20/1/2021.

practical: for now, lecture times stay the same.

Last week: symplectic forms, Darboux theorem.
will revisit

This week: Lie group actions

$\boxed{\text{Smooth group actions}}$

a Lie group G is a set
equipped with a group structure and a mfld str
s.t. the maps $(a,b) \mapsto a \cdot b$ and $a \mapsto a^{-1}$
$G \times G \longrightarrow G$ $\qquad G \to G$

are smooth.

Example $(\mathbb{R}, +)$. $\qquad (S^1)^k$

a torus is a lie group T s.t. $T \cong (S^1)^k$

diffeomorphism & group isomorphism

$GL(n)$. Note: $A \mapsto A^{-1}$ is smooth by Cramer's rule.

Unitary group: $U(n) := \{ A \in \mathbb{C}^{n \times n} \mid AA^* = I \}$

$\Leftrightarrow H(Au, Av) = H(u,v) \quad \forall\, u,v \in \mathbb{C}^n$
where $H(u,v) = \sum \bar{u}_j v_j$

special unitary group: $SU(n) := \{ A \in SU(n) \mid \det A = 1 \}$
(They are submflds of $\mathbb{C}^{n \times n}$ as a consequence of the implicit function theorem)

$$SU(n) \subset U(n) \qquad \lambda I$$

their centres: $\mathbb{Z}_n \subset S^1 \ni \lambda$ s.t. $|\lambda| = 1$, $\lambda \in \mathbb{C}$

$$PU(n) := U(n)/S^1 \cong SU(n)/\mathbb{Z}_n$$

(orientation preserving)
Rotations: $SO(3) := \{ A \in \mathbb{R}^{3\times 3} \mid AA^t = I \}$

= special orthogonal group.

Euclidean transformations:

$$SE(3) := \left\{ \begin{matrix} \text{orientation preserving} \\ \text{isometries } \mathbb{R}^3 \to \mathbb{R}^3 \end{matrix} \right\}$$

$$= \{ x \mapsto Ax + b \mid A \in SO(3),\ b \in \mathbb{R}^3 \}$$

$$= \mathbb{R}^3 \rtimes SO(3)$$

semidirect product.

Linear symplectic group:

$$Sp(\mathbb{R}^{2n}) := \{ A \in \mathbb{R}^{2n\times 2n} \mid \omega(Au, Av) = \omega(u,v)\ \forall u, v \in \mathbb{R}^{2n} \}$$

where $\omega = e_1^* \wedge f_1^* + \dots + e_n^* \wedge f_n^*$
wrt the basis $e_1, f_1, \dots, e_n, f_n$

also write $\omega = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$
on $\mathbb{R}^{2n}$ as a mfld.

a (smooth) action of a Lie group G on a mfld M is a group homomorphism

$$G \longrightarrow \underbrace{\mathrm{Diff}(M)}_{\{\text{diffeomorphisms } M \to M\}}$$

which we write as $g \longmapsto (g : M \to M)$ $g : m \mapsto g \cdot m$

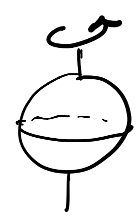
or $g \longmapsto (g_M : M \longrightarrow M)$

s.t. the map $(g, m) \longmapsto g \cdot m$

$G \times M \longrightarrow M$ is smooth

write: $G \circlearrowright M$.

Examples. $S^1 \circlearrowright S^2$



$SO(3) \circlearrowright S^2$

"action" means "left action"

a right action of a Lie gp G on a mfd M is an anti-gp-homomorphism $G \longrightarrow \mathrm{Diff}(M)$ which we write as $g \longmapsto (m \mapsto m \cdot g)$ s.t. the map $M \times G \longrightarrow M$, $(m, g) \mapsto m \cdot g$, is smooth.

Note: $g : m \mapsto m \cdot g^{-1}$ is then a (left) action.

