MAT1344 15/1/2021.

practicalities.

- lecture recording & notes on Quercus.
- Lecture notes on course website
  linked from my website
- Doodle poll on course website:
  time constraints/preferences for lectures?

Vector fields & flows.

Bröcker & Jänich "Intro to differential topology" ch. 8
spivak's "comprehensive intro to differential geometry" vol. 1 ch. 5
John Lee "Intro to smooth mflds" ch. 8, 9.

Let M be a mfld.

a vector field ξ on M is a map

$$\xi : \underset{M}{x} \longmapsto \underset{T_x M}{\xi|_x}$$

s.t. in local coordinates $u^1, \dots, u^n$

$$\xi = \sum \underbrace{c^j}\, \frac{\partial}{\partial u^j}$$

is smooth in $u^1, \dots, u^n$

a <u>flow</u> on M is a smooth one-parameter group of diffeomorphisms

$$F_t : M \longrightarrow M, \qquad t \in \mathbb{R}$$

i.e. $F_0 = \mathrm{Id}$ and $F_{t+s} = F_t \circ F_s$.

i.e. $t \longmapsto F_t$ is a group homomorphism

$\mathbb{R} \longrightarrow \mathrm{Diff}(M) := \{\text{diffeomorphisms } M \to M\}$

s.t. $\mathbb{R} \times M \longrightarrow M$
$(t, x) \mapsto F_t(x)$ is smooth.

$\Longleftrightarrow$: (smooth) $\mathbb{R}$-action on M.

Its $\underline{\text{trajectories}}$ are the curves $t \mapsto F_t(x)$
= flow lines

If $\gamma_1(t)$ and $\gamma_2(t)$ are trajectories
that both pass through some $p \in M$

$$\gamma_1(t_1) = \gamma_2(t_2) = p$$

The $\exists s$ s.t. $\forall t$ $\gamma_2(t) = \gamma_1(t+s)$

$$s = t_1 - t_2$$

so $M = \bigsqcup$ unparametrized trajectories

$\uparrow$ orbits

The $\underline{\text{velocity field}}$ of the flow:
v. field ξ s.t. $\forall$ trajectory γ

$$\underbrace{\frac{d}{dt}\gamma(t) = \xi\Big|_{\gamma(t)}}_{\text{write this as } \frac{d}{dt}F_t = \xi \circ F_t}$$

$\underline{\text{Fundamental thm of ODEs}}$ $\Rightarrow$

$\forall$ vector field ξ generates a $\underline{\text{local}}$ flow:
Given ξ, a v. field on M:

$$\exists D$$

$$\{0\} \times M \subseteq D \underset{\text{open}}{\subseteq} \mathbb{R} \times M$$

of the form $D = \{(t,x) \mid a_x < t < b_x\}$
(a "flow domain")

and $\exists \quad F : D \longrightarrow M$ s.t.

$\forall x \in M$ the curve $\gamma_x(\cdot) := F(\cdot, x) : (a_x, b_x) \longrightarrow M$

satisfies

- $\gamma_x(0) = x \quad \& \quad \frac{d}{dt}\gamma_x(t) = \xi\Big|_{\gamma_x(t)} \quad \forall t$

- $\forall \gamma : (a,b) \longrightarrow M$ s.t. $a < 0 < b$

  if $\gamma(0) = x \quad \& \quad \frac{d}{dt}\gamma(t) = \xi\Big|_{\gamma(t)} \quad \forall t$

  then $(a,b) \subseteq (a_x, b_x)$ and $\gamma = \gamma_x\Big|_{(a,b)}$.

Moreover, $F_t := F(t, \cdot) :$ subset of $M \longrightarrow M$

satisfy: $F_{t+s} = F_t \circ F_s$ wherever these are defined.

Finally, if $b_x < \infty$ then $\forall t_0 \in (a_x, b_x)$

$\gamma_x\big([t_0, b_x)\big)$ is not contained in any compact subset of M

("escape lemma")

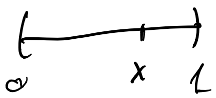
& similarly if $a_x < -\infty$.

