Perverse sheaves learning seminar: Perverse sheaves and intersection homology

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In these notes, we will introduce the notion of t-structures on a triangulated cateogry. The initial motivation behind this definition has to do with the derived category $D^b(\mathscr{A})$ of an abelian category $\mathscr{A}$. What structure do we need to impose on $D^b(\mathscr{A})$ to recover $\mathscr{A}$ inside $D^b(\mathcal{A})$? This is done via introducing the notation of a *t-structure*, which is an additional structure on a triangulated category. The category $D^b(\mathcal{A})$ has a tautological t-structure that gives rise to $\mathcal{A}$.

Let X be an algebraic variety. We introduce the *perverse t-structure* on $D^b(X)$, which are used to define the abelian subcategory of perverse sheaves in the derived category. Perverse sheaves are very closely related to intersection cohomology - which is a cohomology theory defined for singular spaces that has a Poincaré duality. We will give a both a topological definition and a sheaf theoretic definition of intersection homology. Intersection cohomology and perverse sheaves have applications in representation theory. For example, these play a cruicial rolein the proof of Kazhdan-Lustig conjecture on the characters of simple modules in the BGG category $\mathcal{O}$ and in the Geometric Satake correspondence.

1 t-structures and Truncation

Definition 1.1. Let $\mathscr{T}$ be a triangulated category (e.g. $D^b(\mathcal{A})$ for an abelian category $\mathcal{A}$). and let $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ be a pair of strictly full subcategories (subcategories that are full and closed under isomorphism). For $n \in \mathbb{Z}$, let

$$\mathscr{T}^{\leq n} = \mathscr{T}^{\leq 0}[-n], \qquad \mathscr{T}^{\geq n} = \mathscr{T}^{\geq 0}[-n]$$

Then $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ is called a *t-structure* on $\mathscr{T}$ if the following holds:

1. $\mathscr{T}^{\leq -1} \subset \mathscr{T}^{\leq 0}$ and $\mathscr{T}^{\geq -1} \supset \mathscr{T}^{\geq 0}$.
2. If $X \in \mathscr{T}^{\leq -1}$ and $Y \in \mathscr{T}^{\geq 0}$, then $\mathrm{Hom}(X, Y) = 0$.
3. For any $X \in \mathscr{T}$, there is a distinguished triangle $A \to X \to B \to$ with $A \in \mathscr{T}^{\leq -1}$ and $B \in \mathscr{T}^{\geq 0}$.

If this pair is a t-structure, then $\mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq 0}$ is called the *heart*.

Definition 1.2. We say that a $t-$structure is

- *bounded below* if $\forall X \in \mathscr{T}$, $\exists n \in \mathbb{Z}$ such that $X \in \mathscr{T}^{\geq n}$
- *bounded above* if $\forall X \in \mathscr{T}$, $\exists n \in \mathbb{Z}$ such that $X \in \mathscr{T}^{\leq n}$
- *bounded* if it is bounded above and below
- *non-degenerate* if $\bigcap_{n\in\mathbb{Z}} \mathscr{T}^{\leq n} = \bigcap_{n\in\mathbb{Z}} \mathscr{T}^{\geq n} = 0$

Notice that it follows immediately that $\mathscr{T}^{\leq n} \subset \mathscr{T}^{\leq n+1}$ and $\mathscr{T}^{\geq n} \supset \mathscr{T}^{\geq n+1}$ from the second condition.

Example 1.3. Let $\mathscr{A}$ be an abelian category. Consider the subcategories

$$D^b(\mathscr{A})^{\leq 0} = \{X \in D^b(\mathscr{A}) : H^i(X) = 0 \text{ for } i > 0\} \quad D^b(\mathscr{A})^{\geq 0} = \{X \in D^b(\mathscr{A}) : H^i(X) = 0 \text{ for } i < 0\}$$

Then this pair forms a t-structure on $D^b(\mathscr{A})$.

Example 1.4 (Torsion-pair t-structure, see [2])**.** Let $\mathscr{A}$ be an abelian category. A *torsion pair* $(\mathcal{T}, \mathscr{F})$ is a pair of strictly full subcategories such that

1. $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}, F \in \mathscr{F}$
2. For any object $A \in \mathscr{A}$, there exists a short exact sequence

$$0 \to T \to A \to F \to 0$$

where $T \in \mathcal{T}, F \in \mathscr{F}$.

If these conditions hold, we call $\mathcal{T}$ a *torsion class* and $\mathscr{F}$ a *torsion free class.*

Now, we can get a t-structure on $D^b(\mathscr{A})$ by setting

$$D^b(\mathscr{A})^{\leq 0} = \{\mathcal{F} \in D^b(\mathscr{A}) : H^i(\mathcal{F}) = 0 \text{ for } i > 0, H^0(\mathcal{F}) \in \mathcal{T}\}$$
$$D^b(\mathscr{A})^{\geq 0} = \{\mathcal{F} \in D^b(\mathscr{A}) : H^i(\mathcal{F}) = 0 \text{ for } i < -1, H^{-1}(\mathcal{F}) \in \mathscr{F}\}$$

Exercise: Prove this is a t-structure.

Example 1.5 (Special case of Example 1.4)**.** Consider the category whose objects are triples (V, W, T) where V, W are vector spaces and $T : V \to W$ is a linear transformation and the objects are defined in a natural way (i.e. the category of representations of the type A_2 Dynkin quiver). As we saw in Balázs' talk [4], this is equivalent to the category of sheaves on $\mathbb{P}^1$ constructible with respect to the stratification $\{0\}, \mathbb{P}^1 \backslash \{0\}$.

Let $\mathscr{T} = \{V \to 0\}$ and let $\mathcal{F} = \{V \to W \text{ injective }\}$. Then $(\mathscr{T}, \mathcal{F})$ is a torsion pair and this $({}^tD^b(\mathcal{A})^{\leq 0}, {}^tD^b(\mathcal{A})^{\geq 0})$ is the induced t-structure on the derived category.

For the rest of this section, we will always consider a triangulated category $\mathscr{T}$ with t-structure denoted by $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$.

Lemma 1.6.

1. *$X \in \mathscr{T}^{\leq n} \iff \mathrm{Hom}(X, Y) = 0$ for all $Y \in \mathscr{T}^{\geq n+1}$.*
2. *$X \in \mathscr{T}^{\geq n} \iff \mathrm{Hom}(Y, X) = 0$ for all $Y \in \mathscr{T}^{\leq n-1}$*

Proof. The direction ($\implies$) is obvious by the definition of t-structure. For ($\impliedby$), pick a distinguished triangle $A \to X \to B \to$ where $A \in \mathscr{T}^{\leq n}, B \in \mathscr{T}^{\geq n+1}$. Since $X \in \mathscr{T}^{\leq n}$, then $\mathrm{Hom}(X, B) = 0$ so that $X \to B$ is zero. Thus, this distinguished triangle splits and $A \cong X \oplus B[-1]$. Then the projection map $p : A \to B[-1] \in \mathrm{Hom}(A, B[-1])$. But $A \in \mathscr{T}^{\leq n} \subset \mathscr{T}^{\leq n+1}$ and $B[-1] \in \mathscr{T}^{\geq n+2}$ so by the second condition $\mathrm{Hom}(A, B[-1]) = 0$. Then $p = 0$ hence $B[-1] = 0$. Thus $A \cong X$ so that $X \in \mathscr{T}^{\leq n}$.

Part (2) is left as an exercise. □

Definition 1.7. A subcategory $\mathscr{C}$ of $\mathscr{T}$ is called *stable under extensions* if for every $A, B \in \mathscr{C}$ such that there is a distinguished triangle $A \to T \to B \to$, then $T \in \mathscr{C}$.

Lemma 1.8. *For any $n \in \mathbb{Z}$, the categories $\mathscr{T}^{\leq n}$ and $\mathscr{T}^{\geq n}$ are stable under extensions.*

Proof. Exercise. (Hint: Construct a long exact sequence using $Y \in \mathscr{T}^{\geq n+1}$). □

Proposition 1.9 (Truncation)**.**

1. *The inclusion $\mathscr{T}^{\leq n} \hookrightarrow \mathscr{T}$ admits a right adjoint $\tau^{\leq n} : \mathscr{T} \to \mathscr{T}^{\leq n}$.*

2. *The inclusion $\mathscr{T}^{\geq n} \hookrightarrow \mathscr{T}$ admits a left adjoint $\tau^{\geq n} : \mathscr{T} \to \mathscr{T}^{\geq n}$.*
3. *There is a unique natural transformation $\delta : \tau^{\geq n+1} \to \tau^{\leq n}[1]$ s.t. for any $X \in \mathscr{T}$, the diagram*
$$\tau^{\leq n}X \to X \to \tau^{\geq n+1}X \to \tau^{\leq n}X[1]$$
is a distinguished triangle. Any distinguished triangle $A \to X \to B \to$ with $A \in \mathscr{T}^{\leq n}$, $B \in \mathscr{T}^{\geq n+1}$ is canonically isomorphic to this one.

In particular, $\tau^{\leq b}\tau^{\geq a}$ takes values is $\mathscr{T}^{\leq b} \cap \mathscr{T}^{\geq a}$.

Example 1.10. Consider the natural t-structure of $D^b(\mathscr{A})$. Pick $X \in D^b(\mathscr{A})$ represented by a complex
$$\cdots \to X^{-2} \to X^{-1} \to X^0 \to X^1 \to X^2 \to \cdots$$
Then $\tau^{\leq n}(X)$ is given by the chain complex
$$\cdots \to X^{n-2} \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} \ker(d^n) \xrightarrow{d^n} 0 \to 0 \to \dots$$
This chain complex has the same cohomology of X when $i \leq n$ and zero for $i > n$.

Similarly, $\tau^{\geq n}(X)$ is the chain complex
$$\cdots \to 0 \to 0 \xrightarrow{d^{n-1}} \mathrm{coker}(d^{n-1}) \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \to \dots$$
Proof of Proposition 1.9. This proof is done in 4 steps:

Step 1: Define $\tau^{\leq -1}$ and $\tau^{\geq 0}$ and δ on objects.

For each $X \in \mathscr{T}$, fix a distinguished triangle
$$A_X \xrightarrow{g} X \xrightarrow{h} B_X \to$$
where $A_X \in \mathscr{T}^{\leq -1}, B_X \in \mathscr{T}^{\geq 0}$. Set $\tau^{\leq -1}(X) = A_X, \tau^{\geq 0}(X) = B_X$ and $\delta : \tau^{\geq 0}(X) \to (\tau^{\leq -1}X)[1]$ is the third morphism in this triangle.

Step 2: Define $\tau^{\leq -1}, \tau^{\geq 0}, \delta$ as functors by specifying what they do on morphisms.

We still need to describe that these functors do on morphisms, $f : X \to Y$. Consider the distinguished triangle $A_Y \xrightarrow{g'} Y \xrightarrow{g'} B_Y \to$ where A_Y and B_Y are the images of $\tau^{\leq -1}$ and $\tau^{\geq 0}$ respectively. Then we have the diagram
$$\begin{tikzcd} A_X \arrow[r, "g"] \arrow[d, dashed, "p"] & X \arrow[r, "h"] \arrow[d, "f"] & B_X \arrow[r] \arrow[d, dashed, "q"] & {} \\ A_Y \arrow[r, "g'"] & Y \arrow[r, "h'"] & B_Y \arrow[r] & {} \end{tikzcd}$$
By the Lemma of unicity of triangles from Hyungseop's talk [7], we know that there exist a unique p and q that make the diagram commute. So we have defined how $\tau^{\leq -1}$ and $\tau^{\geq 0}$ act on morphisms and in fact, we have shown that δ is a natural transformation

Step 3: Show that $\tau^{\leq -1}$ is the right adjoint of the inclusion map $\mathscr{T}^{\leq -1} \hookrightarrow \mathscr{T}$ and $\tau^{\geq 0}$ is the left adjoint of the inclusion map $\mathscr{T}^{\geq 0} \hookrightarrow \mathscr{T}$.

Consider $Z \in \mathscr{T}^{\leq -1}$. Then we get the long exact sequence
$$\cdots \to \mathrm{Hom}(Z, (\tau^{\geq 0}X)[-1])) \to \mathrm{Hom}(Z, \tau^{\leq -1}(X)) \to \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, \tau^{\geq 0}X) \to \dots$$
The first and last terms are zero, so that $\mathrm{Hom}(Z, \tau^{\leq -1}(X)) \cong \mathrm{Hom}(Z, X)$ and hence $\tau^{\leq -1}$ is the right adjoint to the inclusion.

