Perverse Sheaves Learning Seminar: Derived Categories and its Applications to Sheaves

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1 Derived Categories

Unless otherwise stated, let $\mathscr{A}$ be an abelian category.

Definition 1.1. Let $Ch(\mathscr{A})$ be the category of chain complexes in $\mathscr{A}$. Objects in this category are chain complexes $A^\bullet$, which is a sequence of objects and morphisms in $\mathscr{A}$ of the form

$$\cdots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \longrightarrow \cdots$$

satisfying $d^i \circ d^{i-1} = 0$ for every $i \in \mathbb{Z}$.

A morphism $f : A^\bullet \to B^\bullet$ between two complexes is a collection of morphisms $f = (f^i : A^i \to B^i)_{i\in\mathbb{Z}}$ in $\mathscr{A}$ such that $f^{i+1} \circ d^i_A = d^i_B \circ f^i$ for every $i \in \mathbb{Z}$.

Definition 1.2. A chain complex $A^\bullet$ is said to be **bounded above** if there is an integer N such that $A^i = 0$ for all $i > N$. Similarly, $A^\bullet$ is said to be **bounded below** if there is an integer N such that $A^i = 0$ for all $i < N$. $A^\bullet$ is said to be **bounded** if it is bounded above and bounded below.

Let $Ch^-(\mathscr{A})$ (resp. $Ch^+(\mathscr{A})$, $Ch^b(\mathscr{A})$) denote the full subcategory of $Ch(\mathscr{A})$ consisting of bounded-above (resp. bounded-below, bounded) complexes.

Let $Ch^\circ(\mathscr{A})$ denote any of the four categories above. For a complex $A^\bullet$, let $[1] : Ch(\mathscr{A}) \to Ch(\mathscr{A})$ denote the shift functor where $A[1]^i = A^{i-1}$.

Definition 1.3. A **quasi-isomorphism** in $Ch(\mathscr{A})$ is a chain map $f : A^\bullet \to B^\bullet$ such that the induced maps $H^n(f) : H^n(A) \to H^n(B)$ are isomorphisms for all n.

The derived category for $\mathscr{A}$ can be thought of as a category obtained from $Ch(\mathscr{A})$ by having quasi-isomorphisms be actual isomorphisms. To do this, we localize (= invert) quasi-isomorphisms.

Definition 1.4. Let $\mathscr{A}$ be an additive category and let $\mathscr{S}$ be a class of morphisms in $\mathscr{A}$ closed under composition. Let $\mathscr{A}_\mathscr{S}$ be an additive category and let $L : \mathscr{A} \to \mathscr{A}_\mathscr{S}$ be an additive functor. We say $(\mathscr{A}_\mathscr{S}, L)$ is obtained by **localizing** $\mathscr{A}$ at $\mathscr{S}$ if $\mathscr{A}'$ is an additive category and $F : \mathscr{A} \to \mathscr{A}'$ is an additive functor that sends all morphisms of $\mathscr{S}$ to isomorphisms, then there exists a unique functor $\overline{F} : \mathscr{A}_\mathscr{S} \to \mathscr{A}'$ and a unique isomorphism $\epsilon : \overline{F} \circ L \xrightarrow{\sim} F$.

This is similar to the construction of localizing a ring. However, like in the case of localizing a ring, localizations may not always exist or be nice. Proposition I.6.3 of [2] state that a localization $\mathscr{A}_\mathscr{S}$ exists for $\mathscr{A}$ if $\mathscr{S}$ satisfies the following conditions:

L0 For every $X \in \mathscr{A}$, we have $id_X \in \mathscr{S}$.

L1 Given morphisms $f : X \to Y$ and $s : Z \to Y$ with $s \in \mathscr{S}$, there is a commutative diagram

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "t"'] & Z \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with $t \in \mathscr{S}$.

L2 Given morphisms $g : W \to Z$ and $t : W \to X$ with $t \in \mathscr{S}$, there is a commutative diagram

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "t"'] & Z \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with $s \in \mathscr{S}$.

L3 Given morphisms $f, g : X \to Y$, the following are equivalent:

- There is a morphism $t : Y \to Y'$ with $t \in \mathscr{S}$ such that $t \circ f = t \circ g$.
- There is a morphism $s : X' \to X$ with $s \in \mathscr{S}$ such that $f \circ s = g \circ s$.

The objects of $\mathscr{A}_{\mathscr{S}}$ are the same as $\mathscr{A}$, but the morphisms are "roofs".

Definition 1.5. Let $\mathscr{S}$ be a class of morphisms closed under composition. For $X, Y \in \mathscr{A}$, a **roof diagram** from X to Y is a diagram of morphisms

$$\begin{tikzcd} & W \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

with $s \in \mathscr{S}$. Two roof diagrams $X \xleftarrow{s} W \xrightarrow{f} Y$ and $X \xleftarrow{s'} W' \xrightarrow{f'} Y$ are equivalent if there is a commutative diagram in $\mathscr{A}$

$$\begin{tikzcd} & W \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & U \arrow[l, "u"'] \arrow[u] \arrow[d] & Y \\ & W' \arrow[ul, "s'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

with $u \in \mathscr{S}$.

If $\mathscr{S}$ satisfies L0-L3, one can identify $Hom_{\mathscr{A}_{\mathscr{S}}}(X, Y)$ with equivalence classes of roof diagrams, where composition of roof diagrams $X \xleftarrow{s} W \to Y$ and $Y \xleftarrow{s'} W' \to Z$ is a commutative diagram $X \leftarrow W' \to Z$ of the form

$$\begin{tikzcd} & & W'' \arrow[dl, "s''"'] \arrow[dr] & & \\ & W \arrow[dl, "s"'] \arrow[dr] & & W' \arrow[dl, "s'"'] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

with $s'' \in \mathscr{S}$. The existence of such a diagram follows from L1.

Remark 1.6. Basement diagrams can also be used instead of roof diagrams to describe $Hom_{\mathscr{A}_{\mathscr{S}}}(X, Y)$. These are diagrams of the form

$$\begin{tikzcd} X \arrow[dr] & & Y \arrow[dl, "s"'] \\ & W & \end{tikzcd}$$

where $s \in \mathscr{S}$.

Unfortunately, quasi-isomorphisms in $Ch(\mathscr{A})$ do not satisfy these conditions. To remedy this, instead of working with $Ch(\mathscr{A})$, we work with the homotopy category $K(\mathscr{A})$.

Definition 1.7. The **homotopy category** of $\mathscr{A}$, denoted by $K(\mathscr{A})$, is the category whose objects are those of $Ch(\mathscr{A})$, but whose morphisms are homotopy classes of chain maps. That is, $Hom_{K(\mathscr{A})}(A^\bullet, B^\bullet) := Hom_{Ch(\mathscr{A})}(A^\bullet, B^\bullet)/\sim$, where for two morphisms $f, g : A^\bullet \to B^\bullet$ in $Ch(\mathscr{A})$, we say $f \sim g$ if there exists a collection of morphisms $h^i : A^i \to B^{i-1}$, $i \in \mathbb{Z}$, such that

$$f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i.$$

As in the case of $Ch(A)$, let $K^-(\mathscr{A})$ (resp. $K^+(\mathscr{A})$, $K^b(\mathscr{A})$) denote the full subcategory of $K(\mathscr{A})$ of bounded-above (resp. bounded-below, bounded) complexes.

Let $K^\circ(\mathscr{A})$ denote any of the four homotopy categories.

Remark 1.8. One can show that $K^\circ(\mathscr{A})$ is equivalent to $Ch^\circ(\mathscr{A})$ localized at chain homotopies.

Proposition 1.9. *In $K^\circ(\mathscr{A})$ the class of quasi-isomorphisms satisfies L1-L3.*

Proof. See Section I.6 of *[2]*. □

Definition 1.10. The **derived category** (resp. bounded-above derived category, bounded-below derived category, bounded derived category) of $\mathscr{A}$, denoted $D(\mathscr{A})$ (resp. $D^-(\mathscr{A}), D^+(\mathscr{A}), D^b(\mathscr{A})$) is the category obtained from $K(\mathscr{A})$ (resp. $K^-(\mathscr{A}), K^+(\mathscr{A}), K^b(\mathscr{A})$) by localizing at the quasi-isomorphisms.

Let $D^\circ(\mathscr{A})$ denote any of the four derived categories.

Remark 1.11. For an object $A \in \mathscr{A}$, we can view A as a chain complex $A^\bullet$ where $A^0 = A$ and $A^i = 0$ for $i \neq 0$. This allows us to embed $\mathscr{A}$ into $D^b(\mathscr{A})$ as a full subcategory.

We will now proceed to the important notion of distinguished triangles.

Definition 1.12. Let $f : (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ be a chain map. The **chain-map cone (mapping cone)** of f, denoted by *cone*(f), is the chain complex given by

$$cone(f)^i = A^{i+1} \oplus B^i$$

with differential $d^i : cone(f)^i \to cone(f)^{i+1}$ given by

$$d^i = \begin{bmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{bmatrix}.$$

The inclusion maps $B^i \to cone(f)^i$ and projection maps $cone(f)^i \to A^{i+1}$ give chain maps

$$i_2 : B \to cone(f) \quad \text{and} \quad p_1 : cone(f) \to A[1]$$

Exercise 1.13. Show that the composition $B^\bullet \to cone(f) \to A^\bullet[1]$ is zero and the composition $A^\bullet \to B^\bullet \to cone(f)$ is homotopic to the zero map.

Definition 1.14. A diagram

$$A_1 \to A_2 \to A_3 \to A_1[1]$$

in $K^\circ(\mathscr{A})$ (resp. $D^\circ(\mathscr{A})$) is called a **distinguished triangle** if it is isomorphic in $K^\circ(\mathscr{A})$ (resp. $D^\circ(\mathscr{A})$) to a diagram of the form

$$A \xrightarrow{f} B \xrightarrow{i_2} cone(f) \xrightarrow{p_1} A[1]$$

for some chain map f.

