : |\beta| \le 2$, s *and all* $\psi_1, \psi_2 \in \mathscr{C}_0^\infty(B(0,1))$, *such that* $\mathsf{diam}(\mathsf{supp}(\psi_1) \cup \mathsf{supp}(\psi_2)) \le \epsilon_0 T$ *and* $|\tau| \le \epsilon$.

Proof. Let $u = e^{ith^{-1}H}f$ with arbitrary $f \in \mathscr{L}^2$.

(i) Statement (i) is easily proven by the same arguments as in the proof of Theorem 2.1.2: we consider just usual function $\phi(x)$ and operators of multiplication like $\chi(\phi(x))$ so there are no "bad" commutators due to non-smoothness of A or V.

(ii) Statement (ii) is also proven by the same arguments; however in this case $\phi = \phi(x,\xi)$ so we need to involve "bad" commutators but their contributions are bounded by

$$C\|Q_1u\| \cdot \Big(h^{1+\theta}\big(\|\!|A|\!\|_{\theta+1} + \|\!|V|\!\|_{\theta+1}\big)\|u\| + h^{1+\delta}\|Q'u\|\Big)$$

in the right-hand expression while the left-hand expression is $\epsilon h\|Q_1u\|^2$ where Q, Q_1, and Q' are operators with symbols $\chi(\phi(x,\xi))$, $\chi_1(\phi(x,\xi))$, and $\chi_1(\phi(x,\xi)-\eta)$ respectively, $\eta > 0$ is an arbitrarily small constant (so the latter symbol has a bit larger support than the former one), $\delta > 0$ is a small exponent, $\chi_1(t) = (-\chi'(t))^{\frac12}$, and f.e. $\chi(t) = e^{-|t|^{-1}}$ for $t < 0$, $\chi(t) = 0$ for $t \ge 0$.

Therefore we conclude that

$$\|Qu\| \le Ch^{\theta}\big(\|\!|A|\!\|_{\theta+1} + \|\!|V|\!\|_{\theta+1}\big)\|u\| + Ch^{\delta}\|Q'u\|$$

and similarly we can estimate $\|Q'u\|$ with $\|Q''u\|$ in the right-hand expression etc and thus we conclude that

$$\|Qu\| \le Ch^{\theta}\big(\|\!|A|\!\|_{\theta+1} + \|\!|V|\!\|_{\theta+1}\big)\|u\| + Ch^{s}\|u\|$$

which is what we need.

(iii) Statement (iii) is easily proven by the same arguments as in the proof of Theorem 2.1.2: we consider just usual function $\phi(x)$ and operators of multiplication like $\chi(\phi(x))$ so there are no "bad" commutators due to non-smoothness of A or V. However we need to consider a contribution of u which is not confined to the small vicinity of (y,η) and we need Statement (ii) for this so the last term in the right-hand expression of (27.2.38) is inherited.

We leave easy details to the reader. □

Remark 27.2.12. In Proposition 27.2.11

(i) Statement (i) means the finite propagation speed with respect to x.

(ii) Statement (ii) means the finite propagation speed with respect to ξ and the last term in the right-hand expression of (27.2.36) is due to the non-smoothness of A and V.

(iii) Statement (iii) means that under assumption (27.2.37) there actually is a propagation with respect to x.

(iv) So far we have not assumed that V is very smooth function; we actually do not need it at all: it is sufficient to assume that ∂V is very smooth in the microscale $\varepsilon = h^{1-\delta}$; one can actually invoke more delicate arguments of the proof of Theorem 2.3.1 and deal with microscale $\varepsilon = Ch|\log h|$.

Therefore in the framework of Proposition 27.2.11(iii) estimate

$$|F_{t\to h^{-1}\tau}\chi_T(t)\big((hD_x)^{\alpha}(hD_y)^{\beta}U(x,y,t)\big)\big|_{x=y}| \le Ch^{1-d+s}T^{-s} + CT^2h^{-d+\theta}\big(\|\!|A|\!\|_{\theta+1} + \|\!|V|\!\|_{\theta+1}\big) \tag{27.2.39}$$

holds for all $\alpha : |\alpha| \le 2$, $\beta : |\beta| \le 2$, s for $T = \epsilon$ and $|\tau| \le \epsilon$ where as usual $\chi \in \mathscr{C}_0^\infty([-1,-\frac12]\cup[\frac12,1])$, $\chi_T(t) = \chi(t/T)$.

Let us consider $T \in (Ch, \epsilon)$; then we apply the standard rescaling $t \mapsto tT^{-1}$, $x \mapsto xT^{-1}$, $h \mapsto hT^{-1}$ and assumptions (27.2.33), (27.2.34) are replaced by weaker assumptions

$$T\|\partial V\|_{\mathscr{C}(B(x,1))} \le C_0 \tag{27.2.33}'$$

and

$$T\|\partial A\|_{\mathscr{C}(B(x,1))} \le C_0. \tag{27.2.34}'$$

Further, $\|\!|A|\!\|_{\theta+1}$ and $\|\!|V|\!\|_{\theta+1}$ acquire factor $T^{\theta+1}$.

Furthermore, since $U(x,y,t)$ is a density with respect to y we need to add factor T^{-d} to the right-hand expression and due to $F_{t\to h^{-1}\tau}$ we need to add another factor T and after these substitution and multiplications we arrive to

Proposition 27.2.13. *Let $h \le T \le \epsilon$ and assumptions* (27.2.33)′, (27.2.34)′ *and* (27.2.37) *be fulfilled. Then estimate* (27.2.39) *holds.*

Next we apply our standard arguments:

Proposition 27.2.14. *In the framework of Proposition 27.2.13 let $\bar\chi \in \mathscr{C}_0^\infty([-1,1])$. Then the following estimates hold:*

$$|F_{t\to h^{-1}\tau}\bigl[\bar\chi_T(t)\bigl((hD_x)^\alpha (hD_y)^\beta U(x,y,t)\bigr)\bigr]\big|_{x=y}| \le Ch^{1-d} + CT^2h^{-d+\theta}\bigl(\|\!|A|\!\|_{\theta+1} + \|\!|V|\!\|_{\theta+1}\bigr) \tag{27.2.40}$$

and

$$|\bigl[\bigl((hD_x - A(x))\cdot\boldsymbol{\sigma}\bigr)^\alpha \bigl((hD_y - A(y))\cdot\boldsymbol{\sigma}\bigr)^\beta e(x,y,t)\bigr]\big|_{x=y} - \mathsf{Weyl}_{\alpha,\beta}(x)| \le Ch^{1-d}\bigl(1+\|\partial A\|_{\mathscr{C}(B(x,1))}+\|\partial V\|_{\mathscr{C}(B(x,1))}\bigr) + Ch^{-d+\frac12(\theta+1)}\bigl(\|\!|A|\!\|_{\theta+1}+\|\!|V|\!\|_{\theta+1}\bigr)^{\frac12} \tag{27.2.41}$$

where

$$\mathsf{Weyl}_{\alpha,\beta}(x) := \mathsf{const}\, h^{-d}\int_{\{H(x,\xi)\le\tau\}} \bigl((\xi - A(x))\cdot\boldsymbol{\sigma}\bigr)^{\alpha+\beta}\, d\xi \tag{27.2.42}$$

is the corresponding Weyl expression and

$$H(x,\xi) = \big((\xi - A(x))\cdot\sigma\big)^2 - V(x); \tag{27.2.43}$$

in particular $\mathsf{Weyl}_{\alpha,\beta}(x) = 0$ *for* $|\alpha| + |\beta| = 1$.

Proof. Obviously, the summation of (27.2.39) over $Ch \le |t| \le T$ and a trivial estimate by Ch^{1-d} of the contribution of the interval $|t| \le Ch$ implies (27.2.40).

Then the standard Tauberian arguments imply that the left-hand expression of (27.2.41) does not exceed the right-hand expression of (27.2.40), divided by T, i.e.

$$CT^{-1}h^{1-d} + CTh^{-d+\theta}\big(\|\!|A|\!\|_{\theta+1} + \|\!|V|\!\|_{\theta+1}\big).$$

Optimizing with respect to $T \le \epsilon$, such that $(27.2.33)'$, $(27.2.34)'$ hold, we pick up $T = T^*$ with

$$T^* = \epsilon \min\Big(1,\ \big(\|\partial A\|_{\mathscr{C}(B(x,1))} + \|\partial V\|_{\mathscr{C}(B(x,1))}\big)^{-1},\ h^{-\frac{1}{2}(\theta-1)}\big(\|\!|A|\!\|_{\theta+1} + \|\!|V|\!\|_{\theta+1}\big)^{-\frac{1}{2}}\Big). \tag{27.2.44}$$

Meanwhile the Tauberian formula and (27.2.39) imply that the contribution of the interval $\{t : |t| \asymp T\}$ with $h \le T \le T^*$ to the Tauberian expression does not exceed the right-hand expression of (27.2.39) divided by T, i.e.

$$Ch^{1-d+s}T^{-s-1} + CTh^{-d+\theta}\|\partial A\|_{\mathscr{C}^\theta};$$

summation over $T_* := h^{1-\delta} \le T \le T^*$ results in the right-hand expression of (27.2.41).

So, we need to calculate only the contribution of $\{t : |t| \le T_*\}$ but one can see easily that modulo indicated error it coincides with $\mathsf{Weyl}_{\alpha,\beta}$. □

Remark 27.2.15. For $d \ge 3$ one can skip assumption (27.2.37).

Indeed, we can apply the standard rescaling technique: $x \mapsto x\ell^{-1}$, $h \mapsto \hbar = h\ell^{-\frac{3}{2}}$, $A \mapsto A\ell^{-\frac{1}{2}}$, $V \mapsto V\ell^{-1}$ with $\ell = \max(\epsilon|V|\nu^{-1},\ h^{\frac{2}{3}}\nu^{-\frac{1}{3}})$, $\nu = (1 + |\partial V|_{\mathscr{C}})$; see Section 5.2.

Application

Let us apply developed technique to estimate a minimizer.

Proposition 27.2.16. *Let $\kappa \le c$ and let A be a minimizer. Let*

$$\mu := \|\partial A\|_{\mathscr{C}} \le Ch^{-1+\delta}. \tag{27.2.45}$$

For $d = 2$ let assumption (27.2.37) *be also fulfilled. Then for $\theta \in (1,2)$ estimate*

$$\|\partial A\|_{\mathscr{C}^{\theta-1}} + h^{\theta-1}\|\partial A\|_{\mathscr{C}^{\theta}} \le C\kappa\big(1 + \|V\|_{\mathscr{C}^1} + h^{\frac{1}{2}(\theta-1)}\|V\|^{\frac{1}{2}}_{\mathscr{C}^{\theta+1}}\big) + C\|\partial A\|' \tag{27.2.46}$$

holds with

$$\|\partial A\|' := \sup_y \|\partial A\|_{\mathscr{L}^2(B(y,1))}. \tag{27.2.47}$$

Proof. Consider expression for ΔA. According to equation (27.2.14) and Proposition 27.2.14 $(|\Delta A| + |h\partial\Delta A|)$ does not exceed the right-hand expression of (27.2.41) multiplied by $C\kappa h^{d-1}$ i.e.

$$\|\Delta A\|_{\mathscr{C}} + \|h\partial\Delta A\|_{\mathscr{C}} \le C\kappa\Big(1 + |\partial A|_{\mathscr{C}} + |\partial V|_{\mathscr{C}} + h^{\frac{1}{2}(\theta-1)}\|\partial A\|^{\frac{1}{2}}_{\mathscr{C}^{\theta}} + h^{\frac{1}{2}(\theta-1)}\|\partial V\|^{\frac{1}{2}}_{\mathscr{C}^{\theta}}\Big), \tag{27.2.48}$$

where we replaced $\|\!|A|\!\|_{\theta+1}$ and $\|\!|V|\!\|_{\theta+1}$ by larger $\|\partial A\|_{\mathscr{C}^{\theta}}$ and $\|\partial V\|_{\mathscr{C}^{\theta}}$ respectively.

Then the regularity theory for elliptic equations implies that

(27.2.49) For any $\theta' \in (1,2)$ $h^{\theta'-1}\|\partial A\|_{\mathscr{C}^{\theta'}}$ does not exceed this expression (27.2.48) plus $C\|\partial A\|'$.

Observe that that $\|\partial A\|_{\mathscr{C}}$ does not exceed $(\epsilon\|\partial A\|_{\mathscr{C}^{\theta}} + C'_{\epsilon}\|\partial A\|')$ with arbitrarily small constant $\epsilon > 0$ and therefore

$$h^{\theta-1}\|\partial A\|_{\mathscr{C}^{\theta}} + \epsilon^{-1}\|\partial A\|_{\mathscr{C}} \tag{27.2.50}$$

does not exceed expression (27.2.48) plus $C'_{\epsilon}\|\partial A\|'$ where we used (27.2.49) for $\theta' = \theta$.

Comparing (27.2.50) and (27.2.48) we conclude that for $\kappa \le c$ and sufficiently small constant $\epsilon > 0$ we can eliminate in the derived inequality both contributions of ∂A to (27.2.48) thus we arrive to (27.2.46). □

Having this strong estimate to A allows us to prove

Theorem 27.2.17. *Let $\kappa \le c$, (27.2.45) be fulfilled, and let $d \ge 3$. Assume that*

$$\|V\|_{\mathscr{C}^{\theta+1}} \le c \tag{27.2.51}$$

with $\theta \in (1,2)$. Then

$$\mathsf{E}^* = \mathsf{Weyl}_1 + O(h^{2-d}) \tag{27.2.52}$$

and a minimizer A satisfies

$$\|\partial A\| \le C\kappa^{\frac12} h^{\frac12} \tag{27.2.53}$$

and

$$\|\partial A\|_{\mathscr{C}^{\theta-1}} + h^{\theta-1}\|\partial A\|_{\mathscr{C}^{\theta}} \le C\kappa^{\frac12} h^{\frac12} + C\kappa. \tag{27.2.54}$$

Proof. (a) In virtue of (27.2.40) the Tauberian error when calculating $\mathsf{Tr}(H^-_{A,V})$ does not exceed the right-hand expression of (27.2.40) multiplied by CT^{-2} i.e.

$$Ch^{1-d}T^{-2} + Ch^{-d+\theta}\bigl(\|\!|A|\!\|_{\theta+1} + \|\!|V|\!\|_{\theta+1}\bigr). \tag{27.2.55}$$

Assumption (27.2.51) allows us to simplify this expression and take $T \asymp (1+\mu)^{-1}$; applying estimate (27.2.46) we conclude that the Tauberian error does not exceed

$$C(1+\mu)^2 h^{2-d} + C(\kappa + \|\partial A\|')h^{2-d}. \tag{27.2.56}$$

We claim that

(27.2.57) Weyl error[5)] when calculating $\mathsf{Tr}(H^-_{A,V})$ also does not exceed (27.2.56).

Then

$$\begin{aligned}(27.2.58)\quad \mathsf{E}(A) \ge{}& \\ &\mathsf{Weyl}_1 - C(1+\mu)^2h^{2-d} - C(\kappa + \|\partial A\|')h^{2-d} + \kappa^{-1}h^{1-d}\|\partial A\|^2 \ge \\ &\mathsf{Weyl}_1 - Ch^{2-d} + \frac{1}{2\kappa}h^{1-d}\|\partial A\|^2\end{aligned}$$

[5)] I.e. error when we replace Tauberian expression by Weyl expression.

because $\mu \le C\|\partial A\|' + 1$ due to (27.2.46) and assumption (27.2.51). This implies an estimate of E^* from below and combining with the estimate $\mathsf{E}^* \le \mathsf{E}^*(0) = \mathsf{Weyl}_1 + Ch^{2-d}$ from above we arrive to (27.2.52) and (27.2.53) and then to (27.2.54) due to (27.2.46) and assumption (27.2.51).

(b) To prove (27.2.57) let us plug A_ε instead of A into $e_1(x,x,0)$. Then in virtue of the rough microlocal analysis the contribution to Weyl error of the interval $\{t : T_* \le |t| \le \epsilon\}$ with $T_* = h^{1-\delta}$ would be negligible and the contribution of the interval $\{t : |t| \le T_*\}$ would be $\mathsf{Weyl}_1 + O(h^{2-d})$.

(c) Now let us calculate an error which we made plugging A_ε instead of A into $e_1(x,x,0)$. Obviously it does not exceed $Ch^{-d}\|A - A_\varepsilon\|_{\mathscr{C}}$ and since $\|A - A_\varepsilon\|_{\mathscr{C}} \le C\varepsilon^{\theta+1}\|\partial A\|_{\mathscr{C}^\theta}$ this error does not exceed $Ch^{\theta+1-d-4\delta}\|\partial A\|_{\mathscr{C}^\theta}$, which is marginally worse than what we are looking for.

However it is good enough to recover a weaker version of (27.2.52) and (27.2.53) with an extra factor $h^{-\delta_1}$ in their right-hand expressions. Then (27.2.46) implies a bit weaker version of (27.2.54) and in particular that its left-hand expression does not exceed C.

Knowing this, let us consider the two term approximation. With the above knowledge one can prove easily that the error in two term approximation does not exceed $Ch^{3-d-\delta'}$ with $\delta' = 100\delta$.

Then the second term in the Tauberian expression is

$$\int \big((H_{A,V} - H_{A_\varepsilon,V})e^{\mathsf{T}}_{(\varepsilon)}(x,y,0)\big)\big|_{y=x}\,dx, \tag{27.2.59}$$

where subscript $_{(\varepsilon)}$ means that we plugged A_ε instead of A and superscript $^{\mathsf{T}}$ means that we consider Tauberian expression with $T = T^* = \epsilon$. But then the contribution of the interval $\{t : T_* \le |t| \le T^*\}$ is also negligible and modulo $Ch^{\theta+2-d-4\delta}\|\partial A\|_{\mathscr{C}^\theta}$ we get a Weyl expression. However

$$(H_{A,V} - H_{A_\varepsilon,V}) = -2(\xi - A_\varepsilon)\cdot(A - A_\varepsilon) + |A - A_\varepsilon|^2. \tag{27.2.60}$$

Observe that the first term in the right-hand expression kills the Weyl expression since an integrand is odd with respect to $(\xi - A_\varepsilon)$ while the second term as one can see easily makes it smaller than $Ch^{3-d-\delta'}$. Therefore (27.2.57) has been proven. □

Remark 27.2.18. (i) For $d = 2$ we cannot drop assumption (27.2.37) at this stage we did it for $d \ge 3$. However results of the next section allow us to cure this problem using the partition-and-rescaling technique.

(ii) Actually for $V \in \mathscr{C}^{2,1}$ we have an estimate

$$|\partial A(x) - \partial A(y)| \le C\kappa|x-y|(|\log|x-y|| + 1) + C\mu. \tag{27.2.61}$$

Combining with (27.2.53) we conclude that

$$\|\partial A\|_{\mathscr{C}} \le C\kappa^{(d+1)/(d+2)}|\log h|^{d/(d+2)}h^{1/(d+2)}. \tag{27.2.62}$$

(iii) If (27.2.51) holds for $(h\partial)^m V$ with $m \in \mathbb{Z}^+$ then (27.2.53) and (27.2.53) also hold for $(h\partial)^m A$ instead of A; further, if $(h\partial)^m V \in \mathscr{C}^{2,1}$ then (27.2.61) and (27.2.62) also hold for $(h\partial)^m A$ instead of A.

Classical Dynamics and Sharper Estimates

Now we want to improve the remainder estimate $O(h^{2-d})$ to $o(h^{2-d})$. Sure, we need to impose condition to the classical dynamical system and since $|\partial A| = O(h^\delta)$ with $\delta > 0$ due to (27.2.62), it should be dynamical system associated with the Hamiltonian flow generated by $H_{0,V}$:

(27.2.63) The set of periodic points of the dynamical system associated with Hamiltonian flow generated by $H_{0,V}$ has measure 0 on the energy level 0.

Recall that on the energy level $\{(x,\xi) : H_{0,V}(x,\xi) = \tau\}$ a natural density $d\mu_\tau = dxd\xi : dH|_{H=\tau}$ is defined.

The problem is we do not have a quantum propagation theory for $H_{A,V}$ as A is not a "rough" function (i.e. smooth in microscale ε). However it is a rather regular function, almost $\mathscr{C}^2$, and $(A - A_\varepsilon)$ is rather small: $|A - A_\varepsilon| \le \eta := Ch^{2-3\delta}$ and $|\partial(A - A_\varepsilon)| \le Ch^{1-3\delta}$ and therefore we can apply a method of successive approximations with the unperturbed operator $H_{A_\varepsilon,V}$ as long as $\eta T/h \le h^\sigma$ i.e. as $T \le h^{1-4\delta}$. Here we, however, have no use for such large T and consider $T = O(h^{-\delta})$.

Consider

$$F_{t\to h^{-1}\tau}\chi_T(t)U(x,y,t), \tag{27.2.64}$$

and consider terms of successive approximations. Then if we forget about microhyperbolicity arguments the first term will be $O(h^{-d}T)$, the second term $O(h^{-1-d}\eta T^2) = O(h^{1-d-\delta'})$ and the error term $O(h^{-2-d}\eta^2 T^3) = O(h^{2-d-\delta''})$.

Therefore since our goal is $O(h^{1-d})$ we need to consider the first two terms only. The first term is the same expression (27.2.64) with U replaced by $U_{(\varepsilon)}$.

Consider the second term, it corresponds to $U'_{(\varepsilon)}(x,y,t)$ which is the Schwartz kernel of operator

$$\mathsf{U}'_{(\varepsilon)} := ih^{-1}\int_0^t e^{i(t-t')h^{-1}H_{A_\varepsilon,V}}\big(H_{A,V}-H_{A_\varepsilon,V}\big)e^{it'h^{-1}H_{A_\varepsilon,V}}\,dt' \tag{27.2.65}$$

and then

$$\mathsf{Tr}\big(\mathsf{U}'_{(\varepsilon)}\psi\big) = ih^{-1}\,\mathsf{Tr}\Big(\big(H_{A,V}-H_{A_\varepsilon,V}\big)e^{ih^{-1}tH_{A_\varepsilon,V}}\psi^1(t)\Big) \tag{27.2.66}$$

where

$$\psi^1(t) := \int_0^t e^{ih^{-1}t'H_{A_\varepsilon,V}}\psi e^{-ih^{-1}t'H_{A_\varepsilon,V}}\,dt'$$

is h-pseudodifferential operator with a rough symbol and $\psi^1(t)\sim t$.

Really, one can prove easily studying first the Hamiltonian flow equation and then the transport equations that $\psi_t := e^{ih^{-1}tH_{A_\varepsilon,V}}\psi e^{-ih^{-1}tH_{A_\varepsilon,V}}$ is a h-pseudodifferential operator with a rough symbol and its corresponding norm is bounded.

Note that

$$ih^{-1}F_{t\to h^{-1}\tau}e^{ih^{-1}tH_{A_\varepsilon,V}}\psi^1(t) = (2\pi)\int\big(F_{t\to h^{-1}\tau'}e^{ih^{-1}tH_{A_\varepsilon,V}}\big)\hat f(h^{-1}(\tau-\tau'))\,d\tau'$$

with $\hat f = F_{t\to\tau}f_t$, $f_t = \chi_T(t)\psi^1(t)$ and therefore (27.2.65)–(27.2.66) imply that

$$|F_{t\to h^{-1}\tau}\chi_T(t)\,\mathsf{Tr}\,\mathsf{U}'_{(\varepsilon)}\psi| \le C\eta T^2h^{-d} \tag{27.2.67}$$

where in comparison with the trivial estimate we gained the factor h.

We can plug here $T'\in(T_*,T)$ instead of T and, taking summation by T' from $T_*=\epsilon$ to T, we conclude that estimate (27.2.67) also holds for $\chi_T(t)$ replaced by $\big(\bar\chi_T(t)-\bar\chi_{T_*}(t)\big)$ (provided $\bar\chi=1$ on $(-\frac12,\frac12)$) and since $\eta T^2\le h^{1+\delta}$ for $T\le h^{-\delta}$ we see that the right-hand expression (27.2.67) does not exceed $Ch^{1-d+\delta}$.

On the other hand, our traditional methods imply that

$$|F_{t\to h^{-1}\tau}\chi_T(t)\,\mathsf{Tr}\big(e^{ith^{-1}H_{A_\varepsilon,V}}\psi\big)| \le Ch^{1-d}T\mu(\Pi_{T,\zeta})+C_{T,\zeta}h^{1-d+\delta} \tag{27.2.68}$$

where Π_T is the set of points on energy level 0, periodic with periods not exceeding T, $\Pi_{T,\zeta}$ is its ζ-vicinity, $\zeta>0$ is arbitrarily small; recall that for $d=2$ we assume that condition (27.2.37) is fulfilled.

Here again we can plug any $T' \in (T_*, T)$ instead of T and after summation with respect to T' we conclude that (27.2.68) also holds with $\chi_T(t)$ replaced by $\big(\bar\chi_T(t) - \bar\chi_{T_*}(t)\big)$.

Combining with estimate for $\big(e^{ith^{-1}H_{A_\varepsilon,V}} - e^{ith^{-1}H_{A_\varepsilon,V}}\big)$ we conclude that

$$|F_{t\to h^{-1}\tau}\big(\bar\chi_T(t) - \bar\chi_{T_*}(t)\big)\,\mathsf{Tr}\big(e^{it'h^{-1}H_{A,V}}\psi\big)| \le Ch^{1-d}\,T\mu(\Pi_{T,\zeta}) + C_{T,\zeta}h^{1-d+\delta}$$

and since

$$|F_{t\to h^{-1}\tau}\bar\chi_{T_*}(t)\,\mathsf{Tr}\big(e^{it'h^{-1}H_{A,V}}\psi\big)| \le Ch^{1-d}$$

we conclude that

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)\,\mathsf{Tr}\big(e^{it'h^{-1}H_{A,V}}\psi\big)| \le Ch^{1-d} + Ch^{1-d}\,T\mu(\Pi_{T,\zeta}) + C_{T,\zeta}h^{1-d+\delta}. \tag{27.2.69}$$

Then the Tauberian error does not exceed the right-hand expression of (27.2.69) multiplied by ChT^{-2} and it is less than $CT^{-1}h^{2-d}$.

Consider now the Tauberian expression and again apply the two-term approximation for $e^{ih^{-1}tH_{A,V}}$ considering $e^{ih^{-1}tH_{A_\varepsilon,V}}$ as an unperturbed operator; then the error will be less than $Ch^{2-d+\delta}$.

Consider the second term after taking trace; it is $O(h^{2-d-4\delta})$, so it is just slightly too large. Further, if $\psi = I$ one can calculate it easily and observe that it is $O(h^{2-d+\delta})$ provided $V \in \mathscr{C}^{2,1}$.

Finally, the first term is what we get for $e^{ih^{-1}tH_{A_\varepsilon,V}}$ and in virtue of rough microlocal analysis the contribution of the interval $\{t\colon T_* \le |t| \le T\}$ does not exceed $Ch^{2-d}\mu(\Pi_{T,\zeta}) + C_{T,\zeta}h^{2-d+\delta}$ and the contribution of the interval $\{t\colon |t| \le T_*\}$ is $\mathsf{Weyl}_1 + O(h^{2-d+\delta})$.

Then we arrive to

Theorem 27.2.19. *Let $\kappa \le c$, (27.2.45) and (27.2.51) be fulfilled. Furthermore, let condition (27.2.63) be fulfilled*[6]*. Then*

$$\mathsf{E}^* = \mathsf{Weyl}_1^* + o(h^{2-d}) \tag{27.2.70}$$

where

$$\mathsf{Weyl}_1^* = \mathsf{Weyl}_1 + \varkappa h^{2-d}\int V_+^{\frac{d}{2}}\Delta V\,dx \tag{27.2.71}$$

calculated in the standard way for $H_{0,V}$ and a minimizer A satisfies similarly improved versions of (27.2.53) and (27.2.54).

[6] I.e. $\mu_0(\Pi_\infty) = 0$.

Remark 27.2.20. (i) Recall that for $d = 2$ so far we assume (27.2.37). However we need it only to estimate ∂A. Indeed, even in the case of very degenerate potential V the "magnetic" correction will be small since $|\partial A|$ is small.

(ii) Under stronger assumptions to the Hamiltonian flow one can recover better estimates like $O(h^{2-d}|\log h|^{-2})$ or even $O(h^{2+\delta-d})$ (like in Subsubsection 4.5.4.3 *Sharper Remainder Estimates*).

(iii) We leave to the reader to calculate the numerical constants $\varkappa_*$ here and in (27.2.76) below, $\varkappa = \varkappa_1 - \frac{2}{d}\varkappa_2$.

(iv) However, even if $\psi \neq I$ we can observe that it is sufficient to consider only principal terms and then the second term in approximations is also $O(h^{2-d+\delta})$ provided $V \in \mathscr{C}^{2,1}$ as long as principal symbol of $\psi(x)$ is even with respect to ξ, in particular, if $\psi = \psi(x)$.

27.2.3 Local Theory

Localization and Estimate from above

The results of the previous Subsection 27.2.2 have two critical shortcomings: first, they impose the excessive initial requirement (27.2.21) to κ as we need to start from $M \le ch^{-3}$; second, they are not local. However, curing the second shortcoming, we make the way to addressing the first one as well, using the partition and rescaling technique.

We can localize $\mathsf{Tr}^-(H) = \mathsf{Tr}(H^-)$, which is the first term in $\mathsf{E}(A)$, either in our traditional way as $\mathsf{Tr}(H^-\psi^2)$ or in the way favored by some mathematical physicists[7]: namely, we take $\mathsf{Tr}^-(\psi H\psi)$ where in both cases $\psi \in \mathscr{C}_0^\infty(B(0,\frac{1}{2}))$, $0 \le \psi \le 1$ and some other conditions will be imposed to it later. Observe that

$$\mathsf{Tr}^-(\psi H\psi) \ge \mathsf{Tr}(\psi H^-\psi) = \int e_1(x,x,0)\psi^2(x)\,dx. \tag{27.2.72}$$

Really, let us decompose operator $H = H\theta(-H) + H(1-\theta(-H))$, where $\theta(\tau - H)$ is a spectral projector of H, and therefore in the operator sense $H \ge H^- := H\theta(-H)$ and $\psi H\psi \ge \psi H^-\psi$, and therefore all negative eigenvalues

[7] See f. e. L. Erdös, S. Fournais. and J. P. Solovej [4].

of $\psi H\psi$ are greater than or equal to eigenvalues of the negative operator $\psi H^{-}\psi$, and then

$$\mathsf{Tr}^{-}(\psi H\psi) \ge \mathsf{Tr}(\psi H^{-}\psi) = \mathsf{Tr}\Big(\int_{-\infty}^{0} \tau\, d_{\tau}\theta(\tau - H)\psi^2\Big), \tag{27.2.73}$$

which is exactly the right-hand expression of (27.2.72).

Remark 27.2.21. Each approach has its own advantages.

(i) In particular, no need to localize A (see (ii)) and the fact that Proposition 27.2.5 obviously remains true are advantages of $\mathsf{Tr}^{-}(\psi H\psi)$-localization.

(ii) Further, sinces $\mathsf{Tr}^{-}(\psi H\psi)$ does not depend on A outside of $B(0, \frac{3}{4})$ we may assume that $A = 0$ outside of $B(0,1)$. Really, we can always subtract a constant from A without affecting traces and also cut-off A outside of $B(0,1)$ in a way such that $A' = A$ in $B(0, \frac{3}{4})$ and $\|\partial A'\| \le c\|\partial A\|_{B(0,1)}$; the price is to multiply κ by c^{-1}–as long as principal parts of asymptotics coincide.

(iii) On the other hand, additivity rather than sub-additivity of (27.2.88) and the trivial estimate from the above are advantages of $\mathsf{Tr}(\psi H^{-}\psi)$-localization; therefore it is more advantageous.

(iv) In the next Chapter 28 (in Section 28.2) we will use more $\mathsf{Tr}^{-}(\psi H\psi)$-localization for preliminary estimates from below and simplify many arguments of this Section. We apply these modifications and simplifications to this Section in the final version of the Book.

We will use both methods and here we provide an upper estimate for the larger expression $\mathsf{Tr}^{-}(\psi H\psi)$ and a lower estimate for the lesser expression $\mathsf{Tr}(\psi H^{-}\psi)$. Let us estimate from the above:

Proposition 27.2.22. *Assume that $V \in \mathscr{C}^{2,1}$, $d \ge 2$. Let $\ell(x)$ be a scaling function*[8] *and ψ be a function such that $|\partial^{\alpha}\psi| \le c\psi\ell^{-\sigma|\alpha|}$ for all $\alpha : |\alpha| \le 2$ and $|\psi| \le c\ell^{\sigma(1+\delta)}$ with $\sigma > 1$ and $\delta > 0$*[9].

Then, if $A = 0$,

$$\mathsf{Tr}^{-}(\psi H\psi) = \int \mathsf{Weyl}_1(x)\psi^2(x)\, dx + O(h^{2-d}) \tag{27.2.74}$$

[8] I.e. $\ell \ge 0$ and $|\partial\ell| \le \frac{1}{2}$.

[9] Such compactly supported functions obviously exist.

and under assumption (27.2.63)

$$\mathsf{Tr}^-(\psi H\psi) = \int \mathsf{Weyl}_1^*(x)\psi^2(x)\,dx + o(h^{2-d}) \tag{27.2.75}$$

with

$$\mathsf{Weyl}_1^*(x) = \mathsf{Weyl}_1(x) + \varkappa_1 h^{-1} V_+^{\frac{d}{2}}\Delta V + \varkappa_2 h^{-1} V_+^{\frac{d}{2}-1}|\nabla V|^2 \tag{27.2.76}$$

calculated in the standard way for $H_{0,V}$.

Proof. Let us consider $\tilde{H} = \psi H\psi$ as a Hamiltonian and let $\tilde{e}(x,y,\tau)$ be the Schwartz kernel of its spectral projector. Then

$$\mathsf{Tr}^-(\psi H\psi) = \int \tilde{e}_1(x,x,0)\,dx = \sum_j \int \tilde{e}_1(x,x,0)\psi_j^2\,dx, \tag{27.2.77}$$

where ψ_j^2 form a partition of unity in $\mathbb{R}^d$ and we need to calculate the right hand expression. The problem is that $\tilde{H}$ is not a usual Schrödinger operator because of the degenerating factor ψ on each side.

Consider first an $\epsilon\ell$-admissible partition of unity in $B(0,1)$. Let us consider γ-scale in such element where $\gamma = \epsilon\ell^\sigma$ and we will use 1-scale in ξ. Then after rescaling $x \mapsto x\gamma^{-1}$ the semiclassical parameter rescales $h \mapsto h_{\mathsf{new}} = h\gamma^{-1}$ and the contribution of each γ-subelement to a semiclassical remainder does not exceed $C\psi^2(h/\gamma)^{2-d}$ with $\psi \le \gamma^{1+\delta}$, having the same magnitude over element as long as $\gamma \ge 2h$. Then the contribution of ℓ-element to a semiclassical error does not exceed $C\psi^2(h/\gamma)^{2-d}\times \ell^d\gamma^{2-d} \asymp Ch^{2-d}\psi^2\gamma^{-2}\ell^d \le Ch^{2-d}\ell^{d+2\delta}$.

Note that expression (27.2.77) only increases if we sum only with respect to elements where $\ell^\sigma \ge h$. Therefore we arrive to estimate

$$\mathsf{Tr}^-(\psi H\psi) \le \int \mathsf{Weyl}_1(x)\psi^2(x)\,dx + Ch^{2-d},$$

where integration is taken over a domain $\{x\colon \ell(x) \ge h^{1/\sigma}\}$. Note that we can extend this integral to $\mathbb{R}^d$: indeed, it will add negative term with absolute value not exceeding $Ch^{-d}\times h^{2+\delta}$ as $\psi \le h^{1+\delta}$ there and it is absorbed by the remainder estimate. □

Corollary 27.2.23. *In the framework of Proposition 27.2.22*

$$\mathsf{E}_\psi^* \coloneqq \inf_A \mathsf{E}_\psi(A) \le \int \mathsf{Weyl}_1(x)\psi^2(x)\,dx + Ch^{2-d} \tag{27.2.78}$$

and under assumption (27.2.63)

$$\mathsf{E}^*_\psi \le \int \mathsf{Weyl}_1(x)\psi^2(x)\,dx + Ch^{2-d} \tag{27.2.79}$$

with

$$\mathsf{E}_\psi(A) \coloneqq \mathsf{Tr}^-(\psi H\psi) + \frac{1}{\kappa h^2}\int |\partial A|^2\,dx. \tag{27.2.80}$$

Proof. Indeed, we just pick $A = 0$. □

Estimate from below

Now let us estimate *redefined* $\mathsf{E}_\psi(A)$,

$$\mathsf{E}_\psi(A) \coloneqq \int e_1(x,x,0)\psi^2(x)\,dx + \frac{1}{\kappa h^{d-1}}\int |\partial A|^2\,dx, \tag{27.2.81}$$

from below. However we need an equation for an optimizer and it would be easier for us to deal with even lesser expression involving τ-regularization. Let us rewrite the first term in the right-hand expression in the form

$$\int_{-\infty}^0 \bar\varphi(\tau/L)\tau\, d_\tau e(x,x,\tau) + \int_{-\infty}^0 (1-\bar\varphi(\tau/L))\tau\, d_\tau e(x,x,\tau) \ge \int_{-\infty}^L \Big(\bar\varphi(\tau/L)(\tau-L)\, d_\tau e(x,x,\tau) + (1-\bar\varphi(\tau/L))\tau\, d_\tau e(x,x,\tau)\Big),$$

where $\bar\varphi \in \mathscr{C}_0^\infty([-1,1])$ equals 1 in $[-\frac12,\frac12]$ and let us estimate from below

$$\mathsf{E}'_\psi(A) \coloneqq \int \Big(\int_{-\infty}^L \bar\varphi(\tau/L)(\tau-L) d_\tau e(x,x,\tau)(x) + (1-\bar\varphi(\tau/L))(\tau-L)\, d_\tau e(x,x,\tau)\Big)\psi^2(x)\,dx + \frac{1}{kh^{1-d}}\int |\partial A|^2\,dx. \tag{27.2.82}$$

Let us generalize Proposition 27.2.4:

Proposition 27.2.24. *Let A be a minimizer of $\mathsf{E}'_\psi(A)$. Then*

$$\frac{2}{\kappa h^{1-d}}\Delta A_j(x) = \Phi_j \coloneqq \mathsf{Re}\,\mathsf{tr}\Big[\sigma_j\Big((hD-A)_x\cdot\boldsymbol{\sigma}\mathcal{K}(x,y,\tau) + \mathcal{K}(x,y,\tau)\,{}^t(hD-A)_y\cdot\boldsymbol{\sigma}\Big)\Big]\Big|_{y=x} \tag{27.2.83}$$

with

$$\mathcal{K} = \int_{-\infty}^{L} \mathsf{SK}\Big[\bar{\varphi}(\tau/L)(\tau-L)\,\mathsf{Res}_{\mathbb{R}}(\tau-H)^{-1}\psi^2(\tau-H)^{-1}+ \\ (1-\bar{\varphi}(\tau/L))\tau(\tau-L)\,\mathsf{Res}_{\mathbb{R}}(\tau-H)^{-1}\psi^2(\tau-H)^{-1}\Big](x,y)\,d\tau,$$

where we use a temporary notation $\mathsf{SK}[B](x,y)$ *for the Schwartz kernel of operator* B*.*

Proof. Follows immediately from the proof of Proposition 27.2.4. □

Proposition 27.2.25. *Let* $d=3$ *and assumptions* (27.2.20) *and* (27.2.21) *be fulfilled. Then for* $\tau \le c$

(i) Operator norm in $\mathscr{L}^2$ *of* $(hD)^k(\tau-H)^{-1}$ *does not exceed* $C|\operatorname{Im}\tau|^{-1}$ *for* $k=0,1,2$*;.*

(ii) Operator norm in $\mathscr{L}^2$ *of* $(hD)^2\big((hD-A)\cdot\boldsymbol{\sigma}\big)(\tau-H)^{-1}$ *does not exceed* $C|\operatorname{Im}\tau|^{-1}$ *for* $k=0,1,2$*.*

Proof. Proof follows the same scheme as the proof of Proposition 27.2.6. □

Proposition 27.2.26. *Let* $d=3$ *and assumptions* (27.2.20) *and* (27.2.21) *be fulfilled. Then* $|\Phi(x)| \le Ch^{-3}$*.*

Proof. Let us estimate

$$|\int \tau\varphi(\tau/L)\,\mathsf{Res}_{\mathbb{R}}\,\mathsf{SK}\Big[Q(\tau-H)^{-1}\psi^2(\tau-H)^{-1}\Big](x,y)\,d\tau|, \tag{27.2.84}$$

where $L \le c$ and $\varphi \in \mathscr{C}_0^\infty([-1,1])$ and also a similar expression with a factor $(\tau-L)$ instead of τ; here either $Q=I$, or $Q=(hD_k-A_k)_x$ or $Q=(hD_k-A_k)_y$.

Proposition 27.2.25 implies that the Schwartz kernel of the integrand does not exceed $Ch^{-3}|\operatorname{Im}\tau|^{-2}$ and therefore expression (27.2.84) does not exceed $CL^2\times h^{-3}L^{-2}=Ch^{-3}$.

Then what comes out in Φ from the term with the factor $\bar{\phi}(\tau/h)$ does not exceed Ch^{-3}.

Then, representing $\big(1-\bar{\phi}(\tau/h)\big)$ as a sum of $\varphi(\tau/L)$ with $L=2^nh$ with $n=0,\dots,\lfloor|\log h|\rfloor+c$, we estimate the output of each term by Ch^{-3} and thus the whole sum by $Ch^{-3}|\log h|$.

To get rid of the logarithmic factor we use equality

$$(\tau - H)^{-1}\psi(\tau - H)^{-1} = - \partial(\tau - H)^{-1}\psi + (\tau - H)^{-2}[h, \psi](\tau - H)^{-1}; \tag{27.2.85}$$

if we plug only the second part, we recover a factor h/L, where h comes from the commutator and $1/L$ from the increased singularity; an extra operator factor in the commutator is not essential. Then summation over partition results in Ch^{-3}.

Plugging only the first part we do not use the above decomposition but the equality $\mathsf{Res}_{\mathbb{R}}(\tau - H)^{-1}\, d\tau = d_\tau \theta(\tau - H)$. □

Corollary 27.2.27. *Let $d = 3$, assumptions* (27.2.20) *and* (27.2.21) *be fulfilled and A be a minimizer. Then* (27.2.28) *and* (27.2.29) *hold.*

Proof. Proof follows the proof of Corollary 27.2.8. □

Now we can recover both Proposition 27.2.16 and our both main Theorems 27.2.17 and 27.2.19:

Theorem 27.2.28. *Let $d = 3$ and assumptions* (27.2.20) *and $\kappa \le c$ be fulfilled. Then*

(i) The following estimate holds:

$$\mathsf{E}^*_\psi - \int \mathsf{Weyl}_1(x)\psi^2(x)\, dx = O(h^{2-d}) \tag{27.2.86}$$

and and a minimizer A satisfies (27.2.53) *and* (27.2.54)*.*

(ii) Furthermore, let assumption (27.2.63) *be fulfilled (i.e. $\mu_0(\Pi_\infty) = 0$). Then*

$$\mathsf{E}^*_\psi - \int \mathsf{Weyl}^*_1(x)\psi^2(x)\, dx = o(h^{2-d}) \tag{27.2.87}$$

and a minimizer A satisfies similarly improved versions of (27.2.53) *and* (27.2.54)*.*

27.2.4 Rescaling

Now we apply the rescaling. We consider only $d = 3$ here.

Case $\kappa \le 1$

We already have an upper estimate: see Corollary 27.2.23. Let us prove a lower estimate[10]. Consider an error

$$\Big(\int \mathsf{Weyl}_1(x)\psi^2\,dx - \mathsf{E}_\psi(A)\Big)_+. \tag{27.2.88}$$

Obviously, Tr^- is sub-additive

$$\mathsf{Tr}^-(\sum_j \psi_j H \psi_j) \ge \sum_j \mathsf{Tr}^-(\psi_j H \psi_j), \tag{27.2.89}$$

and therefore so is $\mathsf{E}_\psi(A)$ under assumption that $\psi_j \in \mathscr{C}_0^2(B(x_j, \frac{1}{2}\ell_j))$, where multiplicity of covering by $B(x_j,\ell_j))$ does not exceed C_0 and we are allowed to replace κ by $C_1\kappa$ in the right-hand expression[11].

Then we need to consider each partition element and use a lower estimate for it. While considering partition we use so called *ISM identity*: if

$$\sum_j \psi_j^2 = 1, \tag{27.2.90}$$

we have

$$H = \sum_j \big(\psi_j H \psi_j + \frac{1}{2}[[H,\psi_j],\psi_j]\big) = \sum_j \psi_j \big(H + \frac{1}{2}\sum_k [[H,\psi_k],\psi_k]\big)\psi_j, \tag{27.2.91}$$

where the second equality is due to the fact that $[[H,\psi_j],\psi_j]$ is an ordinary function.

In virtue of Proposition 27.2.5, from the very beginning we need to consider

$$M = \kappa^\beta h^{-1-\alpha} \tag{27.2.92}$$

[10] But only for $\mathsf{E}_\psi(A)$ defined by (27.2.80).

[11] Really, $\|\partial A\|^2 \ge c\sum_j \|\partial A_j\|^2$ with $A_j(x) = (A(x)-A_j(x_j))\psi_j'$ with $\psi_j' \in \mathscr{C}_0^2(B(x_j, \frac{7}{8}\ell_j))$ equal 1 in $B(x_j, \frac{3}{4}\ell_j)$).

with $\alpha = 2$, $\beta = 0$ and $\kappa \le c$. But we need to satisfy precondition (27.2.21) which is then

$$\kappa^{\beta+1} h^{-\alpha} \le c. \tag{27.2.93}$$

Therefore, if condition (27.2.93) is fulfilled with $\alpha = 0$, we conclude that the final error is indeed $O(h^{-1})$ or even $o(h^{-1})$ (under assumption (27.2.63)) without any precondition.

Let precondition (27.2.93) fail. Let us use γ-admissible partition of unity ψ_j^2 with ψ_j satisfying after rescaling assumptions of Proposition 27.2.22.

Note that rescaling $x \mapsto x\gamma^{-1}$ results in $h \mapsto h_{\text{new}} = h\gamma^{-1}$ and after rescaling in the new coordinates $\|\partial A\|^2$ acquires factor γ^{d-2} and thus factor $\kappa^{-1}h^{-2}$ becomes $\kappa^{-1}h^{-2}\gamma^{d-2} = \kappa_{\text{new}}^{-1}h_{\text{new}}^{-2}$ with $\kappa \mapsto \kappa_{\text{new}} = \kappa\gamma$.

Then *after rescaling* precondition (27.2.93) is satisfied provided *before rescaling* $\kappa^{\beta+1}h^{-\alpha}\gamma^{\alpha+\beta+1} \le c$. Thus, let us pick up the largest γ satisfying this: $\gamma = \kappa^{-(\beta+1)/(\alpha+\beta+1)}h^{\alpha/(\alpha+\beta+1)}$. Obviously, if before rescaling condition (27.2.93) fails, then $h \ll \gamma \le 1$.

But then expression (27.2.88) with ψ replaced by ψ_j does not exceed $Ch_{\text{new}}^{-1} = C(h\gamma^{-1})^{-1}$ and the total expression (27.2.88) does not exceed $C(h\gamma^{-1})^{-1}\gamma^{-3} = Ch^{-1}\gamma^{-2} = C\kappa^{\beta'}h^{-1-\alpha'}$ with

$$\beta' = 2(\beta+1)/(\alpha+\beta+1), \qquad \alpha' = 2\alpha/(\alpha+\beta+1).$$

Therefore, actually we can pick up M with α, β replaced by α', β' and we have a precondition (27.2.93) with these new α', β' and we do not need an old precondition. Repeating the rescaling procedure again, we derive a proper estimate with again weaker precondition etc.

One can see easily that $\alpha' + \beta' + 1 = 3$ and therefore on each step $\alpha + \beta + 1 = 3$ and we have a recurrent relation for α': $\alpha' = \frac{2}{3}\alpha$; and therefore we have sequence for α which decays and becomes arbitrarily small. Therefore precondition (27.2.93) has been reduced to a much weaker assumption $\kappa \le h^{\delta}$ and under it estimate $M = O(h^{-1})$ has been established. Furthermore, after this under assumption (27.2.63) we can prove even sharper asymptotics.

To weaken assumption $\kappa \le h^{\delta}$ to $\kappa \le c$ we can use rescaling $x \mapsto x\gamma^{-1}$ with $\gamma = h^{\delta}$. We arrive to the error estimate $O(h^{-1-\delta})$ and therefore optimizer satisfies $\|\nabla \times A\| \le h^{\frac{1}{2}-\delta}$ (where δ is increased if necessary but remains arbitrarily small). Repeating the arguments of the proof of Proposition 27.2.6, instead of $\|\Delta A\|_{\mathscr{L}^\infty} = O(1)$ we arrive to $\|\Delta A\|_{\mathscr{L}^\infty} = O(h^{-\delta})$

and to $\|\partial^2 A\|_{\mathscr{L}^\infty} = O(h^{-\delta})$; then $\|\partial A\|_{\mathscr{L}^\infty} = O(h^{\frac{1}{2}-\delta})$; it is more than sufficient to unleash the microlocal analysis technique without any need to appeal to Proposition 27.2.6 which is the only place where we needed assumption (27.2.21).

Thus we arrive to

Theorem 27.2.29. *Let $d = 3$, $V \in \mathscr{C}^{2,1}$, $\kappa \le c$ and let ψ satisfy assumption of Proposition 27.2.22. Then*

(i) Asymptotics (27.2.86) *holds.*

(ii) Further, if assumption (27.2.63) *is fulfilled then asymptotics* (27.2.86) *holds.*

(iii) If (27.2.86) *or* (27.2.87) *holds for $E_\psi(A)$ (we need only an estimate from below) then $\|\partial A\| = O((\kappa h)^{\frac{1}{2}})$ or $\|\partial A\| = o((\kappa h)^{\frac{1}{2}})$ respectively.*

Case $1 \le \kappa \le h^{-1}$

In this framework we can consider even the case $1 \le \kappa \le h^{-1}$. The simple rescaling-and-partition arguments with $\gamma = \kappa^{-1}$ lead to the following

(27.2.94) If $1 \le \kappa \le h^{-1}$, then the remainder estimate $O(\kappa^2 h^{-1})$ holds and for a minimizer A satisfies $\|\partial A\|^2 \le C\kappa^3 h$.

However we would like to improve it and, in particular, to prove that if κ is moderately large then the remainder estimate is still $O(h^{-1})$ and even $o(h^{-1})$ under non-periodicity assumption.

Theorem 27.2.30. *Let $d = 3$, $V \in \mathscr{C}^{2,1}$, and let ψ satisfy assumptions of Proposition 27.2.22. Then*

(i) For

$$\kappa \le \kappa_h^* := \epsilon h^{-\frac{1}{4}} |\log h|^{-\frac{3}{4}} \tag{27.2.95}$$

asymptotics (27.2.86) *holds.*

(ii) Furthermore, for $\kappa = o(\kappa_h^*)$ and assumption (27.2.63) *is fulfilled then asymptotics* (27.2.87) *holds.*

(iii) If $\kappa_h^* \le \kappa \le ch^{-1}$ the following estimate holds:

$$|\mathsf{E}_\psi^* - \int \mathsf{Weyl}_1(x)\psi^2(x)\,dx| \le Ch^{-3}(\kappa h)^{\frac{8}{3}}|\log \kappa h|^2. \tag{27.2.96}$$

Proof. (i) From (27.2.46) we conclude for $\kappa \ge c$ that

$$h^{1-\theta}|\partial A|_{\mathscr{C}^\theta} \le C\kappa(\kappa + \bar\mu).$$

Then, using arguments of Subsection 27.2.2, one can prove easily that for $\kappa \le h^{\sigma-\frac12}$

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)(hD_x)^\alpha(hD_x)^\beta\big(U(x,y,t) - U_{(\varepsilon)}(x,y,t) - U'_{(\varepsilon)}(x,y,t)\big)| \le Ch^{1-d},$$

where we use the same 2-term approximation, $T = \epsilon\bar\mu^{-1}$. Let us take then $x = y$, multiply by $\varepsilon^{-d}\psi(\varepsilon^{-1}(y-z))$ and integrate over y. Using the rough microlocal analysis technique, one can prove easily that from both $U_{(\varepsilon)}(x,y,t)$ and $U'_{(\varepsilon)}(x,y,t)$ we get $O(h^{-2})$ and in the end of the day we arrive to the estimate $|\Delta A_\varepsilon| \le C\kappa\bar\mu$, which implies

$$|\partial^2 A_\varepsilon| \le C\kappa\bar\mu|\log h| + C\mu, \tag{27.2.97}$$

where obviously one can skip the last term. Here we used the regularity property of the Laplace equation. For our purpose it is much better than the estimate $|\partial^2 A_\varepsilon| \le C\kappa^2|\log h| + C\mu$, which one could derive easily.

Again, using arguments of Subsection 27.2.2, one can prove easily that

$$|\operatorname{Tr}(\psi H^-_{A,V}\psi) - \operatorname{Tr}(\psi H^-_{A_\varepsilon,V}\psi)| \le C\bar\mu^2 h^{2-d} \tag{27.2.98}$$

and therefore

$$|\operatorname{Tr}(\psi H^-_{A,V}\psi) - \int \mathsf{Weyl}_1(x)\psi^2(x)\,dx| \le C\bar\mu^2 h^{2-d} \tag{27.2.99}$$

and finally for an optimizer

$$\|\partial A\|^2 \le C\kappa\bar\mu^2 h. \tag{27.2.100}$$

Here μ and $\bar\mu$ were calculated for A, but it does not really matter since due to the estimate $|\partial^2 A| \le C\kappa^2 h^{-\delta}$ we conclude that $|\partial A - \partial A_\varepsilon| \le C\kappa^2 h^{-\delta}\varepsilon \le C$ due to restriction to κ.

Then, for $d = 3$

$$\mu^2\big(\mu/(\kappa\bar{\mu}|\log h|)\big)^3 \le \kappa\bar{\mu}^2 h \tag{27.2.101}$$

and if $\mu \ge 1$ we have $\bar{\mu} = \mu$ and (27.2.101) becomes $\kappa^{-3}|\log h|^{-3} \le C\kappa h$ which is impossible under (27.2.95).

So, $\mu \le 1$ and (27.2.101) implies (27.2.86) and (27.2.100), (27.2.101) imply that for an optimizer $\|\partial A\| \le C(\kappa h)^{\frac{1}{2}}$ and $\mu \le C\kappa^4 h|\log h|^d$. So Statement (i) is proven.

(ii) Proof of Statement (ii) follows then in virtue of arguments of Subsection 27.2.2.

(iii) If $\kappa_h^* \le \kappa \le h^{-1}$ we apply the partition-and-rescaling technique. Then $h \mapsto h' = h\gamma^{-1}$ and $\kappa \mapsto \kappa' = \kappa\gamma$ and to get into the framework of (27.2.95) we need $\gamma = \epsilon\kappa^{-\frac{4}{3}}h^{-\frac{1}{3}}|\log(\kappa h)|^{-1}$, leading to the remainder estimate $Ch^{-1}\gamma^{-2}$, which proves Statement (iii). □

Problem 27.2.31. Repeat arguments of Subsubsections *27.2.1.2. Preliminary Analysis* and *27.2.1.3. Estimates* and of this Subsection for $d \ne 3$. When they hold?

27.3 Global Trace Asymptotics in the Case of Coulomb-Like Singularities

27.3.1 Problem

We consider the same operator (27.1.4) as before in $\mathbb{R}^3$ but now we assume that V has Coulomb-like singularities. Namely let $\mathsf{y}_m \in \mathbb{R}^3$ ($m = 1, \dots, M$, where M is fixed) be singularities ("nuclei"). We assume that

$$V = \sum_{1\le m\le M} \frac{z_m}{|x - \mathsf{y}_m|} + W(x) \tag{27.3.1}$$

where

$$z_m \ge 0,\ z_1 + \dots + z_M \asymp 1, \tag{27.3.2}$$

and

$$|D^{\alpha}W| \le C_{\alpha} \sum_{1\le m\le M} z_m\big(|x-y_m|+1\big)^{-1}|x-y_m|^{-|\alpha|} \qquad \forall \alpha : |\alpha| \le 2, \tag{27.3.3}$$

but at the first stages we will use some weaker assumptions. Later we assume that $V(x)$ decays at infinity sufficiently fast. Let us define $\mathsf{E}^*)$ and $\mathsf{E}(A)$ by (27.2.2)–(27.2.1). Finally, let $\ell(x)\min_{1\le m\le M}\ell_m(x)$ with $\ell_m(x) := \frac{1}{2}|x-y_m|$.

In this and next Sections we assume that

$$\kappa \in (0,\kappa^*] \text{ where } 0<\kappa^* \text{ is a small constant.} \tag{27.3.4}$$

For $\kappa = 0$ we set $A = 0$ and consider $\mathsf{E} := \mathsf{Tr}^-(H_{A,V})$; then our results are covered by Chapter 25.

27.3.2 Estimates of a Minimizer

Let us consider a Hamiltonian with potential V and let A be a magnetic potential, minimizing expression (27.2.2). We say that A is a *minimizer* and in the framework of our problems we will prove it existence.

Preliminary Analysis

First, we start from the roughest possible estimate:

Proposition 27.3.1. *Let V satisfy* (27.3.1)–(27.3.3) *and let $\kappa \le \kappa^*$. Then the near-minimizer A satisfies*

$$\Big|\int \big(\mathsf{tr}\, e_{A,1}(x,x,0) - \mathsf{Weyl}_1(x)\big)\,dx\Big| \le Ch^{-2} \tag{27.3.5}$$

and

$$\|\partial A\| \le C\kappa^{\frac{1}{2}}. \tag{27.3.6}$$

Proof. Definitely (27.3.5)–(27.3.6) follow from the results of L. Erdös, S. Fournais, and J. P. Solovej [3] but we give an independent easier proof, based on our Subsection 27.2.1.

(a) First, let us pick up $A = 0$ and consider $\mathsf{Tr}\big(\psi_\ell \theta(-H)\psi_\ell\big)$ with cut-offs $\psi_\ell(x) = \psi((x-y_m)/\ell)$ where $\psi \in \mathscr{C}_0^\infty(B(0,1))$ and equals 1 in $B(0,\frac{1}{2})$. Here and below $\theta(\tau - H_{A,V})$ is a spectral projector of H.

Then

$$|\mathsf{Tr}\big(\psi_\ell H_{A,V}^-(0)\psi_\ell\big)| \le Ch^{-2} \qquad \text{for } \ell = \ell_* := h^2. \tag{27.3.7}$$

On the other hand, contribution of $B(x,\ell)$ with $\ell(x) \ge \ell_*$ to the Weyl error does not exceed $C\zeta^2\hbar^{-1} = C\zeta^3\ell h^{-1}$ where $\hbar = h/(\zeta\ell)$ in the rescaling; so after summation over $\ell \ge \ell_*$ we also get $O(h^{-2})$ provided $\zeta^2 \le C\ell^{-1}$. Therefore we arrive to the following rather easy inequality:

$$|\int \big(\mathsf{tr}\, e_{0,1}(x,x,0) - \mathsf{Weyl}_1(x)\big)\,dx| \le Ch^{-2}. \tag{27.3.8}$$

This is what the rescaling method gives us without careful the study of the singularity.

(b) On the other hand, consider $A \ne 0$. Let us prove first that

$$\mathsf{Tr}^-(\psi_\ell H \psi_\ell) \ge -Ch^{-2} - Ch^{-2}\int |\partial A|^2\,dx \qquad \text{for } \ell = \ell_*. \tag{27.3.9}$$

Rescaling $x \mapsto (x-y_m)/\ell$ and $\tau \mapsto \tau/\ell$ and therefore $h \mapsto h\ell^{-\frac{1}{2}} \asymp 1$ and $A \mapsto A\ell^{\frac{1}{2}}$ (because the singularity is Coulomb-like), we arrive to the same problem with the same κ (in contrast to Subsection 27.2.4 where $\kappa \mapsto \kappa\ell$ because of the different scale in τ and h) and with $\ell = h = 1$.

In this case the required estimate follows from L. Erdös, J. P. Solovej [1] (we reproduce Lemma 2.1 of this paper in Appendix 27.A.1.

(c) Consider now function ψ_ℓ as in (c) with $\ell \ge \ell_*$. Then according to Theorem 27.2.29 rescaled

$$\mathsf{Tr}^-\big(\psi_\ell H_{A,V}\psi_\ell\big) - \int \mathsf{Weyl}_1(x)\psi_\ell^2(x)\,dx \ge -C\zeta^3\ell h^{-1} - Ch^{-2}\int_{B(x,2\ell/3)} |\partial A|^2\,dx. \tag{27.3.10}$$

Really, rescaling of the first part is a standard one and in the second part we should have in the front of the integral a coefficient $\kappa^{-1}h^{-2}\zeta^2 \times \zeta^{-2}\ell(h/\zeta\ell)^{-2}$

where factor ζ^2 comes from the scaling of the spectral parameter, factor ζ^{-2} comes from the scaling of the magnitude of A, factor $\ell = \ell^3 \times \ell^{-2}$ comes from the scaling of dx and ∂ respectively, and $\hbar := h/(\zeta\ell)$ is a semiclassical parameter after rescaling. Therefore this expression acquires a factor $\zeta^2\ell \le C$.

Then we conclude that

$$\int \bigl(\operatorname{tr} e_{A,1}(x,x,0) - \mathsf{Weyl}_1(x)\bigr)\,dx \ge -Ch^{-2} - Ch^{-2}\int |\partial A|^2\,dx \tag{27.3.11}$$

and adding the magnetic field energy $\kappa^{-1}h^{-2}\|\partial A\|^2$ we find out that

$$\mathsf{E}(A) - \int \mathsf{Weyl}_1(x)\,dx \ge \mathsf{E}(0) - \int \mathsf{Weyl}_1(x)\,dx + (\kappa^{-1}-C)h^{-2}\|\partial A\|^2 - Ch^{-2} \tag{27.3.12}$$

since $\mathsf{Weyl}_1(x)$ does not depend on A. However due to (a) the right-hand expression is greater than $(\kappa^{-1}-C)h^{-2}\|\partial A\|^2 - Ch^{-2}$.

On the other hand, since A is supposed to be a near-minimizer, the left-hand expression of (27.3.12) should not exceed the same expression for $A=0$ plus Ch^2, i.e. Ch^2 due to (a) again. Then due to)(27.3.4) we arrive to (27.3.5) and (27.3.6). □

Proposition 27.3.2. *Let V satisfy* (27.3.1)–(27.3.3)*. Then there exists a minimizer A.*

Proof. After Proposition 27.3.1 has been proven we just repeat arguments of the proof of Proposition 27.2.2. If $V \in \mathscr{L}^{\frac{5}{2}}$ no change would be required but for $V \notin \mathscr{L}^{\frac{5}{2}}$ one needs to consider modifications as in Remark 27.3.3 below. □

Remark 27.3.3. We are a bit ambivalent about a convergence of $\int \mathsf{Weyl}_1(x)\,dx$ at infinity, since for the Coulomb potential it diverges. To avoid this issue, however, we can either assume in addition that $V \in \mathscr{L}^{\frac{5}{2}}$, or tackle it as in Proposition 27.3.16 below.

Estimates to a Minimizer. I

Let us repeat arguments of Subsubsection *27.2.1.3. Estimates.* However our task now is much more complicated: while we know a priory that $\|\partial A\|^2 \le C\kappa$ we will not be able to improve it significantly (or at all for $\kappa \asymp 1$).

Recall equation (27.2.14) for a minimizer A. After rescaling $x \mapsto x/\ell$, $\tau \mapsto \tau/\zeta^2$, $h \mapsto \hbar = h/(\zeta\ell)$, $A \mapsto A\zeta^{-1}\ell$ this equation becomes

$$\Delta A_j = -2\kappa\zeta^2\ell\hbar^2 \operatorname{Re}\operatorname{tr}\Big[\sigma_j\Big(\big(\hbar Dk - \zeta^{-1}A\big)_x\cdot\boldsymbol{\sigma} e(x,y,\tau) + e(x,y,\tau)\,{}^t\big(\hbar D - \zeta^{-1}A\big)_y\cdot\boldsymbol{\sigma}\Big)\Big]\Big|_{y=x} \tag{27.3.13}$$

and since so far $\zeta^2\ell = 1$ we arrive to

$$\Delta A_j = -2\kappa\hbar^2 \operatorname{Re}\operatorname{tr}\Big[\sigma_j\Big(\big(\hbar D - \zeta^{-1}A\big)_x\cdot\boldsymbol{\sigma} e(x,y,\tau) + e(x,y,\tau)\,{}^t\big(\hbar D - \zeta^{-1}A\big)_y\cdot\boldsymbol{\sigma}\Big)\Big]\Big|_{y=x}. \tag{27.3.14}$$

(a) Plugging $u = \psi\theta(-H)f$ with cut-off function ψ and repeating arguments of Subsubsection *27.2.1.3. Estimates*, we conclude that in the rescaled coordinates

$$\begin{aligned}\|(\hbar D_x\cdot\boldsymbol{\sigma})u\| &\le \|((\hbar D_x - A)\cdot\boldsymbol{\sigma})u\| + C\|A\|_{\mathscr{L}^6}\cdot\|u\|_{\mathscr{L}^3}\\ &\le \|((\hbar D_x - A)\cdot\boldsymbol{\sigma})u\| + C\hbar^{-\frac12}\|A\|_{\mathscr{L}^6}\cdot\|u\|^{\frac12}\cdot\|\hbar D_x u\|^{\frac12}\\ &\le \|((\hbar D_x - A)\cdot\boldsymbol{\sigma})u\| + \frac12\|\hbar D_x u\| + C(\hbar^{-\frac12}\|A\|_{\mathscr{L}^6})^2\|u\|,\end{aligned} \tag{27.3.15}$$

where $\|A\|_{\mathscr{L}^6}$ calculated in the rescaled coordinates is equal to $\|A_{\mathsf{orig}}\|_{\mathscr{L}^6,\mathsf{orig}}$ (where subscripts "orig"means that the norm is calculated in the original coordinates and A) which does not exceed $C\kappa^{\frac12}$ due to (27.3.6)[12] and therefore (since $\|(\hbar D_x\cdot\boldsymbol{\sigma})u\| = \|\hbar D_x u\|$)

$$\|\hbar D_x u\| \le C\big(1+\kappa\hbar^{-1}\big)\|f\|. \tag{27.3.16}$$

Continuing arguments of Subsubsection *27.2.1.3. Estimates*, we conclude that in the rescaled coordinates

$$\|(\hbar D_x)^k u\| \le C\big(1+\kappa\hbar^{-1}\big)^k\|f\|, \tag{27.3.17}$$

$$\|(\hbar D_x)^k((\hbar D_x - A)\cdot\boldsymbol{\sigma})u\| \le C(1+\kappa\hbar^{-1})^k\|f\|, \tag{27.3.18}$$

[12] As usual we assume that the average of A over $B(x,1)$ is 0.

for $k = 0, 1, 2$ and therefore

$$\|\Delta A\|_{\mathscr{L}^\infty(B(x,1))} \le C\kappa\hbar^{-1}(1+\kappa\hbar^{-1})^3. \tag{27.3.19}$$

Here we estimate different norms of A locally. Then either

$$\|\partial A\|_{\mathscr{L}^\infty(B(x,\frac{3}{4}))} + \hbar^\delta \|\partial^2 A\|_{\mathscr{L}^\infty(B(x,\frac{3}{4}))} \le C\kappa\hbar^{-1}(1+\kappa\hbar^{-1})^3 \tag{27.3.20}$$

or

$$\|\partial A\|_{\mathscr{L}^\infty(B(x,1-\epsilon))} + \hbar^\delta \|\partial^2 A\|_{\mathscr{L}^\infty(B(x,1-\epsilon))} \le C\|\partial A\| = C\|\partial A_{\mathsf{orig}}\|_{\mathsf{orig}} \le C\kappa^{\frac{1}{2}} \tag{27.3.21}$$

In the latter case (27.3.21) we have in the original coordinates

$$\|\partial A\|_{\mathscr{L}^\infty(B(x,\ell))} \le C\kappa^{\frac{1}{2}}\ell^{-\frac{3}{2}} \tag{27.3.22}$$

and we are rather happy because then the effective intensity of the magnetic field in $B(x,\ell)$ is $\zeta^{-1}\ell\|\partial A\|_{\mathscr{L}^\infty(B(x,1-\epsilon))} \le C\kappa^{\frac{1}{2}}$.

(b) The former case (27.3.20) is much more complicated because our estimate is really poor for $\kappa \asymp 1$ and we are going to act only in the assumption (27.3.4). Assume that

$$\|\partial A\|_{\mathscr{L}^\infty(B(x,1-\epsilon))} \le \mu \tag{27.3.23}$$

with $\mu \ge \hbar^{-\sigma}$. Selecting $u = \psi\theta(-H)f$ with γ-admissible ψ we conclude that $\|(A\cdot\boldsymbol{\sigma})u\| \le \|A\|_{\mathscr{L}^\infty}\|u\| \le C\mu\gamma\|u\|$ (assuming without any loss of the generality that $A = 0$ at some point of $\mathsf{supp}(\psi)$) and that

$$\|(\hbar D)^k u\| \le C(1+\hbar\gamma^{-1}+\mu\gamma)^k,$$
$$\|(\hbar D)^k((\hbar D - A)\cdot\boldsymbol{\sigma})u\| \le C(1+\hbar\gamma^{-1}+\mu\gamma)^{k+1},$$

and therefore

$$|\Gamma_x(\hbar D_x - A)\cdot\boldsymbol{\sigma})e(.,.,0)| \le C\hbar^{-3}(1+\hbar\gamma^{-1}+\mu\gamma)^{\frac{7}{2}},$$

and then

$$\|\Delta A\|_{\mathscr{L}^\infty(B(x,1-\epsilon))} \le C\hbar^{-1}(1+\hbar\gamma^{-1}+\mu\gamma)^{\frac{7}{2}}.$$

Optimizing with respect to $\gamma = \mu^{-\frac{1}{2}}h^{\frac{1}{2}}$ we conclude that either

$$\|\partial^2 A\|_{\mathscr{L}^\infty(B(x,1-\epsilon))} \le C\hbar^{-1-\delta}(1+\hbar\mu)^{\frac{7}{4}} \tag{27.3.24}$$

or (27.2.21) holds. In the former case of (27.3.24), using the second of estimates

$$\|A\|_{\mathscr{L}^\infty(B(x,1-\epsilon))} \le C\|\partial^2 A\|^{\frac{1}{5}}_{\mathscr{L}^\infty(B(x,1-\epsilon))}\|\partial A\|^{\frac{4}{5}}, \tag{27.3.25}$$

$$\|\partial A\|_{\mathscr{L}^\infty(B(x,1-\epsilon))} \le C\|\partial^2 A\|^{\frac{3}{5}}_{\mathscr{L}^\infty(B(x,1-\epsilon))}\|\partial A\|^{\frac{2}{5}} \tag{27.3.26}$$

we conclude that (27.3.23) holds with $\mu = \mu'$,

$$\mu' := \hbar^{-\frac{3}{5}-\delta}(1+\hbar\mu)^{\frac{21}{20}}$$

and one can see easily that starting from $\mu = \hbar^{-4}$ as given by (27.3.20), we can arrive after number of iterations to $\mu = \hbar^{-\frac{3}{5}-\delta}$ and therefore

$$\|\partial^k A\|_{\mathscr{L}^\infty(B(x,1-\epsilon))} \le C\hbar^{-\frac{1}{5}(1+2k)-\delta} \qquad k = 0,1,2. \tag{27.3.27}$$

(c) This estimate (27.3.27) is good enough to launch our microlocal arguments. Assuming (27.3.23) with $\mu \le h^{\sigma-1}$ we estimate as in Section 27.2

$$|\Gamma_x((\hbar D_x - A)\cdot\boldsymbol{\sigma})e(.,.,0)| \le C\mu\hbar^{-\delta},$$

and then

$$\|\partial^2 A\|_{B(x,1-\epsilon)} \le C\mu h^{-\delta},$$

and therefore

$$\|\partial A\|_{B(x,1-\epsilon)} \le C\mu^{\frac{3}{5}} h^{-\delta},$$

resulting in $\mu := \mu^{\frac{3}{5}} h^{-\delta}$ and after a number of iterations we get $\mu = h^{-\delta}$, and therefore iterating this procedure one more time and taking into account factor κ we arrive to

(27.3.28) Either (27.3.21) holds or

$$\|\partial^2 A\|_{\mathscr{L}^\infty(B(x,1-\epsilon))} \le C\kappa h^{-\delta}. \tag{27.3.29}$$

However to prove that the effective magnetic field is $O(1)$ we need to modify these arguments, and we do it in the next subsubsection.

Estimates to a Minimizer. II

In this step we repeat arguments of Subsubsection *27.2.2.1. Sharp Estimates*, but now we have a problem: we cannot use $\mu = \|\partial A\|_\infty$ since we have domains $\mathcal{X}_r = \{x : \ell(x) \ge r\}$ rather than the whole space. So we get the following analogue of (27.2.48) where A is still rescaled and the norms are calculated in the rescaled coordinates:

$$\|\Delta A\|_{\mathscr{C}(B(x,\frac34))} + \hbar\|\Delta\partial A\|_{\mathscr{C}(B(x,\frac34))} \le C\kappa\Big(1 + |\partial A|_{\mathscr{C}(B(x,1)} + h^{\frac12(\theta-1)}\|\partial A\|_{\mathscr{C}^{\theta}\frac12(B(x,1))}\Big), \tag{27.3.30}$$

which implies

$$\|\partial A\|_{\mathscr{C}(B(x,\frac12))} + \hbar^{\theta-1}\|\partial A\|_{\mathscr{C}^\theta(B,(x,\frac12))} \le \epsilon\hbar^{(\theta-1)}\zeta^{-1}\|\partial A\|_{\mathscr{C}^\theta(B(x,1))} + C\kappa\|\partial A\|_{\mathscr{C}(B(x,1))} + C\|\partial A\|_{\mathscr{L}^2(B(x,1))} \tag{27.3.31}$$

and the last term in the right-hand expression does not exceed $C\kappa^{\frac12}$.

Let $\nu(r) = \sup_{x:\,\ell(x)\ge r} f(x)$, where $f(x)$ is the left-hand expression of (27.3.16), calculated for given x in the rescaled coordinates. Then (27.3.31) implies that for $\kappa \in (0,\kappa^*)$ (where $\kappa^* > 0$ is a small constant)

$$\nu(r) \le \frac12\nu(\frac12 r) + C\kappa^{\frac12},$$

which in turn implies that

$$\nu(r) \le \frac12\nu(2^{-n}r) + 2C\kappa^{\frac12}, \qquad n \ge 1,$$

and therefore

$$\nu(r) \le 4C\kappa^{\frac12} + 4\sup_{C_0h^2\le\ell(x)\le 2C_0h^2} f(x) \le C_1\kappa^{\frac12}$$

due to the rough estimate (because $\hbar \asymp 1$ for $\ell(x) \asymp h^2$). Then returning to the original (not rescaled) coordinates and to the original (not rescaled) potential A we arrive to estimates (27.3.32) and (27.3.33) below:

Proposition 27.3.4. *Let $\kappa \le \kappa^*$, $\zeta = c\ell^{-\frac12}$. Let A be a minimizer. Then for $\ell(x) \ge \ell_* = h^2$ estimate* (27.3.22) *holds and also*

$$|\partial^2 A(x) - \partial^2 A(y)| \le C\kappa^{\frac12}\ell^{-\frac52}|x-y|^\theta \ell^{\theta/2}\ell_*^{-\theta/2} \qquad 0<\theta<1, \tag{27.3.32}$$

and

$$|\partial A(x) - \partial A(y)| \le C\kappa^{\frac12}\ell^{-\frac52}|x-y|(1+|\log|x-y||). \tag{27.3.33}$$

Remark 27.3.5. (i) So far we used only assumption that

$$|\partial^\alpha V| \le C\zeta^2\ell^{-|\alpha|} \qquad \forall \alpha : |\alpha| \le 2 \tag{27.3.34}$$

with $\zeta = \ell^{-\frac12}$ but even this was excessive.

(ii) In this framework however we cannot prove better estimates because (27.3.22) always remains a valid alternative even if $\zeta \ll \ell^{-\frac12}$.

(iii) Originally we need an assumption (27.2.37) $|V| \ge \epsilon_0$, but for $d = 3$ one can easily get rid of it by the standard rescaling technique.

Consider now zone $\{x :\ \ell(x) \le \ell_*\}$:

Proposition 27.3.6. *Let $\kappa \le \kappa^*$, $\zeta \le c\ell^{-\frac12}$. Let A be a minimizer. Then $|\partial A| \le C\kappa^{\frac12}h^{-3}$ for $\ell(x) \le \ell_* = h^2$.*

Proof. Proof is standard, based on the rescaling (then $\hbar = 1$) and equation (27.2.14) for a minimizer A. We leave easy details to the reader. □

Let us slightly improve estimate to a minimizer A. We already know that $|\partial A(x)| \le C_0\beta$ with $\beta = \ell^{-\frac32}$ and using the standard rescaling technique we conclude that

$$|\Delta A| \le C\kappa\zeta^2\beta + C\kappa\zeta^3\ell^{-1}, \tag{27.3.35}$$

which does not exceed $C\kappa\ell^{-\frac52}$ which implies

Proposition 27.3.7. *In our framework*

(i) If $\ell(x) \ge \ell_* \coloneqq h^2$, then

$$|A| \le C\kappa\ell^{-\frac12}, \qquad |\partial A| \le C\kappa\ell^{-\frac32} \tag{27.3.36}$$

and

$$|\partial A(x) - \partial A(y)| \le C_\theta\kappa\ell^{-\frac32-\theta}|x-y|^\theta \qquad \text{as } |x-y| \le \frac12\ell(x) \tag{27.3.37}$$

for any $\theta \in (0,1)$.

(ii) If $\ell(x) \le \ell_*$, then these estimates hold with $\ell(x)$ replaced by ℓ_*.

Remark 27.3.8. (i) Here in comparison with old estimates we replaced factor $\kappa^{\frac12}$ by κ which is an advantage.

(ii) These estimates imply that $\int_{\{x:\,\ell(x)\le 1\}} |\partial A|^2\,dx \le C\kappa^2|\log h|$ while in fact it must not exceed $C\kappa$.

Estimates to a Minimizer. III

Consider now external zone $\mathcal{Y} := \{x\colon \ell(x) \ge 1\}$ and assume that

$$\zeta(x) \le C\ell(x)^{-\nu} \qquad \text{for } \ell(x) \ge 1 \tag{27.3.38}$$

with $\nu > 1$.

Then if also $|\partial A(x)| = O(\ell(x)^{-\nu_1})$ for $\ell(x) \ge 1$ then the right hand expression of (27.3.35) does not exceed $C\kappa(\ell^{-3\nu-1} + \ell^{-\nu_1-2\nu})$ and therefore we *almost* upgrade estimate for $|\partial A(x)|$ to $O(\ell^{-3\nu} + \ell^{-\nu_1-2\nu+1})$ and repeating these arguments sufficiently many times to $O(\ell^{-3\nu})$.

However, there are several obstacles to this conclusion: first, if $\nu > 1$ we conclude that

$$A_j = \sum_m \alpha_{j,m}|x - \mathsf{y}_m|^{-1} + O(\ell^{-1-\delta})$$

with constant $\alpha_{j,m}$; however assumption $\nabla \cdot A = 0$ implies $\alpha_{j,m} = 0$ and we pass this obstacle.

Indeed, let our equation be $\Delta A_j = \Phi_j$ and therefore

$$A_j(x) = -\frac{1}{4\pi}\int |x-y|^{-1}\Phi_j(y)\,dy.$$

Let a be the minimal distance between nuclei, $1 = \sum_{0\le m\le M}\phi_m$ where ϕ_m is supported in $\frac{1}{3}a$-vicinity of y_m and equals 1 in $\frac{1}{4}a$-vicinity of y_m, $m = 1, \dots, 1$. Let

$$I_{j,m} = \int \Phi_j(y)\phi_m(y)\,dy, \qquad \eta = \max_{1\le m\le M}|I_{j,m}|.$$

Then if x belongs to b-vicinity of y_m with $b \le \epsilon a$ one can prove easily that

$$|\partial_{x_k}\int |x-y|^{-1}\Phi_j(y)\phi_{m'}(y)\,dy| \le C\eta a^{-2} + Ca^{-3}$$

for $m' = 0, 1, \dots, M$, $m' \ne m$.

Also one can prove easily that

$$|\partial_{x_j}\Big(\int |x-y|^{-1}\Phi_j(y)\phi_{m'}(y)\,dy - |x-\mathsf{y}_m|^{-1}I_{j,m}\Big)| \le C|x-\mathsf{y}_m|^{-3},$$

and combining with the previous inequality and with equation $\nabla \cdot A = 0$ we conclude that $|I_{m,j}| \le C\eta a^{-2}b^2 + Ca^{-3}b^2 + Cb^{-1}$ for $b \le \epsilon a$. Then selecting

$b = \epsilon_1 a$ with sufficiently small constant ϵ_1, we conclude that $\eta \le Ca^{-1}$ which in turn implies that $|\partial_k A_j(x)| \le C\ell^{-3}$.

The second obstacle

$$A_j = \sum_{k,m} \alpha_{jk,m}(x_k - y_{k,m})|x - y_m|^{-3} + O(\ell^{-2})$$

with constant $\alpha_{jk,m}$ we cannot pass since assumption $\nabla \cdot A = 0$ implies only that modulo gradient $A = \sum_m \beta_m \times \nabla \ell_m^{-1}$ with constant vectors β_m and one cannot do anything about this.

Therefore we upgrade (27.3.36)–(27.3.37) there:

Proposition 27.3.9. *In our framework assume additionally that* (27.3.38) *holds. Then for* $\nu > \frac{4}{3}$

$$|A| \le C\kappa\ell^{-2}, \qquad |\partial A| \le C\kappa\ell^{-3} \tag{27.3.39}$$

and

$$|\partial A(x) - \partial A(y)| \le C_\theta \kappa \ell^{-3-\theta}|x-y|^\theta \qquad \text{as } |x-y| \le \frac{1}{2}\ell(x) \tag{27.3.40}$$

if $\ell(x) \ge 1$ *(for all* $\theta \in (0,1)$*).*

Remark 27.3.10. (i) In application to the ground state energy we are interested in $\nu = 2$.

(ii) Observe that for $a \ge 1$

$$\int_{\{\ell(x) \asymp a\}} |\partial A|^2 \, dx = O(\kappa^2 a^{-3}). \tag{27.3.41}$$

(iii) We were not able to improve (27.3.39)–(27.3.41) no matter how fast ζ decays.

27.3.3 Basic Trace Estimates

Recall that the standard Tauberian theory results in the remainder estimate $O(h^{-2})$. Indeed, since the effective magnetic field intensity is no more than $C\kappa$, the contribution of $B(x, \ell(x))$ to the Tauberian error[13)] does not exceed

[13)] And then to the Weyl error because we will explain transition from the Tauberian to Weyl estimates below.

$C\zeta^2 \times \hbar^{-1} = C\zeta^3\ell h^{-1}$, which for $\zeta \asymp \ell^{-\frac{1}{2}}$ translates into $C\ell^{-\frac{1}{2}}h^{-1}$ and summation over domain $\{x : \ell(x) \ge \ell_* = h^2\}$ results in Ch^{-2}. On the other hand, contribution of the domain $\{x : \ell(x) \le \ell_* = h^2\}$ into asymptotics does not exceed $C\hbar^{-3}\ell_*^{-1} = Ch^{-2}$ for $\hbar = 1$.

However, now we can unleash arguments of V. Ivrii and I. M. Sigal [1]. Recall that we are looking at

$$\mathsf{Tr}(\psi H^-_{A,V}\psi) = \mathsf{Tr}(\phi_1 H^-_{A,V}\phi_1) + \mathsf{Tr}(\phi_2 H^-_{A,V}\phi_2) \tag{27.3.42}$$

where $\psi^2 = \phi_1^2 + \phi_2^2$, $\mathsf{supp}(\phi_1) \subset \{x, |x| \le 2r\}$, $\mathsf{supp}(\phi_2) \subset \{x, r \le |x| \le b\}$ and we compare it with the same expression calculated for H_{A,V^0} with $V^0 = Z_m|x|^{-1}$. Here we assume that

$$a \le 1, \qquad z \asymp 1 \tag{27.3.43}$$

and

$$|D^\alpha(V - V^0)| \le c_0 a^{-1}\ell^{-|\alpha|} \qquad \forall \alpha : |\alpha| \le 3. \tag{27.3.44}$$

The latter assumption is too restrictive and could be weaken. Then if $\phi(x)$ is an ℓ-admissible partition element

$$\mathsf{Tr}\big(\theta(-H^-_{A,V})\phi^2\big) = \int \mathsf{Weyl}(x)\phi^2(x)\,dx + O(rh^{-2}) \tag{27.3.45}$$

and

$$\mathsf{Tr}\big(H^-_{A,V}\phi^2\big) = \int \mathsf{Weyl}_1(x)\phi^2(x)\,dx + O(r^{-\frac{1}{2}}h^{-1}), \tag{27.3.46}$$

where the error estimates are $O(\hbar^{-2})$ and $O(\zeta^2\hbar^{-1})$ respectively. One can justify transition from the Tauberian to Weyl errors by considering Tauberian expressions and considering $H_{A_\varepsilon,V}$ and $H_{A,V}$ as unperturbed and perturbed operators respectively; their difference is $O(\zeta^3\varepsilon^2)$ with $\varepsilon = \hbar^{1-\delta}$.

Then the contribution[14)] of the time interval $\{t\colon t \asymp T\}$ to the Tauberian expression for (27.3.45) of the first term in the approximation does not exceed $C\hbar^{-4}T \times (\hbar T^{-1})^s$, of the second term $C\hbar^{-4}T \times (\hbar T^{-1})^s T\hbar^{-1}\varepsilon^2$, and of the third term $C\hbar^{-4}T \times (\hbar T^{-1})T^2\hbar^{-2}\varepsilon^4$. One can see easily that the end of the day the first term gives us the Weyl expression, the second term turns out to be 0, and the third term is less than the announced error.

Similarly, the contribution[14)] of the time interval $\{t\colon t \asymp T\}$ to the Tauberian expression for (27.3.46) of the first term in the approximation does not

[14)] After standard rescaling $x \mapsto x\ell^{-1}$, $\xi \mapsto \xi\zeta^{-1}$, $h \mapsto \hbar$, $\tau \mapsto \tau\zeta^{-2}$, and $t \mapsto t\zeta\ell^{-1}$.

exceed $C\zeta^2\hbar^{-4}T\times(\hbar T^{-1})^s$, of the second term $C\zeta^2\hbar^{-4}T\times(\hbar T^{-1})^s T\hbar^{-1}\varepsilon^2$ and of the third term $C\hbar^{-4}T\times(\hbar T^{-1})^2T^2\hbar^{-2}\varepsilon^4$. Again, in the end of the day the first tem gives us the Weyl expression, the second term turns out to be 0, and the third term is less than the announced error.

The same estimates also hold for operator H_{A,V^0} and then using ℓ-admissible partition of unity we conclude that

$$\mathsf{Tr}\big(\phi_2(H^-_{A,V}-H^-_{A,V^0})\phi_2\big)= \int \big(\mathsf{Weyl}_1(x)-\mathsf{Weyl}^0_1(x)\big)\,\phi_2^2(x)\,dx + O(r^{-\frac12}h^{-1}), \tag{27.3.47}$$

where Weyl^0_1 and Weyl^0 are calculated for operator with potential V^0. Indeed, we just proved this for each operator $H_{A,V}$ and H_{A,V^0} separately.

On the other hand, considering $V^\eta = V^0(1-\eta)+V\eta = V^0+W\eta$ and following V. Ivrii and I. M. Sigal [1], we can rewrite the similar expression albeit for $\phi_2=1$ as

$$\mathsf{Tr}\Big(\int_0^1 W\theta(-H_{A,V^\eta})\,d\eta\Big) \tag{27.3.48}$$

and applying the semiclassical approximation (under the temporary assumption that W is supported in the domain $\{x:\ |x|\le 4r\}$) one can prove that for $\phi_1=1$

$$\mathsf{Tr}\big(\phi_1(H^-_{A,V}-H^-_{A,V^0})\phi_1\big)= \int \big(\mathsf{Weyl}_1(x)-\mathsf{Weyl}^0_1(x)\big)\,\phi_1^2(x)\,dx + O(a^{-1}rh^{-2}). \tag{27.3.49}$$

Really, due to (27.3.45) and (27.3.44) the contribution of the ball $B(x,\ell(x))$ does not exceed $Ca^{-1}\hbar^{-2}=Ca^{-1}\ell(x)h^{-2}$ and summation with respect to partition with $\ell(x)\le 4r$ returns $Ca^{-1}rh^{-2}$); meanwhile, the contribution of $\{x:\ell(x)\le\ell_*\}$ does not exceed $Ca^{-1}\hbar^{-2}=Ca^{-1}$ since there $\hbar=1$.

One can get easily rid of the temporary assumption and take ϕ_1 supported in $\{x:\ell(x)\le 2r\}$ instead.

Therefore we arrive to

Proposition 27.3.11. *Under assumption* (27.3.44)

$$\mathsf{Tr}\big(\psi(H^-_{A,V}-H^-_{A,V^0})\psi\big)= \int \big(\mathsf{Weyl}_1(x)-\mathsf{Weyl}^0_1(x)\big)\,\psi^2(x)\,dx + O\big(a^{-\frac13}h^{-\frac43}\big). \tag{27.3.50}$$

Really, $a^{-\frac{1}{3}}h^{-\frac{4}{3}}$ is $r^{-\frac{1}{2}}h^{-1}+a^{-1}rh^{-2}$ optimized by $r\asymp r_*:=(ah)^{\frac{2}{3}}$; since $h^2\le a$ we note that $h^2\le r_*\le a$.

Corollary 27.3.12. *(i) For $M=1$ equality* (27.3.50) *remains valid with $\psi=1$ and $a=1$.*

(ii) For $M\ge 2$ and $a\ge h^2$ equality (27.3.50) *becomes*

$$\mathsf{Tr}\big(\psi(H^-_{A,V}-H^-_{A,V^0})\psi\big)= \int \big(\mathsf{Weyl}_1(x)-\mathsf{Weyl}^0_1(x)\big)\,\psi^2(x)\,dx + O\big((a^{-\frac{1}{3}}+1)h^{-\frac{4}{3}}\big), \tag{27.3.51}$$

where we reset case $a\ge 1$ to $a=1$.

27.3.4 Improved Trace Estimates

Improved Tauberian Estimates

Let us apply much more advanced arguments of Section 12.6; recall that these arguments are using the long term propagation of singularities. Unfortunately, using these arguments, we are not able to improve the above results unless $\kappa\ll 1$.

First, let us consider ψ, which is r-admissible partition element located in $\{x:\ell(x)\asymp r\}$, and we need to estimate an absolute value of

$$F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\,\mathsf{Tr}\big(e^{ih^{-1}tH}\psi\big), \tag{27.3.52}$$

and to do it we need to estimate the same expression with $\bar{\chi}_T(t)$ replaced by $\chi_{T'}(t)$ with $t_0\le T'\le T$ where $t_0=\epsilon\ell\zeta^{-1}=\epsilon r^{\frac{3}{2}}$. We can break $\psi=\psi^+ +\psi^-$ with $\psi^\pm=\psi^\pm(x,hD)$ such that the trajectories in the positive (negative) time direction from support of its symbol $\psi^+(x,\xi)$ are going after time Ct_0 in the direction of increased $\ell(x)$, and since we consider the trace we need to consider only ψ^+ and only $\chi\in\mathscr{C}^\infty([\frac{1}{2},1])$.

The trouble is that we have not rough but non-smooth magnetic field[15]; so let us consider $t_0+t_1+\dots+t_n\asymp T'$ where $t_j=\epsilon r_j^{\frac{3}{2}}$, $r_j=c^j r$, $j=0,1,\dots,n$,

[15] More precisely, A is rough but with the roughness parameter $\hbar$ which is a bit too small.

and let us estimate an error appearing when we replace in (modified) (27.3.52) $e^{ih^{-1}tH}\psi^+$ by

$$e^{ih^{-1}(t-t_n)H}\psi^+_{n+1}e^{ih^{-1}t_nH}\psi^+_n\cdots e^{ih^{-1}t_1H}\psi^+_1e^{ih^{-1}t_0H}\psi^+ \tag{27.3.53}$$

with ψ^+_j defined similarly and Hamiltonian flow from $\mathsf{supp}(\psi^+_j)$ for $t=t_j$ is inside $\{(x,\xi)\colon \psi^+_{j+1}(x,\xi)=1\}$. Therefore we need to estimate an error when we insert ψ^+_j.

According to our propagation results (namely, Proposition 27.2.11) after $\psi^+_1,\dots,\psi^+_{j-1}$ were inserted, insertion of ψ^+_j brings a relative error not exceeding $C\big(\hbar_j^\theta \|\partial A\|_{\theta,Y_j}+\hbar_j^{s+1}\big)$, where $\hbar_j=hr_j^{-\frac12}$ and Y_j is an ϵr_j-vicinity of the x-projection of $\mathsf{supp}(\psi_j)$; s is an arbitrarily large exponent.

Recall that $\|\partial A\|_{\theta,Y_j}\le C\kappa\hbar_j^{1-\theta}$ for $\theta\in(1,2)$; therefore this relative error does not exceed $C\hbar_j(\kappa+\hbar_j^s)$. Therefore inserting all ψ^+_j brings a relative error $C\sum_{j\ge0}\hbar_j(\kappa+\hbar_j^s)\asymp C\hbar(\kappa+\hbar^s)$ and since a priory expression (27.3.52) is bounded by $C\hbar^{-3}T$ we conclude that

(27.3.54) The absolute value of expression (27.3.52) with $\bar\chi_T(t)$ replaced by $\chi_{T'}(t)$ with $T_*\asymp r^{\frac32}\le T'\le T^*$ [16] does not exceed $C\hbar^{-2}(\kappa+\hbar^s)T'$.

Then an absolute value of expression (27.3.52) with $\bar\chi_T(t)$ replaced by $(\bar\chi_T(t)-\bar\chi_{T_*})$ does not exceed $C\hbar^{-2}(\kappa+\hbar^s)$ and since expression (27.3.52) with $T=T_*$ does not exceed $C\hbar^{-2}t_0$ we conclude that

(27.3.55) The absolute value of expression (27.3.52) with $T_*\le T\le T^*$ does not exceed $C\hbar^{-2}T_*+C\hbar^{-2}(\kappa+\hbar^s)T$.

Then we conclude that

(27.3.56) An error when we replace $\mathsf{Tr}\big(\theta(-H_{A,V})\psi\big)$ by its Tauberian expression with "time" T does not exceed $C\hbar^{-2}\big(T_*T^{-1}+\kappa+\hbar^s\big)$

and

(27.3.57) An error when we replace $\mathsf{Tr}\big(H^-_{A,V}\psi\big)$ by its Tauberian expression with "time" T does not exceed $C\hbar^{-1}\big(T_*T^{-1}+\kappa+\hbar^s\big)T_*T^{-1}\zeta^2$.

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[16] We discuss the choice of T^* later.

In the latter statement we need to remember how everything scales.

Observe that presence of the magnetic field due to its estimates relatively perturbs dynamics by $O(\kappa)$ and therefore if κ is sufficiently small (i.e. $\kappa \le \kappa^*$) it does not affect T^*. Then assuming that

$$|\nabla^\alpha (V - V^0)| \le \epsilon a^{-1} r^{-|\alpha|} \qquad \forall \alpha : |\alpha| \le 1, \qquad V^0 = Z r^{-1} \text{ with } \quad Z \asymp 1, \quad a \ge h^{2-\delta} \tag{27.3.58}$$

we can take $T^* \asymp a^{\frac{3}{2}}$ and therefore we conclude that the Tauberian error in (27.3.57) does not exceed

$$Ch^{-1} a^{-\frac{3}{2}} r \bigl(r^{\frac{3}{2}} a^{-\frac{3}{2}} + \kappa + h^s r^{-\frac{1}{2}s}\bigr) \tag{27.3.59}$$

and we arrive to Statement (i) in Proposition 27.3.13 below.

Meanwhile, the Tauberian error in $\mathsf{Tr}\bigl((H^-_{A,V} - H^-_{A,V^0})\psi\bigr)$ does not exceed

$$Ca^{-1} h^{-2} r \bigl(r^{\frac{3}{2}} a^{-\frac{3}{2}} + \kappa + h^s r^{-\frac{1}{2}s}\bigr) \tag{27.3.60}$$

and we arrive to Statement (ii) below:

Proposition 27.3.13. *Assume that* (27.3.58) *is fulfilled and let ψ be ℓ-admissible function supported in $\{x\colon \ell(x) \asymp r\}$ with $h^2 \le r \le a$. Let A satisfy minimizer estimate. Then*

(i) The Tauberian error with $T = T^* \asymp a^{\frac{3}{2}}$ in $\mathsf{Tr}\bigl(H^-_{A,V}\psi\bigr)$ does not exceed (27.3.59)*.*

(ii) The Tauberian error with $T = T^* \asymp a^{\frac{3}{2}}$ in $\mathsf{Tr}\bigl((H^-_{A,V} - H^-_{A,V^0})\psi\bigr)$ does not exceed (27.3.60)*.*

Proof. An easy proof following arguments of Section 12.6 is left to the reader. □

Observe that in the deduction of Statement (i) summation with respect to $r : b \le r \le a$ returns $Ch^{-1}a^{-\frac{1}{2}}$ and in the deduction of (ii) summation with respect to $r : h^2 \le r \le b$ returns $Ch^{-2}\bigl(a^{-\frac{5}{2}} b^{\frac{5}{2}} + \kappa a^{-1} b\bigr) + Ca^{-1}$. Note also that Statement (ii) remains true for r-admissible function supported in $\{x\colon \ell(x) \le r\}$ with $r \asymp h^2$. Then we arrive to

Corollary 27.3.14. *Assume that* (27.3.58) *is fulfilled. Then*

(i) Let ϕ_2 be ℓ-admissible function supported in $\{x\colon b \le \ell(x) \le a\}$. Then the Tauberian error with $T = T^* \asymp a^{\frac{3}{2}}$ in $\mathsf{Tr}\big(H^-_{A,V}\phi_2\big)$ does not exceed $Ch^{-1}a^{-\frac{1}{2}}$.

(ii) Let ϕ_1 be ℓ-admissible function supported in $\{x\colon \ell(x) \le b\}$. Then the Tauberian error with $T = T^* \asymp a^{\frac{3}{2}}$ in $\mathsf{Tr}\big((H^-_{A,V} - H^-_{A,V^0})\phi_1\big)$ does not exceed $Ch^{-2}\big(a^{-\frac{5}{2}}b^{\frac{5}{2}} + \kappa a^{-1}b\big) + Ca^{-1}$.

Remark 27.3.15. Obviously we do not need any new assumptions on κ to estimate the sum of expressions obtained in Statements (i) and (ii) of Corollary 27.3.14 by $Ch^{-1}a^{-\frac{1}{2}}$ (as $b \le a^{\frac{1}{2}}h$) here but we need to move from Tauberian expression to Weyl expression.

Improved Weyl Estimates

Note that in virtue of (27.3.54) for element ψ the contribution of the time interval $\{t\colon |t| \asymp T'\}$ to the Tauberian expression for $\mathsf{Tr}\big(H^-_{A,V}\psi\big)$ does not exceed $C\hbar^{-1}\big(\kappa + \hbar^s\big)T_*T'^{-1}\zeta^2$, and therefore, replacing $\bar\chi_T(t)$ by $\bar\chi_{T_*}(t)$ we introduce an error not exceeding

$$C\hbar^{-1}\big(\kappa + \hbar^s\big)\zeta^2 \asymp Ch^{-1}r^{-\frac{1}{2}}\big(\kappa + h^s r^{-\frac{1}{2}s}\big) \tag{27.3.61}$$

and summation with respect to $r : b \le r \le a$ returns

$$Ch^{-1}b^{-\frac{1}{2}}\big(\kappa + h^s b^{-\frac{1}{2}s}\big). \tag{27.3.62}$$

On the other hand, also in virtue of (27.3.54) for element ψ the contribution of the time interval $\{t\colon |t| \asymp T'\}$ to the Tauberian expression for $\mathsf{Tr}\big((H^-_{A,V} - H^-_{A,V^0})\psi\big)$ does not exceed $C\hbar^{-2}a^{-1}\big(\kappa + \hbar^s\big)$, and therefore replacing $\bar\chi_T(t)$ by $\bar\chi_{T_*}(t)$ we introduce an error not exceeding

$$Ch^{-2}a^{-1}r\big(\kappa + h^s r^{-\frac{1}{2}s}\big)|\log T/T_*|.$$

Further, in virtue of (27.3.55) the Tauberian error does not exceed $Ch^{-2}a^{-1}r\big(r^{\frac{3}{2}}T^{-1} + \kappa + h^s r^{-\frac{1}{2}s}\big)$, and adding these two errors together and optimizing their sum by $T \le a^{\frac{3}{2}}$ we get $T \asymp r^{\frac{5}{2}}(\kappa + h^s r^{-\frac{1}{2}s})^{-1}$ and the sum

$$Ch^{-2}a^{-1}r\big(\kappa + h^s r^{-\frac{1}{2}s}\big)\big(|\log(\kappa + h^s r^{-\frac{1}{2}s})| + 1\big) + Ch^{-2}a^{-\frac{5}{2}}r^{\frac{5}{2}}. \tag{27.3.63}$$

Meanwhile repeating arguments of Subsection 27.3.3 one can see easily that

(27.3.64) The difference between the Tauberian expression with $T = T_*$ and the Weyl expression for $\mathsf{Tr}\big(H^-_{A,V}\psi\big)$ does not exceed (27.3.61) with any $s < 2$

and

(27.3.65) The difference between the Tauberian expression with $T = T_*$ and the Weyl expression for $\mathsf{Tr}\big((H^-_{A,V} - H^-_{A,V^0})\psi\big)$ does not exceed

$$Ch^{-2}a^{-1}r\big(\kappa + h^s r^{-\frac{1}{2}s}\big) \tag{27.3.66}$$

with any $s < 2$ and thus does not exceed (27.3.63).

Summation with respect to $r : h^2 \le r \le b$ of (27.3.63) returns

$$Ch^{-2}a^{-1}b\kappa|\log\kappa| + Ch^{-2}b^{\frac{5}{2}}a^{-\frac{5}{2}} + Ca^{-1}; \tag{27.3.67}$$

adding expression (27.3.62) and optimizing the sum by b: $h^2 \le b \le a$ we get $b \asymp (ah|\log\kappa|)^{\frac{2}{3}}$ and expression

$$Ch^{-\frac{4}{3}}a^{-\frac{1}{3}}\kappa|\log\kappa|^{\frac{1}{3}} + Ch^{-1}a^{-\frac{1}{2}}. \tag{27.3.68}$$

Thus we have proven

Proposition 27.3.16. *(i) In the framework of Proposition 27.3.11*

$$\mathsf{Tr}\big(\psi(H^-_{A,V} - H^-_{A,V^0})\psi\big) = \tag{27.3.69}$$
$$\int \big(\mathsf{Weyl}_1(x) - \mathsf{Weyl}^0_1(x)\big)\,\psi^2(x)\,dx + O\big(h^{-\frac{4}{3}}a^{-\frac{1}{3}}\kappa|\log\kappa|^{\frac{1}{3}} + h^{-1}a^{-\frac{1}{2}}\big).$$

(ii) In particular, if

$$\kappa \le ca^{-\frac{1}{6}}h^{\frac{1}{3}}|\log ah^{-2}|^{-\frac{1}{3}} \tag{27.3.70}$$

the error in (27.3.69) *does not exceed* $Ch^{-1}a^{-\frac{1}{2}}$ *exactly as in the case without magnetic field.*

Remark 27.3.17. (i) Obviously we could consider $a = 1$ and then just rescale $x \mapsto xa^{-1}$, $\tau \mapsto \tau a$, $h \mapsto ha^{-\frac{1}{2}}$.

(ii) One may wonder if the same approach works for estimate of A. First of all, there is no improvement for estimate for $|\partial^2 A|$ because it follows from the estimate for $|\Delta A|$ which is a pointwise estimate.

(iii) Still, since ∂A and A are mollifications of ΔA one can improve estimates for them if $\kappa \ll 1$ and $\ell \ll a$; however, there are no improvements if either $\kappa \asymp 1$ or $\ell \ge a$. Since these improvements do not lead to the improvements of our final results we do not pursue them.

27.3.5 Single Singularity

Coulomb Potential

Consider now exactly Coulomb potential: $V = Z|x|^{-1}$. Let us establish the existence of the Scott correction:

Proposition 27.3.18. *Let $V = Z|x|^{-1}$, $h > 0$, $Z > 0$ and $0 < \kappa \le \kappa^*$. Then*

(i) The following limit exists

$$\lim_{r\to\infty}\Biggl(\inf_{A}\Bigl(\mathsf{Tr}\bigl((\phi_r H_{A,V}\phi_r)^-\bigr)+\frac{1}{\kappa h^2}\int |\partial A|^2\,dx\Bigr) - \int \mathsf{Weyl}_1(x)\phi_r^2(x)\,dx\Biggr) =: 2Z^2h^{-2}S(Z\kappa). \tag{27.3.71}$$

(ii) And it coincides with

$$\lim_{\eta\to 0^+}\Biggl(\inf_{A,V}\Bigl(\mathsf{Tr}\bigl((H_{A,V}+\eta)^-\bigr)\Bigr)+\frac{1}{\kappa h^2}\int |\partial A|^2\,dx - \int \mathsf{Weyl}_1(H_{A,V}+\eta,x)\,dx\Biggr), \tag{27.3.72}$$

(iii) And also with

$$\inf_{A}\Biggl(\int \Bigl(e_1(H_{A,V};x,x,0)-\mathsf{Weyl}_1(H_{A,V},x)\Bigr)\,dx+ \frac{1}{\kappa h^2}\int |\partial A|^2\,dx\Biggr). \tag{27.3.73}$$

(iv) We also can replace in Statement (i) $\mathsf{Tr}\big((\phi_r H_{A,V}\phi_r)^-\big)$ *by* $\mathsf{Tr}\big(\phi_r H^-_{A,V}\phi_r\big)$.

Here $\phi \in \mathscr{C}_0^\infty(B(0,1))$, $\phi = 1$ *in* $B(0,\frac12)$, $\phi_r = \phi(x/r)$.

Proof. Observe first that due to scaling $x \mapsto Zh^{-2}x$, $A \mapsto Z^{-1}hA$ and $\partial A \mapsto z^{-2}h^3\partial A$ one needs to consider only $Z = h = 1$; all expressions on the left scale exactly as $Z^2h^{-2}S(Z\kappa)$.

(i) Let us compare

$$Q(r,\kappa,A) \coloneqq \mathsf{Tr}\big((\phi_r H_{A,V}\phi_r)^-\big) - \int \mathsf{Weyl}_1(H_{A,V},x)\phi_r^2(x)\,dx + \frac{1}{\kappa}\int |\partial A|^2\,dx$$

and $Q(r',\kappa,A)$ with $r \ge 1$ and $r' \ge 2r$. Note that

$$\begin{aligned} Q(r',\kappa,A) \ge\ & Q(r,(1+\epsilon)\kappa,A) + \\ & \sum_{1\le j\le J}\Big(\mathsf{Tr}\big((\psi_{2^jr}H_{A,V}\psi_{2^jr})^-\big) - \int \mathsf{Weyl}_1(H_{A,V},x)\psi_{2^jr}^2(x)\,dx + \\ & \qquad \frac{\epsilon}{2\kappa}\int |\partial A|^2\bar\psi_{2^jr}^2(x)\,dx\Big), \end{aligned}$$

where ψ and $\bar\psi$ are smooth compactly supported functions, equal 0 in $B(0,\frac12)$ and $\bar\psi = 1$ in the vicinity of $\mathsf{supp}(\psi)$, $J = \lfloor \log_2 r'/r\rfloor$, $\epsilon > 0$ is arbitrarily small.

Therefore we can replace in the sum A by A_j and $\bar\psi_{2^jr}$ by 1 in $\int |\partial A|^2\bar\psi_{2^jr}^2(x)\,dx$; but then in the virtue of Section 27.2 each term in the sum is bounded from below by $-C'(\epsilon)\int \rho^3\ell^{-1}\bar\psi_t\,dx = -C'(\epsilon)t^{-\frac12}$ with $t = 2^jr$. Then

$$Q(r',\kappa,A) \ge Q(r,(1+\epsilon)\kappa,A) - C'(\epsilon)r^{-\frac12}. \tag{27.3.74}$$

We know that $Q(r,(1+\epsilon)\kappa,A)$ is bounded from below by $-C(r)$ but now we conclude that this bound is uniform with respect to r, which implies that $2S(\kappa) \coloneqq \liminf_{r\to+\infty}\inf_A Q(r,\kappa,A) > -\infty$. Further, (27.3.74) implies that

$$2S(\kappa) + C'(\epsilon)r^{-\frac12} \ge \inf_A Q(r,(1+\epsilon)\kappa,A)$$

and therefore

$$\limsup_{r\to+\infty}\inf_A Q(r,\kappa,A) \le 2S((1+\epsilon)^{-1}\kappa). \tag{27.3.75}$$

Furthermore, plugging $A=0$ we can see easily that $Q(r,\kappa,A)$ is uniformly bounded from above and therefore $2S(\kappa)<+\infty$; also our arguments imply that $\int |\partial A|^2\,dx$ is uniformly bounded for near optimizers and therefore $S(\kappa)$ is continuous with respect to $\kappa<\kappa^*$; combining with (27.3.75) we arrive to Statement (i).

(ii) Similarly, (27.3.74) holds with $H_{A,V}$ replaced by $H_{A,V}+\eta$ and then we can take $r'=\infty$ and apply $\inf_A$ to both sides arriving to

$$\inf_A\Biggl(\mathsf{Tr}\bigl((H_{A,V}+\eta)^-\bigr)-\int \mathsf{Weyl}_1(H_{A,V+\eta},x)\,dx+\frac{1}{\kappa}\int|\partial A|^2\,dx\Biggr)\geq \inf_A\Biggl(\mathsf{Tr}\bigl((\phi_r(H_{A,V}+\eta)\phi_r)^-\bigr)-\int \mathsf{Weyl}_1(H_{A,V+\eta},x)\phi_r^2(x)\,dx+ \frac{1}{(1+\epsilon)\kappa}\int|\partial A|^2\Biggr)\,dx-C'(\epsilon)r^{-\frac12}.$$

After this as $\eta\to+0$ the right-hand expression tends to itself with $\eta=0$; tending $r\to+\infty$ we get there $2S((1+\epsilon)\kappa)$ in virtue of Statement (i) and tending $\epsilon\to+0$ we arrive to

$$\liminf_{\eta\to+0}\inf_A\Biggl(\mathsf{Tr}\bigl((H_{A,V}+\eta)^-V-\int \mathsf{Weyl}_1(H_{A,V+\eta},x)\,dx+ \frac{1}{\kappa}\int|\partial A|^2\,dx\Biggr)\geq 2S(\kappa). \tag{27.3.76}$$

On the other hand, consider

$$\mathsf{Tr}\bigl((H_{A,V}+\eta)^-\bigr)-\int \mathsf{Weyl}_1(H_{A,V+\eta},x)\,dx+\frac{1}{\kappa}\int|\partial A|^2\,dx$$

and replace $\mathsf{Tr}\bigl((H_{A,V}+\eta)^-\bigr)$ by

$$\underbrace{\mathsf{Tr}\bigl((H_{A,V}+\eta)^-\phi_r^2\bigr)+\frac{1}{\kappa}\int|\partial A|^2\,dx}+\mathsf{Tr}\bigl((H_{A,V}+\eta)^-(1-\phi_r^2)\bigr). \tag{27.3.77}$$

Let A be a minimizer of the first expression; then in virtue of Propositions 27.2.24–27.2.26 this minimizer is sufficiently "good" on ϵr-vicinity of $\mathsf{supp}(1-\phi_r^2)$ and also the the difference between the second term and its Weyl expression does not exceed $Cr^{-\frac12}$; one can prove it easily by $\ell(x)$-admissible

partition of unity as in Part (i) of the proof and we leave details to the reader.

Observe that the first term in (27.3.77) is $\inf_A \mathsf{Tr}\big(\phi_r \big(H_{A,V}+\eta\big)^{-}\phi_r\big)$. In this expression we can take limit as $\eta \to +0$ just setting $\eta = 0$ and therefore the left-hand expression in (27.3.76) with $\liminf$ replaced by $\limsup$ does not exceed

$$\inf_A \Big(\mathsf{Tr}\big(\phi_r H^{-}_{A,V}\phi_r\big) - \int \mathsf{Weyl}_1(H_{A,V},x)\phi_r^2(x)\,dx + \frac{1}{\kappa}\int |\partial A|^2\,dx\Big) + Cr^{-\frac{1}{2}};$$

taking limit as $r \to +\infty$ we conclude that the left-hand expression in (27.3.76) with $\liminf$ replaced by $\limsup$ does not exceed

$$\liminf_{r\to+\infty}\inf_A \Big(\mathsf{Tr}\big(\phi_r H^{-}_{A,V}\phi_r\big) - \int \mathsf{Weyl}_1(H_{A,V},x)\phi_r^2(x)\,dx + \frac{1}{\kappa}\int |\partial A|^2\,dx\Big).$$

This expression does not exceed (27.3.71) and therefore combining with (27.3.76) we prove Statements (ii) and (iv).

(iii) Similarly,

$$\int \Big(e_1(H_{A,V};x,x,0) - \mathsf{Weyl}_1(H_{A,V},x)\Big)\phi_r^2(x)\,dx + \frac{1}{\kappa}\int |\partial A|^2\,dx \ge$$
$$\inf_A \Big(\mathsf{Tr}\big((\phi_r H_{A,V}\phi_r)^{-}\big) - \int \mathsf{Weyl}_1(x)\phi_r^2(x)\,dx + \frac{1}{\kappa}\int |\partial A|^2\Big) - Cr^{-\frac{1}{2}}$$

and therefore

$$\inf_A \liminf_{r\to\infty} \int \Big(e_1(H_{A,V};x,x,0) - \mathsf{Weyl}_1(H_{A,V},x)\Big)\phi_r^2(x)\,dx+$$
$$\frac{1}{\kappa}\int |\partial A|^2\,dx \ge 2S(\kappa).$$

On the other hand, as in Part (ii) of the proof, taking A to be a minimizer of the first expression in (27.3.77), we see that

$$\inf_A \limsup_{r\to\infty} \int \Big(e_1(H_{A,V};x,x,0) - \mathsf{Weyl}_1(H_{A,V},x)\Big)\phi_r^2(x)\,dx+$$
$$\frac{1}{\kappa}\int |\partial A|^2\,dx \le 2S(\kappa)$$

and Statement (iii) has been proven. □

Remark 27.3.19. (i) Statements similar to (i), (ii) were proven in L. Erdös, S. Fournais, and J. P. Solovej [3] (see Theorem 2.4 and Lemma 2.5 respectively).

(ii) Again as observed in in L. Erdös, S. Fournais, and J. P. Solovej [3] we do not know if

(a) $S(\kappa) < S(0) = \frac{1}{2}$ for $\kappa > 0$ or just

(b) $S(\kappa) = S(0)$ for $\kappa < \kappa^*$ and $S(\kappa) = -\infty$ for $\kappa > \kappa^*$.

If we knew that the optimizer is unique, then obviously $A = 0$ and it would be relatively easy but rather unexciting the latter case.

(iii) While we assumed that $\kappa < \kappa^*$ with $\kappa^* > 0$ and it is possible that $S(\kappa) = -\infty$ as $\kappa > \kappa^*$ with some $\kappa^* < \infty$ we are not aware about any proof of this, so in fact it could happen that $\kappa^* = +\infty$ and then condition $\kappa < \kappa^*$ is superficial and one needs to study asymptotics of $S(\kappa)$ as $\kappa \to +\infty$.

Proposition 27.3.20. *For* $0 < \kappa < \kappa'$

$$S(\kappa') \le S(\kappa) \le S(\kappa') + C\kappa'(\kappa^{-1} - \kappa'^{-1}). \tag{27.3.78}$$

Proof. Monotonicity of $S(\kappa)$ is obvious.

Let $0 < \kappa < \kappa' < \kappa'' \le \kappa^*$. Then for any $\varepsilon > 0$ if $r = r_\varepsilon$ is large enough then the left-hand expression in (27.3.71) for κ' (without $\inf$ and $\lim$) is greater than $S(\kappa'') - \varepsilon + (\kappa'^{-1} - \kappa''^{-1})\|\partial A\|^2$; also, if A is an almost minimizer there, it is less than $S(\kappa') + \varepsilon$.

Therefore $(\kappa'^{-1} - \kappa''^{-1})\|\partial A\|^2 \le |S(\kappa'') - S(\kappa')| + 2\varepsilon$. But then

$$\begin{aligned} S(\kappa) - \varepsilon \le S(\kappa') + \varepsilon + (\kappa^{-1} - \kappa'^{-1})\|\partial A\|^2 \le \\ S(\kappa') + \varepsilon + C(\kappa^{-1} - \kappa'^{-1})(\kappa'^{-1} - \kappa''^{-1})^{-1}\big(|S(\kappa'') - S(\kappa')| + 2\varepsilon\big) \end{aligned}$$

and therefore

$$(\kappa^{-1} - \kappa'^{-1})^{-1}|S(\kappa) - S(\kappa')| \le (\kappa'^{-1} - \kappa''^{-1})^{-1}|S(\kappa') - S(\kappa'')| \tag{27.3.79}$$

which for $\kappa'' = \kappa^*$ implies (27.3.78). □

Remark 27.3.21. Using global equation (27.2.14) we conclude that for $Z = h = 1$

$$|\partial^\alpha A| \le C\kappa \ell^{-1-|\alpha|} \qquad \text{if } \ell \ge 1,\ |\alpha| \le 1, \tag{27.3.80}$$

$$|\partial^\alpha A| \le C\kappa \qquad \text{if } \ell \le 1,\ |\alpha| \le 1, \tag{27.3.81}$$

$$\|\partial A\|^2 \le C\kappa^2. \tag{27.3.82}$$

Then

$$S'(\kappa) \le C, \qquad |S(\kappa(1+\eta)) - S(\eta)| \le C\kappa\eta. \tag{27.3.83}$$

Main Theorem

In the "atomic" case $M = 1$ we arrive instantly to the following theorem:

Theorem 27.3.22. *Let $M = 1$ and $\kappa \le \kappa^*$. Then*

(i) Asymptotics holds

$$\mathsf{E}^* = \int \mathsf{Weyl}_1(x)\,dx + 2z^2 S(z\kappa)h^{-2} + O(h^{-\frac{4}{3}}\kappa|\log\kappa|^{\frac{1}{3}} + h^{-1}). \tag{27.3.84}$$

(ii) If $\kappa = o(h^{\frac{1}{3}}|\log h|^{-\frac{1}{3}})$, then

$$\mathsf{E}^* = \int \mathsf{Weyl}_1^*(x)\,dx + 2z^2 S(z\kappa)h^{-2} + o(h^{-1}). \tag{27.3.85}$$

Proof. If A satisfies the minimizer properties, then in virtue of Proposition 27.3.16

$$\mathsf{Tr}^-(H_{A,V}) - \int \mathsf{Weyl}_1(x)\,dx \equiv \mathsf{Tr}^-(H_{A,V^0}) - \int \mathsf{Weyl}_1^0(x)\,dx + O(h^{-\frac{4}{3}}\kappa|\log\kappa|^{\frac{1}{3}} + h^{-1}) \tag{27.3.86}$$

and adding magnetic energy and plugging either minimizer for V or for V^0 we get

$$\begin{aligned}\inf_A \Big(\mathsf{Tr}^-(H_{A,V}) - \int \mathsf{Weyl}_1(x)\,dx + \frac{1}{\kappa h^2}\int |\partial A|^2\,dx\Big) \lessgtr \\ \inf_A \Big(\mathsf{Tr}^-(H_{A,V^0}) - \int \mathsf{Weyl}_1^0(x)\,dx + \frac{1}{\kappa h^2}\int |\partial A|^2\,dx\Big) \\ \pm C(h^{-\frac{4}{3}}\kappa|\log\kappa|^{\frac{1}{3}} + h^{-1}).\end{aligned} \tag{27.3.87}$$

Obviously if V (and surely V^0) are not sufficiently fast decaying at the infinity, the left (and for sure the right hand) expression in (27.3.86) should be regularized as in Subsubsection *27.3.5.1. Coulomb Potential*. However for potential decaying fast enough (faster than $|x|^{-2-\delta}$) the regularization is not needed.

For V^0 we have an exact expression which concludes the proof of Statement (i).

The proof of Statement (ii) is similar albeit with the small improvement, based on the behavior of the classical dynamics (without magnetic field) exactly as in Chapter 25. □

27.3.6 Several Singularities

Consider now the "molecular" case $M \ge 2$. Then we need more delicate arguments.

Decoupling of Singularities

Consider partition of unity $1 = \sum_{0\le m\le M} \psi_m^2$ where ψ_m is supported in $\frac{1}{3}a$-vicinity of y_m for $m = 1, \dots, M$ and $\psi_0 = 0$ in $\frac{1}{4}a$-vicinities of y_m ("near-nuclei" and "between-nuclei"partition elements).

Estimate from above. Then

$$\mathsf{Tr}(H^-_{A,V}) = \sum_{0\le m\le M} \mathsf{Tr}(\psi_m H^-_{A,V}\psi_m), \tag{27.3.88}$$

and to estimate E^* from the above we impose an extra condition to A:

$$A = 0 \qquad \text{for } \ \ell(x) \ge \frac{1}{5}a. \tag{27.3.89}$$

Then in this framework we estimate

$$\mathsf{Tr}^-(\psi_0 H^-_{A,V}\psi_0) - \int \mathsf{Weyl}_1(x)\psi_0^2(x)\,dx \le Ch^{-1}a^{-\frac{1}{2}}. \tag{27.3.90}$$

Proof of this inequality is trivial by using ℓ-admissible partition and applying results of the theory without any magnetic field.

Thus, to estimate E^* from above[17] we just need to estimate from above the minimum with respect to A satisfying (27.3.89) of the expression

$$\mathsf{Tr}(\psi_m H^-_{A,V}\psi_m) - \int \mathsf{Weyl}_1(x)\psi_m^2(x)\,dx + \frac{1}{\kappa h^2}\int_{\{\ell_m(x)\le \frac{1}{5}a\}} |\partial A|^2\,dx. \tag{27.3.91}$$

Estimate from below. In this case we use the same partition of unity $\{\psi_m^2\}_{j=0,1,\dots,M}$ and estimate

$$\mathsf{Tr}(H^-_{A,V}) \ge \sum_{0\le m\le M} \mathsf{Tr}^-(\psi_m H_{A,V'}\psi_m) \tag{27.3.92}$$

with

$$V' = V + 2h^2 \sum_{0\le m\le M} (\partial\psi)^2, \tag{27.3.93}$$

and we also use decomposition

$$\int |\partial A|^2\,dx = \sum_{0\le m\le M}\int \omega_m^2 |\partial A|^2\,dx \tag{27.3.94}$$

with

$$\omega_m(x) = 1 \ \text{ if } \ \ell_m(x) \le \frac{1}{10}a, \quad \omega_m(x) \ge 1 - C\varsigma \ \text{ if } \ \ell_m(x) \le \frac{1}{2}a \quad \text{for } \ m = 1, \dots, M, \tag{27.3.95}$$

$$\omega_0 \ge \epsilon_0\varsigma \quad \text{if } \ \ell(x) \ge \frac{1}{5}a. \tag{27.3.96}$$

So far $\varsigma > 0$ is a constant but later it will become a small parameter. Then since

$$\mathsf{Tr}^-(\psi_0 H_{A,V'}\psi_0) - \int \mathsf{Weyl}_1(x)\psi_0^2(x)\,dx + \frac{1}{\kappa h^2}\int \omega_0^2|\partial A|^2\,dx \ge Ch^{-1}a^{-\frac{1}{2}} \tag{27.3.97}$$

(again proven by partition) in virtue of the previous Section 27.2 we are left with the estimates from below for

$$\mathsf{Tr}^-(\psi_m H_{A,V'}\psi_m) - \int \mathsf{Weyl}_1(x)\psi_m^2(x)\,dx + \frac{1}{\kappa h^2}\int \omega_m^2|\partial A|^2\,dx. \tag{27.3.98}$$

[17] Modulo error in (27.3.51).

Remark 27.3.23. (i) Note that the error in Weyl_1 when we replace V' there by V does not exceed $Ch^{-1}(1+a^{-\frac12})$ which is less than the error in (27.3.51). Here we can also assume that A satisfies (27.3.89); we need just to replace ς by $\epsilon_0\varsigma$ in (27.3.95)–(27.3.96).

(ii) We can further go down by replacing $\mathsf{Tr}^-(\psi_m H_{A,V'}\psi_m)$ by $\mathsf{Tr}(\psi_m H^-_{A,V'}\psi_m)$.

(iii) Therefore we basically have the same object for both estimates albeit with marginally different potentials (V in the estimate from above and V' in the estimate from below) and with a weight ω_m^2 satisfying (27.3.95)–(27.3.96); in both cases $\omega=1$ if $\ell(x)\le \frac{1}{10}a$ but in the estimate from above $\omega(x)$ grows to C_0 and in the estimate from below $\omega(x)$ decays to ς if $\ell(x)\ge \frac13 a$ and in both cases condition (27.3.89) could be either imposed or skipped.

(iv) From now on we consider a single singularity at 0 and we skip subscript m. However if there was a single singularity from the beginning, all arguments of this and forthcoming paragraphs would be unnecessary.

Scaling. Now we apply scaling arguments:

(i) We are done as $Z\asymp 1$ but as $Z\ll 1$ [18] we need a bit more fixing. The problem is that $V\asymp Z\ell^{-1}$ only for $|x|\le aZ$; otherwise $V\lesssim a^{-1}$ (where we still assume that $a\le 1$). To deal with this we apply in the zone $\{x\colon aZ\le |x|\le a\}$ the same procedure as before and its contribution to the error will be $Ch^{-1}a^{-\frac12}$ as $\rho=a^{-\frac12}$ here. Actually we also need to keep $|x|\ge Z^{-1}h^2$; so we assume that $Z^{-1}h^2\le Za$, i.e. $Z\ge a^{-\frac12}h$.

Now let us scale $x\mapsto x'=x(aZ)^{-1}$, and multiply $H_{a,V}$ by a and therefore also multiply A by $a^{\frac12}$, so $A\mapsto A'=a^{\frac12}A$, $h\mapsto h'=ha^{-\frac12}Z^{-1}$; then the magnetic energy becomes

$$\kappa^{-1}h^{-2}Z\int \omega^2(x)|\partial' A'|^2\,dx',$$

where factors a^{-1} and aZ come from substitution $A=a^{-\frac12}A'$ and scaling respectively. We need to multiply it by a (since we multiplied an operator); then plugging $h^{-2}=h'^{-2}a^{-1}Z^{-2}$ we get the same expression as before but with $Z'=1$, $a'=1$ and $h'=ha^{-\frac12}Z^{-1}\le 1$ and $\kappa'=\kappa Z$ instead of h and κ.

[18] Since Z denotes Z_m now we assume only that $Z_1+\ldots+Z_M\asymp 1$.

If we prove here an error $O\big(h'^{-1}+\kappa'|\log\kappa'|^{\frac{1}{3}}h'^{-\frac{4}{3}}\big)$, then the final error will be this expression multiplied by a^{-1}, i.e. $O\big(a^{-\frac{1}{2}}Zh^{-1}+\kappa a^{-\frac{1}{2}}Z^{\frac{7}{3}}|\log\kappa Z|^{\frac{1}{3}}h^{-\frac{4}{3}}\big)$, which is less than the same expression with $Z=1$.

(ii) On the other hand, let $Z\le a^{-\frac{1}{2}}h$. Recall, we assume that $a\ge C_0h^2$. Then we can apply the same arguments as before but with $\bar{Z}=a^{-\frac{1}{2}}h$ and we arrive to the same situation as before albeit with $h'=1$, $a'=1$, $\kappa'=\kappa a^{-\frac{1}{2}}h$ and with $Z'=Z\bar{Z}^{-1}$. Then we have the trivial error estimate $O(a^{-1})$ which is less than $O(a^{-\frac{1}{2}}h^{-1})$.

Main Results

Combining results of the previous subsubsections and paragraphs with Proposition 27.3.9 we arrive to

Theorem 27.3.24. *Let $M\ge 2$, $\kappa\le\kappa^*$ and (27.3.38) hold with $\nu>\frac{4}{3}$. Then*

(i) The following asymptotics holds

$$\mathsf{E}^*=\int \mathsf{Weyl}_1(x)\,dx+2\sum_{1\le m\le M} z_m^2 S(z_m\kappa)h^{-2}+O(R_1+R_2) \tag{27.3.99}$$

with

$$R_1=\begin{cases} h^{-1}+\kappa|\log\kappa|^{\frac{1}{3}}h^{-\frac{4}{3}} & \text{if } a\ge 1,\\ a^{-\frac{1}{2}}h^{-1}+\kappa|\log\kappa|^{\frac{1}{3}}a^{-\frac{1}{3}}h^{-\frac{4}{3}} & \text{if } h^2\le a\le 1\end{cases} \tag{27.3.100}$$

and

$$R_2=\kappa h^{-2}\begin{cases} a^{-3} & \text{if } a\ge |\log h|^{\frac{1}{3}},\\ |\log h^2a^{-1}|^{-1} & \text{if } h^2\le a\le |\log h|^{\frac{1}{3}}.\end{cases} \tag{27.3.101}$$

(ii) If $\kappa=o(h^{\frac{1}{3}}|\log h|^{-\frac{1}{3}})$, $\kappa a^{-3}=o(h)$ and $a^{-1}=o(1)$, then

$$\mathsf{E}^*=\int \mathsf{Weyl}_1^*(x)\,dx+2\sum_{1\le m\le M} z_m^2 S(z_m\kappa)h^{-2}+o(h^{-1}). \tag{27.3.102}$$

Proof. To prove theorem we need to prove the following estimate

$$\frac{1}{\kappa}\|\partial A\|^2_{\{b\le\ell(x)\le 2b\}},\le Cb^{-3} \tag{27.3.103}$$

where $r_* \le b \le a$ is a "cut-off". On the other hand, we know that

$$\frac{1}{\kappa}\|\partial A\|^2 = -\frac{\partial S}{\partial \kappa} = O(1) \tag{27.3.104}$$

and we need to recover the last factor in the definition of R_2.

For $a \ge 1$ we can have κa^{-3} because in virtue of (27.3.41) the square of the partial norm in (27.3.104) does not exceed $Ca^{-3}\kappa^2$.

On the other hand, for $h^2 \le r_* \le a$ we can select $b : r_* \le b \le a$ such that the partial norm in (27.3.104) does not exceed $C|\log(a/h^2)|^{-1} \cdot \|\partial A\|^2$. □

Remark 27.3.25. (i) For $a \le |\log h|$ we do not need assumption (27.3.38).

(ii) Following arguments of Section 12.6 estimates (27.3.85) and (27.3.104) could be improved to $O(h^{-1+\delta})$ provided $a \ge h^{-\delta_1}$, $\kappa \le h^{\frac{1}{3}+\delta_1}|\log h|^{-\frac{1}{3}}$ and $\kappa \le a^3 h^{1+\delta_1}$.

Problems and rRemarks

The following problem seems to be really challenging and we have no idea how to approach it:

Problem 27.3.26. (i) For $\kappa \in [0, \kappa^*]$ with small enough $\kappa^* > 0$ does $S(\kappa)$ really depend on κ or $S(\kappa) = S(0)$ (see Remark 27.3.19)?

(ii) If $S(\kappa)$ really depends on κ, what is asymptotic behavior of $S(\kappa) - S(0)$ as $\kappa \to +0$: can one improve $S(\kappa) - S(0) = O(\kappa)$?

(iii) Do we really need an assumption $\kappa \in [0, \kappa^*]$ (again see Remark 27.3.19)?

(iv) Can one improve estimates to minimizer?

27.4 Asymptotics of the Ground State Energy

27.4.1 Problem

Now we are ready to tackle our original object (27.1.2)–(27.1.3). So, let us consider our usual quantum Hamiltonian

$$\mathsf{H} = \sum_{1\le j\le N} H^0_{x_j} + \sum_{1\le j<k\le N} |x_j - x_k|^{-1} \tag{27.4.1}$$

in

$$\mathfrak{H} = \bigwedge_{1\le n\le N} \mathsf{H}, \qquad \mathsf{H} = \mathscr{L}^2(\mathbb{R}^3, \mathbb{C}^2) \tag{27.4.2}$$

with

$$H^0 = \big((i\nabla - A)\cdot \boldsymbol{\sigma}\big)^2 - V(x). \tag{27.4.3}$$

We are interested in the *ground state energy* $\mathsf{E}^*_N(A)$ of our system i.e. in the lowest eigenvalue of the operator H on $\mathfrak{H}$:

$$\mathsf{E}^*_N(0) = \inf \mathsf{Spec}(\mathsf{H}) \qquad \text{on} \quad \mathfrak{H} \tag{27.4.4}$$

if $A = 0$ and more generally in

$$\mathsf{E}^*_N = \inf_A \Big(\inf \mathsf{Spec}_{\mathfrak{H}}(\mathsf{H}) + \frac{1}{\alpha}\int |\nabla \times A|^2\, dx\Big), \tag{27.4.5}$$

where

$$V(x) = \sum_{1\le m\le M} \frac{Z_m}{|x - \mathsf{y}_m|} \tag{27.4.6}$$

$$N \asymp Z \gg 1, \qquad Z := Z_1 + \dots + Z_M, \qquad Z_1 > 0, \dots, Z_M > 0, \tag{27.4.7}$$

M is fixed, under assumption

$$0 < \alpha \le \kappa^* Z^{-1} \tag{27.4.8}$$

with sufficiently small constant $\kappa^* > 0$.

Our purpose is to derive an asymptotics

$$\mathsf{E}^*_N \approx \mathcal{E}^{\mathsf{TF}}_N + \sum_{1\le m\le M} 2Z_m^2 S(\alpha Z_m) \tag{27.4.9}$$

and estimate an error (usually) provided

$$b := \min_{1\le m<m'\le M} |\mathsf{y}_m - \mathsf{y}_{m'}| \ge Z^{-\frac{1}{3}}. \tag{27.4.10}$$

Recall that the Thomas-Fermi potential W^{TF} and the Thomas-Fermi density ρ^{TF} satisfy equations

$$\rho^{\mathsf{TF}} = \frac{1}{3\pi^2}(W^{\mathsf{TF}} + \nu)^{\frac{3}{2}}_+ \tag{27.4.11}$$

and

$$W^{\mathsf{TF}} = V^0 + |x|^{-1} * \rho^{\mathsf{TF}}, \tag{27.4.12}$$

where ν is a chemical potential.

27.4.2 Lower Estimate

Consider corresponding to H quadratic form exactly as in Sections 25.2 and 26.6

$$\begin{aligned}(27.4.13)\quad \langle \mathsf{H}\Psi,\Psi\rangle &= \sum_j (H^0_{x_j}\Psi,\Psi) + (\sum_{1\le j<k\le N} |x_j-x_k|^{-1}\Psi,\Psi) = \\ &\sum_j (H_{x_j}\Psi,\Psi) + ((V-W)\Psi,\Psi) + (\sum_{1\le j<k\le N} |x_j-x_k|^{-1}\Psi,\Psi)\end{aligned}$$

with

$$(27.4.14)\qquad H = \big((i\nabla - A)\cdot\boldsymbol{\sigma}\big)^2 - W(x)$$

where we select W later. By Lieb-Oxford inequality the last term is estimated from below:

$$(27.4.15)\qquad \langle \sum_{1\le j<k\le N} |x_j-x_k|^{-1}\Psi,\Psi\rangle \ge \frac{1}{2}\mathsf{D}(\rho_\Psi,\rho_\Psi) - C\int \rho_\Psi^{\frac{4}{3}}\,dx,$$

where

$$(27.4.16)\qquad \rho_\Psi(x) = N\int |\Psi(x,x_2,\dots,x_N)|^2\,dx_2\cdots dx_N$$

is a spatial density associated with Ψ and

$$(27.4.17)\qquad \mathsf{D}(\rho,\rho') := \iint |x-y|^{-1}\rho(x)\rho'(y)\,dxdy.$$

Therefore again repeating arguments of Section 25.2 we estimate H from below:

$$\begin{aligned}(27.4.18)\quad &\langle \mathsf{H}\Psi,\Psi\rangle \ge \\ &\sum_j (H_{x_j}\Psi,\Psi) - 2((V-W)\Psi,\Psi) + \frac{1}{2}\mathsf{D}(\rho_\Psi,\rho_\Psi) - C\int \rho_\Psi^{\frac{4}{3}}\,dx = \\ &\sum_j (H_{x_j}\Psi,\Psi) - \mathsf{D}(\rho,\rho_\Psi) + \frac{1}{2}\mathsf{D}(\rho_\Psi,\rho_\Psi) - C\int \rho_\Psi^{\frac{4}{3}}\,dx = \\ &\sum_j (H_{x_j}\Psi,\Psi) - \frac{1}{2}\mathsf{D}(\rho,\rho) + \frac{1}{2}\mathsf{D}(\rho-\rho_\Psi,\rho-\rho_\Psi) - C\int \rho_\Psi^{\frac{4}{3}}\,dx\end{aligned}$$

swhere

$$W - V = |x|^{-1} * \rho. \tag{27.4.19}$$

Note that due to antisymmetricity of Ψ

$$\sum_j (H_{x_j}\Psi, \Psi) \geq \sum_{1\leq j\leq N:\ \lambda_j<0} \lambda_j \geq \mathsf{Tr}^-(H), \tag{27.4.20}$$

where λ_j are eigenvalues of H.

To estimate the last term in (27.4.18) we reproduce the proof of Lemma 4.3 from L. Erdös, S. Fournais and J. P. Solovej [3]:

According to magnetic Lieb–Thirring inequality for $U \geq 0$:

$$\sum_{j\leq N} \langle (H^0_{x_j} - U)\Psi, \Psi\rangle \geq -C\int U^{5/2}\,dx - C\gamma^{-3}U^4\,dx - \gamma\int B^2\,dx \tag{27.4.21}$$

where $B = \nabla \times A$, $\gamma > 0$ is arbitrary. Then, selecting $U = \beta\,\mathsf{min}(\rho_\Psi^{5/3}, \gamma\rho_\Psi^{4/3})$ with $\beta > 0$ sufficiently small but independent from γ, we ensure that $\frac{1}{2}U\rho_\Psi \geq CU^{5/2} + C\gamma^{-3}U^4$ and then

$$\sum_{j\leq N} \langle (H^0_{x_j})\Psi, \Psi\rangle \geq \epsilon\int \mathsf{min}(\rho_\Psi^{5/3}, \gamma\rho^{4/3})\,dx, -\gamma\int B^2\,dx, \tag{27.4.22}$$

which implies

$$\int \rho_\Psi^{4/3}dx \leq \gamma^{-1}\int \mathsf{min}(\rho_\Psi^{5/3}, \gamma\rho^{4/3})\,dx + \gamma\int \rho_\Psi\,dx \leq c\gamma^{-1}\sum_{j:\lambda_j<0} \langle (H^0_{x_j})\Psi, \Psi\rangle + c\int B^2\,dx + c\gamma N \tag{27.4.23}$$

where we use $\int \rho_\Psi\,dx = N$.

Remark 27.4.1. As one can prove easily (see also L. Erdös, S. Fournais and J. P. Solovej [3]) that

$$\sum_{j\leq N} \langle (H^0_{x_j})\Psi, \Psi\rangle \leq CZ^{\frac{4}{3}}N \tag{27.4.24}$$

even if $N \not\asymp Z$; then we conclude that

$$\int \rho_\Psi^{4/3} dx \le CZ^{\frac{2}{3}} N + C_1 \int B^2 \, dx. \tag{27.4.25}$$

It is sufficient unless we want to recover Dirac and Schwinger correction terms (given by (25.1.29) and (25.1.30) with $q = 2$) which unfortunately is possible only if $\alpha \ll Z^{-\frac{10}{9}} |\log Z|^{-\frac{1}{3}}$. To recover remainder estimate $o(Z^{\frac{5}{3}})$ (or marginally better) we just apply Theorem 26.A.2. We will do it later (see Theorem 27.4.5).

Therefore skipping the non-negative third term in the right-hand expression of (27.4.18) we conclude that

$$\begin{aligned} \langle \mathsf{H}\Psi, \Psi\rangle + \frac{1}{\alpha}\int |\nabla \times A|^2 \, dx \ge \\ \mathsf{Tr}^-(H) + (\frac{1}{\alpha} - C_1)\int |\nabla \times A|^2 \, dx - \frac{1}{2}\mathsf{D}(\rho,\rho) - CZ^{\frac{5}{3}}. \end{aligned} \tag{27.4.26}$$

Applying Theorem 27.3.24 we conclude that

(27.4.27) The sum of the first and the second terms in the right-hand expression of (27.4.26) is greater than

$$\mathcal{E}^{\mathsf{TF}} + \sum_m 2Z_m^2 S(\alpha Z_m) - CZ^{\frac{4}{3}}(R_1 + R_2)$$

with R_1 and R_2 defined by (27.3.100) and (27.3.101) respectively with $\kappa = \alpha Z$, $h = Z^{-\frac{1}{3}}$ and

$$a \coloneqq Z^{\frac{1}{3}} \min_{1\le m<m'\le M} |\mathsf{y}_m - \mathsf{y}_{m'}|. \tag{27.4.28}$$

To prove this claim one needs just to rescale

$$\begin{aligned} x \mapsto xZ^{\frac{1}{3}}, \quad a \mapsto aZ^{\frac{1}{3}}, \quad W \mapsto Z^{-\frac{4}{3}} W, \\ A \mapsto A, \quad \nabla \times A \mapsto Z^{\frac{1}{3}} \nabla \times A \end{aligned} \tag{27.4.29}_{1-5}$$

and introduce

$$h = Z^{-\frac{1}{3}}, \quad \kappa = \alpha Z. \tag{27.4.30}$$

Observe that due to $(27.4.29)_3$ we need to multiply our estimate by $Z^{\frac{4}{3}}$.

Here one definitely needs the regularity properties like in Section 27.3 but we have them since $\rho = \rho^{\mathsf{TF}}$, $W = W^{\mathsf{TF}}$. Also one can see easily that "$-C_1$" brings correction not exceeding $C_2\alpha Z^2$ as $\alpha Z \le 1$.

Meanwhile for $\rho = \rho^{\mathsf{TF}}$, $W = W^{\mathsf{TF}}$

$$\frac{2}{15\pi^2}\int W^{\frac{5}{2}}\,dx - \frac{1}{2}\mathsf{D}(\rho,\rho) = \mathcal{E}^{\mathsf{TF}}. \tag{27.4.31}$$

Lower estimate of Theorem 27.4.3 below has been proven.

Remark 27.4.2. $\rho = \rho^{\mathsf{TF}}$, $W = W^{\mathsf{TF}}$ delivers the maximum of the right-hand expression of (27.4.31) among ρ, W satisfying (27.4.19).

27.4.3 Upper Estimate

Upper estimate is easy. Plugging as in Section 25.2 Ψ, the *Slater determinant* (25.2.16) of $\psi_1,\dots,\psi_N$, where $\psi_1,\dots,\psi_N$ are eigenfunctions of $H_{A,W}$ we get

$$\begin{aligned}\langle \mathsf{H}\Psi,\Psi\rangle = {}& \mathsf{Tr}^-(H_{A,W}-\lambda_N) + \lambda_N N + \\ & \int (W-V)(x)\rho_\Psi(x)\,dx + \frac{1}{2}\mathsf{D}(\rho_\Psi,\rho_\Psi) - \\ & \frac{1}{2}N(N-1)\iint |x_1-x_2|^{-1}|\Psi(x_1,x_2,x_3,\dots,x_N)|^2\,dx_1\cdots dx_N,\end{aligned} \tag{27.4.32}$$

where we do not care about the last term since we drop it (again as long as we cannot get sharp enough estimate) and the first term in the second line is in fact $-\mathsf{D}(\rho,\rho_\Psi)$, provided (27.4.19) holds. Thus we get

$$\begin{aligned}\mathsf{Tr}^-(H_{A,W}-\lambda_N) + \lambda_N N - \frac{1}{2}\mathsf{D}(\rho,\rho) + \frac{1}{2}\mathsf{D}(\rho_\Psi-\rho,\rho_\Psi-\rho) + \\ \frac{1}{\alpha}\int |\partial A|^2\,dx,\end{aligned} \tag{27.4.33}$$

where we added the magnetic energy. Definitely we have several problems here: λ_N depends on A and there may be less than N negative eigenvalues.

However in the latter case we can obviously replace N by the lesser number $N' := \max(n \le N, \lambda_n \le 0)$ since E^*_N is decreasing function of N. In this case the first term in (27.4.33) would be just $\mathsf{Tr}^-(H_{A,W})$ and the second

would be 0. Then we apply theory of the previous Section 27.3 immediately without extra complications.

Consider A a minimizer (or its mollification) for operator $H_{A,W} - \mu$ with potential $W = W^{\mathsf{TF}}$ and $\mu \le 0$. Then

$$\mathsf{N}(\mu) := \#\{\lambda_k < \mu\} = \int (W+\mu)_+^{\frac{3}{2}}\, dx + O(Z^{\frac{2}{3}}). \tag{27.4.34}$$

One can prove (27.4.34) easily using the regularity properties of A established in the previous Section 27.3 and the same rescaling (27.4.29)–(27.4.30) as before. We leave this easy proof to the reader.

Therefore, repeating arguments of Subsubsection *25.4.2.1. Estimating* $|\lambda_N - \nu|$, we conclude that either $N \ge Z - C_0 Z^{\frac{2}{3}}$ and then $|\nu| \le C_1 Z^{\frac{8}{9}}$ and we can take $\mu = 0$ and $|\lambda_{N'}| \le C_1 Z^{\frac{8}{9}}$ or $N \le Z - C_0 Z^{\frac{2}{3}}$ and then we take $\mu = \nu$, $|\lambda_N| \asymp |\nu| \asymp (Z-N)_+^{\frac{4}{3}}$ and $|\lambda_N - \nu| \le C_1 |\nu|^{\frac{1}{4}} Z^{\frac{2}{3}}$.

After this, following again to arguments of Subsection 25.4.2, we conclude that

(27.4.35) Expression (27.4.33) without term $\frac{1}{2}\mathsf{D}(\rho_\Psi - \rho, \rho_\Psi - \rho)$ does not exceed

$$\mathcal{E}^{\mathsf{TF}} + \sum_m 2Z_m^2 S(\alpha Z_m) + CZ^{\frac{4}{3}}(R_1 + R_2)$$

with R_1 and R_2 defined by (27.3.100) and (27.3.101) respectively with $\kappa = \alpha Z$, $h = Z^{-\frac{1}{3}}$ and a defined by (27.4.28).

Now we need to estimate properly $\mathsf{D}(\rho_\Psi - \rho, \rho_\Psi - \rho)$ which as in i.e. Subsubsection *25.4.2.2. Estimating* D*-Term* does not exceed the sum of

$$\mathsf{D}(e(x,x,\mu) - \mathsf{Weyl}(x,\mu),\ e(x,x,\mu) - \mathsf{Weyl}(x,\mu)), \tag{27.4.36}$$

$$\mathsf{D}(e(x,x,\lambda_N) - \mathsf{Weyl}(x,\lambda_N),\ e(x,x,\lambda_N) - \mathsf{Weyl}(x,\lambda_N)) \tag{27.4.37}$$

and

$$\mathsf{D}(\mathsf{Weyl}(x,\mu) - \mathsf{Weyl}(x,\lambda_N),\ \mathsf{Weyl}(x,\mu) - \mathsf{Weyl}(x,\lambda_N)). \tag{27.4.38}$$

Next, following arguments of Subsubsection *25.4.2.2. Estimating* D*-Term*, one can prove easily that due to regularity properties of A both semiclassical (27.4.36) and (27.4.37) terms do not exceed $CZ^{\frac{5}{3}}$ and due to estimates for $|\lambda_N - \mu|$ the last term does (27.4.38) not exceed $CZ^{\frac{5}{3}}$ as well.

This concludes the proof of the upper estimate in Theorem 27.4.3 below.

27.4.4 Main Theorems

Theorem 27.4.3. *(i) Under assumptions* (27.4.7) *and* (27.4.8) *the following asymptotics holds*

$$\mathsf{E}^*_N = \mathcal{E}^{\mathsf{TF}}_N + \sum_{1\le m\le M} 2Z_m^2 S(\alpha Z_m) + O\big(Z^{\frac{4}{3}}(R_1+R_2)\big) \tag{27.4.39}$$

with R_1 and R_2 defined by (27.3.100) *and* (27.3.101) *respectively with $\kappa = \alpha Z$, $h = Z^{-\frac{1}{3}}$ and a defined by* (27.4.28), *$a = \infty$ for $M = 1$.*

(ii) In particular, under assumption (27.4.10) *the following asymptotics holds*

$$\mathsf{E}^*_N = \mathcal{E}^{\mathsf{TF}}_N + \sum_{1\le m\le M} 2Z_m^2 S(\alpha Z_m) + O\big(\alpha|\log(\alpha Z)|^{\frac{1}{3}} Z^{\frac{25}{9}} + Z^{\frac{5}{3}} + \alpha b^{-3} Z^2\big). \tag{27.4.40}$$

Recall that $\mathcal{E}^{\mathsf{TF}}_N$ is a *Thomas-Fermi energy* and $S(\alpha Z_m)Z_m^2$ are magnetic *Scott correction terms*.

Theorem 27.4.4. *(i) Let assumptions* (27.4.7) *and* (27.4.8) *be fulfilled and let $\Psi = \Psi_{\boldsymbol{A}}$ be a ground state for a near optimizer $\boldsymbol{A}$ of the original multiparticle problem. Then*

$$\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}},\, \rho_\Psi - \rho^{\mathsf{TF}}) \le CZ^{\frac{5}{3}}. \tag{27.4.41}$$

(ii) Furthermore, as $b \ge Z^{-\frac{1}{3}}$

$$\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}},\, \rho_\Psi - \rho^{\mathsf{TF}}) \le CZ^{\frac{5}{3}}\big(Z^{-\delta} + (bZ^{\frac{1}{3}})^{-\delta} + (\alpha Z)^{\delta}\big). \tag{27.4.42}$$

Proof. (i) Note that all the terms in estimates from below and from above are $O(Z^{\frac{5}{3}})$ except the common term

$$\mathsf{Tr}^-(H_{A,W} + \mu) + \frac{1}{\alpha}\int |\nabla \times A|^2\, dx, \tag{27.4.43}$$

where A is a minimizer for this term and therefore estimate (27.4.41) has been proven because estimate from below also contains $\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}},\, \rho_\Psi - \rho^{\mathsf{TF}})$.

(ii) To prove Statement (ii) one needs

(a) To improve estimate (27.4.34) to

$$\mathsf{N}(\mu) = \int (W+\mu)_+^{\frac{3}{2}}\,dx + O\bigl(Z^{\frac{2}{3}}\bigl[Z^{-\delta} + (bZ^{\frac{1}{3}})^{-\delta} + (\alpha Z)^{\delta}\bigr]\bigr); \tag{27.4.44}$$

(b) To estimate terms (27.4.36)–(27.4.38) by the right-hand expression of (25.4.45); and

(c) To accommodate Dirac term in both upper and lower estimates.

Tasks (a), (b) are easy and we leave it to the reader (cf. arguments of Subsection 25.4.3); we use that after rescaling effective magnetic field intensity becomes $O(\alpha Z)$ in the zone $\{x : \ell(x) \asymp Z^{-\frac{1}{3}}\}$ due to already established estimates to A.

To fulfill task (c) observe that in the upper estimate we already have term

$$-\frac{1}{2}\,\mathsf{tr}\iint |x-y|^{-1} e_N(x,y) e_N^{\dagger}(x,y)\,dxdy. \tag{27.4.45}$$

On the other hand, in virtue of Theorem 26.A.2 we replace in the lower estimate term $-C\int \rho_\Psi^{\frac{4}{3}}(x)\,dx$ by (27.4.6) with $O(Z^{\frac{5}{3}-\delta})$ error (again we leave easy details to the reader).

One can prove by the same arguments as as in the non-magnetic case that for $\alpha Z \le \kappa^*$ it is $\mathsf{Dirac} + O(Z^{\frac{5}{3}-\delta})$. □

Finally, combining arguments sketched in the proof of Theorem 27.4.4 with the improved estimate of (27.4.43) (see Theorem 27.3.24(ii)) we arrive to

Theorem 27.4.5. *Let assumptions* (27.4.8) *and* (27.4.10) *be fulfilled, and let* $\alpha \le Z^{-\frac{10}{9}}|\log Z|^{-\frac{1}{3}}$. *Then*

$$\mathsf{E}_N^* = \mathcal{E}_N^{\mathsf{TF}} + \sum_{1\le m\le M} 2Z_m^2 S(\alpha Z_m) + \mathsf{Dirac} + \mathsf{Schwinger} + O\bigl(\alpha|\log(\alpha Z)|^{\frac{1}{3}} Z^{\frac{25}{9}} + Z^{\frac{5}{3}-\delta} + \alpha b^{-3} Z^2\bigr) \tag{27.4.46}$$

where Dirac *and* $\mathsf{Schwinger}$ *are* Dirac *and* Schwinger correction terms *defined exactly as in non-magnetic case by* (25.1.29) *and* (25.1.30) *respectively.*

27.4.5 Free Nuclei Model

Consider now free nuclei model (see Subsubsection *25.4.4.2). Estimates for Distance between Nuclei in the Free Nuclei Model*).

Theorem 27.4.6. *Let us consider* $\mathsf{y}_m = \mathsf{y}_m^*$ *minimizing the full energy*

$$\widehat{\mathsf{E}}_N^* := \mathsf{E}_N^* + \sum_{1\le m<m'\le M} Z_m Z_{m'} |\mathsf{y}_m - \mathsf{y}_{m'}|^{-1}. \tag{27.4.47}$$

Assume that

$$Z_m \asymp N \qquad \forall m = 1, \dots, M. \tag{27.4.48}$$

Then

$$b \ge \min\bigl(Z^{-\frac{5}{21}+\delta},\, Z^{-\frac{5}{21}}(\alpha Z)^{-\delta},\, \alpha^{-\frac{1}{4}} Z^{-\frac{1}{2}}\bigr) \tag{27.4.49}$$

and in the remainder estimates in (27.4.40) *and* (27.4.46) *one can skip* b*-connected terms; so we arrive to*

$$\mathsf{E}_N^* = \mathcal{E}_N^{\mathsf{TF}} + \sum_{1\le m\le M} 2Z_m^2 S(\alpha Z_m) + O\bigl(\alpha |\log(\alpha Z)|^{\frac{1}{3}} Z^{\frac{25}{9}} + Z^{\frac{5}{3}}\bigr) \tag{27.4.50}$$

and

$$\mathsf{E}_N^* = \mathcal{E}_N^{\mathsf{TF}} + \sum_{1\le m\le M} 2Z_m^2 S(\alpha Z_m) + \mathsf{Dirac} + \mathsf{Schwinger} + O\bigl(\alpha |\log(\alpha Z)|^{\frac{1}{3}} Z^{\frac{25}{9}} + Z^{\frac{5}{3}-\delta}\bigr) \tag{27.4.51}$$

respectively and also the same asymptotics with $\widehat{\mathsf{E}}_N^*$ *and* $\widehat{\mathcal{E}}_N^{\mathsf{TF}}$ *instead of* E_N^* *and* $\mathcal{E}_N^{\mathsf{TF}}$.

Proof. Optimization with respect to $\mathsf{y}_1, \dots, \mathsf{y}_M$ implies

$$\mathsf{E}^* + \sum_{1\le m<m'\le M} \frac{Z_m Z_{m'}}{|\mathsf{y}_m - \mathsf{y}_{m'}|} < \sum_{1\le m\le M} \mathsf{E}_m^* \tag{27.4.52}$$

where $\mathsf{E}^* = \mathsf{E}^*(\mathsf{y}_1, \dots, \mathsf{y}_M; Z_1, \dots, Z_m, N)$ and $\mathsf{E}_m^* = \mathsf{E}^*(\mathsf{y}_m, Z_m)$ are calculated for separate atoms. In virtue of Theorem 27.4.3

$$\mathcal{E}^{\mathsf{TF}} + \sum_{1\le m<m'\le M} \frac{Z_m Z_{m'}}{|\mathsf{x}_m - \mathsf{x}_{m'}|} - \sum_{1\le m\le M} \mathcal{E}_m^{\mathsf{TF}} \le C\alpha |\log(\alpha Z)|^{\frac{1}{3}} Z^{\frac{25}{9}} + Z^{\frac{5}{3}} + C\alpha b^{-3} Z^2. \tag{27.4.53}$$

However due to the strong non-binding theorem in Thomas-Fermi theory the left-hand expression is $\gtrsim b^{-7}$ for $b \ge Z^{-\frac{1}{3}}$ and therefore (27.4.53) implies

$$b \gtrsim \min\bigl(Z^{-\frac{5}{21}},\, \alpha^{-\frac{1}{7}}|\log(\alpha Z)|^{-\frac{1}{21}} Z^{-\frac{25}{63}},\, \alpha^{-\frac{1}{4}} Z^{-\frac{1}{2}}\bigr)$$

where the third expression is larger than the second one for sure. Unfortunately, for $\alpha \ge Z^{-\frac{10}{9}-\delta'}$ it is not as good as we claimed in (27.4.49). Still this estimate implies both (27.4.50) and (27.4.51).

To prove estimate (27.4.49) we observe that $b \gg Z^{-\frac{2}{7}}$ [19)] and then we employ arguments used in the proof of Theorem 26.8.11 and prove that

$$|\mathsf{Tr}^-(H_{A,W}+\mu) - \sum_{1\le m\le M} \mathsf{Tr}^-(H_{A,W_m}+\mu) - \int \Bigl(\mathsf{Weyl}(H_{A,W}+\mu;\, x) - \sum_{1\le m\le M} \mathsf{Weyl}(H_{A,W_m}+\mu;\, x)\Bigr)\, dx| \le CZ^{\frac{5}{3}}\bigl(Z^{-\delta}+(\alpha Z)^{\delta}\bigr),$$

where A be a minimizer for "molecular" expression (27.4.45) and W_m are atomic potentials. The same estimate holds if we replace $\mathsf{Tr}^-(H_{A,W_m}+\mu)$ by $\mathsf{Tr}^-(H_{A_m,W_m}+\mu)$ with $A_m = A\phi(b^{-1}|x-\mathsf{y}_m|)$ with $\phi \in \mathscr{C}_0^\infty(B(0,\frac{1}{3})$, equal 1 in $B(0,\frac{1}{4})$. We leave an easy proof to the reader.

Then using the lower estimate for $\inf \mathsf{Spec}_{\mathfrak{H}}(\mathsf{H})$ and upper estimates for both $\inf \mathsf{Spec}_{\mathfrak{H}_m}(\mathsf{H}_m)$ through $\mathsf{Tr}^-(H_{A,W}+\mu)$ and $\mathsf{Tr}^-(H_{A_m,W_m}+\mu')$ respectively (where H_m are associated with H_{A_m,W_m}) we arrive to

$$\inf \mathsf{Spec}_{\mathfrak{H}}(\mathsf{H}) \ge \sum_{1\le m\le M} \inf \mathsf{Spec}_{\mathfrak{H}_m}(\mathsf{H}_m) + \mathcal{E}^{\mathsf{TF}} - \sum_{1\le m\le M} \mathcal{E}_m^{\mathsf{TF}} - CZ^{\frac{5}{3}}\bigl(Z^{-\delta}+(\alpha Z)^{\delta}\bigr)$$

and therefore

$$\mathsf{E}_A \ge \sum_{1\le m\le M} \mathsf{E}_{m,A_m} + \mathcal{E}^{\mathsf{TF}} - \sum_{1\le m\le M} \mathcal{E}_m^{\mathsf{TF}} - CZ^{\frac{5}{3}}\bigl(Z^{-\delta}+(\alpha Z)^{\delta}\bigr) - C\alpha b^{-3}Z^2, \tag{27.4.54}$$

where the last term is due to replacement of $\frac{1}{\alpha}\int |\nabla\times A|^2\, dx$ by "atomized" expressions $\sum_{1\le m\le M} \frac{1}{\alpha}\int |\nabla\times A_m|^2\, dx$.

[19)] There is no binding with $b \le Z^{-\frac{1}{3}}$ because remainder estimate is (better than) CZ^2 and binding energy excess is $\asymp Z^{\frac{7}{3}}$.

The last inequality (27.4.54) then obviously holds with A_m replaced by optimizers for "atomic" expressions (27.4.45) and *now* strong non-binding theorem implies that

$$b^{-7} \le CZ^{\frac{5}{3}}\bigl(Z^{-\delta} + (\alpha Z)^{\delta}\bigr) + C\alpha b^{-3}Z^2$$

which implies (27.4.49) where we change $\delta > 0$ as needed. □

27.5 Miscellaneous Problems

Recall that in our analysis in Sections 25.5 and 25.6 the crucial role was played by an estimate of $\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}})$ and since what we have now (see Theorem 27.4.4) is (almost) as good as we had then, all arguments of these Sections still work with the minimal modifications. We leave most of the easy details to the reader but we need to deal with different magnetic fields for different N.

27.5.1 Excessive Negative Charge

Theorem 27.5.1 [20]**.** *Let condition* (27.4.48) *be fulfilled.*

(i) In the framework of the fixed nuclei model let us assume that $\mathsf{I}^*_N := \mathsf{E}^*_{N-1} - \mathsf{E}^*_N > 0$. *Then*

$$(N-Z)_+ \le CZ^{\frac{5}{7}}. \tag{27.5.1}$$

(ii) In the framework of the free nuclei model let us assume that $\widehat{\mathsf{I}}^*_N := \widehat{\mathsf{E}}^*_{N-1} - \widehat{\mathsf{E}}^*_N > 0$. *Then estimate* (27.5.1) *holds.*

Proof. The proof follows the proof of Theorem 25.5.3; since it is not specific to the case where there is no magnetic field, we find that $(N-Z)_+ \le Q^{\frac{3}{7}}$, where Q is an estimate for $\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}})$, which we established already; recall that the equivalence $\rho^{\mathsf{TF}} \asymp \ell^{-6}$ for $\ell \gtrsim Z^{-\frac{1}{3}}$ also plays an important role.

Here we pick up $A = A_N$ [21] and conclude that

$$\mathsf{I}_N(A) := \mathsf{E}_{N-1}(A) - \mathsf{E}_N(A) \ge \mathsf{E}^*_{N-1} - \mathsf{E}^*_N > 0,$$

and then repeat arguments of the proof of Theorem 25.5.3.

We leave the remaining easy details to the reader. □

[20] Cf. Theorem 25.5.3.

[21] Exactly as in the analysis of free nuclei model we pick up $\underline{y}$ for N-electrons we pick now A for N electrons.

27.5.2 Estimates for Ionization Energy

Theorem 27.5.2 [22]**.** *Let condition* (27.4.48) *be fulfilled.*

(i) Then the following estimate holds:

$$\mathsf{I}^*_N \le C(Z-N)^{\frac{4}{3}}_+ + CZ^{\frac{20}{21}}. \tag{27.5.2}$$

(ii) Further, under assumption

$$a := \min_{1\le m<m'\le M} |\mathsf{y}_m - \mathsf{y}_{m'}| \ge Z^{-\frac{1}{3}} \tag{27.5.3}$$

the following estimate holds

$$\mathsf{I}^*_N \le C(Z-N)^{\frac{4}{3}}_+ + CZ^{\frac{20}{21}}\bigl(Z^{-\delta} + (aZ^{\frac{1}{3}})^{-\delta}\bigr). \tag{27.5.4}$$

(iii) Furthermore, for $M \ge 2$ in the framework of free nuclei model (assuming that the molecule exists; see Theorem 27.5.6)

$$\widehat{\mathsf{I}}^*_N \le CZ^{\frac{20}{21}-\delta'}. \tag{27.5.5}$$

Proof. Recall that Theorem 25.5.4 was proven simultaneously with Theorem 25.5.3; we follow the same scheme here picking up $A = A_N$ [21]. Thus here and in the first part of the proof of Theorem 27.5.3 we estimate from above $\mathsf{I}^*_N(A)$.

Again the remaining easy details a left to the reader. □

Theorem 27.5.3 [23]**.** *Let condition* (25.3.33) *be fulfilled and let $N \le Z - C_0Q^{\frac{3}{7}}$ with $Q \le C_1Z^{\frac{5}{3}}$. Then in the framework of fixed nuclei model*

(i) In the general case $Q = Z^{\frac{5}{3}}$

$$|\mathsf{I}^*_N + \nu| \le C(Z-N)^{\frac{17}{18}}Z^{\frac{5}{18}} \times \log^{\frac{1}{3}}\bigl((Z-N)Q^{-\frac{3}{7}}\bigr). \tag{27.5.6}$$

(ii) Under assumption (27.5.3) *as $Q = Z^{\frac{5}{3}}$*

$$(\mathsf{I}^*_N + \nu)_-| \le C(Z-N)^{\frac{17}{18}}Z^{\frac{5}{18}}. \tag{27.5.7}$$

[22] Cf. Theorem 25.5.4.

[23] Cf. Theorem 25.6.1.

(iii) Further, under assumption (27.5.3) *we can take* $Q = Z^{\frac{5}{3}}\big(Z^{-\delta}+(aZ^{\frac{1}{3}})^{-\delta}\big)$ *and then both estimates* (27.5.6) *and* (27.5.7) *hold with an extra factor* $\big(Z^{-\delta'} + (aZ^{\frac{1}{3}})^{-\delta'}\big)$ *in the right-hand expressions.*

Proof. To estimate $\mathsf{I}^*_N + \nu$ from above we follow exactly the arguments of Subsection 25.6.1 to estimate $\mathsf{I}_N(A_N) \ge \mathsf{I}^*_N$. □

Problem 27.5.4. To prove the same estimate for $(\mathsf{I}^*_N + \nu)_-$.

Remark 27.5.5. To estimate $\mathsf{I}^*_N + \nu$ from below we need to pick up $A = A_{N-1}$ rather than $A = A_N$; then

$$\mathsf{I}_N(A) = \mathsf{E}_{N-1}(A) - \mathsf{E}_N(A) \le \mathsf{E}^*_{N-1} - \mathsf{E}^*_N = \mathsf{I}^*_N$$

and we should follow the arguments of Subsection 25.6.2. However, in contrast to all other proofs of this Section, here we should use the spectral properties of $H_{A,W}$ (or at least an estimate from above for its lowest eigenvalue after localization to $\mathsf{supp}(\theta)$, while in all other results we need an estimate from below for the same lowest eigenvalue after localization to $\mathsf{supp}(\theta)$).

To estimate from above the lowest eigenvalue of $H_{A,W}$ we need some uniform (i.e. with constants which do not depend on N) smoothness estimates for A_{N-1} where A_{N-1} is the minimizer for $\mathsf{E}_{N-1}(A)$ as defined in Sections 27.4 and here (rather than as defined in Sections 27.2–27.3).

While (27.A.1) is an analogue of (27.2.14), it is still not the same, and while it implies some estimate, it is not even remotely as good as we achieved in Sections 27.2 and 27.3.

Sure ρ_Ψ is not very smooth either but it close to rather smooth ρ^{TF}; on the other hand, the minimizer A_N is an almost-minimizer for the one-particle trace problem studied Sections 27.2–27.3 but we don't know how close it to the minimizer (or one of the minimizers) of the latter problem.

27.5.3 Free Nuclei Model: Excessive Positive Charge

Theorem 27.5.6 [24]**.** *Let condition* (25.3.33) *be fulfilled. Then in the framework of free nuclei model with* $M \ge 2$ *the stable molecule does not exist unless*

$$Z - N \le Z^{\frac{11}{13}}. \tag{27.5.8}$$

[24] Cf. Theorem 25.6.3.

Proof. Again we just repeat the proof of Theorem 25.6.3; we leave all easy details to the reader. □

27.A Appendices

27.A.1 Minimizers and Ground States

First, establish a conditional existence of the minimizer[25)] and the corresponding ground state of the original problem:

Theorem 27.A.1. *Let $\alpha Z < \kappa^*$ and let $\mathsf{E}_N^* < \mathsf{E}_{N-1}^*$. Then there exist a minimizer $A = A_N$ for the original multiparticle problem and the corresponding ground state Ψ_N.*

Proof. We know that for $\alpha Z < \kappa^*$ (with $\kappa^* > 0$ which does not depend on Z or positions of the nuclei) $\mathsf{E}^*(A)$ is bounded from below; then $\|\nabla \times A'\|^2$ is bounded from above for a near-minimizer A' (but constants do depend on Z and $(\kappa^* - \alpha Z)$ here). On the other hand, for $A' \in \mathscr{C}_0^\infty$ and $\mathsf{E}_N(A') < \mathsf{E}_{N-1}(A')$ there exists a ground state $\Psi_N(A')$.

Therefore if $A_{(k)} \in \mathscr{C}_0^\infty$ is a minimizing sequence for $\mathsf{E}_N(A)$ we have also a sequence $\Psi_N(A_{(k)})$ with $\|\Psi_N(A_{(k)})\| = 1$. Going if necessary to the subsequence, we can assume that $A_{(k)}$ converges weakly in $\mathscr{H}^1$ and strongly in $\mathscr{L}^p_{\mathsf{loc}}$ for any $p < 6$; let A be its limit.

One can prove easily that $\Psi_N(A_{(k)})$ converge weakly in $\mathscr{H}^1$ and strongly in $\mathscr{L}^2$ to Ψ and

$$(\mathsf{H}_{A,V}\Psi, \Psi) = \lim_{k\to\infty} (\mathsf{H}_{A_{(k)},V}\Psi_N(A_{(k)}), \Psi_N(A_{(k)})),$$

and then

$$(\mathsf{H}_{A,V}\Psi, \Psi) + \frac{1}{\alpha} = \lim_{k\to\infty} \mathsf{E}_N(A_{(k)}),$$

which is E_N^* since $A_{(k)}$ is a minimizing sequence and then Ψ must be a ground state. □

Now in this framework we establish properties of the minimizer and the ground state:

[25)] We do not know if it is unique.

Proposition 27.A.2 [26]**.** *Let $\Psi = \Psi_N$ and $A = A_N$ be a ground state and minimizer with energy $\mathsf{E}^*_N < \mathsf{E}^*_{N-1}$.*

(i) $\Psi \in \mathscr{C}^1$ and $\Psi = O(e^{-\epsilon|\underline{x}|})$ as $|\underline{x}| \to \infty$.

(ii) $A \in \mathscr{C}^1$ and $A = O(|x|^{-2})$, $\nabla \times A = O(|x|^{-3})$ as $|x| \to \infty$.

(iii) Let $N < Z$. Then $V_\Psi - V \in \mathscr{C}^1$ and $V_\Psi = (Z-N)|x|^{-1} + O(|x|^{-2})$, $\nabla V_\Psi = -(Z-N)x|x|^{-3} + O(|x|^{-3})$ as $|x| \to \infty$.

Proof. An obvious proof, using also an equation

$$\frac{2}{\alpha}\Delta A_j = \\ -2N\,\mathsf{Re}\,\mathsf{tr}\int \Psi^\dagger(x,x_2,\dots,x_N)\sigma_j(D-A)_x\cdot\boldsymbol{\sigma}\Psi(x,x_2,\dots,x_N)\,dx_2\cdots dx_N, \tag{27.A.1}$$

is left to the reader. This equation is similar to (27.2.14) and is also derived from variational principles, its right-hand expression is $\frac{\delta\Lambda}{\delta A}$ where Λ is the lowest eigenvalue of $\mathsf{H}_{A,V}$ on Fock' space.

27.A.2 Zhislin's Theorem

Theorem 27.A.3 (Zhislin's theorem) [27]**.** *$\mathsf{E}^*_{N+1} < \mathsf{E}^*_N$ if $N < Z$.*

Proof. An easy proof repeating with obvious modifications proof of Theorem 26.A.8 is left to the reader. □

27.A.3 L. Erdös–J. P. Solovej's Lemma

We reproduce here Lemma 2.1 from L. Erdös, J. P. Solovej [1].

Lemma 27.A.4. *There is a positive universal constant κ^* such that for any Z, α with $Z\alpha \le \kappa^*$ we have*

$$\inf_N \inf_A H_{A,V} \ge -CZ^{\frac{7}{3}}\delta^{1/2} - Z^{\frac{2}{3}}\delta^{-2}$$

if $CZ^{-\frac{2}{3}} \le \delta \le C_1$ with a sufficiently large constant C.

[26] Cf. Proposition 26.A.7.

[27] Cf. Theorem 26.A.8.

Proof. Consider a pair of smooth functions θ_0 and θ_1, such that $\theta_0^2+\theta_1^2=1$, $\mathsf{supp}(\theta_1)\subset B(0,2r)$, $\theta_1=1$ on $B(0,r)$, and $|\nabla\theta_0|$, $|\nabla\theta_1|\le Cr^{-1}$ with $r=\delta Z^{-\frac{1}{3}}$.

Let $\tilde\chi_0$ be a smooth cutoff function, supported on $B(0,3r)$ such that $|\nabla\tilde\chi_0|\le Cr^{-1}$ and $\tilde\chi_0=1$ on $B(0,2r)$. Let $\bar A$ be an average of A over $B(0,3r)$. Let $A_0\coloneqq(A-\bar A)\tilde\chi_0$, $B_0\coloneqq\nabla\times A_0$; then

$$\nabla\otimes A_0=\tilde\chi_0\nabla\otimes A+(A-\bar A)\otimes\nabla\tilde\chi_0.$$

Clearly

$$\int_{\mathbb{R}^3}B_0^2\,dx\le\int_{\mathbb{R}^3}|\nabla\otimes A_0|^2\,dx\le 2\int_{\mathbb{R}^3}\tilde\chi_0^2|\nabla\otimes A|^2\,dx+Cr^{-2}\int_{B(0,3r)}(A-\bar A)^2\,dx$$
$$\le C_1\int_{B(0,3r)}|\nabla\otimes A|^2\,dx$$

for some universal constant C_1, where in the last step we used the Poincaré inequality. Let φ be a real phase such that $\nabla\varphi=\bar A$. Since $\tilde\chi_0\equiv 1$ on the support of θ_1, we have

$$\theta_1 H_{A,0}\theta_1=\theta_1 e^{-i\varphi}H_{A-\bar A,0}e^{i\varphi}\theta_1=\theta_1 e^{-i\varphi}H_{A_0,V}e^{i\varphi}\theta_1.$$

After these localizations, we have

(27.A.2) $H^1_{N,Z;A}\coloneqq$

$$\sum_{j=1}^N\Big[\theta_1\Big(H_{A,0}-\frac{Z}{|x|}-\big(|\nabla\theta_0|^2+|\nabla\theta_1|^2\big)\theta_1\Big]_j+\frac{1}{\alpha}\int_{B(0,3r)}|\nabla\otimes A|^2\,dx$$
$$\ge\sum_{j=1}^N\Big[\theta_1 e^{-i\varphi}\big(H_{A_0,0}-W(x)\big)e^{i\varphi}\theta_1\Big]_j+\frac{1}{2C_1\alpha}\int B_0^2\,dx$$

with

$$W(x)=\Big[\frac{Z}{|x|}+Cr^{-2}\Big]\,\mathbf{1}(|x|\le 2r),$$

where $\mathbf{1}(X)$ is a characteristic function of X.

Now we use the "running energy scale" argument in E. Lieb, M. Loss, M. and J. Solovej [1].

$$\begin{aligned}(27.A.3)\quad \sum_{j=1}^{N}\Big[\theta_1 e^{-i\varphi}\big[H_{A',0}-W\big]e^{i\varphi}\theta_1\Big]_j &\ge -\int_0^\infty \mathsf{N}_{-e}(H_{A',0}-W)\,de\\ &\ge -\int_0^\mu \mathsf{N}_{-e}(H_{A',0}-W)\,de-\int_\mu^\infty \mathsf{N}_0\big(\frac{\mu}{e}H_{A',0}-W+e\big)\,de\\ &\ge -\int_0^\mu \mathsf{N}_{-e}(H_{A',0}-W)\,de-\int_\mu^\infty \mathsf{N}_0\big(H_{A',0}-\frac{e}{\mu}W+\frac{e^2}{\mu}\big)\,de,\end{aligned}$$

where $\mathsf{N}_{-e}(H)$ denotes the number of eigenvalues of a self-adjoint operator H below $-e$.

In the first term we use the bound $H_{A_0,0}\ge (D-A_0)^2-|B_0|$ and the CLR (i.e. Cwikel-Lieb-Rozenblum) bound:

$$\begin{aligned}(27.A.4)\quad \int_0^\mu \mathsf{N}_{-e}(H_{A_0,0}-W)\,de &\le C\int_0^\mu de\int_{\mathbb{R}^3}(W+|B_0|-e)_+^{\frac{3}{2}}\,dx\\ &\le C\int_0^\mu de\int_{\mathbb{R}^3}(W-e/2)_+^{\frac{3}{2}}\,dx+C\int_0^\mu de\int_{\mathbb{R}^3}(|B_0|-e^2/2\mu)_+^{\frac{3}{2}}\,dx\\ &\le C\int_{\mathbb{R}^3}W^{\frac{5}{2}}\,dx+C\mu^{\frac{1}{2}}\int_{\mathbb{R}^3}B_0^2\,dx=CZ^{\frac{5}{2}}r^{\frac{1}{2}}+Cr^{-2}+C\mu^{\frac{1}{2}}\int_{\mathbb{R}^3}B_0^2\,dx.\end{aligned}$$

In the second term of (27.A.3) we use

$$\begin{aligned}H_{A_0,0}-\frac{e}{\mu}W\ge \frac{1}{2}\big[(D-A_0)^2-\frac{2eZ}{\mu|x|}\,\mathbf{1}(|x|\le 2r)\big]\\ +\frac{1}{2}(D-A_0)^2-|B_0|-\frac{Ce}{\mu r^2}\,\mathbf{1}(|x|\le 2r),\end{aligned}$$

and that

$$(D-A_0)^2-\frac{2eZ}{\mu|x|}\,\mathbf{1}(|x|\le 2r)\ge (D-A_0)^2-\frac{4eZ}{\mu|x|}\ge -\big(\frac{2eZ}{\mu}\big)^2$$

i.e.

$$H_{A_0,0}-\frac{e}{\mu}W\ge \frac{1}{2}(D-A_0)^2-2\big(\frac{eZ}{\mu}\big)^2-|B_0|-\frac{Ce}{\mu r^2}\,\mathbf{1}(|x|\le 2r).$$

Let $\mu = 4Z^2$, then using $Ce/\mu r^2 \le e^2/4\mu$ for $\mu \le e$ (i.e. $C \le (\delta Z^{2/3})^2$), we get

$$\begin{aligned}(27.A.5)\quad &\int_\mu^\infty \mathsf{N}_0\bigl(H_{A_0,0} - \frac{e}{\mu}W + \frac{e^2}{\mu}\bigr)\,de \\ &\le \int_\mu^\infty \mathsf{N}_0\bigl(\frac{1}{2}(D-A_0)^2 - |B_0| + \frac{e^2}{4\mu}\bigr)\,de \le C\int_0^\mu de \int_{\mathbb{R}^3} (|B_0| - e^2/4\mu)_+^{3/2}\,dx \\ &\le C\mu^{\frac{1}{2}}\int_{\mathbb{R}^3} B_0^2\,dx.\end{aligned}$$

Note that if $Z\alpha \le \kappa^*$ with some sufficiently small universal constant κ^*, then (27.A.5) can be controlled by the corresponding term in (27.A.2). Combining the estimates (27.A.2), (27.A.3), (27.A.4) and (27.A.5), we obtain

$$H_{A,V} \ge -CZ^{\frac{5}{2}}r^{\frac{1}{2}} - Cr^{-2}$$

and lemma follows. □

Comments

The problem was considered first in several papers of L. Erdös, J. P. Solovej [1] and L. Erdös, S. Fournais, J. P. Solovej [1, 3, 4].

The same problem in the relativistic case was considered in L. Erdös, S. Fournais, J. P. Solovej [2] (under assumption preventing relativistic instability).

Important Remarks

Section 27.5 was rewritten. See Important Remarks to Chapter 25.

Chapter 28

The Case of Combined Magnetic Field

28.1 Introduction

In this Chapter instead of the Schrödinger operator without magnetic field as in Chapter 25, or with a constant magnetic field as in Chapter 26, or with a self-generated magnetic field as in Chapter 27 we consider the Schrödinger operator (27.1.1) with unknown magnetic field A, but then we add to the ground state energy of the atom (or molecule) the energy of the self-generated magnetic field (see selected term in (28.1.1) thus arriving to

$$\mathsf{E}(A) = \inf \mathsf{Spec}(\mathsf{H}_{A,V}) + \underbrace{\alpha^{-1} \int |\nabla \times (A - A^0)|^2 \, dx} \tag{28.1.1}$$

where $A^0 = \frac{1}{2} B(-x_2, x_1, 0)$ is a constant *external magnetic field*.

Then finally

$$\mathsf{E}^* = \inf_{A - A^0 \in \mathscr{H}^1_0} \mathsf{E}(A), \tag{28.1.2}$$

defines a ground state energy with a *combined magnetic field* A while $A' := A - A^0$ is a *self-generated magnetic field*.

Note that

$$\int |\nabla \times (A - A^0)|^2 \, dx = \int \bigl(|\nabla \times A|^2 - |\nabla \times A^0|^2\bigr) \, dx \tag{28.1.3}$$

which seems to be a more "physical" definition.

28.1.1 Plan of the Chapter

First of all, we are lacking so far a *semiclassical local theory* and we are developing it in Sections 28.2–28.4, where we consider a one-particle quantum Hamiltonian (28.2.1) with the external constant magnetic field A^0 of intensity β, $h \ll 1$ and a self-generated magnetic field $(A - A^0)$. Here theory significantly depends if $\beta h \lesssim 1$ or $\beta h \gtrsim 1$ as it was in the case without a self-generated magnetic field.

While Section 28.2 is preparatory and rather functional-analytical, Sections 28.3 and 28.4 are microlocal; they cover cases $\beta h \lesssim 1$ and $\beta h \gtrsim 1$ respectively. These three sections are similar to a single Section 27.2. However in Sections 28.3 and 28.4 various non-degeneracy assumptions play a very significant role, especially for large β.

Then in Section 28.5 we consider a global theory if a potential has Coulomb singularities and (in some statements) behaves like (magnetic) Thomas-Fermi potential both near singularities and far from them.

Finally, in Section 28.6 we apply these results to our original problem of the ground state energy so far assuming that the number of nuclei is 1. One can recover the same results if $M \ge 2$ but the external magnetic field B is weak enough. No surprise that the theory is different in the cases $B \le Z^{\frac{4}{3}}$ and $Z^{\frac{4}{3}} \le B \ll Z^3$ (see Chapter 25 where this difference appears). Since as $M = 1$ the strongest non-degeneracy assumption is surely achieved and as $M \ge 2$ much weaker non-degeneracy assumption is achieved in the border zone (see Chapter 25) our remainder estimates for large B significantly differ in the atomic and molecular cases.

In Appendix 28.A we first generalize Lieb-Loss-Solovej estimate to the case of the combined magnetic field (which is necessary if $\beta h \gtrsim 1$), then establish electrostatic inequality in the current settings and finally study very special pointwise spectral expressions for a Schrödinger operator in $\mathbb{R}^3$ with linear magnetic and scalar potentials (we considered such operators already in Section 16.6)

28.1.2 Unfinished Business

One can apply these results to estimates of the excessive positive and negative charges (the estimates for excessive positive charges, if $M \ge 2$ in the free nuclei framework) and estimates or asymptotics of the ionization energy in the same manner as we did it in Chapters 25–27; however there are no new

ideas but rather tedious calculations and we leave it to those readers who decide to explore these topics, which is clearly serious task.

28.2 Local Semiclassical Trace Asymptotics: Preparation

28.2.1 Toy-Model

Let us consider operator (27.1.4)

$$H = H_{A,V} = \big((hD - A)\cdot \boldsymbol{\sigma}\big)^2 - V(x) \tag{28.2.1}$$

in $\mathbb{R}^3$ where A, V are real-valued functions and $V \in \mathscr{C}^{\frac{5}{2}}$, $A - A^0 \in \mathscr{H}^1_0$. Then operator $H_{A,V}$ is self-adjoint. We are interested in $\mathsf{Tr}^-(H_{A,V}) = \mathsf{Tr}^-(H^-_{A,V})$ (the sum of all negative eigenvalues of this operator). Let

$$\mathsf{E}^* = \mathsf{E}^*_\kappa := \inf_{A-A^0\in\mathscr{H}^1_0} \mathsf{E}(A), \tag{28.2.2}$$

where

$$\mathsf{E}(A) = \mathsf{E}_\kappa(A) := \Big(\mathsf{Tr}^-(H_{A,V}) + \kappa^{-1}h^{-2}\int |\partial(A-A^0)|^2\,dx\Big) \tag{28.2.3}$$

with a matrix $\partial A = (\partial_i A_j)_{i,j=1,2,3}$. Recall that A^0 is a linear potential, $A^0(x) = \frac{1}{2}\beta(-x_2, x_1, 0)$. We consider rather separately cases

$$\beta h \lesssim 1 \qquad \text{and} \qquad \beta h \gtrsim 1 \tag{28.2.4}_{1,2}$$

of the *moderate* and *strong* external magnetic field.

To deal with the described problem we need to consider first a formal semiclassical approximation.

28.2.2 Formal Semiclassical Theory

Semiclassical Theory: $\beta h \lesssim 1$

Let us replace the trace expression $\mathsf{Tr}(H^-_{A,V}\psi)$ by its magnetic semiclassical approximation $-h^{-3}\int P_{Bh}(V)\psi\,dx$ where $B = |\nabla\times A|$ is a scalar intensity of the magnetic field and $P_*(.)$ is a pressure. Then $\mathsf{E}(A) \approx \mathcal{E}(A)$ with

$$\mathcal{E}(A) = \mathcal{E}_\kappa(A) := -h^{-3}\int P_{Bh}(V)\psi\,dx + \frac{1}{\kappa h^2}\int |\partial A'|^2\,dx. \tag{28.2.5}$$

Assuming that $|\partial A'| \ll \beta$, $A' = (A - A^0)$ we find out that

$$
\begin{aligned}
(28.2.6)\quad & -h^{-3}\int P_{Bh}(V)\psi\,dx \\
&\approx -h^{-3}\int \Big(P_{\beta h}(V) - \partial_\beta P_{\beta h}(V)(B-\beta)\psi\Big)\,dx \\
&\approx -h^{-3}\int P_{\beta h}(V)\psi\,dx \\
&\quad - h^{-3}\int \Big[\partial_{x_2}\big(\partial_\beta P_{\beta h}(V)\psi\big)\cdot A'_1 - \partial_{x_1}\big(\partial_\beta P_{\beta h}(V)\psi\big)\cdot A'_2\Big]\,dx,
\end{aligned}
$$

where we used that

$$
B \approx \beta - \partial_{x_2}A'_1 + \partial_{x_1}A'_2 \tag{28.2.7}
$$

and integrated by parts. Then $\mathcal{E}(A) \approx \bar{\mathcal{E}}(A)$ with

$$
\begin{aligned}
&(28.2.8)\quad \bar{\mathcal{E}}(A) = \bar{\mathcal{E}}_\kappa(A) \coloneqq -h^{-3}\int P_{\beta h}(V)\psi\,dx \\
&-h^{-3}\int \Big[\partial_{x_2}\big(\partial_\beta P_{\beta h}(V)\psi\big)\cdot A'_1 - \partial_{x_1}\big(\partial_\beta P_{\beta h}(V)\psi\big)\cdot A'_2\Big]\,dx + \frac{1}{\kappa h^2}\int |\partial A'|^2\,dx
\end{aligned}
$$

and replacing approximate equalities by exact ones and optimizing with respect to A' we arrive to

$$
\begin{aligned}
&(28.2.9)\quad \Delta A'_1 = -\frac{1}{2}\kappa h^{-1}\partial_{x_2}\big(\partial_\beta P_{\beta h}(V)\psi\big), \quad \Delta A'_2 = \frac{1}{2}\kappa h^{-1}\partial_{x_1}\big(\partial_\beta P_\beta(V)\psi\big), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Delta A'_3 = 0
\end{aligned}
$$

and

$$
\mathcal{E}^*_\kappa \coloneqq \inf_{A:\ A-A^0\in \mathscr{H}^1_0} \mathcal{E}_\kappa(A) \approx \bar{\mathcal{E}}^*_\kappa \coloneqq \inf_{A:\ A-A^0\in \mathscr{H}^1_0} \bar{\mathcal{E}}_\kappa(A). \tag{28.2.10}
$$

To justify our analysis we need to justify approximate equality

$$
\begin{aligned}
&(28.2.11)\quad -h^{-3}\int P_{Bh}(V)\,dx + \frac{1}{\kappa h^2}\int |\partial A'|^2\,dx \\
&\qquad \approx -h^{-3}\int P_{\beta h}(V)\psi dx - h^{-3}\int \Big[\partial_\beta P_{\beta h}(V)\psi(-\partial_{x_2}A'_1 + \partial_{x_1}A'_2)\Big]\,dx + \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{1}{\kappa h^2}\int |\partial A'|^2\,dx
\end{aligned}
$$

and estimate an error when we minimize the right-hand expression instead of the left-hand one. To do this observe that (even without assumptions $Bh \lesssim 1$, $\beta h \lesssim 1$)

$$|P_{Bh}(V) - P_{\beta h}(V) - \partial_{\beta h} P_{\beta h}(V)\cdot(B-\beta)h| \le C(B-\beta)^2h^2 + C|B-\beta|^{\frac{3}{2}}\beta h^{\frac{5}{2}}. \tag{28.2.12}$$

Indeed, one can prove it easily recalling that

$$P_{\beta h}(V) = \varkappa_0 \sum_{j\ge 0}(1-\frac{1}{2}\delta_{j0})(V-2j\beta h)_+^{\frac{3}{2}}\beta h \tag{28.2.13}$$

and considering cases $\beta h \gtrsim 1$, $Bh \gtrsim 1$, $|B-\beta| \gtrsim \beta h$, analyzing different terms in (28.2.13) and observing that the last term in (28.2.12) appears only in the case $\beta h \lesssim 1$, $|B-\beta| \lesssim \beta$.

Then since $|B-\beta| \le |B'|$ (where $B' = |\partial(A-A^0)|$) we conclude that the left-hand expression of (28.2.11) is greater than

$$-h^{-3}\int P_{\beta h}(V)\psi\,dx - C\|B'\|\beta h^{-1} - C\|B'\|^{\frac{3}{2}}\beta h^{-\frac{1}{2}} + \kappa^{-1}h^{-2}\|B'\|^2,$$

where we used that

$$|\partial_{\beta h} P_{\beta h}(V)| \le C\beta h \qquad \text{if } V \le c; \tag{28.2.14}$$

then a minimizer for the left-hand expression of (28.2.11) must satisfy

$$\|B'\| \le C\kappa\beta h \tag{28.2.15}$$

and one can observe easily that the same is true and for the minimizer for the right-hand expression as well.

Also observe that

$$B = \beta + \partial_{x_1}A'_2 - \partial_{x_2}A'_1 + O(\beta^{-1}|B'|^2). \tag{28.2.16}$$

Then for both minimizers the differences between the left-hand and right-hand expressions of (28.2.11) do not exceed $C\kappa^{\frac{5}{2}}\beta^{\frac{5}{2}}$ and therefore

$$|\mathcal{E}^* - \bar{\mathcal{E}}^*| \le C\kappa^{\frac{5}{2}}\beta^{\frac{5}{2}}. \tag{28.2.17}$$

One can calculate easily the minimizer for

$$-h^{-3}\int \Big[\partial_\beta P_{\beta h}(V)\psi(-\partial_{x_2}A'_1+\partial_{x_1}A'_2)\Big]\,dx+\frac{1}{\kappa h^2}\int |\partial A'|^2\,dx \tag{28.2.18}$$

and conclude that $A'_j=\kappa\beta h a_j$ with $a_3=0$ and

$$\Delta a_1=-(\beta h)^{-1}\partial x_2\partial_{\beta h}P_{\beta h}(V),\quad \Delta a_2=(\beta h)^{-1}\partial x_1\partial_{\beta h}P_{\beta h}(V) \tag{28.2.19}$$

and the minimum is negative and $O(\kappa\beta^2)$; we call it *correction term*; in fact, $\|\partial A'\|\asymp \kappa\beta h$ and the minimum is $\asymp -\kappa\beta^2$ in the generic case.

Then a minimum of the left-hand expression of (28.2.11) is equal to the minimum of the right-hand expression modulo $O(\kappa^{\frac{3}{2}}\beta^{\frac{5}{2}}h)$.

Remark 28.2.1. (i) One can improve this estimate under non-degeneracy assumptions (28.3.60) or (28.3.65). However even in the general case observe that

$$\begin{aligned}&\mathcal{E}(A'')-\mathcal{E}(A')\\ &\geq -C\beta h^{-\frac{1}{2}}\|B'-B''\|^{\frac{3}{2}}-C\beta h^{-\frac{1}{2}}\|B'\|^{\frac{1}{2}}\cdot\|B'-B''\|+2\epsilon_0\kappa^{-1}h^{-2}\|\partial(A'-A'')\|^2\\ &\geq -C\kappa^2\beta^2h+\epsilon_0\kappa^{-1}h^{-2}\|\partial(A'-A'')\|^2\end{aligned}$$

if A' is the minimizer for $\bar{\mathcal{E}}$ and therefore since $\|B'\|\leq C\kappa\beta h$ we conclude that

$$\mathcal{E}^*\geq \mathcal{E}(A')-C\kappa^2\beta^2h \tag{28.2.20}$$

and

$$\|\partial(A'-A'')\|\leq C\kappa^{\frac{3}{2}}\beta h^{\frac{1}{2}} \tag{28.2.21}$$

if A'' is an almost-minimizer for $\mathcal{E}(A'')$.

(ii) Observe, that picking up $A'=0$ and applying arguments of Chapter 18 we can derive an upper estimate

$$\mathsf{E}^*\leq -\int P_{\beta h}(V)\psi\,dx+O(h^{-1}); \tag{28.2.22}$$

however this estimate is not sharp for $\kappa\beta^2\gg h^{-1}$ because $\mathcal{E}^*$ is less than the main term here with a gap $\asymp\kappa\beta^2$. For $\kappa\asymp 1$ it gives us a proper upper estimate only if $\beta\leq h^{-\frac{1}{2}}$.

Therefore for $\kappa\beta^2\gg h^{-1}$ an upper estimate is not as trivial as in Chapter 26; in the future we pick up as A' a minimizer for $\bar{\mathcal{E}}(A)$ (mollified by x as this minimizer is not smooth enough).

Semiclassical Theory: $\beta h \gtrsim 1$

Consider $\beta h \ge 1$. Without any loss of the generality one can assume that $\|V\|_{\mathscr{L}^\infty} \le \beta h$ and $\|\partial A'\|_{\mathscr{L}^\infty} \le \frac{1}{2}\beta$. Then in the definition (28.2.13) of $P_{\beta h}(V)$ (etc) remains only term with $j = 0$:

$$P_{\beta h}(V) = \frac{1}{2}\varkappa_0 V_+^{\frac{3}{2}} \beta h, \tag{28.2.13)'}$$

which leads to simplifications of $\mathcal{E}(A')$ and $\bar{\mathcal{E}}(A')$; both of them become

$$\frac{1}{2}\varkappa_0 \beta h^{-2} \int V_+^{\frac{3}{2}} \psi \, dx + \frac{1}{2}\varkappa_0 h^{-2} \int V_+^{\frac{3}{2}} \big(\partial_{x_1} A'_2 - \partial_{x_2} A'_1\big)\psi \, dx + \kappa^{-1} h^{-2} \|\partial A'\|^2 \tag{28.2.8)'}$$

modulo $O(\beta^{-1} h^{-2} \|B'\|^2)$ and equations to the minimizer become

$$\Delta A'_1 = -\frac{1}{2}\varkappa_0 \kappa \partial_{x_2}\big(V_+^{\frac{3}{2}}\psi\big), \quad \Delta A'_2 = \frac{1}{2}\varkappa_0 \kappa \partial_{x_1}\big(V_+^{\frac{3}{2}}\psi\big), \quad \Delta A'_3 = 0. \tag{28.2.9)'}$$

Then $\|B'\| \asymp \kappa$ and a correction term is negative and $\asymp -\kappa h^{-2}$ in the generic case. An error $O(\beta^{-1} h^{-2} \|B'\|^2)$ becomes $O(\kappa \beta^{-1} h^{-2})$ (and thus not exceeding microlocal error $O(\beta)$).

28.2.3 Estimate from below

Basic Estimates

Let us estimate $\mathsf{E}(A)$ from below. First we need the following really simple

Proposition 28.2.2 [1]**.** *Consider operator $H_{A,V}$, defined on $\mathscr{H}^2(B(0,1)) \cap \mathscr{H}^1_0(B(0,1))$* [2]*. Let $V \in \mathscr{L}^4$.*

(i) Let $\beta h \le 1$. Then

$$\mathsf{E}^* \ge -Ch^{-3} \tag{28.2.23}$$

and either

$$\frac{1}{\kappa h^2} \int |\partial(A - A^0)|^2 \, dx \le Ch^{-3} \tag{28.2.24}$$

or $\mathsf{E}(A) \ge ch^{-3}$.

---

[1] Cf. Proposition 27.2.1.

[2] I.e. on $\mathscr{H}^2(B(0,1))$ with the Dirichlet boundary conditions.

(ii) Let $\beta h \ge 1$. Then

$$\mathsf{E}^* \ge -C\beta h^{-2} - C\kappa^{\frac{1}{3}}\beta^{\frac{4}{3}}h^{-\frac{4}{3}} \tag{28.2.25}$$

and either

$$\frac{1}{\kappa h^2}\int |\partial(A-A^0)|^2\,dx \le C\beta h^{-2} + C\kappa^{\frac{1}{3}}\beta^{\frac{4}{3}}h^{-\frac{4}{3}} \tag{28.2.26}$$

or $\mathsf{E}(A) \ge C\beta h^{-2} + C\kappa^{\frac{1}{3}}\beta^{\frac{4}{3}}h^{-\frac{4}{3}}$.

(iii) Furthermore, let $\beta h \ge 1$ and

$$\kappa\beta h^2 \le c. \tag{28.2.27}$$

Then

$$\mathsf{E}^* \ge -C\beta h^{-2} \tag{28.2.28}$$

and either

$$\frac{1}{\kappa h^2}\int |\partial(A-A^0)|^2\,dx \le C\beta h^{-2} \tag{28.2.29}$$

or $\mathsf{E}(A) \ge C\beta h^{-2}$.

Proof. Using estimate (28.A.2) [3] we have

$$\begin{aligned}\mathsf{E}(A) \ge{}& -C(1+\beta h)h^{-3}\\ &- C\beta h^{-\frac{3}{2}}\Big(\int |\partial A'|^2\,dx\Big)^{\frac{1}{4}} - Ch^{-\frac{3}{2}}\Big(\int |\partial A'|^2\,dx\Big)^{\frac{3}{4}} + \frac{1}{\kappa h^2}\Big(\int |\partial A'|^2\,dx\Big),\end{aligned} \tag{28.2.30}$$

which implies both Statements (i)–(ii) while Statement (iii) is a special case of Statement (ii). □

Remark 28.2.3. (i) Definitely we would prefer to have an estimate

$$\begin{aligned}\mathsf{E}(A) \ge{}& -C(1+\beta h)h^{-3}\\ &- Ch^{-2}\Big(\int |\partial A'|^2\,dx\Big)^{\frac{1}{2}} + \frac{1}{\kappa h^2}\Big(\int |\partial A'|^2\,dx\Big)\end{aligned} \tag{28.2.31}$$

from the very beginning, but we cannot prove it without some smoothness conditions to A and they will be proven only later under the same assumption (28.2.27).

[3] Magnetic Lieb–Thirring inequality (5) of E. H. Lieb, M. Loss, M. and J. P. Solovej [1]) would be sufficient for $\beta h \le 1$ but will lead to a worse estimate than we claim for $\beta h \ge 1$.

(ii) This assumption (28.2.27) in a bit stronger form (28.2.27)* will be required for our microlocal analysis in the next Section 28.3.

Remark 28.2.4. Using Proposition 28.2.2 one can prove easily the following:

(i) Proposition 27.2.2 (existence of the minimizer) remains valid;

(ii) As in Remark 27.2.3 we do not know if the minimizer is unique. From now on until further notice let A be a minimizer. We also assume that V is sufficiently smooth ($V \in \mathscr{C}^{2+\delta}$). This is be the case for magnetic Thomas-Fermi potential for sure.

(iii) Proposition 27.2.4 (namely, equation (27.2.14) to a minimizer) remains valid for both A and $A' = A - A^0$.

Proposition 28.2.5 [4]**.** *(i) Let $\beta h \le 1$ and $0 < \kappa \le (1-\epsilon_0)\kappa^*$. Assume that*

$$\mathsf{E}^*(\kappa^*, \beta, h) \ge \mathcal{E} - CM, \tag{28.2.32}$$

$$\mathsf{E}^*(\kappa, \beta, h) \le \mathcal{E} + CM \tag{28.2.33}$$

with the same number $\mathcal{E}$ and with $M \ge Ch^{-1} + C\kappa^*\beta^2$. Then

$$\int |\partial A'|^2 \, dx \le C_1 \kappa h^2 M; \tag{28.2.34}$$

(ii) Let $\beta h \ge 1$ and $\kappa^* \beta h \le c$, $0 < \kappa \le (1-\epsilon_0)\kappa^*$. Assume that (28.2.32)–(28.2.33) *are fulfilled with the same number $\mathcal{E}$ and with $M \ge C\beta + C\kappa^* h^{-2}$. Then estimate also* (28.2.34) *holds.*

Proof. Proof obviously follows the proof of Proposition 27.2.5. □

Estimates to a Minimizer: $\beta h \lesssim 1$

Consider first the simpler case $\beta h \le 1$.

[4] Cf. Proposition 27.2.5.

Proposition 28.2.6. *Let $\beta h \le 1$. Then*

$$|e(x,y,\tau)| \le C\big(1+\mu^{\frac12}h^{\frac12}\big)h^{-3} \tag{28.2.35}$$

and

$$|((hD-A)_x\cdot\sigma)e(x,y,\tau)| \le C\big(1+\mu^{\frac12}h^{\frac12}\big)h^{-3}, \tag{28.2.36}$$

where

$$\mu = \|\partial A'\|_\infty. \tag{28.2.37}$$

Proof. Without any loss of the generality one can assume that $\mu h \ge 1$. Consider $\mu^{-\frac12}h^{\frac12}$-element in $\mathbb{R}^3_x$. Without any loss of the generality one can assume that $A^0(z) = A(z) = 0$ in its center z.

