A Brief Tour of Metaplectic-c Prequantization

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Preliminary Definitions

Let (M^{2n}, ω) be a symplectic manifold.

Let (V^{2n}, Ω) be a symplectic vector space, with symplectic group $\mathrm{Sp}(V)$. The metaplectic group $\mathrm{Mp}(V)$ is the connected double cover of $\mathrm{Sp}(V)$.

We view the symplectic frame bundle $\mathrm{Sp}(M, \omega)$ as a principal $\mathrm{Sp}(V)$ bundle over M.

Reminder of Kostant-Souriau Quantization

The Kostant-Souriau quantization procedure with half-form correction requires that (M, ω) admit two objects:

- A **prequantization circle bundle**

$$(Y, \gamma) \to (M, \omega)$$

- A **metaplectic structure**, which is a principal $\mathrm{Mp}(V)$ bundle over M that is compatible with the symplectic frame bundle.

The metaplectic structure and a choice of polarization F give rise to the **half-form bundle** $\bigwedge^{1/2} F$, which is a complex line bundle over M.

Key idea: metaplectic-c quantization replaces the prequantization circle bundle and metaplectic structure with a single object.

Origins: Hess (1981), Robinson and Rawnsley (1989)

Metaplectic-c Prequantization

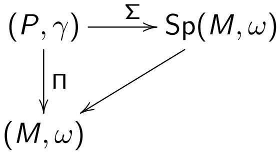
The metaplectic-c group is

$$\mathrm{Mp}^c(V) = \mathrm{Mp}(V) \times_{\mathbb{Z}_2} U(1).$$

It is a circle extension of $\mathrm{Sp}(V)$:

$$1 \longrightarrow U(1) \longrightarrow \mathrm{Mp}^c(V) \longrightarrow \mathrm{Sp}(V) \longrightarrow 1$$

A **metaplectic-c prequantization** for (M, ω) is a triple (P, Σ, γ), where:



- P is a principal $\mathrm{Mp}^c(V)$ bundle over M;
- Σ is an equivariant map from P to $\mathrm{Sp}(M, \omega)$;
- γ is a $\mathfrak{u}(1)$-valued one-form on P, analogous to a connection one-form on a circle bundle.

Now that we have metaplectic-c prequantizations...

what can we do with them?

(M, ω) admits a prequantization circle bundle and a metaplectic structure if the two cohomology classes $\left[\frac{1}{2\pi\hbar}\omega\right]$ and $\frac{1}{2}c_1(TM)$ are both integral.

(M, ω) admits a metaplectic-c prequantization if their sum is integral. So metaplectic-c prequantization applies to a larger class of symplectic manifolds.

Infinitesimal Metaplectic-c Quantomorphisms

Given a prequantization circle bundle $(Y, \gamma) \to (M, \omega)$, let $\mathcal{Q}(Y, \gamma)$ be the Lie algebra of **infinitesimal quantomorphisms**: that is, the vector fields on Y that preserve the connection γ.

Then $C^\infty(M)$ and $\mathcal{Q}(Y, \gamma)$ are isomorphic Lie algebras.

Metaplectic-c analog:

Definition. Given a metaplectic-c prequantization $(P, \gamma) \xrightarrow{\Sigma} \mathrm{Sp}(M, \omega) \to (M, \omega)$, an **infinitesmial metaplectic-c quantomorphism** is a vector field ζ on P that preserves γ and that satisfies $\Sigma_* \zeta = \widetilde{\Pi_* \zeta}$.

Theorem. Let $\mathcal{Q}(P, \Sigma, \gamma)$ be the Lie algebra of infinitesimal metaplectic-c quantomorphisms. Then $\mathcal{Q}(P, \Sigma, \gamma)$ and $C^\infty(M)$ are isomorphic Lie algebras.

Quantized Energy Levels (1)

Consider $H \in C^\infty(M)$, which we interpret as an energy function. What are its quantized energy levels?

Let E be a regular value of H, and let $S = H^{-1}(E)$.

$$\begin{array}{ccccc}
(P,\gamma) & \xleftarrow{\ \supset\ } & (P^S,\gamma^S) & \longrightarrow & (P_S,\gamma_S) \\
\downarrow{\scriptstyle \Sigma} & & \downarrow & & \downarrow \\
\mathrm{Sp}(M,\omega) & \xleftarrow{\ \supset\ } & \mathrm{Sp}(M,\omega;S) & \longrightarrow & \mathrm{Sp}(TS/TS^\perp) \\
\downarrow & & \downarrow & & \downarrow \\
(M,\omega) & \xleftarrow{\ \supset\ } & S & \xrightarrow{\ =\ } & S
\end{array}$$

Construction due to Robinson (1990).

Let H have Hamiltonian vector field ξ_H on M. There is a natural lift to $\widetilde{\xi}_H$ on $\mathrm{Sp}(M,\omega)$, which then descends to $\mathrm{Sp}(TS/TS^\perp)$.

Quantized Energy Levels (2)

Definition. The regular value E of H is a **quantized energy level** for the system (M, ω, H) if the connection one-form γ_S on P_S has trivial holonomy over all closed orbits of $\widetilde{\xi}_H$ on $\mathrm{Sp}(TS/TS^{\perp})$.

Theorem (Dynamical Invariance). Let $H_1, H_2 \in C^{\infty}(M)$ be such that

$$H_1^{-1}(E_1) = H_2^{-1}(E_2)$$

for regular values E_1, E_2 of H_1 and H_2. Then E_1 is a quantized energy level for (M, ω, H_1) if and only if E_2 is a quantized energy level for (M, ω, H_2).

Quantized Energy Levels (3)

Examples.

