AFFINE PUSHFORWARD AND SMOOTH PULLBACK FOR PERVERSE SHEAVES

YEHAO ZHOU

Conventions

In this lecture note, a variety means a separated algebraic variety over complex numbers, and sheaves are $\mathbb{C}$-linear.

1. Homological bounds for pullbacks and pushforwards

In this section, our goal is to understand what nonzero perverse cohomology can appear when applying f^*, $f^!$, f_*, and $f_!$ to a perverse sheaf. The main result is

Theorem 1.1. *Suppose $f : X \to Y$ is a morphism such that the dimensions of fibers of f are $\leq d$. Put perverse t-structures on $D_c(X)$ and $D_c(Y)$, then*

- *$f^*[d]$ and $f_![d]$ are right t-exact, i.e. $\forall i > d$, ${}^p\mathcal{H}^i f^*K = 0$ and ${}^p\mathcal{H}^i f_!L = 0$ for $K \in \mathrm{Perv}(Y)$ and $L \in \mathrm{Perv}(X)$;*
- *$f^![-d]$ and $f_*[-d]$ are left t-exact, i.e. $\forall i < -d$, ${}^p\mathcal{H}^i f^!K = 0$ and ${}^p\mathcal{H}^i f_*L = 0$ for $K \in \mathrm{Perv}(Y)$ and $L \in \mathrm{Perv}(X)$.*

and there are adjoint functors between perverse sheaves

$$ {}^pH^d f^* : \mathrm{Perv}(Y) \rightleftarrows \mathrm{Perv}(X) : {}^pH^{-d} f_* \qquad {}^pH^d f_! : \mathrm{Perv}(X) \rightleftarrows \mathrm{Perv}(Y) : {}^pH^{-d} f^! $$

Specialize to $d = 0$ and we get

Corollary 1.2. *If $f : X \to Y$ is a quasi-finite morphism between varieties, then f^* and $f_!$ are right t-exact, f_* and $f^!$ are left t-exact.*

Before we start proving the main result, let's work out some lemmas on t-exactness, which are standard for ordinary perversity on derived category of sheaves. Suppose that D_1 and D_2 are two triangulated categories with t-structures, $D_1^\heartsuit$ and $D_2^\heartsuit$ are their abelian hearts, $F : D_1 \to D_2$ is a triangulated functor.

Lemma 1.3. *If $G : D_2 \to D_1$ is left adjoint to F, then F is left t-exact if and only if G is right t-exact.*

Proof. Suppose that F is left t-exact, then for any $X \in D_2^{\leq 0}$, we have truncation triangle

$$ \tau^{\leq 0} GX \to GX \to \tau^{\geq 1} GX \to $$

in D_1, by adjunction we have

$$ \mathrm{Hom}_{D_1}(GX, \tau^{\geq 1} GX) \cong \mathrm{Hom}_{D_2}(X, F\tau^{\geq 1} GX) = 0 $$

since $F\tau^{\geq 1}GX \in D_1^{\geq 1}$. It follows that the distinguished triangle splits and $\tau^{\leq 0}GX = GX \oplus \tau^{\geq 1}GX[-1]$, in particular there is a nontrivial map $\tau^{\leq 0}GX \to \tau^{\geq 1}GX[-1]$. However $\mathrm{Hom}_{D_1}(\tau^{\leq 0}GX, \tau^{\geq 1}GX[-1]) = 0$ hence $\tau^{\geq 1}GX[-1] = 0$, i.e. $GX \in D_1^{\leq 0}$.

The converse is the same statement for opposite categories. ■

Lemma 1.4. *If $G : D_2 \to D_1$ is left adjoint to F, and F is left t-exact, then $H^0G : D_2^{\heartsuit} \to D_1^{\heartsuit}$ is left adjoint to $H^0F : D_1^{\heartsuit} \to D_2^{\heartsuit}$.*

Proof. By Lemma 1.3, F is left t-exact. For any $X \in D_2^{\heartsuit}$ and $Y \in D_1^{\heartsuit}$, use the truncation triangle for GX

$$\tau^{\leq -1}GX \to GX \to H^0GX \to$$

and the fact that $\mathrm{Hom}_{D_1}(\tau^{\leq -1}GX, Y) = 0$, we have

$$\mathrm{Hom}_{D_1^{\heartsuit}}(H^0GX, Y) = \mathrm{Hom}_{D_1}(GX, Y)$$

and similarly

$$\mathrm{Hom}_{D_2^{\heartsuit}}(X, H^0FY) = \mathrm{Hom}_{D_2}(X, FY)$$

and the ajointness between H^0G and H^0F follows from the above two isomorphisms and the adjunction between G and F. ■

The next lemma is about the behaviour of t-exactness under a dualizing functor. Recall that a dualizing functor on a triangulated category D with t-structure $(D^{\leq 0}, D^{\geq 0})$ is a t-exact functor $\mathbb{D} : D^{\mathrm{op}} \to D$ such that

- $\mathbb{D}$ maps $D^{\leq 0}$ to $D^{\geq 0}$;
- $\mathbb{D}^2 \cong \mathrm{Id}$.

Note that $\mathbb{D}^2 \cong \mathrm{Id}$ implies that $\mathbb{D}$ is an auto-equivalence, in particular $\mathrm{Hom}(\mathbb{D}X, \mathbb{D}Y) \cong \mathrm{Hom}(Y, X)$.