Since both operators $E(\tau)$ and $((hD-A)_x\cdot\sigma)E(\tau)$ have their operator norms bounded by c in $\mathscr{L}^2$, one can prove easily that both operators $\phi D^\alpha E(\tau)$ and $\phi D^\alpha((hD-A)_x\cdot\sigma)E(\tau)$ have their operator norms bounded by $C\zeta^{|\alpha|}$ with $\zeta = \mu^{\frac12}h^{-\frac12}$ if $\alpha\in\{0,1\}^3$ and ϕ is supported in the mentioned element.

Then operator norms of both operators $\gamma_x E(\tau)$ and $\gamma_x((hD-A)_x\cdot\sigma)E(\tau)$ from $\mathscr{L}^2$ to $\mathbb{C}^q$ do not exceed $C_0\zeta_1\zeta_3^{\frac12}$ and therefore the same is true for adjoint operators; here γ_z is operator of restriction to $x=z$.

Since $E(\tau)^* = E(\tau)^2 = E(\tau)$ we conclude that the left-hand expressions in (28.2.35) and (28.2.36) do not exceed $C\zeta^3$, which is exactly the right-hand expression in both of them.

Then from equation (27.2.14), which remains valid (see Remark 28.2.4(iii)) we conclude that

$$\|\Delta A'\|_{\mathscr{L}^\infty} \le C\kappa\big(1+\mu^{\frac12}h^{\frac12}\big)h^{-1} \tag{28.2.38}$$

and therefore

$$\|\partial^2 A'\|_{\mathscr{L}^\infty} \le C\kappa|\log h|\big(1+\mu^{\frac12}h^{\frac12}\big)h^{-1}. \tag{28.2.39}$$

Further, combining (28.2.39) with (28.2.24), the standard inequality $\|\partial A'\|_{\mathscr{L}^\infty} \le \|\partial^2 A'\|^{\frac35}_{\mathscr{L}^\infty}\cdot\|\partial A'\|^{\frac25}$ and (28.2.37), we conclude that

$$\mu \le C(\kappa|\log h|)^{\frac45}\big(1+\mu^{\frac12}h^{\frac12}\big)^{\frac45}h^{-\frac45}$$

and therefore $\mu h \ll 1$. Then

$$\|\partial A'\|_{\mathscr{L}^\infty} \le C(\kappa|\log h|)^{\frac45}h^{-\frac45} \tag{28.2.40}$$

and therefore due to (28.2.39)

$$\|\partial A'\|_{\mathscr{L}^\infty} \le C\kappa|\log h|h^{-1} \tag{28.2.41}$$

(where for a sake of simplicity we slightly increase power of logarithm) thus arriving to

Proposition 28.2.7. *Let $\beta h \le 1$ and $\kappa \le \kappa^*$. Let A be a minimizer. Then estimates* (28.2.40) *and* (28.2.41) *hold.*

Furthermore, the standard scaling arguments applied to the results of Section 27.2 imply that in fact the left-hand expressions of estimates (28.2.36) and (28.2.24) do not exceed $C(1+\beta+\mu)h^{-2}$ and $C(1+\beta+\mu)^2h^{-1}$ respectively.

Therefore $\|\partial^2 A'\|_{\mathscr{L}^\infty}$ does not exceed $C\kappa|\log h|(1+\beta+\mu)$ and $\|\partial A'\|$ does not exceed $C\kappa^{\frac12}(1+\beta+\mu)h^{\frac12}$, and then $\mu \le C(\kappa|\log h|)^{\frac45}(1+\beta+\mu)^{\frac45}h^{\frac15}$ which implies $\mu \le C(\kappa|\log h|)^{\frac45}(1+\beta)^{\frac45}h^{\frac15}$ and finally we arrive to

Proposition 28.2.8. *Let $\beta h \le 1$ and $\kappa \le \kappa^*$. Then*

$$\|\partial A'\|_{\mathscr{L}^\infty} \le C(\kappa|\log h|)^{\frac45}(1+\beta)^{\frac45}h^{\frac15} \tag{28.2.42}$$

and

$$\|\partial^2 A'\|_{\mathscr{L}^\infty} \le C\kappa|\log h|(1+\beta). \tag{28.2.43}$$

Estimates to a Minimizer: $\beta h \gtrsim 1$

Consider now more complicated case $\beta h \ge 1$.

Proposition 28.2.9. *Let $\beta h \ge 1$. Then the following estimates hold with $\mu = \|\partial A'\|_\infty$:*

$$|e(x,y,\tau)| \le C\big(\beta+\mu\big)\big(1+\mu^{\frac12}h^{\frac12}\big)h^{-2} \tag{28.2.44}$$

and

$$|((hD-A)_x\cdot\sigma)e(x,y,\tau)| \le C\big(\beta+\mu\big)\big(1+\mu^{\frac12}h^{\frac12}\big)h^{-2} \tag{28.2.45}$$

Proof. Without any loss of the generality one can assume that $\mu \le \beta$; otherwise we simply replace β by μ. Consider $(\beta^{-\frac12}h^{\frac12}, \beta^{-\frac12}h^{\frac12}, (\mu+1)^{-\frac12}h^{\frac12})$-box in $\mathbb{R}^3_x$, where recall that $\nabla\times A^0$ is directed along x_3. Without any loss of the generality one can assume that $A^0(z) = A'(z) = 0$ in its center z.

Since both operators $E(\tau)$ and $((hD-A)_x\cdot\boldsymbol{\sigma})E(\tau)$ have their operator norms bounded by c in $\mathscr{L}^2$, one can prove easily that both operators $\phi D^\alpha E(\tau)$ and $\phi D^\alpha((hD-A)_x\cdot\boldsymbol{\sigma})E(\tau)$ have their operator norms bounded by $C\zeta^\alpha$ with $\zeta_1=\zeta_2=\beta^{\frac12}h^{-\frac12}$ and $\zeta_3=(h^{-1}+\mu^{\frac12}h^{-\frac12})$ for $\alpha\in\{0,1\}^3$ and ϕ, supported in the mentioned cube.

Then operator norms of both operators $\gamma_x E(\tau)$ and $\gamma_x((hD-A)_x\cdot\boldsymbol{\sigma})E(\tau)$ from $\mathscr{L}^2$ to $\mathbb{C}^q$ do not exceed $C_0\zeta_1\zeta_3^{\frac12}$ and therefore the same is true for adjoint operators; recall that γ_z is operator of restriction to $x=z$.

Since $E(\tau)^*=E(\tau)^2=E(\tau)$ we conclude that the left-hand expressions in (28.2.44) and (28.2.45) do not exceed $C\zeta_1^2\zeta_3$ which is exactly the right-hand expressions in both of them. □

Then from equation (27.2.14), which remains valid (see Remark 28.2.4(iii)), we conclude that

$$\|\Delta A'\|_{\mathscr{L}^\infty}\le C\kappa\big(\beta+\mu\big)\big(1+\mu^{\frac12}h^{\frac12}\big) \tag{28.2.46}$$

and therefore

$$\|\partial^2 A'\|_{\mathscr{L}^\infty}\le C\kappa|\log\beta|\big(\beta+\mu\big)\big(1+\mu^{\frac12}h^{\frac12}\big). \tag{28.2.47}$$

Let (28.2.27)) be fulfilled. Then combining (28.2.47) with (28.2.29) and $\|\partial A'\|_{\mathscr{L}^\infty}\le \|\partial^2 A'\|_{\mathscr{L}^\infty}^{\frac35}\cdot\|\partial A'\|^{\frac25}$ we conclude that

$$\mu\le C(\kappa|\log\beta|)^{\frac35}\big(\beta+\mu\big)^{\frac35}\big(1+h^{\frac12}\mu^{\frac12}\big)^{\frac35}\times\kappa^{\frac15}\beta^{\frac15}$$

and then <u>either</u>

$$1\le\mu h\le C(\kappa\beta h|\log\beta|)^{\frac67}(\kappa\beta h^2)^{\frac27} \tag{28.2.48}$$

<u>or</u> $\mu h\le 1$. In the former case of (28.2.48)

$$\|\partial A'\|_{\mathscr{L}^\infty}\le C(\kappa\beta|\log\beta|)^{\frac87}h^{\frac37} \tag{28.2.49}$$

and

$$\|\partial^2 A'\|_{\mathscr{L}^\infty}\le C(\kappa\beta|\log\beta|)^{\frac{11}{7}}h^{\frac57}. \tag{28.2.50}$$

Observe that the right-hand expression of (28.2.49) is less than $C\beta$ under assumption

$$\kappa\beta h^2|\log\beta|^K\le c \tag{28.2.27)*}$$

with sufficiently large K; however the right-hand expression of (28.2.50) is not necessarily less than $C\beta$ under this assumption and we need more delicate arguments.

Without any loss of the generality we can assume that $\partial_3 A_3(z) = 0$ (we can reach it by a gauge transformation).

Then considering $(\beta^{-\frac{1}{2}}h^{\frac{1}{2}}, \beta^{-\frac{1}{2}}h^{\frac{1}{2}}, \nu^{-\frac{1}{3}}h^{\frac{1}{3}})$-box in $\mathbb{R}^3$ we can replace factor $(1+\mu^{\frac{1}{2}}h^{\frac{1}{2}})h^{-1}$ by $(1+\nu^{\frac{1}{3}}h^{\frac{2}{3}})h^{-1}$ in all above estimates with $\nu = \|\partial^2 A'\|_{\mathscr{L}^\infty}$ and therefore (28.2.47) is replaced by

$$\nu \le C\kappa|\log\beta|\beta\big(1+\nu^{\frac{1}{3}}h^{\frac{2}{3}}\big)$$

and then under assumption (28.2.27)*

$$\nu = \|\partial^2 A'\|_{\mathscr{L}^\infty} \le C\kappa|\log\beta|\beta \tag{28.2.51}$$

which implies

$$\|\partial A'\|_{\mathscr{L}^\infty} \le C(\kappa|\log\beta|\beta)^{\frac{4}{5}}. \tag{28.2.52}$$

Obviously, if $\mu h \le 1$ we arrive to the same conclusion in easier way. Thus we have proven

Proposition 28.2.10. *Let $\beta h \ge 1$, $\kappa \le \kappa^*$ and (28.2.27)* be fulfilled. Then estimates (28.2.51)–(28.2.52) hold.*

28.3 Microlocal Analysis Unleashed: $\beta h \lesssim 1$

28.3.1 Rough Estimate to a Minimizer

Recall equation (27.2.14) to a minimizer A of $\mathsf{E}(A)$:

$$\frac{2}{\kappa h^2}\Delta A_j(x) = \Phi_j(x) := \tag{27.2.14}$$
$$-\operatorname{Re}\operatorname{tr}\Big[\sigma_j\Big(\big((hD-A)_x\cdot\boldsymbol{\sigma}e(x,y,\tau) + e(x,y,\tau)\,{}^t(hD-A)_y\cdot\boldsymbol{\sigma}\big)\Big)\Big]|_{y=x},$$

where $e(x,y,\tau)$ is the Schwartz kernel of the spectral projector $\theta(\tau - H_{A,V})$. In the current framework this equation should be replaced by

$$\frac{2}{\kappa h^2}\Delta\big(A_j(x) - A_j^0(x)\big) = \Phi_j(x) \tag{28.3.1}$$

with $\Phi_j(x)$ defined above but since $\Delta A_j^0 = 0$ these two equations are equivalent. We assume in this section that $\beta h \lesssim 1$.

Proposition 28.3.1. *Let $\beta h \lesssim 1$ and let*

$$\|\partial A'\|_{\mathscr{L}^\infty} \le \mu \le h^{-1}. \tag{28.3.2}$$

Then for $\theta \in [1,2]$

$$\begin{aligned}\|\Phi_j\|_{\mathscr{L}^\infty} + \|h\partial\Phi_j\|_{\mathscr{L}^\infty} \\ \le Ch^{-2}\Big(1+\beta^{\frac{3}{2}}h^{\frac{1}{2}} + |\log h| + (\beta h)^{\frac{\theta-1}{\theta+1}}\|\partial A'\|_{\mathscr{C}^\theta}^{\frac{1}{\theta+1}} + (\beta h)^{\frac{\theta-1}{\theta+1}}\|\partial V\|_{\mathscr{C}^\theta}^{\frac{1}{\theta+1}}\Big).\end{aligned} \tag{28.3.3}$$

Proof. (i) Assume first that $V \asymp 1$,

$$\beta h^{\frac{1}{3}} \lesssim 1 \qquad \text{and} \qquad \mu = 1. \tag{28.3.4}$$

Note that we need to consider only case $\beta \ge 2$ because otherwise estimate has been proven in Section 27.2 (see Proposition 27.2.16). Then the contribution of the zone $\mathscr{Z}'_\rho := \{|\xi_3 - A_3(x)| \le \rho\}$ with $\rho \ge \rho_* := C_0\beta^{-1}$ to the Tauberian remainder with $T = T_* := \epsilon\beta^{-1}$ does not exceed

$$Ch^{-2}\rho\Big(\beta + h^{\frac{1}{2}(\theta-1)}\|\partial A'\|_{\mathscr{C}^\theta}^{\frac{1}{2}} + h^{\frac{1}{2}(\theta-1)}\|\partial V\|_{\mathscr{C}^\theta}^{\frac{1}{2}}\Big). \tag{28.3.5}$$

Indeed, if Q is h-pseudodifferential operator supported in this zone then exactly as in the proof of (27.2.48) for $T \le T_*$

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma_x\big((hD)^k Q_x U_\varepsilon\big)| \le C\rho h^{-2}$$

and

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma_x\big((hD)^k Q_x (U-U_\varepsilon)\big)| \le C\rho h^{-4}\vartheta T^2,$$

where U and U_ε are Schwartz kernels of $e^{-ih^{-1}tH_{A,V}}$ and $e^{-ih^{-1}tH_{A_\varepsilon,V_\varepsilon}}$ respectively and ϑ is an operator norm of perturbation $\big(H_{A_\varepsilon,V_\varepsilon} - H_{A,V}\big)$, A_ε and V_ε are ε-mollification of A and V respectively and $\varepsilon \ge h$; then

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)\Gamma_x\big((hD)^k Q_x U\big)| \le C\rho(h^{-2} + h^{-4}\vartheta T^2)$$

and therefore the Tauberian error does not exceed $C\rho(h^{-2}T^{-1} + h^{-4}\vartheta T)$.

Optimizing by $T \le T_*$ we get $C\rho(h^{-2}\rho^{-1} + h^{-3}\vartheta^{\frac{1}{2}})$ with $\varepsilon = h$ and $\vartheta = \varepsilon^{\theta+1}\|\partial A'\|_{\mathscr{C}^\vartheta}$, which is exactly the (28.3.5).

Further, following arguments of Section 27.2 we conclude that an error when we pass from the Tauberian expression to the Weyl expression does not exceed

$$C\rho h^{-2}\Big(1 + h^{\frac{1}{2}(\theta-1)}\|\partial A'\|_{\mathscr{C}^\theta}^{\frac{1}{2}} + h^{\frac{1}{2}(\theta-1)}\|\partial V\|_{\mathscr{C}^\theta}^{\frac{1}{2}}\Big). \tag{28.3.6}$$

(ii) On the other hand, the contribution of zone $\mathcal{Z}_\rho = \{|\xi_3 - A_3(x)| \asymp \rho\}$ with $\rho \ge \rho_* = C_0\beta^{-1}$ to the Tauberian remainder with $T = T^* := \epsilon\rho$ does not exceed

$$Ch^{-2}\Big(1 + \rho^{\frac{1}{2}(1-\theta)}h^{\frac{1}{2}(\theta-1)}\|\partial A'\|^{\frac{1}{2}}_{\mathscr{C}^\theta} + \rho^{\frac{1}{2}(1-\theta)}h^{\frac{1}{2}(\theta-1)}\|\partial V\|^{\frac{1}{2}}_{\mathscr{C}^\theta}\Big). \tag{28.3.7}$$

Indeed, if Q is h-pseudodifferential operator, supported in this zone, then for $T \le T^*$

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma_x\big((hD)^k Q_x U_\varepsilon\big)| \le C\rho h^{-2},$$

and for $T_* \le T \le T^*$

$$|F_{t\to h^{-1}\tau}\chi_T(t)\Gamma_x\big((hD)^k Q_x U_\varepsilon\big)| \le C\rho h^{-2}(h/T\rho^2)^s,$$

and then

$$|F_{t\to h^{-1}\tau}\big(\bar{\chi}_T(t) - \bar{\chi}_{T_*}(t)\big)\Gamma_x\big((hD)^k Q_x U_\varepsilon\big)| \le C\rho h^{-2}(\beta h/\rho^2)^s$$

and

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma_x\big((hD)^k Q_x U_\varepsilon\big)| \le C\rho h^{-2},$$

while approximation error is estimated in the same way as before but with $\varepsilon = h\rho^{-1}$ and thus ϑ acquires factor $\rho^{-1-\theta}$.

Then the Tauberian error is estimated and optimized by $T \le T^*$ and it does not exceed (28.3.7).

Following arguments of Section 27.2 we conclude that an error when we pass from the Tauberian expression to the Weyl expression does not exceed (28.3.6).

Then the summation of the Tauberian error with respect to ρ ranging from $\rho = \rho'$ to $\rho = 1$ (where in what follows we use ρ instead of ρ' notation) returns

$$Ch^{-2}\Big(1 + |\log\rho| + \rho^{-\frac{1}{2}(\theta-1)}h^{\frac{1}{2}(\theta-1)}\|\partial A'\|^{\frac{1}{2}}_{\mathscr{C}^\theta} + \rho^{-\frac{1}{2}(\theta-1)}h^{\frac{1}{2}(\theta-1)}\|\partial V\|^{\frac{1}{2}}_{\mathscr{C}^\theta}\Big) \tag{28.3.8}$$

and adding contribution of zone $\mathcal{Z}'_\rho$ we conclude that the total Tauberian remainder does not exceed

$$Ch^{-2}\Big(\beta\rho + |\log\rho| + \rho^{-\frac{1}{2}(\theta-1)}h^{\frac{1}{2}(\theta-1)}\|\partial A'\|^{\frac{1}{2}}_{\mathscr{C}^\theta} + \rho^{-\frac{1}{2}(\theta-1)}h^{\frac{1}{2}(\theta-1)}\|\partial V\|^{\frac{1}{2}}_{\mathscr{C}^\theta}\Big). \tag{28.3.9}$$

Meanwhile, the summation of the Tauberian-to-Weyl error with respect to ρ returns (28.3.9) albeit without logarithmic term. Optimizing with respect to $\rho \ge \rho_*$ we arrive to

$$Ch^{-2}\Big(1+|\log h|+(\beta h)^{\frac{\theta-1}{\theta+1}}\|\partial A'\|_{\mathscr{C}^\theta}^{\frac{1}{\theta+1}}+(\beta h)^{\frac{\theta-1}{\theta+1}}\|\partial V\|_{\mathscr{C}^\theta}^{\frac{1}{\theta+1}}\Big). \tag{28.3.10}$$

Furthermore, observe that

(28.3.11) If in ϵ-vicinity of x inequality $|\nabla V|\le \zeta$ holds (with $\zeta \ge |\log h|^{-1}$), then we can pick up $T^* = \epsilon \min(\zeta^{-1}\rho, 1)$.

Indeed, we can introduce

$$p_3' := \xi_3 - A_3 - \beta^{-1}\alpha_1(\xi_1 - A_1) - \beta^{-1}\alpha_2(\xi_2 - A_2)$$

such that $\{H, p_3'\} = V_{\xi_3} + O(\nu\beta^{-1})$ with $\nu := \|\partial^2 A\|_{\mathscr{C}^\infty}$.

Therefore, in this case the remainder does not exceed

$$Ch^{-2}\Big(1+\zeta|\log h|+(\beta h)^{\frac{\theta-1}{\theta+1}}\|\partial A'\|_{\mathscr{C}^\theta}^{\frac{1}{\theta+1}}+(\beta h)^{\frac{\theta-1}{\theta+1}}\|\partial V\|_{\mathscr{C}^\theta}^{\frac{1}{\theta+1}}\Big). \tag{28.3.12}$$

(iii) Finally, observe that the Weyl expression for Φ_j is just 0. Therefore under assumption (28.3.4) slightly improved estimate (28.3.3) has been proven: $\|\Phi_j\|_{\mathscr{L}^\infty} + \|h\partial\Phi_j\|_{\mathscr{L}^\infty}$ does not exceed expression (28.3.12).

(iv) To get rid of assumption (28.3.4) we scale $x \mapsto x\gamma^{-1}$, $h \mapsto h\gamma^{-1}$, $\beta \mapsto \beta\gamma$ and pick up $\gamma = \min\big((\beta h^{\frac13})^{-\frac32}, \mu^{-1}\big)$; then $\beta h \mapsto \beta h$, and $Ch^{-2} \mapsto Ch^{-2}\gamma^{-1} = C\beta^{\frac32}h^{-\frac32} + C\mu h^{-2}$ (as we need to multiply by γ^{-3}) and both $\|\partial A'\|_{\mathscr{C}^\theta}^{\frac{1}{\theta+1}}$ and $\|\partial V\|_{\mathscr{C}^\theta}^{\frac{1}{\theta+1}}$ acquire factor γ.

Observing that we can take $\zeta = C\gamma$ and that factor γ also pops up in all other terms (except 1) in (28.3.12) we arrive to estimate (28.3.3).

Furthermore, to get rid of assumption $V \asymp 1$ we also can scale with $\gamma = \epsilon|V| + h^{\frac23}$ and multiply operator by γ^{-1}; then $h \mapsto h\gamma^{-\frac32}$, $\beta \mapsto \beta\gamma^{\frac12}$ and estimate (28.3.3) does not deteriorate; we need to multiply by $\gamma^{\frac12}$ which does not hurt. □

Remark 28.3.2. (i) We can use $\theta' \neq \theta$ for norm of V.

(ii) If V is smooth enough we can skip the related term (details later).

(iii) We can take $\theta = 1$ but in this case factor $\rho^{-\frac{1}{2}(\theta-1)}h^{\frac{1}{2}(\theta-1)}$ in (28.3.8) (i.e. after summation) and in (28.3.9) is replaced by $|\log\rho|$; then taking into account (i) we replace (28.3.10) by

$$(28.3.10)' \qquad Ch^{-2}\Big(1+|\log h|+|\log h|\cdot \|\partial A'\|^{\frac{1}{2}}_{\mathscr{C}^1} + (\beta h)^{\frac{\theta'-1}{\theta'+1}}\|\partial V\|^{\frac{1}{\theta'+1}}_{\mathscr{C}^{\theta'}}\Big)$$

and similarly we deal with (28.3.12) and (28.3.3):

$$(28.3.3)' \quad \|\Phi_j\|_{\mathscr{L}^\infty} + \|h\partial\Phi_j\|_{\mathscr{L}^\infty} \leq Ch^{-2}\Big(1+\beta^{\frac{3}{2}}h^{\frac{1}{2}}+|\log h|+|\log h|\|\partial A'\|^{\frac{1}{2}}_{\mathscr{C}^1} + (\beta h)^{\frac{\theta'-1}{\theta'+1}}\|\partial V\|^{\frac{1}{\theta'+1}}_{\mathscr{C}^{\theta'}}\Big).$$

(iv) From the very beginning we could assume that $\mu \leq \beta$; otherwise we could rescale as above with $\gamma = \beta^{-1}$ and apply arguments of Section 27.2 simply ignoring external field.

Corollary 28.3.3. *Let in the framework of Proposition 28.3.1 A' be a minimizer. Then for $\theta, \theta' \in [1,2]$*

$$(28.3.13) \quad |\log h|^{-1}\|\partial A'\|_{\mathscr{C}^1} + h^{\theta-1}\|\partial A'\|_{\mathscr{C}^\theta} \leq C\kappa\Big(\beta^{\frac{3}{2}}h^{\frac{1}{2}}+|\log h|+\mu+(\beta h)^{\frac{1}{2}(\theta'-1)}\|V\|^{\frac{1}{2}}_{\mathscr{C}^{\theta'+1}}\Big) + C\kappa^2|\log h||\log h|^2 + C\|\partial A'\|.$$

Proof. Indeed, the left-hand expression of (28.3.13) does not exceed

$$\|\Delta A'\|_{\mathscr{L}^\infty} + \|h\partial\Delta A'|_{\mathscr{L}^\infty} + C\|\partial A'\|$$

while for a minimizer $\|\Delta A'\|_{\mathscr{L}^\infty} + \|h\partial\Delta A'|_{\mathscr{L}^\infty}$ does not exceed the right-hand expression of (28.3.3) multiplied by $C\kappa h^2$. □

28.3.2 Microlocal Analysis

As long as $\beta \leq C_0 h^{-\frac{1}{3}}$ we are rather happy with our result here, but we want to improve it otherwise. First we will prove that singularities propagate along magnetic lines; however since we do not know a self-generated magnetic field we just consider all possible lines which will be in the cone $\{(x,y)\colon |x'-y'| \leq C_0\mu\beta^{-1}T\}$ where $1 \leq \mu \leq \beta$.

Proposition 28.3.4. *Assume that* $\beta h \lesssim 1$,

$$\|V\|_{\mathscr{C}^1(B(0,2))} \le C_0 \tag{28.3.14}$$

and

$$\|\partial A'\|_{\mathscr{C}(B(0,2))} \le \mu \qquad (1 \le \mu \le \epsilon\beta) \tag{28.3.15}$$

with sufficiently small constant $\epsilon > 0$.

Let $U(x,y,t)$ *be the Schwartz kernel of* $e^{ih^{-1}tH_{A,V}}$. *Then*

(i) For $T \asymp 1$ *estimate*

$$\|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\psi_1(x)\psi_2(y)U\| \le Ch^s \tag{28.3.16}$$

holds for all $\psi_1, \psi_2 \in \mathscr{C}_0^\infty(B(0,1))$, *such that* $\mathsf{dist}(\mathsf{supp}(\psi_1), \mathsf{supp}(\psi_2)) \ge C_0 T$ *and* $\tau \le c_0$; *here* $\|.\|$ *means an operator norm from* $\mathscr{L}^2$ *to* $\mathscr{L}^2$ *and* s *is arbitrarily large.*

(ii) For $\bar{\rho} \le \rho \lesssim 1$ *with* $\bar{\rho} := C_0\mu\beta^{-1}$ *and* $T \asymp \rho$ *estimate*

$$\begin{aligned}\|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\varphi_1(\rho^{-1}p_{3x})\varphi_2(\rho^{-1}p_{3y})\psi_1(x)\psi_2(y)U\| \\ \le C\rho^{1-3s}h^s + C\rho^{2-\theta}h^\theta\big(\|\!|A'|\!\|_{\theta+1} + \|\!|V|\!\|_{\theta+1}\big)\end{aligned} \tag{28.3.17}$$

holds for all all $\varphi_1, \varphi_2 \in \mathscr{C}_0^\infty$, $\psi_1, \psi_2 \in \mathscr{C}_0^\infty(B(0,1))$, *such that* $\mathsf{dist}(\mathsf{supp}(\varphi_1), \mathsf{supp}(\varphi_2)) \ge C_0$, *and* $\tau \le c_0$; *here and below* $p_j = hD_j - A_j$, $p_j^0 = hD_j - A_j^0$.

(iii) For $\bar{\rho} \le \rho \lesssim 1$ *and* $T \asymp \rho \lesssim 1$ *estimate*

$$\begin{aligned}\|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\varphi_1(\rho^{-1}p_{3x})\varphi_2(\rho^{-1}p_{3y})\psi_1(\gamma^{-1}x)\psi_2(\gamma^{-1}y)U\| \\ \le C\rho^{1-s}\gamma^{-s}h^s + Ch^\theta\gamma\rho^{-\theta}\big(\|\!|A'|\!\|_{\theta+1} + \|\!|V|\!\|_{\theta+1}\big)\end{aligned} \tag{28.3.18}$$

holds for all $\varphi_1, \varphi_2 \in \mathscr{C}_0^\infty$, $\psi_1, \psi_2 \in \mathscr{C}_0^\infty$, *such that* $\mathsf{dist}(\mathsf{supp}(\psi_1), \mathsf{supp}(\psi_2)) \ge C_0$, $\gamma = \rho T \ge \beta^{-1}$, $\rho\gamma \ge h$ *and* $\tau \le c_0$.

Proof. Statement (i) claims that the general propagation speed with respect to x is bounded by C_0. Further, Statement (ii) claims that the on distances $\ge C_0\bar{\rho}$ the propagation speed with respect to p_3 is also bounded by C_0. Finally, Statement (iii) claims that on distances $\ge C_0\bar{\rho}$ the propagation speed with respect to x is bounded by $C_0\rho$. Note that from Corollary 28.3.3 we know that $\mu \lesssim \beta$.

(a) Proof follows the proof of Proposition 27.2.11 in the framework of the strong magnetic field. Namely, proof of Statement (i) is a straightforward repetition of the proof of Proposition 27.2.11(i). Since here we do not apply at this stage operators $(hD_x)^\alpha$ and $(hD_y)^{\alpha'}$, no assumption to the smoothness of A is needed.

(b) Assume that $A_3 \equiv 0$ (we will get rid of this assumption on the next step). After Statement (i) has been proven we rescale $t \mapsto t/T$, $x_3 \mapsto x_3/\gamma$ with $\gamma = \rho T$ (since φ_l depend only on ξ_3, all other coordinates are rather irrelevant), $h \mapsto \hbar = h/(\rho\gamma)$, $T \mapsto 1$. Then we apply the arguments used in the proof of Proposition 27.2.11(ii) and conclude that the left-hand expression of (28.3.17) does not exceed

$$T\Big(\hbar^s + C\hbar^\theta \gamma^{\theta+1}\big(\|\!|A'|\!\|_{\theta+1} + \|\!|V|\!\|_{\theta+1}\big)\Big),$$

where factor T is due to the scaling in the Fourier transform and $\gamma^{\theta+1}$ is due to the scaling in $\|\!|\cdot|\!\|$-norms. Plugging $\hbar$, T, and $\gamma = \rho^2$ we get the right-hand expression of (28.3.17).

Then if ψ_l depend only on x_3, y_3 we can follow the proof of Proposition 27.2.11(iii) and prove Statement (iii).

(c) Let A_3 be not necessarily identically 0. To consider ψ_l depending only on x_1 or x_2 we should introduce (in the standard magnetic Schrödinger manner) $x_1' := x_1 + \beta^{-1}p_2^0$ or $x_2' := x_2 - \beta^{-1}p_1^0$ respectively; recall that $p_j^0 = hD_j - A_j^0$ and $p_j = hD_j - A_j$, $j = 1,2,3$.

Then $[p_1^0, p_2^0] = ih\beta^{-1}$, $[p_j^0, x_k] = -ih\delta_{jk}$, $[p_j^0, x_k'] = 0$ and one can see easily that $[p_j, x_k'] = O(\beta^{-1}\mu h)$ (for any $\mu : 1 \le \mu \le \beta$) for $j = 1,2,3$ and $k = 1,2$. Now we can apply the same arguments as above as long as $\rho \ge \bar\rho$.

(d) Next we need to recover Statement (ii) in the general case. Without any loss of the generality we may consider a vicinity of point z where $A'(z) = 0$ and also $\partial A'(z) = 0$. Indeed we can achieve the former by the gauge transformation and the latter by a rotation of coordinates in which case increment of p_3^0 will be $O(\bar\rho)$.

In this case we just repeat the same arguments of Part (b) of our proof.

(e) Finally, the proof of Statement (iii) as ψ_l depend only on x_3 follows from Statement (ii).

We leave all easy details to the reader. □

Proposition 28.3.5. *Let $\beta h \lesssim 1$ and assumptions* (28.3.14) *and* (28.3.15) *be fulfilled. Then*

(i) For $h \le T \le 1$ estimate

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)p_x^{\alpha}p_y^{\alpha'}U| \le CT^{-s}h^{-3+s} \tag{28.3.19}$$

holds for all $\alpha : |\alpha| \le 2$, $\alpha' : |\alpha'| \le 2$, and all $x, y \in B(0,1)$, such that $|x-y| \ge C_0T$ and $\tau \le c_0$.

(ii) In the framework of Proposition 28.3.4(ii) the following estimate holds for all $\alpha : |\alpha| \le 2$, $\alpha' : |\alpha'| \le 2$, and $\tau \le c$:

$$\begin{aligned}|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)p_x^{\alpha}p_y^{\alpha'}\varphi_1(\rho^{-1}p_{3x})\varphi_2(\rho^{-1}p_{3y})\psi_1(x)\psi_2(y)U| \\ \le C\rho h^{-1}\bigl(\beta h^{-1}+\rho^2h^{-2}\bigr)\Bigl(\rho^{1-3s}h^s+\rho^{2-\theta}h^{\theta}\bigl(\|A'\|_{\theta+1}+\|V\|_{\theta+1}\bigr)\Bigr);\end{aligned} \tag{28.3.20}$$

(iii) In the framework of Proposition 28.3.4(iii) the following estimate holds for all $\alpha : |\alpha| \le 2$, $\alpha' : |\alpha'| \le 2$, and $\tau \le c$:

$$\begin{aligned}|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)p_x^{\alpha}p_y^{\alpha'}\varphi_1(\rho^{-1}p_{3x})\varphi_2(\rho^{-1}p_{3y})\psi_1(\gamma^{-1}x)\psi_2(\gamma^{-1}y)U| \\ \le C\rho h^{-1}\bigl(\beta h^{-1}+\rho^2h^{-2}\bigr)\Bigl(\rho^{1-s}\gamma^{-s}h^s+h^{\theta}\gamma\rho^{-\theta}\bigl(\|A'\|_{\theta+1}+\|V\|_{\theta+1}\bigr)\Bigr).\end{aligned} \tag{28.3.21}$$

Proof. Observe that estimates (28.3.16)–(28.3.18) hold if one applies operator $p_x^{\alpha}p_y^{\alpha'}$ under the norm (this follows from equations for U by (x,t) and dual equations by (y,t)). Then estimates (28.3.19)–(28.3.21) hold with $\alpha = \alpha' = 0$.

Really, without any loss of the generality one can assume that $A' = 0$ at some point of $\operatorname{supp}(\psi_1)$; then estimates (28.3.16)–(28.3.18) hold if one applies operator $p_x^{0\alpha}p_y^{0\alpha'}$ instead. Then estimate (28.3.19) holds with $\alpha = \alpha' = 0$; further, estimates (28.3.20)–(28.3.21) also hold with $\alpha = \alpha' = 0$ if one applies an extra factor $\bar{\varphi}_1(\rho^{-1}p_{3x}^0)\bar{\varphi}_2(\rho^{-1}p_{3y}^0)$ under the norm (this follows from the properties of p_j^0, $j = 1,2,3$, in particular, canonical form). However if $\bar{\varphi}_l = 1$ in ϵ-vicinity of $\operatorname{supp}(\varphi_l)$ then we can skip this factor.

Finally, appealing to equations for U by (x,t) and (y,t) again we recover estimates (28.3.19)–(28.3.21) with $|\alpha| \le 2$, $|\alpha'| \le 2$. □

Proposition 28.3.6. *Let* $\beta h \lesssim 1$ *and* (28.3.14) *and* (28.3.15) *be fulfilled. Let* $z \in B(0,1)$. *Then:*

(i) The following estimate

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)p_x^{\alpha}p_y^{\alpha'}\varphi_1(\rho^{-1}p_{3x})\varphi_2(\rho^{-1}p_{3y})U|_{x=y=z}| \leq C\rho T h^{-1}\bigl(\beta h^{-1}+\rho^2h^{-2}\bigr) \tag{28.3.22}$$

holds for all $\alpha : |\alpha| \le 2$, $\alpha' : |\alpha'| \le 2$, *and* $\tau \le c$.

(ii) Let $A_z(x) = A(z) + \langle x-z, \nabla_z\rangle A(z)$, $V_z(x) = V(z) + \langle x-z, \nabla_z\rangle V(z)$ *be linear approximations to* A *and* V *at* z; *let* $H_z = H_{A_z,V_z}$, $U_z(x,y,t)$ *be its Schwartz kernel. Then for* $h^{1-\delta} \le T \le C_0$, $\rho \le C_0$ *estimate*

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)p_x^{\alpha}p_y^{\alpha'}\varphi_1(\rho^{-1}p_{3x})\varphi_2(\rho^{-1}p_{3y})(U-U_z)|_{x=y=z}| \leq C\rho T^2h^{-2}\bigl(\beta h^{-1}+\rho^2h^{-2}\bigr)\nu\gamma^2 \tag{28.3.23}$$

holds with

$$\gamma = \gamma(\rho,T) \coloneqq C_0(\rho+T)T + C_0h\rho^{-1}, \tag{28.3.24}$$

and

$$\nu = \bigl(\|A'\|_2 + \|V\|_2\bigr) + \mu^2. \tag{28.3.25}$$

Proof. (i) Proof of Statement (i) is easy and left to the reader.

(ii) To prove Statement (ii) observe that

$$e^{ith^{-1}H} = e^{ith^{-1}H_z} + ih^{-1}\int_0^t e^{i(t-t')h^{-1}H}(H-H_z)e^{it'h^{-1}H_z}\,dt' = e^{ith^{-1}H_z} + \sum_{0\le k\le K} \underbrace{ih^{-1}\int_0^t e^{i(t-t')h^{-1}H}(H-H_z)\psi_k e^{it'h^{-1}H_z}\,dt'}, \tag{28.3.26}$$

where ψ_0 is a γ-admissible function supported in $B(z,2\gamma)$ and ψ_k are γ_k-admissible functions supported in $B(z,\gamma_k)\setminus B(z,\frac12\gamma_k)$ with $\gamma_k = 2^k\gamma$. Plugging (28.3.26) into the left-hand expression of (28.3.23) we note that the first term is cancelled and we have the sum with respect to $k : 0 \le k \le K$ obtained from this expression when we replace $(U-U_z)$ by the Schwartz kernel of the selected above term.

Further, observe that the term with $k = 0$ does not exceed the right-hand expression of (28.3.23).

Furthermore, terms with $k : 1 \le k \le K$ do not exceed the right-hand expression of (28.3.21) multiplied by $CTh^{-1}\min(\nu\gamma_k^2, 1)$; indeed, we just replace ρ by T if needed. After summation with respect to $k : 0 \le k \le K$ we get

$$C\rho h^{-1}T\big(\beta h^{-1} + \rho^2 h^{-2}\big) \times \Big(\rho^{-s}\gamma^{-s}h^s \min(\nu\gamma^2, 1) + h^2\rho^{-2}\nu \min(\nu, 1)\Big)$$

which again does not exceed the right-hand expression of (28.3.23). □

Remark 28.3.7. Actually Statement (ii) is better than Statement (i) only if $\nu\gamma^2 Th^{-1} \le 1$.

28.3.3 Advanced Estimate to a Minimizer

Now we are going to apply the results of the previous Subsection 28.3.2 to the right-hand expression of (27.2.14).

Tauberian Estimate

Consider different zones (based on the magnitude of $|p_3|$). Recall that $\bar\rho = \beta^{-1}$ [5)] and $\rho^* = (\beta h)^{\frac12}$.

Zone $\{\rho' \lesssim |p_3| \lesssim \rho^*\}$. Observe that Proposition 28.3.6(ii) implies that for $\varphi_j \in \mathscr{C}_0^\infty([-2, -\frac12] \cup [\frac12, 2])$ estimate

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)p_x^\alpha p_y^{\alpha'}\varphi_1(\rho^{-1}p_{3x})\varphi_2(\rho^{-1}p_{3y})U|_{x=y=z}| \le CS(\rho, T) \tag{28.3.27}$$

holds with

$$S(\rho, T) = \big(\beta h^{-1} + \rho^2 h^{-2}\big)\big(\rho^{-1} + \rho h^{-2}\nu\gamma^2 T^2\big), \tag{28.3.28}$$

where $\gamma = \gamma(\rho, T)$ is defined by (28.3.24).

Indeed, one can prove easily that

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)p_x^\alpha p_y^{\alpha'}\varphi_1(\rho^{-1}p_{3x})\varphi_2(\rho^{-1}p_{3y})U_z|_{x=y=z}| \le C(\beta h^{-1}\rho^{-1} + \rho^2 h^{-2}). \tag{28.3.29}$$

[5)] As we assume that $\mu = 1$; otherwise $\bar\rho = \mu\beta^{-1}$.

Let us take $\alpha = \alpha'$, $\varphi_1 = \varphi_2$ and $\rho_1 = \rho_2$. Since in this case expression

$$p_x^{\alpha} p_y^{\alpha'} \varphi_1(\rho_1^{-1} p_{3x}) \varphi_2(\rho_2^{-1} p_{3y}) e(.,.,\tau)|_{x=y} \tag{28.3.30}$$

is a monotone function with respect to τ, then the standard Tauberian arguments (part I–estimates; we leave easy details to the reader) imply that

$$\begin{aligned} |p_x^{\alpha} p_y^{\alpha} \varphi_1(\rho^{-1} p_{3x}) \varphi_1(\rho^{-1} p_{3y}) \big[e(.,.,\tau) - e(.,.,\tau')\big]|_{x=y}| \\ \le C S(\rho, T) \big(T^{-1} + |\tau - \tau'| h^{-1}\big) \end{aligned} \tag{28.3.31}$$

for all $\tau' \le \tau \le c$ and therefore

$$\begin{aligned} &|p_x^{\alpha} p_y^{\alpha'} \varphi_1(\rho_1^{-1} p_{3x}) \varphi_2(\rho_2^{-1} p_{3y}) \big[e(.,.,\tau) - e(.,.,\tau')\big]|_{x=y}| \\ &\le C \big(S(\rho_1, T_1) S(\rho_2, T_2)\big)^{\frac{1}{2}} \Big(T_1^{-\frac{1}{2}} T_2^{-\frac{1}{2}} + |\tau - \tau'|^{\frac{1}{2}} \big(T_1^{-\frac{1}{2}} + T_2^{-\frac{1}{2}}\big) h^{-\frac{1}{2}} + |\tau - \tau'| h^{-1} \Big). \end{aligned} \tag{28.3.32}$$

Then the standard Tauberian arguments (part II–asymptotics, with the minor modifications; again we leave easy details to the reader) imply that expression (28.3.30) is given by the standard Tauberian formula with an error, not exceeding the right-hand expression of (28.3.32) with $|\tau - \tau'|$ replaced by hT^{-1}, which is

$$C \big(S(\rho_1, T_1) S(\rho_2, T_2)\big)^{\frac{1}{2}} \Big(T_1^{-\frac{1}{2}} T_2^{-\frac{1}{2}} + \big(T_1^{-\frac{1}{2}} + T_2^{-\frac{1}{2}}\big) T^{-\frac{1}{2}} + T^{-1} \Big). \tag{28.3.33}$$

Note that for $T \ge \max(T_1, T_2)$ the second factor in (28.3.33) is $\asymp T_1^{-\frac{1}{2}} T_2^{-\frac{1}{2}}$.

In other words, a contribution of the pair (ρ_1, ρ_2) to the Tauberian error does not exceed a square root of $S(\rho_1, T_1) T_1^{-1} \times S(\rho_2, T_2) T_2^{-1}$ with

$$\begin{aligned} S(\rho, T) T^{-1} &= \beta h^{-1} \Big(\rho^{-1} T^{-1} + h^{-2} \rho \nu T \big(T^4 + \rho^2 T^2 + h^2 \rho^{-2}\big) \Big) \\ &\asymp \beta h^{-1} \Big(\rho^{-1} T^{-1} + h^{-2} \nu \rho^3 T^3 + h^{-2} \nu \rho T^5 + \nu \rho^{-1} T \Big) \\ &\qquad \text{for } \ \bar{\rho} \le \rho \le \rho^*. \end{aligned} \tag{28.3.34}$$

Minimizing this expression by $h \le T \lesssim 1$ we get

$$\begin{aligned} &\beta h^{-1} \Big(\rho^{-\frac{3}{4}} (h^{-2} \nu \rho^3)^{\frac{1}{4}} + \rho^{-\frac{5}{6}} (h^{-2} \nu \rho)^{\frac{1}{6}} + \rho^{-1} \nu^{\frac{1}{2}} \Big) \\ &\qquad \asymp \beta h^{-1} \Big(h^{-\frac{1}{2}} \nu^{\frac{1}{4}} + h^{-\frac{1}{3}} \nu^{\frac{1}{6}} \rho^{-\frac{2}{3}} + \nu^{\frac{1}{2}} \rho^{-1} \Big); \end{aligned}$$

then the summation by $\rho \in [\rho', \rho^*]$ returns

$$\beta h^{-1}\Big(h^{-\frac{1}{2}}\nu^{\frac{1}{4}}|\log h| + h^{-\frac{1}{3}}\nu^{\frac{1}{6}}\rho^{-\frac{2}{3}} + \nu^{\frac{1}{2}}\rho^{-1}\Big) \tag{28.3.35}$$

with $\rho = \rho'$ to be selected later.

Zone $\{\rho^* \lesssim |p_3| \lesssim 1\}$. Further, we claim that

(28.3.36) For $h^{\frac{1}{3}} \le \rho \le C_0$, $h \le T \le \epsilon_0\rho$ we can take $\gamma = h\rho^{-1}$.

Indeed, observe that if we use ε-approximation with $\varepsilon = (\rho^{-1}h)^{1-\delta}$ then the contribution of time intervals $\{t\colon T_* \le |t| \le T^*\}$ with $T_* = \rho^{-1}(\rho^{-1}h)^{1-\delta}$, $T^* = \epsilon_0\rho$ is negligible and the transition from $\varepsilon = (\rho^{-1}h)^{1-\delta}$ to $\varepsilon = (\rho^{-1}h)$ is done like in the previous Chapter 26. Again we leave easy details to the reader.

Then

$$S(\rho, T)T^{-1} \asymp \rho h^{-2}\big(T^{-1} + \nu T\big); \tag{28.3.37}$$

minimizing by $T\colon h \le T \le \epsilon\rho$ we get

$$\rho h^{-2}\big(\nu^{\frac{1}{2}} + \rho^{-1} + \nu h\big);$$

then summation by $\rho \in [\rho^*, C_0]$ returns

$$h^{-2}\big(\nu^{\frac{1}{2}} + (1+\nu h)|\log h|\big). \tag{28.3.38}$$

Observe that $\rho^* = (\beta h)^{\frac{1}{2}} \ge h^{\frac{1}{3}}$ as $\beta \ge h^{-\frac{1}{3}}$.

Zone $\{|p_3| \lesssim \rho'\}$. Finally, the remaining zone $\{|p_3| \lesssim \rho'\}$ is covered by a single element $\varphi(\rho^{-1}p_3)$ with $\varphi \in \mathscr{C}_0^\infty([-2,2])$, $\rho = \rho'$.

Then instead of minimized $S(\rho, T)T^{-1}$ we can take $\rho\beta h^{-2}$ which should be added to the sum of expressions (28.3.38) and (28.3.35) which estimate contributions of two other zones resulting in

$$\begin{aligned} &h^{-2}\big(\nu^{\frac{1}{2}} + (1+\nu h)|\log h|\big) \\ &\qquad + \beta h^{-1}\Big[h^{-\frac{1}{2}}\nu^{\frac{1}{4}}|\log h| + \underbrace{h^{-\frac{1}{3}}\nu^{\frac{1}{6}}\rho^{-\frac{2}{3}} + \nu^{\frac{1}{2}}\rho^{-1} + \rho h^{-1}}\Big]. \end{aligned} \tag{28.3.39}$$

Obviously the second term in (28.3.39) should be minimized by $\rho = \rho' \in [\bar{\rho}, \rho^*]$, resulting in

$$\beta h^{-1}\Big[h^{-\frac{1}{2}}\nu^{\frac{1}{4}}|\log h| + h^{-\frac{2}{5}}(h^{-\frac{1}{3}}\nu^{\frac{1}{6}})^{\frac{3}{5}} + h^{-\frac{1}{2}}(\nu^{\frac{1}{2}})^{\frac{1}{2}}\Big] \asymp \beta h^{-1}\Big[h^{-\frac{1}{2}}\nu^{\frac{1}{4}}|\log h| + h^{-\frac{3}{5}}\nu^{\frac{1}{10}} + h^{-\frac{1}{2}}\nu^{\frac{1}{4}})\Big];$$

two terms arising when we set $\rho = \rho^*$ in the terms with negative power of ρ and one term arising when we set $\rho = \bar{\rho}$ in the term with positive power of ρ are absorbed by other terms in (28.3.39), which becomes

$$h^{-2}\big(\nu^{\frac{1}{2}} + (1+\nu h)|\log h|\big) + \beta h^{-1}\Big[h^{-\frac{1}{2}}\nu^{\frac{1}{4}}|\log h| + h^{-\frac{3}{5}}\nu^{\frac{1}{10}} + h^{-\frac{1}{2}}\nu^{\frac{1}{4}}\Big]. \tag{28.3.40}$$

This is an estimate for the whole Tauberian error (with variable $T = T(\rho)$).

Calculating Tauberian Expression

Now we need to consider the Tauberian expression for $p_x^\alpha p_y^{\alpha'} e(.,.,0)|_{x=y=z}$ and estimate an error made when we replace it by the Tauberian expression for $p_x^\alpha p_y^{\alpha'} e_z(.,.,0)|_{x=y=z}$; we will call it the *second error* in contrast to the first (Tauberian) error. Note that we are interested only in the case $|\alpha| + |\alpha'| = 1$.

Let us again consider contribution of pair (ρ_1, ρ_2). First, observe that for $\rho_1 \asymp \rho_2$ this error does not exceed $CS(\rho_2, T)T^{-1}$ due to our standard arguments and therefore we get for such pairs the same contribution to the total error as we already got for the Tauberian error.

Second, consider pairs with $\rho_1 \gg \rho_2$ and in this case redoing previous arguments we observe that the contribution to the first error does not exceed $CS(\rho_1, T)^{\frac{1}{2}} S(\rho_2, T)^{\frac{1}{2}} T^{-1}$ and the contribution to the second error does not exceed

$$C\beta h^{-1} \times \rho_1^{\frac{1}{2}}\rho_2^{\frac{1}{2}} h^{-2}\nu T^3 (T+\rho_2)^2, \tag{28.3.41}$$

where the first term which was $C\beta h^{-1}\rho^{-1}T^{-1}$ in the former case $\rho_1 \asymp \rho_2$ simply disappear. Indeed, it appears only due to the contribution of the time interval $\{|t| \le \rho\}$ where we should take $\rho = \max(\rho_1, \rho_2) = \rho_1$ and estimate an error due to the propagation of singularities.

Similarly, the second term leading to expression (28.3.41) would also disappear unless $\rho_1 \lesssim T$ again due to the propagation of singularities. Therefore the combined contribution of any pair to both errors does not exceed

$$\begin{aligned}(28.3.42)\quad &\Bigl(\rho_1^{-1}T^{-1}+\rho_1^3h^{-2}\nu T^3+\rho_1h^{-2}\nu T^5+\rho_1^{-1}\nu T\Bigr)^{\frac12}\\ &\qquad\times\Bigl(\rho_2^{-1}T^{-1}+\rho_2^3h^{-2}\nu T^3+\rho_2h^{-2}\nu T^5+\rho_2^{-1}\nu T\Bigr)^{\frac12}+\rho_1^{\frac12}\rho_2^{\frac12}h^{-2}\nu T^5,\end{aligned}$$

multiplied by $C\beta h^{-1}$ since we consider at this moment the case of $\rho'\le\rho_2\ll\rho_1\le\rho^*$ while all other cases (namely, $\rho_2\le\rho'\le\rho_1\le\rho^*$; $\rho_2\le\rho'\le\rho^*\le\rho_1$; $\rho'\le\rho_2\le\rho^*\le\rho_1$; and $\rho^*\le\rho_2\ll\rho_1$) are easier and left to the reader.

Opening parenthesis in (28.3.42) and eliminating all smaller terms we arrive to

$$\begin{aligned}&\rho_1^{-\frac12}\rho_2^{-\frac12}T^{-1}+\rho_1^{\frac12}\rho_2^{\frac12}h^{-2}\nu T^5+\rho_1^{\frac32}\rho_2^{\frac12}h^{-2}\nu T^4\\ &\quad+\bigl(\rho_1^{\frac32}\rho_2^{\frac32}h^{-2}\nu+\rho_1^{\frac12}\rho_2^{-\frac12}h^{-1}\nu\bigr)T^3+\bigl(\rho_1^{\frac12}\rho_2^{-\frac12}(h^{-2}\nu)^{\frac12}+\rho_1^{\frac32}\rho_2^{-\frac12}h^{-1}\nu\bigr)T^2\\ &\qquad+\bigl(\rho_1^{-\frac12}\rho_2^{-\frac12}\nu+\rho_1^{\frac32}\rho_2^{-\frac12}(h^{-2}\nu)^{\frac12}\bigr)T+\rho_1^{-\frac12}\rho_2^{-\frac12}\nu^{\frac12};\end{aligned}$$

minimizing by T we get

$$\begin{aligned}&\rho_1^{-\frac13}\rho_2^{-\frac13}(h^{-2}\nu)^{\frac16}+\rho_1^{-\frac{1}{10}}\rho_2^{-\frac{3}{10}}(h^{-2}\nu)^{\frac15}+(h^{-2}\nu)^{\frac14}+\rho_1^{-\frac14}\rho_2^{-\frac12}(h^{-1}\nu)^{\frac14}\\ &\qquad+\rho_1^{-\frac16}\rho_2^{-\frac12}(h^{-1}\nu)^{\frac16}+\rho_1^{-\frac12}\rho_2^{-\frac12}\nu^{\frac12}+\rho_1^{\frac12}\rho_2^{-\frac12}(h^{-2}\nu)^{\frac14}+\underbrace{\rho_1^{-\frac12}\rho_2^{-\frac12}\nu^{\frac12}}.\end{aligned}$$

Observe, that only the last term has ρ_1 in the positive degree. Also observe, that the optimal $T=T(\rho)$ in the Tauberian error is a decreasing function of ρ, so $T_1\le T_2$ where $T_j:=T(\rho_j)$; therefore we consider the Tauberian expression for $T\le T_2$ and thus for $\rho_1\le T\lesssim T_2$.

Therefore T_2 must be an upper bound for ρ_1 and therefore the summation by $\rho_1:\rho_2\le\rho_1\le T_2$ results in all the terms with negative power of ρ_1 in the value as $\rho_1=\rho_2$ and in the exceptional (last) term with $\rho_1=T_2$:

$$\begin{aligned}(28.3.43)\quad &\rho_2^{-\frac23}(h^{-2}\nu)^{\frac16}+\rho_2^{-\frac25}(h^{-2}\nu)^{\frac15}+(h^{-2}\nu)^{\frac14}|\log\rho_2(\rho_2^{-2}h^2\nu^{-1})^{-\frac16}|\\ &\qquad+\rho_2^{-\frac34}(h^{-1}\nu)^{\frac14}+\rho_2^{-1}\nu^{\frac12}+(\rho_2^{-2}h^2\nu^{-1})^{\frac{1}{12}}\rho_2^{-\frac12}(h^{-2}\nu)^{\frac14}\end{aligned}$$

(where we used inequality $T_2 \le (\rho_2^{-2}h^2\nu^{-1})^{\frac16}$) with the last term equal to the first one.

Then the summation by $\rho_2 \ge \rho'$ results in the same expression (28.3.43) calculated for $\rho_2 = \rho'$; adding as usual $\rho' h^{-1}$ (asince $\rho'\beta h^{-2}$ estimates the contribution of zone $\{\rho_2 \le \rho'\}$) and minimizing by $\rho' \ge \bar\rho$, we get, after we multiply by βh^{-1} and add contributions of all other zones and also Tauberian estimate (28.3.40), the following expression:

$$h^{-2}\bigl(\nu^{\frac12} + (\mu+\nu h)|\log h|\bigr) + \beta h^{-1}\Bigl[h^{-\frac12}\nu^{\frac14}|\log h| + h^{-\frac35}\nu^{\frac{1}{10}} + h^{-\frac47}\nu^{\frac17} + h^{-\frac12}\nu^{\frac14}\Bigr]. \tag{28.3.44}$$

Recall that we derived estimate for the difference between $p_x^\alpha p_y^{\alpha'} e(.,.,0)|_{x=y=z}$ and $p_x^\alpha p_y^{\alpha'} e_z(.,.,0)|_{x=y=z}$ and thus for $\mu = 1$ we arrive to Statement (i) of Proposition 28.3.8 below for $\mu = 1$.

Observe, however, that in virtue of Subsection 28.3.1 the same estimate holds for $\beta \le h^{-\frac13}$. Then, if $1 \le \mu \le \beta$, one can scale $x \mapsto \mu x$, $h \mapsto \mu h$, $\nu \mapsto \mu^2\nu$, $\kappa \mapsto \mu\kappa$ and we arrive to the same statement without assumption $\mu = 1$.

Furthermore, in virtue of Propositions 28.A.4 and 28.A.5, expression $|p_x^\alpha p_y^{\alpha'} e_z(.,.,0)|_{x=y=z}|$ does not exceed $C\beta^{\frac12}h^{-2}\|\partial V\|_{\mathscr{L}^\infty}$ if $|\alpha| + |\alpha'| = 1$ [6]. Therefore we arrive to Statement (ii) below:

Proposition 28.3.8. *Let $\beta \le h^{-1}$ and* (28.3.14) *and* (28.3.15) *be fulfilled. Then*

(i) $|p_x^\alpha p_y^{\alpha'}\bigl[e(.,.,0) - e_z(.,.,0)\bigr]|_{x=y=z}|$ does not exceed expression (28.3.44) *for $|\alpha| \le 2$, $|\alpha'| \le 2$.*

(ii) Consider $|\alpha| + |\alpha'| = 1$; then $|p_x^\alpha p_y^{\alpha'} e(.,.,0)|_{x=y=z}|$ does not exceed expression (28.3.44) *plus $C\omega h^{-2}$ with*

$$\omega = \begin{cases} 1 & \text{if } \beta \le h^{-\frac13}, \\ \beta^{\frac32}h^{\frac12} & \text{if } h^{-\frac13} \le \beta \le h^{-\frac12}, \\ \beta^{\frac12} & \text{if } h^{-\frac12} \le \beta \le h^{-1}. \end{cases} \tag{28.3.45}$$

[6] Actually Proposition 28.A.4 provides better estimate for $\|\partial V\|_{\mathscr{L}^\infty} \le \beta^2 h$.

Remark 28.3.9. (i) Observe that for $\beta \le h^{-\frac12}$ we got no improvement over results of Subsection 28.3.1.

(ii) One can replace μ in the definition of $\bar\rho$ by $\nu^{\frac12}$. Indeed, we can assume that $\partial A'(z) = 0$. Then γ-vicinity of z we have $\mu = O(\nu\gamma)$ and scaling we should be concerned only abut this vicinity. We select $\gamma = \nu^{-\frac12}$.

Estimating $|\partial^2 A'|$

Recall that if A' is a minimizer, then it must satisfy (27.2.14) and then for $h^{-\frac12} \le \beta \le h^{-1}$ due to Proposition 28.3.8(ii) $\|\Delta A'\|_{\mathscr{L}^\infty}$ does not exceed $Ch^2\bigl((28.3.44)+\beta^{\frac12}h^{-2}\bigr)$ and then $\|\partial^2 A'\|_{\mathscr{L}^\infty}$ must not exceed this expression multiplied by $C|\log h|$ plus $\|\partial A'\|_{\mathscr{L}^\infty}$ [7]:

$$\begin{aligned}(28.3.46)\quad \|\partial^2 A'\|_{\mathscr{L}^\infty} &\le C\kappa|\log h|\bigl(\nu^{\frac12} + (\mu+\nu h)|\log h|\bigr)\\ &+ C\kappa\beta h|\log h|\bigl(h^{-\frac35}\nu^{\frac{1}{10}} + h^{-\frac47}\nu^{\frac17} + h^{-\frac12}\nu^{\frac14}|\log h|^2\bigr) + C\kappa|\log h|\beta^{\frac12}\|\partial V\|_{\mathscr{L}^\infty}\\ &+ C\|\partial A'\|_{\mathscr{L}^\infty}.\end{aligned}$$

Also recall that we can define $\nu := \max\bigl(\|\partial^2 A'\|_{\mathscr{L}^\infty}, 1\bigr)$; then we arrive to

Proposition 28.3.10. *Let $1 \le \beta \lesssim h^{-1}$ and* (28.3.14) *be fulfilled. Let A' be a minimizer satisfying* (28.3.15).

Then one of the following two cases holds: <u>either</u>

$$\begin{aligned}(28.3.47)\quad \|\partial^2 A'\|_{\mathscr{L}^\infty} &\le C\mu(\kappa|\log h|^2+1)\\ &+ C(\kappa\beta|\log h|)^{\frac{10}{9}}h^{\frac49} + C(\kappa\beta|\log h|)^{\frac76}h^{\frac12} + C(\kappa\beta|\log h|^3)^{\frac43}h^{\frac23}\\ &+ C\kappa|\log h|\omega + C\|\partial A'\|_{\mathscr{L}^\infty}\end{aligned}$$

with the right-hand expressions $\ge C$ <u>or</u>

$$\begin{aligned}(28.3.48)\quad \|\partial^2 A'\|_{\mathscr{L}^\infty} &\le C\mu(\kappa|\log h|^2+1)\\ &+ C\kappa\beta|\log h|h^{\frac25} + C\kappa\beta|\log h|h^{\frac37} + C\kappa\beta|\log h|^3h^{\frac12}\\ &+ C\kappa|\log h|\omega + C\|\partial A'\|_{\mathscr{L}^\infty}\end{aligned}$$

with the right-hand expression $\le C$. Recall that ω is defined by (28.3.45).

[7] Which can be replaced by a different norm, say, $\|\partial A'\|$.

Proof. Indeed, if $\nu \ge C$ we have

$$\nu \le C\kappa\mu|\log h| + C\big(\kappa\beta|\log h|h^{\frac{2}{5}}\big)^{\frac{10}{9}} + C\big(\kappa\beta|\log h|h^{\frac{3}{7}}\big)^{\frac{7}{6}} + C\big(\kappa\beta|\log h|^3h^{\frac{1}{2}}\big)^{\frac{4}{3}} + C\kappa\beta^{\frac{1}{2}}\|\partial V\|_{\mathscr{L}^\infty} + C\|\partial A'\|_{\mathscr{L}^\infty},$$

which leads to (28.3.47); if $\nu \asymp 1$ we have (28.3.48). □

Remark 28.3.11. (i) Observe that the right-hand expressions of (28.3.47) and (28.3.48) are either $\lesssim 1$ or $\gtrsim 1$ simultaneously.

(ii) The second term in the right-hand expression of (28.3.47) (i.e. with the power $\frac{10}{9}$) is always greater than the third and the fourth terms unless $\kappa\beta h \ge |\log h|^{-K}$). Because of this we just take power K of $|\log h|$ in this term and skip two other terms. One can find easily that $K = 4$ is sufficient;

(iii) The second term in the right-hand expression of (28.3.47) is less than the last one as $\beta \le h^{-\frac{8}{11}}(\kappa|\log h|)^{-\frac{20}{11}}$.

(iv) Obviously, in (28.3.48) we can take $\mu = 1$; however we are missing estimate of μ in (28.3.47). For sure, we know that $\mu \le C\nu$ but we will be able to do a better work after we estimate $\|\partial A'\|^2$ in Subsubsection *28.3.5.3. Weak Magnetic Field Approach.*.

28.3.4 Trace Term Asymptotics

General Microlocal Arguments

Now let us consider the trace term. We are not assuming anymore that A' is a minimizer but that it satisfies

$$(28.3.49)_{1,2} \qquad \|\partial A'\|_{\mathscr{L}^\infty} \le \mu, \qquad \|\partial^2 A'\|_{\mathscr{L}^\infty} \le \nu \qquad \text{with } 1 \le \mu \le \nu \le \epsilon.$$

We assume that $V \in \mathscr{C}^2$ uniformly. Later we will impose on V different non-degeneracy assumptions; from now on small constant $\epsilon > 0$ in conditions (28.3.15) and $(28.3.49)_{1,2}$ depends also on the constants in the non-degeneracy assumption.

Let us introduce the scaling function

$$\ell(x) \coloneqq \epsilon_0\big(\min_j |V - 2j\beta h| + |\partial V|^2\big)^{\frac{1}{2}} + \bar{\ell} \tag{28.3.50}$$

with

$$\bar{\ell} \coloneqq C_0 \max\bigl(\nu\beta^{-1},\, \mu\beta^{-1},\, h^{\frac{1}{2}}\bigr). \tag{28.3.51}$$

We need the following

Proposition 28.3.12. *Let $\beta h \lesssim 1$ and $(28.3.49)_{1,2}$ be fulfilled. Consider (γ,ρ)-element with respect to (x,p_3) with $\gamma\rho \ge h$, $\gamma \le \max(\ell,\rho)$ and*

$$\rho \ge \bar{\rho} \coloneqq C_0 \max\bigl(\mu\beta^{-1},\, h^{\frac{1}{2}}\bigr). \tag{28.3.52}$$

Then for

$$T_* \coloneqq h\rho^{-2} \le T \le T^* \coloneqq \epsilon_0 \min(1, \rho\ell^{-1}) \tag{28.3.53}$$

for (γ,ρ)-element $\{(x,\xi_3)\colon x \in B(z,\gamma), |\xi_3 - A_3(z)| \asymp \rho\}$ the following estimate

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\Gamma\bigl(p_x^{\alpha}p_y^{\alpha'}\varphi_1(\rho^{-1}p_{3x})\varphi_2(\rho^{-1}p_{3y})\psi_1(\gamma^{-1}x)\psi_2(\gamma^{-1}y)U\bigr)| \le CS(\rho,T)\gamma^3 \tag{28.3.54}$$

holds with

$$S(\rho,T) = \bigl(\beta h^{-1} + \rho^2 h^{-2}\bigr)\bigl(\rho^{-1} + \rho^{-3}h\nu^2\varepsilon^4 T^3\bigr), \tag{28.3.55}$$

where $\varepsilon = h\rho^{-1}$.

Observe that we redefined $\bar{\rho}$ possibly increasing it.

Proof. The proof is similar to one of Theorem 27.2.17 and is based on $h\rho^{-1}$-approximation. Note first that the propagation speed with respect to x_3 is $\asymp \rho$, the propagation speed with respect to p_3 is $O(\ell)$ and all other propagation speeds are bounded by $\bar{\rho}$. Therefore the shift with respect to x_3 is $\asymp \rho T \lesssim \ell$ as $T \le T^*$ and it is observable for $T \ge T_* = h|\log h|\rho^{-2}$ [8].

Let us apply the three-term approximation. Then since the first term does not includes any error, we can estimate it by

$$C\bigl(\beta h^{-1} + \rho^2 h^{-2}\bigr)\rho\gamma^3 h^{-1} T_* \asymp C\bigl(\beta h^{-1} + \rho^2 h^{-2}\bigr)\rho^{-1}\gamma^3,$$

[8] But in estimates we can skip the logarithmic factor using our standard scaling arguments.

which delivers the first term in $S(\rho, T)\ell^3$.

The second term is linear with respect to the perturbation $(A - A_\varepsilon)$ and asince we consider a shift by x_3 in the estimate of this term, we also can take $T = T_*$. Indeed, contributions of intervals $|t| \asymp T'$ with $T_* \le T' \le T^*$ to this term are negligible if we include a logarithmic factor in T_* [8]. Then this term does not exceed $C\bigl(\beta h^{-1} + \rho^2 h^{-2}\bigr)\rho\ell^3\nu\gamma^2 T_*^2 h^{-2}$ and it is less than the first term in $S(\rho, T)\gamma^3$.

Finally, the third term does not exceed the second term in $S(\rho, T)\gamma^3$.□

After estimate (28.3.54) has been proven we can estimate the contribution of the given element to the Tauberian error by $CS(\rho, T)\gamma^3\rho^2h^2T^{-2}$ [9] which is

$$C\bigl(\beta + \rho^2h^{-1}\bigr)\bigl(\rho T^{-2} + \rho^{-1}\nu^2hT\bigr)\gamma^3. \tag{28.3.56}$$

Consider an error appearing when we replace in the Tauberian expression T by T_*. The first two terms are negligible on intervals $|t| \asymp T'$ with $T_* \le T' \le T^*$ and the third term contributes here

$$C(\beta + \rho^2h^{-1})\rho^{-1}\nu^2hT'\gamma^3,$$

which sums to its value for $T' = T$. Therefore this error does not exceed (28.3.56) as well.

Minimizing expression (28.3.56) by $T \le T^*$ we get

$$\begin{aligned} &C\bigl(\beta + \rho^2h^{-1}\bigr)\bigl(\rho T^{*-2} + \rho^{-\frac13}\nu^{\frac43}h^{\frac23}\bigr)\gamma^3 \\ &\qquad\qquad \asymp C\bigl(\beta + \rho^2h^{-1}\bigr)\bigl(\rho + \rho^{-1}\ell^2 + \rho^{-\frac13}\nu^{\frac43}h^{\frac23}\bigr)\gamma^3, \end{aligned} \tag{28.3.57}$$

where we do not include the last term with $T = T_*$ since then the first term would be larger than $C\beta h^{-2}\rho^3\gamma^3$, which is the trivial estimate.

Now let us sum over the partition. Observe first that

(28.3.58) The contribution of the zone $\{\rho\colon \rho \ge (\beta h)^{\frac12}\}$ does not exceed

$$Q_0 := Ch^{-1} + Ch^{-\frac13}\nu^{\frac43} \tag{28.3.59}$$

since ρ here would be in the positive degree. Consider now the contribution of the zone $\{\rho\colon \rho \le (\beta h)^{\frac12}\}$.

[9] Factors ρ^2 and h^2T^{-2} (rather than hT^{-1}) appear because we consider the trace term.

Strong Non-Degenerate cCase

Here ρ is in the negative degree but we can help it under *strong non-degeneracy assumption*

$$\min_j |V - 2j\beta h| + |\partial V| \ge \epsilon_0 \qquad \text{in } B(0,1), \tag{28.3.60}$$

which later will be relaxed. Indeed, the relative measure of those x-balls with $\gamma = \rho^2$, where operator is non-elliptic is $\rho^2(\beta h)^{-1}$ as $\rho \ge h^{\frac{1}{3}}$. Then the total contribution of such elements does not exceed $\rho^2 h^{-1}\big(\rho^{-1}\ell^2 + \rho^{-\frac{1}{3}}\nu^{\frac{4}{3}}h^{\frac{2}{3}}\big)$ and the summation over ρ results in (28.3.59). Meanwhile, the total contribution of balls with $|\xi_3 - A'_3| \le \bar\rho = h^{\frac{1}{3}}$ does not exceed $C\bar\rho^5 h^{-3}$, which is smaller[10)].

However we have another restriction, namely, $\bar\rho \ge C_0\mu\beta^{-1}$ [11)]. Because of this we need to increase the remainder estimate by $C\bar\rho\beta^2 h^{-1} \times \bar\rho^2(\beta h)^{-1}$ i.e. by

$$Q' = \mu^3\beta^{-2}h^{-2}. \tag{28.3.61}$$

We should not be concerned about the zone $\{\rho\colon \bar\rho \ge \rho \ge (\beta h)^{\frac{1}{2}}\}$ since here we can always use $T \asymp \beta^{-1}$ and its contribution to the remainder will be the same $\mu^3\beta^{-2}h^{-2}$.

Now we need to pass from the Tauberian expression with $T = T_*$ to the magnetic Weyl expression and we need to consider only two first terms in the successive approximations. We can involve our standard methods of Section 18.9: note that $|x-y| \le c\rho T_* = C\varepsilon$ in the propagation and then we consider another unperturbed operator with $V = V(y)$ and $A'_j = A'(y) + \langle \nabla A_j(y), x-y\rangle$ frozen at point y (when we later set $x = y$). Then one can see that these terms modulo an error, not exceeding Q_0, are respectively

$$-h^{-3}\int P_{B_\varepsilon h}(V)\psi\, dx \tag{28.3.62)$_1$}$$

and

$$-h^{-3}\int \big(P_B(V) - P_{B_\varepsilon h}(V)\big)\psi\, dx \tag{28.3.62)$_2$}$$

[10)] These arguments work even if $\beta \le h^{-\frac{1}{3}}$ (and therefore $(\beta h)^{\frac{1}{2}} \le h^{\frac{1}{3}}$): we just set $\gamma = h\rho^{-1}$ for $(\beta h)^{\frac{1}{2}} \le \beta \le h^{\frac{1}{3}}$.

[11)] Here we can take $\mu = \|\partial A'\|_{\mathscr{L}^\infty}$ without resetting it to 1 if the former is smaller.

with $B_\varepsilon = |\nabla \times (A^0 + A'_\varepsilon)|$. Then we arrive to estimate (28.3.64) below.

Observe that non-degeneracy condition (28.3.60) was used only to estimate by $\rho^2(\beta h)^{-1}$ a relative measure of some set. However the same estimate would be achieved under slightly weaker non-degeneracy condition

$$\min_j |V - 2j\beta h| + |\partial V| + |\det(\operatorname{Hess} V)| \ge \epsilon_0 \qquad \text{in } B(0,1); \tag{28.3.63}$$

all arguments including transition to the magnetic Weyl expression work. Therefore under this assumption the same estimate holds and we arrive to

Proposition 28.3.13. *Let $\beta h \lesssim 1$ and conditions $(28.3.49)_{1,2}$ be fulfilled. Then under non-degeneracy assumptions (28.3.60) or (28.3.63) estimate*

$$|\operatorname{Tr}(H^-_{A,V}\psi) + h^{-3}\int P_{Bh}(V)\psi\,dx| \le CQ \tag{28.3.64}$$

holds with $Q = Q_0 + Q'$ with Q_0 and Q' defined by (28.3.59) and (28.3.61).

Remark 28.3.14. We will show that for a minimizer $Q \asymp Q_0$ in both cases (and even under even weaker non-degeneracy assumption (28.3.65)).

Non-Degenerate Case

Assume now that even weaker non-degeneracy condition is fulfilled:

$$\min_j |V - 2j\beta h| + |\partial V| + |\partial^2 V| \ge \epsilon_0 \qquad \text{in } B(0,1). \tag{28.3.65}$$

Then the measure of the degenerate set is $\rho(\beta h)^{-\frac{1}{2}}$ but even this is sufficient to obtain the same sum from the second term. In the first term we get however extra $C\beta|\log h|$ (which is $O(h^{-1})$ provided $\beta \le (h|\log h|)^{-1}$) but we can help with this too: indeed, if we fix $\ell \ge 2\bar\ell$, then the relative measure does not exceed $\rho^2(\beta h\ell)$ and summation results in $O(h^{-1})$. We still need to consider set $\{x\colon \ell(x) \le 2\bar\ell\}$, but its contribution is obviously less than $C\beta\bar\ell|\log h|$ which in turn is $O(h^{-1} + \nu|\log h|)$ (and this is $O(h^{-1})$ for a minimizer).

However contribution of the degenerate set becomes $C\beta h^{-2}\bar\rho^3\bar\ell$ which boils down to the same expression Q'. Then we arrive to

Proposition 28.3.15. *Let $\beta h \lesssim 1$ and conditions $(28.3.49)_{1,2}$ be fulfilled. Then under non-degeneracy assumption (28.3.65) estimate (28.3.63) holds with $Q = Q_0 + Q''$, $Q'' = Q' + \nu|\log h|$ with Q_0 and Q' defined by (28.3.59) and (28.3.61) respectively.*

We leave easy details to the reader.

Degenerate case

Let us derive a remainder estimate without any non-degeneracy assumptions. In comparison with the non-degenerate case we need to sum $\beta\rho^{-\frac{1}{3}}\nu^{\frac{4}{3}}h^{\frac{2}{3}}$ and we sum it over $\rho \ge \rho_*$, resulting in the same expression with ρ replaced by ρ_*; adding the contribution of the degenerate zone, equal to $\beta\rho_*^3$, we get $\beta\rho_*^{-\frac{1}{3}}\nu^{\frac{4}{3}}h^{\frac{2}{3}} + \beta h^{-2}\rho_*^3$, which should be minimized by $\rho_* \ge \bar{\rho}$, resulting in $\beta\nu^{\frac{6}{5}}h^{\frac{2}{5}} + \beta\bar{\rho}^3$ i.e.

$$Q''' = \beta\nu^{\frac{6}{5}}h^{\frac{2}{5}} + \beta h^{-\frac{1}{2}} + \mu^3\beta^{-2}h^{-2} + \nu|\log h|. \tag{28.3.66}$$

Thus we arrive to

Proposition 28.3.16. *Let $\beta h \lesssim 1$ and conditions* (28.3.49)$_{1,2}$ *be fulfilled. Then estimate* (28.3.63) *holds with Q replaced by $Q = Q_0 + Q'''$ with Q_0 and Q''' are defined by* (28.3.59) *and* (28.3.66) *respectively.*

Remark 28.3.17. We are going to apply our results to $V = W_B^{\mathsf{TF}} + \lambda$ with chemical potential λ. We know that

(i) For $M = 1$ (single nucleus case) after rescalings the non-degeneracy condition (28.3.60) is fulfilled everywhere including the *boundary zone* $\{x\colon \epsilon_0\bar{r} \le r(x) \le C_0\bar{r}\}$.

(ii) For $M \ge 2$ (multiple nuclei case) after rescalings the non-degeneracy condition (28.3.60) is fulfilled if $r(x) \le \epsilon d$ where d is the minimal distance between nuclei.

(iii) Further, for $M \ge 2$ and $B \le Z^{\frac{4}{3}}$ after rescalings the non-degeneracy condition (28.3.65) is fulfilled in the zone $\{x\colon Z^{-\frac{1}{3}} \le r(x) \le \epsilon_0\bar{r}\}$ where $r(x)$ is the distance to the closest nuclei and $\bar{r} = \min\bigl(B^{-\frac{1}{4}}, (Z-N)_+^{-\frac{1}{3}}\bigr)$.

(iv) On the other hand, the non-degeneracy condition (28.3.63) is uncalled: while we believe that that this condition is often fulfilled while (28.3.60) fails we have no proof of this.

(v) For $M \ge 2$ in the boundary zone a more delicate scaling needs to be applied to improve remainder estimate which is possible not only because W_B^{TF} is more regular than just $\mathscr{C}^2$ but also has some special properties.

28.3.5 Endgame

Until now in this Section we assumed only that A' satisfies equation to the minimizer locally (and assumptions (28.3.14)–(28.3.15)) but now we assume that A' is a minimizer.

Upper Estimate

Let us first derive an upper estimate for E^*_κ and for this we need to pick-up some A'. First of all, we try $A' = 0$ resulting in

$$\mathsf{E}^*_\kappa \le \mathcal{E}^*_0 + Ch^{-1} \le \mathcal{E}^*_\kappa + Ch^{-1} + C\kappa\beta^2$$

which is a good estimate for $\kappa\beta^2 \lesssim h^{-1}$:

$$\mathsf{E}^*_\kappa \le \mathcal{E}^*_\kappa + Ch^{-1}. \tag{28.3.67}$$

However for $\kappa\beta^2 \gtrsim h^{-1}$ we need to be more subtle. Namely, we pick up a mollified minimizer for the modified functional $\bar{\mathcal{E}}_\kappa(A')$, defined by (28.2.8). More precisely, let A' be the minimizer for $\bar{\mathcal{E}}_\kappa(A')$; then $\mathcal{E}_\kappa(A') = \mathcal{E}^*_\kappa + O(h^{-1})$.

Still it is not a good choice since our approach relies upon $\mathscr{C}^2$-smoothness but A' is only $\mathscr{C}^{\frac{3}{2}}$-smooth.

Proposition 28.3.18. *Let $\beta h \lesssim 1$ and $\kappa\beta^2 h \gtrsim 1$; let A' be a minimizer for the modified functional $\bar{\mathcal{E}}_\kappa(A')$ and let A'_ε be its ε-mollification. Then there exists $\varepsilon > 0$ such that*

$$|\partial A'_\varepsilon| \le C\mu = \kappa\beta h, \quad |\partial^2 A'_\varepsilon| \le C\nu = C\bigl(1 + (\kappa\beta)^{\frac{4}{3}} h^{\frac{2}{3}}\bigr)|\log h| \tag{28.3.68}_{1,2}$$

and

$$\mathcal{E}_\kappa(A'_\varepsilon) = \mathcal{E}^*_\kappa + O(h^{-1}). \tag{28.3.69}$$

Proof. From equation to A' we observe that

$$|\partial(A' - A'_\varepsilon)| \le C\kappa(\varepsilon + \beta h\varepsilon^{\frac{1}{2}}), \quad |\partial^2 A'_\varepsilon| \le C\kappa(1 + \beta h\varepsilon^{-\frac{1}{2}})|\log h| \tag{28.3.70}$$

and

$$|\mathcal{E}_\kappa(A') - \mathcal{E}_\kappa(A'_\varepsilon)| \le C(\varepsilon^2 + \beta h\varepsilon^{\frac{3}{2}})\kappa h^{-3} + C\kappa(\varepsilon + \beta h\varepsilon^{\frac{1}{2}})^2 h^{-2} \tag{28.3.71}$$

because linear with respect to $\partial(A' - A'_\varepsilon)$ terms disappear due to equation to a minimizer. Then for $\varepsilon \ge h$ the right-hand expression of (28.3.71)

is $O(h^{-1}+\kappa\beta h^{-2}\varepsilon^{\frac{3}{2}})$ and we take $\varepsilon = \min\bigl(1, (\kappa\beta)^{-\frac{2}{3}}h^{\frac{2}{3}}\bigr)$ [12], resulting in $(28.3.68)_{1,2}$ and (28.3.69) since $|\partial A'| \le C\kappa\beta h$. □

Then applying Propositions 28.3.12, 28.3.15 and 28.3.16 to operator $H_{A_\varepsilon,V}$ with $A_\varepsilon = A^0 + A'_\varepsilon$ we arrive to

Proposition 28.3.19. *(i) For $\beta h \lesssim 1$ and $\kappa\beta^2 h \lesssim 1$ estimate* (28.3.67) *holds.*

(ii) For $\beta h \lesssim 1$ and $\kappa\beta^2 h \gtrsim 1$ estimate $\mathsf{E}^*_\kappa \le \mathcal{E}^*_\kappa + CQ$ holds with $Q = Q_0$ under non-degeneracy assumption (28.3.65) *and with $Q = Q_0 + Q'''$ in the general case, calculated with $\nu = \bigl(1+(\kappa\beta)^{\frac{4}{3}}h^{\frac{2}{3}}\bigr)|\log h|$, $\bar\rho = h^{\frac{1}{2}}$, and $\bar\ell = \max(h^{\frac{1}{2}}, \beta^{-1}\nu)$.*

Indeed, for A' selected above $\mu \le \kappa\beta h$ and one can check easily that $Q' \le Q_0$. Since ν here is lesser than one derived for a minimizer of $\mathsf{E}^*_\kappa(A')$, we are happy and skip calculation of Q'''.

Lower Estimate

Estimate (28.3.63) implies that

$$\underbrace{\mathsf{Tr}(H^-_{A,V}\psi) + \kappa^{-1}h^2\|\partial A'\|^2}_{=\mathsf{E}_\kappa(A')} \ge \underbrace{-h^{-3}\int P_{Bh}(V)\psi\,dx + \kappa^{-1}h^{-2}\|\partial A'\|^2}_{=\mathcal{E}_\kappa(A')} - CQ$$

and therefore

$$\mathsf{E}^*_\kappa \ge \mathcal{E}^*_\kappa - CQ \qquad \text{with } \mathsf{E}^*_\kappa = \inf_{A'}\mathsf{E}_\kappa(A'),\quad \mathcal{E}^*_\kappa = \inf_{A'}\mathcal{E}_\kappa(A'), \tag{28.3.72}$$

where $A' = A'_\kappa$ is a minimizer of $\mathsf{E}_\kappa(A')$ and Q is defined in Propositions 28.3.12, 28.3.15 and 28.3.16 and ν is a right-hand expression of (28.3.47) or 1 whatever is larger. For a sake of simplicity we replace it by a marginally larger expression

$$\nu = (\mu+1)(\kappa|\log h|^2+1) + (\kappa\beta)^{\frac{10}{9}}h^{\frac{4}{9}}|\log h|^K + \kappa\beta^{\frac{1}{2}}|\log h|; \tag{28.3.47}^*$$

recall that in "old" (28.3.47) $\mu = \|\partial A'\|_{\mathcal{L}^\infty}$ or $\mu = 1$, whatever is larger, so we modified the first term here accordingly.

[12] Then $\varepsilon \gtrsim h$ due to $\kappa\beta^2 h \gtrsim 1$.

Our problem is that so far we know neither $\|\partial A'\|_{\mathcal{L}^\infty}$ nor $\|\partial A'\|$. Observe however that

$$\begin{aligned}\mathsf{E}^*_\kappa = \mathsf{E}_\kappa(A') \ge \mathsf{E}_{2\kappa}(A') + (2\kappa)^{-1}\|\partial A'\|^2 \ge \mathcal{E}^*_{2\kappa} + (2\kappa)^{-1}\|\partial A'\|^2 - CQ\\ \ge \mathcal{E}^*_\kappa(A') - C\kappa\beta^2(2\kappa)^{-1}\|\partial A'\|^2 - CQ\end{aligned}$$

and therefore combining this with an upper estimate we arrive to estimate

$$\|\partial A'\| \le C(\kappa h^2 Q)^{\frac{1}{2}} + C\kappa\beta h, \tag{28.3.73}$$

and we have also

$$\|\partial A'\|_{\mathcal{L}^\infty} \le C\|\partial A'\|^{\frac{2}{5}} \cdot \|\partial^2 A'\|^{\frac{3}{5}}_{\mathcal{L}^\infty}. \tag{28.3.74}$$

We are going to explore what happens if

$$\nu \asymp \mu(\kappa|\log h|^2 + 1), \tag{28.3.75}$$

where the right-hand expression is the sum of all terms in (28.3.47)* containing μ.

Remark 28.3.20. (i) Observe first that remainder estimate $Q = O(\beta^2 h^{-1})$ is guaranteed and therefore $\|\partial A'\| \le C\beta h^{\frac{1}{2}}$. Then $\mu \le C\beta^{\frac{2}{5}} h^{\frac{1}{5}} \nu^{\frac{3}{5}}$ due to (28.3.74).

(ii) Further, if (28.3.75) is fulfilled, then $\mu \le \beta h^{\frac{1}{2}}|\log h|^K$ and the same estimate holds for ν and then $Q_0 \asymp h^{-1}$, $Q''' \le C\beta h^{-\frac{1}{2}}|\log h|^K$; then the rough estimate to Q is improved, and then $\mu \ll \beta h^{\frac{1}{2}}$, $\nu \ll \beta h^{\frac{1}{2}}$ and, finally, $Q \asymp h^{-1}$ under assumption (28.3.65) and $Q \asymp h^{-1} + \beta h^{-\frac{1}{2}}$ in the general case, and we also have nice estimates to μ, ν.

Therefore we can assume that (28.3.75) is not fulfilled, but then ν is defined by the remaining terms and then

$$\nu = \kappa|\log h| + \kappa \min(\beta^{\frac{3}{2}} h^{-\frac{1}{2}},\, \beta^{\frac{1}{2}})|\log h| + (\kappa\beta)^{\frac{10}{9}} h^{\frac{4}{9}}|\log h|^K \tag{28.3.76}$$

(and from now we do not reset to 1 if this expression is smaller).

We *almost* proved the following estimates for Q:

(28.3.77) $Q = Q_0$ under non-degeneracy assumption (28.3.65) and $Q = Q_0 + \beta h^{-\frac{1}{2}} \asymp h^{-1} + \beta h^{-\frac{1}{2}}$ in the general case.

However we need still explore what happens if $Q \asymp \mu^3\beta^{-2}h^{-2}$. In this case $\mu \le (\mu^3\beta^{-2})^{\frac{1}{5}}\nu^{\frac{3}{5}}$ and then $\mu^{\frac{2}{5}} \le \beta^{-\frac{2}{5}}\nu^{\frac{3}{5}}$, and using (28.3.76) one can prove easily (28.3.77) unless $\beta \le h^{-\frac{1}{5}}|\log h|^K$ in which case $\nu = (\kappa|\log h| + 1)$.

Weak Magnetic Field Approach.

Consider now case $\beta \lesssim h^{-\frac{1}{3}}$. Recall that then $\nu \le C\kappa|\log h|$ and $\mu \ll 1$. Then as we study propagation with respect to p_3 we do not need to correct it to p_3' and then we do not need $\bar{\rho}$.

Then we can apply *weak magnetic field approach* (see Section 13.4). Now contribution of a partition element with $\rho \ge C_0\beta^{-1}$ does not exceed $\rho^3 h^{-1} \times \rho^{-2}$ as we use $T \le \epsilon_0\rho$ and the total contribution of such elements does not exceed Ch^{-1}; meanwhile the total contribution of elements with $\rho = C_0\beta^{-1}$ does not exceed $C\beta^2 h^{-1} \times \beta^{-3} \le Ch^{-1}$ as we use $T = \epsilon_0\beta^{-1}$.

Main Theorem

Therefore after we plug ν into Q_0 we have proven our estimate from below and also the main theorem of this Section:

Theorem 28.3.21. *Let $\beta h \lesssim 1$ and $\kappa \le \kappa^*$. Then*

(i) Estimate

$$|\mathsf{E}^*_\kappa - \mathcal{E}^*_\kappa| \le CQ \tag{28.3.78}$$

holds where under non-degeneracy assumption (28.3.65)

$$Q := h^{-1} + \kappa^{\frac{40}{27}}\beta^{\frac{40}{27}}h^{\frac{7}{27}}|\log h|^K, \tag{28.3.79}$$

and in the general case

$$Q := h^{-1} + \beta h^{-\frac{1}{2}}. \tag{28.3.80}$$

(ii) For a minimizer the following estimate holds: $\|\partial^2 A'\|_{\mathscr{L}^\infty} \le C\nu$ with ν, defined by (28.3.76).

We leave as an easy exercise to the reader

Problem 28.3.22. (i) Starting from estimate $\|\partial^2 A'\|_{\mathscr{L}^\infty} \le C\nu$ derive from (28.3.73)–(28.3.74) estimate for $\|\partial A'\|_{\mathscr{L}^\infty}$; consider separately three cases: $1 \le \beta \le h^{-\frac{1}{3}}$, $h^{-\frac{1}{3}} \le \beta \le h^{-\frac{1}{2}}$ and $h^{-\frac{1}{2}} \le \beta \le 1$.

(ii) Prove that for $\kappa\beta^2 h \gtrsim 1$

$$\|\partial(A' - A'')\| \le C(\kappa Q h^2)^{\frac{1}{2}}, \tag{28.3.81}$$

where A' and A'' are minimizers for E_κ and $\bar{\mathcal{E}}_\kappa$ respectively; in particular, observe that $\|\partial(A' - A'')\| \ll \|\partial A'\|$ for $\kappa\beta^2 h \gg 1$;

(iii) Since (28.3.81) holds for A'' replaced by A''_ε as well and since we have estimate $\|\partial^2 A''_\varepsilon\|_{\mathscr{L}^\infty} \le C\nu$, derive from (28.3.81) estimate for $\|\partial(A' - A''_\varepsilon)\|_{\mathscr{L}^\infty}$ and then the same estimate for $\|\partial A'\|_{\mathscr{L}^\infty}$.

Remark 28.3.23. In the following observations we use simpler but less sharp upper estimates to critical β:

(i) Under non-degeneracy assumption (28.3.65) we conclude that $Q \le Ch^{-1}$ and $\nu \le C\beta^{\frac{1}{2}}|\log h|$ for $\beta \le h^{-\frac{2}{3}}$ and therefore $\mu \le 1$ for $\beta \le h^{-\frac{2}{3}}|\log h|^{-K}$.

(ii) In the general case we see that $Q \le Ch^{-1} + C\beta h^{-\frac{1}{2}}$ and $\nu \le C\beta^{\frac{1}{2}}|\log h|$ for $\beta \le h^{-\frac{2}{3}}$ and therefore $\mu \le 1$ fors $\beta \le h^{-\frac{3}{5}}|\log h|^{-K}$; we used here estimate $\mu \le C(\kappa h^2 Q)^{\frac{1}{5}}\nu^{\frac{3}{5}}$.

28.3.6 N-Term Asymptotics

Introduction

In the application to the ground state energy one needs to consider also N-term asymptotics and D-term estimates. Let us start from the former: we consider N-term

$$\int e(x,x,0)\psi(x)\,dx. \tag{28.3.82}$$

Again we consider this asymptotics in the more broad content of assumptions $V \in \mathscr{C}^2$ and $(28.3.49)_{1,2}$ and we will follow arguments employed for a trace term using the same notations. Then the Tauberian error does not exceed $CS(\rho, T)\gamma^3 h T^{-1}$ [13], which is

$$C(\beta + \rho^2 h^{-1})h^{-1}\big(\rho^{-1}T^{-1} + \rho^{-3}\nu^2 h T^{-2}\big)\gamma^3. \tag{28.3.56}'$$

[13] With $S(\rho, T)$ defined by (28.3.28).

Minimizing by $T \le T^*$ we get

$$C(\beta+\rho^2 h^{-1})h^{-1}\bigl(\rho^{-1}+\rho^{-2}\ell+\rho^{-\frac{5}{3}}\nu^{\frac{2}{3}}h^{\frac{1}{3}}\bigr)\gamma^3. \tag{28.3.57}'$$

Then summation over zone $\{\rho\colon \rho^2 \ge \beta h\}$ results in

$$R_0 = C\bigl(h^{-2}+h^{-\frac{5}{3}}\nu^{\frac{2}{3}}\bigr) \tag{28.3.59}'$$

and we need to consider only contributions of elements belonging to the zone $\{\rho\colon \rho^2 \le \beta h\}$

$$\beta h^{-1}\bigl(\rho^{-1}+\rho^{-2}\ell+\rho^{-\frac{5}{3}}\nu^{\frac{2}{3}}h^{\frac{1}{3}}\bigr)\gamma^3. \tag{28.3.83}$$

Strong Non-Degenerate Case

Under non-degeneracy conditions (28.3.60) or (28.3.63) expression (28.3.83) should be multiplied by $\rho^2(\beta h)^{-1}\gamma^{-3}$ and after summation by ρ we get an extra term $Ch^{-2}|\log h|$.

However we can get rid of the logarithmic factor by the standard trick: in one direction time could be improved to $\rho^{1-\delta}\ell^{\delta}$. We leave easy details to the reader.

Further, adding $R' = C\beta h^{-2}\bar\rho\times\bar\rho^2(\beta h)^{-1}$ with $\bar\rho = C_0\max(\mu\beta^{-1},\, h^{\frac{1}{2}})$ [11] i.e.

$$R' = \mu^3\beta^{-3}h^{-3}, \tag{28.3.61}'$$

which is a contribution of the zone $\{\rho\colon \rho\le\bar\rho\}$, we arrive to

Proposition 28.3.24. *Let $\beta h \lesssim 1$ and conditions $(28.3.49)_{1,2}$ be fulfilled. Then under non-degeneracy assumption* (28.3.60) *or* (28.3.63) *estimate*

$$\Bigl|\int \bigl(\operatorname{tr} e(x,x,0) - h^{-3}\int P'_{Bh}(V)\bigr)\psi(x)\,dx\Bigr| \le CR \tag{28.3.84}$$

holds with $R = R_0 + R'$, R_0, R' defined by $(28.3.59)'$ *and* $(28.3.61)'$.

Remark 28.3.25. The above estimate is sufficiently good since the weak magnetic field approach brings remainder estimate $C\mu h^{-2}$ even without any non-degeneracy assumption. We leave easy details to the reader.

Non-Degenerate Case

Under non-degeneracy assumption (28.3.65) we need to apply more subtle arguments than before. Consider first subelements[14] with $\rho \ge \ell$; for them we need to multiply expression (28.3.83) by $\rho\gamma^{-3}$ and sum by $\rho \ge \rho_*$, resulting in $C\beta h^{-1}|\log h| + C\beta h^{-\frac{2}{3}}\nu^{\frac{2}{3}}\rho_*^{-\frac{2}{3}}$.

On the other hand, for subelements with $\rho \le \ell$ we need to multiply (28.3.83) by $\rho^2(\beta h)^{-1}\gamma^{-3}$ and sum by $\rho_* \le \rho \le \ell$ and then by $\ell \ge \rho_*$, resulting in the same expression albeit with a factor $|\log h|^2$ instead of $|\log h|$. Here we get also $Ch^{-2}|\log h|$ term but we deal with it exactly as in the previous Subsubsection *28.3.6.2 Strong Non-Degenerate Case*.

We need also to add contributions of subelements with $\rho_* \le \rho \le \ell \le \bar{\ell}$ and with $\rho \le \rho_*$. For the former subelements we need to consider only term $\beta h^{-1}\rho^{-2}\bar{\ell}\gamma^3$ in (28.3.83) and only if $\bar{\ell} = \nu\beta^{-1}$, resulting in $\beta h^{-1}\rho_*^{-1}\bar{\ell}\gamma^3$ and contribution of the latter subelements we estimate by $\beta h^{-2}\rho_*^2$. So we get

$$C\beta\big(h^{-1}|\log h| + h^{-1}\rho_*^{-1}\bar{\ell} + h^{-\frac{2}{3}}\nu^{\frac{2}{3}}\rho_*^{-\frac{2}{3}} + h^{-2}\rho_*^2\big), \tag{28.3.85}$$

which should be minimized by $\rho_* \ge \bar{\rho}$, resulting in

$$C\beta\big(h^{-1}|\log h|^2 + h^{-\frac{4}{3}}\bar{\ell}^{\frac{2}{3}} + h^{-1}\nu^{\frac{1}{2}} + h^{-2}\bar{\rho}^2\big).$$

Then plugging $\bar{\rho}$ and $\bar{\ell} = \nu\beta^{-1}$ we arrive to

$$R'' = C\beta\big(h^{-1}|\log h|^2 + h^{-\frac{4}{3}}\nu^{\frac{2}{3}}\beta^{-\frac{2}{3}} + h^{-1}\nu^{\frac{1}{2}}\big) + C\mu^2\beta^{-1}h^{-2} \tag{28.3.86}$$

thus proving Proposition 28.3.26(i) below.

Degenerate Case

In the general case we arrive to (28.3.85) albeit with factor ρ_*^{-1}

$$C\beta\big(h^{-1}\rho_*^{-1} + h^{-1}\rho_*^{-2}\bar{\ell} + h^{-\frac{2}{3}}\nu^{\frac{2}{3}}\rho_*^{-\frac{5}{3}} + h^{-2}\rho_*\big), \tag{28.3.87}$$

which should be minimized by $\rho_* \ge \bar{\rho}$, resulting in

$$C\beta\big(h^{-\frac{3}{2}} + h^{-\frac{5}{3}}\bar{\ell}^{\frac{1}{3}} + h^{-\frac{4}{3}}\nu^{\frac{1}{4}} + h^{-2}\bar{\rho}\big).$$

[14] We call them "subelements" but they live in the phase spaces in contrast to elements which live in the coordinate spaces.

Then plugging $\bar{\rho}$ and $\bar{\ell}$ we arrive to

$$R''' = C\beta\big(h^{-\frac{3}{2}} + h^{-\frac{5}{3}}\nu^{\frac{1}{3}}\beta^{-\frac{1}{3}} + h^{-\frac{4}{3}}\nu^{\frac{1}{4}}\big) + C\mu h^{-2} \tag{28.3.88}$$

thus proving Proposition 28.3.26(ii):

Proposition 28.3.26. *Let $\beta h \lesssim 1$ and conditions $(28.3.49)_{1,2}$ be fulfilled. Then*

(i) Under non-degeneracy assumption (28.3.65) *estimate* (28.3.63) *holds with $R = R_0 + R''$, R_0, R'' defined by $(28.3.59)'$ and* (28.3.86).

(ii) In the general case estimate (28.3.63) *holds with $R = R_0 + R'''$, R_0, R''' defined by $(28.3.59)'$ and* (28.3.88).

28.3.7 D-Term Estimate

Consider now D-term

$$\mathsf{D}\big([e(x,x,0) - h^{-3}P'_{Bh}(V)]\psi,\, [e(x,x,0) - h^{-3}P'_{Bh}(V)]\psi\big) \tag{28.3.89}$$

with $\psi \in \mathscr{C}_0^\infty((B(0,1))$.

Proposition 28.3.27. *Let $\beta h \lesssim 1$ and A' satisfy $(28.3.49)_{1,2}$. Then under non-degeneracy assumption* (28.3.60) *D-term* (28.3.89) *does not exceed CR^2 with $R = R_0 + R'$, R_0 and R' defined by $(28.3.59)'$ and $(28.3.61)'$.*

Proof. *Step 1.* Let us apply Fefferman-de Llave decomposition (16.4.1); then we need to consider pairs of elements $B(\bar{x}, r)$ and $B(\bar{y}, r)$ such that $3r \le |\bar{x} - \bar{y}| \le 4r$. If $r \ge \bar{\rho}$, then on each of these elements we should consider (γ, ρ) subelements (we call them "subelements" but they live in the phase spaces in contrast to the elements which live in the coordinate spaces). Then we have three parameters, namely (r, ρ_x, ρ_y).

Observe that for each $\bar{y}$ the number of matching x-elements is $\asymp 1$ and that the summation with respect to $\rho_x \ge \rho^* = (\beta h)^{\frac{1}{2}}$ results in $R_0 r^2$ multiplied by a contribution of (γ, ρ_y)-subelement; after summation by r and then by these (γ, ρ_y)-subelements we get CR_0R. Similarly we are dealing with $\rho_y \ge \rho^*$.

Therefore we need to consider only the case when both ρ_x and ρ_y do not exceed ρ^*. If $r \ge c\rho^*$, then "the relative measure trick" allows us to add factors $\rho_x^2(\beta h)^{-1}$ and $\rho_y^2(\beta h)^{-1}$ even if $\rho_x^2 \ge r$ or $\rho_y^2 \ge r$ and then the total contribution of such subelements also does not exceed CR^2.

Step 2. Consider next $h \le r \le \rho^*$ [15)] and we should look only at $\rho_x \le \rho^*$, $\rho_y \le \rho^*$. Further, if $\rho_x^2 \ge \epsilon r$ or $\rho_y^2 \ge \epsilon r$ we can always inject factor $c\rho_x^2 r^{-1}$ or $c\rho_y^2 r^{-1}$ ending up again with CR^2.

On the other hand, if both $\rho_x^2 \le \epsilon r$ and $\rho_y^2 \le \epsilon r$ but $r \ge c\bar{\rho}$ we can apply the same "relative measure trick" but comparing the measure of ρ_x^2- or ρ_y^2-elements with violated ellipticity assumption to the total measure of $B(z,r)$; then we can inject factors $(\rho_x^2/r)^\theta$ and $(\rho_y^2/r)^\theta$ with arbitrary $0 \le \theta \le 1$ and we select any $\theta : \frac{5}{6} < \theta < 1$ to have positive powers of ρ_x and ρ_y and power of r (counting r^{-1}) greater than -3. We end up again with CR^2.

Observe that these arguments cover also cases $\rho_x \le \bar{\rho}$ or $\rho_y \le \bar{\rho}$.

Step 3. To estimate the contribution of zone $\{(x,y)\colon r \le h\}$ we just estimate $|e(x,x,\tau)| \le Ch^{-3}$. □

For $M \ge 2$ we will need to estimate D-term under non-degeneracy assumptions (28.3.63), or (28.3.65), or without any non-degeneracy assumption.

Proposition 28.3.28. *Let $\beta h \lesssim 1$ and A' satisfy* $(28.3.49)_{1,2}$. *Then* D*-term* (28.3.89) *does not exceed CR^2 where*

(i) Under non-degeneracy assumption (28.3.63) $R = R_0 + R'$, R_0 *and* R' *defined by* $(28.3.59)'$ *and* $(28.3.61)'$.

(ii) Under non-degeneracy assumption (28.3.65) $R = R_0 + R''$, R_0 *and* R'' *defined by* $(28.3.59)'$ *and* (28.3.86).

(iii) In the general case $R = R_0 + R'''$, R_0 and R''' defined by $(28.3.59)'$ *and* (28.3.88).

Proof. Let us use ideas used in the proofs of Proposition 28.3.26, 28.3.24 and 28.3.26. Let us apply Fefferman-de Llave decomposition (16.4.1); then we need to consider pairs of elements $B(\bar{x},r)$ and $B(\bar{y},r)$ with $3r \le |\bar{x}-\bar{y}| \le 4r$. If $r \ge \bar{\rho}$ on each of these elements we should consider (γ,ρ) subelements[14)]. Then again we have three parameters, namely (r,ρ_x,ρ_y). On the other hand, there is a scaling function $\ell(x)$ and covering of $B(0,1)$ by ℓ-elements.

[15)] Observe that we do not need to keep $t \ge \bar{\rho}$ but we need to keep $\rho r \ge h$.

Part 1. Consider case of $\ell_x \lesssim r$ (and therefore $\ell_y \lesssim r$). Then we must assume that $\rho_x\ell_x \gtrsim h$, $\rho_y\ell_y \gtrsim h$. Observe as in *Step 1* of the previous proof that if $\rho_x \gtrsim \rho^* = (\beta h)^{\frac{1}{2}}$, then the relative density of such subelements is $\rho_x^2/(\beta h)$ and therefore the summation over such subelements of the given x-element results in $CR_0\ell_x^3$. Therefore the double summation over corresponding subelements of x- and y-elements results in $CR_0^2\ell_x^3\ell_y^3 r^{-1}$. Finally, after the double summation over x- and y-elements we get $CR_0^2\int |x-y|^{-1}\,dxdy$, which does not exceed CR_0^2.

Therefore in what follows we need to consider only subelements with $\rho_x \lesssim \rho^*$, $\rho_y \lesssim \rho^*$ [16]. Further, observe that the same arguments are applicable if $\ell_x \gtrsim \rho^*$, $\ell_y \gtrsim \rho^*$ and we are left with the pairs of elements with $\ell_x \lesssim \rho^*$, $\ell_y \lesssim \rho^*$ and their subelements with $\rho_x \lesssim \rho^*$, $\rho_y \lesssim \rho^*$ since we will always keep $\ell_x \ge \bar{\rho}$, $\ell_y \ge \bar{\rho}$.

Observe that the summation of (28.3.83) over subelements with $\rho \ge \ell$ of the given element results in

$$C\beta h^{-1}\bigl(\ell^{-1} + \ell^{-\frac{5}{3}}\nu^{\frac{2}{3}}h^{\frac{1}{3}}\bigr)\ell^3. \tag{28.3.90}$$

On the other hand, for $\rho \le \ell$ the relative density of ρ-subelements of the given ℓ-element does not exceed $C\rho^2\ell^{-2}$ and therefore summation over such subelements results in (28.3.90) again.

However in (28.3.83) if $\ell \le \bar{\ell}$ we need to take in the middle term $\ell = \bar{\ell}$ and here we can ignore other options but $\bar{\ell} = \nu\beta^{-1}$.

Then the summation of this term over subelements with $\rho \ge \max(\ell, \rho_*)$ results in $C\beta h^{-1}\min(\rho_*^{-2}, \ell^{-2})\bar{\ell}\ell^3$ and the summation over subelements with $\rho_* \le \rho \le \ell$ results in $C\beta h^{-1}\rho_*^{-2}\bar{\ell}(1 + |\log \rho_*\ell^{-1}|)\ell^3$ (and should be counted as $\ell \ge \rho_*$ only.

Finally, the contribution of subelements with $\rho \le \rho_*$ does not exceed $C\beta h^{-2}\rho_*^3\ell$ and $C\beta h^{-2}\rho_*\ell^3$ if $\rho_* \le \ell$ and $\rho_* \ge \ell$ respectively. So, in the former case the total contribution of all subelements does not exceed

$$C\beta h^{-1}\Bigl[\ell^{-1} + \ell^{-\frac{5}{3}}\nu^{\frac{2}{3}}h^{\frac{1}{3}} + \rho_*^{-2}\bar{\ell}(1 + |\log \rho_*\ell^{-1}|) + h^{-1}\rho_*^3\ell^{-2}\Bigr]\ell^3. \tag{28.3.91}$$

[16] Due to positivity quadratic form $\mathsf{D}(.,.)$ we need to consider only "pure" pairs. We will use this observation many times.

Minimizing this expression by ρ_* we get

(28.3.92)
$$C\beta h^{-1}\Big[\ell^{-1}+\nu^{\frac{2}{3}}h^{\frac{1}{3}}\ell^{-\frac{5}{3}}+\bar{\ell}^{\frac{3}{5}}h^{-\frac{2}{5}}(1+|\log \ell_*\ell^{-1}|)^{\frac{3}{5}}\ell^{-\frac{4}{5}}+h^{-1}\bar{\rho}^3\ell^{-2}\Big]\ell^3,$$

achieved for

$$\rho_*=\rho_*(\ell)\asymp \max((\bar{\ell}h\ell^2|\log h|)^{\frac{1}{5}},\bar{\rho}) \qquad \text{if } \ \ell\ge \ell_*=\max((\bar{\ell}h|\log h|)^{\frac{1}{3}},\ \bar{\rho}). \tag{28.3.93}$$

Then N-term does not exceed

(28.3.94)
$$C\beta h^{-1}\int_{\ell_x\ge\ell_*}\Big[\ell_x^{-1}+\nu^{\frac{2}{3}}h^{\frac{1}{3}}\ell_x^{-\frac{5}{3}}+\bar{\ell}^{\frac{3}{5}}h^{-\frac{2}{5}}(1+|\log \ell_*\ell_x^{-1}|)^{\frac{3}{5}}\ell_x^{-\frac{4}{5}}+h^{-1}\bar{\rho}^3\ell_x^{-2}\Big]\,dx+$$
$$C\beta h^{-1}\Big[\ell_*^{-1}+\nu^{\frac{2}{3}}h^{\frac{1}{3}}\ell_*^{-\frac{5}{3}}+\bar{\ell}^{\frac{3}{5}}h^{-\frac{2}{5}}\ell_*^{-\frac{4}{5}}+h^{-1}\bar{\rho}^3\ell_*^{-2}\Big]\,\mathsf{mes}(\{\ell_x\le\ell_*\})$$

where the first and second terms estimate contributions of elements with $\ell_x\ge\ell_*$ and $\ell_x\le\ell_*$ respectively.

Remark 28.3.29. Observe that

(i) Under non-degeneracy assumption (28.3.60) we get $C(R_0+R')$ as expected and under non-degeneracy assumption (28.3.63) we get $C(R_0+R'|\log h|)$ but this is only because we counted here the contribution of subelements with $\{x\colon \rho_x\le\ell_x\}$ in the less efficient way.

(ii) Under non-degeneracy assumption (28.3.65) we get $C(R_0+R'')$ and and in the general case we get $C(R_0+R''')$ where R_0, R', R'', R''' are defined in Propositions 28.3.24 and 28.3.26.

Similarly, the total contribution of the zone considered here (in *Part I*) to D-term does not exceed

$$C\beta^2h^{-2}\iint_{\ell_x\ge\ell_*,\ell_y\ge\ell_*,|x-y|\ge\max(\ell_x,\ell_y)} \tag{28.3.95}$$

$$\times \Big[\ell_x^{-1} + \nu^{\frac{2}{3}} h^{\frac{1}{3}} \ell_x^{-\frac{5}{3}} + \bar{\ell}^{\frac{3}{5}} h^{-\frac{2}{5}} (1 + |\log \ell_* \ell_x^{-1}|)^{\frac{3}{5}} \ell_x^{-\frac{4}{5}} + h^{-1} \bar{\rho}^3 \ell_x^{-2}\Big]$$
$$\times \Big[\ell_y^{-1} + \nu^{\frac{2}{3}} h^{\frac{1}{3}} \ell_y^{-\frac{5}{3}} + \bar{\ell}^{\frac{3}{5}} h^{-\frac{2}{5}} (1 + |\log \ell_* \ell_y^{-1}|)^{\frac{3}{5}} \ell_y^{-\frac{4}{5}} + h^{-1} \bar{\rho}^3 \ell_y^{-2}\Big]$$
$$\times |x-y|^{-1}\, dxdy$$
$$+C\beta^2 h^{-2}\Big[\ell_*^{-1}+\nu^{\frac{2}{3}} h^{\frac{1}{3}} \ell_*^{-\frac{5}{3}}+\bar{\ell}^{\frac{3}{5}} h^{-\frac{2}{5}} \ell_*^{-\frac{4}{5}}+h^{-1}\bar{\rho}^3 \ell_*^{-2}\Big]^2 \iint_{\ell_x \le \ell_*, \ell_y \le \ell_*} |x-y|^{-1}\, dxdy.$$

Then under non-degeneracy assumption (28.3.60) we get $C(R_0 + R')^2$ as expected while under weaker non-degeneracy assumption (28.3.63) we get $C(R_0 + R'|\log h|)^2$ but this is only because we counted here contribution of subelements with $\{x\colon \rho_x \le \ell_x\}$ in the less efficient way. Using the method employed in the proof of Proposition 28.3.24 we can recover estimate $C(R_0 + R')^2$ as well.

Further, under non-degeneracy assumption (28.3.63) we get $C(R_0 + R'')^2$ and under non-degeneracy assumption (28.3.65) we get $C(R_0 + R''')^2$.

Part 2. Consider the case of $\ell_x \ge Cr$ (and therefore $\ell_y \asymp \ell_x$). Then we apply the same arguments as before albeit with ℓ^2 replaced by ℓr. First, consider the pair of subelements with $\rho_x \ge \rho^*$, $\rho_y \ge \rho^*$. Their contribution to D-term does not exceed the product of expressions (28.3.83) with $\rho = \rho_x$, $\gamma = \gamma_x$ multiplied by $\rho_x^2(\beta h)^{-1}$ and (28.3.83) with $\rho = \rho_y$, $\gamma = \gamma_y$ multiplied by $\rho_x^2(\beta h)^{-1}$, and multiplied by $|x-y|^{-1}$. Then the double summation by ρ_x, ρ_y results in $Ch^{-4}(1 + \nu^{\frac{4}{3}} h^{\frac{2}{3}})\gamma_x^3 \gamma_y^3 |x-y|^{-1}$; and, finally, the double summation over x, y returns $Ch^{-4}(1 + \nu^{\frac{4}{3}} h^{\frac{2}{3}}) \iint |x-y|^{-1}\, dxdy \lesssim CR^2$.

Then we need to consider pairs with $\rho_x \le \rho^*$, $\rho_y \le \rho^*$ and also pairs with $|x-y| \le h \max(\rho_x^{-1}, \rho_y^{-1})$.

Next consider pairs of subelements with $\rho^* \ge \rho_x \ge (\ell r)^{\frac{1}{2}}$, $\rho^* \ge \rho_y \ge (\ell r)^{\frac{1}{2}}$. Their contributions to D-term does not exceed expression

$$C\beta^2 h^{-2}\big(\rho_x^{-1}\rho_y^{-1} + \bar{\ell}^2 \rho_x^{-2}\rho_y^{-2} + \rho_x^{-\frac{5}{3}} \rho_y^{-\frac{5}{3}} \nu^{\frac{4}{3}} h^{\frac{2}{3}}\big)\, \underbrace{\gamma_x^3 \gamma_y^3 |x-y|^{-1}}$$

and the double summation over x, y in $B(z, \ell)$ with $\ell_z = \ell$ results in the same expression with the selected factor replaced by ℓ^5 and then the double summation over ρ_x, ρ_y results in

$$C\beta^2 h^{-2}\Big[\ell^{-2} + \bar{\ell}^2 \ell^{-4} + \ell^{-\frac{10}{3}} \nu^{\frac{4}{3}} h^{\frac{2}{3}}\Big]\ell^5. \tag{28.3.96}$$

Meanwhile, considering pairs of subelements with $(\ell r)^{\frac12} \ge \rho_x \ge \rho_*$ and $(\ell r)^{\frac12} \ge \rho_y \ge \rho_*$ (we use $\rho_* = \rho_*(\ell)$ and ℓ_* introduced in (28.3.93)) we gain factor $\rho_x^2\rho_y^2/(\rho\ell)$ in the summation by subelements and we arrive to the same expression (28.3.96) but with a logarithmic factor at $\bar\ell^2$:

$$C\beta^2 h^{-2}\Big[\ell^{-2} + \bar\ell^2\ell^{-4}(1+|\log \rho_*\ell^{-1}|)^3 + \ell^{-\frac{10}{3}}\nu^{\frac43}h^{\frac23}\Big]\ell^5.$$

However we can get rid of logarithmic factors exactly as in the proof of Proposition 28.3.26 thus getting (28.3.96). Then we need to sum by balls $B(z,\ell_z)$ resulting in the same expression multiplied by ℓ^{-3} and integrated:

$$C\beta^2 h^{-2}\int_{\{\ell_x\ge\ell_*\}}\Big[\ell_x^{-2} + \bar\ell^2\ell_x^{-4} + \ell_x^{-\frac{10}{3}}\nu^{\frac43}h^{\frac23}\Big]\ell_x^2\,dx,$$

which we estimate by

$$C\beta^2 h^{-2}\int_{\ell\ge\ell_*}\Big[\ell^{-2} + \bar\ell^2\ell^{-4} + \ell^{-\frac{10}{3}}\nu^{\frac43}h^{\frac23}\Big]\ell^{2+m}\,d\ell \tag{28.3.97}$$

with $m = 2,1,0$ under non-degeneracy assumptions (28.3.63), (28.3.65) and in the general case respectively. Then we arrive to the terms of integrand multiplied by ℓ and calculated either for $\ell = 1$ or $\ell = \ell_*$. One can see easily that this does not exceed CR^2 with R defined in the corresponding statement of Proposition 28.3.24.

One also can derive easily the same estimate for contributions of the pairs of subelements with $\rho_x \le \rho_*(\ell)$, $\rho_y \le \rho_*(\ell)$, $\ell \ge \ell_*$, and for contributions of the pairs of subelements with $\rho_x \le \ell_*$, $\rho_y \le \ell_*$, $\ell \le \ell_*$, assuming in both cases that $\rho_x r \ge h$, $\rho_y r \ge h$.

Finally, like in the proof of Proposition 28.3.26 we estimate the contribution of zone $\{x,y\colon \rho_x r \le h,\ \rho_y r \le h\}$. We leave easy details to the reader. □

Remark 28.3.30. These arguments also work to estimate

$$\mathsf{D}\big(\Gamma_x(hD-A)_x\cdot\boldsymbol{\sigma}e(.,.,0),\,\Gamma_x\big(hD-A\big)_x\cdot\boldsymbol{\sigma}e(.,.,0)\big). \tag{28.3.98}$$

Indeed, Weyl expression for $\Gamma_x(hD-A)_x\cdot\boldsymbol{\sigma}e(.,.,0)$ is just 0. Therefore we arrive under either of non-degeneracy assumptions (28.3.60), (28.3.63), (28.3.65) and in the general case to estimate

$$\|\nabla A'\| \le C\kappa h^2 R, \tag{28.3.99}$$

which could be better or worse than estimate $\|\nabla A'\| \le C\kappa^{\frac12}$ which we have already. It is not clear if estimate $\|\nabla A'\| \le C\kappa$ holds.

28.4 Microlocal Analysis: $\beta h \gtrsim 1$

Now let us investigate the case of $\beta h \gtrsim 1$. In this case we assume not only that $\kappa \lesssim 1$ but also (28.2.27)*: $\kappa\beta h^2 |\log h|^K \le 1$. We can apply the same arguments as before and in the end of the day we will get the series of the statements; we leave most of the easy details to the reader.

28.4.1 Estimate to a Minimizer

Observe first that

$$\|\partial A'\|^2 \le C\beta h^{-2} \times \kappa h^2 = C\kappa\beta \le Ch^{-2}|\log h|^{-K} \tag{28.4.1}$$

and

$$|\Delta A'| \le C\beta h^{-2} \times \kappa h^2 = C\kappa\beta \le Ch^{-2}|\log h|^{-K},$$

and therefore

$$|\partial^2 A'| \le C\kappa\beta \le Ch^{-2}|\log h|^{-K}. \tag{28.4.2}$$

First of all, repeating arguments leading to Proposition 28.4.8, we arrive to estimate (28.3.46) modified

$$\begin{aligned}\|\partial^2 A'\|_{\mathscr{L}^\infty} \le C\kappa\, \underbrace{\beta h}\, \nu^{\frac12}|\log h|^2 \\ + C\kappa\beta h|\log h|\big(\underbrace{h^{-\frac35}\nu^{\frac{1}{10}}} + \underbrace{h^{-\frac47}\nu^{\frac17}} + h^{-\frac12}\nu^{\frac14}|\log h|^2\big) + C\kappa|\log h|\beta^{\frac12}\|\partial V\|_{\mathscr{L}^\infty} \\ + C\|\partial A'\|_{\mathscr{L}^\infty}.\end{aligned} \tag{28.4.3}$$

Note a new factor βh in the first line, which first becomes

$$C\kappa\beta h|\log h|\big(\nu^{\frac12} + (\mu+\nu h)|\log h|\big). \tag{28.4.4}$$

We prove it first for $\mu = 1$ and then rescale and in virtue of Remark 28.3.9(ii) we can always replace μ by $\nu^{\frac12} \ll h^{-1}$; then, using $\nu := \max\big(\|\partial^2 A'\|_{\mathscr{L}^\infty}, 1\big)$ and assumption (28.2.27)*, we reduce (28.4.4) to its final form in (28.4.3).

Further, under additional super-strong non-degeneracy assumption

$$\min_{x,j\ge 0} |V - 2j\beta h| \asymp 1 \tag{28.4.5}$$

we can skip two selected terms in the second line of (28.4.3), arriving to

$$\|\partial^2 A'\|_{\mathscr{L}^\infty} \le C\kappa\beta h\nu^{\frac12}|\log h|^2 + C\kappa\beta h^{-\frac12}\nu^{\frac14}|\log h|^3 + C\kappa|\log h|\beta^{\frac12}\|\partial V\|_{\mathscr{L}^\infty} + C\|\partial A'\|_{\mathscr{L}^\infty}. \tag{28.4.6}$$

Then we arrive to the following assertion:

Proposition 28.4.1 [17]**.** *Let $\beta h \gtrsim 1$, $\kappa \le \kappa^*$ and (28.2.27)* be fulfilled; let $V \in \mathscr{C}^2$; then*

(i) The following estimates hold:

$$\|\partial^2 A'\|_{\mathscr{L}^\infty} \le \nu \tag{28.4.7}$$

with

$$\nu := C\kappa\beta^{\frac12}|\log h| + C(\kappa\beta)^{\frac{10}{9}}h^{\frac49}|\log h|^K \qquad \text{for } \kappa\beta h \le 1 \tag{28.4.8}$$

and

$$\nu := C\kappa\beta^{\frac12}|\log h| + C(\kappa\beta)^{\frac43}h^{\frac23}|\log h|^K \qquad \text{for } \kappa\beta h \ge 1. \tag{28.4.9}$$

(ii) Moreover, under assumption (28.4.5) *estimate ν is given by* (28.4.9) *even for $\kappa\beta h \lesssim 1$.*

Remark 28.4.2. (i) While case $\beta h \asymp 1$ has been already explored, we missed an important case when non-degeneracy assumption (28.4.5) is fulfilled; so we reexamine this case.

(ii) While technically (28.4.3) and (28.4.6) hold even if assumption (28.2.27)* fails provided $\nu \le \epsilon\beta$, we cannot guarantee in this case that this inequality holds.

28.4.2 Trace Term Asymptotics

Further, continuing our analysis we arrive to the following assertion

Proposition 28.4.3 [18]**.** *Let $\beta h \gtrsim 1$, $\kappa \le \kappa^*$ and $\nu h^2 \le 1$* [19]*. Then*

[17] Cf. Proposition 28.3.10.

[18] Cf. Propositions 28.3.13, 28.3.15 and 28.3.16; only factor βh appears in the definition of Q_0.

[19] In the framework of Proposition 28.4.1 for a minimizer this assumption is due to (28.2.27)*.

(i) Under non-degeneracy assumption (28.3.60), *or* (28.3.63) *remainder estimate*

$$|\operatorname{Tr}(H^{-}_{A,V}\psi) + h^{-3}\int P_{Bh}(V)\psi\,dx| \le Q \tag{28.4.10}$$

holds with $Q = Q_0 + Q'$,

$$Q_0 \coloneqq C\beta + C\beta\nu^{\frac{4}{3}}h^{\frac{2}{3}}, \tag{28.4.11}$$

and Q' *defined by* (28.3.61).

(ii) Under non-degeneracy assumption (28.3.65) *remainder estimate* (28.4.10) *holds with* $Q = Q_0 + Q''$ *with* $Q'' = Q' + \nu|\log h$ *and* Q_0 *and* Q' *defined by* (28.4.10) *and* (28.3.61) *respectively.*

(iii) In the general case remainder estimate (28.4.10) *holds with* $Q = Q_0 + Q'''$ *with* Q_0 *and* Q''' *defined by* (28.4.10) *and* (28.3.66) *respectively.*

Applying Proposition 28.4.1 we arrive to

Corollary 28.4.4. *In the framework of Proposition 28.4.3 let* A' *be a minimizer. Then*

(i) Under non-degeneracy assumption (28.3.60), *or* (28.3.63), *or even* (28.3.65) *estimate* (28.4.10) *holds with*

$$Q = Q_0 = C\beta + C\kappa^{\frac{40}{27}}\beta^{\frac{67}{27}}h^{\frac{34}{27}}|\log h|^{K} \qquad \text{for } \kappa\beta h \le 1 \tag{28.4.12}$$

and

$$Q = Q_0 = C\beta + C\kappa^{\frac{16}{9}}\beta^{\frac{25}{9}}h^{\frac{14}{9}}|\log h|^{K} \qquad \text{for } \kappa\beta h \ge 1. \tag{28.4.13}$$

(ii) Furthermore, under assumption (28.4.5) Q_0 *is defined by* (28.4.13) *even for* $\kappa\beta h \le 1$.

(iii) In the general case estimate (28.4.10) *holds with*

$$Q = Q_0 + \beta h^{-\frac{1}{2}} + \kappa^{\frac{8}{5}}\beta^{\frac{13}{5}}h^{\frac{6}{5}}|\log h|^{K}. \tag{28.4.14}$$

Remark 28.4.5. Observe that

(i) If assumption (28.2.27)* holds, then $Q_0 \lesssim \kappa^{\frac{1}{3}}\beta^{\frac{4}{3}}h^{-\frac{4}{3}} \lesssim \beta h^{-2}$, where the middle expression appears in (28.2.25). On the other hand, if (28.2.27)* fails, then the reverse inequalities hold. In the general case we assume that $\beta h^2 \ll 1$ to get a remainder estimate smaller than the main term.

(ii) Also $\nu \lesssim \beta$ provided (28.2.27)* holds; if $\kappa = 1$ then $\nu \lesssim \beta$ if and only if (28.2.27)* holds.

28.4.3 Endgame

Similarly to Theorem 28.3.21 we arrive to

Theorem 28.4.6. *Let $\beta h \gtrsim 1$, $\kappa \le \kappa^*$ and (28.2.27)* be fulfilled. Then estimate* (28.3.78)

$$|\mathsf{E}^*_\kappa - \mathcal{E}^*_\kappa| \le CQ$$

holds where

(i) Under non-degeneracy assumption (28.3.65) *Q is defined by* (28.4.12) *and* (28.4.13).

(ii) In the general case Q is defined by (28.4.12)*; in particular, $Q = \beta h^{-\frac{1}{2}}$ as $\kappa\beta h \lesssim 1$.*

Problem 28.4.7[20]. In this new settings recover estimates for $\|\partial(A' - A'')\|$, $\|\partial(A' - A'')\|_{\mathscr{L}^\infty}$ and $\|\partial A'\|_{\mathscr{L}^\infty}$ where A'' is a minimizer for $\bar{\mathcal{E}}(A'')$.

28.4.4 N-Term Asymptotics and D-Term Estimates

Repeating arguments of the proofs of Propositions 28.3.24, 28.3.26 we arrive to

Proposition 28.4.8. *Let $\beta h \gtrsim 1$ and conditions* $(28.3.49)_{1,2}$ *be fulfilled. Then*

(i) Under non-degeneracy assumption (28.3.60) *or* (28.3.63) *estimate* (28.3.84) *holds with $R = R_0 + R'$,*

$$R_0 = \beta h^{-1} + \beta h^{-\frac{2}{3}}\nu^{\frac{2}{3}} \tag{28.4.15}$$

and R' defined by $(28.3.61)'$.

[20] Cf. Problem 28.3.22.

(ii) Under non-degeneracy assumption (28.3.63) *estimate* (28.3.84) *holds with* $R = R_0 + R''$, R_0 *and* R'' *defined by* (28.4.15) *and* (28.3.86).

(iii) In the general case estimate (28.3.84) *holds with* $R = R_0 + R'''$, R_0 *and* R''' *defined by* (28.4.15) *and* (28.3.88).

Repeating arguments of the proof of Propositions 28.3.27 and 28.3.27 we arrive to

Proposition 28.4.9. *Let* $\beta h \gtrsim 1$ *and conditions* $(28.3.49)_{1,2}$ *be fulfilled. Then*

(i) Under non-degeneracy assumptions (28.3.60) *or* (28.3.63) D*-term* (28.3.89) *does not exceed* CR^2 *with* $R = R_0 + R'$, R_0 *and* R' *defined by* (28.4.15) *and* $(28.3.61)'$ *respectively.*

(ii) Under non-degeneracy assumption (28.3.65) D*-term* (28.3.89) *does not exceed* CR^2 *with* $R = R_0 + R''$, R_0 *and* R' *defined by* (28.4.15) *and* (28.3.86).

(iii) In the general case D*-term* (28.3.89) *does not exceed* CR^2 *with* $R = R_0 + R'''$, R_0 *and* R' *defined by* (28.4.15) *and* (28.3.88).

Problem 28.4.10. In the general case (without any non-degeneracy assumptions) for $\beta h \lesssim 1$ and for $\beta h \gtrsim 1$ improve the remainder estimates for both the trace term and N-term and estimates for D-term (so, make R''' and Q''' smaller) under assumption $V \in \mathscr{C}^s$ with $s > 2$.

To do this use more advanced partition of unity as in Chapter 25. Most likely, however, it will affect only terms $C\beta h^{-\frac{1}{2}}$ and $C\beta h^{-\frac{3}{2}}$ in Q''' and R''' replacing them by $C\beta h^{(s-4)/(s+2)}$ and $C\beta h^{-1-2/(s+2)}$ respectively.

28.5 Global Trace aAsymptotics in the Case of Thomas-Fermi Potential: $B \le Z^{\frac{4}{3}}$

28.5.1 Introduction

In this Section we consider global trace asymptotics for Thomas-Fermi potential. First we consider the singularity zones where our results would follow from Section 27.3, then we consider their interaction with the regular zone which would lead to the deterioration of the remainder estimates for

$\beta \gg h^{-\frac{1}{2}}$ and finally the boundary zone where non-degeneration properties could be violated (especially for $M \geq 2$), which requires rather subtle analysis and usage of the specific properties of Thomas-Fermi potential.

Remark 28.5.1. Recall that according to Chapter 25 there are two cases:

(a) $B \leq Z^{\frac{4}{3}}$, when the most contributing to both the number of particles and the energy zone is $\{x\colon \ell(x) \asymp r^* = Z^{-\frac{1}{3}}\}$ (where $\ell(x)$ is the distance to the closest nucleus), and then rescaling $x \mapsto xr^{*-1}$, $\tau \mapsto \tau Z^{-\frac{4}{3}}$ we arrive in this zone to $\beta = BZ^{-1}$, $h = Z^{-\frac{1}{3}}$ with $\beta h \leq 1$.

(b) $Z^{\frac{4}{3}} \leq B \leq Z^3$, when the most contributing to both the number of particles and the energy zone is $\{x\colon \ell(x) \asymp r^* = B^{-\frac{2}{5}}Z^{\frac{1}{5}}\}$, and then rescaling $x \mapsto xr^{*-1}$, $\tau \mapsto \tau B^{-\frac{2}{5}}Z^{-\frac{4}{5}}$ we arrive in this zone to $\beta = B^{\frac{2}{5}}Z^{-\frac{1}{5}}$, $h = B^{\frac{1}{5}}Z^{-\frac{3}{5}}$ with $\beta h \geq 1$.

We also recall that in the free (movable) nuclei model the distances between nuclei were greater than ϵr^* (which would be the case in the current settings as well as we show later), so we will assume that it is the case deducting our main results.

28.5.2 Estimates to a Minimizer

Preliminary Analysis

Consider potential V with the Coulomb-like singularities, exactly as in Section 27.3 i.e. satisfying (27.3.1)–(27.3.3).

Proposition 28.5.2 [21]**.** *Let V satisfy* (27.3.1)–(27.3.2) *and*

$$|D^\alpha W| \leq C_\alpha \sum_{1\leq m\leq M} z_m\big(|x-\bar{y}_m|+1\big)^{-4}|x-\bar{y}_m|^{-|\alpha|} \qquad \forall \alpha : |\alpha| \leq 2. \tag{28.5.1}$$

Let $\kappa \leq \kappa^*$ and $\beta h \leq 1$. Then the near-minimizer A satisfies

$$|\operatorname{Tr}(H^-_{A,V}) + \int h^{-3}P_{\beta h}(V(x))\Big)\,dx| \leq Ch^{-2} \tag{28.5.2}$$

and

$$\|\partial A'\| \leq C\kappa^{\frac{1}{2}}. \tag{28.5.3}$$

[21] Cf. Proposition 27.3.1.

Proof. We follow the proof of Proposition 27.3.1. Observe that scaling $x \mapsto (x-\bar{y}_m)\ell^{-1}$, $\tau \mapsto \tau\zeta^{-2}$ leads us to

$$h \mapsto h_1 = h\ell^{-1}\zeta^{-1}, \qquad \beta \mapsto \beta_1 = \beta\ell\zeta^{-1}, \qquad \kappa \mapsto \kappa_1 = \kappa\zeta^2\ell. \tag{28.5.4}$$

Also observe that for $\zeta = \ell^{-\frac{1}{2}}$

$$\kappa_1\beta_1 h_1^2 \gtrsim 1 \implies \ell \gtrsim \ell^* = (\beta+1)^{-\frac{1}{2}}h^{-1} \tag{28.5.5}$$

and for $\kappa \asymp 1$ those are equivalent.

(i) First, we pick up $A' = 0$. Then

$$|\operatorname{Tr}\bigl(H^-_{A^0,V}(0)\bigr) + h^{-3}\int P_{\beta h}(V(x))\,dx| \le Ch^{-2}; \tag{28.5.6}$$

this estimate follows from the standard partition with ℓ-admissible partition elements, supported in $\{x\colon \ell(x) \lesssim \ell\}$ for $\ell = \ell_*$ and and in $\{x\colon \ell(x) \asymp \ell\}$ for $\ell \ge 2\ell_*$.

(ii) On the other hand, consider $A' \ne 0$. Let us prove first that

$$\mathsf{Tr}^-(\psi_\ell H\psi_\ell) \ge -C_\varepsilon h^{-2} - \varepsilon\kappa^{-1}h^{-2}\|\partial A'\|^2 \tag{28.5.7}$$

for $\ell = \ell_* = h^2$ where one can select constant ε arbitrarily small.

Rescaling $x \mapsto (x-\bar{y}_m)/\ell$ and $\tau \mapsto \tau/\ell$ and therefore $h \mapsto h\ell^{-\frac{1}{2}} \asymp 1$ and $A \mapsto A\ell^{\frac{1}{2}}$ (because singularity is Coulomb-like), we arrive to the same problem with the same κ and with $\ell = h = 1$ and with β replaced by βh^3. Then for $\beta h^3 \le 1$ we refer to Appendix 27.A.1 since $H_{A,V} \ge H_{A',V'}$ with $V' = V - \beta^2|x|^2$.

(iii) Consider now ψ_ℓ as in (i) with $\ell \ge \ell_*$. Then according to Theorems 28.3.21 and 28.4.6 for $\beta_1 h_1 \lesssim 1$ and $\kappa\zeta^2\ell \le \kappa^*$

$$\begin{aligned}\mathsf{Tr}^-\bigl(\psi_\ell H_{A,V}\psi_\ell\bigr) + h^{-3}\int P_{\beta h}(V(x))\psi_\ell^2(x)\,dx \\ \ge -C_\varepsilon\zeta^2(h_1^{-1}+\beta_1 h_1^{-1}) - \varepsilon\kappa^{-1}h^{-2}\|\partial A'\|^2.\end{aligned} \tag{28.5.8}$$

Remark 28.5.3. Observe that if ψ_ℓ is supported in $\{x\colon \frac{1}{2}r \le \ell(x) \le 2r\}$, then we can take a norm of $\partial A'$ over $\{x\colon \frac{1}{4}r \le \ell(x) \le 4r\}$. Indeed, we can just replace A' by $A'' = \phi_\ell(A'-\eta)$ with arbitrary constant η and with ϕ_ℓ supported in $\{x\colon \frac{1}{4}r \le \ell(x) \le 4r\}$ and equal 1 in $\{x\colon \frac{1}{3}r \le \ell \le 3r\}$ (and ε by $c\varepsilon$).

Then summation of these norms returns $-C_0\varepsilon\kappa^{-1}h^{-2}\|\partial A'\|^2$.

Furthermore, the first term in the right-hand expression of (28.5.8) is $-C_\varepsilon(\zeta^3\ell^{-1}h^{-1}+\zeta^2\ell^2\beta h^{-1})$ and summation over $\ell \ge h^2$ returns $-C_\varepsilon h^{-2}$ since $\zeta = \min(\ell^{\frac12},\, \ell^{-2})$.

(iv) Consider next zone where $\beta_1 h_1 \ge 1$ (and $\ell \ge 1$) but still $h_1 \le 1$. According to previous Section 28.5 inequality (28.5.8) should be replaced by

$$\mathsf{Tr}^-\bigl(\psi_\ell H_{A,V}\psi_\ell\bigr) + h^{-3}\int P_{\beta h}(V(x))\psi_\ell^2(x)\,dx \ge -C_\varepsilon\zeta^2\beta_1 h_1^{-1}\bigl(1+\nu_1^{\frac43}h_1^{\frac53}\bigr) - \varepsilon\kappa^{-1}h^{-2}\|\partial A'\|^2 \tag{28.5.9}$$

with $\nu_1 = (\kappa_1\beta_1)^{\frac{10}{9}}h_1^{\frac49}|\log h|^{K}$ [22] and $\kappa_1\beta_1 = \kappa\beta\zeta\ell^2 = \kappa\beta$ and thus with $\beta_1 h_1^{-1}\zeta^2 = \beta h^{-1}\zeta^2\ell^2$. Then summation of the first term in the right-hand expression results in its value when ℓ is the smallest i.e. $\beta_1 h_1 = 1$ and one can check easily[23] that this is less than Ch^{-2}.

Further, Remark 28.5.3 remains valid. Then adding this zone does not change inequality in question.

(v) The rest of the proof is obvious. Zone $\{x\colon \ell(x) \ge \ell^*\}$ is considered as a single element and just rough variational estimate is used there to prove that its contribution does not exceed Ch^{-2}. □

Remark 28.5.4. Later we will improve both upper and lower estimates using different tricks: imposing non-degeneracy assumptions, picking for an upper estimate semiclassical self-generated magnetic field, using Scott correction term (derived in Chapter 27). These improvements will lead not only to our final goal, but also to our intermediate one–getting better estimates for a minimizer.

Proposition 28.5.5 [24]**.** *In the framework of Proposition 28.5.2 there exists a minimizer A.*

Proof. After Proposition 28.5.2 has been proven we just repeat arguments of the proof of Proposition 27.2.2. □

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[22] Because $\kappa_1\beta_1 h_1 = \kappa\beta h\ell \le \kappa(\beta h)^{\frac34} \le 1$.

[23] Sufficient to check for $\beta = h^{-1}$, $\ell = 1$ and $\kappa = 1$.

[24] Cf. Proposition 27.3.2.

Estimates to a Minimizer: Interior Zone

Recall equation (27.2.14) for a minimizer A:

$$\frac{2}{\kappa h^2}\Delta A_j(x) = \Phi_j := \\ -\operatorname{Re}\operatorname{tr}\sigma_j\Big(\big(hD-A\big)_x\cdot\boldsymbol{\sigma}e(x,y,\tau)+e(x,y,\tau)\,{}^t\big(hD-A\big)_y\cdot\boldsymbol{\sigma}\Big)\Big|_{y=x}. \tag{27.2.14}$$

After rescaling $x\mapsto x/\ell$, $\tau\mapsto\tau/\zeta^2$, $h\mapsto\hbar=h/(\zeta\ell)$, $A\mapsto A\zeta^{-1}\ell$, $\beta\mapsto\beta\zeta^{-1}\ell$ this equation becomes (27.3.13)

$$\Delta A_j = \\ -2\kappa\zeta^2\ell\hbar^2\operatorname{Re}\operatorname{tr}\sigma_j\Big(\big(\hbar D-\zeta^{-1}A\big)_x\cdot\boldsymbol{\sigma}e(x,y,\tau)+e(x,y,\tau)\,{}^t\big(\hbar D-\zeta^{-1}A\big)_y\cdot\boldsymbol{\sigma}\Big)\Big|_{y=x} \tag{27.3.13}$$

and since we can take $\zeta^2\ell=1$ we arrive to (27.3.14)

$$\Delta A_j = \\ -2\kappa\hbar^2\operatorname{Re}\operatorname{tr}\sigma_j\Big(\big(\hbar D-\zeta^{-1}A\big)_x\cdot\boldsymbol{\sigma}e(x,y,\tau)+e(x,y,\tau)\,{}^t\big(\hbar D-\zeta^{-1}A\big)_y\cdot\boldsymbol{\sigma}\Big)\Big|_{y=x}. \tag{27.3.14}$$

Let us modify arguments of Subsection 27.3.1. First observe that

$$|\partial A'|\le C\kappa^{\frac12}h^{-3},\quad |\partial^2A'|\le C\kappa^{\frac12}h^{-5}\qquad \text{for } \ell\le 2\ell_* \tag{28.5.10}$$

with $\ell_*=h^2$; this follows from above equations rescaled and from $\beta h^3\le\epsilon_0$. Let

$$\mu(r)=\sup_{\ell(x)\ge r}|\partial A'|\ell\zeta^{-1},\qquad \nu(r)=\sup_{\ell(x)\ge r}|\partial^2A'|\ell^2\zeta^{-1}; \tag{28.5.11}$$

then $\nu(r)$ should not exceed[25)]

$$F(\nu)=C\kappa_1\Big(1+\mu+\min\big(\beta_1^{\frac32}h_1^{\frac12},\,\beta_1^{\frac12}\big) \\ +\beta_1h_1\big(\nu^{\frac{1}{10}}h_1^{-\frac35}+\nu^{\frac17}h_1^{-\frac47}+\nu^{\frac14}h_1^{-\frac12}|\log h_1|^2\big)\Big)|\log h_1|+C\kappa^{\frac12}(\ell\zeta^2)^{-\frac12}, \tag{28.5.12}$$

where here $\nu=\nu(\frac12 r)$, $\mu=\mu(r)$, the last term is just an estimate for $\|\partial A'\|$ rescaled and $\ell\asymp r$ in that term. Indeed, (28.5.12) is derived exactly as (28.3.46), but here we cut a hole $\{x\colon \ell(x)\le\frac12 r\}$ in our domain.

[25)] As long as $\beta_1h_1\le 1$.

We also know that $\mu \le C\nu^{\frac{3}{5}}\kappa^{\frac{1}{5}}(\ell\zeta^2)^{-\frac{1}{5}}$. Using (28.5.12) and (28.5.10) one can prove easily that $\nu(r)$ does not exceed solution of the equation $\nu = F(\nu)$ multiplied by C [26], i.e.

$$\nu \le C\kappa_1\Big(1+\min\big(\beta_1^{\frac{3}{2}}h_1^{\frac{1}{2}},\, \beta_1^{\frac{1}{2}}\big)\Big)|\log h_1| + C\Big((\kappa_1\beta_1)^{\frac{10}{9}}h_1^{\frac{4}{9}} + (\kappa_1\beta_1)^{\frac{4}{3}}h_1^{\frac{2}{3}}\Big)|\log h_1|^K + C\kappa^{\frac{1}{2}}(\ell\zeta^2)^{-\frac{1}{2}}. \tag{28.5.13}$$

In particular, scaling back and setting $\zeta = \ell^{-\frac{1}{2}}$ we arrive to

$$|\partial^2 A'| \le C\kappa\Big(\ell^{-\frac{5}{2}} + \min\big(\beta^{\frac{3}{2}}h^{\frac{1}{2}}\ell^{-\frac{1}{2}},\, \beta^{\frac{1}{2}}\ell^{-\frac{7}{4}}\big)\Big)|\log \ell/\ell_*| + C\Big((\kappa\beta)^{\frac{10}{9}}h^{\frac{4}{9}}\ell^{-\frac{19}{18}} + (\kappa\beta)^{\frac{4}{3}}h^{\frac{2}{3}}\ell^{-\frac{5}{6}}\Big)|\log \ell/\ell_*|^K + C\kappa^{\frac{1}{2}}\ell^{-\frac{5}{2}}. \tag{28.5.14}$$

The same arguments work for $\beta_1 h_1 \ge 1$ but now we need to replace $|\log h_1|$ by $|\log \beta_1|$ which however is also $\asymp \ell/\ell_*$ as $\beta h \le 1$.

After this estimate is proven we can remove the last term in the right-hand expression and we arrive to

Proposition 28.5.6. *In the framework of Proposition 28.5.2*

$$|\partial^2 A'| \le C\kappa\Big(\ell^{-\frac{5}{2}} + \min\big(\beta^{\frac{3}{2}}h^{\frac{1}{2}}\ell^{-\frac{1}{2}},\, \beta^{\frac{1}{2}}\ell^{-\frac{7}{4}}\big)\Big)|\log \ell/\ell_*| + C\Big((\kappa\beta)^{\frac{10}{9}}h^{\frac{4}{9}}\ell^{-\frac{19}{18}} + (\kappa\beta)^{\frac{4}{3}}h^{\frac{2}{3}}\ell^{-\frac{5}{6}}\Big)|\log \ell/\ell_*|^K. \tag{28.5.15}$$

Estimates to a Minimizer: Exterior Zone

Let us estimate $|\partial^2 A'|$ as $\ell \ge \ell^*$. Observe that

$$A'_j(x) = -\frac{\kappa h^2}{4\pi}\int |x-y|^{-1}\Phi_j(y)\,dy, \tag{28.5.16}$$

where Φ_j is given by (27.2.14). Then $\partial^2 A'$ is expressed via Φ_j as an integral with a kernel $K(x,y)$, singular when $x = y$ and such that $|K(x,y)| \le c(|x|+|y|)^{-3}$ when $|x-y| \asymp |x|+|y|$. Further, applying representation like in Proposition 27.3.9, we can get an extra factor $|y|(|x|+|y|)^{-1}$ upgrading it to $|K(x,y)| \le c|y|(|x|+|y|)^{-4}$.

[26] As long as a resulting expression rescaled, see (28.5.14) is a decaying function of ℓ.

Then, starting from (28.5.15) and iterating (28.3.46) we arrive to estimate

$$|\partial^2 A'| \le C\kappa\ell^{-4}|\log h| + C(\kappa\beta)^{\frac{10}{9}} h^{\frac{4}{9}}\ell^{-\frac{32}{9}+\delta}|\log h|^K \qquad \text{for } \ell \ge 1$$

with arbitrarily small $\delta > 0$. Furthermore, using arguments of the proof of Proposition 28.5.6 we can make $\delta = 0$ thus arriving to

Proposition 28.5.7. *In the framework of Proposition 28.5.2*

$$|\partial^2 A'| \le C\kappa\ell^{-4}|\log h| + C(\kappa\beta)^{\frac{10}{9}} h^{\frac{4}{9}}\ell^{-\frac{32}{9}}|\log h|^K \qquad \textit{for } \ell \ge 1. \tag{28.5.17}$$

Remark 28.5.8. Sure, as $\ell \ge 1$, $\beta_1 h_1^{\frac{1}{2}}|\log h_1|^K \le 1$ this estimate could be improved but these improvements would not affect our crucial estimates.

28.5.3 Trace aAsymptotics

Before proving trace estimates observe

Remark 28.5.9. (i) All local asymptotics and estimates with mollification with respect to spatial variables [27] proven in Sections 28.3 and 28.4 with unspecified $\nu \le \epsilon\beta$ remain valid in the more general framework of the smooth non-degenerate external field $A^0(x)$: namely

$$\|\partial^4 A^0\| \le C_0\beta, \qquad B^0 = |\nabla \times A^0| \ge \epsilon_0\beta. \tag{28.5.18}_{1,2}$$

Indeed, we use only ε-approximations with $\varepsilon = h$ or $\varepsilon = h\rho^{-1}$ and we can always change coordinate system so magnetic lines are $(x_1, x_2) = \mathsf{const}$. We leave easy arguments to the reader.

(ii) However since we do not have estimates (28.3.47) or (28.4.7)–(28.4.9) in this more general framework[28], we also do not have (28.4.12), (28.4.13) then.

Now consider the trace term assuming that

$$d := \min_{1\le m<m'\le M} |\bar{y}_m - \bar{y}_{m'}| \gtrsim 1. \tag{28.5.19}$$

[27] Thus trace and N-term asymptotics and D-term estimates.

[28] Even if we believe that these estimates are true. So far we have no need in such generalization.

(a) Due to the proofs of Theorem 27.3.22 and Proposition 28.5.2 we can evaluate contribution of the zone $\{x\colon |x-\bar{y}_m|\le \epsilon\}$, provided $\beta\le 1$:

$$|\operatorname{Tr}(H^-_{A,V}\psi_m)-\operatorname{Tr}(H^-_{A,V_m}\psi_m) + h^{-3}\int P_{Bh}(V)\psi_m\,dx - h^{-3}\int P_0(V_m)\psi_m\,dx| \le C\big(h^{-1}+\kappa|\log\kappa|^{\frac13}h^{-\frac43}\big), \tag{28.5.20}$$

where $V_m = z_m|x-\bar{y}_m|^{-1}$ and ψ_m is supported in $\{x : |x-\bar{y}_m|\le\epsilon\}$ and equal 1 in $\{x : |x-\bar{y}_m|\le \frac12\epsilon\}$.

Further, we can replace in this estimate $\operatorname{Tr}(H^-_{A',V_m}\psi_m)+h^{-3}\int P_0(V_m)\psi_m$ by

$$\int\Big(\int_{-\infty}^0 e_{V_m,A'}(x,x,\tau)\,d\tau + h^{-3}P_0(V_m)\Big)\psi_m\,dx \tag{28.5.21}$$

and we can also replace in the latter expression ψ_m by 1.

(b) If $\beta\ge 1$, we can apply estimate (28.5.20) to the zone $\{x\colon |x-\bar{y}_m|\le \epsilon b\}$ with $b=\beta^{-\frac23}$ scaling $x\mapsto (x-\bar{y}_m)b^{-1}$ and $\tau\to\tau$ and $h\mapsto h_1 = hb^{-\frac12}$, $\beta\mapsto \beta b^{\frac32}=1$, $\kappa\mapsto\kappa$; now ψ_m is supported in $\{x : |x-\bar{y}_m|\le\epsilon b\}$ and equals 1 in $\{x : |x-\bar{y}_m|\le\frac12\epsilon b\}$ and the right-hand expression of (28.5.20) becomes

$$Cb^{-1}\big(h_1^{-1}+\kappa|\log\kappa|^{\frac13}h_1^{-\frac43}\big) = C\big(\beta^{\frac13}h^{-1}+\kappa|\log\kappa|^{\frac13}\beta^{\frac29}h^{-\frac43}\big). \tag{28.5.22}$$

If $\beta\ge 1$, let us consider contribution of the zone $\{x\colon \epsilon_0 b\le |x-\bar{y}_m|\le\epsilon_0\}$, where due to assumption (28.5.19) non-degeneracy condition (28.3.60) is automatically satisfied after rescaling; namely before rescaling it is

$$\min_j |V-2j\beta h| + |\nabla V|\ell \asymp \zeta^2. \tag{28.5.23}$$

A contribution of ℓ-element in this zone does not exceed

$$C\zeta^2\big(h_1^{-1}+h_1^{-\frac13}\nu^{\frac43}\big) \tag{28.5.24}$$

with

$$\nu = \sup_{|x-\bar{y}_m|\asymp\ell} |\partial^2 A'|\ell^2\zeta^{-1}, \tag{28.5.25}$$

and plugging (28.5.4) into (28.5.24) we get

$$C\big(\ell^{-\frac12}h^{-1}+(\kappa\beta)^{\frac{40}{27}}h^{\frac{7}{27}}\ell^{\frac{59}{54}}|\log(h\ell^{-\frac12})|^K\big),$$

which sums to

$$C\bigl(\beta^{\frac{1}{3}}h^{-1}+(\kappa\beta)^{\frac{40}{27}}h^{\frac{7}{27}}|\log h|^{K}\bigr), \tag{28.5.26}$$

which obviously does not exceed (28.5.22). Thus after scaling[29)] we arrive to

Proposition 28.5.10. *Let $V=W_B^{\mathsf{TF}}+\lambda$ be Thomas-Fermi potential with $N\le Z$, $N\asymp Z_1\asymp Z_2\asymp ...\asymp Z_M$, $B\lesssim Z^{\frac{4}{3}}$ and*

$$|\mathsf{y}_m-\mathsf{y}_{m'}|\ge d\gtrsim Z^{-\frac{1}{3}}\qquad \forall 1\le m<m'\le M. \tag{28.5.27}$$

Then if ψ_m is supported in ϵr^*-vicinity of y_m

$$\begin{aligned}|\operatorname{Tr}(H^-_{A,V}\psi_m)-\operatorname{Tr}(H^-_{A,V_m}\psi_m)&\\ +\int P_B(V)\psi_m\,dx-\int P_0(V_m)\psi_m\,dx|&\end{aligned} \tag{28.5.28}$$

does not exceed CQ_0 with

$$Q_0\coloneqq\begin{cases}\bigl(Z^{\frac{5}{3}}+\alpha|\log(\alpha Z)|^{\frac{1}{3}}Z^{\frac{25}{9}}\bigr) & \text{for } B\le Z,\\ \bigl(B^{\frac{1}{3}}Z^{\frac{4}{3}}+\alpha|\log(\alpha Z)|^{\frac{1}{3}}B^{\frac{2}{9}}Z^{\frac{23}{9}}\bigr) & \text{for } Z\le B\le Z^{\frac{4}{3}}.\end{cases} \tag{28.5.29}$$

Furthermore, if $B\le Z$, then expression (28.5.28) *does not exceed*

$$C\Bigl(Z^{\frac{5}{3}}[Z^{-\delta}+(BZ^{-1})^{\delta}+(dZ^{\frac{1}{3}})^{-\delta}]+\alpha|\log(\alpha Z)|^{\frac{1}{3}}Z^{\frac{25}{9}}\Bigr). \tag{28.5.30}$$

Here improved estimate (28.5.30) can be proven by our standard propagation arguments.

Further, let us consider the *regular exterior zone* $\{x\colon \epsilon_0 r^*\le |x-\mathsf{y}_m|\le \epsilon\bar{r}\}$ with $\bar{r}\coloneqq \min\bigl(B^{-\frac{1}{4}},(Z-N)_+^{-\frac{1}{3}}\bigr)$. Then due to Thomas-Fermi equation $W_B^{\mathsf{TF}}+\lambda$ satisfies here non-degeneracy condition (28.3.65) after rescaling and $\zeta=\ell^{-2}$, $h_1=\ell(x)$, $\beta_1=B\ell^3$.

Then the contribution of ℓ-element in this zone does not exceed (28.5.25) as long as $\beta_1 h_1\le 1$, $h_1\le 1$ i.e. $\ell(x)\le\min(\bar{r},1)$, and due to (28.5.24) this contribution does not exceed $C\zeta^2\bigl(h_1^{-1}+h_1^{-\frac{1}{3}}\nu^{\frac{4}{3}}\bigr)$, where ν_1 is estimate for $|\partial^2 A'|$ multiplied by $\zeta^{-1}\ell^2$:

$$\nu_1=\kappa|\log h|+(\kappa\beta)^{\frac{10}{9}}h^{\frac{4}{9}}\ell^{\frac{4}{9}}|\log h|^{K}. \tag{28.5.31}$$

[29)] $x\mapsto Z^{\frac{1}{3}}x$, $\tau\mapsto Z^{\frac{4}{3}}\tau$, $1\mapsto h=Z^{-\frac{1}{3}}$, $B\mapsto\beta=Z^{-1}$, $\alpha\mapsto\kappa=\alpha Z$; recall that $\beta h\lesssim 1\iff B\lesssim Z^{-\frac{4}{3}}$.

Then calculating ν_1 and plugging it and $h_1 = h\ell$ into $C\zeta^2\bigl(h_1^{-1} + h_1^{-\frac{1}{3}}\nu^{\frac{4}{3}}\bigr)$ one can see easily that here all terms contain ℓ in the negative powers.

Then summation by ℓ results in the same expression calculated for $\ell = r^*$ and one can observe easily that it does not exceed (28.5.29), (28.5.30) for $B \le Z$, $Z \le B \le Z^{\frac{4}{3}}$ respectively. One can see easily that dealing with terms $C\zeta^2 \times \mu^3\beta^{-2}h^{-2}$ due to (28.3.61) and $C\zeta^2\nu|\log h|$ (see Propositions 28.3.13 and 28.3.15) leads to smaller expressions.

Furthermore, using the standard propagation arguments one can upgrade (28.5.29) to (28.5.30). Therefore we conclude that

(28.5.32) Proposition 28.5.10 remains true for ψ_m supported in the zone $\{x\colon |x - y_m| \le \epsilon\,\mathsf{min}(\bar{r}, 1)\}$.

Consider now the contribution of the *boundary zone* $\{x\colon \ell(x) \ge \mathsf{min}(\bar{r}, 1)\}$.

(a) Let us start from more difficult and interesting case $B \ge 1$ assuming first that $Z = N$. Rescale this zone first $x \mapsto xB^{\frac{1}{4}}$, $\tau \mapsto \tau B^{-1}$, then we have $h_1 = B^{-\frac{1}{4}}$, $\beta_1 = B^{\frac{1}{4}}$. Observe that

(28.5.33) After this rescaling a rescaled magnetic field satisfies $|\partial^2 A'| \le \nu_1$ where ν_1 is given by (28.5.31) for $\ell = \bar{r}$.

As we know from Subsection 26.5.1, after the first scaling there exists the scaling function γ such that

$$|\partial^\alpha V| \le C\gamma^{4-|\alpha|} \qquad |\alpha| \le 4, \tag{28.5.34}$$

$$V \asymp \gamma^4, \qquad |\partial^2 V| \asymp \gamma^2 \tag{28.5.35}$$

and therefore we can use a γ-admissible partition. Then scaling again $x \mapsto x\gamma^{-1}$, $\tau \mapsto \tau\gamma^{-4}$, $h_1 \mapsto h_2 = h\gamma^{-3}$, $\beta_1 \mapsto \beta_2 = \beta_1\gamma^{-1}$ and $\nu_1 \mapsto \nu_2 = \nu_1$ we see that non-degeneracy assumption (28.3.65) is fulfilled and therefore according to Proposition 28.4.3(ii) the contribution of γ-element to the remainder does not exceed $CB\gamma^4\beta_2\bigl(1 + h_2^{\frac{2}{3}}\nu_2^{\frac{4}{3}}\bigr)$ because now $\beta_2 h_2 \ge 1$. Then the total contribution of such elements does not exceed

$$CB \int \beta_2\bigl(1 + h_2^{\frac{2}{3}}\nu_2^{\frac{4}{3}}\bigr)\gamma^{-3}\,dx \tag{28.5.36}$$

with integral taken over zone $\{x\colon \gamma(x) \ge \bar{\gamma} \coloneqq h_1^{\frac{1}{3}}\}$ where $h_2 \le 1$. Plugging β_2, h_2 and $\nu_2 = \nu_1$ we get in the second term γ^{-2} which is not good. Let us apply Remark 28.5.9.

Recall that $A'(x)$ is a solution of the Laplace equation and therefore $A'(x) = \int |x-y|^{-1} F(y)\,dy$; then $A'(x)$ with $x \in B(z, \gamma(z))$ can be decomposed into the sum of two terms; the first one is given by the integral over $\{y : |y-z| \ge \frac{3}{2}\gamma(z)\}$ and therefore is smooth and could be included in $A^0(x)$ while the second is given by integral over $\{y : |y-z| \le 2\gamma(z)\}$ and could be estimated by (28.3.46) with β, h, κ and ν replaced by β_2, h_2, $\kappa_2 = \kappa_1\gamma^5$ and ν_1 and then one can see easily that $\nu_2 \le \nu_1\gamma^{\frac{3}{2}+\delta}$. Therefore we conclude that

Remark 28.5.11. In the boundary zone calculating trace term, N- and D-terms one can take $\nu_2 = \nu_1\gamma^{\frac{3}{2}+\delta}$ with $\delta > 0$.

Plugging this improved ν_2 into (28.5.36) we get everywhere γ in the positive power and therefore expression (28.5.36) does not exceed the integrand for $\gamma = 1$, which is $CB\beta_1\bigl(1 + h_1^{\frac{2}{3}}\nu_1^{\frac{4}{3}}\bigr)$, which we already got when estimating the contribution of the regular zone.

(b) In the zone $\{x\colon \gamma(x) \le \bar{\gamma}\}$ we just reset $\gamma = \bar{\gamma}$ and since $h_2 \asymp 1$ we do not need any non-degeneracy condition here. Thus its contribution does not exceed $CB\beta_1$. Therefore we arrive to Proposition 28.5.12(i) with $Z = N$ and $B \ge 1$.

(c) In the case $B \le 1$ we need no non-degeneracy assumption in the zone $\{x\colon \ell(x) \gtrsim 1\}$ as $h_1 = 1$; Proposition 28.5.12(i) has been proven in this case as well.

(d) Explore now case $N < Z$. Then eventually we will need to take $V = W_B^{\mathsf{TF}} + \lambda$, where λ is a chemical potential. In this case the same arguments hold provided $B\gamma^4 \le |\lambda| \implies \gamma \le h_1^{\frac{1}{3}}$ which is equivalent to $(Z-N)_+ \le B^{\frac{5}{12}}$ since in this case $-\lambda \asymp (Z-N)_+\bar{r}^{-1} = (Z-N)_+B^{\frac{1}{4}}$.

(e) Further, for $M = 1$ we do not need assumption $N = Z$ since we can always refer to non-degeneracy condition (28.5.23), which is fulfilled, and we arrive to Proposition 28.5.12(ii). Indeed, as in Section 26.5 we do not partition further elements where this condition is fulfilled. We arrive to Proposition 28.5.12(i), (ii) below.

(f) Furthermore, consider case $M \ge 2$ and $B^{\frac{5}{12}} \le (Z-N)_+ \le B^{\frac{3}{4}}$. Then $B\bar\gamma^4 = (Z-N)_+ B^{\frac{1}{4}}$ and $\bar\gamma = (Z-N)_+^{\frac{1}{4}} B^{-\frac{3}{16}}$.

In this case the contribution of $\bar\gamma$-element (in the excess what was prescribed before) does not exceed term (28.3.66) multiplied by ζ^2. Note that two last terms in (28.3.66) are not new. Meanwhile, the first term there (i.e. $C\beta\nu^{\frac{6}{5}}h^{\frac{2}{5}}$) becomes $CB\bar\gamma^4 \times B^{\frac{1}{4}}\bar\gamma^{-1} \times \bar\nu^{\frac{6}{5}} \times (B^{-\frac{1}{4}}\bar\gamma^{-3})^{\frac{2}{5}}$ and after multiplication by $\bar\gamma^{-2}|\log\bar\gamma|$ it has $\bar\gamma$ in the negative degree, so it does not exceed the same value for $\bar\gamma = B^{-\frac{1}{12}}$. After easy but tedious calculations one can see that it is less than (28.5.22).

This leaves us with the second term in (28.3.66) (i.e. $C\beta h^{-\frac{1}{2}}$), which becomes $CB\bar\gamma^4 \times B^{\frac{1}{4}}\bar\gamma^{-1} \times B^{\frac{1}{8}}\bar\gamma^{\frac{3}{2}}$ and after multiplication by $\bar\gamma^{-2}|\log\bar\gamma|$ we get

$$CB^{\frac{11}{8}}\bar\gamma^{\frac{5}{2}}|\log\bar\gamma| \asymp CQ' := CB^{\frac{29}{32}}(Z-N)_+^{\frac{5}{8}}\bigl(1+|\log(Z-N)_+B^{-\frac{3}{4}}|\bigr). \tag{28.5.37}$$

(g) Finally, if $(Z-N)_+ \ge B^{\frac{3}{4}}$, we again end up with the term $C\zeta^2\beta h^{-\frac{1}{2}}$ this time with $\zeta^2 = (Z-N)_+^{\frac{4}{3}}$, $h = (Z-N)_+^{-\frac{1}{3}}$, $\beta = B(Z-N)_+^{-1}$ (now $\bar\gamma \asymp 1$); so we arrive to

$$CQ'' := CB(Z-N)_+^{\frac{1}{2}}. \tag{28.5.38}$$

Thus we arrive to Proposition 28.5.12(iii), (iv) below.

Proposition 28.5.12. *Let $V = W_B^{\mathsf{TF}} + \lambda$ be Thomas-Fermi potential with $N \le Z$, $N \asymp Z_1 \asymp Z_2 \asymp \dots \asymp Z_M$, $B \lesssim Z^{\frac{4}{3}}$ and chemical potential λ. Let assumption* (28.5.27) *be fulfilled. Then*

(i) For $Z = N$ ($\lambda = 0$) the trace remainder

$$|\mathsf{Tr}(H^-_{A,V}) + \int P_{Bh}(V)\,dx - \sum_{1\le m\le M}\Bigl(\mathsf{Tr}(H^-_{A,V_m}\psi_m) - \int P_0(V_m)\psi_m\,dx\Bigr)| \tag{28.5.39}$$

does not exceed CQ_0 with Q_0 defined by (28.5.29)*; the same is true for $(Z-N)_+ \le B^{\frac{5}{12}}$.*

(ii) For $M = 1$ *the trace remainder* (28.5.39) *also does not exceed* CQ_0.

(iii) For $M \ge 1$ *and* $B^{\frac{5}{12}} \le (Z-N)_+ \le B^{\frac{3}{4}}$ *the trace remainder* (28.5.39) *does not exceed* $C(Q_0 + Q')$ *with* Q' *defined by* (28.5.37).

(iv) For $M \ge 1$ *and* $(Z-N)_+^{\frac{4}{3}} \ge B$ *the trace remainder* (28.5.39) *does not exceed* $C(Q_0 + Q'')$ *with* Q'' *defined by* (28.5.38).

Remark 28.5.13. (i) For $B \le Z$ expression (28.5.29) could be upgraded to (28.5.30).

(ii) Terms (28.5.38) and (28.5.39) are rather superficial: they do not depend on α and they were not present in Chapter 25. Indeed, using more precise arguments of Chapter 25 one can get rid of them, at least for sufficiently small α.

However in the upper estimate we will need to deal with D-term as well and this would give us a far larger error.

28.5.4 Endgame

Main Theorem: $M = 1$

For $M = 1$ we almost immediately arrive to the following statement which we formulate in "rescaled" terms:

Theorem 28.5.14. *Let* $V = W_B^{\mathsf{TF}} + \lambda$ *be a Thomas-Fermi potential as* $B \le Z^{\frac{4}{3}}$, $N \asymp Z$ *and* $M = 1$. *Then*

$$\mathcal{E}_0^* + 2S(\alpha Z)Z^2 - CZ^{\frac{5}{3}}\big(1+\alpha B\big) \le \mathsf{E}_\alpha^* \le \mathcal{E}_\alpha^* + 2S(\alpha Z)Z^2 + C\big(Z^{\frac{5}{3}} + \alpha B^2 Z^{\frac{1}{3}}\big), \tag{28.5.40}$$

where $S(\alpha Z)Z^2$ *is a Scott correction term derived in Section 27.3.*

Proof: Estimate from above. We already know from (28.5.20) that after the standard rescaling for any magnetic field A' satisfying the same estimates as a minimizer of $\mathsf{E}_\kappa(A')$ the following estimate holds:

$$\mathsf{Tr}(H^-_{A,V}) + \kappa^{-1}h^{-2}\|\partial A'\|^2 \le -h^{-3}\int P_{Bh}(V)\,dx + \underbrace{(28.5.21) + \kappa^{-1}h^{-2}\|\partial A'\|^2}_{} + (28.5.22). \tag{28.5.41}$$

Here the left-hand expression is $\mathsf{E}_\kappa(A') \ge \mathsf{E}^*_\kappa$ and in expression (28.5.21) we take $\psi_1 = 1$.

Let us pick up A' which is a minimizer for the Coulomb potential $V_1(x) = z_1|x - y_1|^{-1}$ without any external magnetic field[30]. Then we can replace the selected sum of two terms by $2h^{-2}S(\kappa z)z^2$ (in virtue of Subsubsection 27.3.5.1 an error does not exceed (28.5.22)).

Unfortunately A' is not a minimizer for a local problem; however we can replace $P_{Bh}(V)$ by

$$P_{\beta h}(V) + \partial_\beta P_{\beta h}(V)\cdot \Phi, \qquad \Phi = \partial_1 A'_2 - \partial_2 A'_1 \tag{28.5.42}$$

and, if we apply partition, then on each partition element an error in (28.5.41) does not exceed $C\kappa_1\zeta^2\beta_1 h_1^{-\frac12} = C\kappa\beta h^{-\frac12}\zeta^{\frac52}\ell^{\frac52}$. Indeed, we know that for a minimizer $\|\partial A'\| \le C(\kappa + \kappa^{\frac12}h^{\frac12})$ (also see in the proof of the estimate from below).

Summation over partition results in $C\kappa\beta^2$ and we arrive to

$$-h^{-3}\int P_{\beta h}(V)\,dx - h^{-3}\int \partial_\beta P_{\beta h}(V)\cdot\Phi\,dx + C(h^{-1}+\kappa\beta^2) \tag{28.5.43}$$

in the right-hand expression where the first term is $\mathcal{E}^*_0$ [31].

The second term is rather unpleasant because we cannot estimate it by anything better than $C\beta h^{-1}$ (see in the estimate from below) but *here* we have a trick[32]: we replace A' by $-A'$ which is also a minimizer for the same Coulomb potential V_1 without external magnetic field. Then Φ and the second term change signs and since nothing else happens we can skip the second term which concludes the proof of the upper estimate.

Scaling back we arrive to the upper estimate in (28.5.40). □

Proof: Estimate from below. Again from (28.5.20) we already know that for a minimizer A' of $\mathsf{E}_\kappa(A')$ estimate (28.5.41) could be reversed

$$\begin{aligned}&\mathsf{Tr}(H^-_{A,V}) + \kappa^{-1}h^{-2}\|\partial A'\|^2\\ &\qquad \ge -h^{-3}\int P_{Bh}(V)\,dx + \underbrace{(28.5.21) + \kappa^{-1}h^{-2}\|\partial A'\|^2} - (28.5.22).\end{aligned} \tag{28.5.44}$$

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[30] Actually since for the Coulomb potential the trace is infinite we take potential $V_1(x)+\tau$ with $\tau<0$, establish estimates and then tend $\tau\to -0$.

[31] I.e. Thomas-Fermi energy calculated for $A'=0$.

[32] Which unfortunately we cannot repeat in estimate from below.

Here the left-hand expression is $\mathsf{E}_\kappa(A') = \mathsf{E}^*_\kappa$ and the selected sum of two terms can be estimated from below by $2h^{-2}S(\kappa z)z^2$.

Again A' is not a minimizer for a local problem; however we can replace $P_{Bh}(V)$ by $P_{\beta h}(V)$ with an error not exceeding

$$Ch^{-3}\int |\partial_\beta P_{\beta h}(V)|\cdot|\partial A'|\,dx + Ch^{-3}\int |\partial^2_\beta P_{\beta h}(V)|\cdot|\partial A'|^2\,dx$$
$$\le Ch^{-3}\|\partial_\beta P_{\beta h}(V)\|\cdot\|\partial A'\| + Ch^{-3}\int |\partial^2_\beta P_{\beta h}(V)|\,dx\times\|\partial A'\|^2$$

with the right-hand expression not exceeding

$$C\beta h^{-1}|\partial A'\| + Ch^{-1}\|\partial A'\|^2. \tag{28.5.45}$$

However we already know that $\mathsf{E}^*_{2\kappa}\ge \mathcal{E}^*_0 - C(\kappa h^{-2}+h^{-1})$ and therefore since $\mathsf{E}_\kappa(A') = \mathsf{E}_{2\kappa}(A') + (2\kappa h^2)^{-1}\|\partial A'\|^2$ we conclude that

$$\|\partial A'\|\le C(\kappa+\kappa^{\frac12}h^{\frac12}); \tag{28.5.46}$$

then expression (28.5.45) does not exceed $C(\kappa\beta+1)h^{-1}$ which concludes the proof of the lower estimate.

Since A' is a minimizer in the presence of the external field, we cannot replace A' by $-A'$ and thus cannot repeat the trick used in the proof of the upper estimate. Thus in the estimate from below we are left with $C\kappa\beta h^{-1}$ rather than with $C\kappa\beta^2$.

Scaling back we arrive to the lower estimate in (28.5.40). □

Remark 28.5.15. (i) It is a very disheartening that our estimate deteriorated here. However it may be that indeed the better estimate does not hold due to the *entanglement* of the singularity and the regular zone via self-generated magnetic field. Still we did not loose Scott correction term as long as $\kappa\beta h\ll 1$.

(ii) Recall that in the Section 27.3 we already had an entanglement of different singularities which obviously remains with us for $M\ge 2$. Surely both of these entanglements matter only if we are looking for the remainder estimate better than $O(\kappa h^{-2})$. Otherwise we can just pick up $A'=0$ near singularities and the Scott correction term equal to $2h^{-2}S(0)z^2$;

(iii) The silver lining is that we do not need all these non-degeneracy conditions for such bad estimate and we expect that our arguments would work for $M \ge 2$. Still for $M \ge 2$ we will need to decouple singularities and to do this we will need to estimate $\|\partial A'\|_{\mathscr{L}^2(\{\ell(x)\asymp d\})}$ where d is the minimal distance between nuclei (see (28.5.19)).

For $\beta \ll 1$, $\kappa \ll h^{\frac{1}{3}}|\log h|^{-\frac{1}{3}}$ we can recover even Schwinger correction term:

Theorem 28.5.16. *Let V be a Thomas-Fermi potential $W_B^{\mathsf{TF}} + \lambda$ rescaled as $B \ll Z$, $N \asymp Z$ and $M = 1$. Let respectively $\beta = BZ^{-1}$, $h = Z^{-\frac{1}{3}}$, and $\kappa = \alpha Z \le \kappa^*$. Then*

$$|\mathsf{E}^*_\alpha - \bigl(\mathcal{E}^*_0 + 2S(\alpha Z)Z^2 + \mathsf{Schwinger}\bigr)| \le CZ^{\frac{5}{3}}\bigl(Z^{-\delta} + B^\delta Z^{-\delta}\bigr) + C\alpha|\log \alpha Z|^{\frac{1}{3}} Z^{\frac{25}{9}} \tag{28.5.47}$$

where $\mathsf{Schwinger}$ *is a Schwinger correction term.*

Proof. The proof is standard like in Section 27.3: we invoke propagation arguments in the zone $\{x\colon (\beta + h)^\sigma \le \ell(x)Z^{\frac{1}{3}} \le (\beta + h)^{-\sigma}\}$. □

28.5.5 N-Term Asymptotics and D-Term Estimate

Consider now N-terms assuming that A' is a minimizer.

Case $M = 1$

Assume first that $M = 1$.

(a) Consider first singular and regular zones (but not the boundary zone). Then after rescaling the contribution of ℓ-element to the remainder does not exceed

$$C\bigl(h_1^{-2} + h_1^{-\frac{5}{3}}\nu_1^{\frac{2}{3}}\bigr) \tag{28.5.48}$$

and summation over zone $\{x\colon \ell(x) \le 1\}$ results in the value of this as $\ell = 1$ i.e.

$$C\bigl(h^{-2} + h^{-\frac{5}{3}}\nu^{*\frac{2}{3}}\bigr), \qquad \nu^* = (\kappa\beta)^{\frac{10}{9}} h^{\frac{4}{9}} |\log h|^K. \tag{28.5.49}$$

Recall that for $\ell \ge 1$ we have $h_1 = h\ell$, $\beta_1 = \beta\ell^3$ and ν_1 is defined by (28.5.31) which is sufficient even without invoking Remark 28.5.9 since all powers of ℓ in (28.5.48) become negative. Therefore the contribution of the regular exterior zone also does not exceed (28.5.49).

(b) Consider now the boundary zone. Let us repeat arguments used in the proof of Proposition 28.5.12: the contribution of γ-element does not exceed $C\beta_2\bigl(h_2^{-1} + h_2^{-\frac{2}{3}}\nu_2^{\frac{2}{3}}\bigr)$ and plugging $\beta_2 = \beta_1\gamma^{-1}$, $h_2 = h_1\gamma^{-3}$ and $\nu_2 = \nu_1\gamma^{\frac{3}{2}+\delta}$ we get $C\beta_1\bigl(h_1^{-1}\gamma^2 + h_1^{-\frac{2}{3}}\nu_1^{\frac{2}{3}}\gamma^{2+\delta}\bigr)$ and therefore the total contribution of the boundary zone to the remainder does not exceed

$$C\beta_1 \int \bigl(h_1^{-1}\gamma^{-1} + h_1^{-\frac{2}{3}}\nu_1^{\frac{2}{3}}\gamma^{-1+\delta}\bigr)\,dx \asymp C\bigl(h_1^{-2}|\log h| + h_1^{-\frac{5}{3}}\nu_1^{\frac{2}{3}}\bigr) \tag{28.5.50}$$

since $\beta_1 = h_1^{-1}$ and from Subsection 26.5.1 we also know that even in the general case

$$\mathsf{D}(\gamma^{-1+s}, \gamma^{-1+s}) \le Cs^{-2} \qquad \text{for } s > 0. \tag{26.5.14}$$

Here we integrate over $\gamma \ge \bar\gamma = h_1^{\frac{1}{3}}$ while the contribution of the zone $\{x\colon \gamma(x) \le \bar\gamma$ does not exceed Ch_1^{-2}.

If $\bar\gamma \ge h_1^{\frac{1}{3}}$ (i.e. $(Z-N)_+ \gtrsim B^{\frac{5}{12}}$) we invoke in the zone $\{x\colon \gamma(x) \le \bar\gamma\}$ the strong non-degeneracy assumption (28.3.60) fulfilled for $M = 1$ and estimate its contribution by the same expression (28.5.50) albeit without the logarithmic term. Similar arguments work also for $(Z-N)_+ \ge B^{\frac{3}{4}}$.

(c) In expression (28.5.50) the logarithmic factor is mildly annoying. However we can get rid of it using our standard propagation arguments like in Subsection 26.6.3; we leave easy details to the reader.

Note that so far in the estimate we also have $P'_{Bh}(V)$ rather than $P'_{\beta h}$; observe however that in the virtue of non-degeneracy assumption (28.3.60) fulfilledfor $M = 1$ the error when we replace $P'_{Bh}(V)$ by $P'_{\beta h}(V)$ does not exceed $C\beta_1 h_1^{-1}$ on the regular elements and Ch_2^{-2} on the boundary elements and summation in both cases results in $O(\beta h^{-1}\bar\ell^2) = O(\beta^{\frac{1}{2}}h^{-\frac{3}{2}})$. Therefore in contrast to the trace estimate there is no deterioration.

Scaling back, we arrive to estimate (28.5.51) below.

(d) In the framework of Theorem 28.5.16 we invoke our standard propagation arguments in the zone $\{x\colon (\beta+h)^{\sigma} \le \ell(x) \le (\beta+h)^{-\sigma}\}$; considering D-terms we invoke these arguments if both elements in the pair belong to this zone. Again, we leave easy details to the reader. Scaling back, we arrive to estimate (28.5.53) below.

(e) We deal with D-term in our usual manner considering double partition and different pairs of partition elements–disjoint when we just apply above arguments since the kernel $|x-y|^{-1}$ is smooth there and non-disjoint when we appeal to the local estimates of D-term. We we leave easy details to the reader.

Proposition 28.5.17. *Let $V = W_B^{\mathsf{TF}} + \lambda$ be a Thomas-Fermi potential for $B \le Z^{\frac{4}{3}}$, $N \asymp Z$ and $M = 1$. Then*

(i)

$$|\int \bigl(\operatorname{tr} e(x,x,0) - P'_B(V(x))\,dx| \le CR \tag{28.5.51}$$

and

$$\mathsf{D}\bigl(\operatorname{tr} e(x,x,0) - P'_B(V(x),\ \operatorname{tr} e(x,x,0) - P'_B(V(x)\bigr) \le CZ^{\frac{1}{3}}R^2 \tag{28.5.52}$$

with

$$R = R_0 \coloneqq DefZ^{\frac{2}{3}} + Z^{\frac{5}{9}}\nu^{*\frac{2}{3}}, \qquad \nu^* = (\alpha B)^{\frac{10}{9}} Z^{-\frac{4}{27}} |\log Z|^K. \tag{28.5.53}$$

(ii) In the framework of Theorem 28.5.16 estimates (28.5.51) *and* (28.5.52) *hold with*

$$R \coloneqq CZ^{\frac{2}{3}}\bigl(Z^{-\delta} + B^{\delta}Z^{-\delta} + (\alpha Z)^{\delta}\bigr). \tag{28.5.54}$$

Case $M \ge 2$

Consider now $M \ge 2$. In comparison with the case $M = 1$ we should get some extra terms because

(a) First, in the regular zone with $\ell(x) \ge \epsilon d$ the strong non-degeneracy assumption (28.3.60) is replaced by the strong assumption (28.3.65).

(b) In the boundary zone with $\gamma(x) \le \bar{\gamma}$ (and with $\bar{\gamma} \ge h_1^{\frac{1}{3}}$) there is no non-degeneracy assumption at all.

Consider a regular zone first. According to (28.3.86) we get an extra term

$$C\beta_1 h_1^{-1}\nu_1^{\frac12} \asymp C\beta h^{-1}(\kappa\beta)^{\frac59}\ell^{\frac{20}{9}}|\log h|^K \tag{28.5.55}$$

since other extra terms are smaller; summation over the regular zone results in the value with $\ell = \bar r$; since $\bar r = (\beta h)^{-\frac14}$ we get $C\kappa^{\frac59}\beta h^{-\frac{14}{9}}|\log h|^K$. Scaling back we arrive to

$$CR'' = C\alpha^{\frac59} BZ^{\frac{2}{27}}|\log Z|^K. \tag{28.5.56}$$

Using γ-partition in the boundary zone and plugging $h_2 = h_1\gamma^{-3}$, $\beta_2 = \beta_1\gamma^{-1}$, $\nu_2 = \nu_1\gamma^{\frac32+\delta}$ (see Remark 28.5.11) and using (28.5.52) we prove easily that if $\bar\gamma = h_1^{\frac13}$ (i.e. $(Z-N)_+ \le B^{\frac{5}{12}}$) the contribution of the boundary zone is smaller than CR''.

On the other hand, in the case $B^{\frac{5}{12}} \le (Z-N)_+ \le B^{\frac34}$ we need to add $C\beta_2 h_2^{-\frac32}\bar\gamma^{-2}(1+|\log\bar\gamma|)$. Plugging $h_2 = h_1\bar\gamma^{-3}$, $\beta_2 = \beta_1\bar\gamma^{-1}$, $\bar\gamma = (Z-N)_+^{\frac14}B^{-\frac{3}{16}}$ and scaling back we arrive to

$$CR''' = C(Z-N)_+^{\frac38}B^{\frac{11}{32}}\bigl(1+|\log(Z-N)_+B^{-\frac34}|\bigr). \tag{28.5.57}$$

Finally if $(Z-N)_+ \ge B^{\frac34}$ we get $C\beta_1 h_1^{-\frac32}$ with $\ell = (Z-N)_+^{-\frac13}Z^{\frac13}$ i.e.

$$CR''' = C(Z-N)_+^{-\frac12}B. \tag{28.5.58}$$

Thus we arrive to Proposition 28.5.18(i). The similar arguments work for D-term and we arrive to Proposition 28.5.18(ii)

Proposition 28.5.18. *Let $V = W_B^{\mathsf{TF}} + \lambda$ be a Thomas-Fermi potential with $B \le Z^{\frac43}$, $N \asymp Z$ and $M \ge 2$. Let $d \ge Z^{-\frac13}$. Then*

(i) Estimate (28.5.51) *holds with $R = R_0 + R''$ and R'' defined by* (28.5.56) *for $(Z-N)_+ \le B^{\frac{5}{12}}$, $R = R_0 + R'' + R'''$ and R''' defined by* (28.5.57) *for $B^{\frac{5}{12}} \le (Z-N)_+ \le B^{\frac34}$, $R = R_0 + R'''$ and R''' defined by* (28.5.58) *for $(Z-N)_+ \ge B^{\frac34}$.*

Furthermore, if $B \le Z$ estimate (28.5.51) *holds with*

$$R = Z^{\frac23}\bigl[Z^{-\delta} + B^\delta Z^{-\delta} + (dZ^{\frac13})^{-\delta} + (\alpha Z)^\delta\bigr]. \tag{28.5.59}$$

(ii) The left-hand expression of (28.5.57) *does not exceed* $CZ^{\frac{1}{3}}R_0^2 + CB^{\frac{1}{4}}R''^2$ *if* $(Z-N)_+ \le B^{\frac{5}{12}}$, $CZ^{\frac{1}{3}}R_0^2 + CB^{\frac{1}{4}}(R''+R''')^2$ *and* R''' *defined by* (28.5.57) *if* $B^{\frac{5}{12}} \le (Z-N)_+ \le B^{\frac{3}{4}}$, $CZ^{\frac{1}{3}}R_0^2 + (Z-N)_+^{\frac{1}{3}}R'''^2$ *and* R''' *defined by* (28.5.58) *if* $(Z-N)_+ \ge B^{\frac{3}{4}}$.

Furthermore, if $B \le Z$ *the left-hand expression of* (28.5.57) *does not exceed* $CZ^{\frac{1}{3}}R^2$ *with* R *defined by* (28.5.59).

28.5.6 More Estimates to a Minimizer

Now we want to provide different kinds of estimates to the minimizer for $\ell(x) \ge Z^{-\frac{1}{3}}$ in the original scale. More precisely, we are looking for

$$\alpha^{-1}\int \phi_r(x)|\partial A'|^2\,dx \tag{28.5.60}$$

with ϕ_r supported in $\{x\colon \ell(x) \asymp r\}$ because it will appear as an error when we decouple singularities for $M \ge 2$ (in this case we should take $r \asymp d$. Due to equation (27.2.14) it is D-type term as well: namely, with the integral taken over $\mathbb{R}^3$ it would be equal to $\alpha Z^{\frac{5}{3}}\mathsf{D}(\phi\Phi_j, \phi\Phi_j)$ calculated in the rescaled coordinates with Φ_j defined by (27.2.14); however with the cut-off the integral kernel $|x-y|^{-1}$ needs to be modified. Recall that the corresponding Weyl expression is 0.

First, using (28.5.16) and decomposition like in the proof of Proposition 27.3.9 we can rewrite (28.5.60) as

$$I \coloneqq \int K(z;x,y)\Phi(x)\Phi(y)\,dxdydz \tag{28.5.61}$$

multiplied by $\alpha Z^{\frac{5}{3}}$; here $K(z;x,y)$ is supported in $\{z\colon \ell(z) \asymp r\}$, singular at $x=z$ and $y=z$ and satisfies

$$|K(z;x,y)| \le |x-z|^{-2}|y-z|^{-2}|x|\cdot|y|(|x|+r)^{-1}(|y|+r)^{-1}. \tag{28.5.62}$$

Here we temporarily replaced r by $Z^{\frac{1}{3}}r$.

Let us make a double ℓ-admissible partition of unity and consider pairs of elements with $\ell(x) \asymp r_1$ and $\ell(y) \asymp r_2$. There are three cases: $r_1 \le \epsilon r$, $r_1 \asymp r$ and $r_1 \ge cr$, and so also for r_2, and we can consider only pure pairs.

Case $M = 1$

Assume first that $M = 1$. Then the contribution of each pair with $r_j \le \epsilon r$ (assuming that they belong to regular zone) does not exceed

$$C\zeta_1 r_1\bigl(h_1^{-2} + h_1^{-\frac{5}{3}}\nu_1^{\frac{2}{3}}\bigr) \times \zeta_2 r_2\bigl(h_2^{-2} + h_2^{-\frac{5}{3}}\nu_2^{\frac{2}{3}}\bigr) \times r^{-3}, \tag{28.5.63}$$

where $\zeta_j = r_j^{-2}$, $h_j = hr_j$, $\nu_j = (\kappa\beta)^{\frac{10}{9}} h^{\frac{4}{9}} r_j^{\frac{4}{9}}$ if $r_j \ge 1$ and $\zeta_j = r_j^{-\frac{1}{2}}$, $h_j = hr_j^{-\frac{1}{2}}$, $\nu_j = (\kappa\beta)^{\frac{10}{9}} h^{\frac{4}{9}}$ if $r_j \le 1$. Double summation returns its value if $r_1 = r_2 = 1$ i.e.

$$Ch^{-4}\bigl(1 + h^{\frac{2}{3}}{\nu^*}^{\frac{4}{3}}\bigr) r^{-3}. \tag{28.5.64}$$

Further, using Fefferman-de Llave decomposition one can prove easily that the contribution of pairs with $r_1 \asymp r_2 \asymp r$ (the only case when we have a singular kernel) does not exceed (28.5.63) calculated as $r_1 = r_2 = r$ which is decaying function of r and therefore does not exceed (28.5.64).

Furthermore, the contribution of each pair with $r_j \ge Cr$ (assuming that they belong to regular zone) does not exceed

$$C\zeta_1 r_1^{-2}\bigl(h_1^{-2} + h_1^{-\frac{5}{3}}\nu_1^{\frac{2}{3}}\bigr) \times \zeta_2 r_2^{-2}\bigl(h_2^{-2} + h_2^{-\frac{5}{3}}\nu_2^{\frac{2}{3}}\bigr) \times r^{3}, \tag{28.5.65}$$

and the double summation returns its value with $r_1 = r_2 = r$ which is the same as (28.5.63) calculated with $r_1 = r_2 = r$ and again does not exceed (28.5.64).

Finally, considering $r_1 \asymp r_2 \asymp \bar{r}$ we apply for $(Z - N)_+ \le B^{\frac{3}{4}}$ secondary partitions with respect to x and y and using our standard arguments we estimate the contribution of this zone by (28.5.65) calculated with $r_1 = r_2 = \bar{r}$.

Therefore we estimated expression (28.5.61) by (28.5.64). In particular, if $\kappa\beta \le h^{-\frac{17}{20}}|\log h|^{-K}$ i.e. $\alpha B \le Z^{\frac{17}{60}}|\log Z|^{-K}$), then $\nu^* \le h^{-\frac{1}{2}}$ and expression (28.5.61) does not exceed $Ch^{-4}r^{-3}$. Plugging $h = Z^{-\frac{1}{3}}$, multiplying by $\alpha Z^{\frac{5}{3}}$ and replacing r by $Z^{\frac{1}{3}}r$ we get $\alpha Z^2 r^{-3}$ thus proving Proposition 28.5.19(i).

On the other hand, for $\kappa\beta \ge h^{-\frac{17}{20}}|\log h|^{-K}$ estimate (28.5.61) by (28.5.64) could be improved. Indeed, let us apply all the above arguments only for $r_j \ge t$ with $1 \le t \le \epsilon r$. Then we get expression (28.5.63) with $r_1 = r_2 = t$ i.e.

$$Ch^{-4} \times {\nu^*}^{\frac{4}{3}} h^{\frac{2}{3}} t^{-\frac{128}{27}} r^{-3} \tag{28.5.66}$$

where we consider only term possibly exceeding $Ch^{-4}r^{-3}$.

To estimate the contribution of zone $\{x,y\colon \ell(x)\le t, \ell(y)\le t\}$ we replace Φ_j by $(\kappa h^{-2})\Delta A'_j$ and using the standard estimate for operator norm in $\mathscr{L}^2$ we conclude that the corresponding part of expression (28.5.61) does not exceed $C\kappa^{-2}h^{-4}\|\partial A'\|^2\times t^3r^{-3}\le Ch^{-4}t^3r^{-3}$ as long as $\|\partial A'\|\le C\kappa$.

Adding this to (28.5.66) and minimizing by $t\lesssim r$ we get $Ch^{-4}(\nu^{*2}h)^{\frac{54}{209}}r^{-3}$ provided $r\ge (\nu^{*2}h)^{\frac{18}{209}}$.

Plugging ν^* and h, multiplying by $\alpha Z^{\frac53}$ and replacing r by $Z^{\frac13}r$ we arrive to Proposition 28.5.19(ii).

Proposition 28.5.19. *Let $V=W_B^{\mathsf{TF}}+\lambda$ be a Thomas-Fermi potential with $B\le Z^{\frac43}$, $N\asymp Z$ and $M=1$. Then*

(i) A minimizer satisfies

$$\alpha^{-1}\int_{\{x\colon \ell(x)\ge r\}} |\partial A'|^2\,dx \le CT_0r^{-3}=C\alpha Z^2r^{-3} \tag{28.5.67}$$

for $r\ge r_*=Z^{-\frac13}$ holds provided

$$\alpha B\le Z^{\frac{17}{60}}|\log Z|^{-K}. \tag{28.5.68}$$

(ii) Otherwise a minimizer satisfies

$$\alpha^{-1}\int_{\{x\colon \ell(x)\ge r\}} |\partial A'|^2\,dx \le CT_0r^{-3}=C\alpha Z^3r_*^3r^{-3} \qquad \text{for } r\ge r_* \tag{28.5.69}$$

with

$$r_*:=(\alpha B)^{\frac{40}{209}}Z^{-\frac{81}{209}}|\log Z|^K\gtrsim Z^{-\frac13}. \tag{28.5.70}$$

Remark 28.5.20. (i) Assumption (28.5.68) means exactly that $r_*\le Z^{-\frac13}$.

(ii) Our usual approach implies that for $B\le Z$ the Tauberian error estimate could be slightly improved but it has no implications here because in contrast to $e(x,x,\tau)$ where the main term in the formal asymptotic decomposition is $h^{-3}P'_{Bh}$ and the next one is $\asymp\beta^2h^{-2}$, in Φ_j the main term is 0 and the next one is ηh^{-2} with the coefficient η depending on A' and trying to calculate it and plug the corresponding term into (28.5.61) instead of Φ_j will certainly result in the identity.

(iii) It may happen that $r_*\ge \bar r$; then it follows from the proof that r_* should be truncated to $\bar r$ in (28.5.69). Moreover, estimate (28.5.69) with $r_*=\bar r$ holds even for $M\ge 2$ if no non-degeneracy assumption is made.

Case $M \ge 2$

Assume now that $M \ge 2$ and that (28.5.19) is fulfilled. Then we need to take into account excess terms in our estimates. In the regular zone such excess term is

$$C\zeta_1 r_1 \beta_1 h_1^{-1} \nu_1^{\frac{1}{2}} \times \zeta_2 r_2 \beta_2 h_2^{-1} \nu_2^{\frac{1}{2}} \times r^{-3} \qquad \text{for } r_1 \le r, r_2 \le r \tag{28.5.71}$$

and

$$C\zeta_1 r_1^{-2} \beta_1 h_1^{-1} \nu_1^{\frac{1}{2}} \times \zeta_2 r_2^{-2} \beta_2 h_2^{-1} \nu_2^{\frac{1}{2}} \times r^{3} \qquad \text{for } r_1 \ge r, r_2 \ge r. \tag{28.5.72}$$

Plugging β_j, h_j and ν_j one observes easily that the former is a growing and the latter is a decaying function of r_j and these expressions coincide at $r_1 = r_2 = r$. To decouple singularities we need to consider $r \le \epsilon d$ where d is the minimal distance between singularities; so we will assume this. Observe that extra terms appear only if $r_j \ge \epsilon d$, so we need to consider only (28.5.72).

Therefore if $d \le \bar{r}$ the summation results in expression (28.5.72) calculated at $r_1 = r_2 = d$ which is

$$C\beta^2 h^{-\frac{14}{9}} (\kappa\beta)^{\frac{10}{9}} |\log h|^K d^{-\frac{32}{9}} r^3,$$

which results in the original settings in[33)]

$$CT' r^3 = C(\alpha B)^{\frac{19}{9}} B d^{-\frac{32}{9}} r^3; \tag{28.5.73}$$

recall that we plug β, h, κ, replace d and r by $Z^{\frac{1}{3}} d$ and $Z^{\frac{1}{3}} r$, and multiply by $\alpha Z^{\frac{5}{3}}$.

Meanwhile, using our standard arguments one can prove easily that the contribution of the boundary zone is less than this provided $(Z-N)_+ \le B^{\frac{5}{12}}$.

On the other hand, for $B^{\frac{5}{12}} \le (Z-N)_+ \le B^{\frac{3}{4}}$ an extra contribution of the boundary zone does not exceed

$$C\beta^2 h^{-3} \bar{\gamma}^{13} \bar{r}^{-5} r^3 (1 + |\log \bar{\gamma}|)^2,$$

which results in the original settings in[33)]

$$CT'' r^3 = C\alpha (Z-N)_+^{\frac{13}{4}} B^{\frac{13}{16}} \big(1 + |\log (Z-N)_+ B^{-\frac{3}{4}}|\big)^2 r^3. \tag{28.5.74}$$

[33)] I.e. after we plug κ, β, h, replace d and r by $Z^{\frac{1}{3}} d$ and $Z^{\frac{1}{3}} r$ and multiply by $\alpha Z^{\frac{5}{3}}$.

Finally, for $(Z-N)_+ \ge B^{\frac{3}{4}}$ an extra contribution of the boundary zone does not exceed

$$C\beta^2 h^{-3}\bar{r}^{-5}r^3,$$

which results in the original settings in[33)]

$$CT''r^3 = C\alpha(Z-N)_+^{\frac{5}{3}}B^2. \tag{28.5.75}$$

Thus we have proven

Proposition 28.5.21. *Let $V = W_B^{\mathsf{TF}} + \lambda$ be Thomas-Fermi potential with $B \le Z^{\frac{4}{3}}$, $N \asymp Z_1 \asymp Z_2 \asymp ... \asymp Z_M$ and $M \ge 2$. Let $r_* \ll r \ll d \le \epsilon\bar{r}$.*

Then expression

$$\alpha^{-1}\int_{\{x:\, \ell(x)\asymp r\}} |\partial A'|^2\,dx \tag{28.5.76}$$

does not exceed $C\big(T_0 r^{-3} + (T'+T'')r^3\big)$ with T_0 defined by (28.5.68) *or* (28.5.69) *and T' defined by* (28.5.73)*, and T'' either 0 (as $(Z-N)_+ \le B^{\frac{5}{12}}$) or defined by* (28.5.74)*, or* (28.5.75)*.*

Remark 28.5.22. (i) Recall that the decoupling error between the singularity and the regular part does not exceed $C\alpha BZ^{\frac{5}{3}}$.

(ii) Meanwhile in these settings $\mathsf{Scott} - \mathsf{Scott}_0 = O(\alpha Z^3)$ and if decoupling error is greater than this there is no point in decoupling.

(iii) Obviously we need to assume that $r_* \le \epsilon\bar{r}$ which implies that

$$(\alpha Z)^{40} B^{\frac{369}{4}} \le \epsilon Z^{121}, \tag{28.5.77}$$

which is just tiny bit stronger than $\alpha Z \lesssim 1$, $B \lesssim Z^{\frac{4}{3}}$. For $(Z-N)_+ \le B^{\frac{3}{4}}$ this condition is also sufficient.

(iv) Decoupling singularities we get an error (28.5.76) with integration over $\{x:\, \ell(x) \asymp r\}$; therefore if $r_* \ll d \le \bar{r}$ then minimizing $T_0 r^{-3} + T'r^3$ by $r : r_* \le r \le d$ we get

$$T_* := \big(T_0(T'+T'')\big)^{\frac{1}{2}} + (T'+T'')r_*^3 + T_0 d^{-3}. \tag{28.5.78}$$

28.5.7 Endgame: $M \ge 2$

For $M \ge 2$ we have two rather different results. In the first (28.5.39) we appeal to the sum of localized trace terms $\sum_{1\le m\le M} \mathsf{Tr}(\psi_m H^-_{A,V_m}\psi_m)$ where ψ_m is supported in $\{x\colon |x-y_m|\le \frac{1}{3}d\}$ (recall that d is the minimal distance between singularities).

In the second one we want to use $2h^{-2}\sum S(\alpha Z_m)Z_m^2$ instead. If $A'=0$ then transition would be immediate. However in our case we need to "decouple" singularities. Therefore in the estimate from below we need results from the previous subsubsection:

Theorem 28.5.23. *Let $V = W_B^{\mathsf{TF}} + \lambda$ be a Thomas-Fermi potential with $B \le Z^{\frac{4}{3}}$, $N \asymp Z$ and $M \ge 2$. Let $\kappa = \alpha Z \le \kappa^*$. Then if $r_* \le d \le \bar r$*

$$|\mathsf{E}^*_\alpha + \int P_B(V)\,dx - 2\sum S(\alpha Z_m)Z_m^2 - \mathsf{Schwinger}| \tag{28.5.79}$$

does not exceed $C(Q+T)$ where Q is the trace estimate obtained in Proposition 28.5.12(iii)–(iv) and T_* is an estimate for expression (28.5.76) *given by* (28.5.78).

Proof. (i) In the estimate from below we just replace A' in $\mathsf{E}_\alpha(A)$ by the sum $\sum_{1\le m\le M} A'\psi_m$ with ψ_m supported in $\{x\colon |x-y_m|\le \frac{1}{3}r\}$ and equal 1 in $\{x\colon |x-y_m|\le \frac{1}{4}r\}$ where r is the minimal distance between singularities, and observe that $\alpha^{-1}\|\partial A'\|^2$ increased by no more than CT.

(ii) In the estimate from above we just plug into $\mathsf{E}_\alpha(A)$ $A' = \sum_{1\le m\le M} A'_m\psi_m$ with A'_m minimizers for a single-singularity potential V_m. □

Remark 28.5.24. (i) Theorem 28.5.23 makes sense only if $r_* \ll d \le \bar r$ and $T_* \ll \alpha Z^3$; if any of these assumptions fails, we observe that $\mathsf{E}_\alpha(A')$ is greater than $\mathcal{E}_0 + \mathsf{Schwinger} - CQ - C\alpha Z^3$ and in this case we can replace $S(\alpha Z_m)$ by $S(0)$; in this case in the upper estimate we pick up $A'=0$.

(ii) In the estimate from above $T = \max_m T_m$ with T_m an estimate for a single-singularity potential V_m delivered by Proposition 28.5.19; thus $T = T_0 d^{-3}$ if $r_* \le d \le \bar r$. Still it is at least $C\alpha B^{\frac{3}{4}}Z^2$ while decoupling error of singularity and the regular zone is $C\alpha BZ^{\frac{5}{3}}$ which is smaller.

28.6 Global Trace Asymptotics in the Case of Thomas-Fermi Potential: $Z^{\frac{4}{3}} \le B \le Z^3$

28.6.1 Trace Estimates

In this Section we consider the case of $Z^{\frac{4}{3}} \le B \le Z^3$ (corresponding to $\beta h \ge 1$ after rescaling[34)]). We start with

Remark 28.6.1. (i) In contrast to the previous Section 28.5 in this case the remainder estimate will be at least $C\kappa h^{-2}$ and therefore there will be no difficulty to decouple between singularities or between singularities and a regular zone and the Scott correction term will be either $\sum 2S(0)Z_m^2$ or even absent.

