Regular Polygons and Constructible Angles

Original Notes adopted from February 26, 2002 (Week 20)

Surds: $Q \subset Q(\sqrt{r}) \subset (Q(\sqrt{r}))(\sqrt{r_1})$, Also Surd = Constructible.
Theorem: If a cubic with rational coefficients has a constructible root, then it has a rational root.

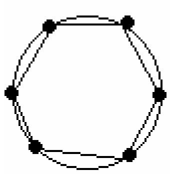
"Duplication of the Cube". Given a cube of side, volume 1, Can you construct a cube of volume 2?

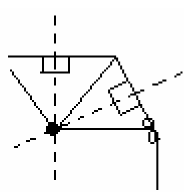
Volume = x^3 . Can construct a solution x to $x^3 = 2$?
$x^3 - 2 = 0$. Has only $2^{1/3}$ not rational, so $x^3 - 2$ has no rational root and thus has no constructible root.

Regular Polygon: Equal sides and Equal angles...

3 Equilateral Triangle
4 Square
5 Pentagon
6 Hexagon

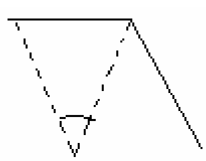
Theorem: Every regular polygon can be inscribed in a circle.
The intersection of the perpendicular bisector of any two sides is the middle.





Given Regular polygon:

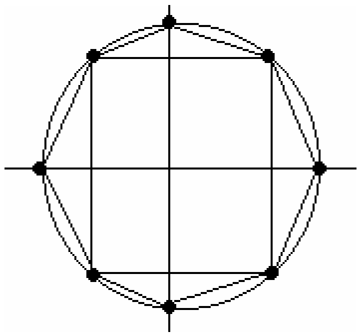
Central angle = 360°/ # of sides.
A Regular Polygon is Constructible if and only if its CENTRAL ANGLE is constructible.



We proved that the Angle of 20° is not constructible

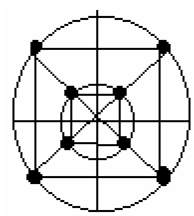
Corollary: A regular polygon of 18 sides is not constructible.

Note: If regular polygon with n sides is constructible, so is regular polygon of 2n sides.
(Bisect central angles).



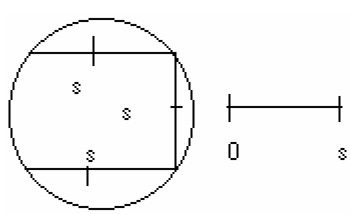
If regular polygon with 2m sides is constructible, so is regular polygon with m sides (m>2).

Consider 10 – sided regular polygon:
If a regular polygon with n sides is constructible, then a regular polygon with n sides can be constructed so that it is inscribed in circle radius 1.

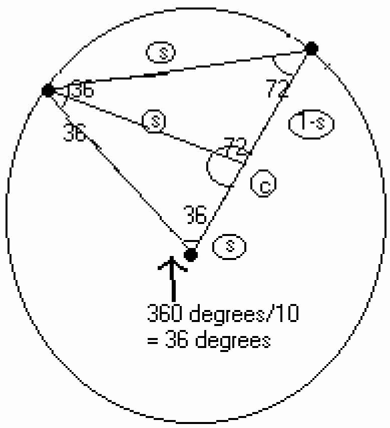


Lemma: A regular n-gon is constructible if and only if the length of the side of a regular n-gon inscribed in a circle of radius 1 is a constructible number.

Proof: If regular n-gon is contructible, then construct such in circle of radius 1. Use compass to measure side, done. Conversely, if side is a constructible number, say, make radius 1.



Suppose we have a 10 sided regular polygon inscribed in circle of radius 1. Let s be the lenght of its side. Is s constructible?



Triangle OAB ~ Triangle CAB (ie. 36°, 36°, 72°)

s/1-s = 1/s
$s^2 = 1 - s$
$s^2 + s - 1 = 0$.

Using Quadratic formula...