Lemma 1.5. *Suppose that D_i has dualizing functors $\mathbb{D}_i$, define $F_! := \mathbb{D}_2 \circ F \circ \mathbb{D}_1$, then $F_!$ is right (resp. left) t-exact if and only if F is left (resp. right) t-exact.*

Proof. Exercise. ■

Proof of Theorem 1.1. We first show that $f^*[d]$ is right t-exact. By definition of perversity, $\forall K \in \mathrm{Perv}(Y)$, $\dim(\mathrm{Supp}(\mathcal{H}^i(f^*K))) \leq \dim(\mathrm{Supp}(\mathcal{H}^i(K))) + d \leq d - i$, i.e. $f^*K[d] \in {}^pD_c^{\leq 0}(Y)$, so $f^*[d]$ is right t-exact. $f_*[-d]$ follows from applying Lemma 1.3 to adjunction $f^*[d] \dashv f_*[-d]$, $f_![d]$ follows from applying Lemma 1.5 to Verdier duality isomorphism $f_![d] \cong \mathbb{D}_Y \circ f_*[-d] \circ \mathbb{D}_X$, and $f^![-d]$ follows from applying Lemma 1.3 to the adjunction $f_![d] \dashv f^![-d]$. Finally, the statement about adjoint functors between perverse sheaves comes from applying Lemma 1.4 to adjunctions $f^*[d] \dashv f_*[-d]$ and $f_![d] \dashv f^![-d]$. ■

2. Affine pushforward

For a general morphism between varieties $f : X \to Y$, one would expect that f_* is left t-exact up to degree shift and according to Lemma 1.5 $f_!$ is right t-exact up to degree shift. Nevertheless, for affine morphisms, more can be said:

Theorem 2.1. *If $f : X \to Y$ is an affine morphism between varieties, then f_* is right t-exact and $f_!$ is left t-exact.*

Combine this theorem with Corollary 1.2 and we get

Corollary 2.2. *If $f : X \to Y$ is a quasi-finite affine morphism between varieties, then f_* and $f_!$ are t-exact.*

Example 2.3. Let $j : \mathbb{C}^\times \to \mathbb{C}$ be the natural open immersion, it's affine since $\mathbb{C}^\times$ is. Then we have seen that $j_!\underline{\mathbb{C}}[1]$ and $j_*\underline{\mathbb{C}}[1]$ are perverse sheaves, which agrees with the theorem. However, let $k : \mathbb{C}^2 - \{0\} \to \mathbb{C}^2$ be the natural embedding, it's not affine because $\mathbb{C}^2 - \{0\}$ is not affine. Then we have distinguished triangles

$$j_!\underline{\mathbb{C}}[2] \to \underline{\mathbb{C}}[2] \to \underline{\mathbb{C}^{\{0\}}}[2] \to \quad ; \quad \underline{\mathbb{C}^{\{0\}}}[-2] \to \underline{\mathbb{C}}[2] \to j_*\underline{\mathbb{C}}[2] \to$$

Suppose that $j_!\underline{\mathbb{C}}[2]$ is perverse, applying ${}^p\mathcal{H}^*$ to the first distinguished triangle, we find that $0 \to \underline{\mathbb{C}^{\{0\}}}[2] \to 0$ is exact, which is absurd, hence $j_!\underline{\mathbb{C}}[2]$ is not perverse; similarly, suppose that $j_*\underline{\mathbb{C}}[2]$ is perverse, then $0 \to \underline{\mathbb{C}^{\{0\}}}[-2] \to 0$ is exact, which implies that $j_*\underline{\mathbb{C}}[2]$ is not perverse.

We will provide the proof of the right t-exactness of f_* in the following subsection and the left t-exactness of $f_!$ follows from the Verdier duality. Roughly speaking, we will first prove a nonvanishing property of $\mathrm{H}^{\dim X}(U, F)$ for some open affine U (Lemma 2.7), use it to deduce a cohomological criterion for an object to lie in ${}^pD^{\leq 0}$ and ${}^pD^{\geq 0}$ (Proposition 2.9), then prove the theorem by this criterion.

Cohomology on affine open subschemes. Before we start, let's recall a result on homological bound in Balazs' talk [Ele18]

Lemma 2.4. *Let X be a variety of dimension d, and let F be a constructible sheaf on X. Then $\mathrm{H}^i(X, F) = \mathrm{H}^i_c(X, F) = 0$ for $i > 2d$; moreover, if X is affine, then $\mathrm{H}^i(X, F) = 0$ for $i > d$.*

As a corollary to the first part of the lemma, we have the following homological bound on pushforward (exercise):

Corollary 2.5. *Let $f : X \to Y$ be a morphism between varieties, F is a constructible sheaf on X, then $\mathcal{H}^i f_* F = \mathcal{H}^i f_! F = 0$ for $i > 2 \dim X$.*

As a corollary to the second part of the lemma, we have the following homological bound of constructible sheaves on affine varieties (exercise):

Corollary 2.6. *Let X be an affine variety, F is a constructible sheaf on X, then $\mathrm{H}^i(X, F) = 0$ for $i > \dim \mathrm{Supp}(F)$.*

Now let's prove a nonvanishing result about constructible sheaves on affine varieties

Lemma 2.7. *Let X be a variety, and let F be a nonzero constructible sheaf on X. If $\dim(\mathrm{Supp}(F)) = n$, then there exists an affine open subset $U \subset X$ such that*

$$\mathrm{H}^n(U, F|_U) \neq 0$$

Proof. Shrinking X if necessary, we can assume that X is affine. Let $Z = \mathrm{Supp}(F)$, so Z is a closed subvariety of X thus it's affine.