Therefore in the estimate from above we just pick up $A' = 0$ both here and in the multiparticle problem and we will need only N-term and D-terms with $A' = 0$ referring to Chapter 25.

(ii) For $\beta h \ge 1$ we have a major dichotomy unrelated to the self-generated magnetic field:

(a) $\beta h^2 \le 1$ (i.e. $Z^{\frac{4}{3}} \le B \le Z^{\frac{7}{4}}$). In this case Scott correction term could be larger than the contribution of zone $\{x\colon \ell(x) \asymp 1\}$ to the remainder estimate which is no better than $O(\beta)$ and one probably needs to include Scott correction term in the final trace asymptotics.

(b) $\beta h^2 \ge 1$ (i.e. $Z^{\frac{7}{4}} \le B \le Z^3$). In this case the opposite is true; then one does not need to include Scott correction term for sure. Recall that condition $C \le Z^3$ is also unrelated to self-generated magnetic field.

(iii) Recall that we need to impose condition $\kappa\beta h^2 |\log \beta|^K \le 1$ which is equivalent to

$$\alpha B^{\frac{4}{5}} Z^{-\frac{2}{5}} |\log Z|^K \le 1. \tag{28.6.1}$$

[34)] Recall that as $Z^{\frac{4}{3}} \lesssim B \lesssim Z^3$ the scaling is $x \mapsto B^{\frac{2}{5}} Z^{-\frac{1}{5}} x$ (and the original distance between nuclei is at least $B^{-\frac{2}{5}} Z^{\frac{1}{5}}$), $\tau \mapsto B^{-\frac{2}{5}} Z^{\frac{4}{5}} \tau$, $\beta = B^{\frac{2}{5}} Z^{-\frac{1}{5}}$, $h = B^{\frac{1}{5}} Z^{-\frac{3}{5}}$ and $B \lesssim Z^3$.

[35)] Cf. Proposition 28.5.2.

Theorem 28.6.2 [35)]**.** *Let V be Thomas-Fermi potential $V \coloneqq W_B^{\mathsf{TF}} + \lambda$ with $Z^{\frac{4}{3}} \le B \le Z^3$, $N \asymp Z_1 \asymp Z_2 \asymp \dots \asymp Z_M$ and $N \le Z$. Let $\alpha Z \le \kappa^*$ and assumption* (28.6.1) *be fulfilled. Let A' be a minimizer. Then*

(i) If <u>either</u> $(Z-N)_+ \lesssim B^{\frac{4}{15}} Z^{\frac{1}{5}}$ <u>or</u> $M = 1$ and $\alpha B^{\frac{3}{5}} Z^{\frac{1}{5}} \gtrsim 1$, then expression

$$|\mathsf{E}_\kappa(A') - \mathsf{Scott}_0 + \int\int P_B(V(x))\,dx| \tag{28.6.2}$$

does not exceed

$$C\Big(B^{\frac{1}{3}} Z^{\frac{4}{3}} + B^{\frac{4}{5}} Z^{\frac{3}{5}} + \alpha Z^3 + \alpha^{\frac{16}{9}} B^{\frac{82}{45}} Z^{\frac{58}{45}} |\log Z|^K\Big). \tag{28.6.3}$$

(ii) If $M = 1$, $(Z-N)_+ \gtrsim B^{\frac{4}{15}} Z^{\frac{1}{5}}$ and $\alpha B^{\frac{3}{5}} Z^{\frac{1}{5}} \lesssim 1$, then expression (28.6.2) *does not exceed*

$$\begin{aligned} C\Big(B^{\frac{1}{3}} Z^{\frac{4}{3}} + B^{\frac{4}{5}} Z^{\frac{3}{5}} + \alpha Z^3 + \alpha^{\frac{16}{9}} B^{\frac{82}{45}} Z^{\frac{49}{45}} |\log Z|^K \\ + \alpha^{\frac{40}{27}} B^{\frac{74}{45}} Z^{\frac{131}{540}} (Z-N)_+^{\frac{85}{108}} |\log Z|^K\Big). \end{aligned} \tag{28.6.4}$$

(iii) If $M \ge 2$ and $(Z-N)_+ \gtrsim B^{\frac{4}{15}} Z^{\frac{1}{5}}$, then expression (28.6.2) *does not exceed*

$$\begin{aligned} C\Big(B^{\frac{1}{3}} Z^{\frac{4}{3}} + B^{\frac{4}{5}} Z^{\frac{3}{5}} + \alpha Z^3 + \alpha^{\frac{16}{9}} B^{\frac{82}{45}} Z^{\frac{49}{45}} |\log Z|^K \\ + B^{\frac{7}{10}} Z^{\frac{11}{40}} (Z-N)_+^{\frac{5}{8}} |\log (Z-N)_+ Z^{-1}|\Big). \end{aligned} \tag{28.6.5}$$

Proof. (a) Observe first that we need to prove only the estimate from below for expression (28.6.2) without absolute value since in the estimate from above we just pick $A' = 0$ and apply results of Chapter 25 producing estimate $CB^{\frac{4}{5}} Z^{\frac{3}{5}}$.

Proof of the estimate from below repeats the proof of Proposition 28.5.2. Namely, we apply an appropriate partition and on each element ψ_ι^2 estimate from below

$$\mathsf{Tr}^-(\psi_\iota H_{A,V} \psi_\iota) + (C_0\alpha)^{-1} \int |\partial A'|^2\,dx + \int\int P_B(V(x))\psi_\iota^2(x)\,dx \tag{28.6.6}$$

for $\iota \ge 1$ and

$$\mathsf{Tr}^-(\psi_0 H_{A,V}\psi_0) + (C_0\alpha)^{-1}\int |\partial A'|^2\,dx + \int\int P_B(V(x))\psi_0^2(x)\,dx - \mathsf{Scott}_0. \tag{28.6.7}$$

For $B \le Z^2$ we separate zone $\mathcal{X}_0 := \{x\colon \ell(x) \le r_* = B^{-\frac{2}{3}}Z^{\frac{1}{3}}\}$ in which after rescaling $x \mapsto r_*^{-1}x$, $\tau \mapsto r_* Z^{-1}\tau$, we have $\beta = 1$, $h = B^{\frac{1}{3}}Z^{-\frac{2}{3}}$ and $\kappa = \alpha Z$. Then for the corresponding partition element ψ_0^2 expression (28.6.7) in virtue of Chapter 26 does not exceed (by an absolute value)

$$C\bigl(h^{-1} + \kappa|\log\kappa|^{\frac{1}{3}}h^{-\frac{4}{3}}\bigr)\times Zr_*^{-1},$$

which does not exceed (28.6.3) without the last term.

For $B \ge Z^2$ we separate zone $\mathcal{X}_0 := \{x\colon \ell(x) \le r_* = Z^{-1}\}$ and after rescaling $x \mapsto r_*^{-1}x$, $\tau \mapsto Z^{-1}r_*\tau$ we have $h = 1$, $\beta = BZ^{-2}$ and apply variational estimates of Appendix 28.A.1 here. Then the contribution of $\mathcal{X}_0$ to the remainder does not exceed $C\beta \times Zr_*^{-1} = CB \le CB^{\frac{4}{5}}Z^{\frac{3}{5}}$.

(b) Further, a contribution of each regular element with $r_* \le \ell(x) \le \epsilon\bar{r}$ (recall that $\bar{r} = B^{-\frac{2}{5}}Z^{\frac{1}{5}}$) does not exceed

$$C\zeta^2 h_1^{-1}\Bigl(1 + (\kappa_1\beta_1)^{\frac{40}{27}}h_1^{\frac{34}{27}}|\log h_1|^K\Bigr) \qquad \text{for } \beta_1 h_1 \le C_0, \tag{28.6.8}$$

$$C\zeta^2\beta_1\Bigl(1 + (\kappa_1\beta_1)^{\frac{16}{9}}h_1^{\frac{14}{9}}|\log h_1|^K\Bigr) \qquad \text{for } \beta_1 h_1 \ge C_0 \tag{28.6.9}$$

with $\zeta = Z^{\frac{1}{2}}\ell^{-\frac{1}{2}}$, $\beta_1 = BZ^{-\frac{1}{2}}\ell^{\frac{3}{2}}$, $h_1 = Z^{-\frac{1}{2}}\ell^{-\frac{1}{2}}$, $\kappa_1 = \alpha Z$. Indeed, if $\beta_1 h_1 \ge C_0$ and $\ell(x) \le \epsilon\bar{r}$, then the super-strong non-degeneracy assumption (28.4.5) is fulfilled.

Observe that the first term in (28.6.8) has ℓ in the negative power and therefore sums to its value at $\ell = r_*$ while the second term has ℓ in the positive power and therefore sums to its value at $\beta_1 h_1 = 1$ (i.e. $\ell = B^{-1}Z$, $h_1 = B^{\frac{1}{2}}Z^{-1}$, $\beta_1 = B^{-\frac{1}{2}}Z$); one can see easily that it is less than αZ^3. Actually this zone appears only as $B \le Z^2$.

On the other hand, both terms in (28.6.9) have ℓ in the positive power and thus sum to their values at $\ell = \bar{r}$, $\beta_1 = B^{\frac{2}{5}}Z^{-\frac{1}{5}}$, $h_1 = B^{\frac{1}{5}}Z^{-\frac{3}{5}}$ and $\kappa_1 = \alpha Z$ which are exactly the second and the fourth terms in (28.6.2).

(c) *Boundary zone* $\{x\colon \epsilon \le \ell(x) \le c\}$ is treated in the same way albeit with $\zeta = B^{\frac{2}{5}} Z^{\frac{4}{5}} \gamma^2$, $\beta_1 = B^{\frac{2}{5}} Z^{-\frac{1}{5}} \gamma^{-1}$, $h_1 = B^{\frac{1}{5}} Z^{-\frac{3}{5}} \gamma^{-3}$ and $\kappa_1 = \alpha Z \gamma^5$ as long as $\gamma \ge C_0 \bar{\gamma}$ (with $\bar{\gamma} = (Z-N)_+^{\frac{1}{4}} Z^{-\frac{1}{4}}$ but reset to $B^{\frac{1}{15}} Z^{-\frac{1}{5}}$ if the latter is larger). Observe that plugging into (28.6.9) we get in both terms γ in the power greater than 2; therefore after summation with respect to partition elements we get expression (28.6.9) with $\gamma = 1$, $\beta_1 = B^{\frac{2}{5}} Z^{-\frac{1}{5}}$, $h_1 = B^{\frac{1}{5}} Z^{-\frac{3}{5}}$ and $\kappa_1 = \alpha Z$.

This proves the lower estimate (28.6.2) in the framework of the first clause of Statement (i) as contribution of the zone $\gamma \le B^{\frac{1}{15}} Z^{-\frac{1}{5}}$ is estimated easily; we leave it to the reader.

(d) Assume now that $(Z-N)_+ \ge B^{\frac{4}{15}} Z^{\frac{1}{5}}$. We do not partition zone $\{x\colon \gamma(x) \le C_0 \bar{\gamma}\}$ further. In this case we need to take

$$\nu_1 = \bigl((\kappa_1\beta_1)^{\frac{4}{3}} h_1^{\frac{2}{3}} + (\kappa_1\beta_1)^{\frac{10}{9}} h_1^{\frac{4}{9}}\bigr) |\log h_1|^K. \tag{28.6.10}$$

For $M = 1$ we should plug it into

$$C\zeta^2 \beta_1 \bigl(1 + h_1^{\frac{2}{3}} \nu_1^{\frac{4}{3}}\bigr) \gamma^{-2} \tag{28.6.11}$$

with $\zeta = B^{\frac{2}{5}} Z^{\frac{4}{5}} \gamma^2$ and $\gamma = \bar{\gamma}$.

For $\kappa\beta h \gtrsim 1$ we estimate it by the same expression with $\bar{\gamma}$ replaced by 1 but then in ν_1 dominates the first term and we arrive to the lower estimate (28.6.2) in the framework of the second clause of Statement (i).

(e) For $M = 1$ and $\kappa\beta h \lesssim 1$ we need to take into account term (28.6.11) with $\nu_1 = (\kappa_1\beta_1)^{\frac{10}{9}} h_1^{\frac{4}{9}} |\log h_1|^K$ which results in the last term in (28.6.4). Indeed, as $\gamma(x) \le C_0 \bar{\gamma}$ super-strong non-degeneracy condition is not fulfilled.

(f) For $M \ge 2$, we need to take into account term $C\zeta^2 \beta_1 h_1^{-\frac{1}{2}} \gamma^{-2} (1 + |\log \gamma|)$ with $\gamma = \bar{\gamma}$ which results in the last term in (28.6.5). This concludes estimate from below. □

Corollary 28.6.3. *In the framework of Theorem 28.6.2(i), (ii), (iii) expression* $\|\partial A'\|^2$ *does not exceed expressions* (28.6.3), (28.6.4) *and* (28.6.5) *respectively, multiplied by* α.

Proof. Indeed, the same estimates hold with α replaced by 2α. □

The same methods lead us to a similar result for $B \lesssim Z^{\frac{4}{3}}$ (we leave the oproof to the reader):

Theorem 28.6.4. *Let V be Thomas-Fermi potential $W_B^{\mathsf{TF}}+\lambda$ with $B \le Z^{\frac{4}{3}}$, $N \asymp Z_1 \asymp Z_2 \asymp ... \asymp Z_M$ and $N \le Z$. Let $\alpha Z \le \kappa^*$. Let A' be a minimizer. Then*

(i) If either $(Z-N)_+ \lesssim B^{\frac{5}{12}}$ or $M=1$, then expression

$$|\mathsf{E}_\kappa(A') - \mathsf{Scott}_0 + \int\int P_B(V(x))\,dx| \tag{28.6.12}$$

does not exceed

$$C\Big(B^{\frac{1}{3}}Z^{\frac{4}{3}} + Z^{\frac{5}{3}} + \alpha Z^3\Big);. \tag{28.6.13}$$

(ii) If $M \ge 2$ and $B^{\frac{5}{12}} \lesssim (Z-N)_+ \lesssim B^{\frac{3}{4}}$, then expression (28.6.12) *does not exceed*

$$C\Big(B^{\frac{1}{3}}Z^{\frac{4}{3}} + Z^{\frac{5}{3}} + \alpha Z^3 + B^{\frac{29}{32}}(Z-N)_+^{\frac{5}{8}}|\log(Z-N)_+B^{-\frac{3}{4}}|\Big). \tag{28.6.14}$$

(iii) If $M \ge 2$ and $(Z-N)_+ \gtrsim B^{\frac{3}{4}}$, then expression (28.6.12) *does not exceed*

$$C\Big(B^{\frac{1}{3}}Z^{\frac{4}{3}} + Z^{\frac{5}{3}} + \alpha Z^3 + B(Z-N)_+^{\frac{1}{2}}\Big). \tag{28.6.15}$$

28.6.2 Estimates to a Minimizer

Observe that only terms $B^{\frac{1}{3}}Z^{\frac{4}{3}}$ and αZ^3 are associated with the singularities and they are definitely smaller than $B^{\frac{4}{5}}Z^{\frac{3}{5}}$ if $B \gtrsim Z^{\frac{7}{4}}$. Therefore for $B \gtrsim Z^{\frac{7}{4}}$ we do not expect estimate for $\mathsf{D}(\rho_\Psi - \rho_B,\, \rho_\Psi - \rho_B)$ better than expressions (28.6.3)–(28.6.5).

However for $B \lesssim Z^{\frac{7}{4}}$ to improve such estimate we need to study a minimizer. We assume that the remainder in Theorem 28.6.4 does not exceed $C\big(B^{\frac{1}{3}}Z^{\frac{4}{3}} + \alpha Z^3\big)$ and therefore

$$\|\partial A'\| \le C\alpha^{\frac{1}{2}}B^{\frac{1}{6}}Z^{\frac{2}{3}} + C\alpha Z^{\frac{3}{2}} \tag{28.6.16}$$

or, after our usual scaling,

$$\|\partial A'\| \le \varsigma := C\big(\kappa + \kappa^{\frac{1}{2}}\beta^{\frac{1}{6}}h^{\frac{1}{2}}\big). \tag{28.6.17}$$

Observe that for $Z^{\frac{4}{3}} \le B \le Z^2$ we have all zones.

Proposition 28.6.5. *Let V be Thomas-Fermi potential $W_B^{\mathsf{TF}}+\lambda$ rescaled with $Z^{\frac43}\le B\le Z^3$, $N\asymp Z_1\asymp Z_2\asymp ...\asymp Z_M$ and $N\le Z$. Further, let $\alpha Z\lesssim 1$ and $\alpha B^{\frac25}Z^{-\frac25}|\log\beta|^K\lesssim 1$. Then under assumption* (28.6.17) *the minimizer A' satisfies*

(i) If $\ell\le r_*=h^2$ (i.e. $h_1\ge 1$) then

$$|\partial^2 A'|\le C\varsigma h^{-5}. \tag{28.6.18}$$

(ii) If $r_*\le\ell\le c(\beta h)^{-1}$ (i.e. $h_1\le 1$, $\beta_1 h_1\le c$), then

$$\begin{aligned}|\partial^2 A'|\le C\kappa\Big(\varsigma\ell^{-\frac52}+\min\big(\beta^{\frac32}h^{\frac12}\ell^{-1},\ \beta^{\frac12}\ell^{-\frac34}\big)\Big)|\log\ell/r_*|\\ +C(\kappa\beta)^{\frac{10}{9}}h^{\frac49}\ell^{-\frac{19}{9}}|\log\ell/r_*|^K.\end{aligned} \tag{28.6.19}$$

(iii) If $c(\beta h)^{-1}\le\ell\le c$, then

$$\begin{aligned}|\partial^2 A'|\le C\varsigma\ell^{-\frac52}+C\kappa\beta^{\frac12}\ell^{-\frac74}|\log\ell/r_*|\\ +C\Big((\kappa\beta)^{\frac{10}{9}}h^{\frac49}\ell^{-\frac{19}{9}}+(\kappa\beta)^{\frac43}h^{\frac23}\ell^{-\frac56}\Big)|\log\ell/r_*|^K.\end{aligned} \tag{28.6.20}$$

Proof. The proof of these two propositions repeats our standard arguments and is left to the reader. □

These propositions may not provide the best D-term estimate as $\kappa\beta h\le 1$ (i.e. $\alpha B^{\frac35}Z^{\frac15}\le 1$) and could be improved in virtue of the super-strong non-degeneracy assumption fulfilled at regular elements with $\ell\ge c(\beta h)^{-1}$ and at border elements with $\gamma\ge C_0\bar\gamma$. We want to improve term containing $(\kappa\beta)^{\frac{10}{9}}h^{\frac48}\ell^{-\frac{19}{9}}$ in (28.6.20). Assume now that $\beta h^2\le 1$ and $\kappa\beta h\le 1$ (case we need to cover). Let us consider zone $\{x:(\epsilon_0\beta h\ge V(x)\ge C_0|\eta|\}$ where in the corresponding scale super-strong non-degeneracy condition is fulfilled and $\eta=\lambda B^{-\frac25}Z^{\frac15}$.

Proposition 28.6.6. *Let conditions of Proposition 28.6.5 be fulfilled. Then*

(i) Estimate

$$|\partial^2 A'(x)|\le C\Big(\varsigma\ell^{-\frac52}+\kappa\beta^{\frac12}\ell(x)^{-\frac74}+(\kappa\beta)^{\frac43}h^{\frac23}\ell(x)^{-\frac56}\Big)|\log h|^K \tag{28.6.21}$$

holds provided

$$\beta h|\log h|^{-\delta}\ge V(x)\ge|\eta|\cdot|\log h|^{\delta} \tag{28.6.22}$$

with arbitrarily small exponent $\delta>0$.

(ii) Furthermore, if $|\eta| \le |\log h|^{-\delta}$*, then estimate*

$$|\partial^2 A'(x)| \le C\Big(\varsigma + \kappa\beta^{\frac12} + (\kappa\beta)^{\frac43} h^{\frac23} + (\kappa\beta)^{\frac{10}{9}} h^{\frac49} \bar\gamma^{\frac{28}{9}}\Big) |\log h|^K \tag{28.6.23}$$

holds provided $|V(x)| \lesssim 1$, $\bar\gamma = |\eta|^{\frac14}$.

Proof. (i) Let

$$\nu_n(t) = \sup_{\mathcal{X}_n(t)} |\partial^2 A'(x)| \ell(x)^{\frac52}, \qquad \mathcal{X}_n(t) = \{e^{-\epsilon n-1} t \le V(x) \le e^{\epsilon n+1} t\}.$$

Here $\epsilon > 0$ is arbitrarily small (but constants may depend on it). Assume that

$$e^{\epsilon n} t \le \epsilon_0, \qquad e^{-\epsilon n} t \ge C_0 |\eta|, \qquad e^{\epsilon n} \le |\log h|. \tag{28.6.24}$$

Here first two conditions assure that in $\mathcal{X}_n(t)$ the super-strong non-degeneracy assumption is fulfilled after rescaling and the last condition assures that $\ell(x)$ remains the same (modulo logarithmic factor) here. Then

$$\nu_n(t) \le C\Big(\varsigma + \kappa\beta_1^{\frac12} + C\kappa_1\beta_1 h_1^{\frac14} (\nu_{n+1}(t))^{\frac14}\Big) |\log h|^{K_1} \tag{28.6.25}$$

with $\kappa_1 = \kappa$, $\beta_1 = \beta r^{\frac32}$, $h_1 = h r^{-\frac12}$, $r = \min(t^{-1}, 1)$. Indeed, $|\Delta A'|$ in $\mathcal{X}_{n+1/2}(t)$ does not exceed the right-hand expression (without term ς and without logarithmic factor) multiplied by $r^{-\frac52}$ and $C\varsigma$ estimates $\mathscr{L}^2$-norm of $\partial A'$ (and we scale it properly). Recall that we scale $x \mapsto x/r$ if $r \le \epsilon_0$ and $x \mapsto x/\gamma$ if $r \asymp 1$ and in the latter case $\beta_1 = \beta\gamma^{-1}$, $h_1 = h\gamma^{-3}$ and $\kappa_1 = \kappa\gamma^5$; the uncertainty due to r or γ defined modulo logarithmic factor compensates by $|\log h|^{K_1}$ in the right-hand expression of (28.6.25).

Therefore

$$F_n(t) \le C\Big(\varsigma r^{-\frac53} + C\kappa\beta^{\frac12} r^{-\frac{11}{12}} + C\kappa\beta h^{\frac12} \times (F_{n+1}(t))^{\frac14}\Big) |\log h|^{K_1}$$

for $F_n(t) = \nu_n(t) r^{-\frac53}$. Iterating we see that

$$F_0(t) \le C\Big(\varsigma r^{-\frac53} + \kappa\beta^{\frac12} r^{-\frac{11}{12}}\Big) |\log h|^K + C(\kappa\beta h^{\frac12})^{\frac43} |\log h|^K \times \underbrace{(F_{n+1}(t))^{\frac{1}{4^n}}}.$$

Since $F_{n+1}(r) \le h^{-L}$ the last factor is bounded by a constant if $2^n \ge |\log h|$ and we can satisfy this and (28.6.24) as long as (28.6.22) holds. This proves Statement (i).

(ii) Consider the remaining zone $\mathcal{Y} = \{x\colon V(x) \le |\eta| \cdot |\log h|^{\delta}\}$ and let $\nu = \sup_{\mathcal{Y}} |\partial^2 A'|$. Observe that in $\mathcal{Y}$ $|\Delta A'|$ does not exceed

$$C\Big(\kappa\beta^{\frac12} + \kappa\beta h^{\frac12}|\nu|^{\frac14} + \kappa\beta h^{\frac25}\bar\gamma^{\frac{14}{5}}\nu^{\frac{1}{10}}\Big)|\log h|^{K}$$

and on its border (28.6.23) is fulfilled. It implies that (28.6.23) is fulfilled in $\mathcal{Y}$ as well. This proves Statement (ii). □

Remark 28.6.7. If $|\eta| \ge |\log h|^{-\delta}$ then Proposition 28.6.5 is sufficiently good in the remaining zone $\mathcal{Y}$.

28.6.3 N-Term Asymptotics and D-Term Estimates

We leave to the reader not complicated but rather tedious and error-prone task

Problem 28.6.8. Estimate remainder in N-term

$$|\int \big(e(x,x,\lambda') - P_B(V(x)+\lambda')\big)\,dx| \tag{28.6.26}$$

and D-term

$$\mathsf{D}\big(e(x,x,\lambda') - P_B(V(x)+\lambda'),\ e(x,x,\lambda') - P_B(V(x)+\lambda')\big). \tag{28.6.27}$$

After usual rescaling one needs to consider the following zones:

(a) Zone $\{x\colon \ell(x) \le (\beta h)^{-1}|\log h|^{\delta}\}$. In this zone one should use $\beta_1 = \beta\ell^{\frac32}$, $h_1 = h\ell^{-\frac12}$ and $\nu_1 = (\kappa\beta_1)^{\frac{10}{9}} h_1^{\frac49}|\log h_1|^{K}$ (other terms are not important here); then its contributions to expressions (28.6.26) and (28.6.27) do not exceed respectively $C\big(h_1^{-2} + h_1^{-\frac53}\nu_1^{\frac23}\big)$ and $C\big(h_1^{-2} + h_1^{-\frac53}\nu_1^{\frac23}\big)^2\ell^{-1}$ (for $\ell(x) \lesssim (\beta h)^{-1}$ but slight extension just adds some logarithmic factor). In the final tally $\ell = (\beta h)^{-1}|\log h|^{\delta}$.

(b) Zone $\{x\colon (\beta h)^{-1}|\log h|^{\delta} \le \ell(x) \le |\log h|^{-\delta}\}$. In this zone we have the same expressions for h_1 and β_1 and $\nu_1 = (\kappa\beta_1)^{\frac43} h_1^{\frac23}|\log h_1|^{K}$; then its contributions to expressions (28.6.26) and (28.6.27) do not exceed respectively $C\beta_1\big(h_1^{-1} + h_1^{-\frac23}\nu_1^{\frac23}\big)$ and $C\beta_1^2\big(h_1^{-1} + h_1^{-\frac23}\nu_1^{\frac23}\big)^2\ell^{-1}$. In the final tally $\ell = 1$.

(c) Zone $\{x\colon \ell(x) \ge |\log h|^{-\delta},\ \gamma(x) \ge \bar{\gamma}|\log h^{\delta}\}$. In this zone $h_1 = h\gamma^{-3}$ and $\beta_1 = \beta\gamma^{-1}$ and $\nu = (\kappa\beta)^{43} h^{\frac{2}{3}} |\log h|^{K}$ but we use according to Remark 28.5.9 instead $\nu_1 = \kappa_1\beta_1 h_1^{\frac{1}{4}}\nu^{\frac{1}{4}}$ with $\kappa_1 = \kappa\gamma^{5}$. Then

(c_1) For $M = 1$ contributions of γ-element to expressions (28.6.26) and (28.6.27) do not exceed $C\beta_1\bigl(h_1^{-1} + h_1^{-\frac{2}{3}}\nu_1^{\frac{2}{3}}\bigr)$ and $C\beta_1^2\bigl(h_1^{-1} + h_1^{-\frac{2}{3}}\nu_1^{\frac{2}{3}}\bigr)^2$ respectively.

(c_2) As $M \ge 2$ contributions of γ-element to expressions (28.6.26) and (28.6.27) do not exceed respectively $C\beta_1 h_1^{-1}\nu_1^{\frac{1}{2}}$ and $C\beta_1^2 h_1^{-2}\nu_1$.

In the final tally $\gamma = 1$ in both cases (c_1) and (c_2).

(d) Then for $M = 1$ contributions of $\bar{\gamma}$-element to expressions (28.6.26) and (28.6.27) do not exceed respectively (in comparison to what we have already) $C\beta_1 h_1^{-\frac{2}{3}}\nu_1^{\frac{2}{3}}$ and $C\beta_1^2 h_1^{-\frac{4}{3}}\nu_1^{\frac{4}{3}}$ with $\nu_1 = (\kappa\beta)^{\frac{10}{9}} h_1^{\frac{4}{9}}\bar{\gamma}^{\frac{58}{27}}|\log h|^{K}$.

On the other hand, for $M \ge 2$ contributions of $\bar{\gamma}$-element to expressions (28.6.26) and (28.6.27) do not exceed respectively (in comparison to what we have already) $C\beta_1 h_1^{-\frac{3}{2}}$ and $C\beta_1^2 h_1^{-3}$.

Let us partially summarize what we got. Let $\kappa\beta h \gtrsim 1$. Then for $M = 1$ the final results are $C\beta\bigl(h^{-1} + h^{-\frac{2}{9}}(\kappa\beta)^{\frac{8}{9}}|\log h|^{K}\bigr)$ and the same expression squared. On the other hand, for $M \ge 2$ the final results from all zones except $\{x\colon \gamma(x) \le C_0\bar{\gamma}\}$ are $C\beta h^{-\frac{1}{3}}(\kappa\beta)^{\frac{2}{3}}|\log h|^{K}$ and the same expression squared.

On the other hand, let $\kappa\beta h \lesssim 1$. Then for $M = 1$ the final results do not exceed $C\beta\bigl(h^{-1} + h^{-\frac{10}{27}}(\kappa\beta)^{\frac{20}{27}}|\log h|^{K}\bigr)$ and the same expression squared.

28.7 Applications to Ground State Energy

28.7.1 Preliminary Remarks

Recall that we are looking for

$$\langle \mathsf{H}\Psi, \Psi\rangle + \frac{1}{\alpha}\int |\partial A'|^2\, dx, \tag{28.7.1}$$

which should be minimized by $\Psi \in \mathfrak{H}$ and A'.

We know (see f.e. Subsection 25.2.1) that

$$\langle \mathsf{H}\Psi, \Psi\rangle \ge \mathsf{Tr}^-(H_{A,W+\lambda'}) + \lambda' N + \frac{1}{2}\mathsf{D}(\rho_\Psi - \rho,\ \rho_\Psi - \rho) - \frac{1}{2}\mathsf{D}(\rho,\ \rho) - C\int \rho_\Psi^{\frac{4}{3}}\,dx, \tag{28.7.2}$$

where $W = V - |x|^{-1} * \rho$, V is a Coulomb potential of nuclei, ρ and $\lambda \le 0$ are arbitrary.

Therefore to derive an estimate from below for expression (28.7.1) we just need to pick up ρ and λ' but we cannot pick up A'. Let us select ρ and λ equal to Thomas-Fermi density $= \rho_B^{\mathsf{TF}}$ and chemical potential λ respectively (but if $N \approx Z$ it is beneficial to pick up $\lambda' = 0$), add $\alpha^{-1}\int |\partial A'|^2\,dx$, and apply trace asymptotics without any need to consider N- or D-terms and we have an estimate from below which includes also "bonus term" $\frac{1}{2}\mathsf{D}(\rho_\Psi - \rho,\ \rho_\Psi - \rho)$ and is as good as a remainder estimate in the trace asymptotics rescaled–provided we estimate properly the last term in (28.7.2).

Thus here we are missing only estimate for $\int \rho_\Psi^{\frac{4}{3}}\,dx$ or more sophisticated estimate if we are interested in Dirac and Schwinger correction terms. We will prove in Appendix 28.A.2 that in the electrostatic inequality for near ground-state one can replace this term $-C\int \rho_\Psi^{\frac{4}{3}}\,dx$ by $-CZ^{\frac{5}{3}}$ for $B \le Z^{\frac{4}{3}}$ and $-CB^{\frac{4}{5}}Z^{\frac{3}{5}}$ for $Z^{\frac{4}{3}} \le B \le Z^3$; further, for $B \le Z$ one can replace it by $\mathsf{Dirac} - CZ^{\frac{5}{3}-\delta}$ thus proving Bach-Graf-Solovej estimate in our current settings.

Therefore due to these arguments estimates from below in Theorems 28.7.4 and 28.7.5 follow immediately from Theorems 28.5.14, 28.5.16 and 28.5.23 for $B \le Z^{\frac{4}{3}}$ and estimates from below in Theorem 28.7.11 follow immediately from Theorems 28.6.2 and 28.6.4 for $Z^{\frac{4}{3}} \le B \le Z^3$.

On the other hand, an estimate from above involves picking up ρ (which we select to be ρ_B^{TF} again) and picking A' as well–which we choose as in upper estimates of Section 28.5 (rescaled) and also picking up Ψ which we select $\phi_1(x_1,\varsigma_1)\cdots\phi_N(x_N,\varsigma_N)$ anti-symmetrized by $(x_1,\varsigma_1),\dots,(x_N,\varsigma_N)$ but we do not select λ' in the trace asymptotics which *must* be equal to λ_N, which is N-th eigenvalue of $H_{W,A}$ or to 0 if there are less than N negative eigenvalues of $H_{W,A}$.

In this case we need to estimate $|\lambda' - \lambda|$ and also

$$\mathsf{D}\bigl(\mathsf{tr}\, e(x,x,\lambda') - \rho_B^{\mathsf{TF}},\ \mathsf{tr}\, e(x,x,\lambda') - \rho_B^{\mathsf{TF}}\bigr), \tag{28.7.3}$$

which required some efforts in Chapters 25–27 but here we will do it rather easily because in Section 28.5 we took either A' equal to one for Coulomb potential and without external magnetic field (as $\beta h^2 \le 1$) or just 0 (as $\beta h^2 \ge 1$).

28.7.2 Estimate from above: $B \le Z^{\frac{4}{3}}$

Recall that now we can select ρ and A' but cannot select λ'.

If $B \le Z^{\frac{4}{3}}$ (or $\beta h \le 1$ after rescaling) we select A' as a minimizer for one-particle operator with Coulomb potential and without external magnetic field. Then after rescaling $x \mapsto Z^{\frac{1}{3}}x$, $\tau \mapsto Z^{-\frac{4}{3}}\tau$, $h = 1 \mapsto h = Z^{-\frac{1}{3}}$, $B \mapsto \beta = BZ^{-1}$, $\alpha \mapsto \kappa = \alpha Z$

$$|\partial A'| \le C\kappa\ell^{-\frac{3}{2}}, \qquad |\partial^2 A'| \le C\kappa\ell^{-\frac{5}{2}}|\log(\ell/h^2)|, \tag{28.7.4}$$

or before it

$$|\partial A'| \le C\alpha Z^{\frac{5}{3}}\ell^{-\frac{3}{2}}, \qquad |\partial^2 A'| \le C\alpha Z^{\frac{5}{3}}\ell^{-\frac{5}{2}}|\log(Z\ell)|. \tag{28.7.5}$$

Let us start from the easy case $M = 1$. We need the following

Proposition 28.7.1. *Let $M = 1$, $N \asymp Z$, $B \lesssim Z^{\frac{4}{3}}$ and $\alpha Z \le \kappa^*$. Assume that A' satisfies* (28.7.5)*. Then*

(i) The remainder in the trace asymptotics does not exceed

$$\begin{cases} C\bigl(Z^{\frac{5}{3}} + \alpha|\log(\alpha Z)|^{\frac{1}{3}} Z^{\frac{25}{9}}\bigr) & \text{for } B \le Z, \\ C\bigl(B^{\frac{1}{3}} Z^{\frac{4}{3}} + \alpha|\log(\alpha Z)|^{\frac{1}{3}} B^{\frac{2}{9}} Z^{\frac{23}{9}}\bigr) & \text{for } B \ge Z. \end{cases} \tag{28.7.6}$$

(ii) The remainder in N*-term asymptotics does not exceed $CZ^{\frac{2}{3}}$.*

(iii) D*-term does not exceed $CZ^{\frac{5}{3}}$.*

Proof. We cannot directly apply previous results because A' is now generated by much slower decaying Coulomb potential. The good news however is that A' is generated without presence of the external magnetic field. Let us scale as we mentioned above.

(i) Consider the remainder estimate in the trace asymptotics.

(ia) A contribution of the near singularity zone $\{x\colon \ell(x) \le \ell^* = \epsilon\,\mathsf{min}(\beta^{-\frac{2}{3}}, 1)\}$ does not exceed

$$\begin{cases} C(h^{-1} + \kappa|\log\kappa|^{\frac{1}{3}} h^{-\frac{4}{3}}) & \text{as } \beta \le 1,\\ C(\beta^{\frac{1}{3}} h^{-1} + \kappa|\log\kappa|^{\frac{1}{3}} \beta^{\frac{2}{9}} h^{-\frac{4}{3}}) & \text{as } \beta \ge 1. \end{cases} \tag{28.7.7}$$

(ib) A contribution of ℓ-element in the regular zone $\{x\colon \ell(x) \ge \ell^*\}$ does not exceed $C\zeta^2\big(h_1^{-1} + h_1^{-\frac{1}{3}}\nu_1^{\frac{2}{3}}\big)$. We plug $h_1 = h/(\zeta\ell)$, and in virtue of (28.7.4) $\nu_1 = \kappa\ell^{-\frac{5}{2}}\cdot \ell^3\zeta^{-1} = \kappa\ell^{\frac{1}{2}}\zeta^{-1}$. Taking a sum we arrive to the same expression at $\ell = \ell^*$ i.e. expression (28.7.7). Scaling back we arrive to (28.7.6).

(ic) We leave analysis of the boundary zone to the reader. It requires just repetition of the corresponding arguments of Section 28.5.

(ii) Next let us consider a remainder estimate in the N-term asymptotics.

(iia) Note that a contribution of ℓ-element in the regular zone does not exceed $C\big(h_1^{-2} + h_1^{-\frac{5}{3}}\nu_1^{\frac{1}{3}}\big)$. Then plugging h_1, ν_1 we arrive after summation to the same expression at $\ell = 1$ i.e. Ch^{-2}. One can prove easily that the contribution of the near singularity zone is $O(h^{-2+\delta})$.

Consider the contribution of the boundary zone. Note that this zone appears only if $B \ge (Z-N)_+^{\frac{4}{3}}$. Let us scale $x \mapsto x\bar\ell^{-1}$, $\tau \mapsto \bar\ell^{-4}\tau$, with $\bar\ell = (\beta h)^{-\frac{1}{4}}$. Then $h \mapsto h_1 = h^{\frac{3}{4}}\beta^{-\frac{1}{4}}$, $\beta \mapsto \beta_1 = h_1^{-1}$ and after scaling A' satisfies $|\partial^2 A'| \le \nu_1 = C\kappa\bar\ell^{\frac{3}{2}}|\log\kappa|$.

(iib) Consider a contribution of γ-element; scaling again $x \mapsto x\gamma^{-1}$, $\tau \mapsto \gamma^{-4}\tau$, $h_1 \mapsto h_2 = h_1\gamma^{-3}$, $\beta_1 \mapsto \beta\gamma^{-1}$, $\nu_1 \mapsto \nu_2 = \nu_1$ we see that it does not exceed $C\beta_2\big(h_2^{-1} + h_2^{-\frac{2}{3}}\nu_2^{\frac{2}{3}}\big) = C\beta_1\big(\gamma^2 h_1^{-1} + \gamma h_1^{-\frac{2}{3}}\nu_1^{\frac{2}{3}}\big)$. This expression must be divided by γ^2 and summed resulting in

$$C\big(h_1^{-2}|\log\bar\gamma| + \bar\gamma^{-1}h_1^{-\frac{5}{3}}\nu_1^{\frac{2}{3}}\big) \tag{28.7.8}$$

with $\bar\gamma \ge h_1^{\frac{1}{3}}$ (equality is achieved if $(Z-N)_+$ is small enough, otherwise partitioning may be cut-off by a chemical potential). One can get rid of the logarithmic factor by our standard propagation arguments; the second term in (28.7.8) does not exceed $Ch_1^{-2}\nu_1^{\frac{2}{3}}$ and plugging h_1, ν_1 we get $O(h^{-2}) = O(Z^{\frac{2}{3}})$.

(iii) D-term is analyzed in the same way. □

Proposition 28.7.2. *In the framework of Proposition 28.7.1 assume that $B \le Z$. Then*

(i) The remainder in the trace asymptotics (with the Schwinger term) does not exceed

$$CZ^{\frac{5}{3}}\bigl(Z^{-\delta} + (BZ^{-1})^{\delta}\bigr) + \alpha|\log(\alpha Z)|^{\frac{1}{3}}Z^{\frac{25}{9}}. \tag{28.7.9}$$

(ii) The remainder in N-term asymptotics does not exceed $CZ^{\frac{2}{3}}\bigl(Z^{-\delta} + (BZ^{-1})^{\delta} + (\alpha Z)^{\delta}\bigr)$.

(iii) D-term does not exceed $CZ^{\frac{5}{3}}\bigl(Z^{-\delta} + (BZ^{-1})^{\delta} + (\alpha Z)^{\delta}\bigr)$.

Proof. Proof includes improved (due to the standard arguments of propagation of singularities) estimates of the contributions of *threshold zone* $\{x\colon h^{\delta'} \le \ell(x) \le h^{-\delta'}\}$ after rescaling. We leave details to the reader. □

Since we now have exactly the same N-term asymptotics and the same remainder term estimate as in Subsection 26.6.3 as if there was no self-generated magnetic field we immediately arrive to the same estimates of $|\lambda_N - \lambda|$ as there and therefore to the same estimates for $|\lambda_N - \lambda|\cdot N$ and not only for two D-terms

$$\mathsf{D}\bigl(\operatorname{tr} e(x,x,\tau) - P'_B(W+\tau),\ \operatorname{tr} e(x,x,\tau) - P'_B(W+\tau)\bigr)$$

with $\tau = \lambda$ and $\tau = \lambda_N$ but also for the third one

$$\mathsf{D}\bigl(P'_B(W+\lambda_N) - P'_B(W+\lambda),\ P'_B(W+\lambda_N) - P'_B(W+\lambda)\bigr).$$

The trace term however is different–with Scott correction term $2S(\alpha Z)Z^2$ instead of $2S(0)Z^2$ and the remainder estimate here includes an extra term related to α.

It concludes the proof of the estimate from above for E^*_N. Combined with estimate from below it concludes the proof of Theorem 28.7.4.

Consider now the case $M \ge 2$. Since we need to decouple singularities in this case we need sufficiently fast decaying magnetic field and thus potential generating it. So we will take $A' = \sum_m A'_m \phi_m$ with A'_m defined by $V = W^{\mathsf{TF}}_m$

(where $\mathsf{W}_m^{\mathsf{TF}}$ corresponds to a single nucleus) without any magnetic field and with $N = Z$ (i.e. $\lambda = 0$) and ϕ_m is supported in $\{x\colon |x - \mathsf{y}_m \le \frac{1}{3}d\}$ and equals 1 in $\{x\colon |x - \mathsf{y}_m \le \frac{1}{4}d\}$. However everywhere else we take $V = W_B^{\mathsf{TF}}$.

Assume first that $(Z - N)_+$ is sufficiently small and we take $Z = N$ even in the definition of W_B^{TF}. Then one can prove easily that

$$|\operatorname{Tr}^-(H_{A,V}) + \alpha^{-1}\int |\partial A'|^2\,dx + \int P_B(V)\,dx - \mathsf{Scott}| \le (28.7.6) + C\alpha Z^2 d^{-3}, \tag{28.7.10}$$

where the last term is due to decoupling; indeed, $|\partial A'|$ and $|\partial^2 A'|$ decay as ℓ^{-3} and ℓ^{-4} if $\ell \ge Z^{-\frac{1}{3}}$.

Moreover, for $B \le Z$ one can replace (28.7.6) by

$$CZ^{\frac{5}{3}}\big(Z^{-\delta} + (BZ^{-1})^{\delta} + (\alpha Z)^{\delta} + (dZ^{\frac{1}{3}})^{-\delta}\big) + C\alpha|\log(\alpha Z)|^{\frac{1}{3}} Z^{\frac{25}{9}} \tag{28.7.11}$$

while including $\mathsf{Schwinger}$ into left-hand expression. This estimate is at least as good as what we got in the estimate from below but probably even better since magnetic field admits now better estimates.

Let us estimate $|\lambda_N|$ if $\lambda_N < 0$. To do this consider

$$\int \big[e(x,x,\lambda) - e(x,x,\lambda_N)\big]\,dx \tag{28.7.12}$$

with non-negative integrand. Then the contribution of ℓ-element into the main part of this expression, namely

$$\int \big[e(x,x,\lambda) - e(x,x,\lambda_N)\big]\psi_\iota^2\,dx \tag{28.7.13}$$

is

$$\int \big[P'(x,V(x)+\lambda) - P'(x,V(x)+\lambda_N)\big]\psi_\iota^2\,dx \tag{28.7.14}$$

and it does not depend on A'. On the other hand, since now $|\partial^2 A'|$ admits so good estimate, the contribution of this element to the remainder estimated as if there was no self-generated magnetic field.

The same is true for the boundary elements as well.

But then $|\lambda_N|$ is estimated exactly as if there was no self-generated magnetic field, i.e. exactly as in Section 26.6. But then all components of the estimate from above, with exception of the trace term, namely $|\lambda_N| \cdot N$

and all three D-terms are estimated as in Section 26.6. Under assumption small $(Z-N)_+$ all of them do not exceed $CZ^{\frac{5}{3}}$ which could be improved to

$$CZ^{\frac{5}{3}}\bigl(Z^{-\delta}+(BZ^{-1})^{\delta}+(\alpha Z)^{\delta}+(dZ^{\frac{1}{3}})^{-\delta}\bigr) \tag{28.7.15}$$

for $B\le Z$.

Similarly, if $(Z-N)_+$ is larger than the corresponding threshold, we need to consider two cases: $0>\lambda>\lambda_N$ and $0>\lambda_N>\lambda$ and estimate from below

$$\int \bigl(e(x,x,\lambda)-e(x,x,\lambda_N)\bigr)\psi_\iota^2\,dx \tag{28.7.16}$$

and

$$\int \bigl(e(x,x,\lambda_N)-e(x,x,\lambda)\bigr)\psi_\iota^2\,dx \tag{28.7.17}$$

respectively, leading to the estimate of $|\lambda_N-\lambda|$ and then $|\lambda_N-\lambda|\cdot N$ and all three D-terms and again here these terms are estimated as if there was no self-generating magnetic field.

This concludes the proof of Theorem 28.7.5.

Remark 28.7.3. For $M\ge 2$ one could be concerned about term coming from $C\beta h^{-\frac{1}{2}}$ in the trace term and $C\beta h^{-\frac{3}{2}}$ or its square in N- and D-terms but assuming that $d\lesssim \bar{r}$ we simply have $A'=0$ due to decoupling there and therefore apply theory of Chapter 25 without any modification.

28.7.3 Main Theorems: $B\le Z^{\frac{4}{3}}$

Ground State Energy Asymptotics

Theorem 28.7.4. *Let $M=1$, $N\asymp Z$, $B\le Z^{\frac{4}{3}}$ and $\alpha\le \kappa^* Z^{-1}$ with small constant κ^*. Then*

(i) If $B\le Z$, then

$$\mathsf{E}^*_N=\mathcal{E}^{\mathsf{TF}}_N+\mathsf{Scott}+O\Bigl(Z^{\frac{5}{3}}+\alpha|\log(\alpha Z)|^{\frac{1}{3}}Z^{\frac{25}{9}}\Bigr) \tag{28.7.18}$$

with $\mathsf{Scott}=2Z^2S(\alpha Z)$.

(ii) Moreover, if $B \ll Z$ *and* $\alpha|\log Z|^{\frac{1}{3}} \ll Z^{-\frac{10}{9}}$*, then this estimate could be improved to*

$$\mathsf{E}^*_N = \mathcal{E}^{\mathsf{TF}}_N + 2Z^2 S(\alpha Z) + \mathsf{Schwinger} + \mathsf{Dirac} + O\Big(Z^{\frac{5}{3}}\big[Z^{-\delta} + B^{\delta} Z^{-\delta}\big] + \alpha|\log(\alpha Z)|^{\frac{1}{3}} Z^{\frac{25}{9}}\Big). \tag{28.7.19}$$

(iii) If $Z \le B \le Z^{\frac{4}{3}}$*, then*

$$\mathsf{E}^*_N = \mathcal{E}^{\mathsf{TF}}_N + 2Z^2 S(\alpha Z) + O\Big(B^{\frac{1}{3}} Z^{\frac{4}{3}} + \alpha|\log(\alpha Z)|^{\frac{1}{3}} B^{\frac{2}{9}} Z^{\frac{23}{9}} + \alpha B Z^{\frac{5}{3}}\Big). \tag{28.7.20}$$

Theorem 28.7.5. *Let* $M \ge 2$*,* $N \asymp Z_1 \asymp Z_2 \asymp \dots \asymp Z_M$*,* $B \lesssim Z^{\frac{4}{3}}$*. Let* $\alpha \le \kappa^* Z^{-1}$*,* $d \ge Z^{-\frac{1}{3}}$ *be a minimal distance between nuclei capped by* $\bar{r} = \min\big(B^{-\frac{1}{4}}, (Z-N)_+^{-\frac{1}{3}}\big)$*. Then*

(i) If $B \le Z$*, then*

$$\mathsf{E}^*_N = \mathcal{E}^{\mathsf{TF}}_N + \mathsf{Scott} + O\Big(Z^{\frac{5}{3}} + \alpha\big[|\log(\alpha Z)|^{\frac{1}{3}} Z^{\frac{25}{9}} + Z^2 d^{-3}\big]\Big) \tag{28.7.21}$$

with $\mathsf{Scott} = 2\sum_{1\le m\le M} Z_m^2 S(\alpha Z_m)$ *and, moreover,*

$$\mathsf{E}^*_N = \mathcal{E}^{\mathsf{TF}}_N + \mathsf{Scott} + \mathsf{Schwinger} + \mathsf{Dirac} + O\Big(Z^{\frac{5}{3}}\big[Z^{-\delta} + (BZ^{-1})^{\delta} + (dZ^{\frac{1}{3}})^{-\delta}\big] + \alpha\big[|\log(\alpha Z)|^{\frac{1}{3}} Z^{\frac{25}{9}} + Z^2 d^{-3}\big]\Big). \tag{28.7.22}$$

(ii) If $B \ge Z$*,* $\alpha B \le Z^{\frac{17}{60}}|\log Z|^{-K}$ *and* $(Z-N)_+ \le B^{\frac{5}{12}}$*, then*

$$\mathsf{E}^*_N = \mathcal{E}^{\mathsf{TF}}_N + \mathsf{Scott} + O\Big(B^{\frac{1}{3}} Z^{\frac{4}{3}} + \alpha\big[|\log\alpha|^{\frac{1}{3}} B^{\frac{2}{9}} Z^{\frac{23}{9}} + BZ^{\frac{5}{3}} + Z^2 d^{-3}\big]\Big); \tag{28.7.23}$$

(iii) If $B \ge Z^{\frac{77}{60}}|\log Z|^{-K}$*,* $\alpha \ge B^{-1} Z^{\frac{17}{60}}|\log Z|^{-K}$*,* $(Z-N)_+ \le B^{\frac{5}{12}}$ *and* $d \ge \bar{d} = (\alpha B)^{\frac{40}{209}} Z^{-\frac{81}{209}}|\log Z|^{K}$*, then*

$$\mathsf{E}^*_N = \mathcal{E}^{\mathsf{TF}}_N + \mathsf{Scott} + O\Big(B^{\frac{1}{3}} Z^{\frac{4}{3}} + \alpha\big[|\log\alpha|^{\frac{1}{3}} B^{\frac{2}{9}} Z^{\frac{23}{9}} + BZ^{\frac{5}{3}} + (\alpha B)^{\frac{120}{209}} Z^{\frac{384}{209}} d^{-3}|\log Z|^{K}\big]\Big). \tag{28.7.24}$$

(iv) In any case

$$\mathsf{E}^*_N = \mathcal{E}^{\mathsf{TF}}_N + \mathsf{Scott}_0 + O\big(B^{\frac{1}{3}}Z^{\frac{4}{3}} + \alpha Z^3\big) \tag{28.7.25}$$

with $\mathsf{Scott}_0 = 2\sum_{1\le m\le M} Z_m^2 S(0)$.

Recall that in the free nuclei model excess energy is $\asymp d^{-7}$ (as $d \le \epsilon B^{-\frac{1}{4}}$) and therefore an error must be greater than ϵd^{-7}. One can see easily that $d \ge \mathsf{min}(Z^{-\frac{5}{21}}, B^{-\frac{1}{4}})$ if $B \le Z$ and then in estimates (28.7.21) and (28.7.22) the last terms (with d^{-3}) could be skipped.

On the other hand, in (28.7.23)–(28.7.25) we can either skip the last terms (with d^{-3}) or assume that $d \asymp B^{-\frac{1}{4}}$ and these terms should be calculated under this assumption and we arrive to

Theorem 28.7.6. *Let $M \ge 2$, $N \asymp Z_1 \asymp Z_2 \asymp ... \asymp Z_M$, $B \lesssim Z^{\frac{4}{3}}$ and $(Z-N)_+ \lesssim B^{\frac{1}{2}}$. Consider a free nuclei model. Then*

(i) If $B \le Z$, then

$$\widehat{\mathsf{E}}^*_N = \widehat{\mathcal{E}}^{\mathsf{TF}}_N + \mathsf{Scott} + O\Big(Z^{\frac{5}{3}} + \alpha\big[|\log(\alpha Z)|^{\frac{1}{3}} Z^{\frac{25}{9}}\big]\Big) \tag{28.7.26}$$

and moreover

$$\widehat{\mathsf{E}}^*_N = \widehat{\mathcal{E}}^{\mathsf{TF}}_N + \mathsf{Scott} + \mathsf{Schwinger} + \mathsf{Dirac} + O\Big(Z^{\frac{5}{3}}\big[Z^{-\delta} + (BZ^{-1})^{\delta}\big] + \alpha\big[|\log(\alpha Z)|^{\frac{1}{3}} Z^{\frac{25}{9}}\big]\Big). \tag{28.7.27}$$

(ii) If $B \ge Z$, $\alpha \le B^{-1}Z^{\frac{17}{60}}|\log Z|^{-K}$, then

$$\widehat{\mathsf{E}}^*_N = \widehat{\mathcal{E}}^{\mathsf{TF}}_N + \mathsf{Scott} + O\Big(B^{\frac{1}{3}}Z^{\frac{4}{3}} + \alpha\big[|\log \alpha|^{\frac{1}{3}} B^{\frac{2}{9}} Z^{\frac{23}{9}} + Z^2 B^{\frac{3}{4}}\big]\Big). \tag{28.7.28}$$

(iii) If $B \ge Z^{\frac{77}{60}}|\log Z|^{-K}$, $B^{-1}Z^{\frac{17}{60}}|\log h|^{-K} \le \alpha \le B^{-\frac{369}{160}}Z^{\frac{81}{40}}|\log Z|^{-K}$, then

$$\mathsf{E}^*_N = \mathcal{E}^{\mathsf{TF}}_N + \mathsf{Scott} \\ O\Big(B^{\frac{1}{3}}Z^{\frac{4}{3}} + \alpha\big[|\log \alpha|^{\frac{1}{3}} B^{\frac{2}{9}} Z^{\frac{23}{9}} + (\alpha B)^{\frac{120}{209}} Z^{\frac{384}{209}} B^{\frac{3}{4}} |\log Z|^{K}\big]\Big). \tag{28.7.29}$$

(iv) In any case

$$\widehat{\mathsf{E}}^*_N = \widehat{\mathcal{E}}^{\mathsf{TF}}_N + \mathsf{Scott}_0 + O\big(B^{\frac{1}{3}}Z^{\frac{4}{3}} + \alpha Z^3\big). \tag{28.7.30}$$

We leave to the reader the following

Problem 28.7.7. In the frameworks of fixed nuclei and free nuclei models consider the case $M \ge 2$, $B \ge Z$ and $(Z-N)_+ \ge B^{\frac{5}{12}}$. Use results of Sections 28.5 and 26.6. Recall that there are two cases: $B^{\frac{5}{12}} \le (Z-N)_+ \le B^{\frac{3}{4}}$ and $(Z-N)_+ \ge B^{\frac{3}{4}}$.

In particular find out for which B this assumption could be skipped without deterioration of the remainder estimates.

Ground State Density Asymptotics

Consider now asymptotics of ρ_Ψ. Apart of independent interest one needs them for estimate of excessive negative charge and estimate or asymptotics of the ionization energy.

Theorem 28.7.8. *Let $M=1$, $N \asymp Z$, $B \lesssim Z^{\frac{4}{3}}$ and $\alpha \le \kappa^* Z^{-1}$, with small constant κ^*. Then*

(i) If $B \le Z$, then

$$\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}},\, \rho_\Psi - \rho^{\mathsf{TF}}) = O\big(Z^{\frac{5}{3}}\big) \tag{28.7.31}$$

and moreover this estimate could be improved to

$$\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}},\, \rho_\Psi - \rho^{\mathsf{TF}}) = O\Big(Z^{\frac{5}{3}}\big[Z^{-\delta} + B^{\delta} Z^{-\delta} + (\alpha Z)^{\delta}\big]\Big); \tag{28.7.32}$$

(ii) If $Z \le B \le Z^{\frac{4}{3}}$, then

$$\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}},\, \rho_\Psi - \rho^{\mathsf{TF}}) = O\Big(Z^{\frac{5}{3}} + (\alpha B)^{\frac{40}{27}} Z^{\frac{101}{81}} |\log Z|^{K}\Big). \tag{28.7.33}$$

Proof. We need to consider only the case when errors in the estimates for E^*_N exceed those announced in (28.7.31)–(28.7.33). Otherwise an estimate for $\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}},\, \rho_\Psi - \rho^{\mathsf{TF}})$ follow from the estimates from above and below for E^*_N as estimates from below contain the "bonus term" $\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}},\, \rho_\Psi - \rho^{\mathsf{TF}})$.

Let in the estimate from below pick up $\lambda' = \lambda_N$ and in the estimate from above pick up A' as a minimizer for a potential $W_B^{\mathsf{TF}} + \lambda'$ with $\lambda' = \lambda_N$; we

do not calculate asymptotics of the trace terms since these terms in both estimates coincide[36]; then we arrive to estimate

$$\begin{aligned}(28.7.34)\quad &\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}},\, \rho_\Psi - \rho^{\mathsf{TF}})\\ &\qquad \le C\mathsf{D}(\operatorname{tr} e(x,x,\lambda_N) - \rho^{\mathsf{TF}},\, \operatorname{tr} e(x,x,\lambda_N) - \rho^{\mathsf{TF}}) + CZ^{\frac{5}{3}}\end{aligned}$$

and we need to estimate the first term in the right-hand expression.

Let us scale as usual. Then $\beta h \lesssim 1$ since $B \le Z^{\frac{4}{3}}$ and our standard arguments estimate this term by $C\bigl(h^{-2}+h^{-\frac{5}{3}}\nu^{\frac{2}{3}}\bigr)^2$ with $\nu = (\kappa\beta)^{\frac{10}{9}}h^{\frac{4}{9}}|\log h|^K$.

Plugging $\beta = BZ^{-1}$, $h = Z^{-\frac{1}{3}}$, $\kappa = \alpha Z$ and multiplying by $Z^{\frac{1}{3}}$ due to the spatial scaling we arrive exactly to (28.7.31) and (28.7.33).

Furthermore, as $\beta \ll 1$ let us consider contribution of the *main zone* $\{x\colon h^\delta \le \ell(x) \le h^{-\delta}+(\beta+\kappa)^\delta\}$ and use propagation arguments and improved electrostatic inequality; then we arrive to estimate $Ch^{-4}(h+\beta+\kappa)^\delta$ which after rescaling becomes (28.7.32).

We leave all easy details to the reader. □

Consider now case $M \ge 2$. Then no matter what is the distance between nuclei (as long as it is greater than $\epsilon Z^{-\frac{1}{3}}$) we need to add one or two more extra terms.

(a) The first one always appears and it is what becomes from $C\beta_1^2 h_1^{-2}\nu_1\ell^{-1}$ as we plug $\beta_1 = \beta\ell^3$, $h_1 = h\ell$, $\kappa_1 = \kappa\ell^{-3}$, $\nu_1 = (\kappa_1\beta_1)^{\frac{10}{9}}h_1^{\frac{4}{9}}|\log h_1|^K \asymp (\kappa\beta)^{\frac{10}{9}}h^{\frac{4}{9}}\ell^{\frac{4}{9}}|\log h|^K$, and multiply by ℓ^{-1} we get ℓ in the positive power and therefore we must plug the largest possible ℓ which in case $(Z-N)_+ \le B^{\frac{3}{4}}$ is $\ell = (\beta h)^{-\frac{1}{4}}$. Also plugging $\kappa = \alpha Z$, $\beta = BZ^{-1}$, $h = Z^{-\frac{1}{3}}$ and $\ell = (\beta h)^{-\frac{1}{4}} = B^{-\frac{1}{4}}Z^{\frac{1}{3}}$ and multiplying by $Z^{\frac{1}{3}}$ due to the scaling (with a possible improvement for $B \ll Z$) we arrive to $\alpha^{\frac{40}{27}}B^{\frac{9}{4}}|\log Z|^K$. One can see easily that this term is larger than the second term in (28.7.33).

For $(Z-N)_+ \ge B^{\frac{3}{4}}$ this term will be smaller than the second extra term.

(b) The second extra term appears only if $(Z-N)_+ \gtrsim B^{\frac{5}{12}}$.

(b)$_1$ For $B^{\frac{5}{12}} \le (Z-N)_+ \le B^{\frac{3}{4}}$ it is what becomes from $\beta_2^2 h_2^{-3}\bar\gamma^{-4}|\log\bar\gamma|^2$ with substitutions $\beta_2 = \beta_1\bar\gamma^{-1}$, $h_2 = h_1\bar\gamma^{-3}$, $\bar\gamma = (Z-N)_+^{\frac{1}{4}}B^{-\frac{3}{16}}$ and with β_1, h_1, ℓ defined above (we still need to multiply by $Z^{\frac{1}{3}}$).

[36] Due to the matching choices of A' and λ'.

(b)$_2$ For $(Z-N)_+ \ge B^{\frac{3}{4}}$ it is what becomes out $\beta_1^2 h_1^{-3}\ell^{-1}$ with $\ell = (Z-N)_+^{-\frac{1}{3}}$.

Thus we arrive to

Theorem 28.7.9. *Let $M \ge 2$, $N \asymp Z_1 \asymp ... \asymp Z_M$, $B \lesssim Z^{\frac{4}{3}}$. Further, let $\alpha \le \kappa^* Z^{-1}$, $d \gtrsim Z^{-\frac{1}{3}}$. Then*

(i) For $B \le Z$ estimate (28.7.31) *holds and, moreover, it could be improved to*

$$\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}}) = O\Big(Z^{\frac{5}{3}}\big[Z^{-\delta} + B^\delta Z^{-\delta} + (\alpha Z)^\delta + (dZ^{\frac{1}{3}})^{-\delta}\big]\Big). \tag{28.7.35}$$

(ii) For $B \ge Z$ and $(Z-N)_+ \le B^{\frac{5}{12}}$

$$\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}}) = O\Big(Z^{\frac{5}{3}} + \alpha^{\frac{10}{9}} B^{\frac{9}{4}} |\log Z|^K\Big). \tag{28.7.36}$$

(iii) For $B \ge Z$ and $B^{\frac{5}{12}} \le (Z-N)_+ \le B^{\frac{3}{4}}$

$$\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}}) = O\Big(Z^{\frac{5}{3}} + \alpha^{\frac{10}{9}} B^{\frac{9}{4}} |\log Z|^K + B^{\frac{15}{16}}(Z-N)_+^{\frac{3}{4}}(1+|\log (Z-N)_+ B^{-\frac{1}{3}}|)^2\Big). \tag{28.7.37}$$

(iv) For $B \ge Z$ and $(Z-N)_+ \le B^{\frac{3}{4}}$

$$\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}}) = O\Big(Z^{\frac{5}{3}} + B^2 (Z-N)_+^{-\frac{2}{3}}\Big). \tag{28.7.38}$$

Corollary 28.7.10. *Estimates* (28.7.31), (28.7.35)– (28.7.38) *hold for a free nuclei model.*

28.7.4 Main Theorems: $Z^{\frac{4}{3}} \le B \le Z^3$

Ground State Energy Asymptotics

For $Z^{\frac{4}{3}} \le B \le Z^3$ we select $A' = 0$ in the estimate from above and therefore just apply an upper estimate E_N^* from Subsection 26.6.3. Combined with estimate from below provided by Theorem 28.6.2 it implies the following

Theorem 28.7.11. *Let* $Z^{\frac{4}{3}} \lesssim B \lesssim Z^3$, $Z_1 \asymp ... \asymp Z_M \asymp N$, *and* $\alpha \le \kappa^* Z^{-1}$ *with small constant* κ^*, *and also* $\alpha \le B^{-\frac{4}{5}} Z^{\frac{2}{5}} |\log Z|^{-K}$. *Then*

(i) For $M = 1$ *and either* $(Z-N)_+ \lesssim B^{\frac{4}{15}} Z^{\frac{1}{5}}$ *or* $\alpha B^{\frac{3}{5}} Z^{\frac{1}{5}} \gtrsim 1$

$$|\mathsf{E}^*_N - \mathcal{E}^{\mathsf{TF}}_N - \mathsf{Scott}_0| \tag{28.7.39}$$

does not exceed

$$C\Big(B^{\frac{1}{3}} Z^{\frac{4}{3}} + B^{\frac{4}{5}} Z^{\frac{3}{5}} + \alpha Z^3 + \alpha^{\frac{16}{9}} B^{\frac{82}{45}} Z^{\frac{49}{45}} |\log Z|^K\Big). \tag{28.6.3}$$

(ii) For $M = 1$ *and* $(Z-N)_+ \gtrsim B^{\frac{4}{15}} Z^{\frac{1}{5}}$ *and* $\alpha B^{\frac{3}{5}} Z^{\frac{1}{5}} \lesssim 1$ *expression* (28.7.39) *does not exceed*

$$C\Big(B^{\frac{1}{3}} Z^{\frac{4}{3}} + B^{\frac{4}{5}} Z^{\frac{3}{5}} + \alpha Z^3 + \alpha^{\frac{16}{9}} B^{\frac{82}{45}} Z^{\frac{49}{45}} |\log Z|^K + \alpha^{\frac{40}{27}} B^{\frac{13}{15}} Z^{\frac{139}{135}} (Z-N)_+^{\frac{28}{27}} |\log Z|^K\Big). \tag{28.6.4}$$

(iii) For $M \ge 2$ *and* $(Z-N)_+ \lesssim B^{\frac{4}{15}} Z^{\frac{1}{5}}$ *expression* (28.7.39) *does not exceed*

$$C\Big(B^{\frac{1}{3}} Z^{\frac{4}{3}} + B^{\frac{4}{5}} Z^{\frac{3}{5}} (1 + |\log BZ^{-3}|)^2 + \alpha Z^3 + \alpha^{\frac{16}{9}} B^{\frac{82}{45}} Z^{\frac{49}{45}} |\log Z|^K\Big). \tag{28.7.40}$$

(iv) For $M \ge 2$ *and* $(Z-N)_+ \gtrsim B^{\frac{4}{15}} Z^{\frac{1}{5}}$ *expression* (28.7.39) *does not exceed*

$$C\Big(B^{\frac{1}{3}} Z^{\frac{4}{3}} + B^{\frac{4}{5}} Z^{\frac{3}{5}} (1 + |\log BZ^{-3}|)^2 + \alpha Z^3 + \alpha^{\frac{16}{9}} B^{\frac{82}{45}} Z^{\frac{49}{45}} |\log Z|^K + B^{\frac{2}{3}} (Z-N)_+ (1 + |\log (Z-N)_+ Z^{-1}|)^2\Big). \tag{28.7.41}$$

Ground State Density Asymptotics

Consider now asymptotics of ρ_Ψ. Apart of independent interest we need them for estimate of excessive negative charge and estimate or asymptotics of the ionization energy. We are interested as usual in $\mathsf{D}\big(\rho_\Psi - \rho^{\mathsf{TF}}_B, \rho_\Psi - \rho^{\mathsf{TF}}_B\big)$ and we know that

Corollary 28.7.12. *In the framework of Theorem 28.7.11(i), (ii), (iii), (iv)* $\mathsf{D}\big(\rho_\Psi - \rho^{\mathsf{TF}}_B, \rho_\Psi - \rho^{\mathsf{TF}}_B\big)$ *does not exceed the corresponding remainder estimate* (28.6.3), (28.6.4), (28.7.40), (28.7.41).

If we want to get rid of $\alpha Z^3 + B^{\frac{1}{3}} Z^{\frac{4}{3}}$ terms (which may be dominant only for $B \le Z^{\frac{7}{4}}$ and $B \le Z^{\frac{11}{7}}$ respectively) we need not to find asymptotics of the trace term but to have trace term in the estimates from above and below more consistent. The only explored option is to take in the estimate from above the same A' as in the estimate from below, which is a minimizer for the corresponding one particle problem.

Theorem 28.7.13. *Let $Z^{\frac{4}{3}} \lesssim B \lesssim Z^3$, $Z_1 \asymp ... \asymp Z_M \asymp N$, and $\alpha \le \kappa^* Z^{-1}$ with small constant κ^*, and also $\alpha \le B^{-\frac{4}{5}} Z^{\frac{2}{5}} |\log Z|^{-K}$.*

(i) Let $M = 1$. Then $\mathsf{D}\big(\rho_\Psi - \rho_B^{\mathsf{TF}},\, \rho_\Psi - \rho_B^{\mathsf{TF}}\big)$ does not exceed

$$CB^{\frac{4}{5}} Z^{\frac{3}{5}} + C\Big(\alpha^{\frac{16}{9}} B^{\frac{82}{45}} Z^{\frac{49}{45}} + \alpha^{\frac{40}{27}} B^{-\frac{10}{9}} Z^{\frac{127}{27}} + \alpha^{\frac{40}{27}} B^{\frac{74}{45}} Z^{\frac{11}{90}} (Z-N)_+^{\frac{29}{54}}\Big) |\log Z|^K. \tag{28.7.42}$$

(ii) Let $M = 2$ and the minimal distance between nuclei $d \gtrsim B^{-\frac{2}{5}} Z^{\frac{1}{5}}$. Then $\mathsf{D}\big(\rho_\Psi - \rho_B^{\mathsf{TF}},\, \rho_\Psi - \rho_B^{\mathsf{TF}}\big)$ does not exceed

$$CB^{\frac{4}{5}} Z^{\frac{3}{5}} + C\Big(\alpha^{\frac{4}{3}} B^{\frac{22}{15}} Z^{\frac{19}{15}} + \alpha^{\frac{40}{27}} B^{-\frac{10}{9}} Z^{\frac{127}{27}} + B^{\frac{3}{5}} Z^{\frac{9}{20}} (Z-N)_+^{\frac{3}{4}}\Big) |\log Z|^K. \tag{28.7.43}$$

Proof. We will use Propositions 28.6.5 and 28.6.6 to estimate (28.6.27). Let us consider for each partition element

$$\Big| \int \big(e(x,x,\lambda') - P_B(V(x)+\lambda')\big)\psi(x)\, dx\Big|. \tag{28.7.44}$$

(a) Zone $\{x\colon \ell(x) \le B^{-1} Z |\log h|^\delta\}$. Here for each ℓ-element expression (28.7.44) does not exceed $R_a = C(h_1^{-2} + \beta_1 h_1^{-1})(1 + \nu_1^{\frac{2}{3}} h_1^{\frac{1}{3}})$ with $\beta_1 = \beta \ell^{\frac{3}{2}}$, $h_1 = h\ell^{-\frac{1}{2}}$ and ν_1 defined according to Proposition 28.6.5(ii). As usual $\beta = B^{\frac{2}{5}} Z^{-\frac{1}{5}}$ and $h = B^{\frac{1}{5}} Z^{-\frac{3}{5}}$.

Then one can prove easily that the total contribution of this zone to

$$\mathsf{D}\big(e(x,x,\lambda') - P_B(V(x)+\lambda'),\, e(x,x,\lambda') - P_B(V(x)+\lambda')\big) \tag{28.7.45}$$

does not exceed $CBZ^{-1}R_a^{2}$ [37] which is the second term in the parenthesis of (28.7.42) and (28.7.43) [38].

(b) Zone $\{x\colon B^{-1}Z|\log h|^{\delta} \le \ell(x) \le \epsilon B^{-\frac{2}{5}}Z^{\frac{1}{5}}\}$ (with the exception of the case $\bar\gamma \ge |\log h|^{-\delta}$ which we leave to the reader). Here for each ℓ-element expression (28.7.44) does not exceed $R_b = C\beta_1 h_1^{-1}(1+\nu_1^{\frac{2}{3}}h_1^{\frac{1}{3}})$ with $\beta_1 = \beta\ell^{\frac{3}{2}}$, $h_1 = h\ell^{-\frac{1}{2}}$ and ν_1 defined according to Proposition 28.6.6(i).

Then one can prove easily that the total contribution of this zone to (28.7.45) does not exceed $CB^{\frac{2}{5}}Z^{-\frac{1}{5}}R_b^{2}$ [37] which is the first term in the parenthesis of (28.7.42) [38].

(c) Zone $\{x\colon \bar\gamma|\log h|^{\delta} \le \gamma(x) \le C_0\}$. Here for each γ-element expression (28.7.44) does not exceed R_c where $R_c = C\beta_2 h_2^{-1}(1+\nu_2^{\frac{2}{3}}h_2^{\frac{1}{3}})$ (as $M=1$) and $R_c = C\beta_2 h_2^{-1}\nu_2^{\frac{1}{2}}$ (as $M \ge 2$) with $\beta_2 = \beta\gamma^{-1}$, $h_2 = h\gamma^{-3}$ and ν_2 defined by Proposition 28.6.6(i) and redefined by Remark 28.5.9.

Then one can prove easily that the total contribution of this zone to (28.7.45) does not exceed $CB^{\frac{2}{5}}Z^{-\frac{1}{5}}R_c^{2}$ [37] which is the first term in the parenthesis of (28.7.42) and (28.7.43) for $M=1$ and $M \ge 2$ respectively [38].

(d) Zone $\{x\colon \gamma(x) \le \bar\gamma|\log h|^{\delta}\}$. Here for each γ-element expression (28.7.44) does not exceed $R_d = C\beta_2 h_2^{-1}(1+\nu_2^{\frac{2}{3}}h_2^{\frac{1}{3}})$ (as $M=1$) and $R_d = C\beta_2 h_2^{-\frac{3}{2}}$ (as $M \ge 2$) with $\beta_2 = \beta\gamma^{-1}$, $h_2 = h\gamma^{-3}$ and ν_2 defined by Proposition 28.6.6(ii).

Then one can prove easily that the total contribution of this zone to (28.7.45) does not exceed $CB^{\frac{2}{5}}Z^{-\frac{1}{5}}R_d^{2}\bar\gamma^{-4}$ [37] which is the third term in the parenthesis of (28.7.42) and (28.7.43) for $M=1$ and $M \ge 2$ respectively [38]. □

28.A Appendices

28.A.1 Generalization of Lieb-Loss-Solovej Estimate

Proposition 28.A.1. *Consider operator H defined by* (27.1.1) *with* $A = A' + A''$, $A' = (A_1'(x'), A_2'(x'), 0)$, $x' = (x_1, x_2)$, *and* $A'' = (A_1''(x), A_2''(x), A_3''(x))$

[37] Calculated for ℓ or γ on its maximum.

[38] Modulo term not exceeding $CB^{\frac{4}{5}}Z^{\frac{3}{5}}$.

on Ω. *Assume that*

$$\int B'^{\,2}\,dx \ge \int B''^{\,2}\,dx \tag{28.A.1}$$

with $B = |\nabla \times A|$, $B' = |\nabla \times A'|$, $B'' = |\nabla \times A''|$. *Then*

$$-\mathsf{Tr}(H^-_{A,V}) \le \tag{28.A.2}$$
$$C\int V_+^{\frac{5}{2}}(x)\,dx + C\Big(\int B^2\,dx\Big)^{\frac{1}{2}}\Big(\int B''^{\,2}\,dx + \int V^2\,dx\Big)^{\frac{1}{4}}\Big(\int V^4\,dx\Big)^{\frac{1}{4}} +$$
$$C\Big(\int B^2\,dx\Big)^{\frac{3}{8}}\Big(\int V^2\,dx\Big)^{\frac{3}{8}}\Big(\int V^4\,dx\Big)^{\frac{1}{4}}.$$

Proof. Without any loss of the generality we can assume that $V \ge 0$. We apply the *moving frame technique* of Lieb-Loss-Solovej [1]. Obviously

$$-\mathsf{Tr}(H^-_{A,V}) = \int_0^\infty \mathsf{N}^-(H_{A,V}+\lambda)\,d\lambda = \tag{28.A.3}$$
$$\int_0^\infty \mathsf{N}^-(H_{A,0}+\lambda-V)\,d\lambda \le \int_0^\infty \mathsf{N}^-\big(H_{A,0}+(\lambda-V)\phi(\lambda)\big)\,d\lambda$$

with $\phi(\lambda) = \mathsf{max}(1, \lambda\mu^{-1})$ since $H_{A,0} \ge 0$. Since

$$H_{A,0} = \big(P\cdot\boldsymbol{\sigma}\big)^2 = \big(P'\cdot\boldsymbol{\sigma}'\big)^2 + P_3^2 + \sum_{j=1,2}[P_j,P_3]\,[\sigma_j,\sigma_3] \ge \tag{28.A.4}$$
$$H'_{A,0} + P_3^2 - B''$$

with

$$H'_{A,0} = \big(P'\cdot\boldsymbol{\sigma}'\big)^2 = \big(\sum_{j=1,2} P_j\cdot\sigma_j\big)^2, \tag{28.A.5}$$

$P_j = D_j - A_j$, we conclude that $-\mathsf{Tr}(H^-_{A,V})$ does not exceed

$$\int_0^\infty \mathsf{N}^-\big(H'_{A,0}+P_3^2-B''+(\lambda-V)\phi(\lambda)\big)\,d\lambda. \tag{28.A.6}$$

Consider this integral over (μ,∞); it is equal to

$$\int_\mu^\infty \mathsf{N}^-\big(H'_{A,0}+P_3^2-B''+(\lambda-V)\lambda\mu^{-1}\big)\,d\lambda \tag{28.A.7}$$

and since $H'_{A,0} \ge 0$ this integral does not exceed

$$\int_{\mu}^{\infty} \mathsf{N}^{-}\bigl(H'_{A,0} + a[P_3^2 - B'' + (\lambda - V)\lambda\mu^{-1}]\bigr)\, d\lambda \tag{28.A.8}$$

with $a \ge 1$; since $H'_{A,0} \ge P_1^2 + P_2^2 - |B|$ this integral (28.A.8) does not exceed

$$\int_{\mu}^{\infty} \mathsf{N}^{-}\bigl(P_1^2 + P_2^2 - B + a[P_3^2 - B'' + (\lambda - V)\lambda\mu^{-1}]\bigr)\, d\lambda \tag{28.A.9}$$

which can be estimated due to CLR inequality after rescaling $x_3 \mapsto a^{\frac12}x_3$, $P_3 \mapsto a^{-\frac12}P_3$ by

$$\begin{aligned} C\int\int_{\mu}^{\infty} a^{-\frac12}\bigl(B + a[B'' + (V-\lambda)\lambda\mu^{-1}]\bigr)_+^{\frac32}\, d\lambda dx \\ \le C\int\int_{\mu}^{\infty} a^{-\frac12}\bigl(B - \frac{1}{3}a\lambda^2\mu^{-1}\bigr)_+^{\frac32}\, d\lambda dx \\ + C\int\int_{\mu}^{\infty} a\bigl(B'' - \frac{1}{3}\lambda^2\mu^{-1}\bigr)_+^{\frac32}\, d\lambda dx \\ + C\int\int_{\mu}^{\infty} a(\lambda\mu^{-1})^{\frac32}\bigl(V - \frac{1}{3}\lambda\bigr)_+^{\frac32}\, d\lambda dx \\ \le Ca^{-1}\mu^{\frac12}\int B^2\, dx + Ca\mu^{\frac12}\int B''^2\, dx + Ca\mu^{-\frac32}\int V^4\, dx, \end{aligned}$$

where we integrated over $[0,\infty]$. Optimizing with respect to $a \ge 1$ we get

$$\begin{aligned} C\Bigl(\int B^2\, dx\Bigr)^{\frac12}\Bigl(\mu\int B''^2\, dx + \mu^{-1}\int V^4\, dx\Bigr)^{\frac12} \\ + C\mu^{\frac12}\int B''^2\, dx + C\mu^{-\frac32}\int V^4\, dx. \end{aligned} \tag{28.A.10}$$

Therefore integral (28.A.7) does not exceed (28.A.10).

Consider integral (28.A.6) over $[0,\mu]$; it is

$$\int_{0}^{\mu} \mathsf{N}^{-}\bigl(H'_{A,0} + P_3^2 - B'' + (\lambda - V)\bigr)\, d\lambda \tag{28.A.11}$$

and exactly as before it does not exceed

$$\begin{aligned} C\int\int_{0}^{\mu} a^{-\frac12}\bigl(B - \frac{1}{3}a\lambda\bigr)_+^{\frac32}\, d\lambda dx + C\int\int_{0}^{\mu} a\bigl(B'' - \frac{1}{3}\lambda\bigr)_+^{\frac32}\, d\lambda dx \\ + C\int\int_{0}^{\mu} a\bigl(V - \frac{1}{3}\lambda\bigr)_+^{\frac32}\, d\lambda dx \end{aligned}$$

with $a \ge 1$; in first two integrals we replace (in parenthesis) λ by $\lambda^2\mu^{-1}$ and expand integral to $[0,\infty]$, arriving to

$$Ca^{-1}\mu^{\frac12}\int B^2\,dx + Ca\mu^{\frac12}\int B''^2\,dx + C\int aV^{\frac32}\min(V,\mu)\,dx;$$

optimizing with respect to $a \ge 1$ we get

$$C\Big(\int B^2\,dx\Big)^{\frac12}\Big(\mu\int B''^2\,dx + \mu^{\frac12}\int V^{\frac32}\min(V,\mu)\,dx\Big)^{\frac12} + C\mu^{\frac12}\int B''^2\,dx + C\int V^{\frac32}\min(V,\mu)\,dx. \tag{28.A.12}$$

Therefore integral (28.A.11) does not exceed (28.A.12) and the whole expression does not exceed

$$C\Big(\int B^2\,dx\Big)^{\frac12}\Big(\mu\int B''^2\,dx + \mu^{-1}\int V^4\,dx + \mu^{\frac12}\int V^{\frac32}\min(V,\mu)\,dx\Big)^{\frac12} + C\mu^{\frac12}\int B''^2\,dx + C\mu^{-\frac32}\int V^4\,dx + C\int V^{\frac32}\min(V,\mu)\,dx;$$

replacing $\min(V,\mu)$ by $V^{\frac12}\mu^{\frac12}$ and V in the first and second lines respectively we get

$$C\Big(\int B^2\,dx\Big)^{\frac12}\Big(\mu\int B''^2\,dx + \mu^{-1}\int V^4\,dx + \underbrace{\mu\int V^2\,dx}\Big)^{\frac12} + C\mu^{\frac12}\int B''^2\,dx + C\mu^{-\frac32}\int V^4\,dx + C\int V^{\frac52}\,dx.$$

We skip the last term since it is already in (28.A.2); temporarily skip monotone increasing by μ selected term; optimizing the rest by $\mu > 0$ we get

$$C\Big(\int B^2\,dx\Big)^{\frac12}\Big(\int B''^2\,dx\Big)^{\frac14}\Big(\int V^4\,dx\Big)^{\frac14}.$$

Now we are left with

$$C\mu^{\frac12}\Big(\int B^2\,dx\Big)^{\frac12}\Big(\int V^2\,dx\Big)^{\frac12} + C\mu^{-\frac12}\Big(\int B^2\,dx\Big)^{\frac12}\Big(\int V^4\,dx\Big)^{\frac12} + C\mu^{-\frac32}\int V^4\,dx.$$

Optimizing by $\mu > 0$ we get

$$C\Big(\int B^2\,dx\Big)^{\frac12}\Big(\int V^2\,dx\Big)^{\frac14}\Big(\int V^4\,dx\Big)^{\frac14}+ \\ C\Big(\int B^2\,dx\Big)^{\frac38}\Big(\int V^2\,dx\Big)^{\frac38}\Big(\int V^4\,dx\Big)^{\frac14}$$

which concludes the proof. □

28.A.2 Electrostatic Inequality

Proposition 28.A.2. *(i) Let* $B \le Z^3$, $\alpha \le \kappa^* Z^{-1}$, $c^{-1}Z \le N \le cZ$. *Further, if* $B \ge Z^{\frac43}$ *then* $\alpha B^{\frac45} Z^{-\frac25} \le \epsilon$. *Then*

$$\sum_{1\le j<k\le M}\int |x_j - x_k|^{-1}|\Psi(x_1,\dots,x_N)|^2\,s dx_1\cdots dx_N \\ \ge \mathsf{D}(\rho_\Psi,\rho_\Psi) - C\big(Z^{\frac53} + B^{\frac25}Z^{\frac{17}{15}} + B\big); \tag{28.A.13}$$

(ii) Further, afor $B \le Z$ *one can replace the last term in* (28.A.13) *by* $\mathsf{Dirac} - CZ^{\frac53-\delta}$.

Remark 28.A.3. Without self-generated magnetic field the last term was $-C(Z^{\frac53} + B^{\frac25}Z^{\frac{17}{15}})$ and probably it holds here but does not give us any advantage; for $B \ge Z$ we need only $C\big(Z^{\frac53} + B^{\frac45}Z^{\frac35}\big)$ estimate.

Proof. Since we prove estimate from below we replace first

$$\langle \sum_{1\le j\le N} (H_{A,V})_{x_j}\Psi,\ \Psi\rangle$$

by

$$\langle \sum_{1\le j\le N} (H_{A,W})_{x_j}\Psi,\ \Psi\rangle + \int (W-V)\rho_\Psi\,dx$$

without changing anything else and then we estimate the first term here from below by $\mathsf{Tr}(H^-_{A,W})$; then in $\mathsf{Tr}(H^-_{A,W}) + \alpha^{-1}\|\partial A'\|^2$ we replace A' by a minimizer for this expression (rather than for the original problem) only decreasing this expression. So we can now consider A' a minimizer of Section 28.5.

Then we follow arguments of Appendix 26.A.1 but now we need to justify magnetic Lieb-Thirring estimate (26.A.12)

$$\mathsf{Tr}(H^-_{A,W}) \ge -C\int P_B(W)\,dx \tag{26.A.12}$$

in the current settings and with $W : CP'(W) = \rho_\Psi$ and then $W \asymp \min(B^{-2}\rho_\Psi^2; \rho_\Psi^{\frac{2}{3}})$.

Estimate (26.A.12) has been proven in L. Erdös [1] (Theorem 2.2) under assumption that intensity of the magnetic field $\vec{B}(x)$ has a constant direction which was the case in Chapter 25 but not here.

However we actually we do not need (26.A.12); we need this estimate but with an extra term $-CR$ in the right-hand expression where in (i) $R = (Z^{\frac{5}{3}} + B^{\frac{4}{5}}Z^{\frac{3}{5}})$ is the last term in (28.A.13) and in (ii) $R = CZ^{\frac{5}{3}-\delta}$.

Further, the same paper L. Erdös [1]) provides an alternative version of Theorem 2.2: as long as $|\partial A'| \le B$ it is sufficient to estimate $|\partial^2 A| \le cB^{\frac{3}{2}}$.

One can check easily that this pointwise estimate holds if either $B \le Z^2$ and $\ell(x) \ge r_* := B^{-\frac{3}{2}}Z^{\frac{1}{3}}$ or $Z^2 \le B \le Z^3$ and $\ell(x) \ge r_* = Z^{-1}$. Introducing partition into two zones $\{x\colon \ell(x) \ge r_*\}$ and $\{x\colon \ell(x) \le 2r_*\}$ adds $\ell^{-2}\phi(x)$ with $\phi(x) = \mathbf{1}_{\{x\colon r_* \le \ell(x) \le 2r_*\}}$, which adds $-CR$ to the right-hand expression of (26.A.12).

Therefore we need to deal with the zone $\{x\colon \ell(x) \le 2r_*\}$. In this zone however we can neglect an external field; indeed, as in Remark 27.4.1 we get the same estimate (27.4.25) but with B intensity of the combined field; however $\int B^2\,dx$ over this zone does not exceed CR. This concludes proof of Statement (i).

Statement (ii) is proven in the same manner as in Appendix 26.A.1. We leave details to the reader. □

28.A.3 Estimates for $(hD_{x_j} - \mu x_j)e(x,y,\tau)|_{x=y}$ for Toy-Model Operator

We will use here notations of Subsection 27.5.1.

Calculations

Let us calculate the required expressions as $X = \mathbb{R}^3$ and $A(x)$ and $V(x)$ are linear. To do this we can consider just Schrödinger operator (acting on

vector-functions) and then replace V by $V \pm \mu h$ where μ is the magnetic intensity; since $\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}$ we have to consider scalar a Schrödinger operator. Let us apply calculations of Subsection 16.6.1 with operator

$$H = h^2 D_1^2 + (hD_2 - \mu x_1)^2 + hD_3^2 - 2\alpha x_1 - 2\beta x_3, \tag{28.A.14}$$

where without any loss of the generality we assume that $\alpha \ge 0$, $\beta \ge 0$.

After rescaling $x \mapsto \mu x$, $y \mapsto \mu y$, $t \mapsto \mu t$, $\mu \mapsto 1$ (but we will need to use old μ in calculations), $h \mapsto \hbar = \mu h$ we have $U(x,y,t) = U_{(1)}(x_3,y_3,t)U_{(2)}(x',y',t)$ where from (16.6.4)

$$U_{(1)}(x_3,y_3,t) = \frac{1}{2}\mu(2\pi\hbar|t|)^{-\frac{1}{2}} \exp\Big(i\hbar^{-1}\big(\mu^{-1}\beta t(x_3+y_3) + \frac{1}{8}t^{-1}(x_3-y_3)^2 + \frac{1}{3}\mu^{-2}\beta^2 t^3\big)\Big); \tag{28.A.15}$$

and repeating (16.2.9)–(28.A.16) we get

$$U_{(2)}(x,y,t) = i(4\pi\hbar)^{-1}\mu^2 \csc(t)\, e^{i\hbar^{-1}\bar{\phi}_{(2)}(x',y',t)} \tag{28.A.16}$$

with

$$\begin{aligned}\bar{\phi}_{(2)} \coloneqq & -\frac{1}{4}\cot(t)(x_1-y_1)^2 \\ & + \frac{1}{2}(x_1+y_1+2\alpha\mu^{-1})(x_2-y_2+2t\alpha\mu^{-1}) \\ & - \frac{1}{4}\cot(t)(x_2-y_2+2t\alpha\mu^{-1})^2 - t\alpha^2\mu^{-2}.\end{aligned} \tag{28.A.17}$$

Then

$$U(x,y,t) = i(2\pi h)^{-\frac{3}{2}}|t|^{-\frac{1}{2}}\mu^{\frac{3}{2}} \csc(t)\, e^{i\hbar^{-1}\bar{\phi}(x,y,t)} \tag{28.A.18}$$

with

$$\begin{aligned}\bar{\phi} \coloneqq & -\frac{1}{4}\cot(t)(x_1-y_1)^2 \\ & + \frac{1}{2}(x_1+y_1+2\alpha\mu^{-1})(x_2-y_2+2t\alpha\mu^{-1}) - \frac{1}{4}\cot(t)(x_2-y_2+2t\alpha\mu^{-1})^2 - t\alpha^2\mu^{-2} + \\ & \mu^{-1}\beta t(x_3+y_3) + \frac{1}{8}t^{-1}(x_3-y_3)^2 + \frac{1}{3}\mu^{-2}\beta^2 t^3\big).\end{aligned} \tag{28.A.19}$$

Therefore applying first $\hbar D_{x_1}$, $\hbar D_{x_2} - \mu x_1$, or $\hbar D_{x_3}$ and setting after this $x = y = 0$ we conclude that

$$(28.A.20)_1 \quad \bigl(\hbar D_{x_1} U\bigr)|_{x=y=0} = i\alpha\mu^{-1}t \times (2\pi h)^{-\frac{3}{2}}|t|^{-\frac{1}{2}}\mu^{\frac{3}{2}} e^{i\hbar^{-1}\varphi(t)},$$

$$(28.A.20)_2 \quad \bigl(\hbar D_{x_2} U\bigr)|_{x=y=0} = i\alpha\mu^{-1}(1 - t\cot(t)) \times (2\pi h)^{-\frac{3}{2}}|t|^{-\frac{1}{2}}\mu^{\frac{3}{2}} e^{i\hbar^{-1}\varphi(t)},$$

$$(28.A.20)_3 \quad \bigl(\hbar D_{x_3} U\bigr)|_{x=y=0} = i\beta\mu^{-1}t \times (2\pi h)^{-\frac{3}{2}}|t|^{-\frac{1}{2}}\mu^{\frac{3}{2}} e^{i\hbar^{-1}\varphi(t)}$$

with

$$\varphi(t) = \alpha^2\mu^{-2}t - \alpha^2\mu^{-2}t^2\cot(t) + \frac{1}{3}\mu^{-2}\beta^2 t^3. \tag{28.A.21}$$

In other words, in comparison with $U|_{x=y=0}$, calculated in Subsection 16.6.1, expressions $\bigl(\hbar D_{x_1} U\bigr)|_{x=y=0}$, $\bigl(\hbar D_{x_2} U\bigr)|_{x=y=0}$ and $\bigl(\hbar D_{x_3} U\bigr)|_{x=y=0}$ acquire factors $\mu^{-1}\alpha t$, $\mu^{-1}\alpha(1 - t\cot(t))$ and $\mu^{-1}\beta t$ respectively.

Recall that we had 2 cases: $\mu^2 h \le \alpha$ and $\mu^2 h \ge \alpha$.

Case $\alpha \ge \mu^2 h$

Then for each k, $1 \le |k| \le C_0\mu\alpha^{-1}$, the k-th tick contributed no more than

$$C\mu h^{-1} \underbrace{(\mu^2 h/\alpha|k|)^{\frac{1}{2}}} \times (\mu/h|k|)^{\frac{1}{2}} \tag{28.A.22}$$

to $F_{t\to\hbar^{-1}\tau} U|_{x=y=0}$ (see Subsection 16.6.2) and then it contributed no more than this multiplied by $|t_k|^{-1}$, i.e.

$$C\mu h^{-1}|k|^{-1} \underbrace{(\mu^2 h/\alpha|k|)^{\frac{1}{2}}} \times (\mu/h|k|)^{\frac{1}{2}} \tag{28.A.23}$$

to the corresponding Tauberian expression. Even when we multiply by $\mu^{-1}|k|$, we get (28.A.22) again proportional to $|k|^{-1}$; then summation with respect to k, $1 \le |k| \le k^* := C_0\mu(\alpha+\beta)^{-1}$ [39] returns its value at $k = 1$, i.e. $\mu^{\frac{3}{2}} h^{-1}\alpha^{-\frac{1}{2}}$ multiplied by logarithm $(1 + |\log k^*|)$ and therefore we arrive to Proposition 28.A.4 below for $j = 1, 3$.

Let $j = 2$. Since $t_k/\cot(t_k) \asymp \alpha^{-1}\mu$ we conclude that contribution of k-th tick does not exceed

$$C\mu h^{-1}|k|^{-1} \underbrace{(\mu^2 h/\alpha|k|)^{\frac{1}{2}}} \times (\mu/h|k|)^{\frac{1}{2}} \tag{28.A.24}$$

[39] As $|t| \ge k^*$ we have $\phi'(t) \ge c_1$ and then integrating by parts there we can recover factor $(t/k^*)^{-n}$ thus effectively confining us to integration over $\{t\colon |t| \le k^*\}$. This observation can also improve some results of Sections 16.6–16.10.

and summation by $|k| \ge 1$ returns its value as $|k| = 1$ i.e. $C\mu^{\frac{5}{2}}h^{-1}\alpha^{-\frac{1}{2}}$ and therefore we arrive to proposition 28.A.4 below for $j = 2$.

Proposition 28.A.4. *Let $\mu h \le \epsilon_0$, $\tau \asymp 1$, $\alpha \ge \mu^2 h$.*

Then

(i) Expression $(hD_{x_2} - \mu x_1)e(x, y, \tau)|_{x=y=0}$ does not exceed $C\mu^{\frac{5}{2}}h^{-1}\alpha^{-\frac{1}{2}}$.

(ii) Expression $hD_{x_j}e(x, y, \tau)|_{x=y=0}$ does not exceed $C\mu^{\frac{3}{2}}h^{-1}\alpha^{\frac{1}{2}}(1+|\log \mu(\alpha+\beta)^{-1}|)$, and $C\mu^{\frac{3}{2}}h^{-1}\beta\alpha^{-\frac{1}{2}}(1+|\log \mu(\alpha+\beta)^{-1}|)$ for $j = 1, 3$ respectively.

Case $\alpha \le \mu^2 h$

If $\mu^2 h \ge \alpha$, then the same arguments work only for $\bar{k} := \mu^2 h\alpha^{-1} \le |k| \le k^*$, resulting in contributions $C\mu^{\frac{3}{2}}h^{-1}\alpha^{\frac{1}{2}}(1 + |\log k^*\bar{k}^{-1}|)$, $C\mu^{\frac{1}{2}}h^{-2}\alpha^{\frac{1}{2}}$, and $C\mu^{\frac{3}{2}}h^{-1}\beta\alpha^{-\frac{1}{2}}(1 + |\log k^*\bar{k}^{-1}|)$ for $j = 1, 2, 3$ respectively if $\bar{k} \le k^*$ (i.e. $\mu h\beta \le \alpha$) or 0 otherwise.

Let $1 \le |k| \le \bar{k}$. We mainly consider the most difficult case $j = 2$ and (as $|t| \ge \epsilon_0$) only term arising from $-\alpha\mu^{-1}t\cot(t)$ factor, namely

$$\alpha\mu^{-1} \times \mu^{\frac{3}{2}}h^{-\frac{3}{2}} \int |t|^{-\frac{1}{2}} \cos(t)(\sin(t))^{-2} e^{ih^{-1}(\varphi(t)-t\tau)}\, dt, \tag{28.A.25}$$

where we took into account that we need to divide by t and skipped a constant factor.

Consider first (28.A.25) with integration over interval $\{t\colon |t - t_k| \le s_k\}$ near t_k. Observe that

$$\phi'(t) = (\sin(t))^{-2}\alpha^2\mu^{-2}t^2 - 2t(\sin(t))^{-1}\alpha^2\mu^{-2}t^2 + \beta^2t^2 \tag{28.A.26}$$

and transform (28.A.25) into

$$\begin{aligned}&\alpha^{-1}\mu^{\frac{7}{2}}h^{-\frac{1}{2}} \int_{t_k-s_k}^{t_k+s_k} |t|^{-\frac{5}{2}} \cos(t)\partial_t\big[e^{ih^{-1}(\varphi(t)-t\tau)}\big]\, dt \\ &+\alpha^{-1}\mu^{\frac{5}{2}}h^{-\frac{3}{2}} \int_{t_k-s_k}^{t_k+s_k} |t|^{-\frac{5}{2}} \cos(t)\big[2t(\sin(t))^{-1}\alpha^2\mu^{-2} - \beta^2t^2 + \tau\big]e^{ih^{-1}(\varphi(t)-t\tau)}.\end{aligned} \tag{28.A.27}$$

Integrating the first term by parts we get a non-integral term

$$\alpha^{-1}\mu^{\frac{7}{2}}h^{-\frac{1}{2}}|t|^{-\frac{5}{2}} \cos(t)e^{ih^{-1}(\varphi(t)-t\tau)}\big|_{t=t_k-s_k}^{t=t_k+s_k} \tag{28.A.28}$$

and we get an integral term

$$\begin{aligned}(28.A.29)\quad &\alpha^{-1}\mu^{\frac{5}{2}}h^{-\frac{3}{2}}\int_{t_k-s_k}^{t_k+s_k}\Big[\mu h\partial_t\big[|t|^{-\frac{5}{2}}\cos(t)\big]\\ &\qquad+|t|^{-\frac{5}{2}}\cos(t)\big[2t(\sin(t))^{-1}\alpha^2\mu^{-2}-\beta^2t^2+\tau\big]\Big]e^{i\hbar^{-1}(\varphi(t)-t\tau)}\,dt\\ &=2\alpha\mu^{\frac{1}{2}}h^{-\frac{3}{2}}\int_{t_k-s_k}^{t_k+s_k}|t|^{-\frac{3}{2}}\cot(t)e^{i\hbar^{-1}(\varphi(t)-t\tau)}\,dt+O\big(\alpha^{-1}\mu^{\frac{5}{2}}h^{-\frac{3}{2}}s_k|k|^{-\frac{5}{2}}\big).\end{aligned}$$

Repeating the same trick we can eliminate the first term in the right-most expression. Therefore we arrive to (28.A.28) with $O\big(\alpha^{-1}\mu^{\frac{5}{2}}h^{-\frac{3}{2}}s_k|k|^{-\frac{5}{2}}\big)$ error. When $s_k\asymp\alpha^2\mu^{-3}h^{-1}k^2$ we get

$$(28.A.30)\qquad C\mu^{\frac{3}{2}}h^{-\frac{3}{2}}\times(\alpha/\mu^2h)|k|^{-\frac{1}{2}}$$

error.

On the other hand, consider integral over $[t_k+s_k,\,t_{k+1}-s_{k+1}]$, $k\neq0$. Decomposing $e^{i\mu^{-1}h^{-1}(\phi(t)-t\tau)}$ into Taylor series with respect to $\alpha^2h^{-1}\mu^{-3}\cot(t)$ one can prove easily that expression in question is

$$\alpha^{-1}\mu^{\frac{7}{2}}h^{-\frac{1}{2}}|t|^{-\frac{5}{2}}\cos(t)\big(e^{i\hbar^{-1}(\varphi(t)-t\tau)}-e^{i\hbar^{-1}(\varphi_{(1)}(t)-t\tau)}\big)\big|_{t=t_k+s_k}^{t=t_{k+1}-s_{k+1}}$$

with $\varphi_{(1)}(t)=\frac{1}{3}\mu^{-2}\beta^2t^3$ and with error not exceeding (28.A.30) multiplied by $(1+|\log s_k|)$:

$$(28.A.31)\qquad C\mu^{\frac{3}{2}}h^{-\frac{3}{2}}\times(\alpha/\mu^2h)|k|^{-\frac{1}{2}}\times(1+|\log(\alpha^2\mu^{-3}h^{-1}k^2)|).$$

So non-integral terms with φ cancel one another because by the similar arguments we can also cover $[0,t_1-s_1]$ and $[t_{-1}+s_{-1}]$ and due to non-singularity of $t^{-1}(1-t\cot(t))\csc(t)$ at $t=0$ there will be no non-integral terms with $k=0$. So we are left with

$$-\alpha^{-1}\mu^{\frac{7}{2}}h^{-\frac{1}{2}}|t|^{-\frac{5}{2}}\cos(t)e^{i\hbar^{-1}(\varphi_{(1)}(t)-t\tau)}\big)\big|_{t=t_k-s_k}^{t=t_k+s_k}$$

and their absolute values do not exceed (28.A.30).

Finally, summation of (28.A.31) by $k:1\le|k|\le\min(\bar k,k^*)$ returns

$$\begin{aligned}(28.A.32)\quad &C\mu^{\frac{3}{2}}h^{-\frac{3}{2}}(\alpha/\mu^2h)^{\frac{1}{2}}\\ &\qquad\times\begin{cases}(1+|\log(\mu h)|) & \beta\mu h\le\alpha,\\ (\alpha/\beta\mu h)^{\frac{1}{2}}(1+|\log(\mu h)|+|\log(\alpha/\beta\mu h)|) & \beta\mu h\ge\alpha.\end{cases}\end{aligned}$$

and we arrive to Proposition 28.A.5 below for $j=2$:

Proposition 28.A.5. *Let $\mu h \le \epsilon_0$, $\tau \asymp 1$, and $\alpha \le \mu^2 h$.*
Then

(i) Expression $(hD_{x_2} - \mu x_1)e(x,y,\tau)|_{x=y=0}$ does not exceed (28.A.32).

(ii) Expression $hD_{x_3}e(x,y,\tau)|_{x=y=0}$ does not exceed

$$C\begin{cases}\mu^{\frac{3}{2}}h^{-1}\beta\alpha^{-\frac{1}{2}}(1+|\log(\alpha/\beta\mu h)|) & \text{if } \beta\mu h \le \alpha,\\ \mu h^{-\frac{3}{2}}\beta^{\frac{1}{2}} & \text{if } \beta\mu h \ge \alpha.\end{cases} \tag{28.A.33}$$

(iii) Expression $hD_{x_1}e(x,y,\tau)|_{x=y=0}$ does not exceed

$$C\begin{cases}\mu^{\frac{3}{2}}h^{-1}\alpha^{\frac{1}{2}}(1+|\log(\alpha/\beta\mu h)|) & \text{if } \beta\mu h \le \alpha,\\ \mu h^{-\frac{3}{2}}\alpha^{\frac{1}{2}} & \text{if } \beta\mu h \ge \alpha.\end{cases} \tag{28.A.34}$$

Case $\mu h \ge \epsilon_0$

If $\mu h \ge \epsilon_0$ we consider a different representation: namely (16.2.15) for a spectral projector in dimension 2 (again after rescaling where we scale e_* as functions rather than Schwartz kernels):

$$e_{(2)}(x',y',\tau) = \tag{28.A.35}$$
$$(2\pi)^{-1}\mu h^{-1}\sum_{m\in\mathbb{Z}^+}\int \upsilon_m\bigl(\eta+\mu^{-\frac{1}{2}}h^{-\frac{1}{2}}(x_1-y_1)\bigr)\upsilon_m\bigl(\eta-\mu^{-\frac{1}{2}}h^{-\frac{1}{2}}(x_1-y_1)\bigr)$$
$$\times\theta\Bigl(\tau-\alpha\mu^{-1}(x_1+y_1)-2\alpha\mu^{-\frac{1}{2}}h^{\frac{1}{2}}\eta-\alpha^2\mu^{-2}-2m\mu h\Bigr)e^{i\mu^{-\frac{1}{2}}h^{-\frac{1}{2}}(x_2-y_2)\eta}\,d\eta,$$

where we also replaced $H_{(2)}$ by $H_{(2)}-\mu h$ and τ by $\mu h+\tau$. Since

$$e(x,y,\tau) = e_{(2)}(x',y',.)\ast_\tau e_{(1)}(x_3,y_3,.), \tag{28.A.36}$$

where $e_{(1)}(x_3,y_3,\tau)$ is a Schwartz kernel of the spectral projector of 1-dimensional operator

$$\mu^2h^2D_3^2 - 2\beta\mu^{-1}x_3, \tag{28.A.37}$$

we conclude that

$$e(x,y,\tau) \tag{28.A.38}$$
$$= (2\pi)^{-1}\mu h^{-1}\sum_{m\in\mathbb{Z}^+}\int \upsilon_m\bigl(\eta+\mu^{-\frac{1}{2}}h^{-\frac{1}{2}}(x_1-y_1)\bigr)\upsilon_m\bigl(\eta-\mu^{-\frac{1}{2}}h^{-\frac{1}{2}}(x_1-y_1)\bigr)$$
$$\times e_{(1)}\Bigl(x_3,y_3,\tau-\alpha\mu^{-1}(x_1+y_1)-2\alpha\mu^{-\frac{1}{2}}h^{\frac{1}{2}}\eta-\alpha^2\mu^{-2}-2m\mu h\Bigr)e^{i\mu^{-\frac{1}{2}}h^{-\frac{1}{2}}(x_2-y_2)\eta}\,d\eta.$$

Then

$$(28.A.39)\quad (\mu hD_{x_2} - x_1)e(x,y,\tau)\big|_{x=y=0} = (2\pi)^{-1}\mu^{\frac{3}{2}}h^{-\frac{1}{2}}\sum_{m\in\mathbb{Z}^+}\int \upsilon_m^2(\eta)\eta\times e_{(1)}\Big(0,0,\tau-2\alpha\mu^{-\frac{1}{2}}h^{\frac{1}{2}}\eta-\alpha^2\mu^{-2}-2m\mu h\Big)\,d\eta$$

and since $\upsilon_m(.)$ is an even (odd) function for even (odd) m we can replace $e_{(1)}(0,0,\tau' - 2\alpha\mu^{-\frac{1}{2}}h^{\frac{1}{2}}\eta)$ by

$$(28.A.40)\qquad e_{(1)}(0,0,\tau' - 2\alpha\mu^{-\frac{1}{2}}h^{\frac{1}{2}}\eta) - e_{(1)}(0,0,\tau' + 2\alpha\mu^{-\frac{1}{2}}h^{\frac{1}{2}}\eta).$$

In virtue of Subsubsection 5.2.1.3 *Asymptotics without Spatial Mollification* we know that an absolute value of this expression does not exceed $Ch^{-\frac{1}{2}}\alpha^{\frac{1}{2}}$ [40] we arrive to estimate[41]

$$(28.A.41)\qquad |(hD_{x_2} - \mu x_1)e(x,y,\tau)\big|_{x=y=0}| \le C\mu^{\frac{3}{2}}h^{-1}\alpha^{\frac{1}{2}}.$$

Further,

$$(28.A.42)\quad \mu hD_{x_1}e(x,y,\tau)\big|_{x=y=0} = i(2\pi)^{-1}\alpha\mu\sum_{m\in\mathbb{Z}^+}\int \upsilon_m^2(\eta)\times\partial_\tau e_{(1)}\big(0,0,\tau-2\alpha\mu^{-\frac{1}{2}}h^{\frac{1}{2}}\eta-\alpha^2\mu^{-2}-2m\mu h\big)\,d\eta$$
$$= i(2\pi)^{-1}\mu^{\frac{3}{2}}h^{-\frac{1}{2}}\sum_{m\in\mathbb{Z}^+}\int \upsilon_m(\eta)\upsilon_m'(\eta)\times e_{(1)}\big(0,0,\tau-2\alpha\mu^{-\frac{1}{2}}h^{\frac{1}{2}}\eta-\alpha^2\mu^{-2}-2m\mu h\big)\,d\eta$$

and using the same arguments we arrive to estimate[41]

$$(28.A.43)\qquad |hD_{x_1}e(x,y,\tau)\big|_{x=y=0}| \le C\mu^{\frac{3}{2}}h^{-1}\alpha^{\frac{1}{2}}.$$

Finally,

$$(28.A.44)\quad \mu hD_{x_3}e(x,y,\tau)\big|_{x=y=0} = (2\pi)^{-1}\mu h^{-1}\sum_{m\in\mathbb{Z}^+}\int \upsilon_m^2(\eta) \times \underbrace{\mu hD_{x_3}e_{(1)}\big(x_3,y_3,\tau-2\alpha\mu^{-\frac{1}{2}}h^{\frac{1}{2}}\eta-\alpha^2\mu^{-2}-2m\mu h\big)\big|_{x_3=y_3=0}}\,d\eta$$

[40] Only in the worst case when $|\tau - 2m\mu h|$ is not disjoint from 0.

[41] In the non-rescaled coordinates.

and again in virtue of Subsubsection 5.2.1.3 *Asymptotics without Spatial Mollification* we know that an absolute value of selected expression does not exceed $Ch^{-\frac{1}{2}}\beta^{\frac{1}{2}}$ [42] and we arrive to estimate[41]

$$|hD_{x_3}e(x,y,\tau)\big|_{x=y=0}| \le C\mu h^{-\frac{3}{2}}\beta^{\frac{1}{2}}. \tag{28.A.45}$$

Therefore we have proven

Proposition 28.A.6. *Let $\mu h \ge 1$, $\alpha \le 1$, $\beta \le 1$, $|\tau| \le c_0$. Then for operator $(H-\mu h)$ estimates* (28.A.41)*,* (28.A.43) *and* (28.A.45) *hold.*

Tauberian Estimates

Remark 28.A.7. Assume now that all assumptions are fulfilled only in $B(0,\ell)$ rather than in $\mathbb{R}^3$. Then there is also a Tauberian estimate which should be added to Weyl estimate. This Tauberian estimate (the same for all $j=1,2,3$) coincides with the Tauberian estimate was calculated in Chapter 16. Namely

(i) For $\mu h \le 1$, $\ell \ge C_0\mu^{-1}$ this Tauberian estimate was calculated in Proposition 16.6.2(ii)[41].

(ii) For $\mu h \ge 1$, $\ell \ge C_0 h$ this Tauberian estimate was calculated in Proposition 16.6.7(i) and corollary 16.6.8(i)[41] and it does not exceed $C\mu h^{-\frac{3}{2}}\ell^{-1}$.

Important Remarks

One can apply estimates of $\mathsf{D}(\rho_\Psi - \rho_B^{\mathsf{TF}},\ \rho_\Psi - \rho_B^{\mathsf{TF}})$ to the following problems we considered in the frameworks of chapters 25, 26 and 27:

1. Excessive negative charge (estimates).
2. Ionization energy (estimates and asymptotics: again we estimate only $(\mathsf{I}^* + \nu)_+$ rather than $|\mathsf{I}^* + \nu|$.
3. Excessive positive charge in the free nuclei model as $M \ge 2$.

We leave it to the reader.

[42] Without applying $(hD_j - \mu A_j(x))$ but it does not matter.

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Part XII

Articles

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Spectral Asymptotics For The Semiclassical Dirichlet To Neumann Operator *,†

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Abstract

Let M be a compact Riemannian manifold with smooth boundary, and let $R(\lambda)$ be the Dirichlet-to-Neumann operator at frequency λ. The semiclassical Dirichlet-to-Neumann operator $R_{\mathsf{scl}}(\lambda)$ is defined to be $\lambda^{-1}R(\lambda)$. We obtain a leading asymptotic for the spectral counting function for $R_{\mathsf{scl}}(\lambda)$ in an interval $[a_1, a_2)$ as $\lambda \to \infty$, under the assumption that the measure of periodic billiards on T^*M is zero. The asymptotic takes the form

$$\mathsf{N}(\lambda; a_1, a_2) = \big(\kappa(a_2) - \kappa(a_1)\big)\,\mathsf{vol}'(\partial M)\lambda^{d-1} + o(\lambda^{d-1}),$$

where $\kappa(a)$ is given explicitly by

$$\kappa(a) = \frac{\omega_{d-1}}{(2\pi)^{d-1}}\bigg(-\frac{1}{2\pi}\int_{-1}^{1}(1-\eta^2)^{(d-1)/2}\frac{a}{a^2+\eta^2}\,d\eta \\ -\frac{1}{4} + H(a)(1+a^2)^{(d-1)/2}\bigg).$$

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1 Introduction

Let M be a Riemannian manifold with boundary. The Dirichlet-to-Neumann operator is a family of operators defined on $L^2(\partial M)$ depending on the parameter $\lambda \geq 0$. It is defined as follows: given $f \in L^2(\partial M)$, we solve the equation (if possible)

$$(\Delta - \lambda^2)u = 0 \quad \text{on} \quad M, \qquad u|_{\partial M} = f. \tag{1.1}$$

Then the Dirichlet-to-Neumann operator at frequency λ is the map

$$R(\lambda) : f \mapsto -\partial_\nu u|_{\partial M}. \tag{1.2}$$

Here ∂_ν is the interior unit normal derivative, and Δ is the positive Laplacian on M.

It is well known that $R(\lambda)$ is a self-adjoint, semi-bounded from below pseudodifferential operator of order 1 on $L^2(\partial M)$, with domain $H^1(\partial M)$. It therefore has discrete spectrum accumulating only at $+\infty$. The Dirichlet-to-Neumann operator and closely related operators are important in a number of areas of mathematical analysis including inverse problems (such as Calderón's problem [Cal]), domain decomposition problems (such as the determinant gluing formula of Burghelea-Friedlander-Kappeler [BFK]), and spectral asymptotics (see e.g. [Fried]).

In this paper, we are interested in the spectral asymptotics of $R(\lambda)$ in the high-frequency limit, $\lambda \to \infty$. Let us recall standard spectral asymptotics for elliptic differential operators, for simplicity in the simplest case of a positive self-adjoint second order scalar operator. Suppose that Q is such an operator on a manifold M of dimension d, with principal symbol q. Then in the case that M is closed, we have an asymptotic for the number $\mathsf{N}(\lambda)$ of eigenvalues of Q (counted with multiplicity) less than λ^2:

$$\begin{aligned} \mathsf{N}(\lambda) &= (2\pi)^{-d}\,\mathsf{vol}\{(x,\xi) \in T^*M \mid q(x,\xi) \leq \lambda^2\} + O(\lambda^{d-1}) \\ &= \big(\frac{\lambda}{2\pi}\big)^d\,\mathsf{vol}\{(x,\xi) \in T^*M \mid q(x,\xi) \leq 1\} + O(\lambda^{d-1}), \end{aligned} \tag{1.3}$$

where the equality of the two expressions on the RHS is a simple consequence of the homogeneity of q. Moreover, if the set of periodic geodesics has measure zero, then there is a two-term expansion of the form

$$\big(\frac{\lambda}{2\pi}\big)^d\,\mathsf{vol}\{(x,\xi) \in T^*M \mid q(x,\xi) \leq 1\} + \frac{\lambda^{d-1}}{(2\pi)^d}\int_{\{q=1\}} \mathsf{sub}(Q) + o(\lambda^{d-1})$$

where $\mathsf{sub}(Q)$ is the subprincipal symbol of Q [DuGu]. This was generalised to the case of manifolds with boundary by the second author [Ivr1]. For simplicity we state the result in the case that $Q = \Delta$ is the (positive) metric Laplacian, which satisfies $\mathsf{sub}(\Delta) = 0$. Then Δ is a self-adjoint operator under either Dirichlet $(-)$ or Neumann $(+)$ boundary conditions, and if the set of periodic generalised bicharacteristics has measure zero, we get a two-term expansion for $\mathsf{N}_\Delta(\lambda)$ of the form

$$\Big(\frac{\lambda}{2\pi}\Big)^d \mathsf{vol}\, B^*M \pm \frac{1}{4}\Big(\frac{\lambda}{2\pi}\Big)^{d-1} \mathsf{vol}\, B^*\partial M + o(\lambda^{d-1}). \tag{1.4}$$

These statements can be generalized to the semiclassical setting. Consider a classical Schrödinger operator on M, $P = h^2\Delta + V(x) - 1$, where $h > 0$ is a small parameter ("Planck's constant") and V is a smooth real-valued function. We consider the asymptotic behaviour $\mathsf{N}_h^-(P)$ of the number of negative eigenvalues of P as $h \to 0$. This is equivalent to the problem above if $h = \lambda^{-1}$ and V is identically zero. Define $p(x,\xi)$ to be the semiclassical symbol of P, i.e. $p = |\xi|^2_{g(x)} + V(x) - 1$. Then, if M is closed, under the assumption that the measure of periodic bicharacteristics of P is zero in T^*M, and that 0 is a regular value for p, we have

$$\mathsf{N}_h^-(P) = (2\pi h)^{-d}\, \mathsf{vol}\{(x,\xi) \in T^*M \mid p(x,\xi) \le 0\} + O(h^{1-d}). \tag{1.5}$$

Moreover, for manifolds with boundary, we have an analogue of (1.4): under either Dirichlet $(-)$ or Neumann $(+)$ boundary conditions, if the set of periodic generalised bicharacteristics has measure zero, we get a two-term expansion for $\mathsf{N}_h^-(P)$ (where here we understand the self-adjoint realization of P with either Dirichlet or Neumann boundary condition) of the form

$$(2\pi h)^{-d}\, \mathsf{vol}\{(x,\xi) \in T^*M \mid p(x,\xi) \le 0\} \pm \frac{1}{4}(2\pi h)^{1-d}\, \mathsf{vol}\, \mathcal{H} + o(h^{1-d}), \tag{1.6}$$

where $\mathcal{H} \subset T^*(\partial M)$ is the hyperbolic region in the boundary, that is, the projection of the set $\{(x,\xi) \mid p(x,\xi) \le 0\} \cap T^*_{\partial M}M$ to $T^*\partial M$.

From the semiclassical point of view, since $R(\lambda)$ is a first order operator, it makes sense to consider $R_{\mathsf{scl}}(\lambda) := \lambda^{-1}R(\lambda)$ (for $\lambda > 0$), which we call the semiclassical Dirichlet-to-Neumann operator. Like $R(\lambda)$, it is a self-adjoint, semi-bounded from below operator on $L^2(\partial M)$, with discrete spectrum accumulating only at $+\infty$. The goal of this paper is to investigate the spectral asymptotics of $R_{\mathsf{scl}}(\lambda)$, that is, the asymptotics of

$$\mathsf{N}(\lambda; a_1, a_2) \coloneqq \#\{\beta :\ \beta \text{ is an eigenvalue of } \ R_{\mathsf{scl}}(\lambda),\ a_1 \le \beta < a_2\}, \tag{1.7}$$

the number of eigenvalues of $R_{\mathrm{scl}}(\lambda)$ in the interval $[a_1, a_2)$, as $\lambda \to \infty$.

Both $R(\lambda)$ and $R_{\mathrm{scl}}(\lambda)$ have the disadvantage that they are undefined whenever λ^2 is a Dirichlet eigenvalue, since then (1.1) is not solvable for arbitrary $f \in H^1(M)$. Indeed, when λ^2 is a Dirichlet eigenvalue, a necessary condition for solvability of (1.1) is that f is orthogonal to the normal derivatives of Dirichlet eigenfunctions at frequency λ. To overcome this issue, we introduce the *Cayley transform* of $R_{\mathrm{scl}}(\lambda)$: we define

$$C(\lambda) = (R_{\mathrm{scl}}(\lambda) - i)(R_{\mathrm{scl}}(\lambda) + i)^{-1}. \tag{1.8}$$

This family of operators is related to impedance boundary conditions: we have $C(\lambda)f = g$ if and only if there is a function u on M satisfying

$$(\Delta - \lambda^2)u = 0 \tag{1.9}$$

and

$$\frac{1}{2}(\lambda^{-1}\partial_\nu u - iu) = f, \qquad \frac{1}{2}(\lambda^{-1}\partial_\nu u + iu) = g. \tag{1.10}_{1,2}$$

As observed in [BH], $C(\lambda)$ is a well-defined analytic family of operators for λ in a neighbourhood of the positive real axis, which is unitary on the real axis. In particular, it is well-defined even when λ^2 is a Dirichlet eigenvalue of the Laplacian on M. As a unitary operator, $C(\lambda)$, $\lambda > 0$, has its spectrum on the unit circle, and as $R_{\mathrm{scl}}(\lambda)$ has discrete spectrum accumulating only at ∞, it follows that the spectrum of $C(\lambda)$ is discrete on the unit circle except at the point 1. Our question can be formulated in terms of $C(\lambda)$: given two angles θ_1, θ_2 satisfying $0 < \theta_1 < \theta_2 < 2\pi$, what is the leading asymptotic for

$$\tilde{\mathsf{N}}(\lambda; \theta_1, \theta_2) := \#\{e^{i\theta} : \ e^{i\theta} \text{ is an eigenvalue of } \ C(\lambda), \ \theta_1 \le \theta < \theta_2\} \tag{1.11}$$

the number of eigenvalues of $C(\lambda)$ in the interval $\{e^{i\theta} : \theta \in [\theta_1, \theta_2)\}$ of the unit circle, as $\lambda \to \infty$. Clearly, we have

$$\tilde{\mathsf{N}}(\lambda; \theta_1, \theta_2) = \mathsf{N}(\lambda; a_1, a_2), \text{ where } \ e^{i\theta_j} = \frac{a_j - i}{a_j + i}, \text{ i.e. } a_j = -\cot\Big(\frac{\theta_j}{2}\Big). \tag{1.12}$$

To answer this question we relate it to a standard semiclassical eigenvalue counting problem on M. To state the next result, we first define the self-adjoint operator $P_{a,h}$ on $L^2(M)$ by

$$\mathfrak{D}(P_{a,h}) = \{u \in H^2(M) : \ (h\partial_\nu + a)u = 0 \ \text{ at } \ \partial M\}, \tag{1.13}$$

$$P_{a,h}(u) = (h^2\Delta - 1)u, \quad u \in \mathfrak{D}(P_{a,h}). \tag{1.14}$$

It is the self-adjoint operator associated to the semi-bounded quadratic form

$$h^2\|\nabla u\|_M^2 - \|u\|_M^2 - ha\|u\|_{\partial M}^2. \tag{1.15}$$

The operator $P_{a,h}$ is linked with the semiclassical Dirichlet-to-Neumann operator as follows: if f is an eigenfunction of $R_{\mathsf{scl}}(\lambda)$ with eigenvalue a, then the corresponding Helmholtz function u defined by (1.1) is in the domain (1.13) of $P_{a,h}$, and $P_{a,h}u = 0$ (where $h = \lambda^{-1}$).

Then we have the following result, proved in Section 2.

Proposition 1.1. *Let $h = \lambda^{-1}$. Assume $0 < \theta_1 < \theta_2 < 2\pi$. Then the number of eigenvalues of $C(\lambda)$ in the interval $J_{\theta_1,\theta_2} := \{e^{i\theta} : \theta \in [\theta_1, \theta_2)\}$ is equal to*

$$\tilde{\mathsf{N}}(\lambda; \theta_1, \theta_2) = \mathsf{N}(\lambda; a_1, a_2) = \mathsf{N}_h^-(a_2) - \mathsf{N}_h^-(a_1), \tag{1.16}$$

where $a_j = -\cot(\theta_j/2)$ and

$$\mathsf{N}_h^-(a) \coloneqq \#\{\mu : \mu \text{ is an eigenvalue of } P_{a,h},\ \mu < 0\}. \tag{1.17}$$

Having thus reduced the problem to a standard question about semiclassical spectral asymptotics, we obtain (after some calculations in Section 3) our main result.

Theorem 1.2. *(i) The following estimate for the quantity (1.11) holds:*

$$\mathsf{N}(\lambda; a_1, a_2) = O(\lambda^{d-1}); \tag{1.18}$$

(ii) Further, if the set of periodic billiards on M has measure 0 then the following asymptotic holds as $\lambda \to +\infty$:

$$\mathsf{N}(\lambda; a_1, a_2) = \big(\kappa(a_2) - \kappa(a_1)\big)\,\mathsf{vol}'(\partial M)\lambda^{d-1} + o(\lambda^{d-1}), \tag{1.19}$$

where $\kappa(a)$ is given explicitly by

$$\kappa(a) = \frac{\omega_{d-1}}{(2\pi)^{d-1}}\bigg(-\frac{1}{2\pi}\int_{-1}^{1}(1-\eta^2)^{(d-1)/2}\frac{a}{a^2+\eta^2}\,d\eta \\ -\frac{1}{4} + H(a)(1+a^2)^{(d-1)/2}\bigg). \tag{1.20}$$

Here $H(\cdot)$ is the Heaviside function, ω_d is the volume of the unit ball in $\mathbb{R}^d$, and $\mathsf{vol}(M)$ and $\mathsf{vol}'(\partial M)$ are d-dimensional volume of M and $(d-1)$-dimensional volume of ∂M respectively.

(iii) In the case $d = 3$*, we can evaluate this integral exactly and we find that*

$$\kappa(a) = \frac{1}{4\pi}\Big(-\frac{1}{4} - \frac{1}{\pi}\operatorname{arccot}(a)(1+a^2) + (1+a^2) + \frac{1}{\pi}a\Big) \tag{1.21}$$

where arccot *has range* $(0,\pi)$*. This is simpler expressed in terms of* θ*. Defining* $\tilde\kappa(\theta) = \kappa(a)$ *where* $a = -\cot(\theta/2) = \cot(\pi - \theta/2)$*, we have (still under the zero-measure assumption on periodic billiards)*

$$\tilde{\mathsf{N}}(\lambda;\theta_1,\theta_2) = \big(\tilde\kappa(\theta_2) - \tilde\kappa(\theta_1)\big)\,\mathrm{vol}'(\partial M)\lambda^2 + o(\lambda^2), \tag{1.22}$$

$$\tilde\kappa(\theta) = \frac{1}{4\pi}\Big(-\frac{1}{4} + \frac{1}{2\pi}\big(\frac{\theta - \sin\theta}{\sin^2(\theta/2)}\big)\Big). \tag{1.23}$$

Remark 1.3. It looks disheartening that we only get an upper bound in case (i) and only a "o" remainder under a global geometric condition, but it is the nature of the beast.

In regards to case (i), consider M a hemisphere; then for $\lambda_n^2 = n(n+d-1)$ with $n \in \mathbb{Z}^+$ the operator $R(\lambda)$ has eigenvalue 0 of multiplicity $\asymp n^{d-1}$, hence $N(\lambda; a_1, a_2)$ jumps by at least $c\lambda^{d-1}$ as a_1 or a_2 crosses zero. Therefore, in this case, we do not have an asymptotic, but only an estimate.

In regards to case (ii), we believe that the "o" remainder is the best that can be achieved using current technology. To justify this, consider the problem of finding the spectral asymptotics of the semiclassical Dirichlet-to-Neumann operator for the operator $\Delta + \lambda^2$, instead of $\Delta - \lambda^2$ (for real λ). In this case, one can readily check that the semiclassical Dirichlet-to-Neumann operator is a semiclassically elliptic pseudodifferential operator on the boundary, with principal symbol $\sqrt{1 + h^2\Delta_{\partial M}}$. Then standard spectral asymptotics hold for this operator, and we would get a remainder term $O(\lambda^{d-2})$. However, for the operator $\Delta - \lambda^2$, the semiclassical Dirichlet-to-Neumann operator is only microlocally elliptic in the region $\{(y,\eta) \mid |\eta|_{g'} > 1\} \subset T^*\partial M$, where g' is the induced metric on the boundary. It is hyperbolic in the region where $|\eta|_{g'} < 1$, and that means there is (currently) no machinery for *directly* tackling its spectral asymptotics. Instead, we proceed by relating it to the spectral asymptotics for the interior problem with a family of boundary conditions depending on the spectral parameter a. This means that the problem is in some sense really d-dimensional, and our $o(\lambda^{d-1})$ remainder is the "ghost" of d-dimensional spectral asymptotics, in which the principal, Weyl term cancels under taking the difference (1.16),

and all we are left with is the second term — and only under the global geometric assumption of measure zero periodic billiard trajectories.

We note that under stronger assumptions on the billiard flow, the remainder could be improved, for example to $O(\lambda^{d-1-\delta})$ for some $\delta > 0$ in the case of a Euclidean ellipse or elliptical annulus — see Section 7.4 of [Ivr4].

Remark 1.4. One can consider eigenvalues of operator $\rho\lambda^{-1}R(\lambda)$ with $\rho > 0$ smooth on ∂M; then estimates (3.1), (1.18) and asymptotics (3.2), (1.19) hold in the frameworks of Statement (i) and (ii) of Theorem 1.2 respectively albeit with $\kappa(a)\,\mathsf{vol}'(\partial M)$ replaced by

$$\int_{\partial M} \kappa(\rho(x')a)\, d\sigma$$

where $d\sigma$ is a natural measure on ∂M; however without this condition $\rho > 0$ problem may be much more challenging; even self-adjointness is by no means guaranteed.

Remark 1.5. Operators of the form $P_{a,h}$ were considered by Frank and Geisinger [FG]. They showed that the trace of the negative part of $P_{a,h}$ has a two-term expansion as $h \to 0$ regardless of dynamical assumptions[1)], and the second term in their expansion (the $L_d^{(2)}$ term of [FG, Theorem 1.1]) is closely related to $\kappa(a)$ — see Remark 3.4.

Remark 1.6. We can rephrase Theorem 1.2 in terms of a limiting measure on the unit circle. For each $\lambda > 0$, let $\mu(h)$, $h = \lambda^{-1}$, denote the atomic measure determined by the spectrum of $C(\lambda)$:

$$\mu(h) = (2\pi h)^{d-1} \sum_{e^{i\theta_j} \in \mathsf{spec} C(h^{-1})} \delta(\theta - \theta_j), \tag{1.24}$$

where we include each eigenvalue according to its multiplicity as usual. Then Theorem 1.2 can be expressed in the following way: the measures $\mu(h)$ converge in the weak-$*$ topology as $h \to 0$ to the measure

$$\omega_{d-1}\,\mathsf{vol}'(\partial M)\frac{d}{d\theta}\tilde{\kappa}(\theta)d\theta \quad \text{on } (0, 2\pi), \text{ that is on } S^1 \setminus \{1\}. \tag{1.25}$$

[1)] The fact that Frank and Geisinger obtain a second term regardless of dynamical assumptions is simply due to the fact that they study $\mathsf{Tr}\, f(P_{a,h})$ with $f(\lambda) = -\lambda H(-\lambda)$ (H is the Heaviside function), which is one order smoother than $f(\lambda) = H(-\lambda)$.

In particular, this measure is absolutely continuous, and finite away from $e^{i\theta} = 1$ with an infinite accumulation of mass as $\theta \uparrow 2\pi$. In this form, we can compare our result with results on the semiclassical spectral asymptotics of scattering matrices. In [DGHH] and [GHZ], the scattering matrix $S_h(E)$ at energy E for the Schrödinger operator $h^2\Delta + V(x)$ on $\mathbb{R}^d$ was studied in the semiclassical limit $h \to 0$. Assuming that V is smooth and compactly supported, that E is a nontrapping energy level, and that the set of periodic trajectories of the classical scattering transformation on T^*S^{d-1} has measure zero, it was shown that the measure $\mu(h)$ defined by (1.24) converged weak-$*$ to a uniform measure on $S^1 \setminus \{1\}$, with an atom of infinite mass at the point 1. On the other hand, for polynomially decaying potentials, it was shown by Sobolev and Yafaev [H-SoYa] in the case of central potentials and by Gell-Redman and the first author more generally [GRH] that there is a limiting measure which is nonuniform, and is qualitatively similar to the measure for $C(h^{-1})$ above in that it is finite away from 1, with an infinite accumulation of mass at 1 from one side.

2 Reduction to Semiclassical Spectral Asymptotics

In this section we prove Proposition 1.1. This result actually follows directly from the Birman-Schwinger principle. As some readers may not be familiar with this, we give the details.

Proof of Proposition 1.1. We begin by recalling that the operator $P_{a,h}$ is the self-adjoint operator associated to the quadratic form (1.15), that is,

$$Q_{a,h}(u) := h^2\|\nabla u\|_M^2 - \|u\|_M^2 - ha\|u\|_{\partial M}^2.$$

We recall the min-max characterization of eigenvalues: the nth eigenvalue $\mu_n(a,h)$ of $P_{a,h}$ is equal to the infimum of

$$\sup_{v\in V, \|v\|=1} Q_{a,h}(v)$$

over all subspaces $V \in H^1(M)$ of dimension n. The monotonicity of $Q_{a,h}$ in a, for fixed h, shows that the eigenvalues are monotone nonincreasing with a. In fact, they are strictly decreasing, which follows from the fact

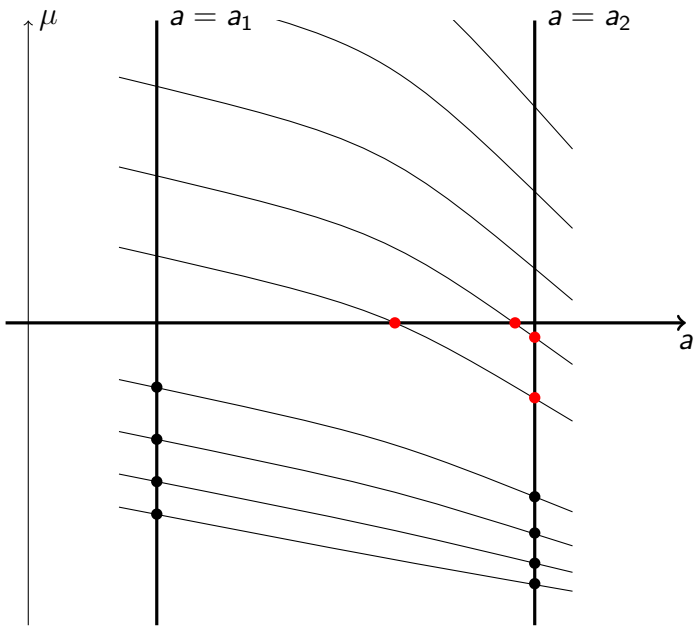

Figure 28.1: Diagram showing the variation of eigenvalues $\mu(a,h)$ of $P_{a,h}$ as a function of a for fixed h. The eigenvalues are strictly decreasing in a. Consequently, the number of negative eigenvalues of $P_{a_2,h}$ is equal to the number of negative eigenvalues of $P_{a_1,h}$ together with the number that cross the a-axis between $a = a_1$ and $a = a_2$.

that eigenfunctions of $P_{a,h}$ cannot vanish at the boundary. Indeed, the eigenfunctions satisfy the boundary condition $h\partial_\nu u = -au$, which shows that if u vanishes at the boundary, so does $\partial_\nu u$, which is impossible.

The eigenvalues $\mu_n(a,h)$ are thus continuous, strictly decreasing functions of a. Let $a_1 < a_2$ be real numbers. The Birman-Schwinger principle [Ivr3, Prop. 9.2.7] says that the number of negative eigenvalues of $P_{a_2,h}$ is equal to the number of negative eigenvalues of $P_{a_1,h}$, plus the number of eigenvalues $\mu_n(a,h)$ of $P_{a,h}$ that change from nonnegative to negative as a varies from a_1 to a_2. A diagram makes this clear: see Figure 28.1.

The strict monotonicity of $\mu(a,h)$ in a shows that the number of eigenvalues $\mu_n(a,h)$ of $P_{a,h}$ that change from nonnegative to negative as a varies from a_1 to a_2 is the same as the number of $\mu(a,h)$ (counted with multiplicity) equal to zero, for $a \in [a_1, a_2)$. Next, we observe that the space of eigenfunctions $u_n(a,h)$ of $P_{a,h}$ with zero eigenvalue, i.e. $\mu_n(a,h) = 0$ is in one-to-one

correspondence with the space of eigenfunctions of $C(\lambda)$, $\lambda = h^{-1}$, with eigenvalue $(a-i)(a+i)^{-1}$, or equivalently $e^{i\theta}$ where $a = -\cot(\theta/2)$. Indeed, whenever u_n is such an eigenfunction of $P_{a,h}$, then

$$f := \frac{1}{2}(h\partial_\nu u - iu)\big|_{\partial M} \tag{2.1}$$

is an eigenfunction of $C(\lambda)$, with eigenvalue $(a-i)(a+i)^{-1}$. Conversely, if f is an eigenfunction of $C(\lambda)$ with eigenvalue $(a-i)(a+i)^{-1}$, then by definition there exists a Helmholtz function u such that u is related to f according to (2.1), and we have $(h\partial_\nu + a)u = 0$ at ∂M. This completes the proof. □

Remark 2.1. We can apply similar arguments for $\lambda^{-\delta}\rho^{-1}R(\lambda)$ as $\rho > 0$ is a smooth function on ∂M and then plug corresponding parameters in the boundary conditions coming again to equality (1.16).

We next digress to prove that the eigenvalues of $C(\lambda)$ are monotonic (that is, they move monotonically around the unit circle) in λ. This plays no role in the remainder of our proof, but is (in the authors' opinion) of independent interest.

Proposition 2.2. *The eigenvalues of $C(\lambda)$ rotate clockwise around the unit circle as λ increases.*

Remark 2.3. This implies that the eigenvalues of $R_{\mathrm{scl}}(\lambda)$ are monotone decreasing in λ.

Proof. As discussed in the previous proof, $C(\lambda)$ has eigenvalue $e^{i\theta}$ if and only if $P_{a,h}$ has a zero eigenvalue, where $a = a(\theta) = -\cot(\theta/2)$. Thus, as a function of $h = \lambda^{-1}$, $\theta(h)$ is defined implicitly by the condition

$$\mu(a(\theta), h) = 0.$$

Since a is a strictly increasing function of θ, and we have just seen that μ is a strictly decreasing function of a, it suffices to show that when $\mu = 0$, μ is a strictly increasing function of h, hence a strictly decreasing function of λ.

We now compute the derivative of μ with respect to h, at a value of a and h where $\mu(a,h)=0$. We have

$$\begin{aligned}\frac{d}{dh}\mu(a,h) &= \frac{d}{dh}((h^2\Delta-1)u(h),u(h))_M \\ &= 2h(\Delta u,u)_M + ((h^2\Delta-1)u'(h),u(h))_M + ((h^2\Delta-1)u(h),u'(h))_M \\ &= 2h(\Delta u,u)_M + ((h^2\Delta-1)u',u)_M - (u',(h^2\Delta-1)u)_M.\end{aligned}$$

In the third line, we used the fact that $(h^2\Delta-1)u=0$ when $\mu(h)=0$. Note the second term is not zero, as u' is not in the domain of the operator due to the changing boundary condition, so we cannot move the operator to the right hand side of the inner product without incurring boundary terms. We use the Gauss-Green formula to express the last two terms as a boundary integral:

$$\begin{aligned}\mu'(h) &= 2h(\Delta u,u)_M + h(h\partial_\nu u',u)_{\partial M} - h(u',h\partial_\nu u)_{\partial M} \\ &= 2h(\Delta u,u)_M + h(h\partial_\nu u',u)_{\partial M} + ha(u',u)_{\partial M}.\end{aligned}$$

Differentiating the boundary condition we find that

$$(h\partial_\nu + a)u' = -h\partial_\nu u \text{ at } \partial M.$$

Substituting that in we get

$$\mu'(h) = 2h(\Delta u,u)_M - h(u,\partial_\nu u)_{\partial M}.$$

Applying Gauss-Green again, we get

$$\begin{aligned}\mu'(h) &= h(\Delta u,u)_M + h\|\nabla u\|^2_M \\ &= h^{-1}\Big(\|u\|^2_{L^2(M)} + \|h\nabla u\|^2_{L^2(M)}\Big) > 0.\end{aligned}$$

3 Semiclassical Spectral Asymptotics

In this section, we prove Theorem 1.2. Essentially, we have arrived at a rather standard semiclassical spectral asymptotics problem and results are due to [Ivr3], Chapter 5 or [Ivr4], Chapter 7. See the appendix to this paper for further discussion.

Proposition 3.1. *(i) Let* $\mathsf{N}_h^-(a)$ *be as in (1.17). The following asymptotic holds as* $h \to +0$*:*

$$\mathsf{N}_h^-(a) = (2\pi h)^{-d}\omega_d \operatorname{vol}(M) + O(h^{1-d}) \tag{3.1}$$

(ii) Further, if the set of periodic billiards on M *has measure* 0 *then as* $h \to +0$*:*

$$\mathsf{N}_h^-(a) = (2\pi h)^{-d}\omega_d \operatorname{vol}(M) + h^{1-d}\kappa(a)\operatorname{vol}'(\partial M) + o(h^{1-d}) \tag{3.2}$$

with $\kappa(a)$ *given by (1.20).*

Proof. One can check easily that the operator $P_{a,h}$ is microhyperbolic at energy level 0 at each point $(x,\xi) \in T^*M$ in the direction ξ; further, the boundary value problem is microhyperbolic at each point $(x';\xi') \in T^*\partial M$ at energy level 0 in the multidirection (ξ', ξ_1^-, ξ_1^+) with $\xi_1 = \xi_1^{\pm}$ roots of $\sum g^{jk}\xi_j\xi_k = 0$; finally, the boundary value problem is elliptic at each point of the elliptic zone ($\subset T^*\partial M$) if $a \le 0$, and either elliptic or microhyperbolic in the direction ξ' at each point of the elliptic zone ($\subset T^*\partial M$) if $a > 0$ — see definitions in Chapters 2, 3 of [Ivr4]. Then statements (1.18), (1.19) follow from Theorems 7.3.11 and 7.4.1 of [Ivr4].

We now assume that the set of periodic billiards on M has measure zero, and compute the second term in the spectral asymptotic explicitly. Similar calculations appear in [FG].

To do this, one can use method of freezing coefficients (see the appendix, or [Ivr4], 7.2) which results in

$$h^{1-d}\kappa(a) = \int_{\mathbb{R}^+} \bigl(e(0,x_1;0,x_1;1) - (2\pi h)^{-d}\omega_d\bigr)\,dx_1 \tag{3.3}$$

where $e(x',x_1;y',y_1;\tau)$ is the Schwartz kernel of the spectral projector $E(\tau)$ of the operator $H_a = h^2\Delta$ in half-space $\mathbb{R}^{d-1}\times\mathbb{R}^+ \ni (x',x_1)$ with domain $\mathfrak{D}(H_a) = \{u \in H^2 :\ (h\partial_{x_1} + a)u|_{x_1=0} = 0\}$.

We obtain this spectral projector by integrating the spectral measure. This in turn is obtained via Stone's formula

$$dE_L(\sigma) = \frac{1}{2\pi i}\Bigl(\bigl(L - (\sigma + i0)\bigr)^{-1} - \bigl(L - (\sigma - i0)^{-1}\Bigr)\,d\sigma. \tag{3.4}$$

Consider the resolvent for H_a, $(H_a - \sigma)^{-1}$, for $\sigma \in \mathbb{C} \setminus \mathbb{R}$. Using the Fourier transform in the x' variables, we can write the Schwartz kernel of this resolvent in the form

$$(2\pi h)^{1-d} \int e^{i(x'-y')\cdot\xi'} (T_a + |\xi'|^2 - \sigma)^{-1}(x_1, y_1)\, d\xi'. \tag{3.5}$$

Here T_a is the one-dimensional operator $T_a = -h^2\partial^2 + |\xi'|^2$ on $L^2(\mathbb{R}_+)$ under the boundary condition $(h\partial + a)u|_{x_1=0} = 0$. The spectral projector $E_{H_a}(1)$ is therefore given by

$$(2\pi h)^{1-d} \int_{-\infty}^{1} \int e^{i(x'-y')\cdot\xi'}\, dE_{T_a}(\sigma - |\xi'|^2)(x_1, y_1)\, d\xi'\, d\sigma. \tag{3.6}$$

Thus, we need to find the spectral measure for T_a. Write $\sigma - |\xi'|^2 = \eta^2$, where we take η to be in the first quadrant of $\mathbb{C}$ for $\operatorname{Im}\sigma > 0$, and in the fourth quadrant for $\operatorname{Im}\sigma < 0$.

Lemma 3.2. *Suppose that* $\operatorname{Im}\eta > 0$ *and* $\operatorname{Re}\eta \geq 0$. *Then the resolvent kernel* $(T_a - \eta^2)^{-1}$ *takes the form*

$$(T_a - \eta^2)(x,y) = \begin{cases} \frac{i}{2h\eta}\left(e^{i\eta(x-y)/h} + \frac{i\eta - a}{i\eta + a} e^{i\eta(x+y)/h}\right), & x > y \\ \frac{i}{2h\eta}\left(e^{i\eta(y-x)/h} + \frac{i\eta - a}{i\eta + a} e^{i\eta(x+y)/h}\right), & x < y. \end{cases} \tag{3.7}$$

If $\operatorname{Im}\eta < 0$ *and* $\operatorname{Re}\eta \geq 0$, *then the resolvent kernel* $(T_a - \eta^2)^{-1}$ *takes the form*

$$(T_a - \eta^2)(x,y) = \begin{cases} -\frac{i}{2h\eta}\left(e^{i\eta(y-x)/h} + \frac{i\eta + a}{i\eta - a} e^{-i\eta(x+y)/h}\right), & x > y \\ -\frac{i}{2h\eta}\left(e^{i\eta(x-y)/h} + \frac{i\eta + a}{i\eta - a} e^{-i\eta(x+y)/h}\right), & x < y. \end{cases} \tag{3.8}$$

Proof. In the regions $x < y$ and $x > y$, the resolvent kernel must be a linear combination of $e^{i\eta x/h}$ and $e^{-i\eta x/h}$. Moreover, for $\operatorname{Im}\eta > 0$, we can only have the $e^{+i\eta x/h}$ term, as $x \to \infty$, as the other would be exponentially increasing. So we can write the kernel in the form

$$\begin{cases} c_1 e^{+i\eta x/h}, & x > y \\ c_2 e^{+i\eta x/h} + c_3 e^{-i\eta x/h}, & x < y. \end{cases} \tag{3.9}$$

We apply the boundary condition, and the two connection conditions at $x = y$, namely continuity, and a jump in the derivative of $-1/h$, in order to

obtain the delta function $\delta(x-y)$ after applying T_a. These three conditions determine the c_i uniquely, and we find that, in the case $\operatorname{Im}\eta > 0$,

$$(3.10)_1 \qquad c_1 = \frac{i}{2h\eta}\big(e^{-i\eta y/h} + \frac{i\eta - a}{i\eta + a}e^{+i\eta y/h}\big),$$

$$(3.10)_{2,3} \qquad c_2 = \frac{i}{2h\eta}\frac{i\eta - a}{i\eta + a}e^{+i\eta y/h}, \qquad c_3 = \frac{i}{2h\eta}e^{+i\eta y/h},$$

yielding (3.7). A similar calculation yields (3.8). □

We now apply (3.4) to find the Schwartz kernel of the spectral measure for T_a.

Lemma 3.3. *The spectral measure $dE_{T_a}(\tau)$ is given by the following.*

(i) For $\tau \geq 0$, $\tau = \eta^2$

$$(3.11) \quad dE_{T_a}(\tau)$$
$$= \frac{1}{4\pi h\eta}\Big(e^{i\eta(x-y)/h} + e^{i\eta(y-x)/h} + \frac{i\eta - a}{i\eta + a}e^{i\eta(x+y)/h} + \frac{i\eta + a}{i\eta - a}e^{-i\eta(x+y)/h}\Big)\, 2\eta d\eta.$$

(ii) For $\tau < 0$, the spectral measure $dE(\tau)$ vanishes for $a \leq 0$, while for $a > 0$

$$(3.12) \qquad dE_{T_a}(\tau) = \frac{2a}{h}e^{-ax/h}e^{-ay/h}\delta(\tau + a^2)d\tau.$$

Proof. This follows directly from Lemma 3.2 and Stone's formula, (3.4). The extra term for $a > 0$ arises from the pole in the denominator, $i\eta + a$ for $\operatorname{Im}\eta > 0$ and $i\eta - a$ for $\operatorname{Im}\eta < 0$ in the expressions (3.7), (3.8), which only occurs for $a > 0$. For τ negative, we need to set $\eta = i\sqrt{-\tau} + 0$ in (3.7) and $\eta = -i\sqrt{-\tau} + 0$ in (3.8), and subtract. Then everything cancels except at the pole, where we obtain a delta function $-2\pi i\delta(\sqrt{-\tau} - a)$, which arises from $(\sqrt{-\tau} + i0 + a)^{-1} - (\sqrt{-\tau} - i0 + a)^{-1}$. This term arises from the negative eigenvalue $-a^2$ which occurs for $a > 0$, corresponding to the eigenfunction $\sqrt{2a/h}\, e^{-ax/h}$. □

Plugging this into (3.6), and making use of the fact that $d\sigma d\xi' = 2\eta d\eta d\xi'$, we find that the Schwartz kernel of $E_{H_a}(1)$ is given by

$$(3.13)\quad (2\pi h)^{-d}\int_0^1\int H(1-|\xi'|^2-\eta^2)e^{i(x'-y')\cdot\xi'} \times\left(e^{i\eta(x_1-y_1)/h}+e^{i\eta(y_1-x_1)/h}+\frac{i\eta-a}{i\eta+a}e^{i\eta(x_1+y_1)/h}+\frac{i\eta+a}{i\eta-a}e^{-i\eta(x_1+y_1)/h}\right)d\xi'\,d\eta$$

for $a\leq 0$ while for $a>0$, it is given by the sum of (3.13) and

$$(3.14)\quad (2\pi h)^{1-d}\frac{2a}{h}e^{-ax_1/h}e^{-ay_1/h}\int_{-\infty}^1\int e^{i(x'-y')\cdot\xi'}\delta(\sigma-|\xi'|^2+a^2)\,d\xi'\,d\sigma.$$

We are actually interested in the value on the diagonal. Setting $x=y$, and performing the trivial ξ' integral, we find that the Schwartz kernel of the spectral projector $E_{H_a}(1)(x,x)$ on the diagonal is given by

$$(3.15)\quad \frac{\omega_{d-1}}{(2\pi h)^d}\int_0^1(1-\eta^2)^{(d-1)/2}\left(2+\frac{i\eta-a}{i\eta+a}e^{2i\eta x_1/h}+\frac{i\eta+a}{i\eta-a}e^{-2i\eta x_1/h}\right)d\eta + H(a)\frac{(d-1)\omega_{d-1}}{(2\pi h)^{d-1}}\frac{a}{h}e^{-2ax_1/h}\int_{-a^2}^1(\sigma+a^2)^{(d-3)/2}\,d\sigma.$$

Since

$$\omega_{d-1}\int_0^1 2(1-\eta^2)^{(d-1)/2}\,d\eta=\omega_d,$$

we see by comparing with (3.3) that this term disappears in the expression for $\kappa(a)$ and we have, after performing the x_1 integral as in (3.3)

$$(3.16)\quad h^{1-d}\kappa(a)=\frac{\omega_{d-1}}{(2\pi h)^d}\int_0^1(1-\eta^2)^{(d-1)/2} \times\left(\frac{i\eta-a}{i\eta+a}\Big(\frac{ih}{2}(\eta+i0)^{-1}\Big)\;-\frac{i\eta+a}{i\eta-a}\Big(\frac{ih}{2}(\eta-i0)^{-1}\Big)\right)d\eta + H(a)\frac{(d-1)\omega_{d-1}}{2(2\pi h)^{d-1}}\int_{-a^2}^1(\sigma+a^2)^{(d-3)/2}\,d\sigma.$$

Simplifying a bit, and performing the σ integral, we have

$$\kappa(a) = -\frac{i\omega_{d-1}}{2(2\pi)^d}\int_{-1}^{1}(1-\eta^2)^{(d-1)/2}\frac{(i\eta-a)^2}{a^2+\eta^2}(\eta+i0)^{-1}\,d\eta + H(a)\frac{\omega_{d-1}}{(2\pi)^{d-1}}(1+a^2)^{(d-1)/2}. \tag{3.17}$$

We further simplify this expression by expanding $(i\eta-a)^2 = a^2 - 2ia\eta - \eta^2$, and noting that the contribution of the $-\eta^2$ term is zero, as this gives an odd integrand in the η integral. A similar statement can be made for the a^2 term, except that there is a contribution from the pole in this case. This leads to the expression

$$\kappa(a) = \frac{\omega_{d-1}}{(2\pi)^{d-1}}\Big(-\frac{1}{2\pi}\int_{-1}^{1}(1-\eta^2)^{(d-1)/2}\frac{a}{a^2+\eta^2}\,d\eta - \frac{1}{4} + H(a)(1+a^2)^{(d-1)/2}\Big). \tag{3.18}$$

Although not immediately apparent, this formula is continuous at $a=0$. In fact, the function $a(a^2+\eta^2)^{-1}$ has a distributional limit $(\operatorname{sgn} a)\pi\delta(\eta)$ as a tends to zero from above or below. The change of sign as a crosses 0 means that the integral in (3.18) has a jump of -1 as a crosses zero from negative to positive. That exactly compensates the jump in the final term.

In odd dimensions, we can compute this integral exactly. In particular, in dimension $d=3$, we find that

$$\kappa(a) = \frac{\omega_2}{(2\pi)^2}\Big(-\frac{1}{4}+\frac{a}{\pi}+(1+a^2)\big(1-\frac{\operatorname{arccot} a}{\pi}\big)\Big). \tag{3.19}$$

Proof of Theorem 1.2. This follows immediately from Proposition 3.1 and Proposition 1.1. □

Remark 3.4. The second term of the expansion in [FG, Theorem 1.1] is obtained by computing

$$(2\pi h)^{1-d}\int_{-\infty}^{1}(1-\sigma)\int e^{i(x'-y')\cdot\xi'}\,dE_{T_a}(\sigma-|\xi'|^2)(x_1,y_1)\,d\xi'\,d\sigma \tag{3.20}$$

instead of (3.6).

4 Relation to Dirichlet Boundary Condition

In this section we observe that the limit $a \to -\infty$ corresponds to the Dirichlet boundary condition. More precisely, we have

Proposition 4.1. *Let* $\mathsf{N}_h^-(-\infty)$ *denote the limit*

$$\mathsf{N}_h^-(-\infty) \coloneqq \lim_{a\to-\infty} \mathsf{N}_h^-(a),$$

where $\mathsf{N}_h^-(a)$ *is given by (1.17). Then we have*

$$\mathsf{N}_h^-(-\infty) = \#\{\lambda_j \le h^{-1} \mid \lambda_j^2 \text{ is a Dirichlet eigenvalue of } \Delta\}. \tag{4.1}$$

Remark 4.2. Because the quadratic form (1.15) is monotone in a, the counting function $\mathsf{N}_h^-(a)$ is monotone in a. Hence the limit above exists.

Proof. We use the min-max characterisation of eigenvalues. Let $\tilde{\mathsf{N}}_D(\lambda)$ denote the number of Dirichlet eigenvalues (counted with multiplicity) less than or equal to $\lambda = h^{-1}$. This is equal to the maximal dimension of a subspace of $H_0^1(M)$ on which the quadratic form Q_D, given by

$$Q_D(u,u) = h^2\|\nabla u\|_2^2 - \|u\|_2^2 \tag{4.2}$$

is negative semidefinite. On the other hand, $\tilde{\mathsf{N}}_h^-(a)$ is equal to the maximal dimension of a subspace of $H^1(M)$ on which the quadratic form Q_a given by (1.15) is (strictly) negative definite.

We first show that $\tilde{\mathsf{N}}_D(h^{-1}) \le \tilde{\mathsf{N}}_h^-(-\infty)$. Let V be the vector space spanned by Dirichlet eigenfunctions with eigenvalue $\le \lambda^2$. Clearly, the quadratic form Q_a is negative *semi*definite on V, and if λ^2 is not a Dirichlet eigenvalue, then it is negative definite, proving the assertion. In the case that λ^2 is a Dirichlet eigenvalue, we perturb V to V_ϵ, a vector space of $H^1(M)$ of the same dimension as V, so that, for for ϵ sufficiently small depending on a, Q_a is negative definite on V_ϵ. For simplicity we only do this in the case that the λ^2-eigenspace is one dimensional, leaving the general case to the reader. To do this, we choose an orthonormal basis of V (with respect to the L^2 inner product) of Dirichlet eigenfunctions $v_1, \dots, v_k$ with eigenvalues $\lambda_1^2 \dots \lambda_k^2$, where $\lambda_k = \lambda$. Then we perturb only v_k, leaving the others fixed. We choose $s \in H_0^1(M)^\perp$, the orthogonal complement of $H_0^1(M)$ in $H^1(M)$ (with respect to the inner product in $H^1(M)$), so that

$$Q_a(v_i, s) = 0, \quad i < k \text{ and } Q_a(v_k, s) > 0. \tag{4.3}$$

We check that this is possible. Notice that $s \in H_0^1(M)^\perp$ implies that $(\Delta + 1)s = 0$ in M. Then as v_i has zero boundary data, we have

$$(\lambda_i^2 + 1)(v_i, s)_M = (\Delta v_i, s)_M - (v_i, \Delta s)_M = \langle \partial_\nu v_i, s\rangle_{\partial M}. \tag{4.4}$$

We choose s so that $\langle \partial_\nu v_i, s\rangle_{\partial M}$ vanishes for $i < k$ and is positive for $i = k$. This is possible: in fact, due to the unique solvability of the boundary value problem

$$(\Delta + 1)s = 0, \quad s|_{\partial M} = f \in H^{1/2}(M), \tag{4.5}$$

for $s \in H^1(M)$, we see that s can have any boundary value in $H^{1/2}(\partial M)$ which is dense in $L^2(\partial M)$. Then using (4.4) we see that $\langle \partial_\nu v_k, s\rangle_{\partial M} > 0$ implies that $(v_k, s)_M > 0$.

We now define V_ϵ to be the span of $v_1, \dots, v_{k-1}$ and $v_k + \epsilon s$. Then we have

$$Q_a(v_i, v_k + \epsilon s) = 0, \ \ i < k \tag{4.6}$$

and

$$\begin{aligned} Q_a(v_k + \epsilon s, v_k + \epsilon s) &= Q_a(v_k, v_k) + 2\epsilon Q_a(v_k, s) + \epsilon^2 Q_a(s, s) \\ &= 2\epsilon Q_a(v_i, s_i) + \epsilon^2 Q_a(s_i, s_i) \\ &= -2\epsilon(h^2 + 1)(v_i, s_i)_M + O(\epsilon^2 a^2) \end{aligned} \tag{4.7}$$

which is strictly negative for ϵa^2 small enough. It follows that Q_a is negative definite on V_k when ϵa^2 is small enough. A similar construction can be made when λ^2 has multiplicity greater than 1.

We next show that $\tilde{\mathsf{N}}_D(h^{-1}) \geq \mathsf{N}_h^-(-\infty)$. We argue by contradiction: if not, then for any a, there is a vector space W of dimension $\geq k + 1$ on which Q_a is negative definite. Then there is a nonzero vector $w \in W$ orthogonal (in the $H^1(M)$ inner product) to V. We can write $w = w' + s$ where $w' \in H_0^1(M)$ and $s \in H_0^1(M)^\perp$. Then w' is a linear combination of Dirichlet eigenfunctions with eigenvalue $\geq \lambda' > \lambda$, where λ' is the smallest eigenvalue larger than λ. We then have

$$\begin{aligned} 0 > Q_a(w' + s, w' + s) &= Q_a(w', w') + 2Q_a(w', s) + Q_a(s, s) \\ &\geq (\lambda' - \lambda)\|w'\|_2^2 - 2(h^2 + 1)\|w'\|_2\|s\|_{L^2(M)} - ha\|s\|_{L^2(\partial M)}^2. \end{aligned} \tag{4.8}$$

However, some standard potential theory shows that $\|s\|_{L^2(M)}$ is bounded by a constant times $\|s\|_{L^2(\partial M)}$. To see this, extend M to a larger manifold $\tilde{M}$ of the same dimension, and let $G(x,y)$ be the Schwartz kernel of the inverse of $(\Delta_{\tilde{M}}+1)^{-1}$ on $L^2(\tilde{M})$, with Dirichlet boundary conditions at $\partial\tilde{M}$. We can write s as $\int_{\partial M} d_{\nu_y} G(x,y)h(y)\,dy$ where $(1/2+D)h = s|_{\partial M}$ and D is the double layer operator on ∂M determined by G. Standard arguments show that $(1/2+D)$ has a bounded inverse on $L^2(\partial M)$ and $d_{\nu_y}G(x,y)$ is a bounded integral operator from $L^2(\partial M)$ to $L^2(M)$. So we can write, for $a<0$,

$$\begin{aligned} 0 > Q_a(w'+s, w'+s) \\ \geq (\lambda'-\lambda)\|w'\|_2^2 - 2C(h^2+1)\|w'\|_2\|s\|_{L^2(\partial M)} + h|a|\|s\|^2_{L^2(\partial M)^2} \end{aligned}$$

and the RHS is clearly positive for $|a|$ large enough, giving us the desired contradiction. □

A Appendix

A.1 Standard Semiclassical Asymptotics

The proof of the standard semiclassical asymptotics (i.e. asymptotics of the number of negative eigenvalues of $H_a := h^2\Delta - 1$ with the boundary condition $(h\partial_{x_1}+a)u|_{\partial M}=0$) is in [Ivr4], Section 8.3 and also in [Ivr3], Section 5.3, but we describe a simplified albeit less general proof. Basically it is a simplified proof of [Ivr1], used also in [LH], Section 29.3.

A.2 Tauberian Theorem

We use the following 'semiclassical' version of the Tauberian theorem in [Ivr1].

Proposition A.1. *Let $e_h(\lambda)$ be an nondecreasing function of λ, depending on the parameter $h>0$, equal to zero for $\lambda \leq \Lambda_0$. Let $\beta \in C_c^\infty(\mathbb{R})$ be a cutoff function with $\beta(t)=1$, $|t|\leq 1/2$, $\beta(t)=0$, $|t|\geq 1$, and $\hat{\beta}(\lambda)>0$. Let $\beta_T(t)=\beta(t/T)$. Assume that for all λ, we have*

$$|e_h(\lambda)| \leq C'(1+|\lambda|)^M h^{-d} \tag{A.1}$$

and, for all $\lambda \in [\Lambda_0, \Lambda_1]$ *we have*

$$\frac{1}{h}\int_{-\infty}^{\infty} \hat{\beta}_T\big(\frac{\lambda-\mu}{h}\big)\, de_h(\mu) = A_0(\lambda)h^{-d} + A_1(\lambda)h^{1-d} + o(h^{1-d}), \quad h \to 0. \tag{A.2}$$

Then for all $\lambda \in [\Lambda_0, \Lambda_1 - \epsilon]$ *we have*

$$\Big| e_h(\lambda) - B_0(\lambda)h^{-d} - B_1(\lambda)h^{1-d} \Big| \leq \frac{C\|A_0\|_{L^\infty([\Lambda_0,\Lambda_1])}}{T} h^{1-d} + o(h^{1-d}), \tag{A.3}$$

where

$$B_i(\lambda) = \int_{-\Lambda_0}^{\lambda} A_i(\mu)\, d\mu$$

and C *depends only on* ϵ, Λ_1, C' *and* β.

This is proved by modifying the proof of the corresponding proposition in [Shub], pp 152-153.

A.3 Propagator

We now fix $a \in \mathbb{R}$ and let $e_h(\lambda)$ be the number of eigenvalues, counting multiplicity, of the operator $P_{a,h}$ that are less than λ, or equivalently, the trace of the spectral projection $E_{a,h}(\lambda)$ for $P_{a,h}$. According to Proposition A.1, it suffices to consider the smoothed spectral projector,

$$\mathrm{Tr}\Big(\frac{1}{h}\int_{-\infty}^{\infty} \hat{\beta}\big(\frac{\lambda-\mu}{h}\big)\, dE_{a,h}(\mu)\Big), \tag{A.4}$$

since it is straightforward to show that the estimate (A.1) holds with $M = d$. By the spectral theorem, this is precisely the trace of the operator

$$\mathrm{Tr}\Big(\hat{\beta}\big(\frac{\lambda - P_{a,h}}{h}\big)\Big). \tag{A.5}$$

If we are only interested in this for λ in some interval $[\Lambda, \Lambda_1]$, then, up to $O(h^\infty)$ errors we can compose with a smooth function $\phi(P_{a,h})$ where $\phi \in C_c^\infty(\mathbb{R})$, and $\phi = 1$ on $[\Lambda - \epsilon, \Lambda_1 + \epsilon]$. Then, using the Fourier transform we can express this operator in terms of the propagator $e^{itP_{a,h}/h}$:

$$\int_{-\infty}^{\infty} e^{-it\lambda/h}\beta(t)\phi(P_{a,h})e^{itP_{a,h}/h}\, dt. \tag{A.6}$$

Since $(hD_t - P_{a,h})e^{itP_{a,h}/h} = 0$, this is the same as

$$\int_{-\infty}^{\infty} e^{-it\lambda/h}\beta(t)\phi(hD_t)e^{itP_{a,h}/h}\,dt. \tag{A.7}$$

The advantage of the spectral cutoff $\phi(hD_t)$ is that the operator $\phi(hD_t)e^{itP_{a,h}/h}$ has finite speed of propagation.

A.4 Propagation of Singularities

Let $u_h(x, y, t)$ be the Schwartz kernel of $e^{ih^{-1}tP_{a,h}}$; then $u_h(x, y, t) = u_h^0(x, y, t)+ u_h^1(x, y, t)$ where $u_h^0(x, y, t)$ is a free space solution and $u_h^1(x, y, t)$ satisfies

$$(hD_t - P_{a,h})u^1 = 0, \quad u^1|_{t=0} = 0, \quad (hD_{x_1} + a)(u^0 + u^1)|_{\partial M} = 0. \tag{A.8}$$

We define

$$\sigma_h^i(t) \coloneqq \int_M \phi(hD_t)u^i(x, x, t)\,dx, \quad i = 0, 1. \tag{A.9}$$

We claim that, for suitable ϕ, $\sigma_h^i(t)$ has an isolated singularity (in the semiclassical sense of nontrivial behaviour as $h \to 0$) at $t = 0$. More precisely, we claim that if $\phi \in C^\infty$ and is supported in $(-\epsilon_0, \epsilon_0)$, then

$$\sigma_h^i(t) = O(h^\infty) \text{ for } \epsilon \le |t| \le \epsilon_0, \tag{A.10}$$

where ϵ_0 is a fixed, sufficiently small constant, and $0 < \epsilon < \epsilon_0$ is arbitrary.

This follows from propagation of singularities arguments. The bicharacteristic flow for $P_{a,h}$ is given by

$$\begin{cases} \dot t = 1, \\ \dot\tau = 0, \\ \dot x^i = 2g^{ij}(x)\xi_j, \\ \dot\xi_j = -2\dfrac{\partial g^{kl}}{\partial x^j}\xi_k\xi_l. \end{cases} \tag{A.11}$$

That is, with respect to the parameter t, (x, ξ) moves along a geodesic at speed $2|\xi|_g$, and τ is fixed. By standard propagation of singularities arguments, the (semiclassical) wavefront set of u^0 is contained in the conormal bundle of $\{t = 0, s = y\}$ together with the forward bicharacteristic

flow from this conormal bundle intersected with the characteristic variety of $hD_t - P_{a,h}$, namely $\{\tau = |\xi|_g^2 - 1\}$. Composing with $\phi(hD_t)$ restricts this wavefront set to be contained in $\{\tau \in (-\epsilon_0, \epsilon_0)\}$. That implies that $\big||\xi|_g - 1\big| \leq 2\epsilon_0$. So for small time, the wavefront set of u_0 is restricted to the set where $|d(x,y) - 2t| \leq 2\epsilon_0 t$. In particular, points with $x = y$ are not in the wavefront set for t in a deleted neighbourhood of 0. This proves (A.10) for $i = 0$.

A similar argument for u^1 shows that $\phi(hD_t)u^1(t,x,y)$ is $O(h^\infty)$ for $|t| \leq \epsilon_0$ unless we have

$$\mathsf{dist}(x, \partial M) + \mathsf{dist}(y, \partial M) \leq 2\epsilon_0(1 + 2\epsilon_0).$$

Thus, we can work in a collar neighbourhood of ∂M. We can choose coordinates $x = (x_1, x')$ so that the boundary is given by $x_1 = 0$, $x_1 \geq 0$ on M, and the metric takes the form $dx_1^2 + g'_{ij}(x_1, x')x'^i x'^j$, that is, Fermi coordinates near the boundary. Now we split the analysis of u^1 into two cases. We write the identity operator in the x' coordinates in the form $\mathsf{Id} = Q_{\mathsf{norm}} + Q_{\mathsf{tan}}$, where Q_* are pseudodifferential operators in the x' variables such that the symbol q_{norm} of Q_{norm} is supported in $\{|\xi'| \leq 2\epsilon_1\}$ and q_{tan} is supported in $\{|\xi'| \geq \epsilon_1\}$.

Correspondingly, write

$$\phi(hD_t)u^1 = \phi(hD_t)Q_{\mathsf{tan}}(x', hD_{x'})u^1 + \phi(hD_t)Q_{\mathsf{norm}}(x', hD_{x'})u^1.$$

For the first term, a standard positive commutator argument in the x' variables only shows that this term is $O(h^\infty)$ unless $\mathsf{dist}(x', y') \geq 2\epsilon_1 t - \epsilon'$ for arbitrary $\epsilon' > 0$; in particular, if $x' = y'$ then this is $O(h^\infty)$ for $t \geq \epsilon$, $\epsilon > 0$ arbitrary. On the other hand, for the Q_{norm} term, then we have $\xi_1^2 = 1 + O(\epsilon_0 + \epsilon_1)$. In particular, this means that $\dot{x}_1 = 1 + O(\epsilon_0 + \epsilon_1)$, so the propagation is transverse (in fact, nearly normal) to ∂M. A standard propagation of singularities argument shows that the wavefront set of $\phi(hD_t)Q_{\mathsf{norm}}u^1$ for $\epsilon \leq |t| \leq \epsilon_0$ is contained in

$$\begin{aligned}\mathsf{WF}_h(\phi(hD_t)Q_{\mathsf{norm}}u^1) \cap \{x = y,\ \epsilon \leq |t| \leq \epsilon_0\} \\ \subset \big\{(t,x,x;\tau,\xi,\eta) \mid \xi_1, \eta_1 = 1 + O(\epsilon_0 + \epsilon_1)\big\},\end{aligned}$$

since the only way to have $x = y$ is to bounce of the boundary, in which case ξ_1, the momentum in the normal direction, changes from being approximately opposite to η_1 to being approximately equal. This wavefront set is killed under restriction to $x = y$ and then integration in x, showing that $\phi(hD_t)Q_{\mathsf{norm}}u^1$ is also $O(h^\infty)$ for $\epsilon \leq |t| \leq \epsilon_0$. This establishes (A.10).

A.5 Method of Successive Approximations

We now observe that (A.10) self-improves to the statement that

$$\sigma_h^i(t) = O(h^\infty) \text{ for } h^{1-\delta} \le |t| \le \epsilon_0, \quad \delta > 0. \tag{A.12}$$

This follows from a simple scaling argument. Fix a base point $y \in \partial M$, and consider the scaled metric (following [Mel])

$$g_{T,y} = g_{ij}(TX + y)dX^i dX^j$$

where T is a small parameter. Let $u_{T,\hbar}(t, X)$ be given by

$$u_{T,\hbar}(t, X) = T^{d+1}\big(\phi(hD_t)u_{T\hbar}\big)(Tt, y + TX, y);$$

then, with $\hbar := h/T$ we have

$$\begin{aligned}\Big(\hbar D_t - \hbar^2 \Delta_{g_{T,y}}\Big) u_{T,\hbar}(t, X) = \\ T^{(d+1)}\Big(hD_t - h^2\Delta_g\Big)u_h(Tt, y + TX, y) = \phi(hD_t)\delta(t)\delta(X),\end{aligned} \tag{A.13}$$

$$(\hbar\partial_{X_1} + a)u_{T,\hbar}(t, X) = 0, \quad X_1 = 0. \tag{A.14}$$

Using (A.10) we see that we have

$$\int_M \phi(\hbar D_t)u_{T,\hbar}(X, X, t)\, dX = O(T^{-(d+1)}h^\infty) \text{ for } \epsilon \le |t| \le \epsilon_0, \tag{A.15}$$

In particular, (A.15) is $O(h^\infty)$ provided that $T \ge h^{1-\delta}$ for arbitrary $\delta > 0$. Unravelling the scaling demonstrates (A.12).

Therefore if we want to construct $\sigma(t)$ for $|t| \le \epsilon_0$ it suffices to construct it for $|t| \le t_* = h^{1-\delta}$. However on this short interval we can construct it by the method of successive approximations. For each $y \in M$ we let P_y be the constant coefficient differential operator $P_{a,h}$ with coefficients frozen at y. Then we regard $P_{h,a}$ as a perturbation of P_y for x close to y. So $K = P_{h,a} - P_y$ is a second order differential operator with coefficients that are $O(x-y)$. The perturbation K is $O(t_*)$ due to finite speed of propagation and each successive term in the approximation acquires a factor not exceeding $Ct_* \cdot t_*/h = Ct_*^2 h$ due to Duhamel's principle. So the construction works for $t_* \le h^{\frac{1}{2}+\delta}$.

If we apply this to the u^0 term then the calculation proceeds as follows. We let $\overline{u^0}$ denote the propagator for the constant coefficient operator P_y with coefficients frozen at $y \in M$ (and taking only the second order derivatives). Then, with E the forward fundamental solution for P_y, we obtain a formula

$$u^0 = \overline{u^0} + EKu^0,$$

leading to a formal series

$$u^0 = \sum_{k=0}^{m} (EK)^k \overline{u^0} + (EK)^{m+1} u^0. \tag{A.16}$$

Applying the Fourier transform in x, we find that the first term is

$$\phi(hD_t)\overline{u^0} = (2\pi h)^{-d} \int e^{i(x-y)\cdot\xi/h} e^{it(|\xi|^2-1)/h} \phi(|\xi|^2 - 1)\, d\xi.$$

If we then plug this term into (A.7) then we find that this term is h^{-d} times a smooth function of λ and a. The method of successive approximations then generates a series of the form $\sum_{n=1}^{\infty} \kappa'_n(a,\lambda) h^{-d+n}$, and the terms corresponding to odd n are given by the integral in ξ of an odd function of ξ, hence vanish.

Using this expansion of u^0, we compute a series for u^1. In this case, the leading term $\overline{u^1}$ is given as follows:

$$\phi(hD_t)\overline{u^1} = (2\pi h)^{-d} \int e^{-i(x_1+y_1)\xi_1/h} e^{i(x'-y')\cdot\xi'/h} e^{it(|\xi|^2-1)/h} \phi(|\xi|^2 - 1)\, \frac{i\xi_1 + a}{i\xi_1 - a} d\xi.$$

We can form a similar formal series for u^1, which converges when $t \leq h^{1/2+\delta}$. As with the case of u^0, we can check that the successive terms in the approximation for u^1, when plugged into (A.7), give a series with the contribution of $\phi(hD_t)\overline{u^1}$ at order h^{1-d} and with successive terms contributing at increasing integer powers of h.

Now, using this together with the Tauberian theorem, Proposition A.1, shows that we get a leading Weyl asymptotic with remainder term $O(h^{1-d})$, as in (1.18).

A.6 Two Term Expansion

If we assume that the measure of periodic generalized bicharacteristics is zero, then we can use the standard method to obtain a two-term expansion of the counting function. We briefly recall here the argument, following [Ivr1].

Let $\epsilon > 0$ be given. Then we decompose the identity operator on $L^2(M)$ as a sum of three terms. The first is multiplication by a cutoff function ζ identically 1 near ∂M and supported in a collar neighbourhood of ∂M, such that

$$\int_M \zeta^2 \leq \epsilon.$$

The second and third are pseudodifferential operators chosen as follows. With $T > 1$ a large constant to be chosen later, we let Λ_T denote the union of points in

$$\{(x, \xi) \in T^*M \mid |\xi|_g \in (1 - \epsilon_0, 1 + \epsilon_0)\}$$

which are either periodic with period $\leq T$ under generalized bicharacteristic flow (for $P_{a,h}$), or for which the generalized bicharacteristic of length T in both directions are not transverse to the boundary. This is a closed set of measure zero so one can find two open sets U_1, U_2 such that $T^*M = U_1 \cup U_2$, $\Lambda_T \subset U_1$, and the measure of U_1 is less than ϵ. Then we choose (semiclassical) pseudodifferential operators Q_1, Q_2 such that Q_i is microsupported in the conic set determined by U_i, and such that $\mathsf{Id} = \zeta^2 + Q_1^* Q_1 + Q_2^* Q_2$. We then define

$$\begin{aligned} e_h^0(\lambda) &= \mathsf{Tr}\, \zeta^2 E_{h,a}(\lambda) = \mathsf{Tr}\, \zeta E_{h,a} \zeta \\ e_h^i(\lambda) &= \mathsf{Tr}\, Q_i^* Q_i E_{h,a}(\lambda) = \mathsf{Tr}\, Q_i E_{h,a}(\lambda) Q_i^*, \quad i = 1, 2; \end{aligned}$$

notice that each of these is nondecreasing in λ, and the sum of the three terms is equal to the counting function for $P_{h,a}$. Correspondingly, we break (A.7) into a sum of three terms, with $i = 0, 1, 2$.

Using the series for u^0 and u^1 sketched above, we compute expansions for the three terms. Applying Proposition A.1 we find that

$$\left| e_h^0(\lambda) - A_0^0(\lambda) h^{-d} - A_1^0(\lambda) h^{1-d} \right| = O(\epsilon h^{1-d}),$$

since in this case $A_0^0(\lambda)$ is $O(\epsilon)$, as it is proportional to $\int_M \zeta^2$. For the term $i = 1$ we similarly find that

$$\left| e_h^1(\lambda) - A_0^1(\lambda) h^{-d} - A_1^1(\lambda) h^{1-d} \right| = O(\epsilon h^{1-d}),$$

since $A_0^1(\lambda)$ is proportional to the integral of $|\sigma(Q_1)|^2$ which is also $O(\epsilon)$. For the third term, we scale β to β_T, exploiting the condition that on the microsupport of Q_2, there are no periodic bicharacteristics up to time T, hence the trace of $\phi(hD_t) e^{itP_{a,h}/h}$ has no singularities for $t \in [-T, T]$ except

at $t = 0$. Hence this term also has an expansion in powers of h, and we find that

$$\left| e_h^2(\lambda) - A_0^2(\lambda) h^{-d} - A_1^2(\lambda) h^{1-d} \right| = O(\frac{h^{1-d}}{T}).$$

Choosing T sufficiently large, this is also $O(\epsilon h^{1-d})$, and we have shown the existence of a two-term expansion.

It only remains to identify the first two terms. But once we know that there *is* a two-term expansion, we can identify the coefficients from the first two terms in the expansion of $\mathsf{Tr}\, \phi(hD_t) e^{itP_{a,h}/h}$ at $t = 0$. From the method of successive approximations we see that these terms arise from the contribution of $\overline{u^0}$ and $\overline{u^1}$. We observe that $\overline{u^0} + \overline{u^1}$ gives precisely the propagator for the $h^2 \Delta_{\mathbb{R}^d_+}$, the flat Laplacian on the half-space $\mathbb{R}^d_+ = \{x_1 \geq 0\}$, with the boundary condition $(h\partial_{x_1} + a)\overline{u}(x, y, t) = 0$ at $x_1 = 0$. It follows that the local densities for each term of the two-term expansion is equal to the local density for the flat half-space model. This justifies the calculations in Section 3 based on this flat model.

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Spectral Asymptotics for Fractional Laplacians*,†

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Abstract

In this article we consider fractional Laplacians which seem to be of interest to probability theory. This is a rather new class of operators for us but our methods works (with a twist, as usual). Our main goal is to derive a two-term asymptotics since one-term asymptotics is easily obtained by R. Seeley's method.

In this article we consider fractional Laplacians. This is a rather new class of operators for us but our methods works (with a twist, as usual). Our main goal is to derive a two-term asymptotics since one-term asymptotics is rather easily obtained by R. Seeley's method.

1 Problem Set-up

Let us consider a bounded domain $X \subset \mathbb{R}^d$ with the smooth boundary $\partial X \in \mathscr{C}^{\infty}$ [1)]. In this domain we consider a fractional Laplacian $\Lambda_m = (\Delta^{m/2})_{\mathfrak{D}}$ with $m > 0$ originally defined on functions $u \in \mathscr{C}^{\infty}(\mathbb{R}^d) : \theta_X u \in \mathscr{H}^{m/2}(\mathbb{R}^d)$ by

$$\Lambda_{m,X} \coloneqq (\Delta^{m/2})_{\mathfrak{D}} u = R_X \Delta^{m/2}(\theta_X u) \tag{1.1}$$

where θ_X is a characteristic function of X, R_X is an operator of restriction to X and $\Delta^{m/2}$ is a standard pseudodifferential operator in $\mathbb{R}^d$ with the Weyl symbol $g(x,\xi)^{m/2}$ where as usual $g(x,\xi)$ is non-degenerate Riemannian metrics.

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[1)] Alternatively consider a bounded domain X on the Riemannian manifold $\mathscr{X}$.

Remark 1.1. (i) We consider $\Lambda_{m,X}$ as an unbounded operator in $\mathscr{L}^2(X)$ with domain $\mathfrak{D}(\Lambda_{m,X}) = \{u \in \mathscr{L}^2(\mathbb{R}^d) : \mathsf{supp}(u) \subset \bar{X}, R_X\Lambda_m \in \mathscr{L}^2(X)\} \subset \mathscr{H}^{m/2}_0(X)$.

(ii) This operator can also be introduced through positive quadratic form with domain $\{u \in \mathscr{H}^m(\mathbb{R}^d), \mathsf{supp}(u) \subset \bar{X}\}$ and is a positive self-adjoint operator which is Friedrichs extension of operator originally defined on $\mathscr{H}^m_0(X)$.

(iii) We can consider this operator as a bounded operator from $\mathscr{H}^{m/2}_0(X)$ to $\mathscr{H}^{-m/2}(X) \coloneqq \mathscr{H}^{m/2\,*}_0(X)$.

(iv) Let $0 < m \notin 2\mathbb{Z}$. Then $\mathfrak{D}(\Lambda_{m,X}) \subset \mathscr{H}^m(X)$ if and only if $m \in (0,1)$; otherwise even eigenfunctions of $\Lambda_{m,X}$ may not belong to $\mathscr{H}^m(X)$.

(v) Since Λ_m does not possess transmission property for $m \notin 2\mathbb{Z}$, we are not in the framework of the Boutet-de-Monvel algebra, but pretty close: Λ_m possess μ-transmission property introduced by L. Hörmander and systematically studied by G. Grubb in [5,6]. We provide definition in Subsection A.2.

We are interested in the asymptotics of the eigenvalue counting function $\mathsf{N}(\lambda)$ for $\Lambda_{m,X}$ as $\lambda \to +\infty$.

2 Preliminary aAnalysis

As usual we reduce problem to a semiclassical one. Let $A = A_h \coloneqq h^m\Lambda_{m,X} - 1$ with $h = \lambda^{-1/m}$, $e_h(x,y,\tau)$ the Schwartz kernel of $\theta(\tau - A_h)$ a spectral projector of A_h, and N^-_h the number of negative eigenvalues of A_h.

Proposition 2.1. *Let $\bar{x} \in X$, $B(\bar{x}, 2\gamma) \subset X$, $\gamma \ge h$. Then*

$$|e_h(x,x,0) - \mathsf{Weyl}(x)| \le Ch^{1-d}\gamma^{-1} \qquad \forall x \in B(\bar{x},\gamma) \tag{2.1}$$

and

$$\Big|\int \psi((x-\bar{x})/\gamma)\Big(e_h(x,x,0) - \mathsf{Weyl}(x)\Big)\,dx\Big| \le Ch^{1-d}\gamma^{d-1}\big(\gamma^{\delta} + h^{\delta}\gamma^{-\delta}\big) \tag{2.2}$$

as $\psi \in \mathscr{C}_0^\infty(B(0,1))$ *and* $\delta > 0$, *where*

$$\mathsf{Weyl}(x) = (2\pi h)^{-d}\,\mathsf{mes}(\{\xi\colon g(x,\xi) \le 1\})$$

is the standard pointwise Weyl expression.

Proof. Estimate (2.1) is easily proven by just rescaling as modulo $O(h^s\gamma^{-s})$ we get a $\hbar$-pseudodifferential operator with $\hbar = h\gamma^{-1}$.

Estimate (2.2) is easily proven by rescaling plus R. Seeley's method as described in Subsection 7.5.1 of [9]. We leave easy details to the reader. □

Then we immediately arrive to

Corollary 2.2. *(i) Contribution of the inner zone* $\{x : \mathsf{dist}(x,\partial X) \ge h\}$ *to the Weyl remainder does not exceed* Ch^{1-d}.

(ii) Contribution of the intermediate strip $\{x : \varepsilon \ge \mathsf{dist}(x,\partial X) \ge \varepsilon^{-1}h\}$ *to the Weyl remainder does not exceed* $\eta(\varepsilon)h^{1-d}$ *with* $\eta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

Here and in what follows $\varepsilon > 0$ is an arbitrarily small constant.

Proposition 2.3. *The following estimate holds:*

$$|e_h(x,x,0)| \le Ch^{-d}. \tag{2.3}$$

Proof. The standard proof we leave to the reader. □

Theorem 2.4. *(i) For operator* A *the Weyl remainder in the asymptotics for* N_h^- *does not exceed* Ch^{1-d}.

(ii) For operator $\Lambda_{m,X}$ *the following asymptotics holds*

$$\mathsf{N}(\lambda) = \kappa_0\lambda^{\frac{d}{m}} + O(\lambda^{\frac{d-1}{m}}) \qquad \text{as} \quad \lambda \to +\infty, \tag{2.4}$$

where $\kappa_0 = (2\pi)^{-d}\varpi_d\,\mathsf{vol}(X)$, ϖ_d *is a volume of the unit ball in* $\mathbb{R}^d$ *and* $\mathsf{vol}(X)$ *means the Riemannian volume of* X.

Proof. Statement (i) follows immediately from Corollary 2.2(i) and Proposition 2.3. Statement (ii) follows immediately from (i) as $d \ge 2$. □

Remark 2.5. (i) Therefore, we extended the result, well-known for $m \in 2\mathbb{Z}^+$ to $m \in \mathbb{R}^+$.

(ii) This was easy but to recover the second term (which is also known for $m \in 2\mathbb{Z}^+$ under non-periodicity condition) is a much more daunting task requiring first to improve the contribution of the *near boundary strip* $\{x : \mathsf{dist}(x, \partial X) \le \varepsilon^{-1}h\}$ and also of the *inner zone* $\{x : \mathsf{dist}(x, \partial X) \ge \varepsilon\}$.

3 Propagation of Singularities near the Boundary

Without any loss of the generality one can assume that

$$X = \{x : x_1 > 0\}, \qquad g^{jk} = \delta_{1j} \quad \forall j = 1, \dots, d. \tag{3.1}$$

First let us study the propagation of singularities along the boundary:

Theorem 3.1. *On the energy level $\tau : |\tau| \le \epsilon_0$*

(i) Singularities (with respect to x') propagate with the speed not exceeding c with respect to (x, ξ').

(ii) For $|\xi'| \asymp \rho \ge Ch^{\frac{1}{2}-\delta}$, and $|t| \le T = \epsilon\rho$, singularities (with respect to x') move from $x' = y'$ with the speed $\asymp \rho$ with respect to x.

Proof. In the terminology of [9] both statements mean that $u = u_h(x, y, t)$, the Schwartz kernel of $e^{-ih^{-1}tA}$, satisfies

$$F_{t\to h^{-1}\tau}\chi_T(t) Q_{1,x}\, u\, {}^tQ_{2,y} = O(h^s), \tag{3.2}$$

where $F_{t\to h^{-1}\tau}$ is h-Fourier transform, $Q_1 = q_1(x, hD')$ and $Q_2 = q_2(x, hD')$ are h-pseudodifferential operators, ${}^tQ_{2,y}$ is a dual operator, acting with respect to y (and we write it to the right of the function, it is applied to), $\chi_T(t) = \chi(t/T)$, where

(a) in the Statement (i) $h^{\frac{1}{2}-\delta} \le T_0$, T_0 is the small constant, the distance between $\mathsf{supp}(q_1)$ and $\mathsf{supp}(q_2)$ is at least cT, and $\chi \in \mathscr{C}_0^\infty([-1, 1])$.

(b) in the Statement (ii) the diameter of $\mathsf{supp}(q_1) \cup \mathsf{supp}(q_2)$ is does not exceed $\epsilon\rho$, ϵ is the small constant, and $\chi \in \mathscr{C}_0^\infty([-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1])$;

s is an arbitrarily large exponent, $\delta > 0$ is arbitrarily small,

The proof is standard, by means of the positive commutator method, like those proofs in the Chapters 2 and 3 of [9], since it involves only pseudodifferential operators $q_j(x, hD')$ and their commutators with A, but one can see easily that those commutators do not bring any troubles as the energy level is $\asymp 1$. We leave all easy details to the reader. □

Corollary 3.2. *(i) Let $\rho \ge Ch^{\frac{1}{2}-\delta}$, $h^{1-\delta} \le \gamma \le \epsilon$. Then the contribution of the zone $\{(x,\xi') : x_1 \le \gamma, |\xi'| \asymp \rho\}$ to the Tauberian remainder with $T \in (T_*(\rho), T^*(\rho))$, where $T_*(\rho) = h^{1-\delta}\rho^{-2}$, $T^*(\rho) = \epsilon\rho$ does not exceed*

$$C\rho^{d-1}h^{-d} \times \gamma \times h^{1-\delta}\rho^{-2} \times \rho^{-1}. \tag{3.3}$$

(ii) The total contribution of the zone $\{(x,\xi') : x_1 \le \gamma = h^{1-\delta}\}$ to the Tauberian error with $T = h^{1-3\delta}$ does not exceed $Ch^{-d+1+\delta}$.

Proof. In the terminology of [9] the *Tauberian error* is the difference between N_h^- and the *Tauberian expression*

$$\mathsf{N}_h^{\mathsf{T}} := h^{-1}\int_{-\infty}^{0}\Big(\int F_{t\to h^{-1}\tau}\bar\chi_T(t)u(x,x,t)\,dx\Big)\,d\tau, \tag{3.4}$$

where $\bar\chi \in \mathscr{C}_0^\infty([-1,1])$, $\bar\chi = 1$ on $[-\frac{1}{2},\frac{1}{2}]$.

The easy and standard proof of Statement (i) is left to the reader; it is like those proofs in Chapter 7 of [9].

Then the contribution of the zone $x_1 \le h^{1-\delta}$, $|\xi'| \ge Ch^\delta$ to the Tauberian error with $T = h^{1-3\delta}$ does not exceed $Ch^{-d+1+\delta}$. Since the contribution of the zone $\{(x,\xi') : x_1 \le \gamma = h^{1-\delta}, |\xi'| \le Ch^{2\delta}\}$ to the asymptotics does not exceed $C\rho^{d-1}h^{-d} \times \gamma \le Ch^{-d+1+\delta}$ we arrive to Statement (ii). □

Therefore in this zone $\{(x,\xi') : x_1 \le \gamma = h^{1-\delta}\}$ all we need is to pass from the Tauberian expression to the Weyl expression.

However the inner zone should be reexamined and we need to describe what happens with the propagation along Hamiltonian trajectory in the zone $\{(x,\xi) : x_1 \le h^{1-\delta}\}$. We can assume that $|\xi_1| \ge \varepsilon$ since the measure of the remaining trajectories is small, here $\varepsilon > 0$ is an arbitrarily small constant.

4 Reflection of Singularities from the Boundary

4.1 Toy-Model

We start from the pilot-model which will be used to prove the main case. Namely, let us consider 1-dimensional operator on half-line $\mathbb{R}^+$ with Euclidean metrics

$$B := B_{m,a,h} = ((h^2 D_x^2 + a^2)^{m/2})_{\mathfrak{D}} \tag{4.1}$$

with $a \geq 0$. We denote $e_{m,a,h}(x_1, y_1, \tau)$ the Schwartz kernel of its spectral projector.

Observe that scaling $x \mapsto x\gamma^{-1}$, $\tau \mapsto \tau\rho^{-m}$ transforms operator to one with $h \mapsto h/(\rho\gamma)$, $\tau \mapsto \tau\rho^{-m}$; because of this we can assume that $h = 1$ and the second scaling implies that we can assume that either $a = 1$ or $\tau = 1$.

Proposition 4.1. *(i) The spectrum of operator $\Lambda_{m,a}$ is absolutely continuos and it coincides with $[a^m, \infty)$.*

(ii) The following equalities hold:

$$e_{m,a,h}(x, y, \lambda) = e_{m,1,ha^{-1}}(ax, ay, \lambda a^{-m}) = \lambda^{1/m} e_{m,a\lambda^{-1/m},h}(\lambda^{1/m}x, \lambda^{1/m}y, 1) = a e_{m,1,h}(ax, ay, \lambda a^{-m}). \tag{4.2}$$

Proposition 4.2. *Let $\psi \in \mathscr{C}_0^\infty([-1,1])$, $\psi_\gamma(x) = \psi(x/\gamma)$ and $\phi \in \mathscr{C}_0^\infty([-1,1])$, $0 \leq a \leq 1 - \epsilon_0$. Then as $\gamma \geq h^{1-\delta}$, $T \geq C_0\gamma$, $h^\delta \geq \eta \geq h^{1-\delta}T^{-1}$*

$$\|\phi(\eta^{-1}(hD_t - 1))\psi_\gamma e^{i(mh)^{-1}tB}\psi_\gamma|_{t=T}\| \leq CT^{-1}\gamma + Ch^{\delta'}. \tag{4.3}$$

Proof. Observe first that if for u supported in $\mathbb{R}^+$ and $L = x_1 hD_1 - ih/2 = L^*$

$$\operatorname{Re} i(BLu, u) = \frac{1}{2}(i[B, L]u, u), \tag{4.4}$$

and then

$$\operatorname{Re}(Lu, u_t - ih^{-1}Bu) = \frac{1}{2}(ih^{-1}[B, L]u, u) + \frac{1}{2}\partial_t \operatorname{Re}(Lu, u) \tag{4.5}$$

and

$$\mathrm{Re}(ktu + Lu, u_t - ih^{-1}Bu) = \frac{1}{2}\partial_t\big(kt\|u\|^2 + (Lu,u)\big) + \frac{1}{2}\big((ih^{-1}[B,L]u,u) - k\|u\|^2\big). \tag{4.6}$$

Let us plug

$$u = \phi(\eta^{-1}(hD_t - 1))e^{ih^{-1}TB}\psi_\gamma v \tag{4.7}$$

with $\|v\| = 1$. Then the left hand expression in (4.6) is 0 and

$$\frac{1}{2}\partial_t\big(kt\|u\|^2 + (Lu,u)\big) \le \frac{1}{2}\big(-(ih^{-1}[B,L]u,u) + k\|u\|^2\big). \tag{4.8}$$

Let us estimate from above the right-hand expression; obviously

$$ih^{-1}[B_m, L] = m(B_m - a^2 B_{m-2}). \tag{4.9}$$

(a) Assume first that $m > 2$. Then since

$$(1 - C\eta)\|u\| \le \|B_m u\| \le (1 + C\eta)\|u\| \tag{4.10}$$

due to cutoff by ϕ and

$$B_{m-2} \le B_m^{(m-2)/m} \tag{4.11}$$

in virtue of Corollary A.2 we conclude that as $k = m(1-a^2)$ the right-hand expression does not exceed $C\eta$ and therefore

$$m(1-a^2)t\|u\|^2 + (Lu,u) \le C\gamma + C\eta T \tag{4.12}$$

since the value of the left-hand expression as $t = 0$ does not exceed $C\gamma$.

Further, observe that on the energy levels from $(1 - C_0\eta, 1 + C_0\eta)$ the singularities propagate with a speed (with respect to x_1) not exceeding $m(1-a^2)^{\frac{1}{2}}(1 + C_0\eta)$. Therefore we conclude that u is negligible as $|x_1| \ge m(1-a^2)^{\frac{1}{2}}(1 + C_0\eta)T + C\gamma$ and therefore since

$$\|D_1 u\| \le (B_2 u, u) \le ((1-a^2)^{\frac{1}{2}} + C_0\eta), \tag{4.13}$$

we conclude using (4.10) and (4.11) that

$$|(Lu,u)| \le m(1 - a^2 + C_0\eta)T + C\gamma - \epsilon_0 T\|\psi_\gamma(x_1)u\|^2 \tag{4.14}$$

and the left-hand expression of (4.12) is greater than $\epsilon_0 T\|\psi_\gamma u\|^2 - C(\eta T + \gamma)$ and we arrive to (4.3).

(b) Assume now that $0 < m < 2$. Then our above proof fails short in both estimating $\operatorname{Re} ih^{-1}([B, L]u, u)$ from below and $|(Lu, u)|$ from above and we need to remedy it.

Note first that away from $x_1 = 0$ only symbols are important and therefore the right-hand expression of (4.8) does not exceed $\frac{m}{2}(a^m - a^2)\|\psi_\sigma u\|^2 + Ch^2\sigma^{-2}$ since $\|B_{m-2}\| \le a^{m-2}$. Indeed, we need just to decompose $1 = \psi_\sigma^2 + \psi_\sigma'^2$ and use our standard arguments to rewrite the right-hand expression of (4.8) as the sum of the same expressions for $\psi_\sigma u$ and $\psi_\sigma' u$ plus $Ch^2\sigma^{-2}\|u\|^2\|$.

Similarly we deal $\operatorname{Re}(Lu, u) = \operatorname{Re}(L\psi_\sigma u, \psi_\sigma u) + \operatorname{Re}(L\psi_\sigma' u, \psi_\sigma' u)$ and the absolute value of the second term does not exceed $m(1 - a^2)^{\frac{1}{2}} T\|B_m^{1/m}\psi_\sigma'\|^2$.

We claim that

$$\operatorname{Re}(L\psi_\sigma u, \psi_\sigma u) \le C\sigma\|\psi_\sigma u\|^2 + C\sigma h^\delta; \tag{4.15}$$

we prove it later but now instead of (4.12) we arrive for $\sigma = \epsilon_0 t$ to

$$((1 - a^2) - \epsilon)P(T) \le Ch^{\delta'} + (a^m - a^2)T^{-1}\int_\gamma^T P(t)\,dt + C\gamma T^{-1}$$

with $P(t) = \|\psi_{\epsilon_0 t} u(., t)\|^2$. Then since $\nu = (a^m - a^2)/((1 - a^2) - \epsilon) < 0$ this inequality implies (4.3) again. □

Proof of (4.15). Indeed, as $h = 1$, $\|B_m^{1/m} u\| \le 1$ we from G. Grubb [5, 6] conclude that $|u(x_1)| \le Cx_1^{(m-1)/2}\|uB_m^{1/m}u\|$ and $|Lu(x_1)| \le Cx_1^{(m-1)/2}$ and therefore $|(Lu, u)| \le C\sigma^m\|B_m^{1/m}u\|^2$. Take $\sigma = 1$.

Scaling returns (4.15) as $\sigma = h$. □

Proposition 4.3. *Let $\Lambda = \Lambda_{m,X}$ be a d-dimensional operator* (1.1) *on the half-space $X = \{x \in \mathbb{R}^d, x_1 > 0\}$ with Euclidean metrics ($d \ge 2$) and $A = h\Lambda^{1/m} - 1$.*

Let $\psi \in \mathscr{C}_0^\infty([-1, 1])$, $\psi_\gamma(x) = \psi(x_1/\gamma)$, $\phi \in \mathscr{C}_0^\infty([-1, 1])$, $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^{d-1})$ supported in $\{|\xi'| \le 1 - \epsilon\}$ with $\epsilon > 0$. Finally, let $\gamma \ge h^{1-\delta}$, $T \ge Ch^{-\delta}\gamma$, $h^\delta \ge \eta \ge h^{1-\delta}T^{-1}$. Then

$$\|\phi(\eta^{-1}hD_t - 1)\varphi(hD')\psi_\gamma(x_1)e^{ih^{-1}tA}\psi_\gamma(x_1)|_{t=T}\| = O(h^s) \tag{4.16}$$

with arbitrarily large s.

Proof. By making Fourier transform $F_{x'\to h^{-1}\xi'}$ we reduce the general case to $d=1$ and operator B.

According to Proposition 4.2 $\|\phi(\eta^{-1}hD_t)\psi_\gamma e^{ih^{-1}TA}\psi\| \le h^\delta$. Thus for $s=\delta'$ (4.16) has been proven (we reduce δ' if necessary).

Without any loss of the generality we assume that $\psi(x_1)=1$ as $x_1\le 1$, $\psi(x_1)=0$ as $x_1\ge 2$. Observe that due to propagation as $t\le T$ and $x_1\ge\gamma$ we see that $\phi(\eta^{-1}hD_t)Q^+(hD_1)(1-\psi_\gamma)e^{ih^{-1}TA}\psi_\gamma$ is negligible where $Q^\pm\in\mathscr{C}^\infty(\mathbb{R})$ is supported in $\{\pm\xi_1>\epsilon\}$. Furthermore, from the standard ellipticity arguments we conclude that $\phi(\eta^{-1}hD_t)Q^0(hD_1)(1-\psi_\gamma)e^{ih^{-1}TA}\psi_\gamma$ is also negligible for $Q^0\in\mathscr{C}_0^\infty([-2\epsilon,2\epsilon])$.