Exercise: Prove that $\tau^{\geq 0}$ is the left adjoint of $\mathscr{T}^{\geq 0} \hookrightarrow \mathscr{T}$.

Step 4: For general $n \in \mathbb{Z}$:

Set $\tau^{\leq n}(X) = (\tau^{\leq -1}(X[n+1]))[-n-1]$ and $\tau^{\geq n}(X) = (\tau^{\geq 0}(X[n]))[-n]$. The proof that these functors are the right and left adjoints of the natural inclusion maps is similar to the above case. ∎

Lemma 1.11. *For any $a, b \in \mathbb{Z}$ such that $a \leq b$, there are natural isomorphisms*

1. $\tau^{\leq a}\tau^{\leq b} \to \tau^{\leq a}$
2. $\tau^{\geq b} \to \tau^{\geq b}\tau^{\geq a}$
3. $\tau^{\geq a}\tau^{\leq b} \to \tau^{\leq b}\tau^{\geq a}$

Theorem 1.12 (Theorem 1.8.10 of [1])**.** *The heart $\mathscr{C} := \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq 0}$ of a t-structure is an abelian category.*

To prove this, we need to show that every morphism has kernel and cokernel, and that every monomorphism is the kernel of its cokernel and the dual claim holds for epimorphisms. We first prove the following lemma, which shows that every morphism in $\mathscr{C}$ has a kernel and cokernel.

Lemma 1.13. *Let $f : X \to Y$ be a morphism in the heart of the t-structure on $\mathscr{T}$. Consider the distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{k} X[1]$$

(i) Then $Z \in \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq -1}$ and $\tau^{\leq 0}(Z[-1])$, $\tau^{\geq 0}Z \in \mathscr{C}$.

(ii) The composition $\tau^{\leq 0}(Z[-1]) \to Z[-1] \xrightarrow{-k[-1]} X$ is the kernel of f in $\mathscr{C}$.

(iii) The composition $Y \xrightarrow{g} Z \to \tau^{\geq 0}Z$ is the cokernel of f in $\mathscr{C}$.

Proof. For (1), we know that $Y \in \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq 0} \subset \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq -1}$ and $X[-1] \in \mathscr{T}^{\leq -1} \cap \mathscr{T}^{\geq -1} \subset \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq -1}$. Since these categories are stable under extensions, then $Z \in \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq -1}$.

Then $Z \in \mathscr{T}^{\leq 0}$ and so $\tau^{\leq 0}Z = Z$. If we apply $\tau^{\geq 0}$, we get that $\tau^{\geq 0}Z = \tau^{\geq 0}\tau^{\leq 0}Z = \tau^{\leq 0}\tau^{\geq 0}Z$, which takes values in $\mathscr{C}$. Similarly, $Z[-1] \in \mathscr{T}^{\leq 1} \cap \mathscr{T}^{\geq 0}$ so $\tau^{\leq 0}Z[-1] = \tau^{\leq 0}\tau^{\geq 0}Z[-1] \in \mathscr{C}$.

For (2), consider $X' \in \mathscr{C}$. Then we obtain the long exact sequence

$$\cdots \to \mathrm{Hom}(X', Y[-1]) \to \mathrm{Hom}(X', Z[-1]) \to \mathrm{Hom}(X', X) \to \mathrm{Hom}(X', Y) \to \dots$$

Since $Y \in \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq 0}$ then $Y[-1] \in \mathscr{T}^{\leq 1} \cap \mathscr{T}^{\geq 1}$ so that $Y[-1] \in \mathscr{T}^{\geq 1}$. Since $X' \in \mathscr{T}^{\leq 0}$ then $\mathrm{Hom}(X, Y[-1]) = 0$.

Suppose $g : X' \to X$ is a morphism of $\mathcal{C}$ such that $f \circ g = 0$. Then since $\mathrm{Hom}(X, Y[-1]) = 0$, g factors through a unique morphism $g' : X' \to Z[-1]$. Since $X' \in \mathscr{T}^{\leq 0}$ then as $\tau^{\leq 0}$ is the adjoint of the inclusion, then there is a unique map $g'' : X \ \to \tau^{\leq 0}(Z[-1])$ such that g' factors through g''.

Then we conclude $\tau^{\leq 0}(Z[-1]) \to Z[-1] \to X$ is the kernel of f.

Similarly, we can find the cokernel as $Y \to Z \to \tau^{\geq 0}Z$. □

Proof of Theorem 1.12. All that is left is to show that is that every monomorphism is the kernel of its cokernel and that every epimorphism is the cokernel of its kernel.

Suppose that $f : X \to Y$ is a monomorphism in $\mathscr{C}$. Complete this morphism to a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

By the previous lemma, $\tau^{\leq 0}(Z[-1]) \to Z[-1] \to X$ is the kernel of f and since f is a monomorphism, this is zero. Thus, $\tau^{\leq 0}(Z[-1]) = 0$ and hence $Z \cong \tau^{\geq 0}Z$. Then $Z \in \mathscr{C}$ and g is a morphism in $\mathscr{C}$, where g is the cokernel of f.

We have the distinguished triangle

$$Y \xrightarrow{g} Z \to X[1] \xrightarrow{-f[1]}$$

so that the kernel of g is the composition $\tau^{\leq 0}(X) \to X \xrightarrow{f} Y$. But $\tau^{\leq 0}(X) = X$ so that the kernel of g is f.

Similarly, we can prove that every epimorphism is the cokernel of its kernel. □

Remark 1.14. One of the motivations of defining a t-structure was to recover the abelian category $\mathscr{A}$ from its derived category $D(\mathscr{A})$. In this case, the heart $\mathscr{C}$ of the natural t-structure on $D(\mathscr{A})$ is equivalent to $\mathscr{A}$.

Definition 1.15. Let $\mathscr{C} = \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq 0}$ be the heart of a t-structure on $\mathscr{T}$. The zeroth *t-cohomology* functor is defined to be

$$^t\mathbf{H}^0 = \tau^{\leq 0}\tau^{\geq 0} : \mathscr{T} \to \mathscr{C}$$

The n^{nt} t-cohomology functor is $^t\mathbf{H}^n(X) = {}^t\mathbf{H}^0(X[n])$.

Example 1.16. Consider the tautological t-structure on $D^b_c(\mathscr{A})$. For a complex $X \in D^b(\mathscr{A})$, then $^t\mathbf{H}^0(X) = \tau^{\leq 0}(\tau^{\geq 0}X)$ is just the complex

$$\cdots \to 0 \to H^0(X) \to 0 \to \ldots$$

Proposition 1.17. *The functor* $^t\mathbf{H}^0 : \mathscr{T} \to \mathscr{C}$ *is a cohomological functor.*

Corollary 1.18. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be two morphisms in* $\mathscr{C}$. *The following conditions are equivalent:*

1. *The sequence* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *is a short exact sequence*
2. *There exists a morphism* $h : Z \to X[1]$ *in* $\mathscr{T}$ *such that* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h}$ *is a distinguished triangle.*

If these conditions hold, then h *is unique.*

This follows since we can obtain 1) from 2) by applying $^t\mathbf{H}^0$ which sends distinguished triangles to exact sequences.

Definition 1.19. A triangulated function $F : \mathscr{T}_1 \to \mathscr{T}_2$ is *left t-exact* if $F(\mathscr{T}_1^{\geq 0}) \subset \mathscr{T}_2^{\geq 0}$ and *right t-exact* of $F(\mathscr{T}_1^{\leq 0}) \subset \mathscr{T}_2^{\leq 0}$. The functor F is called *t-exact* if it is both left and right t-exact.

Lemma 1.20. *Let* $\mathscr{T}_1$ *and* $\mathscr{T}_2$ *be triangulated categories equipped with t-structures and let* $\mathscr{C}_1$ *and* $\mathscr{C}_2$ *denote their hearts. Let* $F : \mathscr{T}_1 \to \mathscr{T}_2$ *be a triangluated functor.*

1. *If* F *is left t-exact, then the functor* $^t\mathbf{H}^0 \circ F : \mathscr{C}_1 \to \mathscr{C}_2$ *is left exact.*
2. *If* F *is right t-exact, then the functor* $^t\mathbf{H}^0 \circ F : \mathscr{C}_1 \to \mathscr{C}_2$ *is right exact.*

Example 1.21 (Triangulated category which is not the derived category of the heart)**.** Consider the category $D^b_{\text{locf}}(\mathbb{P}^1)$. The heart $\mathcal{C}$ of the tautological t-structure is

$$\mathcal{C} \cong \underline{\text{Loc}}(\mathbb{P}^1)$$

But $\underline{\text{Loc}}(\mathbb{P}^1) \cong \text{Rep}(\pi_1(\mathbb{P}^1)) \cong \text{Rep}(\mathbb{Z}) \cong \text{Rep}(\{1\}) \cong \underline{\text{Vect}}_{\mathbb{C}}$. Then $D^b(\mathcal{C}) \cong D^b(\underline{\text{Vect}})$.

But in $D^b_{\text{locf}}(\mathbb{P}^1)$, $\text{Ext}^2_{D^b_c(\mathbb{P}^1)}(\mathbb{C}_{\mathbb{P}^1}, \mathbb{C}_{\mathbb{P}^1}) = R\operatorname{Hom}(\mathbb{C}_{\mathbb{P}^1}, \mathbb{C}_{\mathbb{P}^1}) = R\Gamma(\mathbb{C}_{\mathbb{P}^1}) = H^2(\mathbb{P}^1) = \mathbb{C}$ while $\text{Ext}^2_{\underline{\text{Vect}}}(V, V) = 0$ for every vector space. Thus $D^b_{\text{locf}}(\mathbb{P}^1) \ncong D^b(\mathcal{C})$.

2 Perverse Sheaves

Definition 2.1. Let X be a variety. The *perverse t-structure* on X is the t-structure on $D^b_c(X)$ given by

$$\begin{aligned} {}^pD^b_c(X)^{\leq 0} &= \left\{\mathcal{F} \in D^b_c(X) : \forall i, \dim\operatorname{supp} H^i(\mathcal{F}) \leq -i\right\} \\ {}^pD^b_c(X)^{\geq 0} &= \left\{\mathcal{F} \in D^b_c(X) : \forall i, \dim\operatorname{supp} H^i(\mathbb{D}\mathcal{F}) \leq -i\right\} \end{aligned}$$

The heart of this t-structure is $\text{Perv}(X) = {}^pD^b_c(X)^{\leq 0} \cap {}^pD^b_c(X)^{\geq 0}$, i.e. . Objects in the heart are called *perverse sheaves.*

Remark 2.2. In particular, this means that when $\mathcal{F} \in \mathrm{Perv}(X)$, then $H^i(\mathcal{F}) = 0$ for $i > 0$ while $H^i(\mathbb{D}\mathcal{F}) = 0$ for $i < 0$

We will prove this is indeed a t-structure in Theorem 2.10.

Remark 2.3. Given this definition of the perverse t-structure, then

$$\mathcal{F} \in {}^pD^b_c(X)^{\geq 0} \iff \mathbb{D}\mathcal{F} \in {}^pD^b_c(X)^{\leq 0}$$

Definition 2.4. Let X be a variety. A *good stratification* is a stratification $(X_s)_{s\in\mathscr{S}}$ such that for any local system of finite type $\mathcal{L}$ on X_s, if $j_s : X_S \hookrightarrow X$ is the inclusion, then the object $j_{s*}\mathcal{L} \in D^b(X)$ is constructible with respect to $\mathscr{S}$.

Remark 2.5. Any stratification of X can be refined by a good stratification.

Example 2.6 (Normal crossing stratification)**.** Let $Z \subset X$ be a divisor with simple normal crossings and components $Z_1, \ldots, Z_k$. As in Balázs's talk, the normal crossing stratification under the index set $\mathscr{S} = \{I \subset [k]\}$ is given by $X_I = \{x \in X : x \in Z_i \iff i \in I\}$. This stratification is a good stratification by Lemma 3.5.8 of [1].