Step 1. Assume that $Z = X = \mathbb{C}^n$. Pick a point $i : x \hookrightarrow \mathbb{C}^n$ such that F is a local system in an open neighborhood of x. Up to a translation, we can assume that x is the origin. Denote the coordinate system on $\mathbb{C}^n$ by $\{x_1, x_2, \cdots, x_n\}$, we are going to prove that $\mathrm{H}^n(D(x_1x_2\cdots x_n), F|_{D(x_1x_2\cdots x_n)}) \neq 0$, where $D(f)$ means the open locus where the function f does not vanish. On the one hand, $i_*i^!F = i_*i^*F[-2n] \neq 0$, and the exact sequence

$$\mathrm{H}^{2n-1}(X - x, F) \to \mathrm{H}^{2n}(X, i_*i^!F) \to \mathrm{H}^{2n}(X, F)$$

indicates that $\mathrm{H}^{2n-1}(X-x, F) \neq 0$ (X is affine, so by the cohomology bound of affine variety, the last term is zero). On the other hand, note that $X - x = \bigcup_{i=1}^n D(x_i)$, so we have Čech spectral sequence:

$$E_1^{pq} = \bigoplus_{\substack{I\subset\{1,\dots,n\}\\ |I|=p+1}} \mathrm{H}^q(D(x_I), F) \Longrightarrow_p \mathrm{H}^{p+q}(X - x, F)$$

where $x_I := \prod_{i\in I} x_i$. Since E_1^{pq} is nonzero only for $0 \leq p \leq n-1$ and $0 \leq q \leq n$, the maximal degree term in the LHS is $\mathrm{H}^n(D(x_1x_2\cdots x_n), F)$ of total degree $2n-1$ which must be nonzero.

Step 2. Assume that $Z = X$. By Noether's normalization theorem, there exists a finite morphism $\phi : X \to \mathbb{C}^n$. We have $\mathrm{Supp}(\phi_*F) = \mathbb{C}^n$, so there is an open affine subvariety U of $\mathbb{C}^n$ such that $\mathrm{H}^n(U, (\phi_*F)|_U) \neq 0$, according to step 1. Hence $\mathrm{H}^n(\phi^{-1}U, F|_{\phi^{-1}U}) = \mathrm{H}^n(U, (\phi_*F)|_U) \neq 0$.

Step 3. For the general case $k : Z \hookrightarrow X$ is a closed embedding. In step 2 we have proven that $\exists U \subset Z$ of the form $D(f)$ such that $\mathrm{H}^n(U, (k^*F)|_U) \neq 0$. Lift f to an element g in the coordinate ring of X so $D(g) \bigcap Z = D(f)$, then we have $\mathrm{H}^n(D(g), F) = \mathrm{H}^n(D(f), k^*F)$. ■

Lemma 2.8. *Let X be an affine variety, and let $F \in {}^pD_c^{\leq 0}(X)$, then $R\Gamma(X, F) \in D_c^{\leq 0}(\mathrm{pt})$, equivalently $\mathrm{H}^i(X, F) = 0$ for $i > 0$. Moreover if $F \notin {}^pD_c^{\leq -1}(X)$, then there exists an open affine subvariety $U \subset X$ such that $\mathrm{H}^0(U, F|_U) \neq 0$.*

Proof. For the first part, note that we have a convergent spectral sequence

$$E_2^{pq} := \mathrm{H}^q(X, \mathcal{H}^p(F)) \Longrightarrow_p \mathrm{H}^{p+q}(X, F)$$

By definition of ${}^pD^{\leq 0}$, $\dim \mathrm{Supp}(\mathcal{H}^p(F)) \leq -p$, so according to Corollary 2.6, E_2^{pq} vanishes if $q > -p$, i.e. $p + q > 0$. As a result, $\mathrm{H}^{p+q}(X, F) = 0$ for $p + q > 0$.

For the second part,if $F \notin {}^pD_c^{\leq -1}(X)$, then ${}^p\tau^{\geq 0}F \neq 0$. Note that since $F \in {}^pD_c^{\leq 0}(X)$, ${}^p\tau^{\geq 0}F \in \mathrm{Perv}(X)$. Let us note that $\mathrm{H}^0(U, F|_U) \neq 0$ if $\mathrm{H}^0(U, K|_U) \neq 0$ for a simple quotient K of F. Indeed, consider the short exact sequence

$$0 \to G \to F \to K \to 0$$

where K is a simple perverse sheaf, if there exists an open affine subvariety $U \subset X$ such that $\mathrm{H}^0(U, K|_U) \neq 0$, then $\mathrm{H}^0(U, F|_U) \neq 0$, since $\mathrm{H}^1(U, G|_U) = 0$. So it's enough to prove the case when F is simple, i.e. $F = \mathrm{IC}(V, \mathcal{L})$ where V is smooth subvariety and $\mathcal{L}$ is an irreducible local system on V.