Finally, due to propagation as $t\ge T$ and $x_1\ge\gamma$ we conclude that $\phi(\eta^{-1}hD_t)\psi_\gamma e^{ih^{-1}(t-T)A}Q^-(hD_1)(1-\psi_\gamma)e^{ih^{-1}TA}\psi_\gamma$ is negligible for $t\ge T$.

What is left is $\phi(\eta^{-1}hD_t)\psi_\gamma e^{ih^{-1}(t-T)A}Q^-(hD_1)\psi_\gamma e^{ih^{-1}TA}\psi_\gamma$ and since (4.16) holds for $s=\delta'$ we conclude that it holds for $s=2\delta'$ and T replaced by $2T$.

Continuing this process we see that (4.16) holds for $s=n\delta'$ and T replaced by nT. Therefore, as we redenote nT by T (and T by T/n respectively), we acquire factor $(\gamma n/T)^n$ in our estimate and it is $O(h^s)$ for any s as h is sufficiently small, $\gamma/T\le h^\delta$ and $n=s/\delta'$. □

4.2 General Case

Theorem 4.4. *Let $(\bar x,\bar\xi)$ be a point on the energy level 1. Consider a Hamiltonian trajectory $\Psi_t(\bar x,\bar\xi)$ with $\pm t\in[0,mT]$ (one sign only) with $T\ge\epsilon_0$ and assume that for each t indicated it meets ∂X transversally i.e.*

$$\mathsf{dist}(\pi_x\Psi_t(x,\xi),\partial X)\le\epsilon \implies |\frac{d}{dt}\mathsf{dist}(\pi_x\Psi_t(x,\xi),\partial X)|\ge\epsilon \qquad \forall t:\pm t\in[0,mT]. \tag{4.17}$$

Also assume that

$$\mathsf{dist}(\pi_x\Psi_t(x,\xi),\partial X)\ge\epsilon_0 \qquad \text{as}\quad t=0,\ \pm t=mT. \tag{4.18}$$

Let $\epsilon>0$ be a small enough constant, Q be supported in ϵ-vicinity of (x,ξ) and $Q_1\equiv 1$ in $C_0\epsilon$-vicinity of $\Psi_t(x,\xi)$ as $t=\pm mT$. Then operator $(I-Q_1)e^{-ih^{-1}tH}Q$ is negligible as $t=\pm mT$.

Proof. (a) Obviously without any loss of the generality one can assume that there is just one reflection from ∂X (and this reflection is transversal) and that (3.1) is fulfilled in its vicinity.

Further, without any loss of the generality one can assume that Q is supported in ε-vicinity of $(\bar{x},\bar{\xi})$, $Q_1 \equiv 1$ in ε-vicinity of $\Psi_{mT}(\bar{x},\bar{\xi})$ and $T \asymp \varepsilon$ with $\varepsilon = h^{\frac{1}{2}-\delta'}$. Then both $\bar{x}$ and $\pi_x \Psi_{mT}(\bar{x},\bar{\xi})$ belong to $C_0\varepsilon$-vicinity of ∂X.

Indeed, it follows from the propagation inside of domain.

(b) Then instead of isotropic vicinities we can consider anisotropic ones: ε with respect to (x',ξ'), $h^{1-3\delta'}$ with respect to x_1 and $h^{\delta'}$ with respect to ξ_1. Let now Q and Q_1 be corresponding operators.

In this framework from the propagation inside of domain it follows that without any loss of the generality one can assume that $T \asymp \gamma = h^{1-\delta''}$ and both $\bar{x}$ and $\pi_x \Psi_{mT}(\bar{x},\bar{\xi})$ belong to $C_0\gamma$-vicinity of ∂X.

(c) Then one can employ the method of the successive approximations freezing coefficients at point $\bar{x}$ and in this case the statement of the theorem follows from the construction of Section 7.2 of [9][2)] and Proposition 4.3. We leave easy details to the reader. □

Then we arrive immediately to

Corollary 4.5. *Under standard non-periodicity condition*[3)] N_h^- *is given with* $o(h^{1-d})$*-error by the Tauberian expression with* $T = h^{1-\delta}$.

Proof. Easy details are left to the reader. □

5 Main rResults

5.1 From Tauberian to Weyl Asymptotics

Now we can apply the method of successive approximations as described in Section 7.2 [2)] and prove that for operator A the Tauberian expression with

[2)] Insignificant and rather obvious modifications are required.

[3)] The set of all periodic billiards has measure zero.

$T = h^{1-\delta}$ (with sufficiently small $\delta > 0$) equals to Weyl expression $\mathsf{N}_h^{\mathsf{W}}$ with $O(h^{2-d-\delta''})$ error,

$$\mathsf{N}_h^{\mathsf{W}} = \kappa_0 h^{-d} + \kappa_{1,m} h^{1-d} + o(h^{1-m}) \tag{5.1}$$

with the standard coefficient $\kappa_0 = (2\pi)^{-d}\varpi_d \operatorname{vol}_d(X)$ and with

$$\kappa_{1,m} = (2\pi)^{1-d}\varpi_{d-1}\varkappa_m \operatorname{vol}_{d-1}(\partial X), \tag{5.2}$$

where

$$\varkappa_m = \tag{5.3}$$
$$= \frac{d-1}{m} \iint_1^\infty \lambda^{-(d-1)/m-1}\Big(\boldsymbol{e}_m(x_1, x_1, \lambda) - \pi^{-1}(\lambda-1)^{1/m}\Big)\, dx_1 d\lambda$$

with $\boldsymbol{e}_m(x_1, y_1, \tau) = \boldsymbol{e}_{m,1,1}(x_1, y_1, \tau)$ the Schwartz kernel of the spectral projector of operator $\boldsymbol{a}_m := B_{m,1,1}$ introduced by (4.1):

$$\boldsymbol{a}_m = ((D_x^2+1)^{m/2})_{\mathfrak{D}} \tag{5.4}$$

Recall that vol_d and vol_{d-1} are Riemannian volumes corresponding to metrics g and its restriction to ∂X respectively and $\pi^{-1}(\lambda-1)^{1/m}$ is a Weyl approximation to $\boldsymbol{e}_{m,1}(x_1, x_1, \lambda)$. Thus we arrive to

Theorem 5.1. *Under standard non-periodicity condition the following asymptotics holds:*

$$\mathsf{N}(\tau) = \kappa_0 \tau^{\frac{d}{m}} + \kappa_{1,m}\tau^{\frac{d-1}{m}} + o(\tau^{\frac{d-1}{m}}) \qquad \text{as} \quad \tau \to +\infty. \tag{5.5}$$

Proof. First we establish as described above asymptotics

$$\mathsf{N}_h^- = \kappa_0 h^{-d} + \kappa_{1,m} h^{1-d} + o(h^{1-d}) \qquad \text{as} \quad h \to +0, \tag{5.6}$$

which immediately implies (5.5). □

Remark 5.2. Based on our analysis one can prove easily also the asymptotics for the Riesz means:

$$\Big(\mathsf{N}(\tau) - \sum_{k<r+1} \kappa_{k,m}\tau^{\frac{d-k}{m}}\Big) * \tau_+^{r-1} = O(\tau^{\frac{d-1}{m}}) \qquad \text{as} \quad \tau \to +\infty, \tag{5.7}$$

and under standard non-periodicity condition

$$\Big(\mathsf{N}(\tau) - \sum_{k\le r+1} \kappa_{k,m}\tau^{\frac{d-k}{m}}\Big) * \tau_+^{r-1} = o(\tau^{\frac{d-1}{m}}) \qquad \text{as} \quad \tau \to +\infty, \tag{5.8}$$

with some coefficients $\kappa_{k,m}$.

5.2 Discussion

The following problems seem to be challenging

Problem 5.3. As $m_1 > 0$, $m_2 > 0$ consider

$$K = K_{m_1,m_2,X} \coloneqq \Lambda_{m,X} - \Lambda_{m_1,X}\Lambda_{m_2,X} \tag{5.9}$$

on $\mathfrak{D}(\Lambda_m)$ with $m = m_1 + m_2$. From Corollary A.2 we conclude that this is a non-negative operator. Furthermore, due to [12, 13] it is a positive operator. Obviously singularities of its Schwartz kernel $K(x, y)$ belong to $\partial X \times \partial X$.

(i) Provide an effective estimate for this operator from below.

(ii) Prove that as $X = \{x \in \mathbb{R}^d : x_1 > 0\}$ with Euclidean metrics its Schwartz kernel $K(x, y) = k(x_1, y_1, x' - y')$ which is positive homogeneous of degree $-m - d$ satisfies

$$|D_x^\alpha D_y^\beta K(x, y)| \leq C_{\alpha\beta} x_1^{-\frac{m}{2}-\alpha_1} y_1^{-\frac{m}{2}-\beta_1} (x_1 + y_1 + |z|)^{-d-|\alpha'|+|\beta'|}. \tag{5.10}$$

(iii) In the general case in the local coordinates in which $X = \{x : x_1 > 0\}$ and $x_1 = \mathsf{dist}(x, \partial X)$ not only (5.10) holds but also

$$|D_x^\alpha D_y^\beta \big(K(x, y) - K^0(x, y)\big)| \leq C_{\alpha\beta} (x_1 + y_1)^{-m-\alpha_1-\beta_1} (x_1 + y_1 + |z|)^{-d-|\alpha'|+|\beta'|+1} \tag{5.11}$$

where $K^0(x, y) = k(x_1, y_1, x' - y')$ and $g^{jk} = \delta_{jk}$ at point $(0, \frac{1}{2}(x' + y'))$.

Problem 5.4. For $m > 0$, $n > 0$ consider operator

$$K = K_{m,n,X} \coloneqq \Lambda_{m,X} - \Lambda_{n,X}^{m/n} \tag{5.12}$$

on $\mathfrak{D}(\Lambda_k)$ with $k = \mathsf{max}(m, n)$. Then this is a non-negative (non-positive) operator as $m > n$ ($m < n$ respectively. Furthermore, due to [12, 13] it is a positive (negative) operator respectively.

(i) Provide an effective estimate for this operator from below.

(ii) Prove that if $X = \{x \in \mathbb{R}^d : x_1 > 0\}$ with Euclidean metrics its Schwartz kernel $K(x,y) = k(x_1, y_1, x' - y')$ which is positive homogeneous of degree $-(m+d)$ and satisfies (5.10).

(iii) Prove that in the framework of Problem 5.3(iii) both (5.10) and (5.11) hold.

Problem 5.5. (i) Consider operators $(\Delta^{m/2})_{\mathfrak{D}}$ with $m < 0$ and the asymptotics of eigenvalues tending to $+0$.

(ii) Consider operators with degenerations like $A_{m,X} = h^m \Lambda_{m,X} + V(x)$.

(iii) Consider more general operators where instead of Δ general elliptic (matrix) operator is used.

Problem 5.6. (i) Consider Neumann boundary conditions: having smooth metrics g in the vicinity of $\bar{X}$ for each point $x \notin X$ in the vicinity of ∂X we can assign a mirror point $j(x) \in \bar{X}$ such that x and $j(x)$ are connected by a (short) geodesics orthogonal to Y at the point of intersection. Each u defined in X we can continue to the vicinity of $\bar{X}$ as $Ju(x) = \psi(x) u(j(x))$ with ψ supported in the vicinity of $\bar{X}$ and $\psi = 1$ in the smaller vicinity of $\bar{X}$. Then $\Lambda_m u = R_X \Delta^{m/2} Ju$.

- Establish eigenvalue asymptotics for this operator.
- Surely we need to prove that the choice neither of metrics outside of X nor ψ is important.

(ii) One can also try $Ju(x) = -\psi(x) u(j(x))$ and prove that eigenvalue asymptotics for this operator do not differ from what we got just for continuation by 0.

Problem 5.7. Consider manifolds with all geodesic billiards closed as in Section 8.3 of [9]. To do this we need to calculate the "phase shift" at the transversal reflection point itself seems to be an extremely challenging problem.

6 Global Theory

Let us discuss fractional Laplacians defined by (1.1) in domain $X \subset \mathbb{R}^d$. Then under additional condition

$$\mathsf{dist}(x,y) \le C_0 |x-y| \qquad \forall x, y \in X \tag{6.1}$$

(where $\mathsf{dist}(x,y)$ is a "connected" distance between x and y) everything seems to work. We leave to the reader:

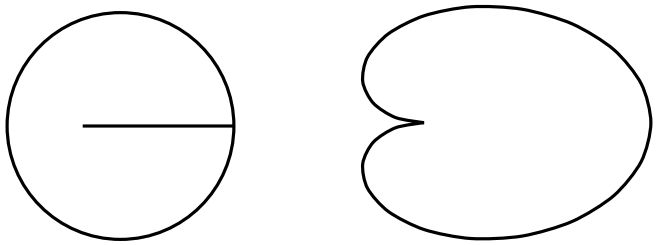
Problem 6.1. Under assumption (6.1)

(i) prove Lieb-Cwikel-Rozeblioum estimate (9.A.11) of [9].

(ii) Restore results of Chapter 9 of [9].

(iii) Reconsider examples of Sections 11.2 and 11.3 of [9].

Remark 6.2. Obviously domains with cuts and inner spikes (inner angles of 2π) do not fit (6.1). On the other hand, in the case of the domain with the cut due to non-locality of Δ^r with $r \in \mathbb{R}^+ \setminus \mathbb{Z}$ both sides of the cut "interact" and at least coefficient in the second term of two-term asymptotics may be wrong; in the case of the inner spike some milder effects are expected.



(a) (b)

Figure 28.1: Domain with a cut (a) and an inner spike (b).

The following problem seems to be very challenging:

Problem 6.3. (i) Investigate fractional Laplacians in domains with cuts and inner spikes and save whatever is possible.

(ii) Generalize these results to higher dimensions.

A Variational Estimates for Fractional Laplacian

A.1 Variational Estimates for Fractional Laplacian

We follow here R. Frank and L. Geisinger [4]. This is Lemma 19 and the next paragraph of their paper:

Lemma A.1. *(i) Let B be a non-negative operator with $\mathsf{Ker}\, B = \{0\}$ and let P be an orthogonal projection. Then for any operator monotone function $\phi : (0,\infty) \to \mathbb{R}$,*

$$P\phi(PBP)P \ge P\phi(B)P. \tag{A.1}$$

(ii) If, in addition, B is positive definite and ϕ is not affine linear, then $\phi(PBP) = P\phi(B)P$ implies that the range of P is a reducing subspace of B.

We recall that, by definition, the range of P is a reducing subspace of a non-negative (possibly unbounded) operator if $(B+\tau)^{-1}\,\mathsf{Ran}\, P \subset \mathsf{Ran}\, P$ for some $\tau > 0$. We note that this is equivalent to $(B+\tau)^{-1}$ commuting with P, and we see that the definition is independent of τ since

$$\begin{aligned}&(B+\tau')^{-1}P - P(B+\tau')^{-1}\\ &\quad = (B+\tau)(B+\tau')^{-1}\left((B+\tau)^{-1}P - P(B+\tau)^{-1}\right)(B+\tau)(B+\tau')^{-1}.\end{aligned}$$

We refer to the proof given there.

Corollary A.2. *The following inequality holds*

$$\Lambda_{m,X} \le \Lambda_{n,X}^{m/n} \qquad \text{as} \quad 0 < m < n. \tag{A.2}$$

Proof. Plugging into (A.1) $B = \Delta^{n/2}$ in $\mathbb{R}^d$, $P = \theta_X(x)$ and $\phi(\lambda) = \lambda^{m/n}$ we get (A.2). □

Repeating arguments of Proposition 20 and following it Subsection 6.4 of R. Frank and L. Geisinger [4] (powers of operators will be different but also negative) we conclude that

Proposition A.3. *Let $d \ge 2$. Then $-\varkappa_m$ is positive strictly monotone increasing function of $m > 0$.*

We leave details to the reader.

A.2 μ-Transmission Property

Proposition 1 of G. Grubb [6] claims that

Proposition A.4. *A necessary and sufficient condition in order that* $R_X P u \in \mathscr{C}^\infty(\bar{X})$ *for all* $u \in \mathscr{E}_\mu(\bar{X})$ *is that* P *satisfies the* μ-transmission condition *(in short: is of type* μ*), namely that*

$$\partial_x^\beta \partial_\xi^\alpha p_j(x,-N) = e^{\pi i(m-2\mu-j-|\alpha|)} \partial_x^\beta \partial_\xi^\alpha p_j(x,N) \qquad \forall x \in \partial\Omega, \tag{A.3}$$

for all j, α, β*, where* N *denotes the interior normal to* ∂X *at* x*,* m *is an order of classical pseudo-differential operator* P *and for* $\mu \in \mathbb{C}$ *with* $\operatorname{Re}\mu > -1$*.*

Here $\mathscr{E}_\mu(\bar{X})$ denotes the space of functions u such that $u = E_X d(x)^\mu v$ with $v \in \mathscr{C}^\infty(\bar{X})$ where E_X is an operator of extension by 0 to $\mathbb{R}^d \setminus X$ and $d(x) = \operatorname{dist}(x, \partial X)$.

Observe that for $\mu = 0$ we have an ordinary transmission property (see Definition 1.4.3) of [9].

Comments

This project started when I learned to my surprise that fractional Laplacians are of the interest to probability theory: which seem to be of interest to probability theory starting from R. M. Blumenthal, R. M. and R. K. Getoor [3] and then by R. Bañuelos and T. Kulczycki [1], R. Bañuelos, T. Kulczycki and B. Siudeja [2], M. Kwaśnicki [11]; some of these authors were interested in "Ivrii-type results" (i.e. generalizations from $m = 2$ to $m \in (0,2)$

Those operators were formulated in the framework of stochastic processes and thus were not accessible for me until I found paper R. Frank and L. Geisinger [4] provided definition we follow here. They showed that the trace has a two-term expansion regardless of dynamical assumptions[4)], and the second term in their expansion paper [4] defined by (3.2)–(3.3) is closely related to $\kappa_{1,m}$. It corresponds to $r = 1$ in Remark 5.2.

Furthermore, I learned that one-term asymptotics for more general operators (albeit without remainder estimate) was obtained by G. Grubb [7].

[4)] The fact that R. Frank and L. Geisinger obtain a second term regardless of dynamical assumptions is simply due to the fact that they study $\operatorname{Tr}(f(\Lambda_{m,X}))$ with $f(\lambda) = -\lambda\theta(-\lambda)$, which is one order smoother than $f(\lambda) = \theta(-\lambda)$.

Very recently I used the ideas of Section 4 to study sharp spectral asymptotics for Dirichlet-to-Neumann operator in [10].

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Spectral Asymptotics for Dirichlet to Neumann Operator in the Domains with Edges[*,†]

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Abstract

We consider eigenvalues of the Dirichlet-to-Neumann operator for Laplacian in the domain (or manifold) with edges and establish the asymptotics of the eigenvalue counting function

$$\mathsf{N}(\lambda) = \kappa_0 \lambda^d + O(\lambda^{d-1}) \qquad \text{as} \quad \lambda \to +\infty,$$

where d is dimension of the boundary. Further, in certain cases we establish two-term asymptotics

$$\mathsf{N}(\lambda) = \kappa_0 \lambda^d + \kappa_1 \lambda^{d-1} + o(\lambda^{d-1}) \qquad \text{as} \quad \lambda \to +\infty.$$

We also establish improved asymptotics for Riesz means.

1 Introduction

Let X be a compact connected $(d+1)$-dimensional Riemannian manifold with the boundary Y, regular enough to properly define operators J and Λ below[1)]. Consider *Steklov problem*

$$\Delta w = 0 \qquad \text{in} \quad X, \tag{1.1}$$

$$(\partial_\nu + \lambda) w|_Y = 0, \tag{1.2}$$

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[1)] Manifolds with edges are of this type.

where Δ is the positive Laplace-Beltrami operator[2)], acting on functions on X, and ν is the unit inner normal to Y. In the other words, we consider eigenvalues of the Dirichlet-to-Neumann operator. For v, which is a restriction to Y of $\mathscr{C}^2$ function, we define $Jv = w$, where $\Delta w = 0$ in X, $w|_Y = v$, and $\Lambda v = -\partial_\nu Jv|_Y$.

Definition 1.1. Λ is called *Dirichlet-to-Neumann operator*.

The purpose of this paper is to consider manifold with the boundary which has edges: i.e. each point $y \in Y$ has a neighbourhood U in $\bar{X} := X \cap Y$, which is the diffeomorphic either to $\mathbb{R}^+ \times \mathbb{R}^d$ (then y is a *regular point*), or to $\mathbb{R}^{+2} \times \mathbb{R}^{d-1}$ (then y is an *inner edge point*) or to $(\mathbb{R}^2 \setminus \mathbb{R}^{-2}) \times \mathbb{R}^{d-1}$ (then y is an *outer edge point*). Let Z_{inn} and Z_{out} be sets of the inner and outer edge points respectively, and $Z = Z_{\mathsf{inn}} \cup Z_{\mathsf{out}}$.

One can prove easily the following proposition:

Proposition 1.2. *(i) Λ is a non-negative essentially self-adjoint operator in $\mathscr{L}^2(Y)$; $\mathsf{Ker}(\Lambda)$ consists of constant functions.*

(ii) Λ has a discrete accumulating to infinity spectrum with eigenvalues $0 = \lambda_0 < \lambda_1 \le \dots$ could be obtained recurrently from the following variational problem:

$$\int_X |\nabla w|^2 \, dx \mapsto \mathsf{min}(=\lambda_n) \quad as \quad \int_Y |w|^2 \, dx' = 1, \qquad \int_Y w w_k^\dagger \, dx' = 0 \quad for \quad k = 0, \dots, n-1. \tag{1.3}$$

Corollary 1.3. *The number of eigenvalues of Λ, which are less than λ, equals to the maximal dimension of the linear space of $\mathscr{C}^2$-functions, on which the quadratic form*

$$\int_X |\nabla w|^2 \, dx - \lambda \int_Y |w|^2 \, dx' \tag{1.4}$$

is negative definite.

Proposition 1.4. *Operator Λ has a domain $\mathscr{H}^1(Y)$ and*

$$\|\Lambda u\|_Y + \|u\|_Y \asymp \|u\|_{\mathscr{H}^1(Y)}, \tag{1.5}$$

where $(.,.)$ and $\|.\|$ denote $\mathscr{L}^2$ inner product and norm.

[2)] Defined via quadratic forms.

Proof. Let $L = \ell \cdot \nabla$, ℓ be a vector field which makes an acute angle with the inner normal (at Z–with both inner normals). Consider

$$0 = -(\Delta w, Lw)_X = (\nabla w, \nabla Lw)_X + (\partial_\nu w, Lw)_Y = \int Q(\nabla w)\, dy + O(\|w\|^2_{\mathscr{H}^1(X)}), \tag{1.6}$$

where

$$Q(\nabla w) = (\nu \cdot \nabla w)(\ell \cdot \nabla w) - \frac{1}{2}\nu \cdot \ell |\nabla w|^2. \tag{1.7}$$

This quadratic form has one positive and d negative eigenvalues. Further, on the subspace orthogonal to ℓ, all eigenvalues are negative. Then

$$\|\partial_\nu w\|^2 + C\|w\|^2_{\mathscr{H}^1(X)} \asymp \|w\|^2_{\mathscr{H}^(Y)}. \tag{1.8}$$

Combined with the estimate for $\|w\|^2_{\mathscr{H}^1(X)} \le C\|w\|^2_{\mathscr{H}^{\frac{1}{2}}(Y)}$ it implies the statement. □

Remark 1.5. (i) If Y is infinitely smooth, then Λ is the first-order pseudodifferential operator on Y with the principal symbol $(g_Y(x,\xi))^{1/2}$, where g_Y is the restriction of the metrics to Y. Then the standard results hold:

$$\mathsf{N}(\lambda) = \kappa_0 \lambda^d + O(\lambda^{d-1}) \qquad \text{as} \quad \lambda \to +\infty \tag{1.9}$$

with the standard coefficient $\kappa_0 = (2\pi)^{-d}\omega_d \,\mathsf{mes}(Y)$, where $\mathsf{mes}(Y)$ means d-dimensional volume of Y, ω_d is the volume of the unit ball in $\mathbb{R}^d$. We also can get two-term asymptotics with the same remainder estimate for $\mathsf{N}(\lambda) * \lambda_+^{r-1}$, $0 < r \le 1$.

(ii) Moreover, if the set of all periodic geodesics of Y has measure 0, then

$$\mathsf{N}(\lambda) = \kappa_0 \lambda^d + \kappa_1 \lambda^{d-1} + o(\lambda^{d-1}) \qquad \text{as} \quad \lambda \to \infty. \tag{1.10}$$

We also can get two-term asymptotics (three-term for $r = 1$) with the same remainder estimate for $\mathsf{N}(\lambda) * \lambda_+^{r-1}$, $0 < r \le 1$. The same asymptotics, albeit with a larger number of terms, hold for $r > 1$.

(iii) "Regular" singularities of the dimension $< (d-1)$ (like conical points in 3D) do not cause any problems for asymptotics of $\mathsf{N}(\lambda)$—we can use a rescaling technique to cover them; moreover, in the framework of this paper they would not matter even combined with edges (like vertices in 3D).

2 Dirichlet-to-Neumann Operator

2.1 Toy-Model: Dihedral Angle

Let $Z = \mathbb{R}^{d-1}$ with the Euclidean metrics, $X = \mathcal{X} \times Z$, $Y = \mathcal{Y} \times Z$, where $\mathcal{X}$ is a planar angle of solution α, $0 < \alpha \le 2\pi$, $\mathcal{Y} = \mathcal{Y}_1 \cup \mathcal{Y}_2$, $\mathcal{Y}_j$ are rays (see Figure .2).

Then one can identify Y with $\mathbb{R}^d$ with coordinates (s, z), where $z \in Z$ and

- $s = \mathsf{dist}(y, Z)$ for for a point $y \in Y_1 = \mathcal{Y}_1 \times Z$,
- $s = -\mathsf{dist}(y, Z)$ for for a point $y \in Y_2 = \mathcal{Y}_2 \times Z$.

Then we have a Euclidean metrics and a corresponding positive Laplacian Δ_Y on Y.

Remark 2.1. (i) We can consider any angle $\alpha > 0$, including $\alpha > 2\pi$ (in which case X could be defined in the polar coordinates, but then we need to address some issues with the domain of operator).

(ii) If $\alpha = \pi$, then $\Lambda = \Delta_Y^{1/2}$.

(iii) We say that X is a *proper angle* if $\alpha \in (0, \pi)$ and that X is a *improper angle* if $\alpha \in (\pi, 2\pi)$. We are not very concerned about $\alpha = \pi,\ 2\pi$ since these cases will be forbidden in the general case.

For this toy-model we can make a partial Fourier transform $F_{z\to\zeta}$ and then study equation in the planar angle:

$$\Delta_2 w + w = 0, \tag{2.1}$$

where Δ_2 is a positive 2D-Laplacian and we made also a change of variables $x'' \mapsto |\zeta| \cdot x''$, $x'' = (x_1, x_2)$. Denote by $\bar{J}$ and $\bar{\Lambda}$ operators J and Λ for (2.1). This problem is extensively studied in Appendix A.1.

Then we can use the separation of variables. Singularities at the vertex for solutions to (2.1) and $w|_Y = 0$, are the same as for $\Delta_2 w = 0$, $w|_Y = 0$ and they are combinations of $r^{\pi n/\alpha} \sin(\pi n\theta/\alpha)$ with $n = 1, 2, ...$, where $(r, \theta) \in \mathbb{R}^+ \times (0, \alpha)$ are polar coordinates.

This show the role of α: if $\alpha \in (0, \pi)$ those functions are in $\mathscr{H}^{\sigma}_{\mathsf{loc}}(\mathcal{X})$ with $\sigma < 1 + \pi n/\alpha$, and $\partial_\nu w|_Y$ belong to $\mathscr{H}^{\sigma-3/2}_{\mathsf{loc}}(\mathcal{Y})$.

One can prove easily the following Propositions 2.2 and 2.3 below:

Proposition 2.2. *The following are bounded operators*

$$\Delta_{\mathsf{D}}^{-1} : \mathscr{H}^{\sigma}(X) \to \mathscr{H}^{\sigma+2}(X), \tag{2.2}$$

$$J : \mathscr{H}^{\sigma+\frac{3}{2}}(Y) \to \mathscr{H}^{\sigma+2}(X), \tag{2.3}$$

$$\Lambda : \mathscr{H}^{\sigma+\frac{3}{2}}(Y) \to \mathscr{H}^{\sigma+\frac{1}{2}}(X), \tag{2.4}$$

where Δ_{D} *is an operator* Δ *with zero Dirichlet boundary conditions on* Y *and*

- $\sigma \in [-\frac{1}{2}, 0]$, *if* $\alpha \in (0, \pi)$, *and*
- $\sigma \in [-\frac{1}{2}, \bar{\sigma})$ *with* $\bar{\sigma} = \pi/\alpha - 1$ *otherwise.*

Proposition 2.3. *For equation* (2.1) *in* $\mathcal{X}$

$$\bar{\Lambda} - (D_s^2 + 1)^{1/2} = \sum_{j+k\le 1} D_s^j \bar{K}_{jk} D_s^k\,, \tag{2.5}$$

where operators $\bar{K}_{jk}$ *have Schwartz kernels* $\bar{K}_{jk}(s, s')$ *such that*

$$|D_s^p D_{s'}^q \bar{K}_{jk}(s, t')| \le C_{pqm} |s|^{-(\bar{\sigma}-p)_-} |s'|^{-(\bar{\sigma}-q)_-} (|s| + |s'|)^{-p-q+(\bar{\sigma}-p)_-+(\bar{\sigma}-q)_-} (|s| + |s'| + 1)^m \tag{2.6}$$

and $l_\pm := \max(\pm l, 0)$ *and* m *is arbitrarily large.*

Then

Corollary 2.4. *For the toy-model in* X

$$\Lambda - \Delta_Y^{1/2} = \sum_{j+k\le 1} D_s^j K_{jk} D_s^k\,, \tag{2.7}$$

where operators K_{jk} *have Schwartz kernels*

$$K_{j,k}(x', s; y', s') = (2\pi)^{1-d} \iint |\xi'|^{2-j-k} \bar{K}_{jk}(s|\xi'|,\, s'|\xi'|) e^{-i\langle x'-y', \xi'\rangle}\, d\xi'. \tag{2.8}$$

2.2 General Case

Consider now the general case. In this case we can again introduce coordinate s on Y and consider Y as a Riemannian manifold, but with the metrics which is only $\mathscr{C}^{0,1}$ (Lipschitz class); more precisely, it is $\mathscr{C}^{\infty}$ on both Y_1 and Y_2, but the first derivative with respect to s may have a jump on Z. It does not, however, prevent us from introduction of Δ_Y and therefore $\Delta_Y^{\frac{1}{2}}$, but the latter would not be necessarily the classical pseudodifferential operator.

We want to exclude the degenerate cases of the angles π and 2π. So, let us assume that

(2.9) $Z = \{x\colon x_1 = x_2 = 0\}$ and $X = Z \times \mathcal{X}$ with a planar angle $\mathcal{X} \ni (x_1, x_2)$, disjoint from half-plane and the plane with a cut.

Definition 2.5. For $z \in Z$ let $\alpha(z)$ be an internal angle between two leaves of Y at point z (calculated in the corresponding metrics). Due to our assumption either $\alpha(z) \in (0, \pi)$ or $\alpha(z) \in (\pi, 2\pi)$. Let Z_j be a connected component of Z.

(i) Z_j is a *inner edge* if $\alpha(z) \in (0, \pi)$ on Z_j, and

(ii) Z_j is an *outer edge* if $\alpha(z) \in (\pi, 2\pi)$ on Z_j.

One can prove easily

Proposition 2.6. *The following are bounded operators*

$$\Delta_{\mathsf{D}}^{-1} : \mathscr{H}^{\sigma}(X) \to \mathscr{H}^{\sigma+2}(X), \tag{2.10}$$

$$J : \mathscr{H}^{\sigma+\frac{3}{2}}(Y) \to \mathscr{H}^{\sigma+2}(X), \tag{2.11}$$

$$\Lambda : \mathscr{H}^{\sigma+\frac{3}{2}}(Y) \to \mathscr{H}^{\sigma+\frac{1}{2}}(X), \tag{2.12}$$

where Δ_{D} *is an operator* Δ *with zero Dirichlet boundary conditions on* Y *and*

(i) $\sigma \in [-\frac{1}{2}, 0]$, *if* $\alpha(z) \in (0, \pi)$ $\forall z \in Z$, *and*

(ii) $\sigma \in [-\frac{1}{2}, \bar{\sigma})$ *with* $\bar{\sigma} = \pi/\bar{\alpha} - 1$, $\bar{\alpha} = \max_{z\in Z} \alpha(z)$ *otherwise.*

One can also prove easily

Proposition 2.7. *In the general case, assuming that* $Z = \{x\colon x_1 = x_2 = 0\}$ *and* $X = Z \times \mathcal{X}$ *with a planar angle* $\mathcal{X} \ni (x_1, x_2)$ *of solution* $\in (0,\pi)\cup(\pi, 2\pi)$

$$\Lambda - \Delta_Y^{1/2} = b + \sum_{j+k\le 1} D_s^j K_{jk} D_s^k\,, \tag{2.13}$$

where b *is a bounded operator and operators* K_{jk} *have Schwartz kernels and*

$$K_{j,k}(x', s; y', s') = (2\pi)^{1-d} \iint |\xi'|^{2-j-k} \bar{K}_{jk}\bigl(\frac{1}{2}(x'+y'), s|\xi'|,\, s'|\xi'|\bigr)\, e^{-i\langle x'-y', \xi'\rangle}\, d\xi'. \tag{2.14}$$

Remark 2.8. On the distances $\gtrsim 1$ from Z, b is a classical 0-order pseudodifferential operator, on the distance $\gtrsim |\xi'|^{-1+\delta}$ it is a rough 0-order pseudodifferential operator[3)].

3 Microlocal Analysis

3.1 Propagation of Singularities near Edge

We are going to consider microlocal analysis near point $(\bar{x}, \bar{\xi}'') \in T^*Z$ under assumption (2.9). In our approach we use definition of operator Λ rather than its description of the previous Section 2. So, let $x = (x''; x') \in \mathbb{R}^2 \times \mathbb{R}^{d-1}$.

Proposition 3.1. *(i) Let* $q_j = q_j(\xi')$ *(*$j = 1, 2$*) be two symbols, constant as* $|\xi'| \ge C$*. Assume that the* $\mathsf{dist}(\mathsf{supp}(q_1), \mathsf{supp}(q_2)) \ge \epsilon$.

Consider h*-pseudodifferential operators* $Q_j = q_j^{\mathsf{w}}(h^{-1}D')$, $j = 1, 2$. *Then the operator norms of*

$$(3.1)_{1,2} \qquad Q_1\Delta_{\mathsf{D}}^{-1}Q_2 : \mathscr{L}^2(X) \to \mathscr{H}^2(X), \qquad Q_1 J Q_2 : \mathscr{H}^{\frac{1}{2}}(Y) \to \mathscr{H}^2(X),$$

$$(3.1)_3 \qquad Q_1 \Lambda Q_2 : \mathscr{H}^1(Y) \to \mathscr{L}^2(Y)$$

do not exceed $C'h^s$ *with arbitrarily large* s *where* Δ_{D} *is an operator* Δ *with zero Dirichlet boundary conditions on* Y.

(ii) Let $Q_j(x')$ *(*$j = 1, 2$*) be two functions. Then operators* $(3.1)_{1-3}$ *are infinitely smoothing by* x'.

[3)] I. e. with the symbols such that $|D_x^\alpha D_\xi^\beta| \le C_{\alpha\beta}\rho^{-|\beta|}\gamma^{-|\alpha}$ with $\rho\gamma \ge h^{1-\delta}$, $\rho\gamma \ge h^{1-\delta}$. Here $\rho = 1$, $\gamma = |x|$.

Proof. (i) Without any loss of the generality one can assume that q_j are constant also in the vicinity of 0. Then the operator norms of $[Q_j, \Delta]\Delta_{\mathsf{D}}^{-1}$ in $\mathscr{L}^2(X)$ do not exceed Ch; replacing Q_j by $Q_j^{(n)}$ with $Q_j^{(0)} = Q_j$ and $Q_j^{(n)} := [Q_j^{(n-1)}, \Delta]\Delta_{\mathsf{D}}^{-1}$ for $j = 1, 2, ...$, we prove by induction that the operator norms of $Q_j^{(n)}$ in in $\mathscr{L}^2(X)$ do not exceed Ch^n. Then one can prove by induction easily that the operator norm of $(3.1)_1$ does not exceed Ch^s.

Then one can prove easily that the operator norm of $(3.1)_{2,3}$ do not exceed Ch^s as well. It concludes the proof of Statement (i).

(ii) Statement (ii) is proven by the same way. □

Let $u(x, y, t)$ be Schwartz kernel of $e^{it\Lambda}$, $x, y \in Y$.

Proposition 3.2. *Consider h-pseudodifferential operator $Q = q^{\mathsf{w}}(x', h^{-1}D')$ where q vanishes $\{|\xi'| \le c_0\}$. Let $\chi \in \mathscr{C}_0^\infty(\mathbb{R})$, $T \ge h^{1-\delta}$. Then operator norms of $F_{t\to\tau}\chi_T(t)Q_x u$ and $F_{t\to\tau}\chi_T(t)u\,{}^tQ_y$ do not exceed $C_T' h^s$ for $\tau \le c$ for $c_0 = c_0(c)$.*

Proof. One need to consider $v = e^{it\Lambda}f$, $f \in \mathscr{H}^1(Y)$, $\|f\|_{\mathscr{L}^2(Y)} = 1$ and observe that it satisfies $(D_t - \Lambda)v = 0$. Using (1.5) we see that operator $(D_t - V)$ is elliptic in $\{|\xi'| \ge c_0, \tau \le c\}$ while Proposition 3.1 ensures its locality. □

Therefore, in what follows

Remark 3.3. Studying energy levels $\tau \le c$ we can always apply cut-out domain $\{|\xi'| \ge c_0\}$.

Now we can study the propagation of singularities. Let us prove that the propagation speed with respects to x and ξ' do not exceed C_0. For this and other our analysis we need the following Proposition 3.4:

Proposition 3.4. *For h-pseudodifferential operator $Q = q^{\mathsf{w}}(x, hD')$ the following formula connecting commutators $[\Delta, Q]$ and $[\Lambda + \partial_\nu, Q]$ holds:*

$$-\operatorname{Re} i([\Delta, Q]Jv, Jv)_X = \operatorname{Re} i(([\Lambda, Q] + [\partial_\nu, Q])v, v)_Y \tag{3.2}$$

Proof. First, consider real valued symbol $q = q(x, \xi')$ and $Q = q^{\mathrm{w}}(x, hD')$ its Weyl quantization. Let v denote just any function on Y and V its continuation as a harmonic function. Then for $w = Jv$

$$\begin{aligned} 0 = (Q\Delta w, w)_X = (\Delta Q w, w) - ([\Delta, Q]w, w)_X = \\ - ([\Delta, Q]w, w)_X + (Qw, \Delta w)_X - (\partial_\nu Qw, w)_Y + (Qw, \partial_\nu w)_Y = \\ - ([\Delta, Q]w, w)_X - (Q\partial_\nu w, w)_Y - ([\partial_\nu, Q]w, w)_Y + (Qw, \partial_\nu w)_Y = \\ - ([\Delta, Q]w, w)_X + (Q\Lambda v, v)_Y - ([\partial_\nu, Q]v, v)_Y - (v, Q\Lambda v)_Y = \\ - ([\Delta, Q]w, w)_X - (\Lambda Q v, v)_Y - ([\partial_\nu, Q]v, v)_Y, \end{aligned}$$

which implies (3.2). □

Now we can prove that at energy levels $\tau \le c$ the propagation speed with respects to x and ξ' do not exceed $C_0 = C_0(c)$.

Proposition 3.5 [4]**.** *Let $Q_j = q_j^{\mathrm{w}}(x, hD')$ and $\mathsf{dist}(\mathsf{supp}(q_1), \mathsf{supp}(q_2)) \ge C_0 T$ with fixed $T > 0$. Let $\chi \in \mathscr{C}_0^\infty([-1, 1])$. Then for $\tau \le c$*

$$|F_{t\to h^{-1}\tau}\big(\chi_T(t) Q_{1x} u\, {}^t Q_{2y}\big)| \le C h^m, \tag{3.3}$$

where here and below m is an arbitrarily large exponent and $C = C_m$.

Proof. (i) The proof is the standard one for propagation with respect to (x', ξ'): we consider $\phi(x', \xi', t)$ and prove that under the microhyperbolicity condition

$$\phi_t - \{|\xi'|, \phi\} \ge \epsilon_0, \tag{3.4}$$

which is equivalent to

$$2\phi_t - |\xi'|^{-1}\{|\xi'|^2, \phi\} \ge 2\epsilon_0 |\xi'|, \tag{3.5}$$

our standard propagation theorem (see Theorem 2.1.2 of [Ivr1]) holds, just repeating arguments of its proof, using equality (3.2) and the fact that $\|Jv\|_{\mathscr{H}^{1/2}(X)} \asymp \|v\|_{\mathscr{L}^2(Y)}$.

Then we plug $\phi(x', \xi', t) = \psi(x', \xi') - t$ with $|\nabla_{x', \xi'}\psi| \le \epsilon_0$, and prove that (3.3) for $q_j = q_j(x', \xi')$.

[4] Cf. Theorem 3.1(i) of [Ivr1].

(ii) We need also prove that the propagation speed with respect to (x_1, x_2) [5] does not exceed C_0, but it is easy since for $|s| \geq \epsilon$, Λ is a first-order pseudodifferential operator with the symbol $|\xi|$. □

Remark 3.6. In fact, it follows from the proof, that the propagation speed with respect to x' do not exceed C_0, and the propagation speed with respect to ξ' does not exceed $C_0|\xi'|$ with C_0, which does not depend on restriction $\tau \leq c$. Meanwhile, the propagation speed with respect (x_1, x_2) does not exceed 1.

Next we prove that at energy levels $\tau = 1$ the propagation speed with respects to x' in the vicinity of $(0, \bar{\xi}')$ with $|\bar{\xi}'| \geq \epsilon_0$ is at least $\epsilon_1 = \epsilon_1(\epsilon_0)$.

Proposition 3.7 [6]**.** *Let* $Q_j = q_j^{\mathsf{w}}(x, hD')$ *and*

$$\mathsf{dist}_{x'}(\mathsf{supp}(q_1), \mathsf{supp}(q_2)) \leq \epsilon_1 T$$

with fixed $T > 0$. *Let* $\chi \in \mathscr{C}_0^\infty([-1, -\frac{1}{2}] \cup [-\frac{1}{2}, -1])$. *Then for* $|\tau - 1| \leq \epsilon_0$ (3.3) *holds.*

Proof. After propagation theorem mentioned in the proof of Proposition 3.5 is proven we just plug $\phi(x', \xi'.t) = \psi(x', \xi') - \epsilon t$ with $\xi' \cdot \nabla_{x'}\psi \geq 1$. □

Corollary 3.8 [7]**.** *In the framework of Proposition 3.5 consider* $|\tau - 1| \leq \epsilon$. *Then*

$$|F_{t\to h^{-1}\tau}\Gamma_x\chi_T(t)u\,{}^tQ_y| \leq Ch^{1-d+m}T^{-m} \tag{3.6}$$

and

$$|F_{t\to h^{-1}\tau}\Gamma_x\big(\bar{\chi}_{T'}(t) - \bar{\chi}_T(t)\big)u\,{}^tQ_y| \leq Ch^{1-d+m}T^{-m}, \tag{3.7}$$

provided $\chi \in \mathscr{C}_0^\infty([-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1])$, $\bar{\chi} \in \mathscr{C}_0^\infty([-1, 1])$, $\bar{\chi} = 1$ *on* $[-\frac{1}{2}, \frac{1}{2}]$, $h \leq T \leq T' \leq T_0$ *with small constant* T_0.

Proof. For small constant T (3.6) follows directly from Proposition 3.7, after one proves easily that we can insert $D_{x''}$, $D_{y''}$ to the corresponding estimate, which is easy.

For $h \leq T \leq T_0$ we use just rescaling like in the proof of Theorem 2.1.19 of [Ivr1]. Finally, (3.7) is obtained by the summation with respect to partition of unity with respect to t. □

[5] Or, equivalently, with respect to s.

[6] Cf. Theorem 3.1(ii) of [Ivr1].

[7] Cf. Corollary 2.2(ii) of [Ivr1].

This implies immediately

Corollary 3.9. $\mathsf{N}_h(\tau)$ *and* $\mathsf{N}_h(\tau)*\tau_+^{\sigma-1}$ *are approximated by the corresponding Tauberian expressions with* $T \asymp h^{1-\delta}$ *with errors* $O(h^{1-d})$ *and* $O(h^{1-d+\sigma})$ *respectively (as* $\tau = 1$ *and* $h \to +0$*).*

3.2 Reflection of Singularities from the Edge

The results of the previous subsubsection are sufficient to prove *sharp spectral asymptotics* (with the remainder estimate $O(\lambda^{d-1})$), which do not require conditions of the global nature, but insufficient to prove sharper spectral asymptotics (with the remainder estimate $o(\lambda^{d-1})$), which require conditions of the global nature.

For this more ambitious purpose we need to prove that the singularities propagate along geodesic billiards on the boundary Y, reflecting and refracting on the edge Z (so billiards will be branching), and the typical singularity (with $|\xi'| < \tau$) does not stick to Z.

To do this we will follow arguments of Subsubsection 4 of [Ivr1]. Assuming (2.9) consider operator $Q = x_1D_1 + x_2D_2 - i/2$, which acts along Y. As an operator in $\mathscr{L}^2(Y)$ it is self-adjoint, as an operator in $\mathscr{L}^2(X)$ it is not, but differs from a self-adjoint operator $Q = x_1D_1 + x_2D_2 - i$ by $i/2$, which does not affects commutators.

As a result, repeating the proof of Proposition 3.4 we arrive to

(3.8) Under assumption (2.9) equality (3.2) also holds for the operator $Q = x_1D_1 + x_2D_2 - i$.

To apply arguments of the proof of Propositions 4.2 and then 4.3 of [Ivr1], we need to check, if operator $i[\Lambda, Q]$ is positive definite, which in virtue of (3.2) is equivalent to the same property for the form in the left:

$$\mathsf{Re}(i[\Delta, Q]w, w) - \mathsf{Re}(i[\partial_\nu, Q]w, w)_Y \ge \epsilon\|\nabla w\|^2 \quad \text{for} \quad w\colon \Delta = 0. \tag{3.9}$$

For the toy-model $i[\Delta, Q] = 2(D_1^2 + D_2^2)$, $i[\partial_\nu, Q] = -\partial_\nu$, and the form on the left coincides with $\|\nabla w\|^2 - 2\|\nabla' w\|^2$ on w in question and therefore after Fourier transform $F_{x'\to\zeta}$ and change of variables $x_{1,2}$ it boils down to the inequality

$$\|\nabla w\|^2 - \|w\|^2 \ge \epsilon(\|\nabla w\|^2 + \|w\|^2) \qquad \text{for} \quad w\colon \Delta_2 w + w = 0 \tag{3.10}$$

for two-dimensional Δ_2, norms and scalar products.

This inequality is explored in Appendix A.1, and in virtue of Proposition A.11 (3.10) holds for $\alpha \in (\pi, 2\pi)$. Meanwhile due to Proposition A.16 (3.10) fails for $\alpha \in (0, \pi)$.

Therefore we arrive to

Proposition 3.10 [8]**.** *Consider two-dimensional toy-model (planar angle) with $\alpha \in (\pi, 2\pi)$.*

Let $\psi \in \mathscr{C}_0^\infty([-1,1])$, $\psi_\gamma(x) = \psi(x/\gamma)$ and $\phi \in \mathscr{C}_0^\infty([-1,1])$, $\tau \ge 1 + \epsilon_0$. Then as $\gamma \ge h^{1-\delta}$, $T \ge C_0\gamma$, $h^\delta \ge \eta \ge h^{1-\delta}T^{-1}$

$$\|\phi(\eta^{-1}(hD_t - \tau))\psi_\gamma e^{it\Lambda}\psi_\gamma|_{t=T}\| \le CT^{-1}\gamma + Ch^{\delta'}. \tag{3.11}$$

Proof. Proof follows the proof of Proposition 4.2 of [Ivr1] with $m = 1$, and uses equality (3.2) to reduce calculation of the commutator $[\Lambda, Q]$ to the calculation of the commutator $[\Delta, Q]$. □

Proposition 3.11 [9]**.** *Consider $(d+1)$-dimensional toy-model (dihedral angle) with $\alpha \in (\pi, 2\pi)$.*

Let $\psi \in \mathscr{C}_0^\infty([-1,1])$, $\psi_\gamma(x) = \psi(x_1/\gamma)$, $\phi \in \mathscr{C}_0^\infty([-1,1])$, $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^{d-1})$ supported in $\{|\xi'| \le 1 - \epsilon\}$ with $\epsilon > 0$. Finally, let $\gamma \ge h^{1-\delta}$, $T \ge Ch^{-\delta}\gamma$, $h^\delta \ge \eta \ge h^{1-\delta}T^{-1}$. Then

$$\|\phi(\eta^{-1}(hD_t - 1))\varphi(hD')\psi_\gamma(x_1)e^{it\Lambda}\psi_\gamma(x_1)|_{t=T}\| = O(h^m) \tag{3.12}$$

with arbitrarily large m.

Proof. Proof follows the proof of Proposition 4.3 of [Ivr1] with $m = 1$. □

Now we can consider the general case. Consider a point $\bar z = (\bar x, \bar\xi') \in T^*Z$, $|\bar x'| < 1$.

We can raise to points $\bar z^\pm = (\bar x, \bar\xi^\pm) \in T^*Y_{1,2}|_Z$ with $|\bar\xi^\pm| = 1$ and $\iota\bar z^\pm = \bar z$, where $\iota(x,\xi) = (x,\xi') \in T^*Z$ for $(x,\xi) \in T^*Y|_Z$.

Consider geodesic trajectories $\Psi_t(\bar z^\pm)$, going from $\bar z^\pm)$ into $T^*Y_{1,2}$ for $t < 0$, $|t| < \epsilon$; this distinguishes these two points.

We also can consider geodesic trajectories $\Psi_t(\bar z^\pm)$, going from $\bar z^\mp$ into $T^*Y_{1,2}$ for $t > 0$, $|t| < \epsilon$.

Let $\iota^{-1}\bar z = \{\bar z^+, \bar z^-\}$ and let $\Psi_t(\iota^{-1}\bar z)$ be obtained as a corresponding union as well[10]. So, for such point $\bar z$ $\Psi_t(\iota^{-1}z)$ with $t < 0$ consists of two

[8] Cf. Proposition 4.2 of [Ivr1].

[9] Cf. Proposition 4.3 of [Ivr1].

[10] So we actually restrict ι to $S^*Y|_Z$ and ι^{-1} to B^*Z.

incoming geodesic trajectories, $\Psi_t(\iota^{-1}z)$ with $t<0$ consists of two outgoing geodesic trajectories. Similarly, for $z\in (T^*(Y\setminus Z))$ we can introduce $\Psi_t(z)$: when trajectory hits Z it branches.

Theorem 3.12 [11]**.** *Consider a point $z=(x,\xi)\in T^*Y$, $|\xi|=1$. Consider a (branching) geodesic trajectory $\Psi_t(z)$ with $\pm t\in[0,T]$ (one sign only) with $T\ge\epsilon_0$ and assume that for each t indicated it meets ∂X transversally i.e.*

$$\mathsf{dist}(\pi_x\Psi_t(x,\xi),\partial X)\le\epsilon \implies |\frac{d}{dt}\mathsf{dist}(\pi_x\Psi_t(x,\xi),\partial X)|\ge\epsilon \qquad \forall t:\pm t\in[0,mT]. \tag{3.13}$$

Also assume that

$$\mathsf{dist}(\pi_x\Psi_t(x,\xi),\partial X)\ge\epsilon_0 \qquad \text{as}\ \ t=0,\ \pm t=T. \tag{3.14}$$

Let $\epsilon>0$ be a small enough constant, Q be supported in ϵ-vicinity of (x,ξ) and $Q_1\equiv 1$ in $C_0\epsilon$-vicinity of $\Psi_t(x,\xi)$ as $t=\pm T$. Then operator $(I-Q_1)e^{-it\mathsf{A}}Q$ is negligible as $t=\pm mT$.

Proof. Proof follows the proof of Theorem 4.4 of [Ivr1] with $m=1$. □

Adapting construction of the "dependence set" to our case, we arrive to the following

Definition 3.13. (i) The curve $z(t)$ in T^*Y is called a *generalized geodesic billiard* if a.e.

$$\frac{dz}{dt}\in K(z), \tag{3.15}$$

where

(a) $K(z)=\{H_g(z)\}$, $g(z)$ is a metric form, if $z\in T^*(Y\setminus Z)$,
(b) $K(z)=\{H_g(z')\colon z'\in\iota^{-1}\iota z\}$, if $z\in T^*Y|_Z$.

(ii) Let $\Psi_t(z)$ for $t\gtrless 0$ be a set of points $z'\in T^*Y$ such that there exists generalized geodesic billiard $z(t')$ with $0\lesseqgtr t'\lesseqgtr t$, such that $z(0)=z$ and $z(t)=z'$. Map $(z,t)\mapsto\Psi_t(z)$ is called a *generalized (branching) billiard flow*.

[11] Cf. Theorem 4.4 of [Ivr1].

(iii) Point $z \in T^*Y$ is *partially periodic* (with respect to Ψ) if for some $t \neq 0$ $z \in \Psi_t(z)$. Point $z \in T^*Y$ is *completely periodic* (with respect to Ψ) if for some $t \neq 0$ $\{z\} = \Psi_t(z)$

Then we arrive immediately to

Corollary 3.14. *Assume that Z consists only of only outer edges. Also assume that the set of all partially periodic points is zero.*

Then $\mathsf{N}_h(\tau)$ and $\mathsf{N}_h(\tau) * \tau_+^{r-1}$ are approximated by the corresponding Tauberian expressions with $T \asymp h^{1-\delta}$ with errors $o(h^{1-d})$ and $o(h^{1-d+r})$ respectively (as $\tau = 1$ and $h \to +0$).

Proof. Easy details are left to the reader. □

4 Main Results

4.1 From Tauberian to Weyl Asymptotics

Now we can apply the method of successive approximations as described in Section 7.2 of [Ivr1], considering an unperturbed operator

(a) As one in $\mathbb{R}^d$, with the metrics, frozen at point y, if $\mathsf{dist}(y, Z) \geq h^{1-\delta}$.

(b) As one in the dihedral edge, with the metrics, frozen at point $(y', 0)$, if $y = (y'; s)$ with $|s| = \mathsf{dist}(y, Z) \leq h^{1-\delta}$,

with the following modification: We calculate Λ also this way, applying successful approximations for both Δ, when we solve $\Delta w = 0$, $w|_Y = v$, and to ∂_ν, when we calculate $\partial_\nu w|_Y$.

Then we prove that for operator $h\Lambda$ the Tauberian expression $\mathsf{N}_h^{\mathsf{T}}(1)$ for $\mathsf{N}_h^-(1)$ with $T = h^{1-\delta}$ coincides modulo $O(h^m)$ with the (generalized) Weyl expression

$$\mathsf{N}_h^{\mathsf{W}} \sim \kappa_0 h^{-d} + \kappa_1 h^{1-d} + \dots, \tag{4.1}$$

with the standard coefficient κ_0 and with $\kappa_1 = \kappa_{1,Y\setminus Z} + \kappa_{1,Z}$, where $\kappa_{1,Y\setminus Z}$ is calculated in the standard way, for the smooth boundary, and

$$\kappa_{1,Z} = (2\pi)^{1-d}\omega_{d-1}\int_Z \varkappa(\alpha(y))\, dy\,, \tag{4.2}$$

$$\varkappa(\alpha) = \int_1^\infty \int_{-\infty}^\infty \lambda^{-d}\Big(e_\alpha(s,s,\lambda) - \pi^{-1}(\lambda-1)\Big)\, ds d\lambda\,, \tag{4.3}$$

$e_\alpha(s,s',\lambda)$ is a Schwartz kernel of the spectral projector of $\hat{\Lambda}$ in the planar angle of solution α and $\pi^{-1}(\lambda-1)$ is a corresponding Weyl approximation.

4.2 Main Theorems

Thus we arrive to the corresponding asymptotics for $N_h^-(1)$ and from them, obviously to asymptotics for $N(\tau)$:

Theorem 4.1. *Let Y be a compact manifold with edges. Then the following asymptotics hold as $\tau \to +\infty$:*

$$N(\tau) = \kappa_0 \tau^d + O(\tau^{d-1}) \tag{4.4}$$

and for $r > 0$

$$N(\tau) * \tau_+^{r-1} = \Big(\sum_{m<r} \kappa_m \tau^{d-r}\Big) * \tau_+^{r-1} + O(\tau^{d-1}). \tag{4.5}$$

Theorem 4.2. *Let Y be a compact manifold with edges. Assume that Z consists only of outer edges and that the set of all points, which are partially periodic with respect to the generalized billiard flow, has a measure 0. Then the following asymptotics hold as $\tau \to +\infty$:*

$$N(\tau) = \kappa_0 \tau^d + \kappa_1 \tau^{d-1} + o(\tau^{d-1}) \tag{4.6}$$

and for $r > 0$

$$N(\tau) * \tau_+^{r-1} = \Big(\sum_{m\le r} \kappa_m \tau^{d-r}\Big) * \tau_+^{r-1} + o(\tau^{d-1}). \tag{4.7}$$

4.3 Discussion

Remark 4.3. (i) Even for standard ordinary non-branching billiards, billiard flow Ψ_t could be multivalued. However, if through point $z \in T^*(Y \setminus Z)$ (where now Y is a manifold, and Z is its boundary) passes an infinitely long in the positive (or negative) time direction billiard trajectory, which always meets Z transversally, and each finite time interval contains a finite number of reflections, then $\Psi_t(z)$ for $\pm t > 0$ is single-valued. Points, which do not have such property, are called *dead-end points*. For ordinary billiards the set of dead-end points has measure zero.

(ii) For branching billiards (with velocities c_1, c_2) we can introduce the notion of the dead-end point as well: it is if at least one of the branches either meets Z non-transversally, or makes an infinite number of reflections on some finite time interval. As it was shown by Yu. Safarov and D. Vassiliev [SaVa], if c_1 and c_2 are not disjoint (our case!), the set of dead-end billiards could have positive measure.

Remark 4.4. (i) Checking non-periodicity assumption is difficult. But in some domains it will be doable. F.e. assume that $Y = Y_1 \cup Y_2$ globally is a domain of revolution, so Z is a $(d-1)$-dimensional sphere. Then the measure of the set of dead-end billiards is 0.

(ii) Assume that neither Y_1, nor Y_2 contains closed geodesics. Let $\varphi_j(\beta)$ be the length of the segment of geodesics in Y_j, with only ends on Z, where β is the reflection angle: Assume that $\varphi_j(\beta)$ are analytic and $\varphi_j(\beta) \to 0$ as

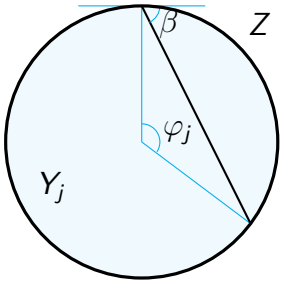


Figure 28.1: Trajectories on the manifold of revolution

$\beta \to +0$. Then the measure of the set of partially periodic billiards is 0.

Remark 4.5. Our arguments hold not only for compact X but also for $X \subset \mathbb{R}^{d+1}$ with the compact complement and with the metrics. stabilizing to Euclidean at infinity.

Remark 4.6. (i) In the next version of this paper we want to prove sharper asymptotics for domains with inner edges. To do this we need to understand, how singularities propagate near inner edges. One can prove that there are plenty of singularities, concentrated in $Z \times \mathbb{R} \ni (x,t)$ and $\{|\xi'| < \tau\}$. This is similar to the Rayleigh waves. And, we hope, exactly like Rayleigh waves, those singularities do not prevent us from the sharper asympotics.

What we need to prove is that the singularities in $\{|\xi'| < \tau\}$, coming from $Y \setminus Z$ transversally to Z, reflect and refract but leave Z instantly. In other words, that these two kinds of waves are completely separate. It is what I am trying to prove now.

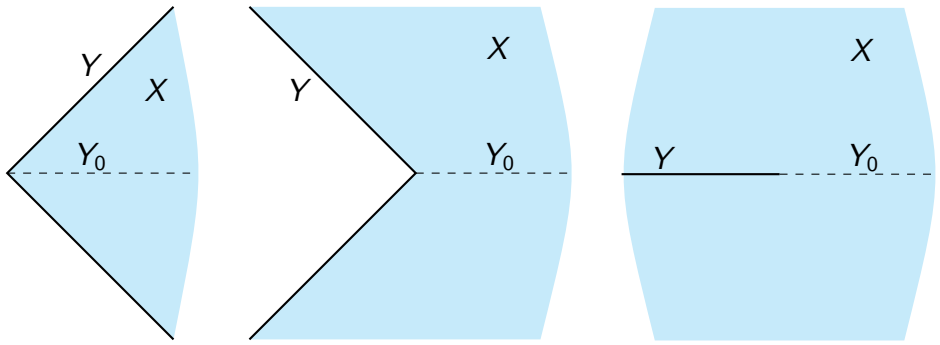
(ii) Let $\mathcal{K}$ be the linear span of the corresponding eigenfunctions. We need to prove that $\|\nabla w\| \geq \|w\|$ holds for $w = \hat{J}v$ with $v \in \mathcal{K}^{\perp}$. One can prove easily that $\|\nabla w\| = \|w\|$ for $w = \hat{J}v$ and eigenfunction v (Proposition A.16).

A Appendix

A.1 Planar Toy-Model

Preparatory Results

Here, in contrast to the whole article, $X = \{x \in \mathbb{R}^2, x_1 \geq |x_2| \operatorname{cot}(\alpha/2)\}$ is a planar angle of solution $\alpha \in (0, 2\pi]$ with a boundary $Y = Y_1 \cup Y_2$, $Y_{1,2} = \{x\colon x_1 = |x_2| \operatorname{cot}(\alpha/2),\ \pm x_2 < 0\}$ and a bisector $Y_0 = \{x\colon x_2 = 0, x_1 > 0\}$, and $\Delta = -\partial_1^2 - \partial_2^2$ is a positive Laplacian (so, for simplicity we do not write "hat").



(a) $\alpha \in (0, \pi)$ (b) $\alpha \in (\pi, 2\pi)$ (c) $\alpha = 2\pi$

Figure .2: Proper and improper angles

Remark A.1. (i) For $\alpha = \pi$ we have a regular half-plane $\{x\colon x_1 > 0\}$, and for $\alpha = 2\pi$ we have a plane with the cut $\{x\colon x_1 \leq 0,\ x_2 = 0\}$.

(ii) One can consider $\alpha > 2\pi$ on the covering of $\mathbb{R}^2$.

Consider real-valued[12] solutions of

$$Lw := (\Delta + 1)w = 0 \tag{A.1}$$

and operators J, Λ: $w = Jv$ solves (A.1) and $w|_Y = v$; $\Lambda v = -\partial_\nu w|_Y$ with $w = Jv$. Recall that ν is an inner normal to Y.

Observe that for any angle[13]

$$2\iint_X Lw \cdot w_{x_1}\, dx_1 dx_2 = \int_{\partial X} \Big((w_{x_1}^2 - w_{x_2}^2 - w^2)\nu_1 + 2w_{x_1}w_{x_2}\nu_2 \Big)\, dr, \tag{A.2}$$

where dr is a Euclidean measure on Y, and the then similar formula holds with x_1 and x_2 permuted. Then for solution of (A.1)

$$\int_{\partial X} \Big((w_{x_1}^2 - w_{x_2}^2)(\nu_1\ell_1 - \nu_2\ell_2) + 2w_{x_1}w_{x_2}(\nu_2\ell_1 + \nu_1\ell_2) - w^2(\nu_1\ell_1 + \nu_2\ell_2) \Big)\, dr = 0. \tag{A.3}$$

If on $\Gamma \subset \partial X$ $\ell_1 = \nu_1$, $\ell_2 = \nu_2$, then we can calculate invariantly as if $\ell_1 = \nu_1 = 0$, $\ell_2 = \nu_2 = 1$:

$$(w_{x_1}^2 - w_{x_2}^2)(\nu_1\ell_1 - \nu_2\ell_2) + 2w_{x_1}w_{x_2}(\nu_2\ell_1 + \nu_1\ell_2) - w^2(\nu_1\ell_1 + \nu_2\ell_2) = w_\nu^2 - w_r^2 - w^2, \tag{A.4}$$

where $w_r = \partial_r w$ and $w_\nu = \partial_\nu w$.

All these formulae hold not only for the original angle, but also for the smaller angle. Then let consider as X an upper half of the symmetric angle, $\partial X = Y_2 \cup Y_0$, on Y_0 the integrand is

$$\mathcal{I} := (w_{x_2}^2 - w_{x_1}^2 - w^2)\ell_2 + 2w_{x_1}w_{x_2}\ell_1 \tag{A.5}$$

with $\ell_1 = \sin(\alpha/2)$, $\ell_2 = -\cos(\alpha/2)$.

Consider different cases:

Antisymmetric case: $w|_{Y_0} = 0$, then $\mathcal{I} = -w_{x_2}^2 \cos(\alpha/2)$ and

$$\int_{Y_2} \big(w_\nu^2 - w_r^2 - w^2\big)\, dr - \cos(\alpha/2) \int_{Y_0} w_{x_2}^2\, dx = 0. \tag{A.6}$$

Symmetric case: $w_{x_2}|_{Y_0} = 0$, then $\mathcal{I} = (w_{x_1}^2 + w^2)\cos(\alpha/2)$ and

$$\int_{Y_2} \big(w_\nu^2 - w_r^2 - w^2\big)\, dr + \cos(\alpha/2) \int_{Y_0} (w_{x_1}^2 + w^2)\, dx_1 = 0. \tag{A.7}$$

[12] For complex-valued solutions then the main inequalities with w^2 replaced by $|w|^2$ follow automatically.

[13] Not necessary symmetric with respect to x_1-axis.

Proposition A.2. *Let w satisfy* (A.1). *Let either $\alpha \in (0, \pi]$ and w is antisymmetric, or $\alpha \in [\pi, 2\pi)$ and w is symmetric. Then*

$$\|\nabla w\|^2 \geq \|w\|^2. \tag{A.8}$$

Proof. In both cases $\int_{Y_2}(|\nabla w|^2 - w^2)\, dr \geq \int_{Y_2}(w_\nu^2 - w^2)\, ds \geq 0$. Applying this inequality to the angle, shifted by t along x_1, and integrating by $t \in (0, \infty)$, we obtain a double integral (divided by $\sin(\alpha/2)$).

Moreover, one can see easily, that this inequality is strict unless $w = 0$.

Similarly, if instead of multiplying by $(\nu_1 w_{x_1} + \nu_2 w_{x_2})$ we multiply by $(x_2 w_{x_1} - x_1 w_{x_2})$, then extra terms in the double integral will be $\pm w_{x_1} w_{x_2}$ and they cancel one another. However, on Y we get $x_2 = \nu_1 r$, $x_1 = -\nu_2 r$ and therefore contribution of Y_2 will be as in above with extra factor r:

$$\int_{Y_1} \left(w_\nu^2 - w_r^2 - w^2\right) r dr. \tag{A.9}$$

On Y_0 we get extra factor $x_1 = r$, but not $\nu_2 = -\cos(\alpha/2)$, and we arrive to

Antisymmetric case: $w|_{Y_0} = 0$, then $\mathcal{I} = w_{x_2}^2 x_1$ and

$$\int_{Y_2} \left(w_\nu^2 - w_r^2 - w^2\right) r dr + \int_{Y_0} w_{x_2}^2\, x_1 dx_1 = 0. \tag{A.10}$$

Symmetric case: $w_{x_2}|_{Y_0} = 0$, then $\mathcal{I} = (-w_{x_1}^2 - w^2)x_1$ and

$$\int_{Y_2} \left(w_\nu^2 - w_r^2 - w^2\right) r dr - \int_{Y_0} (w_{x_1}^2 + w^2)\, x_1 dx_1 = 0. \tag{A.11}$$

Let us explore dependence $\Lambda = \Lambda(\alpha)$ on α. Observe first that

$$\iint \left(\nabla w \cdot \nabla w' + w w'\right) dx d_1 dx_2 = \int Lw \cdot w' - \int_{\partial X} \partial_\nu w \cdot w'\, dr \tag{A.12}$$

where (r, θ) are polar coordinates and therefore dr is an Euclidean measure on Y. It implies

$$(\Lambda v, v')_Y = \iint \left(\nabla w \cdot \nabla w' + w w'\right) dx_1 dx_2, \tag{A.13}$$

for $w = Jv$, $w' = Jv'$. Therefore

(A.14) Λ is symmetric and nonnegative operator in $\mathscr{L}^2(Y)$.

Consider $X = X(\alpha)$, $Y = Y(\alpha)$, $\Lambda = \Lambda(\alpha)$ and keep w independent on α. Let us replace α by $\alpha + \delta\alpha$ etc. Then for a symmetric X we have $\delta v = -r(\partial_\nu w)\delta\alpha = \frac{1}{2} r(\Lambda v)\delta\alpha$ and it follows from (A.13) that

$$((\delta\Lambda)v, v)_Y + 2(\Lambda v, \delta v)_Y = \frac{1}{2}\int_Y \left(|\nabla w|^2 + |w|^2\right) r dr \times \delta\alpha$$

and therefore

$$((\delta\Lambda)v, v)_Y = -\frac{1}{2}\int_Y \left(w_\nu^2 - w_r^2 - |w|^2\right) r dr \times \delta\alpha. \tag{A.15}$$

Combining with (A.10) and (A.13) we arrive to

Proposition A.3. *(i) On symmetric functions $\Lambda(\alpha)$ is monotone increasing function of α.*

(ii) On antisymmetric functions $\Lambda(\alpha)$ is monotone inreasing function of α.

Let us identify Y with $\mathbb{R} \ni s$, $s = \mp r$ on $Y_{1,2}$ respectively.

Proposition A.4. *(i) On symmetric functions $\Lambda(\pi) = (D_s^2 + I)^{\frac{1}{2}}$.*

(ii) On antisymmetric functions $\Lambda(2\pi) \ge (D_s^2 + I)^{\frac{1}{2}}$.

Proof. Statement (i) is obvious. Statement (ii) follows from the fact that on antisymmetric function v $\Lambda(2\pi)v$ coincides with $\Lambda(\pi)v^0$, restricted to $\{x_1 < 0\}$, where v^0 is v, extended by 0 to $\{x_1 > 0\}$. □

Therefore, combining Propositions A.3 and A.4 we conclude that

Corollary A.5. *(i) On symmetric functions $\Lambda(\alpha) \ge (D_s^2 + I)^{\frac{1}{2}}$ for $\alpha \in [\pi, 2\pi]$.*

(ii) On antisymmetric functions $\Lambda(\alpha) \ge (D_s^2 + I)^{\frac{1}{2}}$ for $\alpha \in (0, 2\pi]$.

Remark A.6. One can prove easily, that inequalities are strict for $\alpha \in (\pi, 2\pi]$, $\alpha \in (0, 2\pi]$ respectively.

Now we want to finish general arguments and to prove inequality (A.8) for antisymmetric w and $\alpha \in (\pi, 2\pi]$. It will be more convenient to use polar coordinates (r, θ) and notations $\mathcal{Y}_\beta = \{(r, \theta) \colon : \theta = \beta\}$, $\mathcal{X}_{\beta_1,\beta_2} = \{(r, \theta) \colon : \beta_1 \le \theta \le \beta_2\}$. Here and below $\beta_* \in [-\alpha/2, \alpha/2]$.

Recall that

$$L = -\partial_r^2 - r^{-1}\partial_r - r^{-2}\partial_\theta^2 + 1. \tag{A.16}$$

Proposition A.7. *(i) Let w satisfy equation* (A.1) *in X. Then*

$$\mathcal{I}(\beta) := \int_{\mathcal{Y}_\beta} \Big[r^{-2} w_\theta^2 - w_r^2 - w^2 \Big] \, r dr \tag{A.17}$$

does not depend on β.

(ii) Therefore

$$\mathcal{J}(\beta_1, \beta_2) := \iint_{\mathcal{X}_{\beta_1,\beta_2}} \Big[r^{-2} w_\theta^2 - w_r^2 - w^2 \Big] \, r dr d\theta \tag{A.18}$$

depends only on $\beta_2 - \beta_1$ and therefore is proportional to it.

Proof. One proves (i) by analyzing $-\iint_{\mathcal{X}_{\beta_1,\beta_2}} Lw \cdot \partial_\theta w \, dxdy$ (which actually was done before, since $w_\theta = -x_2 w_{x_1} + x_1 w_{x_2}$.

To prove (ii) observe that $\partial_\beta \mathcal{J}(\beta_1, \beta) = \mathcal{I}(\beta)$. □

Proposition A.8. *(i) Function*

$$\mathcal{J}(\beta_1, \beta_2) := \int_{\mathcal{X}_{\beta_1,\beta_2}} w^2 r^{-1} \, dr d\theta \tag{A.19}$$

with fixed $\beta_{1,2} = \beta \mp \sigma$ is convex with respect to β (if $\sigma > 0$).

(ii) Further, if w is either symmetric or antisymmetric, then it reaches minimum as $\beta = 0$ (i.e. $\mathcal{X}_{\beta_1,\beta_2}$ is symmetric with respect to Y_0).

Proof. (i) Consider

$$0 = \iint_{\mathcal{X}_{\beta_1,\beta_2}} Lw \cdot w \, r dr d\theta = \iint_{\mathcal{X}_{\beta_1,\beta_2}} (w_r^2 + r^{-2} w_\theta^2 + w^2) \, r dr d\theta + \mathcal{I}'(\beta_1) - \mathcal{I}'(\beta_2) \tag{A.20}$$

with

$$\mathcal{I}'(\beta) = \int_{\mathcal{Y}_\beta} w w_\theta \, r^{-1} dr = \partial_\beta \mathcal{I}(\beta), \qquad \mathcal{I}(\beta) := \int_{\mathcal{Y}_\beta} w^2 \, r^{-1} dr. \tag{A.21}$$

Observe that the first term is positive. Then $\mathcal{I}'(\beta_2) - \mathcal{I}'(\beta_1) > 0$; on the other hand, it is the second derivative of $\mathcal{J}(\beta_1, \beta_2)$ with respect to β.

(ii) Moreover, for both symmetric and antisymmetric w $\mathcal{I}(\beta_2) - \mathcal{I}(\beta_1) = 0$. And the difference $\mathcal{I}(\beta_2) - \mathcal{I}(\beta_1) = 0$ for $\beta = 0$. □

Corollary A.9. *Since w_θ satisfies the same equation and is antisymmetric (symmetric) respectively, the same conclusions (i), (ii) hold for $\mathcal{J} := \int_{\mathcal{X}_{\beta_1,\beta_2}} w_\theta^2 r^{-1} \, r dr d\theta$.*

Then in virtue of Proposition A.7(ii) the same conclusions (i), (ii) hold for $\mathcal{J} := \iint_{\mathcal{X}_{\beta_1,\beta_2}} (w_r^2 + w^2) \, r dr d\theta$.

Next, observe that $Lr\partial_r w = 2\Delta w = -2w$ and if we use the same arguments, as in the proof of Proposition A.8(ii) for $r\partial_r w$, then instead of the first term in (A.20) we get

$$\iint_{\mathcal{X}_{\beta_1,\beta_2}} \big((r w_r)_r^2 + w_{r\theta}^2 + (r w_r)^2 - w^2\big) \, r dr d\theta, \tag{A.22}$$

where an additional last term appears as

$$\iint_{\mathcal{X}_{\beta_1\beta_2}} 2w\partial_r w \cdot r^2 dr d\theta = -\iint_{\mathcal{X}_{\beta_1\beta_2}} w^2 \, r dr d\theta.$$

Consider last two terms and skip integration by $d\theta$; plugging $w = r^{-3/2}u$ with $u(0) = 0$, we arrive to

$$\int \big(u_r - \frac{3}{2} r^{-1} u)^2 - r^{-2}u^2\big) \, dr = \int \big(u_r^2 - 3r^{-1}u_r u + \frac{5}{4} r^{-2}u^2\big) \, dr = \int \big(u_r^2 - \frac{1}{4} r^{-2}u^2\big) \, dr$$

which is again nonnegative term. Then we arrive to

Corollary A.10. *The same conclusions (i) and (ii) of Proposition A.8 hold for $\mathcal{J}(\beta_1, \beta_2)$ with w replaced by rw_r, i.e. $\mathcal{J}(\beta_1, \beta_2) := \iint_{\mathcal{X}_{\beta_1,\beta_2}} w_r^2 \, r dr d\theta$.*

Then in virtue of Proposition A.7(ii) the same conclusions (i), (ii) hold for

$$2\iint_{\mathcal{X}_{\beta_1,\beta_2}} |\partial_r w|^2\, r dr d\theta + \iint_{\mathcal{X}_{\beta_1,\beta_2}} (r^{-2}|\partial_\theta w|^2 - |\partial_r w|^2 - |w|^2|)\, r dr d\theta = \iint_{\mathcal{X}_{\beta_1,\beta_2}} (|\nabla w|^2 - |w|^2)\, dxdy. \tag{A.23}$$

Now we can prove

Proposition A.11. *Let $\alpha \in (\pi, 2\pi]$. Then for both symmetric and antisymmetric w* (A.8) *holds.*

Proof. Indeed, assume that it is not the case: $\iint_X (|\nabla w|^2 - w^2)\, dxdy < 0$ for some w. Then due to Corollary A.10 the same is true for X replaced by $\mathcal{X}_{\beta_1,\beta_2}$ with $\beta_{1,2} = \mp(2\pi - \alpha)/2$ and the same w. Then it is true for the sum of these to expressions (with $X == X_{-\alpha/2,\alpha/2}$ and $\mathcal{X}_{\beta_1,\beta_2}$), which is the sum of the same expressions for the half-planes $X_{\alpha/2-\pi,\alpha/2}$ and $X_{-\alpha/2,\pi-\alpha/2}$. However, for half-planes (A.8) holds. □

Let $P_\tau = \theta(\tau - \Lambda)$.

Proposition A.12. *(i) Let $\alpha \in [\pi, 2\pi]$. Then for any $\tau > 1$ for $w = Jv$, $v \in \mathsf{Ran}(I - P_\tau)$,*

$$\|\nabla w\|^2 \ge (1 + \delta)\|w\|^2 \tag{A.24}$$

with $\delta = \delta(\tau) > 0$.

(ii) Let $\alpha \in (0, \pi]$. Then for any $\tau > 1$ for antisymmetric $w = Jv$, $v \in \mathsf{Ran}(I - P_\tau)$, (A.24) *holds.*

Proof. Observe first that

$$\{v \in \mathsf{Ran}(I - P_\tau)\colon \|v\|_Y = 1,\ :\ \|\nabla Jv\|^2 \le (1+\delta')\|Jv\|^2\} \text{ is a compact set in } \mathscr{L}^2(Y) \text{ for } \delta' = \delta'(\tau) > 0. \tag{A.25}$$

Indeed, in the zone $\{x\colon |x| \ge R\}$ we can apply semiclassical arguments with $h := R^{-1}$ after scaling $x \mapsto R^{-1}x$.

Since in both cases (A.8) holds with a strict inequality for $w \ne 0$), we arrive to both Statements (i) and (ii). □

A.2 Spectrum

The above results are sufficient for our needs, for $\alpha \in (\pi, 2\pi]$. However we would like to explore the case of $\alpha \in (0, \pi)$ and even $\alpha \in (\pi, 2\pi]$ in more depth.

Corollary A.13. *(i) Let $\alpha \in [\pi, 2\pi]$. Then $\mathsf{Spec}(\Lambda) = [1, \infty)$ and it is continuous.*

(ii) Let $\alpha \in (0, \pi]$. Then $\mathsf{Spec}(\Lambda_{\mathsf{asym}}) = [1, \infty)$ and it is continuous, where Λ_{sym} and Λ_{asym} denote the restriction of Λ to the spaces of symmetric and antisymmetric functions, correspondingly.

Proof. We already know that the that essential spectrum of Λ is $[1, \infty)$. We also know that in the case (i) $\Lambda > I$ and in the case (ii) $\Lambda_{\mathsf{asym}} > I$. Therefore 1 is not an eigenvalue. Continuity of the spectrum follows from $(i[\Lambda, Q]v, v) \ge \delta\|v\|^2$ for $v \in \mathsf{Ran}(I - P_\tau)$, v is antisymmetric in the case (ii), which is due to Statements (i) and (ii) of Propostion A.12. □

Remark A.14. Paper [S-KP] is dealing mainly with the eigenvalues of Δ_2 in the planar sector under Robin boundary condition $(\partial_\nu + \gamma)w|_Y = 0$, $\gamma > 0$ [14]. Then eigenvalues τ of Λ and eigenvalues μ of that problem are related through Birman-Schwinger principle and scaling: $\mu_k = -\tau_k^{-2}\gamma^2$. Some of the results:

(i) Theorem 3.1 states that $(-\infty, -\gamma^2)$ contains only discrete spectrum of such operator and it is finite.

(ii) Theorem 2.3 states that for $\alpha \in (0, \pi)$ the bottom eigenvalue $-\gamma^2/\sin^2(\alpha/2)$ is simple and the corresponding eigenfunction is $\exp(-\gamma x_1/\sin(\alpha/2))$.

(iii) Theorem 3.6 states that for $\alpha \in [\frac{\pi}{3}, \pi)$ there is no other eigenvalues in $(0, 1)$, while Theorem 4.1 implies that the number of such eigenvalues is $\asymp \alpha^{-2}$ as $\alpha \to 0$ [15].

Then we conclude that

[14] In that paper α is a half-angle, and ν is a unit external normal. Below we refer to this paper using our notations.

[15] In fact, the compete asymptotic expansion of the eigenvalues is derived in Theorem 4.16 of [S-KP].

Corollary A.15. *(i) Interval* $(0,1)$ *contains only discrete spectrum of* Λ_{sym} *which is finite.*

(ii) For $\alpha \in (0,\pi)$ *the bottom eigenvalues is* $\sin(\alpha/2)$ *and the corresponding eigenfunction is* $\exp(-x_1)$.

The discrete spectrum would not prevent us from the extending our main results to $\alpha \in (0,\pi)$. Even (possible) eigenvalue 1 on the edge of the essential spectrum would not be an obstacle. However eigenvalues embedded into $(1,\infty)$ are an obstacle (see Proposition A.16).

Proposition A.16. *If* $w_p = Jv_p$ *where* v_p *are eigenfunctions of* Λ*, corresponding to eigenvalues* τ_p*, and* $\tau_j = \tau_k$*, then*

$$(\nabla w_j, \nabla w_k) - (w_j, w_k) = 0. \tag{A.26}$$

In particular,

$$\|\nabla w_j,\|^2 - \|w_j\|^2 = 0. \tag{A.27}$$

Proof. It follows from equality (3.2) for $Q = x_1D_1+x_2D_2+i/2$ and $([Q,\Lambda]v_j, v_k) = (\Lambda v_j, Qv_jk) - (Qv_j, \Lambda v_k) = 0$ for eigenfunction v_j v_k provided $\tau_j = \tau_k$. □

To extend the main sharp spectral asymptotics to operators in domains with inner edges one needs to prove the first following

Conjecture A.17. *For any* $\tau > 1$ *and for any* $w = Jv$ *with symmetric* $v \in \mathsf{Ran}(I - P_\tau)$ *estimate* (A.24) *holds.*

Remark A.18. (i) Recall that this is true for $\alpha \in [\pi, 2\pi]$ and, also, for $\alpha \in (0,\pi)$ and antisymmetric v. So, only the case of $\alpha \in (0,\pi)$ and symmetric v needs to be covered.

(ii) So far it is unknown, if in the the case of $\alpha \in (0,\pi)$ Λ_{sym} has eigenvalues embedded into continuous spectrum $(1,\infty)$ or on its edge.

(iii) Also it is unknown in any case, if the continuous spectrum is absolutely continuous (i.e. that the singular continuous spectrum is empty).

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To the memory of Michael Solomyak

Asymptotics of the Ground State Energy in the Relativistic Settings*,†

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Abstract

The purpose of this paper is to derive sharp asymptotics of the ground state energy for the heavy atoms and molecules in the relativistic settings, and, in particular, to derive relativistic Scott correction term and also Dirac, Schwinger and relativistic correction terms. Also we will prove that Thomas-Fermi density approximates the actual density of the ground state, which opens the way to estimate the excessive negative and positive charges and the ionization energy.

1 Introduction

The purpose of this paper is to derive sharp asymptotics of the ground state energy for heavy atoms and molecules in the relativistic settings, and, in particular, to derive relativistic Scott correction term and also Dirac, Schwinger and relativistic correction terms. The relativistic Scott correction term was first derived in [SSS] which both inspired our paper and provided necessary functional analytic tools; our improvement is achieved due to more refined microlocal semiclassical technique.

Also we will prove that the Thomas-Fermi density approximates the actual density of the ground state, which opens the way to estimate the excessive negative and positive charges and the ionization energy.

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In the next paper [Ivr2] we will introduce a self-generated magnetic field and improve results of [EFS2].

Multielectron Hamiltonian is given by

$$\mathsf{H} = \mathsf{H}_N := \sum_{1\le j\le N} H_{V,x_j} + \sum_{1\le j<k\le N} \frac{\mathsf{e}^2}{|x_j - x_k|} \tag{1.1}$$

on

$$\mathfrak{H} = \bigwedge_{1\le n\le N} \mathscr{H}, \qquad \mathscr{H} = \mathscr{L}^2(\mathbb{R}^3, \mathbb{C}^q) \simeq \mathscr{L}^2(\mathbb{R}^3 \times \{1, \dots, q\}, \mathbb{C}) \tag{1.2}$$

with

$$H_V = T - \mathsf{e}V(x), \tag{1.3}$$

describing N same type particles in the external field with the scalar potential $-V$ and repulsing one another according to the Coulomb law; e is the charge of the electron, T is an *operator of the kinetic energy*.

In the non-relativistic framework this operator is defined as

$$T = \frac{1}{2\mu}(-i\hbar\nabla)^2. \tag{1.4}$$

In the relativistic framework this operator is defined as

$$T = \left(c^2(-i\hbar\nabla)^2 + \mu^2c^4\right)^{\frac{1}{2}} - \mu^2c^4. \tag{1.5}$$

Here

$$V(x) = \sum_{1\le m\le M} \frac{Z_m\mathsf{e}}{|x - \mathsf{y}_m|} \tag{1.6}$$

and

$$d = \min_{1\le m<m'\le M} |\mathsf{y}_m - \mathsf{y}_{m'}| > 0. \tag{1.7}$$

where $Z_m\mathsf{e} > 0$ and y_m are charges and locations of nuclei.

It is well-known that the non-relativistic operator is always semibounded from below. On the other hand, it is also well-known [Herb, LY] that

Remark 1.1. One particle relativistic operator is semibounded from below if and only if

$$Z_m\beta \le \frac{2}{\pi} \qquad \forall m = 1, \dots, M; \qquad \beta := \frac{\mathsf{e}^2}{\hbar c}. \tag{1.8}$$

We will assume (1.8), sometimes replacing it by a strict inequality:

$$Z_m\beta \le \frac{2}{\pi} - \epsilon \qquad \forall m = 1, \dots, M; \qquad \beta := \frac{\mathsf{e}^2}{\hbar c}. \tag{1.9}$$

We also assume that $d \ge CZ^{-1}$. Then we are interested in $\mathsf{E} := \inf \mathsf{Spec}(\mathsf{H})$.

Remark 1.2. (i) In the non-relativistic theory by scaling with respect to the spatial and energy variables we can make $\hbar = \mathsf{e} = \mu = 1$ while Z_m are preserved.

(ii) In the relativistic theory by scaling with respect to the spatial and energy variables we can make $\hbar = \mathsf{e} = \mu = 1$ while β and Z_m are preserved.

From now on we assume that such rescaling was already made and we are free to use letters $\hbar$, μ and c for other notations.

2 Functional Analytic Arguments

2.1 Estimate from below

In contrast to [SSS] we will start from the more traditional approach. We estimate $\sum_{1\le j<k\le N} \langle |x_j - x_k|^{-1}\Psi, \Psi\rangle$ from below, using Lieb's electrostatic inequality, by $\frac{1}{2}\mathsf{D}(\rho_\Psi, \Psi) - C\int \rho_\Psi^{4/3}\,dx$ where $\langle\cdot,\cdot\rangle$ means the inner product in $\mathfrak{H}$, $\rho_\Psi(x)$ is a one particle density,

$$\mathsf{D}(\rho, \rho') = \iint |x-y|^{-1}\rho(x)\rho'(y)\,dxdy,$$

and we use notations of Chapter 25 of [Ivr1].

The the standard lower estimate (25.2.2) from [Ivr1] holds:

$$\langle \mathsf{H}_N\Psi, \Psi\rangle \ge \sum_{1\le j\le N} \langle H_{V,x_j}\Psi, \Psi\rangle + \frac{1}{2}\mathsf{D}\big(\rho_\Psi, \rho_\Psi\big) - C\int \rho_\Psi^{\frac{4}{3}}(x)\,dx = \tag{2.1}$$

$$\sum_{1\le j\le N} \langle H_{W,x_j}\Psi, \Psi\rangle + \frac{1}{2}\mathsf{D}\big(\rho_\Psi - \rho, \rho_\Psi - \rho\big) - \frac{1}{2}\mathsf{D}\big(\rho, \rho\big) - C\int \rho_\Psi^{\frac{4}{3}}(x)\,dx,$$

where H_W is the one-particle Schrödinger operator (respectively, non-relativistic or relativistic) with the potential

$$W = V - |x|^{-1} * \rho, \tag{2.2}$$

and ρ is an arbitrary chosen real-valued non-negative function. Then again we get

$$\mathsf{E}_N \geq \mathsf{Tr}(H^-_{W+\lambda}) + \lambda N + \frac{1}{2}\mathsf{D}\big(\rho_\Psi - \rho, \rho_\Psi - \rho\big) - \frac{1}{2}\mathsf{D}\big(\rho,\rho\big) - C\int \rho_\Psi^{\frac{4}{3}}(x)\,dx \tag{2.3}$$

with arbitrary λ.

Remark 2.1. As usual, we will need to improve these estimates to recover a remainder estimate better than $O(Z^{\frac{5}{3}})$.

Now we need to prove estimate

$$\int \rho_\Psi^{\frac{4}{3}}(x)\,dx \leq CZ^{\frac{5}{3}} \tag{2.4}$$

for the ground state energy. It follows from

$$\int \rho_\Psi^{\frac{5}{3}}(x)\,dx \leq CZ^{\frac{7}{3}}, \tag{2.5}$$

equality $\int \rho_\Psi\,dx = N$ and assumption $N \lesssim Z$. To prove (2.5) we apply classical arguments of Lieb–Thirring, but replacing the Lieb–Thirring inequality by some relativistic inequalities (see Appendix A). Namely, let $\mathsf{b} := T - KU$ with $U = \rho_\Psi^{\frac{2}{3}}\varphi_< + \beta^{-1}\rho_\Psi^{\frac{1}{3}}\varphi_>$ where $\varphi_{\gtrless}$ is the characteristic function of $\{x\colon \rho_\Psi \gtrless \beta^{-3}\}$.

Consider the multiparticle operator $B = \sum \mathsf{b}_{x_j}$ and its lowest eigenvalue E_0. Obviously,

$$E_0 \leq \langle B\Psi,\Psi\rangle = \sum_j \langle T_{x_j}\Psi,\Psi\rangle - K\int (\rho_\Psi^{\frac{5}{3}}\varphi_< + \beta^{-1}\rho_\Psi^{\frac{4}{3}}\varphi_>)\,dx. \tag{2.6}$$

On the other hand, E_0 does not exceed the sum of negative eigenvalues of b, and due to Daubechies inequality (A.1), the absolute value of this sum does not exceed

$$C_0\int \max(U^{\frac{5}{2}}, \beta^3U^4)\,dx \leq C_0K^{\frac{5}{2}}\int \big(\rho_\Psi^{\frac{5}{3}}\varphi_< + \beta^{-1}\rho_\Psi^{\frac{4}{3}}\varphi_>\big)\,dx. \tag{2.7}$$

Therefore, assuming that $E_0 \leq 0$ we conclude that

$$\sum_j \langle T_{x_j}\Psi,\Psi\rangle - K\int \min(\rho_\Psi^{\frac{5}{3}},\ \beta^{-1}\rho_\Psi^{\frac{4}{3}}) + C_0K^{\frac{5}{2}}\int \big(\rho_\Psi^{\frac{5}{3}}\varphi_< + \beta^{-1}\rho_\Psi^{\frac{4}{3}}\varphi_>\big)\,dx \geq 0$$

and therefore, for small positive constant K, we conclude that

$$\sum_j \langle T_{x_j}\Psi,\Psi\rangle \ge 2\epsilon_0 \int (\rho_\Psi^{\frac53}\varphi_< + \beta^{-1}\rho_\Psi^{\frac43}\varphi_>)\,dx. \tag{2.8}$$

Thus, we proved that for any $\Psi\in\mathfrak{H}$ (2.8) holds. Then

$$\sum_j \langle H_{x_j}\Psi,\Psi\rangle = \sum_j \langle T_{x_j}\Psi,\Psi\rangle - \int V(x)\rho_\Psi(x)\,dx \ge \tag{2.9}$$
$$\int \big(2\epsilon_0\rho_\Psi^{\frac53} - V(x)\rho_\Psi\big)\varphi_<\,dx + \int \big(2\epsilon_0\beta^{-1}\rho_\Psi^{\frac43} - V(x)\rho_\Psi\big)\varphi_>\,dx.$$

We know, that the latter expression must be less than $-c_0Z^{\frac73}$ (it will follow, e. g., from the estimate from above). Observe that for $\ell(x)\ge aZ^{-\frac13}$ we have $V(x)<a^{-1}Z^{\frac43}$ and the integral over this zone of $-V\rho_\Psi$ is greater than $-C_0a^{-1}Z^{\frac43}N$. Here and below $\ell(x) :== \min_j|x-y_j|$. Let us fix a, a large enough constant.

Next,

$$\int_{\{x:\,\ell(x)\le aZ^{-1/3}\}} \big(\epsilon_0\rho_\Psi^{\frac53} - V(x)\rho_\Psi\big)\varphi_<\,dx \ge -C\int_{\{x:\,\ell(x)\le aZ^{-1/3}\}} V^{\frac52}\,dx \ge -C_1Z^{\frac73}$$

and $(\epsilon_0\beta^{-1}\rho_\Psi^{1/3}-V)\varphi_>$ is positive unless $\rho_\psi>\beta^{-3}$ and $V\ge\epsilon_1\beta^{-1}\rho_\Psi^{1/3}\ge\epsilon_1\beta^{-2}$ (and then $\ell(x)\le C_0\beta$).

Therefore, we estimate $\int(\rho_\Psi^{5/3}\varphi_< + \beta^{-1}\rho_\Psi^{4/3}\varphi_<)\,dx$ from above by $CZ^{7/3}$ plus $\int_{\{x:\ell(x)\le C\beta\}} V\rho_\Psi\,dx$, and to obtain (2.4), it is sufficient to estimate this term. Further, it is sufficient to replace V by V_m (since $V=V_m+O(\beta^2)$, provided distances between nuclei are $\ge C\beta$). Also we can replace V_m by $V_m+C\beta^{-2}$.

If $Z_m\beta\le\frac{2}{\pi}-\epsilon$, then we can decompose $H=\eta(H-V^1)+(1-\eta)(H-V^0)$, where $(1-\eta)V^0$ coincides with V in the β-vicinity of y_m and equals 0 outside of the 2β-vicinity of it and $V^1=\eta^{-1}(V-(1-\eta)V^0)$, and apply all above arguments for the operator with $V=V^1$ while simply observing that $H-V^0$ is a positive operator for η sufficiently small. So we have proven that

Proposition 2.2. *Under assumption* (1.9) *for the ground state,*

$$\int \min(\beta^{-1}\rho_\Psi^{\frac43},\rho_\Psi^{\frac53})\,dx \le CZ^{\frac73} \tag{2.10}$$

and (2.4) *holds.*

Then we immediately arrive to Statement (i) below, and Statement (ii) follows from the main theorems of either [Bach] (Theorem 1) or [GS] (Theorem 3):

Corollary 2.3. *Under assumption* (1.9)

(i) The following estimate holds

$$\mathsf{E}_N \ge \mathsf{Tr}(H^-_{W+\lambda}) - \frac{1}{2}\mathsf{D}(\rho,\rho) - C Z^{\frac{5}{3}} + \frac{1}{2}\mathsf{D}(\rho-\rho_\Psi,\, \rho-\rho_\Psi), \tag{2.11}$$

where ρ, λ are arbitrary and $W = V - |x|^{-1} * \rho$.

(ii) Further,

$$\begin{aligned}\mathsf{E}_N \ge \mathsf{Tr}(H^-_{W+\lambda}) - \frac{1}{2}\mathsf{D}(\rho,\rho) - {}& \\ \frac{1}{2}\int |x-y|^{-1}\, \mathsf{tr}\big(e_N^\dagger(x,y)\, e_N(x,y)\big)\, dxdy - C Z^{\frac{5}{3}-\delta} + {}& \\ \frac{1}{2}\mathsf{D}(\rho-\rho_\Psi,\, \rho-\rho_\Psi),&\end{aligned} \tag{2.12}$$

where $e_N(x,y)$ is the Schwartz kernel of the projector to N lower eigestates of H_W and tr *denotes the matrix trace.*

To cover[1)] the critical case[2)] we will use (2.21) from [SSS]

$$\sum_{1\le i<j\le N} |x_i - x_j|^{-1} \ge \sum_{j=1}^{N} (\rho * |x|^{-1} * \Phi_s)(x_j) - \frac{1}{2}\mathsf{D}(\rho,\rho) - CN\varepsilon^{-1}, \tag{2.13}$$

where, again, $\rho \ge 0$ is arbitrary, λ is arbitrary, $\Phi \ge 0$ is spherically symmetric with $\int \Phi\, dx = 1$, $\Phi_\varepsilon(x) = \varepsilon^{-3}\Phi(x/\varepsilon)$. Here the factor $\frac{1}{2}$ is due to the difference in notations and also now

$$W \coloneqq W_\varepsilon = V - |x|^{-1} * \rho * \Phi_\varepsilon \tag{2.14}$$

instead of (2.2) and the term $-CNs^{-1}$ replaces the last term in (2.3):

[1)] Unfortunately, only partially.

[2)] I.e. with the non-strict inequality (1.8) instead of (1.9).

Proposition 2.4. *Under assumption* (1.8)

$$\mathsf{E}_N \ge \mathsf{Tr}(H^-_{W+\lambda}) + \lambda N - \frac{1}{2}\mathsf{D}(\rho,\rho) - CN\varepsilon^{-1}. \tag{2.15}$$

Remark 2.5. (i) Later we set $\varepsilon = Z^{-\frac{2}{3}}$. This would lead to $O(Z^{\frac{5}{3}})$ remainder estimate.

(ii) Proposition 2.4 is weaker than Corollary 2.3 in two ways: there is no improved version of corollary 2.3(ii) and also there is no "bonus term" $\frac{1}{2}\mathsf{D}(\rho-\rho_\Psi,\rho-\rho_\Psi)$ in the right-hand expression.

2.2 Estimate from above

The estimate from above is straight-forward: we simply take Ψ as a Slater determinant of N lower eigenfunctions of H_W. If there are only $N' < N$ negative eigenvalues then we take only N' such eigenvalues, because $\mathsf{E}_N \le \mathsf{E}_{N'}$. Then we arrive to

Proposition 2.6.

$$\begin{aligned}\mathsf{E}_N \le \mathsf{Tr}(H^-_{W+\lambda}) - \frac{1}{2}\mathsf{D}(\rho,\rho)+ \\ |\lambda-\nu|\cdot|\mathsf{N}^-_{W+\nu}-N| + \mathsf{D}(\mathsf{tr}\, e_N(x,x)-\rho,\ \mathsf{tr}\, e_N(x,x,\nu)-\rho)- \\ \frac{1}{2}\int |x-y|^{-1}\,\mathsf{tr}\big(e_N^\dagger(x,y)e_N(x,y)\big)\,dxdy\end{aligned} \tag{2.16}$$

with an arbitrary ρ and any $\nu \le 0$, $W = V - |x|^{-1} * \rho$.

3 Semiclassical Methods

We will need the following semiclassical expressions:

$$P'(w) = (2\pi)^{-3} q \int_{\{\xi: b(\xi)\le w\}} d\xi, \tag{3.1}$$

and its integral

$$P(w) = (2\pi)^{-3} q \int_{\{\xi: b(\xi)\le w\}} b(\xi)\, d\xi, \tag{3.2}$$

where in the non-relativistic case $b(\xi) = \frac{\hbar^2}{2\mu}|\xi|^2$ and, correspondingly for $\mu = \hbar = 1$

$$P^{\mathsf{TF}\prime}(w) = \frac{q}{6\pi^2} w_+^{\frac{3}{2}}, \tag{3.3}$$

$$P^{\mathsf{TF}}(w) = \frac{q}{15\pi^2} w_+^{\frac{5}{2}}, \tag{3.4}$$

while in the relativistic case $b(\xi) = (c^2\hbar^2|\xi|^2 + \mu^2 c^4)^{\frac{1}{2}} - \mu c^2$ and, correspondingly for $\mu = \hbar = 1$

$$P^{\mathsf{RTF}\prime}(w) = \frac{q}{6\pi^2} w_+^{\frac{3}{2}}(1 + \beta^2 w_+)^{\frac{3}{2}}. \tag{3.5}$$

Note that $P^{\mathsf{RTF}}(w)$ is an elementary function as well, and a sadistic Calculus instructor can give it on the test. However it turns out that we really do not need any separate relativistic Thomas-Fermi theory.

After scalings we have a *semiclassical zone* $\mathcal{X}_{\mathsf{scl}} := \{x\colon \ell(x) \geq cZ^{-1}\}$, where the effective semiclassical parameter is $h = 1/\zeta\ell$. Then, from the semiclassical point of view, on the energy levels ≤ 0, the relativistic operator has the same properties as the non-relativistic one.

There is also a *singular zone* $\mathcal{X}_{\mathsf{sing}} := \{x\colon \ell(x) \leq cZ^{-1}\}$ and it covers the *relativistic zone* $\mathcal{X}_{\mathsf{rel}} := \{x\colon \ell(x) \leq c\beta\}$. The important properties are that

$$0 \leq V(x) - W(x) \leq C\zeta^2 := \min(Z^{\frac{4}{3}}, Z\ell^{-1}), \tag{3.6}$$

$$|\partial^\gamma (W - V)| \leq C\zeta^2 \ell(x)^{-|\gamma|} \qquad \forall \gamma : |\gamma| \leq 2. \tag{3.7}$$

3.1 Trace Term

Now the rescaling methods of [Ivr1] allow us to prove the following:

Proposition 3.1. *Let condition* (1.8) *be fulfilled and let* W *satisfy* (3.6) *and* (3.7).

(i) Let $\psi_0(x)$ *be* ℓ*-admissible function*[3)]*, being equal* 1 *in* $\{x\colon \ell(x) \geq 2a\}$ *and supported in* $\{x\colon \ell(x) \geq a\}$. *Then for* $W = W^{\mathsf{TF}}$

$$|\operatorname{Tr}(H^-_{W+\lambda}\psi_0) - \int P^{\mathsf{RTF}}(W+\lambda)\psi_0(x)\,dx| \leq \tag{3.8}$$

$$C\begin{cases} Z^{\frac{3}{2}} a^{-\frac{1}{2}} & a \leq Z^{-\frac{1}{3}}, \\ Z^{\frac{5}{3}}(aZ^{\frac{1}{3}})^{-\delta} & a \geq Z^{-\frac{1}{3}}. \end{cases}$$

[3)] I.e. $\partial^\alpha \psi_0| \leq C_\alpha \ell^{-\alpha}$.

(ii) Let $\psi_m(x)$ be ℓ-admissible, equal 1 in $\{x\colon |x-\mathsf{y}_m|\le a\}$ and supported in $\{x\colon |x-\mathsf{y}_m|\le 2a\}$. Then for $W=V_m=Z_m|x-\mathsf{y}_m|^{-1}$

$$|\int \bigl(\mathsf{tr}(e^1(x,x,0))-P^{\mathsf{RTF}}(V_m)\bigr)(1-\psi_m(x))\,dx|\le Z^{\frac{3}{2}}d^{-\frac{1}{2}}, \tag{3.9}$$

where $e^1(.,.,\tau)=\int_{-\infty}^{\tau} e(.,.,\tau')\,d\tau'$.

Proof. Indeed, the contribution of the ℓ-element of the partition[4)] to the remainder is $O(\zeta^3\ell)$, exactly as in the non-relativistic case. Summation by partition elements results in the right-hand expression. □

Next, we need to consider vicinities of the singularities. Then the methods of Chapter 25 of [Ivr1] allow us to prove the following:

Proposition 3.2. *In the framework of Proposition 3.1 let ϕ_m be equal 1 in $\{x\colon |x-\mathsf{y}_m|\le Z_m^{-1}\}$ and supported in $\{x\colon |x-\mathsf{y}_m|\le 2Z_m^{-1}\}$. Let $|\lambda|\le C_0Zd^{-1}$. Then*

$$\begin{aligned}|\,\mathsf{Tr}(H^-_{W+\lambda}\psi_m(1-\phi_m))-\mathsf{Tr}(H^-_{V_m}\psi_m(1-\phi_m))+\\ \int \bigl(P^{\mathsf{RTF}}(W+\lambda)-P^{\mathsf{RTF}}(V_m)\bigr)\psi_m(x)(1-\phi_m(x))\,dx|\le\\ C\begin{cases} Z^{\frac{3}{2}}d^{-\frac{1}{2}} & d\le Z^{-\frac{1}{3}},\\ Z^{\frac{5}{3}} & d\ge Z^{-\frac{1}{3}},\end{cases}\end{aligned} \tag{3.10}$$

where $d\ge cZ^{-1}$ is the minimal distance between nuclei.

Proof. Indeed, exactly as in the non-relativistic case, using methods of Sections 12.6 and 25.4 of [Ivr1] we estimate the contribution of an ℓ-element to the remainder by $O(\zeta\ell^3\bar\zeta^2\bar\ell^{-2})$ provided $Z^{-1+\delta}\lesssim\ell\lesssim d$, and by $O(\zeta^2\ell^2\bar\zeta^2)$ provided $Z^{-1}\lesssim\ell\lesssim Z^{-1+\delta}$. Here $\bar\ell=d$ and $\bar\zeta=Z^{\frac{1}{2}}d^{-\frac{1}{2}}$. This proves the required remainder estimate. For $d\le Z^{-1+\delta}$ we use a rescaling.

Summation by partition elements results in the right-hand expression.□

Remark 3.3. We need to include the cut-off $(1-\phi_m(x))$ because not only integrals of $P^{\mathsf{RTF}}(W+\lambda)$ and $P^{\mathsf{RTF}}(V_m)$ (of magnitude $\beta^3Z^4\ell^{-4}$) and $P^{\mathsf{RTF}\prime}(W+\lambda)$ are diverging at y_m, but even integral of their difference is logarithmically diverging.

[4)] I.e. ℓ-admissible function $\psi(x)$, supported in $\frac{1}{2}\ell(y)$ for some y; then $\ell(x)\asymp\ell(y)$ on $\mathsf{supp}(\psi)$.

Now we need to consider CZ^{-1} vicinities of y_m and we will use the following Proposition:

Proposition 3.4. *In the framework of Proposition 3.1*

(i) $H_W \ge -C_0Z^2$.

(ii) Further,

$$e(x,x,\lambda) \le CZ^{1-\delta}\ell(x)^{\delta-2} \qquad \text{for } |\lambda| \le c_0Z^2. \tag{3.11}$$

Proof. (a) Assume first that $Z \asymp \beta^{-1}$ (i. e. $Z \ge \epsilon_0\beta^{-1}$); then Statement (i) follows immediately from Lieb-Yau inequality (Theorem A.2): in the operator sense $H \ge \beta^{-1}\sqrt{\Delta} - \beta^{-2} - Z_m r^{-1} \ge -\beta^{-2}$, $r = |x - \mathsf{y}_m|$.

Then $e(x,x,\lambda) \le C\ell(x)^{-3}h^{-3}$ with the semiclassical parameter h, which is $\asymp 1$ for $\ell \lesssim Z^{-1}$, $\lambda \lesssim Z^2$. Therefore,

$$e(x,x,\lambda) \le C\ell(x)^{-3} \qquad \text{for } \lambda \le C_0Z^2,\ \ell(x) \lesssim Z^{-1}. \tag{3.12}$$

Unfortunately this estimate falls short for our needs. Let us shift $\mathsf{y}_m \mapsto 0$, and scale $x \mapsto Zx$, $\tau \mapsto Z^{-2}\tau$. Then we arrive to an operator which modulo $O(1)$ is $\sqrt{\Delta} - Zr^{-1}$. Due to

$$\sqrt{\Delta} - \frac{2}{\pi|x|} \ge A_s(\Delta)^s - B_s \tag{3.13}$$

for any $s \in [0,1/2)$ and $A_s, B_s > 0$ we can "trade" (due to Sobolev embedding theorem) $\ell^{-1+\delta}$ by 1 in the scaled inequality (3.12) and by $Z^{1-\delta}$ in the original one, thus arriving to (3.11).

(b) Let us consider $Z \le \epsilon_0\beta^{-1}$. Observe that in the operator sense

$$H \ge (\frac{1}{4}\beta^{-2}r^{-2} + \beta^{-4})^{1/2} - Zr^{-1} - C\beta^{-2} \ge CZ^{-2};$$

the latter inequality is proven separately for $r \lesssim \beta$ and for $r \gtrsim \beta$.

Moreover, we get $H \ge \epsilon_1 \min(r^{-2}, \beta^{-1}r^{-1})$ for $r \le \epsilon_1 Z^{-1}$ and then we can trade ℓ^{-3} to CZ^3 arriving even to the stronger version of (3.12): namely,

$$e(x,x,\lambda) \le CZ^3. \tag{3.14}$$

Actually estimate (3.14) holds as $Z_m\beta \le 2\pi^{-1} - \sigma$ for $\sigma > 0$ with $C = C(\sigma)$ which could be calculated explicitly. □

Then we immediately conclude that

Corollary 3.5. *In the framework of Proposition 3.1 for* $|\lambda| \le C_0 Z d^{-1}$

$$|\operatorname{Tr}(H^-_{W+\lambda}\phi_m) - \operatorname{Tr}(H^-_{V_m}\phi_m)| \le CZd^{-1}. \tag{3.15}$$

Now we can assemble all these results. However, before doing this we replace P^{RTF} by P^{TF}:

Proposition 3.6. *(i) Estimates* (3.8), (3.9) *and* (3.10) *hold with* P^{RTF} *replaced by* P^{TF}.

(ii) Estimate (3.10) *with* P^{RTF} *replaced by* P^{TF} *also holds with* $\phi_m = 0$.

Proof. Statement (i) follows immediately from

$$P^{\mathsf{RTF}}(w) - P^{\mathsf{TF}}(w) \asymp \beta^2 w^{\frac{7}{2}}, \qquad P^{\mathsf{RTF}\prime}(w) - P^{\mathsf{TF}\prime}(w) \asymp \beta^2 w^{\frac{5}{2}} \quad \text{for } \beta^2 w \lesssim 1 \tag{3.16}$$

due to (3.5). Statement (ii) follows immediately from $P^{\mathsf{TF}}(w) \asymp w^{\frac{5}{2}}$, $P^{\mathsf{TF}\prime}(w) \asymp w^{\frac{3}{2}}$. □

Remark 3.7. Meanwhile,

$$\int \bigl(P^{\mathsf{RTF}}(V+\lambda) - P^{\mathsf{TF}}(V+\lambda)\bigr)\psi(x)\,dx \asymp \beta^2 Z^4, \tag{3.17}$$

which could be as large as Z^2.

Due to the scaling properties of $e(x,x,0)$ for $H = H_V$ and $P^{\mathsf{TF}}(V)$ for $V = V_m$ we conclude that

$$\int \bigl(\operatorname{tr}(e^1(x,x,0)) - P^{\mathsf{RTF}}(V_m)\bigr)\,dx = qZ_m^2 S(Z_m\beta) \tag{3.18}$$

with an unknown function $S(Z_m\beta)$. Indeed, if $\mathsf{y}_m = 0$ then $x \mapsto x/k$ transforms the operator with parameters Z_m, β into the operator with parameters $Z_m k$, βk^{-1}, multiplied by k^{-2}.

Remark 3.8. Obviously, $S(Z_m\beta)$ monotonely decreases as $\beta \to 0+$ and tends to $S(0) = \frac{1}{2}$ for the Schrödinger operator.

Then due to (3.9) for $V = V_m$

$$|\int \bigl(\mathsf{tr}(e^1(x,x,0)) - P^{\mathsf{TF}}(V_m)\bigr)\psi_m(x)\,dx - qZ_m^2 S(Z_m\beta)| \le Z^{\frac{3}{2}} d^{-\frac{1}{2}} \tag{3.19}$$

and we arrive to

Proposition 3.9. *Let* (1.8) *be fulfilled. Then for* $W = W^{\mathsf{TF}}$

$$|\,\mathsf{Tr}(H^-_{W+\lambda}) + \int P^{\mathsf{TF}}(W+\lambda)\,dx - \sum_{1\le m\le M} qZ_m^2 S(Z_m\beta)| \le C\begin{cases} Z^{\frac{3}{2}} d^{-\frac{1}{2}} & d \le Z^{-\frac{1}{3}},\\ Z^{\frac{5}{3}} & d \ge Z^{-\frac{1}{3}}.\end{cases} \tag{3.20}$$

3.2 Trace tTerm. II

Let improve the above results for $d \gg Z^{-\frac{1}{3}}$. Observe first that the in this case the error in (3.8) can be made $O(Z^{\frac{5}{3}}(dZ^{\frac{1}{3}})^{-\delta} + Z^{\frac{5}{3}-\delta})$ provided we include the relativistic Schwinger correction term. Since this term has magnitude $Z^{\frac{5}{3}}$ and the contributions of the zones $\{x\colon \ell(x) \le Z^{-\frac{1}{3}-\delta_1}\}$ and $\{x\colon \ell(x) \ge Z^{-\frac{1}{3}+\delta_1}\}$ in this term are $O(Z^{\frac{5}{3}-\delta})$, the difference between the relativistic and the standard non-relativistic Schwinger terms is $O(Z^{\frac{5}{3}-\delta})$, and we can use the latter

$$\mathsf{Schwinger} = \bigl(\frac{q}{6\pi^2}\bigr)^{\frac{2}{3}} \int \rho^{\mathsf{TF}\,\frac{4}{3}}\,dx. \tag{3.21}$$

Next, consider the relativistic correction term

$$\int \Bigl(-P^{\mathsf{RTF}}(W+\lambda) + P^{\mathsf{RTF}}(V_m) + P^{\mathsf{TF}}(W+\lambda) - P^{\mathsf{TF}}(V_m)\Bigr)\psi_m(1-\phi_m)\,dx. \tag{3.22}$$

Again, one can see easily that we need to consider only the contribution of the *threshold zone* $\mathcal{Y} \coloneqq \{x\colon Z^{-\frac{1}{3}-\delta_1} \le \ell(x) \le Z^{-\frac{1}{3}+\delta_1}\}$ because the contributions of both zones $\{x\colon \ell(x) \le Z^{-\frac{1}{3}-\delta_1}\}$ and $\{x\colon \ell(x) \ge Z^{-\frac{1}{3}+\delta_1}\}$ in this term are $O(Z^{\frac{5}{3}-\delta})$.

One can see easily that in the threshold zone due to (3.5)

$$P^{\mathsf{RTF}}(w) - P^{\mathsf{TF}}(w) = \frac{q}{14\pi^2}\beta^2 w_+^{\frac{7}{2}} + O(Z^{\frac{8}{3}-\delta})$$

(for both $w = W^{\mathsf{TF}} + \lambda$ and $w = V_m$), and therefore modulo the same error expression (3.22) coincides with

$$\mathsf{RCT} \coloneqq \frac{q}{14\pi^2}\beta^2 \int \bigl(-(W^{\mathsf{TF}} + \lambda)_+^{\frac{7}{2}} + V^{\frac{7}{2}}\bigr)\, dx \tag{3.23}$$

with the integral taken over this zone or $\mathbb{R}^3$ (it does not matter). Then we arrive to

Proposition 3.10. *Let* (1.8) *be fulfilled and* $d \ge Z^{-\frac{1}{3}}$. *Then for* $W = W^{\mathsf{TF}}$

$$\begin{aligned} |\operatorname{Tr}(H^-_{W+\lambda}) + \int P^{\mathsf{TF}}(W+\lambda)\, dx - \sum_{1\le m\le M} qZ_m^2 S(Z_m\beta) - \\ \mathsf{Schwinger} - \mathsf{RCT}| \le C\bigl(Z^{\frac{5}{3}}(dZ^{\frac{1}{3}})^{-\delta} + Z^{\frac{5}{3}-\delta}\bigr). \end{aligned} \tag{3.24}$$

3.3 Trace Term. III

Obviously, all these results hold for $W = W_\varepsilon$ defined by (2.14) with $\rho = \rho^{\mathsf{TF}}$. However we need to estimate an error when we replace W_ε by W^{TF}. One can prove easily that

$$|W_\varepsilon - W^{\mathsf{TF}}| \le C_s(Z\ell^{-1})^{\frac{3}{2}}\varepsilon^2(\varepsilon\ell - 1)^s \tag{3.25}$$

with arbitrary s for $\ell \le \epsilon_0 Z^{-\frac{1}{3}}$ and with $s = \frac{1}{2}$ for $\ell \le \epsilon_0 Z^{-\frac{1}{3}}$, and therefore,

$$|\int \bigl(P^{\mathsf{TF}}(W_\varepsilon + \lambda) - P^{\mathsf{TF}}(W^{\mathsf{TF}} + \lambda)|\, dx| \le CZ^3\varepsilon^2; \tag{3.26}$$

adding error $CZ\varepsilon^{-1}$ in (2.15) we get $C(Z^3\varepsilon^2 + Z\varepsilon^{-1})$. It reaches minimum $CZ^{\frac{5}{3}}$ as $\varepsilon \asymp Z^{-\frac{2}{3}}$ and we arrive to

Proposition 3.11. *Let* (1.8) *be fulfilled. Then for* $W = W_\varepsilon$ *with* $\varepsilon = Z^{-\frac{2}{3}}$, (3.20) *holds and the left-hand expression of* (3.26) *is* $O(Z^{\frac{5}{3}})$.

3.4 N- and D-Terms

For these terms (needed for the estimate from above) arguments are simpler; let $\phi_0 = 1 - \phi_1 - \ldots - \phi_M$.

Proposition 3.12. *In the framework of Proposition 3.1*

(i) The following estimates hold

$$|\int \big(e(x,x,\lambda) - P^{\mathsf{RTF}\,\prime}(W+\lambda)\big)\phi_0(x)\,dx| \le CZ^{\frac{2}{3}} \tag{3.27}$$

and for $d \ge Z^{-\frac{1}{3}}$

$$|\int \big(e(x,x,\lambda) - P^{\mathsf{RTF}\,\prime}(W+\lambda)\big)\phi_0(x)\,dx| \le C\big(Z^{\frac{2}{3}}(dZ^{\frac{1}{3}})^{-\delta} + Z^{\frac{2}{3}-\delta}\big). \tag{3.28}$$

(ii) Further,

$$|\int e(x,x,\lambda)\phi_m(x)\,dx| \le C. \tag{3.29}$$

(iii) Finally,

$$|\int \big(P^{\mathsf{RTF}\,\prime}(W+\lambda) - P^{\mathsf{TF}\,\prime}(W+\lambda)\big)\phi_0(x)\,dx| \le CZ^{\frac{1}{3}}. \tag{3.30}$$

Proposition 3.13. *In the framework of Proposition 3.1*

(i) The following estimates hold

$$\mathsf{D}\big((e(x,x,\lambda) - P^{\mathsf{RTF}\,\prime}(W+\lambda))\phi_0,\ (e(x,x,\lambda) - P^{\mathsf{RTF}\,\prime}(W+\lambda))\phi_0\big) \le CZ^{\frac{5}{3}} \tag{3.31}$$

and for $d \ge Z^{-\frac{1}{3}}$

$$\mathsf{D}\big((e(x,x,\lambda) - P^{\mathsf{RTF}\,\prime}(W+\lambda))\phi_0,\ (e(x,x,\lambda) - P^{\mathsf{RTF}\,\prime}(W+\lambda))\phi_0\big) \le CZ^{\frac{5}{3}}(dZ^{\frac{1}{3}})^{-\delta} + CZ^{\frac{5}{3}-\delta}. \tag{3.32}$$

(ii) Further,

$$\mathsf{D}\big(e(x,x,\lambda)\phi_m(x),\ e(x,x,\lambda)\phi_m(x)\big) \le CZ. \tag{3.33}$$

(iii) Finally,

$$\mathsf{D}\big((P^{\mathsf{RTF}\,\prime}(W+\lambda) - P^{\mathsf{TF}\,\prime}(W+\lambda))\phi_0,\ (P^{\mathsf{RTF}\,\prime}(W+\lambda) - P^{\mathsf{TF}\,\prime}(W+\lambda))\phi_0\big) \le CZ. \tag{3.34}$$

Proof of Propositions 3.12 and 3.13. Proof is straightforward:

Statements (i) are proven by the semiclassical scaling technique exactly as in [Ivr1], Chapter 25.

Statements (ii) follow from Proposition 3.4. Statements (iii) follow from (3.5) and properties of W^{TF}. □

3.5 Dirac Term

Finally, consider the term $-\frac{1}{2}\iint \mathsf{tr}\big(e_N^{\dagger}(x,y)e_N(x,y)\big)\,dxdy$. The main contribution to it is delivered by the zone $\mathcal{Y}\times\mathcal{Y}$ where $\mathcal{Y}$ is the threshold zone, and in this zone the non-magnetic approximation delivers the correct expression

$$\mathsf{Dirac} = -\big(\frac{6}{\pi q}\big)^{\frac{1}{3}} q^{\frac{2}{3}} \int \rho^{\mathsf{TF}\,\frac{4}{3}}\,dx, \tag{3.35}$$

with an error $Z^{\frac{5}{3}-\delta}$.

4 Main Theorems

Now repeating arguments of Section 25.4 of [Ivr1] we arrive to our main results:

Theorem 4.1 [5]**.** *Let assumption* (1.8) *be fulfilled. Then*

(i) The following asymptotics holds

$$\mathsf{E}_N = \mathcal{E}_N^{\mathsf{TF}} + \mathsf{Scott} + O\big(Z^{\frac{5}{3}} + Z^{\frac{3}{2}}d^{-\frac{1}{2}}\big). \tag{4.1}$$

Recall that $\mathsf{Scott} = q\sum Z_m^2 S(Z_m\beta)$ *and* d *is the minimal distance between nuclei.*

(ii) Furthermore, let assumption (1.9) *be fulfilled. Then for* $d \ge Z^{-\frac{1}{3}}$

$$\mathsf{E}_N = \mathcal{E}_N^{\mathsf{TF}} + \mathsf{Scott} + \mathsf{Dirac} + \mathsf{Swinger} + \mathsf{RCT} + O\big(Z^{\frac{5}{3}}(dZ^{\frac{1}{3}})^{-\delta} + Z^{\frac{5}{3}-\delta}\big). \tag{4.2}$$

[5] Cf. Theorems 25.4.8 and 25.4.13 of [Ivr1].

Remark 4.2. (i) For the improved upper estimate in (4.2) we do not need assumption (1.9).

(ii) These theorems allow us to consider the free nuclei model and recover Theorem 25.4.14 of [Ivr1], albeit without assumption (1.9) we get only $\delta = 0$.

(iii) We also recover the estimate

$$|\lambda_N - \nu| \le C \begin{cases} Z^{\frac{8}{9}} & (Z-N)_+ \le Z^{\frac{2}{3}}, \\ (Z-N)_+^{\frac{1}{3}} & (Z-N)_+ \ge Z^{\frac{2}{3}}, \end{cases} \tag{4.3}$$

where ν is a chemical potential and λ_N is the N-th lowest eigenvalue of $H_{W^{\mathsf{TF}}}$ (reset to 0 if there are less than N negative eigenvalues). Furthermore, for $d \ge Z^{-\frac{1}{3}}$ one can include the factor $((dZ^{\frac{1}{3}})^{-\delta} + Z^{-\delta})$ into the right-hand expression.

Theorem 4.3 [6)]**.** *Let assumption* (1.9) *be fulfilled. Then*

(i) The following estimate holds:

$$\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}}) \le CZ^{\frac{5}{3}}. \tag{4.4}$$

(ii) Furthermore, for $d \ge Z^{-\frac{1}{3}}$

$$\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}}) \le C(Z^{\frac{5}{3}}(dZ^{\frac{1}{3}})^{-\delta} + Z^{\frac{5}{3}-\delta}). \tag{4.5}$$

Remark 4.4. (i) Estimates (4.4) and (4.5) allow us to consider the excessive negative charge and ionization energy and, repeating arguments of Section 25.5 of [Ivr1], to recover Theorems 25.5.3 and 25.5.4.

(ii) Further, these estimates allow us to consider the excessive positive charge in the free nuclei model and, repeating arguments of Section 25.6 of [Ivr1], to recover Theorems 25.6.1 and 25.6.3.

Remark 4.5. We can even make a poor man version of (4.2) in the critical case, when only assumption (1.8) is fulfilled.

[6)] Cf. Theorem 25.4.15 of [Ivr1].

(i) Consider how our terms depend on q. In the atomic case consider given Z, N and shift to $\mathsf{y}_1 = 0$. Then

$$\rho_q^{\mathsf{TF}}(x) = q^2\rho_1^{\mathsf{TF}}(q^{\frac{2}{3}}x), \qquad W_q^{\mathsf{TF}}(x) = q^{\frac{2}{3}}W_1^{\mathsf{TF}}(q^{\frac{2}{3}}x) \tag{4.6}$$

and $\mathcal{E}^{\mathsf{TF}} \asymp q^{\frac{2}{3}}Z^{\frac{7}{3}}$, $\mathsf{Scott} \asymp qZ^2$, $\mathsf{Dirac} \asymp q^{\frac{1}{3}}Z^{\frac{5}{3}}$, $\mathsf{Schwinger} \asymp q^{\frac{4}{3}}Z^{\frac{5}{3}}$ while $\mathsf{RCT} \asymp q^{\frac{4}{3}}\beta^2Z^{\frac{11}{3}}$.

(ii) Repeating the corresponding arguments in [SSS], one can prove that in the correlation inequality (2.13), the constant is $C(q) \le C_0q^{\frac{2}{3}}$. On the other hand, we use the estimate for $|W - W_\varepsilon| \asymp q\varepsilon^2Z^{\frac{3}{2}}\ell^{-\frac{3}{2}}$ and then the approximation error is $C_0Z^3q^2\varepsilon^2$. Optimizing $Z^3q^2\varepsilon^2 + Zq^{\frac{2}{3}}\varepsilon^{-1}$ by ε we get $Cq^{\frac{10}{9}}Z^{\frac{5}{3}}$ and for a large constant q it is less than $q^{\frac{4}{3}}$. In the "real life" $q = 2$.

A Some Inequalities

We follow [SSS] with some modifications:

The following two inequalities we recall are crucial in many of our estimates. They serve as replacements for the Lieb-Thirring inequality [LT] used in the non-relativistic case.

Theorem A.1 (Daubechies inequality). *(i) One-body case:*

$$\mathsf{Tr}\big[(\beta^{-2}\Delta + \beta^{-4})^{\frac{1}{2}} - \beta^{-2} - V(\mathsf{x})\big]^- \ge -C\int \Big(V_+^{(n+2)/2} + \beta^n V_+^{n+1}\Big)\,dx. \tag{A.1}$$

where $n \ge 3$ is a dimension.

(ii) Many-body case: Let $\Psi \in \bigwedge_{j=1}^N \mathscr{L}^2(\mathbb{R}^3;\mathbb{C}^q)$ and let ρ_Ψ be its one-particle density. Then for $n = 3$

$$\langle \sum_{j=1}^N \big[(\beta^{-2}\Delta_j + \beta^{-4})^{\frac{1}{2}} - \beta^{-2}\big]\Psi, \Psi\rangle \ge \int \min\big(\rho_\Psi^{\frac{5}{3}}, \beta^{-1}\rho_\Psi^{\frac{4}{3}}\big)\,dx. \tag{A.2}$$

This theorem also holds in the non-relativistic limit $\beta = 0$ and operator $(\beta^{-2}\Delta + \beta^{-4})^{\frac{1}{2}} - \beta^{-2}$ replaced by $\frac{1}{2}\Delta$.

Theorem A.2 (Lieb-Yau inequality). *Let $n = 3$. Let $C > 0$ and $R > 0$ and let*

$$H_{C,R} = \Delta^{\frac{1}{2}} - \frac{2}{\pi|x|} - C/R. \tag{A.3}$$

Then, for any density matrix γ and any function θ with support in $B_R = \{x \,|\, |x| \leq R\}$

$$\mathrm{Tr}\big[\bar{\theta}\gamma\theta H_{C,R}\big] \geq -4.4827\, C^4 R^{-1}\{3/(4\pi R^3) \int |\theta(x)|^2\, dx\}. \tag{A.4}$$

Note that when $\theta = 1$ on B_R then the term inside the brackets $\{\}$ equals 1.

Theorem A.3 (Critical Hydrogen inequality). *Let $n = 3$. For any $s \in [0, 1/2)$ there exists constants $A_s, B_s > 0$ such that*

$$\Delta^{\frac{1}{2}} - \frac{2}{\pi|x|} \geq A_s \Delta^s - B_s. \tag{A.5}$$

Theorem A.4 (Hardy-Littlewood-Sobolev inequality). *There exists a constant C such that*

$$D(f) := \iint |x-y|^{-1} f(x) f^\dagger(y)\, dxdy \leq C\, \|f\|^2_{\mathcal{L}^{6/5}}. \tag{A.6}$$

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Asymptotics of the Ground State Energy in the Relativistic Settings and with Self-Generated Magnetic Field[*,†]

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Abstract

The purpose of this paper is to derive sharp asymptotics of the ground state energy for the heavy atoms and molecules in the relativistic settings, with the self-generated magnetic field, and, in particular, to derive relativistic Scott correction term and also Dirac, Schwinger and relativistic correction terms. Also we will prove that Thomas-Fermi density approximates the actual density of the ground state, which opens the way to estimate the excessive negative and positive charges and the ionization energy.

1 Introduction

Multielectron Hamiltonian is defined by

$$\mathsf{H} = \mathsf{H}_N := \sum_{1\le j\le N} H_{V,x_j} + \sum_{1\le j<k\le N} \frac{\mathsf{e}^2}{|x_j - x_k|} \tag{1.1}$$

on

$$\mathfrak{H} = \bigwedge_{1\le n\le N} \mathscr{H}, \qquad \mathscr{H} = \mathscr{L}^2(\mathbb{R}^3, \mathbb{C}^q) \simeq \mathscr{L}^2(\mathbb{R}^3 \times \{1, \dots, q\}, \mathbb{C}) \tag{1.2}$$

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with

$$H_V = T - \mathsf{e}V(x), \tag{1.3}$$

describing N same type particles in the external field with the scalar potential $-V$ and repulsing one another according to the Coulomb law; e is a charge of the electron, T is an *operator of the kinetic energy*. Unless specifically mentioned, $q = 2$.

In the non-relativistic framework this operator is defined as

$$T = \frac{1}{2\mu}(-i\hbar\nabla - \mathsf{e}A)^2, \tag{1.4}_1$$

$$T = \frac{1}{2\mu}\big((i\nabla - \mathsf{e}A)\cdot\boldsymbol{\sigma}\big)^2 \tag{1.4}_2$$

in the magnetic (Schrödinger) and (Schrödinger-Pauli) settings respectively.

In the relativistic framework this operator is defined as

$$T = \Big(c^2(-i\hbar\nabla - \mathsf{e}A)^2 + \mu^2c^4\Big)^{\frac{1}{2}} - \mu^2c^4 \tag{1.5}_1$$

$$T = \Big(c^2\big((-i\hbar\nabla - \mathsf{e}A)\cdot\boldsymbol{\sigma}\big)^2 + \mu^2c^4\Big)^{\frac{1}{2}} - \mu^2c^4 \tag{1.5}_2$$

in the magnetic (Schrödinger) and (Schrödinger-Pauli) settings respectively.

Recall that in non-magnetic settings we have (1.4) and (1.5) of [Ivr2] in the non-relativistic and relativistic settings respectively. Here

$$V(x) = \sum_{1\le m\le M} \frac{Z_m\mathsf{e}}{|x - \mathsf{y}_m|} \tag{1.6}$$

and

$$d = \min_{1\le m<m'\le M} |\mathsf{y}_m - \mathsf{y}_{m'}| > 0. \tag{1.7}$$

where $Z_m\mathsf{e} > 0$ and y_m are charges and locations of nuclei.

It is well-known that the non-relativistic operator is always semibounded from below. On the other hand, it is also well-known [Herb, LY] that one particle relativistic non-magnetic operator is semibounded from below if and only if $Z_m\beta \le \frac{2}{\pi}$ for $m = 1, \dots, M$. In this paper we assume a strict condition:

$$Z_m\beta \le \frac{2}{\pi} - \epsilon \qquad \forall m = 1, \dots, M; \qquad \beta := \frac{\mathsf{e}^2}{\hbar c}. \tag{1.8}$$

In the non-magnetic case we were interested in $\mathsf{E} := \inf \mathsf{Spec}(\mathsf{H})$. In the magnetic case we consider only a self-generated magnetic field, that is we consider

$$\mathsf{E}^* = \inf_{A \in \mathscr{H}^1_0} \mathsf{E}(A), \tag{1.9}$$

where

$$\mathsf{E}(A) = \inf \mathsf{Spec}(\mathsf{H}_{A,V}) + \underbrace{\frac{\mathsf{e}^2}{\alpha\hbar^2} \int |\nabla \times A|^2 \, dx}, \tag{1.10}$$

$$\alpha Z_m \le \kappa^* (2\pi^{-1} - \beta Z_m)^{\frac{3}{2}} \qquad m = 1, \dots, M. \tag{1.11}$$

with a unspecified constant $\kappa^* > 0$. We also assume that $d \ge CZ^{-1}$.

Remark 1.1. (i) In the non-relativistic theory by scaling with respect to the spatial and energy variables we can make $\hbar = \mathsf{e} = \mu = 1$ while α and Z_m are preserved.

(ii) In the relativistic theory by scaling with respect to the spatial and energy variables we can make $\hbar = \mathsf{e} = \mu = 1$ while β, α and Z_m are preserved.

(iii) In the one particle case there are additional scalings with respect to the spatial and energy variables, preserving only $Z_m\alpha$ and $Z_m\beta$ (but not the Z_m, α, β).

From now on we assume that such rescaling was done and we are free to use letters $\hbar$, μ and c for other notations.

The sharp results in the non-relativistic frameworks, without magnetic field and with self-generated magnetic filed were obtained in Chapters 25 and 27 of [Ivr] respectively, and in the relativistic frameworks without magnetic field—in [Ivr2]. The transition from the non-relativistic framework to the relativistic one required mainly modifications of the function-analytic arguments in the *singular zone* $\bigcup_m \{x\colon |x - \mathsf{y}_m| \le cZ_m^{-1}\}$, and it was done in many articles, listed in the references, which we heavily use. On the other hand, transition from the non-magnetic case to the case of the self-generated magnetic field requires microlocal semiclassical arguments of Chapter 27 of [Ivr] in the *semiclassical zone* $\bigcap_m \{x\colon |x - \mathsf{y}_m| \ge cZ_m^{-1}\}$, which we also heavily rely upon. However relativistic settings require modifications of these arguments, and we are providing most of details when such modifications are needed, and are rather sketchy when no modifications are required.

2 Local Semiclassical Trace Asymptotics

2.1 Set-up

This section matches to Section 27.2 of [Ivr]. We consider potential W supported in $B(\mathsf{x}, r)$ (with $r = \ell(\mathsf{x})$ the half-distance to the nearest nucleus), and scale it to $B(0,1)$ with $W \asymp 1$.

Recall that the original non-relativistic operator is

$$\frac{1}{2}((D-A)\cdot\boldsymbol{\sigma})^2 - W, \tag{2.1}$$

which after rescaling $x \mapsto (x-\mathsf{x})/r$, $\tau \mapsto \tau/(Zr^{-1})$ becomes

$$\frac{1}{2}((hD-A')\cdot\boldsymbol{\sigma})^2 - W, \qquad h = Z^{-1/2}r^{-1/2}, \qquad A' = Z^{-1/2}r^{1/2}A, \tag{2.2}$$

while the "penalty" becomes

$$\frac{r}{\alpha}\int |\nabla\times A'|^2\,dx = \frac{1}{\kappa h^2}\int |\nabla\times A'|^2\,dx \tag{2.3}$$

with $\kappa = Z\alpha$ and we assume that $\kappa \le \kappa^*$.

What happens with our relativistic operator? The same scaling transforms $\bigl(\beta^{-2}((D-A)\cdot\boldsymbol{\sigma})^2 + \beta^{-4}\bigr)^{1/2} - \beta^{-2}$ into

$$rZ^{-1}\bigl(\beta^{-2}((r^{-1}D-A)\cdot\boldsymbol{\sigma})^2 + \beta^{-4}\bigr)^{1/2} - rZ^{-1}\beta^{-2} = \bigl(\gamma^{-2}((hD-A')\cdot\boldsymbol{\sigma})^2 + \gamma^{-4}\bigr)^{1/2} - \gamma^{-2} \tag{2.4}$$

with $\gamma = \beta h^{-1} \le 1$.

Exactly like in Subsubsection 27.2.1 of [Ivr] we need to start with the functional-analytic arguments.

2.2 Functional Analytic Arguments

Estimates

Proposition 2.1 [1]. *Let $V \in \mathscr{L}^{\frac{5}{2}} \cap \mathscr{L}^4$. Then*

$$\mathsf{E}^* \ge -Ch^{-3} \tag{2.5}$$

[1] Cf. Proposition 27.2.1 of [Ivr].

and either

$$\frac{1}{\kappa h^2}\int |\partial A|^2\,dx \le Ch^{-3} \tag{2.6}$$

or $\mathsf{E}(A) \ge ch^{-3}$.

Proof. Using Theorem A.1 (magnetic Daubechies inequality rather than magnetic Lieb-Thirring inequality) with $\gamma := \gamma h$, $V := h^{-2}V$, $A := h^{-1}A$ and with multiplication of the result by h^2, we have

$$\begin{aligned}\mathsf{Tr}(H^-_{A,V}) \ge \\ -Ch^{-3}\int \Big(V_+^{5/2} + \gamma^3 V_+^4\Big)\,dx - Ch^{-2}\Big(|\partial A|^2\,dx\Big)^{\frac14}\Big(V_+^4\,dx\Big)^{\frac34}\end{aligned} \tag{2.7}$$

(cf. (27.2.9) of [Ivr]; only the term $\gamma^3 V_+^4$ adds up); then (27.2.10) holds, which completes the proof. □

Proposition 2.2 [2]**.** *Let* $V_+ \in \mathscr{L}^{\frac52} \cap \mathscr{L}^4$, $\kappa \le ch^{-1}$ *and*

$$V \le -C^{-1}(1+|x|)^{\delta} + C. \tag{2.8}$$

Then there exists a minimizer A.

Proof. Let us consider a minimizing sequence A_j. Without any loss of the generality one can assume that $A_j \to A_\infty$ weakly in $\mathscr{H}^1$ and in $\mathscr{L}^6$ and then strongly in $\mathscr{L}^p_{\mathsf{loc}}$ with any $p < 6$ [3]. Then A_∞ is a minimizer.

Really, due to (2.6) and (2.8) negative spectra of $H_{A_j,V}$ are discrete and the number of negative eigenvalues is bounded by $N = N(h)$. Consider ordered eigenvalues $\lambda_{j,k}$ of $H_{A_j,V}$. Without any loss of the generality one can assume that $\lambda_{j,k}$ have limits $\lambda_{\infty,k} \le 0$ (we go to the subsequence if needed).

We claim that $\lambda_{\infty,k}$ are also eigenvalues and if $\lambda_{\infty,k} = ... = \lambda_{\infty,k+r-1}$ then it is eigenvalue of at least multiplicity r.

Indeed, let $u_{j,k}$ be corresponding eigenfunctions, orthonormal in $\mathscr{L}^2$. Then in virtue of A_j being bounded in $\mathscr{L}^6$ and $V \in \mathscr{L}^4$ we can estimate

$$\||D|^{1/2}u_{j,k}\| \le K\|u_{j,k}\|_6^{1-\delta}\cdot\|u_{j,k}\|^{\sigma} \le K\||D|^{1/2}u_{j,k}\|^{1-\delta}\cdot\|u_{j,k}\|^{\delta}$$

[2] Cf. Proposition 27.2.2 of [Ivr].

[3] Otherwise we select a converging subsequence.

with $\delta > 0$ which implies $\||D|^{1/2} u_{j,k}\| \le K$. Also assumption (2.8) implies that $\|(1+|x|)^{\delta/2} u_{j,k}\|$ are bounded and therefore without any loss of the generality one can assume that $u_{j,k}$ converge strongly.

Then

$$\lim_{j\to\infty} \mathsf{Tr}(H^-_{A_j,V}) \ge \mathsf{Tr}(H^-_{A_\infty,V}), \tag{2.9}$$

$$\liminf_{j\to\infty} \int |\partial A_j|^2 \, dx \ge \int |\partial A_\infty|^2 \, dx \tag{2.10}$$

and therefore $\mathsf{E}(A_\infty) \le \mathsf{E}^*$. Then A_∞ is a minimizer and there are equalities in (2.9)–(2.10) and, in particular, there no negative eigenvalues of $H_{A_\infty,V}$ other than $\lambda_{\infty,k}$. □

Properties of a Minimizer

Next, we need to study the minimizer[4)].

Proposition 2.3 [5)]**.** *Let A be a minimizer. Then*

$$\begin{aligned}\frac{2}{\kappa h^2}\Delta A_j(x) = \Phi_j \coloneqq \\ -\mathsf{Re}\,\mathsf{tr}\Big[\int_0^\infty \sigma_j((hD-A)_x\cdot\boldsymbol{\sigma})e^{-\lambda S}e(.,.,0)e^{-\lambda S}\,d\lambda\Big]\Big|_{x=y} \\ -\mathsf{Re}\,\mathsf{tr}\Big[\int_0^\infty \sigma_j e^{-\lambda S}e(.,.,0)e^{-\lambda S\,t}((hD-A)_y\cdot\boldsymbol{\sigma})\,d\lambda\Big]\Big|_{x=y}.\end{aligned} \tag{2.11}$$

where $A=(A_1,A_2,A_3)$, $\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ and $e(x,y,\tau)$ is the Schwartz kernel of the spectral projector $\theta(-H)$ of $H=H_{A,V}$ and tr is a matrix trace;

$$S=\gamma^2(T+\gamma^{-2}) = \big((\gamma^2(hD-A)\cdot\boldsymbol{\sigma})^2+1\big)^{\frac12} \tag{2.12}$$

Proof. Consider variation δA of A and variation of $\mathsf{Tr}(H^-)$ where $H^- = H\theta(-H)$ is a negative part of H. Then, like in the proof of Proposition 27.2.4,

$$\delta\,\mathsf{Tr}(H^-) = \mathsf{Tr}\big((\delta H)\theta(-H)\big). \tag{2.13}$$

[4)] We do not know if it is unique, exactly like in the non-relativistic case; see Remark 27.2.3 of [Ivr].

[5)] Cf. Proposition 27.2.4 of [Ivr]. Observe that (2.11) is more complicated than (27.2.14) of [Ivr].

But we need to find $\delta H = \gamma^{-2}\delta S$, which is a bit more tricky than in the non-relativistic case. Observe that

$$\delta(S^2) = S(\delta S) + (\delta S)S; \tag{2.14}$$

then

$$\delta S = \int_0^\infty e^{-\lambda S}\delta(S^2)e^{-\lambda S}\, d\lambda \tag{2.15}$$

while

$$\gamma^{-2}S^2 = ((hD - A)\cdot\boldsymbol{\sigma})^2 + \gamma^{-2} \tag{2.16}$$

and therefore

$$\delta(\gamma^{-2}S^2) = -\sum_j \Big(\delta A_j\sigma_j((hD - A)\cdot\boldsymbol{\sigma}) - ((hD - A)\cdot\boldsymbol{\sigma})\delta A_j\sigma_j\Big), \tag{2.17}$$

exactly like in non-relativistic case.

Therefore $\mathsf{Tr}(\delta S\theta(-H))$ is equal to the sum of $\int_0^\infty d\lambda$ of

$$\begin{aligned}
&-\mathsf{Tr}\Big(e^{-\lambda S}\delta A_j\sigma_j((hD - A)\cdot\boldsymbol{\sigma})e^{-\lambda S}\theta(-H)\Big)\\
&-\mathsf{Tr}\Big(e^{-\lambda S}((hD - A)\cdot\boldsymbol{\sigma})\delta A_j\sigma_j e^{-\lambda S}\theta(-H)\Big) =\\
&-\mathsf{Tr}\Big(\delta A_j\sigma_j((hD - A)\cdot\boldsymbol{\sigma})e^{-tS}\theta(-H)e^{-\lambda S}\Big)\\
&-\mathsf{Tr}\Big(\delta A_j\sigma_j e^{-\lambda S}\theta(-H)e^{-\lambda S}((hD - A)\cdot\boldsymbol{\sigma})\Big).
\end{aligned}$$

Then $\mathsf{Tr}(\delta L\theta(-H)) = \int \sum_j \Phi_j(x)\delta A_j$, which implies equality (2.11). □

Proposition 2.4 [6]**.** *If for* $\kappa = \kappa^*$

$$\mathsf{E}^* \ge \mathsf{Weyl}_1 - CM \tag{2.18}$$

(2.19)

with $M \ge Ch^{-1}$*, then for* $\kappa \le \kappa^*(1 - \epsilon_0)$

$$\frac{1}{\kappa h^2}\int |\partial A|^2\, dx \le C_1 M. \tag{2.20}$$

[6] Cf. Proposition 27.2.5 of [Ivr].

Proof. Proof is obvious, also based on the upper estimate $\mathsf{E}^* \le \mathsf{E}(0) \le \mathsf{Weyl}_1 + Ch^{-1}$, which is due to [Ivr2]. □

Proposition 2.5 [7]**.** *Let estimate* (2.20) *be fulfilled and let*

$$\varsigma = \kappa M h \le c. \tag{2.21}$$

Then for $\tau \le c$

(i) Operator norm in $\mathscr{L}^2$ *of* $(hD)^k \theta(\tau - H)$ *does not exceed* C *for* $k = 0, 1$.

(ii) Operator norm in $\mathscr{L}^2$ *of* $(hD)^k \big((hD - A)\cdot\boldsymbol{\sigma}\big)\theta(\tau - H)$ *does not exceed* C *for* $k = 0$.

Proof. First, let us repeat of some arguments of the proof of Proposition 27.2.6 of [Ivr]. Let $u = \theta(\tau - H)f$. Then $\|u\| \le \|f\|$ and since

$$\|A\|_{\mathscr{L}^6} \le C\|\partial A\| \le C(\kappa M)^{\frac12} h, \tag{2.22}$$

we conclude that

$$\begin{aligned}\|hDu\| \le \|(hD - A)u\| + \|Au\| \le \|(hD - A)u\| + C\|A\|_{\mathscr{L}^6}\cdot\|u\|_{\mathscr{L}^3} \le \\ \|(hD - A)u\| + C(\kappa M)^{\frac12} h\|u\|^{1/2}\cdot\|u\|_{\mathscr{L}^6}^{1/2} \le \\ \|(hD - A)u\| + C(\kappa M h)^{\frac12}\|u\|^{1/2}\cdot\|hDu\|^{1/2} \le \\ \|(hD - A)u\| + \frac{1}{2}\|hDu\| + C\kappa M h\|u\|;\end{aligned}$$

therefore due to (2.21)

$$\|hDu\| \le 2\|(hD - A)u\| + C\kappa M h\|u\|. \tag{2.23}$$

Further, $\|Tu\| \le c_1\|u\|$ because $H \ge -c$, $|V| \le c$, $|\tau| \le c$; then $\|(T + \gamma^{-2})u \le (c_1 + \gamma^{-2})\|u\|$ and therefore

$$(((T + \gamma^{-2})^2 - \gamma^{-4})u, u) \le ((c_1 + \gamma^{-2})^2 - \gamma^{-4}) = (2c_1\gamma^{-2} + c_1^2)\|u\|^2,$$

and finally

$$(Lu, u) \le C\|u\|^2 \tag{2.24}$$

[7] Cf. Proposition 27.2.6 of [Ivr].

with

$$L := ((hD - A)\cdot\boldsymbol{\sigma})^2. \tag{2.25}$$

Then, again following the same proof, we conclude that

$$\|(hD - A)u\| \le C\|u\| \quad \text{and} \quad \|hDu\| \le C(1+\kappa Mh)\|u\|, \tag{2.26}$$

provided $\kappa M h^{1+\delta} \le c$ for sufficiently small $\delta > 0$. Therefore under assumption (2.21) for $j = 0, 1$ both Statements (i) and (ii) are proven. □

Thus, in contrast to Proposition 27.2.6 of [Ivr], we do not have $k = 2$ in Statement (i), and $k = 1$ in Statement (ii) so far and need some extra arguments.

Proposition 2.6. *Assume that* $\|V\|_{\mathscr{C}^2} \le c$. *Then*

$$\|[S, V]u\| \le Ch\gamma^2(\|L^{\frac12}u\| + \|u\|). \tag{2.27}$$

Proof. Recall that that $S^2 = \gamma^2 L + 1$. Therefore $\gamma^2[L, V] = [S, V]S + S[S, V]$ and then

$$[S, V] = \gamma^2 \int_0^\infty e^{-\lambda S}[L, V]e^{-\lambda S}\, d\lambda. \tag{2.28}$$

Also

$$\|[L, V]w\| \le Ch(\|L^{1/2}w\| + \|w\|) \tag{2.29}$$

and

$$\|e^{-\lambda S}\| \le e^{-\lambda}. \tag{2.30}$$

Proposition 2.7. *(i) Assume that* $\|V\|_{\mathscr{C}^2} \le c$ *and* $|\tau| \le c$. *Then the operator norm of* $(hD)^k\theta(\tau - H)$ *does not exceed* C *for* $k = 0, 1, 2$.

(ii) Assume that $\|V\|_{\mathscr{C}^3} \le c$ *and* $|\tau| \le c$. *Then the operator norm of operators* $(hD)^k((hD - A)_x\cdot\boldsymbol{\sigma})\theta(\tau - H)$ *and* $(hD)^k\hat{\Phi}_j\theta(\tau - H)$ *with*

$$\hat{\Phi}_j = \int_0^\infty \sigma_j((hD - A)_x\cdot\boldsymbol{\sigma})e^{-\lambda S}e(.,.,0)e^{-\lambda S}\, d\lambda \tag{2.31}$$

do not exceed C *for* $k = 0, 1, 2$.

Proof. (i) Let $u = \theta(\tau - H)f$ with $f \in \mathscr{L}^2$. Then u satisfies (2.2) and $\|Tu\| \le C\|u\|$. Also, which implies

$$\gamma^{-2}\|Lu\| = \|(T + 2\gamma^{-2})Tu\| \le C\gamma^{-2}\|u\| + C\gamma^{-1}\|[T, V]u\| \le C_1\gamma^{-2}\|u\|$$

due to (2.27) . Then, repeating arguments of the proof of Propositiob 27.2.6 of [Ivr], we conclude that $\|(hD)^2 u\| \le C\|u\|$, i.e. Statement (i).

(ii) Plugging $(T - V - \tau)u$ instead of u (with $\|(T - V - \tau)u\| \le C\|u\|$) we have $\|L(T - \tau - V)u\| \le C\|u\|$. Then

$$\|TLu\| \le C\|Lu\| + C\|[L, V]u\| \le C(\|Lu\| + \|u\|) \le C_1\|u\|.$$

Again plugging $(T - V - \tau)u$ instead of u we have

$$\gamma^{-2}\|L^2 u\| = \|(T + 2\gamma^{-2})TLu\| \le \\ \|T(T - V - \tau)Lu\| \le C\gamma^{-2}\|Lu\| + C\|T[L, V]u\|.$$

Further, the last term does not exceed $Ch\|TV'L^{\frac12}u\| + Ch^2\|TV''u\|$ where V' are miscellaneous first derivatives of V and $V'' = \Delta V$. Then, the former does not exceed $C\|Lu\|$, while the latter does not exceed $C\gamma^{-2}\|u\| + h\|\nabla(V''u)\|$, which does not exceed $C\gamma^{-2}\|u\|$.

Therefore $\|L^2 u\| \le C\|u\|$, which implies that $\|L((hD - A)\cdot\sigma)u\| \le C\|u\|$, which in turn implies that $\|(hD)^2((hD - A)\cdot\sigma)u\| \le C\|u\|$ and, finally, $\|(hD)^2\hat{\Phi}_j u\| \le C\|u\|$. □

Corollary 2.8. *(i) Assume that $\|V\|_{\mathscr{C}^2} \le c$ and $|\tau| \le c$. Then the operator norm of $\theta(\tau - H)$ from $\mathscr{L}^2$ to $\mathscr{C}^\delta$ does not exceed $Ch^{-3/2-\delta}$ for $0 \le \delta \le \frac12$.*

(ii) Assume that $\|V\|_{\mathscr{C}^3} \le c$ and $|\tau| \le c$. Then the operator norm of $\hat{\Phi}_j$ from $\mathscr{L}^2$ to $\mathscr{C}^\delta$ do not exceed $Ch^{-3/2-\delta}$ for $0 \le \delta \le \frac12$. Then $\|\Phi_j\|_{\mathscr{C}} \le Ch^{-3}$.

Corollary 2.9. *Under assumptions* (2.18)–(2.21) $\|A_j\|_{\mathscr{C}^{2-\delta}} \le C\kappa h^{-1}$ *for any $\delta > 0$.*

2.3 Microlocal Analysis and Local Theory

Microlocal Analysis Unleashed

Then we can apply all arguments of Subsection 27.2.2[8)] of [Ivr], even if expression for Φ_j differs. Indeed, observe first that we can restrict ourselves by $0 \le \lambda \le c|\log h|$. Then, using our standard arguments based on the analysis of the propagation of singularities, we can prove that the Tauberian expression with $T = h^{1-\delta}$ for Φ_j has an error $O(h^{-2})$ provided

$$V(x) \asymp 1. \tag{2.32}$$

Then our standard trick with the freezing coefficients works and with the same $O(h^{-2})$ error we can replace $\Phi_j(x)$ by its Weyl expression, i.e. expression we obtain if replace operators by their symbols, depending on x and ξ, integrating by $d\xi$ and multiplying by $(2\pi h)^{-3}$. However due to skew-symmetry with respect to $\xi - A(x)$, this Weyl expression is 0, and $\Phi_j(x) = O(h^{-2})$.

Finally, we can get rid of assumption (2.32) by the standard rescaling arguments. We leave all the details to the reader.

Local Theory and Rescaling

Then we can apply all arguments of Subsection 27.2.3 [9)] of [Ivr]. As a result we arrive under assumption (2.21) to the trace formula[10)] with the remainder estimate $O(h^{-1})$ and to estimate $\|\partial A\| = O(\kappa^{1/2}h^{1/2})$.

Moreover, under the standard assumption of the global nature we arrive to the trace formula with the remainder estimate $o(h^{-1})$ (but it will have the Schwinger-type correction term) and to estimate $\|\partial A\| = o(\kappa^{1/2}h^{1/2})$.

Finally, we an apply all arguments of Subsection 27.2.4 [11)] of [Ivr] and we weaken assumption (2.21), recovering the same estimates as before. Again, we leave all the details to the reader.

[8)] Namely of Subsubsections *27.2.2.1. Sharp Estimates*, *27.2.2.2. Application* and *27.2.2.3. Classical Dynamics and Sharper Estimates*.

[9)] Namely, Subsubsections *27.2.3.1. Localization and Estimate from above* and *27.2.3.2. Estimate from below*.

[10)] In the trace formula "Weyl_1" is given by the relativistic expression, $-P^{\mathsf{RTF}}(W+\nu)$.

[11)] Namely, Subsubsection *27.2.4.1. Case* $\kappa \le 1$ and Subsubsection *27.2.4.2. Case* $1 \le \kappa \le h^{-1}$.

3 Global Trace Asymptotics in the Case of Coulomb-Like Singularities

3.1 Estimates to a Minimizer

Let us return to the original settings, with Coulomb-like singularities and parameters Z_m, α, β. At the moment we consider the one-particle Hamiltonian. Let us deal first with the vicinity of y_m.

Then we scale like in Section 27.3 of [Ivr]: $x \mapsto Z^{\frac{1}{3}}x$, $\tau \mapsto Z^{-\frac{4}{3}}\tau$, $A \mapsto Z^{-\frac{2}{3}}A$, $\beta \mapsto \beta Z^{\frac{3}{3}}$, arriving to the semiclassical problem with Coulomb singularities $z_m|x-\mathsf{y}_m|^{-1}$ ($z_m = Z_m Z^{-1}$), with $h = Z^{-\frac{1}{3}}$ and with $\kappa = \alpha Z^{\frac{2}{3}}$. In particular, E^* is a minimum with respect to A of

$$\mathsf{E}(A) \coloneqq \mathsf{Tr}(H^-_{A,W+\tau}) + \kappa^{-1}h^{-2}\|\partial A\|^2. \tag{3.1}$$

Let us follow arguments of Subsubsection *27.3.2.1 Preliminary Analysis*. Observe first that the estimate from above is

$$\mathsf{E}^* \le h^{-3}\int \mathsf{Weyl}_1(x)\,dx + Ch^{-2}; \tag{3.2}$$

we simply take $A = 0$ and refer to [Ivr2][12)].

Consider now estimate from below and apply ℓ-admissible partition exactly like in Subsection 27.3.2 of [Ivr]. Then, according to the previous section, for any element of partition with $\ell \ge ch^{-2}$ ($\ell \ge cZ_m^{-1}$ in the original settings) its contribution is estimated from below by the corresponding Weyl expression minus $Ch^{-1}\zeta^2 \times \zeta\ell = Ch^{-1}\zeta^3\ell^{-1}$ and summation with respect to these elements returns $\mathcal{E}^{\mathsf{TF}}$ minus $Ch^{-1}\zeta^3\ell|_{\ell=h^{-2}}$, i.e.

$$h^{-3}\int \mathsf{Weyl}_1(x)\,dx - Ch^{-2} \tag{3.3}$$

because the contribution of the zone $\mathcal{Z}_0 = \{x\colon \ell(x) \le ch^{-2}\}$ to the main term is $O(h^{-2})$.

On the other hand, the contribution of $\mathcal{Z}_0$ is $-Ch^{-2}$. Indeed, scale first $x \mapsto h^{-2}x$, $\tau \mapsto h^2\tau$, $h \mapsto 1$, $A \mapsto hA$, $\beta \mapsto \gamma = \beta h^{-1}$, and the Coulomb

[12)] In the original settings the remainder estimate would be $O(Z^2)$ exactly as in the non-relativistic case.

singularity remains the same while the magnetic energy becomes $\kappa^{-1}\|\partial A\|^2$. Observe that $\gamma = \beta_{\mathsf{orig}} Z_m$ [13], so (1.8), (1.11) become

$$\beta \le 2\pi^{-1} - \epsilon, \qquad \kappa \le \kappa^*\big(2\pi^{-1} - \beta\big). \tag{3.4}$$

Then we can apply a "singular magnetic Daubechies inequality" (A.3) and repeat all arguments of the regular case in a simple case of $h = 1$. There will be an extra terms $O(1)$ and $-C(1-\pi\gamma/2)^{-\frac{3}{2}}\|\partial A\|^2$ and that latter term requires (1.11).

Now we conclude that Proposition 27.3.1 of [Ivr] holds:

Proposition 3.1 [14]**.** *In our framework $\kappa \le \kappa^*$. Then the near-minimizer A satisfies*

$$\Big|\int \big(\operatorname{tr} e_{A,1}(x,x,0) - \mathsf{Weyl}_1(x)\big)\,dx\Big| \le Ch^{-2} \tag{3.5}$$

and

$$\|\partial A\| \le C\kappa^{\frac{1}{2}}. \tag{3.6}$$

It allows us to repeat arguments of the proof Proposition 2.2 and to prove

Proposition 3.2 [15]**.** *In our framework there exists a minimizer A* [4]*.*

Now we can repeat arguments of Subsubsection *27.3.2.2 Estimates to a Minimizer. I* of [Ivr], albeit with the right-hand expression of (27.3.14) given now by (2.11) and to prove the claim (27.3.28), which is marginally stronger than

$$\|\partial^2 A\|_{\mathscr{L}^\infty(B(0,1-\epsilon))} \le C\kappa^{\frac{1}{2}} h^{-\delta}. \tag{3.7}$$

Then we can repeat arguments of Subsubsection *27.3.2.3 Estimates to a Minimizer. II* of [Ivr] and recover Propositions 27.3.4, 27.3.6 and 27.3.7, estimating A and its derivatives as $\ell(x) \lesssim 1$:

---

[13] Considering vicinity of y_m it is more convenient to take the original rescaling with Z replaced by Z_m, and therefore $z_m = 1$.

[14] Cf. Proposition 27.3.1 of [Ivr].

[15] Cf. Proposition 27.3.2 of [Ivr].

Proposition 3.3 [16]**.** *In our framework if* $\ell(x) \ge \ell_* := h^2$*, then*

$$|A| \le C\kappa\ell^{-\frac{1}{2}}, \qquad |\partial A| \le C\kappa\ell^{-\frac{3}{2}} \tag{3.8}$$

and

$$|\partial A(x) - \partial A(y)| \le C_\theta \kappa \ell^{-\frac{3}{2}-\theta}|x-y|^\theta \qquad \text{as } \ |x-y| \le \frac{1}{2}\ell(x) \tag{3.9}$$

for any $\theta \in (0,1)$.

Consider now the non-semiclassical zone $\{x\colon \ell(x) \lesssim \ell_*\}$, which contains the relativistic zone $\{x\colon \ell(x) \lesssim \bar{\ell} := \gamma h\}$. Using arguments of the proof of Proposition 3.4 of [Ivr2], but additionally taking care of the magnetic field using arguments of the proofs of Propositions 2.5, 2.6 and 2.7 (we leave all details to the reader), we arrive to

Proposition 3.4. *In our framework* $H_{W,A} \ge C_0\ell_*^{-1}$ *and* $e(x,x,\lambda) \le C\ell_*^{-3}$ *for* $\ell(x) \le c\ell_*$ *and* $|\lambda| \le Ch^{-2}$.

Remark 3.5. (i) Then in the original settings $H_{W,A} \ge C_0 Z^{-2}$ and $e(x,x,\lambda) \le CZ^3$ for $\ell(x) \le cZ^{-1}$ and $|\lambda| \le C_0 Z^2$.

(ii) We have a better estimate than (3.11) of [Ivr2] due to assumptions (1.8) and (1.11).

Next, we follow arguments of Subsubsection *27.3.2.4 Estimates to a Minimizer. III* of [Ivr] and prove (again, leaving details to the reader)

Proposition 3.6 [17]**.** *In our framework*

$$|A| \le C\kappa\ell^{-2}, \qquad |\partial A| \le C\kappa\ell^{-3} \tag{3.10}$$

and

$$|\partial A(x) - \partial A(y)| \le C_\theta \kappa \ell^{-3-\theta}|x-y|^\theta \qquad \text{as } \ |x-y| \le \frac{1}{2}\ell(x) \tag{3.11}$$

if $\ell(x) \ge 1$ *(for all* $\theta \in (0,1)$*).*

[16] Cf. Proposition 27.3.7(i) of [Ivr].
[17] Cf. Proposition 27.3.9 of [Ivr].

3.2 Trace Estimates

Next we can go after trace asymptotics. Recall that we are dealing with the rescaled operator. Let a be the minimal distance between nuclei (after rescaling), capped by 1; recall that $a \ge \ell_*$.

After we estimated A for $\ell(x) \lesssim 1$ in Proposition 3.1 and $e(x,x,\lambda)$ for $\ell(x) \lesssim \ell_* = h^{-2}$, we can apply arguments of Subsection 27.3.3 of [Ivr] and arrive to

Proposition 3.7 [18]**.** *In our framework let ψ be a-admissible and supported in $\frac{1}{2}a$-vicinity of y_m, let φ be ℓ_*-admissible, supported in $2\ell_*$-vicinity and equal 1 in ℓ_*-vicinity of y_m, and let $V^0 = Z_m|x|^{-1}$. Then*

$$\mathsf{Tr}\big(\psi(H^-_{A,V} - H^-_{A,V^0})\psi\big) = \int \big(\mathsf{Weyl}_1(x) - \mathsf{Weyl}^0_1(x)\big)(1-\varphi(x))\,dx + O\big(a^{-\frac{1}{3}}h^{-\frac{4}{3}}\big). \tag{3.12}$$

Remark 3.8. Here and in Proposition 3.9 Weyl and Weyl_1 are defined for the relativistic operator (i.e. $\mathsf{Weyl} = P^{\mathsf{RTF}\prime}(V)$ and $\mathsf{Weyl} = -P^{\mathsf{RTF}}(V)$), but following arguments of 3.6 of [Ivr2], we can replace it by those for non-relativistic operator (i.e. $\mathsf{Weyl} = P^{\mathsf{TF}\prime}(V)$ and $\mathsf{Weyl} = -P^{\mathsf{TF}}(V)$) and then skip the factor $(1-\varphi(x))$.

Moreover, applying arguments of Subsection 27.3.4 of [Ivr] we arrive to

Proposition 3.9 [19]**.** *(i) In the framework of Proposition 3.7*

$$\mathsf{Tr}\big(\psi(H^-_{A,V} - H^-_{A,V^0})\psi\big) = \int \big(\mathsf{Weyl}_1(x) - \mathsf{Weyl}^0_1(x)\big)\,\psi^2(x)(1-\varphi(x))\,dx + O\big(h^{-\frac{4}{3}}a^{-\frac{1}{3}}\kappa|\log\kappa|^{\frac{1}{3}} + h^{-1}a^{-\frac{1}{2}}\big). \tag{3.13}$$

(ii) In particular, if

$$\kappa \le ca^{-\frac{1}{6}}h^{\frac{1}{3}}|\log ah^{-2}|^{-\frac{1}{3}}, \tag{3.14}$$

then the error in (3.13) *does not exceed $Ch^{-1}a^{-\frac{1}{2}}$ exactly as in the case without magnetic field.*

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[18] Cf. Proposition 27.3.11 of [Ivr].

[19] Cf. Proposition 27.3.16 of [Ivr].

Next consider the case of exactly Coulomb potential $V = Z|x|^{-1}$ and $\nu = 0$. Then

Proposition 3.10 [20]**.** *Let $V = Z|x|^{-1}$, $h > 0$, $Z > 0$, and* (1.8) *and* (1.11) *be fulfilled. Then*

(i) The following limit exists [21]

$$\lim_{r\to\infty}\Biggl(\inf_{A}\Bigl(\mathsf{Tr}\bigl((\phi_r H_{A,V}\phi_r)^-\bigr)+\frac{1}{\kappa h^2}\int |\partial A|^2\,dx\Bigr) - \int \mathsf{Weyl}_1(x)\phi_r^2(x)\,dx\Biggr) =: 2Z^2h^{-2}S(Z\kappa, Z\beta). \tag{3.15}$$

(ii) And it coincides with (27.3.72) *and also with* (27.3.73) *of [Ivr].*

(iii) We also can replace in Statement (i) $\mathsf{Tr}\bigl((\phi_r H_{A,V}\phi_r)^-\bigr)$ *by* $\mathsf{Tr}\bigl(\phi_r H^-_{A,V}\phi_r\bigr)$.

Here $\phi \in \mathscr{C}_0^\infty(B(0,1))$, $\phi = 1$ in $B(0,\frac12)$, $\phi_r = \phi(x/r)$ and Weyl *and* Weyl_1 *are defined for non-relativistic operator.*

Then we also arrive to

Proposition 3.11 [22]**.** *In the framework of Proposition 3.10 for $0 < \kappa < \kappa'$, $\beta < \beta'$*

$$S(\kappa',\beta) \le S(\kappa,\beta) \le S(\kappa',\beta) + C\kappa'(\kappa^{-1}-\kappa'^{-1}), \tag{3.16}$$

$$S(\kappa,\beta) \le S(\kappa',\beta'). \tag{3.17}$$

Then, in the "atomic" case $M = 1$ we arrive instantly to the following theorem:

Theorem 3.12 [23]**.** *Let $M = 1$ and* (1.8) *and* (1.11) *be fulfilled. Then*

(i) The following asymptotics holds

$$\mathsf{E}^* = \int \mathsf{Weyl}_1(x)\,dx + 2z^2S(z\kappa, z\beta)h^{-2} + O(h^{-\frac43}\kappa|\log\kappa|^{\frac13} + h^{-1}). \tag{3.18}$$

[20] Cf. Proposition 27.3.18 of [Ivr].
[21] Cf. (27.3.71) of [Ivr] and (3.18) of [Ivr2].
[22] Cf. Proposition 27.3.20 of [Ivr] and Remark 3.8 of [Ivr2].
[23] Cf. Theorem 27.3.22 of [Ivr] and Propositions 3.9 and 3.10 of [Ivr2].

(ii) If $\kappa = o(h^{\frac{1}{3}}|\log h|^{-\frac{1}{3}})$, *then*

$$\mathsf{E}^* = \int \mathsf{Weyl}_1^*(x)\,dx + 2z^2 S(z\kappa, z\beta)h^{-2} + o(h^{-1}), \tag{3.19}$$

in which case Weyl_1^* *must in addition to* $-h^{-3}P^{\mathsf{TF}}(W+\nu)$ *contain the Schwinger correction, and also the relativistic correction.*

Next, using arguments Subsection 27.3.6 of [Ivr], in particular, decoupling of singularities (which is needed only in the case ofthe self-generated magnetic field), we arrive to

Theorem 3.13 [24]**.** *Let* $M \ge 2$, $\kappa \le \kappa^*$ *and* (1.8) *and* (1.11) *be fulfilled. Then*

(i) The following asymptotics holds

$$\mathsf{E}^* = \int \mathsf{Weyl}_1(x)\,dx + 2\sum_{1\le m\le M} z_m^2 S(z_m\kappa, z_m\beta)h^{-2} + O(R_1+R_2) \tag{3.20}$$

with

$$R_1 = \begin{cases} h^{-1} + \kappa|\log\kappa|^{\frac{1}{3}}h^{-\frac{4}{3}} & \text{if } \ a \ge 1, \\ a^{-\frac{1}{2}}h^{-1} + \kappa|\log\kappa|^{\frac{1}{3}}a^{-\frac{1}{3}}h^{-\frac{4}{3}} & \text{if } \ h^2 \le a \le 1 \end{cases} \tag{3.21}$$

(3.22)

and

$$R_2 = \kappa h^{-2}\begin{cases} a^{-3} & \text{if } \ a \ge |\log h|^{\frac{1}{3}}, \\ |\log h^2 a^{-1}|^{-1} & \text{if } \ h^2 \le a \le |\log h|^{\frac{1}{3}}. \end{cases} \tag{3.23}$$

(ii) If $\kappa = o(h^{\frac{1}{3}}|\log h|^{-\frac{1}{3}})$, $\kappa a^{-3} = o(h)$ *and* $a^{-1} = o(1)$, *then*

$$\mathsf{E}^* = \int \mathsf{Weyl}_1^*(x)\,dx + 2\sum_{1\le m\le M} z_m^2 S(z_m\kappa, z_m\beta)h^{-2} + o(h^{-1}). \tag{3.24}$$

[24] Cf. Theorem 27.3.24 of [Ivr].

4 Main Results

4.1 Asymptotics of the Ground State Energy

Now we can apply arguments of Section 27.4. In addition to (1.8) and (1.11) we assume that

$$d := \min_{1\le m<m'\le M} |\mathsf{y}_m - \mathsf{y}_{m'}| \ge Z^{-1}, \tag{4.1}$$

$$N \asymp Z_1 \asymp ... \asymp Z_M. \tag{4.2}$$

Then the estimates from below follow immediately from the trace asymptotics, while for the estimate from above we need also estimate N and miscellaneous D-terems. We leave all the details to the reader.

Theorem 4.1 [25]**.** *(i) Under assumptions* (1.8), (1.11), (4.1) *and* (4.2) *the following asymptotics holds*

$$\mathsf{E}^*_N = \mathcal{E}^{\mathsf{TF}}_N + \sum_{1\le m\le M} 2Z_m^2 S(\alpha Z_m, \beta Z_m) + O\big(Z^{\frac{4}{3}}(R_1+R_2)\big) \tag{4.3}$$

with R_1 and R_2 defined by (3.21) *and* (3.23) *respectively with $\kappa = \alpha Z$, $h = Z^{-\frac{1}{3}}$ and $a = Z^{\frac{1}{3}}d$, d is defined by* (4.1), *$d = \infty$ for $M = 1$.*

(ii) In particular, under assumption $d \gtrsim Z^{-\frac{1}{3}}$ the following asymptotics holds

$$\mathsf{E}^*_N = \mathcal{E}^{\mathsf{TF}}_N + \sum_{1\le m\le M} 2Z_m^2 S(\alpha Z_m, \beta Z_m) + O\big(\alpha|\log(\alpha Z)|^{\frac{1}{3}} Z^{\frac{25}{9}} + Z^{\frac{5}{3}} + \alpha d^{-3}Z^2\big). \tag{4.4}$$

Theorem 4.2 [26]**.** *(i) Let assumptions* (1.8), (1.11), (4.1) *and* (4.2) *be fulfilled and let $\Psi = \Psi_{\boldsymbol{A}}$ be a ground state for a near optimizer $\boldsymbol{A}$ of the original multiparticle problem. Then*

$$\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}}) \le CZ^{\frac{5}{3}}. \tag{4.5}$$

[25] Cf. Theorem 27.4.3 of [Ivr].

[26] Cf. Theorem 27.4.4 of [Ivr].

(ii) Furthermore, if $d \ge Z^{-\frac{1}{3}}$, *then*

$$\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}},\, \rho_\Psi - \rho^{\mathsf{TF}}) \le CZ^{\frac{5}{3}}\big(Z^{-\delta} + (dZ^{\frac{1}{3}})^{-\delta} + (\alpha Z)^{\delta}\big). \tag{4.6}$$

Theorem 4.3 [27]**.** *Let assumptions* (1.8), (1.11), (4.1) *and* (4.2) *be fulfilled, and let* $\alpha \le Z^{-\frac{10}{9}}|\log Z|^{-\frac{1}{3}}$. *Then*

$$\mathsf{E}^*_N = \mathcal{E}^{\mathsf{TF}}_N + \sum_{1\le m\le M} 2Z_m^2 S(\alpha Z_m, \beta Z_m) + \mathsf{Dirac} + \mathsf{Schwinger} + \mathsf{RCT} + O\big(\alpha|\log(\alpha Z)|^{\frac{1}{3}} Z^{\frac{25}{9}} + Z^{\frac{5}{3}-\delta} + \alpha d^{-3}Z^2\big) \tag{4.7}$$

where Dirac *and* Schwinger *are* Dirac *and* Schwinger correction terms *defined exactly as in non-magnetic non-relativistic case by* (25.1.29) *and* (25.1.30) *of [Ivr] respectively, and* RCT *is relativistic correction term, defined as in the non-magnetic case by* (3.23) *of [Ivr2].*

Theorem 4.4 [28]**.** *Let assumptions* (1.8), (1.11) *and* (4.2) *be fulfilled. Let us consider* $\mathsf{y}_m = \mathsf{y}^*_m$ *minimizing the full energy*

$$\widehat{\mathsf{E}}^*_N := \mathsf{E}^*_N + \sum_{1\le m<m'\le M} Z_m Z_{m'}|\mathsf{y}_m - \mathsf{y}_{m'}|^{-1}. \tag{4.8}$$

Then

$$d \ge \min\big(Z^{-\frac{5}{21}+\delta},\, Z^{-\frac{5}{21}}(\alpha Z)^{-\delta},\, \alpha^{-\frac{1}{4}}Z^{-\frac{1}{2}}\big) \tag{4.9}$$

and in the remainder estimates in (4.4) *and* (4.7) *one can skip* d*-connected terms; so we arrive to*

$$\mathsf{E}^*_N = \mathcal{E}^{\mathsf{TF}}_N + \sum_{1\le m\le M} 2Z_m^2 S(\alpha Z_m, \beta Z_m) + O\big(\alpha|\log(\alpha Z)|^{\frac{1}{3}} Z^{\frac{25}{9}} + Z^{\frac{5}{3}}\big) \tag{4.10}$$

and

$$\mathsf{E}^*_N = \mathcal{E}^{\mathsf{TF}}_N + \sum_{1\le m\le M} 2Z_m^2 S(\alpha Z_m) + \mathsf{Dirac} + \mathsf{Schwinger} + \mathsf{RCT} + O\big(\alpha|\log(\alpha Z)|^{\frac{1}{3}} Z^{\frac{25}{9}} + Z^{\frac{5}{3}-\delta}\big) \tag{4.11}$$

respectively and also the same asymptotics with $\widehat{\mathsf{E}}^*_N$ *and* $\widehat{\mathcal{E}}^{\mathsf{TF}}_N$ *instead of* E^*_N *and* $\mathcal{E}^{\mathsf{TF}}_N$.

[27] Cf. Theorem 27.4.5 of [Ivr].
[28] Cf. Theorem 27.4.6 of [Ivr].

4.2 Related Problems

After Theorems 4.1–4.4 are proven, we can apply arguments of Sections 25.5 and 25.6 of [Ivr].

Theorem 4.5 [29]**.** *Let assumptions* (1.8), (1.11) *and* (4.2) *be fulfilled.*

(i) In the framework of the fixed nuclei model let us assume that $\mathsf{I}^*_N := \mathsf{E}^*_{N-1} - \mathsf{E}^*_N > 0$. *Then*

$$(N-Z)_+ \le CZ^{\frac{5}{7}} \begin{cases} 1 & \text{if } \ d \le Z^{-\frac{1}{3}}, \\ Z^{-\delta} + (dZ^{\frac{1}{3}})^{-\delta} + (\alpha Z)^{\delta} & \text{if } \ d \ge Z^{-\frac{1}{3}}. \end{cases} \tag{4.12}$$

(ii) In particular, for a single atom and for molecule with $d \ge Z^{-\frac{1}{3}+\delta}$

$$(N-Z)_+ \le Z^{\frac{5}{7}}\bigl(Z^{-\delta} + (\alpha Z)^{\delta}\bigr). \tag{4.13}$$

(iii) In the framework of the free nuclei model let us assume that $\widehat{\mathsf{I}}^*_N := \widehat{\mathsf{E}}^*_{N-1} - \widehat{\mathsf{E}}^*_N > 0$. *Then estimate* (4.13) *holds.*

Theorem 4.6 [30]**.** *Let assumptions* (1.8), (1.11) *and* (4.2) *be fulfilled and let* $N \ge Z - C_0 Z^{\frac{5}{7}}$. *Then*

(i) In the framework of the fixed nuclei model

$$\mathsf{I}^*_N \le CZ^{\frac{20}{21}}. \tag{4.14}$$

(ii) In the framework of the free nuclei model with $N \ge Z - C_0 Z^{\frac{5}{7}}\bigl(Z^{-\delta} + \alpha Z^{\delta}\bigr)$

$$\widehat{\mathsf{I}}^*_N := \widehat{\mathsf{E}}^*_{N-1} - \widehat{\mathsf{E}}^*_{N-1} \le Z^{\frac{20}{21}}\bigl(Z^{-\delta'} + (\alpha Z)^{\delta'}\bigr). \tag{4.15}$$

Theorem 4.7 [31]**.** *Let assumptions* (1.8), (1.11) *and* (4.2) *be fulfilled and let* $N \le Z - C_0 Z^{\frac{5}{7}}$. *Then in the framework of the fixed nuclei model under assumption* $b \ge C_1 (N-Z)^{-\frac{1}{3}}$

$$(\mathsf{I}^*_N + \nu)_+ \le C(Z-N)^{\frac{17}{18}} Z^{\frac{5}{18}} \begin{cases} 1 & \text{if } \ d \le Z^{-\frac{1}{3}}, \\ Z^{-\delta} + (dZ^{\frac{1}{3}})^{-\delta} & \text{if } \ d \ge Z^{-\frac{1}{3}}. \end{cases} \tag{4.16}$$

[29] Cf. Theorem 27.5.1 of [Ivr].
[30] Cf. Theorem 27.5.2 of [Ivr].
[31] Cf. Theorem 27.5.3 of [Ivr].

Theorem 4.8 [32]**.** *Let assumptions* (1.8), (1.11). *Then in the framework of free nuclei model with* $M \geq 2$ *the stable molecule does not exist unless*

$$Z - N \leq Z^{\frac{5}{7}}\big(Z^{-\delta} + (\alpha Z)^{\delta}\big). \tag{4.17}$$

A Some iInequalities

In this section we reproduce from [EFS2]: two new Lieb-Thirring type inequalities for the relativistic kinetic energy with a magnetic field.

Theorem A.1 [33]**.** *There exists a universal constant* $C > 0$ *such that for any positive number* $\gamma > 0$, *for any potential* V *with* $V_+ \in \mathscr{L}^{5/2} \cap \mathscr{L}^4(\mathbb{R}^3)$, *and magnetic field* $B = \nabla \times A \in \mathscr{L}^2(\mathbb{R}^3)$, *we have*

$$\begin{aligned}\mathsf{Tr}\Big(\big(\sqrt{\gamma^{-2}(D-A)\cdot\boldsymbol{\sigma})^2+\gamma^{-4}}-\gamma^{-2}-U(x)\big)^{-}\Big) \geq \\ -C\bigg\{\int U_+^{5/2}\,dx+\gamma^3\int U_+^4\,dx+\Big(\int|\nabla\times A|^2\,dx\Big)^{3/4}\Big(\int U_+^4\,dx\Big)^{1/4}\bigg\}.\end{aligned} \tag{A.1}$$

Notice that Theorem A.1 reduces to the well-known Daubechies inequality in the case $A = 0$ [Dau].

For the Schrödinger case, the Daubechies inequality was generalized (and improved to incorporate a critical Coulomb singularity) to non-zero A in [FLS] by using diamagnetic techniques. Theorem A.1 is the generalization of the Daubechies inequality for the Pauli operator, in which case there is no diamagnetic inequality. Moreover, in the $\gamma \to 0$ limit, (A.1) converges to the magnetic Lieb-Thirring inequality for the Pauli operator [LLS] since

$$\sqrt{\gamma^{-2}(D-A)\cdot\boldsymbol{\sigma})^2+\gamma^{-4}}-\gamma^{-2} \to \frac{1}{2}(D-A)\cdot\boldsymbol{\sigma})^2, \qquad \gamma\to 0. \tag{A.2}$$

Theorem A.1 does not cover the case of a Coulomb singularity. The next result shows that for γ smaller than the critical value $2/\pi$, the Coulomb singularity can be included.

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[32] Cf. Theorem 27.5.6 of [Ivr].

[33] Theorem 2.2 of [EFS2].

[34] Theorem 2.3 of [EFS2].

Theorem A.2 [34]**.** *Let ϕ_r be a real function satisfying* $\operatorname{supp}\phi_r \subset \{|x| \le r\}$, $\|\phi_r\|_\infty \le 1$. *There exists a constant $C > 0$ such that if $\gamma \in (0, 2/\pi)$, then*

$$\begin{aligned}(\text{A.3}) \quad & \operatorname{Tr}\Big(\phi_r\big(\sqrt{\gamma^{-2}(D-A)\cdot\boldsymbol{\sigma})^2+\gamma^{-4}}-\gamma^{-2}-\frac{1}{|x|}-U\big)\phi_r\Big)^- \\ & \ge -C\Big\{\eta^{-3/2}\int|\nabla\times A|^2\,dx+\eta^{-3}r^3+\eta^{-3/2}\int U_+^{5/2}\,dx+\eta^{-3}\gamma^3\int U_+^4\,dx \\ & \qquad +\Big(\int|\nabla\times A|^2\,dx\Big)^{3/4}\Big(\int U_+^4\,dx\Big)^{1/4}\Big\},\end{aligned}$$

where $\eta := \frac{1}{10}(1-(\pi\gamma/2)^2)$.

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Complete Semiclassical Spectral Asymptotics for Periodic and Almost Periodic Perturbations of Constant Operators*,†

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Abstract

Under certain assumptions we derive a complete semiclassical asymptotics of the spectral function $e_{h,\varepsilon}(x,x,\lambda)$ for a scalar operator

$$A_\varepsilon(x,hD) = A^0(hD) + \varepsilon B(x,hD),$$

where A^0 is an elliptic operator and $B(x,hD)$ is a periodic or almost periodic perturbation.

In particular, a complete semiclassical asymptotics of the integrated density of states also holds. Further, we consider generalizations.

1 Introduction

1.1 Preliminary Remarks

This work is inspired by several remarkable papers of L. Parnovski and R. Shterenberg [PS1,PS2,PS3], S. Morozov, L. Parnovski and R. Shterenberg [MPS] and earlier papers by A. Sobolev [So1,So2]. I wanted to understand the approach of the authors and, combining their ideas with my own approach, generalize their results.

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In these papers the complete asymptotic expansion of the integrated density of states $\mathsf{N}(\lambda)$ for operators $\Delta+V$ was derived as $\lambda \to +\infty$; here Δ is a positive Laplacian and V is a periodic or almost periodic potential (satisfying certain conditions). In [MPS] more general operators were considered.

Further, in [PS3] the complete asymptotic expansion of $e(x,x,\lambda)$ was derived, where $e(x,y,\lambda)$ is the Schwartz kernel of the spectral projector.

I borrowed from these papers Conditions 1.2–1.6 and the *special gauge transformation* and added the *hyperbolic operator method* (actually non-stationary semiclassical Schrödinger operator method– [Ivr1]) and extremely long propagation of singularities. I believe that this is a simpler and more powerful approach. Also, in contrast to those papers I consider more general semiclassical asymptotics.

Consider a scalar self-adjoint h-pseudo-differential operator $A(x,hD)$ in $\mathbb{R}^d$ with the Weyl symbol $A(x,\xi)$, such that

$$|D_x^\alpha D_\xi^\beta A(x,\xi)| \le c_{\alpha\beta}(|\xi|+1)^m \qquad \forall \alpha,\beta\ \forall x,\xi \tag{1.1}$$

and

$$A(x,\xi) \ge c_0|\xi|^m - C_0 \qquad \forall x,\xi. \tag{1.2}$$

Then it is semibounded from below. Let $e_h(x,y,\lambda)$ be the Schwartz kernel of its spectral projector $E(\lambda)=\theta(\lambda-A)$. We are interested in the semiclassical asymptotics of $e_h(x,x,\lambda)$ and

$$\mathsf{N}_h(\lambda) = \mathsf{M}[e(x,x,\lambda)] \coloneqq \lim_{\ell\to\infty} (\mathsf{mes}(\ell X))^{-1} \int_{\ell X} e(x,x,\lambda)\,dx, \tag{1.3}$$

where $0\in X$ is an open domain in $\mathbb{R}^d$. The latter expression in the cases we are interested in does not depend on X and is called *Integrated Density of States*.

It is well-known that under *ξ-microhyperbolicity condition on the energy level λ*

$$|A(x,\xi,h)-\lambda| + |\nabla_\xi A(x,\xi,h)| \ge \epsilon_0 \tag{1.4}$$

the following asymptotics holds

$$e_h(x,x,\lambda) = \kappa_0(x,\lambda)h^{-d} + O(h^{1-d}) \qquad \text{as}\ \ h\to +0, \tag{1.5}$$

and therefore

$$\mathsf{N}_h(\lambda) = \bar\kappa_0(\lambda)h^{-d} + O(h^{1-d}), \tag{1.6}$$

where here and below

$$\bar{\kappa}_n(\lambda) = \mathsf{M}[\kappa_n(x,\lambda)]. \tag{1.7}$$

For generalization to matrix operators and degenerate scalar operators see Chapters 4 and 5 respectively. Also there one can find slightly sharper two-term asymptotics under non-periodicity conditions.

Also it is known (see Chapter 4) that under microhyperbolicity condition (1.4) for $|\tau-\lambda|<\epsilon$ the following complete asymptotics holds:

$$F_{t\to h^{-1}\tau}\bigl(\bar{\chi}_T(t)\bigl(Q_{2x}u_h(x,y,t)\,{}^tQ_{1y}\bigr)|_{y=x}\bigr) \sim \sum_{n\ge 0}\kappa'_{n,Q_1,Q_2}(x,\tau)h^{1-d+n}, \tag{1.8}$$

where $u_h(x,y,t)$ is the Schwartz kernel of of the *propagator* $e^{ih^{-1}tA}$, $\bar{\chi}\in \mathscr{C}_0^\infty([-1,1])$, $\bar{\chi}(t)=1$ at $[-\frac12,\frac12]$, $T\in[h^{1-\delta},T^*]$, T^* is a small constant here and $Q_j=Q_j(x,hD)$ are h-pseudo-differential operator; we write operators, acting with respect to y on Schwartz kernels to the right of it.

Further, it is known that

$$\mathsf{supp}(Q_1)\cap\mathsf{supp}(Q_2)=\emptyset \implies \kappa'_{n,Q_1,Q_2}(x,\tau)=0, \tag{1.9}$$

where $\mathsf{supp}(Q_j)$ is a support of its symbol $Q_j(x,\xi)$ and

$$\tau\le\tau^*=\inf_{x,\xi}A(x,\xi) \implies \kappa'_{n,Q_1,Q_2}(x,\tau)=0. \tag{1.10}$$

Let

$$\kappa_{n,Q_1,Q_2}(x,\tau)=\int_{-\infty}^{\tau}\kappa'_{n,Q_1,Q_2}(x,\tau')\,d\tau. \tag{1.11}$$

In what follows we skip subscripts $Q_j=I$.

Remark 1.1. This equality (1.8) plus Hörmander's Tauberian theorem imply the remainder estimates $O(h^{1-d})$ for $Q_{2x}e_h(x,y,\tau)\,{}^tQ_{1y}|_{x=y}$. On the other hand, if we can improve (1.8) by increasing T^*, we can improve the remainder estimate to $O(T^{*-1}h^{1-d})$ [1),2)].

Observe that for $A=A(hD)$

$$e_h(x,x,\lambda)=\mathsf{N}_h(\lambda)=\kappa_0(\lambda)h^{-d}. \tag{1.12}$$

[1)] Provided $T^*=O(h^{-M})$ for some M.

[2)] This plus estimate for κ'_0 is a major method for obtaining sharp remainder estimates in [Ivr1].

In this paper we consider

$$A(x, hD) = A^0(hD) + \varepsilon B(x, hD), \tag{1.13}$$

where $A^0(\xi)$ satisfies (1.1), (1.2) and (1.4) and $B(x,\xi)$ satisfies (1.1) and $\varepsilon > 0$ is a small parameter. Later we assume that $B(x, hD)$ is almost periodic and impose other conditions.

First, we claim that for operator (1.13) with $\varepsilon \le \epsilon_0$ the equality (1.8) holds with $T^* = \epsilon_1 \varepsilon^{-1}$ where ϵ_j are small constants and we assume that $\varepsilon \ge h^M$ for some M. Then the remainder estimate is $O(\varepsilon h^{1-d})$ [3].

1.2 Main Theorem

Now we consider the main topic of this work where we will use ideas from [PS1, PS2, PS3, MPS]: the case of an almost periodic operator $B(x, hD)$,

$$B(x,\xi) = \sum_{\theta\in\Theta} b_\theta(\xi) e^{i\langle\theta,x\rangle} \tag{1.14}$$

with discrete (i.e. without any accumulation points) *frequency set* Θ.

Operator B is *quasiperiodic* if Θ is a finite set, *periodic* if Θ is a lattice and *almost periodic* in the general case.

Our goal is to derive (under certain assumptions) complete semiclassical asymptotics:

$$e_{h,\varepsilon}(x,x,\tau) \sim \sum_{n\ge 0} \kappa_{n,\varepsilon x}(x,\tau) h^{-d+n}. \tag{1.15}$$

First, in addition to microhyperbolicity condition (1.4) we assume that $\Sigma_\lambda = \{\xi\colon A^0(\xi) = \lambda\}$ is a *strongly convex surface* i.e.

$$\pm \sum_{j,k} A^0_{\xi_j\xi_k}(\xi)\eta_j\eta_k \ge \epsilon|\eta|^2 \qquad \forall \xi\colon A^0(\xi) = \lambda \ \ \forall \eta\colon \sum_j A^0_{\xi_j}(\xi)\eta_j = 0, \tag{1.16}$$

where the sign depends on the connected component of Σ_λ, containing ξ.

Without any loss of generality we assume that

(1.17) Θ spans $\mathbb{R}^d$, contains 0 and is symmetric about 0.

[3] See Theorem 2.4.

Condition 1.2. *For each $\theta_1, \dots, \theta_d \in \Theta$ either $\theta_1, \dots, \theta_d$ are linearly independent over $\mathbb{R}$ or they linearly dependent over $\mathbb{Z}$.*

Assume also that

Condition 1.3. *For any arbitrarily large L and for any sufficiently large real number ω there are a finite symmetric about 0 set $\Theta' := \Theta'_{(L,\omega)} \subset (\Theta \cap \mathsf{B}(0,\omega))$ (with $\mathsf{B}(\xi, r)$ the ball of the radius r and center ξ) and a "cut-off" coefficients $b'_\theta := b'_{\theta,(L,\omega)}$, such that*

$$B' := B'_{(L,\omega)}(x,\xi) := \sum_{\theta\in\Theta'} b'_\theta(\xi) e^{i\langle\theta,x\rangle} \tag{1.18}$$

satisfies

$$\|D_x^\alpha D_\xi^\beta (B - B')\|_{\mathscr{L}^\infty} \le \omega^{-L}(|\xi|+1)^m \qquad \forall \alpha,\beta\colon |\alpha|\le L, |\beta|\le L. \tag{1.19}$$

Remark 1.4. (i) Then

$$|D_\xi^\beta b_\theta| = O(|\theta|^{-\infty}(|\xi|+1)^m) \qquad \text{as } |\theta|\to\infty \tag{1.20}$$

and

$$|D_\xi^\beta (b_\theta - b'_\theta)| = O(\omega^{-\infty}(|\xi|+1)^m). \tag{1.21}$$

Indeed, one suffices to observe that $b_\theta(\xi) = \mathsf{M}(B(x,\xi)e^{-i\langle\theta,x\rangle})$ etc.

(ii) On the other hand, under additional assumption

$$\#\{\theta\in\Theta,\ |\theta|\le\omega\} = O(\omega^p) \qquad \text{as } \omega\to\infty \tag{1.22}$$

for some p, (1.20) implies Condition 1.3 with $\Theta'_{(L,\omega)} := \Theta\cap\mathsf{B}(0,\omega)$. However we will need $\Theta'_{(L,\omega)}$ in the next condition.

(iii) We need only to estimate the operator norm of the difference between $B(x,hD)$ and $B'(x,hD)$ (from $\mathscr{H}^m$ to $\mathscr{L}^2$); therefore for differential operators we can weaken (1.19): if

$$B = \sum_{\mu,\nu:|\alpha|\le m',|\beta|\le m'} D^\alpha b_{\alpha\beta}(x) D^\beta, \qquad b_{\alpha\beta} = b^\dagger_{\beta\alpha}, \tag{1.23}$$

where we assume that $b_{\alpha\nu}(x)$ and $b'_{\alpha\beta}(x)$ have similar decompositions (1.14) and (1.18) respectively, then (1.19) should be replaced by

$$\|D_x^\alpha (b_{\alpha\beta} - b'_{\alpha\beta})\|_{\mathscr{L}^\infty} \le \omega^{-L} \qquad \forall\alpha. \tag{1.24}$$

(iv) While Condition 1.3 is Condition B of [PS3], adopted to our case, Condition 1.2 and Conditions 1.5, 1.6 below are borrowed without any modifications (except changing notations).

The next condition we need to impose is a version of the Diophantine condition on the frequencies of B. First, we need some definitions. We fix a natural number K (the choice of K will be determined later by how many terms in the asymptotic decomposition of $e(x,x,\lambda)$ we want to obtain) and consider Θ'_K, which here and below denotes the algebraic sum of K copies of Θ':

$$\Theta'_K := \sum_{1\le i\le K} \Theta. \tag{1.25}$$

We say that $\mathfrak{V}$ is a *quasilattice subspace* of dimension q, if $\mathfrak{V}$ is a linear span of q linear independent vectors $\theta_1,\dots,\theta_q \in \Theta'_K \setminus 0$. Obviously, the zero space is a quasilattice subspace of dimension 0 and $\mathbb{R}^d$ is a quasilattice subspace of dimension d.

We denote by $\mathcal{V}_q$ the collection of all quasilattice subspaces of dimension q and also $\mathcal{V} := \bigcup_{q\ge 0} \mathcal{V}_q$.

Consider $\mathfrak{V},\mathfrak{U}\in\mathcal{V}$. We say that these subspaces are *strongly distinct*, if neither of them is a subspace of the other one. Next, let $\widehat{(\mathfrak{V},\mathfrak{U})}\in[0,\pi/2]$ be the angle between them, i.e. the angle between $\mathfrak{V}\ominus\mathfrak{W}$ and $\mathfrak{U}\ominus\mathfrak{W}$, $\mathfrak{W}=\mathfrak{U}\cap\mathfrak{V}$. This angle is positive iff $\mathfrak{V}$ and $\mathfrak{U}$ are strongly distinct.

Condition 1.5. *For each fixed L and K the sets $\Theta'_{(L,\omega)}$ satisfying (1.18) and (1.19) can be chosen in such a way that for sufficiently large ω we have*

$$s(\omega) = s(\Theta'_K) := \inf_{\mathfrak{V},\mathfrak{U}\in\mathcal{V}} \sin\bigl(\widehat{(\mathfrak{V},\mathfrak{U})}\bigr) \ge \omega^{-1} \tag{1.26}$$

and

$$r(\omega) := \inf_{\theta\in\Theta'_K\setminus 0} |\theta| \ge \omega^{-1}, \tag{1.27}$$

where the implied constant (i.e. how large should ω be) depends on L and K.

Let $\mathfrak{V}$ be the span of $\theta_1,\dots,\theta_q\in\Theta'_K\setminus 0$. Then due to Condition 1.2 each element of the set $\Theta'_K\cap\mathfrak{V}$ is a linear combination of $\theta_1,\dots,\theta_q$ with rational coefficients. Since the set $\Theta'_K\cap\mathfrak{V}$ is finite, this implies that the set $\Theta'_\infty\cap\mathfrak{V}$ is discrete and is, therefore, a lattice in $\mathfrak{V}$. We denote this lattice by $\Gamma(\omega;\mathfrak{V})$.

Our final condition states that this lattice cannot be too dense.

Condition 1.6. *We can choose $\Theta'_{(L;\omega)}$, satisfying Conditions 1.3 and 1.5 in such a way that for sufficiently large ω and for each $\mathfrak{V} \in \mathcal{V}$, $\mathfrak{V} \neq \mathbb{R}^d$, we have*

$$\mathsf{vol}(\mathfrak{V}/\Gamma(\omega;\mathfrak{V})) \geq \omega^{-1}. \tag{1.28}$$

Remark 1.7. See Section 2 of [PS3] for discussion of these conditions. In particular, if Θ is a lattice, then Conditions 1.2–1.6 are fulfilled. Further, if Θ is a finite set and Condition 1.2 is fulfilled, then $\Theta_\infty := \bigcup_{K\geq 1} \Theta_K$ is a lattice and Conditions 1.3–1.6 are fulfilled. Furthermore, the same is true, if Θ is an arithmetic sum of a finite set and a lattice.

The main theorem of this paper is

Theorem 1.8. *Let A be a self-adjoint operator* (1.13)*, where A^0 satisfies* (1.1)*,* (1.2)*,* (1.4) *and* (1.16) *and B satisfies* (1.1)*.*

Let Conditions 1.2–1.6 be fulfilled. Then for $|\tau - \lambda| < \epsilon$, $\varepsilon \leq h^\vartheta$, $\vartheta > 0$

$$e_{h,\varepsilon}(x,x,\tau) \sim \sum_{n\geq 0} \kappa_n(x,\tau;\varepsilon) h^{-d+n}. \tag{1.29}$$

Corollary 1.9. *In the framework of Theorem 1.8*

$$\mathsf{N}_{h,\varepsilon}(\tau) \sim \sum_{n\geq 0} \bar{\kappa}_n(\tau;\varepsilon) h^{-d+n}. \tag{1.30}$$

1.3 Plan of the Paper

Section 2 is devoted to the proof of Theorem 1.8. In Subsection 2.1 we make some general remarks, and, in particular, we prove more general albeit far less precise Theorem 2.4. Then, in Subsection 2.2 we describe a gauge transformation.

In Subsection 2.3 we consider a non-resonant zone and justify such transformation, which reduces operator microlocally to a constant symbol operator $A''(hD,h)$. This allows us to study a propagation of singularities with respect to ξ and prove that the singularities do not propagate with respect to ξ [4)]. In Subsection 2.5 we consider a resonant zone and justify such transformation, which reduces operator microlocally to an operator

[4)] For time $T^* = h^{-M}$ with arbitrarily large M.

$A''(x', hD, h)$, where $x' \in \mathfrak{V}$ the corresponding resonant subspace, and prove that the singularities propagate only with respect to ξ'. Then the convexity condition implies that the singularities actually do not propagate with respect to ξ [4)].

In Subsection 2.6 we consider propagation with respect to x and using the results of Subsections 2.3 and 2.5 we prove that the singularities "propagate away" and do not return[4)]. The we apply Tauberian theorem with $T = T^*$ and prove Theorem 1.8.

In Section 3 we generalize Theorem 1.8. First, in Subsection 3.1 we consider matrix operators with the simple eigenvalues of $A^0(\xi)$.

Then, in Subsection 3.2 we consider operators $A^0(hD) + \varepsilon V(x, hD)$ where symbol $V(x, \xi)$ decays as $|x| \to \infty$ and hybrid operators $A^0(hD) + \varepsilon(B(x, hD) + V(x, hD))$ with almost periodic B and decaying V and show that our methods work for them as well.

Finally, in Subsection 3.3 we discuss differentiability of our asymptotics with respect to τ.

