MAT425/1340 Differential Topology, Fall 2016. Problem Set 7.

- You are encouraged to work on the problems with other students.
- You are required to write your solution later, entirely on your own, as if it were a take-home exam.

Due on Wednesday November 16th in class.

(1) (Borsuk Ulam theorem:) *Let* $f\colon S^{n-1} \to \mathbb{R}^{n-1}$ *be a continuous map. Then there exists* $x \in S^{n-1}$ *such that* $f(x) = f(-x)$.

In class we proved this theorem for maps $f$ that are smooth. Prove it for maps $f$ that are continuous. (Hint: find a sequence of smooth maps $S^{n-1} \to \mathbb{R}^{n-1}$ that uniformly converges to $f$.)

(2) A subset of a topological space is of *first Baire Category*, or *meagre*, if it is contained in a countable union of closed sets whose interiors are empty. It is called *residual* if its complement is meagre.

Let $f\colon M \to N$ be a smooth map between manifolds. Show that the set of critical values of $f$ is of first Baire category in $N$.

(3) Let $f_t\colon M \to N$, for $t \in [0,1]$, be a proper smooth homotopy. Let $h_t\colon N \to N$, for $t \in [0,1]$, be a compactly supported smooth family of diffeomorphisms (in other words, an isotopy between the diffeomorphisms $h_0$ and $h_1$). Prove that $h_t \circ f_t\colon M \to N$, for $t \in [0,1]$, is a proper smooth homotopy. (Hint: write a composition $[0,1] \times M \to [0,1] \times N \to N$.)

Solve but don't hand in:

- Fix non-negative integers $n$ and $k$. Let $\phi\colon S^{n-1} \to \mathbb{R}^k$ be a continuous map from the unit sphere in $\mathbb{R}^n$ to $\mathbb{R}^k$. Suppose that $\phi(x) \neq 0$ for all $x \in S^{n-1}$. Show that the following two conditions are equivalent.
  (1) For every continuous map $F\colon B^n \to \mathbb{R}^k$ from the closed unit ball in $\mathbb{R}^n$ to $\mathbb{R}^k$ whose restriction to $S^{n-1}$ coincides with $\phi$, the equation $F(x) = 0$ has a solution.
  (2) The map $\psi := \frac{\phi}{|\phi|}\colon S^{n-1} \to S^{k-1}$ is homotopically non-trivial.
- Let $X$ be a manifold with boundary and $f\colon X \to \mathbb{R}^n$ a continuous map. Show:
  - $f$ is proper if and only if $|f|\colon X \to \mathbb{R}$ is proper.
  - If $g\colon X \to \mathbb{R}^n$ is proper and $|f(x)| \geq |g(x)|$ for all $x \in X$ then $f$ is proper.
  - If $f$ is proper and $A$ is a subset of $\mathbb{R}^n$ then $f|_{f^{-1}(A)}\colon f^{-1}(A) \to A$ is proper.
  - If $C$ is a closed subset of $X$ and $f$ is proper then $f|_C\colon C \to \mathbb{R}^n$ is proper.
  - If $C$ is a closed subset of $X$ and $f|_C\colon C \to \mathbb{R}^n$ is proper, and if the complement of $C$ has compact closure in $X$, then $f$ is proper.