A <u>linear representation</u> is an action of G on a vector space s.t. each $g \in G$ acts as a linear transformation.

Let $G \circlearrowright M$ and $G \circlearrowright N$ be G-actions.
A map $f: M \to N$ is G-<u>equivariant</u> if it intertwines the G-actions:

$$f(a \cdot m) = a \cdot f(m) \qquad \forall a \in G \ \forall m \in M.$$

Let a Lie group G act on a mfld M.

- $\forall$ point $m \in M$, its <u>stabilizer</u> $\equiv$ isotropy subgroup is

$$\mathrm{Stab}(m) := \{ a \in G \mid a \cdot m = m \}$$

It's a closed subgroup of G.

The <u>orbit</u> of $m \in M$ is $G \cdot m := \{ a \cdot m \mid a \in G \}$.
$\subset M$.

<u>Note</u>:
- $M = \bigsqcup$ orbits
- points in the same orbit have conjugate stabilizers.

If $H = \mathrm{Stab}(m)$ then

$$G \circlearrowright G/H \longrightarrow G\cdot m \subset M$$
$$aH \longmapsto a\cdot m$$

is a G-equivariant bijection.

In fact: G/H $\boxed{\text{is}}$ a mfld.

$G\cdot m$ $\boxed{\text{is}}$ a mfld

The map $G/H \longrightarrow G\cdot m$ is a diffeomorphism.

i.e.

There exists a unique mfld str on G/H s.t. $G \longrightarrow G/H$ is a submersion

There exists a unique mfld str. on $G\cdot m$ s.t. $G\cdot m \hookrightarrow M$ is an immersion

Warning on the "figure eight" $\infty \subseteq \mathbb{R}^2$
$\exists$ two different mfld structures s.t. the inclusion map to $\mathbb{R}^2$ is an immersion:

Corollary ∞ is not an orbit of any flow on $\mathbb{R}^2$.

- The action $G \circlearrowright M$ is <u>free</u> if $\forall m \in M$ $\quad \mathrm{Stab}(m) = \{1\}$.

- The action $G \circlearrowright$ is <u>faithful</u> = effective if $G \longrightarrow \mathrm{Diff}(M)$ is one to one

$\iff \quad \bigcap_m \mathrm{Stab}(m) = \{1\}$.

<u>Any</u> G-action on M descends to a faithful action of $G_{\text{eff}} := \boxed{G/K}$

where K is the kernel of the action.

$K \subset G$ is a closed normal subgp

G/K $\boxed{\text{is}}$ a Lie group.

Let G be a Lie group.
The Lie algebra of G is the vector space

$$\mathfrak{g} := T_1 G$$

(wrt the Lie algebra structure that we will recall later.).

The exponential map $\exp : \mathfrak{g} \longrightarrow G$
is characterized by

$\boxed{1}$ $\forall \xi \in \mathfrak{g}$ $\quad t \mapsto \exp(t\xi)$ is a group homo $\mathbb{R} \to G$.

$\boxed{2}$ $\forall \xi \in \mathfrak{g}$ $\quad \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi) = \xi$

If G is a group of matrices then

$$\exp(A) = I + A + \frac{1}{2!}A^2 + \cdots\cdots$$

If $G = T \cong (S^1)^k$ is a torus
and $\mathfrak{t} = \underbrace{\mathrm{Lie}}(T)$ then
its Lie algebra

$\exp : \mathfrak{t} \longrightarrow T$ is a group homomorphism
from $(\mathfrak{t}, +)$ to T

$Z_T := \ker(\exp) \subset \mathfrak{t}$ is a lattice

e.g. $T \cong (S^1)^k$
induces $\mathfrak{t} \cong \mathbb{R}^k$
s.t. $Z_T \longleftrightarrow 2\pi \mathbb{Z}^k$

(Here we identified $Lie(S^1) \cong \mathbb{R}$ by $i\alpha \longleftrightarrow \alpha$)