By replacing X with the Zariski closure of V (which is closed in X thus is also affine), we can assume that V is open and dense in X. Shrink V if necessary, we can assume that V

is affine. Accoring to Lemma 2.7, there exists an affine open subvariety U of V such that $\mathrm{H}^0(U, \mathcal{L}|_U[\dim(V)]) \neq 0$ and we win. ■

Proposition 2.9 (A Criterion for Half Perversity)**.** *X is a variety and $K \in D_c^-(X)$, then following statements are equivalent*

(1) $K \in {}^pD_c^{\leq 0}(X)$
(2) *For all open affine subvariety $U \subset X$, and $R\Gamma(U, K|_U) \in D_c^{\leq 0}(\mathrm{pt})$*

Proof. (1) $\Rightarrow$ (2): This follows from Lemma 2.8;

(2) $\Rightarrow$ (1): Since $K \in D_c^-(X)$, i.e. there exists an integer m such that $K \in D_c^{\leq m}(X)$, thus $\forall i \in \mathbb{Z}$, $\dim(\mathrm{Supp}(\mathcal{H}^i K)) \leq m + d - i$, i.e. $K \in {}^pD_c^{\leq m+d}(X)$. So there exists the smallest integer n such that $K \in {}^pD_c^{\leq n}(X)$. Since $K \notin {}^pD_c^{\leq n-1}(X)$, according to Lemma 2.8, there exists an affine open subvariety $U \subset X$ such that $\mathrm{H}^n(U, K|_U) \neq 0$. Now condition in (2) implies that n must be non-positive and (1) follows. ■

Remark 2.10. If $L \in D_c^+(X)$, apply the Verdier duality to the lemma, then following statements are equivalent

(a) $L \in {}^pD_c^{\geq 0}(X)$
(b) For all open affine subvariety $U \subset X$, and $R\Gamma_c(U, L|_U) \in D_c^{\geq 0}(\mathrm{pt})$

Proof of Theorem 2.1. Suppose $K \in {}^pD_c^{\leq 0}(X)$. Since f_* has finite cohomological dimension (Corollary 2.5), we see that $f_*K \in D_c^-(Y)$. For any open affine subcheme $U \subset Y$,

$$R\Gamma(U, f_*K|_U) = R\Gamma(f^{-1}U, K|_{f^{-1}U})$$

Since f is an affine morphism and U is affine, $f^{-1}U$ is also affine, by Proposition 2.9, $R\Gamma(f^{-1}U, K|_{f^{-1}U}) \in {}^pD_c^{\leq 0}(\mathrm{pt})$, i.e. $R\Gamma(U, f_*K|_U) \in {}^pD_c^{\leq 0}(\mathrm{pt})$. By Proposition 2.9 again, $f_*K \in {}^pD_c^{\leq 0}(Y)$. ■

3. Smooth pullback

Suppose $f : X \to Y$ is a smooth morphism between varieties of relative dimension d, then $f^! = f^*[2d]$, or equivalently $f^*[d] = f^![-d]$. According to Theorem 1.1, the LHS is right t-exact, while the RHS is left t-exact, so we have the following

Theorem 3.1. *If $f : X \to Y$ is a smooth morphism of relative dimension d, then functor $f^*[d] = f^![-d]$ is t-exact.*

Definition 3.2. If $f : X \to Y$ is an smooth morphism of relative dimension d, we define $f^\dagger := f^*[d] : D_c^-(Y) \to D_c^-(X)$ and its right adjoint $f_\dagger := f_*[-d] : D_c^+(X) \to D_c^+(Y)$.

Remark 3.3. Theorem 3.1 tells us that $f^\dagger$ restricts to an exact functor $\mathrm{Perv}(Y) \to \mathrm{Perv}(X)$. Moreover, according to Lemma 1.3 and 1.4, we have

Corollary 3.4. *$f^\dagger$ is t-exact, $f_\dagger$ is left t-exact, and we have a pair of adjoint functors between perverse sheaves:*

$$f^\dagger : \mathrm{Perv}(Y) \rightleftarrows \mathrm{Perv}(X) : {}^p\mathrm{H}^0 f_\dagger$$

Smooth morphisms with connected fibers.

Theorem 3.5. *Suppose $f : Y \to X$ is a smooth surjective morphism with connected fibers, then $f^* : \mathrm{Shv}(X) \to \mathrm{Shv}(Y)$ is fully faithful and sends irreducible local systems to irreducible local systems*

Proof. In order to prove that $\mathrm{Hom}(F, G) \to \mathrm{Hom}(f^*F, f^*G) \cong \mathrm{Hom}(f^!F, f^!G)$ is an isomorphism, it suffices to prove that adjunction

$$\mathcal{H}^0 f_! f^! F \to F$$

is isomorphism. By proper base change theorem (See Roger's talk [Bai18]), it's enough to check the isomorphism property fiberwise:

$$i_x^* \mathcal{H}^0 f_! f^! F \cong \mathrm{H}_c^{2d}(Y_x, \underline{F_x}) \to F_x$$

Now the last homomorphism is an isomorphism because of Poincaré duality and the fact that Y_x is connected. This shows that f^* is fully faithful. Now let us prove that it sends irreducible local systems to irreducible local systems.