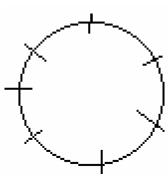
$$s = \frac{-1 \ (+/-) \sqrt{1+4}}{2} = \frac{-1 + \sqrt{5}}{2} = \frac{\sqrt{5} - 1}{2} \Rightarrow \text{Constructible.}$$

∴ a 10 sided regular polygon is constructible
∴ a regular pentagon is constructible.

Constructible polygons: 3,4,5,6,8 (yes because we can do 4), 10(we proved), 12
Not constructible: 9 (by Lemma)

7 sided Polygon?
Recall: The complex solutions to $z^n = 1$ are the verticies of an n-sided regular polygon inscribed in circle of radius 1.

$z = \cos\theta + i\sin\theta$, where $n\theta = 2\pi k$, some k.



The solutions of $z^7 = 1$ are vertices of a regular 7-gon inscribed in circle of radius 1. What are the solutions?

Suppose $z^7 = 1$ & $z \neq 1$
$z^7 - 1 = (z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)$

Thus $z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$
OR $z^3 + z^2 + z + 1 + + 1/z + 1/z^2 + 1/z^3 = 0$ (Dividing by z^3)
$(z + 1/z)^3 = z^3 + 3z^2 (1/z) + 3z(1/z)^2 (1/z)^3 = 0$
$= z^3 + 3z + 3/z + 1/z^3 = 0$
$(z + 1/z)^2 = z^2 + z + 1/z^2$

$z^3 + z^2 + z + 1 + + 1/z + 1/z^2 + 1/z^3 = 0$
$= (z + 1/z)^3 + (z + 1/z)^2 - 3(z + 1/z) - 1 + (z + 1/z)$ (This is an extra step)
$= *(z + 1/z)^3 + (z + 1/z)^2 - 2(z + 1/z) - 1$

Side note:
$z^2 = 1$,

$z \overline{z} = 1$ $\overline{z} = 1/z$

$z + 1/z = z + \overline{z} = 2 \text{ Re } z$

If a regular 7-gon were constructible, then one could be constructed in circle of radius 1, such that 1 vertex is on the x-axis at the point 1. The next vertex up would have a real part (ie. x component) that is constructible (Drop a perpendicular from the point to the x-axis).
Let x be twice that real part.
ie.) $x = z + 1/x$, so * $x^3 + x^2 - 2x - 1 = 0$
ie.) By Theorem $x^3 + x^2 - 2x - 1 = 0$ would have a rational root if regular 7-gon was constructible.

If $x = m/n$ in lowest terms,
$(m/n)^3 + (m/n)^2 - 2(m/n) - 1 = 0$
$m^3 + m^2 n - 2mn^2 - n^3 = 0$

If p prime, if p|m then p|n $\Rightarrow$ no p, no common factor $\Rightarrow$ m +/- 1.
If p|n, then p|m, so no p, so n = +/-1./
$\therefore$ x = m/n = +/- 1.
$1^3 + 1^2 - 2 - 1 \neq 0$.
$(-1)^3 + (-1)^2 + 2 - 1 \neq 0$ $\therefore$ no rational solution.
$\therefore$ Regular 7-gon is not constructible.

Lemma: If x_0 is a root of a polynomial with coefficients in $F(\sqrt{r})$, then x_0 is a root of a polynomial (with twice the degree) with coefficients in F.

Proof: $a_n x_0^n + a_{n-1} x_0^{n-1} + \ldots\ldots a_1 x_0 + a_0 = 0$.
$a_i = b_i + c_i \sqrt{r}$ $b_i, c_i \in F$.
Then $b_n x_0^n + b_{n-1} x_0^{n-1} + \ldots\ldots b_1 x_0 + b_0 = -\sqrt{r} \ (c_n x_0^n + c_{n-1} x_0^{n-1} + \ldots\ldots c_1 x_0 + c_0)$
$(b_n x_0^n + b_{n-1} x_0^{n-1} + \ldots\ldots b_1 x_0 + b_0)^2 = -\sqrt{r} \ (c_n x_0^n + c_{n-1} x_0^{n-1} + \ldots\ldots c_1 x_0 + c_0)^2 = 0$.
r is in F, so all in F

Theorem: Every constructible number is algebraic.