So $T \cong \mathfrak{t}/\mathbb{Z}_T$.

a Lie gp action $G \circlearrowright M$ determines vector fields ξ_M, for $\xi \in \mathfrak{g} = Lie(G)$

by
$$\xi_M\Big|_m := \frac{d}{dt}\Big|_{t=0} \exp(t\cdot\xi)\cdot m$$

i.e. $\forall\, \xi \in Lie(G)$ we get a <u>flow</u>

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\xi \mapsto \exp(t\xi)"'] \arrow[rr, bend left] & G \arrow[r] & \mathrm{Diff}(M) \end{tikzcd}$$

& ξ_M is the velocity vector field of this flow.

2nd hour

Recall. a s. form ω on a mfld M is a differential 2-form ω that is closed & nondeg.

It induces a bijection

$$\begin{Bmatrix}\text{smooth}\\ \text{vector fields}\end{Bmatrix} \longleftrightarrow \begin{Bmatrix}\text{smooth}\\ \text{1-forms}\end{Bmatrix}$$

$$\xi \longmapsto \xi \lrcorner\, \omega$$
$$= \omega(\xi, \cdot)$$
$$= \iota(\xi)\,\omega$$
$$= \iota_\xi\, \omega$$

Moreover it induces a bijection between smooth families of vector fields and smooth families of 1-forms:

i.e. let $U \underset{\text{open}}{\subset} \mathbb{R}^m$ for some m.

$\forall t \in U$ Let ξ_t be a v. field on M & Let α_t be a 1-form on M.

Suppose $\alpha_t = \xi_t \lrcorner\, \omega$ $\forall t$. Then

The family $\{\xi_t\}_{t \in U}$ is smooth iff the family $\{\alpha_t\}_{t \in U}$ is smooth.

true $\forall$ nondeg 2-form (NOT using $d\omega = 0$)

This follows from the following (bi)linear algebra:

Let V be a (real) vector space of dimension $2n$.

$\Lambda^2 V^* := \{2\text{-covectors on } V\} = \left\{\begin{array}{l}\text{antisymmetric} \\ \text{bilinear} \quad V \times V \to \mathbb{R}\end{array}\right\}.$

$$\begin{aligned} \Lambda^2 V^* \times V &\longrightarrow V^* \\ (\omega_v \,,\, \xi) &\longmapsto \omega_v(\xi, \cdot) \end{aligned}$$

is smooth.

$$\left(\Lambda^2 V^*\right)_{\text{nondeg}} := \left\{ \omega_v \in \Lambda^2 V^* \;\middle|\; \begin{array}{l} \xi \mapsto \omega_v(\xi, \cdot) \\ V \to V^* \end{array} \begin{array}{c} \text{is a} \\ \text{linear} \\ \text{isomorphism} \end{array} \right\}$$

$$\begin{aligned} \left(\Lambda^2 V^*\right)_{\text{nondeg}} \times V^* &\longrightarrow V \\ (\omega_v \,,\, \alpha) &\longmapsto \xi \quad \text{s.t. } \omega_v(\xi, \cdot) = \alpha \end{aligned}$$

is also smooth.

In coordinates:

$$(\xi_1 \cdots \xi_{2n}) \begin{bmatrix} \Omega_{ij} \end{bmatrix} \begin{bmatrix} \bullet \\ \vdots \\ \bullet \end{bmatrix} = (\alpha_1, \cdots, \alpha_{2n}) \begin{bmatrix} \bullet \\ \vdots \\ \bullet \end{bmatrix}$$

Solve: $$\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2n} \end{pmatrix} = -\begin{bmatrix} \Omega_{ij} \end{bmatrix}^{-1} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{2n} \end{bmatrix}$$

is smooth by Cramer's rule.

Let (M, ω) be a sympl. mfld and $G \circlearrowright M$ a Lie group action.