2 Proof of the Main Theorem

2.1 Preliminary Analysis

Remark 2.1. (i) It follows from Section 4 that the contribution of the zone $\{\xi\colon |A^0(\xi) - \tau| \ge C_0\varepsilon + h^{1-\varsigma}\}$ to the remainder is negligible. Here and below $\varsigma > 0$ is an arbitrarily small exponent. Namely, let $Q_j = Q_j(hD)$ be operators with the symbols $Q_j(\xi)$, such that

$$\mathsf{supp}(Q_1) \cap \mathsf{supp}(Q_2) \cap \Omega_\tau = \emptyset \tag{2.1}$$

with

$$\Omega_\tau := \{\xi\colon |A^0(\xi) - \tau| \le C_0\varepsilon + h^{1-\varsigma}\} \tag{2.2}$$

and satisfying

$$|D^\alpha Q_j| \le C_\alpha h^{-(1-\varsigma)|\alpha|} \qquad \forall \alpha. \tag{2.3}$$

Then

$$(Q_{2x} e(x, y, \tau)\, {}^t Q_{1y})|_{y=x} = \kappa_{0,Q_1,Q_2} h^{-d} + O(h^\infty) \tag{2.4}$$

with

$$\kappa_{0,Q_1,Q_2} = (2\pi)^{-d} \int \theta(\tau - A^0(\xi)) Q_1(\xi) Q_2(\xi)\, d\xi \tag{2.5}$$

with $\theta(\tau - A^0(\xi))$ equal to either 0 or 1 on each connected component of $\Omega_\tau \cap \mathsf{supp}(Q_1) \cap \mathsf{supp}(Q_2))$.

Therefore we restrict ourself by the analysis in the zone Ω_τ.

(ii) To upgrade (1.8) with $T = T_*$ (a small constant) to (1.8) with $T = T^*$ it is sufficient to prove that

$$|F_{t\to h^{-1}\tau}\big(\chi_T(t)\big(Q_{2x}u_h(x,y,t)\,{}^tQ_{1y}\big)\big|_{y=x}\big)| \le C_s h^{-d+s}, \tag{2.6}$$

for $|\tau - \lambda| \le \epsilon,\ T \in [T_*,\ T^*]$ and $\chi \in \mathscr{C}_0^\infty([-1,-\frac12]\cup[\frac12,1])$, where s is an arbitrarily large exponent.

In the very general setting for $|t| \le h^{-M}$ the propagation speed with respect to ξ does not exceed $C\varepsilon$. More precisely

Proposition 2.2. *Let $A = A^0 + \varepsilon B$ where $A^0(hD)$ and $B(x,hD)$ are matrix operators satisfying* (1.1). *Let $Q_j(hD)$ be operators with symbols satisfying* (2.3). *Further, let $\mathsf{supp}(Q_j) \subset \{\xi\colon |\xi| \le c\}$ and*

$$\mathsf{dist}(\mathsf{supp}(Q_1),\ \mathsf{supp}(Q_2)) \ge \mathsf{max}(C_0\varepsilon T,\ h^{1-\varsigma}) \tag{2.7}$$

with $T \le h^{-M}$. Then for $|t| \le T$

$$\|Q_2 e^{ih^{-1}tA}Q_1\| \le C_{M,s}h^s. \tag{2.8}$$

Proof. One can prove easily by arguments of the proof of Theorem 2.1.2, applied to operator $\varepsilon^{-1}A = \varepsilon^{-1}A^0(hD_x) + B(x,hD)$ and $\phi(\xi,t)$, that the propagation speed with respect ξ does not exceed C_0; presence of the term $\varepsilon^{-1}A^0(hD_x)$ does not matter since it disappears in the commutator with $\phi(hD)$. Changing $t \mapsto \varepsilon t$ we conclude that for operator A the propagation speed with respect to ξ does not exceed $C_0\varepsilon$.

We do not need compactness of the domain in the phase space with respect to x since the propagation speed with respect to x does not exceed C_0 and we have such compactness implicitly. We leave easy details to the reader. □

Proposition 2.3. *In the framework of Proposition 2.2 assume that $A^0(hD)$ is microhyperbolic on the energy level λ* [5)].

Then for $T_* \le T \le T^* = \mathsf{min}(\epsilon_0\varepsilon^{-1}, h^{-M})$ (2.6) *holds.*

[5)] For definition for matrix operators see Definition 2.1.1.

Proof. It is sufficient to prove for $\mathsf{supp}(Q_1)$ contained in the small vicinity of some point $\bar\xi$. Then due to Proposition 2.2 $e^{ih^{-1}tA}Q_1 \equiv Q_2 e^{ih^{-1}tA}Q_1$ modulo operators with $O(h^\infty)$-norms [6] and with Q_2 also supported in the small vicinity of $\bar\xi$ and equal 1 in the vicinity of $\mathsf{supp}(Q_1)$.

Then on $\mathsf{supp}(Q_2)$ operator is microhyperbolic with respect to vector ℓ and we can employ the proof of Theorem 2.1.2 again, this time with $\phi(x,t) = \ell x - \epsilon_0 t$. For further details see Chapter 4. □

Then in virtue of (1.8) with $t = T_*$ (which is also due to the microhyperbolicity condition) (1.8) also holds with $T = T^*$ and applying Hörmander's Tauberian theorem we arrive to the remainder estimate $Ch^{1-d}T^{*-1} = C\varepsilon h^{1-d}$, thus proving the following theorem:

Theorem 2.4. *Let $A = A^0(hD) + \varepsilon B(x, hD)$ with A^0 satisfying conditions* (1.1), (1.2) *and* (1.4) *and B satisfying conditions* (1.1). *Then*

$$e_h(x,x,\tau) = \sum_{0\le n\le M} \kappa_n(x,\tau)h^{-d+n} + O(\varepsilon h^{1-d}) \tag{2.9}$$

provided $\varepsilon \ge h^M$, $|\tau - \lambda| \le \epsilon$.

From now on we discuss only Theorem 1.8.

Remark 2.5. (i) It suffices to prove asymptotics

$$e_h(x,x,\tau) = \sum_{0\le n\le M} \kappa_n(x,\tau)h^{-d+n} + O(h^{-d+M}) \tag{2.10}$$

with arbitrarily large fixed M. To do so we will use the *hyperbolic operator method* (which we implement as *semiclassical Schrödinger operator method*) with maximal time $T^* = h^{-M}$.

(ii) Then we can replace operator B by operator B', provided operator norm of $B - B'$ from $\mathscr{H}^m$ to $\mathscr{L}^2$ does not exceed Ch^{3M}.

Indeed, let $A' = A^0 + \varepsilon B'$. Due to Remark 2.1 we need to compare only $Q_1 e^{ih^{-1}tA'}Q_1$ and $Q_1 e^{ih^{-1}tA}Q_1$. Observe that due to (1.2)

$$\bigl\| e^{ih^{-1}tA}Q_1 - Q_2 e^{ih^{-1}tA}Q_1\bigr\|_k \le C_{k,s}h^s$$

[6] By default, operator norm is from $\mathscr{L}^2$ to $\mathscr{L}^2$.

with arbitrarily large k, s, where $\|\!|.|\!\|_k$ denotes an operator norms from $\mathscr{L}^2$ to $\mathscr{H}^k$ provided $\mathsf{supp}(Q_j) \subset \{\xi\colon A^0(\xi) \le 2jc\}$ and $Q_2 = 1$ in $\{\xi\colon A^0(\xi) < 3c\}$. The same is true for A' as well.

Then equality

$$e^{ih^{-1}tA'} - e^{ih^{-1}tA} = ih^{-1}\int_0^t e^{ih^{-1}(t-t')A}(A'-A)e^{ih^{-1}t'A'}\,dt'$$

and restriction $|t| \le T^*$ imply that $\|\!|\bigl(e^{ih^{-1}tA'} - e^{ih^{-1}tA}\bigr)Q|\!\|_k$ does not exceed $C_{k,s}h^s + Ch^{-1-M}\|\!|Q_2(B-B')Q_2|\!\|_k$.

Finally, observe that $\|\!|Q_2(B-B')Q_2|\!\|_k \le C_k h^{-k-m}\|\!|(B-B')|\!\|'_k$ where $\|\!|.|\!\|'_k$ denotes an operator norm from $\mathscr{H}^m$ to $\mathscr{L}^2$.

(iii) Since $\mathsf{N}_h(\tau)$ could be defined equivalently as

$$\mathsf{N}_h(\lambda) = \lim_{\ell\to\infty} (\mathsf{mes}(\ell X))^{-1}\mathsf{N}_h(\lambda, \ell X)e(x,x,\lambda)\,dx, \tag{2.11}$$

where $\mathsf{N}_h(\lambda, X)$ is an eigenvalue counting function for operator A in X with the Dirichlet (or Neumann–does not matter) boundary conditions on ∂X, for $\mathsf{N}_h(\tau)$ we can arrive to the same conclusion from the variational arguments.

(iv) First such replacement will be $B' := B'_{(L,\omega)}$ from Condition 1.3 with $\omega = h^{-\sigma}$, arbitrarily small $\sigma > 0$ and $L = 3M/\sigma$.

So, from now Θ and B are effectively replaced by $\Theta' := \Theta'_{(L,\omega)}$ and $B'_{(L,\omega)}$ correspondingly.

2.2 Gauge Transformation

Consider now the "gauge" transformation $A \mapsto e^{-i\varepsilon h^{-1}P}Ae^{i\varepsilon h^{-1}P}$ with h-pseudodifferential operator P. Observe that

$$\begin{aligned} e^{-i\varepsilon h^{-1}P}Ae^{i\varepsilon h^{-1}P} &= A - i\varepsilon h^{-1}[P,A] + \sum_{2\le n\le K-1}\frac{1}{n!}(-i\varepsilon h^{-1})^n\,\mathsf{Ad}_P^n(A)\\ &\quad + \int_0^1 \frac{1}{(K-1)!}(1-s)^{K-1}(-i\varepsilon h^{-1})^K e^{-i\varepsilon h^{-1}sP}\,\mathsf{Ad}_P^K(A)e^{i\varepsilon h^{-1}sP}\,ds, \end{aligned} \tag{2.12}$$

where $\mathsf{Ad}_P^0(A) = A$ and $\mathsf{Ad}_P^{n+1}(A) = [P, \mathsf{Ad}_P^n(A)]$ for $n = 0, 1, \ldots$.

Thus *formally* we can compensate εB, taking

$$P = \sum_{\theta} ih\big(A^0(\xi+\theta h/2) - A^0(\xi-\theta h/2)\big)^{-1} b_\theta(\xi) e^{i\langle\theta,x\rangle}, \tag{2.13}$$

so that

$$ih^{-1}[P,A^0] = B \implies ih^{-1}[P,A] = B + i\varepsilon h^{-1}[P,B]. \tag{2.14}$$

Then perturbation εB is replaced by $\varepsilon^2 B'$, which is the right hand expression in (2.12) minus A^0, i.e.

$$B' = -ih^{-1}[P,B] + \sum_{2\le n\le K-1} \frac{1}{n!}\varepsilon^{n-2}(-ih^{-1})^n \operatorname{Ad}_P^n(A), \tag{2.15}$$

where we ignored the remainder.

New perturbation, again formally, has a magnitude of ε^2. Repeating this process we will make a perturbation negligible.

Remark 2.6. However, we need to address the following issues issues:

(i) Denominator $h^{-1}\big(A^0(\xi+\theta h/2) - A^0(\xi-\theta h/2)\big) = \langle\nabla_\xi A^0,\theta\rangle + O(h^{1-\sigma})$ could be small.

(ii) In B' set Θ' increases: $\varepsilon^2 B' = \varepsilon^2 B'_2 + \varepsilon^3 B'_3 + ... + \varepsilon^M B'_M$, where for B'_j the frequency set is Θ'_j (the arithmetic sum of j copies of Θ').

(iii) We need to prove that the remainder is negligible.

(iv) This transformation was used in Section 9 of [PS3] (etc); in contrast to these papers we use Weyl quantization instead of pq-quantization, and have therefore $\big(A^0(\xi+\theta h/2) - A^0(\xi-\theta h/2)\big)$ instead of $\big(A^0(\xi+\theta h) - A^0(\xi)\big)$.

2.3 Non-Resonant Zone

2.4 Gauge Transformation

One can see easily that if inequality

$$|\langle\nabla_\xi A^0(\xi),\theta\rangle| \ge \gamma := \varepsilon^{\frac{1}{2}} h^{-\delta} \tag{2.16}$$

holds for all $\theta \in \Theta'_K \setminus 0$, then the terms could be estimated by $h^{\delta n}$ and our construction works with $K = 3M/\delta$. Here and below without any loss of the generality we assume that $\varepsilon \geq h$; so, in fact,

$$h^{\vartheta} \geq \varepsilon \geq h. \tag{2.17}$$

Indeed, if $P = P(x, hD)$ has the symbol, satisfying

$$|D_\xi^\alpha D_x^\beta P| \leq c_{\alpha\beta}\gamma^{-1-|\alpha|} \qquad \forall \alpha, \beta, \tag{2.18}$$

then $B' = \varepsilon h^{-1}[P, B]$ has a symbol, satisfying

$$|D_\xi^\alpha D_x^\beta B'| \leq c'_{\alpha\beta}\varepsilon\gamma^{-2-|\alpha|} \qquad \forall \alpha, \beta, \tag{2.19}$$

so indeed $\varepsilon' = \varepsilon\gamma^{-2}$.

Then we can eliminate a perturbation completely, save terms with the frequency 0, both old and new. The set of ξ satisfying (2.16) for all $\theta \in \Theta'_K$ we call *non-resonant zone* and denote by $\mathcal{Z}$. Thus, we arrive to

Proposition 2.7. *Let $Q = Q(hD)$ with the symbol supported in $\mathcal{Z} \cap \Omega$ and satisfying* (2.3)

Then there exists a pseudo-differential operator $P = P(x, hD)$ with the symbol, satisfying (2.18) *and such that*

$$\bigl(e^{-i\varepsilon h^{-1}P} A e^{i\varepsilon h^{-1}P} - A''\bigr) Q \equiv 0 \tag{2.20}$$

with

$$A'' = A^0(hD) + \varepsilon B''_0(hD) \tag{2.21}$$

modulo operator from $\mathscr{H}^m$ to $\mathscr{L}^2$ with the operator norm $O(h^{3M})$.

Remark 2.8. (i) This proposition is similar to Lemma 9.3 of [PS3]. However, in contrast to [PS1, PS2, PS3, MPS], after it is proven we do not write asymptotic decomposition there, but simply prove that singularities do not propagate with respect to ξ there.

(ii) It is our second replacement of operator A; recall that the first one was based on Condition 1.3, and now we ignore the remainder after transformation, which is justified by Remark 2.5(i).

Propagation.

Proposition 2.9. *Let $Q_j = Q_j(hD)$ with the symbols, satisfying* (2.3) *and let symbol of Q_1 be supported in $\mathcal{Z} \cap \Omega$.*

Let $\mathsf{dist}(\mathsf{supp}(Q_1), \mathsf{supp}(Q_2)) \geq c\gamma$. *Then*

$$\|Q_2 e^{ih^{-1}tA} Q_1\| = O(h^{2M}) \qquad \text{as} \quad |t| \leq T^* = h^{-M}. \tag{2.22}$$

Proof. One can prove easily that the operator norms of $Q_2 e^{ih^{-1}tA''} Q_1$ and $Q_2 e^{\pm i\varepsilon h^{-1}P} Q_1$ are $O(h^{2M})$. We leave all easy details to the reader. □

2.5 Resonant Zone

Consider now *resonant zone*

$$\Lambda := \bigcup_{\theta \in \Theta'_K \setminus 0} \Lambda(\theta), \tag{2.23}$$

where $\Lambda(\theta)$ is the set of ξ, violating (2.16) for given θ:

$$\Lambda(\theta) = \Lambda_\delta(\theta) := \{\xi \colon |\langle \nabla_\xi A^0(\xi), \theta\rangle| \geq \gamma = c\varepsilon^{\frac{1}{2}} h^{-\delta}\}. \tag{2.24}$$

Case $d = 2$. We start from the easiest case $d = 2$ (in the trivial case $d = 1$ there is no resonant zone). Observe that due to assumption (1.16) for each θ

$$\mathsf{mes}_1(\Lambda(\theta) \cap \Sigma_\lambda) \leq C\gamma. \tag{2.25}$$

Further, $\#\Theta'_K \leq Ch^{-\sigma}$ (as $h \leq h_0(K, \sigma)$) due to Condition 1.5. Thus $\mathsf{mes}_1(\Lambda \cap \Sigma_\lambda) \leq \gamma h^{-\sigma}$. Recall, that $\sigma > 0$ is arbitrarily small.

Since due to Proposition 2.9, the propagation which starts in the non-resonant zone $\mathcal{Z}$ remains there[7)] we conclude that the propagation which is started in some connected component of the resonant zone also remains there[7)].

Thus, $\nabla_\xi A^0(\xi)$ does not change by more than $\gamma h^{-\sigma}$ and since σ ais arbitrarily small we conclude that (2.22) also holds for Q_1, supported in the resonant zone. Therefore

[7)] May be, with different constant c in the definition of γ.

(2.26) Estimate (2.22) holds for all Q_1, Q_2 satisfying (2.3) and

$$\mathsf{dist}(\mathsf{supp}(Q_1), \mathsf{supp}(Q_2)) \ge \gamma. \tag{2.27}$$

Remark 2.10. (i) In the proof of Theorem 1.8 we need only to have estimate (2.22) holding for all Q_1, Q_2 satisfying (2.3) and (2.27) with arbitrarily small constant γ.

(ii) Then for $d = 2$ we can replace assumption (1.16) by

(2.28) $\varkappa(s)$ (a curvature of Σ_λ, naturally parametrized by s) has zeroes only of the finite order.

Indeed, then (2.25) will be replaced by $\mathsf{mes}_1(\Lambda(\theta)\cap\Sigma_\lambda) \le C\gamma^\nu$, $\nu = 1/(q+1)$ with q the maximal order of zeroes of $\varkappa(s)$.

General Case: Gauge Transformation. Consider now the general case $d \ge 2$. In this case due Conditions 1.2, 1.5 and 1.6 we can cover $\Lambda \cap \Omega_\tau$ by Λ^*,

$$\Lambda \cap \Omega_\tau \subset \Lambda^* = \bigcup_{1\le j\le d-1} \Lambda^*_j, \tag{2.29}$$

defined as:

(2.30) Let $\xi \in \Omega_\tau$; then $\xi \in \Lambda^*_j$ iff there exist $\theta_1, \dots, \theta_j \in \Theta'_K$ which are linearly independent and such that $\xi \in \Lambda_{\delta_j}(\theta_k)$ for all $k = 1, \dots, j$,

where $0 < \delta = \delta_1 < \delta_2 < \dots < \delta_{d-1}$ are arbitrarily fixed and we chose sufficiently small $\sigma > 0$ afterwards.

Further, due to Conditions 1.2, 1.5, 1.6 and (1.16) $\Lambda^*_{d-1} \cap \Omega_\tau$ could be covered by no more than γ_{d-1}-vicinities of some points ξ_ι, $\iota = 1, \dots, \omega^g$, $g = g(d)$. Recall that $\Omega_\tau := \{\xi\colon |A^0(\xi) - \tau| \le C_0\varepsilon + h^{1-\varsigma}\}$.

Consider some connected component Ξ of Λ^*_j. Let some point $\bar\xi$ of it belong to $\bigcap_{1\le k\le j} \Lambda_{\delta_j}(\theta_k) \cap \Omega_\tau$ with linearly independent $\theta_1, \dots, \theta_j$. Observe that $\mathsf{diam}(\bigcap_{1\le k\le j} \Lambda_{\delta_j}(\theta_k) \cap \Omega) \le c\gamma_j$ due to strong convexity assumption (1.16). Then this set either intersects or does not intersect with $\Lambda^*_{j+1} \cap \Omega$. In the former case we include it to Λ^*_{j+1} and exclude it from Λ^*_j.

After we redefined Λ^*_j we arrive to the following proposition:

Proposition 2.11. *Equation* (2.29) *still holds where now each connected component* Ξ *of* Λ_j^* *has the following properties:*

(i) $\operatorname{diam}\Xi \le c\gamma_j$.

(ii) There exist linearly independent $\theta_1, \dots, \theta_j \in \Theta'_K$, *such that for each* $\xi \in \Xi$ $|\langle \nabla_\xi A^0(\xi), \theta\rangle| \le c_j\gamma_j$ *for all* $\theta \in \mathfrak{V} \cap (\Theta'_K \setminus 0)$ *and* $|\langle \nabla_\xi A^0(\xi), \theta\rangle| \ge \epsilon_j\gamma_{j+1}$ *for all* $\theta \in \Theta'_K \setminus \mathfrak{V})$ *with* $\mathfrak{V} = \operatorname{Span}(\theta_1, \dots, \theta_j)$.

Now we generalize Proposition 2.7:

Proposition 2.12. *Let* $Q = Q(hD)$ *with the symbol supported in the connected component* Ξ *of* Λ_j^*, *corresponding to subspace* $\mathfrak{V}$, *and satisfying* (2.3). *Then there exists a pseudo-differential operator* $P = P(x, hD)$ *with the symbol, satisfying* (2.18) *and such that*

$$\left(e^{-i\varepsilon h^{-1}P} A e^{i\varepsilon h^{-1}P} - A''\right) Q \equiv 0 \tag{2.31}$$

modulo operator from $\mathscr{H}^m$ *to* $\mathscr{L}^2$ *with the operator norm* $O(h^{3M})$, *where* $A'' = A^0 + \varepsilon B''(x, hD)$, *where* B'' *is an operator with Weyl symbol*

$$B''(x,\xi) = \sum_{\theta \in \Theta'_K \cap \mathfrak{V}} b_{\mathfrak{V},\theta}(\xi) e^{i\langle \theta, x\rangle}. \tag{2.32}$$

Proof. The proof obviously generalizes the proof of Proposition 2.7. We eliminate all $\theta \notin \mathfrak{V}$ exactly in the same way as it was done there. □

General Case: Propagation.

Proposition 2.13. *Let* $Q_j = Q_j(hD)$ *with the symbols, satisfying* (2.3) *and let symbol of* Q_1 *be supported in* Λ_j^* .

Let $\operatorname{dist}(\operatorname{supp}(Q_1), \operatorname{supp}(Q_2)) \ge C_0\gamma_j$. *Then* $\|Q_2 e^{ih^{-1}tA} Q_1\| = O(h^{2M})$ *for* $|t| \le T_* = h^{-M}$.

Proof. In virtue of Proposition 2.9 it is sufficient to consider $\operatorname{supp}(Q_1)$ belonging to the connected component Ξ' of Λ_j^*. Indeed, the values of $\delta_1, \dots, \delta_{d-1}$ are arbitrarily small.

One can prove easily that the operator norm of $Q_2 e^{\pm i\varepsilon h^{-1}P} Q_1$ are $O(h^{2M})$. We need to prove that the operator norm of $Q_2 e^{\pm ih^{-1}tA''} Q_1$ is also $O(h^{2M})$. In the coordinates $(x'; x'') \in \mathfrak{V} \oplus (\mathbb{R}^d \ominus \mathfrak{V})$ we observe that the propagation

speed is only along $\mathfrak{V}$ as long as it remains in $\epsilon\gamma_j$ vicinity of $\mathsf{supp}(Q_1)$. The proof is similar to the proof of Proposition 2.2 and we leave it to the reader.

However propagation is confined to $\Omega'_\tau := \{\xi : |A^0(\xi) - \tau| \le C\varepsilon + 2h^{1-\varsigma}\})$ and due to (1.16) it remains in that vicinity as $\varsigma < \delta$. □

Now we arrive to the following proposition:

Proposition 2.14. *Let* Q_1, Q_2 *satisfy* (2.3) *and* $\mathsf{supp}(Q_1) \subset \Omega$. *Then for* $T_* \le T \le T^*$

$$F_{t\to h^{-1}\tau}\bigl(\chi_T(t) Q_{2x} u(x,y,t)\, {}^t Q_{1y}\bigr) = O(h^{2M}). \tag{2.33}$$

Proof. It is standard, due to Proposition 2.13, microhyperbolicity condition and the results of Chapter 2 of [Ivr1] we conclude that if $|\ell| = 1$ and

$$\langle \ell, \nabla_\xi A^0(\xi)\rangle \ge \epsilon_0 \qquad \forall \xi \in \mathsf{supp}(Q_1) \tag{2.34}$$

and

$$\langle \ell, x-y\rangle \le \epsilon_1 T \qquad \forall x \in \mathsf{supp}(\phi_1),\ y \in \mathsf{supp}(\phi_2), \tag{2.35}$$

then $\|\phi_2 e^{ih^{-1}tA} Q_1 \phi_2\| = O(h^{2M})$ for $T \le t \le 2T$.

This implies (2.33) provided $\mathsf{diam}(\mathsf{supp}(Q_1)) \le \epsilon$. But then for (2.33) we can drop this assumption. □

2.6 End of the Proof

Now we conclude that

$$F_{t\to h^{-1}\tau}\bigl([\bar\chi_T(t) - \bar\chi_{T_*}(t)] Q_{2x} u(x,y,t)\, {}^t Q_{1y}\bigr)\big|_{x=y} = O(h^{2M}) \tag{2.36}$$

and since

$$F_{t\to h^{-1}\tau}\bigl(\bar\chi_T(t) Q_{2x} u(x,y,t)\, {}^t Q_{1y}\bigr)\big|_{x=y} = \sum_{0\le n\le M} \kappa'_n(x,\varepsilon) h^{1-d+n} + O(h^{M+1}) \tag{2.37}$$

holds for $T = T_*$, it also holds for $T = T^*$.

Finally, Hörmander's Tauberian theorem implies Theorem 1.8.

3 Generalizations and Discussion

3.1 Matrix Operators

Consider now $n \times n$-matrix operators A^0 and B; then (1.2) should be understood in the matrix sense. Assume that

(3.1) Symbol $A^0(\xi)$ has only simple eigenvalues $a_1^0(\xi), \dots, a_n^0(\xi)$, which also satisfy (1.4) and (1.16).

Then there exists a unitary transformation $R^0 = R(\xi)$, such that $R^{0\,\dagger}(\xi)A^0(\xi)R^0(\xi) = \mathsf{diag}(a_1^0(\xi), \dots, a_n^0(\xi))$.

Then one can prove easily, that there exists a unitary operator $R(x, hD) = R^0(hD) + \varepsilon R'(x, D)$, such that $R^* A R = \mathsf{diag}(a_1, \dots, a_n)$, where $a_j = a_j(x, hD) = a_j^0(hD) + \varepsilon b_j(x, hD)$ (and we assume as before that (2.17) holds.

If Conditions 1.2–1.6 are fulfilled for $A(x, hD)$, then they are also fulfilled for $a_j(x, hD)$ and we can apply the same propagation arguments as before and Theorem 1.8 extends to such operators provided conditions (1.4) and (1.16) are fulfilled for $a_j(x, hD)$ with $j = 1, \dots, n$.

Let us replace (1.2) by more general ellipticity assumption

$$|A^0(\xi)v| \ge \epsilon |\xi|^m |v| \qquad \forall v \in \mathbb{C}^n \ \forall \xi \colon |\xi| \ge C_0. \tag{3.2}$$

Then we cannot restrict $e(x, y, \lambda)$ to $x = y$ but we can restrict $e(x, y, \lambda, \lambda')$, the Schwartz kernel of the difference of the corresponding projectors.

Theorem 1.8 trivially extends to such operators, if instead of $e(x, x, \lambda)$ we consider $e(x, x, \lambda, \lambda')$ provided conditions (1.4) and (1.16) are fulfilled for $a_j(x, hD)$ with $j = 1, \dots, n$ and for both λ and λ'. It also extends to

$$\int e(x, y, \lambda, \lambda')\phi(\lambda')\, d\lambda', \qquad \phi \in \mathscr{C}_0^\infty(\mathbb{R}), \tag{3.3}$$

provided conditions (1.4) and (1.16) are fulfilled for $a_j(x, hD)$ with $j = 1, \dots, n$ for λ.

Remark 3.1. Our reduction construction fails in the case of a scalar operator A^0 and a matrix operator B unless either $\varepsilon = h^{1+\delta}$ or the principal symbol of B satisfies some very restrictive condition. Therefore for a matrix operator A^0 with the eigenvalues of $A^0(\xi)$ of constant multiplicities our construction works only under similar assumptions.

3.2 Perturbations

Consider operators in question, perturbed by $\varepsilon V(x, hD)$ where $V(x,\xi)$ decays as $|x| \to \infty$. Such perturbations do not affect $\mathsf{N}_h(\lambda)$, but they do affect $e_h(x, x, \lambda)$.

Decaying Perturbations. We start from the easy case

$$A = A^0(hD) + \varepsilon V(x, hD), \tag{3.4}$$

where

$$|D_\xi^\alpha D_x^\beta V(x,\xi)| \le c_{\alpha\beta}(|\xi|+1)^m(|x|+1)^{-\delta-|\beta|} \qquad \forall \alpha,\beta\ \forall x,\xi. \tag{3.5}$$

First of all, we claim that

(3.6) Under assumption (3.7) below the propagation speed with respect to ξ does not exceed $c\varepsilon(|x|+1)^{-\delta}$.

Indeed, note first that due to Proposition 2.2 the propagation speed with respect to ξ does not exceed $c\varepsilon$. Next, consider domain $\{x\colon |x| \asymp r\}$ with $r \ge 1$. Scaling $x \mapsto x/r$, $t \mapsto t/r$ we get a domain $\{x\colon |x| \asymp 1\}$, $h \mapsto \hbar = h/r$ and we need to prove that after this scaling the propagation speed with respect to ξ does not exceed $\nu = c\varepsilon r^{-\delta}$, on the time interval $\{t\colon |t| \le 1\}$.

To prove this we can apply Proposition 2.2 but ewe need to have the microlocal uncertainty principle fulfilled: $\nu \ge \hbar^{1-\sigma}$ with $\sigma > 0$, where ν is a shift with respect to ξ. This inequality is equivalent to $\varepsilon r^{-\delta} \ge h^{1-\sigma} r^{-1+\sigma}$ i.e. $\varepsilon r^{1-\sigma-\delta} \ge h^{1-\sigma}$ and it suffice to have

$$\delta < 1, \qquad \varepsilon \ge h^{1-\sigma} \quad \text{with} \quad \sigma > 0. \tag{3.7}$$

To join different time intervals one can use the technique of Subsection 12.7.6.

Consider now ξ in the vicinity of $\bar\xi$ and x with $|x| \le c$. Then as long as $|\xi - \bar\xi| \le \epsilon$ with small enough constant $\epsilon > 0$, evolution goes away from 0 with the speed $\asymp 1$, so we are in the zone $\{x\colon |x| \asymp |t|\}$ and in this zone the propagation speed with respect to ξ does not exceed $c\varepsilon r^{-1-\delta}$, and therefore $|\xi - \bar\xi| \le c\varepsilon \int_1^\infty t^{-1-\delta}\,dt \le c\varepsilon$ and this is less that $\epsilon/2$ as $\varepsilon \le \epsilon_0$.

We can also consider evolution which starts from x with $|x| \ge 1$. Then the same arguments work albeit with $r \asymp |t - t^*|$ for some t^* with $|t^*| \le c|x|$.

Then we arrive to

Theorem 3.2. *Consider operator* (3.4) *with* V *satisfying* (3.5). *Let microhyperbolicity condition* (1.4) *on the energy level* λ *be fulfilled and* $\varepsilon \le \epsilon_0$. *Then the complete spectral asymptotics* (1.29) *holds.*

Hybrid Perturbations. Now we consider the hybrid operators, containing both εB and εV. However, trying to eliminate εB by the same approach as in Subsubsection 2.5.2, we get an another type of terms, and it is only natural to consider them being in the operator from the beginning:

$$A = A^0(hD) + \varepsilon\big(B(x, hD) + V(x, hD)\big), \tag{3.8}$$

where

$$V(x,\xi) = \sum_{\theta\in\Theta} e^{i\langle\theta,x\rangle} V_\theta(x,\xi), \tag{3.9}$$

$$|D_\xi^\alpha D_x^\beta V(x,\xi)| \le c_{\alpha\beta}(|\xi|+1)^m(|x|+1)^{-\delta} \quad \forall\alpha,\beta\ \forall x,\xi. \tag{3.10}$$

We impose condition

Condition 3.3. *For each* ω *and* L *for the same set* Θ' *as before there exists*

$$V'(x,\xi) = \sum_{\theta\in\Theta'} e^{i\langle\theta,x\rangle} V'_\theta(x,\xi), \tag{3.11}$$

such that

$$\|D_x^\alpha D_\xi^\beta\big(V - V'\big)\|_{\mathscr{L}^\infty} \le \omega^{-L}(|\xi|+1)^m \tag{3.12}$$

and

$$|D_x^\alpha D_\xi^\beta V'_\theta| \le c_{Ls\alpha\beta}(|x|+1)^{-1-\delta-|\alpha|}(|\theta|+1)^{-s} \tag{3.13}$$

$$\forall\alpha,\beta\colon |\alpha|\le L, |\beta|\le L\ \forall s.$$

Non-Resonant Zone. We deal with the purely exponential terms in our standard way and with the hybrid terms as if they were purely exponential (i.e. as if V'_θ were not depending on x), then a new kind of terms will be produced: they acquire factor $h(A^0(\xi+\theta h/2) - A^0(\xi-\theta h/2))^{-1}$ and the derivative with respect to x to V'_θ.

Eventually we end up with the operator of the same type (3.8) with $B(x,\xi)$ replaced by $B''(\xi)$ and with $V_\theta(x,\xi)$ replaced by $V''_\theta(x,\xi)$, such that

$$|D_\xi^\alpha D_x^\beta V''_\theta(x,\xi)| \le C_{n\alpha\beta}\varepsilon^{k+1}\gamma^{-2k-n-|\alpha|}(|x|+1)^{-n-\delta-|\beta|}$$

with $n+k \ge 3K$.

Then

$$|D_\xi^\alpha D_x^\beta \big[V''_\theta(x,\xi)e^{i\langle\theta,x\rangle}\big]| \le C_{s\alpha\beta}\varepsilon^{k+1}\gamma^{-2k-n-|\alpha|}(|x|+1)^{-n-\delta}(|\theta|+1)^{|\beta|};$$

recall that $|\theta| \le CKh^{-\sigma}$.

Let us pick up $\gamma = h^\delta$ with $\delta = \vartheta/6K$. Then, ignoring therms with $k \ge K$ which are negligible, and following the proof of (3.6), we can recover the same statement for the operator after transform, and, finally, to the analogue of Proposition 2.9.

Resonant Zone. If $d = 2$ we arrive to the analogue of Proposition 2.2 in the virtue of the we arguments as in Subsubsection 2.5.1.

If $d \ge 3$ we apply the reduction, similar to one, used in Subsubsection 2.5.3, and arrive again to operator of the type (3.8) with B replaced by $B''(x,\xi')$ and with $V_\theta(x,\xi)$ replaced by $V''_\theta(x,\xi)$.

Then we observe that the shift in direction $\mathbb{R}^d \ominus \mathfrak{V}$ does not exceed $c\varepsilon^{\delta/2}$ and if it is $\ll \gamma^2$ we arrive to the analogue of Proposition 2.13. It is doable by the choice of really small $\sigma_1 < ... < \sigma_{d-1}$. Then we arrive to the analogue of Proposition 2.14 and, finally, to

Theorem 3.4. *Let A be a self-adjoint operator* (3.8)*, where A^0 satisfies* (1.1), (1.2), (1.4) *and* (1.16) *and B satisfies* (1.1)*, V satisfies* (3.9) *and* (3.10).

Let Conditions 1.2–3.3 be fulfilled. Then for $|\tau-\lambda| < \epsilon$, $\varepsilon \le h^\vartheta$, $\vartheta > 0$ asymptotics (1.29) *holds.*

3.3 Differentiability

It also follows from Corollary 1.9 that

$$\frac{1}{\nu}\Big[\mathsf{N}_{h,\varepsilon}(\tau+\nu) - \mathsf{N}_{h,\varepsilon}(\tau)\Big] = \frac{1}{\nu}\Big[\mathcal{N}_{h,\varepsilon}(\tau+\nu) - \mathcal{N}_{h,\varepsilon}(\tau)\Big] + O(h^\infty) \tag{3.14}$$

provided $\nu \ge h^M$, where $\mathcal{N}_{h,\varepsilon}(\tau)$ is the right-hand expression of (1.30).

The question remains, if (3.14) holds for smaller ν, in particular, if it holds in $\nu \to 0$ limit? If the latter holds, then

$$\frac{\partial}{\partial\tau}\mathsf{N}_{h,\varepsilon}(\tau) = \frac{\partial}{\partial\tau}\mathcal{N}_{h,\varepsilon}(\tau) + O(h^\infty) \tag{3.15}$$

and we call the left-hand expression the *density of states*.

It definitely is not necessarily true, at least in dimension 1. From now on we consider only asymptotics with respect to $\tau \to +\infty$. Let $A = \Delta + V(x)$ with periodic V. It is well-known that for $d = 1$ and generic periodic V all spectral gaps are open which contradicts to

$$\frac{\partial}{\partial \tau}\mathsf{N}(\tau) = \frac{\partial}{\partial \tau}\mathcal{N}(\tau) + O(\tau^{-\infty}). \tag{3.16}$$

On the other hand, this objection does not work in case $d \geq 2$ since only several the lowest spectral gaps are open (Bethe-Sommerfeld conjecture, proven in [PS]).

Assume for simplicity, that $A = \Delta + V$ has no negative eigenvalues; then we can apply wave operator method[8]. We consider $u(x, y, t)$, the Schwartz kernel of $\cos(\sqrt{A}t)$,

$$u(x, y, t) = \int \cos(t\tau)\, d_\tau e(x, y, \tau^2). \tag{3.17}$$

Then, for compactly supported V [9]

$$u(x, y, t) = \begin{cases} O(e^{-\epsilon|t|}) & \text{for odd } \ d, \\ O(|t|^{-d}) & \text{for even } \ d \end{cases} \tag{3.18}$$

as $|x| + |y| \leq c$, $|t| \to +\infty$ and $\frac{\partial}{\partial \tau} e(x, x, \tau^2)$ could be completely restored by inverse $\cos$-Fourier transform, without any Tauberian theorem, and we arrive to asymptotics of $\frac{\partial}{\partial \tau} e(x, x, \tau^2)$. Moreover, we can differentiate complete asymptotics of the *Birman-Krein spectral shift function*

$$\xi(\tau) := \int \bigl(e(x, x, \tau^2) - e^0(x, x, \tau^2)\bigr)\, dx \sim \sum_{n \geq 0} \bar{\kappa}_n \tau^{-d+n}, \tag{3.19}$$

with

$$\bar{\kappa}_n := \int (\kappa_n(x) - \kappa_n^0)\, dx, \tag{3.20}$$

where $e^0(x, y, \tau)$ and κ_n^0 correspond to $A^0 = \Delta$. In the case of $A = \Delta$ in the exterior of smooth, compact and non-trapping obstacle and $A^0 = \Delta$ in $\mathbb{R}^d$ such asymptotics was derived in [PP].

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[8] It could be applied without this assumption, but with tweaking.

[9] It, probably could be proven for V, decaying fast enough at infinity

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Complete Differentiable Semiclassical Spectral Asymptotics*,†

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Abstract

For an operator $A \coloneqq A_h = A^0(hD) + V(x, hD)$ with a "potential" V decaying as $|x| \to \infty$ we establish under certain assumptions the complete and differentiable with respect to τ asymptotics of $e_h(x, x, \tau)$ where $e_h(x, y, \tau)$ is the Schwartz kernel of the spectral projector.

1 Introduction

Consider a self-adjoint matrix operator

$$A \coloneqq A_h = A^0(hD) + V(x, hD), \tag{1.1}$$

where

$$|D_\xi^\beta A^0(\xi)| \le c_{\alpha\beta}(|\xi| + 1)^m \qquad \forall \beta\ \forall \xi \tag{1.2}$$

and

$$A^0(\xi) \ge c_0|\xi|^m - C_0 \qquad \forall \xi. \tag{1.3}$$

We assume that $A^0(\xi)$ is ξ-*microhyperbolic at energy level* λ, i.e. for each ξ there exists a direction $\ell(\xi)$ such that $|\ell(\xi)| \le 1$ and

$$(\langle \ell(\xi), \nabla_\xi \rangle A^0(\xi)v, v) + |(A^0(\xi) - \lambda)v| \ge \epsilon_0|v|^2 \qquad \forall v. \tag{1.4}$$

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Further, we assume that $V(x,\xi)$ is a real-valued function, satisfying

$$|D_\xi^\alpha D_x^\beta V(x,\xi)| \le c_{\alpha\beta}(|\xi|+1)^m(|x|+1)^{-\delta-|\beta|} \qquad \forall\alpha,\beta\ \forall x,\xi \tag{1.5}$$

and

$$|D_\xi^\alpha D_x^\beta V(x,\xi)| \le \varepsilon \qquad \forall\alpha,\beta\colon |\alpha|+|\beta|\le 1\ \forall x,\xi. \tag{1.6}$$

Our main theorem is

Theorem 1.1. *Let conditions* (1.2)– (1.4) *and* (1.6) *with sufficiently small constant* $\varepsilon > 0$ *be fulfilled. Then*

(i) The complete spectral asymptotics holds for $\tau\colon |\tau-\lambda|\le\epsilon$*:*

$$e_h(x,x,\tau) \sim \sum_{n\ge 0} \kappa_n(x,\tau) h^{-d+n} \tag{1.7}$$

where $e_h(x,y,\tau)$ *is the Schwartz kernel of the spectral projector* $\theta(\tau - A_h)$ *of* A_h*.*

(ii) This asymptotics is infinitely differentiable with respect to τ*.*

Remark 1.2. (i) Statement (i) was sketched under much more restrictive assumptions in Theorem 3.2 of [Ivr3]; however we provide here more detailed exposition.

(ii) In Theorem 2.8 we provide the dependence of the remainder on $|x|$.

(iii) This asymptotics is also infinitely differentiable with respect to x but it is really easy.

Differentiability and completeness of the spectral asymptotics are really different. F.e. for operators with almost periodic with respect to x perturbation $V(x,hD)$ the spectral asymptotics are complete (see [Ivr3] and references there) but in dimension 1 it is not necessarily differentiable even once due to spectral gaps. Furthermore, if we perturb an operator we study in this paper by an appropriate "negligible" operator (i. e. with $O(h^\infty)$ norm), the absolutely continuous spectrum on the segment $[\lambda_-,\lambda_+]$ with $\lambda_\mp = \lambda + O(h^\infty)$ will be replaced by an eigenvalue of the infinite multiplicity and then the spectral asymptotics will complete albeit non-differentiable even once.

To establish spectral asymptotics we apply the "hyperbolic operator method"; namely, let us consider the Schwartz kernel of the propagator $e^{ih^{-1}tA_h}$:

$$u := u_h(x,y,t) = \int e^{ih^{-1}t\tau}\, d_\tau e_h(x,y,\tau). \tag{1.8}$$

Then under ellipticity and microhyperbolicity conditions (1.3) and (1.4)

$$F_{t\to h^{-1}\tau}\bar\chi_T(t)u_h(x,x,t) \sim \sum_{n\ge 0} \kappa'_n(x,\tau)h^{1-d+n}, \tag{1.9}$$

where here and below $\chi \in \mathscr{C}_0^\infty([-1,-\frac12]\cup[\frac12,1])$, $\bar\chi \in \mathscr{C}_0^\infty([-1,1])$, $\bar\chi(t)=1$ on $[-\frac12,\frac12]$, $\chi_T(t)=\chi(t/T)$ etc, $\kappa'_n(x,\tau)=\partial_\tau\kappa_n(x,\tau)$, and $T=T_*>0$ is a small constant here.

Then, due to Tauberian theorem we arrive to the spectral asymptotics with the remainder estimate $O(h^{1-d})$. Next, under different assumptions one, using propagation of singularities technique, can prove that

$$|F_{t\to h^{-1}\tau}\chi_T(t)u_h(x,x,t)| = O(h^\infty) \tag{1.10}$$

for all $T\in[T_*,T^*]$. Then (1.9) holds with $T=T^*$ and again, due to Tauberian theorem, we arrive to the spectral asymptotics with the remainder estimate $O(T^{*\,-1}h^{1-d})$ (provided $T^*=O(h^{-K})$). In particular, if (1.10) holds provided $T^*=h^{-\infty}$, we arrive to complete spectral asymptotics. This happens f.e. in the framework of [Ivr3].

However we do not have Tauberian theorems for the derivatives (with respect to τ) and we need to use an inverse Fourier transform and its derivatives

$$\partial_\tau^n e_h(x,x,\tau) = (2\pi h)^{-1}\int_{\mathbb{R}} e^{-ih^{-1}\tau t}(-ih^{-1}t)^{n-1}u_h(x,x,t)\,dt \tag{1.11}$$

for $n\ge 1$. If we insert a factor $\bar\chi_T(t)$ into integral, we will get exactly n-th derivative of the right-hand expression of (1.7). However we need to estimate the remainder

$$(2\pi h)^{-1}\int_{\mathbb{R}} e^{-ih^{-1}\tau t}(-ih^{-1}t)^{n-1}(1-\bar\chi_T(t))u_h(x,x,t)\,dt \tag{1.12}$$

and to do this we need to properly estimate the left-hand expression of (1.10) for all $T\ge T_*$ (rather than for $T\in[T_*,T^*]$).

To achieve this we will use a more subtle propagation technique and prove that for $T\ge T_*(R)$ the left-hand expression of (1.10) is $O((h/T)^\infty)$, provided $|x|\le R$.

2 Proofs

2.1 Preliminary Remarks

Observe that, due to assumptions (1.6) and (1.5) a propagation speed with respect to ξ does not exceed $\min\bigl(\varepsilon,\ C(|x|+1)^{-1-\delta}\bigr)$ and one can prove easily, that for for a generalized Hamiltonian trajectory[1)] $(x(t),\xi(t))$ on energy level $\tau \le c$

$$\Sigma_\tau := \{(x,\xi)\colon\ \mathsf{Ker}(A(x,\xi)-\tau) \ne \{0\}\} \tag{2.1}$$

with $A(x,\xi) = A^0(\xi) + V(x,\xi)$ we have $|\xi(t)-\xi(0)| \le \varepsilon'$ for all t with $\varepsilon' = \varepsilon'(\varepsilon) \to 0$ as $\varepsilon \to 0$ and therefore

(2.2) Let conditions (1.2)–(1.4), (1.5) and (1.6) with $\varepsilon = \varepsilon(\varepsilon') > 0$ with arbitrarily small ε' be fulfilled. Then for a generalized Hamiltonian trajectory $(x(t),\xi(t))$ on Σ_τ

$$|\xi(t)-\xi(0)| \le \varepsilon' \quad \text{and} \quad |x(t)-x(0)| \ge \epsilon_2|t| \qquad \forall t \in \mathbb{R}. \tag{2.3}$$

Then we conclude immediately that inequality

$$|F_{t\to h^{-1}\tau}\chi_T(t)u_h(x,x,t)| \le C'_s(T)h^s \tag{2.4}$$

holds for arbitrarily constant $T > 0$.

Combining with (1.9) for small constant T we conclude that

(2.5) Let conditions (1.2)–(1.4), (1.5) and (1.6) with sufficiently small constant $\varepsilon > 0$ be fulfilled. Then asymptotic decomposition (1.9) holds with an arbitrarily large constant T.

2.2 Propagation and Local Energy Decay

First we have the finite speed with respect to x propagation:

Proposition 2.1. *For $\tau \le c$ the following estimate holds*

$$|F_{t\to h^{-1}\tau}\Bigl(\chi_T(t)u(x,y,t)\Bigr)| \le C'_s h^s R^{-s} \tag{2.6}$$
$$\forall x,y\colon |x-y| \ge C_0 T, |x|+|y| \asymp R.$$

[1)] For a definition of the generalized Hamiltonian trajectory see Definition 2.2.8 of [Ivr1].

Proof. In the zone $\{x\colon |x| \asymp R\}$ we can apply scaling $x \mapsto xR^{-1}$, $t \mapsto tR^{-1}$, $h \mapsto hR^{-1}$ and apply the standard theory of Chapter 2 of [Ivr1]. The rest is trivial. □

Next, we consider $R \le \epsilon_1 T$ and apply energy estimate method to prove the local energy decay. Observe that one can select smooth $\ell(\xi)$ in condition (1.4). Consider operator $L^0(x, hF)$ with Weyl symbol $-\langle x, \ell(\xi)\rangle$ and $L(x, hD; t) = L^0 + \varepsilon t$.

Then

$$
\begin{aligned}
(2.7)\quad 2h^{-1}\operatorname{Re} i((hD_t - A)v, Lv)_{\Omega_T} = \\
(Lv, v)\big|_{t=0}^{t=T} - \operatorname{Re} ih^{-1}([hD_t - A, L]v, v)_{\Omega_T} = \\
(Lv, v)\big|_{t=0}^{t=T} - \varepsilon\|v\|^2_{\Omega_T} + \operatorname{Re}(ih^{-1}[A, L]v, v)_{\Omega_T},
\end{aligned}
$$

where $\|.\|_\Omega$ and $(.,.)_\Omega$ are a norm and an inner product in $\mathscr{L}^2(\Omega)$ with $\Omega = \Omega_T = \mathbb{R}^d \times [0, T] \ni (x, t)$. Indeed, writing the left-hand expression as

$$
\begin{aligned}
ih^{-1}\big[((hD_t - A)v, Lv)_{\Omega_T} - (L, (hD_t - A)v)_{\Omega_T}\big] = \\
ih^{-1}\big[(L(hD_t - A)v, v)_{\Omega_T} - ((hD_t - A)L, v)_{\Omega_T}\big] + (Lv, v)\big|_{t=0}^{t=T}
\end{aligned}
$$

because $L^* = L$, we arrive to (2.7).

In virtue of (1.5) and (1.6) for sufficiently small constant ε and for $h \le h_0(\varepsilon_1)$ the operator norm of $h^{-1}[V, L]$ from $\mathscr{H}^m(\mathbb{R}^d)$ to $\mathscr{L}^2(\mathbb{R}^d)$ does not exceed ε_1 with $\varepsilon_1 = \varepsilon_1(\varepsilon) \to 0$ as $\varepsilon \to 0$, and then due to the microhyperbolicity assumption we conclude that

$$
(2.8)\qquad \operatorname{Re}(ih^{-1}[A, L]v, v) \ge (\epsilon_0 - 2\varepsilon_1)\|v\|^2 - C\|(A - \tau)v\|^2
$$

for both $\mathbb{R}^d$ and Ω_T.

Let us plug into (2.7) $v = \varphi_\varepsilon(A - \tau)e^{ihtA}w$ where $\varphi \in \mathscr{C}_0^\infty([-1, 1])$, $0 \le \varphi \le 1$; then for sufficiently small constant $\varepsilon > 0$ we arrive to

$$
(2.9)\qquad \epsilon\|v\|^2_{\Omega_T} + (Lv, v)\big|_{t=T} \le (Lv, v)\big|_{t=0}.
$$

On the other hand,

$$
(2.10)\qquad \operatorname{Re}(Lv, v)|_{t=T} \ge \varepsilon T\|v\|^2 - C\||x|^{\frac{1}{2}}v\|^2
$$

with

$$\||x|^{\frac12}v\|^2 = \||x|^{\frac12}v\|^2_{B(0,R)} + \||x|^{\frac12}v\|^2_{B(0,R')\setminus B(0,R)} + \||x|^{\frac12}v\|^2_{\mathbb{R}^d\setminus B(0,R')}$$

with $R' = C_0T$ and therefore for $R \le \varepsilon T$

$$\operatorname{Re}(Lv, v)|_{t=T} \ge \varepsilon T\|v\|^2 - CR'\|v\|^2_{B(0,R')\setminus B(0,R)} - \||x|^{\frac12}v\|^2_{\mathbb{R}^d\setminus B(0,R')}. \tag{2.11}$$

Observe that

$$|(Lv, v)|_{t=0} \le C\Big(\|v_0\|^2 + \||x|^{\frac12}v_0\|^2\Big)$$

and then (2.9) and (2.10) imply that if $R \le \varepsilon T$ then

$$\|v\|^2_{B(0,r)} \le \sigma\|v_0\|^2 + CT^{-1}\Big(\||x|^{\frac12}v\|^2_{\mathbb{R}^d\setminus B(0,R')} + \|v_0\|^2 + \||x|^{\frac12}v_0\|^2\Big) \tag{2.12}$$

with $\sigma < 1$ and $v|_0 = v|_{t=0}$; recall that $\|v\| = \|v_0\|$.

Recall that $v = e^{ih^{-1}tA}\varphi_\varepsilon(A-\tau)\psi_R(x)w$ where we plugged $\psi_R w$ instead of w, $\psi \in \mathscr{C}_0^\infty(B(0,1))$, $0 \le \psi \le 1$ and $\psi = 1$ in $B(0, \frac12)$.

One can prove easily that $Q = \varphi_\varepsilon(A-\tau)$ is an operator with Weyl symbol $Q(x,\xi)$, satisfying

$$|D_\xi^\alpha D_x^\beta Q| \le C_{\alpha\beta}\varepsilon^{-|\alpha|-|\beta|}(|x|+1)^{-|\beta|}.$$

Then $\|v_0\|_{\mathbb{R}^d\setminus B(0,2R)} \le C(h/R)^s\|w\|$ and therefore $\||x|^{\frac12}v_0\|^2 \le 2R\|w\|^2$. Further, then Proposition 2.1 implies that $\||x|^{\frac12}v\|^2_{\mathbb{R}^d\setminus B(0,R')} \le C(h/T)^s\|w\|$ provided or $R' \ge C_0T$ with sufficiently large C_0 and we arrive to

Proposition 2.2. *In the framework of Theorem 1.1*

$$\|\psi_R e^{ih^{-1}TA}\varphi_\varepsilon(A-\lambda)\psi_R\| < 1 \tag{2.13}$$

provided $\varepsilon T \ge R \ge 1$.

While this statement looks weak, it will lead to much stronger one:

Proposition 2.3. *In the framework of Theorem 1.1*

$$\|\psi_R e^{ih^{-1}TA}\varphi_\varepsilon(A-\lambda)\psi_R\| \le C_sR^sT^{-s} \tag{2.14}$$

provided $T \ge C_0R$, $R \ge 1$.

Proof. We want to prove by induction that

$$\|\psi_R e^{inh^{-1}TA}\varphi_\varepsilon(A-\lambda)\psi_R\| \le C\nu^N + C_s n R^s T^{-s} \tag{2.15}$$

with $\nu < 1$.

Assuming that for n we have (2.15), we apply the previous arguments on the interval $[nt,(n+1)T]$ to $v = e^{ih^{-1}tA}\varphi_\varepsilon(A-\lambda)\psi_{R/2}e^{ih^{-1}nTA}\psi_R w$ and derive an estimate

$$\begin{aligned}\|\psi_R e^{ih^{-1}tA}\varphi_\varepsilon(A-\lambda)\psi_{R/2}e^{ih^{-1}nTA}\psi_R w\| \le \\ \nu\|\varphi_\varepsilon(A-\lambda)\psi_R e^{ih^{-1}nTA}\psi_R w\| + C'_K(h/R)^K\|w\|\end{aligned} \tag{2.16}$$

with $\nu < 1$.

To make a step of induction we weed to estimate the norm of

$$\begin{aligned}\psi_R e^{ih^{-1}tA}\varphi_\varepsilon(A-\lambda)(1-\psi_{R/2})e^{ih^{-1}nTA}\psi_R w = \\ \psi_R e^{ih^{-1}tA}\varphi_\varepsilon(A-\lambda)(1-\psi_{R/2})Q^+ e^{ih^{-1}nTA}\psi_R + \\ \psi_R e^{ih^{-1}tA}\varphi_\varepsilon(A-\lambda)(1-\psi_{R/2})Q^- e^{ih^{-1}nTA}\psi_R,\end{aligned} \tag{2.17}$$

with $Q^\pm = Q^\pm(x,hD)$, $Q^+ + Q^- = I$ to be selected to ensure that

(2.18) Generalized Hamiltonian trajectories on Σ_τ, starting as $t=0$ from $\operatorname{supp}(Q^\pm)\cap\operatorname{supp}(1-\psi_{R/2})$ in the positive (negative) time direction, remain in the zone $\{|x|\ge \epsilon_1 R + \epsilon_2|t|\}$.

Then we show that

$$\|\psi_R e^{ih^{-1}tA}\varphi_\varepsilon(A-\lambda)(1-\psi_{R/2})Q^+\| \le C_s(h/R)^s \tag{2.19}$$

and

$$\|\varphi_\varepsilon(A-\lambda)(1-\psi_{R/2})Q^- e^{ih^{-1}nTA}\psi_R\| \le C_s(h/R)^s. \tag{2.20}$$

To achieve that consider $A^0(\xi)$ and for each ξ in the narrow vicinity $\mathcal{W}$ of Σ_τ let $K^+(\xi)\subset\mathbb{R}^d_x\times\bar{\mathbb{R}}^+$ be a forward propagation cone and $K^-(\xi) = -K^+(\xi)$ be a backward propagation cone. Let

$$\Omega^\pm = \{(x,\xi)\colon x\notin \pi_x K^\pm(\xi)\} \tag{2.21}$$

where π_x is x-projection.

Then $\Omega^{\pm}$ are open sets and since $\pi_x K^+(\xi) \cap \pi_x K^-(\xi) = \{0\}$ we conclude that $\Omega^+ \cup \Omega^- \supset \mathbb{S}^{d-1} \times \mathcal{W}$. We can then find smooth positively homogeneous of degree 0 with respect to x symbols $q^{\pm}(x,\xi)$ supported in $\Omega^{\pm}$ such that $q^+ + q^- = 1$ on $\mathbb{S}^{d-1} \times \mathcal{W}$. Let $q^0 = 1 - (q^+ + q^-)$; then $(A^0 - \tau)$ is elliptic on $\mathsf{supp}(q^0)$.

Finally, let $Q^{\pm}$ and Q^0 be operators with the symbols $q^{\pm}(x,\xi)\phi_R(x)$ and $q^0(x|,\xi)\phi_R(x)$ correspondingly, where $\phi \in \mathscr{C}_0^{\infty}(\mathbb{R}^d \setminus 0)$, equal 1 as $c^{-1} \le |x| \le c$ with large enough constant c. Then

$$Q^+ + Q^- + Q^0 = \phi_R(x), \tag{2.22}$$

where (2.18) holds and $(A - \tau)$ is elliptic on the support of the symbol of Q^0.

Then Proposition 2.4 below implies that for $R \le \varepsilon T$ with sufficiently small constant ε both (2.19) and (2.20) hold. On the other hand, ellipticity of $(A - \tau)$ on $\mathsf{supp}(Q^0)$ implies that

$$\|\varphi_\varepsilon(A - \lambda)(1 - \psi_{R/2})Q^0\| \le C_s(h/R)^s. \tag{2.23}$$

Then we can make an induction step by n and to prove (2.15). After this, let us replace in (2.15) R and T by r and t. Next, for given R, T such that $R \le \varepsilon^3 T$ let us plug into (2.15) $n = (T/R)^{\frac{1}{3}}$, $t = T^{\frac{2}{3}}R^{\frac{1}{3}} = T/n$ and $r = T^{\frac{1}{3}}R^{\frac{2}{3}} = nR$ (obviously $r \le \varepsilon t$). We arrive to (2.14) with a different but still arbitrarily large exponent s. □

As mentioned, we need the following proposition:

Proposition 2.4. *Let conditions of Theorem 1.1 be fulfilled. Let $\bar{x} \in \mathbb{R}^d \setminus 0$, $\bar{\xi} \in \mathcal{W}$ and assume that $0 \notin \bar{x} + \pi_x K^{\mp}(\xi)$. Let $\mathcal{K}^{\mp}$ be a conical η-vicinity of $K^{\mp}(\xi)$ and $\mathcal{V}$ be ηR-vicinity of $\bar{x}$, $R = |x|$. Then*

$$\|Q' e^{\pm ih^{-1}TA} Q\| \le C_s(h/R)^s \tag{2.24}$$

provided $Q = Q(x, hD)$ and $Q' = Q'(x, hD)$ are operators with the symbols satisfying

$$|D_\xi^\alpha D_x^\beta Q| \le c_{\alpha\beta} r^{-|\beta|} \tag{2.25}$$

with $r = T + R$ and $r = R$ respectively, $R \ge R(\eta)$ and support of symbol of Q does not intersect with $\mathcal{V} + \mathcal{K}^{\mp}|_{t=T}$, symbol of Q' is supported in the sufficiently small vicinity of $(\bar{x}, \bar{\xi})$.

Proof. Considering propagation in the zone $\{x\colon |x| \asymp r\}$, we see that the propagation speed with respect to ξ does not exceed $Cr^{-1-\delta}$. To prove this we scale $x \mapsto xr^{-1}$, $t \mapsto tr^{-1}$, $h \mapsto \hbar = hr^{-1}$ and apply the standard energy method (see Chapter 2 of [Ivr1]). We leave the easy details to the reader.

Therefore for time $t \asymp r$ variation of ξ does not exceed $Cr^{-\delta}$. Then, the propagation speed with respect to $\langle x, \ell(\bar{\xi})\rangle$ (which increases) is of magnitude 1 (as long as ξ remains in the small vicinity of $\bar{\xi}$). Again, to prove it we scale and apply the energy method (see Chapter 2 of [Ivr1]).

But then the contribution of the time interval $t \asymp r$ to the variation of ξ does not exceed $Cr^{-\delta}$ and therefore the variation of ξ for a time interval $[0, T]$ with $T \ge 0$ does not exceed $CR^{-\delta} \le \eta$ for $R \ge R(\eta)$. □

Proposition 2.5. *In the framework of Theorem 1.1*

$$\|\psi_R e^{ih^{-1}TA}\varphi_\varepsilon(A-\lambda)\psi_R\| \le C_s h^s, \tag{2.26}$$

provided $T \ge C_0 R$, $R \ge 1$.

Proof. It follows immediately from Proposition 2.4 with the semiclassical parameter hr^{-1} and with r set to its minimal value along the cone of propagation, which is 1. □

Combining Propositions 2.3 and 2.5 we arrive to

Corollary 2.6. *In the framework of Theorem 1.1*

$$\|\psi_R e^{ih^{-1}TA}\varphi_\varepsilon(A-\lambda)\psi_R\| \le C_s h^s R^s T^{-s} \tag{2.27}$$

provided $T \ge C_0 R$, $R \ge 1$.

2.3 Traces and the End of the Proof

Proposition 2.7. *In the framework of Theorem 1.1 the following estimates hold for* $T \ge 1$

$$|F_{t\to\tau}\chi_T(t)u(x,x,t)| \le C_s h^s (|x|+1)^{s+1}((|x|+1)+T)^{-s}, \tag{2.28}$$

$$|F_{t\to\tau}\chi_T(t)\int \psi_R(x)u(x,x,t)\,dx| \le C_s h^s T^{-s}, \tag{2.29}$$

provided $\psi \in \mathscr{C}_0^\infty(B(0,1))$. *Further,*

$$|F_{t\to\tau}\chi_T(t)\int \psi_R(x)u(x,x,t)\,dx| \le C_s h^s R^{-s} T^{-s}, \tag{2.30}$$

provided $\psi \in \mathscr{C}_0^\infty(B(0,1)\setminus B(0,\frac{1}{2}))$.

Proof. Estimate (2.28) follows immediately from (2.26). Estimate (2.30) follows from (2.26) and

$$|F_{t\to\tau}\chi_T(t)\int \psi_R(x)u(x,x,t)\,dx| \le C_s h^s R^{-s} T,$$

which holds because we can chose the time direction on the partition element (see Chapter 4 of [Ivr1]) and we chose the one in which $|x| \gtrsim R$ (which is possible; see the part of proof of Proposition 2.3 dealing with $Q^\pm$ and Q^0).

Finally, estimate (2.29) follows from (2.30). □

Then we immediately arrive to the following theorem, which in turn implies Theorem 1.1:

Theorem 2.8. *In the framework of Theorem 1.1 the following estimates hold*

$$|\partial_\tau^k\Big(e(x,x,\tau) - \sum_{n\le N-1}\kappa_n(x,\tau)h^{-d+n}\Big)| \le C_N h^{-d+N}(|x|+1)^{-N} + C_s h^s(|x|+1)^k \tag{2.31}$$

and for $\psi \in \mathscr{C}_0^\infty(B(0,1))$

$$|\partial_\tau^k\int\Big(e(x,x,\tau) - \sum_{n\le N-1}\kappa_n(x,\tau)h^{-d+n}\Big)\psi_R(x)\,dx| \le C_N h^{-d+N}. \tag{2.32}$$

2.4 Discussion

Corollary 2.9. *Let conditions of Theorem 1.1 be fulfilled. Assume that*

$$|D_x^\alpha V| \le c_\alpha(|\xi|+1)^m(|x|+1)^{-d-|\alpha|-\delta}. \tag{2.33}$$

Then the asymptotics of the Birman-Schwinger spectral shift function

$$\mathsf{N}_h(\tau) := \int\Big(e_h(x,x,\tau) - e_h^0(x,x,\tau)\Big)\,dx \sim \sum_{n\ge 0}\varkappa h^{-d+n} \tag{2.34}$$

is infinitely differentiable with respect to τ. *Here* $e_h^0(x,x,\tau) = \kappa_0^0 h^{-d}$ *and* $e_h^0(x,y,\tau)$ *is the Schwartz kernel of spectral projector for* $A^0(hD)$, *and*

$$\varkappa_n(\tau) = \int \bigl(\kappa_n(x,\tau) - \delta_{n0}\kappa^0\bigr)\,dx. \tag{2.35}$$

Indeed, condition (2.33) guarantees the absolute convergence of integrals in (2.35).

Remark 2.10. Our results could be easily generalized to non-semi-bounded elliptic A^0 (like in Subsection 3.1 of [Ivr3]). Then instead of $e(x,y,\lambda)$ one needs to consider $e(x,y,\lambda,\lambda')$ the Schwartz kernel of $\theta(\lambda - A) - \theta(\lambda' - A)$ and either impose conditions for both λ and λ', or only for λ and mollify with respect to λ'.

It looks strange that the last term in the remainder estimate (2.31) increases as $|x|$ increases, but so far I can neither improve it to the uniform with respect to x in the general case, nor show by the counter-example that such improvement is impossible. However I hope to prove

Conjecture 2.11. *Assume in addition that* A^0 *is a scalar operator and* $\Sigma_\lambda^0 = \{\xi\colon A^0(\xi) = \lambda\}$ *is a* strongly convex surface *i.e.*

$$\pm \sum_{j,k} A^0_{\xi_j\xi_k}(\xi)\eta_j\eta_k \ge \epsilon|\eta|^2 \qquad \forall \xi \in \Sigma_\lambda^0 \quad \forall \eta\colon \sum_j A^0_{\xi_j}(\xi)\eta_j = 0, \tag{2.36}$$

where the sign depends on the connected component of Σ_λ, *containing* ξ.

Then the last term in the right-hand expression of (2.31) *could be replaced by* $C_s h^s(|x|+1)^{k-d}$.

Rationale here is that only few Hamiltonian trajectories from x with $|x| = R \gg 1$ pass close to the origin and even they do not spend much time there.

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Bethe-Sommerfeld Conjecture in Semiclassical Settings*,†

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Abstract

Under certain assumptions (including $d \geq 2$) we prove that the spectrum of a scalar operator in $\mathscr{L}^2(\mathbb{R}^d)$

$$A_\varepsilon(x, hD) = A^0(hD) + \varepsilon B(x, hD),$$

covers interval $(\tau - \epsilon, \tau + \epsilon)$, where A^0 is an elliptic operator and $B(x, hD)$ is a periodic perturbation, $\varepsilon = O(h^\varkappa)$, $\varkappa > 0$.

Further, we consider generalizations.

1 Introduction

1.1 Preliminary Remarks

This work is inspired by a paper [PS3] by L. Parnovski and A. Sobolev, in which a classical Bethe-Sommerfeld conjecture was proven for operators $(-\Delta)^m + B(x, D)$ with operator B of order $< 2m$. In this paper the crucial role was played by a (pseudodifferential) gauge transformation and thorough analysis of the resonant set, both introduced in the papers of L. Parnovski and R. Shterenberg [PSh1, PSh2, PSh3], S. Morozov, L. Parnovski and R. Shterenberg [MPS] and earlier papers by A. Sobolev [So2, So2], devoted to complete asymptotics of the integrated density of states.

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Later in [Ivr3] I used the gauge same transformation in the semiclassical settings, which allowed me to generalize the results and simplify the proofs of those papers[1)]. Now I would like to apply this gauge transform to Bethe-Sommerfeld conjecture in the semiclassical settings. The results obtained are more general (except the smoothness with respect to ξ assumptions in [PS3] are more general than here) and the proofs are simpler.

Consider a scalar self-adjoint h-pseudo-differential operator $A_h := A(x, hD)$ in $\mathbb{R}^d$ with the Weyl symbol $A(x,\xi)$, such that[2)]

$$|D_x^\alpha D_\xi^\beta A(x,\xi)| \le c_{\alpha\beta}(|\xi|+1)^m \qquad \forall \alpha,\beta \tag{1.1}$$

and

$$A(x,\xi) \ge c_0|\xi|^m - C_0 \qquad \forall (x,\xi)\in \mathbb{R}^{2d}. \tag{1.2}$$

Then A_h is semibounded from below. Also we assume that it is Γ-periodic with the *lattice of periods* Γ:

$$A(x+\mathsf{y},\xi) = A(x,\xi) \qquad \forall x\in\mathbb{R}^n \quad \forall \mathsf{y}\in\Gamma. \tag{1.3}$$

We assume that Γ is non-degenerate[3)] and denote by Γ^* the *dual lattice*:

$$\gamma\in\Gamma^* \iff \langle \gamma,\mathsf{y}\rangle \in 2\pi\mathbb{Z} \quad \forall \mathsf{y}\in\Gamma; \tag{1.4}$$

since we use Γ^* and it's elements in the paper much more often, than Γ and it's elements, it is more convenient for us to reserve notation γ for elements of Γ^*.

Also let $\mathcal{O} = \mathbb{R}^d/\Gamma$ and $\mathcal{O}^* = \mathbb{R}^d/\Gamma^*$ be *fundamental domains*; we identify them with domains in $\mathbb{R}^d$.

It is well-known that $\mathsf{Spec}(A)$ has a *band-structure*. Namely, consider in $\mathscr{L}^2(\mathcal{O})$ operator $A_h(\xi) = A(x, hD)$ with the *quasi-periodic boundary condition*:

$$u(x+\mathsf{y}) = e^{i\langle \mathsf{y},\xi\rangle}u(x) \qquad \forall x\in\mathcal{O} \quad \forall \mathsf{y}\in\Gamma \tag{1.5}$$

with $\xi\in\mathcal{O}^*$; it is called a *quasimomentum*. Then $\mathsf{Spec}(A_h(\xi))$ is discrete

$$\mathsf{Spec}(A_h(\xi)) = \bigcup_n \lambda_{n,h}(\xi) \tag{1.6}$$

[1)] The other components of the proof were not only completely different, but in the framework of the different paradigm.

[2)] In fact, we consider $A_h := A(x, hD, h)$.

[3)] I.e. with the $\Gamma = \{\mathsf{y} = n_1\mathsf{y}_1 + \ldots + n_d\mathsf{y}_d,\ (n_1,\ldots,n_d)\in\mathbb{Z}^d\}$ with linearly independent $\mathsf{y}_1,\ldots,\mathsf{y}_d\in\mathbb{R}^d$.

and depends on ξ continuously. Further,

$$\mathsf{Spec}(A_h) = \bigcup_{\xi\in\mathcal{O}^*} \mathsf{Spec}(A_h(\xi)) =: \bigcup_n \Lambda_{n,h}, \tag{1.7}$$

with the *spectral bands* $\Lambda_{n,h} := \bigcup_{\xi\in\mathcal{O}^*}\{\lambda_{n,h}(\xi)\}$.

One can prove that the with of the spectral band near energy level τ is $O(h)$. Spectral bands could overlap but they also could leave uncovered intervals, called *spectral gaps*. It follows from [Ivr3] that in our assumptions (see below) the width of the spectral gaps near energy level τ is $O(h^\infty)$. Bethe-Sommerfeld conjecture in the semiclassical settings claims that there are no spectral gaps near energy level τ (in the corresponding assumptions, which include $d \ge 2$).

1.2 Main Theorem (Statement)

We assume that

$$A_h := A(x, hD) = A^0(hD) + \varepsilon B(x, hD), \tag{1.8}$$

where $A^0(\xi)$ satisfies (1.1), (1.2) and $B(x,\xi)$ satisfies (1.1) and (1.3) and $\varepsilon > 0$ is a small parameter. For $A^0(\xi)$ instead of $\lambda_n(\xi)$ we have

$$\lambda^0_\gamma(\xi) := A^0(h(\gamma+\xi)) \qquad \text{with } \ \gamma\in\Gamma^*. \tag{1.9}$$

Recall that (as in [Ivr3])

$$B(x,\xi) = \sum_{\gamma\in\Gamma} b_\gamma(\xi) e^{i\langle\gamma,x\rangle} \tag{1.10}$$

with $\Theta = \Gamma^*$ where due to (1.1)

$$|D^\beta_\xi b_\gamma(\xi)| \le C_{L\beta}(|\gamma|+1)^{-L}(|\xi|+1)^m \qquad \forall\beta \quad \forall (x,\xi)\in\mathbb{R}^{2d} \tag{1.11}$$

with an arbitrarily large exponent L.

Theorem 1.1. *Let $d \ge 2$ and let operator A_h be given by* (1.8) *with $\varepsilon = O(h^\vartheta)$ with arbitrary $\varkappa > 0$ and with $A^0_h = A^0(hD)$ satisfying* (1.1), (1.2) *and $B(x,\xi)$ satisfying* (1.1) *and* (1.3).

Further, assume that the microhyperbolicity *and* strong convexity *conditions on the energy level τ are fulfilled:*

$$|A^0(\xi)-\tau|+|\nabla_\xi A^0(\xi)|\ge \epsilon_0 \tag{1.12}$$

and

$$\pm\sum_{j,k} A^0_{\xi_j\xi_k}(\xi)\eta_j\eta_k \ge \epsilon_0|\eta|^2 \qquad \forall \xi\colon A^0(\xi)=\tau \quad \forall \eta\colon \sum_j A^0_{\xi_j}(\xi)\eta_j=0. \tag{1.13}$$

Furthermore, assume that there exists $\xi\in\Sigma_\tau$ such that for every $\eta\in\Sigma_\tau$, $\eta\ne\xi$, such that $\nabla_\eta A^0(\eta)$ is parallel to $\nabla_\xi A^0(\xi)$ [4]

(1.14) *Σ_τ, intersected with some vicinity of η and shifted by $(\xi-\eta)$, coincides in the vicinity of ξ with $\{\zeta\colon \zeta_k=g(\zeta_{\hat k}\}$ and Σ_τ coincides in the vicinity of ξ with $\{\zeta\colon \zeta_k=f(\zeta_{\hat k}\}$* [5] *and $\nabla^\alpha(f-g)(0)\ne 0$ for some $\alpha\colon |\alpha|=2$* [6].

Then $\mathsf{Spec}(A_h)\supset[\tau-\epsilon,\tau+\epsilon]$ for sufficiently small $\epsilon>0$.

Remark 1.2. (i) If Σ_τ is strongly convex and connected then for every $\xi\in\Sigma_\tau$ there exists exactly one antipodal point $\eta\in\Sigma_\tau$; then $\nu<0$ and assumption (1.14) is fulfilled. In particular, if $A^0(\xi)=|\xi|^m$, then $\eta=-\xi$ and $\nu=-1$.

(ii) If Σ_τ is is strongly convex and consists of p connected components, then the set $\mathfrak{Z}(\xi)=\{\eta\in\Sigma_\tau,\eta\ne\xi\colon \nabla_\eta A^0(\eta)\parallel\nabla_\xi A^0(\xi)\}$ contains exactly $2p-1$ elements, and for p of them $\nu<0$ and assumption (1.14) is fulfilled for sure, while for $(p-1)$ of them $\nu>0$.

1.3 Idea of the Proof and the Plan of the Paper

One needs to understand, how gaps could appear, why they appear if $d=1$ and why it is not the case if $d\ge 2$. Observe that $\lambda_n(\xi)$ can be identified with some $\lambda^0_\gamma(\xi)$ only locally, if $\lambda^0_\gamma(\xi)$ is sufficiently different from $\lambda^0_{\gamma'}(\xi)$ for any $\gamma'\ne\gamma$.

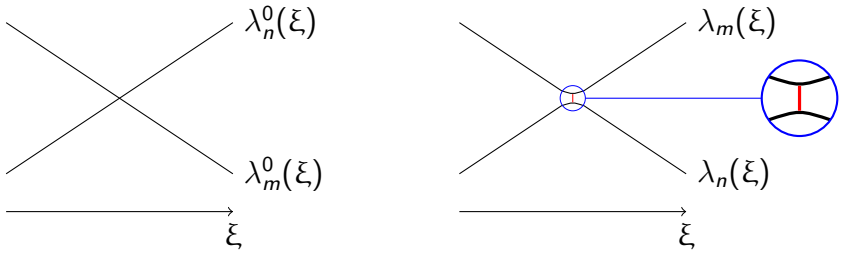
[4] I.e. $\eta A^0(\eta)=\nu\nabla_\xi A^0(\xi)$ with $\nu\ne 0$; we call η *antipodal point.*

[5] With $\zeta_{\hat k}$ meaning all coordinates except ζ_k.

[6] Obviously $\nabla(f-g)(0)=0$. One can prove easily, that if this condition holds at ξ with some $\alpha\colon|\alpha|>2$, then changing slightly ξ, we make it fulfilled with $|\alpha|=2$.

Indeed, in the basis of eigenfunctions of $A^0_\xi(hD)$ [7] perturbation $\varepsilon B(x,hD)$ can contain out-of-diagonal elements $\varepsilon b_{\gamma-\gamma'}(\xi)$ and such identification is possible only if $|\lambda^0_\gamma(\xi)-\lambda^0_{\gamma'}(\xi)|$ is larger than the size of such element.

If $d=1$, $A^0(\xi)=\xi^2$ and $\varepsilon\le\epsilon' h$ with sufficiently small $\epsilon'>0$ and $\tau\asymp 1$, it can happen only if γ' coincides with $-\gamma$ or with one of two adjacent points in Γ^* and $|\xi-\frac12(\gamma+\gamma')|=O(\varepsilon h^\infty)$. This exclude from possible values of either $\lambda^0_\gamma(\xi)$ or $\lambda^0_{\gamma'}(\xi)$ the interval of the width $O(\varepsilon h^\infty)$ and on such interval can happen (and really happens for a generic perturbation) the realignment:



(a) (b)

Figure 28.1: Spectral gap is a red interval

If $d\ge 2$ the picture becomes more complicated: there are much more opportunities for $\lambda^0_\gamma(\xi)$ and $\lambda^0_{\gamma'}(\xi)$ to become close, even if γ and γ' are not that far away; on the other hand, there is a much more opportunities for us to select $\xi=h(\gamma+\xi)\in\Sigma_\tau$ and then to adjust ξ so that $\xi=h(\gamma+\xi)$ remains on Σ_τ but $\eta=h(\gamma'+\xi)$ moves away from Σ_τ sufficiently far away[8] and then tune-up ξ once again so that $\tau\in\mathsf{Spec}(A_h(\xi))$.

In fact, we prove the following statement which together with Theorem 1.1 (which follows from it trivially) are semiclassical analogue of Theorem 2.1 of [PS3]:

Theorem 1.3. *In the framework of Theorem 1.1 there exist n and ξ^* such that $\lambda_n(\xi^*)=\tau$ and $\lambda_n(\xi)$ covers interval $[\tau-\upsilon h,\tau+\upsilon h]$ when ξ runs ball $\mathsf{B}(\xi^*,\upsilon)$ while $|\lambda_m(\xi)-\tau|\ge\epsilon\upsilon h$ for all $m\ne n$ and $\xi\in\mathsf{B}(\xi^*,\upsilon)$. Here*

$$\upsilon=\epsilon\begin{cases} h^{(d-1)^2}\min(1,\ \varepsilon^{-3(d-1)/2}h^{(d-1)+\sigma}) & d\ge 3,\\ h\min(|\log h|^{-1},\ \varepsilon^{-3/2}h^\sigma) & d=2\end{cases}\tag{1.15}$$

[7] Consisting of $\exp(i\langle x,\gamma+\xi\rangle)$.

[8] This will happen if either $\nabla_\eta A^0(\eta)$ differs from $\nu\nabla_\xi A^0(\xi)$, or if coincides with it but (1.14) is fulfilled.

with arbitrarily small exponent $\sigma > 0$.

Proof of Theorem 1.3 occupies two next sections. In Section 2 we reduce operator in the vicinity of Σ_τ to the block-diagonal form and study its structure. To do this we need to examine the structure of the resonant set of the operator. In Section 3 we prove Theorem 1.3 and thus Theorem 1.1.

Finally, in Section 4 we discuss our results and the possible improvements.

2 Reduction of Operator

2.1 Reduction

On this step we reduce A to the block-diagonal form in the vicinity of Σ_τ

$$\Omega_\tau := \{\xi\colon |A^0(\xi) - \tau| \le C\varepsilon h^{-\delta}\}. \tag{2.1}$$

In what follows, we assume that $\varepsilon \ge h$, i.e.

$$h \le \varepsilon \le h^\vartheta, \qquad \vartheta > 0. \tag{2.2}$$

To do this we need just to repeat with the obvious modifications definitions and arguments of Sections 1 and 2 of [Ivr3]. Namely, now $\Theta := \Gamma^*$ is a non-degenerate lattice rather than the pseudo-lattice, as it was in that paper, and all conditions 1.2, 1.3, 1.5, 1.6, and 3.3, are fulfilled with $\Theta' := \Theta \cap B(0,\omega)$ with $\omega = h^{-\varkappa}$ where we select sufficiently small $\varkappa > 0$ later and $\Theta'_K = \Theta \cap B(0, K\omega)$ be an arithmetic sum of K copies of Θ' with sufficiently large K to be chosen later.

We call point ξ *non-resonant* if

$$|\langle \nabla_\xi A^0(\xi), \theta\rangle| \ge \rho \qquad \forall \theta \in \Theta'_K \setminus 0 \tag{2.3}$$

with $\rho \in [\varepsilon^{1/2} h^{-\delta}, h^\delta]$ with arbitrarily small $\delta > 0$. Otherwise we call it *resonant*. More precisely

$$\Lambda := \bigcup_{\theta\in\Theta'_K\setminus 0} \Lambda(\theta), \tag{2.4}$$

where $\Lambda(\theta)$ is the set of ξ, violating (2.3) for given $\theta \in \Theta'_K \setminus 0$.

It obviously follows from the microhyperbolicity and strong convexity assumptions (1.12) and (1.13) that

(2.5) μ_τ-measure[9] of $\Lambda\cap\Sigma_\tau$, does not exceed $C_0 r^{d-1}\rho$ and Euclidean measure of $\Lambda \cap \{\xi\colon |A^0(\xi) - \tau| \le \varsigma\}$ does not exceed $C_0 r^{d-1}\rho\varsigma$, where $r = Kh^{-\varkappa}$.

Indeed, $(d-1)$-dimensional measure of $\{x\colon |x| = 1,\ |\langle x, \theta\rangle| \le \rho\}$ does not exceed $C_0|\theta|^{-1}\rho$ and and due to microhyperbolicity and strong convexity assumptions maps $\Sigma_\tau \to \nabla A^0(\xi)/|\nabla A^0(\xi)| \in \mathbb{S}^{d-1}$ and

$$\{\xi\colon |A^0(\xi) - \tau| \le \varsigma\} \to (\nabla A^0(\xi)/|\nabla A^0(\xi)|, A^0(\xi)) \in \mathbb{R}^d$$

are diffeomorphisms.

Furthermore, according to Proposition 2.7 of [Ivr3] that on the non-resonant set one can "almost" diagonalize $A(x, hD)$. More precisely

Proposition 2.1. *Let assumptions* (1.12) *and* (1.13) *be fulfilled.*

(i) Then there exists a periodic pseudodifferential operator $P = P(x, hD)$ such that

$$\bigl(e^{-i\varepsilon h^{-1}P} A e^{i\varepsilon h^{-1}P} - \mathcal{A}\bigr) Q \equiv 0 \tag{2.6}$$

with

$$\mathcal{A} = A^0(hD) + \varepsilon B'(hD) + \varepsilon B''(x, hD) \tag{2.7}$$

modulo operator from $\mathscr{H}^m$ to $\mathscr{L}^2$ with the operator norm $O(h^M)$ with M arbitrarily large and $K = K(M, d, \delta)$ in the definition of non-resonant point provided $Q = Q(hD)$ has a symbol, supported in $\{\xi\colon |A^0(\xi) - \tau| \le 2\varepsilon h^{-\delta}\}$.

Here $P(x, hD)$, $B'(hD)$ and $B''(x, hD)$ are operator with Weyl symbols of the same form (1.10) *albeit such that*

$$|D_\xi^\alpha D_x^\beta P| \le c_{\alpha\beta}\rho^{-1-|\alpha|} \qquad \forall \alpha, \beta, \tag{2.8}$$

$$|D_\xi^\alpha D_x^\beta B''| \le c'_{\alpha\beta}\rho^{-|\alpha|} \qquad \forall \alpha, \beta, \tag{2.9}$$

and symbol of B' also satisfies (2.9).

(ii) Further,

$$\xi \notin \Lambda(\theta) \implies b''_\theta(\xi) = 0. \tag{2.10}$$

and $B'(\xi)$ coincides with $b_0(\xi)$ modulo $O(\varepsilon\rho^{-2})$.

[9] $\mu_\tau = d\xi : dA^0(\xi)$ is a natural measure on Σ_τ.

In what follows

$$\mathcal{A}^0(hD) \coloneqq A^0(hD) + \varepsilon B'(hD) \qquad \text{and} \qquad \mathcal{B} \coloneqq B''(x, hD). \tag{2.11}$$

Remark 2.2. (i) Eigenvalues of $\mathcal{A}^0$ are

$$\lambda_\gamma(\xi) = \mathcal{A}^0(h(\gamma + \xi)). \tag{2.12}$$

(ii) If $\xi = h(\gamma + \xi)$ is non-resonant, then due to (2.10) in $\epsilon'\rho$-vicinity of ξ this $\lambda_\gamma(\xi)$ is also an eigenvalue of $\mathcal{A}$.

(iii) Without any loss of the generality one can assume that

$$|\theta| \ge \varepsilon h^{-\delta} \implies b''_\theta = 0. \tag{2.13}$$

We assume that this is the case.

(iv) Let us replace operator $\mathcal{A}$ defined by (2.7) by operator

$$\mathcal{A}' = \mathcal{A}^0(hD) + \varepsilon \mathcal{B}'(x, hD), \qquad \mathcal{B}'(x, hD) = T(hD)\mathcal{B}T(hD) \tag{2.14}$$

with $T(hD)$ operator with symbol $T(\xi)$ which is a characteristic function of Ω_τ defined by (2.1) with $C = 6$. Then (2.6) holds.

From now on $\mathcal{A} \coloneqq \mathcal{A}'$ and $\mathcal{B} \coloneqq \mathcal{B}'$.

It would be sufficient to prove Theorem 1.3 for operator $\mathcal{A}$. Indeed,

Proposition 2.3. *(i) For each point* $\lambda \in \mathsf{Spec}(A(\xi)) \cap \{|\lambda - \tau| \le \varepsilon h^{-\delta}\}$ $\mathsf{dist}(\lambda, \mathsf{Spec}(\mathcal{A}(\xi)) \le Ch^M$.

(ii) Conversely, for each point $\lambda \in \mathsf{Spec}(\mathcal{A}(\xi)) \cap \{|\lambda - \tau| \le \varepsilon h^{-\delta}\}$ $\mathsf{dist}(\lambda, \mathsf{Spec}(A(\xi)) \le Ch^M$.

(iii) Furthermore, if $\lambda \in \mathsf{Spec}(A(\xi)) \cap \{|\lambda - \tau| \le \varepsilon h^{-\delta}\}$ *is a simple eigenvalue separated from the rest of* $\mathsf{Spec}(A(\xi))$ *by a distance at least* $2h^{M-1}$, *then there exists* $\lambda' \in \mathsf{Spec}(\mathcal{A}(\xi)) \cap \{|\lambda' - \lambda| \le Ch^M\}$ *separated from the rest of* $\mathsf{Spec}(\mathcal{A}(\xi))$ *by a distance at least* h^{M-1}.

(iv) Conversely, if $\lambda' \in \mathsf{Spec}(\mathcal{A}(\xi)) \cap \{|\lambda - \tau| \le \varepsilon h^{-\delta}\}$ *is a simple eigenvalue separated from the rest of* $\mathsf{Spec}(\mathcal{A}(\xi))$ *by a distance at least* $2h^{M-1}$, *then there exists* $\lambda \in \mathsf{Spec}(A(\xi)) \cap \{|\lambda' - \lambda| \le Ch^M\}$ *separated from the rest of* $\mathsf{Spec}(A(\xi))$ *by a distance at least* h^{M-1}.

Proof. Trivial proof is left to the reader. □

Remark 2.4. One can generalize Statements (iii) and (iv) of Proposition 2.3 from simple eigenvalues to subsets of $\mathsf{Spec}(A(\xi))$ and $\mathsf{Spec}(\mathcal{A}(\xi))$ separated by the rest of the spectra; these subsets will contain the same number of elements.

2.2 Classification of Resonant Points

We start from the case $d = 2$. Then we have only one kind of resonant points $\Xi_1 = \Lambda$. If $d \ge 3$ then there are $(d-1)$ kinds of resonant points. First, following [PS3] consider *lattice spaces* $\mathfrak{V}$ spanned by n linearly independent elements $\theta_1, \dots, \theta_n \in \Gamma^* \cap \mathsf{B}(0, r)$, where we take $r = Kh^{-\varkappa}$. Let $\mathcal{V}_n$ be the set of all such spaces.

It is known [PS3] that

Proposition 2.5. *For $\mathfrak{V} \in \mathcal{V}_n$, $\mathfrak{W} \in \mathcal{V}_m$ either $\mathfrak{V} \subset \mathfrak{W}$, or $\mathfrak{W} \subset \mathfrak{V}$ or the angle*[10] *between $\mathfrak{V}$ and $\mathfrak{W}$ is at least ϵr^{-L} with $L = L(d)$ and $\epsilon = \epsilon(d, \Gamma)$.*

Fix $0 < \delta_1 < \dots < \delta_n$ arbitrarily small and for $\mathfrak{V} \in \mathcal{V}_n$ let us introduce

$$\Lambda(\mathfrak{V}, \rho_n) := \{\xi \in \Omega_\tau : |\langle \nabla_\xi A^0(\xi), \theta\rangle| \le \rho_n |\theta| \quad \forall \theta \in \mathfrak{V}\} \tag{2.15}$$

with $\rho_n = \varepsilon^{\frac{1}{2}} h^{-\delta_n}$.

We define Ξ_n by induction. First, $\Xi_d = \emptyset$. Assume that we defined $\Xi_d, \dots, \Xi_{n+1}$. Then we define

$$\Xi_n := \bigcup_{\mathfrak{V} \in \mathcal{V}_n,\ \xi \in \Lambda(\mathfrak{V}) \cap \Omega_\tau} (\xi + \mathfrak{V}) \cap \Omega_\tau. \tag{2.16}$$

It follows from Proposition 2.5 that

Proposition 2.6. *Let $\varkappa > 0$ in the definition of Θ' and $\delta > 0$ in the definition of Ω_τ be sufficiently small*[11]*. Then for sufficiently small h*

(i) $\Xi_n \subset \bigcup_{\mathfrak{V} \in \mathcal{V}} \Lambda(\mathfrak{V}, 2\rho_n)$.

[10] This angle $\widehat{(\mathfrak{V}, \mathfrak{W})}$ is defined as the smallest possible angle between two vectors $v \in \mathfrak{V} \ominus (\mathfrak{V} \cap \mathfrak{W})$ and $w \in \mathfrak{W} \ominus (\mathfrak{V} \cap \mathfrak{W})$.

[11] They depend on ϑ and $\delta_1, \dots, \delta_n$.

(ii) If $\xi \notin \Xi_{n+1}$ *and* $\xi \in \xi' + \mathfrak{V}$, $\xi \in \xi'' + \mathfrak{W}$ *for* $\xi' \in \Lambda(\mathfrak{V})$, $\xi'' \in \Lambda(\mathfrak{W})$ *with* $\mathfrak{V}, \mathfrak{W} \in \mathcal{V}_n$, *then* $\mathfrak{V} = \mathfrak{W}$.

Corollary 2.7. *Let* $\varkappa > 0$ *in the definition of* Θ' *and* $\delta > 0$ *in the definition of* Ω_τ *be sufficiently small*[11)]. *Let* h *be sufficiently small.*

Then for each $\xi \in \Xi_n \setminus \Xi_{n+1}$ *is defined just one* $\mathfrak{V} = \mathfrak{V}(\xi)$ *such that* $\xi \in \xi' + \mathfrak{V}$ *for some* $\xi' \in \Lambda(\mathfrak{V})$.

(2.17) We slightly change definition of Ξ_n: $\xi = h(\gamma + \xi) \in \Xi_{n,\mathsf{new}}$ iff $h\gamma \in \Xi_n$. From now on $\Xi_n := \Xi_{n,\mathsf{new}}$.

Consider $\xi', \xi'' \in \Xi_n \setminus \Xi_{n+1}$. We say that $\xi' \cong \xi''$ if there exists $\xi \in \mathfrak{V}$, $\mathfrak{V} \in \mathcal{V}$ such that $\xi', \xi'' \in \xi + \mathfrak{V}$ and if $\xi' - \xi'' \in \Gamma$. In virtue of above

(2.18) This relation is reflexive, symmetric and transitive.

For $\xi \in \Xi_n$ we define

$$\mathfrak{X}(\xi) = \{\xi' : \xi' \cong \xi\}. \tag{2.19}$$

Then

$$\mathsf{diam}(\mathfrak{X}(\xi)) \le C\rho_{d-1}. \tag{2.20}$$

2.3 Structure of Operator $\mathcal{A}$

For $\xi \in \Xi_n \setminus \Xi_{n+1}$ denote by $\mathfrak{H}(\xi)$ the subspace $\mathscr{L}^2(\mathcal{O})$ consisting of functions of the form

$$\sum_{\xi' \in \mathfrak{X}(\xi)} c_{\xi'} e^{i\langle x, \xi'\rangle}. \tag{2.21}$$

In virtue of the properties of $\mathcal{A}$ and $\mathcal{B}$ and of resonant sets we arrive to

Proposition 2.8. *Let* $\varkappa > 0$ *in the definition of* Θ' *and* $\delta > 0$ *in the definition of* Ω_τ *be sufficiently small*[11)]. *Let* h *be sufficiently small.*

Then for $\xi \in \Xi_n \setminus \Xi_{n+1}$ *operators* $\mathcal{B}$ *and* $\mathcal{A}$ *transform* $\mathfrak{H}(\xi)$ *into* $\mathfrak{H}(\xi)$.

Let us denote by $\mathcal{A}_\gamma(\xi)$ and $\mathcal{B}_\gamma(\xi)$ restrictions of $\mathcal{A}$ and $\mathcal{B}$ to $\mathfrak{H}(h(\gamma + \xi))$. Here for $n = 0$ we consider Ξ_0 to be the set of all non-resonant points and $\mathfrak{X}(\xi) = \{\xi\}$ for $\xi \in \Xi_0$.

Then due to Propositions 2.5 and 2.8 we arrive to

Proposition 2.9. *(i) For each point* $\lambda \in \mathsf{Spec}(A(\xi)) \cap \{|\lambda - \tau| \le \varepsilon h^{-\delta}\}$ *exists* $\gamma \in \Gamma^*$ *such that* $\xi = h(\gamma + \xi) \in \Omega_\tau$ *and* $\mathsf{dist}(\lambda, \mathsf{Spec}(\mathcal{A}_\gamma(\xi)) \le Ch^M$.

(ii) Conversely, for each point $\lambda \in \mathsf{Spec}(\mathcal{A}_\gamma(\xi)) \cap \{|\lambda - \tau| \le \varepsilon h^{-\delta}\}$ *and* $\xi = h(\xi + \gamma)$, $\mathsf{dist}(\lambda, \mathsf{Spec}(A(\xi)) \le Ch^M$.

(iii) Further, if $\lambda \in \mathsf{Spec}(A_\gamma(\xi)) \cap \{|\lambda - \tau| \le \varepsilon h^{-\delta}\}$ *is a simple eigenvalue separated from the rest of* $\mathsf{Spec}(A(\xi))$ *by a distance at least* $2h^{M-1}$, *then there exist* γ *and* λ', *such that for* $\xi = h(\gamma+\xi)$, $\lambda' \in \mathsf{Spec}(\mathcal{A}(\xi)) \cap \{|\lambda' - \lambda| \le Ch^M\}$, *separated from the rest of* $\mathsf{Spec}(\mathcal{A}_\gamma(\xi))$ *by a distance at least* h^{M-1} *and from* $\bigcup_{\gamma' \in \Gamma^*,\ \gamma' \ne \gamma} \mathsf{Spec}(\mathcal{A}_{\gamma'}(\xi)$ *by a distance at least* h^{M-1} *as well.*

(iv) Conversely, if $\lambda' \in \mathsf{Spec}(\mathcal{A}_\gamma(\xi)) \cap \{|\lambda - \tau| \le \varepsilon h^{-\delta}\}$ *is a simple eigenvalue separated from the rest of* $\mathsf{Spec}(\mathcal{A}_\gamma(\xi))$ *by a distance at least* $2h^{M-1}$, *and also separated from* $\bigcup_{\gamma' \in \Gamma^*,\ \gamma' \ne \gamma} \mathsf{Spec}(\mathcal{A}_{\gamma'}(\xi))$ *by a distance at least* $2h^{M-1}$, *then there exists* $\lambda \in \mathsf{Spec}(A(\xi)) \cap \{|\lambda' - \lambda| \le Ch^M\}$ *separated from the rest of* $\mathsf{Spec}(A(\xi))$ *by a distance at least* h^{M-1}.

Proof. Proof is trivial. □

3 Proof of Theorem 1.3

3.1 Choosing γ^*

The first approximation is $\xi^* \in \Sigma_\tau$ satisfying (1.14). Any $\xi \in \Sigma_\tau$ in ϵ'-vicinity of ξ^* also fits provided $\epsilon' > 0$ is sufficiently small.

(3.1) One can select $\xi^*_{\mathsf{new}} \in \Sigma_\tau$ such that $|\xi^*_{\mathsf{new}} - \xi^*| \le h^\delta$ and ξ^*_{new} satisfies (1.14) with $\rho = \gamma := h^\delta$. Here $\delta > 0$ is arbitrarily small and $\varkappa = \varkappa(\delta)$.

Indeed, it follows from (2.5). From now on $\xi^* := \xi^*_{\mathsf{new}}$.

Then, according to Proposition 2.1 we can diagonalize operator in γ-vicinity of ξ^* and there $\rho = \gamma$. Then there

$$|\nabla^\alpha(\mathcal{A}^0 - A^0)| \le C_\alpha(\varepsilon + \varepsilon^2 \rho^{-2-|\alpha|}) \tag{3.2}$$

and in particular

$$|\nabla^\alpha(\mathcal{A}^0 - A^0)| \le Ch^\delta \qquad \text{for } |\alpha| \le 2. \tag{3.3}$$

Let

$$\Sigma'_\tau = \{\xi\colon \mathcal{A}^0(\xi) = \tau\}. \tag{3.4}$$

Then in the non-resonant points we are interested in functions $\lambda_\gamma(\xi) = \mathcal{A}^0(h(\gamma+\xi))$ rather than in $\lambda^0_\gamma(\xi) = A^0(h(\gamma+\xi))$. One can prove easily the following statements:

Proposition 3.1. *(i) One can select $\xi^* := \xi^*_{\mathsf{new}} \in \Sigma'_\tau$ satisfying* (1.14) *and non-resonant with $\rho = \gamma$.*

(ii) Further, all antipodal to ξ^* points $\xi^*_1,\dots,\xi^*_{2p-1}$ have the same properties.

Let $\xi^* =: h(\gamma^* + \xi^*)$, $\gamma^* \in \Gamma^*$ and $\xi^* \in \mathcal{O}^*$. Then values in the nearby points are sufficiently separated:

$$|\lambda_\gamma(\xi) - \lambda_{\gamma^*}(\xi)| \ge \epsilon h^{1+\delta} \qquad \forall \gamma\colon |\gamma-\gamma^*| \le Kh^{-\varkappa}\ \ \forall \xi \in \mathcal{O}^*. \tag{3.5}$$

Indeed, $|\gamma-\gamma^*| \le Kh^{-\varkappa}$ implies that $(\gamma-\gamma^*) \in \Theta'_K$ and then

$$|\langle \nabla \mathcal{A}^0(\xi^*), \gamma-\gamma^*\rangle| \ge \gamma$$

while

$$|\lambda_\gamma(\xi) - \lambda_{\gamma^*}(\xi) - h\langle \nabla \mathcal{A}^0(\xi^*), \gamma-\gamma^*\rangle| \le Ch^{3-3\varkappa}.$$

3.2 Non-Resonant Points

Consider other non-resonant points (with $\rho = \varepsilon^{1/2}h^{-\delta}$). Let us determine how $\lambda_\gamma(\xi)$ changes when we change ξ. Due to (3.3)

$$\delta\lambda_\gamma := \lambda_\gamma(\xi+\delta\xi) - \lambda_\gamma(\xi) = h\langle \nabla A^0(\xi), \delta\xi\rangle + O(h^2|\delta\xi|^2). \tag{3.6}$$

To preserve $\lambda_{\gamma^*}(\xi) = \tau$ in the *linearized settings* we need to shift ξ by $\delta\xi$ which is orthogonal to $\nabla_\xi \mathcal{A}(\xi^*)$.

Let us take $\delta\xi = t\eta$

$$\ell\colon |\eta| = 1, \qquad \langle \nabla \mathcal{A}^0(\xi^*), \eta\rangle = 0. \tag{3.7}$$

Then in all non-resonant ξ the shift will be $\langle \nabla_\xi \mathcal{A}(\xi), \delta\xi\rangle$ with an absolute value $|\langle \nabla_\xi \mathcal{A}(\xi), \eta\rangle| \cdot |t|$.

Case $d = 2$. Let us start from the easiest case $d = 2$. Without any loss of the generality we assume that ξ^* is strictly inside $\mathcal{O}^*$ (at the distance at least $C\epsilon^*$ from the border). Then there is just one tangent direction η and

$$|\langle \nabla_\xi \mathcal{A}^0(\xi)|_{\xi=h\gamma}, \eta\rangle| \asymp |\sin \varphi(\gamma^*, \gamma)| \asymp h \min_{1\le k\le 2p} |\gamma - \gamma_k^*| \tag{3.8}$$

where $\varphi(\gamma^*, \gamma)$ is an angle between $\nabla_\xi \mathcal{A}^0(\xi)|_{\xi=h\gamma^*}$ and $\nabla_\xi \mathcal{A}^0(\xi)|_{\xi=h\gamma}$, and $\xi_1^*, \dots, \xi_{2p-1}^*$ are antipodal points, and $\xi_{2p}^* = \xi^*$.

As long as $\min_{1\le k\le 2p} |\gamma - \gamma_k^*| \gtrsim h^{1-\varkappa}$ we may replace here $\xi = h(\gamma + \xi)$ by $\xi = h\gamma$ and $\mathcal{A}^0$ by $\mathcal{A}$. In the nonlinear settings to ensure that

$$\lambda_{\gamma^*}(\xi^* + \delta\xi(t)) = \tau \tag{3.9}$$

we need to include in $\delta\xi(t)$ a correction: $\delta\xi(t) = t\eta + O(t^2)$ but still

$$\frac{d}{dt}\lambda_\gamma(\xi^* + \delta\xi(t)) \asymp h\langle \nabla_\xi \mathcal{A}(\xi)|_{\xi=h\gamma}, \eta\rangle^{-1}. \tag{3.10}$$

Then the set $\mathcal{T}(\xi) := \{t\colon |t| \le \epsilon_0,\ |\mathcal{A}^0(\xi(t)) - \tau| \le \upsilon h\}$ is an interval of the length $\asymp \upsilon$ and then the union of such sets over $\xi = h\gamma + \xi$ with indicated γ does not exceed $R\upsilon$ with

$$R := \sum_\gamma |\langle \nabla_\xi \mathcal{A}(\xi)|_{\xi=h\gamma}, \eta\rangle|^{-1}, \tag{3.11}$$

where we sum over set $\{\gamma\colon |\gamma - \gamma^*| \gtrsim h^{-\varkappa}\ \&\ |\lambda_\gamma(h\gamma) - \tau| \le 2Ch\}$. The last restriction is due to the fact that $\mathcal{T}(\xi) \ne \emptyset$ only for points with $|\lambda_\gamma(h\gamma) - \tau| \le 2Ch$.

One can see easily that $R \asymp h^{-1}|\log h|$. Then, as $R\upsilon \le \epsilon'$ the set $[-\epsilon_0, \epsilon_0] \setminus \bigcup_\gamma \mathcal{T}(h(\gamma + \xi))$ contains an interval of the length $\ell = \upsilon$ and for all t, belonging to this interval,

$$|\lambda_\gamma(h(\gamma + \xi + \delta\xi(t))) - \tau| \ge \epsilon\upsilon h. \tag{3.12}$$

Then we need to take $\upsilon = \epsilon R^{-1} = \epsilon h|\log h|^{-1}$ and for $d = 2$ as far as non-resonant are concerned, Theorem 1.3 is almost proven[12)].

[12)] We need to cover almost antipodal points and it will be done in the end of this subsection. We need to consider resonant points and as well, and it will be done in the next subsection.

Case $d \ge 3$. In this case we need to be more subtle and to make $(d-1)$ steps. We start from the point $\xi^* = h(\gamma^* + \xi^*)$; again without any loss of the generality we assume that ξ^* is strictly inside $\mathcal{O}^*$ (at the distance at least $C\epsilon^*$ from the border). Then after each step below it still will be the case (with decreasing constant).

Step I. On the first step we select $\eta = \eta_1$ and consider only γ such that (3.8) holds; more precisely, the left-hand expression needs to be greater than the right-hand expression, multiplied by ϵ [13]. Then $R \asymp h^{1-d}$ and therefore exists ξ^* such that $\lambda_{\gamma^*}(\xi^*) = \tau$ and $|\lambda_\gamma(\xi^*) - \tau| \ge \epsilon\upsilon_1 h$ with $\upsilon_1 = \epsilon h^{d-1}$ for all γ indicated above.

Step II. On the second step we select $\eta = \eta_2$ perpendicular to η_1. To preserve inequality (3.12) (with smaller constant ϵ) for γ, already covered by *Step I*, we need to take $|\delta\xi| \le \epsilon'\upsilon_1$ and consider $\delta\xi = t\eta_2 + O(t^2)$.

Then the same arguments as before results in inequality (3.12) with $\upsilon := \upsilon_2 = \epsilon R^{-1}\upsilon_1$ for a new bunch of points. Then for $d = 3$ as far as non-resonant are concerned, Theorem 1.3 is almost proven[13].

Next steps. Continuing this process, on k-th step we select η_k orthogonal to $\eta_1, \dots, \eta_{k-1}$. Then we get $\upsilon_k = \epsilon R^{-1}\upsilon_{k-1}$ and on the last $(d-1)$-th step we achieve a separation at least $\upsilon_{d-1} = \epsilon R^{1-d}$.

Remark 3.2. In Subsection 4.1 we discuss how to increase υ for $d \ge 3$.

Almost Antipodal Points. We need to cover points with $|\xi - \xi_k^*| \le h^{1-\kappa}$ for $k = 1, \dots, 2p-1$ and as we already know for each k (and fixed ξ) there exists no more than one such point $\xi = h(\gamma + \xi)$ with $|\lambda_\gamma(\xi) - \tau| \lesssim h^{1+\delta}$.

We take care of such points during *Step I*. Observe that during this step we automatically take care of any point with

$$|\nabla_\xi \mathcal{A}^0(\xi), \eta_1\rangle| \ge \epsilon h, \tag{3.13}$$

assuming that $|t| \le \epsilon_0$ with sufficiently small $\epsilon_0 = \epsilon_0(\epsilon)$.

Let us select η_1 so that on η_1 quadratic forms at points $\xi_1^*, \dots, \xi_{2p-1}^*$ in condition (1.14) are different from one at point ξ^* by at least ϵ_0. Then for each $j = 1, \dots, 2p-1$ the the measure of the set

$$\{t\colon |t| \le \epsilon_0,\ |\lambda_{\gamma_j}(\xi + \delta\xi(t))| \le \upsilon h\}$$

[13] One can see easily, that the opposite inequality holds.

does not exceed $Ch^{-1}(\upsilon h)^{\frac{1}{2}}$, and then the measure of the union of such sets (by j) also does not exceed it and therefore for $\upsilon_1 = \epsilon_1 h^{d-1}$ (for $d \ge 3$) and $\upsilon_1 = \epsilon_1 h|\log h|^{-1}$ (for $d = 2$) with sufficiently small ϵ_1 we can find $t\colon |t| \le \epsilon_0$ so that condition (3.8) is fulfilled for all non-resonant points.

3.3 Resonant Points

Next on this step we need to separate $\lambda_{\gamma^*}(\xi)$ from all $\lambda_n(\xi)$ (save one, coinciding with it) by the distance at least υh by choosing ξ. We can during the same steps as described in the previous section: let $\lambda_{\gamma j}(\xi)$ denote eigenvalues of $\mathcal{A}_\gamma(\xi)$ with $j = \#\mathfrak{X}(\gamma h)$.

Observe that both $\mathcal{A}_\gamma(\xi)$ and $\#\mathfrak{X}(\gamma h)$ depend on the equivalency class $[\gamma]$ of γ rather than on γ itself and that

$$\sum_{[\gamma]} \#\mathfrak{X}(\gamma h) = \sum_{1\le n\le d-1} \#(\Xi_n) = O(h^{1-d+\sigma'} + \varepsilon^{3/2} h^{-d-\sigma}), \tag{3.14}$$

where on the left $[\gamma]$ runs over all equivalency classes with $\gamma \in \bigcup_{1\le n\le d-1} \Xi_n$.

We also observe that for resonant points

$$|\sin \varphi(\xi, \xi^*)| \ge \epsilon h^{\delta} \tag{3.15}$$

and therefore for λ'_γ, which are eigenvalues of $\mathcal{A}^0(h(\gamma + \xi))$ [14] (3.10) holds and signs are the same for γ in the same block. On the other hand,

$$|\frac{d}{dt}\mathcal{B}(h(\gamma + \xi^* + \delta\xi(t))| \le C\varepsilon h \ll h^{1+\delta'} \tag{3.16}$$

and therefore for $\lambda_{\gamma,j}(t)$ which are eigenvalues of $\mathcal{A}_{[\gamma]}(\xi)$ (3.10) sill holds.

Therefore the arguments of each *Step I*, *Step II* etc extends to resonant points as well. However the number of *new points* to be taken into account on each step is given by the right-hand expression of (3.14) and therefore R needs to be redefined

$$R := h^{1-d} + \varepsilon^{3/2} h^{-d-\sigma}. \tag{3.17}$$

This leads to the final expression (1.15) for υ. Theorem 1.3 is proven.

[14] Recall, that $\mathcal{A}^0$ is diagonal matrix.

4 Discussion

4.1 Improving υ

Can we improve (increase) expression for υ given by (1.15)? I do not know if one can do anything with the restriction $\upsilon \le \epsilon\varepsilon^{-\frac{3}{2}(d-1)}h^{d(d-1)}$ which is due to resonant points, but restriction $\upsilon \le \epsilon h^{(d-1)^2}$ could be improved for $d \ge 3$, which makes sense only if

$$h \le h \le h^{\frac{2}{3}-\sigma}. \tag{4.1}$$

Indeed, on Step n, $n \ge 2$, we need to take into account only non-resonant points belonging to the set

$$\mathcal{J} := h(\Gamma^* + \xi) \cap \{\xi : |\mathcal{A}^0(\xi) - \tau| \le C\upsilon_{n-1}h\}. \tag{4.2}$$

Determination of upper estimate for such number falls into realm of the Number Theory. I am familiar only with the estimate

$$\#\mathcal{J} \le Ch^{1-d}\upsilon_{n-1} + Ch^{2-d-2/(d+1)}, \tag{4.3}$$

which follows from Theorem at page 224 of [Gui]. Probably it was improved, but those improvement have no value here.

The second term in the right-hand expression of (4.2) is larger, however, the second term in the right-hand expression of (3.17) is still larger and therefore on each Step $n \ge 2$ we have $R := \varepsilon^{3/2}h^{-d-\sigma}$, and this leads us to the following improvement of Theorem 1.3:

Theorem 4.1. *In the framework of Theorem 1.1, under additional assumptions $d \ge 3$ and* (4.1)*, the statement of Theorem 1.3 holds with*

$$\upsilon = \epsilon\varepsilon^{-3(d-2)/2}h^{d^2-d-1-\sigma} \tag{4.4}$$

with arbitrarily small exponent $\sigma > 0$. In particular, $\upsilon = \epsilon h^{d^2-5d/2+3-\sigma}$ for $\varepsilon = h$.

Remark 4.2. One can try to improve further (4.4) for $\varepsilon \le h$. In this case resonant points become the main obstacle. In the definition of resonant points we need to take $\rho = h^{1/2-\delta}$; however only resonant points ξ with

$$\mathfrak{X}(\xi) \cap \{\xi' : |\mathcal{A}^0(\xi') - \tau| \le C(\upsilon_{n-1}h + \varepsilon)\} \ne \emptyset$$

should be taken into account on n-th step.

4.2 Condition (1.14)

We know that for connected component Σ_τ this condition (1.14) is fulfilled automatically.

On the other hand, let $\Sigma_\tau = \bigcup_{1\le j\le p} \Sigma_\tau^{(j)}$ with connected $\Sigma_\tau^{(j)}$. Let $p = 2$ and condition (1.14) be violated at each point of Σ_τ. Does it mean that $\Sigma_\tau^{(2)} = \Sigma_\tau^{(1)} + \eta$ (i.e. $\Sigma_\tau^{(1)}$ shifted by η)?

Next, let $\Sigma_\tau^{(j)} = \Sigma_\tau^{(1)} + \eta_j$ for $j = 2, \dots, q$. Then for these components instead of condition (1.14) one can impose the similar condition involving level surfaces $\sigma_{\tau,\varepsilon}$ of $\mathcal{A}^0(\xi) + \varepsilon B_0(\xi)$. This would affect only *Step I* of our analysis, leading to $\upsilon_{1,\mathsf{new}} := \min(\varepsilon h, \upsilon_1)$ with υ_1 defined without taking into account of antipodal points. Since $\upsilon_1 \le h^{d-1}$ anyway, under assumption $\varepsilon \ge h$ we get the same formulae for υ for $d \ge 3$ as stated in Theorems 1.3 and 4.1, while for $d = 2$ we get

$$\upsilon = h \min(\varepsilon,\ \varepsilon^{-3/2} h^{\sigma}). \tag{4.5}$$

4.3 Differentiability

Definitely our result would follow from the asymptotics of the *density of states*

$$\mathsf{N}'_h(\tau) := \frac{d\mathsf{N}_h(\tau)}{d\tau} = (\kappa'_0(\tau) + o(1))h^{-d} \qquad \text{as} \quad h \to +0, \tag{4.6}$$

where

$$\mathsf{N}_h(\tau) = \int_{\mathcal{O}^*} \mathsf{N}_h(\xi, \tau)\, d\xi \tag{4.7}$$

is an *integrated density of states:*

$$\mathsf{N}_h(\xi, \tau) = \#\{\mu < \tau,\ \mu \ \text{ is an eigenvalue of } \ A_h(\xi)\}. \tag{4.8}$$

However, despite $\mathsf{N}_h(\xi, \tau)$ has a complete asymptotics (see f.e. [Ivr3]), we do not know anything about asymptotics of $\mathsf{N}'_h(\tau)$ (even $\mathsf{N}'_h(\tau) \asymp h^{-d}$ is unknown).

4.4 Bethe-Sommerfeld Conjecture for Almost Periodic Perturbations

While both the proof of Bethe-Sommerfeld conjecture and the statement of Theorem 1.3 rely upon periodicity, the conjecture itself (as stated in

Theorem 1.1) does not. It is only natural to try to prove it for almost periodic perturbations as in [Ivr3].

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Scott and Thomas-Fermi approximations to electronic density *,†,‡

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Abstract

In heavy atoms and molecules, on the distances $a \ll Z^{-1/2}$ from one of the nuclei (with a charge Z_m), we prove that the ground state electronic density $\rho_\Psi(x)$ is approximated in $\mathscr{L}^p$-norm by the ground state electronic density for a single atom in the model with no interactions between electrons.

Further, on the distances $a \gg Z^{-1}$ from all of the nuclei (with a charge $Z_1, \dots, Z_M$) we prove that $\rho_\Psi(x)$ is approximated in $\mathscr{L}^p$-norm, by the Thomas-Fermi density.

We cover both non-relativistic and relativistic cases.

1 Introduction

This paper is a result of my rethinking of one rather old but still remarkable article [IaLS], which I discovered recently and in which the asymptotic of the ground state electronic density on the distances $O(Z^{-1})$ from the nuclei is derived. While there are rather precise results about ground state energy, excessive charges and ionization energy asymptotics[1)] there are relatively few rigorous results about related electronic density.

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1) See f.e. [Ivr1], (in particular, Chapter 25 of [Ivr1], Volume V), and [Ivr2] in non-relativistic and relativistic cases respectively.

The purpose of this paper is to provide a more refined asymptotics (with an error estimate in $\mathscr{L}^p$-norms) and on both small and large distances from the nuclei; we also cover relativistic case.

Multielectron Hamiltonian is given by

$$\mathsf{H} = \mathsf{H}_N := \sum_{1\le j\le N} H_{V,x_j} + \sum_{1\le j<k\le N} \frac{\mathsf{e}^2}{|x_j - x_k|} \tag{1.1}$$

on

$$\mathfrak{H} = \bigwedge_{1\le n\le N} \mathscr{H}, \qquad \mathscr{H} = \mathscr{L}^2(\mathbb{R}^3, \mathbb{C}^q) \simeq \mathscr{L}^2(\mathbb{R}^3 \times \{1, \dots, q\}, \mathbb{C}) \tag{1.2}$$

with

$$H_V = T - \mathsf{e}V(x), \tag{1.3}$$

describing N same type particles (electrons) in the external field with the scalar potential $-V$ and repulsing one another according to the Coulomb law; e is the charge of the electron, T is an *operator of the kinetic energy*.

In the non-relativistic framework this operator is defined as

$$T = \frac{1}{2\mu}(-i\hbar\nabla)^2. \tag{1.4}$$

In the relativistic framework this operator is defined as

$$T = \left(c^2(-i\hbar\nabla)^2 + \mu^2c^4\right)^{\frac{1}{2}} - \mu^2c^4. \tag{1.5}$$

Here

$$V(x) = \sum_{1\le m\le M} \frac{Z_m\mathsf{e}}{|x - \mathsf{y}_m|} \tag{1.6}$$

and

$$d = \min_{1\le m<m'\le M} |\mathsf{y}_m - \mathsf{y}_{m'}| > 0. \tag{1.7}$$

where $Z_m\mathsf{e} > 0$ and y_m are charges and locations of nuclei, μ is the mass of the electron.

It is well-known that the non-relativistic operator is always semibounded from below. On the other hand, it is also well-known [IH, LY] that it this is not necessarily true for the relativistic operator:

Remark 1.1. One particle relativistic operator is semibounded from below if and only if

$$Z_m\beta \le \frac{2}{\pi} \qquad \forall m = 1, \dots, M; \qquad \beta := \frac{\mathsf{e}^2}{\hbar c}. \tag{1.8}$$

We replace (1.8) it by a strict inequality:

$$Z_m\beta \le \frac{2}{\pi} - \epsilon \qquad \forall m = 1, \dots, M; \qquad \beta := \frac{\mathsf{e}^2}{\hbar c}. \tag{1.9}$$

Remark 1.2. (i) In the non-relativistic theory by scaling with respect to the spatial and energy variables we can make $\hbar = \mathsf{e} = 1$, $\mu = \frac{1}{2}$ while Z_m are preserved.

(ii) In the relativistic theory by scaling with respect to the spatial and energy variables we can make $\hbar = \mathsf{e} = 1$, $\mu = \frac{1}{2}$ while β and Z_m are preserved.

(iii) In the corresponding single nuclei theory without electron-to-electron interaction (which is used in the proof of Strong Scott conjecture) only βZ is preserved and we can assume that $Z = 1$.

From now on we assume that such rescaling was already made and we are free to use letters $\hbar$, μ and c for other notations.

We assume that

$$N \asymp Z = Z_1 + \dots + Z_M, \qquad Z_m \asymp Z \qquad \forall m \tag{1.10}$$

and

$$d = \min_{1\le m<m'\le M} |\mathsf{y}_m - \mathsf{y}_{m'}| \ge Z^{-1/3+\sigma} \tag{1.11}$$

with $\sigma \ge 0$.

Our purpose is to consider approximations to the *electronic density*

$$\rho_\Psi(x) = N \int |\Psi(x, x_2, \dots, x_N)|^2 \, dx_2 \cdots dx_N. \tag{1.12}$$

Consider first Scott approximation on the short distances from one of the nuclei.

Theorem 1.3. *Let conditions* (1.9), (1.10) *and* (1.11) *be fulfilled and let* $U \in \mathscr{L}^\infty$ *such that*

$$\mathsf{supp}(U) \subset B(\mathsf{y}_m, a), \qquad |U| \le 1 \tag{1.13}$$

where $a \le Z^{-1/2-\varkappa}$ *with arbitrarily small* $\varkappa > 0$.

Then the following estimate holds:

$$\Big| \int U\big(\rho_\Psi - \rho_{m,\beta}\big)\, dx\Big| \le CF^{1/2}\big((Za)^{3/2}\|\langle Z(x-\mathsf{y}_m)\rangle^{-3/2}U\|_{\mathscr{L}^1}\big)^{1/2} + CG \tag{1.14}$$

with

$$F = \big(Z^{13/6-\delta}a^{-1/2} + Z^4a^3\big)Z^\varkappa \tag{1.15}$$

and

$$G = \big(Z^{7/6-\delta}a^{3/2} + Z^2a^3\big)Z^\varkappa, \tag{1.16}$$

where

$$\rho_{m,\beta}(x) = qZ_m^3\bar\rho_{Z_m\beta}(Z_m(x-\mathsf{y}_m)), \tag{1.17}$$

$\delta = \delta(\sigma) > 0$ *for* $\sigma > 0$ *and* $\delta = 0$ *for* $\sigma = 0$, $\bar\rho_\beta(x) = \bar e_\beta(x,x,0)$, $\bar e_\beta(x,y,\tau)$ *is the Schwartz kernel of the spectral projector* $\theta(\tau - H_{V^0})$ *for toy-model operator* $H_{V^0} \coloneqq H_{\beta,V^0} = T_\beta - |x|^{-1}$ *in* $\mathscr{L}^2(\mathbb{R}^3,\mathbb{C})$ *with*

$$T_\beta = \begin{cases} \Big(-\beta^{-2}\Delta + \dfrac{1}{4}\beta^{-4}\Big)^{\frac12} - \dfrac{1}{2}\beta^{-2} & \beta > 0, \\ -\Delta & \beta = 0. \end{cases} \tag{1.18}$$

Here and below $\langle x\rangle = (|x|^2+1)^{1/2}$.

Corollary 1.4. *Let* <u>*either*</u> $a \asymp Z^{-1}$ *and* $X \subset B(\mathsf{y}_m, a)$ <u>*or*</u> $Z^{-1} \le a \le Z^{-1/2-\varkappa}$ *and* $X \subset B(\mathsf{y}_m, a) \setminus B(\mathsf{y}_m, a/2)$. *Then*

(i) The following estimate holds:

$$\|\rho_\Psi - \rho_{m,\beta}\|_{\mathscr{L}^1(X)} \le CF^{1/2}(\mathsf{mes}(X))^{1/2} + CG. \tag{1.19}$$

(ii) Assume that $|\rho_\Psi| \le \omega$ *in* X *with* $\omega \ge Z^{3/2}a^{-3/2}$ [2)]. *Then for* $p = 2, 3, \dots$ *the following estimate holds:*

$$\|\rho_\Psi - \rho_{m,\beta}\|_{\mathcal{L}^p(X)} \le C\omega^{1-2(1-2^{-p})/p}\Bigl(F^{(1-2^{-p})/p}(\mathsf{mes}(X))^{2^{-p}/p} + F^{(1-2^{1-p})/p}G^{1-2^{1-p}}\Bigr) + C\omega^{1-1/p}G^{1/p}. \tag{1.20}$$

Remark 1.5. (i) F is defined by its first term, as $a \le Z^{-11/21-2/\delta/7}$ and by its second term otherwise. G is defined by its first term as $a \le Z^{-5/9-2\delta/3}$ snd by its second term otherwise.

(ii) Scott approximation definitely is not right for $a \ge Z^{-1/3}$. On the other hand, resztriction $a \le Z^{-1/2-\varkappa}$ is due to requirement of Proposition 3.8. If we do not use this proposition, but all arguments of Section 4 prior to its use, we get estimate (1.14) with

$$F = \bigl(Z^{13/6-\delta} + a^{-1/2} + Z^{7/2}a^{3/2}\bigr)Z^{\varkappa} \tag{1.21}$$

and

$$G = \bigl(Z^{7/6-\delta}a^{3/2} + Z^{5/2}a^{7/2}\bigr)Z^{\varkappa} \tag{1.22}$$

which for $a \ge Z^{-1/2-\varkappa}$ are defined by their second terms.

(iii) On the other hand, while we believe that $\bar\rho_\beta$ is different from $\bar\rho_0$ (the same density in the non-relativistic case) we have no proof of this. We hope to do it later and to estimate the difference from below.

(iv) In [Ivr3] the upper pointwise estimates of ρ_Ψ has been derived in the non-relativistic case but except for Z^{-1}-vicinities of nuclei they are worse than $C\rho^{\mathsf{TF}}$.

Remark 1.6. Obviously

$$\bar\rho_\beta(x) = \frac{1}{4\pi}\sum_{n\ge 1}\sum_{0\le l\le n-1}(2l+1)R^2_{n,l}(|x|;\beta) \tag{1.23}$$

[2)] Such estimates in the non-relativistic case could be found in [Ivr3]: $\omega = Z^3$ for $a \le Z^{-8/9}$, $\omega = Z^{19/9}a^{-1}$ for $Z^{-8/9} \le a \le Z^{-7/9}$, $\omega = Z^{197/90}a^{-9/10}$ for $Z^{-7/9} \le a \le Z^{-1/3}$, $\omega = Z^{17/9}a^{-9/5}$ for $Z^{-1/3} \le a \le Z^{-5/18}$ and $\omega = Z^{19/9}a^{-1}$ for $a \ge Z^{-5/18}$.

with $R_{n,l}(r;\beta)$ defined by (B.5); in particular,

$$\bar\rho_0(0) = \frac{1}{4\pi}\sum_{n\ge 1}\frac{1}{n^3}. \tag{1.24}$$

Consider now Thomas-Fermi approximation on the larger distances from nuclei.

Theorem 1.7. *Let conditions* (1.9), (1.10) *and* (1.11) *be fulfilled and let* $U \in \mathscr{L}^\infty$ *such that*

$$\mathsf{supp}(U) \subset \{x\colon a \le \ell(x) \le 2a\}, \quad \epsilon Z^{-1} \le a \le \epsilon, \quad |U| \le 1\,, \tag{1.25}$$

where here and below

$$\ell(x) = \min_{1\le m\le M} |x - \mathsf{y}_m| \tag{1.26}$$

is the distance from x *to the nearest nucleus,*

$$\zeta = \begin{cases} Z^{1/2}\ell^{-1/2} & \ell \le Z^{-1/3}, \\ \ell^{-2} & \ell \ge Z^{-1/3}, \end{cases} \tag{1.27}$$

Then the following estimate holds:

$$\Big|\int U(x)\big(\rho_\Psi(x) - \rho^{\mathsf{TF}}(x)\big)\,dx\Big| \le C\Big(E\|U\|_{\mathscr{L}^1} + F^{1/3}\|U\|^{2/3}_{\mathscr{L}^1} + G\Big) \tag{1.28}$$

with

$$E = \zeta^2 a^{-1}, \qquad F = \zeta^{4/3}Z^{5/3-\delta}, \qquad G = \zeta^{-2}Z^{5/3-\delta} \tag{1.29}$$

and $\delta = \delta(\sigma)$, $\delta > 0$ *if* $\sigma > 0$ *and* $\delta = 0$ *if* $\sigma = 0$.

Corollary 1.8. *Let* <u>*either*</u> $a \asymp Z^{-1}$ *and* $X \subset B(\mathsf{y}_m, a)$ <u>*or*</u> $Z^{-1} \le a \le 1$ *and* $X \subset B(\mathsf{y}_m, a) \setminus B(\mathsf{y}_m, a/2)$. *Then*

(i) The following estimate holds:

$$\|\rho_\Psi - \rho^{\mathsf{TF}}\|_{\mathscr{L}^1(X)} \le C\Big(E\,\mathsf{mes}(X) + F^{1/3}(\mathsf{mes}(X))^{2/3} + G\Big). \tag{1.30}$$

(ii) Assume that $|\rho_\Psi| \le \omega$ *in* X *with* $\omega \ge Z^{3/2}a^{-3/2\,2)}$. *Then for* $p = 2, 3, ...$ *the following estimate holds:*

$$\|\rho_\Psi - \rho^{\mathsf{TF}}\|_{\mathscr{L}^p(X)} \le C\omega^{1-1/p} J_p^{1/p} \tag{1.31}$$

where J_p *with* $p = 1, 2, ...$ *is a solution to a recurrent relation*

$$J_p = \omega^{-1} E J_{p-1} + (\omega^{-2}F)^{1/3} J_{p-1}^{2/3} + G, \qquad J_0 = \omega\,\mathsf{mes}(X). \tag{1.32}$$

Remark 1.9. (i) As $\|U\|_{\mathscr{L}^(X)} \asymp a^3$ in Theorem 1.7 and $\mathsf{mes}(X) \asymp a^3$ in Corollary 1.8 the estimates of that paper are better than the corresponding estimates of this one, but for relatively small $\|U\|_{\mathscr{L}^1(X)}$ and $\mathsf{mes}(X)$ those of the present paper may be better.

However, as $\|U\|_{\mathscr{L}^1(X)}$ and $\mathsf{mes}(X)$ are too small the right-hand expressions of (1.28), (1.30) and (1.31) may be larger, than $\zeta^3 \int_X |U|\,dx$ or $\zeta^3\,\mathsf{mes}(X)$ respectively in which case those are no more remainder estimates.

(ii) The methods developed in this part paper also work without any modifications in the case of the self-generated magnetic field (see Chapter 27 of [Ivr1], Volume V).

Plan of the paper. In Sections 2 and 3 we consider a one-particle Hamiltonian (relativistic or not) with a potential $V = V^0 + \varsigma U$ where U is supported in $B(0, r)$ and satisfies $|U(x)| \le 1$, $0 < \varsigma r \le \epsilon$ and explore its eigenvalues and projectors, correspondingly.

In Section 4 we prove Proposition 4.1, using arguments of [IaLS], and then, using results of Section 2 and 3, we derive from it Theorem 1.3.

In Section 5 we prove Theorem 1.7. We start from the same estimate (4.1) but then we use microlocal analysis of a one-particle Hamiltonian with a potential $V = W^{\mathsf{TF}} + \varsigma U$ where U is supported in $\mathcal{X}$ and satisfies $|U(x)| \le W^{\mathsf{TF}}$. In Appendix A we estimate eigenfunctions of the non-relativistic hydrogen Hamiltonian and in Appendix B we generalize them to the relativistic hydrogen Hamiltonian.

2 Estimates of eigenvalues

Now we compare negative spectra of operators $H^0 := H_{V^0}$ and $H := H_V$ where $H_V = T - V$, $V^0 = |x|^{-1}$ and $V = V^0 + \varsigma U$ with

$$\mathsf{supp}(U) \subset B(0, r), \qquad r \ge 1, \qquad |U| \le 1 \tag{2.1}$$

and

$$0 < \varsigma r \le \epsilon_0 \tag{2.2}$$

with sufficiently small constant ϵ_0. Then both of these negative spectra are discrete.

According to Corollary B.7 the negative spectrum of H_V^0 consists of the clusters $\mathcal{C}_n^0 = \{\lambda_{n,k}^0(\beta)\}_{k=1,\dots,\nu_k}$ [3]. We denote by $u_{n,k} = u_{n,k}(r,\phi,\theta)$ corresponding eigenfunctios.

(2.3) Let us number eigenvalues of H_V correspondingly by $\lambda_{n,k}$ with with $n = 1, 2, \dots$ and $k = 1, 2, \dots, \nu_n$, so that $\lambda_{n,k} \le \mu_{n',k'}$ for all $n < n'$ and for all corresponding k and k' and $\lambda_{n,k}^0 \le \lambda_{n,k'}^0$ for all $k < k'$.

Proposition 2.1. *Let* (2.1) *and* (2.2) *be fulfilled and let us number eigenvalues of* H_V *and* H_{V^0} *according to* (2.3)*. Furthermore, assume that*

$$\varsigma r^{5/2} \le \epsilon n^3. \tag{2.4}$$

(i) Then the following estimate holds

$$|\lambda_{n,k} - \lambda_{n,k}^0| \le C \begin{cases} \varsigma r^{5/2} n^{-5} + n^{-3} & \text{for } \ r \le C_0 n^2, \\ \varsigma + n^{-3} & \text{for } \ r \ge C_0 n^2, \end{cases} \tag{2.5}$$

(ii) In particular, $-\lambda_{n,k} \asymp n^{-2}$.

Proof. Indeed, for U, which is smooth in r-scale, this estimate follows from the standard semiclassical asymptotics of the eigenvalue counting function:

$$\mathsf{N}(\tau) = \mathcal{N}^{\mathsf{W}}(\tau) + O(|\tau|^{-2}) \qquad \text{for } \ \tau < 0 \tag{2.6}$$

with

$$\mathcal{N}^{\mathsf{W}}(\tau) = \frac{1}{6\pi^2} \int (\tau - V)_+^{3/2} \, dx \asymp |\tau|^{-3} \tag{2.7}$$

[3] More precisely, in the non-relativistic case ($\beta = 0$) $\lambda_{n,k}^0 = \bar{\lambda}_n = -\frac{1}{4n^2}$ of multiplicity n^2, in the relativistic case for $0 < \beta \le \beta_1$ there are also clusters $\mathcal{C}_n^0$ of the width $O(\beta^2 n^{-3})$, containing n^2 eigenvalues and separated by the gaps of the width $\asymp n^{-3}$ and for $\beta_1 < \beta \le \frac{2}{\pi}$ separation into clusters is not unique but allows $O(1)$ number of elements moved to or from the neighbouring clusters, so each cluster contains $\nu_n = n^2 + O(1)$ eigenvalues (see Remark B.8(i)). All "O" are uniform with respect to n, k and β.

for both H_V and H_{V^0} and

$$|\mathcal{N}^{\mathsf{W}}(\tau; H_V) - \mathcal{N}^{\mathsf{W}}(\tau; H_{V^0})| \le C\varsigma r^{5/2} \tag{2.8}$$

for $r \le C_0|\tau|^{-1}$; otherwise in the right-hand expression we need to replace r by n^2.

Then using the standard variational arguments we can drop the smoothness condition. □

Proposition 2.2. *Let condition* (2.1) *be fulfilled and*

$$\varsigma r^{3/2} \le \epsilon_0 \tag{2.9}$$

with sufficiently small constant ϵ_0. *Then for* $n \ge C_0^{-1}\sqrt{r}$

(i) The spectrum of H_V *consists of the clusters* $\mathcal{C}_n = \{\lambda_{n,k}\}_{k=1,\dots,\nu_n}$ *of the width* $\le C_0(\beta^2+\varsigma)n^{-3}$ *separated by gaps of the width* $\asymp n^{-3}$. *Each cluster* $\mathcal{C}_n$ *contains exactly the same number of eigenvalues as* $\mathcal{C}_n^0$ *and they have common borders: both are contained in* $[\eta_n + \epsilon_1 n^{-3}, \eta_{n+1} - \epsilon_1 n^{-3}]$.

In particular, in the non-relativistic case and in the relativistic case with $\beta < \epsilon_1$ *with sufficiently small constant* ϵ_1 *(see Corollary B.7(i))*

$$\mathcal{C}_n \subset [\bar\lambda_n - C_0(\beta^2+\varsigma)n^{-3}, \bar\lambda_n - C_0(\beta^2+\varsigma)n^{-3}]. \tag{2.10}$$

(ii) Furthermore,

$$|\sum_k (\lambda_{n,k} - \lambda^0_{n,k})| \le C\varsigma r \|\langle x\rangle^{-3/2} U\|_{\mathscr{L}^1} n^{-3}. \tag{2.11}$$

Proof. We need to estimate $\lambda_{n,k}$ from above and below.

(a) We start from the easier upper estimate for $\lambda_{n,k}$. Let $\mathsf{N}(\tau)$ be the number of eigenvalues below $\tau \in [\lambda^0_{n,k}, \lambda^0_{n,k} + C_0 n^{-3}]$; we know from Proposition 2.1(ii) that under assumption (2.9) $\lambda_{n,k} \le \lambda^0_{n,k} + C_0 n^{-3}$ for sure. Recall that

$$\mathsf{N}(\tau) = \max \dim \mathcal{L}\,, \tag{2.12}$$

where maximum is taken over subspaces $\mathcal{L} \subset \mathscr{L}^2(\mathbb{R}^3)$ such that

$$(H_V u, u) - \tau\|u\|^2 < 0 \qquad \forall u \in \mathcal{L},\ u \ne 0. \tag{2.13}$$

Therefore we can replace the latter condition by

$$(H_{V^0}u,u)-\tau\|u\|^2+\varsigma(U_-u,u)\|^2<0 \qquad \forall u\in\mathcal{L},\ u\neq 0\,, \tag{2.14}$$

(where here and below $U_\pm \coloneqq \max(\pm U,0)$) but then instead of equality in (2.12) we get an inequality

$$\mathsf{N}(\tau)\ge \max\dim\mathcal{L}. \tag{2.12}_<$$

Let $\lambda=\lambda^0_{n,k}$; let us consider up $\mathcal{L}=\mathcal{L}_1\oplus\mathcal{L}_2$ with $\mathcal{L}_1=\oplus\sum_{n'<n_*}\mathscr{H}_{n',l'}$, $\mathcal{L}_2\subset\oplus\sum_{n_*\le n'\le n}\mathscr{H}_{n',l'}$, where $\mathscr{H}_{n',l'}$ is a linear span of basis functions[4)]

$$\Upsilon_{n',l',m}\coloneqq\Upsilon_{n',l',m}(r,\varphi,\theta;\beta)=R_{n',l'}(r;\beta)Y_{l',m}(x/|x|) \tag{2.15}$$

with $m=-l',\dots,l'$. Let $n_*=n-C_1$. Then (2.14) holds provided

$$\sum_{1\le i\le 2}\Big[(H_{V^0}u_i,u_i)-\tau\|u_i\|^2+2\varsigma(U_-u,u)\Big]<0 \tag{2.16}$$
$$\forall u_i\in\mathcal{L}_i,\ u=u_1+u_2\neq 0.$$

In virtue of Proposition 2.1(ii) under assumption (2.9) the left-hand expression of (2.16) is negative with any $\tau\ge\lambda^0_{n,k}$ for all $u=u_1\in\mathcal{L}_1$, $u\neq 0$. Then we need to consider only $u_2\in\mathcal{L}_2$. Decomposing $\mathcal{L}_2\ni v=\sum_{n',l'}R_{n',l'}(|x|)v_{n',l'}$, where $v_{n',l'}=v_{n',l'}(x/|x|)$ are eigenfunctions of the Laplacian on $\mathbb{S}^2$ corresponding to eigenvalue $-l'(l'+1)$, we estimate

$$\begin{aligned}(U_-u,u)&\le(\phi u,u)\\ &\le\sum_{n',n'';l'}\|v_{n',l'}\|\,\|v_{n'',l'}\|\int\phi(t)|R_{n',l'}(t)|\,|R_{n'',l'}(t)|\,t^2dt\\ &\le C\sum_{n',l'}\|v_{n',l'}\|^2\int\phi(t)|R_{n',l'}(t)|^2\,t^2dt\end{aligned} \tag{2.17}$$

with $\phi(x)\coloneqq\phi_r(x)$, characteristic function of $B(0,r)$; we used that $\phi(x)$ is spherically symmetric, that $w_{n',l'}$ and $w_{n'',l''}$ are orthogonal for $l'\neq l''$, and that n' and n'' are between $n^*=n-C_1$ and n. Here $\|v\|=\|v\|_{\mathscr{L}^2(\mathbb{S}^2)}$.

[4)] In the spherical coordinates (r,φ,θ). Here and below $R_{n,l}(r)$ are defined by (A.2), (B.5) in the non-relativistic and relativistic case respectively, $Y_{l',m}(\varphi,\theta)$ are spherical harmonics and a natural range is $m=-l',\dots,l'$.

Recall that due to Propositions A.8(iii) and B.5 in the non-relativistic and relativistic cases respectively,

$$|R_{n',l}| \le C \begin{cases} \langle x\rangle^{-3/4} n'^{-3/2} & \text{for } \ l \le C_0\langle x\rangle^{1/2}, \\ l^{-s} n'^{-3/2} & \text{for } \ l \ge C_0\langle x\rangle^{1/2}. \end{cases} \tag{2.18}$$

Then

$$(U_- u, u) \le C r^{3/2} n^{-3} \sum_{n',l'} \|v_{n',l'}\|^2 = C r^{3/2} n^{-3} \|u\|^2 \qquad \text{for } \ u \in \mathcal{L}_2\,. \tag{2.19}$$

Therefore we can take $n_* = n$:

$$\lambda_{n',k'} \le \lambda^0_{n,k} \qquad \text{for } \ n' < n,\ k,\ k'. \tag{2.20}_<$$

Further, we conclude that

$$\lambda_{n,k} \le \mu_{n,k} \tag{2.21}_<$$

which are eigenvalues of $\pi^0_n H_{V^0+2\varsigma U_-}\pi^0_n$ where π^0_n is a projector to $\mathcal{L}_2 = \oplus \sum_l \mathscr{H}_{n,l}$. Therefore

$$\sum_k \lambda_{n,k} \le \mathsf{Tr}\big[\pi^0_n H_{V^0-2\varsigma U_-}\pi^0_n\big] = \mathsf{Tr}\big[\pi^0_n H_{V^0}\pi^0_n\big] + 2\varsigma\, \mathsf{Tr}\big[\pi^0_n U_-\pi^0_n\big] \tag{2.22}_<$$

where the first term equals $\sum_k \lambda^0_{n,k}$ and

$$\begin{aligned} \mathsf{Tr}\big[\pi^0_n U_-\pi^0_n\big] &= \sum_{l,m} \int U_-(x) |R_{n,l}(|x|)|^2 |Y_{l,m}(x/|x|)|^2\, dx \\ &= \frac{1}{4\pi} \sum_{l,m} (2l+1) \int U_-(x) |R_{n,l}(|x|)|^2\, dx \le C r \|\langle x\rangle U_-\|_{\mathscr{L}^1} n^{-3} \end{aligned} \tag{2.23}_<$$

due to equality

$$\sum_m |Y_{l,m}|^2 = \frac{1}{4\pi}(2l+1) \tag{2.24}$$

and (2.18). We arrive to

$$\sum_k (\lambda_{n,k} - \lambda^0_{n,k}) \le C\varsigma r \|\langle x\rangle^{-3/2} U_-\|_{\mathscr{L}^1}. \tag{2.11}_<$$

(b) To estimate $\lambda_{n,k}$ from below we need to estimate $\mathsf{N}(\tau)$ from above. To do this we replace (2.16) by

$$\sum_{1\le i\le 2}\Big[(H_{V^0}u_i,u_i)-\tau\|u_i\|^2-2\varsigma(U_+u_i,u_i)\Big]<0 \qquad \forall u_i\in\mathcal{H}_i,\ u=u_1+u_2\neq 0. \tag{2.25}$$

Then $\mathcal{H}_1$ and $\mathscr{H}_2$ become "detached". Let us take $\mathcal{H}_1=\oplus\sum_{n'\le n^*,l'}\mathscr{H}_{n',l'}$ with $n^*=n+C_1$, and $\mathcal{H}_2=\mathscr{L}^2(\mathbb{R}^3)\ominus\mathcal{H}_1$. Then in virtue of Proposition 2.1 for $u=u_2\in\mathcal{H}_2$ expression (2.25) is non-negative and therefore (2.12) holds with maximum taken over $\mathcal{L}\subset\mathcal{H}_1$.

Note that

$$\mathsf{N}(\tau)=N-\tilde{\mathsf{N}}(\tau)\qquad \text{with}\quad \tilde{\mathsf{N}}(\tau)=\max\dim\tilde{\mathcal{L}}\,, \tag{2.26}$$

where maximum is taken over subspaces on which quadratic form

$$(H_{V^0}u,u)-\tau\|u\|^2+2\varsigma(U_+u,u)\ge 0 \tag{2.27}$$

and $N=\dim\mathcal{H}_1$. Now we need to estimate $\tilde{N}(\tau)$ from below can pick-up test functions delivering this inequality. We consider $\tilde{\mathcal{L}}\subset\oplus\sum_{n\le n'\le n^*,l'}\mathscr{H}_{n',l'}$.

Then, repeating arguments of Part (a) we arrive to

$$\lambda_{n',k'}>\lambda^0_{n,k}\qquad \text{for}\quad n'>n,\ k,\ k'. \tag{2.20)$_>$}$$

Further, we conclude that

$$\lambda_{n,k}\ge\mu'_{n,k} \tag{2.21)$_>$}$$

which are eigenvalues of $\pi^0_n H_{V^0-2\varsigma U_+}\pi^0_n$ and

$$\sum_k\lambda_{n,k}\le\mathsf{Tr}\big[\pi^0_nH_{V^0-2\varsigma U_+}\pi^0_n\big]=\mathsf{Tr}\big[\pi^0_nH_{V^0}\pi^0_n\big]-2\varsigma\,\mathsf{Tr}\big[\pi^0_nU_+\pi^0_n\big] \tag{2.22)$_>$}$$

where the first term equals $\sum_k\lambda^0_{n,k}$ and

$$\mathsf{Tr}\big[\pi^0_nU_+\pi^0_n\big]\le Cr\|\langle x\rangle U_+\|_{\mathscr{L}^1}n^{-3} \tag{2.23)$_>$}$$

due to equality (2.24) and (2.18). We arrive to

$$-\sum_k(\lambda_{n,k}-\lambda^0_n)\le C\varsigma r\|\langle x\rangle^{-3/2+\nu}U_+\|_{\mathscr{L}^1}n^{-3}. \tag{2.11)$_>$}$$

We leave details to the reader.

Combining $(2.20)_<$ and $(2.20)_>$ we arrive to Statement (i) and combining $(2.11)_<$ and $(2.11)_>$ we arrive to Statement (ii). □

Proposition 2.3. *Let conditions* (2.1) *and* (2.9) *be fulfilled. Then for* $n \le C_0^{-1}\sqrt{r}$ *the following estimates hold:*

$$|\lambda_{n,k} - \lambda^0_{n,k}| \le C\varsigma\big(\|\langle x\rangle^{-3/2}U\|_{\mathscr{L}^1(B(0,C_0n^2))} + \|\langle x\rangle^{-s}U\|_{\mathscr{L}^1(B(0,r)\setminus B(0,C_0n^2))}\big), \tag{2.28}$$

and

$$\sum_k |\lambda_{n,k} - \lambda^0_{n,k}| \le C\varsigma\big(\|\langle x\rangle^{-1/2}U\|_{\mathscr{L}^1(B(0,C_0n^2))} + \|\langle x\rangle^{-s}U\|_{\mathscr{L}^1(B(0,r)\setminus B(0,C_0n^2))}\big) \tag{2.29}$$

with arbitrarily large exponent s.

Proof. Proof follows the proof of Proposition 2.2 and takes in account that

$$|R_{n,l}(r)| \le Cr^{-s} \qquad \text{for } \ n \le C_0^{-1}\sqrt{r}. \tag{2.30}$$

We leave easy details to the reader. □

3 Estimates of projectors

In this subsection we first consider projectors associated with operators H_{V^0} and H_V and provide some estimate for them; then we combine these results with the results of Section 2. We assume that condition (2.9) is fulfilled. Then due to Proposition 2.2(i) there is a cluster of eigenvalues $\mathcal{C}^0_n = \{\lambda^0_{n,k}\}_{k=1,\dots,\nu_n}$ and $\mathcal{C}_n = \{\lambda_{n,k}\}_{k=1,\dots,\nu_n}$ of operators $H^0 := H_{V^0}$ and $H := H_V$ respectively. Let us denote by π^0_n and π_n the projectors to the corresponding spectral subspaces.

Proposition 3.1. *(i) In the framework of Proposition 2.2 assume that*[5)]

$$\varsigma\big(r^{3/2}(1 + |\log(rn^{-2})|) + n^2\big) \le \epsilon. \tag{3.1}$$

[5)] Obviously this is a stronger assumption than (2.9).

Then the following estimate holds:

$$\|\pi_n - \pi_n^0\| \le C\varsigma\big(r^{3/2}(1+|\log(rn^{-2})|)+n^{1/2}r^{3/4}\big) \times\Big[1+\big(\varsigma\big(r^{3/2}(1+|\log(rn^{-2})|)+n^2\big)\big)^{K-1}\varsigma n^3\Big] \tag{3.2}$$

with arbitrarily large exponent K.

(ii) On the other hand, in the framework of Proposition 2.3 assume that $\mathsf{supp}(U)\subset B(0,r)\setminus B(0,r/2)$. *Then the following estimate holds*

$$\|\pi_n - \pi_n^0\| \le C\varsigma r^{-s} \tag{3.3}$$

with arbitrarily large exponent s.

Proof. Due to Lemma 3.2 below it is sufficient to prove estimates (3.2) and (3.3) for operator $(\pi_n-\pi_n^0)\pi_n^0$.

(i) Note that

$$\pi_n = \frac{1}{2\pi i}\oint_{\gamma_n}(z-H)^{-1}\,dz\,, \tag{3.4}$$

where γ_n is the circle of radius $\asymp n^{-3}$ with counter-clockwise orientation, contains inside cluster of the eigenvalues $\mathcal{C}_n$ and outside all other clusters and $\mathsf{dist}(\gamma_n,\mathsf{Spec}(H))\asymp n^{-3}$. Similar formula holds for π_n^t, corresponding to operator $H^t=H_{V^0+t\varsigma U}$, and we can select a circle which serves to all $|t|\le 1$.

Then

$$\begin{aligned}\pi_n-\pi_n^0 &= \frac{1}{2\pi i}\oint_{\gamma_n}\big[(z-H)^{-1}-(z-H^0)^{-1}\big]\,dz = \\ &\sum_{1\le k\le K} -\frac{\varsigma^k}{2\pi i}\oint_{\gamma_n}(z-H^0)^{-1}U\cdots U(z-H^0)^{-1}\,dz \\ &\quad -\frac{\varsigma^{K+1}}{2\pi i}\oint_{\gamma_n}(z-H)^{-1}U(z-H^0)^{-1}U\cdots U(z-H^0)^{-1}\,dz\,,\end{aligned} \tag{3.5}$$

where each *regular term* (with $k\le K$) contains $(k+1)$ factors $(z-H^0)^{-1}$ and k factors U, while the *remainder term* contains $(K+1)$ factors $(z-H^0)^{-1}$, one factor $(z-H)^{-1}$ (not on the last position) and $(K+1)$ factors U. "Out

of the box" the each factor $(z-H^0)^{-1}$ or $(z-H)^{-1}$ has an operator norm not exceeding Cn^3, so the operator norm of the whole term does not exceed $C(\varsigma n^3)^k$. To improve it, multiply by π^0_n from the right:

$$(\pi_n-\pi^0_n)\pi^0_n = \sum_{1\le k\le K} -\frac{\varsigma^k}{2\pi i}\oint_{\gamma_n}(z-H^0)^{-1}U\cdots U(z-H^0)^{-1}\pi^0_n\,dz \\ -\frac{\varsigma^{K+1}}{2\pi i}\oint_{\gamma_n}(z-H)^{-1}U(z-H^0)^{-1}U\cdots U(z-H^0)^{-1}\pi^0_n\,dz\,. \tag{3.6}$$

Observe that $\|U\pi^0_n\|\le Cr^{3/4}n^{-3/2}$ in virtue of (2.18), so the operator norm of k-th term does not exceed $C(\varsigma n^3)^k r^{3/4}n^{-3/2}$.

To improve this estimate, let us replace j-th copy of $(z-H^0)^{-1}$ (with $j=1,\dots,k$) by

$$\pi^0_{p_j}(z-H^0)^{-1} = \pi^0_{p_j}(z-H^0)^{-1}\pi^0_{p_j}$$

with $p_j\asymp n$. Then each factor U will be sandwiched between $\pi^0_{p_j}$ and $\pi^0_{p_{j+1}}$ and $\|\pi^0_q U\pi^0_p\|\le Cr^{3/2}$ again in virtue of (2.18). Therefore the operator norm of this new k-th term (with $k\le K$) does not exceed

$$C(\varsigma r^{3/2})^k \prod_{1\le j\le k}(|p_j-n|+1)^{-1}$$

because

$$\|\pi^0_p(z-H^0)^{-1}\|\le Cn^3(|p-n|+1)^{-1}.$$

If we take a sum of this expression over $p_j\colon |p_j-n|\le P$ we get

$$C(\varsigma r^{3/2})^k(1+\log(P))^k \tag{3.7}$$

(with $1\le P\le n/2$) and therefore if we insert instead of π_{p_j} their sums

$$\Pi_{n,P} = \sum_{p\colon |p-n|\le P}\pi^0_p \tag{3.8}$$

we get that the operator norm would not exceed (3.7) (for $P\ge 1$).

On the other hand, if instead of $\Pi_{n,P}$ we insert $(I-\Pi_{n,P})$, then the corresponding factor $\varsigma r^{3/2}(1+\log(P)))$ in the estimate should be replaced by ςn^3P^{-1} because $\|(I-\Pi_{n,P})(z-H^0)^{-1}\|\le Cn^3P^{-1}$.

(3.9) For the left-most copy, however of $(z - H^0)^{-1}$, since it serves just one U (on the right of it), we actually get $\varsigma(n^{3/2}r^{3/4}(1 + \log(P)) + n^3P^{-1})$.

Therefore the operator norm of the regular term with $k = 1$ in (3.6) does not exceed $C\varsigma\bigl(r^{3/2}\log(P) + n^{3/2}r^{3/4}P^{-1}\bigr)$ and optimizing this expression by P: $P \le n/2$ we get the first factor in the right-hand expression of (3.2), achieved as $P \asymp \min(n^{3/2}r^{-3/4},\, n)$.

Further, the operator norm of the regular term with $k \ge 2$ in (3.6) does not exceed $C\varsigma^k\bigl(r^{3/2}\log(P) + n^{3/2}r^{3/4}P^{-1}\bigr)\bigl(r^{3/2}\log(Q) + n^3Q^{-1}\bigr)^{k-1}$ because we can take $\Pi_{n,P}$ for the left-most factor $(z - H^0)^{-1}$ and $\Pi_{n,Q}$ for all other factors $(z - H^0)^{-1}$ and we take P the same as above and $Q \asymp \min(n^3r^{-3/2},\, n)$. Then we get the first factor in the right-hand expression of (3.2), multiplied by

$$\bigl(\varsigma\bigl(r^{3/2}(1 + |\log(rn^{-2})|) + n^2\bigr)\bigr)^{k-1}, \tag{3.10}$$

which due to condition (3.1) is less than 1.

The same procedure, applied to the remainder term, results in the expression for $k = K$ multiplied by ςn^3, which means the first factor in (3.2), multiplies by expression (3.10) with $k = K$, and multiplied by ςn^3. It completes the proof of Statement (i).

(ii) Proof of Statement (ii) is trivial due to (properties of $R_{n,l}$)2.18) and assumption $\mathsf{supp}(U) \subset B(0, r) \setminus B(0, r/2)$ with $n \le C_0^{-1}\sqrt{r}$. □

Lemma 3.2. *Let π and π^0 be two orthogonal projectors such that*

$$\|(I - \pi)\pi^0\| \le \varepsilon < \frac{1}{2} \tag{3.11}$$

and

$$\dim \mathsf{Ran}(\pi) = \dim \mathsf{Ran}(\pi^0) < \infty. \tag{3.12}$$

Then for all p: $1 \le p \le \infty$

$$\|\pi - \pi^0\|_p \le 2(1 - \varepsilon)^{-1}\|(I - \pi)\pi^0\|_p. \tag{3.13}$$

Proof. Due to assumption (3.11) operator $\pi : \mathsf{Ran}(\pi^0) \to \mathsf{Ran}(\pi)$ is injective and therefore due to (3.12) it is invertible, and the norm of an inverse operator $\varphi : \mathsf{Ran}(\pi) \to \mathsf{Ran}(\pi^0)$ does not exceed $(1-\varepsilon)^{-1}$ and also

$$\|(I-\varphi)\pi\|_p \le (1-\epsilon)^{-1}\|\pi-\pi^0\|_p.$$

Then the same estimate holds for the corresponding norm of

$$\pi^0(I-\varphi)\pi = \pi^0\pi - \pi^0\varphi\pi = \pi^0\pi - \pi$$

and for adjoint to it: $\|\pi\pi^0-\pi\|_p \le (1-\epsilon)^{-1}\|\pi-\pi^0\|_p$. Since

$$\|\pi^0-\pi\|_p \le \|\pi\pi^0-\pi\|_p + \|\pi\pi^0-\pi^0\|_p$$

we arrive to (3.18). □

Proposition 3.3. *(i) In the framework of Proposition 2.2 the following estimates hold*

$$|\mathsf{Tr}[U\pi_n^0]| \le Cr\|\langle x\rangle^{-3/2}\|_{\mathscr{L}^1} n^{-3} \tag{3.14}$$

and under assumption (3.1) *also*

$$\begin{aligned} |\mathsf{Tr}[U(\pi_n-\pi_n^0)]| &\le C\varsigma r\|\langle x\rangle^{-3/2}\|_{\mathscr{L}^1}\bigl(r^{3/2}n^{-3}(1+|\log(rn^{-2})|)+n^{-1}\bigr) \\ &\quad + C\varsigma r^{5/2}\Bigl(\varsigma\bigl(r^{3/2}(1+|\log(rn^{-2})|)+n^2\bigr)\Bigr)^{K-1}\varsigma n^3\Bigr]. \end{aligned} \tag{3.15}$$

(ii) In the framework of Proposition 3.1(ii) the following estimate holds

$$|\mathsf{Tr}[U\pi_n^0]| + |\mathsf{Tr}[U\pi_n]| \le Cr^{-s}. \tag{3.16}$$

Proof. We actually proved (3.14) (see $(2.23)_<$ and $(2.23)_>$).

To prove (3.15) we apply the same approach as in the proof of (3.2) but we insert factors $\pi^0_{m_j}$ or $(I-\Pi_{n,Q})$ after the last copy of $(z-H_0)^{-1}$ and, due to the trace, the same factor appears in front of the first copy of U. So again, each factor U is sandwiched between two factors of of the type π^0_m or $(I-\Pi_{n,Q})$.

Observe that in the regular terms one of the factors must be π^0_n; otherwise the integrand is holomorphic inside of the contour of integration γ_n and

an integral is 0. And since there is one factor π_n^0, the second appears automatically (on both sides of the same copy of $(z-H^0)^{-1}$.

Therefore two estimate this term for two copies of $U\pi_n^0$ and $\pi_n^0 U$ we can use Hilbert-Schmidt norms. But

$$\|U\pi_n^0\|_2^2 = \|\pi_n^0 U\pi_n^0\|_2^2 = \mathsf{Tr}[\pi_n^0|U|^2\pi_n^0] \le C\|\langle x\rangle^{-3/2}U^2\|_{\mathscr{L}^1} \tag{3.17}$$

and due to assumption $|U| \le 1$ we can replace there U^2 by U. So in comparison with the norm estimate each rregular term acquires two factors $\|\langle x\rangle^{-3/2}U\|_{\mathscr{L}^1}^{1/2}$ (thus, just one factor $\|\langle x\rangle^{-3/2}U\|_{\mathscr{L}^1}$). Then the absolute value of the first term does not exceed

$$C_\varsigma r\|\langle x\rangle^{-3/2}U\|_{\mathscr{L}^1}^{\|}\big(r^3n^{-3}(1+\log(P)) + r^{3/2}P^{-1}\big)\,,$$

and optimizing by $P\colon P\le n$ we arrive to the first term in (3.15).

The k-th regular term is estimated in the same way albeit with an extra factor (3.10), which is less than 1.

While the remainder term could be estimated a bit better than in (3.15), we do not care, since under proper assumption it will be negligible anyway. The same applies to estimate (3.16) as well. □

Proposition 3.4. *The following formula holds*

$$\mathsf{Tr}\big[H_V^- - H_{V^0}^-\big] = -\int_0^\varsigma \mathsf{Tr}\big[U\theta(-H_{V^t})\big]\,dt. \tag{3.18}$$

with $V^t = V^0 + tU$.

Proof. The proof is trivial.

Remark 3.5. (i) One can rewrite the right-hand expression of (3.18) as

$$-\varsigma\,\mathsf{Tr}\big[U\theta(-H_{V^0})\big] - \int_0^\varsigma \mathsf{Tr}\big[U\big(\theta(-H_{V^t}) - \theta(-H_{V^0})\big)\big]\,dt \tag{3.19}$$

with the last term equal

$$\sum_{n\ge 1} -\int_0^\varsigma \mathsf{Tr}\big[U(\pi_n^t - \pi_n^0)\big]\,dt \tag{3.20}$$

with π_n^t associated with H_{V^t}.

(ii) Further, each term in this sum could be rewritten as

$$\sum_{1\le k\le \nu_n} (\lambda_{n,k} - \lambda^0_{n,k}). \tag{3.21}$$

Indeed, Statement (i) is trivial, and to prove Statement (ii) observe that expression (3.21) is equal to

$$\begin{aligned}\frac{1}{2\pi i}\,\mathsf{Tr}\Big[\oint_{\gamma_n} H_{V^t}(z-H_{V^t})^{-1}\,dz\Big]_{t=0}^{t=\varsigma} &= \frac{1}{2\pi i}\,\mathsf{Tr}\Big[\int_0^{\varsigma}\Big(\frac{\partial}{\partial t}\oint_{\gamma_n} H_{V^t}(z-H_{V^t})^{-1}\,dz\Big)\,dt\Big]\\ &= \frac{1}{2\pi i}\,\mathsf{Tr}\Big[\int_0^{\varsigma}\Big(-\oint_{\gamma_n} U(z-H_{V^t})^{-1}\,dz\\ &\quad + \oint_{\gamma_n} \underbrace{H_{V^t}(z-H_{V^t})^{-1}U(z-H_{V^t})^{-1}}\,dz\Big)\,dt\Big].\end{aligned}$$

Observe that because of the trace we can rewrite the selected expression as $H_{V^t}(z-H_{V^t})^{-2}U$ and that $\oint_{\gamma_n}(z-H_{V^t})^{-2}\,dz = 0$. What remains is obviously equal to the term in (3.20).

Proposition 3.6. *In the framework of Proposition 3.1 assume that*

$$r \le \varsigma^{-2/3+\varkappa} \tag{3.22}$$

with arbitrarily small exponent $\varkappa > 0$. Then

$$|\,\mathsf{Tr}\big[H_V^- - H_{V^0}^-\big] + \varsigma\,\mathsf{Tr}\big[U\theta(-H_{V^0})\big]| \le C\varsigma^{2-\varkappa} r \|\langle x\rangle^{-3/2}U\|_{\mathscr{L}^1}. \tag{3.23}$$

Proof. The left-hand expression in (3.22) does not exceed

$$\begin{aligned}&\sum_{n\le N} |\sum_{k\le \nu_n}\big(\lambda_{n,k}-\lambda^0_{n,k}\big) + \varsigma\,\mathsf{Tr}[U\pi^0_n]\,| +\\ &\qquad \sum_{n>N}\Big(\sum_{k\le\nu_n}\underbrace{|\lambda_{n,k}-\lambda^0_{n,k}|} + |\varsigma\,\mathsf{Tr}[U\pi^0_n]|\Big),\end{aligned} \tag{3.24}$$

where N satisfying (3.1) (if we substitute it instead of n) we will chose later.

Then the terms in the first sum due to Proposition 3.3 and Remark 3.5 do not exceed the right-hand expression of (3.15), multiplied by ς, and their sum does not exceed the first term in the expression below (we sum for n: $C_0^{-1}\sqrt{r} \le n \le N$ assuming that $N \ge C_0^{-1}\sqrt{r}$).

Meanwhile, due to Proposition 2.2 the terms in the second sum do not exceed $C\varsigma r\|\langle x\rangle^{-3/2}U\|_{\mathscr{L}^1}n^{-3}$ and their sum does not exceed the second term in the same expression below:

$$C\varsigma r\|\langle x\rangle^{-3/2}U\|_{\mathscr{L}^1}\Big(\varsigma(1+|\log(rN^{-2})|)+N^{-2}\Big). \tag{3.25}$$

We skipped the remainder term which is justified under assumption

$$\varsigma^{1-\varkappa}(r^{3/2}+N^2)\le 1 \tag{3.26}$$

with arbitrarily small exponent $\varkappa>0$; then the remainder term does not exceed ς^s. Condition (3.26) is slightly more restrictive than (3.1) and is fulfilled due to (3.22) and the choice $N=\varsigma^{-(1-\varkappa)/2}$, which almost minimizes (3.25) but also satisfies (3.26).

While Proposition 3.6 will be completely sufficient for our purpose in Section 4 for small distances from the nuclei, for larger distances we will need a bit more complicated setup. Namely, instead of V^0 unperturbed potential will be $V^0+\varepsilon\Phi$, where Φ satisfies the same restriction (2.1) as U.

First, however, we need to modify Proposition 3.3:

Proposition 3.7. *In the framework of Proposition 2.2 assume that*

$$\operatorname{supp}(\Phi)\subset B(0,r),\qquad |\Phi|\le 1. \tag{3.27}$$

and

$$(\varepsilon+\varsigma)^{1-\varkappa}(r^{3/2}+n^2)\le 1 \tag{3.28}$$

wirh arbitrarily small exponent $\varkappa>0$. Then the following estimate holds:

$$\begin{aligned}&|\operatorname{Tr}\big[U(\pi_n(\varepsilon,\varsigma)-\pi_n^0)\big]|+|\operatorname{Tr}\big[\Phi(\pi_n(\varepsilon,\varsigma)-\pi_n^0)\big]|\\ &\qquad\le Cr\big(\varsigma\|\langle x\rangle^{-3/2}U\|_{\mathscr{L}^1}+\varepsilon r^{3/2}\big)\big(r^{3/2}n^{-3}(1+|\log(rn^{-2})|)+n^{-2}\big)\,,\end{aligned} \tag{3.29}$$

where $\pi_n(\varepsilon,\varsigma)$ are projectors corresponding operator to $H_{V^0+\varsigma U+\varepsilon\Phi}$.

Proof. Proof repeats the proof of Proposition 3.3. We leave easy details to the reader. □

Now we can modify Proposition 3.6:

Proposition 3.8. *In the framework of Proposition 2.2 assume that*

$$\varepsilon r^{3/2} \le \epsilon_0 \tag{3.30}$$

and (3.27) *is fulfilled. Then the following estimate holds:*

$$|\operatorname{Tr}\big[H^-_{V^0+\varepsilon\Phi} - H^-_{V^0} - H^-_{V^0+\varsigma U+\varepsilon\Phi} + H^-_{V^0+\varsigma U}\big]| \le C(\varsigma+\varepsilon)^{1-\varkappa} r\big(\varsigma\|\langle x\rangle^{-3/2}U\|_{\mathscr{L}^1} + \varepsilon r^{3/2}\big). \tag{3.31}$$

Proof. Observe that the left-hand expression of (3.31) equals (without traces or absolute value)

$$\begin{aligned} U\int_0^{\varsigma}\Big(\theta(H_{V^0+\varepsilon\Phi+t\varsigma U}) - \theta(H_{V^0+tU})\Big)\,dt \\ = U\int_0^{\varsigma}\Big(\theta(H_{V^0+\varepsilon\Phi+t\varsigma U}) - \theta(H_{V^0})\Big)\,dt \\ - U\int_0^{\varsigma}\Big(\theta(H_{V^0+\varepsilon\Phi}) - \theta(H_{V^0})\Big)\,dt\,, \end{aligned} \tag{3.32}$$

and under assumption (3.28) we can break the last expressions into sums of terms with $\theta(...)$ replaced by the corresponding projectors. We also can combine the first with the second and the third with the fourth terms, rather than the first with the third and the second with the fourth terms.

Then using the corresponding estimates in Proposition 3.7 we can estimate the sum over n: $n \le N$ by the right-hand expression of (3.29), multiplied by ς or by ε (of our choice). Since we take $N = (\varepsilon+\varsigma)^{-\frac{1}{2}-\varkappa}$ as dictated by (3.28), it does not exceed the right-hand expression of (3.31).

Unfortunately, we are not that lucky with $n \ge N$: we have only estimates

$$|\sum_k \big(\lambda_{n,k}(\varepsilon,\varsigma) - \lambda_{n,k}(\varepsilon,0)\big)| \le Cr\big(\varsigma\|\langle x\rangle^{-3/2}U\|_{\mathscr{L}^1} + \varepsilon r^{3/2}\big)n^{-3}, \tag{3.33}$$

and

$$|\sum_k \big(\lambda_{n,k}(\varepsilon,\varsigma) - \lambda_{n,k}(0,\varsigma)\big)| \le Cr\big(\varsigma\|\langle x\rangle^{-3/2}U\|_{\mathscr{L}^1} + \varepsilon r^{3/2}\big)n^{-3} \tag{3.34}$$

which follow from estimate (2.11): namely, we have the same estimate for

$$|\sum_k \big(\lambda_{n,k}(\varepsilon,\varsigma) - \lambda^0_{n,k}\big)|.$$

Then the sum over n: $n \ge N = (\varepsilon+\varsigma)^{-\frac{1}{2}-\varkappa}$ does not exceed the right-hand expression of (3.31). □

4 Electronic Density: Scott approximation

Proposition 4.1. *Under assumption* (1.11) *the following estimate holds:*

$$\varsigma \int U\rho_\Psi \, dx \le \mathsf{Tr}\big[H^-_{W+\nu}\big] - \mathsf{Tr}\big[H^-_{W+\varsigma U+\nu}\big] + CZ^{5/3-\delta} \tag{4.1}$$

with $\delta = \delta(\sigma) > 0$ *as* $\sigma > 0$ *and* $\delta = 0$ *as* $\sigma = 0$ *in* (1.7).

Proof. We know the following upper estimate[6)]

$$\langle \mathsf{H}_{N,Z}\Psi, \Psi\rangle \le \mathsf{Tr}\big[H^-_{W+\nu}\big] + \nu N - \frac{1}{2}\mathsf{D}(\rho^{\mathsf{TF}}, \rho^{\mathsf{TF}}) + \mathsf{Dirac} + CZ^{5/3-\delta}, \tag{4.2}$$

where $\coloneqq W^{\mathsf{TF}}$ and ρ^{TF} are *Thomas-Fermi potential* and *Thomas-Fermi density*[7)], ν is a chemical potential[8)] and $\mathsf{D}(f,g) = \iint |x-y|^{-1} f(x)g(y)\,dxdy$.

On the other hand, similarly to the deduction of the lower estimate in Subsubsection 25.2.1 of [Ivr1], Volume V, for *any potential* $W' = W + U$

$$\begin{aligned}
&\langle \mathsf{H}_{N,Z}\Psi, \Psi\rangle \\
&\ge \sum_n (H_{V,n}\Psi, \Psi) + \frac{1}{2}\mathsf{D}(\rho_\Psi, \rho_\Psi) - CZ^{5/3-\delta} \\
&= \sum_n (H_{W+U,n}\Psi, \Psi) + \int (W + U - V)\rho_\Psi \, dx + \frac{1}{2}\mathsf{D}(\rho_\Psi, \rho_\Psi) - CZ^{5/3-\delta} \\
&= \sum_n (H_{W+U,n}\Psi, \Psi) + \int U\rho_\Psi \, dx + \frac{1}{2}\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}}) \\
&\qquad\qquad - \frac{1}{2}\mathsf{D}(\rho^{\mathsf{TF}}, \rho^{\mathsf{TF}}) - CZ^{5/3-\delta} \\
&\ge \mathsf{Tr}\big[H^-_{W+U+\nu}\big] + \nu N + \int U\rho_\Psi \, dx + \frac{1}{2}\mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}}, \rho_\Psi - \rho^{\mathsf{TF}}) \\
&\qquad\qquad - \frac{1}{2}\mathsf{D}(\rho^{\mathsf{TF}}, \rho^{\mathsf{TF}}) - CZ^{5/3-\delta}.
\end{aligned} \tag{4.3}$$

Therefore combining (4.2), (4.2) we arrive to (4.1). □

[6)] See f.e. Section 25.4 of [Ivr1], Volume V, and [Ivr2] in non-relativistic and relativistic settings respectively.

[7)] See f.e. Subsubsection 25.1.3 of [Ivr1], Volume V.

[8)] See f.e. Subsubsection 25.2.3 of [Ivr1], Volume V.

Remark 4.2. Replacing in (4.1) U by $-U$ and then multiplying by -1, we estimate the left-hand expression of (4.1) from below

$$\varsigma \int U\rho_\Psi \, dx \ge \mathsf{Tr}\big[(H^-_{W-\varsigma U+\nu}\big] - \mathsf{Tr}\big[H^-_{W+\nu}\big] - CZ^{5/3-\delta}. \tag{4.4}$$

In this section we consider ρ_Ψ on rather short distance from the nucleus y_m and want to replace W^{TF} by $V^0_m = Z_m|x-\mathsf{y}_m|^{-1}$ [9].

Let U be supported in $B(\mathsf{y}_m, a)$ with some fixed m and satisfy there $|U| \le Z^2$. Further, let ϕ be a smooth a-admissible function[10], $\phi(x) = 1$ for $|x-\mathsf{y}_m| \le b$ and $\phi(x) = 0$ for $|x-\mathsf{y}_m| \ge 2b$, $b \ge \mathsf{min}(Z^{-1+\delta'}, 2a)$ with small $\delta' > 0$.

Rescaling $x \mapsto (x-\mathsf{y}_m)Z_m$, $\tau \mapsto \tau Z_m^{-2}$ we find ourselves in the framework of Sections 2 and 3. On the other hand, due to Sections 25.4 of citeSS-monsterbook and 12.6 of [Ivr1] (Volumes V and II respectively).

$$|\,\mathsf{Tr}\big[[(1-\phi)H^-_{W+\nu}\big] - \mathsf{Tr}\big[(1-\phi)H^-_{W+\varsigma U+\nu}\big]| \le CZ^{5/3-\delta} \tag{4.5}$$

(with $\delta = \delta(\sigma)$) and we can insert ϕ into right-hand expression of (4.1):

$$\varsigma \int U\rho_\Psi \, dx \le \mathsf{Tr}\big[(\phi H^-_{W+\nu}\big] - \mathsf{Tr}\big[\phi H^-_{W+\varsigma U+\nu}\big] + CZ^{5/3-\delta}. \tag{4.6}$$

Indeed, $W + \varsigma U = W'$ is smooth b-admissible function in the zone $\{x\colon b \le |x-\mathsf{y}_m| \le 9b\}$.

Then since $|V^0_m - W| \le CZ^{4/3}$ in $2b$-vicinity of y_m and $|\nu| \le CZ^{4/3}$ we can replace $W+\nu$ by V^0_m with an error not exceeding $CZ^{4/3} \times r^{3/2}$, where we prefer to use a scaled distance $r = Z_m a$ rather than a:

$$\varsigma \int U\rho_\Psi \, dx \le \mathsf{Tr}\big[(\phi H^-_{V^0_m}\big] - \mathsf{Tr}\big[\phi H^-_{V^0_m+\varsigma U}\big] + CZ^{5/3-\delta} + CZ^{4/3}r^{3/2}. \tag{4.7}$$

Indeed, using arguments of Sections 25.4 and 12.6 of [Ivr1] (volumes V and II respectively), we can first replace W by W_m, coinciding with V^0_m on $B(\mathsf{y}_m, 3b)$ and with W outside of $B(\mathsf{y}_m, 4b)$ [11] and then replace W_m by V^0_m using the same arguments and that $\mathsf{supp}(\phi)$ is "smaller".

---

[9] In Section 5 we use microlocal approach to study ρ_Ψ on the larger distances from all nuclei and do not do such replacement.

[10] I.e. $|D^\alpha \phi| \le C_\alpha b^{-|\alpha|}$ for all α.

[11] This replacement brings an extra error–the last term in the right-hand expression of (4.7). There should be b rather than a but $b \asymp a$ unless $b \asymp Z^{-1+\delta''}$ in which case the previous term is larger.

Note that the last term in the right-hand expression of (4.7) is smaller than the previous one, and we can drop it, as $r \le Z^{2/9-2\delta/3}$, or

$$a \le Z^{-7/9-2\delta/3}. \tag{4.8}$$

Next, using rescaling $x \mapsto (x - \mathsf{y}_m)Z_m$, $U \mapsto \bar{U} = Z_m^{-2}U$ we find ourselves in the framework of Sections 2 and 3 with $r = Z_m a$. Therefore, using decomposition of Remark 3.5(i) and Proposition 3.6, and Remark 4.2, we arrive to

$$|\int \bar{U}(\rho_\Psi - \rho_m)\,dx| \le C\varsigma^{1-\varkappa} r \|\langle x\rangle^{-3/2}\bar{U}\|^*_{\mathscr{L}^1} + C\varsigma^{-1}Z^{-1/3-\delta} \tag{4.9}$$

with an arbitrarily small exponent $\varkappa > 0$ and $\bar{U} = Z^{-2}U$, satisfies (1.13). Here and below $\|.\|^*_{\mathscr{L}^1}$ is calculated *after rescaling*; in comparison with $\|.\|_{\mathscr{L}^1}$, calculated *before rescaling*, it has an extra factor Z^3.

However we can do better than this. Namely, observe that $W'_m := W^{\mathsf{TF}} - V^0_m$ satisfies

$$|W'_m(x) - W'_m(\mathsf{y}_m)| \le CZ_m^{3/2}|x - \mathsf{y}_m|^{1/2} \tag{4.10}$$

before rescaling[12)] and therefore, if we replace V^0_m by $V^0_m + W'_m(0) - \nu$ in (4.7), the last term in the right-hand expression is replaced by CZr^2:

$$\varsigma \int U\rho_\Psi\,dx \le \mathsf{Tr}\big[(\phi H^-_{V^0_m+W'_m(0)-\nu}\big] - \mathsf{Tr}\big[\phi H^-_{V^0_m+W'_m(0)-\nu+\varsigma U}\big] + CZ^{5/3-\delta} + CZr^2. \tag{4.11}$$

The last term in the right-hand expression of (4.11) is smaller than the previous one, and we can drop it as $r \le Z^{1/3-\delta/2}$, or

$$a \le Z^{-2/3-\delta/2}. \tag{4.12}$$

Remark 4.3. Therefore, (4.9) remains true under this assumption. Indeed, adding a constant to V^0 we just shift all eigenvalues of V^0 and $V^0 + \varsigma U$ and preserve all projectors π^0_n.

[12)] It follows from $|\Delta W'| \le CZ_m^{3/2}|x - \mathsf{y}_m|^{-3/2}$ due to Thomas-Fermi equation.

To improve our results on the distances $a \ge Z^{-2/3-\delta/2}$ we will use Proposition 3.8. Due to (4.10) $\varepsilon = Z^{-1/2}a^{1/2}$ and to satisfy $\varepsilon r^{3/2} \le Z^{-\varkappa}$ we assume that

$$a \le Z^{-1/2-\varkappa}. \tag{4.13}$$

Then due to Proposition 3.8 the right-hand expression of (4.9) should be replaced by

$$\begin{aligned} &C\varsigma^{-1}(\varsigma+\varepsilon)\big(\varsigma r\|\langle x\rangle^{-3/2}\bar U\|^*_{\mathscr{L}^1} + \varepsilon r^{5/2}\big)Z^{\varkappa} + C\varsigma^{-1}Z^{-1/3-\delta} \\ &\qquad \asymp C\varsigma r\|\langle x\rangle^{-3/2}\bar U\|^*_{\mathscr{L}^1}Z^{\varkappa} + C\varepsilon r^{5/2}Z^{\varkappa} + C\varsigma^{-1}\big(\varepsilon^2 r^{5/2}Z^{\varkappa} + Z^{-1/3-\delta}\big); \end{aligned}$$

as we will see $\varsigma \ge Z^{-s}$ with sufficiently large exponent s. We rewrite the last expression as

$$\begin{aligned} C\Big(&\varsigma Z^4 a\|\langle Z(x-\mathsf{y}_m)\rangle^{-3/2}\bar U\|_{\mathscr{L}^1} \\ &+ Z^2a^3 + \varsigma^{-1}\big(Z^{3/2}a^{7/2} + Z^{-1/3-\delta}\big)\Big)Z^{\varkappa}, \end{aligned} \tag{4.14}$$

which would lead to the same final estimate (1.14)–(1.16).

Minimizing (4.14) by $\varsigma\colon \varsigma \le Z^{-\varkappa}r^{-3/2} = Z^{-3/2-\varkappa}a^{-3/2}$, we arrive to

$$\begin{aligned} C\Big(&\big(Z^4 a\|\langle Z(x-\mathsf{y}_m)\rangle^{-3/2}\bar U\|_{\mathscr{L}^1}\big)^{1/2}\big(Z^{3/2}a^{7/2} + Z^{-1/3-\delta}\big)^{1/2} \\ &+ Z^2a^3 + Z^{3/2}a^{3/2}\big(Z^{3/2}a^{7/2} + Z^{-1/3-\delta}\big)\Big)Z^{\varkappa}. \end{aligned} \tag{4.15}$$

Therefore writing U instead of $\bar U$, we have

$$\begin{aligned} &\Big|\int U\big(\rho_\Psi - \rho_m\big)\,dx\Big| \le \\ &\quad C\Big(\big(Z^{3/2}a^{3/2}\|\langle Z(x-\mathsf{y}_m)\rangle^{-3/2}U\|_{\mathscr{L}^1}\big)^{1/2}\big(Z^4a^3 + Z^{13/6-\delta}a^{-1/2}\big)^{1/2} \\ &\qquad + Z^2a^3 + Z^3a^5 + Z^{7/6-\delta}a^{3/2}\big)\Big)Z^{\varkappa}. \end{aligned} \tag{4.16}$$

Since $Z^3a^5 \le Z^2a^3$ we arrive to estimate (1.14)–(1.16), which concludes the proof of Theorem 1.3.

Observe that $\mathsf{supp}(U) \subset B(\mathsf{y}_m, a) \setminus B(\mathsf{y}_m, a/2)$ implies

$$Z^{3/2}a^{3/2}\|\langle Z(x-\mathsf{y}_m)\rangle^{-3/2}U\|_{\mathscr{L}^1} \asymp \|U\|_{\mathscr{L}^1}.$$

To prove Corollary 1.4(i) we plug

$$U(x) = \chi_X(x)\,\mathsf{sign}\big(\rho_\Psi(x) - \rho_m(x)\big) \tag{4.17}$$

with characteristic function χ_X of X; then the left-hand expression becomes $\|\rho_\Psi - \rho_m\|_{\mathscr{L}^1(X)}$ and $\|U\|_{\mathscr{L}^1} = \mathsf{mes}(X)$.

To prove Corollary 1.4(ii) we plug

$$U(x) = \epsilon_0 \omega^{1-p} \chi_X(x)\,\mathsf{sign}\big(\rho_\Psi(x) - \rho_m(x)\big)|\rho_\Psi(x) - \rho_m(x)|^{p-1}\,, \tag{4.18}$$

which satisfies (1.13) due to assumptions $\rho_\Psi \le \omega$ in X and estimate $\rho_m \le CZ^{3/2}a^{-3/2} \le C\omega$ and since $\|U\|_{\mathscr{L}^1} = \omega^{-1} J_{p-2}$, we get that

$$J_p \le C\big((\omega^{-1}F)^{1/2} J_{p-1}^{1/2} + G\big) \tag{4.19}$$

with $J_p := \omega^{1-p}\int_X |\rho_\Psi - \rho_m|^p\,dx$ and $J_0 = \omega\,\mathsf{mes}(X)$, and therefore

$$J_p \le C\Big((\omega^{-1}F)^{1-2^{-p}} J_0^{2^{-p}} + (\omega^{-1}F)^{1-2^{1-p}} G^{2^{1-p}} + G\Big),$$

which implies (1.20).

5 Proof of main Theorem 1.7

We consider only non-relativistic case, in the relativistic case proof is the same and we use the same W^{TF} and ρ^{TF} (without any relativistic corrections) and also

$$P'_\beta(W^{\mathsf{TF}}) = P'(W^{\mathsf{TF}}) + O(\beta\ell^{-1}\zeta^3) = \rho^{\mathsf{TF}} + O(\zeta^2\ell^{-1}) \tag{5.1}$$

because $\zeta\beta \le 1$ in the framework of Theorem 1.7.

We assume that

$$\mathsf{supp}(U) \subset \{x\colon a \le \ell(x) \le 2a\}, \qquad \epsilon Z^{-1} \le a \le \epsilon, \qquad |U| \le \zeta^2\,, \tag{5.2}$$

where ℓ and ζ are defined by (1.25) and (1.26) and

$$0 < \varsigma \le \epsilon. \tag{5.3}$$

We rewrite the right-hand expression as

$$-\int_0^\varsigma \mathsf{Tr}\big(U\theta(-H_{W\pm sU+\nu})\big)\,ds. \tag{5.4}$$

After rescaling $x \mapsto xa^{-1}$, $\tau \mapsto \tau\zeta^{-2}$ we are in the semiclassical settings with $h = 1/(\zeta a)$. Condition to a ensures that $h \le 1$.

Due to Tauberian method we can rewrite the integrand as

$$h^{-1} \int_{-\infty}^{0} F_{t\to h^{-1}\tau}\bigl(\bar\chi_T(t)\Gamma(U(x)u_s)\bigr)\, d\tau \tag{5.5}$$

with an error, essentially not exceeding

$$C_\varsigma T^{-1} \sup_{|\tau|\le\epsilon_1} |F_{t\to\tau}\bigl(\bar\chi_T(t)\Gamma(|U(x)|u_s)\bigr)| \tag{5.6}$$

(we will write an exact statement later) where $\bar\chi_T(t) = \bar\chi(t/T)$, $\bar\chi \in \mathscr{C}_0^\infty([-1,1])$, equal 1 on $[-\frac12, \frac12]$, $h \le T \le \epsilon$, and

(5.7) $u_s(x,y,t)$ is a Schwartz kernel of $\exp(-ih^{-1}tH_{W+sU+\nu})$,

$\Gamma v = \int (\Gamma_x v)\, dx$ and $\Gamma_x v = v(x,x,\cdot)$.

Under assumptions (5.2)–(5.4), using two-terms successive approximation method with unperturbed operator $H_{W+\nu}$ and perturbation sU, we can evaluate

$$F_{t\to h^{-1}\tau}\bigl(\bar\chi_T(t)\Gamma(U(x)u_s)\bigr) \tag{5.8}$$

as

$$F_{t\to h^{-1}\tau}\bigl(\bar\chi_T(t)\Gamma(U(x)u_0)\bigr) \tag{5.9}$$

with an error, not exceeding

$$C\|U\|_{\mathscr{L}^1} a^{-3} h^{-1} T^2 \varsigma \times h^{-3}. \tag{5.10}$$

Indeed, one can see easily that the absolute value of n-th term is estimated by

$$Ch^{1-n} T^n \varsigma^{n-1} \times \|U\|_{\mathscr{L}^1} a^{-3} h^{-3} \tag{5.11}$$

for $n \ge 1$ and (5.10) is (5.11) with $n = 1$.

However, we can do better than this:

Proposition 5.1. *Under assumptions* (5.2)–(5.3) *and* $T \le \epsilon$ *the following estimate holds:*

$$|F_{t\to h^{-1}\tau}\bigl(\bar\chi_T(t)\Gamma(U(x)u_0)\bigr)| \le C\|U\|_{\mathscr{L}^1} a^{-3} h^{-2}\,. \tag{5.12}$$

Proof. Estimate (5.12) is due to semiclassical microlocal arguments applied to $\Gamma_x u$, in which case non-smoothness of U plays no role.

Indeed, for $\ell(x) \le \epsilon Z^{-1/3}$ we have $W := W^{\mathsf{TF}} \asymp Z\ell^{-1}$, and $0 \le -\nu \le CZ^{4/3}$, so in this case operator $H_{W+\nu}$ is x-microhyperbolic on the energy level 0.

Furthermore, this operator is also x-microhyperbolic on the energy level 0 for $\ell(x) \le \epsilon_0|\nu|^{1/4} \asymp \epsilon_0(Z-N)_+^{-1/3}$ [13] and elliptic for $\ell(x) \ge C_0|\nu|^{1/4}$.

Finally, as $\epsilon_0|\nu|^{-1/4} \le \ell(x) \le C_0|\nu|^{-1/4}$ x-microhyperbolicity fails but estimate (5.12) follows from the partition-rescaling technique of Subsubsection 5.1.1 of [Ivr1], Volume I, (since dimension is 3). We leave easy details to the reader. □

Thus, in the framework of Proposition 5.1 we can estimate expression (5.8):

$$|F_{t\to h^{-1}\tau}\bigl(\bar\chi_T(t)\Gamma\bigl(U(x)u_s\bigr)\bigr)| \le C\|U\|_{\mathscr{L}^1}a^{-3}\bigl(h^{-2}+\varsigma T^2h^{-4}\bigr)\,, \tag{5.13}$$

where the first term in the right-hand expression is due to (5.12) and the second one is (5.10).

Using this estimate and standard Tauberian arguments we can estimate

$$\begin{aligned}|\,\mathsf{Tr}\bigl[U\theta(-H_{W+sU+\nu})\bigr] &\\ - h^{-1}\int_{-\infty}^{0} F_{t\to h^{-1}\tau}\bigl(\bar\chi_T(t)\Gamma U(x)u_s(.,.,t)\bigr)\,d\tau|&\\ \le C\|U\|_{\mathscr{L}^1}a^{-3}\bigl(T^{-1}h^{-2}+\varsigma Th^{-4}\bigr)\,,&\end{aligned} \tag{5.14}$$

where the right-hand expression is the right-hand expression of (5.13), multiplied by T^{-1}.

On the other hand, without these microlocal arguments and for $\varphi \in \mathscr{C}_0^\infty([-1,-\frac12])$, $LT \ge h$ the following estimate holds:

$$\begin{aligned}|\int_{-\infty}^{0} F_{t\to h^{-1}\tau}\varphi_L(\tau)\bigl(\bar\chi_T(t)\Gamma\bigl(U(x)(u_0-u_s)\bigr)\,d\tau| \le&\\ C_kL\|U\|_{\mathscr{L}^1}a^{-3}h^{-1}T^2\varsigma h^{-3}\times(h/TL)^kL&\end{aligned} \tag{5.15}$$

with arbitrarily large exponent k.

[13] Recall that $|\nu| \asymp (Z-N)_+^{1/4}$.

Therefore, if we replace in the left-hand expression $\varphi_L(\tau)$ by 1, the result would not exceed the same right-hand expression with $L = h/T$, i.e. $C_k\|U\|_{\mathscr{L}^1}a^{-3}T\varsigma h^{-3}$, and therefore

$$h^{-1}\Big|\int_{-\infty}^{0} F_{t\to h^{-1}\tau}\big(\bar{\chi}_T(t)\Gamma(U(x)(u_s-u_0))\big)\,d\tau\Big| \le C\|U\|_{\mathscr{L}^1}a^{-3}\varsigma T h^{-4}. \tag{5.16}$$

Therefore

(5.17) Estimate (5.14) remains true, if we replace there u_s by u_0.

However, then we can replace Tauberian expression by the corresponding Weyl expression

$$\frac{q}{6\pi^2}\int U(W+\nu)_+^{3/2}\,dx = \int U(x)\rho^{\mathsf{TF}}(x)\,dx \tag{5.18}$$

because $W = W^{\mathsf{TF}}$.

Plugging it into (5.14) and minimizing by $T \in [h,1]$ we get

$$\big|\mathsf{Tr}\big[U\big(\theta(-H_{W+sU+\nu})\big]-\int U(x)\rho^{\mathsf{TF}}(x)\,dx\big| \le C\|U\|_{\mathscr{L}^1}a^{-3}\big(\varsigma^{1/2}h^{-3}+h^{-2}\big) \tag{5.19}$$

for $T \asymp \min\big(h\varsigma^{-1/2},\,1\big)$.

Since

$$\mathsf{Tr}\big[H^-_{W+\nu}\big]-\mathsf{Tr}\big[H^-_{W\pm\varsigma U+\nu}\big] = \int_0^{\varsigma}\mathsf{Tr}\big[U\big(\theta(-H_{W+sU+\nu})\big]\,ds \tag{5.20}$$

inequality (4.1) implies, after we divide by $\varsigma\zeta^2$,

$$\Big|\int \bar{U}(x)\big(\rho_\Psi(x)-\rho^{\mathsf{TF}}(x)\big)\,dx\Big| \le C\|\bar{U}\|_{\mathscr{L}^1}a^{-3}\big(\varsigma^{1/2}\zeta^3a^3+\zeta^2a^2\big)+\varsigma^{-1}\zeta^{-2}Z^{5/3-\delta}. \tag{5.21}$$

with $\bar{U} = \zeta^{-2}U$, satisfying (1.25). *From now on we redefine* $U \coloneqq \bar{U}$.

Minimizing the right-hand expression by $\varsigma \le \epsilon$ we arrive to estimate (1.28)–(1.29). Theorem 1.7 has been proven.

Next, picking up

$$U(x) = \mathsf{sign}(\rho_\Psi(x) - \rho^{\mathsf{TF}}(x))\chi_X(x)$$

with χ_X characteristic function of X we arrive to (1.28) with the left-hand expression replaced by $\|\rho_\Psi - \rho^{\mathsf{TF}}\|_{\mathscr{L}^1(X)}$ and the right-hand expression with $\|U\|_{\mathscr{L}^1} = \mathsf{mes}(X)$, which proves Corollary 1.8(i).

Furthermore, in the framework of Corollary 1.8(ii) we pick up

$$U(x) = \omega^{1-p}\,\mathsf{sign}(\rho_\Psi(x) - \rho^{\mathsf{TF}}(x))|(\rho_\Psi(x) - \rho^{\mathsf{TF}}(x))|^{p-1}\chi_X(x)$$

and we conclude that $J_p^* \coloneqq \omega^{1-p}\|\rho_\Psi - \rho^{\mathsf{TF}}\|^p_{\mathscr{L}^p(X)}$ satisfies

$$J_p^* \le C\big(\omega^{-1}EJ_{p-1}^* + (\omega^{-2}F)^{1/3}J_{p-1}^{*\,2/3} + G\big), \qquad J_0^* \le \omega\,\mathsf{mes}(X) \tag{5.22}$$

with E, F and G defined by (1.29) and therefore $J_p^* \le C_p J_p$ with J_p defined as solutions of (1.32). This conclude the proof of Corollary 1.8(ii).

A Properties of eigenfunctions of Coulomb-Schrödinger operator

A.1 General

Consider operator $H^0 = -\Delta - r^{-1}$; $r = |x|^{-1}$. It is known that its eigenfunctions, corresponding to eigenvalues $-\frac{1}{4n^2}$, with norm 1, are (in the spherical coordinates)

$$u_{n,l,m}(r,\varphi,\theta) = R_{n,l}(r)Y_{l,m}(\varphi,\theta) \tag{A.1}$$

where $Y_{l,m}(\varphi,\theta)$ are spherical functions with $m = -l, -l+1, \dots, l-1, l$, $l = 0, 1, \dots, n$ and

$$R_{n,l}(r) = \sqrt{\frac{(n-l-1)!}{2n^4(n+l)!}}\big(\frac{r}{n}\big)^l e^{-r/2n} L^{(2l+1)}_{n-l-1}\big(\frac{r}{n}\big) \tag{A.2}$$

with associated Laguerre polynomials

$$L_n^{(k)}(z) = \frac{1}{n!}z^{-k}e^z\frac{d^n}{dz^n}\big(e^{-z}z^{n+k}\big) = \sum_{0\le j\le n}\frac{(n+k)!}{(n-j)!(k+j)!j!}(-z)^j. \tag{A.3}$$

Then

$$L^{(2l+1)}_{n-l-1}(z) = \sum_{0\le j\le n-l-1} \frac{(n+l)!}{(n-l-1-j)!(2l+1+j)!j!}(-z)^j. \tag{A.4}$$

Then $v := v_{n,l} = R_{n,l}(r)r$ satisfies

$$-v'' + \frac{l(l+1)}{r^2}v - \frac{1}{r}v = \lambda_n v \tag{A.5}$$

and is $(n-l)$-th eigenfunction and λ_n is $(n-l)$-th eigenvalue of such operator na $\mathscr{L}^2(\mathbb{R}^+)$ and the associated variational form is

$$\int \big(v^2 + \frac{l(l+1)}{r^2}v^2 - \frac{1}{r}v^2\big)\, dr. \tag{A.6}$$

A.2 Zeroes

Proposition A.1. *$v_{n,l}$ has exactly $(n-l-1)$ zeroes*

$$l(l+1) < r_* < r_1 < \ldots < r_{n-l-1} < r^* < 4n^2, \tag{A.7}$$

where $r_* < r^*$ are two roots of

$$W(r) := \frac{1}{r} - \frac{l(l+1)}{r^2} = -\lambda_n \tag{A.8}$$

and

$$r_k \asymp (l+k+1)^2. \tag{A.9}$$

Proof. Standard variational methods imply that $v_{n,l}(r)$ has exactly $(n-l-1)$ zeroes.

Further, equation (A.5) and $v(0) = v(\infty) = 0$ imply that all zeroes are simple and satisfy $W(r) > -\lambda_n$, which implies (A.7).

Consider two points $r_* < r' < r'' < r^*$ and observe that $W(r) \ge \min(W(r'), W(r''))$ on (r', r''). Then v must have a zero on (r', r'') as long as $r'' - r' > \pi/\sqrt{\min(W(r'), W(r'')) + \lambda_n}$.

Therefore, if $cl(l+1) \le r \le c^{-1}n^2$, then in $2\pi r^{1/2}$-vicinity of r must be zero of v. Thus the distance between two consecutive zeroes r_k and $r_{k\pm1}$ with $cl(l+1) \le r_k \le c^{-1}n^2$ is $O(r_k^{1/2})$. From this one can prove easily that then

$$|r_k - r_{k\pm1}| = \pi r_k^{1/2}\Big(1 + O\big(\frac{l(l+1)}{r_k} + \frac{r_k}{n^2}\big)\Big)$$

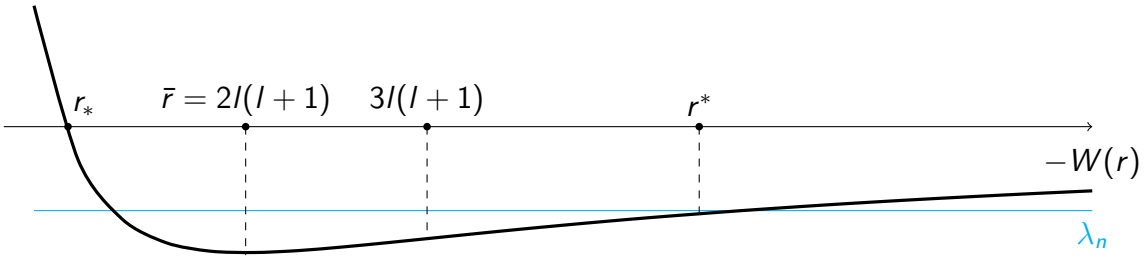

Figure .2: Marked points: left-bound for λ_{l+1}, minimum point, inflection point, right bound for λ_n.

and therefore

$$|r_k^{1/2} - r_{k\pm1}^{1/2}| = \frac{\pi}{2}\Big(1 + O\big(\frac{l(l+1)}{r_k} + \frac{r_k}{n^2}\big)\Big).$$

Further, there must be zeroes to the left of $cl(l+1)$ and to the right of $c^{-1}n^2$.

Finally, observe that $W(r) \ge 1/4l(l+1)$ and therefor e $(r_{k+1} - r_k) \ge 2\pi\sqrt{l(l+1)}$ and therefore $(r_{k+1}^{1/2} - r_k^{1/2}) \ge c^{-1}\pi$ for all k with $r_k \le cl(l+1)$. This implies (A.9). □

Remark A.2. Since $-\partial^2 \ge \frac{1}{4r^2}$ in fact $r_k \ge (l+1/2)^2$; in particular, $r_k \ge \frac{1}{4}$ as $l = 0$.

Let us analyze r_k more carefully. Due to monotonicity of $W(r)$ on $(0, \bar r)$ and $(\bar r, \infty)$ we conclude that

$$\text{(A.10)}\quad r_k \ge \bar r \implies s_{k+1} \ge s_k, \quad \frac{\pi}{\sqrt{W(r_k) + \lambda_n}} \le s_k \le \frac{\pi}{\sqrt{W(r_{k+1}) + \lambda_n}}$$

and

$$\text{(A.11)}\quad r_k \le \bar r \implies s_{k-1} \ge s_k, \quad \frac{\pi}{\sqrt{W(r_k) + \lambda_n}} \le s_k \le \frac{\pi}{\sqrt{W(r_{k-1}) + \lambda_n}}.$$

Consider r_k close to r^*. In this case $W(r) \asymp (r^* - r)/r^{*2}$ and $s_k \asymp r^*/\sqrt{(r^* - r_k)}$ provided $W(r_{k+1}) \asymp W(r_k)$; then $(r^* - r_k) \ge Cs_k$ as $(r^* - r_k) \ge r^*2/3$ and we arrive to Statement (i) below; Statement (ii) is proven the same way[14]. From the same arguments follows Statement (iii).

[14] Assumption $n - l \ge 3$ is needed to have at least 2 zeroes.

Proposition A.3. *Let* $n - l \geq 3$, $n \geq c$. *Then*

(i) As $r_k \asymp r^*$ *and* $(r^* - r_k) \geq Cn^{4/3}$

$$s_k \asymp r^*/\sqrt{(r^* - r_k)} \qquad \text{and} \qquad r^* - r_k \asymp r^{*2/3}(n - l - k)^{2/3}. \tag{A.12}$$

(ii) As $r_k \asymp r_*$ *and* $(r_k - r_*) \geq Cl^{4/3}$

$$s_k \asymp r_*/\sqrt{(r_k - r_*)} \qquad \text{and} \qquad r_k - r_* \asymp r^{*2/3}k^{2/3}. \tag{A.13}$$

(iii) There are no more than C' *zeroes in the zones* $\{r\colon r \leq r_* + Cr_*^{2/3}\}$ *and* $\{r\colon r \geq r^* - Cr^{*2/3}\}$.

A.3 Estimates

From(A.5) it follows that

$$\int_{r_k}^{r_{k+1}} (-2v''v' + \frac{2l(l+1)}{r^2}vv' - \frac{2}{r}vv' - 2\lambda_n vv')\, dr = 0$$

and then

$$-v'^2(r_{k+1}) + v'^2(r_k) = \int_{r_k}^{r_{k+1}} (-\frac{2l(l+1)}{r^3} + \frac{1}{r^2})\, v^2\, dr, \tag{A.14}$$

where $k = 0, \dots, n - l - 1$, and $r_0 := 0$, $r_{n-l} := \infty$.