For a good stratification $\mathscr{S}$ on X, we have an induced t-structure on $D^b_{\mathscr{S}}(X)$:

$$\begin{aligned}{}^pD^b_{\mathscr{S}}(X)^{\leq 0} &= {}^pD^b_c(X)^{\leq 0} \cap D^b_{\mathscr{S}}(X)\\ {}^pD^b_{\mathscr{S}}(X)^{\geq 0} &= {}^pD^b_c(X)^{\geq 0} \cap D^b_{\mathscr{S}}(X)\end{aligned}$$

Example 2.7. For X a smooth connected variety and $\mathcal{S}$ a trivial stratification. Then $\mathrm{Perv}_{\mathcal{S}}(X) = \mathrm{Loc}^{ft}(X)[\dim X]$.

The perverse t-structure of $D^b_{\mathscr{S}}(X)$ is closely related to the standard t-structure on $D^b_{\mathrm{locf}}(X_s)$ of the strata of X in the following way:

Lemma 2.8. *Let X be a variety and let $(X_s)_{s\in\mathscr{S}}$ be a good stratification. For each $s \in \mathscr{S}$, let $j_s : X_s \hookrightarrow X$ be the inclusion map.*

1. *Suppose $\mathcal{F} \in D^b_c(X)$ is constructible with respect to $\mathscr{S}$. Then*

$$\mathcal{F} \in {}^pD^b_c(X)^{\leq 0} \iff j_s^*\mathcal{F} \in D^b_{locf}(X_s)^{\leq -\dim X_s} \quad \forall s \in \mathscr{S}$$

2. *Let $\mathcal{F} \in D^b_c(X)$ and suppose $\mathbb{D}\mathcal{F}$ is constructible with respect to $\mathscr{S}$. Then*

$$\mathcal{F} \in {}^pD^b_c(X)^{\geq 0} \iff j_s^!\mathcal{F} \in D^b_{locf}(X_s)^{\geq -\dim X_s} \quad \forall s \in \mathscr{S}$$

Proof. We will prove (1).

($\Longleftarrow$) Suppose that $j_s^*\mathcal{F} \in D^b_{\mathrm{locf}}(X_s)^{\leq -\dim X_s}$, i.e. $H^i(j_s^*\mathcal{F}) = 0$ for $i > -\dim X_s$ for every $s \in \mathscr{S}$. So if $H^i(j_s^*\mathcal{F}) \neq 0$, then $i \leq -\dim X_s$. Then

$$\dim \operatorname{supp} H^i(\mathcal{F}) = \max\{\dim X_s : j_s^* H^i(\mathcal{F}) \neq 0\}$$

but since j_s^* is exact, then $j_s^*H^i(\mathcal{F}) = H^i(j_s^*\mathcal{F})$ and $\dim \operatorname{supp} H^i(\mathcal{F}) \leq -i$. Hence $\mathcal{F} \in {}^pD^b_c(X)^{\leq 0}$.

For ($\Longrightarrow$), suppose that $\mathcal{F} \in {}^pD^b_c(X)^{\leq 0}$. Then $\dim \operatorname{supp} H^i(\mathcal{F}) \leq -i$ for every i. By definition

$$\dim \operatorname{supp} H^i(j_s^*\mathcal{F}) \leq \dim \operatorname{supp} H^i(\mathcal{F}) \leq -i$$

Since $j_s^*\mathcal{F} \in D^b_{\mathrm{locf}}(X_s)$, then $H^i(j_s^*\mathcal{F})$ is a local system so if it has nonzero support, then

$$\dim \operatorname{supp} H^i(j_s^*\mathcal{F}) = \dim X_s$$

Combining these facts, we get that $\dim \operatorname{supp} H^i(j_s^*\mathcal{F}) = \dim X_s \leq -i$. Thus if $-i < \dim X_s$, this contradicts that $\mathcal{F} \in {}^pD^b_c(X)^{\leq 0}$ so that $H^i(j_s^*\mathcal{F}) = 0$. Thus $\mathcal{F} \in D^b_{\mathrm{locf}}(X_s)^{\leq -\dim X_s}$.

Exercise: prove (2). □

Similarly, we can relate the perverse t-structure of $D_c^b(X)$ to the standard t-structure.

Lemma 2.9. *Let X be a variety. We have*

$$D_c^b(X)^{\leq -\dim X} \subset {}^pD_c^b(X)^{\leq 0} \subset D_c^b(X)^{\leq 0}$$

$$D_c^b(X)^{\geq 0} \subset {}^pD_c^b(X)^{\geq 0} \subset D_c^b(X)^{\geq -\dim X}$$

Proof. If $\mathcal{F} \in D_c^b(X)^{\leq -\dim X}$, we need to show $\dim\operatorname{supp} H^i(\mathcal{F}) \leq -i$. We know $H^i(\mathcal{F}) = 0$ for $i \geq -\dim X$, or equivalently for $-i \leq \dim X$. Then $\dim\operatorname{supp} H^i(\mathcal{F}) = 0 \leq -i$ for $i \geq -\dim X$. If $i < -\dim X$, then $-i > \dim X$ and so $\dim\operatorname{supp} H^i(\mathcal{F}) \leq \dim X < -i$ holds automatically. Thus $\mathcal{F} \in {}^pD_c^b(X)^{\leq 0}$.

For $\mathcal{G} \in {}^pD_c^b(X)^{\geq 0}$, then for all $\mathcal{F} \in {}^pD_c^b(X)^{\leq -1}$, $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) = 0$. In particular, by the first inclusion, $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) = 0$ for all $\mathcal{F} \in D_c^b(X)^{\leq -\dim X - 1}$ and hence by Lemma 1.6, $\mathcal{G} \in D_c^b(X)^{\geq -\dim X}$.

Exercise: Prove ${}^pD_c^b(X)^{\leq 0} \subset D_c^b(X)^{\leq 0}$ and $D_c^b(X)^{\geq 0} \subset {}^pD_c^b(X)^{\geq 0}$. $\square$

Theorem 2.10. *Let X be a variety.*

1. *The pair $({}^pD_c^b(X)^{\leq 0}, {}^pD_c^b(X)^{\geq 0})$ is a t-structure on $D_c^b(X)$.*

2. *Let $(X_s)_{s\in\mathscr{S}}$ be a good stratification on X. Then the pair $({}^pD_{\mathscr{S}}^b(X)^{\leq 0}, {}^pD_{\mathscr{S}}^b(X)^{\geq 0})$ is a t-structure on $D_{\mathscr{S}}^b(X)$.*

To prove this, we need to show the three conditions for being a t-structure hold:

(1) ${}^pD_c^b(X)^{\leq -1} \subset {}^pD_c^b(X)^{\leq 0}$ and ${}^pD_c^b(X)^{\geq -1} \supset {}^pD_c^b(X)^{\geq 0}$

(2) If $\mathcal{F} \in {}^pD_c^b(X)^{\leq -1}$ and $\mathcal{G} \in {}^pD_c^b(X)^{\geq 0}$, then $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) = 0$.

(3) For any $\mathcal{F} \in D_c^b(X)$, there is a distinguished triangle $A \to \mathcal{F} \to B \to$ with $A \in {}^pD_c^b(X)^{\leq -1}$ and $B \in {}^pD_c^b(X)^{\geq 0}$.

The first condition (1) is obvious from the definition. We will use the following lemma to prove (2):

Lemma 2.11. *Let $\mathcal{F} \in D_c^b(X)^{\leq 0}$. For all $\mathcal{G} \in {}^pD_c^b(X)^{\geq 1}$, we have* $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) = 0$.

Proof. We can use an induction argument using the truncation functor to reduce to the case when $\mathcal{F} \cong H^j(\mathcal{F}[-k])$

Since $\mathcal{G} \in {}^pD_c^b(X)^{\geq 1}$, then we can choose a good stratification of X such that $\mathbb{D}\mathcal{G}$ and $\mathcal{F}$ are constructible. Call this stratification $\mathscr{S}$.

Now, we proceed by induction on the size of $\mathscr{S}$. Let $i : X_t \hookrightarrow X$ be the inclusion of the closed stratum and $j : X\backslash X_t \hookrightarrow X$ be the inclusion of the complementary open. Using Theorem 3.34 from Roger's talk, we have the distinguished triangle

$$R\operatorname{Hom}(i^*\mathcal{F}, i^!\mathcal{G}) \to R\operatorname{Hom}(\mathcal{F}, \mathcal{G}) \to R\operatorname{Hom}(j^*\mathcal{F}, j^*\mathcal{G}) \to$$

Then we have the long exact sequence

$$\cdots \to \operatorname{Hom}(i^*\mathcal{F}, i^!\mathcal{G}) \to \operatorname{Hom}(\mathcal{F}, \mathcal{G}) \to \operatorname{Hom}(j^*\mathcal{F}, j^*G) \to \dots$$

By induction, $\operatorname{Hom}(j^*\mathcal{F}, j^*G) = 0$. We will show that the first term is also zero.

If $i^*F = 0$, then $\operatorname{Hom}(i^*\mathcal{F}, i^!\mathcal{G}) = 0$. Suppose that $i^*\mathcal{F} \neq 0$, then $X_t \subset \operatorname{supp} H^i(\mathcal{F})$ and hence $\dim X_t \leq \dim\operatorname{supp} H^i(\mathcal{F}) \leq -k$. Then $\dim X_t \leq -k$.

Since $\mathbb{D}\mathcal{G}$ is constructible and $\mathcal{G} \in {}^pD_c^b(X)^{\geq 1}$ then by Lemma 2.8, $i^!\mathcal{G} \in D_{\mathrm{locf}}^b(X_t)^{\geq -\dim X_t + 1}$. But $i^*\mathcal{F}$ is concentrated in degree k since $\mathcal{F}$ is but $i^!\mathcal{G}$ is above degree $-\dim X_t + 1 > k$ and hence $\operatorname{Hom}(i^*\mathcal{F}, i^!\mathcal{G}) = 0$ so that $\operatorname{Hom}(\mathcal{F}, \mathcal{G})$ is also zero. $\square$

To prove the last condition (3), we will need to construct a distinguished triangle

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$$

where $\mathcal{F}' \in {}^pD_c^b(X)^{\leq -1}$, $\mathcal{F}'' \in {}^pD_c^b(X)^{\geq 0}$. To construct these complexes, we need to understand how open and closed embeddings behave with respect to the perverse t-structure.

Lemma 2.12. *Let $j : U \hookrightarrow X$ be an open embedding and let $i : Z \hookrightarrow X$ be a closed embedding.*

1. $j^*({}^pD_c^b(X)^{\leq 0}) \subset {}^pD_c^b(U)^{\leq 0}$ *and* $j^*({}^pD_c^b(X)^{\geq 0}) \subset {}^pD_c^b(U)^{\geq 0}$.
2. $j_!({}^pD_c^b(U)^{\leq 0}) \subset {}^pD_c^b(X)^{\leq 0}$
3. $j_*({}^pD_c^b(U)^{\geq 0}) \subset {}^pD_c^b(X)^{\geq 0}$
4. $i_*({}^pD_c^b(Z)^{\leq 0}) \subset {}^pD_c^b(X)^{\leq 0}$ *and* $i_*({}^pD_c^b(Z)^{\geq 0}) \subset {}^pD_c^b(X)^{\geq 0}$
5. $i^*({}^pD_c^b(X)^{\leq 0}) \subset {}^pD_c^b(Z)^{\leq 0}$
6. $i^!({}^pD_c^b(X)^{\geq 0}) \subset {}^pD_c^b(Z)^{\geq 0}$.

Proof. For part (1), let $\mathcal{F} \in {}^pD_c^b(X)^{\leq 0}$, $\mathcal{G} \in {}^pD_c^b(X)^{\geq 0}$. Since j^* is t-exact, and doesn't increase the dimension of the support then

$$\dim \operatorname{supp} H^i(j^*\mathcal{F}) = \dim \operatorname{supp} j^*H^i(\mathcal{F}) \leq \dim \operatorname{supp} H^i(\mathcal{F}) \leq -i$$

so that $j^*\mathcal{F} \in {}^pD_c^b(X)^{\leq 0}$.

Since j^* also commutes with $\mathbb{D}$,

$$\dim \operatorname{supp} H^i(\mathbb{D}(j^*\mathcal{G})) = \dim \operatorname{supp} H^i(j^*(\mathbb{D}\mathcal{G})) \leq \dim \operatorname{supp} H^i((\mathbb{D}\mathcal{G})) \leq -i$$

so that $j^*\mathcal{G} \in {}^pD_c^b(X)^{\geq 0}$. Similarly, part (4) holds because $\mathbb{D}i^* = i^*\mathbb{D}$.

