Perverse sheaves learning seminar: Sheaf functors and constructibility

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1 Sheaf functors and constructibility

In this note, we show the six operations of sheaves, namely (π^*, π_*), $(\pi_!, \pi^!)$, and $(\otimes^{\mathsf{L}}, \mathsf{R}\mathcal{H}om)$, map constructible complexes to constructible complexes following the exposition of [Achar], sections 3.8-3.10.

Conventions 1.1. Varieties are (quasiprojective) complex varieties, and we will use their complex topology when dealing with constructibility of sheaves and complexes. Morphisms of varieties are denoted by $\pi, \rho, \ldots$, etc. to distinguish them from functions on spaces, and we generally suppress R for right derived pushforward functors. a_X stands for the structure morphism of X. The coefficient ring for sheaves is $k = \mathbb{C}$.[1]

So far, we have seen from Balazs' talks that the functors π^*, $\otimes = \otimes^{\mathsf{L}}$, π_* for a finite morphism π, and the extension by zeros functor $k_!$ (where k is a locally closed embedding) preserve constructibility. In this section, we prove that the two pushforward functors π_* and $\pi_!$, the functor $\pi^!$, and the derived Hom functor $\mathsf{R}\mathcal{H}om$ preserve constructibility. We first recall some relevant notions and results from algebraic geometry.

Theorem 1.2 (Nagata compactification theorem)**.** Let $X \xrightarrow{\pi} Y$ be a map of varieties. Then there is a commutative diagram of varieties

$$\begin{tikzcd} X \arrow[r, hook, "\jmath"] \arrow[dr, "\pi"'] & \overline{X} \arrow[d, "\widetilde{\pi}"] \\ & Y \end{tikzcd}$$

such that $\overline{X} \xrightarrow{\widetilde{\pi}} Y$ is a proper morphism, and $X \overset{\jmath}{\hookrightarrow} \overline{X}$ is an open embedding.

Theorem 1.3 (Ehresmann fibration theorem)**.** Let $X \xrightarrow{\pi} Y$ be a smooth, surjective and proper map between smooth varieties. Then π is a (differentiably) locally trivial fibration with respect to the complex topology.

[1] This in particular simplifies some proofs. It is indeed safe to assume k is a commutative Noetherian ring (with 1) of gld $k < +\infty$, as all the results remain valid (with slight modification of proofs).

We need a generalized version of this theorem involving normal crossing stratifications on X. We first introduce the following definitions:

Definition 1.4. Let $X \xrightarrow{\pi} Y$ be a map between smooth varieties, and let $Z \overset{\imath}{\hookrightarrow} X$ be a divisor with simple normal crossings with components Z_1,...,Z_k. Denote each stratum of the associated normal crossing stratification by X_I. We say
(1) π is *transverse* to Z if for all $I \subseteq \{1, ..., k\}$, $\pi|_{X_I} : X_I \to Y$ is smooth and surjective.
(2) π is a *transverse locally trivial fibration* if for each $q \in Y$, we have an (analytic) open neighborhood V of q and a diffeomorphism ϕ fitting into a commutative diagram $\begin{tikzcd} \pi^{-1}V \arrow[rr, "\phi", "\approx_{C^\infty}"'] \arrow[dr, "\pi"'] & & V \times \pi^{-1}(q) \arrow[dl, "p_1"] \\ & V & \end{tikzcd}$, such that for all $I \subseteq \{1, ..., k\}$, the restrictions $(\pi^{-1}V) \cap X_I \xrightarrow{\phi} V \times (\pi^{-1}(q) \cap X_I)$ are also diffeomorphisms.

Remark. Note that if π is a transverse locally trivial fibration, then in particular $\pi|_{X_\varnothing} : X\backslash Z \to Y$ is a locally trivial fibration.

Theorem 1.5 (Theorem 3.1.17, see also Ehresmann fibration theorem 1.3)**.** Let $X \xrightarrow{\pi} Y$ be a map of smooth varieties, and let $Z \overset{\imath}{\hookrightarrow} X$ be a divisor with simple normal crossings. If π is a smooth, proper morphism that is *transverse* to Z, then π is a transverse locally trivial fibration.

For the proof of theorem 1.7 and for exercise 1.12, we record the following result from Balazs' talk:

Lemma 1.6 (Lemma 3.5.8)**.** Let X be a smooth variety, and let $Z \hookrightarrow X$ be a divisor with simple normal crossings. If we denote $U = X\backslash Z \overset{\jmath}{\hookrightarrow} X$ for its complement open embedding, then $\jmath_*$ sends finite type local systems to constructible complexes on X with respect to the normal crossing stratification. Moreover, for a local system $\mathscr{L}$ on U, $\operatorname{codim}_X \operatorname{Supp} H^i(\jmath_*\mathscr{L}) \geqslant i$ for all $i \in \mathbb{Z}$.

Theorem 1.7 ($\pi_*, \pi_!$ preserves constructibility, theorem 3.8.1)**.** For a map $X \xrightarrow{\pi} Y$ of varieties, the functors π_* and $\pi_!$ preserve constructibility, i.e., map $\mathsf{D}^b_c(X, k)$ to $\mathsf{D}^b_c(Y, k)$.

First, consider the following special case of the proper pushforward functor:

Lemma 1.8. Let X be a smooth irreducible affine variety, and let $\mathscr{L}$ be a local system on X of finte type. Then $\mathsf{R}\Gamma_c(\mathscr{L})$ is constructible.

Proof. This is proved by theorem 3.7.6 of [Achar] and by Poincaré duality for complex manifolds, both discussed in Balazs' talks. As X with its complex topology is a connected complex manifold, we know $H^i(X, \mathscr{L}^\vee) \underset{\sim}{\to} H^{2n-i}_c(X, \mathscr{L})^\vee$ for each $i \in \mathbb{Z}$ by Poincaré duality (where $n = \dim X$). Since X is an affine variety, the left hand side is finite dimensional and vanishes unless $0 \leqslant i \leqslant n$. Thus, $H^i_c(X, \mathscr{L})$ is finite dimensional and vanishes unless $n \leqslant i \leqslant 2n$. ☐

Proof of theorem 1.7. The proof proceeds by induction on $\dim X$. In Step 1 and Step 2, we explain how one can reduce to the case of (dominant) proper morphisms and open embeddings between irreducible varieties. In Step 3, we prove the case of proper morphisms. In Step 4, we prove the case of open embeddings, finishing the proof.