Then we conclude that

$$|v'(r_k)| > |v'(r_{k+1})| \qquad \text{for} \quad r_k \geq 2l(l+1) \tag{A.15}$$

and

$$|v'(r_k)| < |v'(r_{k+1})| \qquad \text{for} \quad r_{k+1} \leq 2l(l+1). \tag{A.16}$$

Consider first $r_k\colon (1+\epsilon)r_* \leq r_k \leq (1-\epsilon)r^*$ [15].

Then one can see easily that

$$v(r) = v_k'(r_k)\frac{s_k}{\pi}\sin\big(\frac{\pi(r - r_k)}{s_k}\big)\big(1 + O(r_k^{-1/2})\big) \tag{A.17}$$

$$\text{for} \quad r_k \leq r \leq r_{k+1}$$

[15] Which is possible if and only if $l \leq (1 - \epsilon')n$.

and we calculate the right-hand expression of (A.14) arriving to

$$v'^2(r_{k+1}) = v'^2(r_k)\Big[1 - \frac{1}{2}r_k^{-2}\frac{s_k^3}{\pi^2}\Big(1 - \frac{2l(l+1)}{r_k} + O(r_k^{-1/2})\Big)\Big]. \tag{A.18}$$

Then in virtue of (A.11)

$$v'^2(r_{k+1}) = \tag{A.19}$$
$$v'^2(r_k)\Big[1 - \frac{1}{2}r_k^{-1}s_k\Big(1 - \frac{2l(l+1)}{r_k})\Big)\Big(1 - \frac{l(l+1)}{r_k} + \lambda_n\Big)^{-1} + O(r_k^{-1})\Big].$$

Then

$$v'^2(r_{k+1})r_{k+1}^{1/2} = \tag{A.20}$$
$$v'^2(r_k)r_k^{1/2}\Big[1 + \frac{1}{2}r_k^{-1}s_k\Big(\frac{l(l+1)}{r_k} + \lambda_n)\Big)\Big(1 - \frac{l(l+1)}{r_k} + \lambda_n\Big)^{-1} + O(r_k^{-1})\Big] =$$
$$v'^2(r_k)r_k^{1/2}(1+\varepsilon_k)$$

with $\varepsilon_k = O(\frac{l(l+1)}{r_k^{3/2}} + \frac{1}{r_k})$. In virtue of (A.9) $\sum \varepsilon_k \le C$ and therefore we arrive to

$$v'^2(r_k)r_k^{1/2} \asymp v'^2(r_m)r_m^{1/2}. \tag{A.21}$$

Since

$$\max_{r_k \le r \le r_{k+1}} |v(r)| \asymp |v'(r_k)(r_{k+1} - r_k) \tag{A.22}$$

we arrive to

$$r_k^{-1/4} \max_{r_k \le r \le r_{k+1}} |v(r)| \asymp r_m^{-1/4} \max_{r_m \le r \le r_{m+1}} |v(r)|. \tag{A.23}$$

Then taking $m \asymp n$ and using (A.9) we conclude that $p_k \asymp p_m k^{1/2} n^{-1/2}$ with p_k the left-hand expression of (A.21). Since

$$\int_{r_k}^{r_{k+1}} v^2(r)\,dr \asymp \max_{r_k \le r \le r_{k+1}} v^2(r)(r_{k+1} - r_k) \tag{A.24}$$

we conclude that it is $\asymp p_n^2 k^2 n^{-1}$ and therefore their sum is $\asymp p_m^2 n^2$; since it should not exceed 1, we conclude that $p_m \le Cn^{-1}$; applying $p_k \asymp p_m k^{1/2} n^{-1/2}$ again we arrive to

Proposition A.4. *Assume that*

$$l \le (1-\epsilon')n. \tag{A.25}$$

Then for $(1+\epsilon)r_* \le r_k \le (1-\epsilon)r^*$

$$\max_{r_k \le r \le r_{k+1}} |v(r)| \le C' r^{1/4} n^{-3/2}. \tag{A.26}$$

Remark A.5. It follows from the arguments below that actually in (A.26) there is "$\asymp$" sign.

Consider now zone $\{r\colon \max((1-\epsilon)r^* \le r \le r^* - Cr^{*2/3}\}$, again, under assumption (A.26).

Recall, that in this zone A.12) holds and also

$$W(r) = (r^* - r)r^{*-2}\Big(1 + O\big(\frac{r^*}{(r^*-r_k)^{3/2}}\big)\Big) \tag{A.27}$$

and

$$s_k = \frac{\pi r^*}{\sqrt{r^* - r_k}}\Big(1 + O\big(\frac{r^*}{(r^*-r_k)^{3/2}}\big)\Big). \tag{A.28}$$

Then for $r_k \le r \le r_{k+1}$

$$v(r) = v'(r_k)\frac{s_k}{\pi}\sin\big(\frac{\pi(r-r_k)}{s_k}\big)\Big(1 + O\big(\frac{r^*}{(r^*-r_k)^{3/2}}\big)\Big). \tag{A.29}$$

Then (A.14) implies that

$$v'^2(r_{k+1}) - v'^2(r_k) = -\frac{1}{2}v'^2(r_k)\frac{s_k^3}{\pi^2 r^{*2}}\Big(1 - \frac{2l(l+1)}{r^*}\Big)\Big(1 + O\big(\frac{r^*}{(r^*-r_k)^{3/2}}\big)\Big),$$

where (A.25) ensures that the first large parentheses are disjoint 0. Then plugging (A.28) we conclude that

$$v'^2(r_{k+1}) = v'^2(r_k)\Big[1 - \frac{1}{2}\frac{s_k}{(r^*-r_k)}\Big(1 - \frac{2l(l+1)}{r^*}\Big)\Big(1 + O\big(\frac{r^*}{(r^*-r_k)^{3/2}}\big)\Big)\Big]. \tag{A.30}$$

Then

$$v'^2(r_{k+1})(r^* - r_{k+1})^{-1/2} = v'^2(r_k)(r^* - r_{k+1})^{-1/2}\Big[1 + \frac{(r_{k+1}-r_k)}{(r^*-r_k)}\frac{l(l+1)}{r^*} + \varepsilon_k\Big] \tag{A.31}$$

with $\sum_k \varepsilon_k < \infty$.

Therefore

$$v'^2(r_k)(r^* - r_k)^{-1/2+\sigma} \asymp v'^2(r_m)(r^* - r_m)^{-1/2+\sigma},$$

and

$$|v'(r_k)| \asymp |v'(r_m)|(r^* - r_k)^{1/4-\sigma/}(r^* - r_m)^{-1/4+\sigma/2},$$

and finally

$$\max_{r_k<r<r_{k+1}} |v(r)| \asymp \max_{r_m<r<r_{m+1}} |v(r)|(r^* - r_k)^{-1/4-\sigma/2}(r^* - r_m)^{1/4+\sigma/2}.$$

Taking $m \asymp n$ such that $r_m \le (1-\epsilon)r^*$ we arrive in virtue of Proposition A.4 to

Proposition A.6. *Under assumption* (A.25) *for* $(1-\epsilon)r^* \le r < r^* - Cr^{*2/3}$

$$|v(r)| \le C'(r^* - r)^{-1/4-\sigma/2} n^{-1/2+\sigma}, \qquad \sigma = \frac{l(l+1)}{r^*}. \tag{A.32}$$

Consider now zone $\{r\colon r_* + Cr_*^{2/3} \le r \le (1+\epsilon)r_*\}$ provided (A.25). The same arguments lead us to

$$\max_{r_k\le r\le r_{k+1}} |v(r)| \asymp (r_k - r_*)^{-3/4+\sigma'/2}(r_m - r_*)^{3/4-\sigma'/2} \max_{r_m\le r\le r_{m+1}} |v(r)|.$$

and taking $m \asymp l$ such that $r_m \le (1+\epsilon)r_*$ and therefore $r_m \lesssim n^{-3/2}l^{1/2}$, we arrive to

Proposition A.7. *Under assumption* (A.25) *for* $r_* + Cr_*^{2/3} \le r \le (1+\epsilon)r_*$

$$|v(r)| \le C'(r - r_*)^{-3/4+\sigma'/2} n^{-3/2} l^{2-\sigma'}, \qquad \sigma' = \frac{l(l+1)}{r_*}. \tag{A.33}$$

Proposition A.8. *Let assumption* (A.25) *be fulfilled. Then*

(i) The following estimates hold

$$|v(r)| \le C' n^{-5/6-\sigma/3} \qquad \textit{for} \quad r \ge r^* - Cr^{*2/3} \tag{A.34}$$

and

$$|v(r)| \le C' n^{-3/2} l^{1-2\sigma'/3} \qquad \textit{for} \quad r \le r_* + Cr_*2/3. \tag{A.35}$$

(ii) Furthermore, let $b := C_s r^{*2/3}$. *Then*

$$|v(r)| \le C' n^{-5/6-\sigma/3} \Big(\frac{b}{r - r^*}\Big)^s \qquad \text{for } r \ge r^* + b. \tag{A.36}$$

(iii) On the other hand, let $b := C_s r_*^{2/3}$. *Then*

$$|v(r)| \le C' n^{-3/2} \bar{l}^{1-\sigma'/3} \Big(\frac{b}{r_* - r}\Big)^s \qquad \text{for } r \le r_* - b. \tag{A.37}$$

Proof. (i) Estimates (A.34) and (A.35) for $r^* - Cr^{*2/3} \le r \le r^* + Cr^{*2/3}$ and $r_* - Cr_*^{2/3} \le r \le r_* + Cr_*^{*2/3}$ follow from estimates (A.32) and (A.33) and equation (A.5).

(ii) Consider $\phi \in \mathscr{C}^\infty$, $\phi = 0$ on $(-\infty, \frac{1}{2})$ and $\phi = 1$ on $(1, \infty)$ and $\varphi(r) = \phi((r - r^*)/a)$. Then multiplying (A.5) by φv and integrating by parts we get

$$\int \varphi(r) v'^2(r)\, dr + \int \big(W(r) - \lambda_n\big)\varphi(r) v^2(r)\, dr = \frac{1}{2} \int \varphi''(r) v^2(r)\, dr$$

and therefore

$$\int_{r^*+a}^{\infty} r^{-2} v^2(r)\, dr \le Ca^{-3} \int_{r^*+a/2}^{r^*+a} v^2(r)\, dr$$

which implies after iterations estimate

$$\int_{r^*+a}^{\infty} v^2(r)\, dr \le C' a M^2 \Big(\frac{b}{a}\Big)^s, \tag{A.38}$$

for integral from $r^* + a$ to $r^* + 2a$, where M is the right-hand expression of (A.34), which, in turn, implies (A.38) in full measure.

Then the same proof implies that

$$\int_{r^*+a}^{\infty} v'^2(r)\, dr \le C' a^{-1} M^2 \Big(\frac{b}{a}\Big)^s$$

which combined with (A.38) implies (A.36).

(iii) Statement (iii) is proven in the same way. □

Consider now the case $l \geq (1-\epsilon n$. In this case both $r_* \approx r^* \approx 4n^2 \approx r_*/4n^2$ and, $(r^* - r_*) \approx 4n^{3/2}\sqrt{2(n-l)}$, $W(r) \approx (r-r_*)(r^*-r)r^{*-2}$, where $\approx$ means that the ration is close to 1.

Further $Cr^{*,2/3}$ should be replaced by $Cr^*/(r^*-r_*)^{1/3}$ and we want $Cr^*/(r^*-r_*)^{1/3} \leq (r^*-r_*)$ i.e. $(r^*-r_*) \geq Cr^{*\,3/4}$ which is equivalent

$$C_0 \leq (n-l) \leq \epsilon n. \tag{A.39}$$

Proposition A.9. *(i) Let condition* (A.39) *be fulfilled. Then*

$$|v(r)| \leq C' \begin{cases} L^{-1/4+\sigma/2}(r^*-r)^{-1/4-\sigma/2} & r^* - \epsilon L \leq r \leq r^* - Cr^* L^{-1/3}, \\ L^{-1/2} & r_* + \epsilon L \leq r \leq r^* - \epsilon L, \\ L^{1/4-\sigma'/2}(r-r_*)^{-3/4+\sigma'/2} & r_* + Cr_* L^{-1/3} \leq r \leq r_* + \epsilon L \end{cases} \tag{A.40}$$

with $L = r^* - r_*$.

(ii) Further,

$$|v(r)| \leq C' L^{-1/3+2\sigma/3} r^{*\,-1/4-\sigma/2} \begin{cases} 1 & r \geq r^* - Cb, \\ \left(\dfrac{b}{r-r^*}\right)^s & r \geq r^* + Cb \end{cases} \tag{A.41}$$

with $b = r^* L^{-1/3}$.

(iii) Furthermore,

$$|v(r)| \leq C' L^{1/2-2\sigma'/3} r_*^{-3/4+\sigma'} \begin{cases} 1 & r \leq r^* + Cb, \\ \left(\dfrac{b}{r_*-r}\right)^s & r \leq r_* - Cb \end{cases} \tag{A.42}$$

with $b = r_* L^{-1/3}$.

Proof. Statement (i) is proven in the same way as Propositions A.4, A.6 and A.7. Statements (ii) and (iii) are proven in the same way as Proposition A.8. □

Finally, consider the remaining case $1 \leq n - l \leq C$. In the same way

Proposition A.10. *Let* $1 \le n - l \le C$. *Then*

$$|v(r)| \le C' n^{-1/2} \begin{cases} \big(\frac{n^2}{r}\big)^s & r \ge Cn^2, \\ 1 & C^{-1}n^2 \le r \le Cn^2, \\ \big(\frac{\langle r\rangle}{n^2}\big)^s & r \le C^{-1}n^2. \end{cases} \tag{A.43}$$

We would need the following

Corollary A.11. *As* $r \le 2r^*$

$$|v(r)| \le C' r^{1/4} n^{-3/2}. \tag{A.44}$$

Proof. It follows from Propositions A.4–A.10. □

B Properties of eigenvalues and eigenfunctions of Coulomb-Schrödinger operator in relativistic settings

Consider first the negative spectra of operators

$$H^0_\beta := H_{\beta,V^0} = T_\beta - \frac{1}{r} \tag{B.1}$$

in $\mathscr{L}^2(\mathbb{R}^3, \mathbb{C})$ where T_β is defined by (1.18) and $r = |x|$. We assume that

$$0 < \beta \le \frac{2}{\pi} \tag{B.2}$$

and compare it with non-relativistic operator operator $H^0 := -\Delta - \frac{1}{r}$, the eigenvalues of which $\lambda^0_n = -\frac{1}{4n^2}$ (of multiplicity n^2) is well-known.

Note that

(B.3) Negative spectrum of operator H^0_β consists of eigenvalues $\mu^0_{n,l} := \mu^0_{n,l}(\beta)$ $(l = 0, 1, \dots, n-1)$ of multiplicities $(2l+1)$ (some of those could coincide) which are eigenvalues of operator

$$K_l(\beta) = \sqrt{\beta^{-2}\big(-\partial_r^2 + \frac{l(l+1)}{r^2}\big) + \frac{1}{4}\beta^{-4}} - \frac{1}{2}\beta^{-2} - \frac{1}{r} \tag{B.4}$$

in $\mathscr{L}^2(\mathbb{R}^+, \mathbb{C})$ and eigenfunctions of H^0_β are $R_{n,l}(r;\beta)Y_{l,m}(\phi,\theta)$ where

$$R_{n,l}(r;\beta) = r^{-1}w_{n,l}(r;\beta), \tag{B.5}$$

$w_{n,l}(r;\beta)$ are corresponding orthonormal eigenfunctions of $K_l(\beta)$ and $n = l+1, l+2, ...$

Recall that $Y_{l,m}(\phi,\theta)$ are spherical harmonics.

Proposition B.1. *Under assumption* (B.2) *the following hold:*

$$\mu_{n,l}(\beta) < \mu_{n,l}(\beta') \le \lambda^0_n = -\frac{1}{4n^2} \qquad \textit{for} \quad \beta' < \beta, \tag{B.6}$$

$$\mu^0_{n,l}(\beta) < \mu^0_{n+1,l+1}(\beta). \tag{B.7}$$

Proof. Observe that

$$T_l(\beta) \coloneqq \sqrt{\beta^{-2}\Lambda + \frac{1}{4}\beta^{-4}} - \frac{1}{2}\beta^{-2} \tag{B.8}$$
$$= \left(\sqrt{4\beta^2\Lambda + 1} + 1\right)^{-\frac{1}{2}} 2\Lambda \left(\sqrt{4\beta^2\Lambda + 1} + 1\right)^{-\frac{1}{2}}$$

with $\Lambda \coloneqq \Lambda_{n,l} = -\partial_r^2 - \frac{l(l+1)}{r^2}$. Then $T_l(\beta') < T_l(\beta) \le \Lambda$ (B.6) follows from the variational principle, since λ^0_n are eigenvalues of operator

$$K_l(0) \coloneqq \Lambda - \frac{1}{r}. \tag{B.9}$$

Since $T_{l+1} > T_l$, and $(n-l)$ numbers eigenvalues, starting from the first, we have (B.7). □

Proposition B.2. *Let* $l \le \epsilon n$ [16]. *Then the following estimate holds:*

$$\mu^0_{n,l} \ge \lambda^0_n - Cn^{-3}. \tag{B.10}$$

Proof. Consider eigenvalue counting functions $N_{l,\beta}(\lambda)$ and $\mathsf{N}_l(\lambda)$ for 1-dimensional operators $K_l(\beta)$ and $K_l(0)$ respectively, $\lambda < 0$. Under assumption

$$l \le \epsilon|\lambda|^{-1/2} \tag{B.11}$$

[16] We cover case $l \ge \epsilon n$ separately.

the standard semiclassical methods show that

$$\mathsf{N}_{l,\beta}(\lambda) = \mathcal{N}^{\mathsf{W}}_{l,\beta}(\lambda) + O(1), \qquad \mathsf{N}_{l}(\lambda) = \mathcal{N}^{\mathsf{W}}_{l}(\lambda) + O(1), \tag{B.12}$$

where $\mathcal{N}^{\mathsf{W}}_{l,\beta}(\lambda)$ and $\mathcal{N}^{\mathsf{W}}_{l}(\lambda)$ are corresponding Weyl expressions.

Indeed, under assumption (B.11) these operators are microhyperbolic (with effective semiclassical parameter $h \asymp r^{-1/2}$) and the remainder estimate is $C\int_{r_*}^{r^*} r^{-1}\,dr = O(\log(r^*/r_*))$ with $r^* \asymp |\lambda|^{-1}$, $r_* \asymp (l+1)^{-2}$ and it is $O(1)$ as $l \asymp |\lambda|^{-1/2}$.

Further, considering propagation of singularities in the direction of increasing r, one can update the remainder estimate to

$$C\int_{r_*}^{r^*} \bigl(r^{-1+\delta} r^{*\,-\delta} + r^{-s}\bigr)\,dr = O(1)$$

with $\delta > 0$.

Finally, one can see easily that the contribution of the zone $\{r\colon r \le C_0\}$ to both $\mathsf{N}_{l,\beta}(\lambda)$ and $\mathcal{N}^{\mathsf{W}}_{l,\beta}(\lambda)$ does not exceed C.

On the other hand, one can see easily that $\mathcal{N}^{\mathsf{W}}_{l,\beta}(\lambda) = \mathcal{N}^{\mathsf{W}}_{l}(\lambda) + O(1)$ and since $\mathcal{N}^{\mathsf{W}}_{l}(\lambda) \asymp |\lambda|^{-1/2}$ for $r_* \le \epsilon r^*$ we arrive to (B.10) in this case. Here we calculate r^* exactly as in Appendix A for non-relativistic operator $K_l(0)$. One can see easily that assumption (B.11) is equivalent to $l \le \epsilon n$ (with different but still small constant $\epsilon > 0$). We leave easy details to the reader. □

Recall that $w_{n,l} \coloneqq w_{n,l}(r;\beta)$ are eigenfunctions of K_l, $\|w_{n,l}\| = 1$ where $\|.\|$ and $\langle .,.\rangle$ are norm and an inner product in $\mathscr{L}^2(\mathbb{R}^+,\mathbb{C})$.

Proposition B.3. *Under assumption* (B.2) *the following estimates hold with arbitrarily large exponent* s*:*

$$|w_{n,l}(r)| \le Cr^{-s} \qquad \textit{for}\ \ r \ge C_0 n^2 \tag{B.13}$$

and

$$|w_{n,l}(r)| \le C\|\!|w_{n,l}|\!\|_{l;s} \coloneqq C\sum_{k\ge 0} t_k^{-s}\|\phi_{t_k} w_{n,l}\| \qquad \textit{for}\ \ r \le C_0^{-1} l^2\,, \tag{B.14}$$

where $\phi_t(r) = \phi(r/t)$ *and* $\phi \in \mathscr{C}_0^\infty([-1,1])$, $\phi(r) = 1$ *on* $(-\frac12,\frac12)$, $t_k = 2^k l$.

Proof. (a) Proof is standard: observe that for $w := w_{n,l}$

$$|\langle((K_l(\beta)-\mu^0_{n,l})\psi_t w,\ \psi_t w\rangle| \le C|\langle[[K_l(\beta),\psi_t]v,\psi_t]w,\ w\rangle| \le C_0 t^{-2}\|\psi_{t'}v\|^2 + Ct^{-2s}\|w\|^2\,, \tag{B.15}$$

where $1-\psi \in \mathscr{C}_0^\infty([-1,1])$, $\psi(r)=0$ on $(-1-\varepsilon, 1+\varepsilon)$, $t'=(1+2\varepsilon)^{-1}t$.

On the other hand, for $t \ge C_0 n^{-2}$

$$\langle((K_l(\beta)-\mu^0_{n,l})\psi_t w,\ \psi_t w\rangle \ge Cn^{-2}\|\psi_t w\|^2 - Cn{-}2s\|w\|^2\,. \tag{B.16}$$

Combining (B.15) and (B.16) and iterating we arrive to $\|\psi_t w\| \le Ct^{-s}$. Then

$$\|(\beta^{-2}\Lambda+\beta^{-4})^{1/2}\psi_t w\| \le \beta^{-2}\|\phi_t w\| + Ct^{-2s} \implies \langle(\beta^{-2}\Lambda+\beta^{-4})\psi_t w,\ \psi_t w\rangle \le \beta^{-2}\|\psi_t w\|^2 + 2C\beta^{-2}l^{-2s} \implies \langle\Lambda\psi_t w,\ \psi_t w\rangle \le Ct^{-2s}\,.$$

Then

$$\|\partial\psi_t w\| + l\|r^{-1}\psi_t w\| \le Ct^{-s}\,, \tag{B.17}$$

which implies (B.13).

(b) The proof of (B.3) follows the same scheme albeit now $\psi \in \mathscr{C}_0^\infty([-1,1])$, $\psi(r)=0$ on $(-1+\varepsilon, 1-\varepsilon)$, $t'=(1-2\varepsilon)^{-1}t$ and we select $t=C_0^{-1}l^2$. Then (B.15) is replaced by

$$|\langle((K_l(\beta)-\mu^0_{n,l})\psi_t w,\ \psi_t w\rangle| \le C_0 t^{-2}\|\psi_{t'}v\|^2 + C\|\!|w_{n,l}|\!\|_{l;s}\,. \tag{B.18}$$

Then (B.16) is replaced by

$$\langle((K_l(\beta)-\mu^0_{n,l})\psi_t w,\ \psi_t w\rangle \ge Cl^2t^{-1}\|\psi_t w\|^2 - C\|\!|w_{n,l}|\!\|_{l;s}\,, \tag{B.19}$$

and combining (B.18) and (B.19) we arrive to $\|\psi_t w\| \le C\|\!|w_{n,l}|\!\|_{l;s}$. Next two inequalities are preserved, but t^{-2s} is replaced by $\|\!|w_{n,l}|\!\|_{l;s}$ there. □

Proposition B.4. *Under assumption* (B.2) *the following estimate hold swith arbitrarily large exponent* s*:*

$$\|P_{l,s}w_{n,l}\| \le C\|\!|w_{n,l}|\!\|_{l;s}\,, \tag{B.20}$$

with

$$P_{l,s} := -\partial_r^2 + V_s(r,\lambda_n,l,\mu^0_{n,l}) + W_s(r,\lambda_n,l,\mu^0_{n,l})\partial_r\,, \tag{B.21}$$

and

$$V_s(r,\lambda_n,l,\mu) := \frac{l(l+1)}{r^2} - \frac{1}{r} - \mu + \sum_{j,k:\ 2\le j+k\le s} V_{j,k}(r/r_*,\beta)r^{-j}\mu^k\,, \tag{B.22}$$

and

$$W_s(r,\lambda_n,l,\mu) := \sum_{j,k:\ 2\le j+k\le s} W_{j,k}(r/r_*,\beta)r^{-j}\mu^k \tag{B.23}$$

with coefficients which could be decomposed asymptotically in positive powers of β *and non-positive of* r/r_**.*

Proof. Due to Proposition B.3 we need to consider only $r \le C_0 n^2$. Then semiclassical microlocal methods imply that

$$\|(1-\varphi_\tau(D_r))\psi_t(r)w_{n,l}\| \le C\|\!|w_{n,l}|\!\|_{s,l} \tag{B.24}$$

provided $\psi \in \mathscr{C}_0^\infty([\frac12,1))$, $\varphi \in \mathscr{C}_0^\infty([-1,1])$, $\varphi = 1$ on $(-\frac12,\frac12)$ and $\tau = t^{-1/2+\delta}$ with arbitrarily small exponent $\delta > 0$, $t \le C_0 n^2$.

Then $T_l w_{n,l}$ could be approximated in this zone by the (asymptotic) $\sum_{k\ge0} c_k\beta^k\Lambda^{k+1}w_{n,l}$ with $c_0 = 1$, and it equals to

$$\sum_{k\ge0} c_k\beta^k\Lambda^k(\frac{1}{r}+\mu_{n,l})w_{n,l}$$
$$= \sum_{k\ge0} c_k\beta^k(\frac{1}{r}+\mu_{n,l})\Lambda^k w_{n,l} + \sum_{k\ge0} c_k\beta^k[\Lambda^k,(\frac{1}{r}+\mu_{n,l})]w_{n,l}$$

Giving weights $-\frac12$ to ∂_r, -1 to Λ and r^{-1}, and 0 to $l(l+1)/r$, we see that $[\Lambda, r^{-1}]$ has weight of $-\frac52$ and both $[\Lambda,\partial_r]$ and $[r^{-1},\partial_r]$ have weight of -2.

In this process we replace $-\partial_r^2$ by $\Lambda - \frac{l(l+1)}{r^2}$, and when Λ reaches $w_{n,l}$ we replace it by $-(\frac1r+\mu_{n,l})$.

As a result we have

$$\big(K_l - \mu^0_{n,l}\big)w_{n,l} \equiv \big(I+Q\big)P_{l,s} \tag{B.25}$$

with

$$Q = \beta^2 \sum_{j,k:\ 1\le j+k\le s} d_{j,k}(\beta) r^{-j}\mu^k + \beta^2 \sum_{j,k:\ 1\le j+k\le s} d'_{j,k}(\beta) r^{-j}\mu^k \partial_r\,. \tag{B.26}$$

It implies the statement of the proposition. □

Proposition B.5. *Under assumption* (B.2) *the following estimate hold swith arbitrarily large exponent* s*:*

$$|v_{n,l}(x)| \le C r^{1/4} n^{-3/2} \qquad \textit{for}\ \ r_* \le r \le r^* \tag{B.27}$$

with $r_* = \epsilon_0 (l+1)^2$ *and* $r^* = C_0 n^2$.

Proof. We can get rid of the last term in expression (B.21) by the substitution $w_{n,l} = e^{\Phi}\tilde w_{n,l}$ with bounded Φ, because this term is $O(r^{-2}+\mu^2) = O(r^{-2})$ for $r \le r^*$ provided $|\mu| \le c_0 n^{-2}$, which is guaranteed by (B.10) (and it is the only place where we need (B.2). Then we can apply arguments of Appendix A and derive (B.27), uniform by $\beta \le c_0$.

Namely, let us repeat with the minor modifications arguments of Appendix A. Then

(a) Note that we do not claim that $w_{n,l}(r)$ has exactly $(n-l-1)$ zeroes as $v_{n,l}(r)$ has[17] .

(b) On the other hand, as $r \ge C$ (large enough constant) the distance between consecutive zeroes is at least $\epsilon' r^{1/2}$ while for

$$C(l+1)^2 \le r \le (1-\epsilon)^{-1}\mu_{n,l}^{0\,-1}$$

this distance does not exceed $Cr^{1/2}$. More precisely, (A.10) and (A.11) hold[18] and also Proposition A.3, Statements (i) and (ii)[18] hold.

(c) Also Proposition A.4 and Remark A.5 hold[19].

(d) Also Propositions A.6, A.7, A.8 and hold A.9 [20].

(e) Also Propositions A.10 holds. □

[17] See Proposition A.1.
[18] The latter, only as $l \ge C$.
[19] Under extra assumption $r_k \ge C$.
[20] With all statements, referring to r_* only as $r \ge C$.

Remark. We need to remember that equation is fulfilled modulo $O(r^{-s})$ but it would matter only if $|w| + |w'| = O(r^{-s'})$ but then in the process we can prove that it is not the case.

Proposition B.6. *Under assumption* (B.2) *the following estimate hold swith arbitrarily large exponent* s*:*

$$|\mu^0_{n,l}(\beta) - \mu^0_{n,l}(\beta)| \le C|\beta^2 - \beta'^2| n^{-3}(l+1)^{-1}; \tag{B.28}$$

in particular

$$\mu^0_{n,l}(\beta) \le \bar\lambda_n - C\beta^2 n^{-3}(l+1)^{-1}; \tag{B.29}$$

recall that $K_l(\beta) < K_l(\beta') < K_l(0)$ *for* $0 < \beta' < \beta$ *and therefore*

$$\mu^0_{n,l}(\beta) < \mu^0_{n,l}(\beta') < \bar\lambda_n \tag{B.30}$$

Proof.

$$\mu^0_{n,l}(\beta) - \lambda^0_n = \int_{\beta'}^{\beta} \langle (\partial_{\gamma^2} K_{n,l}(\gamma) w_{n,l},\ w_{n,l}\rangle \, d\gamma^2. \tag{B.31}$$

and we use (B.27). □

Corollary B.7. *Consider* $\mu^0_{n,l}$ *as eigenvalues of the corresponding* $3D$*-operator. Then only two cases are possible:*

(i) If $\beta \le \epsilon_1$ *with sufficiently small constant* ϵ_1*, then all these eigenvalues* $\mu^0_{n,l}$ *are in the clusters* $\mathcal{C}^0_n$ *of the width* $\le C_0\beta^2 n^{-3}$*, separated by gaps of the width* $\asymp n^{-3}$*.*

Each cluster contains exactly n^2 *eigenvalues.*

(ii) Otherwise, if $\beta > \epsilon_1$*, we can break* $\mu^0_{n,l}$ *into clusters* $\mathcal{C}^0_n$ *of the width* $\asymp n^{-3}$*, separated by gaps of the width* $\asymp n^{-3}$*. In this case* $\mu^0_{n,l} \in \mathcal{C}_n$ *for* $l \ge C_0$ *but for* $l < C_0$ *it could belong to* $\mathcal{C}^0_m$ *with* $n - C_0 \le m \le n$*.*

Each cluster $\mathcal{C}^0_n$ *contains* $\nu_n \in [n^2 - C_1,\ n^2 + C_1]$ *eigenvalues.*

Remark B.8. (i) We do not know what is the case as (B.2) ever occurs under assumption (B.2) and it would be intersting to learn. However it does not matter for our purposes: we simply change numeration of $\mu^0_{n,l}$ so that $\mu^0_{n,l}$ with $n = l+1, \dots, l+\nu_n$ belongs to $\mathcal{C}^0_n$.

(ii) Probably, in addition to (B.28) the opposite inequality also holds

$$|\mu^0_{n,l}(\beta) - \mu^0_{n,l}(\beta)| \geq \epsilon_0 |\beta^2 - \beta'^2| n^{-3}(l+1)^{-1}. \tag{B.32}$$

(iii) It would be interesting to derive an estimate for $\|w_{n,l}(\beta) - w_{n,l}(\beta')\|$ and a poinwise estimate for $|w_{n,l}(\beta) - w_{n,l}(\beta')|$.

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Upper Estimates for the Electronic Density in Heavy Atoms and Molecules[*,†]

[‡]

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Abstract

We derive an upper estimate for the electronic density $\rho_\Psi(x)$ in heavy atoms and molecules. While not sharp, on the distances $\gtrsim Z^{-1}$ from the nuclei it is still better than the known estimate CZ^3 (Z is the total charge of the nuclei, $Z \asymp N$ the total number of electrons.

1 Introduction

This paper is a result of my rethinking of three rather old but still remarkable papers [HHT, ILS, S1], which I discovered recently. The first of them derives the estimate electronic density $\rho_\Psi(x)$ from above via some integral also containing ρ_Ψ, the second one provides an estimate $\rho_\Psi(x) = O(Z^3)$ where Z is the total charge of nuclei and the third one derives the asymptotic of the averaged electronic density on the distances $O(Z^{-1})$ from the nuclei but its method works also on the larger distances.

The purpose of this paper is to provide a better upper estimate for $\rho_\Psi(x)$ on distances larger than Z^{-1} from the nuclei.

Let us consider the following operator (quantum Hamiltonian)

$$\mathsf{H} = \mathsf{H}_N := \sum_{1\le j\le N} H_{V,x_j} + \sum_{1\le j<k\le N} |x_j - x_k|^{-1} \tag{1.1}$$

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on

$$\mathfrak{H} = \bigwedge_{1\le n\le N} \mathscr{H}, \qquad \mathscr{H} = \mathscr{L}^2(\mathbb{R}^3, \mathbb{C}^q) \tag{1.2}$$

with

$$H_V = -\Delta - V(x) \tag{1.3}$$

describing N identical particles (electrons) in the external field with the scalar potential $-V$ and repelling each other according to the Coulomb law.

Here $x_j \in \mathbb{R}^3$ and $(x_1, \dots, x_N) \in \mathbb{R}^{3N}$. A potential $V(x)$ is assumed to be real-valued. Except when specifically mentioned we otherwise assume that

$$V(x) = \sum_{1\le m\le M} \frac{Z_m}{|x - \mathsf{y}_m|} \tag{1.4}$$

where $Z_m > 0$ and y_m are charges and locations of nuclei.

The mass is equal to $\frac{1}{2}$ and the Plank constant and the charge are equal to 1 here. We assume that $N \asymp Z = Z_1 + \dots + Z_M$.

Our purpose is to derive a pointwise upper estimate for the *electronic density*

$$\rho_\Psi(x) = N \int |\Psi(x, x_2, \dots, x_N)|^2 \, dx_2 \cdots dx_N. \tag{1.5}$$

Let

$$\ell(x) = \min_{1\le m\le M} |x - \mathsf{y}_m| \tag{1.6}$$

is the distance to the nearest nucleus. Our goal is to prove the following theorem:

Theorem 1.1. *Let*

$$\min_{1\le m<m'\le M} |\mathsf{y}_m - \mathsf{y}_{m'}| \ge Z^{-1/3+\sigma} \tag{1.7}$$

with $\sigma > 0$. Then:

(i) For $\ell(x) \le Z^{-1/3}$ the following estimate holds:

$$\rho_\Psi(x) \le C \begin{cases} Z^3 & \text{for } \ell(x) \le Z^{-8/9}, \\ Z^{19/9}\ell^{-1} & \text{for } Z^{-8/9} \le \ell(x) \le Z^{-7/9}, \\ Z^{197/90}\ell^{-9/10} & \text{for } Z^{-7/9} \le \ell \le Z^{-1/3}. \end{cases} \tag{1.8}$$

(ii) For $\ell(x) \ge Z^{-1/3}$ the following estimate holds:

$$\rho_\Psi(x) \le C \begin{cases} Z^{17/9}\ell^{-9/5} & \text{for } \ Z^{-1/3} \le \ell \le Z^{-5/18}, \\ Z^{19/9}\ell^{-1} & \text{for } \ \ell \ge Z^{-5/18}. \end{cases} \tag{1.9}$$

(iii) Furthermore, if $(Z - N) \ge C_0 Z^{5/6}$ then

$$\rho_\Psi(x) \le Z^{19/9}\ell^{-1} \qquad \text{for } \ \ell(x) \ge C_0 (Z-N)_+^{-1/3}. \tag{1.10}$$

Remark 1.2. We would like to prove an estimate $\rho_\Psi(x) \le C\zeta^3$, or to discover that it does not necessarily hold.

Plan of the paper. In Section 2 we prove a more subtle version of the main estimate of [HHT]. In Section 3 we provide upper estimates and asymptotics of ρ_Ψ integrated over small balls. In Section 4 we study the energy of electron-to-electron interaction (it involves a two-point correlation function) and in Section 5 we prove upper estimates for $\rho_\Psi(x)$.

2 Main intermediate inequality

We start from the main intermediate equality.

Proposition 2.1. *Let Ψ be an eigenfunction of H_N with an eigenvalue λ. Let $\phi(|x|)$ be a real-valued spherically symmetric function. Then*

$$\begin{aligned} \rho_\Psi(0) = {} & \\ (2\pi)^{-1} N \iint (\partial_r K(x)) \Psi(x, x_2, \dots, x_N) \Psi^*(x, x_2, \dots, x_N) \phi(|x|)\, dx dx_2 \cdots dx_N & \\ - (8\pi)^{-1} \int \rho_\Psi(x) \phi'''(|x|)\, dx, & \end{aligned} \tag{2.1}$$

where

$$K(x) \coloneqq -\mathsf{H}_N - (\partial_r^2 + 2r^{-1}\partial_r) \tag{2.2}$$

is an operator in the auxiliary space $\mathcal{H} \coloneqq \bigotimes_{n=2,\dots,N} \mathscr{L}^2(\mathbb{R}^3, \mathbb{C}^q) \otimes \mathbb{C}^q$ with an inner product $\langle .,. \rangle$, $\phi'''(r) = \partial_r^3 \phi(r)$ and $x = (r, \theta) \in \mathbb{R}^+ \times \mathbb{S}^2$.

Proof. Let us consider Ψ as a function of $x \in \mathbb{R}^3$ with values in the auxiliary space $\mathcal{H}$, and and let $u = r\Psi$ where (r, θ) are spherical coordinates in $\mathbb{R}^3$. Then similar to (9) of [HHT]

$$\langle \Psi(0), \Psi(0)\rangle = -(2\pi)^{-1} \int \langle \partial_r u, \partial_r^2 u\rangle \phi(r) r^{-2}\, dx - (4\pi)^{-1} \int \langle \partial_r u, \partial_r u\rangle \phi'(r) r^{-2}\, dx \tag{2.3}$$

and since $r^{-1}\partial_r^2 u = r\Delta_r \Psi := r(\partial_r^2 + 2r^{-1}\partial_r)\Psi$, the first term on the right is equal to

$$\begin{aligned} &-(2\pi)^{-1} \iint \langle \partial_r u, r\Delta_r \Psi\rangle \phi(r)\, dr d\theta \\ &\qquad = \pi^{-1} \iint \langle \partial_r u, (K+\lambda)u\rangle \phi(r) dr d\theta = \\ &= -(2\pi)^{-1} \int \langle \Psi, K'\Psi\rangle \phi(r)\, dx - (2\pi)^{-1} \iint \langle \Psi, (K+\lambda)\Psi\rangle \phi'(r)\, dx \end{aligned} \tag{2.4}$$

because $\Delta_r \Psi = -2(K+\lambda)\Psi$, where $-K$ is the rest of the multiparticle Hamiltonian (including $-r^{-2}\Delta_\theta$) and we integrated by parts.

The first term in the latter formula is a corresponding term in [HHT], albeit truncated with ϕ, and we have two new terms

$$-(4\pi)^{-1} \iint \langle \partial_r(r\Psi), \partial_r(r\Psi)\rangle \phi'(r)\, dr d\theta - (2\pi)^{-1} \int \langle \Psi, (W+\lambda)\Psi\rangle \phi'(r)\, dx\,.$$

Integrating the first term by parts we get

$$(4\pi)^{-1} \iint \langle r\Psi, \partial_r^2(r\Psi)\rangle \phi'(r)\, dr d\theta + (4\pi)^{-1} \iint \langle r\Psi, \partial_r(r\Psi)\rangle \phi''(r)\, dr d\theta,$$

where the first term cancels the second term in (2.4), while the second term integrates by parts one more time resulting in the last term in (2.1). □

Applying (2.1) to our problem, and using skew-symmetry of Ψ, we get

$$\rho_\Psi(0) = (2\pi)^{-1} \int \sum_m Z_m \frac{x\cdot(x-\mathsf{y}_m)}{|x|\cdot|x-\mathsf{y}_m|^3}\rho_\Psi(x)\phi(|x|)\,dx \tag{2.5}$$
$$+ (2\pi)^{-1} N(N-1) \int \underbrace{\frac{x_1\cdot(x_2-x_1)}{|x_1|\cdot|x_2-x_1|^3}}\, |\Psi(x_1,x_2,\dots,x_N)|^2 \underbrace{\phi(|x_1|)}\, dx_1\cdots dx_N$$
$$- (2\pi)^{-1} N \int |x|^{-3}|\nabla_\theta \Psi(x,x_2,\dots,x_N)|^2\phi(|x|)\,dx dx_2\cdots dx_N$$
$$- (8\pi)^{-1}\int \rho_\Psi(x)\phi'''(|x|)\,dx.$$

Symmetrizing the second term with respect to x_1 and x_2 e instead of the product of two indicated factors we will get

$$\frac{1}{4}\big(\phi(|x_1|)+\phi(|x_2|)\big)\Big(\frac{x_1\cdot(x_2-x_1)}{|x_1|\cdot|x_2-x_1|^3} - \frac{x_2\cdot(x_2-x_1)}{|x_2|\cdot|x_2-x_1|^3}\Big)$$
$$+\frac{1}{4}\big(\phi(|x_1|)-\phi(|x_2|)\big)\Big(\frac{x_1\cdot(x_2-x_1)}{|x_1|\cdot|x_2-x_1|^3} + \frac{x_2\cdot(x_2-x_1)}{|x_2|\cdot|x_2-x_1|^3}\Big)$$

with the big parenthesis on the first line equal to

$$-\frac{|x_1|+|x_2|}{|x_1-x_2|^3}\Big(1-\frac{x_1\cdot x_2}{|x_1|\cdot|x_2|}\Big) \tag{2.6}$$

and the big parenthesis on the second line equal to

$$-\frac{|x_1|-|x_2|}{|x_1-x_2|^3}\Big(1+\frac{x_1\cdot x_2}{|x_1|\cdot|x_2|}\Big)\,. \tag{2.7}$$

One can see that the former is negative, and the latter, multiplied by $\big(\phi(|x_1|)-\phi(|x_2|)\big)$, is non-negative if ϕ is a non-decreasing function. Let us shift the origin to point x and observe that the first term in (2.5) is equal to

$$(2\pi)^{-1}\int \sum_m Z_m \frac{(x-\mathsf{x})\cdot(x-\mathsf{y}_m)}{|x-\mathsf{x}|\cdot|x-\mathsf{y}_m|^3}\rho_\Psi(x)\phi(|x-\mathsf{x}|)\,dx\,. \tag{2.8}$$

Consider first case $\phi = 1$. Then we get

$$\rho_\Psi(\mathsf{x}) \le (2\pi)^{-1}\int \sum_m Z_m|x-\mathsf{y}_m|^{-2}\rho_\Psi(x)\,dx. \tag{2.9}$$

Indeed, the second term in the right-hand expression of (2.5) is non-positive due to above analysis analysis, so is the third term, and the fourth term vanishes while the first term does not exceed the right-hand expression

Applying Proposition 3.1 below we arrive to the following estimate

$$\rho_\Psi(x) \le CZ^3. \tag{2.10}$$

In the general case we arrive to

Proposition 2.2. *In the framework of Proposition 2.1*

$$\begin{aligned}\rho_\Psi(\mathsf{x}) \le (2\pi)^{-1} \int \sum_m Z_m \frac{(x-\mathsf{x})\cdot(x-\mathsf{y}_m)}{|x-\mathsf{x}|\cdot|x-\mathsf{y}_m|^3}\rho_\Psi(x)\phi(|x-\mathsf{x}|)\,dx \\ + Ct^{-1}\iint_{B(\mathsf{x},t)\times B(\mathsf{x},t)} |x-y|^{-1}\rho^{(2)}_\Psi(x,y)\,dxdy \\ + C\iint_{B(\mathsf{x},t)\times(\mathbb{R}^3\setminus B(\mathsf{x},t))} |y-\mathsf{x}|^{-2}\rho^{(2)}_\Psi(x,y)\,dxdy\,,\end{aligned} \tag{2.11}$$

where

$$\rho^{(2)}_\Psi(x,y) \coloneqq N(N-1)\int |\Psi(x,y,x_3,\dots,x_N)|^2\,dx_3\cdots dx_N \tag{2.12}$$

is a two-point correlation function.

Recall that

$$\int \rho^{(2)}_\Psi(x,y)dy = (N-1)\rho_\Psi(x). \tag{2.13}$$

Remark 2.3. (i) Inequality (2.9) for $M=1$ and $\mathsf{x}=\mathsf{y}_1$ is the main result of [HHT]. Our main achievement so far is an introduction of the truncation ϕ. However it brings three new terms in the right-hand expression of the estimate.

(ii) Estimate (2.10) (with a specified albeit not sharp constant) was proven in [S1] for $x=\mathsf{y}_m$.

(iii) This estimate definitely has the correct magnitude as $|x-\mathsf{y}_m| \lesssim Z^{-1}$ and $Z_m \asymp Z$.

3 Estimates of the averaged electronic density

We will need the following estimate (3.3) from [Ivr2]:

$$\int U\rho_\Psi \, dx \le \mathsf{Tr}(H^-_{W+\nu}) - \mathsf{Tr}(H^-_{W+U+\nu}) + CZ^{5/3-\delta} \tag{3.1}$$

with $\delta = \delta(\sigma)$, $\delta > 0$ for $\sigma > 0$ and $\delta = 0$ for $\sigma = 0$.

First, we use this estimate in the very rough form:

Proposition 3.1. *The following estimate holds:*

$$\int |x - y_m|^{-2} \rho_\Psi(x)\, dx \le CZ^2. \tag{3.2}$$

Proof. Let $\psi(x)$, $\psi_0(x)$ be cut-off functions, $\psi(x) = 0$ in $\{x\colon \ell(x) \le b\}$, $\psi_0(x) = 0$ in $\{x\colon \ell(x) \ge 2b\}$, $\psi + \psi_0 = 1$, $b = Z^{-1}$. Then

$$\mathsf{Tr}(H^-_{W+U+\nu}) = \mathsf{Tr}(H^-_{W+U+\nu}\psi_0) + \mathsf{Tr}(H^-_{W+U+\nu}\psi) \tag{3.3}$$

Using the semiclassical methods of [Ivr1], Section 25.4 in the simplest form, we conclude that for $U = \epsilon|x - y_m|^{-2}$ the second term on the right (with an opposite sign) could be replaced by its Weyl approximation

$$-\frac{2}{5}\kappa \int (W + U + \nu)_+^{5/2}\psi(x)\, dx \tag{3.4}$$

with an error not exceeding CZ^2 where here and below $\kappa = q/(6\pi^2)$. The same is true for $U = 0$. One can see easily that the difference between expression (3.4) and the same expression for $U = 0$ does not exceed

$$C\int \left[(W + \nu)_+^{3/2} U + U^{5/2}\right] dx, \tag{3.5}$$

which does not exceed CZ^2.

Consider the first term in the right-hand expression of (3.3). Using variational methods of [Ivr1], Section 9.1, we can reduce it to the analysis of the same operator in $\mathcal{X}^0 := \{x\colon \ell(x) \le 4b\}$ with the Dirichlet boundary conditions on ∂X. Observing that eigenvalue counting function for such operator is $O(1+\lambda^{3/2}Z^{-3})$ (for ϵ sufficiently small), we conclude that the first term in (3.3) also does not exceed CZ^2. Estimate (3.2) has been proven. □

Let us return to (3.1) and consider $U = \zeta^2\phi_t(x;\mathsf{x})$ where x is a fixed point with

$$\ell(x) \coloneqq \min_m |x - \mathsf{y}_m| \ge Z^{-1}, \tag{3.6}$$

$$\zeta(x) \coloneqq \max\bigl(Z^{1/2}\ell(x)^{-1/2},\, \ell(x)^{-2}\bigr) \tag{3.7}$$

and $\phi_t(x;\mathsf{x}) = \phi_0(t^{-1}|x-\mathsf{x}|)$, $\phi \in \mathscr{C}_0^\infty([-1,1])$, $0 \le \phi \le 1$. We assume that

$$\zeta^{-1} \le t \le \frac{\ell}{2} \tag{3.8}$$

with $\ell = \ell(\mathsf{x})$, $\zeta = \zeta(\mathsf{x})$, where the last inequality allows us to apply semiclassical methods. Consider with $0 \le \varsigma \le 1$

(3.9)
$$\mathsf{Tr}(H^-_{W+\nu}) - \mathsf{Tr}(H^-_{W+\varsigma U+\nu}) = \int_0^\varsigma \mathsf{Tr}\Bigl(U\bigl[\theta(-H_{W+sU+\nu}) - \theta(-H_{W+\nu})\bigr]\Bigr)\,ds$$

and apply semi-classical methods to the right-hand expression. Then we get

$$\begin{aligned}(3.10)\quad &\mathsf{Tr}(H^-_{W+\nu}) - \mathsf{Tr}(H^-_{W+\varsigma U+\nu})\\ &\quad= \kappa\int\int_0^\varsigma \Bigl(U\bigl[(W+sU+\nu)_+^{3/2} - (W+\nu)_+^{3/2}\bigr]\Bigr)\,dxds + O(\varsigma\zeta^4t^2)\\ &\quad= \frac{2}{5}\kappa\int\Bigl((W+\varsigma U+\nu)_+^{5/2} - (W+\nu)_+^{5/2}\Bigr)\,dx + O(\varsigma\zeta^4t^2).\end{aligned}$$

Indeed, factor U is $O(\zeta^2)$ and therefore the semiclassical error is $O(\zeta^4t^2)$ since the effective semiclassical parameter is $h = (\zeta t)^{-1}$. Observe that the principal part in the right-hand expression does is $O(\varsigma\zeta^5t^3)$.

Then after division by $\varsigma\zeta^2$ (3.1) becomes

$$\begin{aligned}(3.11)\quad \int \phi_t(x;\mathsf{x})\rho_\Psi(x)\,dx \le \int \phi_t(x;\mathsf{x})\rho(x)\,dx\\ + C\Bigl(\zeta^2t^2 + \varsigma\zeta^3t^3 + \varsigma^{-1}Z^{5/3-\delta}\Bigr).\end{aligned}$$

Replacing $\phi_t(x;\mathsf{x})$ by $-\phi_t(x;\mathsf{x})$ in this inequality and minimizing by $\varsigma \in (0,1]$ we arrive to the first statement of the following proposition:

Proposition 3.2. *(i) Under assumptions* (3.6)–(3.8)

(3.12)
$$|\int (\rho_\Psi(x)-\rho(x))\phi_t(x;\mathsf{x})\,dx| \le C\Big(\zeta^2 t^2+\zeta^{1/2}t^{3/2}Z^{5/6-\delta/2}+\zeta^{-2}Z^{5/3-\delta}\Big).$$

(ii) Further,

$$|\int \rho_\Psi(x)\phi_t(x;\mathsf{x})\,dx| \le C\Big(\zeta^3 t^3 + t^{6/5}Z^{1-3\delta/5}\Big). \tag{3.13}$$

(iii) Furthermore, if $N < Z$ then

$$|\int \rho_\Psi(x)\phi_t(x;\mathsf{x})\,dx| \le Ct^{6/5}Z^{1-3\delta/5} \qquad \textit{for } \ell(\mathsf{x}) \ge C_0(Z-N)_+^{-1/3}. \tag{3.14}$$

To prove the second statement, we consider $\varsigma > 0$ (without restriction $\varsigma \le 1$); then instead of (3.11) we have

$$\int \phi_t(x;\mathsf{x})\rho_\Psi(x)\,dx \le C\Big(\zeta^3 t^3 + \varsigma^{3/2}\zeta^3 t^3 + \varsigma^{-1}\zeta^{-2}Z^{5/3-\delta}\Big) \tag{3.15}$$

and we optimize it by $\varsigma > 0$.

The third statement follows from the same arguments and the fact that that $\rho^{\mathsf{TF}}(x) = 0$ for $\ell(x) \ge C_0(Z-N)_+^{-1/3}$ and therefore (3.15) holds without the first term in the right-hand expression.

4 Estimates of the correlation function

We will need the following Proposition 25.5.1 from [Ivr1] (first proven in [RS]):

Proposition 4.1. *Let $\theta \in \mathscr{C}^\infty(\mathbb{R}^3)$, such that*

$$0 \le \theta \le 1. \tag{4.1}$$

Let $\chi \in \mathscr{C}^\infty(\mathbb{R}^6)$ and

$$\mathcal{J} = |\int \Big(\rho_\Psi^{(2)}(x,y)-\rho(y)\rho_\Psi(x)\Big)\theta(x)\chi(x,y)\,dxdy| \le \tag{4.2}$$
$$C\sup_x \|\nabla_y\chi\|_{\mathscr{L}^2(\mathbb{R}^3_y)}\Big((Q+\varepsilon^{-1}N+T)^{\frac12}\Theta + P^{\frac12}\Theta^{\frac12}\Big) + C\varepsilon N\|\nabla_y\chi\|_{\mathscr{L}^\infty}\Theta$$

with

$$Q = \mathsf{D}(\rho_\Psi - \rho^{\mathsf{TF}},\, \rho_\Psi - \rho^{\mathsf{TF}}), \tag{4.3}$$

$$\Theta = \Theta_\Psi \coloneqq \int \theta(x)\rho_\Psi(x)dx, \tag{4.4}$$

$$T = \sup_{\mathsf{supp}(\theta)} W, \tag{4.5}$$

$$P = \int |\nabla \theta^{\frac{1}{2}}|^2 \rho_\Psi \, dx. \tag{4.6}$$

respectively and arbitrary $\varepsilon \le Z^{-\frac{2}{3}}$.

We cannot apply it directly to estimate the second to the last term in (2.11) because of singularities. Let us consider

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-y|^{-1}\rho^{(2)}_\Psi(x,y)\, dxdy. \tag{4.7}$$

Let us make an ℓ-admissible partition of unity ϕ_ι in with ℓ-admissible $\phi_\iota^{1/2}$. We set $\ell(x) = Z^{-1}$ if $|x - \mathsf{y}_m| \le Z^{-1}$. Let us consider first

$$\int |x-y|^{-1}\rho^{(2)}_\Psi(x,y)\phi_\iota(x)\phi_\varkappa(y)\, dxdy \tag{4.8}$$

in the case of ϕ_ι and ϕ_κ having disjoint supports. Without any loss of the generality we can consider $\ell_x \le \ell_y$, where subscripts x, y are referring to supports of ϕ_ι, ϕ_κ respectively.

Let $\theta(x) = \phi_\iota(x)$ and

$$\chi(x,y) = \bar\chi(x,y) \coloneqq |x-y|^{-1}\bar\phi_\iota(x)\phi_\kappa(y) \tag{4.9}$$

where $\bar\phi_\iota$ which are ℓ-admissible and equal 1 in the ℓ-vicinity of $\mathsf{supp}(\phi_\iota)$. Then $T = \zeta_x$ and for $\ell_x \le Z^{-5/21}$ in virtue of Proposition 3.2 [1)]

$$\Theta_\Psi \asymp \zeta_x^3\ell_x^3, \qquad P \asymp \zeta_x^3\ell_x \tag{4.10}$$

and

$$\|\nabla_y\chi\|_{\mathscr{L}^2(\mathbb{R}^3_y))} \asymp d_{x,y}^{-1}\ell_y^{1/2}, \qquad \|\nabla\chi\|_{\mathscr{L}^\infty} \asymp d_{x,y}^{-1}\ell_y^{-1} \tag{4.11}$$

[1)] Indeed, $\zeta^3\ell^3 \ge Z^{5/3-\delta}\zeta^{-2} \iff \ell \le Z^{-5/21+\delta/7}$.

where $d_{x,y} \ge \ell_y$ is the distance between supports of ϕ_ι and $\phi_\varkappa$. Then the right-hand expression of (4.2) is

$$C\zeta_x^3\ell_x^3\Big(\ell_y^{-1/2}(Z^{5/6}+\zeta_x)+\varepsilon^{-1/2}\ell_y^{-1/2}Z^{1/2}+\varepsilon Z\ell_y^{-2}+\ell_y^{-1/2}\ell_x^{-1}\Big)$$

and minimizing by $\varepsilon \le Z^{-2/3}$ we get

$$C\zeta_x^3\ell_x^3\Big(\ell_y^{-1/2}(Z^{5/6-\delta}+\zeta_x)+Z^{2/3}\ell_y^{-1}+\ell_y^{-1/2}Z^{5/6}+\ell_y^{-1/2}\ell_x^{-1}\Big).$$

Observe that all powers of ℓ_y are negative. Therefore summation over all elements of the y-partition results in the same expression albeit with ℓ_y replaced by $\ell_x = \ell$:

$$C\zeta^3\ell^2\Big(\ell^{1/2}(Z^{5/6}+\zeta)+Z^{2/3}+\ell^{-1/2}+\ell^{1/2}Z^{5/6}\Big).$$

For $\ell \le Z^{-1/3}$ we have $\zeta = Z^{1/2}\ell^{-1/2}$ and all powers are positive with the exception of one term, where the power is 0, and for $\ell \ge Z^{-1/3}$ we have $\zeta = \ell^{-2}$ and all powers are negative. Therefore summation over all elements of x-partition results in the same expression albeit with $\ell = Z^{-1/3}$, $\zeta = Z^{2/3}$, with the exception of one term which gains a logarithmic factor. We get CZ^2. Then

$$|\sum_{\iota,\varkappa}\iint\Big(\rho_\Psi^{(2)}(x,y)-\rho(y)\rho_\Psi(x)\Big)\phi_\iota(x)\phi_\kappa(y),dxdy| \le CZ^2 \tag{4.12}$$

with summation over indicated pairs of elements of the partition (disjoint, with $\ell_x \le \min(Z^{-5/21+\delta/7},\ \ell_y)$).

Let us prove that

(4.13) Estimate (4.12) also holds with $\rho_\Psi(x)$ replaced by $\rho(x)$ and therefore it holds for a sum over pairs of elements with $\min(\ell_x,\ \ell_y) \le \ell^* = Z^{-5/21+\delta/7}$.

Indeed, in virtue of the proof of Proposition 3.2 (before minimizing by ς) the error

$$|\iint\big(\rho_\Psi(x)-\rho(x)\big)\rho(y)\phi_\iota(x)\phi_\kappa(y)\,dxdy| \tag{4.14}$$

on each pair of elements does not exceed $C\zeta_y^3\ell_y^2$ with all powers of ℓ_y positive for $\ell_y \le Z^{-1/3}$ and negative for $\ell_y \ge Z^{-1/3}$. Then summation with respect to y-partition (recall, that $\ell_y \ge \ell_x$) results in

$$C\big(\zeta_x^2\ell_x^2 + \varsigma\zeta_x^3\ell_x^3 + \varsigma^{-1}\zeta_x^{-2}Z^{5/3-\delta}\big) \times \begin{cases} Z^{4/3} & \text{for } \ell_x \le Z^{-1/3}, \\ \zeta_x^3\ell_x^2 & \text{for } \ell_x \ge Z^{-1/3}, \end{cases}$$

with the first line corresponding to $\ell_y = Z^{-1/3}$, $\zeta_y = Z^{2/3}$ and the second line corresponding to $\ell_y = \ell_x$, $\zeta_y = \zeta_x$.

Powers of ℓ_x are positive for $\ell_x \le Z^{-1/3}$ and negative for $\ell_x \ge Z^{-1/3}$, and summation with respect to x-partition results in the value as $\ell_x = Z^{-1/3}$, $\zeta_x = Z^{2/3}$, which is

$$CZ^2 + C\varsigma Z^{7/3} + C\varsigma^{-1}Z^{5/3-\delta}.$$

Minimizing by $\varsigma = Z^{-1/3}$ we conclude that the sum of expressions (4.14) over required pairs does not exceed CZ^2, which in turn implies (4.12).

Consider now the case when the supports of elements are not disjoint. Then we take

$$\chi(x,y) = \bar{\chi}(x,y)\eta(|x-y|/s) = |x-y|^{-1}\bar{\phi}_\iota(x)\phi_\kappa(y)\eta(|x-y|/s) \tag{4.15}$$

with $\eta(t)$ smooth function, equal 0 at $(0, \frac{1}{2})$ and 1 at $(1,\infty)$; $s \le Z^{-1/3}$ will be selected later[2)]. Then while (4.10) is preserved, (4.11) should be replaced by

$$\|\nabla_y\chi\|_{\mathscr{L}^2(\mathbb{R}^3_y))} \asymp s^{-1/2}, \qquad \|\nabla_y\chi\|_{\mathscr{L}^\infty} \asymp s^{-2}. \tag{4.16}$$

Then the right-hand expression (4.2) is

$$Cs^{-1/2}\zeta^3\ell^3\Big((Z^{5/6}+\zeta) + \varepsilon^{-1/2}Z^{1/2} + \varepsilon s^{-3/2}Z + \ell^{-1}\Big),$$

and minimizing by $\varepsilon \le Z^{-2/3}$ we get

$$\begin{aligned} |\int \phi_\iota(x)\chi(x,y)\big(\rho^{(2)}_\Psi(x,y) - \rho_\Psi(x)\rho(y)\big)\,dxdy| \\ \le Cs^{-1/2}\zeta^3\ell^3\Big(Z^{5/6} + \zeta + s^{-1/2}Z^{2/3} + \ell^{-1}\Big). \end{aligned} \tag{4.17}$$

[2)] Since in this case $\ell_x = \ell_y$ and $\zeta_x = \zeta_y$ we skip subscripts.

Note that summation of (4.17) over partition returns its value as $\ell = Z^{-1/3}$, namely, $Cs^{-1}Z^{5/3}$.

Consider for $t\colon s \le t \le \ell$ zone $\{(x,y)\colon |x-y| \asymp y\}$ and make there a t-admissible subpartition with respect to x, y (this means that the radii of the supports of subelements ϕ_* do not exceed t, and $|D^\alpha \phi_*| \le c_\alpha t^{-|\alpha|}$). Then contribution of each pair of subelements to

$$|\int \phi_\iota(x)\chi(x,y)\bigl(\rho_\Psi(x)-\rho(x)\bigr)\rho(y)\,dxdy|$$

does not exceed

$$C\bigl(\zeta^2 t^2 + \varsigma\zeta^3 t^3 + \varsigma^{-1}\zeta^{-2}Z^{5/3-\delta}\bigr)\zeta^3 t^2$$

and since there are $\asymp \ell^3 t^{-3}$ of such pairs, we get

$$C\bigl(\zeta^2 t^2 + \varsigma\zeta^3 t^3 + \varsigma^{-1}\zeta^{-2}Z^{5/3-\delta}\bigr)\zeta^3 \ell^3 t^{-1}.$$

Then summation over $t\colon s \le t \le \ell$ returns

$$C\bigl(\zeta^2 \ell + \varsigma\zeta^3 \ell^2 + \varsigma^{-1}s^{-1}\zeta^{-2}Z^{5/3-\delta}\bigr)\zeta^3 \ell^3$$

and summation over over the partition returns its value as $\ell = Z^{-1/3}$, namely

$$C\bigl(Z^2 + \varsigma Z^{7/3} + \varsigma^{-1}s^{-1}Z^{4/3-\delta}\bigr).$$

Minimizing by $\varsigma = (sZ)^{-1/2}$ we get $C\bigl(Z^2 + s^{-1/2}Z^{11/6-\delta}\bigr)$.

On the other hand,

$$\iint \phi_\iota(x)\bigl(\bar\chi(x,y)-\chi(x,y)\bigr)\rho(x)\rho(y)\,dxdy \asymp s^2\zeta^6\ell^3\,,$$

and summation over $\ell \ge Z^{-1/3}$ returns its value at $Z^{-1/3}$, which is Cs^2Z^3, but summation over $\ell \le Z^{-1/3}$ returns $s^2Z^3 \log Z$. To remedy this we replace for $\ell \le Z^{-1/3}$ constant s by $s_x = s(\ell_x Z^{1/3})^{\delta'}$ with small $\delta' > 0$. It will not affect our previous estimates.

Consider the sum of these three right-hand expressions

$$C\bigl(Z^2 + s^{-1}Z^{5/3} + s^{-1/2}Z^{11/6-\delta} + s^2Z^3\bigr)$$

and minimize it by s; we get $CZ^{19/9}$ achieved as $s = Z^{-4/9}$.

Since we want $s \le \ell$ we finally set

$$s_x = \begin{cases} Z^{-4/9} & \text{for } \ell_x \ge Z^{-1/3}, \\ \min(Z^{-4/9}(\ell_x Z^{1/3})^{\delta'},\ \ell_x) & \text{for } \ell_x \le Z^{-1/3}. \end{cases} \tag{4.18}$$

Observe that

$$\iint_{\{x:\ \ell_x \ge Z^{-5/21}\}} |x-y|^{-1}\rho(x)\rho(y) \asymp Z^{41/21}$$

and we arrive to

$$\iint_{\Omega} |x-y|^{-1}\big(\rho^{(2)}_\Psi(x,y) - \rho(x)\rho(y)\big)\,dxdy \le CZ^{19/9} \tag{4.19}$$

and

$$\iint_{\mathbb{R}^3_x\times\mathbb{R}^3_y\setminus\Omega} |x-y|^{-1}\rho(x)\rho(y)\,dxdy \le CZ^{19/9} \tag{4.20}$$

with

$$\Omega = \{(x,y)\colon \ell_x \le Z^{-5/21},\ |x-y| \ge s_y\}. \tag{4.21}$$

Therefore

$$\int_{\Omega} |x-y|^{-1}\rho^{(2)}_\Psi(x,y)\,dxdy \ge \int_{\mathbb{R}^6} |x-y|^{-1}\rho(x)\rho(y)\,dxdy - CZ^{19/9}. \tag{4.22}$$

However we know that (see, f.e. Section 25.2 of [Ivr1])

$$E_N \ge \mathsf{Tr}\big((H_W-\nu)^-\big) - \mathsf{D}(\rho_\Psi,\rho) + \frac{1}{2}\int |x-y|^{-1}\rho^{(2)}_\Psi(x,y)\,dxdy \tag{4.23}$$

and

$$E_N \le \mathsf{Tr}\big((H_W-\nu)^-\big) - \frac{1}{2}\mathsf{D}(\rho,\rho) + CZ^{5/3} \tag{4.24}$$

with $\rho = \rho^{\mathsf{TF}}$, $W = W^{\mathsf{TF}}$, $\mathsf{D}(f,g)\colon \iint |x-y|^{-1}f(x)g(x)\,dx$. Then

$$\int |x-y|^{-1}\rho^{(2)}_\Psi(x,y)\,dxdy \le 2\mathsf{D}(\rho_\Psi-\rho,\rho) + \mathsf{D}(\rho,\rho) + CZ^{5/3}$$

and from

$$|\mathsf{D}(\rho_\Psi-\rho,\rho)| \le \mathsf{D}(\rho_\Psi-\rho,\rho_\Psi-\rho)^{1/2}\mathsf{D}(\rho,\rho)^{1/2} \le CZ^{5/6}\times Z^{7/6} = CZ^2$$

we conclude that

$$\int |x-y|^{-1}\rho^{(2)}_\Psi(x,y)\,dxdy \le \mathsf{D}(\rho,\rho) + CZ^2. \tag{4.25}$$

Combining with (4.22) we conclude that

$$\int_{\mathcal{Z}} |x-y|^{-1}\rho^{(2)}_\Psi(x,y)\,dxdy \le CZ^{19/9} \tag{4.26}$$

for $\mathcal{Z} = \mathbb{R}^3_x \times \mathbb{R}^3_y \setminus \Omega$.

5 Proof of Theorem 1.1

Now in the last two terms

$$\begin{aligned} &Ct^{-1}\iint_{B(\mathsf{x},t)\times B(\mathsf{x},t)} |x-y|^{-1}\rho^{(2)}_\Psi(x,y)\,dxdy\\ +&C\iint_{B(\mathsf{x},t)\times (\mathbb{R}^3\setminus B(\mathsf{x},t))} |y-\mathsf{x}|^{-2}\rho^{(2)}_\Psi(x,y)\,dxdy, \end{aligned}$$

in (2.11) we replace $\rho^{(2)}_\Psi(x,y)$ by $\rho(x)\rho(y)$ and get

$$\begin{aligned} &Ct^{-1}\iint_{B(\mathsf{x},t)\times B(\mathsf{x},t)} |x-y|^{-1}\rho(x)\rho(y)\,dxdy\\ +&C\iint_{B(\mathsf{x},t)\times (\mathbb{R}^3\setminus B(\mathsf{x},t))} |\mathsf{x}-y|^{-2}\rho(x)\rho(y)\,dxdy \end{aligned} \tag{5.1}$$

and the first term does not exceed $C\zeta^6 t^4$, while the second term does not exceed

$$C\zeta^3 t^3 \int_{\mathbb{R}^3\setminus B(\mathsf{x},t)} |\mathsf{x}-y|^{-2}\rho(y)\,dy. \tag{5.2}$$

The largest error comes from the first term when integral is taken over $B(\mathsf{x},t)\times B(\mathsf{x},t)\cap\mathcal{Z}$ and in virtue of of (4.26) it does not exceed $Ct^{-1}Z^{19/9}$, all other errors are lesser (to prove it we need just to repeat arguments of the previous subsection).

Observe that for $\ell := \ell_{\mathsf{x}} \le Z^{-1/3}$ the largest contribution to the integral in (5.2) comes from the layer $\{y\colon \ell_y \asymp \ell_{\mathsf{x}}\}$ and it is of magnitude $\zeta^3\ell_x$. On

the other hand, for $\ell_{\mathsf{x}} \ge Z^{-1/3}$ the largest contribution to the integral in (5.1) comes from the layer $\{y\colon \ell_y \asymp Z^{-1/3}\}$ and it is of magnitude $Z\ell^{-2}$; the first term in (5.1) is smaller.

Therefore we estimate two last terms in (2.11) by

$$Ct^{-1}Z^{19/9} + C\begin{cases} Z^3\ell^{-2}t^3 & \text{for } \ell \le Z^{-1/3},\\ Z\ell^{-8}t^3 & \text{for } \ell \ge Z^{-1/3}.\end{cases} \tag{5.3}$$

Consider the second term in (2.11):

$$(2\pi)^{-1}\int \sum_m Z_m \frac{(x-\mathsf{x})\cdot(x-\mathsf{y}_m)}{|x-\mathsf{x}|\cdot|x-\mathsf{y}_m|^3}\rho_\Psi(x)\phi(|x-\mathsf{x}|)\,dx.$$

We replace in the integral in the right-hand expression $\rho_\Psi(x)$ by $\rho(x)$ and get

$$(2\pi)^{-1}\sum_m Z_m \int \frac{(x-\mathsf{x})\cdot(x-\mathsf{y}_m)}{|x-\mathsf{x}|\cdot|x-\mathsf{y}_m|^3}\rho(x)\phi(|x-\mathsf{x}|)\,dx \tag{5.4}$$

with an error

$$(2\pi)^{-1}\sum_m Z_m \int \frac{(x-\mathsf{x})\cdot(x-\mathsf{y}_m)}{|x-\mathsf{x}|\cdot|x-\mathsf{y}_m|^3}(\rho_\Psi(x)-\rho(x))\phi(|x-\mathsf{x}|)\,dx \tag{5.5}$$

and one can see easily that (5.4) does not exceed $CZ\zeta^3\ell^{-3}t^4$ [3)].

To estimate (5.5) we make a partition in $B(\mathsf{x},t)$ with subelements supported in the layers $\{x\colon |x-\mathsf{x}| \asymp t'\}$ with $p < t' \le t$ and in $B(\mathsf{x},p)$ with $\zeta^{-1} \le p \le t$. According to (3.12) the contribution of each layer does not exceed $CZ\ell^{-2}\bigl(\zeta^2 t'^2 + \zeta^{-2}Z^{5/3-\delta}\bigr)$ and summation over layers returns its value as $t' = t$, with $\zeta^{-2}Z^{5/3-\delta}$ acquiring logarithmic factor with we compensate by decreasing δ:

$$CZ\ell^{-2}\bigl(\zeta^2 t^2 + \zeta^{-2}Z^{5/3-\delta}\bigr). \tag{5.6}$$

Meanwhile, contribution of the ball $B(\mathsf{x},p)$ into (5.5) does not exceed $CZ\ell^{-2}\|\rho_\Psi - \rho\|_{\mathscr{L}^1(B(\mathsf{x},p))}$ and to estimate it we use Theorem 1.1 of [Ivr3] with

[3)] Indeed, it suffices to take a half-sum of the integrand in (5.4) with its value at symmetric about x point, because both $|x-\mathsf{y}_m|$ and $\rho(x)$ satisfy $|\nabla f| \le Cf\ell^{-1}$.

$a = \ell$ and $\mu = p^3\ell^{-3}$:

$$(5.7)\quad \|\rho_\Psi - \rho^{\mathsf{TF}}\|_{\mathscr{L}^1(B(x,s))}$$
$$\le C \begin{cases} p^2\ell^{-2/3}Z^{11/9-\delta/3} + p^3 Z\ell^{-2} & \text{as } \ p \ge Z^{-5/18-\delta}\ell^{5/6}, \quad Z^{-1} \le \ell \le Z^{-1/3}, \\ p^2\ell^{-8/3}Z^{5/9-\delta/3} & \text{as } \ p \ge Z^{5/9-\delta}\ell^{10/3}, \quad Z^{-1/3} \le \ell \le Z^{-5/21}. \end{cases}$$

Therefore (2.11) implies

$$(5.8)\quad \rho_\Psi(x) \le C\Big(Z\zeta^3\ell^{-3}t^4 + Z^{19/9}t^{-1} + Z\zeta^2\ell^{-2}t^2\Big) + C\Big(\zeta^3 + Z^{8/3-\delta}\zeta^{-2}\ell^{-2} + Z\ell^{-2}\|\rho_\Psi - \rho\|_{\mathscr{L}^1(B(x,p))}\Big),$$

where only first line depends on t. One can see easily that the third term in the first line does not exceed the sum of two first terms. Further, the second term there is larger than $Z^{19/9}\ell^{-1}$ which is larger than CZ^3 as $\ell \le Z^{-8/9}$ and since we already have the estimate (2.10), we should consider only $\ell \ge Z^{-8/9}$. Furthermore, $Z^{19/9}\ell^{-1} \ge \zeta^3$.

Finally, optimizing remaining two terms in the first line of (5.8) by t: $\zeta^{-1} \le t \le \ell$, we get

$$(5.9)\quad \rho_\Psi(x) \le C\Big(Z^{17/9}\zeta^{3/5}\ell^{-3/5} + Z\zeta^{-1}\ell^{-3} + Z^{19/9}\ell^{-1}\Big) + C\Big(Z^{8/3-\delta}\zeta^{-2}\ell^{-2} + Z\ell^{-2}\|\rho_\Psi - \rho\|_{\mathscr{L}^1(B(x,p))}\Big).$$

Let us compare terms there.

(i) Let $Z^{-8/9} \le \ell \le Z^{-1/3}$. Then one can see easily that the first line is defined by the third term $Z^{19/9}\ell^{-1}$ for $Z^{-8/9} \le \ell \le Z^{-7/9}$ and by the first term, which is $Z^{197/90}\ell^{-9/10}$, for $Z^{-7/9} \le \ell \le Z^{-1/3}$.

One can see easily that the first term in the second line of (5.9) is smaller than $Z\ell^{-1}$. Using the first case in (5.7) with $p = \max(\zeta^{-1}, \ell^{5/6}Z^{-5/18})$ and $\delta = 0$, we see that the second term in the second line is smaller than the first line as well. Thus we arrive to Theorem 1.1, Statement (i).

(ii) Let $\ell \ge Z^{-1/3}$. Then one can see easily that the first line of (5.9) is defined by the first term, which is $Z^{17/9}\ell^{-9/5}$ for $Z^{-1/3} \le \ell \le Z^{-5/18}$ and by $Z\ell^{-1}$ for $\ell \ge Z^{-5/18}$.

Consider the second line and impose condition $\ell \le Z^{-2/9}$. Then the first line dominates the first term here. Using the second case in (5.7) with $p = \max(\zeta^{-1}, \ell^{10/3} Z^{5/9})$ and $\delta = 0$, we see that the first line dominates the last term in the second line as well.

Further, let $\ell \ge Z^{-2/9}$. Recall that the second line (except $C\zeta^3$) was a result of the estimate of the second term in (2.11), which, however, could be estimated by

$$CZ\ell^{-2} \int_{B(x,\ell(x))} \rho_\Psi(x)\, dx. \tag{5.10}$$

It is well known that $\int \rho_\Psi \le CZ$ and therefore (5.10) does not exceed $CZ^2\ell^{-2}$ which covers $\ell \ge Z^{-1/9}$.

Furthermore, in the remaining range $Z^{-2/9} \le \ell \le Z^{-1/9}$ we can use Proposition 3.2(ii) to show, that the first term in the second line does not exceed $Z^{19/9}\ell^{-1}$ while the second term there is estimated again by the second case in (5.7). Thus we arrive to Theorem 1.1, Statement (ii).

(iii) Finally, using Proposition 3.2(iii) we prove Theorem 1.1, Statement (iii).

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Pointwise Spectral Asymptotics out of the Diagonal near Boundary[*,†]

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Abstract

We establish (with respect to x, y) semiclassical asymptotics and estimates for the Schwartz kernel $e_h(x,y,\tau)$ of spectral projector for a second order elliptic operator on the manifold with a boundary. While such asymptotics for its restriction to the diagonal $e_h(x,x,\tau)$ and, especially, for its trace $\mathsf{N}_h(\tau) = \int e_h(x,x,\tau)\,dx$ are well-known, the out-of-diagonal asymptotics are much less explored, especially uniform ones.

Our main tools: microlocal methods, improved successive approximations and geometric optics methods.

Our results would also lead to *classical* asymptotics of $e_h(x,y,\tau)$ for fixed h (say, $h=1$) and $\tau \to \infty$.

1 Introduction

Asymptotics away from the boundary. Let us start from the case of asymptotics when points x and y are disjoint from the boundary. While sharp classical asymptotics of $e(x,x,\tau)$ for elliptic operators are known from L. Hörmander [Hö], and could be traced to B. M. Levitan [Lev1, Lev2] and V. G. Avakumovič [Av1], I could not find asymptotics of $e(x,y,\tau)$, despite it could be easily derived by the method of Fourier integral operators combined with Tauberian theorem.

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Theorem 1.1. *Let $A = A^{\mathsf{w}}(x, hD, h)$ be a self-adjoint scalar pseudo-differential operator with $\mathscr{C}^K$-symbol ($K = K(d)$) which is ξ-microhyperbolic in $B(0,1) \subset X$ on the energy level τ* [1)]*. Further, let $\{\xi\colon a(x,\xi) = \tau\}$ be strongly convex. Then*

$$e_h(x,y,\tau) = e_h^{\mathsf{W}}(x,y,\tau) + O(h^{1-d}) \qquad \forall x, y \in B(0,\epsilon) \tag{1.2}$$

where

$$e_h^{\mathsf{W}}(x,y,\tau) = (2\pi h)^{-d} \int_{\{a(\frac{1}{2}(x+y),\xi)<\tau\}} e^{ih^{-1}\langle x-y,\xi\rangle}\, d\xi \tag{1.3}$$

is corresponding Weyl expression.

We will prove this theorem and discuss its generalizations and also more precise results in Section 2.

Asymptotics near the boundary. Classical asymptotics of $e(x,x,\tau)$ for second order elliptic operators are known from R. Seeley [Se1, 2], and sharp ones from V. Ivrii. The semiclassical version would carry remainder estimate $O(h^{1-d}\nu(x)^{-\frac{1}{2}})$ and $O(h^{1-d})$ respectively while the trivial remainder estimate would be $O(h^{1-d}\nu(x)^{-1})$ where $\nu(x) = \mathsf{dist}(x, \partial X)$.

Let

$$\ell^0(x,y) \coloneqq |x-y|,$$
$$\ell\,(x,y) \coloneqq |x-y| + \nu(x) + \nu(y).$$

Theorem 1.2. *Let*

$$A = \sum_{j,k} \big(hD_j - V_j(x)\big) g^{jk}(x) \big(hD_k - V_k(x)\big) + V(x), \qquad g^{jk} = g^{kj}, \tag{1.4}$$

be a self-adjoint scalar operator which is elliptic second order differential operator with $\mathscr{C}^K$-coefficients ($K = K(d,\delta)$), and it is ξ-microhyperbolic in $B(0,1)$ on the energy level τ, with either Dirichlet or Neumann boundary condition on $\partial X \cap B(0,1)$, and $\partial X \in \mathscr{C}^K$ [2)]*. Then*

[1)] Which means that

$$|a(x,\xi) - \tau| + |\nabla_\xi a(x,\xi)| \ge \epsilon_0 \tag{1.1}$$

where $a(x,\xi) \coloneqq A(x,\xi,0)$.

[2)] More general boundary conditions could be also considered.

(i) For $d \geq 3$ asymptotics (1.2) *holds where now for Dirichlet or Neumann boundary condition*

$$e_h^{\mathsf{W}}(x,y,\tau) = e_h^{0,\mathsf{W}}(x,y,\tau)\ [\mp]\ e_h^{0,\mathsf{W}}(x,\tilde{y},\tau) \tag{1.5}$$

respectively, $e_h^{0,\mathsf{W}}(x,y,\tau)$ defined by (1.3) *and $\tilde{y}$ is a reflected point.*

(ii) For $d = 2$ under one of the following assumptions

(a) $\ell(x,y) \leq h^{\frac{1}{3}+\delta}$,

(b) $\ell(x,y) \geq h^{\frac{1}{3}-\delta}$,

(c) $\nu(x)+\nu(y) \geq C_0(\ell^2 + h^{\frac{2}{3}-2\delta})$

where here and below $\delta > 0$ is an arbitrarily small exponent, asymptotics

$$e_h(x,y,\tau) = e_h^{\mathsf{W}}(x,y,\tau) + e_{h,\mathsf{corr}}(x,y,\tau) + O(h^{1-d}) \quad \forall x,y \in B(0,\epsilon) \tag{1.6}$$

holds with correction term $e_{h,\mathsf{corr}}(x,y,\tau)$ to be defined by (5.53) *later.*

(iii) Finally, for $d = 2$ if neither of condition (a)–(c) is fulfilled

$$e_h(x,y,\tau) = O(h^{-1-\delta}). \tag{1.7}$$

Remark 1.3. (i) According to [Ivr2], Theorem 8.1.6 asymptotics (1.2) holds for $x = y$ in the coordinate system such that x_1 is the distance from x to ∂X in the metrics $g^{jk}(\tau - V)^{-1}$. Such precision is needed for $d = 2$ only.

(ii) The leading term $e_h^{\mathsf{W}}(x,y,\tau)$ is $\asymp h^{-\frac{d-1}{2}}\ell^{-\frac{d+1}{2}}$. In particular it is $O(h^{1-d})$ as $\ell \gtrsim h^{\frac{d-1}{d+1}}$.

(iii) The correction term (as $d = 2$) is $O(h^{-\frac{3}{2}}\ell^{\frac{1}{2}})$ if $\ell \leq h^{\frac{1}{2}}$ and $O(h^{-\frac{1}{2}}\ell^{-\frac{3}{2}})$ if $h^{\frac{1}{2}} \leq \ell \leq \epsilon$.

(iv) The trivial estimate (in much more general settings) is

$$|e_h(x,y,\tau) - e_h^{\mathsf{W}}(x,y,\tau)| \leq Ch^{1-d}(1+\ell^{-1}(x,y)). \tag{1.8}$$

Ideas of proofs. I. Consider $x, y \in B(0, \frac{1}{2})$. We already know that under pretty general assumptions (which will be discussed later), in particular, in the frameworks of Theorems 1.1 and 1.2, the following estimate holds:

$$e(x, x, \tau' + h) - e(x, x, \tau') \le Ch^{1-d} \qquad \text{for } |\tau' - \tau| \le \epsilon_0. \tag{1.9}$$

Then

$$|e(x, y, \tau) - e(x, y, \tau')| \le Ch^{-d}|\tau - \tau'| + h^{1-d} \tag{1.10}$$

and therefore due to Tauberian methods

$$|e_h(x, y, \tau) - e^{\mathsf{T}}_{T,h}(x, y, \tau)| \le CT^{-1}h^{1-d} \tag{1.11}$$

where

$$e^{\mathsf{T}}_{T,h}(x, y, \tau) = h^{-1} \int_{-\infty}^{\tau} F_{t \to h^{-1}\tau'}\big(\bar\chi_T(t) u_h(x, y, t)\big)\, d\tau' \tag{1.12}$$

is the *Tauberian expression*, $u_h(x, y, t)$ is the Schwartz kernel of the propagator $e^{ih^{-1}At}$, $h \le T \le \epsilon_1$, ϵ_1 is a small constant and $\bar\chi_T(t) = \bar\chi(t/T)$ and $\bar\chi \in \mathscr{C}_0^\infty([-1, 1])$, $\bar\chi(t) = 1$ on $(-\frac{1}{2}, \frac{1}{2})$.

Assume first that $B(0, 1) \subset X$. Then for a small constant T we can construct $u_h(x, y, t)$ as an oscillatory integral and we can construct $e^{\mathsf{T}}_h(x, y, \tau)$ even as $d = 2$ or without strong convexity assumption. However the result could be simplified to $e^{\mathsf{W}}_h(x, y, \tau)$ in the framework of Theorem 1.1 (for scalar operators under strong convexity assumption). This is done in Section 2.

Ideas of proofs. II. Construction of the propagator is much more complicated close to the boundary, because there could be many generalized billiard rays with at least one reflection from ∂X, from x to y on the energy level τ. If

$$\ell(x, y) \coloneqq |x - y| + \nu(x) + \nu(y) \le \epsilon_1 \tag{1.13}$$

the length of all such rays is $\asymp \ell(x, y)$. Further, if

$$\nu(x) + \nu(y) \ge \sigma \ell(x, y) \tag{1.14}$$

with $\sigma \ge C_0\ell(x, y)$ there is exactly one such ray and it has exactly one reflection, and the reflection angle $\ge \epsilon_1(\nu(x) + \nu(y))/\ell(x, y)$. It allows us in the framework of Theorem 1.2 under assumption (1.14) with $\sigma =$

$h^{\frac12-\delta}\ell(x,y)^{-\frac12}+C_0\ell(x,y)$ to construct a reflected wave as an oscillatory integral and simplify it. It will be done in Section 5.

So, we should consider a case when (1.14) is violated and we will do it in Section 4. As $|x-y|\le Ch^{\frac12-\delta}$ the propagator could be constructed by the standard method of successive approximations with unperturbed constant coefficients operator.

Furthermore, in the framework of Theorem 1.2 under certain assumptions, which are complementary to (1.14) as $\ell\le h^{\frac13+\delta}$, the reflected wave could be constructed also by the method of the successive approximations, but with unperturbed the operator

$$\bar{A}(z,hD_z)=h^2D_1^2+A(0,w',hD'_z), \tag{1.15}$$

where $x=(x_1;x')=(x_1;x_2,\dots,x_d)$ and $X\cap B(0,1)=\{x\colon x_1>0\}\cap B(0,1)$ and we make a change of variables $w'=\frac12(x'+y')$ and $z'=\frac12(x'-y')$, leaving x_1 and y_1 unchaned.

Ideas of proofs. III. Finally, to estimate the Tauberian expression rather than to find its asymptotics we apply in Section 3 only methods of propagations. Let us illustrate this away from the boundary. Let $Q_{1,2}$ be pseudodifferential operators with the symbols equal 1 in ρ-vicinity of $\{\xi\in\Sigma(w,\tau)\colon \nabla_\xi a(w,\xi)\parallel \bar{x}-\bar{y}\}$; this set consists of just two points. Then

$$e_h^{\mathsf{T}}(\bar{x},\bar{y},\tau)=Q_{2x}e_h^{\mathsf{T}}(x,y,\tau)\,{}^tQ_{1y}|_{x=\bar{x},y=\bar{y}}+O(h^s) \tag{1.16}$$

provided microlocal uncertainty principle $\rho\times\rho\ell\ge h^{1-\delta}$ is fulfilled. It allows us to upgrade $|e_h^{\mathsf{T}}(x,y,\tau)|\le Ch^{1-d}\ell^{-1}(x,y)$ to

$$|e_h^{\mathsf{T}}(x,y,\tau)|\le C\rho^{d-1}h^{1-d}\ell^{-1}(x,y). \tag{1.17}$$

We need also $\rho\ge C_0\ell$. We need also $\rho\ge C_0\ell$.

2 Asymptotics inside domain

2.1 Tauberian asymptotics

As we already mentioned, asymptotics

$$e_h(x,x,\tau)=\kappa_0(x,\tau)h^{-d}+O(h^{1-d}) \tag{2.1}$$

is known under much more general assumptions. It holds for matrix operators under ξ-microhyperbolicity condition at point x on the energy level τ and for Schrödinger operator without any condition in dimension $d \geq 3$ (see [Ivr2], Sections 5.4 and 5.3 correspondingly).

Asymptotics (2.1) implies (1.9) and then (1.10). Then the standard Tauberian arguments (see f.e. [Ivr2], Section 5.2) imply (1.11)–(1.12) with small constant T.

2.2 Weyl asymptotics

While asymptotics (2.1) holds in much more general assumptions, to construct $u_h(x,y,t)$ we need to impose some restrictions. Let A be a scalar operator[3)]. While construction of propagator as an oscillatory integral is possible even if ξ-microhyperbolicity condition is violated, we assume that it is fulfilled from the beginning, as we need it anyway.

Without any loss of the generality one can assume that

$$a(y,0) < 0. \tag{2.2}$$

Indeed, otherwise we can achieve it by a corresponding gauge transformation. Then the strong convexity of $\Sigma(y,\tau) = \{\theta\colon a(y,\theta) = \tau\}$ implies that

$$\langle \nabla_\theta a(y,\theta), \theta\rangle \geq \epsilon_0 \qquad \forall \theta \in \Sigma(y,\tau). \tag{2.3}$$

Proposition 2.1. *Let A be a scalar operator, ξ-microhyperbolic in $B(\bar{x},2\epsilon) \subset X$ on energy level τ and let $x,y \in B(\bar{x},\epsilon)$. Let* (2.3) *be fulfilled. Then*

$$u_h(x,y,t) \equiv \int e^{ih^{-1}\Phi(x,y,t,\theta)} B(x,y,t,\theta,h)\, d\theta \tag{2.4}$$

modulo functions such that

$$F_{t\to h^{-1}\tau'}\bar{\chi}_T(t) v_h(x,y,t) = O(h^s) \qquad \forall x,y \in B(0,\epsilon) \tag{2.5}$$

as $T \leq \epsilon_1$, $|\tau'-\tau| \leq \epsilon_1$ where

$$\Phi(x,y,t,\theta) = \varphi(x,y,\theta) + ta(y,\theta), \tag{2.6}$$

[3)] Construction also works for matrix operators with characteristic roots of constant multiplicity.

$\varphi(x,y,\theta)$ *satisfies stationary eikonal equation*

$$a(x,\nabla_x\varphi)=a(y,\theta) \tag{2.7}$$

and

$$\varphi(x,y,\theta)|_{\langle x-y,\theta\rangle=0}=0, \tag{2.8}$$

$$\nabla_x\varphi(x,y,\theta)|_{x=y}=\theta \tag{2.9}$$

and

$$B(x,y,t,\theta,h)\sim\sum_{n\ge 0}B_n(x,y,t,\theta)h^n, \tag{2.10}$$

and amplitudes B_{jn} are defined from the Cauchy problems for transport equations; all functions are uniformly smooth; here and below s is an arbitrarily large exponent.

Proof. The proof is original L. Hörmander's construction (see M. Shubin [Shu], Theorem 20.1). □

While in [Hö, 3] this decomposition was used to calculate $e_h^{\mathsf{T}}(x,y,\tau)$ as $x=y$ (actually, it's classical variant), we will use it without this restriction. Due to ξ-microhyperbolicity

$$\begin{aligned} &F_{t\to h^{-1}\tau'}\bigl(\bar\chi_T(t)u_h(x,y,t)\bigr)\\ &\qquad\sim(2\pi h)^{-d}h\int_{\Sigma(y,\tau')}e^{ih^{-1}\varphi(x,y,\theta)}B'(x,y,\theta)\,d\theta : da(y,\theta)\end{aligned} \tag{2.11}$$

with $\Sigma(y,\tau)=\{\theta\colon a(y,\theta)=0\}$, $d\theta : da(y,\theta)$ a natural density on $\Sigma(y,\tau')$ and $B'(x,y,\theta)$ also allowing (2.10) decomposition with uniformly smooth B'_n.

Proposition 2.2. *Let in $B(\bar x,2\epsilon_1)$ both ξ-microhyperbolicity and strong convexity conditions are fulfilled on energy level τ. Then for $|\tau'-\tau|\le\epsilon_1$ and $x\ne y$ $\varphi(x,y,\theta)$ has exactly two stationary points $\theta^*_\pm(x,y,\tau')$ on $\Sigma(y,\tau')$, defined from*

$$\nabla_\theta\varphi(x,y,\theta)=-t_\pm\nabla_\theta a(y,\theta),\qquad a(y,\theta)=\tau',\qquad \pm t_\pm>0. \tag{2.12}$$

These points are non-degenerate and $t_\pm\asymp|x-y|$ in (2.12).

Furthermore,

$$\theta^*_\pm(x,y,\tau') = \bar\theta_\pm(x,y,\tau') + O(|x-y|), \tag{2.13}$$

where $\bar\theta_\pm(x,y,\tau')$ *are defined from*

$$x-y = -t'_\pm \nabla_\theta a(y,\theta), \qquad a(y,\theta)=\tau', \quad \pm t'_\pm > 0 \tag{2.14}$$

and also $t'_\pm = t_\pm + O(|x-y|^2)$.

Proof. Proof follows trivially from

$$\varphi(x,y,\theta) = \langle x-y,\theta\rangle + O(|x-y|^2), \tag{2.15}$$

which follows from (2.8), (2.9). □

Proposition 2.3. *In the framework of Proposition 2.2 the following asymptotics hold*

$$F_{t\to h^{-1}\tau}\bigl(\bar\chi_T(t)u_h(x,y,t)\bigr) \sim (2\pi)^{-d}h^{\frac{1-d}{2}}\sum_{\varsigma=\pm} e^{ih^{-1}S_\varsigma(x,y,\tau)}B'_\varsigma(x,y,\tau) \tag{2.16}$$

and with $c\ell \le T \le \epsilon$

$$e^{\mathsf{T}}_{T,h}(x,y,\tau) \sim (2\pi)^{-d}h^{\frac{1-d}{2}}\sum_{\varsigma=\pm} e^{ih^{-1}S_\varsigma(x,y,\tau)}B''_\varsigma(x,y,\tau) \tag{2.17}$$

with

$$S_\varsigma(x,y,\tau) = \varphi(x,y,\theta^*_\varsigma(x,y,\tau)) \tag{2.18}$$

and B'_ς, B''_ς *also decomposed into asymptotic series albeit with* $B'_{\varsigma n}\ell^{n+\frac{d-1}{2}}$ *and* $B''_{\varsigma n}\ell^{n+\frac{d+1}{2}}$ *uniformly smooth.*

Proof. Asymptotics (2.17) follows from Proposition 2.2 and stationary phase principle with effective semiclassical parameter $\hbar = h\ell^{-1}$, $\ell = |x-y|+h$.

To prove (2.18) we first rewrite (1.12) as the sum of two integrals with cut-off functions $\phi_1(\tau')$ and $\phi_2)(\tau')$; $\phi_j \in \mathscr{C}_0^\infty$, $\mathsf{supp}(\phi_1) \subset (\tau-\epsilon,\tau+\epsilon)$, $\mathsf{supp}(\phi_2) \subset (-\infty,\tau-\epsilon/2)$. In the integral with ϕ_1 we plug (2.4), observe that there are no stationary points in $\{\theta\colon \tau-\epsilon < a(y,\theta) < \tau\}$ and use Proposition 2.2 and stationary phase principle with effective semiclassical parameter $\hbar = h\ell^{-1}$. In the integral with ϕ_2 we simply observe that it has a complete asymptotics because it contains a mollification by τ' (see, f.e. Proposition 3.5 below). □

Proposition 2.4. *In the framework of Proposition 2.2*

(i) The following estimates hold:

$$|F_{t\to h^{-1}\tau}\Big(\bar{\chi}_T(t)u_h(x,y,t)\Big)| \le Ch^{\frac{1-d}{2}}\ell^{\frac{1-d}{2}} \tag{2.19}$$

and as $c\ell \le T \le \epsilon$

$$|e^{\mathsf{T}}_{T,h}(x,y,\tau)| \le Ch^{\frac{1-d}{2}}\ell^{\frac{-1-d}{2}}. \tag{2.20}$$

(ii) The following estimate holds as $\max(c\ell,\ h^{1-\delta}) \le T \le \epsilon$*:*

$$|e^{\mathsf{T}}_{T,h}(x,y,\tau) - e^{\mathsf{W}}_{h}(x,y,\tau)| \le Ch^{1-d}. \tag{2.21}$$

Proof. (a) Estimates (2.19)–(2.20) follow from (2.16)–(2.17).

(b) To prove Statement (ii) for $d \ge 3$ one can observe that both $|e^{\mathsf{T}}_{T,h}(x,y,\tau|$ and $|e^{W}(x,y,\tau)|$ do not exceed Ch^{1-d} as $h^{\frac12} \le \ell \le \epsilon$, while for $\ell \le h^{\frac12}$, due to (2.15), replacing $\Phi(x,y,t,\theta)$ by

$$\bar{\Phi}(x,y,\theta) = \langle x-y,\theta\rangle + ta(\frac{1}{2}(x+y),\theta) \tag{2.22}$$

brings an error not exceeding the right-hand expression of (2.20) multiplied by $C\ell^2h^{-1}$, that is $Ch^{-\frac{1+d}{2}}\ell^{\frac{3-d}{2}} \le Ch^{1-d}$.

Moreover, if we also replace $B(x,y,t,\theta,h)$ by 1 then the error will not exceed the right-hand expression of (2.20) multiplied by $C\ell$, that is $Ch^{\frac{1-d}{2}}\ell^{\frac{1-d}{2}} \le Ch^{1-d}$.

Finally, with this choice of the phase function $\bar{\Phi}(x,y,t,\theta)$ and amplitude $B=1$, we get $e^{\mathsf{T}}_{T,h}(x,y,\tau) = e^{\mathsf{W}}(x,y,\tau) + O(h^{\infty})$.

(c) As $d=2$ we need more subtle arguments (which would also work for $d\ge 3$). Indeed, estimate (2.20) implies that $|e^{\mathsf{T}}_{T,h}(x,y,\tau)| \le Ch^{-1}$ as $h^{\frac13} \le \ell \le \epsilon_1$ and estimate (2.21) leads to $O(h^{-1})$ error only as $\ell \asymp h$.

We refer to Proposition 5.13(i). □

Finally, Theorem 1.1 follows from estimate (1.11) and Proposition 2.4.

2.3 Improvements and generalizations

Two-term asymptotics. We know that if A is a scalar operator on the closed manifold, ξ-microhyperbolic at point x and if *non-looping condition*

$$\mu_{x,\tau}\{\xi \in \Sigma(x,\tau)\colon \exists t \neq 0, \pi_x e^{tH_a(x,\xi)} = x\} = 0 \tag{2.23}$$

is fulfilled at point x then two-term asymptotics

$$e_h(x,x,\tau) = \kappa_0(x,\tau)h^{-d} + \kappa_1(x,\tau)h^{1-d} + o(h^{1-d}) \tag{2.24}$$

holds; here $\Sigma(x,\tau) = \{\xi\colon a(x,\xi) = \tau\}$, $\mu_{x,\tau}$ is a natural measure on $\Sigma(x,\tau)$, corresponding to a density $dxd\xi : d_\xi a(x,\xi)$, H_a is a Hamiltonian field generated by a and e^{tH_a} is a Hamiltonian flow (see, f.e. [Ivr2], Section 5.3). Can we get a two-term asymptotics for $e_h(x,y,\tau)$?

First of all, we need to improve Tauberian estimate (1.11). Asymptotics (2.24) at point x implies

$$|e(x,x,\tau + T^{-1}h) - e(x,x,\tau')| \le CT^{-1}h^{1-d} + o_T(h^{1-d}) \tag{2.25}$$

and then

$$\begin{aligned}|e(x,y,\tau) - e(x,y,\tau')| \\ \le C|\tau-\tau'|h^{-d} + C|\tau-\tau'|^{\frac12}h^{\frac12-d} + CT^{-\frac12}h^{1-d} + o_T(h^{1-d})\end{aligned}$$

with arbitrarily large T. Here condition (2.23) is fulfilled at one of points x, y. Then the standard Tauberian arguments imply

$$|e_h(x,y,\tau) - e^{\mathsf{T}}_{T,h}(x,y,\tau)| \le CT^{-\frac12}h^{1-d} + o_T(h^{1-d})$$

and propagation results combined with non-looping condition at one of points x, y imply that in the left hand expression one can replace T by $T' = T(x,y)$ while still having arbitrary T in the right hand-expression, and then we arrive to

$$e_h(x,y,\tau) = e^{\mathsf{T}}_{T,h}(x,y,\tau) + o(h^{1-d}) \qquad \text{with} \quad T = T(x,y). \tag{2.26}$$

Further, if we add another condition

$$\mu_{x,\tau}\{\xi \in \Sigma(x,\tau)\colon \exists t \neq 0, \pi_x e^{tH_a(x,\xi)} = y\} = 0 \tag{2.27}$$

or a similar condition, obtained by permutation of x and y (let's call it (2.26)′), then from (2.26) and propagation results we obtain that

(a) If $\ell(x,y) \ge \epsilon$ then $e_h(x,y,\tau) = o(h^{1-d})$,

(b) If $\ell(x,y) \le \epsilon$, then we can take $T(x,y) = c\ell(x,y)$ in (2.26).

Weyl asymptotics. In particular, in the framework of Theorem 1.1, under extra assumptions (2.23) and either (2.26) or (2.26)′, we can apply the machinery developed in Subsection 2.2 (with the only exception $d = 2$ and neither $\ell \gg h^{\frac{1}{3}}$ nor $\ell \ll h^{\frac{1}{3}}$). One can prove easily that then

$$\big(e(x,y,\tau) = e^{\mathsf{W}}(x,y,\tau)\big) + o(h^{1-d}) \qquad \forall x,y \in B(0,\epsilon) \tag{2.28}$$

where $e^{\mathsf{W}}_h(x,y,\tau)$ is defined by (1.3) with a replaced by $a + ha_1$ where a_1 is the subprincipal symbol of A, that is with the main term defined by (1.3) and with the second term

$$-(2\pi h)^{1-d} \int_{\Sigma(z,\tau)} a_1(z,\theta) e^{i\langle x-y,\theta\rangle}\, d\theta : d_\theta a(z,\theta), \qquad z = \frac{1}{2}(x+y). \tag{2.29}$$

Generalizations.

(i) We can consider the case when $a(x,\xi)$ has characteristic roots $\lambda_j(x,\xi)$ of constant multiplicity; and we need only to assume that it happens as $|\lambda_j(x,\xi) - \tau| \le \epsilon$.

(ii) If we consider only $y\colon y - x \in \Gamma$ where Γ is a cone in $\mathbb{R}^d$ with a vertex at 0, then in virtue of propagation results (see [Ivr2], Section 2.2) we need the previous assumption only for ξ such that

$$\Gamma \cap \big(K(x,\xi) \cup -K(x,\xi)\big) \neq \emptyset \tag{2.30}$$

where $K(x,\xi)$ is a cone dual to $K'(x,\xi)$ which is a connected component of

$$\{\eta\colon ((\eta \cdot \partial_\xi a(x,\xi))v, v) \ge 0\ ; \forall v\colon \|(a-\tau)v\| \le \epsilon\|v\|\}.$$

(iii) Also, instead of strong convexity we need to assume only that at points ξ satisfying (2.30), the matrix of the curvatures of the surface $\{\xi\colon \lambda(x,\xi) = \tau\}$ has at least two eigenvalues, disjoint from 0. Then it would be similar to the case $d \ge 3$ in Proposition 2.4. Probably, it would suffice to have just one eigenvalue, to recover more delicate arguments of the case $d = 2$ in Proposition 2.4.

3 Microlocal methods

3.1 Propagation

Recall that we consider Schrödinger operator (1.4) and for such operator under ξ-microhyperbolicity near boundary condition on the energy level τ instead of on-diagonal asymptotics (2.1) asymptotics with a boundary-layer type term and remainder estimate $O(h^{1-d})$ holds; see [Ivr2], Section 8.1. In particular, under Dirichlet or Neumann boundary conditions (and we consider only those for simplicity) asymptotics (1.5) holds as $x = y$ and also (1.9) holds. Then (1.10)–(1.12) hold and all we need is to rewrite Tauberian expression (1.12) in more explicit terms.

Let us decompose

$$u_h(x,y,t) = u_h^0(x,y,t) + u_h^1(x,y,t) \tag{3.1}$$

where $u_h^0(x,y,t)$ is the solution for the "free space" and $u_h^1(x,y,t)$ is a reflected wave, satisfying

$$hD_t u_h^1 = A(x,hD,h)u_h^1, \tag{3.2}$$

$$B(x,hD,h)u_h^1|_{\partial X} = -B(x,hD,h)u_h^0|_{\partial X}, \tag{3.3}$$

where $B(x,hD,h)$ is a boundary operator. Then

$$e_h^{\mathsf{T}}(x,y,\tau) = e_h^{0,\mathsf{T}}(x,y,\tau) + e_h^{1,\mathsf{T}}(x,y,\tau), \tag{3.4}$$

where $e_h^{0,\mathsf{T}}(x,y,\tau)$ and $e_h^{1,\mathsf{T}}(x,y,\tau)$ are derived by Tauberian expression (1.12) from $u_h^0(x,y,t)$ and $u^1(x,y,t)$ correspondingly with $T = \epsilon_1$.

While $u_h^0(x,y,t)$ can be constructed as an oscillatory integral, our purpose is to construct $u_h^1(x,y,t)$ in some way. To do this, in this section we apply the improved method of successive approximations near boundary, in the same way as the standard method was applied in [Ivr2], Section 7.2 while in the next Section 5 we use the geometric optics method.

Recall that now A is a second order elliptic operator and $X \cap B(0,1) = \{x\colon \nu(x) > 0\} \cap B(0,1)$ with $|\nabla\nu| \ge \epsilon_0$. Without any loss of the generality one can assume that

$$X \cap B(0,1) = \{x\colon x_1 > 0\} \cap B(0,1) \text{ with } V_1(x) = 0,\ g^{1k} = \delta_{1k} \tag{3.5}$$

and therefore

$$\nu(x) = x_1, \qquad \ell(x,y) = |x' - y'| + x_1 + y_1 \tag{3.6}$$

and A is ξ-microhyperbolic on the energy level τ at x as long as

$$|V(x) - \tau| \ge \epsilon_0, \tag{3.7}$$

and in this case $\{\xi\colon a(x,\xi) = \tau\}$ is strongly convex.

Proposition 3.1. *(i) As $T \le \epsilon\ell(x,y)$ the following estimate holds:*

$$|F_{t\to h^{-1}\tau}\Big(\bar\chi_T(t)u^1_h(x,y,t)\Big)| \le Ch^{-d}\Big(\frac{h}{T}\Big)^s. \tag{3.8}$$

(ii) Let condition (3.1) *be fulfilled. Then for $\mathsf{max}(C\ell(x,y),\, h^{1-\delta}) \le T \le T' \le \epsilon_1$ the following estimate holds:*

$$|F_{t\to h^{-1}\tau}\Big(\big(\bar\chi_{T'}(t) - \bar\chi_T(t)\big)u^1_h(x,y,t)\Big)| \le Ch^{-d}\Big(\frac{h}{T}\Big)^s. \tag{3.9}$$

Proof. Statement (i) follows from the finite speed of propagation and Statement (ii) follows from the fact that the speed of propagation with respect to x is disjoint from 0 (either along ∂X, if $|\xi'|$ is disjoint from 0 or in direction of x_1 if ξ_1 is disjoint from 0)–see [Ivr2], Chapter 3. □

Corollary 3.2. *Let $e^{1,\mathsf{T}}_h(x,y,\tau)$ be defined by* (1.12) *with $u^1_h(x,y,t)$ instead of $u_h(x,y,t)$.*

(i) Then in $e^{1,\mathsf{T}}_h(x,y,\tau)$ one can replace $T = \epsilon_1$ by $T = C\,\mathsf{max}(\ell(x,y),\, h^{1-\delta})$.

(ii) Further, in the framework of Proposition 3.1(ii) and $\ell(x,y) \ge h^{1-\delta}$ one can replace $\bar\chi_T(t)$ with $T = \epsilon_1$ by $\big(\bar\chi_T(t) - \bar\chi_{T'}(t)\big)$ with $T = C_0\ell(x,y)$ and $T' = C_0^{-1}\ell(x,y)$.

We need to be more precise about propagation especially with respect to x_1 and ξ_1 as $\ell^0 \asymp \ell$. Let us $\bar x$, $\bar y$ and $\bar w$ be fixed final values of x, y and w respectively and we need to consider case $\ell := \ell(\bar x, \bar y) \ge h^{1-\delta}$.

Proposition 3.3. *Let* $h^{1-\delta} \le T \le \epsilon_1$, $T = C_0\ell$ *and*

$$\rho T \ge h^{1-\delta}, \qquad \rho \ge T^2. \tag{3.10}_{1,2}$$

Let $b(\bar{y}', \bar{\xi}') = \tau$ *and*

$$|\nabla_{y'} b(y', \bar{\xi})_{y'=\bar{y}'}| \le \rho T^{-1} \tag{3.11}$$

where here and below

$$b(x', \xi') := a(x, \xi)|_{x_1=\xi_1=0}; \tag{3.12}$$

and let $Q_1(y, \eta')$ *have a symbol supported in* (T, ρ)*-vicinity of* $(\bar{y}, \bar{\xi}')$.

(i) Let $Q_2(x, \xi')$ *have a symbol, equal* 0 *in* $\{(x, \xi')\colon |\xi' - \bar{\xi}'| \le C_1\rho\}$. *Then*

$$F_{t\to h^{-1}\tau}\Bigl(\bar{\chi}_T(t) Q_{2x} u_h^1(x, y, t)\, {}^t Q_{1y}\Bigr) = O(h^s). \tag{3.13}$$

(ii) Assume instead that $Q_2(x, hD')$ *has a symbol, equal* 0 *as* $x_1 \le C_1\sigma T$ *where here and below*

$$\sigma = (\rho^{\frac{1}{2}} + T). \tag{3.14}$$

Then (3.13) *holds.*

(iii) Assume instead that $Q_2(x, hD')$ *has a symbol, equal* 0 *in*

$$\{(x', \xi')\colon |x' - \bar{y}' + t\nabla_{\xi'} b(\bar{y}', \xi')|_{\xi=\bar{\xi}'}| \le C_1\rho T\} \qquad \forall t\colon \epsilon T \le \pm t \le C_0 T. \tag{3.15}$$

Assume also that

$$\rho^2 T \ge h^{1-\delta}, \qquad \rho \ge C_0 T. \tag{3.16}$$

Then

$$F_{t\to h^{-1}\tau}\Bigl(\chi_T^{\pm}(t) Q_{2x} u_h^1(x, y, t)\, {}^t Q_{1y}\Bigr) = O(h^s). \tag{3.17}$$

where $\chi^{\pm} \in \mathscr{C}_0^{\infty}(\mathbb{R})$ *is supported in* $\{t\colon \epsilon \le \pm t \le 1\}$.

Proof. (i) Observe that under assumptions $(3.10)_2$ and (3.11) the propagation speed with respect to ξ' does not exceed $C_1(\rho+T)$ in

$$\{(x,\xi')\colon |x'-\bar{y}'|+|x_1|+|y_1|\le C_1T,\ |\xi'-\bar{\xi}'|\le C_1\rho\}.$$

This implies Statement (i) by the standard methods of propagation; see Subsection 5.1.2 of [Ivr2]. Here $(3.10)_1$ is the microlocal uncertainty principle).

(ii) Due to Statement (i) the propagation is confined to $\{(x,\xi')\colon |b(x',\xi')-\tau|\le C_0\sigma^2\}$ and thus to $|\xi_1|\le C\sigma$ and then to $x_1\le C_1\sigma\ell$. The microlocal uncertainty principle with respect to (x_1,ξ_1), that is $\sigma\times\sigma\ell\ge h^{1-\delta}$ is fulfilled due to $(3.10)_1$ and therefore we can consider propagation with respect to (x_1,ξ_1) as $x_1\ge\sigma T$.

(iii) Due to Statement (i) the propagation in the time direction $\pm t>0$ is confined to the domain (3.15) and here (3.16) is a microlocal uncertainty principle. □

Remark 3.4. (i) We can achieve (3.11) by gauge transformation, not affecting x_1,ξ_1. Further, if

$$\rho\ge T \tag{3.18}$$

then (3.11) is fulfilled automatically.

(ii) Obviously, condition (3.11) is important only as $\rho\le T$, which is not the case as $T\le h^{\frac12}$. Therefore we can cover also $\ell\le h^{1-\delta}$ with $T=h^{1-\delta}$.

(iii) Both statements of Proposition 3.3 hold for $\tau'\colon |\tau-\tau'|\le\sigma^2=(\rho+\ell^2)$.

3.2 Spectral estimates

The following proposition will be useful in the proof of Theorem 1.2:

Proposition 3.5. *(i) Let $L\ell^0(x,y)\ge h^{1-\delta}$. Let $\phi\in\mathscr{C}_0^\infty(\mathbb{R})$. Then*

$$\int \phi((\tau-\tau')/L)F_{t\to h^{-1}\tau'}\Big((\bar{\chi}_T(t)u_h(x,y,t)\Big)\,d\tau'=O(h^s). \tag{3.19}$$

(ii) Let $L\ell(x,y)\ge h^{1-\delta}$. Let $\phi\in\mathscr{C}_0^\infty(\mathbb{R})$. Then

$$\int \phi((\tau-\tau')/L)F_{t\to h^{-1}\tau'}\Big((\bar{\chi}_T(t)u_h^1(x,y,t)\Big)\,d\tau'=O(h^s). \tag{3.20}$$

Proof. The left-hand expressions are just $\phi((hD_t-\tau)/L)u_h(x,y,t)\big|_{t=0}$ and $\phi((hD_t-\tau)/L)u_h^1(x,y,t)\big|_{t=0}$ correspondingly and we apply finite speed of propagation. □

Proposition 3.6. *Under ξ-microhyperbolicity condition on the energy level τ estimates* (1.8) *and*

$$|e_h(x,y,\tau)| \le Ch^{1-d}\bigl(1+\ell^{0\,-1}(x,y)\bigr) \tag{3.21}$$

hold.

Proof. In virtue of Subsection 2.1 for $h \le \mathsf{T} \le \epsilon$

$$\begin{aligned} e_h(x,y,\tau) &= e_h^{\mathsf{T}}(x,y,\tau) + O(\mathsf{T}^{-1}h^{1-d}) \\ &= e_h^{0,\mathsf{T}}(x,y,\tau) + e_h^{1,\mathsf{T}}(x,y,\tau) + O(\mathsf{T}^{-1}h^{1-d}) \end{aligned}$$

and for $T = \epsilon_1\ell(x,y)$

$$e_h^{0,\mathsf{T}}(x,y,\tau) = e^{0,\mathsf{W}}(x,y,\tau) + O(h^{-d}(h\ell^{-1})^s)$$

and

$$e_h^{1,\mathsf{T}}(x,y,\tau) = O(h^s)$$

due to the finite speed of propagation, which implies (1.8). Estimate (3.21) is proven in the same way. □

As $h^{1-\delta} \le \ell(\bar{x},\bar{(}y)) \le \epsilon_1$ let us introduce $\bar{\xi}'^{\pm}$:

$$\nabla_{\xi'} b(\bar{w}',\xi')|_{\xi'=\bar{\xi}'^{\pm}} = t(\bar{y}-\bar{x}), \qquad b(\bar{w}',\bar{\xi}'^{\pm}) = \tau, \qquad \pm t > 0 \tag{3.22}$$

with $w' = \frac{1}{2}(\bar{x}'+\bar{y}')$. Due to strong convexity $\bar{\xi}'^{\pm}$ are defined uniquely and $t \asymp \ell(\bar{x},\bar{(}y))$.

Proposition 3.7. *Let $h^{1-\delta} \le T \le \epsilon_1$, $T = C_0\ell$,* (3.18) *and* (3.16) *be fulfilled. Then*

$$e_h^{1,\mathsf{T}}(\bar{x},\bar{y},\tau) = Q_{2x}e_h^{1,\mathsf{T}}(x,y,\tau)\,{}^tQ_{1y}|_{x=\bar{x},y=\bar{y}} + O(h^s) \tag{3.23}$$

where Q_1, Q_2 are operators with symbols equal 1 *in $(C_1\ell, C_1\rho)$ vicinities of $(\bar{w}',\bar{\xi}'^{\pm})$.*

Proof. Due to Proposition 3.5

$$e_h^{1,\mathsf{T}}(x,y,\tau) = h^{-1}\int_{-\infty}^{\tau} \bar\phi((\tau-\tau')L^{-1})F_{t\to h^{-1}\tau'}\bigl(\bar\chi_{\mathsf{T}}(t)u_h^1(x,y,t)\bigr)\,d\tau' + O(h^s) \tag{3.24}$$

where $\bar\phi \in \mathscr{C}^\infty(\mathbb{R})$, $\bar\phi(\tau)=1$ as $\tau\le 1$ and $\bar\phi(\tau)=0$ as $\tau\ge 2$, $L\ell \ge h^{1-\delta}$.

Due to Proposition 3.1(ii) we can replace $\bar\chi_{\mathsf{T}}(t)$ by $\chi_{\mathsf{T}}(t)=\bar\chi_{\mathsf{T}}(t)-\bar\chi_{\epsilon\mathsf{T}}(t)$. Then Proposition 3.3(iii) implies that if $\psi_1,\psi_2\in\mathscr{C}_0^\infty$ are supported in $\epsilon\ell$-vicinities of $\bar y$ and $\bar x$ respectively and $Q'=Q'(hD')$ is 0 in $C\rho$-vicinities of $\bar\xi'^{\pm}$ then as $|\tau'-\tau|\le L = h^{1-\delta}L^{-1}$

$$F_{t\to h^{-1}\tau'}\Bigl(\chi_{\mathsf{T}}(t)Q'_x\psi_2(x)u_h^1(x,y,t)\psi_1(y)\Bigr) = O(h^s),$$
$$F_{t\to h^{-1}\tau'}\Bigl(\chi_{\mathsf{T}}(t)\psi_2(x)u_h^1(x,y,t)\psi_1(y)\,{}^tQ'_y\Bigr) = O(h^s)$$

Indeed condition now are stronger than those of Proposition 3.3(iii).

Then these equalities and (3.24) imply (3.23). □

Proposition 3.8. *In the framework of Proposition 3.7*

$$|e_{T,h}^{1,\mathsf{T}}(x,y,\tau)| \le Ch^{1-d}\ell^{-1}\sigma\rho^{d-2} \tag{3.25}$$

with $\sigma = h^{\frac12-\delta'}\ell^{-\frac12}+\ell$.

Proof. Due to Propositions 3.1(ii) and 3.7 we need to estimate the right-hand expression of (3.24) with symbols of Q_1 and Q_2 supported in Ω, the union of $(C_1\ell, C_1\rho)$-vicinities of $(\bar w', \bar\xi'^{+})$ and $(\bar w', \bar\xi'^{-})$ intersected with

$$\{\xi\colon |b(x',\xi')-\tau|\le C\sigma^2\},$$

and with $\bar\chi_{\mathsf{T}}(t)$ replaced by $\chi_{\mathsf{T}}(t)=\bar\chi_{\mathsf{T}}(t)-\bar\chi_{\epsilon\mathsf{T}}(t)$. Observe that modulo $O(h^s)$

$$e_h^{1,\mathsf{T}}(x,y,\tau) \equiv h^{-1}\int_{-\infty}^{\tau} F_{t\to h^{-1}\tau'}\Bigl(\chi_{\mathsf{T}}(t)u_h^1(x,y,t)\Bigr)\,d\tau' = T^{-1}F_{t\to h^{-1}\tau}\Bigl(\beta_{\mathsf{T}}(t)u_h^1(x,y,t)\Bigr)$$

with $\beta(t)=t^{-1}\chi(t)$.

Since $\ell \asymp \ell^0$ it sufficient to prove it for $e^{\mathsf{T}}_{T,h}(x,y,\tau)$. Expressing $u_h(x,y,t) = \int e^{ih^{-1}t\tau}\, d_\tau e_h(x,y,\tau)$ we see that expression in question equals

$$\int \hat{\beta}\big(\frac{(\tau-\tau')\,T}{h}\big)\, d_{\tau'} Q_{2x} e_h(x,y,\tau')\, {}^t Q_{1y}$$

which does not exceed

$$C \sup_{\tau' :\, |\tau'-\tau|\le L} |Q_{2x} e_h(x,y,\tau' + hT^{-1},\tau')\, {}^t Q_{1y}| + C' h^s.$$

Since $e_h(x,y,\tau)$ is the Schwartz kernel of the orthogonal projector this does not exceed

$$C \sup_{\tau' :\, |\tau'-\tau|\le L} |Q_{2x} e_h(x,z,\tau' + hT^{-1},\tau')\, {}^t Q_{2z}|_{z=x}|^{\frac{1}{2}} \times |Q_{1z} e_h(z,y,\tau' + hT^{-1},\tau')\, {}^t Q_{1y}|_{z=y}|^{\frac{1}{2}} + C' h^s.$$

Applying Tauberian estimate for $x = y$ we see that the right-hand expression does not exceed

$$Ch^{1-d}\ell^{-1} \sup_{\tau' :\, |\tau'-\tau|\le L} \int_{\Sigma(x,\tau')\cap(\Omega\times\mathbb{R}_{\xi_1})} d\xi : d_\xi a.$$

The integral in the right-hand expression does not exceed $C\sigma\rho^{d-2}$. Finally, recall that $T \asymp \ell$. □

Taking $\rho = \max((h^{1-\delta'}\ell^{-1})^{\frac{1}{2}},\, \ell)$ to satisfy (3.16) and (3.18), and thus $\sigma = \rho$ we get

Corollary 3.9. *The following estimates hold as* $|x_1| + |y_1| \le C\sigma\ell$

$$e^{1,\mathsf{T}}_h(x,y,\tau) = O(h^{1-d}) \quad \textit{for} \quad \begin{cases} \ell \ge h^{\frac{1}{2}+\delta} & d \ge 4, \\ \ell \ge h^{\frac{1}{2}-\delta} & d = 3, \\ \ell \ge h^{\frac{1}{3}-\delta} & d = 2. \end{cases} \tag{3.26}$$

Remark 3.10. (i) Then $\sigma \ge h^{\frac{1}{3}-\delta'}$.

(ii) The similar arguments work under assumption $|x'-y'| \ge \epsilon|x_1| + |y_1|$ instead of $|x_1| + |y_1| \le C\sigma\ell$.

(iii) The similar but simpler arguments work for $e^{0,\mathsf{T}}_h(x,y,\tau)$ as $\rho^2\ell^0 \ge h^{1-\delta}$.

4 Successive approximations

4.1 Successive approximations inside domain

Standard successive approximations

In this Section we are going to apply method of successive approximations to derive asymptotics of $e_h(x,y,\tau)$ near boundary. However we start from the partial proof of Theorem 1.1 by this method away from the boundary. According to [Ivr2], Section 5.3 (which is our standard reference here), we consider problem for propagator $u_h(x,y,t)$

$$(hD_t - A)u_h = 0, \qquad u|_{t=0} = \delta(x-y)$$

with $A = A(x, hD_x, h)$ and rewrite for $u_h^{\pm}(x,y,t) := u_h(x,y,t)\theta(\pm t)$. We have equation

$$(hD_t - A)u_h^{\pm} = \mp ih\delta(x-y)\delta(t) \tag{4.1}$$

which we are going to solve by the successive approximations with unperturbed operator $\bar{A} = a(y, hD_x)$. Then (4.1) yields the equality

$$(hD_t - \bar{A})u_h^{\pm} = \mp ih\delta(x-y)\delta(t) + Ru_h^{\pm} \tag{4.2}$$

and hence

$$u_h^{\pm} = \mp ih\bar{G}^{\pm}\delta(x-y)\delta(t) + \bar{G}^{\pm}Ru_h^{\pm}, \tag{4.3}$$

where $\bar{G}^{\pm}$ and $G^{\pm}$ are parametrices of the problems

$$(hD_t - \bar{A})v = f, \qquad \operatorname{supp}(v) \subset \{\pm(t-t_0) \ge 0\} \tag{4.4}$$

and

$$(hD_t - A)v = f, \qquad \operatorname{supp}(v) \subset \{\pm(t-t_0) \ge 0\} \tag{4.5}$$

respectively with $\operatorname{supp}(f) \subset \{\pm(t-t_0) \ge 0\}$ for some $t_0 \in \mathbb{R}$; $R = A - \bar{A}$.

Moreover, equation (4.1) yields that

$$u_h^{\pm} = \mp ihG^{\pm}\delta(x-y)\delta(t). \tag{4.6}$$

Iterating (4.3) N times and then substituting (4.6) we obtain the equality

$$u_h^{\pm} = \mp ih \sum_{0\le n\le N-1} (\bar{G}^{\pm}R)^n\bar{G}^{\pm}\delta(x-y)\delta(t) \mp ih(\bar{G}^{\pm}R)^nG^{\pm}\delta(x-y)\delta(t). \tag{4.7}$$

Finally, we apply ${}^tQ_y = {}^tQ(y, hD_y)$ to the right of (4.7), where Q is an operator with compactly supported symbol, equal 1 as $a(y,\xi) \le \tau + \epsilon$. After this cut-off, according to [Ivr2], Section 5.3, norms of the terms in (4.7) in the strip $\{(x,t)\colon |t| \le T\}$ do not exceed $Ch^{-M-n}T^{2n}$ and as $T \le h^{\frac12+\delta}$ we can ignore the remainder term. Since we need to consider $T \le c\ell(x,y)$ in the end we arrive to

$$\max(c\ell,\ h^{1-\delta}) \le T \le h^{\frac12+\delta}. \tag{4.8}$$

Then as it was shown (see [Ivr2], simplified (4.3.33))

$$\begin{aligned} &F_{t\to h^{-1}\tau}\Big(\bar\chi_T u_h^{\pm}(x,y,t)\,{}^tQ_y\Big) \\ &\equiv \mp i \sum_{0\le n\le N} (2\pi)^{-d-1} h^{-d+n} \int e^{ih^{-1}\langle x-y,\xi\rangle} F_n(y,\xi,\tau)[q_2(y,\xi,h)]\,d\xi, \quad \pm \operatorname{Im}\tau < 0, \end{aligned} \tag{4.9}$$

where $F_n = \sum_{\beta\colon |\beta|\le n} F_{n,\beta}\partial_\xi^\beta$ are differential operators applied to q_2, with coefficients $F_{n\beta}$ holomorphic with respect to ξ,τ as $\operatorname{Im}\tau \ne 0$, and have poles as $\tau \in \mathbb{R}$; denominators are $(\tau - a(y,\xi))^{2n+1+|\beta|}$. Then

$$\begin{aligned} &F_{t\to h^{-1}\tau}\Big(\bar\chi_T u_h(x,y,t)\Big) \\ &\qquad \equiv \sum_{0\le n\le N} (2\pi)^{-d-1} h^{-d+n} \int e^{ih^{-1}\langle x-y,\xi\rangle} \mathcal{F}_n(y,\xi,\tau)\,d\xi, \qquad \tau\in\mathbb{R} \end{aligned} \tag{4.10}$$

where $\mathcal{F}_n(y,\xi,\tau) = i\big(F_{n0}(y,\xi,\tau - i0) - F_{n0}(y,\xi,\tau+i0)\big)$. Then the principal term in $e^{\mathsf{T}}_{T,h}(x,y,\tau)$ can be rewritten as $e^{\mathsf{W}}_h(x,y,\tau)$ while all other terms as

$$h^{-d+n-|\alpha|} \sum_{\alpha\colon |\alpha|\le 2n-1} (x-y)^\alpha \int_{\Sigma(y,\tau)} W_{n,\alpha}(y,\xi) e^{ih^{-1}\langle x-y,\xi\rangle}\,d\xi \tag{4.11}$$

with $n \ge 1$ and smooth $W_{n,\alpha}$.

Indeed, taking ξ-partition and using ξ-microhyperbolicity, we can rewrite on each element $\mathcal{F}_n$ as a sum of $W'_{n,\alpha}(y,\xi)\partial_\xi^\alpha \delta'(\tau - a(y,\xi))$. Then, integrating by parts we will get (4.11) for terms in $e^{\mathsf{T}}_{T,h}(x,y,\tau)$.

Now due to strong convexity and stationary phase principle these terms do not exceed

$$Ch^{-d+n-|\alpha|}(h/\ell)^{\frac{d-1}{2}}\ell^{|\alpha|} \le Ch^{-\frac{d+1}{2}+n-|\alpha|}\ell^{-\frac{d-1}{2}+|\alpha|}, \tag{4.12}$$

which is $O(h^{1-d})$ as $d \ge 3$ and $O(h^{-\frac32}\ell^{\frac12})$ as $d = 2$. Recall that $\ell \le h^{\frac12+\delta}$.

Improved successive approximations

To improve this approach we need to freeze symbol at point $w = \frac{1}{2}(x+y)$ and to do this let us observe that

$$\begin{aligned}&\bigl(hD_t - A(x, hD_x, h)\bigr)u_h(x,y,t) = 0,\\ &\bigl(hD_t - A(y, -hD_y, h)\bigr)u_h(x,y,t) = 0, \qquad u_h|_{t=0} = \delta(x-y)\end{aligned}$$

and therefore denoting $v_h(w,z,t) = u_h(w+\frac{1}{2}z, w-\frac{1}{2}z, t)$ we have

$$\bigl(hD_t - \mathfrak{A}(w,z,hD_z,hD_w,h)\bigr)v_h(w,z,t) = 0, \qquad v_h|_{t=0} = \delta(z) \tag{4.13}$$

with

$$\mathfrak{A} = \frac{1}{2}\bigl(A(w+\frac{1}{2}z, hD_z + \frac{1}{2}hD_w, h) + A(w-\frac{1}{2}z, hD_z - \frac{1}{2}hD_w, h)\bigr). \tag{4.14}$$

Taking as unperturbed operator

$$\bar{\mathfrak{A}} = A(w, hD_z, h) \tag{4.15}$$

we get $R = \mathfrak{A} - \bar{\mathfrak{A}}$ a a sum of $z^\alpha(h\partial_w)^\beta$ with $|\alpha| + |\beta| \ge 2$.

With this modification we can write formula, similar to (4.7). Due to the propagation results we can add to R factor $\bar{\chi}_{cT}(h\partial_w)$ and then each new term in this formula adds an extra factor $C(T^2+h)Th^{-1}$ instead of CT^2h^{-1} and we can replace (4.8) by a weaker condition

$$\max(c\ell,\ h^{1-\delta}) \le T \le h^{\frac{1}{3}+\delta}. \tag{4.16}$$

Calculations, similar to those in the standard method lead us to the main term $e^{\mathsf{W}}(x,y,\tau)$ and to (4.11) replaced by

$$h^{-d+n-|\alpha|}\sum_{\alpha:\ 2|\alpha|\le 3n-1}(x-y)^\alpha \int_{\Sigma(w,\tau)} W_{n,\alpha}(w,\xi)e^{ih^{-1}\langle z,\xi\rangle}\,d\xi. \tag{4.17}$$

and to the same estimate (4.12) albeit with a restrictions $|\alpha| \le (3n-1)/2$, $\ell \le h^{\frac{1}{3}}$, and we get $O(h^{1-d})$ as $d \ge 1$.

4.2 Successive approximations near boundary

Standard successive approximations

We know (see [Ivr2], Section 7.2) that $u_h^0(x,y,t)$ and then $u_h^1(x,y,t)$ for $|t| \le T = h^{\frac{1}{2}+\delta}$, $x_1 + y_1 \le C_0 T$ could be constructed by the method of

successive approximations with unperturbed operator $\bar{A} = a(0, y'; hD_x)$ (so the principal part of A is frozen at point $(0, y')$): it leads us to the expression for $e_h^{\mathsf{T}}(x, y, \tau)$.

In this case unperturbed term would be as for $\bar{A}$ in the half-space and it will not exceed $Ch^{-d}(h/\ell)^{d+1/2}$ and and one can prove that the perturbed term would acquire factor T^2/h with $T \asymp \ell$ and it does not exceed (4.12) (with redefined $\ell(x, y)$ now) and it is $O(h^{1-d})$ as $d \geq 3$ and $Ch^{-\frac{3}{2}}\ell^{\frac{1}{2}}$ as $d = 2$.

Since we assumed that the boundary conditions are either Dirichlet or Neumann, we arrive to the expression (1.3) for $e_h^{1,\mathsf{T}}(x, y, \tau)$ as described in Theorem 1.2.

Improved successive approximations

Let us improve this construction in the same manner as we did inside domain. Consider problem for u_h^1:

$$(hD_t - A)u_h^1 = 0, \qquad u_h^1|_{t=0} = 0, \tag{4.18}$$

$$\eth B u_h^1 = -\eth B u_h^0|_{x\in\partial X}, \tag{4.19}$$

where $u_h^0(x, y, t)$ satisfies the same equation in the "whole space" and $\eth$ is an operator restriction to $\partial X \ni x$. Then

$$(hD_t - A)u_h^{1,\pm} = 0, \tag{4.20}$$

$$\eth B u_h^{1\pm} = -\eth B u_h^{0\pm} \tag{4.21}$$

where again $u^{j\pm} = \theta(\pm t)u_h^j(x, y, t)$ and

$$(hD_t - \bar{A})u_h^{1\pm} = Ru^{1\pm}, \tag{4.22}$$

$$\eth B u_h^{1\pm} = -Bu_h^{0\pm}, \tag{4.23}$$

where $u_h^{0,\pm}$ satisfies (4.2); recall that $B = I$ or $B = D_1$ for Dirichlet and Neumann boundary conditions correspondingly[4] Then like in [Ivr2], Section 7.2

$$u_h^{1\pm} = -\bar{G}'^{\pm}\eth B u_h^{0\pm} + \bar{G}^{\pm}Ru_h^{1\pm}, \tag{4.24}$$

where $\bar{G}^{\pm}$ are parametrices for the problems

$$(hD_t - \bar{A})v = f, \qquad \eth Bv = 0, \qquad \mathsf{supp}(v) \subset \{\pm(t - t_0) \geq 0\} \tag{4.25}$$

[4] For more general boundary conditions we would need to replace B by $\bar{B} + R_1$.

and $\bar{G}'^{\pm}$ are parametrices for the problems

$$(hD_t - \bar{A})v = 0, \qquad \eth Bv = f, \qquad \operatorname{supp}(v) \subset \{\pm(t - t_0) \geq 0\} \tag{4.26}$$

respectively with $\operatorname{supp}(f) \subset \{\pm(t - t_0) \geq 0\}$ for some $t_0 \in \mathbb{R}$; $R = A - \bar{A}$. Moreover,

$$u_h^{1\pm} = -G'^{\pm}\eth Bu_h^{0\pm} \tag{4.27}$$

where $G^{\pm}$ and $G'^{\pm}$ are parametrices for the problems (4.25) and (4.26) but for operator A. Iterating (4.24) N times and then substituting (4.27) we arrive to formula similar to (4.7)

$$u_h^{1\pm} = -\sum_{0\leq n\leq N-1} (\bar{G}^{\pm}R)^n \bar{G}'^{\pm}\eth Bu_h^{0\pm} - (\bar{G}^{\pm}R)^N G'^{\pm}\eth Bu_h^{0\pm}. \tag{4.28}$$

What is more, we plug $u_h^{0\pm}$ given by (4.7) with $\bar{G}^{\pm}$ and $G^{\pm}$ replaced by $\bar{G}^{0\pm}$ and $G^{0\pm}$ which are parametrices for (4.4) and (4.5) in the "whole space".

Finally, we apply ${}^tQ_1 = {}^tQ(y_1, z', hD'_z)$ to the right of (4.27) and (4.7), where Q_1 is an operator with the symbol, supported in

$$\Omega_{T,\rho,\sigma} := \{(x_1, z', \zeta) \colon |z'| \leq T,\ |\zeta - \bar{\xi}'| \leq \rho,\ |x_1| \leq \sigma T\} \tag{4.29}$$

where $|a(0, \bar{x}', 0, \bar{\xi}') - \tau)| \leq c\rho^2$. Then due to Proposition 3.3, under assumptions (3.11) and $(3.10)_{1,2}$ we can insert $Q_2(x_1, hD'_z)$ to the right of each copy of R with symbol supported in $\Omega_{3T,3\rho,3\sigma}$ and equal 1 in $\Omega_{2T,2\rho,2\sigma}$.

So far we we followed exactly [Ivr2], Section 7.2, except there we had $\rho = c$. However now instead of $\bar{A} = a(0, y'; hD_x)$ we take

$$\bar{A} = a(w, hD_x), \qquad w = (0, w'), \qquad w' = \frac{1}{2}(x' + y'). \tag{4.30}$$

Then, the norm of $\bar{G}^{\pm}RQ_2$ does not exceed $C(T^2 + \sigma T + h)Th^{-1}$ and we can replace (4.8) by a weaker condition $(T^2 + \sigma T + h)T \leq h^{1+\delta}$, which is equivalent to (4.16) plus

$$\sigma\ell^2 \leq h^{1+\delta}. \tag{4.31}$$

Remark 4.1. Recall that σ is defined by (3.14) and with $\rho = h^{1-\delta}\ell^{-1}$ condition (4.31) is also equivalent to (4.16). However, for us it is more important which value of the reflection angle is allowed in this method.

Thus we arrive to the following statement:

Proposition 4.2. *Let $Q = Q(x_1, z', hD'_z)$ be an operator with the symbol $q(x_1, z', \zeta')$ supported in $\Omega_{T,\rho,\sigma}$ where condition* (3.11) *is fulfilled and let conditions* (4.16) *and* (4.31) *be also fulfilled.*

Then we can skip remainder terms in both (4.27) *and modified* (4.7)*, which are $O(h^s)$, leaving us only with $\bar{G}^{\pm}$, $\bar{G}'^{\pm}$ and $\bar{G}^{0\pm}$.*

Thus (4.28) without the last term becomes an asymptotic series. Now let us calculate $e_h^{1,\mathsf{T}}(x, y, \tau)$ under these assumptions.

Proposition 4.3. *In the framework of Proposition 4.2*

$$
\begin{aligned}
(4.32)\quad & F_{t\to h^{-1}\tau}\Big(\bar{\chi}_T(t)u_h^{1\pm\,{}^t}Q_1\Big)\\
&\sim \sum_{n\ge 0} h^{-d+n}\int T\hat{\bar{\chi}}\big(\frac{(\tau-\tau')T}{h}\big)\,d\tau'\int d\xi'\int_{\gamma_+} d\xi_1\int_{\gamma_-} d\eta_1\\
&\qquad\qquad \times e^{ih^{-1}(x_1\xi_1-y_1\eta_1+\langle x'-y',\xi'\rangle)}F_n(w,\xi',\xi_1,\eta_1,\tau')[q(w,\xi')]
\end{aligned}
$$

for $\tau\in\mathbb{C}_{\mp}$, where $\gamma_{\pm}$ are closed contours in $\mathbb{C}_{\pm}$ passing once around the poles of $\big(\tau-a(z,\xi_1)\big)^{-1}$ lying in $\mathbb{C}_{\pm}$ and not passing around the poles lying in $\mathbb{C}_{\mp}$; for real z and $\operatorname{Im}\tau\neq 0$ there is no real pole, and F_n are sums of the terms

$$
\begin{aligned}
(4.33)\quad & \xi_1^j\eta_1^k W_{n,l,m,p,j,k}(w,\xi',\tau)\\
&\qquad\times\big(\tau-a(w,\xi_1,\xi')\big)^{-l}\big(\tau-a(w,\eta_1,\xi')\big)^{-m}\big(\tau-b(w,\xi')\big)^{-p}
\end{aligned}
$$

with

$$
b(w,\xi') = a(w,0,\xi'), \tag{4.34}
$$

$$
l\ge 1,\ m\ge 1,\ 2(l+m+p)\le 3n+j+k+3,\ j\le l,\ k\le m \tag{4.35}
$$

and uniformly smooth $W_{n,l,m,p,j,k}$.

Proof. Proof follows the proof of Theorem 7.2.13 of [Ivr2] with the some modifications, mainly introduced in Subsection 4.1. We rewrote A as $\mathfrak{A}$ by (4.14) and defined $\bar{\mathfrak{A}}$ by (4.15) with the exception that w' and z' are now $(d-1)$-dimensional variables (see (4.30)) and we preserve x_1 and y_1.

The main part of the perturbation R is quadratic with respect to z', hD'_w and linear with respect to x_1. In comparison with (7.2.60) of [Ivr2] formula (4.33) is simpler, because we have scalar factors here and products collapse into powers. On the other hand, instead of claiming that W of that proof are holomorphic satisfying (7.2.38) of [Ivr2] here we do it in much explicit way since now we need to follow very carefully what are relations between j, l, l, m, p and n.

We have operators $a(w, hD_1, hD'_z, hD'_w)$ with symbols which do not depend on x. Let us make h-Fourier transform by x' and t.

Consider first $u^0_h(x, y, t)$. We claim that

$$\begin{aligned}(4.36)\quad & F_{t\to h^{-1}\tau}\Big(\bar\chi_T(t)\eth(hD_{x_1})^r u^{0\pm}_h {}^tQ_1\Big)\\ &\sim \sum_{n\ge 0} h^{1-d+n}\int T\hat{\bar\chi}\big(\frac{(\tau-\tau')T}{h}\big)\,d\tau'\int d\xi'\int_{\gamma_-} d\eta_1\\ &\qquad\qquad \times e^{ih^{-1}(-y_1\eta_1+\langle x'-y',\xi'\rangle)}F^0_n(w,\xi',\eta_1,\tau')[q(w,\xi')]\end{aligned}$$

for $\tau\in\mathbb{C}_\mp$, $r=1$ and F^0_n are sums of the terms

$$(4.37)\qquad \eta_1^k W^0_{n,m,j,k}(w,\xi',\tau)\big(\tau-a(w,\xi_1,\xi')\big)^{-l}\big(\tau-a(w,\eta_1,\xi')\big)^{-m}$$

with $2m\le 3n+k+2-r$ and $k<m$ with the only exception $n=0$, $m=1$, $k=r=1$.

(a) Like in that proof of [Ivr2], each factor z_r and hD_{w_r} with $r=2,\dots,d$ is moved to the right towards $\delta(z')$, using commutator relations

$$(4.38)\qquad z_r\bar G^{0\pm}=\bar G^{0\pm}z_r-\bar G^{0\pm}[\bar{\mathfrak{A}},z_r]\bar G^{0\pm}$$

which also hold for hD_{w_r}. So, each such commuting adds factor h and increases either m or p by 1. However, each pair of those factors are accompanied by $\bar G^{0\pm}$, and therefore in this process increment of the power of h by 1 is "paid" by increment of m with the factor $3/2$ [5)]. On the other hand, commuting with ${}^tQ_1$, each increment power of h by 1 is "paid" by increment of $(m+p)$ with the factor 1.

[5)] Compare with the standard method, when the factor was 2.

(b) Also recall that each factor x_1 is moved to the left, towards $\eth B$, also using (4.38) but for x_1. In this commutator factor $h\eta_1$ is also gained and m is increased by 1 and also $\bar{G}^{0\pm}$ is applied so in the end m is increased by 2. When x_1 reaches $\eth B$ the term disappears for Dirichlet boundary condition, or this factor cancels with $B = h\partial_{x_1}$ for Neumann boundary condition (and factor h is gained). process each increment power of $h\eta_1$ by 1 is "paid" by increment of m with the factor 2. Therefore (4.36)–(4.37) has been proven.

(c) Then for

$$F_{t\to h^{-1}\tau}\Bigl(\bar{\chi}_T(t)\bar{G}'^{\pm}\eth B u_h^{0\pm\,t}Q_1\Bigr) \tag{4.39}$$

formula (4.33) holds with $l = 1$, $p = 0$ and $j = 1, 0$ for Dirichlet and Neumann boundary conditions respectively.

(d) Again each factor z_r and hD_{w_r} with $r = 2, \dots, d$ is moved to the right towards $\delta(z')$, using commutator relations (4.38) for $\bar{G}^{\pm}$ rather than for $\bar{G}^{0\pm}$ and also

$$z_r\bar{G}'^{\pm} = \bar{G}'^{\pm}z_r - \bar{G}^{\pm}[\bar{\mathfrak{A}}, z_r]\bar{G}'^{\pm} \tag{4.40}$$

which holds also for hD_{w_r}. Observe that that p can increase but again as in (a) increment of the power of h by 1 is "paid" by increment of $m + p$ with the factor $3/2$ [6].

(e) Also recall that each factor x_1 which is to the left from $\bar{G}'^{\pm}\eth B$ is moved to the right, towards $\bar{G}'^{\pm}\eth B$ but instead of (4.38) we use

$$x_1\bar{G}_*^{\pm} = \bar{G}_{\mathsf{D}}^{\pm}x_1 - \bar{G}_{\mathsf{D}}^{\pm}[\bar{\mathfrak{A}}, x_1]\bar{G}_*^{\pm}, \tag{4.41}$$

$$x_1\bar{G}_*'^{\pm} = -\bar{G}_{\mathsf{D}}^{\pm}[\bar{\mathfrak{A}}, x_1]\bar{G}_*'^{\pm} \tag{4.42}$$

with $* = \mathsf{D}, \mathsf{N}$ (in particular, for $\bar{G}_{\mathsf{D}}$ (4.38) holds).

Again, similarly to (c) either $l+p$ is increased by 2 and j does not change, or j is decreased by 1 and $l + p$ is increased by 1.

[6] Indeed, it is sufficient to consider boundary value problem for ODE $(\tau - D^2)v = f$ with $f(x) = \int_{\gamma_+}\xi^j(\tau-\xi^2)^{-l}e^{ix\xi}\,d\xi$. One can see easily that

$$v(x) = \int_{\gamma_+}\Bigl(\xi^j(\tau-\xi^2)^{-l-1} - \kappa\xi^{j'}\tau^{(j-j')/2-l}(\tau-\xi^2)^{-1}\Bigr)e^{ix\xi}\,d\xi$$

satisfies this equation and $v(0) = 0$ for some constant $\kappa = \kappa_{\mathsf{D}jl}$ and $j' \equiv j \mod 2$ and $j' = 0, 1$. Also $v'(0) = 0$ with some other constant $\kappa = \kappa_{\mathsf{N}jl}$.

(f) Then we have $n = n' + j + k$ with $l + p = l' + 2j$, $m = m' + 2k$ and $l' + m' \le \frac{3}{2}n' + 2$. Those are "the worst case scenarios" when we consider the main part of perturbation R, quadratic by $(x' - y')$ and linear by x_1. For the rest of perturbation increments of $l + p, m$ are relatively smaller. In the end we arrive to (4.35).

(g) One can see easily that for the main term we have $l = m = 1$, $p = 0$ and $j + k = 1$ with $j = 0, 1$ for Dirichlet and Neumann problems respectively and $W_* = \mathsf{const}$. □

Proposition 4.4. *In the framework of Proposition 4.2*

$$\begin{aligned}(4.43)\quad & F_{t\to h^{-1}\tau}\Bigl(\bar\chi_T(t)u_h^{1\pm\, t}Q_1\Bigr)\\ &\sim \sum_{n\ge 0, j\ge 0, k\ge 0} h^{-d+n}\int T\hat{\bar\chi}\bigl(\frac{(\tau-\tau')T}{h}\bigr)\,d\tau'\int d\xi'\int_{\gamma_+} d\xi_1\\ &\qquad\qquad \times e^{ih^{-1}((x_1+y_1)\xi_1+\langle x'-y',\xi'\rangle)}F_{n,j,k}(w,\xi',\xi_1,\tau')[q(w,\xi')]\end{aligned}$$

for $\tau \in \mathbb{C}_\mp$ where $F_{n,j,k}$ are sums of the terms

$$(4.44)\qquad \xi_1^j\eta_1^k W_{n,l,j,k}(w,\xi',\tau)\bigl(\tau - a(y',\xi_1,\xi')\bigr)^{-l}$$

with $1 \le l \le \frac{3}{2}n + j + k + 1$ and uniformly smooth $W_{n,l,j,k}$ and also

$$\begin{aligned}(4.45)\quad & F_{t\to h^{-1}\tau}\Bigl(\bar\chi_T(t)u_h^{1\, t}Q_1\Bigr)\\ &\sim \sum_{n\ge 0, j\ge 0, k\ge 0} h^{-d+n}\int T\hat{\bar\chi}\bigl(\frac{(\tau-\tau')T}{h}\bigr)\,d\tau'\int d\xi\\ &\qquad\qquad \times e^{ih^{-1}((x_1+y_1)\xi_1+\langle x'-y',\xi'\rangle)}\mathcal{F}_{n,j,k}(w,\xi',\xi_1,\tau')[q(w,\xi')],\end{aligned}$$

where $\mathcal{F}_{n,j,k}(w,\xi',\xi_1,\tau') = F_{n,j,k}(w,\xi',\xi_1,\tau' - i0) - F_{n,j,k}(w,\xi',\xi_1,\tau' + i0)$.

Proof. (4.43)–(4.44) follows from Proposition 4.3 by calculating residues at zeroes of $(\tau - a(x,\xi_1,\xi'))$ and $(\tau - a(x,\eta_1,\xi'))$ in (4.33) and (4.43).

Next we observe that in (4.43) we can replace integral along γ_+ by integral along $\mathbb{R}$. Then (4.45) follows trivially. □

Proposition 4.5. *Let ξ-microhyperbolicity condition be fulfilled at energy level τ and let*

$$\nu(x)+\nu(y)\le \sigma\ell \tag{4.46}$$

with

$$\ell(x,y)\le h^{\frac{1}{3}+\delta},\quad \sigma = h^{\frac{1}{2}+\delta}\ell^{-\frac{1}{2}}. \tag{4.47}$$

Let $T = T^*\ge C_0\ell(x,y)$. Then for any T: $T^*\le T\le \epsilon$

(i) As $d\ge 3$ estimate

$$|e^{1,\mathsf{T}}_{T,h}(x,y,\tau) - [\mp]e^{0,\mathsf{W}}_h(x,\tilde y,\tau)|\le Ch^{1-d} \tag{4.48}$$

with sign $[\mp]$ for Dirichlet and Neumann boundary conditions respectively.

(ii) As $d=2$ estimate

$$|e^{1,\mathsf{T}}_{T,h}(x,y,\tau) - [\mp]e^{0,\mathsf{W}}_h(x,\tilde y,\tau) - e_{h,\mathsf{corr}}(x,y,\tau)|\le Ch^{-1} \tag{4.49}$$

with $e_{h,\mathsf{corr}}(x,y,\tau)$ given by (4.54) *below.*

Proof. (a) Consider first $\ell\le h^{\frac{1}{2}-\delta}$. Let us take $\rho = \min((h/\ell)^{\frac{1}{2}}h^{-\delta}, C_2)$. Then $\rho\ge C_0\ell$, $\sigma=\rho^{\frac{1}{2}}$ and

$$\sigma\ell^2\le h^{1+\delta}\qquad \text{as}\quad \ell\le h^{\frac{3}{7}+\delta} \tag{4.50}$$

and the method of successive approximations works. Let us take take ρ-admissible partition of unity by ζ' in ρ-vicinity of $\Sigma(w,\tau)$. In this case $\rho\ge\ell$ and $\sigma=\rho^{\frac{1}{2}}$. Moreover, as $\rho = C_2$ it covers the whole domain $\{\zeta'\colon b(w,\zeta')< 2\tau\}$ and as $\rho<C_2$ and therefore $\ell\ge h^{1-2\delta}$ and also $\rho^2\ell\ge h^{1-2\delta}$ we can use the fact that

(4.51) If $\rho\ell\ge h^{1-\delta}$, $\rho\ge C_0\ell^2$, (3.11) is fulfilled and Q_1 is an operator with the symbol equal 0 in ρ-vicinity of $\Sigma(w,\tau)$ and $\ell\ge h^{1-\delta}$ then

$$e^{1,\mathsf{T}}_h(x,y,\tau)\,{}^tQ_{1,z} = O(h^s)$$

which follows from Section 3. Therefore we can take cut-off operator $Q_1 = I$.

Then the main term of the final expression equals to $e_h^{0,\mathsf{W}}(x,\tilde{y},\tau)$ and is $\asymp h^{-d}(h\ell^{-1})^{(d+1)/2}$ and n-th term does not exceed

$$Ch^{-d-\delta'}\bigl(\frac{h}{\ell}\bigr)^{(d+1)/2}\times\bigl(\rho^{\frac{1}{2}}\ell^2h^{-1}\bigr)^{n-1} \tag{4.52}$$

due to the same microlocal arguments as above. One can see easily that expression (4.52) is $O(h^{1-d+\delta''})$ as $d\ge 2$ and $n\ge 3$ and $\ell\le h^{\frac{1}{3}+\delta}$. Furthermore, if we consider terms $O(\ell^2)$ in R then the corresponding second term does not exceed

$$Ch^{-d-\delta'}\bigl(\frac{h}{\ell}\bigr)^{(d+1)/2}\times\ell^3h^{-1})^{n-1} = O(h^{1-d+\delta''}). \tag{4.53}$$

Therefore we need to consider only the second term in the successive approximations. It is equal to

$$\begin{aligned}&[\mp]e_{h,\mathsf{corr}}(x,y,\tau)\\ &\qquad := -\frac{1}{2}(2\pi h)^{-d}\int_{\Sigma(w,\tau)}\lambda(w',\xi')(x_1+y_1)e^{ih^{-1}\langle x-\tilde{y},\xi\rangle}\,d\xi : d_\xi a(w,\xi)\end{aligned} \tag{4.54}$$

with

$$\lambda(w',\xi') = a_{x_1}(x_1,w',\xi)|_{x_1=\xi_1=0}. \tag{4.55}$$

Indeed, one can see easily that the second term in the successive approximation for $[\mp]F_{t\to h^{-1}\tau}\bigl(\bar{\chi}_T(t)u_h^{1\pm}(x,y,t)\bigr)$ is equal to

$$\begin{aligned}&\pm 2h^3(2\pi h)^{-1-d}\int_{\gamma_+\times\mathbb{R}}\lambda(w',\xi')\xi_1(\tau-a(x,\xi))^{-3}e^{i\langle x-\tilde{y},\xi\rangle}\,d\xi\\ &=\mp\frac{i}{2}h^2(2\pi h)^{-1-d}\int_{\gamma_+\times\mathbb{R}}\lambda(w',\xi')(x_1+y_1)(\tau-a(x,\xi))^{-2}e^{i\langle x-\tilde{y},\xi\rangle}\qquad \mp\mathsf{Im}\,\tau<0;\end{aligned}$$

then the second term in decomposition for $[\mp]F_{t\to h^{-1}\tau}\bigl(\bar{\chi}_T(t)u_h^1(x,y,t)\bigr)$ is equal to

$$-\frac{1}{2}h(2\pi h)^{-d}\int\lambda(w',\xi')(x_1+y_1)e^{ih^{-1}\langle x-\tilde{y},\xi\rangle}\delta'(\tau-a(w,\xi))\,d\xi$$

which implies (4.55).

(b) Due to stationary phase integral in 4.54) does not exceed $Ch^{-d}\bigl(\frac{h}{\ell}\bigr)^{(d-1)/2}$, and expression (4.54) is $O(h^{1-d})$ as $d \ge 3$, $x_1 + y_1 \le \sigma\ell$.

Due to Corollary 3.9 as $d \ge 3$ we need to consider only this case $\ell \le h^{\frac{1}{2}-\delta}$. Therefore Statement (i) has been proven.

(c) Let $d = 2$. Then we need to consider also $h^{\frac{1}{2}-\delta} \le \ell \le h^{\frac{1}{3}+\delta}$ and with selected $\rho = (h/\ell)^{\frac{1}{2}}h^{-\delta}$ condition $\sigma\ell^2 \le h^{1+\delta}$ may fail. We need to consider lesser ρ. Let us pick up $\rho = h^{1-\delta}\ell^{-1}$. Then $\rho \ge C_0\ell^2$ and $\sigma\ell^2 \le h^{1+\delta}$ as $\ell \le h^{\frac{1}{3}-\delta}$.

However condition $\rho \ge C_0\ell$ may fail and then we need to take care of (3.11) at each point of $\Sigma(w,\tau) \cap \{\xi_1 = 0\}$.

The good news is that this set consists of two points $\bar{\xi}'^{\pm}$ and we do not need $\rho^2\ell \ge h^{1-\delta}$ because we can deal with each of these two points separately. Therefore we can replace ρ-admissible operator Q_1 with the symbol, equal 1 in ρ-vicinity of $\bar{\xi}'^{\pm}$ by operator with the symbol equal 1 in ϵ-vicinity of this point. Observing that the required gauge transformations are $\xi_2 \mapsto \xi_2 - h^{-1}\alpha_{\pm}z_2$, that are multiplications by $e^{ih^{-1}\alpha_{\pm}z_2^2/2}$ which do not affect $e(x,y,\tau)$ as $z_2 = \frac{1}{2}(x_2 - y_2)$. Therefore again we can take $Q_2 = I$.

Then n-th term does not exceed (4.52) which is $O(h^{-1+\delta''})$ as $n \ge 3$ in current settings. Again, if we consider $O(\ell^2)$ terms in R then the corresponding second term does not exceed (4.53). Thus again we are left with expression (4.54) but this time we cannot claim that it is $O(h^{-1})$ even for $\ell \le h^{\frac{1}{2}+\delta}$ and $\rho = h^{1-\delta'}\ell^{-1}$. Therefore Statement (ii) has been proven. □

5 Geometric optics method

5.1 Constructing solution

Proposition 5.1. *Let*

$$\nu(x) + \nu(y) \ge C_0\ell^2(x,y) \tag{5.1}$$

and $\ell(x,y) \le \epsilon$. TThen on the energy level τ there exists a single generalized Hamiltonian billiard of the length $\le c\epsilon$ with at least one reflection from ∂X, from y to x and one from x to y, and each has exactly one reflection and the incidence angles are $\asymp \bigl(\nu(x) + \nu(y)\bigr)\ell^{-1}(x,y)$ (so, they are standard Hamiltonian billiards*).*

Proof. The proof is obvious. □

Remark 5.2. Assumption (5.1) is essential. Indeed, if ∂X is strongly concave[7)] then for $\nu(x)+\nu(y)\le C_0\ell^2(x,y)$ there could be multiple reflections and if ∂X is strongly convex[7)] then the incidence angle may be 0. In the former case there could be also multiple reflections and multiple billiard rays from x to y on the same energy level τ while in the latter case there could be no rays at all.

Proposition 5.3. *(i) Let A be ξ-microhyperbolic on the energy level $\tau = a(y,\theta)$ and*

$$a(x,0) < \tau - \epsilon.$$

Let $\varphi^0(x,y,\theta)$ be a solution of the stationary eikonal equation (2.7) *satisfying* (2.8) *and* (2.9)*. Then*

$$|\partial_x^\alpha\partial_y^\beta\partial_\theta^\gamma\varphi^0(x,y,\theta)| \le C_{\alpha\beta\gamma}\begin{cases} T & \alpha=\beta=0,\\ 1 & |\alpha|+|\beta|\ge 1\end{cases} \qquad \forall x,y\colon |x-y|\le T \tag{5.2}$$

provided $T\le\epsilon$ with sufficiently small constant $\epsilon>0$.

(ii) Let $\varphi(x,y)$ be another solution of the same equation satisfying

$$\varphi|_{x_1=0} = \varphi^0|_{x_1=0}, \qquad \partial_{x_1}\varphi|_{x_1=0} = -\partial_{x_1}\varphi^0|_{x_1=0}. \tag{5.3}$$

Assume that

$$\sigma = |\partial_{x_1}\varphi|_{x_1=0}| \ge C_0 T. \tag{5.4}$$

Then as $|x-y|\le T$ the following inequalities hold:

$$|\partial_x^\alpha\partial_y^\beta\partial_\theta^\gamma\varphi| \le C_{\alpha\beta\gamma}\begin{cases} T & \alpha=\beta=0,\\ 1 & \alpha_1\le 1,\\ \sigma^{3-\alpha_1-|\alpha|-|\beta|-|\gamma|} & \alpha_1\ge 2.\end{cases} \tag{5.5}$$

(iii) Furthermore,

$$|\partial_x^\alpha\partial_y^\beta\partial_\theta^\gamma\varphi| \le C_{\alpha\beta\gamma}\sigma^{4-\alpha_1-|\alpha|-|\beta|-|\gamma|} \qquad \alpha_1\ge 2,\ |\gamma'|\ge 1. \tag{5.6}$$

[7)] With respect to Hamiltonian trajectories.

Proof. (a) Consider first $\varphi = \varphi^0$. Then the left-hand expression of (5.2) does not exceed $C_{\alpha\beta\gamma}$ and we need to consider only $\alpha = \beta = 0$. Applying ∂_θ^γ to eikonal equation we see that $\frac{d}{dt}\partial_\theta^\gamma\varphi$ is bounded where here and below

$$\frac{d}{dt} \coloneqq \bigl(\partial_t - \sum_k a^{(k)}(x, \nabla_x\varphi)\bigr). \tag{5.7}$$

Since $|\partial_\theta^\gamma\varphi^0(x,y,t,\theta)| \le C_\gamma T$ as $t = 0$ and $|x-y| \le T$ we conclude that the same is true for $|t| \le T$.

(b) First, we provide this estimate as $x_1 = 0$; from (5.2) and (5.3) we conclude that (5.5) holds as $\alpha_1 = 0, 1$. Consider

$$\varphi_{x_1}^2 = a(y,\theta) + b(x, \nabla_{x'}\varphi), \tag{5.8}$$
$$b(x, \nabla_{x'}\varphi) \coloneqq -V - \sum_{j\ge 2} g^{jk}(\varphi_{x_j} - V_j)(\varphi_{x_k} - V_k).$$

Differentiating once by x_1 we get

$$2\varphi_{x_1}\varphi_{x_1x_1} = b_{x_1}(x, \nabla_{x'}\varphi) + \sum_{k\ge 2} b^{(k)}(x, \nabla_{x'})\varphi_{x_kx_1} \tag{5.9}$$

which in virtue of (5.5) with $\alpha_1 \le 1$ does not exceed C and therefore

$$|\varphi_{x_1x_1}| \le C\sigma^{-1}. \tag{5.10}$$

Further, applying to (5.8) $\partial_x^{\alpha-\varepsilon}\partial_y^\beta\partial_\theta^\gamma$ with $\varepsilon = (1,0,\dots,0)$ and $\alpha_1 = 2$ and plugging $x_1 = 0$ we see that $2\varphi_{x_1}\varphi_{\alpha\beta\gamma}$ [8] is a linear combination with bounded coefficients of $\varphi_{\alpha^1\beta^1\gamma^1}$ with $|\alpha^1| + |\beta^1| + |\gamma^1| < |\alpha| + |\beta| + |\gamma|$ plus bounded terms, and then by induction by $|\alpha| + |\beta| + |\gamma|$ we arrive to (5.5) for $\alpha_1 = 2$.

Consider induction by α_1. Assume that (5.5) is proven for lesser α_1 and for arbitrary α', β, γ.

Applying to (5.8) $\partial_{x,t}^{\alpha-\varepsilon}\partial_y^\beta\partial_\theta^\gamma$ and plugging $x_1 = 0$ we get linear combinations with bounded coefficients of products

$$\prod_{1\le j\le m} \varphi_{\alpha^j\beta^j\gamma^j} \tag{5.11}$$

where $\sum_j |\alpha^j| \le |\alpha| + m - 1$, $\sum_j \beta^j \le \beta$, $\sum_j \gamma^j \le \gamma$. These terms appear when

[8] Here and below we use notation $\varphi_{\alpha\beta\gamma} = \partial_{x,t}^\alpha\partial_y^\beta\partial_\theta^\gamma\varphi$.

(A) We differentiate $b(x, \nabla_{x'}\varphi)$, in this case $\sum_j \alpha_1^j < \alpha_1$, $\sum_j |\alpha^j| \le |\alpha|$ and these terms due to induction assumption are $O(\sigma^{4-\alpha_1-|\alpha|-|\beta|-|\gamma}$).

(B) We differentiate $\varphi_{x_1}^2$ and $m = 2$, $|\alpha^1| + |\alpha^2| = |\alpha| + 1$, $\alpha_1^1 + \alpha_1^2 = \alpha_1 + 1$, $\alpha_1^j < \alpha$ and these terms due to induction also are $O(\sigma^{4-\alpha_1-|\alpha|-|\beta|-|\gamma|})$.

(C) We differentiate $\varphi_{x_1}^2$ and $m = 1$, $\alpha_1^1 = \alpha_1$, $\alpha_1^1 < \alpha_1$. Then we apply induction by $|\alpha|+|\beta|+|\gamma|$, and these terms due to induction assumption are $O(\sigma^{4-\alpha_1-|\alpha|-|\beta|-|\gamma|})$ as well.

Then dividing by φ_{x_1} and we get $O(\sigma^{4-\alpha_1-|\alpha|-|\beta|-|\gamma|})$. We need base of the induction in (C). However as $|\alpha| = \alpha_1$, $\beta = \gamma = 0$ terms of type (C) do not appear.

(c) We need to expand these estimates to $x_1 > 0$. Applying to equation (5.8) $\partial_{x,t}^{\alpha}\partial_y^{\beta}\partial_\theta^{\gamma}$ with $|\alpha| + |\beta| + |\gamma| = 1$ we see that $\dfrac{d}{dt}\varphi_{\alpha\beta\gamma}$ is bounded and therefore in this case we extend (5.5) from $x_1 = 0$ to $x_1 > 0$.

Next, for $|\alpha|+|\beta|+|\gamma| = 2$ we get that $\dfrac{d}{dt}\varphi_{\alpha\beta\gamma}$ does not exceed $C(S+1)$ with

$$S = \sum_{\alpha,\beta,\gamma:=|\alpha|+|\beta|+|\gamma|=2} |\varphi_{\alpha\beta\gamma}|^2 \tag{5.12}$$

and therefore $|\dfrac{dS^{\frac12}}{dt}| \le C(S+1)$; since $S|_{x_1=0} = O(\sigma^{-1})$ due to (5.10) we conclude that for $T \le \epsilon\sigma$ estimate $S^{\frac12} \le C\sigma^{-1}$ holds. Then

$$|\varphi_{\alpha\beta\gamma}| \le C\sigma^{-1} \qquad \text{for } \ |\alpha| + |\beta| + |\gamma| = 2. \tag{5.13}$$

But then, taking $\alpha_1 \le 1$ we get that $\dfrac{d}{dt}\varphi_{\alpha\beta\gamma}$ does not exceed $C\sigma^{-1}(S^{\frac12}+1)$ with S defined by (5.12) with summation restricted to $\alpha_1 \le 1$ and then due to (5.5) for $x_1 = 0$ we conclude that

$$|\varphi_{\alpha\beta\gamma}| \le C \qquad \text{for } \ |\alpha| + |\beta| + |\gamma| = 2, \ \alpha_1 \le 1. \tag{5.14}$$

And then, taking $\alpha_1 = 0$ we get that $\dfrac{d}{dt}\varphi_{\alpha\beta\gamma}$ does not exceed C and then due to (5.5) for $x_1 = 0$ we conclude that

$$|\varphi_{\alpha\beta\gamma}| \le CT \qquad \text{for } \ \alpha = \beta = 0, \ |\gamma| = 2. \tag{5.15}$$

So, (5.5) holds as $|\alpha| + |\beta| + |\gamma| \le 2$, $x_1 > 0$.

(d) Assume that for $|\alpha|+|\beta|+|\gamma|<p$ estimate (5.5) has been proven. Applying to equation (5.8) $\partial_{x,t}^{\alpha}\partial_y^{\beta}\partial_\theta^{\gamma}$ with $|\alpha|+|\beta|+|\gamma|=p\ge 3$ we again get a linear combination with bounded coefficients of (5.11) products (plus bounded terms) where this time $\sum_j|\alpha^j|\le|\alpha|+m$. Then, as $\alpha_1=q$, $|\alpha|+|\beta|+|\gamma|=p$ we see from the same analysis as in (ii) that

$$|\frac{d}{dt}\varphi_{\alpha\beta\gamma}|\le C\sigma^{-1}S^{\frac12}+C\sigma^{3-p-q}+C$$

and then (5.5) is extended from $x_1=0$ to $x_1>0$, but for $\alpha=\beta=0$, when we proved so far that $\partial_\theta^{\gamma}=O(1)$. But then $|\frac{d}{dt}\varphi_\gamma|\le C$ and (5.5) is again extended from $x_1=0$ to $x_1>0$.

(e) To prove Statement (iii) we again start from estimates at $x_1=0$. Then, exactly like in (b) we prove see that

$$\varphi_{x_1}\varphi_{x_1x_1\theta_j}+\varphi_{x_1\theta_j}\varphi_{x_1x_1}=O(1)$$

and since $\varphi_{x_1\theta_j}=-\varphi^0_{x_1\theta_j}=O(\sigma)$ as $j\ge 2$, and we know that $\varphi_{x_1x_1}=O(\sigma^{-1})$, we conclude that $\varphi_{x_1x_1\theta_j}=O(\sigma^{-1})$. Again, like in (b), using double induction by $|\alpha|+|\beta|+|\gamma|$ and α_1 we prove (5.6) as $x_1=0$.

Finally, like in (c) and (d) we expand this estimate to $x_1>0$. We leave easy details to the reader. □

Proposition 5.4. *Let $\varphi^0(x,y,\theta)$ and $\varphi(x,y,\theta)$ be defined in Proposition 5.3(i) and (ii) respectively. Consider asymptotic solution*

$$u_h^0(x,y,t)\sim(2\pi h)^{-d}\int e^{i\Phi^0(x,y,t,\theta)}\sum_{n\ge0}B_n^0(x,y,t,\theta)h^n\,d\theta \tag{5.16}$$

where $\Phi^0(x,y,t,\theta)=\varphi^0(x,y,\theta)+ta(y,\theta)$ and

$$\Phi(x,y,t,\theta)=\varphi(x,y,\theta)+ta(y,\theta), \tag{5.17}$$

and B_n^0 are uniformly smooth and satisfying corresponding transport equations with initial conditions such that $u_h^0(x,y,0)=\delta(x-y)$:

$$|\partial_{x,t}^{\alpha}\partial_y^{\beta}\partial_\theta^{\gamma}B_n^0(x,y,t,\theta)|\le C_{\alpha\beta\gamma}\quad\forall x,y,t\colon|x-y|+|t|\le T\ \forall\alpha,\beta,\gamma. \tag{5.18}$$

Consider formal asymptotic solution

$$U_h^1(x,y,t) = (2\pi h)^{-d} \int e^{i\varphi(x,y,t,\theta)} \sum_{n\ge 0} B_n(x,y,t,\theta) h^n \, d\theta \tag{5.19}$$

where B_n satisfy corresponding transport equations and one of the boundary conditions

$$B_n = -B_n^0 \qquad \text{as} \quad x_1 = 0, \tag{5.20}$$

$$\varphi_{x_1} B_n - i\partial_{x_1} B_{n-1} = \varphi_{x_1} B_n^0 + i\partial_{x_1} B_{n-1}^0 = 0 \qquad \text{as} \quad x_1 = 0, \tag{5.21}$$

corresponding to $u_h^1 = -u_h^0$ as $x_1 = 0$ and $\partial_{x_1} u_h^1 = -\partial_{x_1} u_h^0$ as $x_1 = 0$ respectively. Then the following inequalities hold:

$$|\partial_{x,t}^\alpha \partial_y^\beta \partial_\theta^\gamma B_n| \le C_{n\alpha\beta\gamma} \begin{cases} 1 + T\sigma^{-1-|\alpha|-|\beta|-|\gamma|} & \alpha_1 = n = 0, \\ \sigma^{-3n-\alpha_1-|\alpha|-|\beta|-|\gamma|} & \alpha_1 + n \ge 1, \end{cases} \tag{5.22}$$

and if $|\gamma| \ge 1$

$$|\partial_{x,t}^\alpha \partial_y^\beta \partial_\theta^\gamma B_n| \le C_{n\alpha\beta\gamma} \begin{cases} 1 + T\sigma^{-|\alpha|-|\beta|-|\gamma|} & \alpha_1 = n = 0, \\ \sigma^{1-3n-\alpha_1-|\alpha|-|\beta|-|\gamma|} & \alpha_1 + n \ge 1. \end{cases} \tag{5.23}$$

Proof. (a) Observe that the transport equation is

$$\Big(\frac{d}{dt} + f\Big) B_n = \mathcal{L} B_{n-1}, \tag{5.24}$$

where f is a linear combination with the smooth coefficients of second derivatives of Φ, $\mathcal{L} = \mathcal{L}(x, D_x)$ is a second order differential operator with the smooth coefficients, and with the coefficient 1 at $D_{x_1}^2$, and $B_{-1} = 0$. In virtue of (5.5)

$$|D_{x,t}^\alpha D_y^\beta D_\theta^\gamma f| \le \sigma^{-1-\alpha_1-|\alpha|-|\beta|-|\gamma|}. \tag{5.25}$$

Consider first $n = 0$. Then equation (5.24) has a right-hand expression 0 and boundary condition (5.21) becomes $B_0 = \mp B_0^0$ as $x_1 = 0$. Let us establish first (5.22) as $x_1 = 0$. As $\alpha_1 = 0$ these estimates follow from the above boundary condition and (5.18). Consider transport equation (5.24):

$$\varphi_{x_1} B_{0,x_1} + \sum_{k\ge 2} b^{(k)}(x, \nabla_{x'}\varphi) B_{0,x_k} - \frac{1}{2} B_{0,t} + \frac{1}{2} f B_0 = 0 \tag{5.26}$$

and set $x_1 = 0$. Then all terms in (5.26) are smooth, except the first and the last one, and the latter satisfies

$$|\partial_{x,t}^{\alpha-\varepsilon}\partial_y^\beta\partial_\theta^\gamma(f\,B_0)| \le C\sigma^{-|\alpha|-|\beta|-|\gamma|} \qquad \alpha_1 = 1$$

due to (5.21) and smoothness of B_0 at $x_1 = 0$. Sinnce $\partial_{x,t}^{\alpha}\partial_y^\beta\partial_\theta^\gamma\varphi$ is obtained by division by φ_{x_1} we arrive to (5.22) as $n = 0$, $\alpha_1 = 0$.

Next we apply a double induction by α_1 and $|\alpha| + |\beta| + |\gamma|$ exactly as in the proof of Proposition 5.3, Part (b) with the following modifications:

(A) We use transport equation (5.24) rather than eikonal equation.

(B) We observe that $\partial_{x,t}^{\alpha-\varepsilon}\partial_y^\beta\partial_\theta^\gamma$, applied to the first term in (5.26) equals to $\varphi_{x_1}\varphi_{\alpha\beta\gamma}$ plus a linear combination of

$$\prod_{1\le j\le m} \varphi_{\alpha^j\beta^j\gamma^j}\partial_{x,t}^{\alpha^0}\partial_y^{\beta^0}\partial_\theta^{\gamma^0}B_0 \tag{5.27}$$

with $m = 1$ and $\sum_j \alpha^j = \alpha + \varepsilon$, $\sum_j \beta^j = \beta$, $\sum_j \gamma^j = \gamma$ [9] and $|\alpha^0| + |\beta^0| + |\gamma^0| < |\alpha| + |\beta| + |\gamma|$.

(C) We observe that $\partial_{x,t}^{\alpha-\varepsilon}\partial_y^\beta\partial_\theta^\gamma$, applied to the second term in (5.26) also is a linear combination with bounded coefficients of (5.27) with $\sum_j \beta^j = \beta$, $\sum_j \gamma^j = \gamma$, $\sum_j |\alpha^j| \le |\alpha| + m$, $\sum_j \alpha_1^j < \alpha_1$ [9] and $|\alpha^j| + |\beta^j| + |\gamma^j| \ge 2$ for $j \ge 1$; it is possible that $m = 0$ here.

We leave simple but tedious arguments to the reader.

(b) Then we extend these estimates to $x_1 > 0$, $T \le \sigma$. We apply arguments of Parts (c), (d) of the proof of Proposition 5.3. First, transport equation (5.24) and boundary condition $B_0 = B_0^0$ as $x_1 = 0$ imply that $|B_0| \le C$ for $T \le C_0^{-1}\sigma$.

Then applying $\partial_{x,t}^{\alpha}\partial_y^\beta\partial_\theta^\gamma$ with $|\alpha| + |\beta| + |\gamma| = 1$ to transport equation (5.24) we get

$$|\frac{d}{dt}B_{0,\alpha\beta\gamma}| \le C\sigma^{-1}S^{\frac12} + C\sigma^{-3}, \qquad S := \Big(\sum_{\alpha,\beta,\gamma}|B_{0,\alpha\beta\gamma}|^2\Big)^{\frac12} \tag{5.28}$$

[9] With summation over $0 \le j \le m$

due to estimates $|\partial^2_{x,y,\theta}\varphi| \le C\sigma^{-1}$ and $|\partial_{x,t,y,\theta}f| \le C\sigma^{-3}$ ($B_{n,\alpha\beta} := \partial^\alpha_{x,t}\partial^\beta_y\partial^\gamma_\theta B_n$) and then for $T \le C_0^{-1}\sigma$ we get from this estimate and estimate $|B_{0,\alpha\beta\gamma}| \le C\sigma^{-2}$ at $x_1 = 0$ to estimate $|B_{0,\alpha\beta\gamma}| \le C\sigma^{-2}$. So far the only restriction is $|\alpha| + |\beta| + |\gamma| = 1$–here and in summation in the definition of S.

However, if we add an extra restriction $\alpha_1 = 0$, then due to this estimate (without restriction) and estimates $|\partial^2_{x,y,\theta}\varphi| \le C$ and $|\partial_{x,y,t,\theta}f| \le C\sigma^{-2}$ if there is only one derivative with respect to x_1, we arrive to

$$|\frac{d}{dt}B_{0,\alpha\beta\gamma}| \le C\sigma^{-1}S^{\frac12} + C\sigma^{-2}$$

where S is defined by (5.28) with summation under the same restriction. Then due to this estimate and estimate $|B_{0,\alpha\beta\gamma}| \le C$ as $x_1 = 0$ and $\alpha_1 = 0$ we arrive to the estimate $|B_{0,\alpha\beta\gamma}| \le C + CT\sigma^{-2}$.

We apply induction by $q := |\alpha| + |\beta| + |\gamma|$. Assuming that for lesser values it is proven, we due to this induction assumption and estimates (5.5) and (5.25) arrive to

$$|\frac{d}{dt}B_{0,\alpha\beta\gamma}| \le C\sigma^{-1}S^{\frac12} + C\sigma^{p-1-2|\alpha|+|\beta|+|\gamma|}, \quad S := \Big(\sum_{\alpha,\beta,\gamma} |B_{0,\alpha\beta\gamma}|^2\Big)^{\frac12} \tag{5.29}$$

under restrictions

$$|\alpha| + |\beta| + |\gamma| = q, \quad \alpha_1 \le |\alpha| - p \tag{5.30}$$

with $p = 0$ and then we recover estimate

$$|B_{0,\alpha\beta\gamma}| \le C\sigma^{p-2|\alpha|-|\beta|-|\gamma|} \tag{5.31}$$

under these restrictions. Based on it we arrive to (5.29) then to (5.31) now under restriction (5.30) with $p = 1$; again due to (5.31) at $x_1 = 0$, and so on for $p = 2, ...$ until we reach $p = |\alpha|$ but in the latter case we use $|B_{0,\alpha\beta\gamma}| \le C_{\alpha\beta\gamma}$ at $x_1 = 0$ as $\alpha_1 = 0$ and thus we achieve some improvement over (5.31): namely we get (5.22) as $n = \alpha_1 = 0$. We leave easy but tedious details to the reader.

(c) Next we apply induction by $n \ge 1$ to estimate $B_{n,\alpha\beta\gamma}$ at $x_1 = 0$. However, first we consider $n = 1$ to establish the pattern. Observe first that under condition (5.20) $B_n = -B_n^0$ and therefore

$$|B_{n,\alpha\beta\gamma}| \le C_{\alpha\beta\gamma} \qquad \text{at } \ x_1 = 0 \ \text{ as } \ \alpha_1 = 0, \tag{5.32}$$

but under condition (5.21)

$$B_n = -B_n^0 + i\varphi_{x_1}^{-1}\big(B_{n-1,x_1} + B_{n-1,x_1}^0\big) \qquad \text{at } \ x_1 = 0$$

and therefore estimate (5.22) holds as $n = 1$, $\alpha_1 = 0$, $x_1 = 0$. It follows from estimates for $|\partial_{x_1} B_{0,\alpha\beta\gamma}| \le C_{\alpha\beta\gamma}\sigma^{-2-|\alpha|-|\beta|-|\gamma|}$ as $\alpha_1 = 0$, $x_1 = 0$ (but we divide by φ_{x_1} which brings an extra factor σ^{-1}).

Recall that the transport equation is

$$(\frac{d}{dt} + f)B_n = G_n, \qquad G_n \coloneqq \mathcal{L}B_{n-1}, \tag{5.33}$$

and due to results of Part (b)

$$|G_{n,(\alpha-\varepsilon)\beta\gamma}| \le C\sigma^{1-3n-\alpha_1-|\alpha|-|\beta|-|\gamma|} \tag{5.34}$$

and therefore in both cases estimate (5.22) holds (as $n = 1$, $\alpha_1 = 1$ and $x_1 = 0$).

Applying the same argument as in Part (b) we can extend (5.34) and (5.22) to arbitrary α_1 as $n = 1$, and using induction by n to arbitrary n as well. Again, we leave easy but tedious details to the reader.

(d) Using the same arguments as in Part (e) of the proof of Proposition 5.3, we can improve these estimates to (5.23) as $|\gamma| \ge 1$. □

Remark 5.5. Under assumption (5.20) we can further improve estimates (5.22) and (5.23) for $n \ge 1$. However it would not improve our final result.

Remark 5.6. One can wonder how sharp are our estimates.

(i) Due to (5.2) and (5.3) $\varphi_{x_1x_1} \asymp \sigma^{-1}$ as $b_{x_1}(x, \nabla_{x'}\varphi) \asymp 1$. Indeed

$$2\varphi^0_{x_1}\varphi^0_{x_1x_1} + \sum_{k\ge 2} b^{(k)}(x, \nabla_{x'}\varphi^0)\varphi^0_{x_kx_1} - \varphi^0_{x_1t} + b_{x_1}(x, \nabla_{x'}\varphi^0) = 0,$$
$$2\varphi_{x_1}\varphi_{x_1x_1} + \sum_{k\ge 2} b^{(k)}(x, \nabla_{x'}\varphi)\varphi_{x_kx_1} - \varphi_{x_1t} + b_{x_1}(x, \nabla_{x'}\varphi) = 0,$$

and since φ^0 is smooth function, we conclude that

$$\sum_{k\ge 2} b^{(k)}(x, \nabla_{x'}\varphi^0)\varphi^0_{x_kx_1} - \varphi^0_{x_1t} + b_{x_1}(x, \nabla_{x'}\varphi^0) = O(\sigma)$$

and then due to (5.3) that

$$\varphi_{x_1}\varphi_{x_1x_1} + b_{x_1}(x, \nabla_{x'}\varphi) = O(\sigma) \qquad \text{as} \quad x_1 = 0.$$

Therefore estimates (5.5) cannot be improved, at least for derivatives only with respect to x_1.

(ii) Repeating construction of Proposition 5.3, we conclude that under the same assumption $b_{x_1}(x, \nabla_{x'}\varphi) \asymp 1$, $\partial_{x_1}^k \varphi \asymp \sigma^{3-2k}$ for all $k \ge 2$. Then from transport equation (5.24) we we conclude that $B_{0,x_1} \asymp \sigma^{-2}$ and repeating construction of Proposition 5.4 we conclude that $\partial_{x_1}^k B_0 \asymp \sigma^{-2k}$ for all $k \ge 1$ and finally $\partial_{x_1}^k B_n \asymp \sigma^{-3n-2k}$ for all $n \ge 0$, $k \ge 1$.

Proposition 5.7. *(i) In the framework of Proposition 5.3(i) as* $|\alpha| \le 1$

$$|\partial_{x,t}^{\alpha}\partial_{\theta}^{\gamma}\bigl(\Phi^0 - \bar{\Phi}^0\bigr)| \le C_\gamma T^{2-|\alpha|} \quad \text{with} \quad \bar{\Phi}^0 := \langle x-y, \theta\rangle + ta(y,\theta). \tag{5.35}$$

(ii) In the framework of Proposition 5.3(ii) as $|\alpha| \le 1$, $\alpha_1 = 0$

$$|\partial_{x,t}^{\alpha}\partial_{\theta}^{\gamma}\bigl(\Phi \ - \bar{\Phi}\ \bigr)| \le C_\gamma\bigl(\sigma^{1-|\alpha|-|\gamma|}T^2 + T^{2-|\alpha|}\bigr) \quad \text{with} \quad \bar{\Phi} := \langle x-\tilde{y}, \theta\rangle + ta(0,y',\theta), \tag{5.36}$$

where $\tilde{y} = (-y_1, y')$ *as* $y = (y_1, y')$.

Proof. Proof of both statements follows from decomposition of $\partial_{x,t}^{\alpha}\partial_{\theta}^{\gamma}\Phi^0$ into Taylor series with quadratic error. First we prove (i) using estimates (5.2) and then (ii) using estimates (5.5).

For $d = 2$ we will need a better approximation; it will be proven later.□

Then we have immediately

Corollary 5.8. *(i) In the framework of Proposition 5.3(i) for* $Ch \le \ell^0(x,y) + |t| \le \epsilon$

$$a(x, \nabla_x\varphi^0) = \tau, \quad \nabla_\theta\Phi^0(x,y,t,\theta) = 0 \tag{5.37}$$

has no more than a single solution θ *and if it has, then* $|t| \asymp \ell^0$ *and* $\nabla_\theta^2\Phi^0$ *is a positive definite matrix. This solution* $\theta = \bar{\theta} + O(T)$ *where* $\bar{\theta}$ *is a solution to* $t\nabla_\theta a(x,\theta) = y - x$.

(ii) In the framework of Proposition 5.3(ii) for $Ch \le \ell(x,y)+|t| \le \epsilon$

$$a(x,\nabla_x\varphi)=\tau, \quad \nabla_\theta\Phi(x,y,t,\theta)=0 \tag{5.38}$$

has has no more than a single solution θ *and if it has then* $|t| \asymp \ell$ *and* $\nabla^2_\theta\Phi$ *is a positive definite matrix. This solution* $\theta=\bar\theta+O(T)$ *where* $\bar\theta$ *is a solution to* $t\nabla_\theta a(x,\theta)=\tilde y-x$.

Proposition 5.9. *In the framework of Proposition 5.3(ii) under extra assumption*

$$\sigma^2\ell \ge h^{1-\delta} \tag{5.39}$$

the following estimate holds

$$|F_{t\to h^{-1}\tau}\Big(\bar\chi_{T'}(t)u^1_h(x,y,t)-\bar\chi_T(t)U^1_h(x,y,t)\Big)| \le Ch^s \tag{5.40}$$

with $U^1_h(x,y,t)$ *defined by* (5.19) *with* $B_n(x,y,t,\theta)$ *described above and multiplied by* $\phi(\theta)$ *supported in* $2\epsilon_3\rho$*-vicinity and equal* 1 *in* $\epsilon_3\rho$*-vicinity of* θ *described in Corollary 5.8(ii) and with* $Ch \le T \le T' \le \epsilon$, $T=c\ell$, $\rho=\sigma^2$.

Proof. Let us fix $x=\bar x$ and $y=\bar y$ in (5.40) and use x as a variable. Due to Proposition 5.1 there is a single generalized billiard of the length $\le \epsilon$ with reflections at ∂X from $\bar x$ to $\bar y$. It has the length $\asymp \ell$, exactly one reflection and a reflection angle $\asymp \rho=(\nu(\bar x)+\nu(\bar y))\ell^{-1} \gtrsim h^{\frac13-\delta}$.

Further, Proposition 5.4(ii) implies that for $\sigma \ge h^{\frac13-\delta}$ (which is due to (5.39) and $\sigma \ge C_0\ell$) (5.19) is a proper asymptotic series as $|t| \le T$ and we define $U^1_h(x,y,t)$ through it (with cut-off).

Furthermore, Corollary 5.8 implies that then $U^1_h(x,y,t)$ is negligible outside $\Omega_{2T,3\rho,3\sigma}$ (by (x_1,x',ξ')) of this billiard while the standard propagation results imply that $u^1_h(x,y,t)\,{}^tQ_y$ is also negligible outside of this vicinity provided symbol of Q is supported in the similar vicinity of $(\bar y,\bar\theta')$, $\bar\theta'$ corresponds to this billiard. Let symbol of Q be 1 in the similar vicinity of $(\bar y,\bar\xi')$. Then $u^1_h(x,y,t)\,{}^tQ_y \equiv U^1_h(x,y,t)$ for $-T \le t \le T$ with $0<T=C_0\ell$.

On the other hand, in this case $u^1_h(x,y,t)(I-{}^tQ_y)\equiv 0$ in $(\epsilon\rho\ell,\epsilon\ell)$-vicinity of $\bar x$ as $-T \le t \le \epsilon T$. Case of $-\epsilon T \le t \le T$ is considered in the same way, albeit with the billiard from $\bar x$ to $\bar y$. Then we arrive to (5.40) with $T'=T$.

Finally, the standard propagation results imply that

$$|F_{t\to h^{-1}\tau}\Big(\big(\bar\chi_{T'}(t)-\bar\chi_T(t)\big)u^1_h(x,y,t)\Big)| \le Ch^s \tag{5.41}$$

as $T \le T' \le \epsilon$. Then (5.40) expands to T': $T \le T' \le \epsilon$. □

5.2 Spectral asymptotics

Proposition 5.10. *In the framework of Proposition 5.3(ii) under extra assumption*

$$\sigma^2\ell \ge h^{1-\delta}, \qquad \ell \ge h^{\frac{3}{5}} \tag{5.42}$$

the following asymptotics holds

$$e_h^{1,\mathsf{T}}(x,y,\tau) = (2\pi h)^{-d}\int_{a(y,\theta)<\tau} e^{ih^{-1}\varphi(x,y,\theta)}\,d\theta + O(h^{1-d}). \tag{5.43}$$

Proof. Let us plug U_h^1 instead of u_h^1 into Tauberian expression $e_h^{1,\mathsf{T}}(x,y,\tau)$ with $\mathsf{T}=\epsilon$. Let us decompose $B_n(x,y,t,\theta)$ into asymptotic series by t^k.

(a) Then terms with $k=0$ become

$$(2\pi h)^{-d}\int_{a(y,\theta)<\tau}\sum_{n\ge 0} e^{ih^{-1}\varphi(x,y,\theta)}B_n(x,y,0,\theta)h^n\,d\theta \tag{5.44}$$

and we claim that these terms do not exceed $C\sigma^{-3n}h^{-d+n}(h\ell^{-1})^{(d+1)/2}$. Indeed, we know from Proposition 5.3(ii) that φ is uniformly infinitely smooth by θ and from Corollary 5.8 that there are no stationary points by θ and restriction of $\varphi(x,y,\theta)$ to $\Sigma(y,\tau)$ has two non-degenerate critical points. The leading terms would be of this magnitude, while all other terms would have extra factors not exceeding $h/\sigma^2\ell$ due to (5.22).

We want to estimate all extra terms by Ch^{1-d}. One can see easily, that $d=2$ is the worst case but even then therms with $n\ge 1$ are $O(h^{1-d})$ and terms with $h/\sigma^2\ell$ appear when we differentiate $B_0(x,y,0,\theta)$ by θ. Then due to (5.23) these terms do not exceed

$$Ch^{-d}(h\ell^{-1})^{\frac{d+3}{2}}(1+\ell\sigma^{-2})$$

and this does not exceed Ch^{1-d} under assumption (5.32).

Finally, since $B_0(x,y,0,\theta)=1+O(\ell\sigma^{-1})$ we can replace $B_0(x,y,0,\theta)$ by 1 resulting in the main part of asymptotics (5.43).

(b) Consider now terms with $k\ge 1$ in the decomposition of $B_n(x,y,t,\theta)$. We can rewrite these terms as

$$(2\pi h)^{1-d}\partial_\tau^{k-1}\int_{\Sigma(y,\tau)} B'_{n,k}(x,y,\theta)h^{n+k-1}\,d\theta : d_\theta a(y,\theta)$$

and using ξ-microhyperbolicity rewrite it as

$$(2\pi h)^{1-d}\int_{\Sigma(y,\tau)} B''_{n,k}(x,y,\theta)h^n(h\ell^{-1})^k\,d\theta : d_\theta a(y,\theta). \tag{5.45}$$

with $B''_{n,k}$ coming from B_n with k derivatives by t and no more than $(k-1)$ by θ. Then using (5.22)–(5.23) we can estimate these terms by $Ch^{-\frac{d-1}{2}}\ell^{-\frac{d-1}{2}} \le Ch^{1-d}$. □

Remark 5.11. The similar asymptotics

$$e_h^{0,\mathsf{T}}(x,y,\tau) = (2\pi h)^{-d}\int_{a(y,\theta)<\tau} e^{ih^{-1}\varphi^0(x,y,\theta)}\,d\theta + O(h^{1-d}) \tag{5.46}$$

is well-known.

Remark 5.12. Consider in the framework of Proposition 5.3(i), (ii) integrals $I_h(\varphi)$ and $I_h(\varphi^0)$ in the right-hand expressions of (5.46) and (5.43) correspondingly:

$$I_h(\varphi) := (2\pi h)^{-d}\int_{a(y,\theta)<\tau} e^{ih^{-1}\varphi(x,y,\theta)}\,d\theta. \tag{5.47}$$

(i) Let $d \ge 3$. Then $I_h(\varphi^0) = I_h(\bar\varphi^0) + O(h^{1-d})$ and $I_h(\varphi) = I_h(\bar\varphi) + O(h^{1-d})$ with $\bar\varphi^0(x,y,\theta) = \langle x-y,\theta\rangle$, $\bar\varphi^0(x,y,\theta) = \langle x-\tilde y,\theta\rangle$.

(ii) Let $d = 2$. Then $I_h(\varphi^0) \equiv I_h(\bar\varphi^0)$ with an error $O(h^{-1})$ as $\ell^0 \ge h^{\frac13}$, $O(h^{-1})$ as $\ell^0(x,y) \ge h^{\frac13}$, $O(h^{-\frac12}\ell^{-\frac32})$ as $h^{\frac12} \le \ell^0(x,y) \le h^{\frac13}$ and $O(h^{-\frac32}\ell^{\frac12})$ as $h \le \ell^0(x,y) \le h^{\frac12}$.

Also $I_h(\varphi) \equiv I_h(\bar\varphi)$ with an error $O(h^{-1})$ as $\ell^0 \ge h^{\frac13}$, $O(h^{-1})$ as $\ell^0(x,y) \ge h^{\frac13}$, $O(h^{-\frac12}\ell^{-\frac32})$ as $h^{\frac12} \le \ell(x,y) \le h^{\frac13}$ and $O(h^{-\frac32}\ell^{\frac12})$ as $h \le \ell(x,y) \le h^{\frac12}$.

Proof. Observe first that due to the stationary phase principle $I_h(\varphi) = Ch^{-d}(h\ell^{-1})^{(d+1)/2}$ and the same is true for $I_h(\bar\varphi)$. Therefore for $\ell \ge h^{\frac12}$ we have simply estimates rather than the error estimates.

On the other hand due to proposition 5.7 $\varphi(x,y,\theta) = \langle x-\tilde y,\theta\rangle + O(\ell^2)$, for $\ell \le h^{\frac12}$ the error does not exceed $Ch^{-d}(h\ell^{-1})^{(d+1)/2} \times \ell^2 h^{-1}$ which is $O(h^{1-d})$.

The same is true for $I_h(\varphi^0)$ and $I_h(\bar\varphi^0)$ albeith with ℓ^0 rather than ℓ. □

Now we need to deal with $d = 2$ and $\ell^0 \le h^{\frac{1}{3}}$ or $\ell \le h^{\frac{1}{3}}$.

Proposition 5.13. *Let $d = 2$. Then*

(i) In the framework of Proposition 5.3(i) for $h^{1-\delta} \le \ell^0(x,y) \le h^{\frac{1}{3}}$

$$\int_{a(y,\theta)<\tau} e^{ih^{-1}\varphi^0(x,y,\theta)}\,d\theta = \int_{a(w,\xi)<\tau} e^{ih^{-1}\langle x-y,\xi\rangle}\,d\xi + O(h) \tag{5.48}$$

with $w = \frac{1}{2}(x+y)$.

(ii) In the framework of Proposition 5.3(ii) for $h^{1-\delta} \le \ell(x,y) \le h^{\frac{1}{3}}$

$$\int_{a(y,\theta)<\tau} e^{ih^{-1}\varphi(x,y,\theta)}\,d\theta = \int_{a(w,\xi)<\tau} e^{ih^{-1}(\langle x-\tilde{y},\xi\rangle + \frac{1}{2}\varkappa(w',\xi',\tau)(x_1^2+y_1^2))}\,d\xi + O(h) \tag{5.49}$$

with $w = (0, \frac{1}{2}(x'+y'))$ and with

$$\varkappa(x',\xi',\tau) = (\tau - b(x_2,\xi_2))^{-\frac{1}{2}} a_{x_1}(x,\xi)|_{x_1=\xi_1=0}; \tag{5.50}$$

recall that $b(w_2,\xi_2) = a(x,\xi)|_{x_1=\xi_1=0}$.

Proof. (i) Consider $(z,\bar{\xi})$: $a(\bar{w},\bar{\xi}) = \tau$ and Hamiltonian trajectory $\Psi_t(\bar{w},\bar{\xi})$ of $a(x,\xi)$, $\Psi_0(\bar{w},\bar{\xi}) = (\bar{w},\bar{\xi})$. Then as $(y(t),\theta(t)) = \Psi_t(\bar{w},\bar{\xi})$ and $(x(t),.) = \Psi_{-t}(\bar{w},\bar{\xi})$, $|t| \le C_0\ell$ we have

$$\varphi^0(x(t),y(t),\theta(t)) = \langle x(t)-y(t),\bar{\xi}\rangle + O(\ell^3) \tag{5.51}$$

where in this part of the proof for simplicity we write ℓ rather than ℓ^0.

Observe that as $\ell \ge h^{1-\delta}$ we can reduce integrals in (5.48) to $\Sigma(y,\tau)$ and $\Sigma(\bar{w},\tau)$ respectively, gaining factor $h\ell^{-1}$. Consider first those reduced integrals over ϵ-vicinities of $\theta(t)$ and $\bar{\xi}$. We see that there is a diffeomorphism of these vicinities and

$$\varphi^0(x(t),y(t),\theta(\bar{w},\xi(s),t)) = \langle x(t)-y(t),\xi\rangle + O(\ell^3 + s^2\ell)$$

provided $a_\theta \parallel \theta$ at $(\bar{w},\bar{\xi})$ (we can assume it without any loss of the generality), s is an angle parameter on $\Sigma(\bar{w},\tau)$, $\xi(0) = \bar{\xi}$. Making change of variables

we estimate an error by $C(h\ell^{-1})^{\frac{3}{2}} \times (\ell^3 h^{-1} + \varepsilon^2 \ell h^{-1} + \ell)$ when we integrate over $|s| \le \varepsilon$, and when we integrate over $|s| \ge \varepsilon$, we can multiply it by an arbitrary large power of $h/\varepsilon^2\ell$. Taking $\varepsilon = (h\ell^{-1})^{\frac{1}{2}}$ we make both errors $O(h)$.

On the other hand, fix $\bar{x}$ and $\bar{y}$ and find corresponding $\bar{w}$, $\bar{\xi}$ and t. One can see easily that $\bar{w} = w + O(\ell^2)$ and the error in the integral in the right hand expression of (5.48) when we redefine w does not exceed $C(h\ell)^{\frac{1}{2}}\ell^2 = O(h)$.

(ii) Making change of variables $\mathsf{x}_1 = x_1 + \frac{1}{2}\varkappa(w_2, \xi', \tau)x_1^2$ and therefore $\mathsf{y}_1 = y_1 + \frac{1}{2}\varkappa(w_2, \xi', \tau)y_1^2$ we would arrive to operator with the symbol which, after division by $(a - \tau)$, modulo $O(\ell^2)$ does not depend on x_1 and therefore with the corresponding phase function

$$\varphi = (\mathsf{x}_1 + \mathsf{y}_1)\xi_1 + (x_2 - y_2)\xi_2 + O(\ell^3).$$

Then we can use the the method of reflection and after this, Statement (i). So we get

$$\int_{a(w,\xi)<\tau} e^{ih^{-1}((\mathsf{x}_1+\mathsf{y}_1)\xi_1+(x_2-y_2)\xi_2}\, d\xi + O(h) \tag{5.52}$$

with $w = (0, \frac{1}{2}(x_2 + y_2))$; which is exactly the right-hand expression of (5.49). □

Remark 5.14. (i) Therefore in Theorem 1.2

$$\begin{aligned} &e_{h,\mathsf{corr}}(x, y, \tau) \\ &\quad = [\mp](2\pi h)^{-1} \int_{\{a(\frac{1}{2}(x+y),\xi)<\tau\}} e^{ih^{-1}\langle x-\tilde{y},\xi\rangle} \left(e^{ih^{-1}\frac{1}{2}\varkappa(\frac{1}{2}(x_2+y_2),\xi_2,\tau)} - 1\right) d\xi \end{aligned} \tag{5.53}$$

with $\varkappa(x_2, \xi_2, \tau)$ defined by (5.50).

(ii) One can see easily that in the framework of Proposition 4.5 expressions (5.53) and (4.49) coincide modulo $O(h^{-1})$.

6 Synthesis and final remarks

Proof of Theorems 1.1 and 1.2. We know that for $\ell(x,y) \le \epsilon$ in the frameworks of Theorems 1.1 and1.2 estimate (1.8) holds.

$$e_h(x,y,\tau) = e^{\mathsf{T}}_{T,h}(x,y,\tau) + O(h^{1-d}).$$

(i) We also know from Section 2 that in the framework of Theorem 1.1 $e^{\mathsf{T}}_{T,h}(x,y,\tau)$ can be defined as an oscillatory integral and Proposition 5.13(i) implies that modulo $O(h^{1-d})$ this oscillatory integral can be rewritten as $e^{\mathsf{W}}_h(x,y,\tau)$ defined by (1.3). This proves Theorem 1.1.