An additive category with a shift functor (automorphism) and distinguished triangles (collection of diagrams) satisfying some natural axioms is called a triangulated category. The homotopy category $K^\circ(\mathscr{A})$ and the derived category $D^\circ(\mathscr{A})$ are natural examples.

Remark 1.15. If we have a distinguished triangle $X \to Y \to Z \to X[1]$, then it gives us a long exact sequence in cohomology

$$\cdots \longrightarrow H^k(X) \longrightarrow H^k(Y) \longrightarrow H^k(Z) \longrightarrow H^{k+1}(X) \longrightarrow \cdots.$$

2 Derived Functors

Definition 2.1. Let $\mathscr{T}$ and $\mathscr{T}'$ be triangulated categories (e.g. $D^b(\mathscr{A})$ and $D^b(\mathscr{A}')$). A **triangulated functor** is an additive functor $F : \mathscr{T} \to \mathscr{T}'$ with a natural isomorphism

$$F(X[1]) \cong F(X)[1]$$

such that for any distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathscr{T}$,

$$F(X) \to F(Y) \to F(Z) \to F(X)[1]$$

is a distinguished triangle in $\mathscr{T}'$.

Lemma 2.2. *If $F : \mathscr{A} \to \mathscr{B}$ is an additive functor of additive categories, the induced functor $F : K^\circ(\mathscr{A}) \to K^\circ(\mathscr{B})$ is a triangulated functor. If in addition F is an exact functor of abelian categories, the induced functor $F : D^\circ(\mathscr{A}) \to D^\circ(\mathscr{B})$ is a triangulated functor.*

Proof. Easy exercise. □

Recall that a complex $A^\bullet$ in $Ch^\circ(\mathscr{A})$ or $K^\circ(\mathscr{A})$ is called **acyclic** $H^i(A^\bullet) = 0$ for all i. If we have a functor F that is not exact, the image of an acyclic complex may not be acyclic, or it may not send quasi-isomorphisms to quasi-isomorphisms.

In the case of an exact functor F, we obtain a triangulated functor $\overline{F} : D^\circ(\mathscr{A}) \to D^\circ(\mathscr{B})$ and a natural isomorphism $\theta : L_{\mathscr{B}} \circ F \xrightarrow{\sim} \overline{F} \circ L_{\mathscr{A}}$ where $L_{\mathscr{A}} : K^\circ(\mathscr{A}) \to D^\circ(\mathscr{A})$ is the localization functor. Then in the case where F is not exact, the next best thing is to have a natural transformation in one direction.

Definition 2.3. Let $F : K^\circ(\mathscr{A}) \to K^\circ(\mathscr{B})$ be a triangulated functor. A **right derived functor** of F is a triangulated functor $RF : D^\circ(\mathscr{A}) \to D^\circ(\mathscr{B})$ with a natural transformation

$$\epsilon : L_{\mathscr{B}} \circ F \to RF \circ L_{\mathscr{A}}$$

that is universal in the following sense: if $G : D^\circ(\mathscr{A}) \to D^\circ(\mathscr{B})$ is another triangulated functor with a natural transformation $\phi : L_{\mathscr{B}} \circ F \to G \circ L_{\mathscr{A}}$, then there exists a unique functor morphism $\tilde{\phi} : RF \to G$ such that $\phi = (\tilde{\phi} L_{\mathscr{A}}) \circ \epsilon$, where $\tilde{\phi} L_{\mathscr{A}} : RF \circ L_{\mathscr{A}} \to G \circ L_{\mathscr{A}}$.

Similarly, a **left derived functor** of F is a triangulated functor $LF : D^\circ(\mathscr{A}) \to D^\circ(\mathscr{B})$ together with a natural transformation

$$\eta : LF \circ L_{\mathscr{A}} \to L_{\mathscr{B}} \circ F$$

that is universal in the following sense: if $G : D^\circ(\mathscr{A}) \to D^\circ(\mathscr{B})$ is another triangulated functor with a natural transformation $\phi : G \circ L_{\mathscr{A}} \to L_{\mathscr{B}} \circ F$, then there exists a unique functor morphism $\tilde{\phi} : G \to LF$ such that $\phi = \eta \circ (\tilde{\phi} L_{\mathscr{A}})$ where $\tilde{\phi} L_{\mathscr{A}} : G \circ L_{\mathscr{A}} \to LF \circ L_{\mathscr{A}}$.

Definition 2.4. Let $\mathscr{A}$ be an abelian category and $\mathscr{Q} \subset \mathscr{A}$ a full subcategory. $\mathscr{Q}$ is said to be **large enough on the right** if for any object $A \in \mathscr{A}$, there is an injective map $A \to X$ with $X \in \mathscr{Q}$.

Similarly, $\mathscr{Q}$ is said to be **large enough on the left** if for any object $A \in \mathscr{A}$, there is a surjective map $X \to A$ with $X \in \mathscr{Q}$.

Definition 2.5. Let $\mathscr{Q} \subset \mathscr{A}$ be a full subcategory.

1. Given $A \in Ch^\circ(\mathscr{A})$, a **right $\mathscr{Q}$-resolution** of A is a quasi-isomorphism $q : A \to Q$ such that $Q \in Ch^\circ(\mathscr{Q})$. For $A \in Ch^+(\mathscr{A})$, such a right resolution is said to be **strict** if $A \in Ch(\mathscr{A})^{\geq n}$ and $Q \in Ch(\mathscr{A})^{\geq n}$ for a fixed n.

2. Given $A \in Ch^\circ(\mathscr{A})$, a **left $\mathscr{Q}$-resolution** of A is a quasi-isomorphism $q : Q \to A$ such that $Q \in Ch^\circ(\mathscr{Q})$. For $A \in Ch^-(\mathscr{A})$, such a right resolution is said to be **strict** if $A \in Ch(\mathscr{A})^{\leq n}$ and $Q \in Ch(\mathscr{A})^{\leq n}$ for a fixed n.

Example 2.6. Consider the case of $A \in \mathscr{A}$ as a sequence $A^\bullet$ where $A^i = 0$ for $i \neq 0$ and $A^0 = A$. Then a strict right $\mathscr{Q}$-resolution $Q^\bullet$ of A is the same as giving an exact sequence

$$0 \longrightarrow A^0 \xrightarrow{q} Q^0 \xrightarrow{d_Q^0} Q^1 \xrightarrow{d_Q^1} \cdots$$

The map $q : A^0 \to Q^0$ is called the **augmentation map**.

Proposition 2.7. *Let $\mathscr{A}$ be an abelian category and let $\mathscr{Q} \subset \mathscr{A}$ be a full subcategory.*

1. *If $\mathscr{Q}$ is large enough on the right, then every object in $Ch^+(\mathscr{A})$ admits a strict right $\mathscr{Q}$-resolution.*
2. *If $\mathscr{Q}$ is large enough on the left, then every object in $Ch^-(\mathscr{A})$ admits a strict left $\mathscr{Q}$-resolution.*

Proof. 1) Exercise. Hint: Take an injection $q^0 : A^0 \to Q^0$ where $Q^0 \in \mathscr{Q}$. To construct Q^1, let $r : A^0 \to Q^0 \oplus A^1$ be given by $r = \begin{bmatrix} q^0 \\ -d_A^0 \end{bmatrix}$. Then choose an injection coker $r \to Q^1$ with $Q^1 \in \mathscr{Q}$. The map $q^1 : A^1 \to Q^1$ is the composition

$$A^1 \hookrightarrow Q^0 \oplus A^1 \twoheadrightarrow \operatorname{coker} r \to Q^1$$

and the differential $d_Q^0 : Q^0 \to Q^1$ is the composition

$$Q^0 \hookrightarrow Q^0 \oplus A^1 \twoheadrightarrow \operatorname{coker} r \to Q^1.$$

[WLOG, assume $A \in Ch^+(\mathscr{A})^{\geq 0}$. We want to construct a quasi-isomorphism $q = (q^i) : A^\bullet \to Q^\bullet$ where $Q^\bullet \in Ch^+(\mathscr{Q})^{\geq 0}$. As $\mathscr{Q}$ is large enough, we have an injection $q^0 : A^0 \to Q^0$. Suppose we have already constructed $Q^\bullet$ and maps $q^\bullet$ up to the ith step. Let $p : Q^{i-1} \twoheadrightarrow \operatorname{coker} d_Q^{i-2}$ be the quotient map. Let $r : A^{i-1} \to \operatorname{coker} d_Q^{i-2} \oplus A^i$ be the map given by $r = \begin{bmatrix} pq^{i-1} \\ -d_A^{i-1} \end{bmatrix}$. Let $s : \operatorname{coker} d_Q^{i-2} \oplus A^i \twoheadrightarrow \operatorname{coker} r$ be the quotient. Choose an injection $u : \operatorname{coker} r \to Q^i$ with $Q^i \in \mathscr{Q}$. Define $d_Q^{i-1} = u \circ s \circ i_1 \circ p$ and $q^i = u \circ s \circ i_2$ as in the diagram:

$$\begin{tikzcd}
A^{i-1} \arrow[rrrr, "d_A^{i-1}"] \arrow[drr, "r"] \arrow[ddd, "q^{i-1}"'] & & & & A^i \arrow[dll, "i_2"'] \arrow[ddd, "q^i"] \\
& & \operatorname{coker} d_Q^{i-2} \oplus A^i \arrow[dr, "s"] & & \\
& \operatorname{coker} d_Q^{i-2} \arrow[ur, "i_1"] & & \operatorname{coker} r \arrow[dr, "u"] & \\
Q^{i-1} \arrow[ur, "p"] \arrow[rrrr, "d_Q^{i-1}"] & & & & Q^i
\end{tikzcd}$$

Then check that $Q^\bullet$ is a complex and q is a chain map and quasi-isomorphism.] $\square$

Definition 2.8. Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor of abelian categories. A full subcategory $\mathscr{Q} \subset \mathscr{A}$ is said to be a **right adapted class** for F if it satisfies the following conditions:

1. The class $\mathscr{Q}$ is large enough on the right.
2. If $0 \to X' \to X \to X'' \to 0$ is a short exact sequence with $X', X \in \mathscr{Q}$, then $X'' \in \mathscr{Q}$.
3. For any short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathscr{Q}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.