[Step 1]. Reduction to the case of constructible sheaves, X being irreducible, and π being dominant. The first two reductions were explained in Balazs' talk [Elek], proposition 3.18 and lemma 3.19. By decomposing π as $X \to \operatorname{im}\pi \overset{\mathrm{cl}}{\hookrightarrow} Y$, we know it is enough to consider the part $X \to \operatorname{im}\pi$, i.e., the case where $\pi(X)$ is dense in the target (π being dominant), since we know $\imath_* = \imath_!$ preserves constructibility. As X was irreducible, $Y = \overline{\pi(X)}$ is also irreducible.

[Step 2.] By Nagata's compactification theorem 1.2, we know it is enough to treat the case of proper morphisms and open embeddings separately for $*$-pushforwards.

[Step 3.] The case of proper morphisms. Suppose $X \xrightarrow{\pi} Y$ is proper. We proceed by induction on $\dim X$. Note that the initial case $\dim X = 0$ is trivial. We will eventually reduce to the case of lemma 1.8 and the case involving locally trivial fibrations between smooth varieties, even if the latter morphisms are not necessarily proper.

(3-1) Reduction to the case of Y being smooth.

By taking a dense and smooth open subset $V \overset{\jmath}{\hookrightarrow} Y$, we have a recollement distinguished triangle

$$\jmath_!\jmath^!\pi_*\mathscr{F} \longrightarrow \pi_*\mathscr{F} \longrightarrow \imath_*\imath^*\pi_*\mathscr{F} \xrightarrow{+1} .$$

By adopting the abbreviation $\pi_V := \pi|_{\pi^{-1}V} : \pi^{-1}V \to V$ for the base change of π, we know by proper base change the above distinguished triangle takes the following form:

$$\jmath_!(\pi_V)_*(\mathscr{F}|_{\pi^{-1}V}) \longrightarrow \pi_*\mathscr{F} \longrightarrow \imath_*(\pi_{Y\backslash V})_*(\mathscr{F}|_{\pi^{-1}(Y\backslash V)}) \xrightarrow{+1} .$$

The third object is constructible due to induction hypothesis, as $\pi^{-1}(Y\backslash V) \neq X$. Thus it is enough to prove the case of π_V, i.e., the case of morphisms with smooth target.

(3-2) Reduction to the case of $\mathscr{F} = \jmath_!\mathscr{L}$, where $\mathscr{L}$ is a local system of finite type on a smooth (nonempty) affine open subset $U \overset{\jmath}{\hookrightarrow} X$.

Take a smooth affine open subset U making $\mathscr{F}|_U =: \mathscr{L}$ a local system. The following image of the recollement distinguished triangle

$$\begin{array}{ccccc} \pi_*\jmath_!\jmath^!\mathscr{F} \longrightarrow & \pi_*\mathscr{F} & \longrightarrow & \pi_*\imath_*\imath^*\mathscr{F} & \xrightarrow{+1} \\ = \pi_*(\jmath_!\mathscr{L}) & & & = (\pi|_Z)_*(\mathscr{F}|_Z). & \end{array}$$

and the induction hypothesis applied to the proper morphism $\pi|_Z : Z \to Y$ reduces us to the case of $\mathscr{F} = \jmath_!\mathscr{L}$.

(3-3) Reduction to the case $Z = X\backslash U \overset{\imath}{\hookrightarrow} X$ is a divisor with simple normal crossings.

Apply resolution of singularities to $Z \overset{\imath}{\hookrightarrow} X$ to have a proper morphism $\widetilde{X} \xrightarrow{p} X$ from a smooth variety, such that $p^{-1}(Z) \overset{\tilde{\imath}}{\hookrightarrow} \widetilde{X}$ is a divisor with simple normal crossings and its complement $\pi^{-1}(U) \overset{\tilde{\jmath}}{\hookrightarrow} \widetilde{X}$ satisfies $p \circ \tilde{\jmath} = \jmath$ via identification $p^{-1}(U) = U$. Then $\tilde{\pi} = \pi \circ p$ is still a proper morphism with $\dim \widetilde{X} = \dim X$, and we have $\tilde{\pi}_*\tilde{\jmath}_!\mathscr{L} = \pi_*p_*\tilde{\jmath}_!\mathscr{L} = \pi_*\jmath_!\mathscr{L}$.

(3-4) The case of $\dim Y = 0$.

We have $Y = *$, so $\pi_*\jmath_!\mathscr{L} = (\pi \circ \jmath)_!\mathscr{L} = \mathsf{R}\Gamma_c(\mathscr{L})$. The right hand side is constructible by lemma 1.8.

(3-5) Reduction to the case of locally trivial fibrations when $\dim Y > 0$.

Apply generic smoothness on target to each $\pi|_{X_I} : X_I \to Y$, $I \subseteq \{1, ..., k\}$. Here, k is the number of irreducible components of Z. Then, we have a single dense open subset $V \hookrightarrow Y$ making $\pi|_{\pi^{-1}(V)\cap X_I} : \pi^{-1}(V) \cap X_I \to V$ smooth for all $I \subseteq \{1, ..., k\}$. Thus if we let $\pi' := \pi^{-1}(V) \xrightarrow{\pi_V} V$, then π' is a smooth proper map transverse to the divisor $\pi^{-1}(V) \cap Z$ with simple normal crossings in $\pi^{-1}(V)$, hence is a transverse locally trival fibration by theorem 1.5.

We check the constructibility of $\pi_*\jmath_!\mathscr{L}$. By applying the argument of (3-1) on $V \hookrightarrow Y$, we know it is enough to check the constructibility of $\pi'_*\left((\jmath_!\mathscr{L})|_{\pi^{-1}(V)}\right)$. As $(\jmath_!\mathscr{L})|_{\pi^{-1}(V)} \cong \left(\jmath_{\pi^{-1}(V)}\right)_!\left(\mathscr{L}|_{\pi^{-1}(V)\cap U}\right)$ by proper base change (where $\jmath_{\pi^{-1}(V)}$ is the base change of $\jmath$ by $\pi^{-1}(V) \hookrightarrow X$), we have $\pi'_*\left((\jmath_!\mathscr{L})|_{\pi^{-1}(V)}\right) \cong \pi'_*\left(\left(\jmath_{\pi^{-1}(V)}\right)_!\left(\mathscr{L}|_{\pi^{-1}(V)\cap U}\right)\right) \cong \left(\pi'|_{\pi^{-1}(V)\cap U}\right)_!\left(\mathscr{L}|_{\pi^{-1}(V)\cap U}\right)$. Note that we used $\pi_* = \pi_!$, and also note that $\pi'|_{\pi^{-1}(V)\cap U}$ is not necessarily proper. As $\pi'|_{\pi^{-1}(V)\cap U}$ is a locally trivial fibration, we know $\pi'_*\left((\jmath_!\mathscr{L})|_{\pi^{-1}(V)}\right) \cong \left(\pi'|_{\pi^{-1}(V)\cap U}\right)_!\left(\mathscr{L}|_{\pi^{-1}(V)\cap U}\right) \in \mathsf{D}^+_{\mathrm{loc}}(V, k)$ by theorem 2.12.3 of [Achar] explained in

