Reduction and coherent states

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Summary:

1. Symplectic reduction can be used to construct interesting symplectic manifolds
2. There is a quantum analogue of symplectic reduction, perhaps not as well-known or utilized
3. We use quantum reduction to construct interesting wave functions, *squeezed coherent states* on $\mathbb{CP}^{N-1}$
4. They have a *symbol*, a Schwartz function describing them micro-locally
5. We prove that they propagate nicely, as do their symbols.

I. Symplectic reduction

(M, ω) symplectic , $\mathcal{L} \to M$ pre-quantum line bundle

$\mu : M \to \mathbb{R}$ moment map of S^1 action on $\mathcal{L} \to M$.

$$X = \mu^{-1}(0)/S^1 \qquad \text{or} \quad X = M//S^1.$$

X inherits pre-quantization $\mathcal{L}_X \to X$,

$$\begin{array}{ccccc} & & & & \mathcal{L} \\ & & & & \downarrow \\ & & \mu^{-1}(0) & \hookrightarrow & M \\ & & \downarrow & & \\ \mathcal{L}_X & \to & X & & \end{array}$$

Assume now M is Kähler, $\mathcal{L} \to M$ holomorphic and S^1 acts by isometries $\mathcal{L}_X \to X$ is also holomorphic / Kähler.

Bargman spaces

Let:

$$\mathcal{B}_M = H^0(M, \mathcal{L}) \cap L^2(M, \mathcal{L})$$

with the natural Hilbert inner product

Then S^1 acts linearly on $\mathcal{B}_M$, by translations.
Define

$$(\mathcal{B}_M)^{S^1} := \text{the space of invariant vectors}$$

$[Q, R] = 0$ (Atiyah, Guillemin-Sternberg)

- Similarly, one defines $\mathcal{B}_X$.
- Quantization commutes with reduction:

$$\mathcal{B}_X \cong (\mathcal{B}_M)^{S^1}, \qquad X = M//S^1.$$

The isomorphism is just restriction–push forward.

It will be important to do this for all tensor powers

$$\mathcal{L}^k \to M, \qquad k = 1, 2, \cdots, \hbar = \frac{1}{k}$$

$$\mathcal{B}_M^{(k)} = H^0(M, \mathcal{L}^k) \cap L^2(M, \mathcal{L}^k).$$

Then

$$\mathcal{B}_X^{(k)} \cong \left(\mathcal{B}_M^{(k)}\right)^{S^1}.$$

II. Quantum reduction

By this I mean the operator(s)

$$\mathcal{R}_k : \mathcal{B}_M^{(k)} \to \mathcal{B}_X^{(k)},$$

which are just the composition

$$\mathcal{R}_k :\ \mathcal{B}_M^{(k)} \xrightarrow{\Pi_k} \left(\mathcal{B}_M^{(k)}\right)^{S^1} \cong \mathcal{B}_X^{(k)}$$

Π_k being orthogonal projection (averaging).

Theorem: This operator quantizes the canonical relation

$$\left\{(x, m) \in X \times M\ ;\ m \in \mu^{-1}(0) \text{ and } \pi(m) = x\right\} \subset X \times M.$$

Example:

$$M = \mathbb{C}^N, \quad \omega = \frac{1}{i}\, dz \wedge d\,\overline{z}, \quad \mu = |z|^2 - 1$$

Action of S^1: $\quad e^{it} \cdot z = e^{-it} z.$

$$\mathcal{B}^{(k)} = \left\{ \psi = f(z) e^{-k|z|^2/2} \; ; \; \bar{\partial} f = 0 \right\}.$$

Representation of S^1 on $\mathcal{B}^{(k)}$:

$$\rho(e^{it})(\psi)(z) = e^{-ikt} \psi(e^{it} z).$$

$$\left(\mathcal{B}^{(k)}\right)^{S^1} = \left\{ \psi = f(z) e^{-k|z|^2/2} \; ; \; f \text{ homog. polyn. degree } k \right\}.$$

Example:

$$X = \mathbb{CP}^{N-1}, \quad \mathcal{B}_X^{(k)} = \{f|_{S^{2N-1}} \; ; \; f \text{ homog. polyn. degree } k\}.$$

$$\begin{array}{ccccc} & & & & \mathcal{L} = \mathbb{C}^N \times \mathbb{C} \\ & & & & \downarrow \\ & & S^{2N-1} & \hookrightarrow & \mathbb{C}^N \\ & & \downarrow & & \\ \mathcal{L}_X & \to & \mathbb{CP}^{N-1} & & \end{array}$$

Reduction operator:

$$\forall z \in S^{2N-1} \qquad \mathcal{R}_k(\psi)(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt}\, \psi(e^{ikt} z)\, dt$$

Lemma: The reduction of $\psi(z) = f(z)e^{-k|z|^2/2} \in \mathcal{B}^{(k)}$, is

$$\mathcal{R}_k(\psi) = e^{-k}\, f_k(z),$$

where

$$f_k = \text{sum of the terms of degree } k$$
$$\text{in the power series expansion of } f.$$

III. Gaussian coherent states on $\mathbb{C}^n$

The standard coherent state centered at $w \in \mathbb{C}^N$, $e_w \in \mathcal{B}^{(k)}$, is

$$e_w(z) = \left(\frac{k}{\pi}\right)^N e^{kz\overline{w}}\, e^{-k|w|^2/2}\, e^{-k|z|^2/2}$$
$$= \left(\frac{k}{\pi}\right)^N e^{-k\,|z-w|^2/2}\, e^{ik\,\omega(z,w)}.$$

$$e(z,w) := e_w(z)$$

is the kernel of the orthogonal projection $L^2(\mathbb{C}^N) \to \mathcal{B}^{(k)}$.
Husimi function: $|e_w|^2$:

$$|e_w|^2 = \left(\frac{k}{\pi}\right)^{2N} e^{-k\,|z-w|^2/2}$$

Squeezed Gaussian coherent states:

$$\psi_{A,w}(z) := e^{kQ_A(z-w)/2}\, e_w(z)$$

$$Q_A(z) = zAz^T$$

$A \in \mathcal{D}$ where

$$\mathcal{D} := \{A\ ;\ \ A \text{ is an } N \times N \text{ symmetric matrix such that } A^*A < I\}$$

$$A^*A < I \quad \Rightarrow \quad \psi_{A,w}(z) \in L^2.$$

These are necessary to describe the quantum evolution of standard states. (Among other things.)

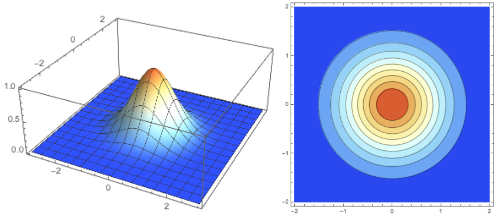

Figure: Husimi function of a standard coherent state

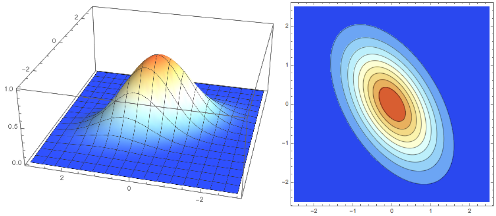

Figure: Husimi function of a squeezed state in $N = 1$, $A = -\frac{1}{4} + \frac{i}{2}$

IV. Reduction of coherent states to $\mathbb{CP}^{n-1}$

1. Reducing the standard coherent states

$$e_w(z) = \left(\frac{k}{\pi}\right)^N e^{kz\overline{w}}\, e^{-|w|^2/2}\, e^{-|z|^2/2}$$

   gives

$$\mathcal{R}_k(e_w)(z) = \text{Const.}\ (z\overline{w})^k,$$

   the standard coherent states of $\mathbb{CP}^{N-1}$.

2. Reducing squeezed Gaussian states gives what?

Exact formulae, $N = 2$

Lemma The reduction of the squeezed Gaussian C.S. is

$$\Psi_{A,w}(z) = \left(\frac{k}{\pi}\right)^N e^{-k}\, e^{kQ_A(w)} \times \sum_{\ell \geq k/2}^{k} \frac{k^\ell}{(k-\ell)!(2\ell-k)!} \left(\frac{1}{2} Q_A(z)\right)^{k-\ell} \left(z(\overline{w} - Aw^T)\right)^{2\ell-k}.$$

with different notation...

Orthonormal basis of $\mathcal{B}^{(k)}_{\mathbb{CP}^1}$: $0 \le n \le k$

$$|n\rangle = \frac{k^{k/2+1}}{\pi} \frac{1}{\sqrt{n!(k-n)!}} z_1^n z_2^{k-n}$$

then if $w = (1, 0)$ the reduction is

$$|o, \mu, k\rangle (1 + O(1/\sqrt{k})), \quad \text{where}$$

$$\mu = b - \frac{c^2}{1+a}, \quad A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

and

$$|o, \mu, k\rangle := \frac{k^{k/2+1}}{\pi} \sum_{0 \le \ell \le k/2} \left(\frac{1}{2k}\right)^{\ell} \frac{1}{\sqrt{(k-2\ell)!}} \sqrt{\binom{2\ell}{\ell}} \mu^{\ell} |k - 2\ell\rangle$$

V. Local picture

The previous formulae are opaque. What is going on?

Introduce the notion of *symbol* of a coherent state.

Easiest to define in adapted coordinates and trivialization, whatever that means.

In those coords:
Given a center w, define the symbol of a coherent state φ_w by:

$$\sigma(\eta) = \lim_{k\to\infty} \varphi_w\left(w + \frac{\eta}{\sqrt{k}}\right),$$

this is a Schwartz function in the Bargmann space of T_wM.
It is a well-defined object.

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After the construction, results:

Theorem: The symbol of the reduction is the reduction of the symbol

What "reduction of the symbol" means is an interesting question re: quantization of symplectic vector spaces.

Theorem Under a quantum Hamiltonian, the reduced C.S. evolve (to leading order) to C.S. in the same class, and their symbols evolve according to the metaplectic representation.

There is a well-defined class of squeezed coherent states on any Kähler-quantized manifold, a special case of "isotropic states".

Thank you for your attention