(ii) In the framework of Theorem 1.2 we established in Section 3

$$e^{1,\mathsf{T}}_{T,h}(x,y,\tau) = O(h^{1-d})$$

as $d \ge 3$ and $\ell(x,y) \ge h^{\frac12-\delta}$ and as $d=2$ and $\ell(x,y) \ge h^{\frac13-\delta}$. Further, we proved there that $e^{1,\mathsf{T}}_{T,h}(x,y,\tau) = O(h^{1-d-\delta})$ as $d=2$ and $\ell(x,y) \ge h^{\frac13+\delta}$. In both cases we also assume there that $|x'-y'| \ge \epsilon\ell(x,y)$. However it follows from Section 5 that this latter condition is not necessary. Combined with Remark 1.3(ii) it proves Theorem 1.2 in this case.

Let $d \ge 3$. In Section 4 we proved asymptotics

$$e^{1,\mathsf{T}}_{T,h}(x,y) = [\mp] e^{0,\mathsf{W}}_h(x,\tilde{y},\tau) + O(h^{1-d}) \tag{6.1}$$

if either $\ell(x,y) \le h^{\frac12+\delta}$ or $h^{\frac12+\delta} \le \ell(x,y) \le h^{\frac12-\delta}$ and

$$\nu(x)+\nu(y) \le \sigma\ell(x,y) \tag{6.2}$$

with $\sigma = h^{2\delta}$. In Section 5 we proved the same asymptotics as

$$\nu(x)+\nu(y) \ge \sigma\ell(x,y) \tag{6.3}$$

with $\sigma = h^{\frac12-\delta}\ell^{-\frac12} + C_0\ell$. These domains overlap and cover all values of $\ell \le h^{\frac12-\delta}$ and σ.

(iii) Let $d=2$. Then instead of asymptotics (6.1) we have

$$e^{1,\mathsf{T}}_{T,h}(x,y) = [\mp] e^{0,\mathsf{W}}_h(x,\tilde{y},\tau) + e_{h,\mathsf{corr}}(x,y,\tau) + O(h^{1-d}) \tag{6.4}$$

and Sections 4 and 5 cover cases (6.2) with $\sigma = h^{1+\delta}\ell^{-2}$ and (6.3) with $\sigma = h^{\frac13-\delta}$, $\ell(x,y) \le h^{\frac13+\delta}$. These domains overlap and cover all values $\ell \le h^{\frac13+\delta}$ and σ. Here we are left with domain $h^{\frac13+\delta} \le \ell \le h^{\frac13-\delta}$, $\sigma = h^{\frac13-\delta}$ where we have less precise estimate $e^{1,\mathsf{T}}_{T,h}(x,y,\tau) = O(h^{1-d-\delta})$. □

Remark 6.1. (i) We would like to derive remainder estimate $O(h^{-1})$ as $d = 2$ and $\ell(x,y) \le \epsilon$, thus removing the gap $h^{\frac{1}{3}+\delta} \le \ell \le h^{\frac{1}{3}-\delta}$, $\nu(x)+\nu(y) \le h^{\frac{1}{3}-\delta}$.

(ii) For Schrödinger operator (1.4) can get rid of ξ-microhyperbolicity assumption by rescaling method using scaling function

$$\gamma_x = (\epsilon |V(x) - \tau| + h^{\frac{2}{3}}). \tag{6.5}$$

Then using arguments of Proposition 3.2 and Remark 4.1 of [Ivr3] we have a remainder estimate $O(h^{1-d}\gamma_x^{(d-3)/2}\gamma_y^{(d-3)/2})$ which as $d \ge 3$ is $O(h^{1-d})$; however as $d = 2$ it is as bad as $O(h^{-\frac{4}{3}})$.

(iii) As $d = 2$ away from ∂X remainder estimate $O(h^{-1})$ in the regular zone and $O(h^{-\frac{16}{15}})$ or better in the singular zone has been derived in [Ivr3]. However it required the analysis of the Hamiltonian trajectories and introduction of the correction term. This does not look feasible near the boundary especially because both the boundary and degeneration define correction term.

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Pointwise Spectral Asymptotics out of the Diagonal near Degeneration*,†

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Abstract

We establish uniform (with respect to x, y) semiclassical asymptotics and estimates for the Schwartz kernel $e_h(x,y;\tau)$ of spectral projector for a second order elliptic operator inside domain under microhyperbolicity (but not ξ-microhyperbolicity) assumption. While such asymptotics for its restriction to the diagonal $e_h(x,x,\tau)$ and, especially, for its trace $\mathsf{N}_h(\tau)=\int e_h(x,x,\tau)\,dx$ are well-known, the out-of-diagonal asymptotics are much less explored, especially uniform ones.

Our main tools: microlocal methods, improved successive approximations and geometric optics methods.

1 Introduction

In this paper we consider a self-adjoint scalar operator which is elliptic second order differential operator with $\mathscr{C}^K$-coefficients ($K=K(d,\delta)$)

$$A=\sum_{j,k}\bigl(hD_j-V_j(x)\bigr)g^{jk}(x)\bigl(hD_k-V_k(x)\bigr)+V(x),\qquad g^{jk}=g^{kj}, \tag{1.1}$$

and under microhyperbolicity condition

$$|V(x)-\tau|+|\nabla V(x)|\ge\epsilon_0\qquad \forall x\in B(0,1)\subset\mathbb{R}^d,\ d\ge 2, \tag{1.2}$$

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we establish sharp (with $O(h^{1-d})$ remainder) asymptotics of the Schwartz kernel of the spectral projector $e_h(x,y,\tau)$ as $h \to +0$.

Under ξ-microhyperbolicity assumption

$$|V(x)-\tau| \ge \epsilon_0 \qquad \forall x \in B(0,1) \subset \mathbb{R}^d \tag{1.3}$$

such asymptotics of $e_h(x,x,\tau)$ is well-known for much more general operators including matrix ones (with a proper definition of ξ-microhyperbolicity). However out-of-diagonal asymptotics, especially uniform with respect to x,y, are much less explored and for simple main part and sharp remainder estimate in addition to ξ-microhyperbolicity

$$|a(x,\xi)-\tau| + |\nabla_\xi a(x,\xi)| \ge \epsilon_0 \tag{1.4}$$

where $a(x,\xi)$ is the principal symbol one needs a strong convexity of the energy surface $\{\xi\colon a(x,\xi)=\tau\}$; see [Ivr3], Theorem 1.1.

On the other hand, under only microhyperbolicity condition

$$|a(x,\xi)-\tau| + \nabla_{x,\xi} a(x,\xi)| \ge \epsilon_0 \tag{1.5}$$

(which could be generalized for matrix operators) sharp asymptotics of

$$\mathsf{N}_h(\tau) = \int e_h(x,x,\tau)\,dx \tag{1.6}$$

are well-known (see [Ivr2], Chapter 4) [1]. However in this case even uniform asymptotics of $e_h(x,x,\tau)$ are much less explored and only for operator (1.1) (see Subsection 5.3.1, Subsubsection "Asymptotics without Spatial Mollification") because while microhyperbolicity conditions does not allow *short periodic trajectories*, it allows *short loops*[2].

Theorem 1.1. *Consider elliptic self-adjoint operator* (1.1) *with smooth coefficients. Assume that microhyperbolicity condition* (1.2) *is fulfilled. Then asymptotics*

$$e_h(x,y,\tau) = e_h^{\mathsf{W}}(x,y,\tau) + \begin{cases} O(h^{1-d}) & d \ge 3,\\ O(h^{-\frac{4}{3}}) & d=2 \end{cases} \tag{1.7}$$

[1] And even near the boundary (see [Ivr2], Chapter 7).

[2] But ξ-microhyperbolicity does not allow short loops.

holds for all $x, y \in B(0,\epsilon)$ *with*

$$e_h^{\mathsf{W}}(x,y;\tau) = (2\pi h)^{-d} \int_{\{a(\frac{1}{2}(x+y),\xi)<\tau\}} e^{ih^{-1}\langle x-y,\xi\rangle}\, d\xi, \tag{1.8}$$

$$a(x,\xi) = \sum_{j,k} g^{jk}(x)\xi_j\xi_k + V(x). \tag{1.9}$$

Theorem 1.2. *Consider elliptic self-adjoint operator* (1.1) *with smooth coefficients. Assume that microhyperbolicity condition* (1.2) *is fulfilled. Let* $d = 2$ *and* $x, y \in B(0, \frac{1}{2})$. *Then*

(i) Asymptotics

$$e_h(x,y,\tau) = e_h^{\mathsf{W}}(x,y,\tau) + O(h^{-1}) \tag{1.10}$$

holds if <u>*either*</u> *condition* (1.3) *is fulfilled,* <u>*or*</u> $\ell(x,y) \ge h^{\frac{1}{3}}$, <u>*or*</u> $\ell(x,y) \ge h^{\frac{2}{5}}$ *and*

$$\bigl|4V(x-\tau)V(y-\tau) - |x-y|^2_{(g_{jk})}|\nabla V|^2_{(g^{jk})} + \langle x-y, \nabla V\rangle^2\bigr| \ge \epsilon\ell(x,y)^2 \tag{1.11}$$

where

$$\ell(x,y) = \max(|x-y|,\, |V(x)-\tau|,\, |V(y)-\tau|,\, h^{\frac{2}{3}}), \tag{1.12}$$

$(g_{jk}) = (g_{jk})^{-1}$ *and* ∇V *is calculated at* $\frac{1}{2}(x+y)$.

(ii) Without condition (1.11) *for* $h^{\frac{1}{3}} \ge \ell(x,y) \ge h^{\frac{2}{5}}$ *the following estimate holds:*

$$e_h(x,y,\tau) = O(h^{-\frac{2}{3}}\ell(x,y)^{-1}). \tag{1.13}$$

(iii) Further, for $\ell(x,y) \le h^{\frac{2}{5}}$ *asymptotics*

$$e_h(x,y,\tau) = e_h^{\mathsf{W}}(x,y,\tau) + e_{\mathsf{corr},k,h}(x,y,\tau) + O(h^{-1}) \tag{1.14}$$

holds with correction term $e_{\mathsf{corr},k,h}(x,y,\tau)$ *to be defined by* (2.46) *later if* <u>*either*</u> *condition* (1.11) *is fulfilled* <u>*or*</u> $\ell(x,y) \le h^{\frac{4}{9}}$.

(iv) Without condition (1.11) *for* $h^{\frac{2}{5}} \ge \ell(x,y) \ge h^{\frac{4}{9}}$ *the following estimate holds:*

$$e_h(x,y,\tau) = e_{\mathsf{corr},k,h}(x,y,\tau) + O(h^{-\frac{5}{3}}\ell(x,y)^{\frac{3}{2}}). \tag{1.15}$$

(v) Finally, as $x = y$ *one can replace* k *by* 0 *while preserving remainder estimate and define* $e_{\mathsf{corr},0,h}(x,y,\tau)$ *as*

$$2h^{-\frac{8}{3}} \int_{-\infty}^{\tau} (\tau - \tau')\, \mathsf{Ai}(-2h^{-\frac{2}{3}}(\tau' - V(x))\, d\tau' - \frac{1}{2\pi h^2}(\tau - V(x))_+ \tag{1.16}$$

provided $|\nabla V|_{g^{jk}} = 1$ *where* $\mathsf{Ai}(\cdot)$ *denotes Airy function.*

Remark 1.3. As $d = 1, 2$ and $x = y$ asymptotics are already known: see [Ivr2], Subsection 5.3.1, Subsubsection Asymptotics without Spatial Mollification and Short Loops.

Plan of the paper. *General idea.* Our main method is to prove (as $d \ge 2$) *Tauberian asymptotics*

$$e_h(x,y,\tau) = e^{\mathsf{T}}_{T,h}(x,y,\tau) + O(h^{1-d}) \tag{1.17}$$

with the *Tauberian expression*

$$e^{\mathsf{T}}_{T,h}(x,y,\tau) = \frac{1}{h}\int_{-\infty}^{\tau} F_{t\to h^{-1}\tau}\Big(\bar\chi_T(t) u_h(x,y,t)\Big)\, d\tau \tag{1.18}$$

where $u_h(x,y,t)$ is the *propagator* (the Schwartz kernel of $e^{ih^{-1}tA_h}$), and $\bar\chi_T(t) = \bar\chi(T^{-1}t)$ is the appropriate cut-off. Then from Tauberian expression we pass to Weyl expression and a correction term if needed. So far our arguments are similar to those of [Ivr3] where we considered points near boundary but assumed ξ-microhyperbolicity.

To prove Tauberian asymptotics we need to prove that Fourier transform in (1.18) with $T \asymp 1$ is $O(h^{1-d})$.

Section 2. We start from the toy-model (2.1), study Hamiltonian trajectories and then do all calculations explicitly. We arrive to the oscillatory integral with the phase function which has four stationary points, corresponding to two trajectories from x to y and two trajectories from y to x on the energy level τ, provided we are in the *regular zone* where the

left-hand expression in (1.11) without absolute value is positive: two are short trajectories and two are long ones. If the same expression is negative (*shadow zone*) there are neither any trajectories nor stationary points, and if (1.11) fails (*singular zone*) these points almost coincide.

Then we repeat these steps for the generalized toy-model (2.31).

Section 3. We follow the same path, but we get semi-explicit expression: using special coordinates and x_1-microhyperbolicity we construct in the standard way $F_{x_1\to h^{-1}\xi_1, y_1\to -h^{-1}\eta_1} u_h(x,y,t)$ as an oscillatory integral and then make inverse Fourier transform. Then we prove (1.17) and either pass from Tauberian expression to Weyl expression (as $\ell(x,y)\ge h^{\frac{2}{5}}$) and estimate the error, or use the perturbation method (as $\ell(x,y)\le h^{\frac{2}{5}}$) and estimate

$$\bigl(e^{\mathsf{T}}_{T,h}(x,y,\tau)-\bar{e}^{\mathsf{T}}_{T,k,h}(x,y,\tau)\bigr)-\bigl(e^{\mathsf{W}}_{h}(x,y,\tau)-\bar{e}^{\mathsf{W}}_{k,h}(x,y,\tau)\bigr)$$

where $\bar{e}^{\mathsf{T}}_{T,k,h}(x,y,\tau)$ and $\bar{e}^{\mathsf{W}}_{k,h}(x,y,\tau)$ correspond to an appropriate generalized toy-model. Then we define correction term as

$$e_{\mathsf{corr},k,h}=\bigl(\bar{e}^{\mathsf{T}}_{T,k,h}(x,y,\tau)-\bar{e}^{\mathsf{W}}_{k,h}(x,y,\tau)\bigr).$$

Section 4. Here we show how simple rescaling arguments allows to get rid off assumption (1.3). We also discuss easier case $d=1$.

2 Toy-model

Let us consider a toy-model operator with $d\ge 2$, $g^{jk}=\delta_{jk}$, $V_j(x)=0$ and $V(x)=-x_1$:

$$\bar{A}_h := \frac{1}{2}h^2D^2-x_1=\frac{1}{2}h^2D_1^2+\frac{1}{2}h^2D'^2-x_1 \tag{2.1}$$

By shift $x_1\mapsto x_1+\tau$ we can reduce $\tau\in\mathbb{R}$ to $\tau=0$. After this by rescaling $x\mapsto h^{\frac{2}{3}}$, $\tau\mapsto h^{\frac{2}{3}}$ we can reduce $h>0$ to $h=1$.

2.1 Hamiltonian trajectories

Proposition 2.1. *Consider toy-model operator* (2.1)*. Then*

(i) Hamiltonian trajectories from $(\bar{x},\bar{\xi})=(\bar{x},\bar{\xi}_1,\bar{\xi}')$, $\bar{x}_1>0$ *on the energy level* 0 *(so,* $\bar{\xi}_1=\mp\sqrt{2\bar{x}_1-|\bar{\xi}'|^2}$*) are*

$$\xi'=\bar{\xi}',\ x_1=\bar{x}_1+\frac{t^2}{2}\mp t\sqrt{2\bar{x}_1-|\bar{\xi}'|^2},\ x'=\bar{x}'+\bar{\xi}'t,\ \xi_1=t\mp\sqrt{2\bar{x}_1-|\bar{\xi}'|^2}.$$

(ii) If $d = 2$ *and* $\pm\bar\xi_1 < 0$ *their projections to* x*-space are parabolas*

$$x_1 = \frac{1}{2\bar\xi_2^2}\Big(x_2 - \bar x_2 \mp \bar\xi_2\sqrt{2\bar x_1 - \bar\xi_2^2}\Big)^2 + \frac{\bar\xi_2^2}{2}$$

which at $t = \pm\frac{2\bar x_1}{\sqrt{2\bar x_1 - \bar\xi_2^2}}$ *are tangent to the parabola* $\Gamma = \{x_1 = \frac{1}{4\bar x_1}|x' - \bar x_2|^2\}$.

(iii) Any point x *above* Γ *(that is with* $x_1 > \frac{1}{4\bar x_1}(x_2 - \bar x_2^2)$*) is covered by two such rays, one of them touches* Γ *between* $\bar x$ *and* x*, and another outside of this segment. Any point* x *below* Γ *(that is with* $x_1 < \frac{1}{4\bar x_1}(x_2 - \bar x_2)^2$*) is not reachable from* $\bar x$ *by rays on the energy level* 0*, and any point on* Γ *is reachable by just one ray (when point above* Γ *tends to* Γ *both rays tend to the same limit).*

(iv) The vertices of these parabolas are on the ellipse $L = \{(2x_1 - \bar x_1)^2 + (x_2 - \bar x_2)^2 = \bar x_1^2\}$*; between* Γ *and* L *along both rays* ξ_1 *have the same sign (opposite to one of* $\bar x_1$*) while inside* L *(brown on Figure 28.3) this is true only for a long ray, while for a short ray sign of* ξ_1 *coincides with the sign of* $\bar\xi_1$.

Proof. Easy proof by direct calculations is left to the reader. □

Remark 2.2. Observe that for a trajectory between points x and y

(i) For a long trajectory $\ell(x,y) \asymp |x'-y'|+\nu(x)+\nu(y)$ with $\nu(x) = |V(x)-\tau|$ and $T(x,y) \asymp \ell^{\frac12}(x,y)$ is a time to reach.

(ii) Either $\ell(x,y) \asymp \ell^0(x,y) := |x-y|$ and $T^0(x,y) \asymp T(x,y)$ where $T^0(x,y)$ is time to reach along short trajectory, or $|\xi_{\mathsf{long}} - \xi_{\mathsf{short}}| \asymp \nu(x)^{\frac12}$ and $|\eta_{\mathsf{long}} - \eta_{\mathsf{short}}| \asymp \nu(y)^{\frac12}$, where ξ_* and η_* correspond to these trajectories at x and y for $|t| \asymp T(x,y)$ or $T^0(x,y)$.

(iii) Further, since $\xi'_{\mathsf{short}}\ell^0 = \xi'_{\mathsf{long}}\ell$, we conclude that

$$\ell^0 < \epsilon\ell \implies |\xi'_{\mathsf{short}}| > \epsilon^{-1}|\xi'_{\mathsf{long}}|. \tag{2.2}$$

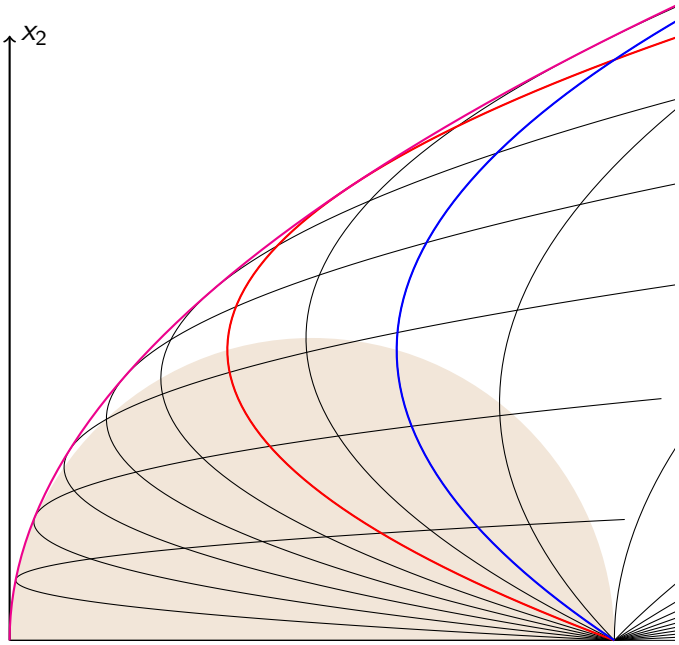

Figure 28.3: $d = 2$. Hamiltonian rays from $\bar{x}$ as $\bar{\xi}_2 > 0$, $t > 0$, scaled.

2.2 Calculations

As usual, consider $u_h^{\pm}(x, y, t) = u_h(x, y, t)\theta(\pm t)$ with Heaviside function θ; then

$$\Big(hD_t - \frac{1}{2}h^2 D_1^2 - \frac{1}{2}h^2|D'|^2 + x_1\Big)u_h^{\pm} = \mp ih\delta(t)\delta(x-y); \tag{2.3}$$

making h-Fourier transform

$$\widehat{u_h^{\pm}}(\xi, y, \tau) := F_{t\to h^{-1}\tau, x\to h^{-1}\xi} u_h^{\pm} = \iint e^{-ih^{-1}(t\tau+\langle x,\xi\rangle)} u_h^{\pm}(x, y, t)\, dxdt$$

we get as $\mp \operatorname{Im}\tau > 0$

$$\Big(\tau - \frac{1}{2}\xi_1^2 - \frac{1}{2}|\xi'|^2 + ih\partial_{\xi_1}\Big)\widehat{u_h^{\pm}} = \mp ie^{ih^{-1}(-y_1\xi_1 - \langle y',\xi'\rangle)}; \tag{2.4}$$

rewriting it as

$$\partial_{\xi_1}\Big(e^{-ih^{-1}((\tau-|\xi'|^2/2)\xi_1 - \xi_1^3/6)}\widehat{u_h^{\pm}}\Big) = \mp e^{ih^{-1}(-y_1\xi_1 - \langle y',\xi'\rangle - (\tau-|\xi'|^2/2)\xi_1 + \xi_1^3/6)},$$

we arrive to

$$\widehat{u_h^{\pm}}(\xi, y, \tau) = \mp e^{ih^{-1}((\tau-|\xi'|^2/2)\xi_1-\xi_1^3/6)} \int_{\mp\infty}^{\xi_1} e^{ih^{-1}(-y_1\eta_1-\langle y',\xi'\rangle-(\tau-|\xi'|^2/2)\eta_1+\eta_1^3/6)}\, d\eta_1 \tag{2.5}$$

and making partial inverse h-Fourier transform

$$F_{t\to h^{-1}\tau} u_h^{\pm} = \mp(2\pi h)^{-d} \iint \Big(\int_{\mp\infty}^{\xi_1} e^{ih^{-1}(x_1\xi_1-y_1\eta_1+\langle x'-y',\xi'\rangle+(\tau-|\xi'|^2/2)(\xi_1-\eta_1)-(\xi_1^3-\eta_1^3)/6))}\, d\eta_1\Big)\, d\xi_1 d\xi'. \tag{2.6}$$

Then as $\tau \in \mathbb{R}$

$$F_{t\to h^{-1}\tau} u_h = (2\pi h)^{-d}\times \iint \Big(\int_{-\infty}^{\infty} e^{ih^{-1}(x_1\xi_1-y_1\eta_1+\langle x'-y',\xi'\rangle+(\tau-|\xi'|^2/2)(\xi_1-\eta_1)-(\xi_1^3-\eta_1^3)/6))}\, d\eta_1\Big)\, d\xi_1 d\xi' = h^{(1-2d)/3} J\big(h^{-\frac{2}{3}}\tau, h^{-\frac{2}{3}}x_1, h^{-\frac{2}{3}}y_1, h^{-\frac{2}{3}}(x'-y')\big), \tag{2.7}$$

with

$$J(\tau, x_1, y_1, z') := (2\pi)^{-d}\times \iiint e^{i(x_1\xi_1-y_1\eta_1+\langle z',\xi'\rangle+(\tau-|\xi'|^2/2)(\xi_1-\eta_1)-(\xi_1^3-\eta_1^3)/6))}\, d\xi_1 d\eta_1 d\xi'. \tag{2.8}$$

We can see easily that the phase function

$$\Phi(x_1, y_1, z'; \xi_1, \eta_1, \xi'; \tau) = x_1\xi_1 - y_1\eta_1 + \langle z', \xi'\rangle + (\tau - \frac{1}{2}|\xi'|^2)(\xi_1-\eta_1) - \frac{1}{6}(\xi_1^3-\eta_1^3) \tag{2.9}$$

has four stationary points (with respect to (ξ_1, η_1, ξ')): on all of them

$$2x_1 = -2\tau + \xi_1^2 + |\xi'|^2, \quad 2y_1 = -2\tau + \eta_1^2 + |\xi'|^2, \quad z' = \xi'(\xi_1-\eta_1), \tag{2.10}$$

and for corresponding trajectories on the energy τ

$$T(x, y) = |\xi_1 - \eta_1| \tag{2.11}$$

but

(a) on two of them $|\xi_1 - \eta_1|$ is the smallest (short rays) and $\ell^0(x,y) \asymp |x_1 - y_1| \asymp |x - y|$ with $x_1 \ge 0$, $y_1 \ge 0$;

(b) on the other two $|\xi_1 - \eta_1|$ are largest (long rays) ξ_1 and η_1 have opposite signs and $\ell(x,y) \asymp x_1 + y_1 \asymp |x - y| + x_1 + y_1$ with $x_1 \ge 0$, $y_1 \ge 0$.

Then we arrive easily to the following proposition:

Proposition 2.3. *For a toy-model operator* (2.1) *with* $d \ge 1$ *the following estimate holds:*

$$|e^{\mathsf{T}}_{\epsilon_1,h}(x,y,\tau) - e^{\mathsf{T}}_{T,h}(x,y,\tau)| \le Ch^{\frac{1-2d}{3}} \tag{2.12}$$

with $T = \max(\epsilon_2\sqrt{\ell(x,y)},\, h^{\frac{1}{3}-\delta})$.

In particular, the right-hand expression is $O(h^{1-d})$ *for* $d \ge 3$, $O(h^{-\frac{4}{3}})$ *for* $d = 2$ *and* $O(h^{-\frac{2}{3}})$ *for* $d = 1$.

It shows that case $d = 2$ requires special consideration. In this case

$$\partial_\tau J(\tau, x_1, y_1, z_2) = \frac{i}{4\pi^2}\iiint (\xi_1 - \eta_1) e^{i(x_1\xi_1 - y_1\eta_1 + z_2\xi_2 + (\tau - \xi_2^2/2)(\xi_1 - \eta_1) - (\xi_1^3 - \eta_1^3)/6)}\, d\xi_1 d\eta_1 d\xi_2$$

which after substitution $\xi_1 = \alpha + \beta$, $\eta_1 = \alpha - \beta$ becomes

$$\frac{i}{\pi^2}\iiint \beta e^{i\beta((x_1+y_1+2\tau) - \xi_2^2 - \alpha^2 - \beta^2/3) + iz_2\xi_2 + i(x_1 - y_1)\alpha}\, d\alpha d\beta d\xi_2$$

and calculating integrals with respect α and ξ_2 we arrive to

$$\partial_\tau J(\tau, x_1, y_1, z_2) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{i(\beta(x_1+y_1+2\tau) + \frac{1}{4\beta}((x_1-y_1)^2 + z_2^2) - \frac{1}{3}\beta^3)}\, d\beta. \tag{2.13}$$

One can see easily that

(2.14) Stationary points

$$\beta^2 = \frac{1}{2}\Big(x_1 + y_1 + 2\tau) \pm \sqrt{4x_1y_1 - z_2^2}\Big)$$

are non-degenerate as $4(x_1 + \tau)(y_1 + \tau) > z_2^2$ and there are no real stationary points as $4(x_1 + \tau)(y_1 + \tau) < z_2^2$.

Proposition 2.4. *Consider toy-model operator* (2.1) *with* $d = 2$. *Let* $x_1 \ge 0$, $y_1 \ge 0$, $\ell(x,y) \coloneqq \max(|x-y|,\, x_1,\, y_1) \ge C_0 h^{\frac{2}{3}}$.

(i) If $\ell^0(x,y) \coloneqq |x-y| \le \epsilon' \ell(x,y)$ *with* $\epsilon' = \epsilon'(\epsilon) > 0$ *then the following estimate holds:*

$$|e^{\mathsf{T}}_{\epsilon_1,h}(x,y,0) - e^{\mathsf{T}}_{T,h}(x,y,0)| \le Ch^{-\frac{1}{2}} \ell(x,y)^{-\frac{5}{4}} \tag{2.15}$$

with $T = C_0 \sqrt{\ell^0(x,y)}$.

(ii) Further, if $\ell^0(x,y) \ge \epsilon \ell(x,y)$ *then the following estimates holds:*

$$|e^{\mathsf{T}}_{\epsilon_1,h}(x,y,0)| \le Ch^{-\frac{2}{3}} \ell(x,y)^{-1}. \tag{2.16}$$

(iii) More precisely, if $\ell^0(x,y) \ge \epsilon \ell(x,y)$ *and*

$$(x_2-y_2)^2 \le 4x_1y_1 - \epsilon \ell(x,y)^2 \tag{2.17}$$

then

$$|e^{\mathsf{T}}_{\epsilon_1,h}(x,y,0)| \le Ch^{-\frac{1}{2}} \ell(x,y)^{-\frac{5}{4}}. \tag{2.18}$$

(iv) On the other hand, if

$$(x_2-y_2)^2 \ge 4x_1y_1 + \epsilon \ell(x,y)^2 \tag{2.19}$$

then this estimate and all similar estimates below acquire factor $(\ell(x,y)h^{-\frac{2}{3}})^{-s}$*:*

$$|e^{\mathsf{T}}_{\epsilon_1,h}(x,y,0)| \le Ch^{(2s-4)/3} \ell(x,y)^{-s}. \tag{2.20}$$

Proof. Integration by τ and multiplication by h^{-1} results in the formal expression

$$\frac{1}{8\pi^2 h^2 i} \iiint (\xi_1-\eta_1)^{-1} e^{ih^{-1}\Phi(x_1,y_1,z';\xi_1,\eta_1,\xi';0)}\, d\xi_1 d\eta_1 d\xi' \tag{2.21}$$

and we need to apply the stationary phase principle counting only points (in $\mathbb{C}^3$) corresponding to $|\xi_1-\eta_1| \gtrsim \sqrt{\ell}$. Then

(a) Due to above analysis in the *regular zone* described in Statements (i) and (iii) all three eigenvalues of $\mathsf{Hess}_{\xi_1,\eta_1,\xi_2}\Phi$ are real and $\asymp \sqrt{\ell}$ and therefore the left-hand expressions in (2.15) and (2.18) do not exceed $Ch^{-2}\ell^{-\frac{1}{2}}(h\ell^{-\frac{1}{2}})^{\frac{3}{2}}$.

(b) On the other hand, in the *singular zone* where

$$|4x_1y_1 - (x_2 - y_2)^2| \le \epsilon\ell(x,y)^2 \tag{2.22}$$

this is true only for two eigenvalues and therefore the left-hand expression in (2.16) does not exceed $Ch^{-2}\ell^{-\frac12}(h\ell^{-\frac12})^{\frac22} \times h^{\frac13}$.

(c) In the *shadow zone* described in Statement (iv) again all eigenvalues of Hessian are $\asymp \sqrt{\ell}$ but only two are real and the third has imaginary part $\asymp \sqrt{\ell}$ and therefore the left-hand expression in (2.18) does not exceed $Ch^{-2}(h\ell^{-\frac12})^s$. □

Remark 2.5. (i) We can be more precise in the analysis in the singular zone.

(ii) Statements (ii)–(iv) remain true as $|x_1| \le h^{\frac23}$ or $|y_1| \le h^{\frac23}$.

(iii) Further, if either $x_1 < -h^{\frac23}$, or $y_1 < -h^{\frac23}$, or both $x_1 < -h^{\frac23}$ and $y_1 < -h^{\frac23}$ then stationary points move to imaginary axis and estimates in Statements (ii) and (iii) and all similar estimates below acquire correspondingly either factor $h^{\frac{2s}{3}}|x_1|^s$, or $h^{\frac{2s}{3}}|y_1|^s$, or both.

(iv) The right-hand expression of (2.15) is $O(h^{-1})$ as $\ell(x,y) \ge h^{\frac25}$ and the right-hand expression of (2.16) is $O(h^{-1})$ as $\ell(x,y) \ge h^{\frac13}$.

(v) Improvements similar to Proposition 2.4(i)–(iv) are possible for $d \ge 3$ as well.

Now we want to replace $e^{\mathsf{T}}_{T,h}(x,y,0)$ in the left-hand expression of (2.15) by $e^{\mathsf{W}}_h(x,y,0)$ described by (1.8).

Proposition 2.6. *(i) For a toy-model operator* (2.1) *with* $d \ge 3$ *and* $\ell(x,y) \le 1$

$$e^{\mathsf{T}}_{\epsilon_1,h}(x,y,0) = e^{\mathsf{W}}_h(x,y,0) + O(h^{1-d}). \tag{2.23}$$

(ii) For a toy-model operator (2.1) *with* $d = 2$

$$e^{\mathsf{T}}_{\epsilon_1,h}(x,y,0) = e^{\mathsf{W}}_h(x,y,0) + e_{\mathsf{corr},h}(x,y,0) + O(h^{-1}) \tag{2.24}$$

with the correction term

$$e_{\mathsf{corr},h}(x,y,0) = \frac{1}{h^2}\int_{-\infty}^{0} J\big(h^{-\frac{2}{3}}\tau, h^{-\frac{2}{3}}x_1, h^{-\frac{2}{3}}y_1, h^{-\frac{2}{3}}(x_2-y_2)\big)\,d\tau - e_h^{\mathsf{W}}(x,y,0). \tag{2.25}$$

Proof. (a) As $\ell(x,y) \le h^{\frac{2}{3}}$ we have both $e^{\mathsf{T}}_{\epsilon_1,h}(x,y,0) = O(h^{-\frac{2d}{3}})$ and $e_h^{\mathsf{W}}(x,y,0) = O(h^{-\frac{2d}{3}})$. Those are $O(h^{-\frac{4}{3}})$ for $d=2$ and $O(h^{1-d})$ for $d \ge 3$.

(b) As $\ell(x,y) \ge h^{\frac{2}{3}}$ due to rescaling $x \mapsto \ell^{-1}x$, $y \mapsto \ell^{-1}y$, $\tau \mapsto \ell^{-1}\tau$ and $h \mapsto \hbar = \ell^{-\frac{3}{2}}h$ we have

$$e^{\mathsf{T}}_{\epsilon_1,h}(x,y,0) - e_h^{\mathsf{W}}(x,y,0) = O\big(\hbar^{1-d}\times \ell^{-d}\big) = O\big(h^{1-d}\ell^{\frac{d-3}{2}}\big) \tag{2.26}$$

which is $O(h^{-\frac{4}{3}})$ for $d=2$, $O(h^{-2})$ for $d=3$ and $O(h^{1-d})$ for $d \ge 4$, $\ell \le 1$. Statement (i) has been proven.

(c) As $d=2$ for a toy-model operator we can replace with $O(h^s)$ error $e^{\mathsf{T}}_{\epsilon_1,h}(x,y,0)$ by $e^{\mathsf{T}}_{\infty,h}(x,y,0)$ which due to (2.7) is exactly the first term in the right-hand expression in (2.25); so Statement (ii) is just a definition. □

Remark 2.7. (i) Due to (2.13) the first term in the right-hand expression of (2.25) can be rewritten as

$$-\frac{2}{\pi}\int_{-\infty}^{0}\int_{0}^{\infty} \tau\cos\Big(\beta(x_1+y_1+2\tau) + \frac{1}{4\beta}\big((x_1-y_1)^2+z_2^2\big) - \frac{1}{3}\beta^3\Big)\,d\beta d\tau. \tag{2.27}$$

As $x=y$ it becomes

$$-\frac{2}{\pi}\int_{-\infty}^{0}\int_{0}^{\infty} \tau\cos\big(2\beta(x_1+\tau) + \frac{1}{3}\beta^3\big)\,d\beta d\tau = -2\int_{-\infty}^{0}\tau\,\mathsf{Ai}(-2(x_1+\tau))\,d\tau \tag{2.28}$$

with Airy function $\mathsf{Ai}(\cdot)$.

(ii) One can prove easily that

$$e^{\mathsf{W}}(x,y,\tau) = \frac{1}{4\pi^2 r^2}\int_0^{2\pi}\int_0^{h^{-1}r\rho} \cos(\sigma\cos(\theta))\,\sigma\,d\sigma\,d\theta \tag{2.29}$$
$$\text{with}\quad \rho = \sqrt{2(\tau V((x+y)/2))},\quad r=|x-y|$$

and in the framework of Proposition 2.4(ii)

$$|e^{\mathsf{W}}_h(x,y,0)| \le Ch^{-\frac{1}{2}}\ell(x,y)^{-\frac{5}{4}}. \tag{2.30}$$

(iii) In Proposition 3.6 in much more general settings will be proven that in the framework of Proposition 2.4(i) $e^{\mathsf{W}}(x,y,0)$ provides a good approximation for $e^{\mathsf{T}}_{T,h}(x,y,0)$ with $T=\epsilon'\sqrt{\ell(x,y)}$.

2.3 Generalized toy-model

Let us consider now a generalized toy-model operator with $d=2$, $g^{jk}=\delta_{jk}$, $V_2(x)=0$, $V_1=-kx_1$ and $V(x)=-x_1-\frac{1}{2}k^2x_1^2$:

$$\bar{A}_{k,h} := \frac{1}{2}h^2D_1^2+\frac{1}{2}h^2(D_2-kx_1)^2+V(x) \tag{2.31}$$
$$= \frac{1}{2}h^2D_1^2+\frac{1}{2}h^2D_2-(1+khD_2)x_1$$

As in Subsection 2.2 considering equation to propagator and making h-Fourier transform we get equation similar to (2.4):

$$\Bigl(\tau-\frac{1}{2}\xi_1^2-\frac{1}{2}\xi_2^2+ih(1+k\xi_2)\partial_{\xi_1}\Bigr)\widehat{u^{\pm}_h} = \mp ie^{ih^{-1}(-y_1\xi_1-y_2,\xi_2)} \tag{2.32}$$

as $\mp\mathsf{Im}\,\tau>0$; rewriting it as

$$\partial_{\xi_1}\Bigl(e^{-ih^{-1}(1+k\xi_2)^{-1}((\tau-\xi_2^2/2)\xi_1-\xi_1^3/6)}\widehat{u^{\pm}_h}\Bigr) =$$
$$\mp(1+k\xi_2)^{-1}e^{-ih^{-1}(1+k\xi_2)^{-1}((\tau-\xi_2^2/2)\xi_1-\xi_1^3/6)-ih^{-1}(y_1\xi_1+y_2\xi_2)},$$

we arrive to expression similar to (2.5)

$$\widehat{u^{\pm}_h}(\xi,y,\tau) = \mp(1+k\xi_2)^{-1}e^{ih^{-1}((\tau-x_2^2/2)\xi_1-\xi_1^3/6)} \tag{2.33}$$
$$\times\int_{\mp\infty}^{\xi_1} e^{-ih^{-1}(1+k\xi_2)^{-1}((\tau-\xi_2^2/2)\eta_1-\eta_1^3/6)-ih^{-1}(y_1\eta_1+y_2\xi_2)}\,d\eta_1$$

and making partial inverse h-Fourier transform we arrive to the final answer similar to (2.6):

$$(2.34)\quad F_{t\to h^{-1}\tau}u_h^{\pm} = \mp(2\pi h)^{-2}\iint(1+k\xi_2)^{-1}$$
$$\times\int_{\mp\infty}^{\xi_1} e^{ih^{-1}(1+k\xi_2)^{-1}((\tau-\xi_2^2/2)(\xi_1-\eta_1)-(\xi_1^3-\eta_1^3)/6)+ih^{-1}(x_1\xi_1-y_1\eta_1+(x_2-y_2)\xi_2)}\,d\eta_1 d\xi_1 d\xi_2.$$

Then as $\tau\in\mathbb{R}$ we get (cf. (2.7))

$$(2.35)\quad F_{t\to h^{-1}\tau}u_h = (2\pi h)^{-2}\iiint(1+k\xi_2)^{-1}$$
$$e^{ih^{-1}(1+k\xi_2)^{-1}((\tau-\xi_2^2/2)(\xi_1-\eta_1)-(\xi_1^3-\eta_1^3)/6)+ih^{-1}(x_1\xi_1-y_1\eta_1+(x_2-y_2)\xi_2)}\,d\eta_1 d\xi_1 d\xi_2.$$

We can see easily that for $|k|\le C$ and $\min(|x_1|,|y_1|)\le\epsilon$ the phase function

$$(2.36)\quad \Phi_k(x_1,y_1,z_2;\xi_1,\eta_1,\xi_2;\tau)$$
$$= x_1\xi_1 - y_1\eta_1 + z_2\xi_2 + (1+k\xi_2)^{-1}[(\tau-\frac{1}{2}\xi_2^2)(\xi_1-\eta_1)-\frac{1}{6}(\xi_1^3-\eta_1^3)]$$

has four stationary points (with respect to $(\xi_1,\eta_1,\xi_2)\in\mathbb{C}^3$): on all of them $|\xi_2|\le C_0\sqrt{\epsilon}$,

$$(2.37)\qquad 2x_1 = (1+k\xi_2)^{-1}[-2\tau+\xi_1^2+\xi_2^2],$$
$$(2.38)\qquad 2y_1 = (1+k\xi_2)^{-1}[-2\tau+\eta_1^2+\xi_2^2],$$

and

$$(2.39)\quad z_2 = \xi_2(1+k\xi_2)^{-1}(\xi_1-\eta_1)$$
$$+k(1+k\xi_2)^{-2}\big[(\tau-\frac{1}{2}\xi_2^2)(\xi_1-\eta_1)-\frac{1}{6}(\xi_1^3-\eta_1^3)\big]$$

and for corresponding trajectories on the energy level τ

$$(2.40)\qquad T(x,y) = (1+k\xi_2)^{-1}|\xi_1-\eta_1|.$$

Then the same conclusion as for $k=0$ holds:

(a) Proposition 2.3 remains true;

(b) All statements of Proposition 2.4 remain true with regular, singular and shadow zones defined exactly by (2.17), (2.22) and (2.19).

(c) All statements of Remark 2.5 remain true.

Again consider $\partial_\tau J_k(\tau, x_1, y_1, z_2)$ this time based on (2.35):

$$\frac{i}{4\pi^2 i}\iiint (1+\kappa\xi_2)^{-2}(\xi_1-\eta_1) \times e^{i(1+\kappa\xi_2)^{-1}(\tau-\xi_2^2/2)(\xi_1-\eta_1)-(\xi_1^3-\eta_1^3)/6)+ih^{-1}(x_1\xi_1-y_1\eta_1+z_2\xi_2)}\, d\eta_1 d\xi_1 d\xi_2.$$

Plugging $\xi_1 = \alpha+\beta$, $\eta_1 = \alpha-\beta$ and then substituting $\beta := (1+\kappa\xi_2)\beta$ we arrive to

$$\frac{i}{\pi^2}\iiint d\alpha d\beta d\xi_2 \times \tag{2.41}$$
$$\beta \exp\Big[i\Big((2\tau-\xi_2^2-\alpha^2)\beta-\frac{1}{3}(1+\kappa\xi_2)\beta^3+(1+\kappa\xi_2)(x_1+y_1)\beta+z_2\xi_2+(x_1-y_1)\alpha\Big)\Big].$$

Then we can calculate integrals with respect to α and ξ_2, arriving to formula similar to (2.27):

$$-\frac{1}{\pi}\int_{-\infty}^{\infty} d\beta \times \tag{2.42}$$
$$\exp\Big[i\Big((2\tau+x_1+y_1)\beta-\frac{1}{3}\beta^3+\frac{1}{4\beta}(x_1-y_1)^2+\frac{1}{4\beta}\big(z_2-\underbrace{\frac{1}{3}\kappa\beta^3-\kappa(x_1+y_1)\beta}\big)^2\Big)\Big].$$

(a) Consider first $\beta \asymp \sqrt{\ell}$. Observe that selected terms in the phase in (2.43) are $O(\ell^{\frac{3}{2}})$ and squared and divided by β they are $O(\ell^{\frac{5}{2}})$ and therefore assuming that $\ell \le h^{\frac{2}{5}}$ we arrive to the following conclusion: if we neglect their square the error will not exceed

$$\begin{cases} Ch^{-\frac{1}{2}}\ell^{-\frac{5}{4}} \times \ell^{\frac{5}{2}}h^{-1} = O(h^{-1}) & \text{in the regular zone,}\\ Ch^{-\frac{2}{3}}\ell^{-1} \times \ell^{\frac{5}{2}}h^{-1} = O(h^{-\frac{5}{3}}\ell^{\frac{3}{2}}) & \text{in the singular zone,}\\ Ch^{(2s-2)/3}\ell^{-s} & \text{in the shadow zone.}\end{cases} \tag{2.43}$$

(b) On the other hand, consider $\beta \le \epsilon\sqrt{\ell}$ and therefore $\ell^0 := |z_2|+|x_1-y_1| \le \ell$, and then one can estimate the error in the cases $\ell^0\sqrt{\ell} \ge h$ and $\ell^0\sqrt{\ell} \le h$ and in both cases the error will be smaller than given by the first line of (2.43).

But then (2.42) becomes

(2.44)
$$\partial_\tau J_\kappa(\tau,x_1,y_1,z_2) \equiv -\frac{2}{\pi}\int_0^\infty \cos\Big((2\tau+x_1+y_1)\beta-\frac{1}{3}\beta^3+\frac{1}{4\beta}[(x_1-y_1)^2+z_2^2]\Big) \times \exp\Big[i\Big(-\frac{1}{6}\kappa z_2\beta^2-\frac{1}{2}\kappa z_2(x_1+y_1)\Big)\Big]\,d\beta$$

and and scaling back we arrive to

Proposition 2.8. *Consider generalized toy-model operator* (2.31) *in dimension* $d=2$. *Then as* $h^{\frac{2}{3}}\le \ell \le h^{\frac{2}{5}}$ *with an error* (2.42)

$$e^{\mathsf{T}}_{\epsilon_1,h}(x,y,0) \equiv e^{\mathsf{W}}_h(x,y,0)+e_{\mathsf{corr},k,h}(x,y,0) \tag{2.45}$$

with the correction term

(2.46)
$$e_{\mathsf{corr},k,h}(x,y,0) = \frac{1}{h^2}\int_{-\infty}^0 J_{h^{\frac{1}{3}}k}\big(h^{-\frac{2}{3}}\tau, h^{-\frac{2}{3}}x_1, h^{-\frac{2}{3}}y_1, h^{-\frac{2}{3}}(x_2-y_2)\big)\,d\tau - e^{\mathsf{W}}_h(x,y,0)$$

where $\partial_\tau J_\kappa(\tau,x_1,y_1,z_2)$ *defined by* (2.44) *and* $e^{\mathsf{W}}_h(x,y,0)$ *by* (2.29).

Remark 2.9. Obviously $J_\kappa(\tau,x_1,y_1,0)=J_0(\tau,x_1,y_1,0)$.

3 General case

In the framework of Theorems 1.1 and 1.2 without any loss of the generality one can assume that

$(3.1)_{1,2}$ $V(x)=x_1W(x)$, $W(0)=1$,

$(3.2)_{1-3}$ $\bar{x}=(\bar{x}_1,0)$ $\bar{y}=(\bar{y}_1,\bar{y}')$, $\bar{y}_1\le\bar{x}_1$,

$(3.3)_{1-4}$ $g^{j1}=\delta_{j1}$, $g^{jk}(0)=\delta_{jk}$, $V_1=0$, $V_j(0)=0$,

where $\bar{x}$ and $\bar{y}$ are two "target" points. Indeed, we can reach this by a change of variables. We will assume that these assumptions are fulfilled until the end of this section. Then according to (1.12)

$$\ell(x,y)\asymp \max(|x_1|,\,|y_1|,\,|x'-y'|,\,h^{\frac{2}{3}}). \tag{3.4}$$

3.1 Preliminary remarks

Proposition 3.1. *(i) The following estimate holds*

$$|F_{t\to h^{-1}\tau}\Big(\chi_T(t)u_h(x,y,t)\Big)| \le Ch^{1-d}\ell^{(d-2)/3}\hbar^s \tag{3.5}$$

with $|\tau| \le \epsilon T^2$, $T = \max(C_0\ell^{\frac12},\, h^{\frac13})$ *and* $\hbar = T^{-3}h$.

(ii) Furthermore, if $\ell^0 = \ell^0(x,y) := |x-y| \ge \epsilon_0\ell$ *and* $\ell \ge h^{\frac23}$, *then in the same framework*

$$|F_{t\to h^{-1}\tau}\Big(\bar\chi_T(t)u_h(x,y,t)\Big)| \le Ch^{1-d}\ell^{(d-2)/3}\hbar^s. \tag{3.6}$$

Here and below $\ell = \ell(x,y)$, $\chi \in \mathscr{C}_0^\infty([-1,-\frac12]\cup[\frac12,1])$, $\bar\chi \in \mathscr{C}_0^\infty([-1,1])$ *equal* 1 *on* $[-\frac12,\frac12]$, s *is an arbitrarily large exponent and* $\delta > 0$ *is an arbitrarily small exponent.*

Proof. (a) Consider first T which is a small constant: $T = \epsilon_3$ (which we can assume is 1; otherwise we achieve it by scaling) and assume only that $T \ge C_0\sqrt{\ell}$. Then Statement (i) follows from the fact that the propagation speed with respect to ξ_1 is disjoint from 0 plus ellipticity arguments.

Statement (ii) follows from the fact that he propagation speeds with respect to x and ξ are bounded.

(b) Then we scale $x \mapsto T^{-2}x$, $y \mapsto T^{-2}y$, $\ell \mapsto T^{-2}\ell$, $\tau \mapsto T^{-2}\tau$ and $h \mapsto \hbar = T^{-3}h$. □

Proposition 3.2. *The following estimates hold for* $\tau\colon |\tau| \le \epsilon$, $T\colon h^{\frac23} \le T \le \epsilon_1$*:*

$$|F_{t\to h^{-1}\tau}\Big(\bar\chi_T(t)u_h(x,y,t)\Big)| \le Ch^{1-d}, \tag{3.7}$$

$$|e_h(x,y,\tau+h) - e_h(x,y,\tau) \le Ch^{1-d}, \tag{3.8}$$

and for $T \ge T^* := C_0\max(\sqrt{\ell},\, h^{\frac13})$

$$|e_h(x,y,0) - e^{\mathsf{T}}_{T,h}(x,y,0)| \le Ch^{-2d/3}\hbar^s + Ch^{1-d} \tag{3.9}$$

again with $\hbar = T^{-3}h$.

Proof. (a) It is sufficiently to prove for $\tau = 0$ since shift of V by $\tau\colon |\tau| \le \epsilon_2$ preserves conditions[3)]. Simple scaling shows that if $\ell^0(x,y) \le \epsilon\ell(x,y)$ then with $T = T^*$ the left-hand expression of (3.7) with $T = T^*$ does not exceed $Ch^{1-d}T^{*\,2(d-2)/3}$.

On the other hand, considering partition by $T\colon T^* \le T \le \epsilon_1$ we see from Proposition 3.1(i) that the same left-hand expression but with $\bar\chi_T(t)$ replaced by $\bar\chi_T(t) - \bar\chi_{T^*}(t)$ also does not exceed this. So, with indicated T the same left-hand expression does not exceed this, which is $O(h^{1-d})$.

Finally, for $\ell^0(x,y) \ge \epsilon\ell(x,y)$ the same estimate (3.7) holds again due to (3.5).

(b) In particular, taking $x = y$ and $T = \epsilon_1$ and applying standard Tauberian arguments, Part I, we conclude that (3.8) holds with $x = y$. And then it holds for $x \ne y$ due to

$$|e_h(x,y,\tau+h) - e_h(x,y,\tau)| \le |e_h(x,x,\tau+h) - e_h(x,x,\tau)|^{\frac12}|e_h(y,y,\tau+h) - e_h(y,y,\tau)|^{\frac12}.$$

(c) Then (3.9) holds with $T = \epsilon_1$ due to standard Tauberian arguments, Part II. Then due to Proposition 3.1(i)

$$|e^{\mathsf T}_{2T,h}(x,y,0) - e^{\mathsf T}_{T,h}(x,y,0)| \le CT'^{\,d-3}\hbar^s$$

and summation by partition over $[T,\epsilon_1]$ results in the same answer which implies (3.9). □

Therefore

(3.10) To evaluate $e_h(x,y,\tau)$ modulo $O(h^{1-d})$ it is enough to consider only $T \le C_0 \max(\sqrt{\ell(x,y)},\ h^{\frac13-\delta})$.

3.2 Propagator as an oscillatory integral

Proposition 3.3. *Let $x, y \in B(0,\epsilon)$, $C_0\sqrt{\ell} \le T \le C_0\epsilon$) and $|\tau| \le \epsilon$. Then*

$$(3.11)\quad F_{t\to h^{-1}\tau}\Big(\bar\chi_T(t)u_h(x,y,t)\Big) = (2\pi h)^{-d}\iint e^{ih^{-1}\varphi(x,y,\xi_1,\eta_1,\xi',\tau)}b(x,y,\xi_1,\eta_1,\xi',\tau,h)\,d\xi_1 d\eta_1 + O(h^s)$$

[3)] Albeit changes x_1 and $\ell(x,y)$.

where phase function $\varphi(x,y,\xi_1,\eta_1,\xi',\tau)$ *differs from the phase function* $\bar\varphi(x,y,\xi_1,\eta_1,\xi',\tau)$ *in* (2.7) *by* $O(T^4)$ *and amplitude* $b(x,y,\xi_1,\eta_1,\xi',\tau,h)$ *differs from the amplitude* $\bar b_k(x,y,\xi_1,\eta_1,\xi',\tau,h)$ *there by* $O(T)$.

Proof. (a) Let us start from the formal construction. Let us apply h-Fourier transform with respect to x_1, y_1: $\hat u_h(\xi_1,x',\eta_1,y',t) = F_{x_1\to h^{-1}\xi_1,y_1\to -h^{-1}\eta_1}u_h$. Then the problem

$$(hD_t - A(x,hD_x))u_h(x,y,t) = 0, \tag{3.12}$$

$$u_h(x,y,0) = \delta(x-y) \tag{3.13}$$

becomes

$$(hD_t - A(-hD_{\xi_1},x',\xi_1,hD_{x'}))\hat u_h(\xi_1,x',\eta_1,y',t) = 0, \tag{3.14}$$

$$\hat u_h(\xi_1,x',\eta_1,y',t) = 2\pi h\delta(\xi_1-\eta_1)\delta(x'-y')\,. \tag{3.15}$$

This operator is x_1-microhyperbolic where x_1 is a variable dual to ξ_1. Then using the standard arguments[4] we can construct $\hat u_h(\xi_1,x',\eta_1,y',t)$ near energy level 0 for $x,y\in B(0,\epsilon_1)$ as a "simple" oscillatory integral

$$\begin{aligned}&\hat U_h(\xi_1,x',\eta_1,y',t)\\ &\quad = (2\pi h)^{1-d}\int e^{ih^{-1}(\psi(\xi_1,x',\eta_1,y',\theta,t)+a(-\theta_1,y',\theta')t)}b(\xi_1,x',\eta_1,y',\theta,h)\,d\theta,\end{aligned} \tag{3.16}$$

with the phase satisfying

$$a(-\partial_{\xi_1}\psi,x',\partial_{x'}\psi) = a(-\theta_1,y',\theta'), \tag{3.17}$$

$$\psi|_{\xi_1=\eta_1} = \langle x'-y',\theta'\rangle, \tag{3.18}$$

$$\nabla_{\xi_1,x'}\psi|_{\xi_1=\eta_1,x'=y'} = \theta \tag{3.19}$$

and with amplitude $b(\xi_1,x',\eta_1,y',\theta,h)$ decomposing into asymptotic series

$$b(\xi_1,x',\eta_1,y',\theta,h)\sim\sum_{n\ge 0}b_n(\xi_1,x',\eta_1,y',\theta)h^n. \tag{3.20}$$

Then for $|\tau|\le\epsilon$

$$\begin{aligned}&F_{t\to h^{-1}\tau}\hat U_h(x,y,t)\\ &\quad = (2\pi h)^{2-d}\int_{\Sigma(\xi_1,y',\tau)} e^{ih^{-1}(\psi(\xi_1,x',\eta_1,y',\theta,\tau))}b(\xi_1,x',\eta_1,y',\theta,h)\,d\theta : d_\theta a\end{aligned} \tag{3.21}$$

[4] See f.e. [Shu], Theorem 20.1 or [Ivr3], Section 2.

where $\Sigma(\xi_1, y', \tau) = \{\theta\colon a(-\theta_1, y', y', \theta') = \tau\}$. Due to x_1-microhyperbolicity we can express θ_1 as function of $\xi_1, \eta_1, x', y', \tau$ and rewrite the right-hand expression as an integral over $\theta' \in \mathbb{R}^{d-1}$.

Finally, making inverse h-Fourier transform by ξ_1, η_1 and plugging $\theta' = \xi'$ we arrive to (3.11) for u_h with

$$\varphi(x, y, \xi_1, \eta_1, \xi', \tau) = \psi(\xi_1, x', \eta_1, y', \theta_1(\xi_1, x', \eta_1, y', \xi'\tau), \xi', \tau) + x_1\xi_1 - y_1\eta_1.$$

(b) To justify (3.11) for u_h we observe that for given $x, y \in B(0, \epsilon)$ and $T \le C_0\epsilon$ behaviour of$a(x, \xi)$ for $|x| \ge c\epsilon$ does not matter.

(c) Finally, since $\varphi = O(T^3)$ and relative error of A_h in comparison to generalized toy-model operator $\bar{A}_h$ is $O(T)$ we conclude that the relative errors in φ and B_h are also $O(T)$ and therefore absolute errors are $O(T^4)$ and $O(T^2)$ correspondingly. □

Therefore we arrive immediately to the following statements:

Proposition 3.4. *Let operator A_h satisfy* $(3.1)_{1,2}$, $(3.2)_{1-3}$ *and* $(3.3)_{1-4}$. *Then*

(i) In the framework of Proposition 2.3 estimate (2.12) *holds.*

(ii) In the framework of Proposition 2.4(i) estimate (2.15) *holds.*

(iii) In the framework of Proposition 2.4(ii) estimate (2.16) *holds.*

(iv) In the framework of Proposition 2.4(iii) estimate (2.18) *holds.*

(v) In the framework of Proposition 2.4(iv) estimate (2.20) *holds.*

Also observe that all statements of Remark 2.5 remain valid.

Corollary 3.5. *Let $d \ge 3$. Then $e_h(x, y, \tau) = e_h^{\mathsf{W}}(x, y, \tau) + O(h^{1-d})$. Furthermore $e_h(x, y, 0) = O(h^{1-d})$ unless $\nu(x) \asymp \nu(y) \ge h^{\frac{2}{3}}$ and $\ell^0(x, y) \le \epsilon\nu(x)$.*

Proof. If $\ell^0(x, y) \le \epsilon\ell(x, y)$, $\ell(x, y) \ge h^{\frac{2}{3}}$ we prove estimate (1.7) by simple rescaling. Otherwise the simple rescaling shows that $|e_{T,h}^{\mathsf{T}}(x, y, 0)| \le Ch^{1-d}$ and therefore $|e_h(x, y, 0)| \le Ch^{1-d}$ and one can see easily that in this case also $|e_h^{\mathsf{W}}(x, y, 0)| \le Ch^{1-d}$ due to stationary phase principle. □

So, Theorem 1.1 has been proven. From now we consider only $d = 2$.

Proposition 3.6. *Let $d = 2$, $\ell(x,y) \le \epsilon_1$ and $\ell^0(x,y) \le \epsilon\ell(x,y)$. Then*

$$|e^{\mathsf{T}}_{T_*,h}(x,y,0) - e^{\mathsf{W}}_h(x,y,0)| \le Ch^{-1} + C\ell^{-2}. \tag{3.22}$$

Proof. Following arguments of [Ivr3], Section 2 one can prove easily that after rescaling the left-hand expression does not exceed

$$C\hbar^{-\frac12}\ell^{0\,-\frac32}\min\Big(1, \frac{\ell^{0\,3}\ell}{\hbar}\Big) + C\hbar^{-1}\ell + C \tag{3.23}$$

where the first term is the difference between

$$(2\pi\hbar)^{-2}\iint e^{i\hbar^{-1}(it\tau-\varphi^0(x,y,\theta))}B_0(x,y,\theta,t)\chi_{T_*/\tau}(t)\,d\theta dt$$

and $e^{\mathsf{W}}_h(x,y,0)$. Further, the second term comes from the amplitude $B_1\hbar$ in the same expression (and acquires factor ℓ due to scaling of V_j) and the third term comes from the amplitude $B_2\hbar^2$ in the same expression, which are in $e^{\mathsf{T}}_{T_*,h}(x,y,0)$ but are skipped in $e^{\mathsf{W}}_h(x,y,0)$ [5] .

The first term in (3.23) does not exceed $C\hbar^{-1}\ell^{\frac12}$, so the left-hand expression of (3.22) does not exceed $C\hbar^{-1}\ell^{\frac12} + C$. Scaling back (that is multiplying by ℓ^{-2} and plugging $\hbar = \ell^{-\frac32}h$) we get $Ch^{-1} + C\ell^{-2}$. □

Combining Propositions 3.6 and 3.4(ii) we arrive to

Corollary 3.7. *In the framework of Proposition 3.6*

$$|e_h(x,y,0) - e^{\mathsf{W}}_h(x,y,0)| \le Ch^{-\frac12}\ell^{-\frac54} + C\ell^{-2} + Ch^{-1}. \tag{3.24}$$

In particular, for $\ell \ge h^{\frac25}$ the right-hand expression does not exceed Ch^{-1}.

3.3 Perturbation methods

Now we compare $e^{\mathsf{T}}_{T,h}(x,y,0)$ with the same function for generalized toy-model operator (2.31) assuming that

$$V_{x_1}(0) = -1, \qquad V_1 = 0, \qquad V_{2\,x_1}(0) = k \tag{3.25}_{1-3}$$

because we can achieve $(3.25)_1$ by scaling and $(3.25)_3$ as a definition. Then

[5] These amplitudes $B_n(x,y,\theta,t)$ should not be confused with the amplitudes b_n in the proof of Proposition 3.3.

(3.26) The phase function $\varphi(x,y,\xi_1,\eta_1,\xi',\tau)$ in expression (3.11) differs from the phase function $\bar\varphi(x,y,\xi_1,\eta_1,\xi',\tau)$ in (2.35) by $O(T^5)$ and amplitude $b(x,y,\xi_1,\eta_1,\xi',\tau,h)$ differs from the amplitude $\bar b_k(x,y,\xi_1,\eta_1,\xi',\tau,h)$ there by $O(T^2)$.

Indeed, in comparison to the Part (c) of the proof of Proposition 3.4 now the relative error in A_h in comparison to generalized toy-model operator $\bar A_{k,h}$ is $O(T^2)$ we conclude that the relative errors in φ and B_h are also $O(T^2)$.

Let us assume that

$$\ell(x,y)\le h^{\frac25}. \tag{3.27}$$

Proposition 3.8. *Let* (3.25) *and* (3.27) *be fulfilled. Then*

(a) Asymptotics

$$\bigl(e^{\mathsf T}_{T^*,h}(x,y,0)-e^{\mathsf W}_h(x,y,0)\bigr)-\bigl(\bar e^{\mathsf T}_{k,h}(x,y,0)-\bar e^{\mathsf W}_{k,h}(x,y,0)\bigr)=O(h^{-1}) \tag{3.28}$$

holds for all $x,y\in B(0,\epsilon)$: $|x-y|\le \epsilon\ell(x,y)$.

(b) Estimates

$$|e^{\mathsf T}_{T^*,h}(x,y,0)-\bar e^{\mathsf T}_{k,T^*,h}(x,y,0)| \tag{3.29}$$
$$\le\begin{cases} Ch^{-\frac12}\ell^{-\frac54}\times\ell^{\frac52}h^{-1}=O(h^{-1}) & \text{in the regular zone,}\\ Ch^{-\frac23}\ell^{-1}\times\ell^{\frac52}h^{-1}=O(h^{-\frac53}\ell^{\frac32}) & \text{in the singular zone,}\\ Ch^{(2s-2)/3}\ell^{-s} & \text{in the shadow zone}\end{cases}$$

and

$$e^{\mathsf W}_h(x,y,0)-\bar e^{\mathsf W}_{k,h}(x,y,0)=O(h^{-1}) \tag{3.30}$$

hold for all $x,y\in B(0,\epsilon)$: $|x-y|\ge\epsilon\ell(x,y)$ *where* $\bar e_{k,h}(x,y,\tau)$, $\bar e^{\mathsf W}_{k,h}(x,y,\tau)$ *and* $\bar e^{\mathsf T}_{k,T,h}(x,y,\tau)$ *are defined for a generalized toy-model operator* (2.31).

Proof. (a) Observe that $A_h-\bar A_{k,h}=O(\ell^2)$. Indeed, when we replace V by x_1 the error is $O(\ell^2)$ due to $(3.25)_1$ and when we replace V_1 by kx_1 the error is $O(\ell^{\frac52})$ due to $(3.25)_{2-3}$ where extra factor $\ell^{\frac12}$ comes from factor ξ_2.

Therefore as $|x-y| \le \epsilon\ell(x,y)$ the following estimate holds

$$|\big(e^{\mathsf{T}}_{T^*,h}(x,y,0) - e^{\mathsf{T}}_{T_*,h}(x,y,0)\big) - \big(\bar{e}^{\mathsf{T}}_{k,T^*,h}(x,y,0) - \bar{e}^{\mathsf{T}}_{k,T_*,h}(x,y,0)\big)| \le Ch^{-1} \tag{3.31}$$

and as $|x-y| \ge \epsilon\ell(x,y)$ estimate (3.30) holds, where we recall that $T^* = C_0\sqrt{\ell}$ and $T_* = \epsilon'\sqrt{\ell}$ with arbitrarily small constant ϵ'. The proof is similar to the proof of Proposition 2.8 and the right-hand expression in (3.29) is exactly expression (2.43).

(b) Further, if $|x-y| \le \epsilon_1\ell(x,y)$ then combining (3.31) and (3.22) we conclude that the left-hand expression of (3.27) does not exceed $Ch^{-1} + C\ell^{-2}$ which implies (3.27) for $\ell \ge h^{\frac{1}{2}}$.

However, for $\ell(x,y) \le h^{\frac{1}{2}}$ we need more subtle arguments. Namely, recall that $C\ell^{-2}$ comes from $\hbar^0 \times \ell^{-2}$ which in turn comes from the decomposition (3.16) with $B_n = O(\ell^{-\frac{3n}{2}})$. However since operators A_h and $\bar{A}_{k,h}$ after rescaling differ by $O(\ell^2)$ rather than $O(\ell)$ we can estimate $B_n - \bar{B}_{k,n} = O(\ell^{1-\frac{3n}{2}})$ and therefore

$$|\big(e^{\mathsf{T}}_{T_*,h}(x,y,0) - e^{\mathsf{W}}_{h}(x,y,0)\big) - \big(\bar{e}^{\mathsf{T}}_{T_*,k,h}(x,y,0) - \bar{e}^{\mathsf{W}}_{k,h}(x,y,0)\big)| \le Ch^{-1} + C\ell^{-\frac{3}{2}} \le Ch^{-1}. \tag{3.32}$$

(c) On the other hand, if $\ell^0(x,y) \ge \epsilon_1\ell(x,y)$ then one can prove easily

$$|e^{\mathsf{W}}_{h}(x,y,0) - \bar{e}^{\mathsf{W}}_{k,h}(x,y,0)| \le Ch^{-1} + C\ell^{-\frac{3}{2}} \le Ch^{-1}. \tag{3.33}$$

□

Remark 3.9. (i) One can prove easily that

$$e^{\mathsf{W}}_{h}(x,y,0) - \bar{e}^{\mathsf{W}}_{k,h}(x,y,0) = O(h^{-2}\ell^2) \tag{3.34}$$

even as $|x-y| \ge \epsilon\ell(x,y)$. Therefore

$$e_h(x,y,0) = \bar{e}_{k,h}(x,y,0) + O(h^{-1}) \qquad \text{as } \ell(x,y) \le h^{\frac{1}{2}}. \tag{3.35}$$

However, one can easily note that it may fail as $\ell(x,y) \gg h^{\frac{1}{2}}$. Thus, as $h^{\frac{1}{2}} \le \ell(x,y) \le h^{\frac{2}{5}}$ we need to use both terms $e^{\mathsf{W}}_{h}(x,y,0)$ and $e_{\mathsf{corr},k,h}(x,y,0)$ in the asymptotics.

(ii) We can replace k by $k=0$ and preserve remainder estimate $O(h^{-1})$ if and only if $|k||x_2-y_2| \lesssim \ell^{\frac14} h^{\frac12}$. In particular, we can do it as $x_2=y_2$.

(iii) We know that in any dimension $d \ge 2$ $e_h(x,y,0)$ is $O(h^s)$ if either $x_1 < -h^{\frac23-\delta}$ or $y_1 < -h^{\frac23-\delta}$ or $4x_1y_1 \le (1-\epsilon)|x'-y'|^2 - h^{\frac43-\delta}$ but it is not the case for $e_h^{\mathsf{W}}(x,y,0)$!

3.4 Proof of Theorem 1.2

Proof of Theorem 1.2. Without any loss of the generality one can assume that $\tau=0$, and assumptions $(3.1)_{1,2}$, $(3.2)_{1-3}$, $(3.3)_{1-4}$ and $(3.25)_{1-3}$ are fulfilled. Then the left-hand expression in (1.11) is $(x_1-x_2)^2-4x_1y_1$ modulo $O(\ell^3)$.

As $\ell(x,y) \ge h^{\frac23-\delta}$ we can replace in the Tauberian expressions $T=C_0\sqrt{\ell}$ by $T=\epsilon_1$ with $O(h^s)$ error. Then

(a) Statements (i) and (ii) of Theorem 1.2 follow immediately from Proposition 3.4.

(b) Statements (iii) and (iv) of Theorem 1.2 follow immediately from Proposition 3.8.

(c) Further, as $\ell(x,y) \le h^{\frac23-\delta}$ we can take $T^*=h^{\frac13-\delta}$ and apply the same arguments.

(d) Finally, Statement (v) of Theorem 1.2 follows from (2.44) and (2.28). □

4 Generalizations and final remarks

Remark 4.1. (i) In any dimension $d \ge 1$ without assumption (1.3) the simple rescaling with the scaling function

$$\gamma_x = (\epsilon|V(x)-\tau| + h^{\frac23}) \tag{4.1}$$

results in the estimate

$$|e_h(x,y,\tau) - e_h^{\mathsf{W}}(x,y,\tau)| \le Ch^{1-d}\gamma_x^{(d-3)/2} \tag{4.2}$$

as long as $\ell(x,y) \le \gamma_x$ (and then $\gamma_y \asymp \gamma_x$) [6].

(ii) To explore the case $\ell(x,y) \ge \gamma_x$ observe that the standard Tauberian method, Part I, implies

$$e_h(x,x,\tau + h\gamma_x^{-\frac12}) - e_h(x,x,\tau) \le Ch^{1-d}\gamma_x^{(d-3)/2} \tag{4.3}$$

Then

$$|e_h(x,y,\tau + h\gamma_*^{-\frac12}) - e_h(x,y,\tau)| \le Ch^{1-d}\gamma^{*(d-1)/4}\gamma_*^{(d-5)/4} \tag{4.4}$$

with $\gamma^* = \max(\gamma_x,\, \gamma_y)$ and $\gamma_* = \min(\gamma_x,\, \gamma_y)$.

Applying standard Tauberian methods, Part II, we arrive to the estimate

$$\begin{aligned} |e_h(x,y,\tau)| &\le CT_{x,y}^{-1}\gamma_*^{\frac12} \times h^{1-d}\gamma^{*(d-1)/4}\gamma_*^{(d-5)/4} \\ &\le Ch^{1-d}\gamma^{*(d-3)/4}\gamma_*^{(d-3)/4} = Ch^{1-d}\gamma_x^{(d-3)/4}\gamma_y^{(d-3)/4} \end{aligned} \tag{4.5}$$

where $T_{x,y} \gtrsim |x-y|^{\frac12} \ge \gamma^{*\frac12}$ is the minimal propagation time between x and y on the energy level τ.

Therefore, in the worst case remainder estimate is $O(h^{-\frac23})$ for $d=1$, $O(h^{-\frac43})$ for $d=2$ and $O(h^{1-d})$ for $d \ge 3$.

Remark 4.2. (i) Under assumption (1.3) following arguments of Proposition 3.2 we upgrade (4.3) to

$$e_h(x,x,\tau + h) - e_h(x,x,\tau) \le Ch^{1-d}\gamma_x^{(d-2)/2} \tag{4.6}$$

and then

$$|e_h(x,y,\tau) - e^{\mathsf{T}}_{\epsilon_1,h}(x,y,\tau)| \le Ch^{1-d}\gamma_x^{(d-2)/4}\gamma_y^{(d-2)/4}. \tag{4.7}$$

Therefore, in the worst case remainder estimate is $O(h^{-\frac13})$ for $d=1$, $O(h^{-1})$ for $d=2$ and $O(h^{1-d})$ for $d \ge 3$.

(ii) However to replace $e^{\mathsf{T}}_{\epsilon_1,h}(x,y,\tau)$ by $e^{\mathsf{W}}_h(x,y,\tau)$ and preserve this remainder estimate we may need to add a correction term (in any dimension!)

[6] As $d=1$ Weyl expression should be modified so it contains a true eikonal rather than $\langle x-y,\theta\rangle - a(\frac12(x+y),\theta)$ approximation.

which in the regular zone is of magnitude $h^{(1-d)/2}\ell(x,y)^{-(d+3)/4}$ and in the singular zone is $O\big(h^{(2-3d)/6}\ell(x,y)^{-(d+2)/4}\big)$ [7]. For toy-model (2.1) it is

$$e_{\mathrm{corr},h}(x,y,\tau)=\frac{2}{(2\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2}+1)h^{d+1}}\int_{-\infty}^{\tau}\int_{0}^{\infty}(\tau-\tau')^{\frac{d}{2}} \tag{4.8}$$
$$\times\cos\Big[h^{-1}\Big(\beta(x_1+y_1+2\tau)+\frac{1}{4\beta}\big((x_1-y_1)^2+|z'|^2\big)-\frac{1}{3}\beta^3\big)\Big)-\frac{\pi}{4}(d+2)\Big]\,d\beta d\tau$$
$$-e_h^{\mathsf{W}}(x,y,\tau)$$

with $z'=x'-y'$. Then $e_{\mathrm{corr},h}(x,x,\tau)$ can be expressed through Airy function.

Remark 4.3. To get rid off condition (1.3) we can use the simple rescaling with the scaling function

$$\gamma_{2x}=(\epsilon|V(x)-\tau|+|\nabla V|^2+h)^{\frac{1}{2}}; \tag{4.9}$$

then $\hbar=\gamma_{2x}^{-2}h$ and the remainder estimates are not spoiled in dimensions $d\ge 2$, but in dimension $d=1$ it becomes $O(\hbar^{-\frac{1}{3}}\gamma_{2x}^{-1})=O(h^{-\frac{1}{3}}\gamma_{2x}^{-\frac{1}{3}})$ which is $O(h^{-\frac{1}{2}})$ in the worst case.

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[7] As $d=1$ there is no shadow zone but there still is a singular zone where $\gamma_x\not\asymp\gamma_y$.

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Pointwise Spectral Asymptotics near Singularity *,†

Victor Ivrii‡

Abstract

We establish semiclassical asymptotics and estimates for the $e_h(x,x,\tau)$ where $e_h(x,y,\tau)$ is the Schwartz kernel of the spectral projector for a second order elliptic operator inside domain with power singularity in the origin. While such asymptotics for its trace $\mathsf{N}_h(\tau) = \int e_h(x,x,\tau)\,dx$ are well-known, the poinwise asymptotics are much less explored.

Our main tools: microlocal methods, improved successive approximations and geometric optics methods.

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Chapter 29

Introduction

In this paper we consider semiclassical diagonal poinwise asymptotics of $e_h(x,x,\tau)$ where $e_h(x,y,\tau)$ is the Schartz kernel of the spectral projector for a *toy-model operator*

$$A_h^0 = -\frac{1}{2}h^2\Delta - |x|^{-2\alpha} \qquad \text{with } \ 0<\alpha<1 \tag{0.1}$$

or a *perturbed toy-model operator*

$$A_h = \sum_{j,k} g^{jk}(hD_j - V_j)(hD_k - V_k) + V, \tag{0.2}$$

assuming that

$$D^\sigma(g^{jk} - \delta_{jk}) = O(|x|^{\beta-|\sigma|}), \tag{0.3}$$

$$D^\sigma V_j(x) = O(|x|^{-\alpha+\beta-|\sigma|}) \tag{0.4}$$

and

$$D^\sigma(V(x) + |x|^{-2\alpha}) = O(|x|^{-2\alpha+\beta-|\sigma|}) \qquad \forall \sigma\colon |\sigma|\le K \tag{0.5}$$

with $\beta>0$ and sufficiently large K.

1 Toy-model operator

Let $A_h := A_h^0$ be toy-model operator (0.1) in $\mathbb{R}^d$. It is self-adjoint in $\mathscr{L}^2(\mathbb{R}^d))$. Let $E_h^0(\tau)$ be its spectral projector and $e_h^0(x,y,\tau)$ be a Schwartz kernel of $E_h^0(\tau)$.

Theorem 1.1. *Consider toy-model operator A_h^0. Then*

(i) For $h \in (0,1]$, $|x| \le 1$, $|y| \le 1$, $T \ge C_0$ the following estimate holds

$$|e_h(x,y,0) - e_{T,h}^{\mathsf{T}}(x,y,0)| \le Ch^s T^{-s} \tag{1.1}$$

with the Tauberian expression

$$e_{T,h}^{\mathsf{T}}(x,y,\tau) = h^{-1}\int_{-\infty}^{\tau} F_{t\to h^{-1}\tau'}\Big(\bar\chi_T u_h(x,y,t)\Big)\,d\tau' \tag{1.2}$$

where here and below $u_h(x,y,t)$ is the Schwartz kernel of $e^{ith^{-1}A_h}$, $\bar\chi \in \mathscr{C}_0^{(}[-1,1])$, $\chi^- \in \mathscr{C}_0([-1,-\frac12])$, $\chi^+ \in \mathscr{C}_0([\frac12,1])$, $\chi = \chi^+ + \chi^-$, $\chi_T(t) = \chi(t/T)$ etc.

(ii) For x: $|x| = 1$

$$|e_{T,h}^{\mathsf{T}}(x,x,0) - e_{T',h}^{\mathsf{T}}(x,x,0)| \le Ch^{\frac32-d} + C\bar\rho^{d-1}h^{1-d} \tag{1.3}$$

with

$$\bar\rho = \begin{cases} h^{\frac12-\delta} & \text{as } (1-\alpha)^{-1} \notin 2\mathbb{Z},\\ h^{\frac13-\delta} & \text{as } (1-\alpha)^{-1} \in 2\mathbb{Z}. \end{cases} \tag{1.4}$$

with an arbitrary small exponent $\delta > 0$ and an arbitrarily small constant $T' > 0$.

(iii) Furthermore, as $0 < \alpha \le \frac12$ one can skip the first term in the right-hand expression of (1.3).

Remark 1.2. We know that for small enough constant $T_* > 0$

$$e_{T_*,h}^{\mathsf{T}}(x,x,\tau) = e_h^{\mathsf{W}}(x,x,\tau) + O(h^{2-d}) \tag{1.5}$$

with the Weyl expression

$$e_h^{\mathsf{W}}(x,x,\tau) = (2\pi h)^{-d}\varpi_d(2(\tau - V(x)))^{\frac{d}{2}} \tag{1.6}$$

with ϖ_d a volume of $B(0,1) \subset \mathbb{R}^d$.

2 General case

Consider general operator (0.2) in $\mathscr{L}^2(X)$ with $0 \in X \subset \mathbb{R}^d$. If $\partial X \neq \emptyset$, assume that there are boundary conditions on ∂X.

Theorem 2.1. *Consider general operator* (0.2) *in* $\mathscr{L}^2(X)$ *with* $0 \in X \subset \mathbb{R}^d$. *Assume that conditions* (0.3)–(0.5) *are fulfilled in* $B(0,2)$ *and that*

(2.1) A_h *is a self-adjoint operator.*

Then

(i) For $h \in (0,1]$, $\epsilon_0 r \le |x| \le r$, $\epsilon_0 r \le |y| \le r$, $\bar{r} \le r \le r_0$, $T \ge C_0 r_0^{1+\alpha}$ *where* r_0 *is a small constant and* $\bar{r} = h^{1/(1-\alpha)}$, *the following estimates hold*

$$|e_h(x,y,0) - e^{\mathsf{T}}_{T,h}(x,y,0)| \le C h^{1-d} r^{2\alpha-\alpha d} \tag{2.2}$$

and

$$|e_h(x,y,0) - e^0_h(x,x,0) - e^{\mathsf{T}}_{T,h}(x,y,0) + e^{0\,\mathsf{T}}_{T,h}(x,y,0)| \le C h^{1-d} r^{2\alpha-\alpha d+\beta} \tag{2.3}$$

with the Tauberian expression $e^{\mathsf{T}}_{T,h}(x,y,\tau)$ *defined by* (1.2).

(ii) For x: $r_* = h^{1/(1-\alpha)-\delta} \le r = |x| \le r_0$

$$\begin{aligned} &|e^{\mathsf{T}}_{T,h}(x,x,0) - e^{\mathsf{T}}_{T',h}(x,x,0)| \\ &\quad \le C h^{\frac{3}{2}-d} r^{-\frac{3}{2}(1-\alpha)-\alpha d} + C\bar{\rho}^{d-1} h^{1-d} r^{-1+\alpha-\alpha d} + C h^{1-d} r^{-1+\alpha-\alpha d+\beta d-\beta} \end{aligned} \tag{2.4}$$

with

$$\bar{\rho} = \begin{cases} \hbar^{\frac{1}{2}-\delta} & \text{as } (1-\alpha)^{-1} \notin 2\mathbb{Z}, \\ \hbar^{\frac{1}{3}-\delta} & \text{as } (1-\alpha)^{-1} \in 2\mathbb{Z}, \end{cases} \qquad \hbar = h r^{-1+\alpha} \tag{2.5}$$

with an arbitrary small exponent $\delta > 0$, $K = K(s,\delta)$ *in* (0.3)–(0.5) *and* $T' = \epsilon' r^{1+\alpha}$ *with an arbitrarily small constant* $\epsilon' > 0$.

(iii) Further, for $r_* \le r \le r^* = h^{1/(1-\alpha+\beta)}$

$$\begin{aligned} &|e^{\mathsf{T}}_{T,h}(x,x,0) - e^{\mathsf{T}}_{T',h}(x,x,0) - e^{0\,\mathsf{T}}_{T,h}(x,x,0) + e^{0\,\mathsf{T}}_{T',h}(x,x,0)| \\ &\qquad \le C h^{\frac{1}{2}-d} r^{-\frac{1}{2}(1-\alpha)-\alpha d+\beta} + C\bar{\rho}^{d-1} h^{-d} r^{\beta-\alpha d} + C h^{-d} r^{-\alpha d+\beta d} \end{aligned} \tag{2.6}$$

(iv) Furthermore, as $0 < \alpha \le \frac{1}{2}$ *one can skip the first terms in the right-hand expressions of* (2.4) *and* (2.6).

Remark 2.2. (i) We know that in the framework of Theorem 2.1(ii)

$$|e^{\mathsf{T}}_{T',h}(x,x,0) - e^{\mathsf{W}}_{h}(x,x,0))| \le Ch^{2-d}r^{-2+2\alpha-\alpha d} + Ch^{1-d}r^{-\alpha d+\alpha-1+\beta}. \tag{2.7}$$

and in the framework of Theorem 2.1(ii)

$$|e^{\mathsf{T}}_{T',h}(x,x,0) - e^{\mathsf{W}}_{h}(x,x,0) - e^{0\,\mathsf{T}}_{T',h}(x,x,0) + e^{0\,\mathsf{W}}_{h}(x,x,0)| \le Ch^{2-d}r^{-2+2\alpha+\beta-\alpha d}. \tag{2.8}$$

(ii) Estimate (2.4) without the last term is estimate (1.3) scaled.

(iii) Combining estimates (2.2), (2.4) and (2.7) we can estimate

$$e_h(x,x,0) - e^{\mathsf{W}}_{h}(x,x,0)$$

and combining estimates (2.2), (2.6) and (2.8) we can estimate

$$e_h(x,x,0) - e^{\mathsf{W}}_{h}(x,x,0) - e^{0}_{h}(x,x,0) + e^{0\,\mathsf{W}}_{h}(x,x,0).$$

3 Plan of the paper.

In Section 30 we study a toy-model operator A^0_h. First, in Subsection 1 we study corresponding Hamiltonian trajectories. In Subsection 2 we study propagation of singularities, especially long-range propagation and propagation near $\{(x,\xi)\colon |x\times\xi| = 0\}$, and derive microlocal and local pointwise spectral asymptotics with the improved remainder estimates. Finally, in Subsection 3 we scale the previous results.

In Section 31 we study operator A_h which is perturbed a toy-model operator A^0_h. In Subsection 1 we study propagation of singularities, especially long-range propagation and propagation near $\{(x,\xi)\colon |x\times\xi| = 0\}$, and derive microlocal and local pointwise spectral asymptotics with the improved remainder estimates. In Subsection 2 by the metod of successive approximations the difference $e_h(x,x,\tau) - e^0_h(x,x,\tau)$. Finally, in Subsection 3 we consider cases $|x| \le \bar{r} = h^{1/(1-\alpha)}$ and $\bar{r} \le |x| \le r_* = h^{1/(1-\alpha)-\delta}$.

In the next paper (or the next variant of this paper) we consider out of diagonal asymptotics for such operators.

Chapter 30

Toy-model operator

1 Hamiltonian trajectories

Geometry: $\mathbf{d = 2}$. As $d = 2$ trajectories on the energy level 0 are defined in the polar coordinates by

$$\frac{dr}{dt} = \pm\sqrt{2r^{-2\alpha} - M^2r^{-2}}, \tag{1.1}$$

$$\frac{d\theta}{dt} = Mr^{-2} \tag{1.2}$$

with *angular momentum* M. Then

$$d\theta = \pm\frac{Mr^{-2}\,dr}{\sqrt{2r^{-2\alpha} - M^2r^{-2}}} = \pm\frac{M\rho^{-2}\,d\rho}{\omega\sqrt{2\rho^{-1} - M^2\rho^{-2}}}, \tag{1.3}$$

with $\rho = r^\omega$ with $\omega = 2 - 2\alpha$, $0 < \omega < 2$. The latter describes Kepler's parabolas as $M \neq 0$ in (θ, ρ) coordinates and therefore we need to perform a conformal transform $\rho \mapsto r = \rho^{1/\omega}$, $\theta \mapsto \omega^{-1}\theta$. Since increment of θ along parabola is $\pm 2\pi$, increment of θ along original Hamiltonian trajectory is $2\pi\omega^{-1}$ and therefore

(1.4) Every Hamiltonian trajectory with $M \neq 0$ has $n := -1 - \lceil -\omega^{-1}\rceil = -1 - \lceil -(2-2\alpha)^{-1}\rceil$ self intersections. In particular, the number of self-intersections is 0 for $0 < \alpha \le \frac{1}{2}$, 1 for $\frac{1}{2} < \alpha \le \frac{3}{4}$, etc.

All these trajectories are similar and obtained from one with $M = 1$ by scaling and rotation. Because of this

(1.5) Let $\frac{1}{2} < \alpha < 1$. Then through every point $x \neq 0$ there pass exactly n Hamiltonian trajectories with $M = M_k^+(x) = \mu_k |x|^{2-2\alpha}$, $0 < \mu_1 < ... < \mu_n$, returning to x once after k winding around 0 in the counter-clockwise direction, and also n Hamiltonian trajectories with $M = M_k^-(x) = -\mu_k |x|^{1-\alpha}$, returning to x once after k winding around 0 in the clockwise direction.

(1.6) Let $\frac{1}{2} < \alpha < 1$. Consider trajectory with $M = 1$. Denote by γ_k, $k = n, n-1, ..., 1$ the radius of $(n-k+1)$-th intersection counting from ininity; then $\gamma_n > \gamma_{n-1} > ... > \gamma_1$ and therefore $\mu_k = \gamma_k^{\alpha-1}$ satisfy $\mu_1 > ... > \mu_n$.

Let $\mu_0 = 0$.

(1.7) Let $x, y \in \mathbb{R}^2 \setminus 0$. There are n trajectories from x to y, winding around 0 in the counter-clockwise direction either $1, 2, ..., n$ or $0, 1, ..., n-1$ times correspondingly, and n trajectories from x to y windingaround 0 in the clockwise direction either $0, 1, ..., n-1$ or $1, 2, ..., n$ times correspondingly.

Indeed, consider two trajectories, issued from x in the corresponding direction, with the angular momenta M_1 and M_2, parametrized by θ. Then $|r_1(\theta) - |x|| > |r_2(\theta) - |x||$ provided $\theta \gtrless 0$ and $|M_1| < |M_2|$.

The trajectory is (rotated)

$$r^\omega = \frac{M^2}{(1 - \cos(\omega\theta))}. \tag{1.8}$$

Indeed, it follows from the fact that as $\omega = 1$ the trajectory is a parabola $r = \frac{A}{1-\cos(\theta)}$ and therefore in the general case $r^\omega = \frac{A}{(1-\cos(\omega\theta))}$. Applying dynamic equations

$$\frac{dr}{dt} = \pm\sqrt{2r^{\omega-2} - M^2 r^{-2}}, \tag{1.9}$$

$$\frac{d\theta}{dt} = Mr^{-2} \tag{1.10}$$

we conclude that $A^\omega = M^2$. Then self-intersections are achieved as

$$2\theta_k + 2\pi k = 2\pi\omega^{-1} \implies \theta_k = \pi\omega^{-1} - \pi k, \tag{1.11}$$

$$\gamma_k^\omega = \frac{1}{(1 - \cos(\omega\pi k))}, \qquad \mu_k = \gamma_k^{-\omega/2} = (1 - \cos(\omega\pi k))^{\frac{1}{2}}. \tag{1.12}$$

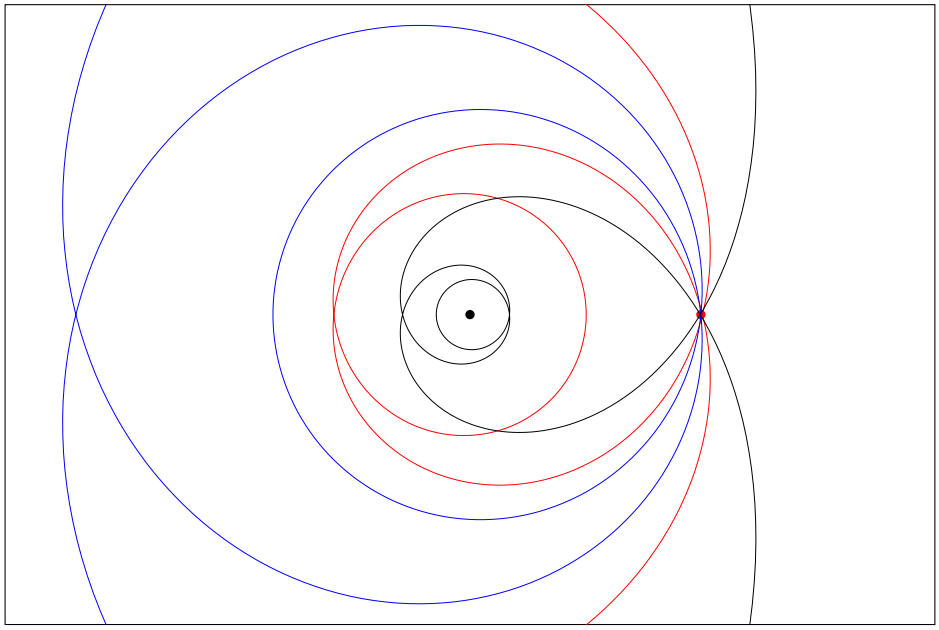

Figure 30.1: This figure shows several loops in the same (red) point, around origin (black point).

Dynamics: $\mathbf{d}=\mathbf{2}$. Consider trajectory passing through x with x being k-th points of self-intersection. Then $T_k(x) = t_k|x|^{1+\alpha}$. Meanwhile

$$
\begin{aligned}
(1.13)\quad dt = r^2 d\theta = M^{4/\omega}(1-\cos(\beta\theta))^{-2/\omega}\, d\theta& \\
\implies t_k =& 2\mu_k^{4/\omega}\int_{\theta_k}^{\pi}(1-\cos(\omega\theta))^{-2/\omega}\, d\theta \\
=& 2(1-\cos(\omega\theta_k))^{2/\omega}\int_{\theta_k}^{\pi}(1-\cos(\omega\theta))^{-2/\omega}\, d\theta.
\end{aligned}
$$

Remark 1.1. (i) We do not know how t_k are ordered and even if they are ordered at all.

(ii) There is however the purely radial movement with $M=0$ which is the limit of trajectories as $M\to 0$. We denote its time by

$$
(1.14)\qquad t_0 = 2\int_0^1 r^{\alpha}\, dr = \frac{2}{1+\alpha}.
$$

$d \geq 3$.

(1.15) As $x = ky \neq 0$, due to rotational symmetry (along straight line passing through 0 and x) instead of n separate curves we get n $(d-2)$-parameter families.

(1.16) On the other hand, if $x \neq ky$ since the trajectory from x to y belongs to a single 2-dimensional plane, passing through $x, y, 0$ the number of trajectories is as for $d = 2$.

2 Estimates

Proposition 2.1. *Consider toy-model operator* (0.1). *Then*

(i) For $T \geq \epsilon$, $|y| \leq C_0 T^{1/(1+\alpha)}$ *and* $|x| \geq 2C_0 T^{1/(1+\alpha)}$

$$|F_{t\to h^{-1}\tau}\big(\bar{\chi}_T(t) u_h(x,y,t)\big)| \leq Ch^s T^{-s} \qquad \forall \tau\colon |\tau| \leq \epsilon_1 T^{-2\alpha/(1+\alpha)}. \tag{2.1}$$

(ii) For $T \geq C_0$, $|x| \leq \epsilon T^{1/(1+\alpha)}$ *and* $|y| \leq \epsilon T^{1/(1+\alpha)}$

$$|F_{t\to h^{-1}\tau}\big(\chi_T(t) u_h(x,y,t)\big)| \leq Ch^s T^{-s} \qquad \forall \tau\colon |\tau| \leq \epsilon_1 T^{-2\alpha/(1+\alpha)}. \tag{2.2}$$

Proof. Let $\zeta \in \mathscr{C}^\infty(\mathbb{R})$, $\zeta(t) = 0$ as $t \geq 1$ and $\zeta(t) = 1$ as $t \leq 0$, $\zeta(t) < 1$ as $0 < t < 1$.

(i) Assume first that $|y| \leq 1$, $|x| \geq 2$ and $T = \epsilon$. Then the proof is standard, based for $\pm t > 0$ on $\phi(x,\xi,t) := \zeta(C_0 t - |x| + c)$ with an appropriate constant c; then

$$|\{\tfrac{1}{2}|\xi|^2 - |x|^{-2\alpha},\, |x|\}| \leq C_0 \qquad \text{for} \quad |\xi| \leq 1. \tag{2.3}$$

One can see easily that in the commutator of $hD_t - A$ with ϕ all next terms acquire factors h but not any negative powers of x.

After (2.1) is proven in this special case we prove it in the general case by scaling $x \mapsto xR^{-1}$, $\xi \mapsto \xi R^{\alpha}$, $t \mapsto tR^{-1+\alpha}$, $h \mapsto hR^{-1+\alpha}$.

(ii) Again consider first $T \asymp 1$, $|y| \le 1$. Let us use our standard method with $\phi(x,t,\xi) \coloneqq \zeta(-\epsilon_0 t + \langle x,\xi\rangle + c)$. Observe that

$$\{\tfrac12|\xi|^2 - |x|^{-2\alpha},\ \langle x,\xi\rangle\} = |\xi|^2 - 2\alpha|x|^{-2\alpha} \tag{2.4}$$

which on the energy level 0 equals $2(1-\alpha)|x|^{-2\alpha}$. We select c such that $\phi = 0$ as $|x| \le 1$, $|\xi|^2 - 2|x|^{-2\alpha} = 0$. Then according to (i) we conclude that

$$|F_{t\to h^{-1}\tau}\bigl(\chi_T^+(t)(I-Q^+)u_h(x,y,t)\bigr)| \le Ch^s \qquad \forall \tau\colon |\tau| \le \epsilon_2 \tag{2.5}$$

where $Q^+ = q^{+\mathsf{w}}$, $\mathsf{supp}(q^+) \subset \{(x,\xi)\colon \langle x,\xi\rangle \ge \epsilon \ge \epsilon_2 T^{(1-\alpha)/(1+\alpha)}\}$, which implies

$$|F_{t\to h^{-1}\tau}\bigl(\chi_T^+(t)u_h(x,y,t)\bigr)| \le Ch^s \quad \forall x\colon |x| \le \epsilon_3 T^{1/(1+\alpha)}\ \forall \tau\colon |\tau| \le \epsilon_3.$$

Like in Part (i) one can see easily that in the commutator of $hD_t - A$ with ϕ all next terms acquire factors h but not any negative powers of x.

Also, after (2.2) is proven in this special case we prove it in the general case by the same scaling as in Part (i). □

Proof of Theorem 1.1(i). It follows by standard Tauberian method from

$$|F_{t\to h^{-1}\tau}\bigl(\bar\chi_T u_h(x,y,\tau)\bigr)| \le Ch^{1-d} \qquad \forall \tau\colon |\tau| \le \epsilon \tag{2.6}$$

with $x = y$ and small constant T that

$$|e(x,x,\tau) - e(x,x,\tau')| \le Ch^{-d}|\tau-\tau'| + Ch^{1-d} \qquad \forall \tau,\tau'\colon |\tau| \le \epsilon_1, |\tau'| \le \epsilon_1 \tag{2.7}$$

which in turn implies (2.6) with arbitrarily large T and $C = C_T$; then we take $T = T_0$ and applying it we conclude that (2.6) holds for arbitrarily large T with C independent on T and with arbitrary $x\colon |x| \le 1$, $y\colon |y| \le 1$, which in turn implies (1.1). □

Now we need to consider $x,y\colon |x| \le 1$, $|y| \le 1$ and $T \le T_0$.

Proposition 2.2. *Consider toy-model operator* (0.1). *Let* $Q_l = q_l^{\mathsf{w}}(x, hD)$ *be operators such that*

$$\mathsf{supp}(q_1) \subset \{(x,\xi)\colon |x| \le 1,\ |x\times\xi| \ge (1+\epsilon)\rho\} \tag{2.8}$$

and

$$\mathsf{supp}(q_2) \subset \{(x,\xi)\colon |x| \le 1, \quad |x\times\xi| \le (1-\epsilon)\rho\} \tag{2.9}$$

where here and until further notice $\rho > 0$ *is an arbitrarily small constant. Then for* $T \ge \epsilon_0$

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)Q_{2x}u_h(x,y,t)\,{}^tQ_{1y}| \le C_T h^s \qquad \forall \tau\colon |\tau| \le \epsilon'. \tag{2.10}$$

Proof. Easy proof by standard microlocal methods based on that operators $(x_kD_j - x_jD_k)$ commute with A, is left to the reader. □

(2.11) Note that in the framework of (2.8) Hamiltonian trajectories passing through $\mathsf{supp}(q_1)$ are contained in $\{(x,\xi)\colon |x| \ge \epsilon''\}$ and one can describe $u_h(x,y,t)\,{}^tQ_{1y}$ as semiclassical Fourier integral distribution corresponding to the Lagrangian manifold $\Lambda = \{(x,\xi,t;y,-\eta,\tau)\colon (y,\eta) = \Phi_t(x,\xi),\ \tau = a(x,\xi)\}$ where $\Phi_t = \Phi_t(x,y)$ denotes a Hamiltonian flow generated by $a(x,\xi)$.

Proposition 2.3. *Consider toy-model operator* (0.1) *with* $d \ge 2$. *Let* $Q_1 = q_1^{\mathsf{w}}(x,hD)$ *be operator with the symbol satisfying* (2.8).

(i) Then for $T\colon \epsilon_0 \le T \le T^*$

$$|F_{t\to h^{-1}\tau}\chi_T(t)u_h(x,y,t)\,{}^tQ_{1y}| \le Ch^{\frac{3}{2}-d} \qquad \forall \tau\colon |\tau| \le \epsilon'. \tag{2.12}$$

(ii) Furthermore, if $|x\times y| \ge \rho$ *then*

$$|F_{t\to h^{-1}\tau}\chi_T(t)u_h(x,y,t)\,{}^tQ_{1y}| \le Ch^{\frac{1}{2}(1-d)} \qquad \forall \tau\colon |\tau| \le \epsilon'. \tag{2.13}$$

Proof. Let us fix $(\bar{x},\bar{\xi})$ and assume that $\Phi_t(\bar{x},\bar{\xi}) = (\bar{y},\bar{\eta})$ for $t = \bar{t} \ne 0$. Consider in the vicinity of $(\bar{\xi},\bar{t})$ the set $\{\xi\colon \Phi_t(\bar{x},\xi) = \bar{y}\}$. We claim that

(2.14) If $\bar{x}\times\bar{y} = 0$ then $\mathsf{rank}(d_{\xi,t}\pi_x\Phi_t(\bar{x},\xi))|_{\xi=\bar{\xi},t=\bar{t}} = 2$

and

(2.15) If $\bar{x}\times\bar{y} \ne 0$ then $\mathsf{rank}(d_{\xi,t}\pi_x\Phi_t(\bar{x},\xi))|_{\xi=\bar{\xi},t=\bar{t}} = d$.

The easy proof of these claims we leave to the reader.

Then combined (2.11) with (2.14) and (2.15) imply Statements (i) and (ii) respectively. □

Remark 2.4. (i) Let $0 < \alpha \le \frac{1}{2}$. Then the lack of self-intersections as $|x \times \xi| > 0$ implies that if $Q_1 = q_1^{\mathsf{w}}(x, hD)$ satisfies (2.8) then for $T\colon \epsilon_0 \le T \le T^*$

$$|F_{t\to h^{-1}\tau}\chi_T(t)u_h(x,y,t)\,{}^tQ_{1y}\big|_{x=y}| \le C_T h^s \qquad \forall \tau\colon |\tau| \le \epsilon'. \tag{2.16}$$

(ii) Let $\frac{1}{2} < \alpha < 1$. Then the lack self-intersections as $0 < |x \times \xi| < (1+\epsilon)\rho$ with small enough constant $\rho > 0$ implies that if $Q_1 = q_1^{\mathsf{w}}(x, hD)$ satisfies

$$\mathsf{supp}(q_1) \subset \{(x,\xi)\colon |x| \le 1,\ (1-\epsilon)\rho' < |x \times \xi| < (1+\epsilon)\rho\} \tag{2.17}$$

with sufficiently small constant $\rho > 0$ and arbitrarily small constant $\rho'\colon 0 < \rho' < \rho$ then (2.16) holds.

(iii) Let $h^{\delta'} < \rho' < \rho \le \epsilon'$ with sufficiently small exponent $\delta' > 0$. Then we still can apply standard microlocal methods and prove Proposition 2.2. Furthermore, we can use long-range propagation[1),2)] and conclude that singularities propagate along Hamiltonian trajectories. Then Statements (i) and (ii) of this Remark hold.

Now we need to consider the same expressions albeit with q_1 satisfying (2.17) with $h^{1-\delta} < \rho' < \rho < h^{\delta'}$. We want to to prove first that the propagation is confined to the strip

$$\{(x,\xi)\colon (1-\epsilon)\rho < |x \times \xi| < (1+\epsilon)\rho\}$$

and that there it happens along Hamiltonian trajectories. The most precise results (with the least restrictions on ρ) can be achieved in spherical coordinates $x = (\theta, r) \in \mathbb{S}^{d-1} \times \mathbb{R}^+$. Let $\xi = (\vartheta, \varrho) \in T^*_\theta \mathbb{S}^{d-1} \times \mathbb{R}$ be dual coordinates in the phase space. Then in $\mathscr{L}^2(\mathbb{S}^{d-1} \times \mathbb{R}^+)$ operator is transformed to

$$A = h^2\Big(D_r^2 + \frac{1}{4}(d-1)(d-3)r^{-2} + r^{-2}\Delta_\theta\Big) + V(r) \tag{2.18}$$

where Δ_θ is a positive Laplacian on $\mathbb{S}^{d-1}$. Then instead of Proposition 2.2 we have

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[1)] See Subsection 2.4.2, [Ivr2].

[2)] Because $D\Phi_t$ has a polynomial growth.

Proposition 2.5. *Consider toy-model operator* (0.1). *Let* $Q_l = q_l^{\mathsf{w}}(\theta, hD_\theta)$ *be operators such that*

$$\mathsf{supp}(q_1) \subset \{(x,\xi)\colon\ |\vartheta| \ge (1+\epsilon)\rho\} \tag{2.19}$$

and

$$\mathsf{supp}(q_2) \subset \{(x,\xi)\colon |x| \le 1,\quad |\vartheta| \le (1-\epsilon)\rho\} \tag{2.20}$$

where $h^{1-\delta} \le \rho \le h^{\delta'}$. *Then for* $x = (\theta, r)$, $y = (\theta', r')$ *and* $T \ge \epsilon_0$

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)Q_{2x}u_h(x,y,t)\,{}^tQ_{1y}| \le C_T h^s \qquad \forall \tau\colon |\tau| \le \epsilon'. \tag{2.21}$$

Proof. Proof is by the standard elliptic microlocal arguments. □

Next we want to prove that propragation is along the Hamiltonian trajectories. Since along such trajectory the minimum of $|x$ is $\asymp r(\rho) \coloneqq \rho^{1/(1-\alpha)}$ we need to assume that $r(\rho) \gtrsim r_* = h^{1/(1-\alpha)-\delta}$ that is $\rho \ge h^{1-\delta}$ with an arbitrarily small exponent $\delta > 0$. This assumption we already made.

As $r \asymp r(\rho)$ standard scaling and long-range propagation results[1)] imply this if we take (after scaling) $\mathcal{W}$ which is $\hbar^{\frac12-\delta'}$-vicinity by (y,η) singularities there for time $t\colon\ :|t| \le T = \hbar^{-\delta''}$ propagate to $\Phi_t(\mathcal{W})$. Here $\hbar = h\rho^{-1}$ [3)].

Therefore we need to explore propagation from (y,η) to (x,ξ) and from (x,ξ) to (y,η) assuming that $|y| \asymp 1$, $|\eta| \asymp 1$, $|x| \asymp r(\rho)\hbar^{-\delta'}$ and $|\xi| \asymp |x|^{-\alpha}$.

Consider first corresponding Hamiltonian flow in the spherical coordinates. For fixed $\rho = |\vartheta|$ we have

$$\frac{dr}{dt} = \varrho = \bigl(r^{-2\alpha} + 2\tau - r^{-2}\rho^2\bigr)^{\frac12} = r^{-\alpha}\bigl(1 + 2r^{2\alpha}\tau - r^{2\alpha-2}\rho^2\bigr)^{\frac12} \tag{2.22}$$
$$\implies t = (1+\alpha)^{-1}r^{1+\alpha} + O\bigl(r^{3\alpha+1}|\tau| + r^{3\alpha-1}\rho^2\bigr)$$

as long as

$$C_0 r(\rho) \le r \le \epsilon_0 \tau_-^{-1/2\alpha}; \tag{2.23}$$

we count t from the point with the minimal $|x|$. On the other hand, we see that in the framework of (2.23)

$$\Phi_t(x,\xi) = (r(t), \varrho(t); \Psi_s(\theta, \vartheta/\rho)) \tag{2.24}$$

[3)] Since scaling translates $x \to x^{-1}r^{-1}(\rho)$, $\xi \to \xi r^{\alpha}(\rho)$ and $h \to \hbar = hr^{-1+\alpha}(\rho) = h\rho^{-1}$.

where Ψ_s is the geodesic flow on the $\mathbb{S}^*(\mathbb{S}^{d-1})$ and $s = s(t)$

$$ds = \rho r^{-2} dt = \rho r^{-2+\alpha}\bigl(1 + 2r^{2\alpha}\tau - r^{2\alpha-2}\rho^2\bigr)^{-\frac{1}{2}} dr \tag{2.25}$$
$$\implies s = s_0 - (1-\alpha)^{-1}\rho r^{-1+\alpha} + O\bigl(r^{3\alpha-1}\rho|\tau| + r^{3\alpha-3}\rho^3\bigr).$$

Remark 2.6. According to the proof of long-term propagation theorem [2)]

$$e^{ih^{-1}tA_h} Q = Q(t) e^{ih^{-1}tA_h} \tag{2.26}$$

provided $Q = q^{\mathsf{w}}(x, hD)$ with the symbol satisfying

$$\mathsf{supp}(q) \subset \{(x,\xi)\colon 1 \le |x| \le 2,\ 1 \le |\xi| \le 2,\ \langle x,\xi\rangle > 0\}, \tag{2.27}$$

$$|\partial_x^\mu \partial_\xi^\nu q| \le C_{\mu\nu} h^{-\frac{1}{2}(1-\delta')(|\mu|+\nu|)}, \tag{2.28}$$

$$|D\Phi_t| \le h^{-\delta''} \quad \text{and} \quad T \le h^{-\delta''} \tag{2.29}$$

with arbitrarily small exponent $\delta' > 0$ ($\langle x,\xi\rangle > 0$ means that the propagation goes in the direction away from the origin). Here $Q(t) = q^{\mathsf{w}}(x, hD_x; t)$ is the solution of

$$Q_t(t) - ih^{-1}[A_h, Q(t)] = 0 \tag{2.30}$$

and its symbol $q(t)$ satisfies

$$|\partial_x^\mu \partial_\xi^\nu q_2| \le C_{\mu\nu} h^{-\frac{1}{2}(1-\delta''')(|\mu|+\nu|)} \tag{2.31}$$

with $\delta'' = \delta''(\delta') > 0$, $\delta''' = \delta'''(\delta') > 0$ and

$$\mathsf{supp}\, q(t) = \Phi_t(\mathsf{supp}(q)). \tag{2.32}$$

Applying the standard scaling we conclude that for symbol q_n supported in $\{(x,\xi)\colon |x| \asymp r_n, |\xi| \asymp r_n^{-\alpha}\}$ with $r_n \ge r_*$ we can take $T_n = r_n^{(1+\alpha)} h_n^{-\delta''}$ with $h_n = h r_n^{-1+\alpha}$ and thus we arrive to a symbol $q'_{n+1} = q'(t_{n+1} - t_n)$ supported in $\{(x,\xi)\colon |x| \asymp r_{n+1},\ |\xi| \asymp r_{n+1}^{-\alpha}\}$ with $r_{n+1} = r_n h_n^{-\delta_4}$, so $h_{n+1} = h_n^{1+\delta_5}$.

Based on (2.22)–(2.25) we see that if q satisfies in the spherical coordinates after corresponding rescaling (2.28) then $q(t)$ satisfies in the corresponding scaling (2.28) again, with the same exponent δ'. Therefore with $N = N(\delta, \delta_5)$ jumps we can go from $r_1 \asymp r(\rho)$ to $r_N \asymp 1$.

We want to do it in order to have

$$\begin{aligned}(2.33)\quad & e^{ih^{-1}tA_h}Q_1 \equiv e^{ih^{-1}tA_h}Q_1'Q_1 \equiv e^{ih^{-1}(t-T_1)A_h}Q_2e^{ih^{-1}T_1}Q_1\\ & \equiv e^{ih^{-1}(t-T_1)A_h}Q_2'Q_2e^{ih^{-1}T_1}Q_1 \equiv e^{ih^{-1}(t-T_1-T_2)A_h}Q_3e^{ih^{-1}T_2A_h}Q_2e^{ih^{-1}T_1}Q_1\\ & \equiv Q_{N+2}'(t)e^{ih^{-1}(t-T_1-T_2-\dots-T_N)A_h}Q_{N+1}e^{ih^{-1}T_NA_h}\cdots Q_3e^{ih^{-1}T_3A_h}Q_2e^{ih^{-1}T_1}Q_1\end{aligned}$$

with $Q_n = q_n^{\mathsf{w}}(x, hD)$, $Q_n' = q_n'^{\mathsf{w}}(x, hD)$ with symbols satisfying after rescalings (2.27) and (2.31) respectively and $(I - Q_n')Q_n \equiv 0$.

Finally, $|t - T_1 - \dots - T_N| \le T_{N+1}$. This equality (2.33) will imply the required propagation result, but we need to look at $\mathsf{supp}(q_n)$ and $\mathsf{supp}(q_n')$.

Remark 2.7. As mentioned, we do it in the spherical coordinates (r, θ). This will lead to the condition

$$\rho \ge h^{\frac12-\delta} \tag{2.34}$$

which is stronger than the original assumption

$$\rho \ge h^{1-\delta}. \tag{2.35}$$

Proposition 2.8. *Consider toy-model operator* (0.1). *Let* $(\bar x, \bar\xi) = \Psi_t(\bar y, \bar\eta)$ *and*

$$(1-\epsilon)\rho \le |\bar x \times \bar\xi| = |\bar y \times \bar\eta| = (1+\epsilon)\rho \tag{2.36}$$

with $\rho \ge h^{\frac12-\delta}$. *Let* $Q_1 = q_1^{\mathsf{w}}(x, hD)$ *be operator with the symbol* q_1 *supported in* $\hbar^{\frac12-\delta'}(r_y, r_y^{-\alpha})$*–vicinity of* $(\bar y, \bar\eta)$ *and* $Q_2 = q_2^{\mathsf{w}}(x, hD)$ *be an operator with the symbol* q_2 *equal* 1 *in* $C_0\hbar^{\frac12-\delta'}(r_x, r_x^{-\alpha})$*–vicinity of* $(\bar x, \bar\xi)$, *where* $r_x = |\bar x|$ *and* $r_y = |\bar y|$.

Then

$$e^{ih^{-1}tA_h}Q_1 \equiv Q_2e^{ih^{-1}tA_h}Q_1. \tag{2.37}$$

Proof. (i) Assume first that $|\pi_x\Psi_{t'}|$ only increases as t' changes from 0 to t. Consider in the scaled spherical coordinates $\hbar^{\frac12-\delta'}$-vicinity $\mathcal{U}$ of $(\bar y, \bar\eta)$. Then $\Psi_t(\mathcal{U}) \subset \mathcal{V}$ which is $C_0\hbar^{\frac12-\delta'}$-vicinity of $\Phi_t(\bar x, \bar\xi)$ again in the scaled spherical coordinates. Combined with (2.33) implies (2.37). Here we can take any $\hbar \ge h^{1-\delta}r_y^{\alpha-1}$.

(ii) Similarly we can consider a case when $|\pi_x\Psi_{t'}|$ only decreases as t' changes from 0 to t. Here we can take any $\hbar \ge h^{1\delta}r_x^{\alpha-1}$.

(iii) Therefore such statement remains true if $|\pi_x \Psi_{t'}|$ first decreases and then increases as t' changes from 0 to t. Here we must take $\hbar \ge h^{1-\delta}\rho^{-1}$. □

Proposition 2.9. *Consider toy-model operator* (0.1). *Let*

$$|\bar{x}| = 1, \qquad a(\bar{x}, \bar{\xi}) = 0, \qquad 0 < \rho = |\bar{x} \times \bar{\xi}| \le \epsilon, \tag{2.38}$$

with sufficiently small constant $\epsilon = \epsilon(\epsilon_0) > 0$. *Then*

(i) If $(1-\alpha)^{-1} \notin 2\mathbb{Z}$ *then*

$$\min_{t:\ |t|\ge\epsilon_0} |\pi_x \Psi_t(\bar{x}, \bar{\xi}) - \bar{x}| \asymp 1. \tag{2.39}$$

(ii) If $(1-\alpha)^{-1} \in 2\mathbb{Z}$ *then*

$$\min_{t:\ |t|\ge\epsilon_0} |\pi_x \Psi_t(\bar{x}, \bar{\xi}) - \bar{x}| \asymp \rho. \tag{2.40}$$

Proof. (i) If $(1-\alpha)^{-1} \notin 2\mathbb{Z}$ then the rotation angle when $\rho \to 0$ tends to $\pi(1-\alpha)^{-1} \notin 2\pi\mathbb{Z}$, which implies (2.39).

(ii) If $(1-\alpha)^{-1} = 2n$, $n \in \mathbb{Z}$ then the rotation angle when $\rho \to 0$ tends to $2\pi n$. Further, if $n = 1$ then the trajectory is a parabola with the curvature radius $\asymp r_\rho = \rho^2$ near 0 and then on the distance $\asymp 1$ from 0 the distance between branches is $\asymp \rho$. For $n \ge 2$ instead of this parabola we get its image under transformation $r \mapsto r^n$, $\theta = n\theta$ (if $d = 2$) and then the distance between branches $\asymp r_\rho^{1/2n} = \rho$ again.

Thus we arrive to (2.40). □

Proposition 2.10. *Consider toy-model operator* (0.1). *Let* $Q = q^{\mathrm{w}}(x, hD)$ *be an operator with the symbol* q *supported in* $\epsilon''\rho$*-vicinity of* $(\bar{x}, \bar{\xi})$ *satisfying* (2.38) *with sufficiently small constant* $\epsilon > 0$. *Then*

(i) If $(1-\alpha)^{-1} \notin 2\mathbb{Z}$ *and* $\rho \ge h^{\frac{1}{2}-\delta}$ *then*

$$F_{t\to h^{-1}\tau}\Big(\chi_T(t) u_h(x, y, t)\, {}^t Q_{1y}\Big)\Big|_{y=x} = O(h^s) \qquad \forall T \in [\epsilon_1, T_1]\ \forall \tau\colon |\tau| < \epsilon_2. \tag{2.41}$$

with an arbitrarily small constant $\epsilon_1 > 0$, *arbitrarily large constant* T_1 *and* $\epsilon_2 = \epsilon_2(T_1)$.

(ii) If $(1-\alpha)^{-1} \in 2\mathbb{Z}$ *and*

$$\rho \ge h^{\frac{1}{3}-\delta} \tag{2.42}$$

then (2.41) *holds.*

Proof. In virtue of Proposition 2.8 we need to check that the left-side expression of (2.39) is larger than $C\hbar^{\frac{1}{2}-\delta'}$. Recall that $\hbar = h/\rho$.

(i) $(1-\alpha)^{-1} \notin 2\mathbb{Z}$ we need to check $\hbar^{\frac{1}{2}-\delta'} \le \epsilon_3$ which follows from $\rho \ge h^{\frac{1}{2}-\delta}$ which we need for Proposition 2.8.

(ii) $(1-\alpha)^{-1} \in 2\mathbb{Z}$ we need to check $\hbar^{\frac{1}{2}-\delta'} \le \epsilon_3\rho$ which is equivalent to (2.42). □

Now we need to check the contribution of zone $\{(x,\xi)\colon |x\times\xi| < \rho\}$ with $\rho = h^{\frac{1}{2}-\delta}$ in the framework of Proposition 2.10(i) and $\rho = h^{\frac{1}{3}-\delta}$ in the framework of Proposition 2.10(ii).

Proposition 2.11. *Consider toy-model operator* (0.1). *Let* $Q = q^{\mathrm{w}}(x, hD)$ *be an operator with the symbol* q *supported in* $\epsilon''\rho$*-vicinity of* $(\bar{x},\bar{\xi})$ *satisfying*

$$|\bar{x}| = 1, \qquad a(\bar{x},\bar{\xi}) = 0, \qquad |\bar{x}\times\bar{\xi}| \le \rho, \tag{2.43}$$

Then

$$|F_{t\to h^{-1}\tau}\Big(\chi_T(t)u_h(x,y,t)\,{}^tQ_{1y}\Big)\Big|_{y=x}| \le C\rho^{d-1}h^{1-d} \qquad \forall T\in[\epsilon_1, T_1]\ \forall \tau\colon |\tau| < \epsilon_2 \tag{2.44}$$

with an arbitrarily small constant $\epsilon_1 > 0$, *arbitrarily large constant* T_1 *and* $\epsilon_2 = \epsilon_2(T_1)$.

Proof. Proof follows from standard arguments and the standard estimate (2.44) with $\chi_T(t)$ replaced by $\bar{\chi}_{T'}(t)$ with small enough constant T'. □

Taking the sum with respect to partition of unity we arrive to

Proposition 2.12. *Consider toy-model operator* (0.1). *Then*

(i) For $|x|=1$

$$|F_{t\to h^{-1}\tau}\Big(\chi_T(t)u_h(x,x,t)\Big)| \le Ch^{\frac{3}{2}-d} + C\bar\rho^{d-1}h^{1-d} \qquad \forall T\ge \epsilon_0\ \forall \tau\colon |\tau|<\epsilon_2 T^{-2\alpha/(1+\alpha)} \tag{2.45}$$

(ii) Furthermore, as $0<\alpha\le \frac12$ *one can skip the first term in the right-hand expression of* (2.45).

Proof. Indeed, the first term in the right-hand expression of (2.45) is due to loops (which are absent as $0<\alpha\le\frac12$ while the second term is a contribution of zone $\{\xi\colon |x\times\xi|\le\bar\rho\}$ with $\bar\rho$ defined by (1.4).. □

Remark 2.13. If we want to get rid off the first term in the right-hand expression of (2.45) as $\frac12<\alpha<1$, we need to calculate contribution of loops (in the zone $\{(x,\xi)\colon |x\times\xi|>\epsilon\}$. One can do it, arriving to the estimate

$$|F_{t\to h^{-1}\tau}\Big((\bar\chi_T(t)-\bar\chi_{T'}(t))u_h(x,x,t)\Big) - \sum_{1\le k\le n}\sum_{\pm}\sum_{m\ge 0} \varkappa^{\pm}_{k,m}(x,\tau)\exp(\pm ih^{-1}\phi_k(x,\tau))h^{\frac{3}{2}-d+m}| \le C\bar\rho^{d-1}h^{1-d} \tag{2.46}$$

for all $T\ge C_0$, $T'=\epsilon_0$ and $\tau\colon |\tau|<\epsilon_2 T^{-2\alpha/(1+\alpha)}$.

Proof of Theorem 1.1. Statements (ii) and (iii) of Theorem 1.1 follow from Statements (i) and (ii) of Proposition 2.12. □

3 Scaling

Now we assume that

(3.1) Operator A_h in question coincides with toy-model (0.1) as $|x|\le R$, $R\ge 2$.

Then, assuming (1.12) (that is, A_h is a self-adjoint operator) we estimate for $|x|\asymp|y|\asymp 1$

$$|F_{t\to h^{-1}\tau}\bar\chi_T(t)u_h(x,y,t)|\le Ch^{1-d} \qquad \text{as}\quad T\le \epsilon R^{1+\alpha},\ |\tau|\le \epsilon R^{-2\alpha}.$$

Taking $x = y$ we apply standard Tauberian arguments (Part I)[4] to get estimate

$$|e_h(x,x,\tau) - e_h(x,x,\tau')| \le C\big(h^{1-d}R^{-1-\alpha} + h^{-d}|\tau-\tau'|\big) \quad \text{as } |\tau|+|\tau'| \le \epsilon R^{-2\alpha}.$$

Now take $|x| = r \le 1$. Then scaling $x \mapsto r^{-1}|x|$, $\tau \mapsto \tau r^{2\alpha}$, $R \mapsto r^{-1}R$ and $h \mapsto \hbar := hr^{\alpha-1}$ we arrive to

$$|e_h(x,x,\tau) - e_h(x,x,\tau')| \le Cr^{-(d-2)\alpha}\big(h^{1-d}R^{-1-\alpha} + h^{-d}|\tau-\tau'|\big) \quad \text{as } |\tau|+|\tau'| \le \epsilon R^{-2\alpha}$$

which for $R = 2$ becomes

$$|e_h(x,x,\tau) - e_h(x,x,\tau')| \le Cr^{-(d-2)\alpha}\big(h^{1-d} + h^{-d}|\tau-\tau'|\big) \quad \text{as } |\tau|+|\tau'| \le \epsilon \tag{3.2}$$

provided

$$\bar{h} := h^{1/(1-\alpha)} \le r \le 1 \tag{3.3}$$

where the last assumption means exactly that $\hbar \le 1$. Then (3.2) implies

$$\begin{aligned} |e_h(x,y,\tau) - e_h(x,y,\tau')| &\le Cr_x^{-(d-2)\alpha/2} r_y^{-(d-2)\alpha/2}\big(h^{1-d} + h^{-d}|\tau-\tau'|\big) \\ &\text{as } |\tau|+|\tau'| \le \epsilon,\ \ \bar{r} \le r_x = |x| \le 1,\ \ \bar{r} \le r_y = |y| \le 1 \end{aligned} \tag{3.4}$$

Apply standard Tauberian arguments (Part II)[4] we arrive to

Proposition 3.1. *Consider self-adjoint operator coinciding with the toy-model operator* (0.1) *as* $|x| \le 2$. *Then*

$$|e_h(x,y,\tau) - e^{\mathsf{T}}_{T,h}(x,y,\tau)| \le Cr_x^{-(d-2)\alpha/2} r_y^{-(d-2)\alpha/2} h^{1-d} \tag{3.5}$$

as $|\tau| \le \epsilon,\ \ \bar{r} \le r_x = |x| \le 1,\ \ \bar{r} \le r_y = |y| \le 1,\ \ T = C_0(r_x + r_y)^{1+\alpha}$

with the Tauberian expression

$$e^{\mathsf{T}}_{T,h}(x,y,\tau) = h^{-1}\int_{-\infty}^{\tau} \Big(F_{t\to h^{-1}\tau}\bar{\chi}_T(t)u_h(x,y,t)\Big)\,d\tau. \tag{3.6}$$

[4] See [Ivr2], Chapter 4.

From now $x = y$. Then scaling results of Theorem 1.1 and Remark 1.2 and combining with (3.5) we arrive to

Proposition 3.2. *Consider self-adjoint operator coinciding with the toy-model operator* (0.1) *as* $|x| \le 2$. *Then for* $\bar{r} \le r = |x| \le 1$

(i) Estimate holds

$$|e_h(x,x,0) - e^{\mathsf{T}}_{T_*,h}(x,x,0)| \le Cr^{-(d-2)\alpha}h^{1-d} + Cr^{-\frac{3}{2}-\alpha(d-\frac{3}{2})}h^{\frac{3}{2}-d} + C\bar{\rho}^{d-1}r^{-1-\alpha(d-1)}h^{1-d} \tag{3.7}$$

with

$$\bar{\rho} = \begin{cases} \hbar^{\frac{1}{2}-\delta} & \text{as } (1-\alpha)^{-1} \notin 2\mathbb{Z}, \\ \hbar^{\frac{1}{3}-\delta} & \text{as } (1-\alpha)^{-1} \in 2\mathbb{Z} \end{cases} \tag{3.8}$$

with $\hbar = hr^{\alpha-1}$ *and with* $T_* = \epsilon r^{1+\alpha}$.

(ii) Further, as $0 < \alpha \le \frac{1}{2}$ *one can skip the second term in the right-hand expression of* (3.7).

(iii) Furthermore,

$$e^{\mathsf{T}}_{T_*,h}(x,x,\tau) = e^{\mathsf{W}}_h(x,x,\tau) + O(r^{-2-(d-2)\alpha}h^{2-d}) \tag{3.9}$$

with Weyl expression $e^{\mathsf{W}}_h(x,x,0)$ *defined by* (1.6).

Remark 3.3. Scaling, rotational symmetry and Remark 2.13 imply that for $\bar{r} \le r = |x|$ and $T_* = \epsilon_0 r^{1+\alpha}$

$$e^{\mathsf{T}}_{T_*,h} \sim \sum_{j=1}^{\infty} \kappa^0_j(\tau|x|^{2\alpha})h^{-d+2j}|x|^{-\alpha d-2j(1-\alpha)}. \tag{3.10}$$

and for $T^* = C_0 r^{1+\alpha}$

$$e^{\mathsf{T}}_{T,h} - e^{\mathsf{T}}_{T_*,h} \sim \sum_{1\le k\le n}\sum_{\pm}\sum_{j\ge 0} \kappa^{0\pm}_{k,j}(\tau|x|^{2\alpha})h^{\frac{3}{2}-d+j}|x|^{-\alpha d-(\frac{3}{2}+j)(1-\alpha)} \exp\bigl(\pm ih^{-1}|x|^{1-\alpha}\phi^0_k(\tau|x|^{2\alpha})\bigr). \tag{3.11}$$

Chapter 31

General case

1 Standard results for perturbed operator

Consider now operator A_h defined by (0.2) and satisfying (0.3)–(0.5) in $B(0,R) \subset X$. Then in $B(0,R)$

$$|D^\sigma (g^{jk} - \delta_{jk})| \le c_\sigma \varepsilon |x|^{-|\sigma|}, \tag{1.1}$$

$$|D^\sigma V_j(x)| \le c_\sigma \varepsilon |x|^{-\alpha-|\sigma|}, \tag{1.2}$$

$$|D^\sigma (V(x) + |x|^{-2\alpha})| \le c_\sigma \varepsilon |x|^{-2\alpha-|\sigma|} \qquad \forall \sigma : |\sigma| \le K \tag{1.3}$$

with $\varepsilon = R^\beta$. We assume that

$$r_* := h^{1/(1-\alpha)-\delta} \le r \le R \tag{1.4}$$

with an arbitrarily small exponent $\delta > 0$.

Remark 1.1. Usual scaling $x \mapsto xR^{-1}$, $\xi \mapsto \mathsf{x} R^\alpha$, $\tau \mapsto \tau R^{2\alpha}$ and $h \mapsto \hbar = hR^{-1+\alpha}$ produces operator (0.2) satisfying (1.1)–1.3) with $\varepsilon = R^\beta$ in $B(0,R)$. In contrast to Subsection 3 here $R \le \epsilon$.

However below we consider operator (0.2) under assumptions (1.1)–(1.4) for $R \ge r_*$ and then scale as necessary. This leads to different ε in different applications.

Repeating proof of Proposition 2.1 we arrive to

Proposition 1.2. *For operator* (0.2) *under assumptions* (1.1)–(1.3) *in* $B(0,R)$ *both Statements (i) and (ii) of Propositions 2.1 remain true as* $r_*^{1+\alpha} \le T \le \epsilon_2 R^{1+\alpha}$, $\varepsilon \le \epsilon_0$ *where* $\epsilon_0 > 0$, ϵ_2 *are small enough constants.*

Then similarly to Theorem 1.1(i) and Proposition 3.1 we arrive to

Corollary 1.3. *Consider operator* (0.2) *under assumptions* (1.1)–(1.3) *in* $B(0,2)$. *Then* (3.5) *remains true.*

Proof of Theorem 2.1(i). We take $R = 2r_0$ so that $\varepsilon \le \epsilon_0$. □

Similarly to Proposition 2.2 we have

Proposition 1.4. *Consider operator* (0.2) *under assumptions* (1.1)–(1.3) *in* $B(0,2)$. *Let assumptions* (2.8)–(2.9) *be fulfilled with*

$$\rho \ge C\varepsilon \tag{1.5}$$

where here ρ *is a small constant and* $C = C(\epsilon)$ *with* ϵ *from* (2.8)–(2.9). *Then* (2.10) *remains true for* $|x| \le 1$, $|y| \le 1$, $\epsilon_1 \le T \le c_0$.

Proof. In contrast to the toy-model operator when $\{a(x,\xi), x_j\xi_k - x_k\xi_j\} = 0$ we have now that on the energy level τ: $|\tau| \le \epsilon_0 |x|^{-2\alpha}$

$$|\{a(x,\xi), x_j\xi_k - x_k\xi_j\}| \le C_0\varepsilon |x|^{-2\alpha}. \tag{1.6}$$

Then we can apply standard propagation results. More sophisticated arguments will be required when both ρ and ε are small parameters. □

Repeating proof of Proposition 2.3 we arrive to

Proposition 1.5. *Consider operator* (0.2) *under assumptions* (1.1)–(1.3) *in* $B(0,R)$. *Let* $Q_1 = q_1^{\mathrm{w}}(x, hD)$ *be operator with the symbol satisfying* (2.8) *with an arbitrarily small constant* ρ.

Then both Statements (i) and (ii) of Proposition 2.3 remain true for $T \ge \epsilon_1$ *provided* ε *satisfies* (1.5).

Remark 1.6. (i) Under assumption (1.5) Statements (i)–(ii) of Remark 2.4 remain true. Furthermore, Statement (iii) remains true due to long-range propagation of singularities [1)] and Proposition 1.7 below.

(ii) For operator (0.2) instead of (2.14) and (2.15) we have respectively

(1.7) If $|\bar{x} \times \bar{y}| \le C_0\varepsilon$ then $\mathsf{rank}(d_{\xi,t}\pi_x \Phi_t(\bar{x},\xi))|_{\xi=\bar{\xi},t=\bar{t}} \ge 2$

and

(1.8) If $|\bar{x}\times\bar{y}|\ge C_0\varepsilon$ then $\mathsf{rank}\big(d_{\xi,t}\pi_x\Phi_t(\bar{x},\xi)\big)|_{\xi=\bar{\xi},t=\bar{t}}=d$.

(iii) I believe that for generic perturbations $\mathsf{rank}\big(d_{\xi,t}\pi_x\Phi_t(\bar{x},\xi)\big)|_{\xi=\bar{\xi},t=\bar{t}}=d$ even as $|\bar{x}\times\bar{y}|\le C_0\varepsilon$; morover, the largest absolute value of $d\times d$–minors of $d_{\xi,t}\pi_x\Phi_t(\bar{x},\xi)\big)$ is $\asymp\varepsilon^{d-2}$.

Proposition 1.7. *Consider operator* (0.2) *under assumptions* (1.1)–(1.3) *in* $B(0,2)$. *Then in the zone* $\{(x,\xi)\colon |x\times\xi|\ge\rho\}$ *with* $\rho\ge C\varepsilon$ *Hamiltonian flow* Φ_t *satisfies*

$$|D_{x,\xi}\Phi_t(x,\xi)|\le C\Big(1+\frac{|t|}{r_x^{1+\alpha}}\Big)^N. \tag{1.9}$$

Proof. It follows easily from

$$\frac{d}{dt}x|x|^{-1}=\sqrt{2}|x|^{-1-\alpha}\Big(1+O(\rho+\varepsilon)\Big). \tag{1.10}$$

Combining with long-term propagation of singularities[1)] we arrive to

Proposition 1.8. *Consider operator* (0.2) *under assumptions* (1.1)–(1.3) *in* $B(0,2)$. *Then*

(i) Statement of Proposition 2.5 remains true in the zone $\{(x,\xi)\colon |x\times\xi|\ge\rho\}$ *with*

$$\rho\ge h^{1-\delta}+C\varepsilon. \tag{1.11}$$

(ii) Statement of Remark 2.6 remains true in the zone $\{(x,\xi)\colon |x\times\xi|\ge\rho\}$ *with*

$$\rho\ge h^{\frac{1}{2}-\delta}+C\varepsilon. \tag{1.12}$$

Continuing, we arrive to

Proposition 1.9. *Consider operator* (0.2) *under assumptions* (1.1)–(1.3) *in* $B(0,2)$. *Let condition* (1.11) *be fulfilled. Then*

(i) Statements of Remark 2.7 and Proposition 2.8 remain true.

(ii) Both Statements (i) and (ii) of Proposition 2.9 remain true.

(iii) Both Statements (i) and (ii) of Proposition 2.10 remain true.

(iv) Statement of Proposition 2.11 remains true.

Taking the sum with respect to partition of unity (like in Proposition 2.12) we see that estimate (2.45) holds with $\bar{\rho}$ replaced by $\bar{\rho}_\varepsilon := \bar{\rho} + C\varepsilon$:

Proposition 1.10. *Consider operator* (0.2). *Assume that* (1.1)–(1.3) *with small parameter* ε *are fulfilled in* $B(0,2)$. *Also assume that the same assumptions albeit with* ε *replaced by* ϵ_2 *which is a small constant are fulfilled in* $B(0,R)$ *with* $R \ge 2$. *Then for* $|x| = 1$,

$$|F_{t\to h^{-1}\tau}\Big(\chi_T(t)u_h(x,x,t)\Big)| \le Ch^{\frac{3}{2}-d} + C\bar{\rho}^{d-1}h^{1-d} + C\varepsilon^{d-1}h^{1-d} \qquad \forall T \ge \epsilon_0\ \forall \tau\colon |\tau| < \epsilon_2 T^{-2\alpha/(1+\alpha)} \tag{1.13}$$

for $\epsilon_1 \le T \le T^* := \epsilon_1 R^{1/(1+\alpha)}$ *with* $\bar{\rho}$ *defined by* (1.4).

Furthermore, as $0 < \alpha \le \frac{1}{2}$ *one can skip the first term in the right-hand expression of* (1.13).

Remark 1.11. Here we need assumptions (1.1)–(1.3) twice:

(i) In $B(0,2)$ with small parameter ε to estimate the contribution to the left-hand expression of (2.44) with χ_T replaced by $\chi_T^{\pm}$ of zone

$$\{\xi\colon |x\times\xi| \le \bar{\rho},\ \pm\langle x,\xi\rangle < 0\}$$

where singularities propagate first towards 0,

(ii) In $B(0,R)$ with the small constant ϵ_2 and $R \gg 1$ to ensure that the singularities propagating away from 0 will continue this until time T^*.

(iii) If the my conjecture (see Remark 1.6(iii)) is correct, then for generic perturbations with $\varepsilon \ge h$ one can replace the first term in the right-hand expression of (1.13) by $Ch^{(1-d)/2}\varepsilon^{1-d/2}$. This improvement will be carried out to estimates (1.14) and (2.4).

Standard Tauberian arguments imply

Corollary 1.12. *Consider operator* (0.2). *Let assumptions of Proposition 1.10 be fulfilled. Then for* x: $|x| = 1$

$$|e_h(x,x,0) - e^{\mathsf{T}}_{T_*,h}(x,x,0)| \le CT^{*-1}h^{1-d} + Ch^{\frac{3}{2}-d} + C\bar\rho^{d-1}h^{1-d} + C\varepsilon^{d-1}h^{1-d} \tag{1.14}$$

with the Tauberian expression (3.4), $T^* = \epsilon_1 R^{1/(1+\alpha)}$ *and arbitrarily small constant* $T_* > 0$.

Furthermore, as $0 < \alpha \le \frac{1}{2}$ *one can skip the second term in the right-hand expression of* (1.14).

Proof of Theorem 2.1(ii), (iv). Now standard scaling and setting $\varepsilon = C_0 r^\beta$ imply estimate (2.4) (that is Theorem 2.1(ii)).

Furthermore, we see that as $0 < \alpha \le \frac{1}{2}$ one can skip the first term in the right-hand expression of this estimate (that is, part of Theorem 2.1(iv)). □

Remark 1.13. We know that for $|x| \asymp 1$ and small enough constant $T_* > 0$ and $|x| = 1$

$$e^{\mathsf{T}}_{T_*,h}(x,x,\tau) = e^{\mathsf{W}}_h(x,x,\tau) + O(h^{2-d} + \varepsilon h^{1-d}) \tag{1.15}$$

with the Weyl expression

$$e^{\mathsf{W}}_h(x,x,0) = (2\pi h)^{-d}\varpi_d g(x)^{-\frac{1}{2}}(2(\tau - V(x))^{\frac{d}{2}}, \qquad g = \det(g^{jk}). \tag{1.16}$$

Now standard scaling and setting $\varepsilon = C_0 r^\beta$ imply estimate (2.7) (that is Remark 2.2(i)).

2 Perturbations of $e_h(x,y,0)$

Let us denote unperturbed operator as A^0_h and corresponding spectral function and propagator as $e^0_h(x,y,\tau)$ and $u^0_h(x,y,t)$.

Consider in the framework of Proposition 1.10 for $|x| \asymp |y| \asymp 1$

$$F_{t\to h^{-1}\tau}\bigl(\bar\chi_T(t)u_h(x,y,t)\bigr) = F_{t\to h^{-1}\tau}\bigl(\bar\chi_{T'}(t)u_h(x,y,t)\bigr) + F_{t\to h^{-1}\tau}\bigl((\bar\chi_T(t) - \bar\chi_{T'}(t))u_h(x,y,t)\bigr) \tag{2.1}$$

with $T = T^* = C_0$ and $T' = T_* = \epsilon_0$ and corresponding formula for A^0_h and $u^0_h(x,y,t)$.

Let us integrate both by τ and multiply by h^{-1}. Then taking the difference we arrive to

$$e^{\mathsf{T}}_{T,h}(x,y,\tau) - e^{0,\mathsf{T}}_{T,h}(x,y,\tau) = e^{\mathsf{T}}_{T',h}(x,y,\tau) - e^{0,\mathsf{T}}_{T',h}(x,y,\tau) + h^{-1}\int_{-\infty}^{\tau} F_{t\to h^{-1}\tau'}\Big(\big(\bar{\chi}_T(t)-\bar{\chi}_{T'}(t)\big)\big(u_h(x,y,t)-u^0_h(x,y,t)\big)\Big)\,d\tau'. \tag{2.2}$$

Proposition 2.1. *Consider operator* (0.2). *Assume that* (1.1)–(1.3) *are fulfilled in* $B(0,2)$. *Then for* $|x|=1$ *and arbitrarily small constant* T'

$$|e^{\mathsf{T}}_{T',h}(x,y,\tau) - e^{0,\mathsf{T}}_{T',h}(x,y,\tau) - e^{\mathsf{W}}_{h}(x,y,\tau) + e^{0,\mathsf{W}}_{h}(x,y,\tau)| \le C\varepsilon h^{2-d} + C'h^{s} \qquad \forall \tau\colon |\tau|\le \epsilon. \tag{2.3}$$

Proof. Let $A'_h = \varepsilon^{-1}(A_h - A^0_h)$. Then $A_h \coloneqq A_{\varepsilon,h} = A^0_h + \varepsilon A'_h$ and ecomposition

$$e^{\mathsf{T}}_{T',h}(x,y,\tau) \sim \sum_{m\ge 0} \varkappa_m(x,\varepsilon,\tau) h^{-d+2m} \tag{2.4}$$

and similar decomposition for $e^{0\mathsf{T}}_{T',h}(x,y,\tau)$ together with

$$\varkappa_m(x,\varepsilon,\tau) \sim \sum_{j\ge 0} \kappa_{m,j}(x,\tau)\varepsilon^{j} \tag{2.5}$$

imply (2.3). □

Proposition 2.2. *Consider operator* (0.2). *Assume that* (1.1)–(1.3) *with small parameter* $\varepsilon \le h$ *are fulfilled in* $B(0,2)$. *Also assume that the same assumptions albeit with* ε *replaced by* ϵ_2 *which is a small constant are fulfilled in* $B(0,R)$ *with* R. *Then for* $|x|=1$,

$$|h^{-1}\int_{-\infty}^{\tau} F_{t\to h^{-1}\tau'}\Big(\big(\bar{\chi}_T(t)-\bar{\chi}_{T'}(t)\big)\big(u_h(x,y,t)-u^0_h(x,y,t)\big)\big|_{y=x}\Big)\,d\tau'| \le C\varepsilon h^{\frac12-d} + C\varepsilon\bar{\rho}^{d-1}h^{-d} + C'h^{s} \tag{2.6}$$

for $\epsilon_1 \le T \le T^* \coloneqq \epsilon_1 R^{1/(1+\alpha)}$ *with* $\bar{\rho}$ *defined by* (1.4).

Furthermore, as $0<\alpha\le \frac12$ *one can skip the first term in the right-hand expression of* (2.6).

Proof. (i) Let $Q = q^{\mathrm{w}}(x, hD)$ be an operator with the symbol supported in $\{x\colon 1-\epsilon_3 \le |x| \le 1+\epsilon_3,\ |x\times\xi| \ge \epsilon_3\}$. Then

$$h^{-1}\int_{-\infty}^{\tau} F_{t\to h^{-1}\tau'}\Big((\bar\chi_T(t)-\bar\chi_{T'}(t))u_h(x,y,t)\,{}^tQ_y\big|_{y=x}\Big)\,d\tau' \sim \sum_k\sum_{m\ge 0} e^{ih^{-1}\varphi_k(x,\varepsilon,\tau)}\varkappa_{k,m}(x,\varepsilon,\tau)h^{\frac{3}{2}-d+m} \tag{2.7}$$

and the same is true for $u_h^0(x,y,t)$. Moreover

$$\varphi_k(x,\varepsilon,\tau)\sim\sum_{j\ge 0}\phi_{k,j}(x,\tau)\varepsilon^j,\qquad \varkappa_{k,m}(x,\varepsilon,\tau)\sim\sum_{j\ge 0}\kappa_{k,m,j}(x,\tau)\varepsilon^j \tag{2.8}$$

and therefore the difference between expression (2.7) and (2.7) with $\varepsilon = 0$ does not exceed $C\varepsilon h^{\frac12-d}+C'h^s$.

(ii) Let $Q = q^{\mathrm{w}}(x, hD)$ be an operator with the symbol supported in $\{x\colon 1-\epsilon_3 \le |x| \le 1+\epsilon_3,\ \bar\rho \le |x\times\xi| \le \epsilon_3\}$. Then the same expressions are $O(h^s)$.

(iii) Finally, let $Q = q^{\mathrm{w}}(x, hD)$ be an operator with the symbol supported in $\{x\colon 1-\epsilon_3 \le |x| \le 1+\epsilon_3,\ |x\times\xi| \le \bar\rho\}$. Then from the standard perturbation method[1)] follows that

$$|h^{-1}\int_{-\infty}^{\tau} F_{t\to h^{-1}\tau'}\Big((\bar\chi_T(t)-\bar\chi_{T'}(t))\big(u_h(x,y,t)-u_h^0(x,y,t)\big)\,{}^tQ_y\big|_{y=x}\Big)\,d\tau'|$$

does not exceed $C\varepsilon h^{-1}\times\bar\rho^{d-1}h^{1-d}$ which is the second term in the right-hand expression of (2.6). □

Corollary 2.3. *Let $\varepsilon \le h$. Then*

$$|h^{-1}\int_{-\infty}^{\tau} F_{t\to h^{-1}\tau'}\Big((\bar\chi_T(t)-\bar\chi_{T'}(t))\big(u_h(x,y,t)-u_h^0(x,y,t)\big)\big|_{y=x}\Big)\,d\tau'| \le C\varepsilon h^{2-d}+C\varepsilon h^{\frac12-d}+C\varepsilon\bar\rho^{d-1}h^{-d}+C'h^s. \tag{2.9}$$

Furthermore, as $0<\alpha\le\frac12$ one can skip the second term in the right-hand expression of (2.9).

[1)] See [Ivr2].

Conjecture 2.4. *Estimate (2.3) and (2.6) hold without $C'h^s$ in their right-hand expression.*

Proof of Theorem 2.1(iii), (iv). Now standard scaling and setting $\varepsilon = C_0 r^\beta$ imply estimate (2.6) (that is Theorem 2.1(iii)).

Furthermore, we see that as $0 < \alpha \le \frac{1}{2}$ one can skip the first term in the right-hand expression of this estimate (that is, remaining part of Theorem 2.1(iv)). □

3 Near singularity

Now we need to investigate zone $\{x\colon |x| \le r_* = h^{1/(1-\alpha)-\delta}\}$ which we divide into non-semiclassical subzone $\{x\colon |x| \le \bar r = h^{1/(1-\alpha)}\}$ and intermediate subzone $\{x\colon \bar r \le |x| \le r_*\}$.

3.1 Non-semiclassical subzone

Proposition 3.1. *As $|x| \le \bar r$, $|y| \le \bar r$ and $\tau \le C_0 \bar r^{-2}$*

$$|e_h(x,y,0)| \le C\bar r^{-d} \tag{3.1}$$

and

$$|e_h(x,y,0) - e_h^0(x,y,0)| \le C\bar r^{\beta-d} \tag{3.2}$$

Proof. (i) Let $E_h(\tau)$ be a spectral projector of A_h. Then $\|E_h(\tau)f\|_{\mathcal{L}^2} \le \|f\|_{\mathcal{L}^2}$. Then $\|A_h E_h(\tau)\| \le |\tau|$ implies that for $\tau \le c\bar r^{-2}$, $\hbar \asymp 1$ and $\|f\|_{\mathscr{L}^2} = 1$

$$\|DE_h(\tau)f\|_{\mathscr{L}^2} \le C\bar r^{-1} \tag{3.3}$$

which in virtue Sobolev inequality implies that

(a) $d = 1 \implies \|E_h(\tau)f\|_{\mathscr{C}} \le C\bar r^{-\frac{1}{2}}$,

(b) $d \ge 2 \implies \|E_h(\tau)f\|_{\mathscr{L}^p} \le C\bar r^{-1}$ with any $p < \infty$ for $d = 2$ and $p = 2d/(d-2)$ for $d \ge 3$.

In the latter case using inequalities $\|Vv\|_{\mathscr{L}^q} \le C\|V\|_{\mathscr{L}^s}\|v\|_{\mathscr{L}^p}$ as $q^{-1} = s^{-1} + p^{-1}$ and $\|V\|_{\mathscr{L}^s(B(0,r))} \le Cr^{d/s-2\alpha}$ as $d/s > 2\alpha$ we upgrade (3.3) to

$$\|D^2 E_h(\tau)f\|_{\mathscr{L}^q} \le C\bar r^{-2} \tag{3.4}$$

which in virtue Sobolev inequality implies that

(c) $\|E_h(\tau)f\|_{\mathscr{C}} \le C\bar r^{-d/2}$ as either $d=2$ or $d\ge 3$ and $1/p<0$,

(d) $\|E_h(\tau)f\|_{\mathscr{L}^{p'}} \le C\bar r^{-d/2+d/p'}$ as $d\ge 3$ and $1/p'>0$

with $1/p'=1/p-\delta>0$ with δ which does not depend on p. Repeating these arguments we decrease $1/p$ until it becomes negative and arrive to $\|E_h(\tau)f\|_{\mathscr{C}} \le C\bar r^{-d/2}$ for all f with $\|f\|_{\mathscr{L}^2}=1$ which is equivalent to

$$\|e_h(x,y,\tau)\|_{\mathscr{L}^2_y} \le C\bar r^{-d/2} \qquad \forall x\in B(0,\bar r) \tag{3.5}$$

which implies (3.2).

(ii) Further, standard perturbation methods imply that

$$\|(E_h(\tau)-E_h^0(\tau))f\|_{\mathcal{L}^2(B(0,2\bar r))} \le C\varepsilon\|f\|_{\mathcal{L}^2} \quad \text{as} \quad \operatorname{supp}(f)\subset B(0,2\bar r)$$

with $\varepsilon=\bar r^{\beta}$.

Then applying standard perturbation arguments[1)] and combining with arguments of Part (i) of the proof one can prove that

(e) $\|\big(E_h(\tau)-E_h^0(\tau)\big)f\|_{\mathscr{C}} \le C\bar r^{\beta-\frac12}$as either $d=1$,

(f) $\|\big(E_h(\tau)-E_h^0(\tau)\big)f\|_{\mathscr{C}} \le C\varepsilon\bar r^{\beta-d/2}$ as either $d=2$ or $d\ge 3$ and $1/p<0$,

(g) $\|\big(E_h(\tau)-E_h^0(\tau)\big)f\|_{\mathscr{L}^{p'}} \le C\varepsilon\bar r^{\beta-d/2+d/p'}$ as $d\ge 3$ and $1/p'>0$

with $\varepsilon=C\bar r^{\beta}$.

Repeating these arguments we decrease $1/p$ until it becomes negative and arrive to $\|\big(E_h(\tau)-E_h^0(\tau)\big)f\|_{\mathscr{C}} \le C\bar r^{-d/2}$ for all f with $\|f\|_{\mathscr{L}^2}=1$ which is equivalent to

$$\|\big(e_h(x,y,\tau)-e_h^0(x,y,\tau)\big)\|_{\mathscr{L}^2_y} \le C\bar r^{\beta-d/2} \qquad \forall x\in B(0,\bar r) \tag{3.6}$$

which together with 3.5) for both $e_h(x,y,\tau)$ and $e_h^0(x,y,\tau)$ implies (3.3).

We leave details to the reader. □

3.2 Intermediate subzone

Remark 3.2. In the framework of Theorem 2.1 in the intermediate zone $\{x\colon \bar{r} \le |x| \le r_*\}$

(i) Estimates (2.2) and (2.3) with $r_*^{1+\alpha} \le T \le \epsilon_0$ remain true.

(ii) Estimates

$$|e^{\mathsf{T}}_{T,h}(x,x,0) - e^{\mathsf{T}}_{T'',h}(x,x,0)| \le Ch^{-d+s} r^{-\alpha d-(1-\alpha)s} \tag{3.7}$$

and

$$|e^{\mathsf{T}}_{T,h}(x,x,0) - e^{0\mathsf{T}}_{T,h}(x,x,0) - e^{\mathsf{T}}_{T',h}(x,x,0) + e^{0\mathsf{T}}_{T',h}(x,x,0)| \le Ch^{-d+s} r^{-\alpha d-(1-\alpha)s+\beta} \tag{3.8}$$

with $T' = C_0 r^{1+\alpha}$ follow from the similar estimates by rescaling.

(iii) Estimates (2.4) and (2.6) with $T' = \epsilon_0 r^{1+\alpha}$ follow from the same estimates by rescaling.

(iv) Furthermore, as $0 < \alpha \le \frac{1}{2}$ corresponding terms in the right-hand expressions could be skipped.

(v) Estimates (2.7) and (2.8) with $T' = \epsilon_0 r^{1+\alpha}$ follow from the same estimates by rescaling.

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100 years of Weyl's Law

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Abstract

We discuss the asymptotics of the eigenvalue counting function for partial differential operators and related expressions paying the most attention to the sharp asymptotics. We consider Weyl asymptotics, asymptotics with Weyl principal parts and correction terms and asymptotics with non-Weyl principal parts. Semiclassical microlocal analysis, propagation of singularities and related dynamics play crucial role.

We start from the general theory, then consider Schrödinger and Dirac operators with the strong magnetic field and, finally, applications to the asymptotics of the ground state energy of heavy atoms and molecules with or without a magnetic field.

1 Introduction

A Bit of History

In 1911, Hermann Weyl, who at that time was a young German mathematician specializing in partial differential and integral equations, proved the following remarkable asymptotic formula describing distribution of (large) eigenvalues of the Dirichlet Laplacian in a bounded domain $X \subset \mathbb{R}^d$:

$$\mathsf{N}(\lambda) = (2\pi)^{-d}\omega_d \,\mathsf{vol}(X)\lambda^{d/2}(1+o(1)) \qquad \text{as} \quad \lambda \to +\infty, \tag{1.1}$$

where $\mathsf{N}(\lambda)$ is the number of eigenvalues of the (positive) Laplacian, which are less than λ [1)], ω_d is a volume of the unit ball in $\mathbb{R}^d$, $\mathsf{vol}(X)$ is the

[1)] $\mathsf{N}(\lambda)$ is called the *eigenvalue counting function*.

volume of X. This formula was actually conjectured independently by Arnold Sommerfeld [Som] and Hendrik Lorentz [Lor] in 1910 who stated the *Weyl's Law* as a conjecture based on the book of Lord Rayleigh "The Theory of Sound" (1887) (for details, see [ANPS]).

H. Weyl published several papers [2–5, W1](1911–1915) devoted to the eigenvalue asymptotics for the Laplace operator (and also the elasticity operator) in a bounded domain with regular boundary. In [W4], he published what is now known as *Weyl's conjecture*

$$\mathsf{N}(\lambda) = (2\pi)^{-d}\omega_d \,\mathsf{vol}(X)\lambda^{d/2} \mp \frac{1}{4}(2\pi)^{1-d}\omega_{d-1}\,\mathsf{vol}'(\partial X)\lambda^{(d-1)/2} \quad \text{as} \quad \lambda \to +\infty \tag{1.2}$$

for Dirichlet and Neumann boundary conditions respectively where $\mathsf{vol}'(\partial X)$ is the $(d-1)$-dimensional volume of $\partial X \in \mathscr{C}^\infty$. Both these formulae appear in the toy-model of a rectangular box $X = \{0 < x_1 < a_1, \dots, 0 < x_d < a_d\}$ and then $\mathsf{N}(\lambda)$ is the number of integer lattice points in the part of ellipsoid $\{z_1^2/a_1^2 + \dots + z_d^2/a_d^2 < \pi^2\lambda\}$ with $z_j > 0$ and $z_j \ge 0$ for Dirichlet and Neumann boundary conditions respectively[2)].

After his pioneering work, a huge number of papers devoted to spectral asymptotics were published. Among the authors were numerous prominent mathematicians.

After H. Weyl, the next big step was made by Richard Courant [Cour](1920), who further developed the variational method and recovered the remainder estimate $O(\lambda^{(d-1)/2}\log\lambda)$. The variational method was developed further by many mathematicians, but it lead to generalizations rather than to getting sharp remainder estimates and we postpone its discussion until Section 3. Here we mention only Mikhail Birman, Elliott Lieb and Barry Simon and their schools.

The next development was due to Torsten Carleman [C1, C2](1934, 1936) who invented the *Tauberian method* and was probably the first to consider an arbitrary spacial dimension (H. Weyl and R. Courant considered only dimensions 2 and 3) followed by Boris Levitan [Lev1](1952) and V. G. Avakumovič [Av](1956) who, applied *hyperbolic operator method* (see Section 1) to recover the remainder estimate $O(\lambda^{(d-1)/2})$, but only for closed manifolds and also for $e(x, x, \lambda)$ away from the boundary[3)].

[2)] Finding sharp asymptotics of the number of the lattice points in the inflated domain is an important problem of the number theory.

[3)] Where here and below $e(x, y, \lambda)$ is the Schwartz kernel of the spectral projector.

After this, Lars Hörmander [Hör1, Hör2](1968, 1969) applied Fourier integral operators in the framework of this method. Hans Duistermaat and Victor Guillemin [DG](1975) recovered the remainder estimate $o(\lambda^{(d-1)/2})$ under the assumption that

(1.3) The set of all periodic geodesics has measure 0

observing that for the sphere neither this assumption nor (1.2) hold. Here, we consider the phase space T^*X equipped with the standard measure $dxd\xi$ where X is a manifold[4]. This was a very important step since it connected the sharp spectral asymptotics with classical dynamics.

The main obstacle was the impossibility to construct the parametrix of the hyperbolic problem near the boundary[5]. This obstacle was partially circumvented by Robert Seeley [See1, See2](1978, 1980) who recovered remainder estimate $O(\lambda^{(d-1)/2})$; his approach we will consider in Subsection 4. Finally the Author [Ivr1](1980), using very different approach, proved (1.2) under assumption that

(1.4) The set of all periodic geodesic billiards has measure 0,

which obviously generalizes (1.3). Using this approach, the Author in [Ivr2] (1982) proved (1.1) and (1.2) for elliptic systems on manifolds without boundary; (1.2) was proven under certain assumption similar to (1.3).

The new approaches were further developed during the 35 years to follow and many new ideas were implemented. The purpose of this article is to provide a brief and rather incomplete survey of the results and techniques. Beforehand, let us mention that the field was drastically transformed.

First, at that time, in addition to the problem that we described above, there were similar but distinct problems which we describe by examples:

(b) Find the asymptotics as $\lambda \to +\infty$ of $\mathsf{N}(\lambda)$ for the Schrödinger operator $\Delta + V(x)$ in $\mathbb{R}^d$ with potential $V(x) \to +\infty$ at infinity;

(c) Find the asymptotics as $\lambda \to -0$ of $\mathsf{N}(\lambda)$ for the Schrödinger operator in $\mathbb{R}^d$ with potential $V(x) \to -0$ at infinity (decaying more slowly than $|x|^{-2}$);

[4] In fact the general scalar pseudodifferential operator and Hamiltonian trajectories of its principal symbol were considered.

[5] Or even inside for elliptic systems with the eigenvalues of the principal symbol having the variable multiplicity.

(d) Find the asymptotics as $h \to +0$ of $\mathsf{N}^-(h)$ the number of the negative eigenvalues for the Schrödinger operator $h^2\Delta + V(x)$.

These four problems were being studied separately albeit by rather similar methods. However, it turned out that the latter problem (d) is more fundamental than the others which could be reduced to it by the variational *Birman-Schwinger principle*.

Second, we should study the *local semiclassical spectral asymptotics*, i.e. the asymptotics of $\int e(x,x,0)\psi(x)\,dx$ where $\psi \in \mathscr{C}_0^\infty$ supported in the ball of radius 1 in which[6] V is of magnitude 1[7]. By means of scaling we generalize these results for ψ supported in the ball of radius γ in which[6] V is of magnitude ρ with $\rho\gamma \ge h$ because in scaling $h \mapsto h/\rho\gamma$. Then in the general case we apply partition of unity with *scaling functions* $\gamma(x)$ and $\rho(x)$.

Third, in the *singular zone* $\{x : \rho(x)\gamma(x) \le h\}$b we can apply variational estimates and combine them with the semiclassical estimates in the *regular zone* $\{x : \rho(x)\gamma(x) \ge h\}$. It allows us to consider domains and operators with singularities.

Some further developments will be either discussed or mentioned in the next sections. Currently, I am working on the Monster book [Ivr4] which summarizes this development. It is almost ready and is available online and we will often refer to it for details, exact statements and proofs.

Finally, I should mention that in addition to the variational methods and method of hyperbolic operator, other methods were developed: other Tauberian methods (like the method of the heat equation or the method of resolvent) and the almost-spectral projector method [ST]. However, we will neither use nor even discuss them; for survey of different methods, see [RSS].

Method of the Hyperbolic Operator

The method of the hyperbolic operator is one of the Tauberian methods proposed by T. Carleman. Applied to the Laplace operator, it was designed as follows: let $e(x,y,\lambda)$ be the Schwartz kernel of a spectral projector and

[6] Actually, in the proportionally larger ball.

[7] Sometimes, however, we consider *pointwise semiclassical spectral asymptotics*, i.e. asymptotics of $e(x,x,0)$.

let

$$u(x,y,t)=\int_0^\infty \cos(\lambda t)\, d_\lambda e(x,y,\lambda^2); \tag{1.5}$$

observe, that now λ^2 is the spectral parameter. Then, $u(x,y,t)$ is a *propagator* of the corresponding wave equation and satisfies

$$u_{tt}+\Delta u=0, \tag{1.6}$$

$$u|_{t=0}=\delta(x-y),\quad u|_{t=0}=0 \tag{1.7}$$

(recall that Δ is a positive Laplacian).

Now we need to construct the solution of (1.6)–(1.7) and recover $e(x,y,t)$ from (1.5). However, excluding some special cases, we can construct the solution $u(x,y,t)$ only modulo smooth functions and only for $t: |t|\le T$, where usually T is a small constant. It leads to

$$F_{t\to\tau}\big(\bar\chi_T(t)u(x,x,t)\big)=T\int \widehat{\bar\chi}((\lambda-\tau)T)\, d_\lambda e(x,x,\lambda^2)= c_0(x)\lambda^{d-1}+c_1(x)\lambda^{d-2}+O(\lambda^{d-3}) \tag{1.8}$$

where F denotes the Fourier transform, $\bar\chi\in \mathscr{C}_0^\infty(-1,1)$, $\bar\chi(0)=1$, $\bar\chi'(0)=0$ and $\bar\chi_T(t)=\bar\chi(t/T)$ [8].

Then using Hörmander's Tauberian theorem[9], we can recover

$$e(x,x,\lambda^2)=c_0(x)d^{-1}\lambda^d+O(\lambda^{d-1}T^{-1}). \tag{1.9}$$

To get the remainder estimate $o(\lambda^{d-1})$ instead, we need some extra arguments. First, the asymptotics (1.8) holds with a cut-off:

$$F_{t\to\tau}\big(\bar\chi_T(t)(Q_x u)(x,x,t)\big)=T\int \widehat{\bar\chi}((\lambda-\tau)T)\, d_\lambda (Q_x e)(x,x,\lambda^2)= c_{0Q}(x)\lambda^{d-1}+c_{1Q}(x)\lambda^{d-2}+O_T(\lambda^{d-3}) \tag{1.10}$$

where $Q_x=Q(x,D_x)$ is a 0-order pseudo-differewntial operator (acting with respect to x only, before we set $x=y$; and $T=T_0$ is a small enough constant. Then the Tauberian theory implies that

$$(Q_x e)(x,x,\lambda^2)=c_{0Q}(x)d^{-1}\lambda^d+c_{1Q}(x)(d-1)^{-1}\lambda^{d-1}+ O\big(\lambda^{d-1}T^{-1}\mu(\operatorname{supp}(Q))\big)+o_{Q,T}\big(\lambda^{d-1}\big) \tag{1.11}$$

[8] In fact, there is a complete decomposition.

[9] Which was already known to Boris Levitan.

where $\mu = \frac{dxd\xi}{dg}$ is a natural measure on the *energy level surface* $\Sigma = \{(x,\xi) : g(x,\xi) = 1\}$ and we denote by $\mathsf{supp}(Q)$ the support of the symbol $Q(x,\xi)$.

On the other hand, propagation of singularities (which we discuss in more details later) implies that if for any point $(x,\xi) \in \mathsf{supp}(Q)$ geodesics starting there are not periodic with periods $\le T$ then asymptotics (1.10) and (1.11) hold with T.

Now, under the assumption (1.4), for any $T \ge T_0$ and $\varepsilon > 0$, we can select Q_1 and Q_2, such that $Q_1 + Q_2 = I$, $\mu(\mathsf{supp}(Q_1)) \le \varepsilon$ and for $(x,\xi) \in \mathsf{supp}(Q_2)$ geodesics starting from it are not periodic with periods $\le T$. Then, combining (1.11) with Q_1, T_0 and with Q_2, T, we arrive to

$$e(x,x,\lambda^2) = c_0(x)d^{-1}\lambda^d + c_1(x)(d-1)^{-1}\lambda^{d-1} + O\big(\lambda^{d-1}(T^{-1}+\varepsilon)\big) + o_{\varepsilon,T}(\lambda^{d-1}) \tag{1.12}$$

with arbitrarily large T and arbitrarily small $\varepsilon > 0$ and therefore

$$e(x,x,\lambda^2) = c_0(x)d^{-1}\lambda^d + c_1(d-1)^{-1}\lambda^{d-1} + O(\lambda^{d-1}T^{-1}). \tag{1.13}$$

holds. In these settings, $c_1 = 0$.

More delicate analysis of the propagation of singularities allows under certain very restrictive assumptions to the geodesic flow to boost the remainder estimate to $O(\lambda^{d-1}/\log\lambda)$ and even to $O(\lambda^{d-1-\delta})$ with a sufficiently small exponent $\delta > 0$.

2 Local Semiclassical Spectral Asymptotics

Asymptotics Inside the Domain

As we mentioned, the approach described above was based on the representation of the solution $u(x,y,t)$ by an oscillatory integral and does not fare well in (i) domains with boundaries because of the trajectories tangent to the boundary and (ii) for matrix operators whose principal symbols have eigenvalues of variable multiplicity. Let us describe our main method. We start by discussing matrix operators on closed manifolds.

So, let us consider a self-adjoint elliptic matrix operator $A(x,D)$ of order m. For simplicity, let us assume that this operator is semibounded from below and we are interested in $N(\lambda)$, the number of eigenvalues not exceeding λ, as $\lambda \to +\infty$. In other words, we are looking for the number $\mathsf{N}^-(h)$ of

negative eigenvalues of the operator $\lambda^{-1}A(x,D) - I = H(x,hD,h)$ with $h = \lambda^{-1/m}$ [10].

Propagation of Singularities

Thus, we are now dealing with the semiclassical asymptotics. Therefore, instead of individual functions, we should consider families of functions depending on the *semiclassical parameter* h [11] and we need a *semiclassical microlocal analysis*. We call such family *temperate* if $\|u_h\| \le Ch^{-M}$ where $\|\cdot\|$ denotes usual $\mathscr{L}^2$-norm.

We say that $u := u_h$ is *s-negligible* at $(\bar{x},\bar{\xi}) \in T^*\mathbb{R}^d$ if there exists a symbol $\phi(x,\xi)$, $\phi(\bar{x},\bar{\xi}) = 1$ such that $\|\phi(x,hD)u_h\| = O(h^s)$. We call the *wave front set of* u_h the set of points at which u_h is not negligible and denote by $\mathsf{WF}^s(u_h)$; this is a closed set. Here, $-\infty < s \le \infty$.

Our first result is rather trivial: if $P = P(x,hD,h)$,

$$WF^s(u) \subset WF^s(Pu) \cup \mathsf{Char}(P) \tag{2.1}$$

where $\mathsf{Char}(P) = \{(x,\xi), \det P^0(x,\xi) = 0\}$; we call $P^0(x,\xi) := P(x,\xi,0)$ the *principal symbol* of P and $\mathsf{Char}(P)$ the *characteristic set* of L.

We need to study the propagation of singularities (wave front sets). To do this, we need the following definition:

Definition 2.1. Let P^0 be a Hermitian matrix. Then P is *microhyperbolic at* (x,ξ) *in the direction* $\ell \in T(T^*\mathbb{R}^d)$, $|\ell| \asymp 1$ if

$$\langle (\ell P^0)(x,\xi)v, v\rangle \ge \epsilon|v|^2 - C|P^0(x,\xi)v|^2 \qquad \forall v \tag{2.2}$$

with constants $\epsilon, C > 0$ [12].

Then we have the following statement which can be proven by the *method of the positive commutator*:

Theorem 2.2. *Let $P = P(x,hD,h)$ be an h-pseudodifferential operator with a Hermitian principal symbol. Let $\Omega \Subset T^*\mathbb{R}^d$ and let $\phi_j \in \mathscr{C}^\infty$ be real-valued functions such that P is microhyperbolic in Ω in the directions $\nabla^{\#}\phi_j$,*

[10] If operator is not semi-bounded we consider the number of eigenvalues in the interval $(0,\lambda)$ (or $(-\lambda,0)$) which could be reduced to the asymptotics of the number of eigenvalues in the interval $(-1,0)$ (or $(0,1)$) of $H(x,hD,h)$.

[11] Which in quantum mechanics is called Planck constant and usually is denoted by $\hbar$.

[12] Here and below ℓP^0 in is the action of the vector field ℓ upon P^0.

$j = 1, \dots, J$ *where* $\nabla^{\#}\phi = \langle(\nabla_\xi\phi), \nabla_x\rangle - \langle(\nabla_x\phi), \nabla_\xi\rangle$ *is the Hamiltonian field generated by* ϕ.

Let u *be tempered and suppose that*

$$\mathrm{WF}^{s+1}(Pu) \cap \Omega \cap \{\phi_1 \le 0\} \cap \dots \cap \{\phi_J \le 0\} = \emptyset, \tag{2.3}$$

$$\mathrm{WF}^{s}(u) \cap \partial\Omega \cap \{\phi_1 \le 0\} \cap \dots \cap \{\phi_J \le 0\} = \emptyset. \tag{2.4}$$

Then,

$$\mathrm{WF}^{s}(u) \cap \Omega \cap \{\phi_1 \le 0\} \cap \dots \cap \{\phi_J \le 0\} = \emptyset. \tag{2.5}$$

Proof. This is Theorem 2.1.2 from [Ivr4]. See the proof and discussion there. □

The above theorem immediately implies:

Corollary 2.3. *Let* $H = H(x, hD, h)$ *be an* h*-pseudodifferential operator with a Hermitian principal symbol and let* $P = hD_t - H$. *Let us assume that*

$$|\partial_{x,\xi} H^0 v| \le C_0|v| + C|(H^0 - \bar\tau)v| \qquad \forall v. \tag{2.6}$$

Let $u(x, y, t)$ *be the Schwartz kernel of* $e^{ih^{-1}tH}$.

(i) For a small constant $T^* > 0$,

$$WF(u) \cap \{|t| \le T^*,\ \tau = \bar\tau\} \subset \{|x-y|^2 + |\xi+\eta|^2 \le (C_0 t)^2\}. \tag{2.7}$$

(ii) Assume that H *is microhyperbolic in some direction* $\ell = \ell(x, \xi)$ *at the point* (x, ξ) *at the energy level* $\bar\tau$ [13]. *Then for a small constant* $T^* > 0$,

$$WF(u) \cap \{0 \le \pm t \le T^*,\ \tau = \bar\tau\} \subset \{\pm(\langle \ell_x, x-y\rangle + \langle \ell_\xi, \xi+\eta\rangle) \ge \pm\epsilon_0 t\}. \tag{2.8}$$

Proof. It is sufficient to prove the above statements for $t \ge 0$. We apply Theorem 2.2 with

(i) $\phi_1 = t$ and $\phi_2 = t - C_0^{-1}(|x - \bar x|^2 + \epsilon^2)^{\frac12} + \varepsilon$,

(ii) $\phi_1 = t$ and $\phi_2 = \big(\langle \ell_x, x-y\rangle + \langle \ell_\xi, \xi+\eta\rangle\big) - \epsilon_0 t + \varepsilon$,

[13] Which means that $H - \bar\tau$ is microhyperbolic in the sense of Definition 2.1.

where $\varepsilon > 0$ is arbitrarily small. □

Corollary 2.4. *(i) In the framework of Corollary 2.3(ii) with $\ell = (\ell_x, 0)$, the inequality*

$$|F_{t\to h^{-1}\tau}\chi_T(t)(Q_{1x}u\,{}^tQ_{2y})(x,x,t)| \le C_s h^{-d}(h/|t|)^s \tag{2.9}$$

holds for all s, $\tau : |\tau - \bar\tau| \le \epsilon$, $h \le |t| \lesssim T \le T^*$ where $Q_{1x} = Q_1(x, hD_x)$, $Q_{2y} = Q_2(y, hD_y)$ are operators with compact supports, ${}^tQ_2$ is the dual rather than the adjoint operator and we write it to the right of the function, $\chi \in \mathscr{C}_0^\infty([-1,-\frac12]\cup[\frac12,1])$, $\chi_T(t) = \chi(t/T)$, and ϵ, T^* are small positive constants.

(ii) In particular, we get the estimate $O(h^s)$ as $T_* := h^{1-\delta} \le |t| \le T \le T^*$.

(iii) More generally, when $\ell = (\ell_x, \ell_\xi)$, the same estimates hold for the distribution $\sigma_{Q_1,Q_2}(t) = \int (Q_{1x}u\,{}^tQ_{2y})(x,x,t)\,dx$.

Proof. (i) If $t \asymp 1$, (2.9) immediately follows from Corollary 2.3(ii). Consider $t \asymp T$ with $h \le T \le T^*$ and make the rescaling $t \mapsto t/T$, $x \mapsto (x-y)/T$, $h \mapsto h/T$. We arrive to the same estimate (with $T^{-d}(h/T)^s$ in the right-hand expression where the factor T^{-d} is due to the fact that $u(x,y,t)$ is a density with respect to y). The transition from $|t| \asymp T$ to $|t| \lesssim T$ is trivial.

(ii) Statement (ii) follows immediately from Statement (i).

(iii) Statement (iii) follows immediately from Statements (i) and (ii) if we apply the metaplectic transformation $(x,\xi) \mapsto (x - B\xi, \xi)$ with a symmetric real matrix B. □

Therefore under the corresponding microhyperbolicity condition, we can construct $(Q_{1x}u\,{}^tQ_{2y})(x,x,t)$ or $\sigma_{Q_1,Q_2}(t)$ for $|t| \le T_*$ and then we automatically get it for $|t| \le T^*$. Since the time interval $|t| \le T_*$ is very short, we are able to apply the successive approximation method.

Successive Approximation Method

Let us consider the propagator $u(x,y,t)$. Recall that it satisfies the equations

$$(hD_t - H)u = 0, \tag{2.10}$$

$$u|_{t=0} = \delta(x-y)I \tag{2.11}$$

and therefore,

$$(hD_t - H)u^{\pm}\,{}^tQ_{2y} = \mp ih\delta(t)\delta(x-y)\,{}^tQ_{2y}, \tag{2.12}$$

where $u^{\pm} = u\theta(\pm t)$, θ is the Heaviside function, I is the unit matrix, $Q_{1x} = Q_1(x, hD_x)$, $Q_{2y} = Q_2(y, hD_y)$ have compact supports, ${}^tQ$ is the *dual operator*[14] and we write operators with respect to y on the right from u in accordance with the notations of matrix theory.

Then,

$$(hD_t - \bar{H})u^{\pm}\,{}^tQ_{2y} = H'u \mp ih\delta(t)\delta(x-y)\,{}^tQ_{2y}I \tag{2.13}$$

with $\bar{H} = H(y, hD_x, 0)$ obtained from H by freezing $x = y$ and skipping lower order terms and $H' = H'(x, y, hD_x, h) = H - \bar{H}$. Therefore,

$$u^{\pm}\,{}^tQ_{2y} = \bar{G}^{\pm}ihH'u^{\pm}\,{}^tQ_{2y} \pm ih\bar{G}^{\mp}\delta(t)\delta(x-y)\,{}^tQ_{2y}I. \tag{2.14}$$

Iterating, we conclude that

$$u^{\pm}\,{}^tQ_{2y} = \sum_{0\le n\le N-1} (\bar{G}^{\pm}ihH')^n \bar{u}^{\pm}\,{}^tQ_{2y} + (\bar{G}^{\pm}ihH')^N u^{\pm}\,{}^tQ_{2y}, \tag{2.15}$$

$$\bar{u}^{\pm} = \mp ih\bar{G}^{\pm}\delta(t)\delta(x-y)\,{}^tQ_{2y}I \tag{2.16}$$

where $\bar{G}^{\pm}$ is a parametrix of the problem

$$(ihD_t - \bar{H})v = f, \qquad \operatorname{supp}(v) \subset \{\pm t \ge 0\} \tag{2.17}$$

and $G^{\pm}$ is a parametrix of the same problem problem albeit for H.

Observe that

$$H' = \sum_{1\le|\alpha|+m\le N-1} (x-y)^{\alpha}h^m R_{\alpha,m}(y, hD_x) + \sum_{|\alpha|+m=N} (x-y)^{\alpha}h^m R_{\alpha,m}(x, y, hD_x); \tag{2.18}$$

therefore due to the finite speed of propagation, its norm does not exceed CT as long as we only consider strips $\Pi_T^{\pm} := \{0 \le \pm t \le T\}$. Meanwhile, due to the Duhamel's integral, the operator norms of $G^{\pm}$ and $\bar{G}^{\pm}$ from $\mathscr{L}^2(\Pi_T^{\pm})$ to $\mathscr{L}^2(\Pi_T^{\pm})$ do not exceed $Ch^{-1}T$ and therefore each next term in the successive

[14] I.e. ${}^tQv = (Q^*v^{\dagger})^{\dagger}$ where $v^{\dagger}$ is the complex conjugate to v.

approximations (2.15) acquires an extra factor $Ch^{-1}T^2 = O(h^\delta)$ as long as $T \le h^{\frac12(1+\delta)}$ and the remainder term is $O(h^s)$ if N is large enough.

To calculate the terms of the successive approximations, let us apply h-Fourier transform $F_{(x,t)\to h^{-1}(\xi,\tau)}$ with $\xi \in \mathbb{R}^d$, $\tau \in \mathbb{C}_\mp := \{\tau : \mp \operatorname{Im}\tau > 0\}$ and observe that $\delta(t)\delta(x-y) \mapsto (2\pi)^{-d-1}e^{-ih^{-1}\langle y,\eta\rangle}$, ${}^tQ_{2y}$ and $R_{\alpha,m}$ become multiplication by $Q_2(y,\eta)$ and $R_{\alpha,m}(y,\xi)$ respectively, and $\bar G^\pm$ becomes multiplication by $(\tau - H^0(y,\xi))^{-1}$. Meanwhile, $(x_j - y_j)$ becomes $-ih\partial_{\xi_j}$.

Therefore the right-hand expression of (2.15) without the remainder term becomes a sum of terms $\mp i\mathcal{F}_m(y,\xi,\tau)h^{m+1}e^{-ih^{-1}\langle y,\eta\rangle}$ with $m \ge 0$ and $\mathcal{F}_m(y,\xi,\tau)$ the sum of terms of the type

$$(\tau - H^0(y,\xi))^{-1}b_*(y,\xi)(\tau - H^0(y,\xi))^{-1}b_*(y,\xi)\cdots b_*(y,\xi)\times (\tau - H^0(y,\xi))^{-1}Q_2(y,\eta) \tag{2.19}$$

with no more than $2m+1$ factors $(\tau - H^0(y,\xi))^{-1}$. Here, the b_* are regular symbols. In particular,

$$\mathcal{F}_0(y,\xi,\tau) = (2\pi)^{-d-1}(\tau - H^0(y,\xi))^{-1}Q_2(y,\eta). \tag{2.20}$$

If we add the expressions for u^+ and u^- instead of $\mathcal{F}_m(y,\xi,\tau)$ with $\tau \in \mathbb{C}_\mp$, we get the distributions $\bigl(\mathcal{F}_m(y,\xi,\tau+i0) - \mathcal{F}_m(y,\xi,\tau-i0)\bigr)$ with $\tau \in \mathbb{R}$.

Applying the inverse h-Fourier transform with respect to x, operator Q_{1x}, and setting $x = y$, we cancel the factor $e^{-ih^{-1}\langle y,\eta\rangle}$ and gain a factor of h^{-d}. Thus we arrive to the Proposition 2.5(i) below; applying Corollary 2.4(ii) and (iii), we arrive to its Statements (ii) and (iii). We also need to use

$$u(x,y,t) = \int e^{ih^{-1}t\tau}\, d_\tau e(x,y,\tau). \tag{2.21}$$

Proposition 2.5. *(i) As* $T_* = h^{1-\delta} \le T \le h^{\frac12+\delta}$ *and* $\bar\chi \in \mathscr{C}_0^\infty([-1,1])$

$$T\int \widehat{\bar\chi}\bigl((\lambda-\tau)Th^{-1}\bigr)\, d_\tau(Q_{1x}e\,{}^tQ_{2y})(y,y,\tau) \sim \sum_{m\ge 0} h^{-d+m}T\int \widehat{\bar\chi}\bigl((\lambda-\tau)Th^{-1}\bigr)\kappa'_m(y,\tau)d\tau, \tag{2.22}$$

where $\widehat{\bar\chi}$ *is the Fourier transform of* $\bar\chi$ *and*

$$\kappa'_m(y) = \int \bigl(\mathcal{F}_m(y,\xi,\tau+i0) - \mathcal{F}_m(y,\xi,\tau-i0)\bigr)\, d\eta. \tag{2.23}$$

(ii) If H is microhyperbolic on the energy level $\bar\tau$ on $\mathsf{supp}(Q_2)$ in some direction ℓ with $\ell_x = 0$ then (2.21) *holds with $T_* \le T \le T^*$, $|\lambda - \bar\tau| \le \epsilon$, where T^* is a small constant.*

(iii) On the other hand, if $\ell_x \ne 0$, then (2.21) *still holds with $T \le T^*$, albeit only after integration with respect to y:*

$$T\int \widehat{\bar\chi}\big((\lambda-\tau)Th^{-1}\big)\,d_\tau\Big(\int (Q_{1x}e^{\,t}Q_{2y})(y,y,\tau)\,dy\Big) \sim \sum_{m\ge 0} h^{-d+m}T\int \widehat{\bar\chi}\big((\lambda-\tau)Th^{-1}\big)\varkappa'_m(\tau)\,d\tau \tag{2.24}$$

with

$$\varkappa'_m(\tau) = \iint \big(\mathcal{F}_m(y,\xi,\tau+i0) - \mathcal{F}_m(y,\xi,\tau-i0)\big)\,dyd\eta. \tag{2.25}$$

For details, proofs and generalizations, see Section 4.3 of [Ivr4].

Recovering Spectral Asymptotics

Let $\alpha(\tau)$ denote $(Q_{1x}e^{\,t}Q_{2y})(y,y,\tau)$ (which may be integrated with respect to y) and $\beta(\tau)$ denote the convolution of its derivative $\alpha'(\tau)$ with $T\widehat{\bar\chi}(\tau T/h)$. To recover $\alpha(\tau)$ from $\beta(\tau)$, we apply *Tauberian methods*. First of all, we observe that under the corresponding microhyperbolicity condition the distribution $\kappa'_m(y,\tau)$ or $\varkappa'_m(\tau)$ is smooth and the right-hand side expression of (2.22) or (2.24) does not exceed Ch^{-d+1}.

Let us take $Q_1 = Q_2$; then $\alpha(y,\tau)$ or $\alpha(\tau)$ is a monotone non-decreasing matrix function of τ. We choose a Hörmander function[15] $\bar\chi(t)$ and estimate the left-hand expressions of (2.22) or (2.24) from below by

$$\epsilon_0 T\big(\alpha(\lambda+hT^{-1}) - \alpha(\lambda-hT^{-1})\big),$$

which implies that $\big(\alpha(\lambda+hT^{-1})-\alpha(\lambda-hT^{-1})\big) \le CT^{-1}h^{-d+1}$ and therefore

$$|\alpha(\lambda)-\alpha(\mu)| \le Ch^{-d+1}|\lambda-\mu| + CT^{-1}h^{-d+1} \tag{2.26}$$

as $\lambda,\mu \in (\bar\tau-\epsilon, \bar\tau+\epsilon)$. Then (2.26) automatically holds, even if Q_1 and Q_2 are not necessarily equal.

[15] I.e. a compactly supported function with positive Fourier transform.

Further, (2.26) implies that

$$|\alpha(\lambda)-\alpha(\mu)-h^{-1}\int_{\mu}^{\lambda}\beta(\tau)\,d\tau|\le CT^{-1}h^{-d+1} \tag{2.27}$$

and therefore

$$|\int\Big(\alpha(\lambda)-\alpha(\mu)-h^{-1}\int_{\mu}^{\lambda}\beta(\tau)\,d\tau\Big)\phi(\mu)\,d\mu|\le CT^{-1}h^{-d+1} \tag{2.28}$$

if $\bar\chi=1$ on $[-\frac12,\frac12]$, $\lambda,\mu\in(\bar\tau-\epsilon,\bar\tau+\epsilon)$ and $\phi\in\mathscr{C}_0^\infty((\bar\tau-\epsilon,\bar\tau+\epsilon))$ with $\int\phi(\tau)\,d\tau=1$.

On the other hand, even without the microhyperbolicity condition, our successive approximation construction is not entirely useless. Let us apply $\varphi_L(hD_t-\lambda)$ with $\varphi\in\mathscr{C}_0^\infty([-1,1])$ and $L\ge h^{\frac12-\delta}$, and then set $t=0$. We arrive to

$$\int\varphi((\tau-\lambda)L^{-1})\big(\alpha'(\tau)-\beta(\tau)\big)\,d\tau=O(h^\infty). \tag{2.29}$$

This allows us to extend (2.28) to $\phi\in\mathscr{C}_0^\infty(bR))$ with $\int\phi(\tau)\,d\tau=1$. For full details and generalizations, see Section 4.4 of [Ivr4].

Thus, we have proved:

Theorem 2.6. *Let $H=H(x,hD,h)$ be a self-adjoint operator. Then,*

(i) The following asymptotics holds for $L\ge h^{\frac12-\delta}$:

$$\int\phi((\tau-\lambda)L^{-1})\Big(d_\tau(Q_{1x}e^{\,t}Q_{2y})(y,y,\tau)-\sum_{m\ge0}h^{-d+m}\kappa'(y,\tau)\,d\tau\Big)=O(h^\infty). \tag{2.30}$$

(ii) Let H be microhyperbolic on the energy level $\bar\tau$ in some direction ℓ with $\ell_x=0$. Then for $|\lambda-\bar\tau|\le\epsilon$,

$$(Q_{1x}e^{\,t}Q_{2y})(y,y,\lambda)=h^{-d}\kappa_0(y,\lambda)+O(h^{-d+1}) \tag{2.31}$$

with $\kappa_m(y,\lambda):=\int_{-\infty}^{\lambda}\kappa_m'(y,\tau)\,d\tau$.

(iii) Let H be microhyperbolic on the energy level $\bar\tau$ in some direction ℓ. Then for $|\lambda-\bar\tau|\le\epsilon$,

$$\int (Q_{1x}e\,{}^tQ_{2y})(y,y,\lambda)\,dy = h^{-d}\varkappa_0(\lambda)+O(h^{-d+1}) \tag{2.32}$$

with $\varkappa_m(y) \coloneqq \int_{-\infty}^{\lambda}\varkappa_m'(\tau)\,d\tau$.

(iv) In particular, it follows from (2.20) *that*

$$\kappa_0(\lambda,x) = (2\pi)^{-d}\int q_1^0(x,\xi)\theta(\lambda-H^0(x,\xi))q_2^0(x,\xi)\,d\xi \tag{2.33}$$

and

$$\varkappa_0(\lambda) = (2\pi)^{-d}\int q_1^0(x,\xi)\theta(\lambda-H^0(x,\xi))q_2^0(x,\xi)\,dxd\xi \tag{2.34}$$

Remark 2.7. (i) So far we have assumed that Q_1, Q_2 had compactly supported symbols in (x,ξ). Assuming that these symbols are compactly supported with respect to x only, in particular when $Q_1=\psi(x)$, $Q_2=1$, with $\psi\in\mathscr{C}_0^\infty(X)$, we need to assume that

(2.35) $\{\xi : \exists x\in X : \operatorname{Spec} H^0(x,\xi)\cap(-\infty,\lambda+\epsilon_0]\ne\emptyset\}$ is a compact set.

(ii) If we assume only that

(2.36) $\{\xi : \exists x\in X : \operatorname{Spec} H^0(x,\xi)\cap(\mu-\epsilon_0,\lambda+\epsilon_0]\ne\emptyset\}$ is a compact set,

instead of (2.31) and (2.32), we get

$$|(Q_{1x}e\,{}^tQ_{2y})(y,y,\lambda,\mu)-h^{-1}\kappa_0(y,\lambda,\mu)|\le Ch^{-d+1} \tag{2.37}$$

and

$$\Big|\iint (Q_{1x}e\,{}^tQ_{2y})(y,y,\lambda,\mu)\,dy - h^{-1}\varkappa_0(\lambda,\mu)\Big|\le Ch^{-d+1}, \tag{2.38}$$

where $\mu\le\lambda$, $e(x,y,\lambda,\mu)\coloneqq e(x,y,\lambda)-e(x,y,\mu)$, $\kappa_m(y,\lambda)\coloneqq\int_\mu^\lambda\kappa_m'(y,\tau)\,d\tau$, $\varkappa_m(y,\lambda)\coloneqq\int_\mu^\lambda\varkappa_m'(y,\tau)\,d\tau$ and we assume that the corresponding microhyperbolicity assumption is fulfilled on both energy levels μ and λ.

(iii) If $H^0(x,\xi)$ is an elliptic symbol which is positively homogeneous of degree $m > 0$ with respect to ξ, then the microhyperbolicity condition is fulfilled with $\ell = (0, \pm\xi)$ on energy levels $\tau \neq 0$. Furthermore, the compactness condition of (ii) is fulfilled, and if $H^0(x,\xi)$ is also positive-definite, then the compactness condition of (i) is also fulfilled.

Second Term and Dynamics

Propagation of Singularities To derive two-term asymptotics, one can use the scheme described in Section 1, albeit one needs to describe the propagation of singularities. For matrix operators, this may be slightly tricky.

Let us introduce the *characteristic symbol* $g(\mathsf{x},\xi) := \det(\tau - H^0(\mathsf{x},\xi))$ where $\mathsf{x} = (x_0, x)$, $\xi = (\xi_0,\xi)$ etc; then $\mathsf{Char}(\xi_0 - H(x,\xi)) = \{(x,\xi) : g(x,\xi) = 0\}$. Let ξ_0 be a root of multiplicity r of $g(x,\xi_0,\xi)$; then $g^{(\alpha)}_{(\beta)}(x,\xi) = 0$ for all $\alpha,\beta : |\alpha| + |\beta| < r$. Let us consider the r-jet of g at such a point:

$$g_{(x,\xi)}(y,\eta) := \sum_{\alpha,\beta:|\alpha|+|\beta|<r} \frac{1}{\alpha!\beta!} g^{(\alpha)}_{(\beta)}(x,\xi) y^\beta \eta^\alpha; \tag{2.39}$$

it is a hyperbolic polynomial with respect to η_0. Consider its *hyperbolicity cone* $K(x,\xi)$, which is the connected component of $\{(\mathsf{y};\eta) \in \mathbb{R}^{2d+2} : g_{(x,\xi)}(\mathsf{y},\eta) \neq 0\}$ containing $\{(y,\eta) : \eta_0 = 1, y = \eta = 0\}$ and the *dual hyperbolicity cone*

$$K^\#(x,\xi) = \{(\mathsf{y}',\eta') : \langle \mathsf{y}',\eta\rangle - \langle \mathsf{y},\eta'\rangle > 0\} \subset \{y_0 = 0\}. \tag{2.40}$$

Definition 2.8. (i) An absolutely continuous curve $(\mathsf{x}(t), \xi(t))$ (with $x_0 = t$) is called a *generalized Hamiltonian trajectory* if a.e.

$$(1, \frac{dx}{dt}; \frac{d\xi}{dt}) \in K^\#(x,\xi_0,\xi). \tag{2.41}$$

Note that $\xi_0 = \tau$ remains constant along the trajectory.

(ii) Let $\mathcal{K}^\pm(x,\xi)$ denote the union of all generalized Hamiltonian trajectories issued from (x,ξ) in the direction of increasing/decreasing t.

If $g = \alpha g_1^r$ where $\alpha \neq 0$ and $g_1 = 0 \implies \nabla g_1 \neq 0$, the generalized Hamiltonian trajectories are just (ordinary) Hamiltonian trajectories of g_1 and $\mathcal{K}^{\pm}(x,\xi)$ are just half-trajectories[16].

The following theorem follows from Theorem 2.2:

Theorem 2.9. *If $u(x,y,t)$ is the Schwartz kernel of $e^{ih^{-1}tH}$, then*

$$\mathsf{WF}(u) \subset \{(x,\xi;y,-\eta;t,\tau) : \pm t > 0, (t,x;\tau,\xi) \in K^{\pm}(0,y;\tau,\eta)\}. \tag{2.42}$$

Then, we obtain:

Corollary 2.10. *In the framework of Theorem 2.9,*

$$\mathsf{WF}(\sigma_{Q_1,Q_2}(t)) \subset \{(t,\tau) : \exists (x,\xi) : (t,x;\tau,\xi) \in \mathcal{K}^{\pm}(0,x;\tau,\xi)\}, \tag{2.43}$$

and for any x,

$$\mathsf{WF}(Q_{1x} u\, {}^tQ_{2y}) \subset \{(t,\tau) : \exists \xi,\eta : (t,x;\tau,\xi) \in \mathcal{K}^{\pm}(0,x;\tau,\eta)\}. \tag{2.44}$$

Definition 2.11. (i) A *periodic point* is a point (x,ξ) which satisfies $(t,x;\tau,\xi) \in \mathcal{K}^{\pm}(0,x;\tau,\xi)$ for some $t \neq 0$.

(ii) A *loop point* is a point x which satisfies $(t,x;\tau,\xi) \in \mathcal{K}^{\pm}(0,x;\tau,\eta)$ for some $t \neq 0$, ξ,η; we call η a *loop direction*.

Application to Spectral Asymptotics Combining Corollary 2.10 with the arguments of Section 1, we arrive to

Theorem 2.12. *(i) In the framework of Theorem 2.6(ii) let for some x the set of all loop directions at point x on energy level λ have measure 0*[17]*. Then,*

$$(Q_{1x} e\, {}^tQ_{2y})(y,y,\lambda) = h^{-d}\kappa_0(y,\lambda) + h^{1-d}\kappa_1(y,\lambda) + o(h^{-d+1}). \tag{2.45}$$

(ii) In the framework of Theorem 2.6(iii), suppose that the set of all periodic points on energy level λ has measure 0[18]*. Then,*

$$\int (Q_{1x} e\, {}^tQ_{2y})(y,y,\lambda)\, dy = h^{-d}\varkappa_0(\lambda) + h^{1-d}\varkappa_1(\lambda) + o(h^{-d+1}). \tag{2.46}$$

[16] Since $e^{ih^{-1}tH}$ describes evolution with revert time, time is also reverted along (generalized) Hamiltonian trajectories.

[17] There exists a natural measure $\mu_{\lambda,x}$ on $\{\xi : \det(\lambda - H^0(x,\xi)) = 0\}$.

[18] There exists a natural measure μ_λ on $\{(x,\xi) : \det(\lambda - H^0(x,\xi)) = 0\}$.

Remark 2.13. (i) When studying propagation, we can allow H to also depend on $x_0 = t$; for all details and proofs, see Sections 2.1 and 2.2 of [Ivr4].

(ii) Recall that $e(x, y, \lambda)$ is the Schwartz kernel of $\theta(\lambda - H)$. We can also consider $e_\nu(x, y, \tau)$ which is the Schwartz kernel of $(\lambda - H)^\nu_+ := (\lambda - H)^\nu \theta(\lambda - H)$ with $\nu \ge 0$. Then in the Tauberian arguments, $h^{-d} \times (h/T)$ is replaced by $h^{-d} \times (h/T)^{1+\nu}$ and then in the framework of Theorem 2.6(ii) and (iii) remainder estimates are $O(h^{-d+1+\nu})$ and in the framework of Theorem 2.6(i) and (ii), the remainder estimates are $o(h^{-d+1+\nu})$; sure, in the asymptotics one should include all the necessary terms $\kappa_m h^{-d+m}$ or $\varkappa_m h^{-d+m}$ [19].

(iii) Under more restrictive conditions on Hamiltonian trajectories instead of T an arbitrarily large constant, we can take T depending on h [20]; see Section 2.4 of [Ivr4]. Usually, we can take $T = \epsilon |\log h|$ or even $T = h^{-\delta}$.

Then in the remainder estimate, the main term is

$$C\big(\mu(\Pi_{T,\gamma}) h^{-d+1} + h^{-d+1+\nu} T^{-1-\nu}\big),$$

where $\Pi_{T,\gamma}$ is the set of all points $z = (x, \xi)$ (on the given energy level) such that $\operatorname{dist}(\Psi_t(z), z) \le \gamma$ for some $t \in (\epsilon, T)$ and $\gamma = h^{1/2-\delta'}$. Here, however, we assume that either H^0 is scalar or its eigenvalues have constant multiplicities and apply the Heisenberg approach to the long-term evolution.

Then the remainder estimates could be improved to $O(h^{-d+1+\nu} |\log h|^{-1-\nu})$ or even to $O(h^{-d+1+\nu+\delta})$ respectively. As examples, we can consider the geodesic flow on a Riemannian manifold with negative sectional curvature (log case) and the completely integrable non-periodic Hamiltonian flow (power case). For all details and proofs, see Section 4.5 of [Ivr4].

Rescaling Technique

The results we proved are very uniform: as long as we know that operator in question is self-adjoint and that the smoothness and non-degeneracy conditions are fulfilled uniformly in $B(\bar{x}, 1)$, then all asymptotics are also uniform (as $x \in B(\bar{x}, \frac{1}{2})$ or $\operatorname{supp}(\psi) \subset B(\bar{x}, \frac{1}{2})$). Then these results could self-improve.

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[19] Here, we need to assume that H is semi-bounded from below; otherwise some modifications are required.

[20] Usually these restrictions are $T \le h^{-\delta}$ and $|D\Psi_t(z)| \le h^{-\delta}$ with sufficiently small $\delta > 0$.

Here we consider only the Schrödinger operator away from the boundary; but the approach could be generalized for a wider class of operators. For generalizations, details and proofs, see Chapter 5 of [Ivr4].

Proposition 2.14. *Consider the Schrödinger operator. Assume that* $\rho\gamma \ge h$ *and in* $B(\bar{x},\gamma)\subset X$,

$$|\partial^\alpha g^{jk}| \le c_\alpha \gamma^{-|\alpha|}, \qquad |\partial^\alpha V| \le c_\alpha \rho^2 \gamma^{-|\alpha|}. \tag{2.47}_{1,2}$$

Then,

(i) In $B(\bar{x},\frac{1}{2}\gamma)$,

$$e(x,x,0) \le C\rho^d h^{-d}. \tag{2.48}$$

(ii) If in addition $|V| + |\nabla V|\gamma \ge \epsilon\rho^2$, *then for* $\mathsf{supp}(\psi)\subset B(\bar{x},\frac{1}{2}\gamma)$ *such that* $|\partial^\alpha\psi| \le c_\alpha\gamma^{-|\alpha|}$,

$$\left|\int \bigl(e(x,x,0) - \kappa_0 V_-^{d/2}\bigr)\,dx\right| \le C\rho^{d-1}\gamma^{d-1}h^{1-d}; \tag{2.49}$$

(iii) If in addition $|V| \ge \epsilon\rho^2$ *in* $B(\bar{x},\gamma)$ *then*

$$|e(x,x,0) - \kappa_0 V_-^{d/2}| \le C\rho^{d-1}\gamma^{-1}h^{1-d}; \tag{2.50}$$

(iv) If in addition $V \ge \epsilon\rho^2$ *in* $B(\bar{x},\gamma)$, *then for any* s,

$$|e(x,x,0)| \le C\rho^{d-s}\gamma^{-s}h^{s-d}. \tag{2.51}$$

Proof. Indeed, we have already proved this in the special case $\rho=\gamma=1$, $h\le 1$. In the general case, we can reduce the problem to the special case by rescaling $x\mapsto x\gamma^{-1}$, $\tau\mapsto\tau\rho^{-2}$ (so we multiply operator by ρ^{-2}) and then automatically $h\mapsto \hbar = h\rho^{-1}\gamma^{-1}$. Recall that $e(x,y,\tau)$ is a function with respect to x but a density with respect to y so an extra factor γ^{-d} appears in the right-hand expressions. □

Let us assume that the conditions $(2.47)_{1,2}$ are fulfilled with $\rho=\gamma=1$. We want to get rid of the non-degeneracy assumption $|V|\asymp 1$ in the pointwise asymptotics. Let us introduce the *scaling function* $\gamma(x)$ and also $\rho(x)$

$$\gamma(x) = \epsilon|V(x)| + \bar{\gamma} \quad \text{with} \quad \bar{\gamma} = h^{\frac{2}{3}}, \qquad \rho(x) = \gamma(x)^{\frac{1}{2}}. \tag{2.52}$$

One can easily see that

$$|\nabla\gamma| \le \frac{1}{2}, \qquad \rho\gamma \ge h, \tag{2.53}$$

$(2.47)_{1,2}$ are fulfilled and either $|V| \ge \epsilon\rho^2$ or $\rho\gamma \asymp h$ and therefore (2.50) holds ($\hbar \asymp 1$ as $\rho\gamma \asymp h$ and no non-degeneracy condition is needed). Note that for $d \ge 3$, the right-hand expression of (2.50) is $O(h^{1-d})$ and for $d = 1, 2$, it is $O(h^{-\frac{2}{3}d})$. So, we got rid of the non-degeneracy assumption $|V| \asymp 1$, and the remainder estimate deteriorated only for $d = 1, 2$.

