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The twistor space of $\mathbb{R}^{4n}$ and Berezin-Toeplitz quantization.

(joint w/A. Tomberg)

L. Coburn '92: $H^2(\mathbb{C}^n, d\mu_k)$ $d\mu_k = \left(\frac{k}{\pi}\right)^n e^{-k|z|^2} d\mu(z)$

for $g \in L^2(d\mu_k)$ Berezin-Toeplitz operator $T_g^{(k)}$ is defined by

$$(T_g^{(k)} h)(z) = \int g(w) h(w) e^{k z \cdot \bar{w}} d\mu_k(w)$$

Berezin-Toeplitz quantization $g \mapsto T_g^{(k)}$

Theorem (Coburn) Let $f \in C_c^{n+3}(\mathbb{C}^n)$, $g \in BC^{2n+6}(\mathbb{C}^n)$

Then $\exists\, C > 0$ s.t.

$$\| T_f^{(k)} T_g^{(k)} - T_{fg}^{(k)} - \frac{1}{k} T^{(k)}_{\sum_j \frac{\partial f}{\partial z_j} \frac{\partial g}{\partial \bar{z}_j}} \| \leq \frac{C}{k^2} \text{ for all } k > 0$$

Cor.

$$\| ik [T_f^{(k)}, T_g^{(k)}] - T^{(k)}_{\{f,g\}} \| \leq \frac{C}{k}$$

$\mathbb{C}^{2n}$ is $\mathbb{R}^{4n}$ with the standard linear complex structure I

$\mathbb{R}^{4n}$ is hyperkähler standard flat metric g

I, J, K $\quad I^2 = J^2 = K^2 = IJK = -1$

A question in the hyperkähler case: is there a quantization that "incorporates" the three quantizations (for I, J, K) ?

More generally, there is a Berezin-Toeplitz quantization as above for each linear c.s. $aI + bJ + cK$ $\quad a^2 + b^2 + c^2 = 1$, $a, b, c \in \mathbb{R}$

S^2 family of quantizations

The twistor space of $\mathbb{R}^{4n}$ is

$\mathbb{R}^{4n} \times S^2$ with integrable almost complex structure, complex manifold,

at $(x, \overset{\omega}{A})$ $\quad T_x\mathbb{R}^{4n} \oplus T_A\mathbb{CP}^1 \to T_x\mathbb{R}^{4n} \oplus T_A\mathbb{CP}^1$

$\quad\ \ \|$

(a,b,c) $\quad (X, V) \mapsto (AX, I_{\mathbb{CP}^1} V)$

$aI + bJ + cK$

Over one chart

$$\mathbb{R}^{4n} \times (S^2 - \{pt\}) \cong \mathbb{C}^{2n+1}$$

$\underbrace{v, \xi}_{\mathbb{C}^{2n}}, \underset{\uparrow\ \mathbb{CP}^1 - \{pt\}}{\zeta}$ complex coordinates (see e.g. Hitchin's papers on hyperholomorphic line bundle)

We define a map (linear, injective)

$$C_c^\infty(\mathbb{R}^{4n}) \to L^2\left(\mathbb{C}^{2n+1}, \left(\frac{k}{\pi}\right)^{2n} e^{-k(v\cdot\bar{v} + \xi\cdot\bar{\xi})} \frac{1}{\pi(1+|\zeta|^2)^2} d\mu(v,\xi,\zeta)\right)$$

$$f \mapsto \tilde{f}$$

Theorem (T.B., A. Tomberg)

For $f, g \in C_c^\infty(\mathbb{R}^{4n})$ $\exists C > 0$ s.t.

$$\| ik [T^{(k)}_{\tilde{f}}, T^{(k)}_{\tilde{g}}] - T^{(k)}_{\{\tilde{f}, \tilde{g}\}} \| \le \frac{C}{k} \quad \text{for all } k > 0.$$

Thus, we replaced the S^2 family of quantizations (minus 1pt) by one quantization, on the twistor space (with one fiber removed)

Comments:
- choice of point $\leftrightarrow$ $SU(2)$ action
- $\mathbb{R}^{4n} \times (\mathbb{CP}^1 - \{pt\}) \cong \mathbb{R}^{4n} \times \mathbb{C}$ is not "the same" as Verbitsky's degenerate twistor space manifold $\times\ \mathbb{C}$
- Penrose: twistor quantization
  Minkowski space
  CCR
  correspondence principle $\{,\}$ $[,]$