If $\mathcal{L}$ is an irreducible local system on X, and suppose that there is a surjective homomorphism of local systems:

$$\phi : f^*\mathcal{L} \twoheadrightarrow \mathcal{M}$$

such that $\mathcal{M}$ is not zero. We are about to show that ϕ is an isomorphism. Pushing forward ϕ to the base, we have morphism

$$\psi : \mathcal{H}^{2d} f_! f^* \mathcal{L} \cong \mathcal{H}^{2d} f_! \underline{\mathbb{C}} \otimes \mathcal{L} \cong \mathcal{L} \twoheadrightarrow \mathcal{H}^{2d} f_! \mathcal{M}$$

which is surjective since $\mathcal{H}^{2d+1} f_! \ker(\phi) = 0$ by the cohomological bound (Corollary 2.5). By proper base change theorem, it is enough to prove that this epimorphism is an isomorphism fiberwise. We find $(\mathcal{H}^{2d} f_! \mathcal{M})_x \cong \mathrm{H}^{2d}(Y_x, \mathcal{M}|_{Y_x})$. Notice that $\phi_x : f_x^* \mathcal{L}_x \twoheadrightarrow \mathcal{M}|_{Y_x}$ is surjective, in particular, $\mathcal{M}|_{Y_x}$ is a constant local system on Y_x, therefore $\mathrm{H}^{2d}(Y_x, \mathcal{M}|_{Y_x})$ has the same rank with $\mathcal{M}$, so $\mathcal{H}^{2d} f_! \mathcal{M}$ has constant rank. Now surjectivity of ψ implies that $\mathcal{H}^{2d} f_! \mathcal{M}$ is also a local system, but $\mathcal{L}$ is irreducible, so ψ is an isomorphism. ∎

Remark 3.6. If X is smooth, then there is an easier way to prove the second part of the theorem: the statement is equivalent to saying that $\pi_1(Y) \twoheadrightarrow \pi_1(X)$. For proper f, this follows from Ehresmann's fibration theorem and the exact sequence of homotopy group associated to fibration. It's clear for open embeddings as well (this is the place where we need smoothness). For general case, we use Nagata's compactification and embed Y as an open subvariety into $\bar{Y}$ which is proper over X, then we apply resolution of singularity to $\bar{Y}$ (recall that this can be done in a way that Y is unchanged: by a sequence of blowing up of subvarieties inside the singular locus $\mathrm{Sing}(\bar{Y}) \subset \bar{Y} - Y$), so we can assume that $\bar{Y}$ is smooth. By generic smoothness, there exists an open subvariety $U \subset X$ such that $\bar{Y} \times_X U \to U$ is smooth. Combining the open embedding case and locally trivial fibration case, we have

$$\pi_1(Y \times_X U) \twoheadrightarrow \pi_1(\bar{Y} \times_X U) \twoheadrightarrow \pi_1(U) \twoheadrightarrow \pi_1(X)$$

which factors through $\pi_1(Y \times_X U) \to \pi_1(Y) \to \pi_1(X)$, hence $\pi_1(Y) \to \pi_1(X)$ is surjective.

Remark 3.7. In general, f^* is not fully faithful as a functor between derived categories. For example, if f is the quotient by $\mathbb{G}_m$: $\mathbb{C}^2 - \{0\} \to \mathbb{P}^1$, then $\mathrm{Hom}_{\mathbb{P}^1}(\underline{\mathbb{C}}, \underline{\mathbb{C}}[3]) = 0$ but $\mathrm{Hom}_{\mathbb{C}^2-\{0\}}(f^*\underline{\mathbb{C}}, f^*\underline{\mathbb{C}}[3]) = \mathbb{C}$, $\mathrm{Hom}_{\mathbb{P}^1}(\underline{\mathbb{C}}, \underline{\mathbb{C}}[2]) = \mathbb{C}$ but $\mathrm{Hom}_{\mathbb{C}^2-\{0\}}(f^*\underline{\mathbb{C}}, f^*\underline{\mathbb{C}}[2]) = 0$, so f^* is neither full nor faithful.

Theorem 3.8. *Suppose $f : Y \to X$ is a smooth surjective morphism with connected fibers, then $f^\dagger$ is fully faithful and sends simple perverse sheaves to simple perverse sheaves, in fact, for smooth scheme Z with immersion $i : Z \hookrightarrow X$ and $\mathcal{L} \in \mathrm{Loc}(Z)$*

$$f^\dagger \,\mathrm{IC}(Z, \mathcal{L}) = \mathrm{IC}(f^{-1}Z, (f|_{f^{-1}Z})^*\mathcal{L})$$

Proof. By definition, $\mathrm{IC}(Z, \mathcal{L})$ is the intermediate extension of $\mathcal{L}$, i.e. there is a sequence of perverse sheaves

$${}^p\mathcal{H}^0 i_!\mathcal{L}[\dim Z] \twoheadrightarrow \mathrm{IC}(Z, \mathcal{L}) \hookrightarrow {}^p\mathcal{H}^0 i_*\mathcal{L}[\dim Z]$$

Apply the t-exact functor $f^\dagger$ to this sequence and we get a sequence of perverse sheaves

$$f^\dagger\, {}^p\mathcal{H}^0 i_!\mathcal{L}[\dim Z] \twoheadrightarrow f^\dagger \,\mathrm{IC}(Z, \mathcal{L}) \hookrightarrow f^\dagger\, {}^p\mathcal{H}^0 i_*\mathcal{L}[\dim Z]$$

By proper base change and smooth base change theorems [Bai18], the sequence can be rewritten as

$${}^p\mathcal{H}^0\tilde{i}_!((f|_{f^{-1}Z})^*\mathcal{L})[\dim f^{-1}Z] \twoheadrightarrow f^\dagger \,\mathrm{IC}(Z, \mathcal{L}) \hookrightarrow {}^p\mathcal{H}^0\tilde{i}_*((f|_{f^{-1}Z})^*\mathcal{L})[\dim f^{-1}Z]$$

where $\tilde{i} : f^{-1}Z \hookrightarrow Y$ is the pullback of i. Now by definition of IC-sheaf, $f^\dagger \,\mathrm{IC}(Z, \mathcal{L})$ is exactly the intermediate extension of $(f|_{f^{-1}Z})^*\mathcal{L}$, i.e.