Similarly, for a right exact functor $F : \mathscr{A} \to \mathscr{B}$, a full subcategory $\mathscr{Q} \subset \mathscr{A}$ is said to be a **left adapted class** for F if it satisfies the following conditions:

1. The class $\mathscr{Q}$ is large enough on the left.

2. If $0 \to X' \to X \to X'' \to 0$ is a short exact sequence with $X, X'' \in \mathscr{Q}$, then $X' \in \mathscr{Q}$.

3. For any short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathscr{Q}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.

Example 2.9. Let A be an algebra and A-mod be the category of A-modules. Then the full subcategory of projective modules $\mathcal{P}$ is large enough on the left and the full subcategory of injective modules $\mathcal{I}$ is large enough on the right.

Let M be an A-module. Then $Hom(M, -)$ is left exact with $\mathcal{I}$ as a right adapted class and $M \otimes -$ (or equivalently $- \otimes M$) is right exact with $\mathcal{P}$ as a left adapted class.

Lemma 2.10. *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories.*

1. *Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor, and let $\mathscr{Q}$ be a right adapted class for F. If $Q \in Ch^+(\mathscr{Q})$ is acyclic, then $F(Q)$ is acyclic. If $f : X \to Y$ is a quasi-isomorphism in $Ch^+(\mathscr{Q})$, then $F(f)$ is a quasi-isomorphism.*

2. *Let $F : \mathscr{A} \to \mathscr{B}$ be a right exact functor, and let $\mathscr{Q}$ be a left adapted class for F. If $Q \in Ch^-(\mathscr{Q})$ is acyclic, then $F(Q)$ is acyclic. If $f : X \to Y$ is a quasi-isomorphism in $Ch^-(\mathscr{Q})$, then $F(f)$ is a quasi-isomorphism.*

Proof. Suppose F is left exact and let $Q \in Ch^+(\mathscr{Q})$ be acyclic. Let $K^i = \text{im } d^{i-1} = \ker d^i$. Any left exact functor preserves kernels, then $F(K^i) \cong \ker F(d^i)$. Using induction, suppose im $F(d^{i-2}) = F(K^{i-1})$ and $K^{i-1} \in \mathscr{Q}$. We have a short exact sequence

$$\eta : 0 \to K^{i-1} \to Q^i \overset{d^{i-1}}{\to} K^i \to 0.$$

As $\mathscr{Q}$ is an adapted class, we have $K^i \in \mathscr{Q}$ and $F(\eta)$ is an exact sequence, so im $F(d^{i-1}) \cong F(K^i)$. Thus $F(Q)$ is acyclic.

Suppose $f : X \to Y$ is a quasi-isomorphism. Extend f to a distinguished triangle $X \overset{f}{\to} Y \to K \to$ in $K^+(\mathscr{Q})$. Note that f is a quasi-isomorphism if and only if K is acyclic, so apply the above result (apply cohomology to the triangle to get a long exact sequence of cohomology). ☐

Theorem 2.11. *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories.*

1. *If $F : \mathscr{A} \to \mathscr{B}$ is a left exact functor that admits a right adapted class, then it admits a right derived functor $RF : D^+(\mathscr{A}) \to D^+(\mathscr{B})$.*

2. *If $F : \mathscr{A} \to \mathscr{B}$ is a right exact functor that admits a left adapted class, then it admits a left derived functor $LF : D^-(\mathscr{A}) \to D^-(\mathscr{B})$.*

We will describe what the functor RF does on objects and morphisms. For $X \in Ch^+(\mathscr{A})$, choose a right $\mathscr{Q}$-resolution $q_X : X \to Q_X$. Define

$$RF(X) = F(Q_X).$$

Let $f : X \to Y$ be a morphism. As q_X is a quasi-isomorphism, we can form $\tilde{f} = q_Y \circ f \circ q_X^{-1} : Q_X \to Q_Y$. As a basement, $\tilde{f}$ can be represented by the diagram

$$\begin{tikzcd} Q_X \arrow[dr, "h"] & & Q_Y \arrow[dl, "s"'] \\ & W & \end{tikzcd}$$

where s is a quasi-isomorphism. Then $q_W \circ s : Q_Y \to Q_W$ is a quasi-isomorphism so $F(q_W \circ s)$ is a quasi-isomorphism. Define $RF(f) : RF(X) \to RF(Y)$ to be the basement diagram

$$\begin{tikzcd} RF(X) = F(Q_X) \arrow[dr, "F(q_W \circ h)"] & & RF(Y) = F(Q_Y) \arrow[dl, "F(q_W \circ s)"'] \\ & F(Q_W) & \end{tikzcd}$$

Define the natural transformation $\epsilon : L_{\mathscr{B}} \circ F \to RF \circ L_{\mathscr{A}}$ where for $X \in K^+(\mathscr{A})$, let ϵ_X be the map

$$L_{\mathscr{B}}(F(X)) \xrightarrow{L_{\mathscr{B}}(F(q_X))} L_{\mathscr{B}}(F(Q_X)) = RF(L_{\mathscr{A}}(X)).$$

Exercise 2.12. Check the above is well-defined. In particular, check that the definition does not depend on which $\mathscr{Q}$-resolution is taken and does not depend on which basement diagram is taken.

Proposition 2.13. *Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ be left exact functors. Suppose that F and G have right adapted classes $\mathscr{Q} \subset \mathscr{A}$ and $\mathscr{S} \subset \mathscr{B}$, respectively, such that $F(\mathscr{Q}) \subset \mathscr{S}$. Then there is a canonical isomorphism $R(G \circ F) \xrightarrow{\sim} RG \circ RF$. Similarly for right exact functors.*

Proof. Exercise. ☐

3 Sheaves

As the category of sheaves of $\mathbb{C}$-vector spaces on X, $Sh(X)$, is abelian, we can form its derived category $D^\circ(X) := D^\circ Sh(X)$.

Proposition 3.1. *$Sh(X)$ has enough injectives.*

Proof. For M a $\mathbb{C}$-vector space, as shown in Example 2.2.4 of Stefan's talk, we have $Hom_{\mathbb{C}}(\mathcal{G}_x, M) \cong Hom_{Sh(X)}(\mathcal{G}, \underline{M}^x)$ natural in $\mathcal{G}$, where $\underline{M}^x$ is the skyscraper sheaf at x. As all vector spaces are injective objects, then $Hom_{\mathbb{C}}(-, M)$ is an exact functor so $Hom_{Sh(X)}(-, \underline{M}^x)$ is also exact. Thus $\underline{M}^x$ is an injective sheaf. Using the universal property of the product, $\prod_{x\in X}(\underline{M}^x)$ is also an injective sheaf.

Let $\mathcal{F}$ be a sheaf. There is a sheaf map $\varphi : \mathcal{F} \to \underline{(\mathcal{F}_x)}^x$ with $\varphi_x : \mathcal{F}_x \to \mathcal{F}_x$ the identity. By the universal property of the product, we obtain an injective sheaf map $\theta : \mathcal{F} \to \prod_{x\in X}\underline{(\mathcal{F}_x)}^x$. ☐

By the proposition, all left exact functors have derived functors. However, $Sh(X)$ may not have enough projectives.

As the pullback functor is exact, for $f : X \to Y$, let $f^* : D^\circ(Y) \to D^\circ(X)$ denoted the induced functor. Since it is exact, we have $(g \circ f)^*\mathcal{F} \cong f^*g^*\mathcal{F}$ for $\mathcal{F} \in D^\circ(X)$, by Proposition 2.1.5 of Stefan's talk, and Proposition 2.13.

As the push-forward $^\circ f_*$ is left exact, it has a derived functor denoted by $f_* : D^+(X) \to D^+(Y)$.

Proposition 3.2. *The push-forward functor $^\circ f_*$ sends injectives to injectives.*

Proof. Exercise. Use the fact that $^\circ f_*$ is a right adjoint to f^* (Proposition 2.2.2 of Stefan's talk) and f^* is exact. ☐

Corollary 3.3. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then for $\mathcal{F} \in D^+(X)$, we have $g_*f_*\mathcal{F} \cong (g \circ f)_*\mathcal{F}$.*

Definition 3.4. Let $A \in K^-(\mathscr{A})$ and $B \in K^+(\mathscr{A})$. Their **Hom chain-complex**, denoted $chHom(A, B)$ is the chain complex in $(\mathrm{Vect}_{\mathbb{C}})$ whose terms are

$$chHom(A, B)^n = \bigoplus_{j-i=n} Hom(A^i, B^j)$$

and differential given by

$$d(f) = d_B \circ f + (-1)^{j-i+1} f \circ d_A$$

for $f \in Hom(A^i, B^j)$.

As $Sh(X)$ has enough injectives, we can form the **derived Hom functor** (in the second variable) $RHom : D^-(X)^{op} \times D^+(X) \to D^+(\mathrm{Vect}_{\mathbb{C}})$.

