A K-homology cycle via perturbation by Dirac operator along orbits

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§ Background & Motivation

Loizides – Song's construction

$LG \curvearrowright \mathcal{M}$: proper Hamiltonian LG-space.

$\cup$

$^\exists W$, $^\exists M$

Clifford module bundle & finite dim'nl non-cpt mfd. transverse to LG-orbits (Loizides-Meinrenken-Song)

want to get its "Quantization" from the view point of index theory or K-theory

- M carries natural $T \times \Lambda$-action.

  ($T \subset G$: max. torus, $\Lambda := \ker(\mathfrak{t} \to T)$)

M carries natural $T \times \Lambda$-action s.t.

$\Lambda \curvearrowright M$: free

& W : (Λ, T)-admissible

guarantees to have well-defined

equiv. index $\in R^{-\infty}(T) = \mathrm{Hom}(R(T), \mathbb{Z})$

(or cycle in $K^0(C^*(T, C(M/\Lambda))))$

main ingredient of their construction.

Def. $\begin{cases} H : \text{cpt Lie gp}, & \Gamma : \text{discrete gp.} \\ \hat{\Gamma} : U(1)\text{-central extension of } \Gamma. \end{cases}$

$\begin{cases} X : \text{complete Riem. mfd. with } H\times\Gamma\text{-action} \\ E \to X : \text{Clifford module bdle with} \\ \qquad H \ltimes \hat{\Gamma}\text{-action.} \end{cases}$

E is (Γ, H)-admissible if

$$\|\varphi \cdot \hat{\gamma} \cdot s\|_{L^2} \to 0 \text{ as } \underset{\Gamma}{\gamma} \to \infty$$

for $\forall \varphi \in C^\infty(H)$, $\forall s \in L^2(E)$.

Thm (Loizides – Song '18)

If E is a (Γ, H)-admissible bdle, then Dirac op. $D : L^2(E) \circlearrowleft$ gives a K-homology cycle $[D] \in K^0(C^*(H, C_0(X/\Gamma)))$ and an equiv. index

$$K^0(C^*(H)) = R^{-\infty}(H) \ni \mathrm{index}(D) : R(H) \longrightarrow \mathbb{Z}.$$

- $C^*(H, C_0(X/\Gamma))$: a completion of $C(H, C_0(X/\Gamma))$
- K-homology cycle = pair of Hilbert sp. and bdd. op. with some compactness.

Example

$M = T^*S^1 = S^1 \times \mathbb{R}$ (with std. str.)

D = Dolbeault Dirac op.

$\rightsquigarrow \operatorname{index}(D) = \ker D^+ - \ker D^-$

$$= \bigoplus_{n\in\mathbb{Z}} \mathbb{C}_{(n)} \quad ; \quad \mathbb{C}_{(m)} \longmapsto 1$$

$\operatorname{index}(D) \in \operatorname{Hom}(R(S^1), \mathbb{Z})$

$=$

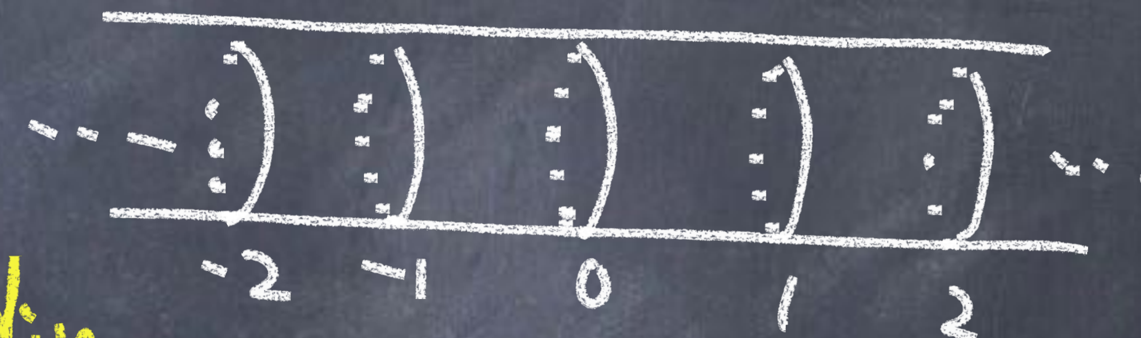
$R^{-\infty}(S^1) \ni$ infinite dim.

$=$

$K^0(C^*(S^1))$ ker. with finite multiplicity.

$=$

$KK(C^*(S^1), \mathbb{C})$



Rem.

- The "Heisenberg commutation relations"

$$\hat{f} \cdot h \cdot \hat{f}^{-1} \cdot h^{-1} = h^{t}$$

implies (Λ, T)-admissibility.

- (Λ, T)-admissibility implies the properness of the moment map.

Aim of this talk

We would like to construct the same equivariant index / K-homology cycle based on some other machinery instead of (Λ, T)-admissibility.

$$\left(\|\varphi \cdot \hat{f} \cdot S\|_{L^2} \to 0\right)$$

We use :

$\boxed{\text{perturbation by Dirac operator along the orbits.}}$

↑

could expect to find localization to lattice points (real polarization / BS condition).

c.f. (F-Furuta-Yoshida, 2010~2016)

- RR = BS
- $[Q, R] = 0$

using the above.