Roger's talk [Bai]. For each point q of V, $(\pi')_*\left((\jmath_!\mathscr{L})|_{\pi^{-1}(V)}\right)_q \cong (\mathsf{a}_{\pi^{-1}(q)})_*\left((\jmath_!\mathscr{L})|_{\pi^{-1}(q)}\right)$, and as $\dim\pi^{-1}(q) = \dim\pi^{-1}V - \dim V = \dim X - \dim Y < \dim X$ by smoothness of π', we can apply induction hypothesis on each $\mathsf{a}_{\pi^{-1}(q)}$ to ensure $(\pi')_*\left((\jmath_!\mathscr{L})|_{\pi^{-1}(V)}\right) \in \mathsf{D}^b_c(V,k)$ indeed.

[Step 4.] The case of open embeddings. Suppose $X \overset{\jmath}{\hookrightarrow} Y$ is an open embedding. Again we proceed by induction on $\dim X$, and the initial case $\dim X = 0$ is again trivial. We first prove the case of local systems on smooth X using [Step 3], and then prove the general case.

(4-1) The case of X being smooth and $\mathscr{F} = \mathscr{L}$ being a local system of finite type on X.

For convenience, identify X with its image in Y. Apply resolution of singularities to $Y\backslash X \subseteq Y$ to obtain a proper morphism $\widetilde{Y} \xrightarrow{p} Y$ from a smooth variety identifying the open subset $p^{-1}(X) \overset{\tilde{\jmath}}{\hookrightarrow} \widetilde{Y}$ with X, hence making $\jmath_*\mathscr{L} = p_*\tilde{\jmath}_*\mathscr{L}$. Since $\tilde{\jmath}_*\mathscr{L}$ is constructible by lemma 1.6, and since p_* preserves constructibility by [Step 3], we know $\jmath_*\mathscr{L}$ is constructible.

(4-2) The general case.

Take a smooth open subset $U \overset{k}{\hookrightarrow} X$ so $\mathscr{F}|_U =: \mathscr{L}$ is a local system. Let $X\backslash U \overset{\imath}{\hookrightarrow} X$ be its complement. From the recollement distinguished triangle $\imath_!\imath^!\mathscr{F} \to \mathscr{F} \to k_*\mathscr{L}$ and (4-1), we know $\imath_!\imath^!\mathscr{F}$ is constructible. Hence $\imath^*\left(\imath_!\imath^!\mathscr{F}\right) \cong \imath^!\mathscr{F}$ is also constructible. Now, consider the image of the above recollement distinguished triangle $(\jmath|_{X\backslash U})_*(\imath^!\mathscr{F}) \to \jmath_*\mathscr{F} \to (\jmath\circ k)_*\mathscr{L} \xrightarrow{+1}$ by $\jmath_*$. By induction hypothesis, $(\jmath|_{X\backslash U})_*(\imath^!\mathscr{F})$ is constructible. By (4-1), $(\jmath\circ k)_*\mathscr{L}$ is also constructible, and we conclude $\jmath_*\mathscr{L}$ is constructible. □

Using above theorem 1.7 for the case of open embeddings, we can now prove $\pi^!$ preserves constructibility as follows.

Proposition 1.9 ($\pi^!$ preserves constructibility, cf. corollary 3.9.13)**.** Let $X \xrightarrow{\pi} Y$ be a morphism of varieties. Then $\pi^!$ sends $\mathsf{D}^b_c(Y,k)$ to $\mathsf{D}^b_c(X,k)$.

Proof. We proceed by Noetherian induction on X. Let $\mathscr{G} \in \mathsf{D}^b_c(Y,k)$.

[Step 1.] The initial case $X = *$.
π becomes a closed embedding, so let us denote $\pi = \imath : * \hookrightarrow Y$ and use $\jmath$ for its complementary open embedding. For $\mathscr{G} \in \mathsf{D}^b_c(Y,k)$, we have a recollement distinguished triangle $\imath_!\imath^!\mathscr{G} \to \mathscr{G} \to \jmath_*\jmath^*\mathscr{G} \xrightarrow{+1}$. As $\mathscr{G}$ and $\jmath_*\jmath^*\mathscr{G}$ are constructible, $\imath_!\imath^!\mathscr{G}$ is constructible, and hence $\imath^*\imath_!\imath^!\mathscr{G} = \imath^!\mathscr{G}$ is also constructible.

[Step 2.] Reduction to the case of smooth morphisms.
Take any dense open subset $U' \hookrightarrow X$ which is smooth. Applying the generic smoothness on target to $\pi|_{U'}$, we obtain a dense open subset $V \hookrightarrow Y$ such that the base change $U := \pi^{-1}(V) \cap U' \xrightarrow{\pi_U} V$ is a smooth morphism. Denote the open embedding $U \subseteq X$ by $\jmath$, and its complementary closed embedding by $\imath$. Then the recollement distinguished triangle gives

$$\begin{array}{ccccc} \imath_!\imath^!\pi^!\mathscr{G} & \longrightarrow & \pi^!\mathscr{G} & \longrightarrow & \jmath_*\jmath^*\pi^!\mathscr{G} \xrightarrow{\quad+1\quad} \\ = \imath_!(\pi|_{X\backslash U})^!\mathscr{G} & & & & = \jmath_*\pi_U^!(\mathscr{G}|_V). \end{array}$$

The first object is constructible by induction hypothesis. Hence it is enough to prove the third object is constructible, and this reduces us to the case of smooth morphisms.

[Step 3.] Case of smooth morphisms.
Recall that for a smooth morphism $X \xrightarrow{\pi} Y$ of relative dimension d, $\pi^! \cong \pi^*[2d]$. As π^* preserves constructibility, we know $\pi^!$ also preserves constructibility. □

Our next target is $\mathsf{R}\mathscr{H}om$, whose proof below requires proposition 1.9 for closed embeddings and theorem 1.7 for open embeddings.

Proposition 1.10 (R$\mathcal{H}om$ preserves constructibility)**.** For each variety X, R$\mathcal{H}om_{k_X}$ preserves constructible complexes, i.e., induces R$\mathcal{H}om$: $\mathsf{D}^b_c(X,k)^{\mathrm{op}} \times \mathsf{D}^b_c(X,k) \to \mathsf{D}^b_c(X,k)$.