Proposition 3.5. *For $A \in D^-(X)$ and $B \in D^+(X)$, there is a natural isomorphism*

$$Hom_{D(X)}(A, B) \cong H^0(RHom(A, B)).$$

Theorem 3.6. *Let $f : X \to Y$ be a continuous map. For $\mathcal{F} \in D^+(Y)$ and $\mathcal{G} \in D^+(X)$, there are natural isomorphisms*

$$RHom_{D^+(X)}(f^*\mathcal{F}, \mathcal{G}) \cong RHom_{D^+(Y)}(\mathcal{F}, f_*\mathcal{G})$$

$$Hom_{D^+(X)}(f^*\mathcal{F}, \mathcal{G}) \cong Hom_{D^+(Y)}(\mathcal{F}, f_*\mathcal{G})$$

Proof. Replace $\mathcal{G}$ by an injective resolution. The first claim reduces to the claim that there is a natural isomorphism $chHom(f^*\mathcal{F}, \mathcal{G}) \cong chHom(\mathcal{F}, ^\circ f_*\mathcal{G})$, which follows from the fact that f^* is adjoint to $^\circ f_*$ in the abelian case. The second claims follows from fact that the 0th cohomology of $RHom$ is Hom. □

Remark 3.7. Let $X, Y \in Sh(X)$. For $n \in \mathbb{Z}$, the nth **Ext group** of X and Y, denoted by $Ext^n_{Sh(X)}(X, Y)$ or $Ext^n(X, Y)$, is given by

$$Ext^n(X, Y) := Hom_{D(X)}(X, Y[n]) = H^n(RHom(X, Y)).$$

3.1 Flabby Sheaves

Definition 3.8. A sheaf $\mathcal{F}$ is said to be **flabby** if for every open set $U \subset X$, the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.

Example 3.9. If X has the discrete topology, for a sheaf $\mathcal{F}$ on X and $U \subset X$, we have, by the gluing axiom,

$$\mathcal{F}(U) = \prod_{x \in U} \mathcal{F}(\{x\}).$$

Then the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective. Thus every sheaf on a discrete space is flabby.

Lemma 3.10. *Let $f : X \to Y$ be a continuous map.*

1. *Flabby sheaves form an adapted class for $^\circ f_*$.*
2. *If $\mathcal{F}$ is flabby, then $^\circ f_*(\mathcal{F})$ is flabby.*

Proof. 1) Exercise.

2) Let $U \subset Y$ be open. Then $^\circ f_*(\mathcal{F})(Y) \to ^\circ f_*(\mathcal{F})(U)$ is the map $\mathcal{F}(X) \to \mathcal{F}(f^{-1}(U))$, which is surjective as $\mathcal{F}$ is flabby. □

Remark 3.11. We will later show that injective sheaves are flabby, but the converse is not true in general.

Let X_{disc} be the set X but with the discrete topology. Let $i : X_{disc} \to X$ be the obvious map. For any sheaf $\mathcal{F}$, the sheaf $i^*\mathcal{F}$ is flabby, as shown in the example above. Then by 2) of the lemma, $^\circ i_* i^*\mathcal{F}$ is flabby. This gives a map $\mathcal{F} \hookrightarrow ^\circ i_* i^*\mathcal{F}$, which can be shown to be an injection. Thus we can embed any sheaf as a subsheaf of a flabby sheaf. Iterating this, we construct the following flabby resolution:

$$0 \to \mathcal{F} \xrightarrow{\epsilon} i_* i^*\mathcal{F} \xrightarrow{d^0} i_* i^*(\mathrm{coke}\epsilon) \xrightarrow{d^1} i_* i^*(\mathrm{coke}d^0) \to \cdots$$

We can do this for any $\mathcal{F} \in Ch^+(Sh(X))$ as well. This resolution is called the **Godement resolution** of $\mathcal{F}$.

Remark 3.12. As Γ is push-forward to a point, flabby sheaves are also an adapted class for Γ.

As an application of flabby sheaves, we will show how Leray-Cartan cohomology (sheaf cohomology) is related to singular cohomology.

Definition 3.13. Let X be a topological space, and let $\mathcal{F} \in D^+(X)$. The nth **hypercohomology** of $\mathcal{F}$ is

$$\mathbf{H}^n(X, \mathcal{F}) := H^n(R\Gamma(\mathcal{F})).$$

The nth **hypercohomology with compact support** of $\mathcal{F}$ is

$$\mathbf{H}^n_c(X, \mathcal{F}) := H^n(R\Gamma_c(\mathcal{F})).$$

Definition 3.14. Let X be a topological space. The nth **Leray-Cartan cohomology group (sheaf cohomology)** of X with coefficients in $\mathbb{C}$ is the nth hypercohomology of the constant sheaf $\underline{\mathbb{C}}_X$:

$$\mathbf{H}^n(X;\mathbb{C}) \coloneqq \mathbf{H}^n(X, \underline{\mathbb{C}}_X).$$

Similarly, the nth **Leray-Cartan cohomology group with compact support** is

$$\mathbf{H}^n_c(X;\mathbb{C}) \coloneqq \mathbf{H}^k_c(X, \underline{\mathbb{C}}_X).$$

Theorem 3.15. *If X is a locally contractible topological space, then there is a natural isomorphism*

$$\mathbf{H}^k(X;\mathbb{C}) \cong \mathbf{H}^k_{sing}(X;\mathbb{C}).$$

In addition, if X is also locally compact, then there is a natural isomorphism

$$\mathbf{H}^k_c(X;\mathbb{C}) \cong \mathbf{H}^k_{sing,c}(X;\mathbb{C}).$$

Example 3.16. Let X be locally contractible and let $f : X \to pt$. Then $H^i(f_*\underline{\mathbb{C}}_X) = \mathbf{H}^i_{sing}(X, \mathbb{C})$.

Example 3.17. Let $f : X \to Y$ be a locally trivial fibration of locally contractible spaces. Then $f_*(\underline{\mathbb{C}}_X)$ is a direct sum of local systems whose underlying vector spaces are the cohomologies of the fibers, that is, $H^\bullet(f^{-1}(y))$ for $y \in Y$. This can be deduced using a later result (Theorem 6.3).

Proof of Theorem. Let $\mathscr{S}^i(X)$ be the group of singular i-cochains with coefficients in $\mathbb{C}$, that is, $\mathscr{S}^i(X) = Hom_{\mathbb{Z}}(C^i(X), \mathbb{C})$ where $C^i(X)$ is the free abelian group with basis generated by continuous maps $\Delta^i \to X$. Let

$$\mathscr{S}^\bullet(X) = (\cdots \to 0 \to \mathscr{S}^0(X) \xrightarrow{d} \mathscr{S}^1(X) \xrightarrow{d} \mathscr{S}^2(X) \xrightarrow{d} \cdots),$$

a chain complex of vector spaces. Then $\mathbf{H}^k_{sing}(X;\mathbb{C}) = H^i(\mathscr{S}^\bullet(X))$.

For $U \subset V$, we have a restriction map $\mathscr{S}^i(V) \to \mathscr{S}^i(U)$ so the map $U \mapsto \mathscr{S}^i(U)$ makes $\mathscr{S}^i$ into a presheaf on X. We then get a sequence of presheaves

$$0 \to \underline{\mathbb{C}}_{X,pre} \xrightarrow{\eta} \mathscr{S}^0 \xrightarrow{d} \mathscr{S}^1 \to \cdots$$

where η is obtained from the fact that X is locally contractible and the 0th cohomology of a point is $\mathbb{C}$. Sheafify to get the sequence

$$0 \to \underline{\mathbb{C}}_X \xrightarrow{\eta} \mathscr{S}^{0,+} \xrightarrow{d} \mathscr{S}^{1,+} \to \cdots$$

We will show that this is a flabby resolution of $\underline{\mathbb{C}}_X$. First, we will show $\mathscr{S}^i$ satisfies the gluing axiom. Let $(U_\alpha)_{\alpha\in I}$ be an open cover of an open set $U \subset X$ and let $(s_\alpha \in \mathscr{S}^i(U_\alpha))_{\alpha\in I}$ be a collection of singular cochains satisfying $s_\alpha|_{U_\alpha\cap U_\beta} = s_\beta|_{U_\alpha\cap U_\beta}$. Define $s \in \mathscr{S}^i(U)$ as follows: for a singular simplex $c : \Delta^i \to U$, let

$$s(c) = \begin{cases} s_\alpha(c) & \text{if } im(c) \subset U_\alpha \\ 0 & \text{if } im(c) \not\subset U_\alpha \text{ for all } \alpha \in I \end{cases}$$

It is clear that $s|_{U_\alpha} = s_\alpha$ for all α so $\mathscr{S}^i$ satisfies the gluing axiom. Let $\mathscr{S}^i_{loc}(U) = \{s \in \mathscr{S}^i(U) :$ for some open cover $(U_\alpha)_{\alpha\in I}$ of U, we have $s|_{U_\alpha} = 0$ for all $\alpha \in I\}$. We will show that

$$\mathscr{S}^{i,+}(U) \cong \mathscr{S}^i(U)/\mathscr{S}^i_{loc}(U).$$

Consider the natural sheafification morphism $+ : \mathscr{S}^i(U) \to \mathscr{S}^{i,+}(U)$ sending $\varphi \mapsto +(\varphi) =: \varphi^+$. Then $\varphi^+ = 0$ if and only if $\varphi|_{U_\alpha} = 0$ for some open cover $(U_\alpha)_{\alpha\in I}$ of U. Thus the kernel is $\mathscr{S}^i_{loc}(U)$. The gluing axiom shown above gives surjectivity.

As $\mathscr{S}^{i,+}(U)$ is a quotient of $\mathscr{S}^i(U)$, to prove it is flabby, it suffices to prove the restriction $\mathscr{S}^i(X) \to \mathscr{S}^i(U)$ is surjective. Let $s \in \mathscr{S}^i(U)$. For a singular simplex $c : \Delta^i \to X$ define $\tilde{s} \in \mathscr{S}^i(X)$ by

$$\tilde{s}(c) = \begin{cases} s(c) & im(c) \subset U \\ 0 & im(c) \not\subset U \end{cases}$$

Then $\tilde{s}|_U = s$.