$$f^\dagger \,\mathrm{IC}(Z, \mathcal{L}) = \mathrm{IC}(f^{-1}Z, (f|_{f^{-1}Z})^*\mathcal{L})$$

and $\mathrm{IC}(f^{-1}Z, (f|_{f^{-1}Z})^*\mathcal{L})$ is simple if and only if $(f|_{f^{-1}Z})^*\mathcal{L}$ is irreducible, and $(f|_{f^{-1}Z})^*\mathcal{L}$ is irreducible when $\mathcal{L}$ is irreducible (Theorem 3.5), i.e. when $\mathrm{IC}(Z, \mathcal{L})$ is simple.

Next, we are proceeding to prove that $f^\dagger$ is fully faithful by showing that

$$\mathcal{F} \to {}^p\mathcal{H}^0 f_\dagger f^\dagger \mathcal{F}$$

is isomorphism. Notice that for every point $x \in X$, there is an étale morphism $U \to X$ such that x is contained in the image and $f_U : Y \times_X U \to U$ has a section $\sigma_U : U \to Y \times_X U$. This implies that the composition $\mathcal{F}_U \to {}^p\mathcal{H}^0 f_{U\dagger} f_U^\dagger \mathcal{F}_U \to {}^p\mathcal{H}^0 f_{U\dagger}\sigma_{U*}\sigma_U^* f_U^\dagger \mathcal{F}_U \cong \mathcal{F}_U$ is identity, so

$${}^p\mathcal{H}^0 f_{U\dagger} f_U^\dagger \mathcal{F}_U \cong \mathcal{F}_U \oplus K$$

for some $K \in \mathrm{Perv}(U)$. To prove that $\mathcal{F} \to {}^p\mathcal{H}^0 f_\dagger f^\dagger \mathcal{F}$ is isomorphism, it suffices to prove the analogous statement for F_U and f_U on every étale morphism $U \to X$. Replacing X by U, we assume that f has a section.

Step 1. We prove the case when X is smooth and $\mathcal{F} = \mathcal{L}[\dim X]$ where $\mathcal{L}$ is a local system on X.

By previous discussion and assumption, there is a decomposition ${}^p\mathcal{H}^0 f_\dagger f^\dagger \mathcal{F} \cong \mathcal{F} \oplus K$. Shrink X to an open subvariety $V \subset X$, we can assume that $\mathcal{H}^i(f|_{f^{-1}V})_\dagger(f|_{f^{-1}V})^\dagger\mathcal{F}|_V$ are local systems, since they are already constructible and there are only finitely many nonzero terms (Corollary 2.5). This implies that

$${}^p\mathcal{H}^0(f|_{f^{-1}V})_\dagger(f|_{f^{-1}V})^\dagger\mathcal{F}|_V = \mathcal{H}^0(f|_{f^{-1}V})_\dagger(f|_{f^{-1}V})^\dagger\mathcal{F}|_V = {}^\circ f_* f^*\mathcal{L}[\dim X]$$

By Theorem 3.5, $\mathcal{H}^0(f|_{f^{-1}V})_*(f|_{f^{-1}V})^*\mathcal{L}|_V[\dim X] \cong \mathcal{L}|_V[\dim X]$ hence we have

$$\mathcal{F}|_V \cong {}^p\mathcal{H}^0(f|_{f^{-1}V})_\dagger(f|_{f^{-1}V})^\dagger\mathcal{F}|_V$$

i.e. $K|_V = 0$. As a result, $\mathrm{Supp}(f^\dagger K) \neq Y$, so $\mathrm{Hom}(f^\dagger K, f^\dagger\mathcal{F}) = 0$ since $f^\dagger\mathcal{F} \in \mathrm{Loc}(Y)[\dim Y]$ which is a Serre subcategory of $\mathrm{Perv}(Y)$. This implies by adjunction that $\mathrm{Hom}(K, {}^p\mathcal{H}^0 f_\dagger f^\dagger\mathcal{F})$ is trivial, in particular, the canonical embedding $K \to {}^p\mathcal{H}^0 f_\dagger f^\dagger\mathcal{F}$ is trivial and we conclude that $\mathcal{F} \to {}^p\mathcal{H}^0 f_\dagger f^\dagger\mathcal{F}$ is isomorphism.

Step 2. We prove the case that X is arbitrary and $\mathcal{F}$ is a simple perverse sheaf, i.e. $\mathcal{F} = \mathrm{IC}(Z,\mathcal{L})$ where $\mathcal{L}$ is an irreducible local system on a smooth locally closed subvariety $i : Z \hookrightarrow X$.

Frist of all, we observe that since $\mathrm{IC}(Z,\mathcal{L})$ is supported on $\bar{Z}$, ${}^p\mathcal{H}^0 f_\dagger f^\dagger \mathrm{IC}(Z,\mathcal{L})$ is also supported on $\bar{Z}$.

Then, apply $i^!$ to $\mathrm{IC}(Z,\mathcal{L}) \to {}^p\mathrm{H}^0 f_\dagger f^\dagger \mathrm{IC}(Z,\mathcal{L})$ and we get

$$\mathcal{L}[\dim Z] \to {}^p\mathrm{H}^0(f|_{f^{-1}Z})_\dagger(f|_{f^{-1}Z})^\dagger\mathcal{L}[\dim Z]$$

which is an isomorphism by step 1, hence $i^!K = 0$. As a result, K is supported on $\bar{Z} - Z$.