Proof. As the functor R$\mathcal{H}om$ is triangulated on each of the arguments, we can apply induction on the number of nonvanishing cohomologies on each arguments to reduce to the case of both input complexes $\mathscr{F}$ and $\mathscr{G}$ being constructible sheaves. We then proceed by Noetherian induction. Take a smooth irreducible open subset U making $\mathscr{F}|_U =: \mathscr{L}$ a finite type local system. The associated recollement distinguished triangle is

$$\imath_!\imath^!\mathsf{R}\mathcal{H}om(\mathscr{F},\mathscr{G}) \longrightarrow \mathsf{R}\mathcal{H}om(\mathscr{F},\mathscr{G}) \longrightarrow \jmath_*\jmath^*\mathsf{R}\mathcal{H}om(\mathscr{F},\mathscr{G}) \xrightarrow{+1} .$$

We have $\imath^!\mathsf{R}\mathcal{H}om(\mathscr{F},\mathscr{G}) \cong \mathsf{R}\mathcal{H}om(\imath^*\mathscr{F},\imath^!\mathscr{G})$ by dual projection formula isomorphism in [Achar], proposition 2.10.10. Thus the distinguished triangle takes the following form:

$$\imath_!\mathsf{R}\mathcal{H}om(\imath^*\mathscr{F},\imath^!\mathscr{G}) \longrightarrow \mathsf{R}\mathcal{H}om(\mathscr{F},\mathscr{G}) \longrightarrow \jmath_*\mathsf{R}\mathcal{H}om(\mathscr{L},\mathscr{G}|_U) \xrightarrow{+1} .$$

By induction hypothesis and the fact that both $\imath_!$ and $\imath^!$ preserves constructibility, we know the first object is constructible. As $\mathsf{R}\mathcal{H}om(\mathscr{L},\mathscr{G}|_U) \cong \mathscr{L}^\vee \otimes \mathscr{G}|_U$ is constructible, the third object is also constructible by theorem 1.7. Hence $\mathsf{R}\mathcal{H}om(\mathscr{F},\mathscr{G})$ is constructible. □

We finish this section with two cohomology vanishing results for constructible sheaves. Roughly, they say the vanishing pattern is the same as vanishing of Betti cohomologies of complex manifolds.

Exercise 1.11 (Theorem 3.8.4, compact support case)**.** Let X be a variety of dimension n, and let $\mathscr{F}$ be a constructible sheaf on X. Show the following statement: $H^i_c(X,\mathscr{F})$ is finite dimensional over k for all $i \in \mathbb{Z}$, and $H^i_c(X,\mathscr{F}) = 0$ unless $0 \leqslant i \leqslant 2n$.

Here is an outline of the proof. As $(\mathsf{a}_X)_!\mathscr{F}$ is constructible, the finite dimensionality statement is already proved with 1.7. We prove the vanishing claim by Noetherian induction.
(1) Case of $X = *$ is immediate, and the dimension 0 case follows.
(2) For $n > 0$, take a smooth n-dimensional open subset $U \overset{\jmath}{\hookrightarrow} X$ with complement Z, and use an appropriate recollement triangle to form a distinguished triangle $\mathsf{R}\Gamma_c(\mathscr{F}|_U) \to \mathsf{R}\Gamma_c(\mathscr{F}) \to \mathsf{R}\Gamma_c(\mathscr{F}|_Z) \xrightarrow{+1}$. Show that it is enough to verify the statement [$H^i_c(U,\mathscr{F}|_U) = 0$ unless $0 \leqslant i \leqslant 2n$].
(3) Prove the vanishing statement for X smooth and $\mathscr{F} = \mathscr{L}$ being a (finite type) local system. Note that X is a complex manifold in complex topology, so one may use de Rham resolution.

Exercise 1.12 (Theorem 3.8.5)**.** Let X be a variety of dimension n, and let $\mathscr{F}$ be a constructible sheaf on X. Show that $H^i(X,\mathscr{F})$ is finite dimensional over k for all $i \in \mathbb{Z}$, and $H^i(X,\mathscr{F}) = 0$ unless $0 \leqslant i \leqslant 2n$.
(1) One has to show $H^i(X,\mathscr{F}) = 0$ for $i > 2n$. Check the vanishing for the case of X smooth and $\mathscr{F} = \mathscr{L}$ being a local system. Again, one may use de Rham resolution.
(2) General case. Proceed by induction on $\dim X$, and on each fixed dimension, use Noetherian induction. Observe that the initial case $X = *$ is trivial. Otherwise, take a smooth irreducible open $U \overset{\jmath}{\hookrightarrow} X$ making $\mathscr{F}|_U =: \mathscr{L}$ a local system. Use recollement triangle to check that it is enough to show [$H^i(X,\jmath_!\mathscr{L}) = 0$ for $i > 2n$].
(3) Reductions: show that we may assume $X = \overline{U}$, and that we may assume X is smooth and $X \backslash U \overset{\imath}{\hookrightarrow} X$ is a divisor with simple normal crossings.
(4) Use recollement triangle $\jmath_!\mathscr{L} \to \jmath_*\mathscr{L} \to \imath_*\imath^*\jmath_*\mathscr{L} \xrightarrow{+1}$ and lemma 1.6 to have $\operatorname{codim}_X \operatorname{Supp} H^i(\imath_*\imath^*\jmath_*\mathscr{L}) \geqslant i$ for $i \geqslant 1$. Hence, $\operatorname{codim}_Z \operatorname{Supp} H^i(\imath^*\jmath_*\mathscr{L}[1]) \geqslant i$ for all $i \in \mathbb{Z}$.
(5) Show the following statement:
Suppose the statement on the vanishing is true for all closed subvarieties of X, and suppose we are given

a complex $\mathscr{G} \in \mathsf{D}^b_c(X, k)$ satisfying [$\operatorname{codim}_X \operatorname{Supp} H^i(\mathscr{G}) \geqslant i$ for $i \leqslant n$, and $H^i(\mathscr{G}) = 0$ for $i > 2n$]. Then, $H^i(X, \mathscr{G}) = 0$ for $i > 2n$.
(6) Rotate the standard recollement triangle to have a distinguished triangle $\mathsf{R}\Gamma(\imath^* \jmath_* \mathscr{L}[-1]) \to \mathsf{R}\Gamma(\jmath_! \mathscr{L}) \to \mathsf{R}\Gamma(\mathscr{L})$. Use this triangle, the smooth local system case (2), the conclusion of (4), and step (5) to deduce the vanishing for $H^i(X, \jmath_! \mathscr{L})$.