Exactness in degree > 0 follows from X being locally contractible: for $U \subset X$ contractible, we have an exact sequence $\mathscr{S}^{i-1}(U) \to \mathscr{S}^i(U) \to \mathscr{S}^{i+1}(U)$ so taking the direct limit using a basis of contractible neighbourhoods, we get an exact sequence of stalks $\mathscr{S}^{i-1}_x \to \mathscr{S}^i_x \to \mathscr{S}^{i+1}_x$. We have exactness in degree 0 by construction.

It turns out that $\mathscr{S}^\bullet$ is quasi-isomorphic to $\mathscr{S}^{\bullet,+}$. Thus we have a quasi-isomorphism $\underline{\mathbb{C}}_X \to \mathscr{S}^{\bullet,+}$. As flabby sheaves are an adapted class for Γ, we have

$$R\Gamma(\underline{\mathbb{C}}_X) \cong \Gamma(\mathscr{S}^{\bullet,+}) = \mathscr{S}^{\bullet,+}(X) \overset{qis}{\leftarrow} \mathscr{S}^\bullet(X).$$

Then the result follows after taking cohomology. A similar proof is done for the compact support case. $\square$

Exercise 3.18. Let $j : \mathbb{C}^\times \hookrightarrow \mathbb{C}$ be the natural inclusion. Let $\mathcal{F} = j_*\underline{\mathbb{C}}$. Show that $H^0(\mathcal{F}) = \underline{\mathbb{C}}$, a constant sheaf, and $H^1(\mathcal{F}) = \underline{\mathbb{C}}^0$, a skyscraper sheaf.

3.2 C-Soft Sheaves

For this section, assume all spaces are locally compact.

As proper push-forward is left exact, the right derived functor for the proper push-forward ${}^\circ f_!$ is $f_! : D^+(X) \to D^+(Y)$. Unlike the push-forward, the proper push-forward does not map injectives to injectives. It requires a different adapted class called c-soft sheaves.

Definition 3.19. A sheaf $\mathcal{F}$ on X is said to be ***c*-soft** if for every compact $K \subset X$, the natural map $\Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}|_K)$ is surjective.

Let $f : X \to Y$ be a continuous map of locally compact spaces. A sheaf $\mathcal{F}$ on X is said to be **relatively *c*-soft for f**, or **f-relative *c*-soft**, if for each $y \in Y$, the sheaf $\mathcal{F}|_{f^{-1}(y)}$ is c-soft.

Exercise 3.20. Let $f : X \to Y$ be a continuous map between locally compact topological spaces. Show any c-soft sheaf on X is relatively c-soft for f.

Lemma 3.21. *Every flabby sheaf on X is c-soft.*

Proof. Let $\mathcal{F}$ be a flabby sheaf on X. Let $K \subset X$ be compact and $s \in \Gamma(\mathcal{F}|_K)$. Using the fact that

$$\Gamma(\mathcal{F}|_K) \cong \lim_{\substack{V \subset X \text{ open} \\ V \supset K}} \mathcal{F}(V)$$

we have s to be the restriction of a section $\tilde{s} \in \mathcal{F}(U)$ for some open set U containing K. As $\mathcal{F}$ is flabby, $\tilde{s}$ is the restriction of a global section in $\Gamma(\mathcal{F})$, hence so is s. Therefore $\mathcal{F}$ is c-soft. $\square$

Lemma 3.22. *Let $f : X \to Y$ be a continuous map of locally compact spaces. If $\mathcal{F}$ is c-soft, then ${}^\circ f_!(\mathcal{F})$ is c-soft.*

Lemma 3.23. *Let $f : X \to Y$ be a continuous map of locally compact spaces. The class of c-soft sheaves and the class of f-relative c-soft sheaves are both adapted class for ${}^\circ f_!$.*

Proposition 3.24. *Let $f : X \to Y$ and $g : Y \to Z$ be a continuous maps of locally compact spaces. For $\mathcal{F} \in D^+(X)$, there is a natural isomorphism $g_! f_! \mathcal{F} \cong (g \circ f)_! \mathcal{F}$.*

Proof. Follows from the same for result for ${}^\circ g_!, {}^\circ f_!$ (Proposition 4.0.3 in Stefan's talk), Proposition 2.13, and Lemma 3.22. $\square$

Theorem 3.25 (Proper Base Change)**.** *Suppose we have a Cartesian square of continuous maps between locally compact spaces*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Then there is a natural isomorphism of functors $g^* \circ f_! \xrightarrow{\sim} f'_! \circ (g')^*$.

Proof. Stefan proved the abelian version of this in his talk (Theorem 4.1.1). From that we obtain an isomorphism $R(g^* \circ f_!) \xrightarrow{\sim} R(\circ f'_!(g')^*)$. As the pullback functor is exact, we have $R(g^{*\circ} f_!) \cong R(g^*) \circ R(\circ f_!) = g^* \circ f_!$.

Suppose $\mathcal{F} \in Sh(X)$ is c-soft. Then $\mathcal{F}|_{f^{-1}(g(y))}$ is c-soft. As $(f')^{-1}(y) \cong f^{-1}(g(y))$, the sheaf $(g')^*\mathcal{F}$ is relatively c-soft for f'. Then as relatively c-soft sheaves for f' form an adapted class for $(f')_!$ and using Proposition 2.13, we have $R(\circ f'_!(g')^*) \cong f'_!(g')^*$. $\square$

3.3 Derived Functor of Restriction with Supports

Let $h : Y \to X$ be the inclusion of a locally closed subset. Recall that the definition of $\circ h^!$ is

$$\circ h^!(\mathcal{F})(U) = \lim_{\substack{V \subset X \text{ open} \\ V \cap \overline{Y} = U}} \{s \in \mathcal{F}(V) : \text{supp } s \subset U\}$$

and that it is left exact.

Lemma 3.26. *$\circ h^!$ takes injectives to injectives.*

Proof. Let $\mathcal{F}$ be an injective sheaf. As shown in Stefan's talk (Lemma 5.0.1), $h_!$ is exact when h is the inclusion of a locally closed subspace. Then using adjointness (Proposition 5.1.2 of Stefan's talk), we have $Hom(-,^\circ h^!\mathcal{F}) \cong Hom(h_!(-), \mathcal{F})$ which is the composition of the two exact functors $Hom(-, \mathcal{F})$ and $h_!$. Thus $Hom(-,^\circ h^!\mathcal{F})$ is exact, so $\circ h^!\mathcal{F}$ is injective. $\square$

Let the right derived functor of $\circ h^!$ be denoted by $h^! : D^+(X) \to D^+(Y)$.

Theorem 3.27. *Let $h : Y \to X$ be a locally closed inclusion. For $\mathcal{F} \in D^+(Y)$ and $\mathcal{G} \in D^+(X)$, there is a natural isomorphism*

$$Hom_{D^+(X)}(h_!\mathcal{F}, \mathcal{G}) \cong Hom_{D^+(Y)}(\mathcal{F}, h^!\mathcal{G}).$$

Proof. Exercise.

[Follows from corresponding statement on abelian level for $h_!,^\circ h^!$, proved in Stefan's lecture (Proposition 5.1.2), and the fact that $\circ h^!$ takes injective to injectives.] $\square$

Remark 3.28. We will later construct $f^!$ for any f and show it is adjoint to $f_!$.

Proposition 3.29. *Let $k : W \to Y$ and $h : Y \to X$ be locally closed inclusions. For $\mathcal{F} \in D^+(W)$, there is a natural isomorphism $h_!k_!\mathcal{F} \cong (h \circ k)_!\mathcal{F}$. For $\mathcal{G} \in D^+(X)$, there is a natural isomorphism $k^!h^!\mathcal{G} \cong (h \circ k)^!\mathcal{G}$.*

Proof. The first assertion follows from similar statements on level of abelian categories (for locally closed inclusions, $h_!$ and $k_!$ come from exact functors). For the second assertion, the right adjoint to $h_! \circ k_!$ is $k^! \circ h^!$ and the right adjoint to $(h \circ k)_!$ is $(h \circ k)^!$, then use uniqueness of adjoint functors. $\square$

Lemma 3.30. *Injective sheaves are flabby.*

Proof. Let $j : U \hookrightarrow X$ be the inclusion of an open subset of X. It's an exercise to show that the sheaf $\circ j_!\underline{\mathbb{C}}_U$ is given by

$$(\circ j_!\underline{\mathbb{C}}_U)(V) = \{\text{locally constant functions } s : V \to \mathbb{C} \text{ such that } s|_{X \setminus U} = 0\}$$

for $V \subset X$ open. Thus we can identify $\circ j_!\underline{\mathbb{C}}_U$ with a subsheaf of $\underline{\mathbb{C}}_X$.

Let $\mathcal{F}$ be an injective sheaf on X. Apply the exact functor $Hom(-, \mathcal{F})$ to the injection $j_!\underline{\mathbb{C}}_U \to \underline{\mathbb{C}}_X$ to obtain a surjection $Hom(\underline{\mathbb{C}}_X, \mathcal{F}) \to Hom(j_!\underline{\mathbb{C}}_U, \mathcal{F})$. We have $Hom(\underline{\mathbb{C}}_X, \mathcal{F}) \cong \Gamma(\mathcal{F})$, using Example 2.2.3 from Stefan's talk. Note that as j is an open inclusion, we have $j^! = j^*$, as in Section 5.1 of Stefan's talk. We have

$$Hom(j_!\underline{\mathbb{C}}_U, \mathcal{F}) \cong Hom(\underline{\mathbb{C}}_U, j^*\mathcal{F}) \cong \Gamma(j^*\mathcal{F}) \cong \mathcal{F}(U).$$

Thus $\Gamma(\mathcal{F}) \to \mathcal{F}(U)$ is surjective, so $\mathcal{F}$ is flabby. $\square$

Proposition 3.31. *Let $h : Y \to X$ be a locally closed inclusion. For any $\mathcal{F} \in D^+(Y)$, the natural maps*

$$h^! h_* \mathcal{F} \to h^* h_* \mathcal{F} \to \mathcal{F}$$

$$\mathcal{F} \to h^! h_! \mathcal{F} \to h^* h_! \mathcal{F}$$

are isomorphisms.