§ Our construction

Main Set-up

$$\begin{tikzcd} & (L, \nabla) \arrow[d] & \text{: pre-quantizing b'dle} \\ S^1 \arrow[ur, bend left] \arrow[r, bend left] & (M, \omega) \arrow[d, "\mu"] & \text{: symplectic mfd. (non-cpt)} \\ & (\mathrm{Lie}S^1)^* = \mathrm{Lie}S^1 \cong \mathbb{R} & \end{tikzcd}$$

(or T)

μ : moment map

- ξ : generator of $\mathrm{Lie}S^1$ with $|\xi| = 1$
- $\underline{\xi}$: induced vector field on M.
- $\mathcal{L}_{\xi}$: infinitetimal action by $\underline{\xi}$.

Assumptions

1. Fixed pt set M^{S^1} is compact.
2. $\forall n \in \mathbb{Z}$, $\mu^{-1}(n)$ is compact.
3. $\exists J$ s.t. $g(\cdot,\cdot) = \omega(J\cdot,\cdot)$ is complete.
4. $\|\underline{\xi}\|_\infty < \infty$
(5. Some uniformality on the end of M)

$W_L := \Lambda^\bullet T_{\mathbb{C}}M \otimes L$: $\mathbb{Z}/2$-gr. Clifford module bdle.

with Clifford action $c = \wedge + \wedge^*$ on W_L.

$$\boxed{D_{S^1} := c(\xi)\left(\mathcal{L}_\xi - \sqrt{-1}\mu\right) \;;\; \Gamma(W_L) \circlearrowleft}$$

key D_{S^1} is a Dirac op. along orbits and

$\mu(x) \neq n \Longrightarrow \ker\left(D_{S^1}|_{S^1 \cdot x}\right)^{(n)} = 0 \quad (\forall n)$

$\hookrightarrow$ D_{S^1} is non-deg outside $\mu^{-1}(n) \cup M^{S^1}$.

Notation $V^{(n)} := \mathrm{Hom}(\mathbb{C}_{(n)}, V) \otimes \mathbb{C}_{(n)}$

the isotypic component of weight n for representation V of S^1.

Fix $n \in \mathbb{Z}$.

Take $D : \Gamma(W_L) \circlearrowleft$: Dirac op. &

$$\rho_n : M \longrightarrow [0, \infty) \text{ s.t } \begin{cases} \bullet\ \rho_n|_{\mu^{-1}(n) \cup M_{S^1}} \equiv 0 \\ \bullet\ \rho_n \nearrow \infty \\ \bullet\ \|d\rho_n\|_\infty < \infty \end{cases}$$

<u>Def.</u> $\hat{D}_n := D + \rho_n D_{S^1} : L^2(W_L)^{(n)} \circlearrowleft$

Prop $\hat{D}_n : L^2(W_c)^{(n)} \circlearrowleft$ is a Fredholm op.

$\hookrightarrow$ $\mathrm{index} : R(S^1) \longrightarrow \mathbb{Z}$

$\cup$

$R^{-\infty}(S^1)$ is defined.

$\mathbb{C}_{(n)} \longmapsto \dim\ker(\hat{D}_n^+) - \dim\ker(\hat{D}_n^-)$

key $D_{S^1}^2 = (c(\xi)(\mathcal{L}_\xi - \sqrt{-1}\mu))^2$

$= |\xi|^2 (n-\mu)^2$ on $L^2(W_c)^{(n)}$.

$\Rightarrow \hat{D}_n^2$ has big spectral gap.

Moreover;

Prop. Put $F_n := \dfrac{\hat{D}_n}{\sqrt{1+\hat{D}_n^2}} : L^2(W_L)^{(n)} \circlearrowleft$ and $F := \bigoplus_{n\in\mathbb{Z}} F_n$. Then $(L^2(W_L), F)$ gives a K-homology cycle of $K^0(C^*(S^1))$

$$\left(\begin{array}{c} \parallel \\ R^{-\infty}(S^1) \end{array}\right)$$

cf. A similar equiv. index ($\in R^{-\infty}(S^1)$) was constructed by myself (2016). But a K-homology cycle was not yet.

§ Further discussions

$$D_{S^1} = c(\underline{\xi})\left(\mathcal{L}_\xi - \sqrt{-1}\,\underline{\mu}\right) \qquad \left(\underline{\mu} = \mu_{\underline{\xi}}\right)$$

$$= c(\underline{\xi})\mathcal{L}_\xi - \sqrt{-1}\, c(\underline{\mu})$$

= Kasparov's orbital Dirac op

+ Braverman's perturbation

Prop If μ is proper, then

$$\begin{array}{lll} & \text{index of } \boxed{D + f_n D_{S^1}} & \text{(our)} \\ R^{-\infty}(S^1) \ni & \qquad \| & \\ & \text{index of } \boxed{D - f_n \sqrt{-1}\, c(\underline{\mu})} & \text{(Braverman's)} \end{array}$$

- Our index $\overset{?}{=}$ Loizides–Song's index

- Can we remove compactness of M^{S^1} ?

(Loizides–Song do not assume.)

Hamiltonian LS^1-case

$$\left(\begin{array}{ccccc} \mathcal{M} \hookrightarrow M & \xrightarrow{/\Lambda} & \overline{M} = M/\Lambda \\ \cup & & \cup \\ M^{S^1} & & \overline{M}^{S^1} \end{array}\right)$$

Summary

Non-cpt symp. mfd.

(Λ, T)-adm. by Loizides-Song

equiv. ind. $\in R^{-\infty}(S^1)$

$\parallel$ $\parallel$

K-homology class $\in K^0(C^*(S^1))$

Today

$\boxed{\text{perturbation by } D_{S^1}}$

Thank you for your attension !!