MAT1344 13 Jan 2021

Introduction to Symplectic Geometry

Plan: emphasize Hamiltonian group actions.

* ☆ Please interrupt me in class
  nag me on emails.
* ☆ lecture times & office hrs: will negotiate ("doodle")

$\boxed{\text{Credit}}$

→ problem sets
→ individual projects
  - oral presentation
  - written report

Course website: linked from my website.
(google "karshon")

(also course website from 2018–2019)

Review: manifolds & differential forms.

quick: Guillemin & Pollack ch. 1, 4

thorough: John Lee "Introduction to smooth mflds".

a manifold M is a Hausdorff & second countable topological space equipped with an equivalence class of atlases.

an atlas is a set of homeomorphisms

$$\{\varphi_i : \underset{\substack{\cap\ \text{open} \\ M}}{U_i} \longrightarrow \underset{\substack{\cap\ \text{open} \\ \mathbb{R}^n}}{\mathcal{O}_i}\}$$

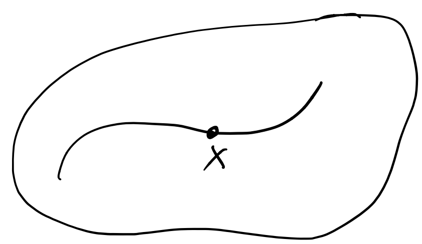
s.t. all the transition maps $\varphi_j \circ \varphi_i^{-1}$ are smooth. and $\bigcup_i U_i = M$

Two atlases are equivalent if their union is an atlas.

$C^\infty(M) = \{\text{smooth } f: M \longrightarrow \mathbb{R}\}$

$\gamma : \mathbb{R} \longrightarrow M$ smooth curve

$x = \gamma(0)$



$\rightsquigarrow D_\gamma : f \mapsto \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t))$

$D_\gamma : C^\infty(M) \longrightarrow \mathbb{R}$

is a "derivation at x" $:\Leftrightarrow$
- ☆ linear
- ☆ Leibniz:

$$D_\gamma(fg) = (D_\gamma f) g(x) + f(x)(D_\gamma g)$$

Lemma: Every derivation at x comes from a curve through x

$$T_x M := \{\text{derivations at } x\}$$

$$f \in C^\infty(M) \rightsquigarrow df\Big|_x \in T_x^* M = (T_x M)^*$$

a covector

$$df\Big|_x : \underset{T_x M}{\underset{\cap}{v}} \mapsto vf$$

Local coordinates $\varphi : U \longrightarrow \mathcal{O} \subset \mathbb{R}^n$

$$\varphi = (u^1, \dots, u^n)$$

$$\rightsquigarrow \text{derivations: } \underbrace{\frac{\partial}{\partial u^1}\Big|_x , \dots, \frac{\partial}{\partial u^n}\Big|_x}_{\text{basis for } T_x M} .$$

Lemma. The dual basis in $T_x^* M$ is $du^1\Big|_x , \dots, du^n\Big|_x$

In coordinates: $df = \sum_j \frac{\partial f}{\partial u^j} du^j$

Differential forms

0-form : $\iff$ a smooth function

1-form : α

$x \in M \longmapsto \alpha|_x \in T_x^* M$

example: $\alpha = df$ for $f \in C^\infty(M)$

in coordinates: $\alpha = \sum_j \underbrace{c_j}\, du^j$

$u^1, \ldots, u^n$

require $c_j = c_j(u^1, \ldots, u^n)$ to be smooth in $u^1, \ldots, u^n$.

2-form:

$\alpha : \quad x \in M \longmapsto \alpha|_x : T_x M \times T_x M \longrightarrow \mathbb{R}$

bilinear
antisymmetric

in local coordinates: $\alpha = \sum_{i,j} \underbrace{c_{i,j}}_{\text{smooth}}\, du^i \wedge du^j$

If $\xi = \sum_\ell \xi^\ell \frac{\partial}{\partial u^\ell}$ and $\eta = \sum_s \eta^s \frac{\partial}{\partial u^s}$

then $(du^i \wedge du^j)(\xi, \eta) := \det \begin{pmatrix} \xi^i & \eta^i \\ \xi^j & \eta^j \end{pmatrix}$

a 2-form α is nondegenerate if, at each $x \in M$, "every nonzero vector has a friend":

$\forall\ 0 \neq \xi \in T_x M \quad \exists \eta$ s.t. $\alpha(\xi, \eta) \neq 0$.