2 Verdier duality

Definition 2.1. Let X be a variety.
(1) The dualizing complex of X is the complex $\omega_X := \mathsf{a}^!_X k$.
(2) The *Verdier duality functor* $\mathbb{D} = \mathbb{D}_X : \mathsf{D}^b_c(X, k)^{\mathrm{op}} \to \mathsf{D}^b_c(X, k)$ is the functor given by $\mathbb{D} := \mathsf{R}\mathscr{H}om(\cdot, \omega_X)$.

By proposition 1.9, we know $\omega_X \in \mathsf{D}^b_c(X, k)$, and hence by 1.10, $\mathsf{R}\mathscr{H}om(\mathscr{F}, \omega_X)$ is constructible for any $\mathscr{F} \in \mathsf{D}^b_c(X, k)$.

Theorem 2.2. The Verdier duality functor $\mathbb{D} : \mathsf{D}^b_c(X, k)^{\mathrm{op}} \to \mathsf{D}^b_c(X, k)$ satisfies the following properties:
(1) For $X = \operatorname{Spec} \mathbb{C} = *$, $\mathbb{D} = \mathsf{R}\mathrm{Hom}(\cdot, k)$.
(2) $\mathbb{D}^2 \cong id_{\mathsf{D}^b_c(X,k)}$.
(3) $\mathbb{D}$ commutes with pushforwards and pullbacks intertwining $*$ and !, i.e, we have isomorphisms

$$\pi_* \circ \mathbb{D} \cong \mathbb{D} \circ \pi_!, \quad \pi^! \circ \mathbb{D} \cong \mathbb{D} \circ \pi^*, \text{ and}$$
$$\pi_! \circ \mathbb{D} \cong \mathbb{D} \circ \pi_*, \quad \pi^* \circ \mathbb{D} \cong \mathbb{D} \circ \pi^!.$$

(4) There are functorial isomorphisms $\mathsf{R}\mathscr{H}om(\mathscr{F}, \mathscr{G}) \cong \mathbb{D}(\mathscr{F} \otimes^{\mathsf{L}} \mathbb{D}\mathscr{G}) \cong \mathsf{R}\mathscr{H}om(\mathbb{D}\mathscr{G}, \mathbb{D}\mathscr{F})$ for $\mathscr{F}, \mathscr{G} \in \mathsf{D}^b_c(X, k)$.

This is our main theorem of this section. Note that the property (1) is obvious, and the property (2) generalizes the fact that the double dual of a finite dimensional vector space is canonically isomorphic to the original vector space. The following exercise shows the above properties characterize $\mathbb{D}$:

Exercise 2.3. Let X be a variety, and let $\mathbb{D}$ be a functor satisfying the properties listed in theorem 2.2. Show the followings:
(1) By using the properties (1), (2), and (3), show that we have an isomorphism $\mathsf{R}\Gamma(\mathscr{F}) \underset{\sim}{\to} \mathsf{R}\mathrm{Hom}(\mathsf{R}\Gamma_c(\mathbb{D}\mathscr{F}), k)$ functorial in $\mathscr{F} \in \mathsf{D}^b_c(X, k)$.
(2) Show that for each open subset $U \hookrightarrow X$, we have an isomorphism $\mathsf{R}\Gamma((\mathbb{D}\mathscr{F})|_U) \underset{\sim}{\to} \mathsf{R}\Gamma(\mathsf{R}\mathscr{H}om(\mathscr{F}, \mathsf{a}^!_X k)|_U)$ functorial in $\mathscr{F}$. Use the isomorphism from (1) and the property (2) and (3) (for open embeddings).
(3) Conclude that if there was a morphism $\mathbb{D}\mathscr{F} \to \mathsf{R}\mathscr{H}om(\mathscr{F}, \mathsf{a}^!_X k)$ functorial in $\mathscr{F}$, then it must be an isomorphism in $\mathsf{D}^b_c(X, k)$.

Proof of Theorem 2.2. We show the properties (2), (3) and (4).

[Step 1.] The evaluation morphism $id \to \mathbb{D}^2$.
We explain the evaluation morphism $id_{\mathsf{D}^b_c(X,k)} \to \mathbb{D}^2$, which will be the isomorphism in property (2). First, note that we have a canonical morphism of complexes $\mathscr{G} \xrightarrow{\mathrm{ev}} \mathsf{ch}\mathscr{H}om(\mathsf{ch}\mathscr{H}om(\mathscr{G}, \mathscr{Q}), \mathscr{Q})$ of sheaves functorial in $\mathscr{G}$ and $\mathscr{Q}$ in $\mathsf{Comp}(\mathrm{Sh}(X, k))$. This is a complex version of the familiar evaluation map $\mathscr{G} \to \mathscr{H}om(\mathscr{H}om(\mathscr{G}, \mathscr{Q}), \mathscr{Q}) = s \mapsto (\phi \mapsto \phi(s))$ for sheaves.

Exercise 2.4. Figure out the sign rule for the complex version of the evaluation map.
Answer: for each $k \in \mathbb{Z}$ and a local section $s \in \mathscr{G}^k(U)$, the i-th component of $\mathrm{ev}^k(s)$ sends each local section ϕ of $\mathscr{H}om^i(\mathscr{G}|_U, \mathscr{Q}|_U)$ to $(-1)^{ik}\phi^k(s)$. This choice of signs respects differentials of each complexes.

This functorial map of complexes induces a morphism $\mathscr{F} \xrightarrow{\mathrm{ev}} \mathsf{R}\mathcal{H}om(\mathsf{R}\mathcal{H}om(\mathscr{F}, \omega_X), \omega_X)$ in $\mathsf{D}^b_c(X, k)$ functorial in $\mathscr{F}$, by taking an injective resolution $\omega_X \underset{\sim}{\xrightarrow{\mathrm{qis}}} \mathscr{Q}$ and using the evaluation map for complexes $\mathscr{F}$ and $\mathscr{Q}$.