The action is <u>symplectic</u> if $\forall a \in G$

$$a_M{}^* \omega = \omega \qquad \text{also write } a^*\omega = \omega$$

$$\left(\begin{array}{l} \text{where } \quad a \longmapsto (a_M : M \to M) \\ \qquad\qquad G \longrightarrow \mathrm{Diff}(M) \end{array} \right)$$

It follows that $\forall \xi \in \mathfrak{g} = \mathrm{Lie}(G)$

$$\mathcal{L}_{\xi_M} \omega = 0 .$$

Indeed, let $\varphi_t := \exp(t\xi)_M : M \longrightarrow M$

By assumption $\varphi_t^* \omega = \omega$.

$$\mathcal{L}_{\xi_M} \omega = \frac{d}{dt}\Big|_{t=0} \underbrace{\varphi_t^* \omega}_{\text{all } = \omega} = 0 .$$

Recall: $\mathcal{L}_{\xi_M} \omega = d\, \iota_{\xi_M} \omega + \iota_\xi \underbrace{d\omega}_{=0}$

$$= d\, \iota_{\xi_M} \omega$$

$\therefore \quad \iota_{\xi_M} \omega$ is a <u>closed</u> 1-form.

Recall:

$$\begin{array}{ccc}
\{\text{v. fields}\} & \xleftrightarrow[\quad]{\text{uses: } \omega \text{ is nondeg}} & \{\text{1-forms}\} \\
 & \xi \longleftrightarrow \iota_\xi \omega & \\
\cup & & \cup \\
\left\{\begin{array}{c}\text{symplectic} \\ \text{v. fields}\end{array}\right\} & \xleftrightarrow[\text{also uses: } \omega \text{ is closed}]{\quad} & \left\{\begin{array}{c}\text{closed} \\ \text{1-forms}\end{array}\right\} \\
\xi \text{ s.t. } \mathcal{L}_\xi \omega = 0 & &
\end{array}$$

Notations: $\xi \lrcorner \omega = \iota_\xi \omega = \iota(\xi)\omega$

Note. Let ξ be the velocity v. field of a flow $\{\varphi_t : M \to M\}_{t \in \mathbb{R}}$. Then ξ is a sympl. v. field iff the flow is symplectic.

shown: if $\varphi_t^* \omega = \omega \ \forall t$ then $\mathcal{L}_\xi \omega = 0$.

The other direction: assume $\mathcal{L}_\xi \omega = 0$.

Since when $t = 0$ $\varphi_0 = \mathrm{Id}$ so $\varphi_0^* \omega = \omega$, it's enough to show $\left.\frac{d}{dt}\right|_{t=t_0} \varphi_t^* \omega = 0$. $\forall t_0$

Indeed: $\left.\frac{d}{dt}\right|_{t=t_0} \varphi_t^* \omega = \left.\frac{d}{dt}\right|_{t=0} \varphi_{t+t_0}^* \omega$

$$= \left.\frac{d}{dt}\right|_{t=0} (\varphi_t \varphi_{t_0})^* \omega = \left.\frac{d}{dt}\right|_{t=0} \left(\varphi_{t_0}^* \varphi_t^* \omega\right)$$

$$= \varphi_{t_0}^* \Big(\underbrace{\left.\frac{d}{dt}\right|_{t=0} \varphi_t^* \omega}_{= \mathcal{L}_\xi \omega \underset{\text{assumption}}{=} 0} \Big) = 0 .$$

ξ is a symp. v. field

$:\Longleftrightarrow \quad \mathcal{L}_\xi \omega = 0 \quad \underset{\text{shown}}{\Longleftrightarrow} \quad \xi \lrcorner \omega$ is a closed 1-form.

ξ is a <u>Hamiltonian</u> v. field

$:\Longleftrightarrow \quad \xi \lrcorner \omega$ is an <u>exact</u> 1-form

$\Longleftrightarrow \quad \exists f \in C^\infty(M) := \{\text{smooth } M \to \mathbb{R}\}$ s.t.

$$\boxed{df = -\xi \lrcorner \omega}$$ Hamilton's equation.