a family of k-forms: α_t for $t \in \mathbb{R}$ or $t \in [0,1]$ or $t \in$ some mfld

$x \in M \longmapsto$ a k-covector on $T_x M$

i.e. alternating multilinear form

$$\underbrace{T_x M \times \dots \times T_x M}_{k} \longrightarrow \mathbb{R}$$

locally $\alpha_t = \sum_{i_1, \dots, i_k} \underbrace{c_{i_1, \dots, i_k}}_{\text{smooth in } t, u^1, \dots, u^n} du^{i_1} \wedge \dots \wedge du^{i_k}$

Exterior derivative:

$$d\alpha = \sum \frac{\partial c_{i_1 \dots i_k}}{\partial u^j} du^j \wedge du^{i_1} \wedge \dots \wedge du^{i_k}$$

α is closed if $d\alpha = 0$

exact if $\exists \beta$ s.t. $d\beta = \alpha$

de Rham cohomology: $H^k_{dR}(M) = \dfrac{\{\text{closed } k\text{-forms}\}}{\{\text{exact } k\text{-forms}\}}$

Poincaré lemma: $\mathcal{O} \subset \mathbb{R}^n$ star shaped

α closed k-form on $\mathcal{O}$ $k \geq 1$

Then $\exists$ $(k-1)$-form β on $\mathcal{O}$ s.t. $d\beta = \alpha$

Def. a symplectic form on a manifold M is a 2-form on M that is closed and non-degenerate.

13/1/2021, 2nd hour:

a <u>symplectic manifold</u> is a pair (M, ω)

a manifold → M

a symplectic form → ω

<u>Example</u>. $M = \mathbb{R}^{2n}$ w/ coordinates $x_1, y_1, \ldots, x_n, y_n$

$\omega_{std} = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$

<u>Darboux's theorem</u>. Let (M, ω) be a sympl. manifold. Then near each point $\exists$ a coordinate chart

$$\varphi : U \longrightarrow \mathcal{O} \subset \mathbb{R}^{2n} \quad \text{s.t.}$$

$$\varphi = (x_1, y_1, \ldots, x_n, y_n)$$

$$\omega = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n .$$

i.e. $\omega = \varphi^* \omega_{std}$

<u>Def</u>. such local coordinates are called

symplectic coordinates

or: Darboux coordinates

or: Canonical coordinates

Linear algebra version.

Let V be a finite dim'l vector space over $\mathbb{R}$.

and ω a 2-covector on V

i.e. antisymmetric bilinear form $V \times V \longrightarrow \mathbb{R}$.

Lemma $\exists$ a basis $e_1, f_1, \dots, e_n, f_n, \partial_1, \dots, \partial_s$ of V s.t.

$$\omega = e_1^* \wedge f_1^* + \dots + e_n^* \wedge f_n^*$$

where $e_1^*, f_1^*, \dots, e_n^*, f_n^*, \partial_1^*, \dots, \partial_s^*$ is the dual basis to V^*.

Here: for any two covectors $\alpha, \beta \in V^*$

$$\alpha \wedge \beta := \alpha \otimes \beta - \beta \otimes \alpha$$

and $\alpha \otimes \beta$ takes (ξ, η) to $\alpha(\xi)\beta(\eta)$.

i.e. $$(\alpha \wedge \beta)(\xi, \eta) = \det \begin{pmatrix} \alpha(\xi) & \alpha(\eta) \\ \beta(\xi) & \beta(\eta) \end{pmatrix}.$$

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i.e. ω is represented by the matrix

$$\begin{array}{c} \left(\begin{array}{cccc|c} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & 0 & & \\ & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & & 0 \\ 0 & & \ddots & & \\ & & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \\ \hline & & 0 & & 0 \end{array} \right) \end{array} \begin{array}{l} \left.\vphantom{\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}}\right\} 2n \\ \}\ s \end{array}$$

Note: ω is nondegenerate iff $s=0$
$\dim V = 2n$

Note. Null ω := { vectors that have no friends }
$= \{ v \mid \omega(u, \xi) = 0 \ \forall \xi \}$
$= \operatorname{span}\{\partial_1, \dots, \partial_s\}$

sketch of
proof of the lemma by induction on $\dim V$.

If $\omega \equiv 0$ take $\partial_1, \dots, \partial_s$ to be any basis for V.

otherwise take e_1, f_1 s.t. $\omega(e_1, f_1) = 1$.