[Step 2.] Property (2), case of smooth X and $\mathscr{F} \in \mathsf{D}^b_{\mathrm{locf}}(X, k)$.
(2-1) Reduction to the case of local systems.
By induction on the number of nonvanishing cohomologies, we may assume $\mathscr{F} = \mathscr{L}$ is a local system of finite type on X.
(2-2) Proof of the case of local systems.
Note that by smoothness of X, $\mathbb{D}(\mathscr{L}) = \mathsf{R}\mathcal{H}om(\mathscr{L}, \omega_X) \cong \mathscr{L}^\vee[2n] \in \mathsf{D}^b_{\mathrm{locf}}(X, k)$, where $\mathscr{L}^\vee = \mathcal{H}om(\mathscr{L}, k_X)$ is the dual local system. Hence $\mathbb{D}^2(\mathscr{L}) \cong (\mathscr{L}^\vee)^\vee$, and we only have to verify the evaluation map $\mathscr{L} \xrightarrow{\mathrm{ev}} (\mathscr{L}^\vee)^\vee$ is an isomorphism. As contractible open subsets form a base of X, it is enough to prove that its restriction to a contractible open subset U such that $\mathscr{L}|_U \cong M_U$ is an isomorphism. The restriction takes the form $M_U \xrightarrow{\mathrm{ev}} \mathcal{H}om(\mathcal{H}om(M_U, k_U), k_U)$, and as U is contractible, we know it is enough to verify the map $M \xrightarrow{\mathrm{ev}} \mathrm{Hom}(\mathrm{Hom}(M, k), k)$ is an isomorphism. As M is a finite dimensional k-vector space, we know it is true.

[Step 3.] Property (3), easy cases.
(3-1) $\pi_* \circ \mathbb{D} \cong \mathbb{D} \circ \pi_!$.
Compute $\mathbb{D}\pi_!\mathscr{F} = \mathsf{R}\mathcal{H}om(\pi_!\mathscr{F}, \omega_Y) \underset{\sim}{\xrightarrow{\text{adj.}}} \pi_*\mathsf{R}\mathcal{H}om(\mathscr{F}, \pi^!\omega_Y) = \pi_*\mathsf{R}\mathcal{H}om(\mathscr{F}, \omega_X) = \pi_*\mathbb{D}\mathscr{F}$.
(3-2) $\pi^! \circ \mathbb{D} \cong \mathbb{D} \circ \pi^*$.
Use the dual projection formula isomorphism $\pi^!\mathsf{R}\mathcal{H}om(\mathscr{G}, \mathscr{H}) \cong \mathsf{R}\mathcal{H}om(\pi^*\mathscr{G}, \pi^!\mathscr{H})$ with $\mathscr{H} = \omega_Y$.

[Step 4.] Proof of the isomorphism $\mathbb{D}(\jmath_*\mathscr{F}) \cong \jmath_!\mathbb{D}(\mathscr{F})$, where $U \overset{\jmath}{\hookrightarrow} X$ is an open embedding of a smooth irreducible open subset into a variety X and $\mathscr{F} \in \mathsf{D}^b_{\mathrm{locf}}(U, k)$.
(4-1) We first state the following vanishing property:
Let $(\mathbb{C}^\times)^n \overset{\jmath}{\hookrightarrow} \mathbb{C}^n$ be the canonical open embedding. Then for $\mathscr{G} \in \mathsf{D}^+_{\mathrm{loc}}((\mathbb{C}^\times)^n, k)$, $\mathsf{R}\Gamma_c(\jmath_*\mathscr{G}) = 0$.
We refer to lemma 3.5.5 and lemma 3.5.7 of [Achar] for the proof.
(4-2) Reduction to the case of $X = \overline{U}$ and X being irreducible.
By factoring
$$\begin{tikzcd} U \arrow[r, hook, "\bar{\jmath}"] \arrow[rr, bend right, "\jmath"'] & \overline{U} \arrow[r, hook, "\imath"] & X \end{tikzcd}$$
, we are reduced to the case of $\jmath = \bar{\jmath}$ being dominant, as $\imath_* = \imath_!$ commutes with $\mathbb{D}$ by [Step 3]. Thus, we may also assume X is irreducible.
(4-3) Reduction to the case of $Z := X \backslash U$ being a divisor with simple normal crossings inside a smooth X.
By applying resolution of singularities to $Z = X \backslash U \hookrightarrow X$, we have a proper morphism $\widetilde{X} \xrightarrow{p} X$ from a smooth variety, so $p^{-1}(Z) \hookrightarrow \widetilde{X}$ is a simple normal crossing divisor with complementary open subset $p^{-1}(U) \overset{\tilde{\jmath}}{\hookrightarrow} \widetilde{X}$ identified with U, hence $\jmath_*\mathscr{F} = p_*\tilde{\jmath}_*\mathscr{F}$. $\mathbb{D}$ commutes with $p_! = p_*$ by [Step 3], so we are reduced to the case of $\tilde{\jmath}$.
(4-4) Reduction to the computation $(\mathbb{D}(\jmath_*\mathscr{F}))|_Z = 0$.
Consider the recollement distinguished triangle

$$\jmath_!\jmath^!\mathbb{D}(\jmath_*\mathscr{F}) \longrightarrow \mathbb{D}(\jmath_*\mathscr{F}) \longrightarrow \imath_*\imath^*\mathbb{D}(\jmath_*\mathscr{F}) \xrightarrow{+1} .$$

$\jmath^!\mathbb{D}(\jmath_*\mathscr{F}) \cong \mathbb{D}(\jmath^*\jmath_*\mathscr{F}) = \mathbb{D}(\mathscr{F})$ by [Step 3], so it remains to check $(\mathbb{D}(\jmath_*\mathscr{F}))|_Z = 0$.
(4-5) Computation of stalks using normal crossing stratifications.
We check $\mathbb{D}(\jmath_*\mathscr{F})_p = 0$ for each $p \in Z$. We have

$$H^i(\mathbb{D}(\jmath_*\mathscr{F}))_p \cong \mathrm{colim}_{p \in V}\, H^i(V, \mathsf{R}\mathcal{H}om(\jmath_*\mathscr{F}|_V, \omega_V)) \cong \mathrm{colim}_{p \in V}\, H^i\mathsf{RHom}((\jmath_V)_*(\mathscr{F}|_{V \cap U}), k_V(n)[2n]),$$

and as V is a normal crossing polydisk chart when sufficiently small, it is enough to compute the vanishing $\mathsf{RHom}((\jmath_V)_*(\mathscr{F}|_{V\cap U}), k_V) = 0$ for such V. If $p \in X_J \cap Z$ for some $\varnothing \neq J \subseteq \{1, ..., k\}$ with $j = |J|$, then we have the following identifications of topological spaces by the left commutative square:

$$\begin{tikzcd}
V \cap U \arrow[r, "\sim"'] \arrow[d, hook, "\jmath_V"'] & (\mathbb{C}^\times)^j \times \mathbb{C}^{n-j} \arrow[d, hook, "u \times id"] & (\mathbb{C}^\times)^j \arrow[l, hook'] \arrow[d, hook, "u"] \\
V \arrow[r, "\sim"'] & \mathbb{C}^n = \mathbb{C}^j \times \mathbb{C}^{n-j} & \mathbb{C}^j \arrow[l, hook']
\end{tikzcd}$$

