Metaplectic-c Quantization

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Outline

This is an ongoing project with Yael Karshon.
Our approach of metaplectic-c quantization is based on Herald Hess [Hess, 1981]. Without using the metaplectic representation, this approach is different from the approach of Robinson and Rawsley [Robinson and Rawnsley, 1989].

The analog of half form bundles in mp-c case

Partial connections

Pairing maps and Blattner's formula

Review of KS theory of geometric quantization

The Konstant-Souriau recipe of geometric quantization:

- A prequantizable metaplectic manifold (M, ω).

Then one proceeds to define partial connections, inner products and then polarized sections and quantum Hilbert spaces etc.
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- The quantization line bundle is defined as $L \otimes \delta_F^{-1}$ whose sections are L-valued half forms normal to F.

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Quantization line bundle in mpc case

- A metaplectic-c manifold (M, ω) equipped with a principal mp-c bundle $(\tilde{P}, \tilde{\gamma})$ and a polarization F with typical fiber $\mathbb{F}$.

This version of quantization line bundles coincides with the one in KS theory if we start with a metaplectic manifold.

Quantization line bundle in mpc case

- A metaplectic-c manifold (M, ω) equipped with a principal mp-c bundle $(\tilde{P}, \tilde{\gamma})$ and a polarization F with typical fiber $\mathbb{F}$.
- Reduce the symplectic frame bundle P to P_F which is a $\mathrm{Sp}_{\mathbb{F}}$-principal bundle. Pull it back to $\tilde{P}$ and denote the pullback by $\tilde{P}_F$ which is a $\mathrm{Mp}^c_{\mathbb{F}}$-bundle.

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- There is a unique homomorphism $\chi_{\mathbb{F}} : \mathrm{Mp}_{\mathbb{F}} \to \mathbb{C}^\times$ such that $(\chi_{\mathbb{F}} \circ \mathrm{proj})^2 = \det \circ \mathrm{res}$, where $\mathrm{res} : \mathrm{Sp}_{\mathbb{F}} \to \mathrm{GL}(\mathbb{F}, \mathbb{C})$ is the restriction map to $\mathbb{F}$. Define $\chi^c_{\mathbb{F}} : \mathrm{Mp}^c_{\mathbb{F}} \to \mathbb{C}^\times$ by $\chi^c_{\mathbb{F}}([g, z]) = \chi_{\mathbb{F}}(g) z$.

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- Define the quantization line bundle as the associated line bundle to $\tilde{P}_F$ and $\chi^c_{\mathbb{F}}$:

$$Q_F := \tilde{P}_F \times_{\chi^c_{\mathbb{F}}} \mathbb{C}.$$

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Partial connections

We want to define a F-connection on Q_F. Let me explain the construction of partial connections in a simplified case: we assume F has a complement polarization G, i.e. $F \oplus G = TM^{\mathbb{C}}$.
The goal is to construct a $\mathfrak{mp}^c_{\mathbb{F}}$-valued connection one form θ on $\tilde{P}_F$ such that the induced covariant derivative on Q_F along F does not depend on the choice of G.

Sketch of the construction

- The pullback $\tilde{\gamma}_F$ of $\tilde{\gamma}$ to $\tilde{P}_F$ via $\tilde{P}_F \hookrightarrow \tilde{P}$ serves as the $\mathfrak{u}(1)$-component of θ.

Lemma
$\nabla_X^{F,G}$, $X \in F$ is independent of the choices of G. Hence we get a well-defined partial connection on Q_F.

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- Use the complement G and Bott connetions to define a symplectic connection on $TM^{\mathbb{C}}$ such that $\nabla_{TM^{\mathbb{C}}}(\Gamma(F)) \subset \Gamma(F)$, $\nabla_{TM^{\mathbb{C}}}(\Gamma(G)) \subset \Gamma(G)$. Equivalently, we obtain a principal connection on P which can be reduced to P_F. Let's denote its pullback to $\tilde{P}_F$ by $A_{F,G}$ which serves as the $\mathfrak{sp}_{\mathbb{F}}$-component of θ.

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- $\theta_{F,G} = \tilde{\gamma}_F + A_{F,G}$ is an ordinary connection one form on $\tilde{P}_F$. As a result, we obtain a covariant derivative $\nabla^{F,G}$ on Q_F.

Lemma

$\nabla^{F,G}_X$, $X \in F$ is independent of the choices of G. Hence we get a well-defined partial connection on Q_F.

Pairing maps

The polarization F on M^{2n} we take into account satisfies the following conditions:

1. Positivity: $i\omega(u, \bar{u}) \geq 0$ for all $u \in F$.
2. $F \cap \bar{F}$ has constant rank.

A pairing of polarizations (F_1, F_2) we take into account further satisfies $F_1 \cap \bar{F}_2 = D^{\mathbb{C}}$ has a constant rank.

Theorem (Pairing maps)

There is a pairing map

$$Q_{F_1} \times_M Q_{F_2} \to \mathscr{D}^1(TM/D)$$

Note that if $F_1 = F_2 = F$, we obtain a pairing of Q_F itself.

Sketch of the proof

We consider the further reduced bundle

$$P_{1,2} = P_{F_1} \cap P_{F_2}$$

consisting of symplectic frames
$(e_1, \cdots, e_d, u_1, \cdots, u_r, f_1, \cdots, f_d, i\overline{v_1}, \cdots, i\overline{v_r})$ such that
$(e_1, \cdots, e_d) \in \mathscr{F}(D)$, $(e_1, \cdots, e_d, u_1, \cdots, u_r) \in \mathscr{F}(F_1)$ and
$(e_1, \cdots, e_d, v_1, \cdots, v_r) \in \mathscr{F}(F_2)$.
Then

$$Q_{F_1} = \tilde{P}_{1,2} \times_{\chi^c_{\mathbb{F}_1}} \mathbb{C},$$

$$Q_{F_2} = \tilde{P}_{1,2} \times_{\chi^c_{\mathbb{F}_2}} \mathbb{C}.$$

For $(\alpha, \beta) \in Q_1 \times_M Q_2$ and $e \in \mathscr{F}(D)$. Lift e to $\tilde{e} \in P_{1,2}$. Assume $\alpha(\tilde{e}) = \lambda$ and $\beta(\tilde{e}) = \mu$. Then define

$$\langle \alpha, \beta \rangle(e) := \lambda\bar{\mu}.$$

Blattner's formula

[Blattner, 1977] For $X \in D$, $\alpha \in \Gamma(F_1)$, and $\beta \in \Gamma(F_2)$,

$$\mathcal{L}_X\langle\alpha,\beta\rangle = \langle\nabla_X\alpha,\beta\rangle + \langle\alpha,\nabla_X\beta\rangle + \kappa_{F_1+\bar{F}_2}(X)\langle\alpha,\beta\rangle,$$

where κ is an invariant defined on a differential system associated to $F_1+\bar{F}_2$.
As a corollary, we have

Corollary

If $F_1+\bar{F}_2$ is integrable, then for polarized sections $\alpha \in \Gamma(F_1)$ and $\beta \in \Gamma(F_2)$, the function $\langle\alpha,\beta\rangle$ is constant along leaves of D. As a result, $\langle\alpha,\beta\rangle$ descends to a 1-density on M/D.

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