Note: e_1, f_1 are linearly indep.

Consider $\operatorname{span}\{e_1, f_1\}$.

Let $V' = \{ \xi \mid \omega(\xi, e_1) = \omega(\xi, f_1) = 0 \}$.

exercise: $V = \operatorname{span}\{e_1, f_1\} \oplus V'$

induction hypothesis $\Rightarrow$ basis $e_2, f_2, \dots, e_n, f_n, \partial_1, \dots, \partial_s$ of V' s.t. $\omega|_{V' \times V'} = e_2^* \wedge f_2^* + \dots + e_n^* \wedge f_n^*$

exercise: $\omega = e_1^* \wedge f_1^* + e_2^* \wedge f_2^* + \dots + e_n^* \wedge f_n^*$

$\square$

Note. We can view V as a manifold and ω as a differential 2-form with constant coefficients.

Next: Weinstein's proof of Darboux's theorem, using Moser's method.

Theorem (Weinstein) (special case)

ω_0, ω_1 differential 2-forms near the origin 0, on a vector space V,

s.t. $\omega_0\big|_0 = \omega_1\big|_0$ is a nondegenerate 2-covector on $T_0 V = V$

Then $\exists$ diffeomorphism $F :$ nbhd of 0 $\to$ nbhd of 0

s.t. $F(0) = 0$

and s.t. $F^* \omega_1 = \omega_0$.

Corollary - Darboux's theorem.

(M, ω) a sympl. mfld, $x \in M$.

a centred coordinate chart near x

$$\mathcal{U} \longrightarrow \mathcal{O}, \quad x \mapsto 0$$

Identifies $\mathcal{U}$ w/ an open subset $\mathcal{O}$ of $\mathbb{R}^{2n}$

and identifies $T_x M$ with $\mathbb{R}^{2n}$

$\Rightarrow$ identifies $\mathcal{U}$ w/ an open subset of $T_x M$ st. $x \mapsto 0$.

the inverse: get

$$\begin{array}{ccc} \text{nbhd of } 0 \quad \mathcal{O} & \xrightarrow[\text{diffeo}]{\psi} & \mathcal{U} \quad \text{nbhd of } x \\ \cap & 0 \mapsto x & \cap \\ T_x M & & M \end{array}$$

s.t. $\quad d\psi\big|_0 = Id_{T_x M}$

Let $\quad \omega_1 := \psi^* \omega \quad$ sympl. form on $\mathcal{O}$

$\boxed{\omega_0 := \omega\big|_x}$ 2-covector on $T_x M$ viewed as a 2-form on $T_x M$ w/ const. coefficients.

Then $\quad \omega_0\big|_0 = \omega_1\big|_0 \quad$ as 2-covectors on $T_0(T_x M) = T_x M$

and are nondeg.

By Weinstein's thm: $\exists$ diffeo

$$F : \text{nbhd of } 0 \longrightarrow \text{nbhd of } 0$$

s.t. $\quad F(0) = 0 \quad$ and s.t. $F^* \omega_1 = \omega_0$ near 0

$$\begin{array}{ccccc} V & & V = T_xM & & M \\ \cup & & \cup & & \cup \\ \mathcal{O}' & \xrightarrow{F} & \mathcal{O} & \xrightarrow{\Psi} & \mathcal{U} \\ \omega_0 & & \omega_1 & & \omega \end{array}$$

$$\Psi \circ F : \mathcal{O}' \xrightarrow{\text{diffeo}} \mathcal{U}' \subset \mathcal{U} \subset M$$

$$\text{s.t.} \quad (\Psi \circ F)^* \omega = \omega_0$$

ω_0 has const coeff

$\Longrightarrow$ $\exists$ coordinates $x_1, y_1, \cdots, x_n, y_n$

(w.r.t. the basis $e_1, f_1, \cdots, e_n, f_n$ as before)

s.t. $\quad \omega_0 = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$

i.e.

$$\begin{array}{ccc} & & \mathcal{O}' \\ & & \cap \\ \mathbb{R}^{2n} & \xrightarrow[\text{linear iso}]{L} & V = T_xM \\ \omega_{std} & & \omega_0 \end{array}$$

Then nbhd of 0 in $\mathbb{R}^{2n}$ $\xrightarrow{\Psi \circ F \circ L}$ nbhd of x in M

s.t. the pullback of ω is ω_{std}