As two horizontal arrows of the right commutative square are homotopy equivalences, we are reduced to the verification of $\mathsf{RHom}(u_*\mathscr{G}, k_{\mathbb{C}^j}) = \mathsf{RHom}(u_*\mathscr{G}, \omega_{\mathbb{C}^j}[-2j]) = 0$, where $(\mathbb{C}^\times)^j \overset{u}{\hookrightarrow} \mathbb{C}^j$ is the inclusion and $\mathscr{G} \in \mathsf{D}^b_{\text{locf}}((\mathbb{C}^\times)^j, k)$. By adjunction $\mathsf{RHom}(u_*\mathscr{G}, \omega_{\mathbb{C}^j}) \cong \mathsf{RHom}(\mathsf{R}\Gamma_c(u_*\mathscr{G}), k)$, so by (4-1), we are done.

[Step 5.] Proof of $id \underset{\sim}{\overset{\text{ev}}{\to}} \mathbb{D}^2$, general case.

We use recollement distinguished triangles, [Step 4], and the unicity of triangles lemma in (5-1) below to prove property (2).

(5-1) We first state the following lemma:

Lemma 2.5 (Unicity of triangles, lemma 1.1.12)**.** Suppose we are given the following two distinguished triangles and a morphism b in a triangulated category D:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, dotted, "a"'] & B \arrow[r, "v"] \arrow[d, "b"] & C \arrow[r, "+1"] \arrow[d, dotted, "c"] & {} \\
A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \arrow[r, "+1"'] & {}
\end{tikzcd}.$$

If $v'bu = 0$, then there exists a map $A \overset{a}{\to} A'$, and hence exists a map $C \overset{c}{\to} C'$, making the diagram commutative. If moreover $\mathrm{Hom}(A, C'[-1]) = 0$, then a and c are unique.

Exercise 2.6. (1) Prove lemma 2.5. Hint: use the fact that $\mathrm{Hom}(A, \cdot)$ and $\mathrm{Hom}(\cdot, C')$ are cohomological functors.

(2) Let X be a variety and let $U \overset{\jmath}{\hookrightarrow} X$ be an open embedding with complement $Z \overset{\imath}{\hookrightarrow} X$. Using the unicity of triangles lemma 2.5, show that if $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \xrightarrow{+1}$ is a distinguished triangle in $\mathsf{D}^b_c(X, k)$ such that $\mathscr{F}'|_Z = 0$ and $\mathscr{F}''|_U = 0$, then it is isomorphic to the recollement distinguished triangle $\jmath_!\jmath^!\mathscr{F} \to \mathscr{F} \to \imath_*\imath^*\mathscr{F} \xrightarrow{+1}$.

(5-2) We proceed by Noetherian induction on X. The initial case $X = *$ was treated in [Step 2]. Suppose $U \overset{\jmath}{\hookrightarrow} X$ is a smooth irreducible open subset such that $\mathscr{F}|_U \in \mathsf{D}^b_{\text{locf}}(U, k)$, and set $Z := X \backslash U \overset{\imath}{\hookrightarrow} X$. Consider the following diagram:

$$\begin{tikzcd}
\jmath_!\jmath^!\mathscr{F} \arrow[r] \arrow[d, "a"] \arrow[dd, bend right=60, "f"', "\cong"] & \mathscr{F} \arrow[r] \arrow[d, "b", "\cong?"'] & \imath_*\imath^*\mathscr{F} \arrow[r, "+1"] \arrow[d, "c"', "\cong"] \arrow[dd, bend right=60, "g"'] & {} \\
\mathbb{D}^2\jmath_!\jmath^!\mathscr{F} \arrow[r] \arrow[d, dotted, "d"] & \mathbb{D}^2\mathscr{F} \arrow[r] \arrow[d, equal] & \mathbb{D}^2\imath_*\imath^*\mathscr{F} \arrow[r, "+1"] \arrow[d, dotted, "e"] & {} \\
\jmath_!\jmath^!\mathbb{D}^2\mathscr{F} \arrow[r] & \mathbb{D}^2\mathscr{F} \arrow[r] & \imath_*\imath^*\mathbb{D}^2\mathscr{F} \arrow[r, "+1"] & {}
\end{tikzcd}.$$

Rows of the diagram are induced from recollement distinguished triangles. Morphisms a, b, and c are evaluation maps, and f and g are induced from evaluation maps. The undotted maps form commutative diagrams by construction. By induction hypothesis and [Step 3], $\imath_*(\mathscr{F}|_Z) \underset{\sim}{\to} \imath_*(\mathbb{D}^2(\mathscr{F}|_Z)) \cong \mathbb{D}^2\left(\imath_*(\mathscr{F}|_Z)\right)$, so the arrow c is an isomorphism. Similarly by [Step 2] and [Step 4], $\mathscr{F}|_U \underset{\sim}{\to} \mathbb{D}^2(\mathscr{F}|_U) \cong (\mathbb{D}^2\mathscr{F})|_U$, so the

arrow f is an isomorphism. Our goal is to prove b is an isomorphism.
(5-3) Application of (5-1) to the diagram of (5-2).

(i) Consider the second and the third rows of the diagram. We have $\mathrm{Hom}(\mathbb{D}^2\jmath_!\jmath^!\mathscr{F}, \imath_*\imath^*\mathbb{D}^2\mathscr{F}[k]) = 0$ for all $k \in \mathbb{Z}$, hence in particular for $k = 0, -1$. Indeed, $\mathbb{D}^2\jmath_!(\mathscr{F}|_U) \cong \mathbb{D}\jmath_*\mathbb{D}(\mathscr{F}|_U) \cong \jmath_!\mathbb{D}^2(\mathscr{F}|_U)$ by [Step 4], so by adjunction and $\jmath^!\imath_* = \jmath^*\imath_! = 0$, we have $\mathrm{Hom}(\mathbb{D}^2\jmath_!\jmath^!\mathscr{F}, \imath_*\imath^*\mathbb{D}^2\mathscr{F}[k]) \cong \mathrm{Hom}(\mathbb{D}^2(\mathscr{F}|_U), \jmath^!\imath_*\imath^*\mathbb{D}^2\mathscr{F}[k]) = 0$. By lemma 2.5, morphisms d and e uniquely exist, and make the lower squares of the diagram commutative.