$\therefore$ If ξ is compactly supported (eg. if M is compact)

Then $D = \mathbb{R} \times M$.

Example $\quad M = (0,1) \subseteq \mathbb{R}$ w/ coordinate x

$\xi = \dfrac{\partial}{\partial x}$



$(a_x, b_x) = (-x\ ,\ 1-x)$

Time-dependent version.

Let ξ_t, for $t \in [0,1]$ be

a time-dependent v. field on a mfld M.

It generates a local isotopy.

An isotopy = timedependent flow

is a family of diffeomorphisms

$$F_t : M \longrightarrow M \quad , \quad t \in [0,1]$$

s.t. $\quad F_0 = \text{Identity}$

and $\quad (t, x) \mapsto F_t(x)$

$[0,1] \times M \to M \quad$ is smooth.

an isotopy determines a time-dependent vector field

$\xi_t \quad$ s.t. $\quad \dfrac{d}{dt} F_t = \xi_t \circ F_t$

i.e. the velocity vector of $\ t \mapsto F_t(x)\ $ at time t

is $\ \xi_t \big|_{F_t(x)}$.

The time-dependent v.field ξ_t generates a local isotopy:

$$F_t(x) = F(t, x)$$

$$F : \underset{\substack{\cup \\ \text{dom} \times M}}{\text{open subset of } [0,1] \times M} \longrightarrow M$$

s.t. $$\frac{d}{dt} F_t = \xi_t \circ F_t .$$

If ξ_t is compactly supported then $F_t(x)$ is defined $\forall (t,x) \in [0,1] \times M$

If $\xi_t \big|_p = 0 \quad \forall t$ for $p \in M$

then $\exists$ nbhd U of p s.t.

$F_t : U \longrightarrow M$ is well defined on U $\forall\, t \in [0,1]$.

Recall Weinstein's theorem.

ω_0, ω_1 closed 2-forms near p in M

(our case: we had $M = V =$ a v. space, $p = 0$)

$\omega_0\big|_p = \omega_1\big|_p$ as 2-covectors on T_pM and are nondeg at p. on T_pM.

Then $\exists$ diffeo F: nbhd of p $\longrightarrow$ nbhd of p s.t.

$F(p) = p$ and $F^*\omega_1 = \omega_0$ near p.

Idea: Moser's method.

$\omega_t := (1-t)\,\omega_0 + t\,\omega_1$, $t \in [0,1]$.

Smooth family of closed 2-forms near p

$\omega_t\big|_p$ are the same $\forall t$

ω_t are nondeg at p hence near p.

Seek F_t: nbhd of p $\longrightarrow$ nbhd of p, $t \in [0,1]$

s.t. $F_0 = Id$ & $F_t^*\omega_t = \omega_0$ $\forall t$

Need $F_0 = Id$ & $\frac{d}{dt} F_t^* \omega_t = 0 \quad \forall t$

We'll obtain F_t by solving an ODE
= integrating a time-dependent v. field ξ_t.

If $\frac{d}{dt} F_t = \xi_t \circ F_t$, then

$$\frac{d}{dt}\left(F_t^* \omega_t\right) \underset{\text{later}}{=} \boxed{F_t^*\Big(\underbrace{\frac{d\omega_t}{dt}}_{\omega_1 - \omega_0} + \underbrace{\mathcal{L}_{\xi_t}}_{\text{Lie derivative}} \omega_t \Big)}$$

RHS

write $\omega_1 - \omega_0 = d\beta$ near p (by Poincaré lemma)

subtract a 1-form w/ const coef $\Rightarrow$ WLG $\beta|_p = 0$

$$\mathcal{L}_{\xi_t} \omega_t \underset{\text{Cartan}}{=} d\, \iota_{\xi_t} \omega_t + \iota_{\xi_t} \underbrace{d\omega_t}_{=0} = d\, \iota_{\xi_t} \omega$$

So our $\text{RHS} = F_t^* \left(d\beta + d\, \iota_{\xi_t} \omega_t \right)$

$$= F_t^* d \left(\beta + \iota_{\xi_t} \omega_t \right)$$

So: Define ξ_t

by: $\beta + \iota_{\xi_t} \omega_t = 0$ ← can solve b/c ω_t is nondeg.

office hour.