Finally, the embedding $K \hookrightarrow {}^p\mathcal{H}^0 f_\dagger f^\dagger \mathrm{IC}(Z,\mathcal{L})$ induced a nontrivial map

$$f^\dagger K \to f^\dagger \mathrm{IC}(Z,\mathcal{L}) = \mathrm{IC}(f^{-1}Z, (f|_{f^{-1}Z})^*\mathcal{L})$$

But $f^\dagger K$ is supported on $f^{-1}\bar{Z} - f^{-1}Z$, which contradicts with the property of IC-sheaves. Hence $K = 0$.

Step 3. We prove the general case by induction on length. For an arbitrary perverse sheaf $\mathcal{F}$, take a short exact sequence $0 \to \mathcal{G} \to \mathcal{F} \to \mathrm{IC}(W,\mathcal{E}) \to 0$ and apply the adjunction to this sequence:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathrm{IC}(W,\mathcal{E}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {}^p\mathcal{H}^0 f_\dagger f^\dagger\mathcal{G} \arrow[r] & {}^p\mathcal{H}^0 f_\dagger f^\dagger\mathcal{F} \arrow[r] & {}^p\mathcal{H}^0 f_\dagger f^\dagger \mathrm{IC}(W,\mathcal{E}) &
\end{tikzcd}$$

By induction and step 2, the first and the third vertical maps are isomorphism, so by the snake lemma the middle vertical map is also an isomorphism and we are done. ■

Introduction to descent theory. Suppose that $f : Y \to X$ is a surjective morphism, then we can ask a general question: Given a homomorphism between f^*F and f^*G, does it come from a morphism $\phi : F \to G$? We have seen that this is always true if f is smooth with connected fibers. For more general f, descent theory is developed to answer this question. In descent theory, a commutative diagram of morphisms is considered:

$$\begin{tikzcd}
Y \times_X Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, "f_2"] & Y \arrow[d, "f"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

In this situation, we have following results:

Theorem 3.9. *Given two constructible sheaves F and G on a variety X, $f : Y \to X$ is a surjective smooth morphism, then there is an exact sequence*

$$0 \to \mathrm{Hom}_X(F, G) \to \mathrm{Hom}_Y(f^*F, f^*G) \to \mathrm{Hom}_{Y\times_X Y}(f_2^*F, f_2^*G)$$

where the third map is given by the difference of two maps pr_1^* *and* $\mathrm{pr}_2^* : \mathrm{Hom}_Y(f^*F, f^*G) \to \mathrm{Hom}_{Y\times_X Y}(f_2^*F, f_2^*G)$.

Theorem 3.10. *Given two perverse sheaves F and G on a variety X, $f : Y \to X$ is a surjective smooth morphism of relative dimension d, then there is an exact sequence*

$$0 \to \mathrm{Hom}_X(F, G) \to \mathrm{Hom}_Y(f^\dagger F, f^\dagger G) \to \mathrm{Hom}_{Y\times_X Y}(f_2^\dagger F, f_2^\dagger G)$$

where the third map is given by the difference of two maps pr_1^* *and* $\mathrm{pr}_2^* : \mathrm{Hom}_Y(f^\dagger F, f^\dagger G) \to \mathrm{Hom}_{Y\times_X Y}(f_2^\dagger F, f_2^\dagger G)$.

Remark 3.11. Roughly speaking, Theorem 3.9 and 3.10 tells us that maps between constructible sheaves or between perverse sheaves can be glued along a smooth surjective morphism $Y \to X$, if and only if they agree on the product $Y \times_X Y$. A classical situation is when $Y \to X$ is a covering of open subvarieties, then a system of maps on individual open subvarieties can be glued to a global one if and only if they agree on the overlap.

Sketch of proof to 3.9. Since f is smooth, $\mathrm{Hom}(f^*F, f^*G) \cong \mathrm{Hom}(f^!F, f^!G)$. By adjunction, it suffices to prove that

$$\mathcal{H}^0 f_{2!} f_2^! F \to \mathcal{H}^0 f_! f^! F \to F \to 0$$

is exact. By proper base change, it's enough to prove it fiberwise, i.e.

$$\mathrm{H}_c^{4d}(Y_x \times Y_x, \underline{F_x}) \to \mathrm{H}_c^{2d}(Y_x, \underline{F_x}) \to F_x \to 0$$

is exact. Observe that $\mathrm{H}_c^{2\dim Y}(Y, \underline{\mathbb{C}}) = \mathbb{C}^{|\text{connected components of Y}|}$. Since connected components of $Y_x \times Y_x$ one to one correspond to product of connected components of Y_x, then we can assume that Y_x is disjoint union of finite points and the exactness follows easily. ■

Sketch of proof to 3.10. It suffices to prove that

$$0 \to G \to {}^p\mathcal{H}^0 f_\dagger f^\dagger G \to {}^p\mathcal{H}^0 f_{2\dagger} f_2^\dagger G$$

is exact. Since a sequence of perverse sheaves is exact if and only if it's exact after being pulled back to an étale cover, we can replace X by any étale cover. We choose an étale cover $U \twoheadrightarrow X$ such that f_U has a section $\sigma : U \to Y \times_X U$. From now on we assume that f has a section.