Proof. Apply the same argument as in the abelian case (Proposition 5.1.5 of Stefan's talk) for an injective resolution of $\mathcal{F}$. □

Theorem 3.32. *Let $i : Z \hookrightarrow X$ be a closed inclusion and let $j : U \hookrightarrow X$ be the complementary open inclusion.*

1. *We have $i^* \circ j_! = 0$, $i^! \circ j_* = 0$, and $j^* \circ i_* = 0$.*
2. *For any $\mathcal{F} \in D^+(X)$, there is a natural distinguished triangle*

$$j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to .$$

3. *For any $\mathcal{F} \in D^+(X)$, there is a natural distinguished triangle*

$$i_* i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F} \to .$$

In each of these triangles, the first two maps are adjunction maps.

Proof. 1) As $i^*, j_!, j^*, i_*$ come from exact functors, the first and third equalities follow from Stefan's talk (Theorem 5.1.6). We have $i^! \circ j_*$ is right adjoint to $j^* \circ i_! \cong j^* \circ i_* = 0$, so it vanishes too.

2) Let $\mathcal{F} \in Sh(X)$ and $x \in X$. Consider the maps on stalks induced by adjunction maps:

$$(j_! j^* \mathcal{F})_x \to \mathcal{F}_x \to (i_* i^* \mathcal{F})_x.$$

If $x \in U$, the first map is an isomorphism and the last term is 0. If $x \in Z$, the first term is 0 and the second map is an isomorphism. Thus we get a short exact sequence of sheaves

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0.$$

For $\mathcal{F} \in D^+(X)$, use the above short exact sequence to get a short exact sequence of chain complexes. Let $f : j_! j^* \mathcal{F} \to \mathcal{F}$ and $g : \mathcal{F} \to i_* i^* \mathcal{F}$ be the corresponding maps. Define $\theta : cone(f) \to i_* i^* \mathcal{F}$ by $\theta^i : (j_! j^* \mathcal{F})^{i+1} \oplus \mathcal{F}^i \to (i_* i^* \mathcal{F})^i$ is the map $\theta^i = \begin{bmatrix} 0 & g^i \end{bmatrix}$.

Exercise: θ is a chain map and a quasi-isomorphism.

Let $i_2 : \mathcal{F} \to cone(f)$ and $p_1 : cone(f) \to (j_! j^* \mathcal{F})[1]$ be the natural inclusion and projection maps. Then

$$j_! j^* \mathcal{F} \xrightarrow{f} \mathcal{F} \xrightarrow{\theta \circ i_2} i_* i^* \mathcal{F} \xrightarrow{p_1 \circ \theta^{-1}} (j_! j^* \mathcal{F})[1]$$

is a distinguished triangle.

3) Similar to 2): use adjunction maps to get a sequence of sheaves. Show it's a short exact sequence when $\mathcal{F}$ is an injective sheaf. □

4 Tensor Product and Sheaf Hom

4.1 Tensor Product

Since we are working over $\mathbb{C}$, all sheaves are flat, so tensor product is an exact functor in both variables. Then it induces a functor $\otimes : D(X) \times D(X) \to D(X)$.

Proposition 4.1. *Let $f : X \to Y$ be a continuous map. For $\mathcal{F}, \mathcal{G} \in D(X)$, there is a natural isomorphism $f^*(\mathcal{F} \otimes \mathcal{G}) \cong f^* \mathcal{F} \otimes f^* \mathcal{G}$.*

Proof. Use the sheaf level result (Lemma 6.1.1 of Stefan's talk). $\square$

Proposition 4.2 (Projection Formula). *Let $f : X \to Y$ be a continuous map of locally compact spaces. For $\mathcal{F} \in D^+(X)$ and $\mathcal{G} \in D^+(Y)$, there is a natural isomorphism $f_!\mathcal{F} \otimes \mathcal{G} \xrightarrow{\sim} f_!(\mathcal{F} \otimes f^*\mathcal{G})$.*

To prove this, we first need a few results.

Lemma 4.3. *Let X be a locally compact space. Then $\mathcal{F} \in Sh(X)$ is c-soft if and only if for every closed subset $Z \subset X$, the natural map $\Gamma_c(\mathcal{F}) \to \Gamma_c(\mathcal{F}|_Z)$ is surjective.*

Proof. Exercise. (Proposition 2.5.6 of [2]). $\square$

Lemma 4.4. *Let $f : X \to Y$ be a continuous map of locally compact spaces. For $\mathcal{F} \in Sh(X)$ and $\mathcal{G} \in Sh(Y)$, there is a natural morphism ${}^\circ f_!\mathcal{F} \otimes \mathcal{G} \to {}^\circ f_!(\mathcal{F} \otimes f^*\mathcal{G})$.*

Proof. Let $U \subset Y$ be open. By definition we have

$$({}^\circ f_!\mathcal{F} \otimes_{pre} \mathcal{G})(U) = \{s \in \mathcal{F}(f^{-1}(U)) : f|_{\text{supp } s} \text{ is proper}\} \otimes \mathcal{G}(U)$$

$${}^\circ f_!(\mathcal{F} \otimes_{pre} f^*_{pre}(\mathcal{G}))(U) = \{u \in \mathcal{F}(f^{-1}(U)) \otimes (f^*_{pre}\mathcal{G})(f^{-1}(U)) : f|_{\text{supp } u} \text{ is proper}\}$$

For $t \in ({}^\circ f_!\mathcal{F} \otimes_{pre} \mathcal{G})(U)$, we can write t as a finite sum $\sum_i s_i \otimes s_i'$ where $s_i \in ({}^\circ f_!\mathcal{F})(U)$ and $s_i' \in \mathcal{G}(U) = (f^*_{pre}\mathcal{G})(f^{-1}(U))$. Then we can regard each $s_i \otimes s_i'$ as an element of ${}^\circ f_!(\mathcal{F} \otimes_{pre} f^*_{pre}\mathcal{G})(U)$. We have $\text{supp}(s_i \otimes s_i')$ to be a closed subset of supp s_i and in general, supp $\sum_i s_i \otimes s_i'$ is a closed subset of $\bigcup_i$ supp s_i so

$$f|_{\text{supp} \sum s_i \otimes s_i'} : \text{supp} \sum_i s_i \otimes s_i' \to Y$$

is proper. This gives us a map $({}^\circ f_!\mathcal{F} \otimes_{pre} \mathcal{G})(U) \to {}^\circ f_!(\mathcal{F} \otimes_{pre} f^*_{pre}\mathcal{G})(U)$ so we get a presheaf map ${}^\circ f_!\mathcal{F} \otimes_{pre} \mathcal{G} \to {}^\circ f_!(\mathcal{F} \otimes_{pre} f^*_{pre}\mathcal{G})$. Composing with the sheafification maps $f^*_{pre}\mathcal{G} \to f^*\mathcal{G}$ and $\mathcal{F} \otimes_{pre} f^*\mathcal{G} \to \mathcal{F} \otimes f^*\mathcal{G}$, we get a map ${}^\circ f_!\mathcal{F} \otimes_{pre} \mathcal{G} \to {}^\circ f_!(\mathcal{F} \otimes \mathcal{G})$. The universal property of sheafification gives us the desired map ${}^\circ f_!\mathcal{F} \otimes \mathcal{G} \to {}^\circ f_!(\mathcal{F} \otimes f^*\mathcal{G})$. $\square$

Lemma 4.5. *Let X be a locally compact space. If $\mathcal{F} \in Sh(X)$ is c-soft and $M \in Vect_{\mathbb{C}}$, then there is a natural isomorphism $\Gamma_c(\mathcal{F}) \otimes M \to \Gamma_c(\mathcal{F} \otimes \underline{M}_X)$.*

Proof. Since M is a direct sum of copies of $\mathbb{C}$, it suffices to show that Γ_c commutes with arbitrary direct sums (c.f. Lemma 2.1.34 of Pramod's text). $\square$

Proof of Projection formula. Suppose $\mathcal{F} \in Sh(X)$ is c-soft and $M \in \text{Vect}_{\mathbb{C}}$. We will first show that the morphism in Lemma 4.4 is an isomorphism by showing it induces an isomorphism on stalks. Let $y \in Y$. Using Lemma 6.1.1 of Stefan's talk, we have

$$({}^\circ f_!\mathcal{F} \otimes \mathcal{G})_y \cong ({}^\circ f_!\mathcal{F})_y \otimes \mathcal{G}_y \cong \Gamma_c(\mathcal{F}|_{f^{-1}(y)}) \otimes \mathcal{G}_y$$

and

$$({}^\circ f_!(\mathcal{F} \otimes f^*\mathcal{G}))_y \cong \Gamma_c((\mathcal{F} \otimes f^*\mathcal{G})|_{f^{-1}(y)}) \cong \Gamma_c(\mathcal{F}|_{f^{-1}(y)} \otimes g^*\mathcal{G}_y)$$

where $g : f^{-1}(y) \to pt$. These two are isomorphic by Lemma 4.5.