The time-dependent v.field ξ_t generates a local isotopy:

$$F_t(x) = F(t,x)$$

$$F : \underset{\substack{\cup \\ \{0\} \times M}}{\text{open subset of } [0,1] \times M} \longrightarrow M$$

s.t. $\frac{d}{dt} F_t = \xi_t \circ F_t$.

Given ξ_t, $t \in [0,1]$, get:

$$\{0\} \times M \subseteq D \underset{\text{open}}{\subseteq} [0,1] \times M$$

of the form $\left\{ (t,x) \;\middle|\; \begin{array}{l} t \in I_x \\ I_x = [0,1] \text{ or } [0,b_x) \end{array} \right\}$

$F : D \longrightarrow M$ s.t. $\forall x \in M$

- $\gamma_x(t) := F(t,x)$, $\gamma_x : I_x \to M$,

satisfies:

$$\gamma_x(0) = x \qquad \& \qquad \frac{d}{dt}\gamma_x(t) = \xi_t\Big|_{\gamma_x(t)}$$

$$\forall t \in I_x$$

- $\forall\, I = [0,1]$ or $[0,b)$

If $\gamma : I \longrightarrow M$ s.t.

$$\gamma(0) = x \qquad \& \qquad \frac{d}{dt}\gamma(t) = \xi_t\Big|_{\gamma(t)} \quad \forall t \in I$$

Then $I \subset I_x$ and $\gamma = \gamma_x\big|_I$.

Assume $\forall t$ $\xi_t\Big|_p = 0$.

Then $\gamma : [0,1] \longrightarrow M$

$$\gamma(t) = p \quad \forall t$$

satifies $\gamma(0) = p$ & $\frac{d}{dt}\gamma(t) = \xi_t\Big|_p$ $\forall t$

So $D \supseteq [0,1] \times \{p\}$.

Since $\mathcal{D} \subseteq [0,1] \times M$ is open

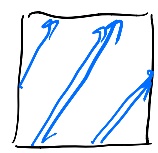
and $[0,1]$ is compact

$\exists$ nbhd U of p in M s.t.

$$\mathcal{D} \supseteq [0,1] \times U.$$

Each orbit is a weakly embedded submfld but not necessarily embedded.

Example: $M = \mathbb{R}^2 / \mathbb{Z}^2$



coordinates x, y mod 1

$$\xi = \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial y} \qquad \alpha \notin \mathbb{Q}.$$

$$F_t([x,y]) = [x+t, \; y+\alpha t]$$

$\gamma(t) = (t, \alpha t)$ is a trajectory

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$$

injective immersion

image is dense in $\mathbb{R}^2/\mathbb{Z}^2$

NOT an embedding.

the orbit $\{(t, \alpha t) \mid t \in \mathbb{R}\}$ with the subset topology is not locally compact.

But γ is a weak embedding:

$\rightarrow$ an injective immersion

$\rightarrow$ th $\forall$ $U \underset{\text{open}}{\subset} \mathbb{R}^n$

$\forall$ $p : U \to \mathbb{R}$

The composition $\gamma \circ p : U \to \mathbb{R}^2/\mathbb{Z}^2$ is smooth

iff $p : U \to \mathbb{R}$ is smooth.

$$U \xrightarrow{p} \mathbb{R} \xrightarrow[\gamma(t) = [t, \alpha t]]{\gamma} \mathbb{R}^2/\mathbb{Z}^2$$