Remark 2.15. (i) We can improve the estimates for $d = 1, 2$ to $O(h^{\frac{1}{3}-\frac{2}{3}d})$, but then we will need to add some correction terms first under the assumption $|V| + |\nabla V| \asymp 1$ and then get rid of it by rescaling; these correction terms are of boundary-layer type (near $V = 0$) and are $O(h^{-\frac{2}{3}d})$ and are due to short loops. For details, see Theorems 5.3.11 and 5.3.16 of [Ivr4].

(ii) If $d = 2$, then under the assumption $|V| + |\nabla V| \asymp 1$, the weight $\rho^{-1}\gamma^{-1}$ is integrable, and we arrive to the local asymptotics with the remainder estimate $O(h^{1-d})$.

(iii) We want to get rid of the non-degeneracy assumption $|V| + |\nabla V| \asymp 1$ in the local asymptotics. We can do it with the scaling function

$$\gamma(x) = \epsilon\big(|V(x)| + |\nabla V|^2\big)^{\frac{1}{2}} + \bar\gamma \quad \text{with} \quad \bar\gamma = h^{\frac{1}{2}}, \qquad \rho(x) = \gamma(x). \tag{2.54}$$

Then for $d = 2$, we recover remainder the estimate $O(h^{-1})$; while for $d = 1$, the remainder estimate $O(h^{-\frac{1}{2}})$ which could be improved further up to $O(1)$ under some extremely weak non-degeneracy assumption or to $O(h^{-\delta})$ with an arbitrarily small exponent $\delta > 0$ without it.

(iv) If $d \ge 2$, then in the framework of Theorem 2.12(i), we can get rid of the non-degeneracy assumption as well. This is true for the magnetic Schrödinger operator as well if $d \ge 2$; when $d = 2$, some modification of the statement is required; see Remark 5.3.4 of [Ivr4].

(v) Furthermore, if we consider asymptotics for $\mathsf{Tr}\big((\lambda - H)^{\nu}_{+}\psi\big)$ (see Remark 2.13(ii)) with $\nu > 0$ then in the local asymptotics, we get the remainder estimate $O(h^{1-d+s})$ without any non-degeneracy assumptions. For details, see Theorem 5.3.5 of [Ivr4].

Operators with Periodic Trajectories

Preliminary Analysis Consider a scalar operator H. For simplicity, assume that X is a compact closed manifold. Assume that all the Hamiltonian trajectories are periodic (with periods not exceeding $C(\mu)$ on the energy levels $\lambda \le \mu$). Then the period depends only on the energy level and let $T(\lambda)$ be the minimal period such that all trajectories on the energy level λ are $T(\lambda)$ periodic[21].

Without any loss of the generality, one can assume that $T(\lambda) = 1$. Indeed, we can replace H by $f(H)$ with $f'(\lambda) = 1/T(\lambda)$. Then,

$$e^{\nabla^{\#} H^0} = I \tag{2.55}$$

and therefore,

$$e^{ih^{-1}H} = e^{i\varepsilon h^{-1}B}, \tag{2.56}$$

where $B = B(x, hD, h)$ is an h-pseudo-differential operator which could be selected to commute with H, at this point, $\varepsilon = h$. Then, $H_0 = H - \varepsilon B$ satisfies

$$e^{ih^{-1}H_0} = I \implies \mathsf{Spec}(H_0) \subset 2\pi h\mathbb{Z}; \tag{2.57}$$

we call this *quantum periodicity* in contrast to the *classical periodicity* (2.55).

We can calculate the multiplicity $\mathsf{N}_{k,h} = O(h^{1-d})$ of the eigenvalue $2\pi hk$ with $k \in \mathbb{Z}$ modulo $O(h^{\infty})$. The formula is rather complicated especially since subperiodic trajectories[21] cause the redistribution of multiplicities between eigenvalues (however, this causes no more than $O(h^{1-d+r})$ error).

We consider $H := H_\varepsilon = H_0 + \varepsilon B$ as a perturbation of H_0 and we assume only that $\varepsilon \ll 1$. If $\varepsilon \le \epsilon_0 h$, the spectrum of H consists of *eigenvalue clusters* of the width $C_0\varepsilon$ separated by *spectral gaps* of the width $\asymp h$, but if $\varepsilon \ge \epsilon_0 h$, these clusters may overlap.

Long Range Evolution Consider

$$e^{ih^{-1}tH} = e^{ih^{-1}tH_0} e^{-ih^{-1}t\varepsilon B} = e^{ih^{-1}t'H_0} e^{ih^{-1}t''B} \tag{2.58}$$

[21] However, there could be *subperiodic trajectories*, i.e. trajectories periodic with period $T(\lambda)/p$ with $p = 2, 3, \dots$. It is known that the set Λ_p of subperiodic trajectories with subperiod $T(\lambda)/p$ is a union of symplectic submanifolds $\Lambda_{p,r}$ of codimension $2r$.

with $t'' = \varepsilon t$, $t' = t - \lfloor t \rfloor$. We now have a *fast evolution* $e^{ih^{-1}t'H_0}$ and a *slow evolution* $e^{ih^{-1}t''B}$ and both t', t'' are bounded as $|t| \le T^* := \varepsilon^{-1}$. Therefore, we can trace the evolution up to time T^*.

Let the following non-degeneracy assumption be fulfilled:

$$|\nabla_{\Sigma(\lambda)} b| \ge \epsilon_0, \tag{2.59}$$

where b is the principal symbol of B, $\Sigma(\lambda) := \{(x,\xi) : H^0(x,\xi) = \lambda\}$ and $\nabla_{\Sigma(\lambda)}$ is the gradient along $\Sigma(\lambda)$. Then using our methods, we can prove that

$$|F_{t\to h^{-1}\tau}\chi_T(t) \int u(x,x,t)\psi(x)\,dx| \le CTh^{1-d}(h/\varepsilon T)^s, \tag{2.60}$$

and therefore

$$F_{t\to h^{-1}\tau}\bar\chi_T(t) \int u(x,x,t)\psi(x)\,dx| \le Ch^{1-d}(\varepsilon^{-1}h+1) \tag{2.61}$$

for $\epsilon_0(\varepsilon^{-1}h+1) \le T \le \epsilon_0\varepsilon^{-1}$; recall that $\chi \in \mathscr{C}_0^\infty([-1,-\frac12]\cup[\frac12,1]$ and $\bar\chi \in \mathscr{C}_0^\infty([-1,1]$, $\bar\chi = 1$ on $[-\frac12,\frac12]$.

Then the Tauberian error does not exceed the right-hand expression of (2.61) multiplied by $T^{*-1} \asymp \varepsilon$, i.e. $Ch^{1-d}(\varepsilon + h)$. In the Tauberian expression, we need to take $T = \epsilon_0(\varepsilon^{-1}h^{1-\delta}+1)$.

Calculations We can pass from Tauberian expression to a more explicit one. Observe that the contribution to the former are produced only by time intervals $t \in [n-h^{1-\delta}, n+h^{1-\delta}]$ with $|n| \le T_*$; contribution of the remaining interval will be either negligible (if there are no subperiodic trajectories) or $O(h^{2-d})$ (if such trajectories exist). Such an interval with $n=0$ produces the standard Weyl expression.

Consider $n \ne 0$. Then the contribution of such intervals lead to a correction term

$$\mathsf{N}_{\mathsf{corr},Q_1,Q_2}(\lambda) := (2\pi)^{-d}h^{1-d}\int_{\Sigma_\tau} q_1^0\Upsilon_1\bigl(h^{-1}(H^0-\varepsilon b)\bigr)\,d\mu_\tau q_2^0, \tag{2.62}$$

where $\Upsilon_1(t) = 2\pi\lceil\frac{t}{2\pi}\rceil - t + \frac12$.

Theorem 2.16. *Under assumptions* (2.55), (2.56), (2.59) *and* $\varepsilon \ge h^M$,

$$\begin{aligned}\int (Q_{1x}e^{\,t}Q_{2y})(y,y,\lambda)\,dy &= h^{-d}\varkappa_{0,Q_1,Q_2}(\lambda)\\ &\quad + h^{1-d}\varkappa_{1,Q_1,Q_2}(\lambda) + \mathsf{N}_{\mathsf{corr},Q_1,Q_2}(\lambda) + O\bigl(h^{1-d}(\varepsilon+h)\bigr).\end{aligned} \tag{2.63}$$

For a more general statement with (2.59) replaced by a weaker non-degeneration assumption, see Theorem 6.2.24 of [Ivr4]. Further, we can skip a correction term (2.62) if $\varepsilon \ge h^{1-\delta}$; while if $h^M \le \varepsilon \le h^{1-\delta}$, this term is $O(h^{1-d}(h/\varepsilon)^s)$ for $\varepsilon \ge h$ and of magnitude h^{1-d} for $h^M \le \varepsilon \le h$.

For further generalizations, details and proofs, see Sections 6.2 and 6.3 of [Ivr4]. For related spectral asymptotics for a family of commuting operators, see Section 6.1 of [Ivr4].

One can also consider the case when there is a massive set of periodic trajectories, yet non-periodic trajectories exist. For details, see [SV1] and Subsection 6.3.7 of [Ivr4].

Boundary Value Problems

Preliminary Analysis

Let X be a domain in $\mathbb{R}^d$ with boundary ∂X and H an h-differential matrix operator which is self-adjoint in $\mathscr{L}^2(X)$ under the h-differential boundary conditions. Again, we are interested in the local and pointwise spectral asymptotics, i.e. those of $\int e(x,x,0)\psi(x)\,dx$ with $\psi \in \mathscr{C}_0^\infty(B(0,\frac12))$ and of $e(x,x,0)$ with $x \in B(0,\frac12)$.

Assume that in $B(0,1)$, everything is good: ∂X and coefficients of H are smooth, H is ξ-microhyperbolic on the energy levels $\lambda_{1,2}$ ($\lambda_1 < \lambda_2$) and also H is elliptic as a differential operator, i.e.

$$\|H(x,\xi)v\| \ge (\epsilon_0|\xi|^m - C_0)\|v\| \qquad \forall v \quad \forall x \in B(0,1)\ \forall \xi. \tag{2.64}$$

Then,

$$e(x,x,\lambda_1,\lambda_2) = \kappa_0(x,\lambda_1,\lambda_2)h^{-d} + O(h^{1-d}\gamma(x)^{-1}) \tag{2.65}$$

for $x \in B(0,\frac12)$ and $\gamma(x) \ge h$,

$$\kappa_0(x,\lambda_1,\lambda_2) = (2\pi)^{-d}\int \bigl(\theta(\lambda_2 - H^0(x,\xi)) - \theta(\lambda_1 - H^0(x,\xi))\bigr)\,d\xi \tag{2.66}$$

and $\gamma(x) = \frac12 \mathsf{dist}(x,\partial X)$.

Indeed, the scaling $x \mapsto (x-y)/\gamma$ and $h \mapsto \hbar/\gamma$ brings us into the framework of Theorem 2.6(ii) because ξ-microhyperbolicity (in contrast to

the (x,ξ)-microhyperbolicity) survives such rescaling. Then,

$$\int_{\{x:\,\gamma(x)\ge h\}} \bigl(e(x,x,\lambda_1,\lambda_2) - h^{-d}\kappa_0(x,\lambda_1,\lambda_2)\bigr)\psi(x)\,dx = O(h^{1-d}\log h), \tag{2.67}$$

since $\int_{\{x:\,\gamma(x)\ge h\}} \gamma(x)^{-1}\,dx \asymp |\log h|$.

One can easily show that if the boundary value problem for H is elliptic then

$$e(x,x,\lambda_1,\lambda_2) = O(h^{-d}) \tag{2.68}$$

and therefore,

$$\int \bigl(e(x,x,\lambda_1,\lambda_2) - h^{-d}\kappa_0(x,\lambda_1,\lambda_2)\bigr)\psi(x)\,dx = O(h^{1-d}\log h). \tag{2.69}$$

To improve this remainder estimate, one needs to improve (2.67) rather than (2.65) but to get sharper asymptotics, we need to improve both. We will implement the same scheme as inside the domain.

Propagation of Singularities

Toy-Model: Schrödinger Operator Let us consider the Schrödinger operator

$$H := h^2\Delta + V(x) \tag{2.70}$$

with the boundary condition

$$\bigl(\alpha(x)h\partial_\nu + \beta(x)\bigr)v|_{\partial X} = 0, \tag{2.71}$$

where

$$\Delta = \sum_{j,k} D_j g^{jk} D_k, \qquad \partial_\nu = \sum_j g^{j1}\partial_j \tag{2.72}$$

is derivative in the direction of the inner normal ν (we assume that $X = \{x : x_1 > 0\}$ locally[22)]), α and β are real-valued and do not vanish simultaneously. Without any loss of the generality, we can assume that locally

$$g^{j1} = \delta_{j1}. \tag{2.73}$$

[22)] I.e. in intersection with $B(0,1)$.

First of all, near the boundary, we can study the propagation of singularities using the same scheme as in Subsection 2 as long as $\phi_j(x,\xi) = \phi_j(x,\xi')$ do not depend on the component of ξ which is "normal to the boundary". The intuitive way to explain why one needs this is that at reflections, ξ_1 changes by a jump.

For the Schrödinger operator, it is sufficient for our needs: near *glancing points* (x,ξ') (which are points such that $x_1 = 0$ and the set $\{\xi_1 : H^0(x',\xi',\xi_1) = \tau\}$ consists of exactly one point), we can apply this method. On the other hand, near other points, we can construct the solution by traditional methods of oscillatory integrals.

It is convenient to decompose $u(x,y,t)$ into the sum

$$u = u^0(x,y,t) + u^1(x,y,t), \tag{2.74}$$

where $u^0(x,y,t)$ is a *free space solution* (without boundary) which we studied in Subsection 2 and $u^1 := u - u^0$ is a *reflected wave*.

Observe that even for the Schrödinger operator, we cannot claim that the singularity of $u(x,x,t)$ at $t = 0$ is isolated. The reason are *short loops* made by trajectories which reflect from the boundary in the normal direction and follow the same path in the opposite direction. However, these short loops affect neither $u(x,x,t)$ at the points of the boundary nor $u(x,x,t)$ integrated in any direction transversal to the boundary (and thus do not affect $\sigma_\psi(t)$ defined below).

Furthermore, they do not affect $(Q_{1x}u\,{}^tQ_{2y})(x,x,t)$ as long as at least one of operators $Q_j = Q_j(x,hD',hD_t)$ cuts them of. Then we get the estimate (2.9). Consider $Q_1 = Q_2 = 1$. Then, if $V(x) - \lambda > 0$, we get the same estimate at the point $x \in \partial X$. On the other hand, if either $V(x) - \lambda < 0$ or $V(x) = \lambda,\ \nabla_{\partial X} V(x) \neq 0$ (where $\nabla_{\partial X}$ means "along ∂X") at each point of $\mathsf{supp}(\psi)$, we get the same estimate for $\sigma_\psi(x) = \int u(x,x,t)\,dx$. As usual, λ is an energy level.

Moreover, $\sigma^1_\psi(t) = \int u^1(x,x,t)\psi(x)\,dx$ satisfies

$$|F_{t\to h^{-1}\tau}\chi_T(t)\sigma^1_\psi(t)| \le C_s h^{1-d}(h/|t|)^s. \tag{2.75}$$

In contrast to the Dirichlet ($\alpha = 0$, $\beta = 1$) or Neumann ($\alpha = 1$, $\beta = 0$) conditions, under the more general boundary condition (2.71), the classically forbidden level λ (i.e. with $\lambda < \inf_{B(0,1)} V$) may be not forbidden after all. Namely, in this zone, the operator $hD_t - H$ is elliptic and we can construct

the *Dirichlet-to-Neumann operator* $L : v|_{\partial X} \to h\partial_1 v|_{\partial X}$ as $(hD_t - H)v \equiv 0$. This is an h-pseudo-differential operator on ∂X with principal symbol

$$L^0(x', \xi', \tau) = -\big(V + \sum_{j,k\ge 2} g^{jk}\xi_j\xi_k - \tau\big)^{\frac{1}{2}}. \tag{2.76}$$

Then the boundary condition (2.71) becomes

$$Mw := (\alpha L + \beta)w \equiv 0, \qquad w = v|_{\partial X}. \tag{2.77}$$

The energy level $\lambda < V(x)$ is indeed forbidden if the operator M is elliptic as $\tau = \lambda$, i.e. if $M^0(x', \xi', \lambda) = \alpha L^0(x', \xi', \lambda) + \beta \ne 0$ for all ξ'; it happens as either $\alpha^{-1}\beta < 0$ or $W := V - \alpha^{-2}\beta^2 > \lambda$. Otherwise, to recover (2.75)[23], we assume that M is either ξ'-microhyperbolic or (x', ξ')-microhyperbolic ($W > \lambda$ and $W = \lambda \implies \nabla_{\partial X} W \ne 0$ respectively).

General Operators For more general operators and boundary value problems, we use similar arguments albeit not relying upon the representation of $u(x, y, t)$ via oscillatory integrals. It follows from (2.74) that

$$(hD_t - H)u^{1\pm} = 0, \tag{2.78}$$

$$Bu^{1\pm}|_{x_1=0} = -Bu^{0\pm}|_{x_1=0}, \tag{2.79}$$

where as before, $u^{k\pm} = u^k\theta(\pm t)$, $k = 0, 1$. Assuming that H satisfies (2.64), we reduce (2.78)–(2.79) to the problem

$$\mathcal{A}U^{1\pm} := (\mathcal{A}_0 hD_1 + \mathcal{A}_1)U^{1\pm} \equiv 0, \tag{2.80}$$

$$\mathcal{B}U^{1\pm}|_{x_1=0} = -\mathcal{B}U^{0\pm} \tag{2.81}$$

with $\mathcal{A}_k = \mathcal{A}_k(x, hD', hD_t)$, $\mathcal{B} = \mathcal{B}(x, hD', hD_t)$ and $U = S_x u\, {}^tS_y$ with $S = S(x, hD_x, hD_t)$ etc[24].

In a neighbourhood of any point $(\bar{x}', \bar{\xi}', \lambda)$, the operator $\mathcal{A}$ could be reduced to the block-diagonal form with blocks $\mathcal{A}_{kj}$ ($k = 0, 1$, $j = 1, \dots, N$) such that

(a) For each $j = 1, \dots, N-1$, the equation $\det(\mathcal{A}^0_{0j}\eta + \mathcal{A}^0_{1j}) = 0$ has a single real root η_j (at the point $(\bar{x}', \bar{\xi}', \lambda)$ only), η_j are distinct, and

[23] With d replaced by $d - 1$.

[24] If H is a $\mathsf{D} \times \mathsf{D}$ matrix operator of order m then $\mathcal{A}$ and S are $m\mathsf{D} \times m\mathsf{D}$ and $m\mathsf{D} \times \mathsf{D}$ matrix operators, B and $\mathcal{B}$ are $\frac{1}{2}m\mathsf{D} \times \mathsf{D}$ and $\frac{1}{2}m\mathsf{D} \times m\mathsf{D}$ matrix operators respectively.

(b) $\mathcal{A}_{kN} = \begin{pmatrix} 0 & \mathcal{A}'_{kN} \\ \mathcal{A}''_{kN} & 0 \end{pmatrix}$ with $\det(\mathcal{A}'^0_{0N}\eta + \mathcal{A}'^0_{1N}) = 0$ and $\det(\mathcal{A}''^0_{0N}\eta + \mathcal{A}''^0_{1N}) = 0$ has only roots with $\operatorname{Im}\eta < 0$ and with $\operatorname{Im}\eta > 0$ respectively.

We can prove a statement similar to Theorem 2.2, but instead of functions $\phi_*(x,\xi)$, we now have arrays of functions $\phi_{*j}(x,t,\xi',\tau)$ $(j = 1,\dots,N-1)$ coinciding with $\phi_{*N}(x',t,\xi',\tau)$ as $x_1 = 0$. Respectively, instead of microhyperbolicity of the operator in the direction $\ell \in T(T^*(X\times\mathbb{R}))$, we now have the *microhyperbolicty of the boundary value problem in the multidirection* $(\ell',\nu_1,\dots,\nu_{N-1}) \in T(T^*(\partial X\times\mathbb{R}))\times\mathbb{R}^{N-1}$; see Definition 3.1.4 of [Ivr4]. It includes the microhyperbolicity of $\mathcal{A}_j$ in the direction $(\ell',0,\nu_j)$ for $j = 1,\dots,N-1$ and a condition invoking $\mathcal{A}_N$ and $\mathcal{B}$ and generalizing the microhyperbolicity of operator M for the Schrödinger operator. Respectively, instead of the microhyperbolicity of an operator in the direction $\nabla^{\#}\phi_*$, we want the microhyperbolicity in the multidirection $(\nabla'^{\#}\phi_*, \partial_1\phi_{*1},\dots,\partial_1\phi_{*(N-1)})$.

As a corollary, under the microhyperbolicity assumption on the energy level λ, we prove estimates (2.9) for $\sigma^0_\psi(t)$, $\sigma_\psi(t)$ and (2.75) for $\sigma^1_\psi(t)$ as τ is close to λ. Furthermore, if the operator H is elliptic on this energy level then $\sigma^0_\psi(t)$ is negligible and (2.75) holds for $\sigma^1_\psi(t)$ and $\sigma_\psi(t)$.

For details, proofs and generalizations, see Chapter 3 of [Ivr4].

Successive Approximations Method

After the (2.9) and (2.75)-type estimates are established, we can apply the successive approximations method like in Subsection 2 but with some modifications: to construct $Bu^{0\pm}|_{x_1=0}$ and from it to construct $u^{1\pm}$, we freeze coefficients in $(y',0)$ rather than in y. As a result, we can calculate all terms in the asymptotics and under microhyperbolicity in the multidirection condition, we arrive to the formulae (2.24) for $e^0(.,.,\tau)$, $e^1(.,.,\tau)$ and $e(.,.,\tau)$ [25] with $m \ge 1$ for $e^1(.,.,\tau)$.

The formulae for $\varkappa^1_m(\tau)$ (and thus for $\varkappa_m(\tau) = \varkappa^0_m(\tau)+\varkappa^1_m(\tau)$ are however rather complicated and we do not write them here. For the Schrödinger operator with $V = 0$ and boundary condition (2.71), the calculation of $\varkappa^1_1(\tau)$ is done in [HA].

Similar formulae also hold if we take $x_1 = y_1 = 0$ and integrate over ∂X (but in this case $m \ge 0$ even for $e^1(.,.,\tau)$).

Furthermore, if $\ell'_x = 0$ and $\nu_1 = \dots = \nu_{N-1}$ in the condition of microhyperbolicity, we are able to get formulae for $e^0(x,x,\tau)$, $e^1(x,x,\tau)$ and

[25] With the obvious definitions of $e^0(.,.,\tau)$ and $e^1(.,.,\tau)$.

$e(x,x,\tau)$ without setting $x_1=0$ and without integrating but $e^1(x,x,\tau)$ is a boundary-layer type term.

For details and proofs, see Section 7.2 of [Ivr4].

Recovering Spectral Asymptotics

Repeating the arguments of Subsection 2, we can recover the local spectral asymptotics:

Theorem 2.17. *(i) Let an operator H be microhyperbolic on $\mathsf{supp}(\psi)$ on the energy levels λ_1 and λ_2 ($\lambda_1<\lambda_2$) and the boundary value (H,B) problem be microhyperbolic on $\mathsf{supp}(\psi)\cap\partial X$ on these energy levels. Then,*

$$\int_X e(y,y,\lambda_1,\lambda_2)\psi(y)\,dy = h^{-d}\int_X \kappa_0(y,\lambda_1,\lambda_2)\psi(y)\,dy + O(h^{-d+1}). \tag{2.82}$$

(ii) Suppose that an operator H is elliptic on $\mathsf{supp}(\psi)$ on the energy levels λ_1 and λ_2 ($\lambda_1<\lambda_2$)[26] *and the boundary value (H,B) problem is microhyperbolic on $\mathsf{supp}(\psi)\cap\partial X$ on these energy levels. Then,*

$$\int_X e(y,y,\lambda_1,\lambda_2)\psi(y)\,dy = h^{1-d}\int_X \kappa_1(y,\lambda_1,\lambda_2)\psi(y)\,dy + O(h^{-d+2}). \tag{2.83}$$

On the other hand, for the Schrödinger operator, we can calculate the contributions of near normal trajectories explicitly and then we arrive to:

Theorem 2.18. *Let (H,B) be the Schrödinger operator* (2.70)–(2.71) *and let $|V|\neq\lambda$ on $\mathsf{supp}(\psi)$. Then,*

$$e(y,y,\lambda) = h^{-d}\bigl(\kappa_0(x,\lambda)+\mathcal{Q}(x',\lambda;h^{-1}x_1)\bigr)+O(h^{-d+1}) \tag{2.84}$$

where $\mathcal{Q}$ depends on the "normal variables" (x',λ) and a "fast variable" $s=h^{-1}x_1$ and decays as $O(s^{-d+1/2})$ as $s\to+\infty$. Here, $x_1=\mathsf{dist}(x,\partial X)$.

For details, exact statement and proofs, see Section 8.1 of [Ivr4].

[26] Then, it is elliptic on all energy levels $\tau\in[\lambda_1,\lambda_2]$.

Second Term and Dynamics

As in Subsection 2, we can improve our asymptotics under certain conditions to the dynamics of propagation of singularities. However, in the case that the manifold has a non-empty boundary, propagation becomes really complicated. For Schrödinger operators, we can prove that singularities propagate along Hamiltonian billiards unless they "behave badly" that is become tangent to ∂X at some point or make an infinite number of reflections in finite time. However, the measure of *dead-end points*[27] is 0.

Thus, applying the arguments of Section 1 we arrive to

Theorem 2.19. *Let $d \geq 2$, $|V-\lambda| + |\nabla V| \neq 0$ on $\mathsf{supp}(\psi)$ and $|V-\lambda| + |\nabla_{\partial X} V| \neq 0$ on $\mathsf{supp}(\psi) \cap \partial X$. Further, assume that the measure of periodic Hamiltonian billiards passing through points of $\{H^0(x,\xi)=0\} \cap \mathsf{supp}(\psi)$ is 0*[28]*. Then,*

$$\int e(y,y,\lambda)\psi(y)\,dy = h^{-d}\int \kappa_0(y,\lambda)\psi(y)\,dy + o(h^{-d+1}). \tag{2.85}$$

Remark 2.20. If we are interested in the propagation of singularities without applications to spectral asymptotics, the answer is "singularities propagate along the generalized Hamiltonian billiards" (see Definition 3.2.2 in [Ivr4]).

One can easily show:

Theorem 2.21. *Let $d \geq 3$. Assume that we are in the framework of Theorem 2.17(ii). Further assume that the set of periodic trajectories of the Schrödinger operator on ∂X with potential W introduced after* (2.77) *has measure 0. Then,*

$$\int e(y,y,\lambda)\psi(y)\,dy = h^{1-d}\varkappa_{1,\psi}(\lambda) + h^{2-d}\varkappa_{2,\psi}(\lambda) + o(h^{-d+2}). \tag{2.86}$$

Remark 2.22. Analysis becomes much more complicated for more general operators even if we assume that the inner propagation is simple. For example, if the operator in question is essentially a collection of m Schrödinger operators intertwined through boundary conditions then every incidence ray after reflection generates up to m reflected rays and we have *branching*

[27] I.e. points $z \in \Sigma(\lambda)$ the billiard passing through which behaves badly.

[28] There is a natural measure $dxd\xi : dH^0$.

Hamiltonian billiards. Here, a *dead-end point* is a point $z \in \Sigma(\lambda)$ such that *some of the branches* behave badly and a *periodic point* is a point $z \in \Sigma(\lambda)$ such that *some of the branches* return to it.

Assume that the sets of all periodic points and all dead-end points on the energy level $\Sigma(\lambda)$ have measure 0 (as shown in [SV2], the set of all dead-end points may have positive measure). Then, the two-term asymptotics could be recovered. However, the investigation of branching Hamiltonian billiards is a rather daunting task.

Rescaling Technique

The rescaling technique could be applied near ∂X as well. Assume that $\lambda = 0$. Then to get rid of the non-degeneracy assumption $V(x) \le -\epsilon$, we use scaling functions $\gamma(x)$ and $\rho(x)$ as in Subsection 2. It may happen that $B(x, \gamma(x)) \subset X$ or it may happen that $B(x, \gamma(x))$ intersects ∂X. In the former case, we are obviously done and in the latter case we are done as well because in the condition (2.71) we scale $\alpha \mapsto \alpha\rho\nu$, $\beta \mapsto \beta\nu$ where $\nu > 0$ is a parameter of our choice. Thus, in the pointwise asymptotics, we can get rid of this assumption for $d \ge 3$, and in the local asymptotics for $d \ge 2$ assuming that $|V| + |\nabla V| \asymp 1$ because the total measure of the balls of radii $\le \gamma$ which intersect ∂X is $O(\gamma)$. For details, exact statements and proofs, see Section 8.2 of [Ivr4].

Operators with Periodic Billiards

Simple Billiards Consider an operator on a manifold with boundary. Assume first that all the billiard trajectories (on energy levels close to λ) are *simple* (i.e. without branching) and periodic with a period bounded from above; then the period depends only on the energy level. Example: the Laplace-Beltrami operator on the semisphere. Under some non-degeneracy assumptions similar to (2.59), we can derive asymptotics similar to (2.63) but with two major differences:

(i) We assume that $\varepsilon \asymp h$ and recover remainder estimate only $O(h^{1-d+\delta})$; it is still good enough to have the second term of the non-standard type.

(ii) We can consider $b(x, \xi)$ (which is invariant with respect to the Hamiltonian billard flow) as a phase shift for one period. Now, however, it could

be a result not only of the quantum drift as in Subsection 2, but also of an instant change of phase at the moment of the reflection.

For exact statements, details and proofs, see Subsection 8.3.2 of [Ivr4].

Branching Billiards with "Scattering" We now assume that the billiard branches but only one ("main") branch is typically periodic. For example, consider two Laplace-Beltrami operators intertwined through boundary conditions: one of them is an operator on the semisphere X_1 and another on the disk X_2 with ∂X_1 and ∂X_2 glued together. Then all billiards on X_1 are periodic but there exist nowhere dense sets $\Lambda_j(\lambda)$ of measure 0, such that the billiards passing through $\Sigma_j(\lambda) \setminus \Lambda_j(\lambda)$ and containing at least one segment in X_2 are not periodic. Assume also that the boundary conditions guarantee that at reflection, the "observable" part of energy escapes into X_2. Then to recover the sharp remainder estimates, we do not need a phase shift because for time $T \gg 1$, we have

$$|F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\, d_\tau e(y,y,\tau)\, dy| \le C_0 h^{1-d} \sum_{|n|\le T} q^n + o_T(h^{1-d}), \tag{2.87}$$

where $q \le 1$ estimates from above the "portion of energy" which goes back to X_1 at each reflection; if $q < 1$, as we have assumed the right-hand expression does not exceed $C_1 h^{1-d} + o_T(h^{1-d})$ and we recover asymptotics similar to (2.63) with the remainder estimate $o(h^{1-d})$.

For exact statements, details and proofs, see Subsection 8.3.3 of [Ivr4].

Two Periodic Billiards We can also consider the case when the billiards flows in X_1 and X_2 are both periodic but "magic" happens at reflections. For exact statements, details and proofs, see Subsection 8.3.4 of [Ivr4].

3 Global Asymptotics

In this section, we consider global spectral asymptotics. Here we are mainly interested in the asymptotics with respect to the spectral parameter λ. We consider mainly examples.

Weyl Asymptotics

Regular Theory

We start from examples in which we apply only the results of the previous Chapter 2 which may be combined with *Birman-Schwinger principle* and the rescaling technique.

Simple Results

Example 3.1. Consider a self-adjoint operator A with domain $\mathfrak{D}(A) = \{u : Bu|_{\partial X} = 0\}$. We assume that A is elliptic and the boundary value problem (A, B) is elliptic as well.

(i) We are interested in $\mathsf{N}(0, \lambda)$, the number of eigenvalues of A in $[0, \lambda)$. Instead we consider $\mathsf{N}(\lambda/2, \lambda)$, which is obviously equal to $\mathsf{N}_h(\frac{1}{2}, 1)$, the number of eigenvalues of $A_h = \lambda^{-1}A$ that lie in $[\frac{1}{2}, 1)$, with $h = \lambda^{-1/m}$ where m is the order of A. In fact, more is true: the principal symbols of semiclassical operators A_h and B_h coincide with the senior symbols of A and B. Then the microhyperbolicity conditions are satisfied and the semiclassical asymptotics with the remainder estimate $O(h^{1-d})$ hold which could be improved to two-term asymptotics under our standard non-periodicity condition. As a result, we obtain

$$\mathsf{N}(0, \lambda) = \varkappa_0 \lambda^{\frac{d}{m}} + O(\lambda^{\frac{d-1}{m}}) \tag{3.1}$$

and

$$\mathsf{N}(0, \lambda) = \varkappa_0 \lambda^{\frac{d}{m}} + \varkappa_1 \lambda^{\frac{d-1}{m}} + o(\lambda^{\frac{d-1}{m}}), \tag{3.2}$$

as $\lambda \to +\infty$ in the general case and under the standard non-periodicity condition respectively. Here,

$$\varkappa_0 = (2\pi)^{-d} \iint \boldsymbol{n}(x, \xi)\, dx d\xi \tag{3.3}$$

where $\boldsymbol{n}(x, \xi)$ is the number of eigenvalues of $A^0(x, \xi)$ in $(0, 1)$ and $m = m_A$ is the order of A.

(ii) Suppose that A_B is positive definite (then $m_A \geq 2$) and V is an operator of the order $m_B < m_A$, symmetric under the same boundary conditions. We are interested in $\mathsf{N}(0, \lambda)$, the number of eigenvalues of VA_B^{-1} in (λ^{-1}, ∞).

Using the Birman-Schwinger principle, we can again reduce the problem to the semiclassical one with $H = h^{m_A}A - h^{m_V}V$, $h = \lambda^{-1/m}$, $m = m_A - m_V$. The microhyperbolicity condition is fulfilled automatically unless $\xi = 0$ and $V^0(x,\xi)$ is degenerate. Still under certain appropriate assumptions about V^0, we can ensure microhyperbolicity (for $m_B = 0, 1$ only). Then (3.1) and (3.2) (the latter under standard non-periodicity condition) hold with $\boldsymbol{n}(x,\xi)$ the number of eigenvalues of $V^0(x,\xi)(A^0(x,\xi)^{-1})$ in $(1,\infty)$.

(iii) Alternatively, we can consider the case when V is positively defined (and A_B may be not).

(iv) For scalar operators, one can replace microhyperbolicity by a weaker non-degeneracy assumption. Furthermore, without any non-degeneracy assumption we arrive to one-term asymptotics with the remainder estimate $O(\lambda^{(d-1+\delta)/m})$.

(v) Also one can consider operators whose all Hamiltonian trajectories are periodic; in this case the oscillatory correction term appears.

(vi) Suppose the operator A_B has negative definite principal symbol but A_B is not semi-bounded from above and V is positive definite. Then instead of (3.1) or (3.2), we arrive to

$$\mathsf{N}(0,\lambda) = \varkappa_1\lambda^{\frac{d-1}{m}} + O(\lambda^{\frac{d-2}{m}}) \tag{3.4}$$

and

$$\mathsf{N}(0,\lambda) = \varkappa_1\lambda^{\frac{d-1}{m}} + \varkappa_2\lambda^{\frac{d-2}{m}} + o(\lambda^{\frac{d-2}{m}}), \tag{3.5}$$

(the latter under an appropriate non-periodicity assumption).

Fractional Laplacians The fractional Laplacian $\Lambda_{m,X}$ appears in the theory of stochastic processes. For $m > 0$, it is defined first on $\mathbb{R}^d$ as $\Delta^{m/2}$, and in a domain $X \subset \mathbb{R}^d$, it is defined as $\Lambda_{m,X}u = R_X\Delta^{m/2}(\theta_X u)$ where R_X is the restriction to X and θ_X is the characteristic function of X. It differs from the $m/2$-th power of the Dirichlet Laplacian in X and for $m \notin 2\mathbb{Z}$, it does not belong to the Boutet de Monvel's algebra. In particular, even if X is a bounded domain with $\partial X \in \mathscr{C}^\infty$ and $u \in \mathscr{C}^\infty(\bar{X})$, $\Lambda_{m,X}u$ does not necessarily belong to $\mathscr{C}^\infty(\bar{X})$ (smoothness is violated in the direction normal to ∂X).

Then the standard Weyl asymptotics (3.1) and (3.2) hold (the latter under standard non-periodicity condition) with the standard coefficient $\varkappa_0 = (2\pi)^{-d}\omega_{d-1}\,\mathsf{vol}_d(X)$ and with

$$\varkappa_{1,m} = (2\pi)^{1-d}\omega_{d-1}\sigma_m\,\mathsf{vol}_{d-1}(\partial X), \tag{3.6}$$

$$\sigma_m = \frac{d-1}{m}\iint_1^\infty \tau^{-(d-1)/m-1}\Big(e_m(x_1,x_1,\tau) - \pi^{-1}(\tau-1)^{1/m}\Big)\,dx_1\,d\tau \tag{3.7}$$

where $e_m(x_1,y_1,\tau)$ is the Schwartz kernel of the spectral projector of operator

$$a_m = ((D_x^2+1)^{m/2})_{\mathsf{D}} \tag{3.8}$$

on $\mathbb{R}^+$. To prove this, we need to redo some analysis of Chapter 2. While tangent rays are treated exactly as for the ordinary Laplacian, normal rays require some extra work. However, we can show that the singularities coming along transversal rays do not stall at the boundary but reflect according the standard law. For exact statements, details and proofs, see [Ivr5].

Semiclassical Dirichlet-to-Neumann Operator Consider the Laplacian Δ in X. Assuming that λ is not an eigenvalue of Δ_{D}, we can introduce the Dirichlet-to-Neumann operator $L_\lambda : v \mapsto \lambda^{-\frac{1}{2}}\partial_\nu u|_{\partial X}$ where u is defined as $(\Delta-\lambda)u = 0$, $u|_{\partial X} = v$ and ν is the inner unit normal. Here, L_λ is a self-adjoint operator and we are interested in $\mathsf{N}_\lambda(a_1,a_2)$, the number of its eigenvalues in the interval $[a_1,a_2)$. Due to the Birman-Schwinger principle, it is equal to $N_h^-(a_1) - N_h^-(a_2)$ where $N_h^-(a)$ is the number of the negative eigenvalues of $h^2\Delta - 1$ under the boundary condition $(h\partial_\nu - a)u|_{\partial X} = 0$ and then we arrive to

$$\mathsf{N}_\lambda(a_1,a_2) = O(\lambda^{\frac{d-1}{m}}) \tag{3.9}$$

and

$$\mathsf{N}_\lambda(a_1,a_2) = \varkappa_1(a_1,a_2)\lambda^{\frac{d-1}{2}} + o(\lambda^{\frac{d-1}{2}}) \tag{3.10}$$

(the latter under a standard non-periodicity condition). For exact statements, details and proofs, see [HA].

Rescaling Technique We are interested in the asymptotics of either

$$\mathsf{N}^-(\lambda) = \int e(x,x,\lambda)\,dx \tag{3.11}$$

or

$$\mathsf{N}(\lambda_1,\lambda_2) = \int e(x,x,\lambda_1,\lambda_2)\,dx \qquad \text{with} \quad \lambda_1 < \lambda_2 : \tag{3.12}$$

with respect to either the spectral parameter(s), or semiclassical parameter(s), or some other parameter(s). We assume that there exist *scaling functions* $\gamma(x)$ and $\rho(x)$ satisfying

$$|\nabla\gamma| \le \frac{1}{2}, \qquad |x-y| \le \gamma(y) \implies c^{-1} \le \rho(x)/\rho(y) \le c, \tag{3.1.13)$_{1,2}}$$

such that after rescaling $x \mapsto x/\gamma(y)$ and $\xi \mapsto \xi/\rho(y)$ in $B(y,\gamma(y))$, we find ourselves in the framework of the previous chapter with an *effective semiclassical parameter* $\hbar \le 1$ [29].

To avoid non-degeneracy assumptions, we consider only the Schrödinger operator (2.70) in $\mathbb{R}^d$, assuming that $g^{jk} = g^{kj}$,

$$|\nabla^\alpha g^{jk}| \le c_\alpha \gamma^{-|\alpha|}, \qquad |\nabla^\alpha V| \le c_\alpha \rho^2 \gamma^{-|\alpha|} \tag{3.14}$$

and

$$\sum_{j,k} g^{jk}\xi_j\xi_k \ge \epsilon_0 |\xi|^2 \qquad \forall x,\xi. \tag{3.15}$$

In the examples below, $h \asymp 1$.

Example 3.2. (i) Suppose the conditions (3.14), (3.15) are fulfilled with $\gamma(x) = \frac{1}{2}(|x|+1)$ and $\rho(x) = |x|^m$, $m > 0$. Further, assume that the *coercivity condition*

$$V(x) \ge \epsilon_0 \rho^2 \tag{3.16}$$

holds for $|x| \ge c_0$. Then if $|x| \le C\lambda^{1/2m}$, for the operator $H - \lambda$, we can use $\rho_\lambda(x) = \lambda^{1/2}$ and then the contribution of the ball $B(x,\gamma(x))$ to the remainder does not exceed $C\lambda^{(d-1)/2}\gamma^{d-1}(x)$; summation over these balls results in $O\big(\lambda^{(d-1)(m+1)/2m}\big)$.

[29] In purely semiclassical settings, $\hbar = h/\rho\gamma$ and we assume $\rho\gamma \ge h$.

On the other hand, if $|x| \le C\lambda^{\frac{1}{2m}}$, for the operator $H-\lambda$ we can use $\rho(x)=\gamma^m(x)$ but there the ellipticity condition is fulfilled and then the contribution of the ball $B(x,\gamma(x))$ to the remainder does not exceed $C\gamma^{-s}$; summation over these balls results in $o\big(\lambda^{(d-1)(m+1)/2m}\big)$. Then we arrive to

$$\mathsf{N}(\lambda) = c_0 h^{-d}\int (\lambda - V(x))_+^{\frac{d}{2}} + O\big(\lambda^{(d-1)(m+1)/2m}\big) \tag{3.17}$$

as $\lambda \to +\infty$. Obviously the main part of the asymptotics is $\asymp \lambda^{d(m+1)/2m}$.

(ii) Suppose instead $0 > m > -1$. We are interested in its eigenvalues tending to the bottom of the continuous spectrum (which is 0) from below. We no longer require the assumption (3.16).

We use the same $\gamma(x)$ but now $\rho_\lambda(x)=\gamma(x)^m$ for $|x| \le C|\lambda|^{1/2m}$. Then the contribution of the ball $B(x,\gamma(x))$ to the remainder does not exceed $C\gamma(x)^{(d-1)(m+1)}$; summation over these balls results in $O\big(|\lambda|^{(d-1)(m+1)/2m}\big)$.

On the other hand, if $|x| \ge C|\lambda|^{1/2m}$, for the operator $H-\lambda$ we can use $\rho_\lambda(x)=|\lambda|^{\frac{1}{2}}$, but there the ellipticity condition is fulfilled and then the contribution of the ball $B(x,\gamma(x))$ to the remainder does not exceed $C|\lambda|^{-s}\gamma^{d-s}$; summation over these balls results in $o\big(|\lambda|^{(d-1)(m+1)/2m}\big)$. Then we arrive to asymptotics (3.17) again as $\lambda \to -0$.

Obviously the main part of the asymptotics is $O\big(|\lambda|^{d(m+1)/2m}\big)$ and under the assumption $V(x) \le -\epsilon\rho(x)^2$, in some cone it is $\asymp |\lambda|^{d(m+1)/2m}$.

(iii) In both cases (i) and (ii), the main contribution to the remainder is delivered by the zone $\{\varepsilon < |x||\lambda|^{-1/2m} < \varepsilon^{-1}\}$ and assuming that $g^{jk}(x)$ and $V(x)$ stabilize as $|x| \to +\infty$ to $g^{jk0}(x)$ and $V^0(x)$, positively homogeneous functions of degrees 0 and $2m$ respectively, and that the set of periodic trajectories of the Hamiltonian $\sum_{j,k} g^{jk}(x)\xi_j\xi_k + V^0(x)$ on energy level 1 in (i) or -1 in (ii) has measure 0, we can improve the remainder estimates to $o\big(|\lambda|^{(d-1)(m+1)/2m}\big)$.

Example 3.3. Consider the Dirac operator

$$H = \sum_{1\le j\le d} \sigma_j D_j + M\sigma_0 + V(x), \tag{3.18}$$

where σ_j ($j=0,\dots,d$) are Pauli matrices in the corresponding dimension and $M>0$. Let $V(x)\to 0$ as $|x|\to\infty$. Then the essential spectrum of H

is $(-\infty,-M]\cup[M,\infty)$ and for V as in Example 3.2(ii), we can get similar results for the asymptotics of eigenvalues tending to $M-0$ or $-M+0$: so instead of $\mathsf{N}(\lambda)$, we consider $\mathsf{N}(0,M-\eta)$ or $\mathsf{N}(M+\eta,0)$ with $\eta\to+0$.

Example 3.4. Consider the Schrödinger operator, either in a bounded domain $X\ni 0$ or in $\mathbb{R}^d$ like in Example 3.2(i) and assume that $g^{jk}(x)$ and $V(x)$ have a singularity at 0 satisfying there (3.14)–(3.16) with $\gamma(x)=|x|$ and $\rho(x)=|x|^m$ with $m<-1$.

Consider the asymptotics of eigenvalues tending to $+\infty$. As in Example 3.2(i), we take $\gamma(x)=\frac{1}{2}|x|$ and $\rho_\lambda(x)=\lambda^{1/2}$ for $|x|\ge\epsilon_0\lambda^{1/2m}$ (then the contribution of $B(x,\gamma(x))$ to the remainder does not exceed $\lambda^{(d-1)/2}|x|^{d-1}$) and $\rho_\lambda(x)=|x|^m$ as $|x|\le\epsilon_0\lambda^{1/2m}$ (then due to the ellipticity the contribution of $B(x,\gamma(x))$ to the remainder does not exceed $\rho^{-s}\gamma^{-s-d}$). We conclude that the contribution of $B(0,\epsilon)$ to the remainder does not exceed $C\lambda^{(d-1)/2}\epsilon^{d-1}$ which means that this singularity does not prevent remainder estimate as good as $o(\lambda^{(d-1)/2})$. However, this singularity affects the principal part which should be defined as in (3.17).

Example 3.5. (i) When analyzing the asymptotics of the large eigenvalues, we can consider a potential that is either rapidly increasing (with $\rho=\exp(|x|^m)$, $\gamma(x)=|x|^{1-m}$, $m>0$), or slowly increasing (with $\rho=(|\log x|)^m$, $\gamma(x)=|x|$, $m>0$) which affects both the magnitude of the main part and the remainder estimate.

(ii) When analyzing the asymptotics of the eigenvalues tending to the bottom of the essential spectrum, we can consider a potential that is either rapidly decreasing (with $\rho=|x|^{-1}(\log|x|)^m$ with $m>0$, $\gamma(x)=|x|$, $m>0$) or slowly decreasing (with $\rho=(|\log x|)^m$, $\gamma(x)=|x|$, $m<0$) which affects both the magnitude of the main part as well as the remainder estimate.

Remark 3.6. We can consider the same examples albeit assuming only that $h\in(0,1)$; then the remainder estimate acquires the factor h^{-d+1}.

Singularities

Let us consider other types of singularities when there is *a singular zone* where after rescaling $\hbar\le 1$ [29]. Still, it does not prevent us from using the approach described above to get an estimate from below for (3.11) or (3.12): we only need to decrease these expressions by inserting ψ ($0\le\psi\le 1$) that

is supported in *the regular zone* (aka *the semiclassical zone*) defined by $\hbar \le 2\hbar_0$ after rescaling and equal to 1 for $\hbar \le \hbar_0$ and applying the rescaling technique there.

Let us discuss an estimate from above. If there was no regular zone at all, we would have no estimate from below at all but there could be some estimate from above of variational nature. The best known is the LCR (Lieb-Cwikel-Rosenblum) estimate

$$\mathsf{N}^-(0) \le Ch^{-d} \int V_-^{\frac{d}{2}}\, dx \tag{3.19}$$

for the Schrödinger operator with Dirichlet boundary conditions as $d \ge 3$. For $d = 2$, the estimate is marginally worse (see [Roz1] for the most general statement for arbitrary orders of operators and dimensions and [Shar] for the most general results for the Schrödinger operator in dimension 2).

It occurs that we can split our domain into an overlapping *regular zone* $\{x :\ \rho(x)\gamma(x) \ge h\}$ and a *singular zone* $\{x :\ \rho(x)\gamma(x) \le 3h\}$, then evaluate the contribution of the regular zone using the rescaling technique and the contribution of the singular zone by the variational estimate as if on the inner boundary of this zone (i.e. a part of its boundary which is not contained in ∂X) the Dirichlet boundary condition was imposed, and we add these two estimates:

$$\begin{aligned} -Ch^{1-d} \int_{\{\rho\gamma\ge h, V\le\epsilon\rho^2\}} \rho^{d-1}\gamma^{-1}\sqrt{g}\, dx& \\ \le \mathsf{N}^-(0) - (2\pi)^{-d}\omega_d h^{-d} \int_{\{\rho\gamma\ge h\}} V_-^{d/2}\, dx& \\ \le Ch^{1-d} \int_{\{\rho\gamma\ge h, V\le\epsilon\rho^2\}} \rho^{d-1}\gamma^{-1}\, dx + Ch^{-d} \int_{\{\rho\gamma\le h, V\le\epsilon\rho^2\}} \rho^d\, dx.& \end{aligned} \tag{3.20}$$

See Theorems 9.1.7 and 9.1.13 of [Ivr4] for more general statements. Further, similar statements could be proven for operators which are not semi-bounded (see Theorems 10.1.7 and 10.1.8 of [Ivr4]).

In particular, we have:

Example 3.7. (i) Let

$$\int \rho^{d-1}\gamma^{-1}\, dx < \infty. \tag{3.21}$$

Then,

$$\mathsf{N}^-(0) = (2\pi)^{-d}\omega_d h^{-d}\int V_-^{d/2}\sqrt{g}\,dx + O(h^{1-d}). \tag{3.22}$$

(ii) If in addition the standard non-periodicity condition is satisfied then

$$\mathsf{N}^-(0) = (2\pi)^{-d}\omega_d h^{-d}\int V_-^{d/2}\sqrt{g}\,dx - \frac{1}{4}(2\pi)^{1-d}\omega_{d-1}h^{1-d}\int V_-^{(d-1)/2}\,dS + o(h^{1-d}), \tag{3.23}$$

where dS is a corresponding density on ∂X.

Example 3.8. Consider the Dirichlet Laplacian in a domain X assuming that there exists scaling function $\gamma(x)$ such that (3.14) holds and

(3.24) For each $y \in X$, the connected component of $B(y,\gamma(x))\cap X$ containing y coincides with $\{x \in B(0,1), x_1 \le f(x')\}$, where $x' = (x_2,\dots,x_d)$ and

$$|\nabla^\alpha f| \le C_\alpha \gamma^{1-|\alpha|} \qquad \forall \alpha, \tag{3.25}$$

where we rotate the coordinate system if necessary[30)].

(i) Then the principal part of asymptotics is

$$c_0 \lambda^{\frac{d}{2}} h^{-d}\int_{\{x:\gamma(x)\ge\lambda^{-\frac12}\}} dx \tag{3.26}$$

and the remainder does not exceed

$$C\lambda^{\frac{d-1}{2}}h^{1-d}\int_{\{x:\gamma(x)\ge\lambda^{-\frac12}\}}\gamma(x)^{-1}\,dx + C\lambda^{\frac{d}{2}}h^{-d}\int_{\{x:\gamma(x)\le\lambda^{-\frac12}\}} dx. \tag{3.27}$$

(ii) In particular, if

$$\int_X \gamma(x)^{-1}\,dx < \infty, \tag{3.28}$$

[30)] It is precisely the condition that we need to impose on the boundary for the rescaling technique to work.

then the standard asymptotics with the remainder estimate $O(\lambda^{(d-1)/2}h^{1-d})$ hold. Moreover, under the standard condition (1.3), we arrive to the two-term asymptotics (1.2).

These conditions are satisfied for domains with vertices, edges and conical points. In fact, we may allow other singularities including outer and inner spikes and cuts.

Furthermore, these conditions are satisfied for unbounded domains with cusps (exits to infinity) provided these cusps are thin enough (basically having finite volume and area of the boundary).

(iii) These results hold under the Neumann or mixed Dirichlet-Neumann boundary condition, but then we need to assume that the domain satisfies the cone condition; for the two-term asymptotics, we also need to assume that the border between the parts of ∂X with the Dirichlet and Neumann boundary conditions has $(d-1)$-dimensional measure 0.

Example 3.9. (i) Suppose that the potential is singular at $0 \in X$ like $|x|^{2m}$ with $m \in (-1, 0)$. Then this singularity does not affect the asymptotics of large eigenvalues.

(ii) Let us consider Example 3.2(i) albeit allow $V \ge 0$ to vanish along certain directions. Then we have *canyons* and $\{x : V(x) \le \lambda\}$ are cusps. If the canyons are narrow and steep enough then the same asymptotics (3.17) hold. Moreover, under the non-periodicity condition, the remainder estimate is "o".

(iii) Let us consider Example 3.2(ii) albeit allow $V \ge 0$ to be singular along certain directions. Then we have *gorges* and $\{X : V(x) \le \lambda\}$ are cusps. If the gorges are narrow and shallow enough then the same asymptotic (3.17) hold. Moreover, under the non-periodicity condition, the remainder estimate is "o".

Example 3.10. We can consider also multiparameter asymptotics, for example with respect to $h \to +0$ and λ. In addition to what we considered above, the following interesting possibility appears: $\lambda \searrow \lambda_* := \inf V(x)$ which is either finite or $-\infty$. Then if λ tends to λ_* slowly enough so that $\mathsf{N}_h^-(\lambda) \to +\infty$, we get interesting asymptotics.

In particular, as either $V(x) \asymp |x|^{2m}$ with $m > 0$ and $\lambda \to +0$ or $V(x) \asymp |x|^{2m}$ with $0 > m > -1$ and $\lambda \to -\infty$, then this condition is $h = o(|\lambda|^{(m+1)/2m})$.

Remark 3.11. We can also consider $\mathsf{Tr}((-H)^{\nu}\theta(-H))$ with $\nu > 0$. Then in the estimates above, $\rho^d \mapsto \rho^{d+2\nu}$ and $\rho^{d-1} \mapsto \rho^{d+\nu-1}\gamma^{-\nu-1}$.

For full details, proofs and generalizations, see Chapter 11 of [Ivr4] which covers also non-semibounded operators.

Non-Weyl Asymptotics

Partially Weyl Theory

Analyzing the examples of the previous section, one can observe that for some values of the exponents, the condition (3.21) (or it special case (3.28)) fails but the main term of the asymptotics is still finite and has the same rate of the growth as it had before, while for other values of the exponent, it is infinite. In the former case, we get Weyl asymptotics but with a worse remainder estimate, in the latter case, all we can get is an estimate rather than the asymptotics. Can one save the day?

In many cases, the answer is positive and we can derive either the Weyl asymptotics but with a non-Weyl correction term or completely non-Weyl asymptotics. The main but not the only tool is the spectral asymptotics for operators with operator-valued symbols. Namely, in some zone of the phase space, we separate the variables[31] $x = (x'; x'')$ and $(\xi'; \xi'')$ respectively and consider the variables (x', ξ') as *semiclassical variables* (or *Weyl variables*), similar to (x, ξ) in the previous scheme. So we get an operator $\mathsf{H}(x', D')$ with an operator-valued symbol which we can study in the same way as the operator H before.

One can say that we have a matrix operator but with a twist: first, instead of finite-dimensional matrices, we have unbounded self-adjoint operators in the auxilary infinite-dimensional Hilbert space $\mathbb{H}$ (usually $\mathscr{L}^2$ in the variables x''); second, we are interested in the asymptotics

$$\int \mathsf{tr}_{\mathbb{H}}\bigl(\hat{e}(x', x', \lambda)\bigr)\, dx', \tag{3.29}$$

[31] After some transformation, the transformations and separations in the different zones are not necessarily the same.

rather than in the asymptotics without trace where $\hat{e}(x', y'; \lambda)$ is an operator in $\mathbb{H}$ (with Schwartz kernel $e(x', y'; x'', y''; \lambda)$); and, finally, the main term in asymptotics is

$$\int \mathsf{tr}_{\mathbb{H}}\big(\mathsf{e}(x', \xi'; \lambda)\big)\, d\xi' dx', \tag{3.30}$$

where $\mathsf{e}(x', \xi', \lambda)$ is a spectral projector (in $\mathbb{H}$) of $\mathsf{H}(x', \xi')$. Here, we need to assume that $\mathsf{H}(x', \xi')$ is microhyperbolic with respect to (x', ξ'). Since the operator $\mathsf{tr}_{\mathbb{H}}$ is now unbounded, both the main term of the asymptotics of (3.29) and the remainder estimate *may* have magnitudes different from what they would be without $\mathsf{tr}_{\mathbb{H}}$.

Since the operator $\mathsf{H}(x', \xi')$ is rather complicated, we want to replace it by some simpler operator and add some easy to calculate correction terms.

We consider multiple examples below. Magnetic Schrödinger, Schrödinger-Pauli and Dirac operators studied in Sections 2 and 3 are also of this type.

Domains with Thick Cusps

This was done in Section 12.2 of [Ivr4] for operators in domains with thick cusps of the form $\{x : x'' \in f(x')\Omega\}$ where Ω is a bounded domain in $\mathbb{R}^{d''}$ with smooth boundary, defining the cusp crossection. Here again we consider for simplicity the Dirichlet Laplacian. Assume first that the metric is Euclidean and the domain $X = \{x = (x', x'') : x' \in X' := \mathbb{R}^{d'}, x'' \in f(x')\Omega\}$. Then, the change of variables $x'' \mapsto x''/f(x')$ transforms Δ to the operator

$$P = \sum_{1\le j\le d'} \Big(D_j + g_{x_j} L + \frac{id''}{2} g_{x_j}\Big)\Big(D_j + g_{x_j} L - \frac{id''}{2} g_{x_j}\Big) + \frac{1}{f^2}\Delta'' \tag{3.31}$$

in $\mathscr{L}^2(X' \times \Omega) = \mathscr{L}^2(\mathbb{R}^{d'}, \mathbb{H})$ where $L = \langle x'', D''\rangle$, $g = -\log f$, $\mathbb{H} = \mathscr{L}^2(\Omega)$, Δ' is a Laplacian in X', and $\Delta'' = \Delta''_{\mathsf{D}}$ is a Dirichlet Laplacians in Ω, and we simultaneously multiply u by $f^{-d''/2}$ to have the standard Euclidean measure rather than the weighted one $f^{d''} dx$. We consider the operator (3.31) as a perturbation of the operator

$$\bar{P} := \Delta' + \frac{1}{f^2}\Delta'', \tag{3.32}$$

which is a direct sum of d'-dimensional Schrödinger operators $P_n = \Delta' + \mu_n f^{-2}$ in X' where $\mu_n > 0$ are the eigenvalues of Δ''_{D}. Assuming that

$$f \asymp |x|^{-m}, \qquad |\nabla f| \asymp |x|^{-m-1} \qquad \text{for } |x'| \ge c, \tag{3.33}$$

we can ensure that the microhyperbolicity condition (with respect to $(x';\xi')$) is fulfilled for P_n, $\bar{P}$, as well as for P.

Then according to the previous section, for P_n the eigenvalue counting function is

$$\mathsf{N}_n(\lambda) = c_{d'} \int \big(\lambda - \mu_n f^{-2}(x')\big)_+^{\frac{d'}{2}}\, dx' + O\big(\lambda^{(d'-1)(m+1)/2m}\mu_n^{-(d'-1)/2m}\big), \tag{3.34}$$

where the remainder estimate is uniform with respect to n. Observe that for $\bar{P}$ the eigenvalue counting function is $\bar{\mathsf{N}}(\lambda) = \sum_n \mathsf{N}_n(\lambda)$. Using $\mu_n \asymp n^{2/d''}$, we arrive to

$$\mathsf{N}_n(\lambda) = c_{d'} \iint (\lambda - \mu f^{-2}(x'))_+^{\frac{d'}{2}}\, dx' d_\mu \boldsymbol{n}(\mu) + O(R(\lambda)) \tag{3.35}$$

with

$$R(\lambda) = \lambda^{\frac{1}{2}(d-1)} + \lambda^{\frac{m+1}{2m}(d'-1)} + \delta_{(d'-1),md''}\lambda^{\frac{1}{2}(d-1)}\log\lambda, \tag{3.36}$$

where $\boldsymbol{n}(\mu)$ is the eigenvalue counting function for Δ''_{D}.

We show, moreover, that the same asymptotics holds for our original operator (3.31). Furthermore, if the first term in (3.35) is dominant, then under the standard non-periodicity assumption we can replace $O(\lambda^{(d-1)/2})$ by $o(\lambda^{(d-1)/2})$; we need to add the standard boundary term to the right-hand expression in (3.35).

On the other hand, if the second term in (3.35) dominates, then assuming that f stabilizes as $|x'| \to \infty$ to a positively homogeneous function f_0, under the corresponding non-periodicity assumption (now in $T^*\mathbb{R}^{d'}$) for $|\xi'|^2 + f_0^{-2}(x')$, we can replace $O(\lambda^{(m+1)(d'-1)/2m})$ by $o(\lambda^{(m+1)(d'-1)/2m})$. Finally, if both powers coincide then under the stabilization condition, the remainder estimate is $o(\lambda^{(d-1)/2}\log\lambda)$ but we need to add the modified boundary term to the right-hand expression in (3.35).

Obviously, the principal part in (3.35) is of the magnitude

$$S(\lambda) = \lambda^{\frac{1}{2}d} + \lambda^{\frac{m+1}{2m}d'} + \delta_{d',md''}\lambda^{\frac{1}{2}d}\log\lambda. \tag{3.37}$$

If X is not exactly of the same form and the metric only stabilizes (fast enough) at infinity to $g^{jk0} := \delta_{jk}$, then we can recover the same remainder estimate and reduce the principal part to

$$\begin{aligned} c_{d'} \iint (\lambda - \mu f^{-2}(x'))_+^{\frac{d'}{2}} \phi(x')\, dx' d_\mu \boldsymbol{n}(\mu) \\ + c_d \lambda^{d/2} \int_X (\sqrt{g} - \sqrt{g^0}\phi(x'))\, dx, \end{aligned} \tag{3.38}$$

where $\mathsf{supp}(\phi) \subset \{|x'| \ge c\}$, $\phi = 1$ in $\{|x'| \ge c\}$. Here, the first part is exactly as above and the second term is actually the sum of two terms; one of them $c_d\lambda^{d/2}\int \sqrt{g}(1-\phi(x'))\,dx$ is the contribution of the "finite part of the domain" (without the cusp) and the second $c_d\lambda^{d/2}\int(\sqrt{g}-\sqrt{g^0})\phi(x')\,dx$ is a contribution of the cusp in the correction.

Note that now to get the remainder estimate $o(\lambda^{(d-1)/2})$, one needs to include the standard boundary term in the second part of (3.38).

The crucial part of our arguments is a *multiscale analysis*. As long as $r \le c\lambda^{1/2m-\delta}$, we can scale $x \mapsto xr^m$ and consider $\sigma_0(t) = \mathsf{Tr}\big(e^{ih^{-1}tH}\phi(x'/r)\big)$; here $H = \lambda^{-1}P$, $h = \lambda^{-1/2}r^m$. From the propagation with respect to (x,ξ), we know that on energy level 1, the time interval $(h^{1-\delta},\epsilon)$ contains no singularities of $\sigma_0(t)$.

On the other hand, for $r \ge c$, we can scale $x \mapsto x/r$ and consider $\sigma_1(t) = \mathsf{Tr}\big(e^{i\hbar^{-1}tH}\phi(x'/r)\big)$; here $\hbar = \lambda^{-1/2}r^{-1}$. From the propagation with respect to (x',ξ'), we know that on energy level 1, the time interval $(\hbar^{1-\delta},\epsilon)$ contains no singularities of $\sigma_1(t)$.

Observe first that $\sigma_1(t) = \sigma_0(r^{-1-m}t)$ and therefore the time interval $(h^{1-\delta},\epsilon r^{m+1})$ contains no singularities of $\sigma_0(t)$. This allows us to improve the remainder estimate in the full Weyl asymptotics but we need to include many terms which are difficult to calculate.

On the other hand, for $\lambda^\delta \le r \le c\lambda^{1/2m}$, we can consider H as a perturbation of $\bar{H} = \lambda^{-1}\bar{P}$. We do it first in the framework of the theory of operators with operator-valued symbols. Then we consider all perturbation terms and apply to them "full Weyl theory" and due to the stabilization assumption, the error is less than (3.36). This gives us another asymptotics, also with many terms which are difficult to calculate.

Comparing these two asymptotics in their common domain $\lambda^\delta \le r \le \lambda^{1/2m-\delta}$, we conclude that all terms but those present in both must be 0; it allows us to eliminate almost all the terms and sew these asymptotics resulting in (3.38).

Using the same approach, we can consider higher order operators, the case when X' is a conical set and there are several cusps X_k which may have different dimensions d_k' and rates of decay (then both the principal part and the remainder estimate should be modified accordingly).

Neumann Laplacian in Domains with Ultra-Thin Cusps

Consider the Neumann Laplacian in domains with cusps. Recall that since these domains do not satisfy the cone condition, we so far have no results even if the cusp is thin. Applying the same arguments as before, we hit two obstacles. The first (minor) obstacle is that the Neumann boundary condition for the operator (3.31) coincides with the same condition for Δ'' only asymptotically. The second (major) obstacle is that $\mu_1 = 0$ and $P_1 = \Delta'$ has a continuous spectrum. In fact, we should not reduce P to $\bar{P}$; from (3.31) we conclude that

$$P_1 = \sum_{1\le j\le d'} \Big(D_j + \frac{id''}{2} g_{x_j}\Big)\Big(D_j - \frac{id''}{2} g_{x_j}\Big) = \Delta' + W \tag{3.39}$$

with

$$W = \frac{d''^2}{4}|\nabla g|^2 + \frac{d''}{2}\Delta' g. \tag{3.40}$$

Still this operator may have a continuous spectrum unless $|\nabla g| \to \infty$ as $|x| \to \infty$. We need to assume that f has superexponential decay: $f = e^{-g}$ with

$$|\nabla^\alpha g| \le c_\alpha |x|^{1+m-|\alpha|} \qquad \forall \alpha, \tag{3.41}$$

$$g \asymp |x'|^{m+1}, \quad |\nabla g| \asymp |x'|^m \qquad \text{for } |x'| \ge c, \tag{3.42}$$

$$|\nabla |\nabla g|^2| \asymp |x|^{2m-1} \qquad \text{for } |x'| \ge c, \tag{3.43}$$

where $m > 0$ and (3.43) is a microhyperbolicity condition for P_1. Then one can prove easily that when $d'' \ge 2$,

$$\mathsf{N}(\lambda) = c_d \lambda^{d/2} \int_X \sqrt{g}\, dx + c_{d'} \int (\lambda - W)_+^{d'/2}\, dx' + O(R(\lambda)) \tag{3.44}$$

with

$$R(\lambda) = \lambda^{\frac{1}{2}(d-1)} + \lambda^{\frac{m+1}{2m}(d'-1)}. \tag{3.45}$$

Moreover, if the first term in (3.35) dominates, then under the standard non-periodicity assumption, we can replace $O(\lambda^{(d-1)/2})$ by $o(\lambda^{(d-1)/2})$ (simultaneously including the standard boundary term); if the second term dominates, then assuming that W stabilizes as $|x'| \to \infty$ to a positively homogeneous function W_0, under the corresponding assumption for $|\xi'|^2 + W_0(x')$ we can replace $O(\lambda^{(m+1)(d'-1)/2m})$ by $o(\lambda^{(m+1)(d'-1)/2m})$.

One can see easily that $\mathsf{N}(\lambda) \asymp S(\lambda) = \lambda^{\frac{1}{2}d} + \lambda^{\frac{m+1}{2m}d'}$. Observe that in contrast to (3.36) and (3.37), even if the exponents coincide, a logarithmic factor does not appear.

The case $d'' = 1$ is special since even an ultra-thin cusp is also thick (according to the classification of the previous Subsection 3) and the corresponding formulae should include a modified boundary term containing the double logarithm of λ. For this and other generalizations, see Section 12.7 of [Ivr4]. Also one can consider *spikes* with $\mathsf{supp}(f) = \{|x'| \le L\}$, in which case the standard Weyl asymptotics holds.

Operators in $\mathbb{R}^d$

The scheme of Subsection 3 is repeated in many similar cases.

First, consider eigenvalues tending to $+\infty$ for the Schrödinger operator with potential V which generically is $\asymp |x|^{2m}$ but vanishes along some directions.

For example, consider the toy-model $V = |x|^{2m-2m'}|x''|^{2m'}$ with $m > m' > 0$. Let $X' = \mathbb{R}^{d'} \ni x'$ and $X'' = \mathbb{R}^{d''} \ni x''$. Consider only the conical vicinity of X' and here we instead consider the potential $V = |x'|^{2m-2m'}|x''|^{2m'}$. Consider only the part of operator which is related to x'': $\Delta'' + |x'|^{2m-2m'}|x''|^{2m'}$ and after the change of variables $x'' \mapsto x''|x'|^k$ with $k = (m-m')/(m'+1)$, it becomes $|x'|^{2k}L$ with $L = \Delta'' + U(x'')$, $U = |x''|^{2m''}$. The condition $m'' > 0$ ensures that the spectrum of L is discrete and accummulates to $+\infty$.

So basically we have a mixture of the Schrödinger operator on $\mathbb{R}^d$ with a potential growing as $|x|^{2m}$ and the Schrödinger operator with the operator-valued symbol on $\mathbb{R}^{d''}$ with a potential growing as $|x|^{2k}$ and we recover the asymptotics with the remainder estimate $O(R(\lambda))$, where

$$R(\lambda) = \lambda^{\frac{m(d-1)}{(m+1)}} + \lambda^{\frac{k(d'-1)}{(k+1)}} + \delta_{\frac{m(d-1)}{(m+1)},\frac{k(d'-1)}{(k+1)}}\lambda^{\frac{m(d-1)}{(m+1)}}\log\lambda \tag{3.46}$$

and the principal part is $\asymp S(\lambda)$, where

$$S(\lambda) = \lambda^{\frac{md}{(m+1)}} + \lambda^{\frac{kd'}{(k+1)}} + \delta_{\frac{md/(m+1)}{,}\frac{kd'}{(k+1)}}\lambda^{\frac{md}{(m+1)}}\log\lambda. \tag{3.47}$$

In a rather general situation, this principal part is similar to the one in (3.38) where $\boldsymbol{n}(\mu)$ is the eigenvalue counting function for L. Further, under similar

non-periodicity assumptions, we can replace "O" by "o". For generalizations, details and proofs, see Section 12.3 of [Ivr4].

Second, consider eigenvalues tending to -0 for the Schrödinger operator with a potential V which generically is $\asymp |x|^{2m}$ with $m \in (-1, 0)$ but is singular in some directions. Again, consider a toy-model $V = -|x|^{2m-2m'}|x''|^{2m'}$ with $-1 < m < m' < 0$. Again, $L = \Delta'' + U(x'')$, $U = -|x''|^{2m''}$ and its negative spectrum is discrete and accummulates to -0. The formulae (3.46) and (3.47) remain valid (albeit $\lambda \to -0$). For generalizations, details and proofs, see Section 12.4 of [Ivr4].

Maximally Hypoelliptic Operators

Third, consider the eigenvalues tending to $+\infty$ for *maximally hypoelliptic operators with a symplectic manifold of degeneration*. Consider the toy-model $P = \Delta'' + |x''|^{2m}\Delta'$. In this case, after the partial Fourier transform, we get $\Delta'' + |x''|^{2m}|\xi'|^2$ and after the change of variables $x'' \mapsto |\xi'|^k x''$, we get $|\xi'|^{2k} L$, $L = \Delta'' + |x''|^{2m}$ and $k = 1/(m+1)$.

This toy-model is maximally hypoelliptic as the spectrum of L is discrete and accummulates to $+\infty$. So basically we have a blend of operator of order 2 on $\mathbb{R}^d$ and of order $2k$ on $\mathbb{R}^{d'}$ and we recover the asymptotics with remainder estimate $O(R(\lambda))$ with

$$R(\lambda) = \lambda^{\frac{(d-1)}{2}} + \lambda^{\frac{(d'-1)}{2k}} + \delta_{d-1,(d'-1)/k}\lambda^{\frac{(d-1)}{2}} \log \lambda \tag{3.48}$$

and principal part $\asymp S(\lambda)$ with

$$S(\lambda) = \lambda^{\frac{d}{2}} + \lambda^{\frac{d'}{2k}} + \delta_{d,d'/k}\lambda^{\frac{d}{2}} \log \lambda. \tag{3.49}$$

Further, under similar non-periodicity assumptions, we can replace "O" by "o". For generalizations, details and proofs, see Section 12.5 of [Ivr4].

Trace Asymptotics for Operators with Singularities

Here, we also consider only one example (albeit the most interesting one) of a Schrödinger operator $H := h^2\Delta - V(x)$ in $\mathbb{R}^3$ with potential $V(x)$ at 0 stabilizing to a positive homogeneous function V_0 of degree -1:

$$|\nabla^\alpha (V - V_0)| \le c_\alpha |x|^{-|\alpha|} \qquad \forall \alpha. \tag{3.50}$$

We assume that $V(x)$ decays fast enough at infinity and we are interested in the asymptotics of $\mathsf{Tr}(H_-)$, which is the sum of the negative eigenvalues of H. While generalizations are considered in Section 12.6 of [Ivr4], exactly this problem with $V_0 = |x|^{-1}$ arises in the asymptotics of the ground state energy of heavy atoms and molecules.

It follows from Section 3 that N_h^- has purely Weyl asymptotics with the remainder estimate $O(h^{-2})$ and[32] it could be improved to $o(h^{-2})$ but we have a different object and if the potential had no singularities, the remainder estimate would be $O(h^{-1})$ or even $o(h^{-1})$ [32),33)].

Therefore considering the contribution of the ball $B(x, \gamma(x))$ with $\gamma(x) = \frac{1}{2}|x|$, we have a contribution to the Weyl expression

$$-ch^{-3} \int V_+^{\frac{5}{2}}\, dx \tag{3.51}$$

of magnitude $C\rho^2(h/\rho\gamma)^{-3} = Ch^{-3}\rho^5\gamma^3$, while the contribution to the remainder does not exceed $C\rho^2(h/\rho\gamma)^{-1} = Ch^{-1}\rho\gamma$ with $\rho = |x|^{-\frac{1}{2}}$. We see that the former converges at 0 and the latter diverges. This analysis could be done for $\rho\gamma \ge h$ i.e. if $|x| \ge h^2$. Then we conclude that the contribution of the zone $\{x : |x| \ge a\}$ to the remainder does not exceed $Ch^{-1}a^{-\frac{1}{2}}$ which as $a = h^2$ is $O(h^{-2})$. On the other hand, one can easily prove that the contribution of $B(0, h^2)$ to the asymptotics is also $O(h^{-2})$.

To improve this estimate, we analyze $B(0, a)$ in more detail. In virtue of (3.50), we can easily prove that the contribution of $B(x, \gamma)$ to $\mathsf{Tr}(H_- - H_{0-})$ (with $H_0 = h^2\Delta - V_0$) does not exceed $C(h/\rho\gamma)^{-2} = Ch^{-2}\rho^2\gamma^2$ and therefore the contribution of $B(0, a)$ to the remainder is $O(h^{-2}a)$. Minimizing the total error $h^{-2}a + h^{-1}a^{-\frac{1}{2}}$ in a, we get $a = h^{\frac{2}{3}}$ and the remainder $O(h^{-\frac{4}{3}})$, which is better than $O(h^{-2})$ but not as good as $O(h^{-1})$.

But then we need to include in the asymptotics the extra term

$$\int \bigl(e_0^1(x, x, 0) - cV_+^{\frac{5}{2}}(x)\bigr)\psi(a^{-1}x)\, dx, \tag{3.52}$$

where $e(\cdot, \cdot, \lambda)$ is the Schwartz kernel of the spectral projectors for H, $e^1(\cdot, \cdot, 0) = \int_{-\infty}^0 \lambda\, d_\lambda e(\cdot, \cdot, \lambda)$ and the subscript 0 means that it is for H_0 and $\psi \in \mathscr{C}_0^\infty(B(0, 2))$ and equals 1 in $B(0, 1)$.

[32] Under the standard non-periodicity condition.

[33] But then the principal part of asymptotics should include the third term ch^{-1} while the second term vanishes.

Basically, all that we achieved so far was to replace H by H_0 in (3.52). The same arguments allow us to replace ψ by 1 in this expression with the same error $O(h^{-1}a^{-\frac{1}{2}})$. This time, we cannot decompose it as the difference of two integrals because each of them is diverging at infinity (since V_0 decays there not fast enough). Further, due to the homogeneity of V_0, one can prove that this remodelled expression (3.52) is homogeneous of degree -2 with respect to h and thus is equal to κh^{-2}. Here, κ is some unknown constant, but for $V_0 = |x|^{-1}$, it could be calculated explicitly.

Therefore, we conclude that with the remainder estimate $O(h^{-\frac{4}{3}})$, $\mathsf{Tr}(H_-)$ is given by the Weyl expression plus the *Scott corretion term* κh^{-2}.

To improve this remainder estimate, we should carefully study the propagation of singularities. We can prove that if $h^{2-\delta} \le \gamma \le 1$, then the singularities do not come back "in real time" $\asymp 1$, which is a vast improvement over $\asymp \gamma\rho^{-1} \asymp \gamma^{\frac{3}{2}}$. Then the contribution of $B(x,\gamma)$ to the "trace remainder" does not exceed $Ch^{-1}\rho^2\gamma^3$ but then the principal part of asymptotics should have a lot of terms; the n-th term is of the magnitude $h^{-3+2n}\rho^{2-2n}\gamma^{3-2n}$; however, using (3.50) we conclude that the difference between such terms for H and H_0 is $O(h^{-3+2n}\rho^{-2n}\gamma^{3-2n})$ which leads to the estimate

$$\left|\int \bigl(e^1(x,x,0) - e_0^1(x,x,0) - cV_+^{\frac{5}{2}}(x) + cV_{0+}^{\frac{5}{2}}(x)\bigr)\,dx\right| \le Ch^{-1}. \tag{3.53}$$

This estimate implies that with the remainder estimate $O(h^{-1})$, $\mathsf{Tr}(H_-)$ is given by the Weyl expression plus κh^{-2}. Moreover, this estimate could be further improved to $o(h^{-1})$ [32),33)].

Similar results hold for other singularities (including singularities of the boundary), dimensions and $\mathsf{Tr}(H_-^\nu)$ with $\nu > 0$. However, note that there could be more than one such correction term.

Periodic Operators

Finally, consider an operator $H_0 = H_0(x,D)$ with periodic coefficients (with the lattice of periods Γ). Then its spectrum is usually absolutely continuous and consists of *spectral bands* $\{\lambda_k(\xi) : \xi \in \mathcal{Q}'\}$ separated by *spectral gaps*. Here, λ_k are the eigenvalues of operator H_0 with *quasiperiodic boundary conditions*

$$u(x+n) = T^n e^{i\langle n,\xi\rangle}(x) \qquad \forall n \in \Gamma, \tag{3.54}$$

Γ^* is the dual lattice[34], $\mathcal{Q}$ and $\mathcal{Q}'$ are corresponding *elementary cells*[35]; ξ is called the *quasimomentum*. Here, $T = (T_1, \dots, T_d)$ is a family of commuting unitary matrices.

Let us consider an operator $H_t = H_0 - tW(x)$ with $W(x) > 0$ decaying at infinity. Then, while the essential spectra of H and H_t are the same, H_t can have discrete eigenvalues in the spectral gaps and these eigenvalues decrease as t increases.

Let us fix an *observation point* E belonging to either the spectral gap or its boundary and introduce $\mathsf{N}_E(\tau)$, the number of eigenvalues of H_t crossing E as t changes between 0 and τ. We are interested in the asymptotics of $\mathsf{N}_E(\tau)$ as $t \to \infty$.

Then using Gelfand's transform,

$$\mathcal{F}u(\xi, x) = (2\pi)^{\frac{d}{2}}(\mathsf{vol}(\mathcal{Q}'))^{-1} \sum_{n\in\Gamma} T^n e^{-i\langle n-x,\xi\rangle} u(x-n) \tag{3.55}$$

with $(x, \xi) \in \mathcal{Q} \times \mathcal{Q}'$, this problem is reduced to the problem for operators with operator-valued symbols on $\mathscr{L}^2(\mathcal{Q}', \mathbb{H}_{\xi,\{T\}})$ where $\mathbb{H}_{\xi,\{T\}}$ is the space of functions satisfying (3.54).

After that, different results are obtained in three essentially different cases: when E belongs to the spectral gap, E belongs to the bottom of the spectral gap, and E belongs to the top of the spectral gap. For exact results, proofs and generalizations, see Section 12.8 of [Ivr4].

4 Non-Smooth Theory

So far we have considered operators with smooth symbols in domains with smooth boundaries. Singularities were possible but only on "lean" sets. However, it turns out that many results remain true under very modest smoothness assumptions.

Non-Smooth Symbols and Rough Microlocal Analysis

To deal with non-smooth symbols, we approximate them by *rough* symbols $p \sim \sum_m p_m$, depending on a small *mollification parameter* ε and satisfying

$$|\nabla_\xi^\alpha \nabla_x^\beta p_m(x,\xi)| \le C_{m\alpha\beta} \boldsymbol{\rho}^{-\alpha} \boldsymbol{\gamma}^{-\beta} \varepsilon^{-m} \tag{1.1}$$

[34] I.e. if $\Gamma = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \dots \oplus \mathbb{Z}e_d$ then $\Gamma^* = \mathbb{Z}e'_1 \oplus \mathbb{Z}e'_2 \oplus \dots \oplus \mathbb{Z}e'_d$ with $\langle e_j, e'_k\rangle = \delta_{jk}$.

[35] I.e. $\mathcal{Q} = \{x_1e_1 + \dots + x_de_d : \ x \in [0,1]^d\}$ and $\mathcal{Q}'* = \{\xi_1e'_1 + \dots + \xi_de'_d : \ \xi \in [0,1]^d\}$.

with

$$\min_j \rho_j\gamma_j \ge \varepsilon \ge Ch^{1-\delta} \tag{1.2}$$

(*microlocal uncertainty principle*), which could be weakened to

$$|\nabla_\xi^\alpha\nabla_x^\beta p_m(x,\xi)| \le C^{|\alpha|+|\beta|+m+1}\alpha!\beta!m!\boldsymbol{\rho}^{-\alpha}\boldsymbol{\gamma}^{-\beta}\varepsilon^{-m} \tag{1.3}$$
$$\forall \alpha,\beta,m : |\alpha|+|\beta|+2m \le N = C|\log h|^{-1}$$

with

$$\min_j \rho_j\gamma_j \ge \varepsilon \ge Ch|\log h| \tag{1.4}$$

(*logarithmic uncertainty principle*). At this point, microlocal analysis ends: the assumptions cannot be weakened any further.

Assuming that

$$|\nabla_\xi^\alpha\nabla_x^\beta\nabla p_0(x,\xi)| \le C^{|\alpha|+|\beta|+1}\alpha!\beta!\boldsymbol{\rho}^{-\alpha}\boldsymbol{\gamma}^{-\beta} \tag{1.5}$$

and

$$|\nabla_\xi^\alpha\nabla_x^\beta\nabla p_m(x,\xi)| \le C^{|\alpha|+|\beta|+m+1}\alpha!\beta!m!\boldsymbol{\rho}^{-\alpha}\boldsymbol{\gamma}^{-\beta}\varepsilon^{1-m} \quad (m\ge 1), \tag{1.6}$$

we can restore Theorem 2.2 (see Theorem 2.3.2 of [Ivr4]) and therefore also the Corollaries 2.3 and 2.4, assuming ξ-microhyperbolicity instead of the usual microhyperbolicity. For proofs and details, see Section 2.3 of [Ivr4].

After this, we can than use the successive approximation method like in Subsection 2 (definitely some extra twisting required) and then recover the spectral asymptotics – originally only for operators which are ξ-microhyperbolic.

To consider non-smooth symbols, we can bracket them between rough symbols: for example, for the Schrödinger operator $p^-(x,\xi,h) \le p(x,\xi,h) \le p^+(x,\xi,h)$ where $p^\pm = p_\varepsilon \pm C\nu(\varepsilon)$ and p_ε is the symbol p, ε-mollified and $\nu(\varepsilon)$ is the modulus of continuity of the metric and potential; $\varepsilon = Ch|\log h|$.

Then for $\nu(\varepsilon) = O(\varepsilon|\log\varepsilon|^{-1})$ [36], we can recover the remainder estimate $O(h^{1-d})$; under even weaker regularity conditions by rescaling, we can recover weaker remainder estimates. On the other hand, if $\nu(\varepsilon) = o(\varepsilon|\log\varepsilon|^{-1})$,

[36] Which means that the first partial derivatives are continuous with modulus of continuity $\nu_1(\varepsilon) = \nu(\varepsilon)\varepsilon^{-1}$.

we can recover the remainder estimate $o(h^{1-d})$ under the standard non-periodicity condition[37]. For proofs and details, see Section 4.6 of [Ivr4]. There is an alternative to the bracketing construction based on perturbation theory, which works better for the trace asymptotics and also covers the pointwise asymptotics. For an exposition, see Section 4.6 of [Ivr4].

Further, for scalar and similar operators, the rescaling technique allows us to replace ξ-microhyperbolicity by microhyperbolicity under really weak smoothness assumptions; here we also use ε depending on the point so that we can consider scalar symbols under weaker and weaker non-degeneracy assumptions albeit stronger and stronger smoothness assumptions. See Section 5.4 of [Ivr4].

Non-Smooth Boundaries

Let us consider a domain with non-smooth boundary (with the Dirichlet boundary condition). Here, the standard trick to flatten out the boundary by the change of variables $x_1 \mapsto x_1 - \phi(x')$ works very poorly: the operator principal symbol contains the first partial derivatives ϕ and therefore we need to require $\phi \in \mathscr{C}^2$. Fortunately, the method of R. Seelley [See1] can help us. This method was originally developed for the Laplacian with a smooth metric and a smooth boundary.

Here, we consider only the Schrödinger operator; assume first that the metric and potential are smooth. Consider a point $\bar{x} \in X$ and assume that the metric is Euclidean at $\bar{x}$ and nearby, X looks like $\{x : x_1 \ge \phi(x')\}$ with $\nabla'\phi(\bar{x}') = 0$. Observe that these assumptions do not require any smoothness beyond $\mathscr{C}^1$.

Consider a trajectory starting from $(\bar{x}, \xi)$. If $|\xi_1| < \rho := Ch|\log h|/\gamma$, the trajectory starts parallel to ∂X and ∂X can "catch up" only at time at least $T = \sigma(\gamma)$ where $\gamma = \frac{1}{2}\mathsf{dist}(x, \partial X)$ and σ is the inverse function to ν, which is a modulus of continuity for ϕ[36].

If $\xi_1 > \rho$ then this trajectory "runs away from ∂X" and ∂X can "catch up" only at time at least $T = \sigma(\gamma) + \sigma_1(\xi_1)$ where σ_1 is the inverse function to ν_1[36]. On the other hand, if $\xi_1 < -\rho$, then we can revert the trajectory (which works only for local but not pointwise spectral asymptotics).

[37] However, even for the Schrödinger operator without boundary, the dynamic equations do not satisfy the Lipschitz condition and thus the flow could be multivalued.

These arguments allow us to estimate the contribution of $B(x,\gamma(x))$ to the remainder by $Ch^{1-d}\gamma^d h|\log h|\sigma(\gamma)^{-1}$ and then the total remainder by $Ch^{1-d}\int \sigma(\gamma)^{-1}\,dx$. The latter integral converges for $\nu(t) = t|\log t|^{-1-\delta}$.

Sure, this works only when $\gamma \ge \bar\gamma = Ch|\log h|$. However, if we smoothen the boundary with a smoothing parameter $C\bar\gamma$, for $\gamma \le \bar\gamma$, we will be in the framework of the smooth theory after rescaling and we can take $T = \bar\gamma$. The contribution of this strip to the remainder does not exceed $Ch^{1-d}\bar\gamma T^{-1}$ as its measure does not exceed $C\bar\gamma$. One can easily check that the variation of $\mathsf{vol}(X)$ due to the smoothing of the boundary is Ch^{1-d} and we can use the bracketing of X as well.

We can even improve the remainder estimate to $o(h^{1-d})$ under the standard non-periodicity condition.

Furthermore, if the metric and potential are not smooth, we need to mollify them, taking the mollification parameter ε larger near ∂X and taking $\rho = Ch|\log h|/\varepsilon$, but it works. For systems, we can exploit the fact that most of the cones of dependence are actually trajectories. For exact statements, proofs and details, see Section 7.5 of [Ivr4].

Aftermath

After the non-smooth local theory is developed, we can use all the arguments of Section 3 and consider "stronger but more concentrated" singularities added on the top of the weaker ones.

2 Magnetic Schrödinger Operator

Introduction

This Section is entirely devoted to the study of the *magnetic Schrödinger operator*

$$H = (-ih\nabla - \mu A(x))^2 + V(x) \tag{2.1}$$

and of the Schrödinger-Pauli operator

$$H = ((-ih\nabla - \mu A(x))\cdot \boldsymbol{\sigma})^2 + V(x) \tag{2.2}$$

with a small semiclassical parameter h and large *magnetic intensity parameter* (coupling constant) responsible for the interaction of a particle with the

magnetic field μ. Here, $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_d)$ where $\sigma_1, \dots, \sigma_d$ are Pauli $\mathsf{D} \times \mathsf{D}$-matrices and A is *magnetic vector potential*. We are interested in the two-parameter asymptotics (with respect to h and μ) as well as related asymptotics.

Standard Theory

Preliminaries

For a detailed exposition, generalizations and proofs, see Chapter 13 of [Ivr4].

Consider the most interesting cases $d = 2, 3$ with smooth $V(x)$ and $A(x)$. If $d = 2$, the magnetic field could be described by a single (pseudo)scalar $F_{12} = \partial_1 A_2 - \partial_2 A_1$ and by a scalar $F = |F_{12}|$. If $d = 3$, the magnetic field could be described by a (pseudo)vector $\boldsymbol{F} = \nabla \times A$ (*vector magnetic intensity*) and by a scalar $F = |\boldsymbol{F}|$ (*scalar magnetic intensity*). As a toy-model, we consider an operator in $\mathbb{R}^d$ with constant V and $\boldsymbol{F}$. Then canonical form of the operator (2.1) is

$$H = h^2 D_1^2 + (hD_2 - \mu F x_1)^2 + \underbrace{h^2 D_3^2} + V, \tag{2.3}$$

with the third term omitted when $d = 3$. Then we can calculate

$$e(x, x, \tau) = h^{-d} \mathcal{N}_d^{\mathsf{MW}}(\tau - V, \mu h F) \tag{2.4}$$

with

$$\mathcal{N}_d^{\mathsf{MW}}(\tau, F) = \kappa_d \sum_{n \ge 0} \big(\tau - (2n+1)F\big)_+^{\frac{1}{2}(d-2)} F, \tag{2.5}$$

where $\kappa_2 = 1/(2\pi)$, $\kappa_3 = 1/(2\pi^2)$. In particular, if $d = 2$, $F \neq 0$ this operator has a pure point spectrum of infinite multiplicity. Eigenvalues $(2m+1)\mu h F$ are called *Landau levels*. If $d = 3$, this operator has an absolutely continuous spectrum.

In these cases, the operator (2.2) is a direct sum of $\mathsf{D}/2$ operators H_- and $\mathsf{D}/2$ operators H_+ where $H_\mp = H_0 \mp \mu h F$, H_0 is the operator (2.1); then

$$\mathcal{N}_d^{\mathsf{MW}}(\tau - V, \mu h F) := \kappa_d D \Big(\frac{1}{2} \tau_+^{\frac{1}{2}(d-2)} + \sum_{n \ge 1} \big(\tau - 2nF\big)_+^{\frac{1}{2}(d-2)} \Big) F \tag{2.6}$$

Classical dynamics are different as well: when $d = 2$, the trajectories are *magnetrons*–circles of radii $(\mu F)^{-1}$, while if $d = 3$, there is also free movement along *magnetic lines*–integral curves of $\boldsymbol{F}$, so the trajectories are solenoids.

Canonical Form

Using the $\hbar$-Fourier transform, we can reduce the magnetic Schrödinger operator to its microlocal canonical form

$$\mu^2 \sum_{n\ge 0} B_n(x_1, \hbar D_1, \mu^{-2}, \hbar)\mathcal{L}_0^n \qquad \text{for } d = 2, \tag{2.7}$$

$$h^2 D_2^2 + \mu^2 \sum_{n\ge 0} B_n(x', \hbar D_1, \mu^{-2}, \hbar)\mathcal{L}_0^n \qquad \text{for } d = 3, \tag{2.8}$$

with $\hbar = \mu^{-1}h$, $\mathcal{L}_0 = x_d^2 + \hbar^2 D_d^2$, $x' = x_1$ and $x' = (x_1, x_2)$ when $d = 2, 3$ respectively. Further, the principal symbols of the operators B_0 and B_1 are $\mu^{-2}V \circ \Psi$ and $F \circ \Psi$ respectively, where Ψ is a diffeomorphism $(x', \xi_1) \to x$.

This canonical form allows us to study both the classical trajectories and the propagation of singularities in the general case. When $d = 2$, there is still movement along the magnetrons but magnetrons are drifting with the velocity $v = \mu^{-1}(\nabla((V - \tau)/F_{12}))^{\perp}$ where $^{\perp}$ denotes the counter-clockwise rotation by $\pi/2$. If $d = 3$, trajectories are solenoids winding around magnetic lines and the movement along magnetic lines is described by an 1-dimensional Hamiltonian but there there is also side-drift as in $d = 2$.

We can replace then $\mathcal{L}_0$ by its eigenvalues which are $(2j + 1)\hbar$, thus arriving to a family of $\hbar$-pseudodifferential operators with respect to x_1 if $d = 2$ and to a family of $\hbar$-pseudodifferential operators with respect to x_1 which is also a Schrödinger operator with respect to x_2.

Asymptotics: Moderate Magnetic Field

We can always recover the estimate $O(\mu h^{1-d})$ with the standard Weyl principal part simply by using the scaling $x \to \mu x$, $h \mapsto \mu h$, $\mu \mapsto 1$. On the other hand, for $d = 2$, we cannot in general improve it as follows from the example with constant F and V.

However, under a(THE?) non-degeneracy assumption, the remainder estimate is much better:

Theorem 2.1. *Let $d = 2$, $F \asymp 1$, $\mu h \lesssim 1$ and*

$$|\nabla VF^{-1}| + |\det \operatorname{Hess} VF^{-1}| \ge \epsilon. \tag{2.9}$$

Then

$$\int \Big(e(x, x, 0) - h^{-2}\mathcal{N}_2^{\mathsf{MW}}(x, -V, \mu hF)\Big)\psi(x)\, dx = O(\mu^{-1}h^{-1}). \tag{2.10}$$

The explanation is simple: each of the non-degenerate 1-dimensional $\hbar$-pseudodifferential operators contributes $O(1)$ to the remainder estimate and there is $\asymp (\mu h)^{-1}$ of them which should be taken into account. Another explanation is that under the non-degeneracy assumption, the drift of the magnetrons destroys the periodicity but we can follow the evolution for time $T^* = \epsilon\mu$, so the remainder estimate is $O(T^{*-1}h^{-1})$.

If $d = 3$, we cannot get the local remainder estimate better than $O(h^{-2})$ without global non-periodicity conditions due to the evolution along magnetic lines. On the other hand, we do not need strong non-degeneracy assumptions:

Theorem 2.2. *Let $d = 3$, $F \asymp 1$ and $\mu h \lesssim 1$. Then,*

$$\int \Big(e(x,x,0) - h^{-3}\mathcal{N}_3^{\mathsf{MW}}(x,-V,\mu hF)\Big)\psi(x)\,dx = O(h^{-2} + \mu h^{-1-\delta}) \tag{2.11}$$

in the general case and

$$\int \Big(e(x,x,0) - h^{-3}\mathcal{N}_3^{\mathsf{MW}}(x,-V,\mu hF)\Big)\psi(x)\,dx = O(h^{-2}), \tag{2.12}$$

provided

$$\sum_{\alpha:\, 1\le|\alpha|\le K} |\nabla^\alpha V F^{-1}| \ge \epsilon. \tag{2.13}$$

Further, in the general case, as(FOR?) $\mu \le h^{-\frac{1}{3}}$, we can replace the magnetic Weyl expression $\mathcal{N}_3^{\mathsf{MW}}$ by the standard Weyl expression $\mathcal{N}_3$.

Asymptotics: Strong Magnetic Field

Let us now consider the strong magnetic field case $\mu h \gtrsim 1$. Then the remainder estimates (2.10), (2.11) and (2.12) acquire a factor of μh^{-1}:

Theorem 2.3. *Let $d = 2$, $F \asymp 1$ and $\mu h \gtrsim 1$. Then for the operator* (2.3),

(i) Under the assumption

$$|\tau V - (2j+1)\mu hF| \ge \epsilon_0 \qquad \forall j \in \mathbb{Z}^+, \tag{2.14}$$

the following asymptotics holds:

$$e(x,x,\tau) - h^{-2}\mathcal{N}_2^{\mathsf{MW}}(x,\tau - V,F) = O(\mu^{-s}h^s). \tag{2.15}$$

(ii) Under the assumption

$$|\tau V - (2j+1)\mu hF| + |\nabla((V-\tau)F^{-1})| + |\det \mathsf{Hess}((V-\tau)F^{-1})| \ge \epsilon_0 \qquad \forall j \in \mathbb{Z}^+, \tag{2.16}$$

the following asymptotics holds:

$$\int \Big(e(x,x,\tau) - h^{-2}\mathcal{N}_2^{\mathsf{MW}}(x,\tau - V, F)\Big)\psi(x)\,dx = O(1). \tag{2.17}$$

Remark 2.4. If $d = 2$, we only need that $\mu^{-1}h \ll 1$ rather than $h \ll 1$.

Theorem 2.5. *Let $d = 3$, $F \asymp 1$ and $\mu h \gtrsim 1$. Then for the operator* (2.3),

$$\int \Big(e(x,x,0) - h^{-3}\mathcal{N}_3^{\mathsf{MW}}(x,-V,\mu hF)\Big)\psi(x)\,dx = O(\mu h^{-1-\delta}) \tag{2.18}$$

in the general case and

$$\int \Big(e(x,x,0) - h^{-3}\mathcal{N}_3^{\mathsf{MW}}(x,-V,\mu hF)\Big)\psi(x)\,dx = O(\mu h^{-1}) \tag{2.19}$$

under the assumption

$$|V + (2j+1)\mu hF| + \sum_{\alpha:\, 1\le|\alpha|\le K} |\nabla^\alpha V F^{-1}| \ge \epsilon \qquad \forall j \in \mathbb{Z}^+. \tag{2.20}$$

Remark 2.6. (i) $\mathcal{N}_d^{\mathsf{MW}} = O(\mu h)$ for $\mu h \gtrsim 1$.

(ii) For the Schrödinger-Pauli operator (2.4), one only needs to replace "$(2j+1)$" by "$2j$" in the assumptions above.

2D case, Degenerating Magnetic Field

Preliminaries

Since μF plays such a prominent role when $d = 2$, one may ask what happens if F vanishes somewhere? Obviously, one needs to make certain assumptions; it turns out that in the generic case

$$|F| + |\nabla F| \asymp 1, \tag{2.21}$$

the *degeneration manifold* $\Sigma := \{x : F(x) = 0\}$ is a smooth manifold and the operator is modelled by

$$h^2 D_1^2 + (hD_2 - \mu x_1^2/2)^2 + V(x_2), \tag{2.22}$$

which we are going to study. We consider the local spectral asymptotics for ψ supported in a small enough vicinity of Σ. Under the assumption (2.21) (or, rather more a general one), the complete analysis was done in Chapter 14 of [Ivr4].

Moderate and Strong Magnetic Field

We start from the case $\mu h \lesssim 1$. Without any loss of the generality, one can assume that $\Sigma = \{x : x_1 = 0\}$. Then, the scaling $x \mapsto x/\gamma(\bar{x})$ (with $\gamma(x) = \frac{1}{2}\mathsf{dist}(x, \Sigma)$), brings us to the case of the non-degenerate magnetic field with $h \mapsto h_1 = h/\gamma$ and $\mu \mapsto \mu_1 = \mu\gamma^2$ as long as $\gamma \ge \mu^{-\frac{1}{2}}$. Then the contribution of $B(x, \gamma(x))$ to the remainder does not exceed $C\mu_1^{-1}h_1^{-1} = C\mu^{-1}h^{-1}\gamma^{-1}$ and the total contribution of the *regular zone* $\mathcal{Z} = \{\gamma(x) \ge C_0\mu^{-\frac{1}{2}}\}$ does not exceed $C\int \mu^{-1}h^{-1}\gamma^{-3}\,dx = Ch^{-1}$.

On the other hand, in the *degeneration zone* $\mathcal{Z}_0 = \{\gamma(x) \le C_0\mu^{-\frac{1}{2}}\}$, we use $\gamma = \mu^{-\frac{1}{2}}$ and the contribution of $B(x, \gamma(x))$ does not exceed $Ch_1^{-1} = Ch^{-1}\mu^{-\frac{1}{2}}$ and the total contribution of this zone also does not exceed Ch^{-1}.

Thus we conclude that the left-hand expression of (2.10) is now $O(h^{-1})$. Can we do any better than this?

Analysis of the evolution and propagation in the zone $\mathcal{Z}$ shows that there is a drift of magnetic lines along Σ with speed $C\mu^{-1}\gamma^{-1}$ which allows us to improve $T^* \asymp \mu\gamma^2$ to $T^* \asymp \mu\gamma$ (both before rescaling) and improve the estimate of the contribution of $B(x, \gamma(x))$ to $C\mu^{-1}h^{-1}$ and the total contribution of this zone to $C\int \mu^{-1}h^{-1}\gamma^{-2}\,dx = C\mu^{-\frac{1}{2}}h^{-1}$.

Analysis of evolution and propagation in the zone $\mathcal{Z}_0$ is more tricky. It turns out that there are *short periodic trajectories* with period $\asymp \mu^{-\frac{1}{2}}$, but there are not many of them which allows us to improve the remainder estimate in this zone as well.

Theorem 2.7. *Let $d = 2$ and suppose the condition* (2.21) *is fulfilled. Let $\Sigma := \{F = 0\} = \{x_1 = 0\}$ and $-V \asymp 1$. Let $\mu \le h^{-1}$. Then,*

(i) The left-hand expression of (2.17) *is $O(\mu^{-\frac{1}{2}}h^{-1} + h^{-1}(\mu^{\frac{1}{2}}h|\log h|)^{\frac{1}{2}})$. In particular, for $\mu \lesssim (h|\log h|)^{-\frac{2}{3}}$, it is $O(\mu^{-\frac{1}{2}}h^{-1})$.*

(ii) Further,

$$\int \Big(e(x,x,0) - h^{-2}\mathcal{N}_2^{\mathsf{MW}}(x,-V,\mu hF)\Big)\psi(x)\,dx - h^{-1}\int \mathcal{N}_{\mathsf{corr}}^{\mathsf{MW}}(x_2,0)\psi(x_2,0) = O(\mu^{-\frac12}h^{-1}+h^{-\delta}) \tag{2.23}$$

with

$$h^{-1}\mathcal{N}_{\mathsf{corr}}^{\mathsf{MW}} := (2\pi h)^{-1}\int \boldsymbol{n}_0(\xi_2,-W(x_2),\hbar)\,d\xi_2 - h^{-2}\int \mathcal{N}^{\mathsf{MW}}(-W(x_2),\mu hF(x_1))\,dx_1 \tag{2.24}$$

where $\boldsymbol{n}_0(\xi_2,\tau,\hbar)$ *is an eigenvalue counting function for the operator* $\boldsymbol{a}_0(\xi_2,\hbar) = \hbar^2 D_1^2 + (\xi_2 - x_1^2/2)^2$ *on* $\mathbb{R}^1 \ni x_1$, $\hbar = \mu^{\frac12}h$ *and* $W(x_2) = V(0,x_2)$.

(iii) Furthermore, under the non-degeneracy assumption

$$\sum_{1\le k\le m} |\partial_{x_2}^k W| \asymp 1 \tag{2.25}$$

(in the framework of assumption (2.22)*), one can take* $\delta = 0$ *in* (2.23).

Remark 2.8. (i) Under some non-degeneracy assumptions, Theorem 2.7(i) could also be improved.

(ii) Theorem 2.7 remains valid for $h^{-1} \le \mu \lesssim h^{-2}$ as well but then the zone $\{x : \gamma(x) \ge C(\mu h)^{-1}\}$ is forbidden, contribution to the principal part is delivered by the zone $\{x : \gamma(x) \lesssim (\mu h)^{-1}\}$ and it is $\lesssim \mu^{-1}h^{-3}$.

(iii) As $\mu \ge Ch^2$, the principal part is 0 and the remainder is $O(\mu^{-s})$.

For further details, generalizations and proofs, see Section 14.6 of [Ivr4].

Strong and Superstrong Magnetic Field

Assume now that $\mu \gtrsim h^{-1}$ and replace V by $V - (2j+1)\mu hF_{12}$ with $j \in \mathbb{Z}^+$. Then the zone $\{x : \gamma(x) \ge C(\mu h)^{-1}\}$ is no longer forbidden, the principal part of asymptotics is of the magnitude μh^{-1} (cf. Remark 2.8(ii) and the remainder estimate becomes $O(\mu^{-\frac12}h^{-1}+h^{-\delta})$ (and under the non-degeneracy assumption one can take $\delta = 0$).

Furthermore, the case $\mu \ge Ch^{-2}$ is no longer trivial. First, one needs to change the correction term by replacing $\boldsymbol{a}_0$ with $\boldsymbol{a}_{2j+1} \coloneqq \boldsymbol{a}_0 - (2j+1)\hbar x_1$. Second, the non-degeneracy condition should be relaxed by requiring (2.25) only if

$$|\hbar^{2/3}\lambda_{j,n}(\eta) + W(x_2)| + |\partial_\eta \lambda_{j,n}(\eta) \ge \epsilon \qquad \forall \eta, \tag{2.26}$$

fails, where $\lambda_{j,n}$ are the eigenvalues of $\boldsymbol{a}_{2j+1}$ with $\hbar = 1$.

Furthermore, if

$$|\hbar^{2/3}\lambda_{j,n}(\eta) + W(x_2)| \ge \epsilon \qquad \forall \eta, \tag{2.27}$$

then 0 belongs to the spectral gap and the remainder estimate is $O(\mu^{-s})$.

For further details, exact statements, generalizations and proofs, see Sections 14.7 and 14.8 of [Ivr4].

2D Case, near the Boundary

Moderate Magnetic Field

We now consider the magnetic Schrödinger operator with $d = 2$, $F \asymp 1$ in a compact domain X with $\mathscr{C}^\infty$-boundary. While the dynamics inside the domain do not change, the dynamics in the boundary layer of the width $\asymp \mu^{-1}$ are completely different. When the magnetron hits ∂X, it reflects according to the standard "incidence angle equals reflection angle" law and thus the "particle" propagates along ∂X with speed $O(1)$ rather than $O(\mu^{-1})$. Therefore, physicists distinguish between *bulk* and *edge particles*. Note however that in general, this distinction is not as simple as in the case of constant F and V. Indeed, a drifting inner trajectory can hit ∂X and become a *hop trajectory*, while the latter could leave the boundary and become an inner trajectory.

It follows from Section 2 that the contribution of $B(x, \gamma(x))$ with $\gamma(x) = \frac{1}{2}\mathsf{dist}(x, \partial X) \ge \bar\gamma = C\mu^{-1}$ to the remainder is $O(\mu^{-1}h^{-1}\gamma^2 T(x)^{-1})$, where $T(x)$ is the length of the drift trajectory inside the *bulk zone* $\{x \in X : \gamma(x) \ge \bar\gamma\}$. Then the total contribution of this zone to the remainder does not exceed $C\mu^{-1}h^{-1}\int T(x)^{-1}\,dx = O(\mu^{-1}h^{-1})$ since $T(x) \gtrsim \gamma(x)^{\frac{1}{2}}$ (in the proper direction).

On the other hand, due to the rescaling $x \mapsto x/\bar\gamma$, the contribution of $B(x, \bar\gamma)$ with $\gamma(x) \le \bar\gamma$ to the remainder does not exceed $C\mu h^{-1}\bar\gamma^2$ and the

total contribution of the *edge zone* $\{x \in X : \gamma(x) \le \bar\gamma\}$ does not exceed $C\mu h^{-1}\bar\gamma = Ch^{-1}$ and the total remainder is $O(h^{-1})$. Thus, if we want a better estimate, we need to study propagation along ∂X.

Theorem 2.9. *Under the non-degeneracy assumption* $|\nabla_{\partial X}(VF^{-1})| \asymp 1$ *on* $\mathsf{supp}(\psi)$ *(contained in the small vicinity of* ∂X*) for* $\mu h \lesssim 1$

$$
\begin{aligned}
(2.28)\quad \int_X \Big(e(x,x,0) - h^{-2}\mathcal{N}^{\mathsf{MW}}(x,0,\mu h)\Big)\psi(x)\,dx - \\
h^{-1}\int_{\partial X} \mathcal{N}^{\mathsf{MW}}_{*,\mathsf{bound}}(x,0,\mu h)\psi(x)\,ds_g = O(\mu^{-1}h^{-1}),
\end{aligned}
$$

where $\mathcal{N}^{\mathsf{MW}}_{*,\mathsf{bound}}$ *is introduced in* $(5.4.2)_{\mathsf{D}}$ *or* $(5.4.2)_{\mathsf{N}}$ *below for the Dirichlet or Neumann boundary conditions respectively with* $\hbar = \mu h F(x)$ *and* τ *replaced by* $-V(x)$.

Here,

$$
\begin{aligned}
&(5.4.2)_{\mathsf{D}} \quad \mathcal{N}^{\mathsf{MW}}_{\mathsf{D},\mathsf{bound}}(\tau,\hbar) := \\
&(2\pi)^{-1}\int_0^\infty \sum_{j\ge 0}\Big(\int \theta\big(\tau - \hbar\lambda_{\mathsf{D},j}(\eta)\big)\upsilon^2_{\mathsf{D},j}(x_1,\eta)\,d\eta - \theta\big(\tau-(2j+1)\hbar\big)\Big)\hbar^{\frac12}\,dx_1
\end{aligned}
$$

and

$$
\begin{aligned}
&(5.4.2)_{\mathsf{N}} \quad \mathcal{N}^{\mathsf{MW}}_{\mathsf{N},\mathsf{bound}}(\tau,\hbar) := \\
&(2\pi)^{-1}\int_0^\infty \sum_{j\ge 0}\Big(\int \theta\big(\tau-\hbar\lambda_{\mathsf{N},j}(\eta)\big)\upsilon^2_{\mathsf{N},j}(x_1,\eta)\,d\eta-\theta\big(\tau-(2j+1)\hbar+\big)\Big)\hbar^{\frac12}\,dx_1,
\end{aligned}
$$

where $\lambda_{\mathsf{D},j}(\eta)$ *and* $\lambda_{\mathsf{N},j}(\eta)$ *are eigenvalues and* $\upsilon_{\mathsf{D},j}$ *and* $\upsilon_{\mathsf{N},j}$ *are eigenfunctions of operator*

$$
(2.30)\qquad \mathsf{a}(\eta,x_1,D_1) = D_1^2 + x_1^2 \qquad as \quad x_1 < \eta
$$

with the Dirichlet or Neumann boundary conditions respectively at $x_1 = \eta$.

Remark 2.10. Under weaker non-degeneracy assumptions $|\nabla VF^{-1}| \asymp 1$ and $\nabla_{\partial X} VF^{-1} = 0 \implies \pm\nabla^2_{\partial X} VF^{-1} \ge \epsilon$, less sharp remainder estimates are derived. The sign in the latter inequality matters since it affects the dynamics. It also matters whether Dirichlet or Neumann boundary conditions are considered: for the Dirichlet boundary condition we get a better remainder estimate.

For exact statements, generalizations and proofs, see Sections 15.2 and 15.2 of [Ivr4].

Strong Magnetic Field

We now consider the strong magnetic field $\mu h \gtrsim 1$ and for simplicity assume that $F = 1$. In this case, we need to study an auxillary operator $D_1^2+(x_1-\eta)^2$ as $x_1 < 0$ with either the Dirichlet or Neumann boundary conditions at $x_1 = 0$ or equivalently the operator (2.30) and our operator is basically reduced to a perturbed operator

$$a(\hbar D_2, x_1, D_1) - (2j+1) - (\mu h)^{-1} W(x_2) \qquad \hbar = \mu^{-1} h, \tag{2.31}$$

with $W(x_2) = V(0, x_2)$. Then we need to analyze either $\lambda_{\mathsf{D},j}(\eta)$ or $\lambda_{\mathsf{N},j}(\eta)$ more carefully. It turns out that:

Proposition 2.11. *$\lambda_{\mathsf{D},n}(\eta)$ and $\lambda_{\mathsf{N},n}(\eta)$, $n = 0, 1, 2, \dots$ are real analytic functions with the following properties:*

(i) $\lambda_{\mathsf{D},k}(\eta)$ are monotone decreasing for $\eta \in \mathbb{R}$; $\lambda_{\mathsf{D},k}(\eta) \nearrow +\infty$ as $\eta \to -\infty$; $\lambda_{\mathsf{D},k}(\eta) \searrow (2k+1)$ as $\eta \to +\infty$; $\lambda_{\mathsf{D},k}(0) = (4k+3)$.

(ii) $\lambda_{\mathsf{N},k}(\eta)$ are monotone decreasing for $\eta \in \mathbb{R}^-$; $\lambda_{\mathsf{N},k}(\eta) \nearrow +\infty$ as $\eta \to -\infty$; $\lambda_{\mathsf{N},k}(\eta) < (2n+1)$ as $\eta \ge (2n+1)^{\frac{1}{2}}$; $\lambda_{\mathsf{N},k}(0) = (4n+1)$.

(iii) $\lambda_{\mathsf{N},k}(\eta) < \lambda_{\mathsf{D},k}(\eta) < \lambda_{\mathsf{N},(k+1)}(\eta)$; $\lambda_{\mathsf{D},k}(\eta) > (2k+1)$, $\lambda_{\mathsf{N},n}(\eta) > (2k-1)_+$.

Proposition 2.12. *(i) $\partial_\eta \lambda_{\mathsf{D},k}(\eta) < 0$.*

(ii) $\partial_\eta \lambda_{\mathsf{N},n}(\eta) \gtreqless 0$ if and only if

$$\lambda_{\mathsf{N},k}(\eta) \lesseqgtr \eta^2. \tag{2.32}$$

(iii) $\lambda_{\mathsf{N},k}(\eta)$ has a single stationary point[38] *η_k, it is a non-degenerate minimum, and at this point* (2.32) *holds.*

(iv) In particular, $(\lambda_{\mathsf{N},k}(\eta) - \eta^2)$ has the same sign as $(\eta_k - \eta)$.

[38] And it must have one due to Proposition 2.11.

We see the difference between the Dirichlet and Neumann cases because we need non-degeneracy for $\lambda_{*,k}(\hbar D_2) - (2j+1) + W(x_2)$. It also means the difference in the propagation of singularities along ∂X: in the Dirichlet case, all singularities move in one direction (constant sign of $\lambda'_{\mathsf{D},k}$), while in the Neumann case, some move in the opposite direction (variable sign of $\lambda'_{\mathsf{N},k}$); this effect plays a role also in the case when $\mu h \lesssim 1$.

Assume here that τ is in an "inner" spectral gap:

$$|(2j+1)\mu h + V - \tau| \geq \epsilon_0 \mu h \qquad \forall x\ \forall j \in \mathbb{Z}^+. \tag{2.33}$$

Theorem 2.13. *Suppose that $\mu h \gtrsim 1$ and the condition* (2.33) *is fulfilled. Then,*

(i) In case of the Dirichlet boundary condition, the left-hand expression in (2.28) *is $O(1)$.*

(ii) In case of the Neumann boundary condition, assume additionally that

$$\begin{aligned}(5.4.7)_{\pm}\quad & |\bigl(\lambda_{\mathsf{N},j}(\eta)\mu h + V - \tau| \leq \epsilon_0 \mu h, \\ & |\lambda'_{\mathsf{N},j}(\eta)| + |\partial_{x_2} V| \leq \epsilon_0 \implies \pm \partial^2_{x_2} V \geq \epsilon_0 \qquad \forall j = 0,1,2,...\end{aligned}$$

Then, the left-hand expression in (2.28) *is $O(1)$ under the assumption $(2.34)_+$ and $O(\log h)$ under the assumption $(2.34)_-$.*

Remark 2.14. (i) If (2.33) is fulfilled for all $\tau \in [\tau_1, \tau_2]$, then the asymptotics is "concentrated" in the boundary layer.

(ii) For a more general statement when (2.33) fails (i.e. when τ is no longer in the "inner" spectral gap) and is replaced with the condition

$$|V + (2j+1)\mu h - \tau| + |\nabla V| \asymp 1 \tag{2.35}$$

on $\mathsf{supp}(\psi)$, see Theorem 15.4.18 of [Ivr4].

Pointwise Asymptotics and Short Loops

We are now interested in the pointwise asymptotics inside the domain. Surprisingly, it turns out that the standard Weyl formula for this purpose is better than the Magnetic Weyl formula.

Case $d = 2$

We start from the case $d = 2$, $F = 1$, $|\nabla V| \asymp 1$. One can easily see that in classical dynamics, short loops of the lengths $\asymp \mu^{-1} n$ with $n = 1, \dots, N$, $N \asymp \mu$ appear. We would like to understand how these loops affect the asymptotics in question.

Theorem 2.15. *For the magnetic Schrödinger operator which satisfies the above assumptions in a domain $X \subset \mathbb{R}^2$, with $B(0,1) \subset X$, the following estimates hold at a point $x \in B(0, \frac{1}{2})$*

(i) For $1 \le \mu \le h^{-\frac{1}{2}}$,

$$|e(x,x,\tau) - h^{-2}\mathcal{N}^{\mathsf{W}}_x(\tau)| \le C\mu^{-1}h^{-1} + C\mu^{\frac{1}{2}}h^{-\frac{1}{2}} + C\mu^2 h^{-\frac{1}{2}} \tag{2.36}$$

and

$$\begin{aligned} \mathsf{R}^{\mathsf{W}}_{x(r)} := |e(x,x,0) - h^{-2}\big(\mathcal{N}^{\mathsf{W}}_x(0) + \mathcal{N}^{\mathsf{W}}_{x,\mathsf{corr}(r)}(0)\big)| \le \\ C\mu^{-1}h^{-1} + C\mu^{\frac{1}{2}}h^{-\frac{1}{2}} + C\mu h^{-1}\big(\mu^2 h\big)^{r+\frac{1}{2}} + \\ C \begin{cases} \Big(h^{-1}\big(h\mu^{\frac{5}{2}}\big)^{r+\frac{1}{2}} + \mu^{\frac{1}{3}}h^{-\frac{2}{3}}\Big) & \text{as} \quad \mu \le h^{-\frac{2}{5}}, \\ \mu^{\frac{5}{3}}h^{-\frac{1}{3}} & \text{as} \quad \mu \ge h^{-\frac{2}{5}} \end{cases} \end{aligned} \tag{2.37}$$

where $\mathcal{N}^{\mathsf{W}}_{x,\mathsf{corr}(r)}$ is the r-term stationary phase approximation to some explicit oscillatory integral (see Section 16.3 of [Ivr4]).

(ii) For $h^{-\frac{1}{3}} \le \mu \le h^{-1}$,

$$\begin{aligned} \mathsf{R}^{\mathsf{W}\prime\prime}_{x(r)} := \\ |\Big(e(x,x,\tau) - h^{-2}\mathcal{N}_{x,\mathsf{corr}(r)} - \bar{e}_x(x,x,\tau) + h^{-2}\bar{\mathcal{N}}_{x,\mathsf{corr}(r)}\Big)| \le \\ C\mu^{\frac{1}{2}}h^{-\frac{1}{2}} + C \begin{cases} \mu^{-2}h^{-2}(\mu^2 h)^{r+\frac{1}{2}} & \text{for} \ \ \mu \le h^{-\frac{1}{2}} \\ h^{-1} & \text{for} \ \ \mu \ge h^{-\frac{1}{2}} \end{cases} \\ + C \begin{cases} h^{-1}\big(\mu^{\frac{5}{2}}h\big)^{r+\frac{1}{2}} + \mu^{\frac{1}{3}}h^{-\frac{2}{3}} & \text{for} \ \ \mu \le h^{-\frac{2}{5}} \\ \mu^{\frac{5}{3}}h^{-\frac{1}{3}} & \text{for} \ \ h^{-\frac{2}{5}} \le \mu \le h^{-\frac{1}{2}} \\ \mu^{-\frac{1}{3}}h^{-\frac{4}{3}} & \text{as} \ \ \mu \ge h^{-\frac{1}{2}} \end{cases} \end{aligned} \tag{2.38}$$

while for $r = 0$,

$$\mathsf{R}^{\mathsf{W}\prime\prime}_{x(0)} := |e(x,x,\tau) - \bar{e}_x(x,x,\tau)| \le C\mu^{\frac12}h^{-\frac12} + C\begin{cases} h^{-1}\mu^{\frac12} & \text{for } \mu \le h^{-\frac12},\\ \mu^{-\frac12}h^{-\frac32} & \text{for } \mu \ge h^{-\frac12}\end{cases} \tag{2.39}$$

where here and in (iii), $\bar{e}_y$ *is constructed for the toy-model in* y *(with* $F = F(y)$ *and* $V(x) = V(y) + \langle \nabla V(y), x-y\rangle$*).*

(iii) For $\mu \ge h^{-1}$, $\tau \le c\mu h$,

$$|e(x,x,\tau) - \bar{e}_x(x,x,\tau)| \le C\mu^{\frac12}h^{-\frac12}. \tag{2.40}$$

Case $d = 3$

For $d = 3$, we cannot expect a remainder estimate better than $O(h^{-1})$. On the other hand, the purely Weyl approximation has a better chance to succeed as the loop condition now includes returning free movement along the magnetic line in addition to the returning circular movement. We formulate only one theorem out of many from Section 16.7 of [Ivr4]:

Theorem 2.16. *Let* $d = 3$. *Then,*

(i) In the general case,

$$|e(x,x,\tau) - h^{-3}\mathcal{N}^{\mathsf{W}}_x(x,x,\tau)| \le Ch^{-2} + C\mu^{\frac32}h^{-\frac32}. \tag{2.41}$$

(ii) Under non-degeneracy condition

$$|\nabla_{\perp \boldsymbol{F}}(V-\tau)/F| \asymp 1, \tag{2.42}$$

where $\nabla_\perp$ *is the component of the gradient perpendicular to* $\boldsymbol{F}$, *for* $\mu \le h^{-\frac12}$, *we have the estimates*

$$|e(x,x,\tau) - h^{-3}\mathcal{N}^{\mathsf{W}}_x(x,x,\tau)| \le Ch^{-2} + C\mu^{\frac52}h^{-1} \tag{2.43}$$

and

$$|e(x,x,\tau) - h^{-3}\mathcal{N}^{\mathsf{W}}_x(x,x,\tau) - h^{-3}\mathcal{N}_{x,\mathsf{corr}(r)}| \le Ch^{-2} + C\mu^{\frac32}h^{-\frac32}(\mu^2h)^{r+\frac12}. \tag{2.44}$$

Here we use stationary phase approximations again.

Remark 2.17. One can also consider the cases $h^{-\frac12} \lesssim \mu \lesssim h^{-1}$ and $\mu \gtrsim h^{-1}$. But then one needs to include the toy-model expression (with constant $\boldsymbol{F}$ and ∇V) into the approximation.

Related Asymptotics

Apart from the pointwise asymptotics, one can consider the related asymptotics of

$$\int \omega(\frac{1}{2}(x+y), x-y) e(x,y,\tau) e(y,x,\tau)\, dxdy \tag{2.45}$$

and estimates of

$$\int \omega(\frac{1}{2}(x+y), x-y)\big(e(x,x,\tau) - e^{\mathsf{W}}(x,x,\tau)\big)\times \big(e(y,y,\tau) - e^{\mathsf{W}}(y,y,\tau)\big)\, dxdy. \tag{2.46}$$

For all the details, see Chapter 16 of [Ivr4]. These expressions play important role in Section 4.

Magnetic Dirac oOperators

We discuss the magnetic Dirac operators

$$H = ((-ih\nabla - \mu A(x)) \cdot \boldsymbol{\sigma}) + \sigma_0 M + V(x) \tag{2.47}$$

and

$$H = ((-ih\nabla - \mu A(x)) \cdot \boldsymbol{\sigma}) + V(x). \tag{2.48}$$

If $d = 3$, for the second operator, we can consider 2×2 matrices rather than 4×4 matrices.

If $V = 0$ then H^2 equals to the Schrödinger-Pauli operator (plus M^2) and therefore the theory of magnetic Dirac and Schrödinger operators are closely connected. If $V = 0$ and $0 \neq \boldsymbol{F}$ is constant then the operator for $d = 2$ has a pure point spectrum of infinite multiplicity consisting of $\pm\sqrt{M^2 + 2j\mu h F}$ with $j \in \mathbb{Z}^+$ [39] and for $d = 3$, this operator has absolutely continuous spectrum $(-\infty, -M] \cup [M, \infty)$. Thus we get corresponding *Landau levels*.

The results similar to those of Section 2 hold (for details, exact statements and proofs, see Chapter 17 of [Ivr4]. Further, the results of Sections 2 and 2 probably are not difficult to generalize, and maybe the results of Section 2 as well under correctly posed boundary conditions.

[39] However, one of the points $\pm M$ is missing depending on whether $F_{12} \gtrless 0$ and $\sigma_1\sigma_2\sigma_3 = \pm i$.

3 Magnetic Schrödinger Operator. II

Higher Dimensions

General Theory

We can consider a magnetic Schrödinger (and also Schrödinger-Pauli and Dirac) operators in higher dimensions. In this case, the magnetic intensity is characterized by the skew-symmetric matrix $F_{jk} = \partial_j A_k - \partial_k A_j$ rather than by a pseudo-scaler F_{12} or a pseudo-vector $\boldsymbol{F}$. As a result, the magnetic Weyl expression becomes more complicated; as before, it is exactly $e(x,x,\tau)$ for the operator in $\mathbb{R}^d$ if F_{jk} and V constant:

$$h^{-d}\mathcal{N}_d^{\mathsf{MW}}(\tau) \coloneqq (2\pi)^{-r}\mu^r h^{-r} \sum_{\alpha\in\mathbb{Z}^{+r}} \Big(\tau - \sum_j (2\alpha_j+1) f_j \mu h - V\Big)_+^{\frac{1}{2}(d-2r)} f_1\cdots f_r\sqrt{g}, \tag{3.1}$$

where $2r = \mathsf{rank}(F_{jk})$ and $\pm i f_j$ (with $j = 1,\dots,r$, $f_j > 0$) are its eigenvalues[40] which are not 0; recall that $z_+^0 = \theta(z)$.

One can see that H has pure point of infinite multiplicity spectrum when $d = 2r$ and H has an absolutely continuous spectrum when $d > 2r$. In any case, the bottom of the spectrum is $\mu h(f_1 + \dots + f_r)$.

We are interested in the asymptotics with the sharpest possible remainder estimate like the one for $d = 2$ or $d = 3$ in the cases $d = 2r$ and $d > 2r$ respectively[41]. Asymptotics without a remainder estimate were derived in [Rai1, Rai2].

As we try to reduce the operator to the canonical form, we immediately run into problem of *resonances* when $f_1 m_1 + \dots + f_r m_r = 0$ at some point with $m \in \mathbb{Z}^d$; $|m_1| + \dots + |m_r|$ is *an order of resonance*. If the lowest order of resonances is k, then we can reduce the operator to its canonical form modulo $O(\mu^{-k})$ for $\mu h \le 1$ (when $\mu h \ge 1$, this problem is less acute).

It turns out, however, that we can deal with an incomplete canonical form (with a sufficiently small remainder term).

Another problem is that we cannot in general assume that the f_j are constant (for $d = 2, 3$, we could achieve this by multiplying the operator by $f_1^{-1/2}$) and the microhyperbolicity condition becomes really complicated.

---

[40] In the general case, the eigenvalues of $(F_k^j) = (g^{jl})(F_{lk})$ where (g^{jk}) is a metric.

[41] We assume that $(F_{jk}(x))$ has constant rank.

Case $2r = d$ In this case, only the resonances of orders $2, 3$ matter as we are looking for an error $O(\mu^{-1}h^{1-d})$ when $\mu h \le 1$. If the magnetic field is weak enough, we use the incomplete canonical form only to study propagation of the singularities and if the magnetic field is sufficiently strong, the omitted terms $O(\mu^{-4})$ in the canonical form are small enough to be neglected.

As a result, the indices $j = 1, \dots, r$ are broken into several groups (indices j and k belong to the same group if they "participate" in the resonance of order 2 or 3 after all the reductions). Then under a certain non-degeneracy assumption called $\mathfrak{N}$-microhyperbolicity (see Definition 19.2.5 of [Ivr4]) in which this group partition plays a role, we can recover the remainder estimate $O(\mu^{-1}h^{1-d})$ for $\mu h \lesssim 1$ (in which case, the principal part has magnitude h^{-d}) and $O(\mu^{r-1}h^{1-d+r})$ for $\mu h \gtrsim 1$ (in which case, the principal part has magnitude $\mu^r h^{r-d}$). If we ignore the resonances of order 3, we get a partition into smaller groups and we need a weaker non-degeneration assumption called microhyperbolicity (see Definition 19.2.4 of [Ivr4]) but the remainder estimate would be less sharp.

In an important special case of constant $f_1, \dots, f_r$, both these conditions are equivalent to $|\nabla V|$ disjoint from 0 (on $\mathsf{supp}(\psi)$) but it could be weakened to V having only non-degenerate critical points (if there are saddles, we need to add a logarithmic factor to the remainder estimate).

On the other hand, without any non-degeneracy assumptions, the remainder estimate can be as bad as $O(\mu h^{1-d})$ for $\mu h \lesssim 1$ and as bad as the principal part itself for $\mu h \gtrsim 1$.

For exact statements, details, proofs and generalizations, see Chapter 19 of [Ivr4].

Case $2r < d$ In this case, only the resonances of order 2 matter since we are expecting a larger error than in the previous case. As a result, the indices $j = 1, \dots, r$ are broken into several groups (indices j and k belong to the same group if $f_j = f_k$).

Then under a certain non-degeneracy assumption called microhyperbolicity (see Definition 20.1.2 of [Ivr4]) in which this group partition plays a role, we can recover the remainder estimate $O(h^{1-d})$ for $\mu h \lesssim 1$ (then the principal part is of the magnitude h^{-d}) and $O(\mu^r h^{1-d+r})$ for $\mu h \gtrsim 1$ (then the principal part is of the magnitude $\mu^r h^{r-d}$).

In an important special case of constant $f_1, \dots, f_r$, this condition is equiv-

alent to $|\nabla V|$ being disjoint from 0 (on $\mathsf{supp}(\psi)$) but it could be weakened to "$\nabla V = 0 \implies \mathsf{Hess}\, V$ has a positive eigenvalue"; if we assume only that "$\nabla V = 0 \implies \mathsf{Hess}\, V$ has a non-zero eigenvalue", but we need to add a logarithmic factor to the remainder estimate.

As expected, for $2r = d-1$, we can recover less sharp remainder estimates even without any non-degeneracy assumptions and for $2r \le d-2$, we do not need any non-degeneracy conditions at all. For exact statements, details, proofs and generalizations, see Chapter 20 of [Ivr4].

Case $d = 4$: More Results

This case is simpler than the general one since we have only f_1 and f_2 and resonance happens if either $f_1 = f_2$ or $f_1 = 2f_2$ (or $f_2 = 2f_1$).

If we assume that the magnetic intensity matrix (F_{jk}) has constant rank 4, this case is simpler than the general one ($d = 2r$) and we can recover sharp remainder estimates under less restrictive conditions. For exact statements, details, proofs and generalizations, see Chapter 22 of [Ivr4].

On the other hand, if we consider (F_{jk}) of variable rank, then in the generic case, it has the eigenvalues $\pm if_1$ and $\pm if_2$ and $\Sigma = \{x : f_1(x) = 0\}$ is a $\mathscr{C}^\infty$ manifold of dimension 3, $\nabla f_1 \ne 0$ on Σ, while f_2 is disjoint from 0. It is similar to a 2-dimensional operator which we considered in Section 2, although with a twist: the symplectic form restricted to Σ has rank 2 everywhere except on a 1-dimensional submanifold Λ where it has rank 0.

Our results are also similar to those of Section 2 with rather obvious modifications but the proofs are more complicated.

For exact statements, details, proofs and generalizations, see Chapter 21 of [Ivr4].

Non-Smooth Theory

As in Chapter 4, we do not need to assume that the coefficients are very smooth. As before, we bracket the operator in question between two "rough" operators with the same asymptotics and with sharp remainder estimates. However, the lack of sufficient smoothness affects the reduction to the canonical form: it will be incomplete even if there are no resonances. Because of this, to get as sharp asymptotics as in the smooth case, we need to request more smoothness than in Chapter 4.

Case $d = 2$ For $d = 2$, we require smoothness of F_{12}, g^{jk} and V marginally larger than $\mathscr{C}^2$ to recover the same remainder estimate as in the smooth case, but there is a twist: unless the smoothness is $\mathscr{C}^3$, a correction term needs to be included. This is due to the fact that $V(x)$ and $W(x)$ differ and a more precise formula should use $W(x)$ rather than $V(x)$. Here, W is V averaged along a magnetron with center x. In fact, it is possible to consider V of the lesser smoothness than $\mathscr{C}^2$ (but marginally better than $\mathscr{C}^1$), but one gets a worse remainder estimate. For exact statements, details and proofs, see Chapter 18 of [Ivr4] and especially Section 18.5.

Case $d = 3$ Results are similar to those in the smooth case. However, in this case, if we assume no non-degeneracy conditions then the exponent δ in the estimates (2.11) and (2.18) depends on the smoothness and if we assume a non-degeneracy condition (2.13) or (2.20) then obviously K depends on the smoothness. Under the non-degeneracy assumption with $K = 1$, we need smoothness marginally better than $\mathscr{C}^1$ but again unless the smoothness is $\mathscr{C}^2$, we need to use the averaged potential $W(x)$ rather than $V(x)$.

For exact statements, details and proofs, see Chapter 18 of [Ivr4] and especially Section 18.9.

Case $d \geq 4$ Basically, the results are similar to those for $d = 2$ (if $\mathsf{rank}(F_{jk}) = d$) or for $d = 3$ (if $\mathsf{rank}(F_{jk}) < d$), but we cannot recover the sharp remainder estimate if the smoothness of V is less than $\mathscr{C}^3$ or $\mathscr{C}^2$ respectively because we cannot replace $V(x)$ by its average $W(x)$ in the canonical form.

For exact statements, details and proofs, see Chapters 19 (if $\mathsf{rank}(F_{jk}) = d$) and Chapters 20 (if $\mathsf{rank}(F_{jk}) < d$) of [Ivr4].

Global Asymptotics

For magnetic Schrödinger and Dirac operators, one can derive results similar to those of Sections 3 and 3. We describe here only some results which are very different from those already mentioned and only for the Schrödinger operator.

Case $d = 2r$

Assume that $F_{jk} = \mathsf{const}$, $\mathsf{rank}(F_{jk}) = 2r = d$, $\mu = h = 1$ and V decays at infinity. Then instead of an eigenvalue of infinite multiplicity $\lambda_{j,\infty} = (2j_1+1)f_1 + ... + (2j_r+1)f_r$ (with $j \in \mathbb{Z}^{+r}$), we have a sequence of eigenvalues $\lambda_{j,n}$ tending to $\lambda_{j,\infty}$ as $n \to \infty$ and we want to consider the asymptotics of $\mathsf{N}_j^-(\eta)$ which is the number of eigenvalues in $(\lambda_{j,\infty} - \epsilon, \lambda_{j,\infty} - \eta)$ and $\mathsf{N}_j^+(\eta)$ which is the number of eigenvalues in $(\lambda_{j,\infty} + \eta, \lambda_{j,\infty} + \epsilon)$, as $\eta \to +0$.

It turns out that in contrast to the Schrödinger operator without a magnetic field, there are meaningful results no matter how fast V decays.

Theorem 3.1. *Let us consider a Schrödinger operator in $\mathbb{R}^2$ satisfying the above conditions, $\mu = h = 1$ and*

$$|\nabla^\alpha V| \le c_\alpha \rho^2 \gamma^{-|\alpha|}, \tag{3.2}$$

with $\rho = \langle x\rangle^m$, $\gamma = \langle x\rangle$, $m < 0$. Let

$$\mp V \ge -\epsilon\rho^2 \implies |\nabla V| \ge \epsilon\rho^2\gamma^{-1} \qquad \text{for } |x| \ge c. \tag{6.3.2}_\mp$$

Then,

(i) The asymptotics

$$\mathsf{N}_j^\mp(\eta) = \mathcal{N}^\mp(\eta) + O(\log \eta) \tag{6.3.3}_\pm$$

hold with

$$\mathcal{N}^\mp(\eta) = \frac{1}{2\pi} \int_{\{\mp V > \eta\}} F\, dx \tag{6.3.4}_\pm$$

and in our conditions $\mathcal{N}^\mp(\eta) = O(\eta^{1/m})$. Moreover, $\mathcal{N}^\mp(\eta) \asymp \eta^{1/m}$ provided that $\mp V \ge \epsilon\rho^2$ for $x \in \Gamma$, where Γ is a non-empty open sector (cone) in $\mathbb{R}^2$.

(ii) Furthermore, if

$$\mp V \ge \epsilon\rho^2 \qquad \text{for } |x| \ge c, \tag{6.3.5}_\pm$$

then the remainder estimate is $O(1)$. In this case, the points $(2j+1)F \pm 0$ are not limit points of the discrete spectrum.

This theorem is proved by rescaling the results of Subsection 2 which do not require $h \ll 1$, but only $\mu h \gtrsim 1$ and $\mu^{-1}h \ll 1$ (see Remark 2.4); in our case, after rescaling $\mu = 1/\rho\gamma$ and $\mu = \gamma/\rho$, so that $\mu h = 1/\rho^2$ and $\mu^{-1}h = \gamma^{-2}$. Therefore, the remainder does not exceed $\int \gamma^{-2}\, dx$, where we integrate over $\{x : |V(x)| \ge (1-\epsilon)\eta\}$ in the general case and over $\{x : (1+\epsilon)\eta \ge |V(x)| \ge (1-\epsilon)\eta\}$ under the assumption $(6.3.5)_\pm$.

Remark 3.2. (i) Similar results hold in the d-dimensional case ($d \ge 4$) when $(F) = \mathsf{const}$ and $d = \mathsf{rank}(F)$: the remainder is $O(\eta^{(d-2)/2m})$ and the principal part is $O(\eta^{d/2m})$.

(ii) One can consider the case $\rho = \exp(-\langle x\rangle^m)$, $\gamma = \langle x\rangle^{1-m}$, $0 < m < 1$, and recover remainder estimate $O(|\log\eta|^{(d-2)/m+2})$ in the general case and $O(|\log\eta|^{(d-2)/m+1})$ under the assumption $(6.3.5)_{\pm}$ with $\mathcal{N}^{\mp}(\eta) = O(|\log\eta|^{d/m})$.

(iii) On the other hand, the cases when V decays like $\exp(-2\langle x\rangle)$ or faster, or is compactly supported, are out of reach of our methods but the asymptotics (without a remainder estimate) were obtained in [MR, RT, RW].

For exact statements, details, proofs and generalizations for arbitrary $\mathsf{rank}(F_{jk}) = d = 2r$, see Subsection 23.4.1 of [Ivr4].

Case $d > 2r$. I

This case is less "strange" than case $d = 2$. Here, we can discuss only the eigenvalue counting function $\mathsf{N}_0^-(\eta)$.

Theorem 3.3. *Let us consider a Schrödinger operator in $\mathbb{R}^3$ satisfying $F = \mathsf{const}$ and (3.2) with $\rho = \langle x\rangle^m$, $\gamma = \langle x\rangle$, $m \in (-1, 0)$. Then,*

(i) The asymptotics

$$\mathsf{N}_0^-(\eta) = \mathcal{N}^-(\eta) + O(\eta^{\frac{1}{m}-\delta}) \tag{6.3.6}_-$$

hold with

$$\mathcal{N}^-(\eta) = \frac{1}{2\pi^2}\int F(-V-\eta)_+^{\frac{1}{2}}\,dx \tag{3.8}$$

and arbitrarily small $\delta > 0$, and furthermore, $\mathcal{N}^-(\eta) = O(\eta^{\frac{3}{2m}+\frac{1}{2}})$. Moreover, $\mathcal{N}^-(\eta) \asymp \eta^{\frac{3}{2m}+\frac{1}{2}}$, provided $-V \ge \epsilon\rho^2$ for $x \in \Gamma$ where Γ is a non-empty open cone in $\mathbb{R}^3$.

(ii) Further, under the assumption

$$\sum_{|\alpha|\le K} |\nabla^\alpha V|\cdot\gamma^{|\alpha|} \ge \epsilon\rho^2 \qquad \text{for } |x| \ge c, \tag{3.9}$$

the asymptotics (3.8) hold with $\delta = 0$.

Remark 3.4. (i) Similar results hold in the d-dimensional case when $(F_{jk}) = \mathsf{const}$ and $d > 2r = \mathsf{rank}(F_{jk})$: the remainder estimate is $O(\eta^{(d-2r-1)/2+(d-1)/2m-\delta})$ [42] and $\mathcal{N}^-(\eta) = O(\eta^{(d-2r)/2+d/2m})$.

(ii) Observe that for $m = -1$, both the principal part and the remainder estimate have magnitude η^{-r}.

(iii) One can also consider $\rho = \langle x\rangle^{-1}|\log\langle x\rangle|^{\alpha}$ with $\alpha > 0$.

For exact statements, details, proofs and generalizations, see Subsection 24.4.1 of [Ivr4].

Case $d > 2r$. II

We now discuss faster decaying potentials. Assume that $d = 2r+1$ (otherwise there will be no interesting results). Assume for simplicity that $g^{jk} = \delta_{jk}$ and $F_{dk} = 0$. Further, one can assume that $A_d(x) = 0$; otherwise one can achieve it by a gauge transformation. Then, $A_j = A_j(x')$ with $x' = (x_1, \dots, x_{2r})$ and the operator is of the form

$$D_d^2 + V(x) + H_0', \qquad \text{with} \quad H_0' \coloneqq \sum_{1\le j\le d-1} (D_j - A_j(x'))^2. \tag{3.10}$$

For any fixed $x' : |x'| \ge c$, consider the one-dimensional operator

$$L \coloneqq D_t^2 + V(x'; t) \tag{3.11}$$

on $\mathbb{R} \ni t$. It turns out that under the assumption

$$|V(x'; t)| \le \varepsilon t^{-2}, \tag{3.12}$$

with $\varepsilon \le (\frac{1}{4} - \epsilon)$, this operator has no more than one negative eigenvalue $\lambda(x')$; moreover, it has exactly one negative eigenvalue

$$\lambda(x') = -\frac{1}{4} W(x')^2 + O(\varepsilon^3), \tag{3.13}$$

provided

$$W(x') \coloneqq \int_{\mathbb{R}} V(x'; t) < 0 \qquad \text{and} \quad -W(x') \asymp \varepsilon. \tag{3.14}$$

[42] Where $\delta = 0$ if either $d \ge 2r + 2$ or the assumption (3.9) is fulfilled.

Furthermore, in this case $\lambda(x')$ nicely depends on x'.

Let

$$|\nabla^{\alpha} V| \le c_{\alpha}\rho^{2}\gamma_{1}^{-|\alpha'|}\gamma^{-\alpha_{d}}, \tag{3.15}$$

with $\rho = \langle x\rangle^{l}\langle x'\rangle^{k}$, $\gamma = \langle x\rangle$, $\gamma_1 = \langle x'\rangle$ and $l \le -2$, $m \coloneqq 2l + 2k + 1 < 0$ and if

$$W(x') < 0, \qquad W(x') \asymp \rho', \qquad \rho' \coloneqq \langle x'\rangle^{m}, \tag{3.16}$$

then we are essentially in the $(d-1)$-dimensional case of an operator $H' \coloneqq H'_0 + \lambda(x')$, and for $\mathsf{N}^{-}(\eta)$, we have the corresponding asymptotics of Subsubsection 3.

For exact statements, details, proofs and generalizations, see Subsections 24.4.2 of [Ivr4]. For improvements for slower decaying potentials, see Subsections 24.4.3 of [Ivr4].

4 Applications to Multiparticle Quantum Theory

Problem Set-up

In this Chapter, we discuss an application to Thomas-Fermi Theory. Consider a large (heavy) atom or molecule; it is described by a *Multiparticle Quantum Hamiltonian*

$$\mathsf{H}_N = \sum_{1\le n\le N} H_V(x_n) + \sum_{1\le n<k\le N} \frac{1}{|x_n - x_k|} \tag{4.1}$$

where H is a *one-particle quantum Hamiltonian*, Planck constant $\hbar = 1$, electron mass $= \frac{1}{2}$, electron charge $= -1$. This operator acts on the space $\mathfrak{H} \coloneqq \wedge_{1\le j\le N}\mathscr{L}^2(\mathbb{R}^3, \mathbb{C}^2)$ of totally antisymmetric functions $\Psi(x_1, \varsigma_1; \dots; x_N, \varsigma_N)$ because the electrons are fermions, $x_n = (x_n^1, x_n^2, x_n^3)$ is a coordinate and $\varsigma_n \in \{-\frac{1}{2}, \frac{1}{2}\}$ is the *spin* of n-th particle. We identify the $\mathbb{C}^2$-valued function $\psi(x)$ on $\mathbb{R}^3$ with a scalar-valued function $\psi(x, \varsigma)$.

If the electrons did not interact between themselves, but the field potential was $-W(x)$, then they would occupy the lowest eigenvalues and the ground state wave functions would be the anti-symmetrized product

$\phi_1(x_1,\varsigma_1)\phi_2(x_2,\varsigma_2)\dots\phi_N(x_N,\varsigma_N)$, where ϕ_n and λ_n are the eigenfunctions and eigenvalues of H_W respectively.

Then the local electron density would be $\rho_\Psi = \sum_{1\le n\le N}|\phi_n(x)|^2$ and according to the pointwise Weyl law (if there is no magnetic field)

$$\rho_\Psi(x) \approx \frac{1}{3\pi^2}(W+\nu)_+^{\frac{3}{2}}, \tag{4.2}$$

where $\nu = \lambda_N$. We first assume that there is no magnetic field and therefore, $H_V = -\Delta - V(x)$.

This density would generate the potential $-|x|^{-1}*\rho_\Psi$ and we would have $W \approx V - |x|^{-1}*\rho_\Psi$.

Replacing all approximate equalities by strict ones, we arrive to the *Thomas-Fermi equations*:

$$V - W^{\mathsf{TF}} = |x|^{-1}*\rho^{\mathsf{TF}}, \tag{4.3}$$

$$\rho^{\mathsf{TF}} = \frac{1}{3\pi^2}(W^{\mathsf{TF}}+\nu)_+^{\frac{3}{2}}, \tag{4.4}$$

$$\int \rho^{\mathsf{TF}}\,dx = N, \tag{4.5}$$

where $\nu \le 0$ is called the *chemical potential* and in fact approximates λ_N.

Considering atoms and molecules, we assume that

$$V(x) = \sum_{1\le m\le M}\frac{Z_m}{|x-\mathsf{y}_m|}, \tag{4.6}$$

where y_m is the *position* and Z_m is the *charge* of the m-th nuclei, M is fixed and $Z_1 \asymp Z_2 \asymp \dots \asymp Z_M \asymp N \to \infty$.

Thomas-Fermi theory has been rigorously justified (with pretty good error estimates) and we want to explain how.

Reduction to One-Particle Problem

Estimate from below

We start from the estimate from below. The *ground state energy* $\mathsf{E}_N := \inf\langle \mathsf{H}_N\Psi,\Psi\rangle$, taken over all $\Psi\in\mathfrak{H}$ with $\|\Psi\|=1$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathfrak{H}$. Classical mathematical physics provides a wealth of results. One of them is the *electrostatic inequality* due to

E. H. Lieb [L]:

$$(4.7)\quad \sum_{1\le j<k\le N}\int |x_j-x_k|^{-1}|\Psi(x_1,\dots,x_N)|^2\,dx_1\cdots dx_N \ge \frac{1}{2}\mathsf{D}(\rho_\Psi,\rho_\Psi)-C\int \rho_\Psi^{\frac{4}{3}}(x)\,dx,$$

with ρ_Ψ defined by (4.2). This inequality holds for all (not necessarily antisymmetric) functions Ψ with $\|\Psi\|_{\mathscr{L}^2(\mathbb{R}^{3N})}=1$. Therefore,

$$(4.8)\quad \langle \mathsf{H}_N\Psi,\Psi\rangle \ge \sum_{1\le j\le N}\langle H_{V,x_j}\Psi,\Psi\rangle+\frac{1}{2}\mathsf{D}\big(\rho_\Psi,\rho_\Psi\big)-C\int\rho_\Psi^{\frac{4}{3}}(x)\,dx = \sum_{1\le j\le N}\langle H_{W,x_j}\Psi,\Psi\rangle+\frac{1}{2}\mathsf{D}\big(\rho_\Psi-\rho,\rho_\Psi-\rho\big)-\frac{1}{2}\mathsf{D}\big(\rho,\rho\big)-C\int\rho_\Psi^{\frac{4}{3}}(x)\,dx$$

and H_W is a one-particle Schrödinger operator with potential

$$(4.9)\qquad W=V-|x|^{-1}*\rho,$$

where ρ is an arbitrarily chosen real-valued non-negative function and therefore,

$$(4.10)\qquad (V-W,\rho_\Psi)=-\mathsf{D}(\rho,\rho_\Psi).$$

The physical sense of the second term in W is transparent: it is a potential created by the charge $-\rho$. Skipping the positive second term in the right-hand expression of (4.8) and believing that the last term is not very important for the ground state function Ψ [43)], we see that we need to estimate from below the first term. Since the first term is simply the sum of operators acting with respect to different variables, we can estimate it from below by

$$(4.11)\qquad \langle (H_{W,x_j}-\nu)\Psi,\Psi\rangle+\lambda N$$

with arbitrary ν; therefore, it is bounded from below by $\mathsf{Tr}((H_W-\nu)_-)$, where $(H_W-\nu)_-$ denotes the negative part of the operator $(H_W-\nu)$, and hence its trace is the sum of the negative eigenvalues.

[43)] When we derive the upper estimate for E, we will get an upper estimate for this term as a bonus.

Here, the assumption that Ψ is antisymmetric is crucial. Namely, for general (or symmetric–does not matter) Ψ, the best possible estimate is $\lambda_1 N$ where λ_1 is the lowest eigenvalue of H_W (we always assume that there are sufficiently many eigenvalues below the bottom of the essential spectrum of H_W) and we cannot apply semiclassical theory.

Thus we arrive to

$$\mathsf{E}_N \ge \mathsf{Tr}((H_W-\nu)_-) + \nu N - \frac{1}{2}\mathsf{D}\big(\rho,\rho\big) - CN^{\frac{5}{3}}, \tag{4.12}$$

where we used another result of E. H. Lieb [L]: $\int \rho_\Psi^{\frac{4}{3}}(x)\,dx \le CN^{\frac{5}{3}}$ for the ground state Ψ.

Estimate from above

Here, we simply plug in a test function Ψ which is an (anti-symmetrized) product $\phi_1(x_1,\varsigma_1)\phi_2(x_2,\varsigma_2)\dots\phi_N(x_N,\varsigma_N)$ where ϕ_n and λ_n are eigenfunctions and eigenvalues of H_W respectively, and we pick W later. It may happen, however, that H_W does not have N negative eigenvalues, then we can reduce N and use the inequality $\mathsf{E}_N \le E_{N'}$ as $N' \le N$.

Then, E_N is estimated from above by

$$\begin{aligned}\langle \mathsf{H}_N\Psi,\Psi\rangle = {}& \sum_n (H_{W,x_j}-\lambda)\Psi,\Psi\rangle + \lambda N - (V-W,\rho_\Psi) + \frac{1}{2}\mathsf{D}(\rho_\Psi,\rho_\Psi)\\ & - \frac{1}{2}\sum_n \iint |x-y|^{-1}|\psi_n(x)|^2|\psi_n(y)|^2\,dxdy\end{aligned} \tag{4.13}$$

and therefore recalling (4.10), we obtain

$$\mathsf{E}_N \le \mathsf{Tr}((H_W-\lambda)_-) + \lambda N + \frac{1}{2}\mathsf{D}(\rho_\Psi-\rho,\rho_\Psi-\rho) - \frac{1}{2}\mathsf{D}(\rho,\rho) \tag{4.14}$$

and $\rho_\Psi = \mathsf{tr}\, e_N(x,x)$ where $e_N(x,y)$ and $e(x,y,\lambda)$ are the Schwartz kernels of the projector to the N lowest eigenvalues of H_W and of the operator $\theta(\lambda - H_W)$ respectively; here tr denotes the matrix trace, and $\lambda = \lambda_N$ if $\lambda_N < 0$ and $\lambda = 0$ otherwise. Finally, we conclude that

$$\begin{aligned}\mathsf{E}_N \le {}& \mathsf{Tr}((H_W-\nu)_-) + \nu N + |\lambda-\nu|\cdot|\mathsf{N}^-(H_W-\nu) - N|\\ & + \frac{1}{2}\mathsf{D}(\mathsf{tr}\, e_N(x,x)-\rho, \mathsf{tr}\, e_N(x,x)-\rho) - \frac{1}{2}\mathsf{D}(\rho,\rho)\end{aligned} \tag{4.15}$$

with arbitrary $\nu \le 0$.

Semiclassical Approximation

Estimate from below

In the estimate from below (4.12), we replace $\mathsf{Tr}((H_W-\nu)_-)$ by its semiclassical approximation

$$\mathsf{Tr}((H_W-\nu)_-) \approx -\int P(W+\nu)\,dx \tag{4.16}$$

with

$$P(W+\nu) := \frac{2}{5\pi^2}(W+\nu)_+^{\frac{5}{2}}, \tag{4.17}$$

and also plug in $\rho = \frac{1}{4\pi}\Delta(W-V)$; then we obtain the functional

$$\Phi_*(W,\nu) = -\int P(W+\nu)\,dx - \frac{1}{8\pi}\|\nabla(W-V)\|^2 + \nu N; \tag{4.18}$$

maximizing it, we arrive to the Thomas-Fermi equations and its maximal value is $\mathcal{E}_N^{\mathsf{TF}}$, delivered by Thomas-Fermi theory. Then, we need to understand the semiclassical error. To do this, we use the properties of the Thomas-Fermi potential and rescale $x \mapsto xN^{\frac{1}{3}}$ and $\tau \mapsto N^{-\frac{4}{3}}\tau$ (so, $\nu \mapsto N^{-\frac{4}{3}}\nu$) with

$$H_W = -h^2\nabla^2 - W, \tag{4.19}$$

where near y_m, the rescaled potential is Coulomb-like: $W \sim z_m|x-y|^{-1}$ with $z_m = Z_m N^{-1}$.

Then, we can apply results Subsection 3 (see (3.53)): for the operator (4.19),

$$\mathsf{Tr}((H_W-\nu)_-) = -h^{-3}\int P(W+\nu)\,dx + \kappa h^{-2} + O(h^{-1}), \tag{4.20}$$

where in this case, the numerical value of $\kappa = 2\sum_m z_m^2$ is well-known. Scaling back, we obtain $\mathcal{E}_N^{\mathsf{TF}} + \mathsf{Scott} + O(N^{\frac{5}{3}})$ where the leading term is of magnitude $N^{\frac{7}{3}}$ and the Scott correction term $\mathsf{Scott} = 2\sum_m Z_m^2$. Here, we need to assume that $|\mathsf{y}_m - \mathsf{y}_{m'}| \gtrsim 1$ after rescaling (and $|\mathsf{y}_m - \mathsf{y}_{m'}| \gtrsim N^{-\frac{1}{3}}$ before it).

Indeed, after rescaling, we get an operator which is uniformly in the framework of Subsection 3 due to the following properties of the Thomas-Fermi potential:

(4.21) Before rescaling, $W^{\mathsf{TF}} = Z_m|x-\mathsf{y}_m|^{-1} + O(N)$ for $|x-\mathsf{y}_m| \lesssim N^{-\frac{1}{3}}$ and $W^{\mathsf{TF}} \asymp \sum_m \big(|x-\mathsf{y}_m|^{-4} + (Z-N)_+|x-\mathsf{y}_m|^{-1}\big)$ for $|x-\mathsf{y}_m| \gtrsim 1$ for all $m = 1, \dots, M$.

In fact, the analysis of Subsection 3 was mainly motivated by this problem.

Estimate from above

Again, using the semiclassical approximation (4.16) for $\mathsf{Tr}((H_W-\nu)^-)$ and also $e_N(x,x) \approx P'(W+\nu)$ with $P' = \frac{1}{3\pi^2}(W+\nu)^{\frac{3}{2}}$ the derivative of $P(W+\nu)$, we arrive to the functional

$$\Phi^*(W,\nu) = -\int P(W+\nu)\,dx - \frac{1}{8\pi}\|\nabla(W-V)\|^2 + \nu N + \mathsf{D}(P'(W+\nu) - \frac{1}{4\pi}\Delta(W-V),\ P'(W+\nu) - \frac{1}{4\pi}\Delta(W-V)); \tag{4.22}$$

minimizing it, we again arrive to the Thomas-Fermi equations and the minimal value is $\mathcal{E}_N^{\mathsf{TF}}$, again delivered by Thomas-Fermi theory.

However, in addition to the semiclassical error for the trace, we have other errors from (4.15):

$$|\lambda-\nu|\cdot|\mathsf{N}^-(H_W-\nu) - N|, \tag{4.23}$$

$$\mathsf{D}(\mathsf{tr}\, e(x,x,\nu) - P'(W+\nu),\ \mathsf{tr}\, e(x,x,\nu) - P'(W+\nu)) \tag{4.24}$$

and

$$\mathsf{D}(\mathsf{tr}\, e_N(x,x) - \mathsf{tr}\, e(x,x,\nu),\ \mathsf{tr}\, e_N(x,x) - \mathsf{tr}\, e(x,x,\nu)). \tag{4.25}$$

The expression (4.24) is the semiclassical error and after rescaling it, we can estimate it by $O(h^{-4})$ (due to the pointwise spectral asymptotics). When scaling back, we gain the factor $N^{\frac{1}{3}}$, resulting in $O(N^{\frac{5}{3}})$.

Expressions (4.23) and (4.25) can be also estimated by $O(N^{\frac{5}{3}})$ based on another semiclassical error

$$\mathsf{N}^-(H_W-\sigma) - \int P'(W+\sigma)\,dx = O(h^{-2}), \tag{4.26}$$

(for $\sigma \le 0$) after rescaling and thus, $O(N^{\frac{2}{3}})$ in the original scale, due to the definitions of λ and ν. One needs to consider four cases depending on whether $\lambda < 0$ (i.e. $N^-(H_W) \ge N$) or $\lambda = 0$ (i.e. $N^-(H_W) < N$) and whether $\nu < 0$ (i.e. $N < Z$) or $\nu = 0$ (i.e. $N \ge Z$), where $Z = Z_1 + \dots + Z_M$ is the total charge of the nuclei.

More Precise Estimates

If we want to improve the remainder estimate $O(N^{\frac{5}{3}})$, then we need to improve the semiclassical remainder estimates and also deal with $O(N^{\frac{5}{3}})$ in Lieb's electrostatic inequality (4.7).

The first task could be done under the assumption

$$a := \min_{m\neq m'} |\mathsf{y}_m - \mathsf{y}_{m'}| \gg \bar{a} := N^{-\frac{1}{3}}, \tag{4.27}$$

which is completely reasonable (see Section 4). In this case, in each zone $\mathcal{Y}_m := \{x : |x-\mathsf{y}_m| \le a^{1-\eta}\bar{a}^{\eta}\}$, with $\eta > 0$, both ρ^{TF} and W^{TF} are close to those of a single atom which are spherically symmetric. Then one can prove easily that the standard conditions to the trajectories are fulfilled and we may use the improved remainder estimates. On the other hand, contributions of the "outer" zone $\mathcal{Y}_0 := \{x : |x-\mathsf{y}_m| \ge a^{1-\eta}\bar{a}^{\eta}\ \forall m = 1,\dots,M\}$ to these remainders is smaller.

Therefore all remainder estimates acquire the factor $(h^{\delta} + b^{-\delta})$ with $b = a\bar{a}^{-1}$ before scaling back, i.e. $(N^{\frac{1}{3}\delta} + (aN^{\frac{1}{3}})^{-\delta})$ after it. However, the trace asymptotics should also include the term $-\kappa_1 h^{-1}$ before scaling back or $-\kappa_1 N^{\frac{5}{3}}$ after it; for the potential W^{TF}, it is numerically equal to $\mathsf{Schwinger} = -c_1 \int \rho^{\mathsf{TF}\,\frac{4}{3}}\,dx$ which is called the *Schwinger correction term*.

The second task requires an improvement in Lieb's electrostatic inequality due to [GS] and [Ba]: one can replace the last term in (4.8) for the ground state energy Ψ by

$$-\frac{1}{2}\iint |x-y|^{-1}\,\mathsf{tr}\big(e_N^{\dagger}(x,y)e_N(x,y)\big)\,dxdy - O(N^{\frac{5}{3}-\delta}), \tag{4.28}$$

where the first term coincides with the last term in (4.13) (the estimates from above) and again modulo $O(N^{\frac{5}{3}-\delta})$ can be rewritten as

$$-\frac{1}{2}\iint |x-y|^{-1}\,\mathsf{tr}\big(e^{\dagger}(x,y,\nu)e(x,y,\nu)\big)\,dxdy. \tag{4.29}$$

So far, we have not explored such expressions but we can handle them.

For this expression, after rescaling, we can derive the asymptotics with principal term $-\kappa_2 h^{-4}$ and with remainder estimate as good as $O(h^{-3})$, which after scaling back becomes $O(N^{\frac{4}{3}})$ (which is an overkill). Here, we use the representation of $e(x,y,\nu)$ by an oscillatory integral modulo a term whose $\mathscr{L}^2(\mathbb{R}^6)$ norm does not exceed Ch^{-2}.

To calculate κ_2, we can consider the operator with constant potential W, and for this operator, we calculate $-\frac{1}{2}\int |x-y|^{-1}\,\mathsf{tr}\big(e^{\dagger}(x,y,\nu)e(x,y,\nu)\big)\,dy$ obtaining $-\mathsf{const}(W+\nu)^2h^{-4}$, then plug in $W=W(x)$ and integrate over x. For $W=W^{\mathsf{TF}}$, after scaling back, we arrive to $\mathsf{Dirac}=-c_2\int \rho^{\mathsf{TF}\,\frac{4}{3}}\,dx$ which is called the *Dirac correction term*.

Despite having completely different origins, these correction terms differ only by numerical constants.

We arrive to the theorem:

Theorem 4.1. *As* $Z=Z_1+\ldots+Z_M\asymp N\to\infty$, M *remains bounded,* $a=\min_{m\neq m'}|\mathsf{y}_m-\mathsf{y}_{m'}|\gtrsim N^{-\frac{1}{3}}$ *and*

$$\mathsf{E}_N=\mathcal{E}_N^{\mathsf{TF}}+\mathsf{Scott}+\mathsf{Schwinger}+\mathsf{Dirac}+O(R) \qquad (4.30)$$
$$\text{with}\quad R=N^{\frac{5}{3}}\big(N^{-\delta}+(aN^{\frac{1}{3}})^{-\delta}\big)$$

where $\delta>0$ *is unspecified.*

As a byproduct of the proof, we obtain

$$\mathsf{D}(\rho_\Psi-\rho^{\mathsf{TF}},\,\rho_\Psi-\rho^{\mathsf{TF}})=O(R). \qquad (4.31)$$

For details and proofs, see Sections 25.1–25.4 of [Ivr4].

Ramifications

First, instead of the *fixed nuclei model*, we can consider the *free nuclei model* where we add to both E_N and $\mathcal{E}_N^{\mathsf{TF}}$ the energy of nuclei-to-nuclei interaction

$$\sum_{m<m'} Z_mZ_{m'}|\mathsf{y}_m-\mathsf{y}_{m'}|^{-1} \qquad (4.32)$$

and minimize the results by the position of nuclei $(\mathsf{y}_1,\ldots,\mathsf{y}_m)$; denote the results by $\widehat{\mathsf{E}}_N$ and $\widehat{\mathcal{E}}_N^{\mathsf{TF}}$ respectively.

Combining (4.30) with the non-binding theorem in Thomas-Fermi theory[44], we obtain that in the free nuclei model (with $Z_1\asymp\ldots\asymp Z_M\asymp Z\asymp N$),

$$a=\min_{m\neq m'}|\mathsf{y}_m-\mathsf{y}_{m'}|\gtrsim N^{-\frac{5}{3}+\delta} \qquad (4.33)$$

[44] In the Thomas-Fermi theory, molecules do not exist.

and then (4.30) and (4.31) hold with $R = N^{\frac{5}{3}-\delta}$.

Next, using methods already developed by mathematical physicists before asymptotics (4.30) and (4.31) were derived, we can answer several questions with far better precision than before; for simplicity, we assume that $a \ge N^{-\frac{1}{3}+\delta}$.

(i) How many extra electrons can the system bind? In other words, if $\mathsf{E}_N < \mathsf{E}_{N-1}$, what we can say about $N - Z$? According to a classical theorem due to G. Zhislin, the system can bind at least Z electrons. Our answer: $(N-Z)_+ = O(N^{\frac{5}{7}-\delta})$, based on the fact that in the Thomas-Fermi theory, negative ions do not exist.

(ii) What we can say about the *ionization energy* $\mathsf{I}_N = \mathsf{E}_{N-1} - \mathsf{E}_N$? Our answer: $\mathsf{I}_N = O(N^{\frac{20}{21}-\delta})$ if $N-Z \ge -CN^{\frac{5}{7}-\delta}$ and $\mathsf{I}_N = -\nu + O((Z-N)^{\frac{17}{18}} Z^{\frac{5}{18}-\delta}$ if $N - Z \le -CN^{\frac{5}{7}-\delta}$; if $N \le Z$ $\nu \asymp (Z-N)^{\frac{4}{3}}$.

(iii) In the free nuclei model (with $M \ge 2$), what can we say about $N-Z > 0$ if a stable configuration exists? Our answer: $Z - N \le CN^{\frac{5}{7}-\delta}$ (again based on the non-binding theorem).

For details and proofs, see Sections 25.5 and 25.6 of [Ivr4].

Adding Magnetic Field

Adding External Magnetic Field

Consider the Schrödinger-Pauli operator with magnetic field

$$H_{A,V} = ((-ih\nabla - \mu A(x)) \cdot \boldsymbol{\sigma})^2 - V(x). \tag{4.34}$$

Then instead of $P(w)$ defined by (4.17), we need to define it according to (2.5) by

$$P(w) = \frac{2}{\pi^2}\Big(\frac{1}{2} w_+^{\frac{3}{2}} B + \sum_{j=1}^{\infty} (w - 2jB)_+^{\frac{3}{2}} B\Big), \tag{4.35}$$

where B is the scalar intensity of the magnetic field. This changes both the Thomas-Fermi theory and properties of the Thomas-Fermi potential W^{TF} and Thomas-Fermi density ρ^{TF}.

Case $B \lesssim Z^{4/3}$ For $B \lesssim Z^{4/3}$, the main contributions to the (approximate) electronic charge $\int \rho^{\mathsf{TF}}\,dx$ and the energy $\mathcal{E}^{\mathsf{TF}}$ come from the zone $\{x : d(x) \asymp Z^{-1/3}\}$ ($d(x) = \min_m |x - \mathsf{y}_m|$), exactly as for $B = 0$.

Furthermore, $W^{\mathsf{TF}} \asymp Z_m d(x)^{-1}$ if $d(x) \lesssim Z^{-1/3}$ and (for $Z = N$) $W^{\mathsf{TF}} \asymp d(x)^{-4}$ if $Z^{-1/3} \le d(x) \lesssim B^{-1/4}$ but $\rho^{\mathsf{TF}} = 0$ if $d(x) \ge C_0 B^{-1/4}$ [45)].

Finally, as we using scaling to bring our problem to the standard one, we get that in the zone $\{x : d(x) \asymp Z^{-1/3}\}$, the effective semiclassical parameter is $h_{\mathsf{eff}} = Z^{-1/3}$ which leads to $\mathcal{E}^{\mathsf{TF}} \asymp Z^{7/3}$ again exactly as for $B = 0$.

As a result, assuming that $M = 1$, we can recover asymptotics for the ground state energy E with the Scott correction term but with the remainder estimate $O(Z^{5/3} + Z^{4/3}B^{1/3})$. For $M \ge 2$ and $N \ge Z$, our estimates are almost as good (provided $a = \min_{m \ne m'} |\mathsf{y}_m - \mathsf{y}_{m'}| \ge Z^{-1/3}$), but deteriorate when both $(Z - N)_+$ and B are large.

Moreover, for $B \ll Z$ assuming (4.27), we can marginally improve these results and include the Schwinger and Dirac correction terms.

The main obstacles we need to overcome are that now W^{TF} is not infinitely smooth but only belongs to the class $\mathscr{C}^{5/2}$ and that for $M \ge 2$, the nondegeneracy assumption ($|\nabla W| \asymp 1$ after rescaling) fails.

Case $B \gtrsim Z^{4/3}$ On the other hand, for $B \gtrsim Z^{4/3}$, the the main contributions to the (approximate) electronic charge and the energy $\mathcal{E}^{\mathsf{TF}}$ come from the zone $\{x : d(x) \asymp B^{-1/4}\}$ and (for $Z = N$) $W^{\mathsf{TF}} \asymp Z d(x)^{-1}$ if $d(x) \lesssim B^{-1/4}$ but $W^{\mathsf{TF}} = 0$ if $d(x) \ge C_0 B^{-1/4}$. In this case, $\mathsf{E}^{\mathsf{TF}} \asymp B^{2/5} Z^{9/5}$.

Further, as we using scaling to bring our problem to the standard one, we see that in the zone $\{x : d(x) \asymp B^{-1/4}\}$, the effective semiclassical parameter is $h_{\mathsf{eff}} = B^{1/5} Z^{-3/5}$ and therefore unless $B \ll Z^3$, the semiclassical approximation fails and the correct answer should be expressed in completely different terms [LSY1].

As a result, assuming that $M = 1$ if $Z^{4/3} \le B \le Z^3$, we can recover the asymptotics for E with the Scott correction term but with the remainder estimate $O(B^{4/5} Z^{3/5} + Z^{4/3} B^{1/3})$.

For $M \ge 2$ and $N \ge Z$, our estimates are almost as good (provided $a = \min_{m \ne m'} |\mathsf{y}_m - \mathsf{y}_{m'}| \ge B^{-1/4}$), but deteriorate when $(Z - N)_+$ is large.

Again the main obstacles we need to overcome are that now W^{TF} is not infinitely smooth but only belongs to $\mathscr{C}^{5/2}$ and that for $M \ge 2$, the nondegeneracy assumption ($|\nabla W| \asymp 1$ after rescaling) fails.

[45)] So, the radii of atoms in Thomas-Fermi theory are $\asymp \min(B^{-1/4}, (Z - N)_+^{-1/3})$.

For details, exact statement and proofs, see Sections 26.1 and 26.6 of [Ivr4]. We also estimate the left-hand expression of (4.20) and are able to obtain results similar to those mentioned in Section 4. For details and proofs, see Sections 26.7–26.8 of [Ivr4].

Adding Self-Generated Magnetic Field

Let

$$\mathsf{E}(A) = \inf \mathsf{Spec}(\mathsf{H}_{A,V}) + \underline{\alpha^{-1} \int |\nabla \times A|^2 \, dx} \tag{4.36}$$

and

$$\mathsf{E}^* = \inf_{A \in \mathscr{H}^1_0} \mathsf{E}(A), \tag{4.37}$$

where A is an unknown magnetic field and the underlined term is its energy. One can prove that an "optimal" magnetic field exists (for given parameters $Z_1, \dots, Z_M, \mathsf{y}_1, \dots, \mathsf{y}_M, N$) but we do not know if it is unique[46].

Using the same arguments as before, we can reduce this problem to the one-particle problem with $\inf \mathsf{Spec}(\mathsf{H}_{A,V})$ replaced by $\mathsf{Tr}((H_{A,W} + \nu)_-)$ plus some other terms. However, in the estimate from below, most of the terms do not depend on A and in the estimate from above, we pick up A.

Then after the usual rescalings, the problem is reduced to the problem of minimizing

$$\mathsf{Tr}((H_{A,W} + \nu)_-) + \underline{\frac{1}{\kappa h^2} \int |\nabla \times A|^2 \, dx} \tag{4.38}$$

and then the optimal magnetic potential A must satisfy

$$\begin{aligned} &\frac{2}{\kappa h^2} \Delta A_j(x) = \Phi_j := \\ &\quad - \operatorname{Re} \mathsf{tr} \Big(\sigma_j \Big((hD - A)_x \cdot \boldsymbol{\sigma} e(x,y,\tau) + e(x,y,\tau)\, {}^t(hD - A)_y \cdot \boldsymbol{\sigma} \Big) \Big|_{y=x} \Big), \end{aligned} \tag{4.39}$$

where $e(x,y,\tau)$ is the Schwartz kernel of the spectral projector $\theta(-H)$ of $H = H_{A,W}$ and tr is the matrix trace. As usual, we are mainly interested in $h = Z^{-1/3}$ (and then $\kappa = \alpha Z$).

[46] If it was unique, then for $M = 1$, the spherical symmetry would imply that $A = 0$.

First, (4.39) allows us to claim a certain smoothness of A. Second, the right-hand expression is something we studied in pointwise spectral asymptotics, and the Weyl expression here is 0, so the right-hand expression of (4.39) is something that we could estimate. Surely, it is not that simple but improving our methods in the case of smooth W, we are able to prove that A is so small that the ordinary asymptotics with remainder estimates $O(h^{-2})$ and $O(h^{-1})$ would hold in both the pointwise asymptotics and the trace asymptotics. Moreover, under standard conditions, we would be able to get the remainder estimates $o(h^{-2})$ and $o(h^{-1})$ in the eigenvalue counting and the trace asymptotics respectively.

However, in reality, the above is not exactly true since W has Coulomb-like singularities $W \sim z_m|x-\mathsf{y}_m|^{-1}$ with $z_m \asymp 1$. If $M=1$, $z_m=1$, a singularity leads us to the Scott correction term $S(\kappa)h^{-2}$ derived in the same way as without a self-generated magnetic field. However, we do not have an explicit formula for $S(\kappa)$; we even do not know its properties except that it is non-increasing function of $\kappa \in [0,\kappa^*)$; we even do not know if we can take $\kappa^*=\infty$. If the optimal magnetic potential A was unique, then $A=0$ and $S(\kappa)=S(0)$, which corresponds to this term without a magnetic field.

Then as $M \ge 2$, the Scott correction term is $\sum_{1\le m\le M} S(\kappa z_m) z_m^2 h^{-2}$ in the general case. However, as $M \ge 2$ we need to decouple singularities as all of them are served by the same A and it leads to decoupling errors depending on the internuclei distance.

For details, exact statements and proofs, see Sections 27.2–27.3 of [Ivr4].

As a result, we derive the ground state asymptotics with the Scott correction term $\sum_{1\le m\le M} S(\alpha Z_m) Z_m^2$. We also estimate the left-hand expression of (4.20) and are able to obtain results similar to those mentioned in Section 4. For details, exact statements and proofs, see Sections 27.3–27.4 of [Ivr4].

Combining External and Self-Generated Magnetic Fields

We can also combine a constant strong external magnetic field and a self-generated magnetic field. Results are very similar to those of Subsection 4, but this time, the Scott correction term and the decoupling errors are like in Subsection 4. For details, exact statements and proofs, see Chapter 28 of [Ivr4].

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Presentations

[1] *Sharp spectral asymptotics for irregular operators*

[2] *Sharp spectral asymptotics for magnetic Schrödinger operator*

[3] *25 years after*

[4] *Spectral asymptotics for 2-dimensional Schrödinger operator with strong degenerating magnetic field*

[5] *Magnetic Schrödinger operator: geometry, classical and quantum dynamics and spectral asymptotics*

[6] *Spectral asymptotics and dynamics*

[7] *Magnetic Schrödinger operator near boundary*

[8] *2D- and 3D-magnetic Schrödinger operator: short loops and pointwise spectral asymptotics*

[9] *100 years of Weyl's law*

[10] *Some open problems, related to spectral theory of PDOs*

[11] *Large atoms and molecules with magnetic field, including self-generated magnetic field (results: old, new, in progress and in perspective)*

[12] *Semiclassical theory with self-generated magnetic field*

[13] *Eigenvalue asymptotics for Dirichlet-to-Neumann operator*

[14] *Eigenvalue asymptotics for Fractional Laplacians*

[15] *Asymptotics of the ground state energy for relativistic atoms and molecules*

[16] *Etudes in spectral theory*

[17] *Eigenvalue asymptotics for Steklov's problem in the domain with edges*

[18] *Complete semiclassical spectral asymptotics for periodic and almost periodic perturbations of constant operators*

[19] *Complete Spectral Asymptotics for Periodic and Almost Periodic Perturbations of Constant Operators and Bethe-Sommerfeld Conjecture in Semiclassical Settings*

[20] *Heavy atoms and molecules: Thomas-Fermi and Scott approximations*

[21] *Pointwise spectral asymptotics out of the diagonal*

Index

A-bound, 866
A-bounded, 866
A-compact, 866
r-term semiclassical approximation, 1686

action, 538, 1682
adjoined boundary value problem, 117
adjoint operator
  pseudodifferential, 12
admissible
  box, 94
  function, 11, 94
  operator, 94
almost analytic function, 399
amplitude, 19, 69
angular momentum, 994
approximaion
  error, 287
  Tauberian, 286
approximation
  method of successive, 321
  Weyl, 287
approximation error, 376
argument
  Tauberian, 299
asymptotic Euclidean metric, 2506
asymptotics
eigenvalues tending to -0
  non-homogeneous potential, 1026
  non-periodic Hamiltonian flow, 1020
  periodic Hamiltonian flow, 1021
  power decay potential, 1017
  quickly decreasing potential, 1026
  slowly decreasing potential, 1029
eigenvalues tending to border of spectral gap
  Dirac operator, 1031
eigenvalues tending to point of essential spectrum
  magnetic Schrödinger operator, 2425
large eigenvalue
  cusp, 938
  elliptical domain, 690
  exponential singularity, 984
  indefinite operator, 968
  magnetic Schrödinger with singularities, 2410, 2463
  maximally hypoelliptic operator, 1152
  periodic trajectory, 957

polyhedron, 690
potential strongly degenerate at infinity, 1114
power growth potential, 988
power growth potential, degenerate, 1000
power growth potential, nonperiodic, 993
power growth potential, periodic, 997
power singularity, 981
quasihomogeneous potential, 1007
quasipolygonal domain, 933
quasipolyhedral domain, 933
quickly growing potential, 1009
slowly growing potential, 1013
sphere Laplace-Beltrami operator, 960
multiparameter, 1037
semiclassical
conical exit to infinity, 916
conical singularity, 906
exit to infinity, 914
quasiconical exit to infinity, 917
quasiconical singularity, 913
spectral
local, 445
microlocal, 445
two-term, lxxii
averaging, 286, 616
averaging function, 311

bicharacteristic
boundary, 236
generalized, 151
semi-, 155
bicharacteristic billiard
generalized, 228
bicharacteristic loop, 358
bicharacteristically concave, 237
bicharacteristically convex, 236
bicharacteristically flat, 770
bicharacteristics, lxxviii
generalized, lxxviii
limiting, 160
periodic, 358
billiard
generalized bicharacteristic, 228
Birkhoff normal form, 1278
Birman-Schwinger principle, 847, 869
blunt quasiconical singularity, 941
Bohr-Sommerfeld condition, 1487
Bohr-Sommerfeld quantization condition, 76
boundary
characteristic, 197
boundary bicharacteristic, 236
boundary wave front set, 200
boundary zone, 1556
Boutet de Monvel algebra, 110
box
admissible, 94
bracket, 375

canonical form
microlocal, 1278
partial reduction, 1882
symplectic, 1278
canonical graph, 80
Cauchy problem
multi-time, 143
causality, 102
characteristic

boundary, 197
point of a problem, 200
point of an operator, 200
polynomial, 201
symbol, 149
charge term, 2637
chemical potential, 2521
pseudodifferential operator, 53
classically allowed zone, 1444
cluster
spectral, 540
coercivity condition, 975
coherent states, 78
commutator
pseudodifferential operators, 13
commuting operators
spectral asymptotics, 520
compact part of a set, 1066
concave
bicharacteristically, 237
condition
no critical point, 411
non-periodicity, 950
cone condition, 836
configuration space, 63
conical exit to infinity
semiclassical asymptotics of, 916
conical singularity
semiclassical asymptotics of, 906
constant, 93
continuity modulus, 375
continuous with continuity modulus, 375
convex
bicharacteristically, 236
manifold, 687
corner zone, 1632
correction
Scott, 1155
correction term, 1666
creeping rays, 237
cusp, 1066
base, 1066
form defining function, 1066
large eigenvalue asymptotics of, 938
section, 1066
thick, 1079
cusp zone
far cusp zone, 1071

$\mathfrak{d}$ (variation), 1376
Dauge-Helffer formulae, 1641
degeneration
simple, 2087
degeneration line, 1421
degeneration zone, 1422
density
classical, 2551
smeared, 2551
spacial, 2515
Thomas-Fermi, 2521
Dirac
matrix, 514
operator, 461
Dirac correction term, 593
Dirac operator, xciv
asymptotics of eigenvalues tending to border of spectral gap, 1031
generalized, 462, 918
magnetic, xciv
massless, 461
principally massless, 1804
direct term, 2567
Dirichlet-Neumann bracketing, lxx

dissipative problem, 198
distribution
  Lagrangian, 69
domain
  bicharacteristically concave, 237
  bicharacteristically convex, 236
  elliptical, 690
  essentially unbounded, 988
  exterior, 2417
  polyhedral, 690
  quasipolygonal
    large eigenvalue asymptotics of, 933
  quasipolyhedral
    large eigenvalue asymptotics of, 933
domination, 313
drift, 541
  of singularities, 551
dual elementary cell, 1211
dual operator, 139
dynamics
  classical, 63

Egorov's theorem, 82
eigenvalue counting function, lxix
eigenvalues accumulating to the border of the essential spectrum, 1016
eigenvalues tending to $-\mathsf{0}$
  asymptotics of
    non-homogeneous potential, 1026
    non-periodic Hamiltonian flow, 1020
    periodic Hamiltonian flow, 1021
    power decay potential, 1017
    quickly decreasing potential, 1026
    slowly decreasing potential, 1029
eigenvalues tending to a point of essential spectrum
  asymptotics of
    magnetic Schrödinger operator, 2425
eigenvalues tending to border of spectral gap
  asymptotics of
    Dirac operator, 1031
eikonal, 63
  equation, 63
electrostatic inequality, 2519
elementary cell, 1211
elliptic
  point of a problem, 200
  point of an operator, 200
  zone, 235
elliptic boundary value problem, 114
elliptic component, 233
elliptic Green operator, 111
elliptic operator, 40
elliptical
  annulus, 690
  domain, 690
entanglement, 2884
error
  approximation, 491
  combined, 491
escape condition, 1173
essentially supported
  function, 95
  operator, 95
essentially unbounded domain, 988

estimate
variational
zone of, 805
Euclidean metric
asymptotic, 2506
exit to infinity
conical
semiclassical asymptotics of, 916
quasiconical
semiclassical asymptotics of, 917
semiclassical asymptotics of, 914
exponent, 93
exponential geodesic flow with reflection, 950
exponential singularity
large eigenvalue asymptotics of, 984
exterior domain, 2417

families of commuting operators
spectral asymptotics, 520
family of commuting operators
propagator, 522
spectral projector, 521
fermion, 2513
field
Hamiltonian, 87
magnetic
scalar intensity of, 1244
tensor intensity of, 1242
vector intensity of, 1244
finite speed of propagation of singularities, 135, 239
fixed nuclei model, 2514, 2602
fixed object, 93
flow
geodesic
exponential with reflection, 950
non-periodic exponential, 950
non-periodic power, 950
power with reflection, 950
Hamiltonian, 87, 179
generated by a Hamiltonian field, 87
magnetic, 1371
Fock space, 2513
forbidden zone, 1443
form
canonical
microlocal, 1278
symplectic, 1278
symplectic
skew-symmetric, 88
formerly forbidden zone, 1518
Fourier integral operator, 80
Fourier transform, 5
inverse, 5
free nuclei model, 2514, 2602
funcion
averaging, 286
function
admissible, 94
almost analytic, 399
averaging, 311
cusp form defining, 1066
essentially supported in a box, 95
Hörmander, 299
negligible, 94
in a box, 94
of a self-adjoint pseudodifferential operator, 45
phase

non-degenerate, 68
r-almost analytic, 399
roughness, 488
scaling, 411
momentum, 804
spatial, 804
tempered, 94

gap
spectral, 1350
semiclassical, 293, 579
Gårding inequality, 30
gauge transformation, 34, 446, 1241, 1801
Gelfand' transformation, 1215
generalized
bicharacteristic, 151
bicharacteristic billiard, 228
ρ-bicharacteristic, 151
generalized Dirac operator, 918
generalized metaplectic operator, 90
geodesic flow
exponential
with reflection, 950
non-periodic exponential, 950
non-periodic power, 950
power
with reflection, 950
germ of an operator, 92
glancing
zone, 235
gliding rays, 237
grazing rays, 237
Green operators, 110
ground state energy, 2514, 2601
group
orthogonal, 1809
special orthogonal, 1809

h-Fourier transform
partial, 72, 81
h-pseudodifferential operator, lxxii
Hörmander function, 299
multidimensional, 524
Hamilton-Jacobi equation, 63
Hamiltonian, 63, 179
cluster, 2576, 2730
dynamics, 63
field, 63, 87
flow, 63, 87, 179
generated by a Hamiltonian field, 87
non-periodic, 1020
periodic, 1021
intercluster, 2576, 2730
trajectory, lxxiii
Hamiltonian dynamics with the reflections, 271
Hamiltonian dynamics with the reflections and branching, 271
Hamiltonian flow
generalized
multitime, 531
heat equation method, lxx
Heisenberg representation, 177
Hessian
skew, 88
hop, 1558
length, 1558
movement, 1558
speed, 1558
hyperbolic
zone, 235
hyperbolic operator method, lxx

incoming component, 232

incoming point, 232
indefinite operator, 968
index
  Maslov, 75
indirect term, 2567
inequality
  Gårding, 30
inner core, 2640
inner zone, 1555
integrated density of states, 3119
intensity of a magnetic field
  scalar, 1244
  tensor, 1242
  vector, 1244
intermediate case, 2605
intermediate magnetic field, 1607
ionization energy, 2514, 2602
ISM identity, 2768
Ivrii's conjecture, 696

$K^{\#}(p,z)$, 147
$K^{\pm}(p,z,\Omega)$, 151
$K^{\#}_{\rho\varepsilon}(p,z)$, 147
$K^{\pm}_{\rho\varepsilon}(p,z,\Omega)$, 151
$K_{\rho}(p,z)$, 146

(l,r)-germ of an operator, 92
Lagrangian distribution, 69
  principal symbol of, 74
Lagrangian manifold, 70
  generated by a phase function, 69
Lagrangian submanifold, 65
Landau level, 1243, 1608
Landau levels, 1986
Laplace-Beltrami operator
  on the sphere, 960
large eigenvalue asymptotics
  cusp, 938
  elliptical
    annulus, 690
    domain, 690
  exponential singularity, 984
  indefinite operator, 968
  magnetic Schrödinger with singularities, 2410, 2463
  maximally hypoelliptic operator, 1152
  periodic trajectory, 957
  polyhedron, 690
  potential strongly degenerate at infinity, 1114
  power growth potential, 988
    degenerate, 1000
    non-periodic, 993
    periodic, 997
  quasihomogeneous potential, 1007
  quasipolygonal domain, 933
  quasipolyhedral domain, 933
  quickly growing potential, 1009
  slowly growing potential, 1013
  sphere Laplace-Beltrami operator, 960
large eigenvalues asymptotics
  power singularity, 981
elliptic operator, 40
parametrix, 40
left parametrix of boundary value problem, 115
Legendre transformation, 66
level
  Landau, 1243
line
  magnetic, 1371
local spectral asymptotics, 445
uncertainty principle, 49

magnetic
field
scalar intensity of, 1244
tensor intensity of, 1242
vector intensity of, 1244
flow, 1371
line, 1371
magnetic drift lines, 1363
magnetic field
combined, 2819
external, 2819
moderate, 2821
strong, 2821
intermediate, 1349, 1607
self generated, 2742
self-generated, 2819
strong, 1349
superstrong, 1420, 2071
very strong, 1420, 2071
magnetic field intensity tensors, 1984
Magnetic Schrödinger-Pauli operator, 1350
Magnetic Weyl expression, 1243
modified, 1249
manifold
convex, 687
Lagrangian, 70
generated by a phase function, 69
Maslov
canonical operator, 75
constant, 539
index, 75, 539
Maslov canonical operator, 65
matrix
Dirac, 514
Pauli, 514, 516
matrix signature, 66
maximal excessive negative charge, 2514, 2602
maximal excessive positive charge, 2515, 2602
maximally hypoelliptic operator, 1125
large eigenvalue asymptotics, 1152
mean
spectral
smooth, 341
Melin inequality, 39
metaplectic operator, 90
generalized, 90
method
approximation
successive, 286
heat equation, lxx
hyperbolic operator, lxx
of successive approximations, 321
partition-rescaling, lxxxiii
rescaling, lxxxiii
resolvent, lxx
Tauberian, lxx
variational, lxx
ζ-function, lxx
metric
asymptotic Euclidean, 2506
micro-averaging
anisotropic, 1732
isotropic, 1732
microhyperbolic
problem, 201
under restriction, 791
microhyperbolic symbol, 127
microhyperbolic under constrain
problem, 795

microhyperbolicity, 2107
under restriction, 791
microhyperbolicity cones
cones, 147
microlocal
spectral asymptotics, 445
microlocal canonical form, 1278
microlocalization, 92
anisotropic, 92
microlocalization of an operator, 1125
moderate magnetic field case, 2604
mollification
with respect to non-spectral parameters, 612
momentum scaling function, 804
multi-time Cauchy problem, 143
multidirection, 201
multiparameter asymptotics, 1037
multiplicator, 6

negligible
double symbol, 11
function, 11, 94
operator, 11, 94
negligible function, 94
negligible operator, 94
Neumann boundary condition, 1188
no critical point condition, 411
non-degeneracy condition
improved, 905
non-homogeneous potential
asymptotics of eigenvalues tending to $-\mathsf{0}$, 1026
non-periodic
exponential geodesic flow, 950
Hamiltonian flow
asymptotics of eigenvalues tending to $-\mathsf{0}$, 1020
power geodesic flow, 950
non-periodicity condition, 950
non-resonance condition, 1270
non-spectral parameter
mollification with respect to, 612
non-strictly microhyperbolic, 131
normal singularity property, 141, 245

$\mathsf{O}(d)$, 1809
observation point, 1213
operator
admissible, 94
commuting families
spectral asymptotics, 520
Dirac, 461
asymptotics of eigenvalues tending to border of spectral gap, 1031
generalized, 918
essentially supported in a box, 95
Fourier integral, 80
h-pseudodifferential, lxxii
indefinite, 968
Laplace-Beltrami
on the sphere, 960
magnetic Schrödinger
asymptotics of eigenvalues tending to point of essential spectrum, 2425
magnetic Schrödinger with singularities
asymptotics of large eigenvalues, 2410, 2463

Maslov canonical, 75
maximally hypoelliptic, 1125
large eigenvalue asymptotics, 1152
metaplectic, 90
generalized, 90
microlocalization of, 1125
negligible, 94
in a box, 94
periodic, 1210
with period lattice, 1210
pilot-model, 1728
relatively compact, 847
scalar
with periodic Hamiltonian flow, 536
Schrödinger, 446
tempered, 94
toy-model, 1655
operator localization, 2062
pseudodifferential operator, 53
ordinary differential inclusions, 154
orthogonal group, 1809
outgoing component, 232
outgoing point, 232
outward spike, 1208

parameter, 93
averaging, 286
coupling, 1239
magnetic, 1239
non-spectral
mollification with respect to, 612
roughness, 488
parameterized family of objects, 93
parametrix, 39
parametrix of boundary value problem, 116
parametrix of operator in domain with boundary, 116
wave front set, 135
partial h-Fourier transform, 72, 81
partition-rescaling method, lxxxiii
Pauli
matrix, 514
Pauli matrix, 516
period lattice, 1210
periodic
classical dynamics., 538
periodic Hamiltonian flow
asymptotics of eigenvalues tending to -0, 1021
phase function, 19
non-degenerate, 68
phase space, 63
pilot-model
generalized, 1744
point
absolutely periodic, 587
antipodal, 772
characteristic
of a problem, 200
of an operator, 200
elliptic
of a problem, 200
of an operator, 200
regular, 71
points
diffractive, 259
elliptic, 259
glancing, 259
degenerate, 259
diffractive, 259
gliding, 259

grazing, 259
gliding, 259
grazing, 259
hyperbolic, 259
subperiodic, 537
Poisson
relation, lxxiii
Poisson brackets, 13
Poisson operator, 106
polarization wave front set, 194
polyhedron, 690
polynomial
characteristic, 201
positive Lagrangian ideal, 78
potential
effective, 995
non-homogeneous
asymptotics of eigenvalues tending to -0, 1026
power decay
asymptotics of eigenvalues tending to -0, 1017
power growth
large eigenvalue asymptotics, 988
power growth (degenerate)
large eigenvalue asymptotics, 1000
power growth (non-periodic)
large eigenvalue asymptotics, 993
power growth (periodic)
large eigenvalue asymptotics, 997
quasihomogeneous
large eigenvalue asymptotics, 1007
quickly decreasing
asymptotics of eigenvalues tending to -0, 1026
quickly growing
large eigenvalue asymptotics, 1009
slowly decreasing
asymptotics of eigenvalues tending to -0, 1029
slowly growing
large eigenvalue asymptotics, 1013
smeared, 2552
strongly degenerate at infinity
large eigenvalue asymptotics, 1114
Thomas-Fermi, 2521
power decay potential
asymptotics of eigenvalues tending to -0, 1017
power geodesic flow with reflection, 950
power growth potential
degenerate
large eigenvalue asymptotics, 1000
large eigenvalue asymptotics, 988
non-periodic
large eigenvalue asymptotics, 993
periodic
large eigenvalue asymptotics, 997
power singularity
large eigenvalue asymptotics of, 981
pressure, 2516, 2603
principal symbol

of a Lagrangian distribution, 74
principle
Birman-Schwinger, 847, 869
problem
dissipative, 198
microhyperbolic, 201
microhyperbolic under constrain, 795
multi-time Cauchy, 143
strictly dissipative, 198
symmetric, 243
procedure
sandwich, 327
product of operators
pseudodifferential, 13
propagation cones
cones, 147
propagation of singularities, lxxiii
finite speed of, 135, 239
propagator, 177
property
normal singularity, 141, 245
pseudodifferential operator, 7, 15
adjoint, 12
applied to exponent, 21
classical, 53
function of, 42
Hilbert-Schmidt norm, 16
order, 53
product of, 13
Schwartz kernel, 8
self-adjoint
function of, 45
trace of, 16

quantization, 4
pq-quantization, 4, 7
qp-quantization, 4, 7
abstract, 15
symmetric quantization, 7
Weyl quantization, 7
quantum correlation function, 2552
quasiconical exit to infinity
semiclassical asymptotics of, 917
quasiconical singularity
blunt, 941
semiclassical asymptotics of, 913
sharp, 940
quasihomogeneous potential
large eigenvalue asymptotics, 1007
quasiperiodic with quasimomentum, 1211
quasiperiodic with quasimomentum and transformation matrices, 1211
quasipolygonal domain
large eigenvalue asymptotics of, 933
quasipolyhedral domain
large eigenvalue asymptotics of, 933
quickly decreasing potential
asymptotics of eigenvalues tending to -0, 1026
quickly growing potential
large eigenvalue asymptotics, 1009

r-almost analytic function, 399
rays
creeping, 237
gliding, 237
grazing, 237
reference theorem, 803

regular
set, 895
zone, 805, 895
regular point, 71
regular zone, 1422, 1439
redefined, 1515
regularity conditions
improved, 905
relation
Poisson, lxxiii
relatively compact operator, 847
representation, 6
coordinate, 6
momentum, 6
spinor, 1811
Rescaling, 29
rescaling, 142, 407, 412
method, lxxxiii
with variable scale parameters, 407
resolvent method, lxx
resonances
2-nd order, 1989
3-rd order, 1989
m-th order, 1988
(ρ,ε)-bicharacteristic
generalized, 151
elliptic operator, 40
parametrix, 40
right parametrix of boundary value problem, 116

sandwich procedure, 327
scalar
intensity of a magnetic field, 1244
operator
with periodic Hamiltonian flow, 536
scalar intensity of the magnetic field, 1303
scaling
functions, 408
parameters, 407
scaling function, 35, 411
absolute, 2653
admissible, 411
momentum, 804
relative, 2653
spatial, 804
Schrödinger operator
magnetic, xciv
microhyperbolic, 1999
$\mathfrak{N}$-microhyperbolic, 1999
Schrödinger-Pauli operator, xciv, 1986
magnetic, xciv
Schrödinger operator, 446
magnetic
asymptotics of eigenvalues tending to point of essential spectrum, 2425
magnetic with singularities
asymptotics of large eigenvalues, 2410, 2463
Scott correction term, 1155–1170
Scott correction terms, 2807
self-adjoint boubdary value problem, 118
self-adjoint pseudodifferential operator
function of, 45
semibicharacteristic, 155
semiclassical asymptotics
conical exit to infinity, 916

conical singularity, 906
exit to infinity, 914
quasiconical exit to infinity, 917
quasiconical singularity, 913
zone of, 805
semiclassical parameter, 93
semiclassical spectral gaps, 293, 579
senior symbol, 634
set
regular, 895
singular, 895
wave front, 95
boundary, 200
set-valued function, 146
continuous, 146
lower semicontinuous, 146
upper semicontinuous, 146
sharp quasiconical singularity, 940
singular
set, 895
zone, 805, 895
singular Green operator of order operator, 107
singularities
propagation of, lxxiii
singularity
conical
semiclassical asymptotics of, 906
drift of, 551
exponential
large eigenvalue asymptotics of, 984
normal, 141, 245
normal type, 404
power
large eigenvalue asymptotics of, 981
propagation of
finite speed of, 135, 239
quasiconical
blunt, 941
semiclassical asymptotics of, 913
sharp, 940
skew-Hessian, 38, 88
Slater determinant, 2522
slowly decreasing potential
asymptotics of eigenvalues tending to -0, 1029
slowly growing potential
large eigenvalue asymptotics, 1013
smooth spectral mean, 341
$\mathsf{SO}(d)$, 1809
spacial density, 2603
spatial scaling function, 804
special orthogonal group, 1809
spectral
asymptotics
commuting operators, 520
local, 445
microlocal, 445
bands, 556
gap, 1350
semiclassical, 293, 579
gaps, 539, 556
mean
smooth, 341
spectral asymptotics, lxxx
local, lxxx
microlocal, lxxx
pointwise, lxxxiii
spectral band, 1212
spectral gap, 1212
spectral zone, 1212

spin variable, 2513
spinor representation, 1811
stabilization condition, 982
standard set of assumptions
  first, 295
  second, 295
standard Weyl expression, 609
stationary phase method, 20
strictly dissipative problem, 198
strip
  boundary, 706
strong magnetic field case, 2604
strongly degenerate at infinity potential
  large eigenvalue asymptotics, 1114
successive approximation
  method of, 321
successive approximation method, 286
superstrong magnetic field, 1531
symbol
  pq-symbol, 11
  qp-symbol, 11
  characteristic, 149
  double, 7
  microhyperbolic, 127
  principal, 12
    of a Lagrangian distribution, 74
  senior, 634
  simple, 7
  Weyl symbol, 11
symmetric in the principal, 197
symmetric problem, 243
symmetric product, 91
symplectic, 37
  symplectic form, 65
symplectic canonical form, 1278
symplectic form
  skew-symmetric, 88
symplectomorphism, 63
system
  symmetric in the principal, 197

Tauberian
  arguments, 299
Tauberian approximaion, 286
Tauberian error, 286
Tauberian formula
  alternative, 311
  main, 310
  mollified, 311
Tauberian method, lxx
Tauberian technique
  simplified, 298
technique
  Tauberian, 299
tempered
  function, 94
  operator, 94
tensor intensity of a magnetic field, 1242
term
  correction, 452
  Scott correction, 1155
theorem
  Egorov's, 82
thick cusp, 1079
Thomas-Fermi
  density, 2521
  potential, 2521
Thomas-Fermi energy, 2514
  magnetic, 2602
trace operator, 107
trace term, 2640

trajectory
  Hamiltonian, lxxiii
transform
  partial $\hbar$-Fourier, 81
transformation
  gauge, 446, 1241
transformation matrices, 1210
transitional zone, 1518
transmission property, 102, 103
transport equation, 64, 178
two-term asymptotics, lxxii

uncertainty principle, lxxvii, 30
  logarithmic, lxxvii, 30, 49
  microlocal, lxxvii, 30
  standard, lxxvii

variational
  estimate
    zone of, 805
  method, lxx
vector intensity of a magnetic field, 1244

wave front set, 95
  boundary, 200
weak magnetic field, 1566
Weyl approximation, 287
Weyl error, 287
Weyl expression
  cut, 897
Weyl expression, Magnetic, 1243
quantization, 4
Weyl's law, lxix
  local, lxix
  pointwise, lxix
WF^s, 95
$\mathsf{WF}^{s\prime}$, 135

ζ-function method, lxx
Zoll-Tanner manifold, 967
zone
  boundary, 1556, 1559
  exterior, 1903
  external, 2667, 2673
  gliding, 1559
  inner, 706, 1555, 1559, 1903, 2115, 2141, 2534
  intermediate, 2130
  near-equator, 1660
  near-equator singular, 1660
  near-pole, 1660
  near-pole singular, 1660
  of semiclassical asymptotics, 805
  of variational estimates, 805
  outer, 2111
  quasiregular, 822
  regular, 805, 895, 1660, 2198
  singular, 805, 895, 2197, 2198
  standard, 2162
  transitional, 1558
zone spectrum, 1212