Suppose $\mathcal{F}$ is c-soft and $\mathcal{G}$ is a sheaf. Then we will show $\mathcal{F} \otimes f^*\mathcal{G}$ is relatively c-soft for f. Let $y \in Y$. Consider the commutative diagram

$$\begin{tikzcd}
\Gamma_c(\mathcal{F}) \otimes \mathcal{G}_y \arrow[r] \arrow[d, "\sim"'] & \Gamma_c(\mathcal{F}|_{f^{-1}(y)}) \otimes g^*_{f^{-1}(y)}\mathcal{G}_y \arrow[d, "\sim"] \\
\Gamma_c(\mathcal{F} \otimes f^*\mathcal{G}) \arrow[r] & \Gamma_c((\mathcal{F} \otimes f^*\mathcal{G})|_{f^{-1}(y)})
\end{tikzcd}$$

where the vertical maps are isomorphisms by Lemma 4.5. The top horizontal map is surjective as $\mathcal{F}$ is c-soft so the bottom map is also surjective. Thus $\mathcal{F} \otimes f^*\mathcal{G}$ is relatively c-soft for f by Lemma 4.3.

Let $\mathcal{F} \in D^+(X)$ and $\mathcal{G} \in D^+(Y)$. Replace $\mathcal{F}$ by a c-soft resolution. Then $\mathcal{F} \otimes f^*\mathcal{G}$ is relatively c-soft for f, an adapted class for $f_!$. Then conclude the result using the isomorphism obtained at the beginning of this proof and Proposition 2.13. $\square$

We will now introduce the external tensor product functor.

Definition 4.6. Let $\mathcal{F} \in Sh(X)$ and $\mathcal{G} \in Sh(Y)$. Their **external tensor product** $\mathcal{F} \boxtimes \mathcal{G}$ is the sheaf on $X \times Y$ given by $\mathcal{F} \boxtimes \mathcal{G} = p_1^* \mathcal{F} \otimes p_2^* \mathcal{G}$.

As the pullback functor and tensor product functor are exact, we obtain a derived external tensor product functor $\boxtimes : D^-(X) \times D^-(Y) \to D^-(X \times Y)$.

4.2 Sheaf Hom

Recall that the sheaf hom functor $\mathscr{H}om$ is left exact in both variables. For $\mathcal{F} \in Ch^-(Sh(X))$ and $\mathcal{G} \in Ch^+(Sh(X))$, we can form the chain complex $ch\mathscr{H}om(\mathcal{F}, \mathcal{G})$, in a way similar to $chHom$. As we have enough injectives, we can form the derived functor (in the second variable)

$$R\mathscr{H}om : D^-(X) \times D^+(X) \to D^+(X).$$

Explicitly, $R\mathscr{H}om(\mathcal{F}, \mathcal{G})$ is computed by replacing $\mathcal{G}$ with an injective resolution, and then applying $ch\mathscr{H}om$. In general, there is no adapted class for $\mathscr{H}om$ in the first variable.

Lemma 4.7. *For $\mathcal{F} \in D^+(X)$, there is a natural isomorphism*

$$R\mathscr{H}om(\underline{\mathbb{C}}_X, \mathcal{F}) \cong \mathcal{F}.$$

Proof. If $\mathcal{F}$ is a sheaf, by Example 2.2.3 of Stefan's talk, we have $Hom(\underline{\mathbb{C}}_X, \mathcal{F}) \cong \Gamma(\mathcal{F})$. Applying this to $\mathscr{H}om$, we get $\mathscr{H}om(\underline{\mathbb{C}}, \mathcal{F})(U) \cong \Gamma(\mathcal{F}|_U) = \mathcal{F}(U)$ so $\mathscr{H}om(\underline{\mathbb{C}}, -)$ is the identity functor. Then $R\mathscr{H}om(\underline{\mathbb{C}}, -)$ is also isomorphic to the identity functor. $\square$

Lemma 4.8. *Let $U \subset X$ be an open subset. For any $\mathcal{F} \in D^-(X)$ and $\mathcal{G} \in D^+(X)$, there is a natural isomorphism*

$$R\mathscr{H}om(\mathcal{F}, \mathcal{G})|_U \cong R\mathscr{H}om(\mathcal{F}|_U, \mathcal{G}|_U).$$

Proof. Use the abelian version, which is clear from the definition, and the fact that restriction of an injective sheaf to an open subset is injective. The latter fact follows from results in Section 5 of Stefan's talk, namely if $j : U \hookrightarrow X$ is an open inclusion, then $j^! = j^*$ and $j_!$ is exact, so for an injective sheaf $\mathcal{F}$ on X, we have $Hom(-, \mathcal{F}|_U) \cong Hom(-, j^*\mathcal{F}) \cong Hom(j_!(-), \mathcal{F})$, which is a composition of exact functors. Hence $\mathcal{F}|_U$ is injective. $\square$

Proposition 4.9. *For $\mathcal{F} \in D^-(X)$ and $\mathcal{G} \in D^+(X)$, there is a natural isomorphism*

$$R\Gamma(R\mathscr{H}om(\mathcal{F}, \mathcal{G})) \xrightarrow{\sim} RHom(\mathcal{F}, \mathcal{G}).$$

Proof. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, we have $\Gamma(\mathscr{H}om(\mathcal{F}, \mathcal{G})) = Hom(\mathcal{F}, \mathcal{G})$ by definition. Suppose $\mathcal{G}$ is an injective sheaf. We will show $\mathscr{H}om(\mathcal{F}, \mathcal{G})$ is flabby so that $ch\mathscr{H}om(\mathcal{F}, \mathcal{G})$ is a chain complex of flabby sheaves, which is an adapted class for Γ. Let $j : U \to X$ be an open subset. We have an injection $j_! j^* \mathcal{F} \to \mathcal{F}$ from Theorem 3.29. Applying the exact functor $Hom(-, \mathcal{G})$ and using $j^* \cong j^!$, we get a surjection

$$Hom(\mathcal{F}, \mathcal{G}) \twoheadrightarrow Hom(j_! j^* \mathcal{F}, \mathcal{G}) \cong Hom(j^* \mathcal{F}, j^* \mathcal{G}) = Hom(\mathcal{F}|_U, \mathcal{G}|_U).$$

Thus $\Gamma(\mathscr{H}om(\mathcal{F}, \mathcal{G})) \to \mathscr{H}om(\mathcal{F}, \mathcal{G})(U)$ is surjective, so $\mathscr{H}om(\mathcal{F}, \mathcal{G})$ is flabby. For the derived case, conclude using Proposition 2.13. $\square$

Theorem 4.10. *For $\mathcal{F}, \mathcal{G} \in D^-(X)$ and $\mathcal{H} \in D^+(X)$, there is a natural isomorphism*

$$RHom(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \cong RHom(\mathcal{F}, R\mathscr{H}om(\mathcal{G}, \mathcal{H})).$$

Proof. Exercise.

[Replace $\mathcal{H}$ by an injective resolution. Note that for an injective sheaf $\mathcal{I}$, we have $Hom(-, \mathscr{H}om(\mathcal{T}, \mathcal{I})) \cong Hom(- \otimes \mathcal{T}, \mathcal{I})$, the composition of $- \otimes \mathcal{T}$ and $Hom(-, \mathcal{I})$, so it is exact. Thus $\mathscr{H}om(\mathcal{T}, \mathcal{I})$ is injective. Then the result follows from the abelian version.] $\square$

Summarizing our results, we obtain this table:

Functor	Exactness	Adapted Classes
$f^*, \otimes, \boxtimes$	exact	
$f_*, R\Gamma$	left	injective, flabby
$f_!$	left	injective, flabby, c-soft, relatively c-soft
$R\Gamma_c$	left	injective, flabby, c-soft
$R\mathscr{H}om$	left	injective

5 Right Adjoint to Proper Push-Forward

In general, functors may not have adjoints. There are general categorical theorems (Adjoint Functor Theorem and Special Adjoint Functor Theorem) detailing when adjoints exist, and they involve some finiteness conditions on the category and the functor (preserves small colimits).

Definition 5.1. Let X be a locally compact topological space. If there is a nonnegative integer n such that every sheaf $\mathcal{F} \in Sh(X)$ admits a c-soft resolution of length at most n, then X is said to have **finite c-soft dimension**. In that case, the smallest such integer n is called the **c-soft dimension** of X.

Theorem 5.2. *Let $f : X \to Y$ be a continuous map of locally compact spaces of finite c-soft dimension. There exists a triangulated functor $f^! : D^+(Y) \to D^+(X)$ that is right adjoint to $f_! : D^+(X) \to D^+(Y)$. Moreover, for $\mathcal{F} \in D^+(X)$ and $\mathcal{G} \in D^+(Y)$, there is a natural isomorphism*

$$R\mathscr{H}om(f_!\mathcal{F}, \mathcal{G}) \cong f_* R\mathscr{H}om(\mathcal{F}, f^!\mathcal{G}).$$

Proof. Idea: Suppose $f^!$ has been constructed and $f^!\mathcal{G}$ is a sheaf. For an open $U \subset X$, let $j_U : U \hookrightarrow X$ be the inclusion. Then

$$(f^!\mathcal{G})(U) \cong \Gamma(j_U^! f^! \mathcal{G}) \cong Hom(\underline{\mathbb{C}}_U, j_U^! f^! \mathcal{G}) \cong Hom(f_! j_{U!} \underline{\mathbb{C}}_U, \mathcal{G}).$$

In general, given a sheaf $\mathcal{F}$ on X, the sheaf $\mathscr{H}om(\mathcal{F}, f^!\mathcal{G})$ is described by

$$\mathscr{H}om(\mathcal{F}, f^!\mathcal{G})(U) = Hom(\mathcal{F}|_U, (f^!\mathcal{G})|_U) \cong Hom(j_{U!} j_U^* \mathcal{F}, f^!\mathcal{G}) \cong Hom(f_!(\mathcal{F} \otimes j_{U!}\underline{\mathbb{C}}_U), \mathcal{G})$$

where the last equality is due to the projection formula. Thus we can describe $\mathscr{H}om(\mathcal{F}, f^!\mathcal{G})$ without needing $f^!$.