(ii) By exercise 2.6 (2), d and e are isomorphisms. Indeed by [Step 3] and [Step 4], $(\mathbb{D}^2\jmath_!\jmath^!\mathscr{F})|_Z \cong \imath^*\jmath_!\mathbb{D}^2(\mathscr{F}|_U) = 0$ and $(\mathbb{D}^2\imath_*\imath^*\mathscr{F})|_U \cong \mathbb{D}^2\jmath^*\imath_*\imath^*\mathscr{F} = 0$.

(iii) Consider the first and the third rows of the diagram. Arguing as (i), we know $\mathrm{Hom}(\jmath_!\jmath^!\mathscr{F}, \imath_*\imath^*\mathbb{D}^2\mathscr{F}[k]) = 0$ for $k = 0, -1$. Hence by lemma 2.5, the diagram of (5-2) including the dotted arrows is commutative. In particular, $d \circ a = f$. As both f and d are isomorphisms, a is an isomorphism. This implies b is an isomorphism.

[Step 6.] Proof of the remaining isomorphisms in property (3) and property (4).
As we already have established property (2), we can conjugate $\mathbb{D}$ on both sides of the isomorphisms from [Step 2] to obtain the remaining isomorphisms in property (3). For example, $\mathbb{D}\pi^! \cong \mathbb{D}(\pi^!\mathbb{D})\mathbb{D} \underset{\sim}{\to} \mathbb{D}(\mathbb{D}\pi^*)\mathbb{D} \cong \pi^*\mathbb{D}$.

By using property (2), definition of $\mathbb{D}$, and the derived tensor-Hom adjunction, we can compute $\mathsf{R}\mathscr{H}om(\mathscr{F}, \mathscr{G}) \cong \mathsf{R}\mathscr{H}om(\mathscr{F}, \mathbb{D}^2\mathscr{G}) = \mathsf{R}\mathscr{H}om(\mathscr{F}, \mathsf{R}\mathscr{H}om(\mathbb{D}\mathscr{G}, \omega_X)) \cong \mathsf{R}\mathscr{H}om(\mathbb{D}\mathscr{G} \otimes \mathscr{F}, \omega_X) \cong \mathbb{D}(\mathscr{F} \otimes \mathbb{D}\mathscr{G})$.

Exercise 2.7. Finish [Step 6], by establishing the remaining isomorphism $\mathbb{D}(\mathscr{F} \otimes \mathbb{D}\mathscr{G}) \cong \mathsf{R}\mathscr{H}om(\mathbb{D}\mathscr{G}, \mathbb{D}\mathscr{F})$.

□

3 Compatibility of $\boxtimes$ with six operations and $\mathbb{D}$

The moto is: $\boxtimes = \boxtimes^{\mathsf{L}}$ commutes with the six operations and $\mathbb{D}$ on constructible derived categories in reasonable ("expectable") ways. We record the natural isomorphisms here without proof.

Proposition 3.1 (Proposition 3.10.1)**.** For maps of varieties $X \xrightarrow{\pi} X'$, $Y \xrightarrow{\rho} Y'$ and $\mathscr{F} \in \mathsf{D}^b_c(X, k)$, $\mathscr{G} \in \mathsf{D}^b_c(Y, k)$, we have natural isomorphisms

$\pi_!\mathscr{F} \boxtimes \rho_!\mathscr{G} \underset{\sim}{\to} (\pi \times \rho)_!(\mathscr{F} \boxtimes \mathscr{G})$, and

$\pi_*\mathscr{F} \boxtimes \rho_*\mathscr{G} \underset{\sim}{\to} (\pi \times \rho)_*(\mathscr{F} \boxtimes \mathscr{G})$.

Proposition 3.2. For maps of varieties $X \xrightarrow{\pi} X'$, $Y \xrightarrow{\rho} Y'$ and $\mathscr{F} \in \mathsf{D}^b_c(X', k)$, $\mathscr{G} \in \mathsf{D}^b_c(Y', k)$, we have natural isomorphisms

$\pi^*\mathscr{F} \boxtimes \rho^*\mathscr{G} \underset{\sim}{\to} (\pi \times \rho)^*(\mathscr{F} \boxtimes \mathscr{G})$, and

$\pi^!\mathscr{F} \boxtimes \rho^!\mathscr{G} \underset{\sim}{\to} (\pi \times \rho)^!(\mathscr{F} \boxtimes \mathscr{G})$.

Remark. The first compatibility of $\pi_!$ with $\boxtimes$ in proposition 3.1 is often called the *Kunneth formula*, and it is valid already for the complexes in D^+. Likewise, the first compatibility of π^* with $\boxtimes$ in proposition 3.2 is also true for the complexes in D^+.

Example 3.3. Taking a_X and a_Y as π and ρ in the second isomorphism of proposition 3.2, we in particular have $\omega_X \boxtimes \omega_Y \cong \omega_{X \times Y}$.

Proposition 3.4 (Proposition 3.10.3 and corollary 3.10.5)**.** For varieties X and Y, denote $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$ for the projections. Then for $\mathscr{F}, \mathscr{F}' \in \mathsf{D}^b_c(X, k)$ and $\mathscr{G} \in \mathsf{D}^b_c(Y, k)$, we have a natural isomorphism

$\mathsf{R}\mathscr{H}om(\mathscr{F}, \mathscr{F}') \boxtimes \mathscr{G} \cong \mathsf{R}\mathscr{H}om(p^*\mathscr{F}, \mathscr{F}' \boxtimes \mathscr{G})$.

Exercise 3.5. Let $X \xrightarrow{\pi} X'$ and $Y \xrightarrow{\rho} Y'$ be maps of varieties. Using proposition 3.4, show that we have the following natural isomorphisms:

(1) $\mathsf{R}\mathcal{H}om(\mathscr{F}, \mathscr{F}') \boxtimes \mathsf{R}\mathcal{H}om(\mathscr{G}, \mathscr{G}') \cong \mathsf{R}\mathcal{H}om(\mathscr{F} \boxtimes \mathscr{G}, \mathscr{F}' \boxtimes \mathscr{G}')$, $\quad \mathscr{F}, \mathscr{F}' \in \mathsf{D}^b_c(X,k), \mathscr{G}, \mathscr{G}' \in \mathsf{D}^b_c(Y,k)$.

(2) $(\mathbb{D}\mathscr{F}) \boxtimes (\mathbb{D}\mathscr{G}) \cong \mathbb{D}(\mathscr{F} \boxtimes \mathscr{G})$, $\quad \mathscr{F} \in \mathsf{D}^b_c(X,k), \mathscr{G} \in \mathsf{D}^b_c(Y,k)$.

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