The composition $\mathrm{Hom}_X(F, G) \to \mathrm{Hom}_Y(f^\dagger F, f^\dagger G) \to \mathrm{Hom}_X(\sigma^* f^\dagger F, \sigma^* f^\dagger G) \cong \mathrm{Hom}_X(F, G)$ is identity, so $\mathrm{Hom}_X(F, G) \to \mathrm{Hom}_Y(f^\dagger F, f^\dagger G)$ is injective. Suppose $\phi \in \ker(\mathrm{pr}_1^* - \mathrm{pr}_2^*)$, then we can obtain $\sigma^*(\phi) \in \mathrm{Hom}_X(F, G)$. We are going to prove that $\phi = f^*\sigma^*(\phi)$, or equivalently, if $\sigma^*(\phi) = 0$, then $\phi = 0$.

This is done by the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_X(F,G) & \xrightarrow{f^*} & \mathrm{Hom}_Y(f^\dagger F, f^\dagger G) & \xrightarrow{\sigma^*} & \mathrm{Hom}_X(F,G) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \mathrm{pr}_2^*} & & \downarrow{\scriptstyle f^*} \\
\mathrm{Hom}_Y(f^\dagger F, f^\dagger G) & \xrightarrow{\mathrm{pr}_1^*} & \mathrm{Hom}_{Y\times_X Y}(f_2^\dagger F, f_2^\dagger G) & \xrightarrow{\sigma_1^*} & \mathrm{Hom}_Y(f^\dagger F, f^\dagger G)
\end{array}$$

then $\sigma_1^* \operatorname{pr}_2^*(\phi) = f^*\sigma^*(\phi) = 0$. Since $\operatorname{pr}_1^*(\phi) = \operatorname{pr}_2^*(\phi)$ by assumption, we obtain $\sigma_1^* \operatorname{pr}_1^*(\phi) = 0$, but $\operatorname{pr}_1 \circ \sigma_1 = \mathrm{Id}_Y$ hence $\phi = 0$. ■

Example 3.12. We don't expect that statement in Theorem 3.10 will be true for the whole derived category: Consider $X = \mathbb{P}^1$, take a nontrivial element $\alpha \in \mathrm{H}^2(\mathbb{P}^1, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{P}^1}(\underline{\mathbb{C}}, \underline{\mathbb{C}}[2])$. Restricting to every open subvariety $U \subsetneqq \mathbb{P}^1$, there is no nontrivial homomorphism between $\underline{\mathbb{C}}_U$ and $\underline{\mathbb{C}}_U[2]$, since U being affine implies that $\mathrm{H}^2(U, \mathbb{C}) = 0$. Take Y to be disjoint union of two copies of $\mathbb{C}$ and $f : Y \to X$ is the canonical covering, then $\mathrm{Hom}_X(\underline{\mathbb{C}}, \underline{\mathbb{C}}[2]) \to \mathrm{Hom}_Y(f^\dagger\underline{\mathbb{C}}, f^\dagger\underline{\mathbb{C}}[2])$ is not injective.

Having seen the gluing of maps, we can ask the question about the gluing of constructible sheaves or perverse sheaves.First of all we define what it means to be "agree" on the product $Y \times_X Y$.

Definition 3.13. Suppose that $f : Y \to X$ is a smooth surjective morphism. Let $F \in \mathrm{Shv}_c(Y)$. A **descent datum** of F with respect to f is an isomorphism $\phi : \operatorname{pr}_2^* F \cong \operatorname{pr}_1^* F \in \mathrm{Shv}_c(Y \times_X Y)$, such that the following cocycle condition is satisfied:

$$\operatorname{pr}_{13}^*\phi = \operatorname{pr}_{12}^*\phi \circ \operatorname{pr}_{23}^*\phi : \ \operatorname{pr}_3^* F \to \operatorname{pr}_1^* F$$

A descent datum (F, ϕ) is called **effective** if there is a constructible sheaf $G \in \mathrm{Shv}_c(X)$ such that $F = f^*G$ and $\phi : \operatorname{pr}_2^* f^*G \cong \operatorname{pr}_1^* f^*G$ is the natural isomorphism.

Definition 3.14. Same notation as above. Let $F \in \mathrm{Perv}(Y)$. A **descent datum** of F with respect to f is an isomorphism $\phi : \operatorname{pr}_2^\dagger F \cong \operatorname{pr}_1^\dagger F \in \mathrm{Perv}(Y \times_X Y)$, such that the following cocycle condition is satisfied:

$$\operatorname{pr}_{13}^\dagger\phi = \operatorname{pr}_{12}^\dagger\phi \circ \operatorname{pr}_{23}^\dagger\phi : \ \operatorname{pr}_3^\dagger F \to \operatorname{pr}_1^\dagger F$$

A descent datum (F, ϕ) is called **effective** if there is a perverse sheaf $G \in \mathrm{Perv}(X)$ such that $F = f^\dagger G$ and $\phi : \operatorname{pr}_2^\dagger f^\dagger G \cong \operatorname{pr}_1^\dagger f^\dagger G$ is the natural isomorphism.

Following theorem answers the question about gluing of sheaves (proof omitted):

Theorem 3.15. *Same notation as above, then every descent datum for constructible sheaves or perverse sheaves is effective.*

References

[Bai18] Yuguang (Roger) Bai. Perverse sheaves learning seminar: Derived categories and its applications to sheaves. 2018.

[Ele18] Balázs Elek. Perverse sheaves learning seminar: Smooth maps and constructible sheaves. 2018.