For a c-soft sheaf $\mathcal{K}$ on X, $\mathcal{F} \in Sh(X)$ and $\mathcal{G} \in Sh(Y)$, define the sheaf $E_{\mathcal{K}}(\mathcal{F}, \mathcal{G})$ by

$$E_{\mathcal{K}}(\mathcal{F}, \mathcal{G})(U) = Hom(f_!(\mathcal{F} \otimes j_{U!}(\mathcal{K}|_U)), \mathcal{G}).$$

The functor $E_{\mathcal{K}}(-,-)$ is left exact. There is a natural isomorphism $E_{\mathcal{K}}(\mathcal{F}, \mathcal{G}) \cong \mathscr{H}om(\mathcal{F}, E_{\mathcal{K}}(\underline{\mathbb{C}}_X, \mathcal{G}))$. Let r be the c-soft dimension of X. Then take a c-soft resolution of the constant sheaf:

$$0 \to \underline{\mathbb{C}}_X \to \mathcal{K}^0 \to \cdots \to \mathcal{K}^r \to 0.$$

Let $\mathcal{K}^\bullet$ be the complex

$$0 \to \mathcal{K}^0 \to \cdots \to \mathcal{K}^r \to 0.$$

For $\mathcal{F} \in Ch^-(Sh(X))$ and $\mathcal{G} \in Ch^+(Sh(Y))$, define a complex of sheaves $\mathcal{E}(\mathcal{F}, \mathcal{G})$ by

$$\mathcal{E}(\mathcal{F}, \mathcal{G})^n = \bigoplus_{k-(i+j)=n} E_{\mathcal{K}^j}(\mathcal{F}^i, \mathcal{G}^k)$$

with differential similar to the construction of $chHom$ so that

$$\mathcal{E}(\mathcal{F}, \mathcal{G}) \cong ch\mathscr{H}om(\mathcal{F}, \mathcal{E}(\underline{\mathbb{C}}_X, \mathcal{G})).$$

The functor $\mathcal{E}(-,-)$ has a right derived functor and defining $f^!(\mathcal{G}) = R\mathcal{E}(\underline{\mathbb{C}}_X, \mathcal{G})$ gives us the result.

To show the isomorphism, let $\mathcal{F} \in D^+(X)$ and $\mathcal{G}, \mathcal{H} \in D^+(Y)$. Then using tensor-Hom adjunction and the projection formula, we have

$$\begin{aligned} Hom(\mathcal{H}, R\mathscr{H}om(f_!\mathcal{F}, \mathcal{G}) &\cong Hom(\mathcal{H} \otimes f_!\mathcal{F}, \mathcal{G}) \\ &\cong Hom(f_!(f^*\mathcal{H} \otimes \mathcal{F}), \mathcal{G}) \\ &\cong Hom(f^*\mathcal{H} \otimes \mathcal{F}, f^!\mathcal{G}) \\ &\cong Hom(f^*\mathcal{H}, R\mathscr{H}om(\mathcal{F}, f^!\mathcal{G})) \\ &\cong Hom(\mathcal{H}, f_*R\mathscr{H}om(\mathcal{F}, f^!\mathcal{G})) \end{aligned}$$

Then using Yoneda's lemma, we get the required isomorphism. ☐

Proposition 5.3. *Suppose we have a Cartesian square of continuous maps between locally compact spaces of finite c-soft dimension:*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Then there is a natural isomorphism of functors $g'_* \circ (f')^! \xrightarrow{\sim} f^! \circ g_*$.

Proof. Let $\mathcal{F} \in D^+(X)$ and $\mathcal{G} \in D^+(Y)$. Proper base change gives us

$$Hom(f'_!(g')^*\mathcal{F}, \mathcal{G}) \cong Hom(g^*f_!\mathcal{F}, \mathcal{G}).$$

Adjunction gives us

$$Hom(\mathcal{F}, g'_*(f')^!\mathcal{G}) \cong Hom(\mathcal{F}, f^!g_*\mathcal{G}).$$

As this is for any $\mathcal{F}$, the result follows from Yoneda's lemma. ☐

Proposition 5.4. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of locally compact spaces of finite c-soft dimension. There is a natural isomorphism $f^! \circ g^! \cong (g \circ f)^!$.*

Proposition 5.5. *Let $f : X \to Y$ be a continuous map of locally compact spaces of finite c-soft dimension. For $\mathcal{F} \in D^b(Y)$ and $\mathcal{G} \in D^+(Y)$, there is a natural isomorphism*

$$f^!R\mathscr{H}om(\mathcal{F}, \mathcal{G}) \cong R\mathscr{H}om(f^*\mathcal{F}, f^!\,\mathcal{G}).$$

Proof. Let $\mathcal{H} \in D^+(Y)$. Then using tensor-Hom adjunction and the projection formula, we have

$$\begin{aligned} Hom(\mathcal{H}, f^!R\mathscr{H}om(\mathcal{F}, \mathcal{G})) &\cong Hom(f_!\mathcal{H}, R\mathscr{H}om(\mathcal{F}, \mathcal{G})) \\ &\cong Hom(f_!\mathcal{H} \otimes \mathcal{F}, \mathcal{G}) \\ &\cong Hom(f_!(\mathcal{H} \otimes f^*\mathcal{F}), \mathcal{G}) \\ &\cong Hom(\mathcal{H} \otimes f^*\mathcal{F}, f^!\mathcal{G}) \\ &\cong Hom(\mathcal{H}, R\mathscr{H}om(f^*\mathcal{F}, f^!\mathcal{G})) \end{aligned}$$

Then using Yoneda's lemma, we get the required isomorphism. ☐

6 Base Change for Locally Trivial Fibrations

Definition 6.1. Let X be a locally contractible space. Define $D^\circ_{loc}(X)$ to be the full subcategory of $D^\circ(X)$ given by

$$D^\circ_{loc}(X) = \{\mathcal{F} \in D^\circ(X) | H^k(\mathcal{F}) \in Loc(X) \text{ for all } k \in \mathbb{Z}\}.$$

Lemma 6.2. *Let X be a topological space and let $x \in X$. For $\mathcal{F} \in D^+(X)$, there is a natural isomorphism*

$$H^n(\mathcal{F}_x) \cong \lim_{U \ni x} \mathbf{H}^n(U, \mathcal{F}|_U).$$

Proof. (Skip proof) WLOG, $\mathcal{F}$ is a complex of flabby sheaves (Godement resolution). It is clear from the definition that restriction of flabby is flabby. Thus

$$R\Gamma(\mathcal{F}|_U) = \Gamma(\mathcal{F}|_U) = \mathcal{F}^\bullet(U).$$

As direct limits of k-modules are exact, $\lim_{\to} H^k(R\Gamma(\mathcal{F}|_U))$ can instead by computed as the kth cohomology of the chain complex

$$\cdots \to \lim_{U\ni x} \mathcal{F}^{k-1}(U) \to \lim_{U\ni x} \mathcal{F}^k(U) \to \lim_{U\ni x} \mathcal{F}^{k+1}(U) \to \cdots$$

which is the chain complex for $\mathcal{F}_x$. □

Theorem 6.3. *Let $f : X \to Y$ be a locally trivial fibration of locally contractible spaces.*

1. *For $\mathcal{F} \in D^+_{loc}(X)$ and any point $y \in Y$, there is a natural isomorphism $(f_*\mathcal{F})_y \cong R\Gamma(\mathcal{F}|_{f^{-1}(y)})$.*
2. *Suppose we have a Cartesian square of locally contractible spaces:*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

For any $\mathcal{F} \in D^+_{loc}(X)$, there is a natural isomorphism $g^* f_* \mathcal{F} \xrightarrow{\sim} f'_*(g')^*\mathcal{F}$.

Proof. 1) Let $y \in Y$. We have

$$H^k(f_*\mathcal{F})_y \cong \lim_{U\ni y} \mathbf{H}^k(U, f_*\mathcal{F}|_U) \cong \lim_{U\ni y} \mathbf{H}^k(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)})$$

$$\cong \lim_{U\ni y} Hom(\underline{\mathbb{C}}_{f^{-1}(U)}, \mathcal{F}|_{f^{-1}(U)}[k]).$$

As Y is locally contractible, this limit can be computed using contractible neighbourhoods of y. As f is a locally trivial fibration, for U a sufficiently small neighbourhood of y, we have $f^{-1}(U) \cong U \times f^{-1}(y)$. If U is also contractible, we get that the inclusion $f^{-1}(y) \hookrightarrow f^{-1}(U)$ is a homotopy equivalence. Use this to deduce that

$$H^k(f_*\mathcal{F})_y \cong \lim_{U\ni y} Hom(\underline{\mathbb{C}}_{f^{-1}(U)}, \mathcal{F}|_{f^{-1}(U)}[k])$$

$$\cong Hom(\underline{\mathbb{C}}_{f^{-1}(y)}, \mathcal{F}|_{f^{-1}(y)}[k]) \cong \mathbf{H}^k(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)}).$$

Thus the base change map $(f_*\mathcal{F})_y \to R\Gamma(\mathcal{F}|_{f^{-1}(y)})$ induces isomorphisms in cohomology, so it is an isomorphism.

2) In Stefan's talk (Theorem 4.1.1), a map $g^*f_*\mathcal{F} \to f'_*(g')^*\mathcal{F}$ was constructed on the sheaf-theoretic level. Replacing $\mathcal{F}$ with a flabby resolution, we get the corresponding derived version. It is enough to show the result on stalks, which is 1). □

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