For (2), let $\mathcal{F} \in {}^pD_c^b(X)^{\leq 0}$. Since $j_!$ is t-exact and does not change the dimension of the support, then

$$\dim \operatorname{supp} H^i(j_!\mathcal{F}) = \dim \operatorname{supp} H^i(\mathcal{F}) \leq -i$$

so that $j_!\mathcal{F} \in {}^pD_c^b(X)^{\leq 0}$. Similarly, part (5) holds.

For (3), suppose $\mathcal{F} \in {}^pD_c^b(X)^{\geq 0}$. Then by Theorem 2.2 part (3) of [7], $\mathbb{D} \circ j_* = j_! \circ \mathbb{D}$ so that $\mathbb{D}j_*\mathcal{F} = j_!\mathbb{D}\mathcal{F}$. So by using part (2),

$$\dim \operatorname{supp} H^i(\mathbb{D}j_*\mathcal{F}) = \dim \operatorname{supp} H^i(j_!\mathbb{D}\mathcal{F}) = \dim \operatorname{supp} H^i(\mathbb{D}\mathcal{F}) \leq -i$$

Hence $j_*\mathcal{F} \in {}^pD_c^b(X)^{\leq 0}$. Similarly, part (6) holds. □

We are now ready to prove the theorem.

Proof of Theorem 2.10. To prove the theorem, all that is left to show is condition (3).

We will first construct an object $\mathcal{G}$ which we use to construct $\mathcal{F}'$. Then, using an octahedral diagram, we will construct $\mathcal{F}''$. Finally, we prove that these objects $\mathcal{F}', \mathcal{F}''$ form the distinguished triangle we want.

We proceed by noetherian induction. Let $\mathscr{S}$ be a good stratification such that $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$ are constructible. Let $j : X_t \hookrightarrow X$ be an open stratum and $i : X \backslash X_t \hookrightarrow X$ be the closed complement. Denote $\dim X_t$ by d_t.

Step 1: Construct $\mathcal{G}$.

$D^b_{\mathrm{locf}}(X_t)$ has a natural t-structure. By definition, $j_t^*\mathcal{F} \in D^b_{\mathrm{locf}}(X_t)$. Then using the t-structure, we have the distinguished triangle

$$\tau^{\leq -d_t - 1} j_t^*\mathcal{F} \to j_t^*\mathcal{F} \to \tau^{\geq -d_t} j_t^*\mathcal{F} \to$$

Since $j_{t!}$ is t-exact for this t-structure, then we obtain the exact sequence

$$j_{t!}\tau^{\leq -d_t-1}j_t^*\mathcal{F} \to j_{t!}j_t^*\mathcal{F} \to j_{t!}\tau^{\geq -d_t}j_t^*\mathcal{F} \to$$

By composing with the natural map $j_{t!}j_t^*\mathcal{F} \to \mathcal{F}$ defined in Roger's talk [3], we can complete the distinguished triangle to obtain $\mathcal{G} \in D_c^b(X)$.

$$j_{t!}\tau^{\leq -d_t-1}j_t^*\mathcal{F} \to \mathcal{F} \to \mathcal{G} \to$$

If we apply j_t^* to this exact sequence, then we get

$$\tau^{\leq -\dim X_t-1}j_t^*\mathcal{F} \to j_t^*\mathcal{F} \to j_t^*\mathcal{G} \to$$

and by the uniqueness of the distinguished triangle, then $j_t^*\mathcal{G} \cong \tau^{\geq -\dim X_t}j_t^*\mathcal{F}$.

Step 2: Construct $\mathcal{F}''$.

Since $\mathcal{G} \in D_c^b(X)$, then $i^!\mathcal{G} \in D_c^b(Z)$. By induction, $({}^pD_c^b(Z)^{\leq 0}, {}^pD_c^b(Z)^{\geq 0})$ is a t-structure. In particular, we have the distinguished triangle

$${}^p\tau^{\leq -1}i^!\mathcal{G} \to i^!\mathcal{G} \to {}^p\tau^{\geq 0}i^!\mathcal{G} \to$$

By applying the t-exact functor i_*, we have

$$i_*\,{}^p\tau^{\leq -1}i^!\mathcal{G} \to i_*i^!\mathcal{G} \to i_*\,{}^p\tau^{\geq 0}i^!\mathcal{G} \to$$

Then composing with the natural map $i_*i^!\mathcal{G} \to \mathcal{G}$, we find $\mathcal{F}'' \in D_c^b(X)$ such that the following is a distinguished triangle

$$i_*\,{}^p\tau^{\leq -1}i^!\mathcal{G} \to \mathcal{G} \to \mathcal{F}''$$

Apply $i^!$ to conclude that $i^!\mathcal{F}'' \cong {}^p\tau^{\geq 0}i^!\mathcal{G}$.

Now, we can compose the morphism $\mathcal{F} \to \mathcal{G} \to \mathcal{F}''$ to get a map $\mathcal{F} \to \mathcal{F}''$. We will show that $\mathcal{F}'' \in {}^pD_c^b(X)^{\geq 0}$ in Step 4.

Step 3: Construct $\mathcal{F}'$.

There exists a $\mathcal{F}'$ such that we can complete $\mathcal{F} \to \mathcal{F}''$ to the distinguished triangle

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$$

We will now use the octahedral axiom of a triangulated category to show that $j_{t!}\tau^{\leq \dim d_t-1}j_t^*\mathcal{F} \to \mathcal{F}' \to i_*\,{}^p\tau^{\leq -1}i^!\mathcal{G} \to$ is a distinguished triangle.

Octahedral axiom Consider three distinguished triangles

$$X \xrightarrow{a} Y \xrightarrow{a'} A \xrightarrow{a''}$$
$$X \xrightarrow{b} Z \xrightarrow{b'} B \xrightarrow{b''}$$
$$Y \xrightarrow{c} Z \xrightarrow{c'} C \xrightarrow{c''}$$

such that $b = c \circ a$. Then there exist morphisms $f : A \to B$, $g : B \to C$ such that

$$A \xrightarrow{f} B \xrightarrow{g} B \xrightarrow{a'[1]\circ c''} A[1]$$

is a distinguished triangle and $b'' \circ f = a''$, $g \circ b' = c'$, $b' \circ c = f \circ a'$, $a[1] \circ b'' = c'' \circ g$. This gives the octahedral diagram:

$$\begin{tikzcd}
A \arrow[dd, squiggly, "a''"'] & & C \arrow[ll, squiggly, "a'[1]c''"'] \arrow[dl, squiggly, "c''"] \\
& Y \arrow[ul, "a'"'] \arrow[dr, "c"] & \\
X \arrow[ur, "a"] \arrow[rr, "b"] & & Z \arrow[uu, "c'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[dd, squiggly, "a''"'] \arrow[dr, "f"] & & C \arrow[ll, squiggly, "a'[1]c''"'] \\
& B \arrow[ur, "g"] \arrow[dl, squiggly, "b''"] & \\
X \arrow[rr, "b"] & & Z \arrow[uu, "c'"'] \arrow[ul, "b'"']
\end{tikzcd}$$

where $\rightsquigarrow$ are the maps $+1$.

Then our distinguished triangles fit inside the diagram

$$
\begin{tikzcd}
j_{t!}\tau^{\leq -d_t-1}j_t^*\mathcal{F} \arrow[dd, squiggly] \arrow[dr] & & i_*\,{}^p\tau^{\leq -1}i^!\mathcal{G} \arrow[ll, squiggly] \\
& \mathcal{G} \arrow[ur] \arrow[dl, squiggly] & \\
\mathcal{F} \arrow[rr] & & \mathcal{F}'' \arrow[ul] \arrow[uu]
\end{tikzcd}
\qquad
\begin{tikzcd}
j_{t!}\tau^{\leq -d_t-1}j_t^*\mathcal{F} \arrow[dd, squiggly] & & i_*\,{}^p\tau^{\leq -1}i^!\mathcal{G} \arrow[ll, squiggly] \arrow[dl, squiggly] \\
& \mathcal{F}' \arrow[ul] \arrow[dr] & \\
\mathcal{F} \arrow[rr] \arrow[ur] & & \mathcal{F}'' \arrow[uu]
\end{tikzcd}
$$

so that the following is a distinguished triangle:

$$j_{t!}\tau^{\leq -\dim X_t-1}j_t^*\mathcal{F} \to \mathcal{F}' \to i_*\,{}^p\tau^{\leq -1}i^!\mathcal{G}$$

Step 4: Show that $\mathcal{F}' \in {}^pD_c^b(X)^{\leq -1}$.

First, consider

$$j_{t!}\tau^{\leq -\dim X_t-1}j_t^*\mathcal{F} \to \mathcal{F}' \to i_*\,{}^p\tau^{\leq -1}i^!\mathcal{G}$$

Then since ${}^p\tau^{\leq -1}i^!\mathcal{G} \in {}^pD_c^b(Z)^{\leq -1}$ by Lemma 2.12, then the right term is in ${}^pD_c^b(X)^{\leq -1}$. Similarly, $\tau^{\leq -\dim X_t-1}j_t^*\mathcal{F} \in D^b_{\text{locf}}(X)^{\leq -\dim X_t -1}$ so by Lemma 2.8 then $j_{t!}\tau^{\leq -\dim X_t-1}j_t^*\mathcal{F} \in {}^pD_c^b(X)^{\leq -1}$.

Claim: ${}^pD_c^b(X)^{\leq -1}$ is stable under extensions (see Definition 1.7).

Consider $A, C \in {}^pD_c^b(X)^{\leq -1}$ and consider the distinguished triangle $A \to B \to C$. Then for the long exact sequence of cohomology

$$\cdots \to H^{i-1}(C) \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots$$

Since $\dim\operatorname{supp} H^i(A) \leq -i-1$ and $\dim\operatorname{supp} H^i(C) \leq -i-1$, by exactness $\dim\operatorname{supp} H^i(B)$ must also be $\leq i-1$.

Therefore, $\mathcal{F}' \in {}^pD_c^b(X)^{\leq -1}$.

Step 5: Show that $\mathcal{F}'' \in {}^pD_c^b(X)^{\geq 0}$.

We will show that $j_s^!\mathcal{F}'' \in D^b_{\text{locf}}(X_s)^{\geq -\dim X_s}$ for every $s \in \mathscr{S}$. Then by Lemma 2.8 $\mathcal{F}'' \in {}^pD_c^b(X)$.

First, for $s = t$, we have the distinguished triangle

$$i_*\,{}^p\tau^{\leq -1}i^!\mathcal{G} \to \mathcal{G} \to \mathcal{F}'' \to$$

By applying $j_t^!$, the left term becomes zero so that $j_t^!\mathcal{F}'' \cong j_t^!\mathcal{G} = j_t^*\mathcal{G}$ but we showed $j_t^*\mathcal{G} \cong \tau^{\geq -\dim X_t}j_t^*\mathcal{F}$ which is in $D^b_{\text{locf}}(X_t)^{\geq -\dim X_t}$.

For $s \neq t$, then j_s factors as $X_s \overset{j_s'}{\hookrightarrow} X\backslash X_t \overset{i}{\hookrightarrow} X$. Then $j_s! = (j_s')^!i^!$ so

$$j_s^!\mathcal{F}'' = (j_s')^!i^!\mathcal{F}'' \cong (j_s')^!\,{}^p\tau^{\geq 0}i!\mathcal{G}$$

Now we can apply induction to show the $(j_s')^!\,{}^p\tau^{\geq 0}i!\mathcal{G} \in D^b_{\text{locf}}(X)^{\geq -\dim X_s}$. $\square$

Definition 2.13. A *Serre subcategory* $\mathscr{T}$ of an abelian category $\mathscr{A}$ is a full subcategory such that for any exact sequence $0 \to M' \to M \to M'' \to 0$, we have $M \in \mathscr{T} \iff M', M'' \in \mathscr{T}$.

Proposition 2.14. *Let $i : Z \hookrightarrow X$ be the inclusion of a closed subvariety. The functor i_* induces an equivalence of categories*

$$\operatorname{Perv}(Z) \to \{\mathcal{F} \in \operatorname{Perv}(X) : \operatorname{supp}\mathcal{F} \subset Z\}$$

Moreover, the right-hand side is a Serre subcategory of $\operatorname{Perv}(X)$.