- The n-dimensional harmonic oscillator: $M = \mathbb{R}^{2n}$, Cartesian coordinates (q, p), $\omega = \sum_{j=1}^{n} dq_j \wedge dp_j$, $H = \frac{1}{2}(p^2 + q^2)$.
  Quantized energy levels:
  $$E_N = \hbar\left(N + \frac{n}{2}\right), \quad N \in \mathbb{Z}, \quad E_N > 0.$$

- The hydrogen atom: $M = \dot{\mathbb{R}}^3 \times \mathbb{R}^3$, $\omega = \sum_{j=1}^{3} dq_j \wedge dp_j$,
  $H = \frac{1}{2m_e}p^2 - \frac{k}{|q|}$, $m_e, k > 0$. Negative quantized energy levels:
  $$E_N = -\frac{m_e k^2}{2\hbar^2 N^2}, \quad N \in \mathbb{N}.$$

Quantized Energy Levels (4)

Consider k Poisson-commuting functions $H = (H_1, \dots, H_k)$, and a regular level set $S = H^{-1}(E)$ where $E \in \mathbb{R}^k$.

There is an analogous construction of

$$(P_S, \gamma_S) \to \mathrm{Sp}(TS/TS^{\perp}) \to S$$

Definition. The regular value E is a **quantized energy level** for (M, ω, H) if γ_S has trivial holonomy over all curves in $\mathrm{Sp}(TS/TS^{\perp})$ with tangent vectors in the span of $\widetilde{\xi}_{H_1}, \dots, \widetilde{\xi}_{H_k}$.

This definition satisfies a generalized dynamical invariance property.

In the special case $k = n$, it is equivalent to a Bohr-Sommerfeld condition.

Equivariant Metaplectic-c Prequantizations (1)

Let (M, ω) have a Hamiltonian G-action with momentum map $\Phi : M \to \mathfrak{g}^*$. Each $\xi \in \mathfrak{g}$ generates vector fields ξ_M on M and $\widetilde{\xi}_M$ on $\mathrm{Sp}(M, \omega)$.

A metaplectic-c prequantization $(P, \Sigma, \gamma) \to (M, \omega)$ is **equivariant** if there is a G-action on P, lifting that on $\mathrm{Sp}(M, \omega)$, such that for all $\xi \in \mathfrak{g}$,

$$\gamma(\xi_P) = -\frac{1}{i\hbar}\Pi^*\Phi^\xi.$$

For Hamiltonian torus actions:

Fact. Let (M, ω) have an effective Hamiltonian T^k action with momentum map Φ and a fixed point z. Given a metaplectic-c prequantization $(P, \Sigma, \gamma) \to (M, \omega)$, it is always possible to shift the momentum map Φ such that (P, Σ, γ) is equivariant.

Equivariant Metaplectic-c Prequantizations (2)

Fix a Delzant polytope

$$\Delta = \{x \in \mathbb{R}^{n*} : \langle x, v_j \rangle \leq \lambda_j,\ 1 \leq j \leq N\}$$

where v_j are primitive outward-pointing normals to the N facets and λ_j are real numbers.

Define $\pi_* : \mathbb{R}^N \to \mathbb{R}^n$ by $\pi_* e_j = v_j$.

Let $K = \ker \pi$, and let d be the dimension of K. Short exact sequences:

$$1 \to K \xrightarrow{i} T^N \xrightarrow{\pi} T^n \to 1$$

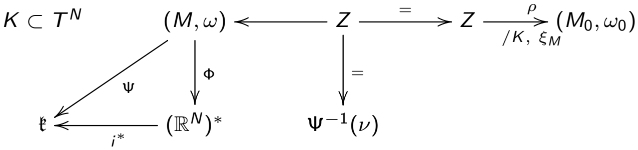
$$0 \to \mathfrak{k} \xrightarrow{i_*} \mathbb{R}^N \xrightarrow{\pi_*} \mathbb{R}^n \to 0$$

$$0 \to \mathbb{R}^{n*} \xrightarrow{\pi^*} \mathbb{R}^{N*} \xrightarrow{i^*} \mathfrak{k}^* \to 0$$

Let $\nu = i^*(-\lambda + \frac{h}{2}\mathbf{1}) \in \mathfrak{k}^*$.

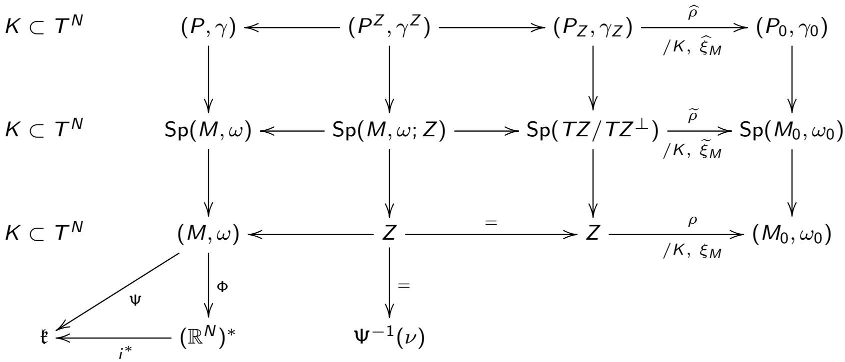
Equivariant Metaplectic-c Prequantizations (3)

Let $M = \mathbb{R}^{2N}$, with the standard action of T^N. The Delzant construction...



Equivariant Metaplectic-c Prequantizations (3)

...extends to a metaplectic-c equivariant Delzant construction...



...when $i^* \left(-\lambda + \frac{h}{2}\mathbf{1}\right) \in h\mathbb{Z}^{d*}$.