MAT425/1340 Differential Topology, Fall 2016. Problem Set 3.

- You are encouraged to work on the problems with other students.
- You are required to write your solution later, entirely on your own, as if it were a take-home exam.

Due by Thursday October 6th at 11 am:

(1) Let M be a manifold. Show that there exists a proper embedding of M into some Euclidean space

("Embedding" is a diffeomorphism with a subset of the target. "Proper" means that preimages of compact sets are compact.)

Hint: show that the function $\sum n\rho_n$, where $\{\rho_n\}$ is a partition of unity on M with $\operatorname{supp}\rho_n$ compact for all n, is proper and non-negative. Use this function to get from an embedding of M into a Euclidean space (which we already know exists) to a proper embedding of M into a Euclidean space.

(2) Recall that an $n \times n$ matrix A is called orthogonal if its action on $\mathbb{R}^n$ preserves the standard inner product, i.e., if $\langle Au, Av\rangle = \langle u, v\rangle$ for all $u, v \in \mathbb{R}^n$. The orthogonal group $\mathrm{O}(n)$ consists of the set of orthogonal matrices (with the operation of matrix multiplication). Express $\mathrm{O}(n)$ as a regular level set of the function $f(A) = A^T A$ from the vector space of all $n \times n$ matrices to the vector space of symmetric $n \times n$ matrices; conclude that $\mathrm{O}(n)$ is an embedded submanifold of the vector space of all $n \times n$ matrices. (Hint: $df|_A(B) = \frac{d}{dt}|_{t=0} f(A+tB)$ takes a matrix B to the symmetric matrix $B^T A + A^T B$.)

(3) Let M and N be manifolds. Describe a manifold structure on $M \times N$.

(4) Show that the set $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 = y^2 + z^2 \text{ and } x \geq 0\}$ is not an embedded submanifold of $\mathbb{R}^3$. (Hint: consider tangents to curves in this set.)