Proof. The inverse of i_* is given by i^* then the equivalence is a consequence of Lemma 2.12. To show $\mathrm{Perv}(Z)$ is a Serre subcategory, let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence.

Suppose $\mathcal{F}', \mathcal{F}'' \in \mathrm{Perv}(Z)$. Then clearly $\operatorname{supp} \mathcal{F} \subset Z$.

Suppose $\mathcal{F} \in \mathrm{Perv}(Z)$. Let $U = X \backslash Z$ and let $i : U \hookrightarrow X$ be the open embedding. Then by Lemma 2.12, i^* is t-exact so the following sequence is a short exact sequence in $\mathrm{Perv}(U)$:

$$0 \to i^*\mathcal{F}' \to i^*\mathcal{F} \to i^*\mathcal{F}'' \to 0$$

But $i^*\mathcal{G} = \mathcal{G}|_U$ and $\mathcal{F}|_U = 0$ since it is supported on Z. Then it follows that $\mathcal{F}'|_U = \mathcal{F}''|_U = 0$ so that $\mathcal{F}, \mathcal{F} \in \mathrm{Perv} Z$. ☐

Lemma 2.15. *For any two objects $\mathcal{F} \in {}^pD^b_c(X)^{\leq 0}$ and $\mathcal{G} \in {}^pD^b_c(X)^{\geq 0}$, we have $R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \in D^b_c(X)^{\geq 0}$.*

Proof. We will prove this using noetherian induction.

Choose a stratification of X such that both $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$ are constructible. Let $j : X_s \hookrightarrow X$ be an inclusion of an open stratum of this stratification. Let $i : X \setminus X_s \hookrightarrow X$. Then by Lemma 2.8, we have that $j^*\mathcal{F} \in D^b_{\mathrm{locf}}(U)^{\leq -\dim U}$ and $j^*\mathcal{G} \in D^b_{\mathrm{locf}}(U)^{\geq -\dim U}$. Then $R\mathscr{H}om(j^*\mathcal{F}, j^*\mathcal{G}) \in D^b_c(U)^{\geq 0}$.

Consider the distinguished triangle

$$i_*i^!R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \to R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \to j_*j^*R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \to$$

By Lemma 4.8 of Roger's talk, we know that $j^*R\mathscr{H}om(\mathcal{F}, \mathcal{G}) = R\mathscr{H}om(j^*\mathcal{F}, j^*\mathcal{G})$, and since j is an open inclusion, $j^! \cong j^*$. Using the dual projection formula from Hyungseop's talk [7], we can get the distinguished triangle

$$i_*R\mathscr{H}om(i^*\mathcal{F}, i^!\mathcal{G}) \to R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \to j_*R\mathscr{H}om(j^*\mathcal{F}, j^*\mathcal{G}) \to$$

But j_* is the right derived functor of j^* and $R\mathscr{H}om(j^*\mathcal{F}, j^*\mathcal{G}) \in D^b_c(U)^{\geq 0}$ so that $j_*R\mathscr{H}om(j^*\mathcal{F}, j^*\mathcal{G}) \in D^b_c(X)^{\geq 0}$ as well.

By Lemma 2.12 we have $i^*\mathcal{F} \in {}^pD^b_c(X \setminus X_s)^{\leq 0}$ and $i^!\mathcal{G} \in {}^pD^b_c(X \setminus X_s)^{\geq 0}$. Here we use induction to conclude that $R\mathscr{H}om(i^*\mathcal{F}, i^!\mathcal{G}) \in D^b_c(X \setminus X_s)^{\geq 0}$. Since i_* is the right derived functor of i_*, then $i_*R\mathscr{H}om(i^*\mathcal{F}, i^!\mathcal{G}) \in D^b_c(X)^{\geq 0}$ as well. Hence the middle term of the distinguished triangle is also in $D^b_c(X)^{\geq 0}$, which is what we wanted to prove. ☐

The next four results tell us which functors on $D^b(X)$ are t-exact for the perverse t-structure.

Lemma 2.16. *The Verdier duality functor $\mathbb{D} : D^b_c(X)^{op} \to D^b_c(X)$ is a t-exact for the perverse t-structure.*

Proof. We need to show that $\mathbb{D}\left({}^pD^b_c(X)^{op\geq 0}\right) \subset {}^pD^b_c(X)^{\geq 0}$ and $\mathbb{D}\left({}^pD^b_c(X)^{op\leq 0}\right) \subset {}^pD^b_c(X)^{\leq 0}$. But by definition

$$\mathcal{F} \in {}^pD^b_c(X)^{\geq 0} \iff \mathbb{D}\mathcal{F} \in {}^pD^b_c(X)^{\leq 0}$$

so that $\mathbb{D}$ is obviously left t-exact.

But if $\mathbb{D}\mathcal{F} \in {}^pD^b_c(X)^{\geq 0}$, then $\mathbb{D}^2\mathcal{F} \in {}^pD^b_c(X)^{\leq 0}$ but $\mathbb{D}^2\mathcal{F} \cong \mathcal{F}$ so that the Verdier duality functor is also right t-exact. ☐

Proposition 2.17. *Let $f : X \to Y$ be a finite morphism. The functor $f_* : D^b_c(X) \to D^b_c(Y)$ is t-exact for the perverse t-structure.*

Proof. Exercise. ☐

Lemma 2.18. *Let X be a variety and let $\mathcal{L}$ be a local system of finite type on X. The functor $(-) \otimes \mathcal{L} : D^b_c(X) \to D^b_c(X)$ is t-exact for the perverse t-structure.*

Proof. Exercise. ☐

Lemma 2.19. *The functor $\boxtimes$ is t-exact for the perverse t-structure*

Proof. Exercise. $\square$

Example 2.20. As in Example 1.4, let $\mathcal{S}$ be the stratification of $\mathbb{P}^1$ where $\mathcal{S} = \{\{0\}, \mathbb{A}^1\}$ and let $j_0 : \{0\} \hookrightarrow \mathbb{P}^1$ and $j_{\mathbb{A}^1} : \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ be the inclusions. We will show that the torsion-pair t-structure on $D(Sh_{\mathcal{S}}(\mathbb{P}^1))$ is equivalent to the perverse t-structure.

The heart $\mathcal{C}$ of the torsion-pair t-structure is

$$\mathcal{C} = \left\{(V_\bullet \to W_\bullet) : H^0\left(V_\bullet \to W_\bullet\right) = V \to 0, H^1\left(V_\bullet \to W_\bullet\right) = V \hookrightarrow W\right\}$$

As objects, these are just the diagrams

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & V_{-1} \arrow[r] \arrow[d, hook] & V_0 \arrow[r] \arrow[d] & 0 \arrow[r] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & W_{-1} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

and the morphisms are still quasi-isomorphism of the chains.

Consider the perverse t-structure on $\mathbb{P}^1$ and suppose $\mathcal{F} \in \mathrm{Perv}(\mathbb{P}^1)$.

Then $\mathcal{F} \in {}^pD^b_{\mathcal{S}}(\mathbb{P}^1)^{\leq 0}$ so by Lemma 2.8, then

$$j_0^*\mathcal{F} \in D^b_{\mathrm{locf}}(\{0\})^{\leq 0} \qquad\qquad j_{\mathbb{A}^1}^*\mathcal{F} \in D^b_{\mathrm{locf}}(\mathbb{A}^1)^{\leq -1}$$

Since $j_0^*\mathcal{F} = V_\bullet$, then we have that $H^i(V_\bullet) = 0$ for $i > 0$. Since $j_{\mathbb{A}^1}^*\mathcal{F} = W_\bullet$, then $H^i(W) = 0$ for $i > -1$.

Again, as $\mathcal{F} \in {}^pD^b_{\mathcal{S}}(\mathbb{P}^1)^{\geq 0}$ so by Lemma 2.8, then

$$j_0^!\mathcal{F} \in D^b_{\mathrm{locf}}(\{0\})^{\geq 0} \qquad\qquad j_{\mathbb{A}^1}^!\mathcal{F} \in D^b_{\mathrm{locf}}(\mathbb{A}^1)^{\geq -1}$$

Since $j_{\mathbb{A}^1}$ is an open embedding, $j_{\mathbb{A}^1}^! = j_{\mathbb{A}^1}^*$. Hence $H^i(W) = 0$ if $i < -1$.

Claim: For $\mathcal{F} = (V_\bullet \to W_\bullet) \in D^b_{\mathcal{S}}(\mathbb{P}^1)$, $H^i(j_0^!\mathcal{F}) = \ker(H^i(V_\bullet) \to H^i(W_\bullet)) \oplus \mathrm{coker}(H^{i+1}(V_\bullet) \to H^{i+1}(W_\bullet))$.

Proof. Exercise: Hint use adjunction or spectral sequences. $\square$

Since $H^i(V) = 0$ for $i > 0$, then $H^i(j_0^!\mathcal{F}) = 0$ for $i > 0$ and hence the only possibly nonzero term is $H^0(j_0^!\mathcal{F})$ since $j_0^!\mathcal{F} \in D^b_{\mathrm{locf}}(\{0\})^{\geq 0}$.

Consider $H^{-1}(j_0^!\mathcal{F}) = \ker(H^{-1}(V_\bullet) \to H^{-1}(W_\bullet)) = 0$. Then $H^{-1}(V_\bullet) \hookrightarrow H^{-1}(W_\bullet)$ must be injective. Thus these t-structures are equivalent.

2.1 Intersection homology theory

2.1.1 Intersection Homology following [5]

Following [5] and [8], we will define the intersection homology of algebraic varieties. This homology theory is a homology on singular spaces that satifies the Poincaré duality, whereas the usual singular homology does not.

Let X be a variety. Let $(X_s)_{s\in\mathscr{S}}$ be a stratification of X and suppose that X has a triangulation such that each X_s is a union of simplices. We will consider simplicial chains on X.

Let $C_i^T(X)$ be the set of all locally finite simplicial i-chains on X with respect to the triangulation T, which is the set of formal linear combinations $\xi = \sum_{\sigma \text{ an } i\text{-chain}} \xi_\sigma \sigma$ where for every $x \in X$, there exists an open set U_x with $x \in U_x$ such that the set $\{\xi_\sigma : \xi_\sigma \neq 0, \sigma^{-1}(U_x) \neq \emptyset\}$ is finite.

Definition 2.21. For a locally finite $\xi = \sum_\sigma \xi_\sigma \sigma \in C_i^T(S)$, the *support* $|\xi|$ of ξ is the union of the closures of all such σ such that $\xi_\sigma \neq 0$.

An intersection i-chain on X with respect to T is a $(\dim_{\mathbb{C}} X - i)$-chain $\xi \in C^T_{(\dim_{\mathbb{C}} X - i)}(X)$ such that

$$\dim_{\mathbb{R}}(|\xi| \cap X_s) \leq -i + \dim_{\mathbb{C}} X_s - 1$$
$$\dim_{\mathbb{R}}(|\partial\xi| \cap X_s) \leq -i + \dim_{\mathbb{C}} X_s - 2$$

for $s \in \mathscr{S}$ such that $\dim_{\mathbb{C}} X_s \leq \dim_{\mathbb{C}} X - 1$.

The set of all intersection i-chains with respect to T is denoted $\mathrm{IC}^i_T(X) \subset C^T_{(\dim_{\mathbb{C}} X - i)}(X)$. By taking the direct limit under refinement of triangulations of X, we obtain the space $\mathrm{IC}^i(X)$ the set of intersection i-chains of X.

The boundary maps in for the simplicial chains restrict to maps on the intersection chains so that we have a complex $\mathrm{IC}^\bullet(X)$. Now we can define the intersection homology groups.

Definition 2.22. The i^{th} intersection homology group of X is

$$IH^i(X) = \frac{\ker(IC^i(X) \xrightarrow{\partial} IC^{i-1})}{\operatorname{coker}(IC^{i+1}(X) \xrightarrow{\partial} IC^i)}$$

These groups are nonzero for $i = -\dim_{\mathbb{C}} X, \ldots, \dim_{\mathbb{C}} X$.

Remark 2.23. Intersection homology theory can be defined in a more general setting, in particular when X is a *psuedomanifold*.

2.1.2 The IC sheaf defined in [6]

In their next paper [6], Goresky and MacPherson defined the IC sheaf as an element of $D^b(X)$, we will denote it by IC_X.

We have defined the complex $\mathrm{IC}_\bullet(X)$. To turn this into a complex of sheaves, for every $V \subset U$ open we need a restriction map from $\mathrm{IC}^i_X(U) \to \mathrm{IC}^i_X(V)$. The natural map goes from $\mathrm{IC}^i(V) \to \mathrm{IC}^i(U)$ so we need to use barycentric subdivision to define this restriction map.

Let $V \subset U$ and consider an i-simplex $\sigma \in \mathrm{IC}^i(U)$. We want to define $\sigma|_V = \sum_{\tau \in J} \tau$ for some set of simplices $J \subset \mathrm{IC}^i(V)$.

If $\mathrm{im}(\sigma) \subset V$ set $J = \{\sigma\}$. If $\mathrm{im}(\sigma) \not\subset V$, preform a barycentric subdivision of σ and consider every τ in the subdivision of σ. If $\mathrm{im}(\tau) \subset V$, then add τ to the set J. If $\mathrm{im}(\tau) \not\subset V$, then preform a barycentric subdivision of τ and repeat the process of adding i-simplicies of this subdivion to J if their image is in V and preforming a barycentric subdivision if their images are not in V.

In this way, we define J and so we set $\sigma|_V = \sum_{\tau \in J} \tau$. For arbitrary $\xi \in \mathrm{IC}^\bullet(X)$, let $\xi|_V = \sum_{\sigma \in \mathrm{IC}_i(X)} \xi_\sigma \sigma|_V$.

We can define the IC_X sheaf by

$$\mathrm{IC}^i_X(U) = \mathrm{IC}^i(U)$$

with restriction map defined above. This commutes with the boundary map on chains $(\partial\xi)|V = \partial(\xi|_V)$ so that we have a boundary operator on this sheaf. Thus we have a complex of sheaves $\mathrm{IC}^\bullet_X$.

Lemma 2.24. *The IC sheaves* IC^i_X *are soft for all* $i \leq 0$ *so that*

$$\mathbf{H}^\bullet(X, \mathrm{IC}^\bullet_X) = IH^\bullet(X, \mathbb{C})$$

In the following section, we will we that $\mathcal{IC}^\bullet(X)$ lies in $\mathrm{Perv}(X)$ when X is complex algebraic variety.

2.1.3 IC sheaves

In this section, we give a sheaf theoretic definition of the IC sheaf, without using topology.

Definition 2.25. Let $h : Y \hookrightarrow X$ be a locally closed embedding. The *intermediate-extension functor* is the functor

$$h_{!*} : \mathrm{Perv}(Y) \to \mathrm{Perv}(X)$$

given by

$$h_{!*}(\mathcal{F}) = \mathrm{im}({}^p\mathbf{H}^0(h_!\mathcal{F}) \to {}^p\mathbf{H}^0(h_*\mathcal{F}))$$

Remark 2.26. By definition, there are morphisms ${}^p\mathbf{H}^0(h_!\mathcal{F}) \to h_{!*}(\mathcal{F})$ and $h_{!*}(\mathcal{F}) \to {}^p\mathbf{H}^0(h_*\mathcal{F})$.

Definition 2.27. Let X be a variety, $Y \subset X$ a smooth, connected, locally closed subvariety and let $h : Y \hookrightarrow X$ be the inclusion. Let $\mathcal{L}$ be a local system of finite type on Y. The *intersection cohomology complex* associated to $(Y, \mathcal{L})$ is the perverse sheaf

$$\mathrm{IC}(Y, \mathcal{L}) = h_{!*}(\mathcal{L}[\dim Y])$$

Remark 2.28. Recall that $\mathcal{L}[\dim Y]$ is perverse so that $h_{!*}$ is indeed a map from $\mathrm{Perv}(Y)$.

The main goal of this section is to prove the following theorem.

Theorem 2.29 (Theorem 4.2.17 of [1])**.**

1. *If $Y \subset X$ is a smooth, connected locally closed subvariety and $\mathcal{L}$ is an irreducible local system on Y, the* $\mathrm{IC}(Y, \mathcal{L})$ *is a simple object in* $\mathrm{Perv}(X)$*. We will call these simple intersection cohomology complexes.*
2. *Every perverse sheaf admits a finite filtration whose subquotients are simple intersection cohomology complexes.*
3. *Every simple object in* $\mathrm{Perv}(X)$ *is of the form described in 1.*

Example 2.30. Consider X smooth, connected and consider the local system $\underline{\mathbb{C}}_X$. Then $\mathrm{IC}(X, \underline{\mathbb{C}}_X) = \underline{\mathbb{C}}_X[\dim X]$.

Example 2.31. Consider $\mathbb{P}^1$ with the standard stratification $\{\{0\}, \mathbb{A}^1\}$. We consider the locally closed subvarieties $\{0\}, \mathbb{A}^1$. Then we have the simple intersection cohomology complexes:

$$\mathrm{IC}(\{0\}, \mathbb{C}) = j_{0!*}(\mathbb{C}) = \underline{\mathbb{C}}^0 \text{ since } j_0 \text{ is proper}$$
$$\mathrm{IC}(\mathbb{A}^1, \mathbb{C}_{\mathbb{A}^1}) = \mathbb{C}_{\mathbb{P}^1}[1] \text{ by Example 2.30}$$

which are, respectively, the following objects

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, hook] & \mathbb{C} & \mathbb{C} \arrow[r] \arrow[d, hook, "1"] & 0 \\ 0 & & \mathbb{C} & \end{tikzcd}$$

Example 2.32. Let G be an algebraic group, let $B \subset G$ be a borel subgroup. Then G/B has the Bruhat stratification, with strata $BwB/B \cong \mathbb{C}^{\ell(w)}$. The simple cohomology complexes are $\mathrm{IC}(BwB/B, \mathbb{C}) = h_{!*}\mathbb{C}[\ell(w)]$. The category $\mathrm{Perv}(X)$ is equivalent to the BGG category $\mathcal{O}$.

The following two lemmas tell us why we should consider the intermediate-extension functor.

Lemma 2.33. *Let $h : Y \hookrightarrow X$ be a locally closed embedding.*

1. *For $\mathcal{F} \in$ $\mathrm{Perv}(Y)$, there is a natural isomorphism $h^* h_{!*}\mathcal{F} \cong \mathcal{F}$.*
2. *For $\mathcal{F} \in$ $\mathrm{Perv}(Y)$, the object $h_{!*}\mathcal{F}$ has no nonzero subobjects or quotients supported on $\overline{Y} \backslash Y$.*

Proof. For (1), let $\mathcal{F} \in \mathrm{Perv}(Y)$. Without loss of generality, we can assume that $X = \overline{Y}$ as $\operatorname{supp} h_*\mathcal{F}, \operatorname{supp} h_!\mathcal{F} \subset \overline{Y}$. Now h is an open embedding.

$$\begin{aligned} h^* h_{!*}\mathcal{F} &= h^*(\mathrm{im}({}^p\mathbf{H}^0(h_!\mathcal{F}) \to {}^p\mathbf{H}^0(h_*\mathcal{F}))) \\ &= \mathrm{im}(h^*\,{}^p\mathbf{H}^0(h_!\mathcal{F}) \to h^*\,{}^p\mathbf{H}^0(h_*\mathcal{F})), \ \text{[since } h^* \text{ is } t\text{-exact by Lemma 2.12]} \\ &= \mathrm{im}({}^p\mathbf{H}^0(h^*h_!\mathcal{F}) \to {}^p\mathbf{H}^0(h^*h_*\mathcal{F})) \end{aligned}$$

Since h is an open embedding, then $h^*h_!\mathcal{F} \cong h^*h_* \cong \mathcal{F}$ and ${}^p\mathbf{H}^0(\mathcal{F}) \cong \tau^{\geq 0}\tau^{\leq 0}\mathcal{F} \cong \mathcal{F}$ since $\mathcal{F}$ is a perverse sheaf. So

$$h^*h_{!*}\mathcal{F} \cong \mathrm{im}(\mathcal{F} \xrightarrow{id} \mathcal{F}) = \mathcal{F}$$

For (2), we will let $Z = X \backslash Y$ and let $i : Z \hookrightarrow X$ be the inclusion map. We will show that $h_{!*}\mathcal{F}$ has no quotients supported on Z, and the case of subobjects will follow from Verdier duality.

Suppose that $\mathcal{G} \in \mathrm{Perv}(X)$ such that $\mathcal{G}$ is a quotient of $h_{!*}\mathcal{F}$, i.e. there is a surjective map $h_{!*}\mathcal{F} \to \mathcal{G}$. Consider the natural map ${}^p\mathbf{H}^0(h_!\mathcal{F}) \to h_{!*}\mathcal{F}$, then we have a surjective map ${}^p\mathbf{H}^0(h_!\mathcal{F}) \to \mathcal{G}$. But by Lemma 2.12, $h_!\mathcal{F} \in {}^pD^b_c(X)^{\leq 0}$ so that $\tau^{\leq 0}h_!\mathcal{F} \cong h_!\mathcal{F}$ and hence ${}^p\mathbf{H}^0(h_!\mathcal{F}) \cong \tau^{\geq 0}h_!\mathcal{F}$. Then this surjective map ${}^p\mathbf{H}^0(h_!\mathcal{F}) \to \mathcal{G}$ is in $\mathrm{Hom}(\tau^{\geq 0}h_!\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(h_!\mathcal{F}, \mathcal{G})$ where the latter isomorphism holds by the adjunction property of $\tau^{\geq 0}$.

But by Prop 2.14, there exists $\mathcal{H} \in \mathrm{Perv}(Z)$ such that $\mathcal{G} \cong i_*\mathcal{H}$ so that this map is in

$$\mathrm{Hom}(h_!\mathcal{F}, i^*\mathcal{H}) \cong \mathrm{Hom}(\mathcal{F}, h^!i_*\mathcal{H})$$

by the adjunction of $h_!$. But $h^!i_*\mathcal{H} = 0$, which contradicts that $\mathcal{G} \neq 0$. □

Lemma 2.34. *Let $h : Y \hookrightarrow X$ be a locally closed embedding.*

The functor $h_{!*} : \mathrm{Perv}(Y) \to \mathrm{Perv}(X)$ is fully faithful. For $\mathcal{F} \in \mathrm{Perv}(Y)$ the object $h_{!*}\mathcal{F}$ is the unique perverse sheaf (up to isomorphism) with the following properties:

1. *It is supported on $\overline{Y}$.*
2. *Its restriction to Y is isomorphic to $\mathcal{F}$.*
3. *It has no nonzero subobjects or quotients supported on $\overline{Y} \backslash Y$*

Proof. Without loss of generality, let $X = \overline{Y}$. Let $i : Z \hookrightarrow X$ be the complementary closed embedding of h. Let $\mathrm{Perv}^\circ(X)$ be the set of perverse sheaves on X which have no nonzero subobjects or quotients supported on Z.

We will show that $h^* : \mathrm{Perv}^\circ(X) \to \mathrm{Perv}(Y)$ is fully faithful, then by the previous lemma, $h^*h_{!*}\mathcal{F} \cong \mathcal{F}$ so that $h_{!*}$ is a right inverse of h^* and hence also fully faithful. So all we need to show is that $\mathrm{Hom}_{\mathrm{Perv}^\circ(X)}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{Perv}(Y)}(h^*\mathcal{F}, h^*\mathcal{G})$.

Step 1: Obtain a long exact sequence with $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ and $\mathrm{Hom}(h^*\mathcal{F}, h^*\mathcal{G})$.

Let $\mathcal{F} \in \mathrm{Perv}^\circ(X)$ and consider the distinguished triangle

$$h_!h^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to$$

Let $\mathcal{G} \in \mathrm{Perv}^\circ(X)$ and apply $\mathrm{Hom}(-, \mathcal{G})$ to get the long exact sequence

$$\begin{aligned} \cdots \to \mathrm{Hom}(h_!h^*\mathcal{F}[1], \mathcal{G}) \to \mathrm{Hom}(i_*i^*\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(h_!h^*\mathcal{F}, \mathcal{G}) \to \\ \to \mathrm{Hom}(i_*i^*\mathcal{F}[-1], \mathcal{G}) \to \cdots \end{aligned}$$

By adjunction and using $h^* = h^!$ and $i_* = i_!$, this complex coincides with

$$\begin{aligned} \cdots \to \mathrm{Hom}(h^*\mathcal{F}, h^*\mathcal{G}[-1]) \to \mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(h^*\mathcal{F}, h^*\mathcal{G}) \to \\ \to \mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{G}[1]) \to \cdots \quad (1) \end{aligned}$$

We want to show all the terms except $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ and $\mathrm{Hom}(h^*\mathcal{F}, h^*\mathcal{G})$ are zero.

Step 2: For $\mathcal{F} \in \mathrm{Perv}^\circ(X)$, show $i^*\mathcal{F} \in {}^pD^b_c(Z)^{\leq -1}$ and $i^!\mathcal{F} \in {}^pD^b_c(X)^{\geq 1}$.

Since $\mathcal{F} \in \mathrm{Perv}(X)$, then using Lemma 2.12, we see that $i_*i^*\mathcal{F} \in {}^pD^b_c(X)^{\leq 0}$ and $h_!h^*\mathcal{F} \in {}^pD^b_c(X)^{\leq 0}$. By applying ${}^p\mathbf{H}^0$, we get the long exact sequence,

$$\cdots \to {}^p\mathbf{H}^0(\mathcal{F}) \to {}^p\mathbf{H}^0(i_*i^*\mathcal{F}) \to {}^p\mathbf{H}^0(h_!h^*\mathcal{F}[1]) \to \cdots$$

But $h_!h^*\mathcal{F}[1] \in {}^pD^b_c(X)^{\leq -1}$ so that $\tau^{\geq 0}h_!h^*\mathcal{F}[1] = 0$. Then ${}^p\mathbf{H}^0(h_!h^*\mathcal{F}[1]) = 0$ and so ${}^p\mathbf{H}^0(\mathcal{F}) \to {}^p\mathbf{H}^0(i_*i^*\mathcal{F})$ is surjective. But ${}^p\mathbf{H}^0(i_*i^*\mathcal{F})$ is supported on Z so since ${}^p\mathbf{H}^0(\mathcal{F}) = \mathcal{F}$ has no nonzero quotients supported on Z, then ${}^p\mathbf{H}^0(i_*i^*\mathcal{F}) = 0$, which means $i_*i^*\mathcal{F} \in {}^pD^b_c(X)^{\leq -1}$. By Proposition 2.14, i_* is an equivalence of categories so i_* is t-exact and fully faithful so in fact $i^*\mathcal{F} \in {}^pD^b_c(Z)^{\leq -1}$.

Similarly, by using the distinguished triangle $i_*i^!\mathcal{F} \to \mathcal{F} \to h_*h^*\mathcal{F} \to$, we show that $i^!\mathcal{F} \in {}^pD^b_c(Z)^{\geq 1}$ for $\mathcal{F} \in \mathrm{Perv}^\circ(X)$.

Step 3: Show $\mathrm{Hom}(h^*\mathcal{F}, h^*\mathcal{G}[-1]) = 0$

By Lemma 2.12, h^* is t-exact so $h^*\mathcal{F} \in {}^pD^b_c(X)^{\leq 0}$, and $h^*\mathcal{G}[-1] \in {}^pD^b_c(X)^{\geq 1}$. Then it follows that $\mathrm{Hom}(h^*\mathcal{F}, h^*\mathcal{G}[-1]) = 0$.

Step 4: Show $\mathrm{Hom}(h^*\mathcal{F}, i^!\mathcal{G}) = 0$

By Lemma 2.12, $i^*\mathcal{F} \in {}^pD^b_c(Z)^{\leq 0}$ and Step 2, $i^!\mathcal{G} \in {}^pD^b_c(Z)^{\geq 1}$ so that $\mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{G}) = 0$.

Step 5: Show $\mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{G}[1])0$

By Step 2, $i^*\mathcal{F} \in {}^pD^b_c(Z)^{\leq -1}$ and $i^!\mathcal{G}[1] \in {}^pD^b_c(Z)^{\geq 0}$ so that $\mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{G}[1]) = 0$.

Hence, (1) gives that $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(h^*\mathcal{F}, h^*\mathcal{G})$ so that h^* is indeed fully faithful. □

Corollary 2.35. *Let $h : Y \hookrightarrow X$ be a locally closed embedding. The functor $h_{!*}$ takes injective maps to injective maps and surjective maps to surjective maps.*

Proof. Exercise. □

Proposition 2.36. *Let X be a smooth, connected variety of dimension n. The category of Loc$(X)[n]$ is a Serre subcategory* $\mathrm{Perv}(X)$.

Proof. For proof, see Proposition 4.2.12 on page 280 of [1]. □

Lemma 2.37. *Let $\mathcal{F} \in \mathrm{Perv}(X)$ and let $i : Z \hookrightarrow X$ be the inclusion of a closed subvariety.*

1. *${}^p\mathbf{H}^0(i_*i^!\mathcal{F})$ is the largest subobject of $\mathcal{F}$ supported on Z.*
2. *${}^p\mathbf{H}^0(i_*i^*\mathcal{F})$ is the largest quotient of $\mathcal{F}$ supported on Z.*

Proof. For part (1), let $j : U \to X$ be the complementary open inclusion. Consider the the distinguished triangle $i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F} \to$. By Lemma 2.12, j^* and i_* are t-exact and $i^!$ and j_* are left t-exact for the perverse t-structure, so all these terms are in ${}^pD^b_c(X)^{\geq 0}$. By applying the perverse zero$^{\text{th}}$ cohomology functore, we get a long exact sequence which starts with ${}^p\mathbf{H}^0(i_*i^!\mathcal{F})$, so that ${}^p\mathbf{H}^0(i_*i^!\mathcal{F})$ is a subobject.

Suppose $\mathcal{G} \in \mathrm{Perv}(X)$ such that $\mathrm{supp}\,\mathcal{G} \subset Z$. Then by Proposition 2.14, there is some $\mathcal{H} \in \mathrm{Perv}(Z)$ such that $i_*\mathcal{H} = \mathcal{G}$. By applying $\mathrm{Hom}(\mathcal{G}, -)$ to the distinguished triangle above,

$$\cdots \to \mathrm{Hom}(i_*\mathcal{H}, j_*j^*\mathcal{F}[-1]) \to \mathrm{Hom}(\mathcal{G}, i_*i^!\mathcal{F}) \to \mathrm{Hom}(\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}(i_*\mathcal{H}, j_*j^*\mathcal{F}) \to \cdots$$

By adjunction for j_*, the first and last terms become $\mathrm{Hom}(j^*i_*\mathcal{H}, j^*\mathcal{F}[-1])$ and $\mathrm{Hom}(j^*i_*\mathcal{H}, j^*\mathcal{F})$ respectively. But $j^*i_* = 0$ so that these terms are zero. Then each map $\mathcal{G} \to \mathcal{F}$ factors uniquely through $\mathcal{G} \to i_*i^!\mathcal{F}$. Since $\mathcal{G}, \mathcal{F} \in \mathrm{Perv}(X)$ and $i_*i^!\mathcal{F} \in {}^pD^b_c(X)^{\geq 0}$, then when we apply ${}^p\tau^{\leq 0}$ to these maps, we have a unique map $\mathcal{G} \to {}^p\tau^{\leq 0}i_*i^!\mathcal{F} = {}^p\mathbf{H}^0(i_*i^!\mathcal{F})$ which makes the diagram in the lemma commute.

For part (2), the proof is analogous. □

Example 2.38. Let X be smooth and let $\mathcal{L}$ be a local system on X. Then for $h : U \hookrightarrow X$ a dense open subspace, by Lemma 2.34, $h_{!*}(\mathcal{L}[\dim X])|_U \cong \mathcal{L}[\dim X]|_U$ and it has no nonzero subobjects or quotients supported on $X \setminus U$. By the universal property in Lemma 2.37, the largest subobject supported on $X \setminus U$ is ${}^p\mathbf{H}^0(i_*i^!\mathcal{L}[\dim X]) = 0$ where $i : X \setminus U \hookrightarrow X$. Similarly, there are no nonzero quotients on $\mathcal{L}[\dim X]$ supported on $X \setminus U$ so that $\mathrm{IC}(U, \mathcal{L}|_U) = \mathcal{L}[\dim X]$.

Lemma 2.39. *Let X be an irreducible variety. Let $j : U \hookrightarrow X$ be the inclusion map of an open subset. Let $i : Z \hookrightarrow$ be the complementary closed subset. Let $\mathcal{F} \in Perv(X)$.*

1. *If $\mathcal{F}$ has no nonzero quotients supported on Z, then there is a natural short exact sequence*

$$0 \to {}^p\mathbf{H}^0(i_*i^!\mathcal{F}) \to \mathcal{F} \to j_{!*}(\mathcal{F}|_U) \to 0$$

2. *If $\mathcal{F}$ has non nonzero subobject supported on Z, then there is a natural short exact sequence*

$$0 \to j_{!*}(\mathcal{F}|_U) \to \mathcal{F} \to {}^p\mathbf{H}^0(i_*i^*\mathcal{F}) \to 0$$

Proof. For part (1), suppose $\mathcal{F}$ has no nonzero quotients supported on Z. Consider the injective map ${}^p\mathbf{H}^0(i_*i^!\mathcal{F}) \to \mathcal{F}$ given in Lemma 2.37 and let $\mathcal{K}$ be the cokernel of this map. We want to show that $j_{!*}(\mathcal{F}|_U) \cong \mathcal{K}$.

Since $\mathcal{K}$ is a quotient of $\mathcal{F}$, it is not supported on Z and has no quotients supported on Z. By the universal property of the injective map, $\mathcal{K}$ has no nonzero subobjects supported on Z. Finally, since $\mathcal{K}|_U \cong \mathcal{F}|_U$ then by Lemma 2.34, this object must be $j_{!*}\mathcal{F}|_U$. □

Lemma 2.40. *Let $Y \subset X$ be a smooth, connect, locally closed subvariety. Let $0 \to \mathcal{L}' \to \mathcal{L} \to \mathcal{L}'' \to 0$ by a short exact sequence of local systems on Y. Then $\mathrm{IC}(Y, \mathcal{L})$ admits a three step filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 = \mathrm{IC}(Y, \mathcal{L})$$

such that $\mathcal{F}_1 \cong \mathrm{IC}(Y, \mathcal{L}')$, $\mathcal{F}_3/\mathcal{F}_2 \cong \mathrm{IC}(Y, \mathcal{L}'')$ such that $\mathcal{F}_2/\mathcal{F}_1$ is supported on $\overline{Y} \setminus Y$.

Proof. Let $h : Y \hookrightarrow X$ be the inclusion. By Corollary 2.35, $h_{!*}$ takes injective maps to injective maps and surjective maps to surjective maps so that we have the maps

$$\mathrm{IC}(Y, \mathcal{L}') \xhookrightarrow{f} \mathrm{IC}(Y, \mathcal{L}) \xrightarrow{g} \mathrm{IC}(Y, \mathcal{L}'')$$

Since these maps come from a short exact sequence and $h_{!*}$ is a functor, then $g \circ f = 0$ and hence $\mathrm{im}(f) \subset \ker g$. Set $\mathcal{F}_1 = \mathrm{im}(F)$ so that $\mathcal{F}_1 \cong \mathrm{IC}(Y, \mathcal{L}')$ and set $\mathcal{F}_2 = \ker(g)$ so that $\mathrm{IC}(Y, \mathcal{L})/\mathcal{F}_2 \cong \mathrm{IC}(Y, \mathcal{L}'')$. All that is left is to show that $\mathcal{F}_2/\mathcal{F}_1$ is supported on $\overline{Y} \setminus Y$. Clearly, the support is contained in $\overline{Y}$. We want to show that $(\mathcal{F}_2/\mathcal{F}_1)|_Y = h^*(\mathcal{F}_2/\mathcal{F}_1) = 0$. But when we apply h^* to the sequence above, since $h^*h_{!*}\mathcal{F} \cong \mathcal{F}$, then

$$\mathcal{L}'[\dim Y] \hookrightarrow \mathcal{L}[\dim Y] \twoheadrightarrow \mathcal{L}''[\dim Y]$$

which is the original short exact sequence so that $\ker(g)|_Y = \mathrm{im}(f)|_Y$. Then it follows that $(\mathcal{F}_2/\mathcal{F}_1)|_Y = \ker(g)|_Y/\mathrm{im}(f)|_Y = 0$.

□

Proposition 2.41. *Every perverse sheaf admits a finite filtration whose subquotients are intersection cohomology complexes.*

Proof. We proceed by Noetherian induction.

Let $\mathcal{F} \in \mathrm{Perv}(X)$. Let $\mathscr{S}$ be a stratification that $\mathcal{F}$ is constructible with respect to. Let U be an open stratum of $\mathscr{S}$ and $j : U \hookrightarrow X$ be the inclusion. Let $i : Z \hookrightarrow X$ be the complementary closed inclusion.

Then $j^*\mathcal{F} = \mathcal{F}|_U \in \mathrm{Perv}(U)$ so by Example 2.7, since $j^*\mathcal{F}$ is constructible with respect to the trivial stratification, then $j^*\mathcal{F} \cong \mathcal{L}[\dim U]$ for some local system on U. Apply Lemma 2.37 to consider the short exact sequence

$$0 \to {}^p\mathbf{H}^0(i_*i^!\mathcal{F}) \to \mathcal{F} \to \mathcal{K}$$

where $\mathcal{K}$ is the cokernel of the injective map ${}^p\mathbf{H}^0(i_*i^!\mathcal{F}) \to \mathcal{F}$. Since ${}^p\mathbf{H}^0(i_*i^!\mathcal{F})$ is supported on Z, then we can apply the induction assumption to see that ${}^p\mathbf{H}^0(i_*i^!\mathcal{F})$ has a finite filtration whose subquotients are intersection cohomology complex. We will now show that the cokernel $\mathcal{K}$ also has such a filtration and thus $\mathcal{F}$ does as well.

Since ${}^p\mathbf{H}^0(i_*i^!\mathcal{F})$ is supported on Z, when we apply j^* to the distinguished triangle above, we get

$$0 \to j^*\mathcal{F} \to j^*\mathcal{K} \to 0$$

Hence $j^*\mathcal{K} \cong j^*\mathcal{F} \cong \mathcal{L}[\dim U]$. By the universal property from Lemma 2.37, $\mathcal{K}$ has no subobjects supported on Z and so we may apply Lemma 2.39 to $\mathcal{K}$.

$$0 \to j_{!*}(\mathcal{K}|_U) \to \mathcal{K} \to {}^p\mathbf{H}^0(i_*i^*\mathcal{K}) \to 0$$

Again, by induction ${}^p\mathbf{H}^0(i_*i^*\mathcal{K})$ has a finite filtration whose subquotients are intersection cohomology complexes and the first term is a intersection cohomology complex and thus $\mathcal{K}$ has such a filtration as well. □

Remark 2.42. Proposition 2.41 implies that every simple object in $\mathrm{Perv}(X)$ is an IC sheaf of a local system.

Proof of part 1) of Theorem 2.29. We want to show that $\mathrm{IC}(Y, \mathcal{L})$ is a simple object in $\mathrm{Perv}(X)$, i.e. we need to show there are no non-trivial subobjects.

Suppose that $\mathcal{F} \subset \mathrm{IC}(Y, \mathcal{L})$ is a nonzero subobject. Then by Proposition 2.14, we can assume that $X = \overline{Y}$ so that $h : Y \to X$ is an open embedding. Since $\mathcal{F}$ is not supported on $X \setminus Y$, then $h^*\mathcal{F} \neq 0$. h^* is t-exact so $h^*\mathcal{F}$ is a subobject in $\mathcal{L}[\dim X]$.

By Proposition 2.36, $\mathrm{Loc}(X)[\dim X]$ is a Serre subcategory and thus closed under taking subobjects. So $h^*\mathcal{F}$ is a sub-local system of $\mathcal{L}[\dim X]$ and hence it coincides with $\mathcal{L}[\dim X]$. Then the cokernel of $\mathcal{F}$ inside $\mathrm{IC}(Y, \mathcal{L})$ is supported on $X \setminus Y$, but $\mathrm{IC}(Y, \mathcal{L})$ has no subquotients supported on $X \setminus Y$ by Lemma 2.34 so that the cokernel must be zero. Hence $\mathcal{F} \cong \mathrm{IC}(Y, \mathcal{L})$.

□

Proof part 2) of Theorem 2.29. We proceed by Noetherian induction. We assume that the statement is true for all proper closed subvarieties of X.

By Proposition 2.41, we only need to prove that $\mathrm{IC}(Y, \mathcal{L})$ has a finite filtration whose subquotients are simple intersection cohomology complexes.

Pick $y_0 \in Y$. Then by the correspondence between local systems and representations of $\pi_1(Y, y_0)$, $\mathcal{L}$ correspondes to a finite dimensional representation M. Since M is a representation, it has a finite filtration whose subquotients are irreducible representations and thus by this correspondence, $\mathcal{L}$ has a finite filtration by sub local systems such that the quotients are irreducible local systems. Now we apply Lemma 2.40 multiple times to get quotients that are either IC sheaves of irreducible local systems or quotients supported on $\overline{Y} \setminus Y$. The former are simple IC sheaves while the later have the desired filtration by induction. □

Proof of part 3) of Theorem 2.29. This follows from Remark 2.42. □

Lemma 2.43 (Duality)**.** *Let $Y \subset X$ be a smooth, connected, locally closed subvariety and let $\mathcal{L}$ be a local system on Y. Then there is a natural isomorphism*

$$\mathbb{D}(\mathrm{IC}(Y, \mathcal{L})) \cong \mathrm{IC}(Y, \mathcal{L}^\vee)$$

Finally consider the following lemma:

Lemma 2.44 (Lemma 4.2.8 of [1]). *Let $\mathcal{F} \in \mathrm{Perv}(X)$. Let $(X_s)_{s\in\mathscr{S}}$ be a stratifications such that both $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$ are constructible with respect to $\mathscr{S}$. Let $u \in \mathscr{S}$ and let $\mathcal{L}$ be a local system of finite type on X_u. The following conditions are equivalent:*

1. *$\mathcal{F} \cong IC(X_u, \mathcal{L})$*
2. *We have $\operatorname{supp}\mathcal{F} \subset \overline{X_u}$ and $\mathcal{F}|_{X_u} \cong \mathcal{L}[\dim X_u]$ and for each stratum $X_t < X_u$ if $j_t : X_t \hookrightarrow X$ we have*

$$j_t^*\mathcal{F} \in D^b_{locf}(X_t)^{\leq -\dim X_t - 1} \qquad\qquad j_t^!\mathcal{F} \in D^b_{locf}(X_t)^{\geq -\dim X_t + 1}$$

Proof. For proof, see Lemma 4.2.8 of [1]. □

Remark 2.45. This lemma shows us that for IC sheaves, the requirements for cohomology vanishing is one degree stricter than for general of perverse sheaves.

Theorem 2.46. $\mathrm{IC}_X \cong \mathrm{IC}(Y, \mathbb{C})$*, where Y is any smooth subset of X and IC_X is as defined in section 2.1.2.*

Sketch of proof. We prove that IC_X satisfies part 2. of Lemma 2.44 when $\mathcal{L} = \mathbb{C}_{X_u}$. For convenience, let $n = \dim_{\mathbb{C}} X$.

Step 1: Prove that $\mathrm{IC}_X|_{X_u} \cong \mathbb{C}[n]$.

Consider $\mathrm{IC}_X|_{X_u}$. Then there are no added conditions on $|\xi| \cap X_u$ by definition, so every i-intersection chain is just an $(n-i)$-chain. Let $C^\bullet$ be the sheaf of locally finite i-chains on X_u so that $\mathrm{IC}_X|_{X_u} = C^\bullet$.

Let $S^i(X_u)$ be the group of singular i-cochains with coefficients in $\mathbb{C}$ and let $S^\bullet$ be the chain complex. Since X_u is locally contractible, then using the proof of Theorem 3.15 from Roger's talk [3], $S^{\bullet,+} \cong \underline{\mathbb{C}}_{X_u}$ are quasi-isomorphic. For $U \subset X_u$ open, we have a map $S^{2n-i}(U) \to C_i(U)$ by $\alpha \mapsto [U] \cap \alpha$, where $[U]$ is the fundemental class of U. This induces a map $S^{2n-1,+}(U) \to C_i(U) = \mathrm{IC}^{n-i}(U)$. By Poincaré duality, this is a quasi-isomorphism and hence $S^{\bullet,+}[n] \cong \mathrm{IC}_X\,|_{X_u}$ so that it follows that $\underline{\mathbb{C}}_{X_u}[n] \cong \mathrm{IC}_X|_{X_u}$.

Remark 2.47. It is a more general fact that $\omega_X \cong C^\bullet$.

Step 2: Prove that $H^i(j_t^*\,\mathrm{IC}_X) = 0$ for all $i > -\dim X_t - 1$.

For this proof, we follow [6]. Consider X_t and let $i_x : \{x\} \to X$ be the inclusion of some $x \in X_t$.

fact: For X a complex variety and $X_s \subset X$ smooth, there exists an $N \subset X$ open such that $x \in N$ and there exists some L_x such that $\mathbb{R}^{2\dim X_s} \times \mathrm{cone}(L_x) \cong N$.

Claim: The stalk at $x \in X_t$ of $H^\bullet(\mathrm{IC}_X)$ is given by

$$i_x^* H^i(\mathrm{IC}_X) = \begin{cases} IH^{-i-\dim X_t - 1}(L_x) & i \leq -\dim X_t - 1 \\ 0 & i \geq -\dim X_t \end{cases}$$

Idea of proof: If $i > -\dim X_t - 1$, then $-i < \dim X_t + 1$, then any intersection i-chain ξ will have

$$\dim_{\mathbb{R}}(|\xi| \cap X_t) \leq -i + \dim X_t - 1 < \dim_{\mathbb{R}} X_t$$

So ξ intersections X_t in a subset of dimension less that $\dim_{\mathbb{R}} X_t$. By transversailty, it can be moved away from $\{x\}$. Thus $\xi = 0$ in the local homology group.

If $i \leq -\dim X_t - 1$, the idea is that $\dim_{\mathbb{R}}(|\xi| \cap X_t) \geq \dim_{\mathbb{R}} X_t$ so that any cycle that contains $\{x\}$ also contains a neighborhood of $\{x\}$. This means that locally this intersection homology looks like $\mathbb{R}^{\dim X_t} \times \mathrm{cone}(L_x)$.

So we know for every $x \in X_t$, $i_x^* H^i(\mathrm{IC}_X) = 0$ for $i \geq -\dim X_t$ or, equivalently, for $i < -\dim X_t - 1$. As $H^i(i_x^*\,\mathrm{IC}_X) = i_x^* H^i(\mathrm{IC}_X)$, then we have for every $x \in X_t$, $H^i(i_x^*\,\mathrm{IC}_X) = 0$ when $i < \dim X_t - 1$. But since this is true for every $x \in X_t$, then $H^i(j_t^*\,\mathrm{IC}_X) = 0$ for all $i < -\dim X_t - 1$.

Step 3: Prove that $H^i(j_t^!\,\mathrm{IC}_X) = 0$ for all $i < -\dim X_t + 1$.

We will omit this step. □

Remark 2.48. Recall that $\mathbf{H}^i(X, \mathrm{IC}_X^\bullet)) = IH_i(X, \mathbb{C})$, where $\mathrm{IC}_X^\bullet \cong \mathrm{IC}(Y, \mathbb{C})$ for some smooth Y. Define $IH_c(X) := H^i R\Gamma_c(\mathrm{IC}_X^\bullet)$. Then $\mathbb{D}(\mathrm{IC}(Y,\mathbb{C})) \cong \mathrm{IC}(Y,\mathbb{C}))$ so that $\mathbb{D}(\mathrm{IC}_X) \cong \mathrm{IC}_X$. Also, we know $f_! \circ \mathbb{D} = \mathbb{D} \circ f_*$ so since $R\Gamma = a_{X*}$ and $R\Gamma_c = a_{X!}$ then

$$\begin{aligned} IH_c^i(X) &\cong \mathbf{H}_c^i(\mathrm{IC}_X) \\ &= H^i R\Gamma_c \mathbb{D}(\mathrm{IC}_X) \\ &= H^i \mathbb{D} R\Gamma(\mathrm{IC}_X) \\ &= H^{-i} R\Gamma(\mathrm{IC}_X)^* \\ &= IH^{-i}(X)^* \end{aligned}$$

so that intersection homology obeys Poincaré duality.

2.2 References

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