<u>warning</u> the sign convention is not consistent in the literature.

In symplectic coordinates: $q_1, p_1, \ldots, q_n, p_n$

$$\omega = dq_1 \wedge dp_1 + \cdots + dq_n \wedge dp_n$$

$$\xi = \sum_j \dot{q}_j \frac{\partial}{\partial q_j} + \dot{p}_j \frac{\partial}{\partial p_j}$$

(e.g. if $\gamma(t) = (q_j(t), p_j(t))$ is a trajectory of the flow of ξ then $\forall t$ $\xi\big|_{\gamma(t)} = \sum \dot{q}_j(t) \frac{\partial}{\partial q_j} + \dot{p}_j(t) \frac{\partial}{\partial p_j}$

where $\dot{q}_j(t) = \frac{dq_j}{dt}$ and $\dot{p}_j(t) = \frac{dp_j}{dt}$)

$$\frac{\partial}{\partial q_j} \lrcorner\, \omega = dp_j \quad , \quad \frac{\partial}{\partial p_j} \lrcorner\, \omega = -dq_j$$

So $$\xi \lrcorner\, \omega = \sum_j \dot{q}_j \, dp_j - \dot{p}_j \, dq_j$$

$$df = \sum_j \frac{\partial f}{\partial q_j} dq_j + \frac{\partial f}{\partial p_j} dp_j$$

So $df = \xi \lrcorner\, \omega$ iff

$$\boxed{\dot{q}_j = \frac{\partial f}{\partial p_j} \quad \text{and} \quad \dot{p}_j = -\frac{\partial f}{\partial q_j}}$$ Hamilton's equations

physics: $f = \frac{1}{2} m v^2 + U(q) = \frac{1}{2m} p^2 + U(q)$ $\quad (p = mv)$

So $\frac{\partial f}{\partial p} = \frac{p}{m} = v = \dot{q}$

a <u>momentum map</u> for a sympl. group action $G \circlearrowright (M, \omega)$ is

a <u>G-equivariant</u> map

will explain

$$\mu : M \longrightarrow \mathfrak{g}^* = (\text{Lie}\, G)^*$$

whose coordinates $\mu^\xi := \langle \mu, \xi \rangle : M \longrightarrow \mathbb{R}$

$\forall \xi \in \mathfrak{g}$

satisfy Hamilton's equation

$$\boxed{d\mu^\xi = -\imath_{\xi_M} \omega} \, .$$

Note if μ exists then it is unique up to adding a $\underbrace{\text{constant}}_{G\text{-invariant}}$ in $\mathfrak{g}^*$

a <u>Hamiltonian G-action</u> is a triple (M, ω, μ)

where (M, ω) is a sympl. mfld equipped w/ a sympl. G-action

and $\mu : M \longrightarrow \mathfrak{g}^*$ is a momentum map.

after-class discussion

Suppose $\eta = f\xi$ for $f: M \to \mathbb{R}_{>0}$.

so the flows of η & ξ have the same orbits.

Suppose $\mathcal{L}_\xi \omega = 0$.

$$\mathcal{L}_\eta \omega = \mathcal{L}_{f\xi} \omega = d\, \iota_{f\xi} \omega + \iota_{f\xi} \underbrace{d\omega}_{0}$$

$$= d(f \iota_\xi \omega)$$

$$= \underbrace{df \wedge \iota_\xi \omega}_{?} + \underbrace{f\, d\iota_\xi \omega}_{=0 \text{ by assumption}}$$

So we could get $\mathcal{L}_\eta \omega \neq 0$.

eg. dim = 2. $\xi = \frac{\partial}{\partial x}$ on $\mathbb{R}^2_{x,y}$.

Rescale: